Informatics

A General Framework for Choice Logics

DIPLOMARBEIT

zur Erlangung des akademischen Grades

Diplom-Ingenieur

im Rahmen des Studiums

Logic and Computation

eingereicht von

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Wien, 25. August 2020

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DIPLOMA THESIS

submitted in partial fulfillment of the requirements for the degree of

Diplom-Ingenieur

in

Logic and Computation

by

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Vienna, 25^{th} August, 2020

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Michael Bernreiter, BSc

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Wien, 25. August 2020

Michael Bernreiter

Acknowledgements

There are many people without whom I could not have written this thesis. First and foremost, I want to thank my supervisors Stefan Woltran and Jan Maly for their support, suggestions, and patience. I am grateful to all of my friends that accompanied me during my studies. It would have been no fun without them. Special thanks go to Matthias König for his feedback, and David Penz, who started to work on choice logics with me. Of course, I am also grateful to my family, especially my mother and my father. They have provided for me, were always there for me, and have accepted all of my decisions. Last but not least, I want to thank Yol, who I could always rely on.

Kurzfassung

Eine Vielzahl an Forschungsbereichen, darunter auch die der Informatik und der künstlichen Intelligenz, beschäftigen sich mit Präferenzen. Zwei bereits bestehende formale Systeme, welche sich mit Präferenzen auseinandersetzen, sind Qualitative Choice Logic (QCL) und Conjunctive Choice Logic (CCL). In beiden Logiken schreiben Interpretationen Formeln anstatt eines Wahrheitswertes eine natürliche Zahl, auch Satisfaction Degree genannt, zu. Dabei werden jene Interpretationen bevorzugt, die in einem geringstmöglichen Satisfaction Degree resultieren. In dieser Arbeit werden QCL und CCL durch die Einführung eines formalen Frameworks für Choice Logics generalisiert. Es wird gezeigt dass sowohl QCL, als auch CCL, Teil des Frameworks sind. Neue Choice Logics, welche auf neuen, nichtklassischen Konnektiven basieren, werden eingeführt. Da das Framework nicht sehr restriktiv definiert wird, und deswegen viele verschiedene Choice Logics mit unterschiedlichen Eigenschaften spezifiziert werden können, werden mehrere Klassen von Choice Logics eingeführt und untersucht. Ein Begriff der starken Äquivalenz zwischen den Formeln einer Choice Logic wird definiert, und anderen Äquivalenzbegriffen gegenübergestellt. Dabei wird bewiesen, dass der von uns eingeführte Begriff der starken Äquivalenz bezüglich QCL und CCL ident mit einem von Brewka et al. verwendeten Äquivalenzbegriff ist. Schlußendlich wird eine Komplexitätsanalyse durchgeführt. Dadurch ergeben sich auch neue Resultate für QCL und CCL, etwa dass das Hauptproblem bezüglich bevorzugten Modellen für beide Logiken $\Theta_2\mathsf{P}$-vollständig ist. Das selbe Problem ist für eine andere, im Zuge dieser Arbeit eingeführte, Logik $\Delta_2\mathsf{P}$-vollständig.

Abstract

The topic of preferences is of importance in many areas of research, including computer science, and, more specifically, artificial intelligence. Two formal systems in the literature that are designed for preference handling are Qualitative Choice Logic (QCL) and Conjunctive Choice Logic (CCL). Both of these logics extend classical propositional logic by a non-classical choice connective, with which preferences can be expressed. Instead of evaluating formulas to true or false, formulas in QCL and CCL are ascribed a satisfaction degree, by which interpretations are ranked. In this thesis, QCL and CCL are generalized by the formal introduction of a choice logic framework. Besides showing that QCL and CCL are captured by this framework, several new choice logics, based on new choice connectives, are introduced. Since the specified framework is not very restrictive, and therefore a multitude of different choice logics can be expressed, several classes of choice logics are defined and examined. A notion of strong equivalence between the formulas of a choice logic is introduced and related to other notions of equivalence. In the course of this analysis, it is proven that for QCL and CCL, our notion of strong equivalence is interchangeable with another notion of equivalence introduced by Brewka et al. in the original QCL paper. Lastly, the computational complexity of different reasoning tasks relevant for choice logics is examined. Although this complexity analysis is conducted in a general manner, it also yields new results regarding QCL and CCL. For example, the main decision problem regarding preferred models is $\Theta_2\mathsf{P}$-complete in both of these logics. For the same problem, we show $\Delta_2\mathsf{P}$-completeness for Lexicographic Choice Logic (LCL), a logic introduced in this thesis.

Contents

CHAPTER 1

Introduction

In a broad sense, this thesis is concerned with two main topics, namely logic, and preferences. The study of logic is relevant to many fields of research, including computer science [Gen12]. Two prominent logic formalisms are those of classical propositional logic and classical first-order logic. However, there exist many other logics, generally referred to as non-classical logics, which often extend (propositional or first-order) classical logic by additional functionalities.

The concept of preferences is a point of interest in many research areas such as economics, psychology, philosophy, but also computer science. For example, the fields of artificial intelligence and databases often deal with analyzing "human choice behavior", and are therefore concerned with preferences [PTV16].

There are many formalisms that are designed to handle preferences, including various non-classical logics. In this thesis, we will investigate and generalize two such logics, namely Qualitative Choice Logic (QCL) and Conjunctive Choice Logic (CCL) [BBB04, BB16].

