Determination of the derivative of the tangent stiffness matrix with respect to the load parameter

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In order to solve the so-called consistently linearized eigenproblem in the frame of the Finite Element Method (FEM), the derivative of the tangent stiffness matrix $\tilde{\mathbf{K}}_T$ with respect to the load parameter λ needs to be calculated. In this work, three schemes for calculation of $\dot{\tilde{\mathbf{K}}}_T$ are presented. The first scheme is based on an analytical expression for the first derivative of the element tangent stiffness matrix $\tilde{\mathbf{K}}_T^e$ with respect to λ for the special case of a co-rotational beam element. The second one is a finite difference approach for computation of $\dot{\tilde{\mathbf{K}}}_T := d\tilde{\mathbf{K}}_T/d\lambda$. The third one is also a finite difference approach. However, it is based on a directional derivative of $\tilde{\mathbf{K}}_T$. An elastic beam, subjected to a compressive axial force and a small transverse uniform load, is chosen as a numerical example. The effectiveness and the accuracy of the three schemes are compared. The third scheme is found to be not only *very practical* but also *more effective* than the two competing schemes.

1 Introduction

The so-called consistently linearized eigenproblem (CLE), originally proposed in [1], plays a pivotal role in a new concept of categorization of buckling of structures by means of spherical geometry [2]. Its mathematical formulation reads as

$$[\tilde{\mathbf{K}}_T + (\lambda_1^* - \lambda)\dot{\tilde{\mathbf{K}}}_T] \cdot \mathbf{v}_1^* = \mathbf{0} \tag{1}$$

where $\tilde{\mathbf{K}}_T$ denotes the tangent stiffness matrix of a structure, in the frame of the Finite Element Method (FEM), evaluated along the primary path;

$$\dot{\tilde{\mathbf{K}}}_T := \frac{d\tilde{\mathbf{K}}_T}{d\lambda}, \tag{2}$$

where λ stands for a dimensionless load factor, and $(\lambda_1^* - \lambda, \mathbf{v}_1^*)$ is the first eigenpair. To solve the CLE, $\dot{\tilde{\mathbf{K}}}_T$ needs to be calculated. The effectiveness of the calculation depends on the analysis method. In this work, three schemes for calculation of $\dot{\tilde{\mathbf{K}}}_T$ are presented. An elastic beam, subjected to an axial force and a small transverse uniform load, serves as the numerical basis for a comparison of the potential of these schemes.

2 Analytical expression for $\dot{\tilde{\mathbf{K}}}_T$, considering co-rotational beam elements

Concerning the first scheme, an analytical expression for the first derivative of the element tangent stiffness matrix $\tilde{\mathbf{K}}_T^e$ with respect to λ, denoted as $\dot{\tilde{\mathbf{K}}}_T^e$, is derived for the special case of a co-rotational beam element [3]. It is obtained as

$$\begin{aligned}\dot{\tilde{\mathbf{K}}}_T^e =& \dot{\mathbf{X}}^T\bar{\mathbf{K}}_T^e\mathbf{X} + \mathbf{X}^T\dot{\bar{\mathbf{K}}}_T^e\mathbf{X} + \mathbf{X}^T\bar{\mathbf{K}}_T^e\dot{\mathbf{X}} + \frac{(\dot{\mathbf{z}}\mathbf{z}^T + \mathbf{z}\dot{\mathbf{z}}^T)\hat{l} - \mathbf{z}\mathbf{z}^T\dot{\hat{l}}}{\hat{l}^2}\bar{N} + \frac{\mathbf{z}\mathbf{z}^T}{\hat{l}}\dot{\bar{N}} \\ &+ \frac{(\dot{\mathbf{r}}\mathbf{z}^T + \mathbf{r}\dot{\mathbf{z}}^T + \dot{\mathbf{z}}\mathbf{r}^T + \mathbf{z}\dot{\mathbf{r}}^T)\hat{l}^2 - 2(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)\hat{l}\dot{\hat{l}}}{\hat{l}^4}(\bar{M}_1 + \bar{M}_2) + \frac{\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T}{\hat{l}^2}(\dot{\bar{M}}_1 + \dot{\bar{M}}_2)\end{aligned} \tag{3}$$

where $\mathbf{X}$ is a matrix required for transformations from local to global coordinates; $\hat{l}$ denotes the length of the deformed beam; $\mathbf{r}$,$\mathbf{z}$ are vectors that represent abbreviations of lengthy expressions; $\bar{\mathbf{K}}_T^e$ is referred to the local coordinate system; $\bar{N}$, $\bar{M}_1$, $\bar{M}_2$ denote the components of the force vector in the local coordinate system. $\mathbf{X}$, $\hat{l}$, $\mathbf{r}$,$\mathbf{z}$ are purely geometrical relationships. The expressions for $\bar{\mathbf{K}}_T^e$, $\bar{N}$, $\bar{M}_1$, $\bar{M}_2$ depend on the chosen finite element. Herein, the axial displacement u is assumed to be linear, and the deflection w is taken as cubic. The element is based on the Euler-Bernoulli theory. After determination of $\dot{\tilde{\mathbf{K}}}_T^e$, the element matrices are assembled to the global matrix $\dot{\tilde{\mathbf{K}}}_T$.