1.1 A Brief History of Choice Logics

QCL was first described in 2004 by Brewka, Benferhat, and Le Berre [BBB04]. It extends classical propositional logic by an additional binary connective $\vec{\times}$, called ordered disjunction. Let A and B be propositional formulas. Then the intuitive meaning of $A \vec{\times} B$ is that if possible, A should be satisfied. If this is not possible, then it is still acceptable, but less preferable, to satisfy only B. Satisfying neither A nor B is not acceptable. In this way, ordered disjunction is based on classical disjunction, but additionally, it has the capability to express preferences. More specifically, the satisfaction relation of QCL ascribes a natural number (called satisfaction degree) to a formula, given some interpretation. We then prefer those interpretations that result in the smallest satisfaction degree for a given formula.

Alternative satisfaction relations for QCL formulas have been proposed in 2008 by Benferhat and Sedki in order to address issues with "negated and conditional preferences" [BS08b]. These alternative semantics do not alter the meaning ascribed to ordered disjunction ($\vec{\times}$), but rather change how the classical connectives of negation ($\neg$), conjunction ($\wedge$), and disjunction ($\vee$) operate with respect to satisfaction degrees.

In CCL, which was introduced in 2016 by Boudjelida and Benferhat [BB16], the ordered disjunction of QCL is replaced by another binary connective $\vec{\odot}$, called ordered conjunction. The intuitive meaning of $A\vec{\odot}B$ is that if possible, A and B should be satisfied. If this is not possible, then at least A should be satisfied. Like in QCL, the formulas of CCL are ascribed a satisfaction degree by interpretations. The semantics of the classical connectives ($\neg, \wedge, \vee$) in CCL are based on the original definition of QCL rather than on the alternative satisfaction relations introduced by Benferhat and Sedki.

One possible application for both QCL and CCL is to reason about preferences provided by users within a system, for example, a travel planner. When booking a journey, e.g. from Vienna to Amsterdam, a user could specify their preferred mode of transportation. One might wish to go by train in order to limit the environmental impact of their travel, but still prefer a flight to a bus journey for reasons of comfort. This could be expressed as a QCL formula as follows:

$$train \vec{\times} plane \vec{\times} bus.$$

Further constraints, such as the maximum travel time or price, can be encoded as classical formulas. The inference relation of QCL can then be used to automatically find the most preferable form of transport which is compatible with the additional constraints. For example, a train journey from Vienna to Amsterdam might be too expensive, and the best solution could be to take a plane.

In contrast to QCL, CCL is designed to encode preferences where, if possible, all options should be satisfied. For example, when deciding on which restaurant to go to with a group of people, different desires have to be taken into account. It would, of course, be best to go to a restaurant that satisfies the need of everyone in the group. But such an establishment might not exist. Therefore, one might want to encode the following preference list as a CCL formula:

$$vegetarian \vec{\odot} fish \vec{\odot} meat.$$

Again, further constraints, such as available restaurants, or which restaurant offers which kind of food, can be encoded as classical formulas.

In addition to reasoning about user preferences, it has been proposed that QCL can be employed when dealing with other practical problems such as alert correlation or database querying [BS08a, LHR14].

Of course, one could define logics similar to QCL and CCL, based on new non-classical connectives. In some cases, given a formula $A \circ B$, it might be sensible to prefer solutions satisfying A and B to those that satisfy only A, which in turn would be preferred to solutions satisfying only B. Note that this is neither the behavior of ordered disjunction nor ordered conjunction. Another reasonable connective would be one that prefers A over B but does not allow both A and B to be true at the same time. This might be useful when specifying preferences over items that are in conflict, for example time slots. Moreover, since the semantics of these non-classical connectives may also be based on satisfaction degrees, i.e. natural numbers, they can be implemented in several different ways. This leaves us with a myriad of non-classical connectives, and therefore a wide variety of choice logics, to be defined.

1.2 Problem Definition and Main Contributions

This thesis aims to generalize QCL and CCL by formally defining and examining a framework for logics in which preferences can be expressed by extending propositional logic with additional non-classical connectives. A logic of this framework will be referred to as a choice logic.

Both QCL and CCL, but also propositional logic itself, must be captured by this framework. It will also be possible to represent the alternative satisfaction relations for QCL introduced by Benferhat and Sedki. In general, the framework is intended to be as nonrestrictive as possible. This might mean that our framework will feature somewhat insensible choice logics, but it will also give us the freedom to define new choice logics, which will be equipped to deal with other forms of preference than those featured in QCL or CCL.

The framework's properties will be formally examined. This includes an investigation of the impact of non-occurring atoms in formulas, and the renaming of variables. Furthermore, several classes of choice logics will be defined and investigated.

An analysis of strong equivalence between choice logic formulas, in the sense of replaceability [FTW13a] with respect to preferred models, will be conducted. It will be shown that for some choice logics, including QCL and CCL, this notion of strong equivalence is identical to the notion of strong equivalence defined by Brewka et al. in the original QCL paper.

Lastly, the computational complexity of choice logics with polynomial-time computable inference relations will be examined by defining suitable decision problems and providing membership- and hardness proofs for these problems with respect to applicable complexity classes. This will be done by specifying algorithms and certificate relations to show membership, and by providing reductions from other decision problems, whose complexity is known, to show hardness. For example, we will prove that the complexity of the main decision problem regarding preferred models can vary, depending on the choice logic, from NP-complete (propositional logic) to $\Theta_2\mathsf{P}$-complete (QCL/CCL) to $\Delta_2\mathsf{P}$-complete (a newly introduced choice logic).

To our knowledge, neither QCL nor CCL has yet been analyzed in detail in terms of strong equivalence or computational complexity. Therefore, our results do not only apply to choice logics in general but also yield new insights regarding QCL and CCL. A lower bound for the complexity of QCL was given by Lang [Lan04]. There, it is stated that certain decision problems related to QCL are "at least as hard" as those associated with lexicographically minimal orderings, but possibly harder.

1.3 Structure

In Chapter 2, the formal preliminaries necessary for this thesis will be established. This includes a brief establishment of some notation, a formal introduction to classical propositional logic, QCL, and CCL, and a section on complexity theory, where the complexity classes needed in this thesis are introduced.

The definition of the choice logic framework is contained in Chapter 3. The syntax and semantics of choice logics are specified in Section 3.1, and some important properties of the framework are established in Section 3.2. Proofs for how QCL and CCL fit into our framework, and examples for new choice logics that can be defined within the framework are contained in Section 3.3. In Section 3.3.3, we explain how two choice logics can be combined to create a completely new choice logic.

Chapter 4 establishes various classes of choice logics and examines these classes with respect to certain properties. This includes a class of choice logics whose non-classical connectives are related in a natural way to classical binary connectives (Section 4.5).

In Chapter 5, the notion of strong equivalence between the formulas of a given choice logic will be considered and compared to other forms of equivalence. We will first do this for QCL and CCL (Sections 5.1 and 5.2), and then for choice logics in general (Section 5.3).

The computational complexity of choice logics is examined in Chapter 6. First, we introduce the idea of tractable choice logics, before defining relevant decision problems and investigating their complexity.

Lastly, in Chapter 7, we will summarize our findings, provide an overview of work related to choice logics, and discuss possible future work.

1.4 Published Work

Part of this work has been accepted for publication at the ASPOCP 2020 workshop under the title *Encoding Choice Logics in ASP* [BMW20]. In that paper, the choice logic framework is defined in more or less the same way as in this thesis. However, most of the results provided in this thesis, including classification and complexity analysis, are novel. In the paper, it is also shown how choice logics can be encoded using Answer Set Programming (ASP), which will not be part of this thesis.

CHAPTER 2

Preliminaries

This chapter aims to explain and formally define some concepts necessary for the remaining text. This includes Propositional Logic, Qualitative Choice Logic, Conjunctive Choice Logic, and some basic complexity theory.

Before doing so, we will establish some notation which might be misunderstood without clarification. In this text, zero will not be considered a natural number, i.e. $\mathbb{N} = \{1, 2, \ldots\}$. The power set of a set S is written as 2^S. We will also make use of the special symbol ∞.

Definition 1. *The symbol ∞ is given the meaning that $k < \infty$ for all $k \in \mathbb{N}$.*

The functions *max* and *min*, which return the maximum/minimum number from a given set, can be extended to use ∞: Let $S \subseteq 2^{\mathbb{N}} \cup \{\infty\}$. Then $max(S) = \infty$ if $\infty \in S$ and $min(S) = \infty$ if $S = \{\infty\}$. For the purpose of readability, we write $max(k_1, \ldots, k_n)$ for $max(\{k_1, \ldots, k_n\})$, and likewise for *min*.

2.1 Propositional Logic

Classical propositional logic (PL) is a tool to reason about propositions, i.e. statements that are either true or false [Hod13]. For example, *"The sky is blue"* is a proposition, and it depends on the reality we are reasoning about whether the statement is true (e.g. on Earth) or false (e.g. on Mars). Historically, PL has its roots as far back as ancient Greece, and was first described in a mathematical system in the 19th century by George Boole [Smu14]. In what follows, the syntax and semantics of PL will be defined. Several sources were used to this end, namely [Hod13], [Smu14], and [Gen12].

2.1.1 Syntax

Formally, elementary propositions are captured with propositional variables, also referred to as atoms. These variables can be joined together with the help of logical connectives to create formulas, which are more complex propositions.

Definition 2. *The set of all propositional variables, also called the universe, is denoted as* $\mathcal{U} = \{a_i \mid i \in \mathbb{N}\}$*.*

Note that, by the above definition, there are countably-infinitely many propositional variables. Lower case letters such as x, y, or z will be used to refer to arbitrary propositional variables. As for connectives, we will make use of $\neg$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction).

Definition 3. *The set of formulas of propositional logic* $\mathcal{F}_{PL}$ *is defined inductively as follows:*

1. *if* $x \in \mathcal{U}$*, then* $x \in \mathcal{F}_{PL}$*.*
2. *if* $F \in \mathcal{F}_{PL}$*, then* $(\neg F) \in \mathcal{F}_{PL}$*.*
3. *if* $F, G \in \mathcal{F}_{PL}$*, then* $(F \wedge G) \in \mathcal{F}_{PL}$ *and* $(F \vee G) \in \mathcal{F}_{PL}$*.*

For example, $((a_1 \vee a_2) \wedge (\neg a_3))$ is a formula of PL, while $(a_1 \neg \wedge a_2)$ is not. Upper case letters such as F, G, or H will be used to refer to arbitrary formulas. If parenthesis are omitted, we follow the convention that $\neg$ takes precedence over $\wedge$, and $\wedge$ takes precedence over $\vee$. This means that the formula $\neg F \wedge G \vee H$ is to be read as $(((\neg F) \wedge G) \vee H)$.

We will also be interested in which variables (and connectives) appear in a formula.