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3 Numerical approximations of $\dot{\tilde{\mathbf{K}}}_T$

The second scheme is based on a one-sided two-point finite difference approach for computation of $\dot{\tilde{\mathbf{K}}}_T$ according to Eq.(2). It reads as

$$\dot{\tilde{\mathbf{K}}}_T \approx \frac{\tilde{\mathbf{K}}_T(\lambda+\epsilon) - \tilde{\mathbf{K}}_T(\lambda)}{\epsilon} \tag{4}$$

where ϵ describes a small increment of λ. The third scheme is also based on a one-sided two-point finite difference approach. However, the finite difference expression for $\dot{\tilde{\mathbf{K}}}_T$ is obtained from a directional derivative, defined as

$$\dot{\tilde{\mathbf{K}}}_T := \tilde{\mathbf{K}}_{T,\mathbf{u}} \cdot \dot{\mathbf{q}} = \left.\frac{d\tilde{\mathbf{K}}_T(\mathbf{q}+\epsilon\dot{\mathbf{q}})}{d\epsilon}\right|_{\epsilon=0}, \tag{5}$$

where $\mathbf{q} := \mathbf{q}(\lambda)$ is the vector of nodal displacements in the frame of the FEM. The finite difference approximation of the directional derivative is given as

$$\dot{\tilde{\mathbf{K}}}_T \approx \frac{\tilde{\mathbf{K}}_T(\mathbf{q}+\epsilon\dot{\mathbf{q}}) - \tilde{\mathbf{K}}_T(\mathbf{q})}{\epsilon} \tag{6}$$

where $\epsilon\dot{\mathbf{q}}$ denotes a small change of the displacement vector.

4 Numerical verification

The three schemes are coded in *FEMv2*, which is a nonlinear finite element program, developed by the first author. The beam (see Fig. 1a) is discretized by 100 2-node elements. $\dot{\tilde{\mathbf{K}}}_T$ is calculated by means of the three aforementioned schemes. The computer times required by the three schemes have been compared. Fig. 1b indicates that scheme 3 is faster than scheme 2. The reason for this is the *Newton-Raphson* iteration, needed in scheme 2 to obtain $\tilde{\mathbf{K}}_T(\lambda+\epsilon)$ (see Eq.(4)), but not in scheme 3. Although scheme 3 is slower than scheme 1, it represents an element-independent approach, and, hence, is more practical. Fig. 1c shows that the results obtained by scheme 3 are more accurate than those obtained by scheme 2. This is the consequence of evaluating $\tilde{\mathbf{K}}_T$ at $\mathbf{q}+\epsilon\dot{\mathbf{q}}$ (see Eq. (6)) which is "closer" to $\mathbf{q}$ than $\mathbf{q}(\lambda+\epsilon)$, for which $\tilde{\mathbf{K}}_T$ is evaluated in scheme 2. Hence, it can be concluded that scheme 3 is not only *very practical* but also *more effective* for calculation of $\dot{\tilde{\mathbf{K}}}_T$ in the frame of the FEM than the two competing schemes.

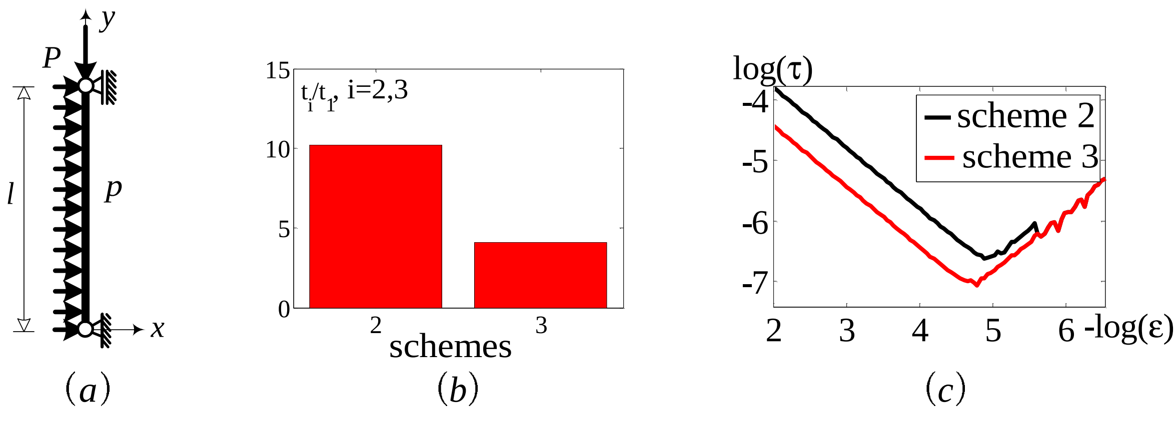

Fig. 1 Numerical example: (a) configuration of the beam subjected to an axial compressive force and a small transverse uniform load, (b) relative computer time of schemes 2 and 3, (c) accuracy of results obtained by means of schemes 2 and 3

References

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