Definition 4. *The set of all propositional variables occurring in a formula* F *is denoted as* $var(F)$*.*

Consider, for instance, $F = (x \vee \neg y)$. Then $var(F) = \{x, y\}$. As for connectives, $\vee$ and $\neg$ occur in F, while $\wedge$ does not. Observe that any formula consists of only finitely many variables and connectives.

2.1.2 Semantics

The semantics of PL is based on interpretations, which assign a value of either *true* or *false* to propositional variables. For example, an interpretation could set a_1 to *true* and a_2 to *false*. Then, under this interpretation, the formula $(a_1 \vee a_2)$ would be *true* and $(a_1 \wedge a_2)$ would be *false*, since $(F \wedge G)$ is *true* exactly when F and G are *true*. As for negation and disjunction, we have that $\neg F$ is *true* if and only if F is *false*, and $(F \vee G)$ is *true* if and only if F or G (or both) are *true*.

One way to formally describe an interpretation is to define a corresponding function that takes propositional variables as input, and outputs either *true* or *false*. Another possibility is to characterize an interpretation by the set of exactly those propositional variables that are assigned a value of *true* by this interpretation. The latter approach is used by both [BBB04] and [BB16], and will be used by us as well.

Definition 5. *An interpretation $\mathcal{I}$ is a set of propositional variables, i.e. $\mathcal{I} \subseteq \mathcal{U}$.*

For example, $\{x, y\}$ is an interpretation that sets x and y to *true*, and all other propositional variables to *false*. Interpretations will most often be denoted with the letters $\mathcal{I}$ and $\mathcal{J}$. We can now define a function that ascribes a truth value to every formula, given some interpretation.

Definition 6. *The truth value of a PL formula under any interpretation $\mathcal{I}$ is given by the function $v\colon 2^{\mathcal{U}} \times \mathcal{F}_{PL} \to \{true, false\}$ such that*

1. $v(\mathcal{I}, x) = true$ *iff* $x \in \mathcal{I}$*, for every* $x \in \mathcal{U}$.
2. $v(\mathcal{I}, \neg F) = true$ *iff* $v(\mathcal{I}, F) = false$.
3. $v(\mathcal{I}, F \wedge G) = true$ *iff* $v(\mathcal{I}, F) = true$ *and* $v(\mathcal{I}, G) = true$.
4. $v(\mathcal{I}, F \vee G) = true$ *iff* $v(\mathcal{I}, F) = true$ *or* $v(\mathcal{I}, G) = true$.

Consider the formula $(x \wedge \neg y)$, and the two interpretations $\{x\}$ and $\{x, y\}$. Then $v(\{x\}, x) = v(\{x, y\}, x) = true$, while $v(\{x\}, \neg y) = true$ and $v(\{x, y\}, \neg y) = false$. Therefore, we have that $v(\{x\}, x \wedge \neg y) = true$ and $v(\{x, y\}, x \wedge \neg y) = false$.

We also write $\mathcal{I} \models F$ for $v(\mathcal{I}, F) = true$, and $\mathcal{I} \not\models F$ for $v(\mathcal{I}, F) = false$. If $\mathcal{I} \models F$ holds, we say that F is (classically) satisfied by $\mathcal{I}$, or that $\mathcal{I}$ is a model of F.

Another point of interest are so called logically equivalent formulas. These are formulas that behave in the same way under all interpretations. For example, $(x \wedge y)$ and $\neg(\neg x \vee \neg y)$ are not the same formula, but the have the same truth values, given any interpretation.

Definition 7. *PL-formulas F and G are logically equivalent, written as $F \equiv G$, if for all interpretations $\mathcal{I}$ we have that $v(\mathcal{I}, F) = v(\mathcal{I}, G)$.*

We defined the formulas of PL with the use of two binary connectives, namely $\wedge$ and $\vee$. Other binary connectives, such as $\to$ (implication), are not used here, but can be expressed with the use of $\neg$, $\wedge$, and $\vee$. Indeed, $\{\neg, \wedge, \vee\}$ is a functionally complete set of operators [Wer42]. In total, classical propositional logic has 16 different binary connectives, which represent the 16 possibilities for how the truth values of two formulas can be mapped to a single truth value. We call these 16 connectives the classical binary connectives.

Since choice logics introduce new non-classical binary connectives, and since we would like to compare these new connectives to the classical binary connectives, we listed all 16 classical binary connectives, and how they can be expressed via $\neg$, $\wedge$, and $\vee$, in Table 2.1. Note that the constant truth and falsity functions are often denoted by the symbols $\top$ and $\bot$ respectively.

Name	**Notation**	**Logically Equivalent to**
Conjunction, AND	$F \wedge G$	$F \wedge G$
Disjunction, OR	$F \vee G$	$F \vee G$
Implication, IF	$F \rightarrow G$	$\neg F \vee G$
Biconditional, IFF	$F \leftrightarrow G$	$(F \wedge G) \vee (\neg F \wedge \neg G)$
Joint Denial, NOR	$F \downarrow G$	$\neg(F \vee G)$
Alternative Denial, NAND	$F \uparrow G$	$\neg(F \wedge G)$
Exclusive Disjunction, XOR	$F \oplus G$	$(F \wedge \neg G) \vee (\neg F \wedge G)$
Negated Implication	$F \not\rightarrow G$	$(F \wedge \neg G)$
Inverse Implication	$F \leftarrow G$	$(F \vee \neg G)$
Negated Inverse Implication	$F \not\leftarrow G$	$(\neg F \wedge G)$
First Projection	$proj_1(F, G)$	F
Second Projection	$proj_2(F, G)$	G
Negated First Projection	$\overline{proj_1}(F, G)$	$\neg F$
Negated Second Projection	$\overline{proj_2}(F, G)$	$\neg G$
Constant Truth Function	$f_T(F, G)$	$(F \vee \neg F)$
Constant Falsity Function	$f_F(F, G)$	$(F \wedge \neg F)$

Table 2.1: The 16 classical binary connectives

Of course, all connectives can be written in prefix notation if required, e.g. $(F \rightarrow G)$ can be written as $(\rightarrow (F, G))$. Conversely, all connectives can also be written in infix notation. For convenience, we will sometimes use the classical binary connectives as shorthand notation for the equivalent PL-formula. For example, we might write $(x \rightarrow (y \oplus z))$ instead of $(\neg x \vee ((y \wedge \neg z) \vee (\neg y \wedge z)))$.

2.1.3 Properties

When investigating whether a formula is satisfied by some interpretation, it suffices to look at those variables in the interpretation that actually occur in the formula. For example, whether z is part of some interpretation $\mathcal{I}$ or not makes no difference when determining whether $\mathcal{I}$ satisfies the formula $(x \wedge \neg y)$.

Lemma 1. *Let $\mathcal{I}$ be an interpretation. If a propositional variable x does not occur in a PL-formula F, then $\mathcal{I} \models F$ iff $(\mathcal{I} \setminus \{x\}) \models F$.*

Proof. We prove the statement by structural induction.

- Base case: Let y be a propositional variable such that $x \neq y$. Thus,

$$\mathcal{I} \models y \iff y \in \mathcal{I} \iff y \in (\mathcal{I} \setminus \{x\}) \iff (\mathcal{I} \setminus \{x\}) \models y.$$

- Induction step: Assume x does not occur in F and G. Then, by the I.H., $\mathcal{I} \models F$ iff $(\mathcal{I} \setminus \{x\}) \models F$ and $\mathcal{I} \models G$ iff $(\mathcal{I} \setminus \{x\}) \models G$. It follows that

$$\mathcal{I} \models \neg F \iff \mathcal{I} \not\models F \iff (\mathcal{I} \setminus \{x\}) \not\models F \iff (\mathcal{I} \setminus \{x\}) \models \neg F,$$

$$\begin{aligned} \mathcal{I} \models F \wedge G &\iff \mathcal{I} \models F \text{ and } \mathcal{I} \models G \\ &\iff (\mathcal{I} \setminus \{x\}) \models F \text{ and } (\mathcal{I} \setminus \{x\}) \models G \\ &\iff (\mathcal{I} \setminus \{x\}) \models F \wedge G, \end{aligned}$$

$$\begin{aligned} \mathcal{I} \models F \vee G &\iff \mathcal{I} \models F \text{ or } \mathcal{I} \models G \\ &\iff (\mathcal{I} \setminus \{x\}) \models F \text{ or } (\mathcal{I} \setminus \{x\}) \models G \\ &\iff (\mathcal{I} \setminus \{x\}) \models F \vee G. \end{aligned}$$

□

Some properties that will be shown to be true in this thesis are concerned with the replacement (or substitution) of subformulas within formulas. For instance, when considering a formula F in which A occurs as a subformula, one might want to also examine a formula identical to F, except that A has been replaced by B.

Definition 8. *Let F, A, and B be formulas of PL. Then $F[A/B]_i$ is the formula obtained by replacing the i-th occurrence of A in F by B.*

For example, in the formula

$$F = ((x \vee y) \wedge (x \vee z)) \vee (x \wedge \neg x)$$

there are 4 occurrences of x. Replacing the second occurrence of x in F by x' yields

$$F[x/x']_2 = ((x \vee y) \wedge (x' \vee z)) \vee (x \wedge \neg x).$$

If F has less than i occurrences of A, then it is convenient to simply let $F[A/B]_i = F$. Since we will be interested in proving that certain properties hold even if *arbitrary* occurrences of subformulas are replaced, we will use the notation $F[A/B]$ to mean that $F[A/B]$ is a formula obtained from F by replacing an arbitrary occurrence of A in F by B. If F contains no occurrence of A, then $F[A/B] = F$.

An important fact regarding replacement in propositional logic is that if two formulas are logically equivalent, then substituting one for the other has no impact with respect to truth values.

Lemma 2. *Let A, B, and F be PL-formulas. If $A \equiv B$, then $F \equiv F[A/B]$.*

Proof. Assume $A \equiv B$. Note that if A is not contained in F, then trivially $F[A/B] = F$, and therefore $F \equiv F[A/B]$. The case that A does appear in F can be proven by structural induction. This induction is similar to the proof of Lemma 1, and will not be given explicitly here. □

2.2 Qualitative Choice Logic

Qualitative Choice Logic (QCL) extends PL by adding to it a new binary connective $\vec{\times}$, called ordered disjunction. As we will see, this enables us to express preferences about which propositional variables should be satisfied. The syntax and semantics of QCL were first defined by Brewka et al. [BBB04].

2.2.1 Syntax

QCL-formulas are built in the same way as PL-formulas, except that $\vec{\times}$ is part of the inductive definition.

Definition 9. *Let $\mathcal{F}_{PL}$ be the set of formulas of classical propositional logic, defined over the connectives $\wedge$, $\vee$, and $\neg$. Then the set of QCL-formulas $\mathcal{F}_{CCL}$ is defined inductively as follows:*

1. *if $F \in \mathcal{F}_{PL}$, then $F \in \mathcal{F}_{QCL}$.*
2. *if $F \in \mathcal{F}_{QCL}$, then $(\neg F) \in \mathcal{F}_{QCL}$.*
3. *if $F, G \in \mathcal{F}_{QCL}$, then $(F \circ G) \in \mathcal{F}_{QCL}$ for $\circ \in \{\wedge, \vee, \vec{\times}\}$.*

By the above definition, every PL-formula is also a QCL-formula. The non-classical formulas of QCL are those that contain an occurrence of ordered disjunction, e.g. $\neg(x \vec{\times} z)$ or $((x \vec{\times} y) \wedge z)$. In [BBB04, p. 206], it is stated that "all classical connectives have stronger bindings than $\vec{\times}$". This means that the formula $x \wedge y \vec{\times} z$ is to be read as $((x \wedge y) \vec{\times} z)$. We will follow this convention.

2.2.2 Semantics

The semantics of QCL is based on satisfaction degrees: Interpretations ascribe a natural number to formulas. The lower this number (satisfaction degree) is, the more preferable this interpretation is for that particular formula. Before we can define the inference relation $\mid\!\sim_k^{\mathrm{QCL}}$ for satisfaction degrees, we need to define the concept of optionality, which expresses the number of satisfaction degrees that a formula can possibly have.

Definition 10. *The optionality of a QCL-formula is defined as follows:*

1. $opt(x) = 1$, *for every* $x \in \mathcal{U}$.
2. $opt(\neg F) = 1$.
3. $opt(F \wedge G) = max(opt(F), opt(G))$.
4. $opt(F \vee G) = max(opt(F), opt(G))$.
5. $opt(F \overrightarrow{\times} G) = opt(F) + opt(G)$.

The optionality of a propositional variable is 1, since there is only a single way for *how* a variable can be satisfied. However, $(x \overrightarrow{\times} y)$ can be satisfied in two different ways: Either x is satisfied (preferable) or x is not satisfied, but y is satisfied (less preferable). Therefore, we have that $opt(x \overrightarrow{\times} y) = 2$. Now we can define the notion of satisfaction degrees formally.

Definition 11. *The satisfaction degree* $k \in \mathbb{N}$ *of a QCL-formula under any interpretation* $\mathcal{I}$ *is defined as follows:*

1. $\mathcal{I} \mid\!\sim_k^{QCL} x$ *iff* $k = 1$ *and* $x \in \mathcal{I}$, *for every* $x \in \mathcal{U}$.
2. $\mathcal{I} \mid\!\sim_k^{QCL} \neg F$ *iff* $k = 1$ *and for no* $m \in \mathbb{N}$: $\mathcal{I} \mid\!\sim_m^{QCL} F$.
3. $\mathcal{I} \mid\!\sim_k^{QCL} F \wedge G$ *iff* $\mathcal{I} \mid\!\sim_m^{QCL} F$ *and* $\mathcal{I} \mid\!\sim_n^{QCL} G$ *and* $k = max(m, n)$.
4. $\mathcal{I} \mid\!\sim_k^{QCL} F \vee G$ *iff* $[\mathcal{I} \mid\!\sim_m^{QCL} F$ *or* $\mathcal{I} \mid\!\sim_n^{QCL} G]$ *and* $k = min(\{r \mid \mathcal{I} \mid\!\sim_r^{QCL} F \text{ or } \mathcal{I} \mid\!\sim_r^{QCL} G\})$.
5. $\mathcal{I} \mid\!\sim_k^{QCL} F \overrightarrow{\times} G$ *iff* $\mathcal{I} \mid\!\sim_k^{QCL} F$ *or* $[\mathcal{I} \mid\!\sim_1^{QCL} \neg F$, $\mathcal{I} \mid\!\sim_m^{QCL} G$, *and* $k = m + opt(F)]$.

If there is a $k \in \mathbb{N}$ such that $\mathcal{I} \mid\!\sim_k^{\mathrm{QCL}} F$, we say that $\mathcal{I}$ satisfies F with a degree of k. Otherwise, we say that F is not satisfied under $\mathcal{I}$ (or that $\mathcal{I} \mid\!\sim_1^{\mathrm{QCL}} \neg F$). As an example, let us consider the formula $F = ((x \wedge y) \overrightarrow{\times} x)$. Intuitively, F expresses that it is preferable to satisfy both x and y. If this is not possible, then it is still acceptable to satisfy only x. More formally, we have that $\{x\} \mid\!\sim_1^{\mathrm{QCL}} x$ as well as $\{x, y\} \mid\!\sim_1^{\mathrm{QCL}} x$. However, while $\{x, y\} \mid\!\sim_1^{\mathrm{QCL}} (x \wedge y)$ holds, there is no $k \in \mathbb{N}$ such that $\{x\} \mid\!\sim_k^{\mathrm{QCL}} (x \wedge y)$. This means that $\{x, y\} \mid\!\sim_1^{\mathrm{QCL}} F$ and $\{x\} \mid\!\sim_2^{\mathrm{QCL}} F$. Also note that $\emptyset$ does not satisfy F.

QCL has the following important properties, which are also contained as Lemma 1 and 2 in [BBB04]:

Lemma 3. *If* $\mathcal{I} \mid\!\sim_k^{QCL} F$ *and* $\mathcal{I} \mid\!\sim_j^{QCL} F$, *then* $i = j$.

Lemma 4. *If* $\mathcal{I} \mid\!\sim_k^{QCL} F$, *then* $k \leq opt(F)$.

Furthermore, as we know from Proposition 2 in [BBB04], $\vec{\times}$ is associative, which means that for arbitrary F, G, and H, the formulas $((F\vec{\times}G)\vec{\times}H)$ and $(F\vec{\times}(G\vec{\times}H))$ have the same optionality and the same satisfaction degrees under all interpretations.

Lastly, we will discuss preferred models in the context of QCL. In the above example, we had that $\{x, y\} \mid\!\sim_1^{\mathrm{QCL}} F$ and $\{x\} \mid\!\sim_2^{\mathrm{QCL}} F$, which means that the interpretation $\{x, y\}$ is preferable to $\{x\}$ when considering the formula F. However, both $\{x, y\}$ and $\{x\}$ satisfy F (with some satisfaction degree), i.e. both $\{x, y\}$ and $\{x\}$ are models of F. Since the purpose of QCL is to express preferences, we are most often interested in those models that satisfy a formula with minimal satisfaction degree.

Definition 12. *Let $\mathcal{I}$ be an interpretation, and F be a QCL-formula. Then $\mathcal{I}$ is a preferred model of F if $\mathcal{I} \mid\!\sim_k^{QCL} F$ and for all other interpretations $\mathcal{J}$ we have $\mathcal{J} \mid\!\sim_m^{QCL} F$ with $k \leq m$.*

2.2.3 Alternative Definitions

There are two alternative definitions of QCL, namely Prioritized Qualitative Choice Logic (PQCL) and Positive Qualitative Choice Logic (QCL+). Both were described by Benferhat and Sedki in [BS08b] and introduce alternative semantics for the classical connectives $\neg$, $\wedge$, and $\vee$. The semantics of ordered disjunction remains unchanged in both PQCL and QCL+.

In PQCL, the optionalities of the classical connectives are defined as follows:

1. $opt(\neg F) = opt(F)$.
2. $opt(F \wedge G) = opt(F) \cdot opt(G)$.
3. $opt(F \vee G) = opt(F) \cdot opt(G)$.

The satisfaction degree regarding the classical connectives in PQCL is given as:

1. $\mathcal{I} \mid\!\sim_k^{\mathrm{PQCL}} \neg F$ iff one of the following holds:
   a) $F = x$ and $k = 1$ and $x \notin \mathcal{I}$,
   b) $F = (G \wedge H)$ and $\mathcal{I} \mid\!\sim_k^{\mathrm{PQCL}} (\neg G \vee \neg H)$,
   c) $F = (G \vee H)$ and $\mathcal{I} \mid\!\sim_k^{\mathrm{PQCL}} (\neg G \wedge \neg H)$,
   d) $F = (G\vec{\times}H)$ and $\mathcal{I} \mid\!\sim_k^{\mathrm{PQCL}} (\neg G\vec{\times}\neg H)$.
2. $\mathcal{I} \mid\!\sim_k^{\mathrm{PQCL}} F \wedge G$ iff $\mathcal{I} \mid\!\sim_i^{\mathrm{PQCL}} F$ and $\mathcal{I} \mid\!\sim_j^{\mathrm{PQCL}} G$ and $k = (i-1) \cdot opt(G) + j$.
3. $\mathcal{I} \mid\!\sim_k^{\mathrm{PQCL}} F \vee G$ iff one of the following holds:
   a) $\mathcal{I} \mid\!\sim_1^{\mathrm{PQCL}} F$ or $\mathcal{I} \mid\!\sim_1^{\mathrm{PQCL}} G$ and $k = 1$,

b) $(\exists i\colon i > 1$ and $\mathcal{I} \mathrel{|\!\!\sim}_i^{\text{PQCL}} F)$ and $(\nexists\, j\colon \mathcal{I} \mathrel{|\!\!\sim}_j^{\text{PQCL}} G)$ and $k = (i-1) \cdot \mathit{opt}(G)+1$,

c) $((\exists i\colon i > 1$ and $\mathcal{I} \mathrel{|\!\!\sim}_i^{\text{PQCL}} F)$ or $(\nexists\, l\colon \mathcal{I} \mathrel{|\!\!\sim}_l^{\text{PQCL}} F))$ and $(\exists j\colon \mathcal{I} \mathrel{|\!\!\sim}_j^{\text{PQCL}} G)$ and $k = j$.

The idea behind PQCL is to deal with prioritized preferences via the classical connectives. For example, the intuitive meaning of $F \wedge G$ in PQCL is that while both F and G must be satisfied, if F and G can not both be satisfied to a minimal degree, then it is preferable to satisfy F with a lower degree rather than G. This has as a consequence that, in contrast to QCL, neither conjunction nor disjunction are commutative in PQCL. Let $\mathcal{I}$ be an interpretation and let F and G be formulas s.t. $\mathcal{I} \mathrel{|\!\!\sim}_2^{\text{PQCL}} F$, $\mathcal{I} \mathrel{|\!\!\sim}_3^{\text{PQCL}} G$, and $\mathit{opt}(F) = \mathit{opt}(G) = 3$. Then $\mathcal{I} \mathrel{|\!\!\sim}_6^{\text{PQCL}} (F \wedge G)$, but $\mathcal{I} \mathrel{|\!\!\sim}_8^{\text{PQCL}} (G \wedge F)$. Similarly, $\mathcal{I} \mathrel{|\!\!\sim}_3^{\text{PQCL}} (F \vee G)$, but $\mathcal{I} \mathrel{|\!\!\sim}_2^{\text{PQCL}} (G \vee F)$.

Regarding QCL+, the satisfaction relation for optionalities and satisfaction degrees of the classical connectives is not given explicitly in [BS08b], but the semantics are provided via a normal form transformation to so called basic QCL-formulas. Basic choice formulas are a concept that we will introduce for our choice logic framework in Section 3.1.1 (see Definition 27), and investigate further in Section 4.2. From the normal form transformation mentioned above we can infer that negation in QCL+ is defined in the same way as in PQCL, and that disjunction is defined in the same way as in QCL. The optionality of conjunction in QCL+ is given by

$$\mathit{opt}(F \wedge G) = \mathit{max}(\mathit{opt}(F), \mathit{opt}(G)),$$

and for the satisfaction degree we have that

$$\begin{aligned} \mathcal{I} \mathrel{|\!\!\sim}_k^{\text{QCL+}} (F \wedge G) \text{ iff } & (\mathcal{I} \mathrel{|\!\!\sim}_k^{\text{QCL+}} F \text{ and } \mathcal{I} \mathrel{|\!\!\sim}_k^{\text{QCL+}} G) \text{ or} \\ & (\mathcal{I} \mathrel{|\!\!\sim}_k^{\text{QCL+}} F \text{ and } k > \mathit{opt}(G)) \text{ or} \\ & (\mathcal{I} \mathrel{|\!\!\sim}_k^{\text{QCL+}} G \text{ and } k > \mathit{opt}(F)). \end{aligned}$$

QCL+ is designed to work with positive preferences, i.e. preferences in which solutions are simply ranked, but not excluded.

In this thesis, we will focus on the original semantics of QCL. In Section 3.1.2 we discuss why we prefer the semantics of QCL for the classical connectives rather than the alternative semantics of PQCL or QCL+, and in Section 3.3 we explain how PQCL and QCL+ can still fit into the general framework of choice logics that we define in Section 3.1.

2.3 Conjunctive Choice Logic

Conjunctive Choice Logic (CCL) was described by Boudjelida and Benferhat in [BB16]. Similarly to QCL, CCL extends PL by adding a new binary connective $\vec{\odot}$, called ordered conjunction.

2.3.1 Syntax

The formulas of CCL are defined analogously to those of QCL.

Definition 13. *Let $\mathcal{F}_{PL}$ be the set of formulas of classical propositional logic, defined over the connectives $\wedge$, $\vee$, and $\neg$. Then the set of CCL-formulas $\mathcal{F}_{CCL}$ is defined inductively as follows:*

1. *if $F \in \mathcal{F}_{PL}$, then $F \in \mathcal{F}_{CCL}$.*
2. *if $F \in \mathcal{F}_{CCL}$, then $(\neg F) \in \mathcal{F}_{CCL}$.*
3. *if $F, G \in \mathcal{F}_{CCL}$, then $(F \circ G) \in \mathcal{F}_{CCL}$ for $\circ \in \{\wedge, \vee, \vec{\odot}\}$.*

For example, $((x \vec{\odot} y) \wedge z)$ is a CCL-formula. We will follow the convention that, when omitting brackets, the classical connectives bind more strongly than $\vec{\odot}$, analogously to QCL.

2.3.2 Semantics

The semantics of CCL is based on that of QCL. Optionality was renamed to *npsd*, which simply stands for "number of possible satisfaction degrees".

Definition 14. *The optionality (or npsd) of a CCL-formula is defined as follows:*

1. *$npsd(x) = 1$, for every $x \in \mathcal{U}$.*
2. *$npsd(\neg F) = 1$.*
3. *$npsd(F \wedge G) = max(npsd(F), npsd(G))$.*
4. *$npsd(F \vee G) = max(npsd(F), npsd(G))$.*
5. *$npsd(F \vec{\odot} G) = npsd(F) + npsd(G)$.*

As one can see, *npsd* in CCL is defined analogously to how *opt* is defined in QCL. As for satisfaction degree, CCL uses the symbol ∞ when a formula is not satisfied to some (finite) satisfaction degree. Otherwise, the inference relation for the classical connectives remains unchanged between CCL and QCL. Of course, the semantics of $\vec{\odot}$ differs to that of $\vec{\times}$.

Definition 15. *The satisfaction degree k of a CCL-formula under any interpretation $\mathcal{I}$ is defined as follows:*

1. *$\mathcal{I} \mid\!\sim_k^{CCL} x$, for every $x \in \mathcal{U}$, with*

$$k = \begin{cases} 1 & \text{if } x \in \mathcal{I} \\ \infty & \text{otherwise} \end{cases}$$

2. $\mathcal{I} \mid\!\sim_k^{CCL} \neg F$ *with*

$$k = \begin{cases} 1 & \text{if } \mathcal{I} \mid\!\sim_\infty^{CCL} F \\ \infty & \text{otherwise} \end{cases}$$

3. $\mathcal{I} \mid\!\sim_k^{CCL} F \wedge G$ *with*

$$k = \begin{cases} max(m,n) & \text{if } \mathcal{I} \mid\!\sim_m^{CCL} F \text{ and } \mathcal{I} \mid\!\sim_n^{CCL} G \\ & \text{with } m \neq \infty \text{ and } n \neq \infty. \\ \infty & \text{otherwise} \end{cases}$$

4. $\mathcal{I} \mid\!\sim_k^{CCL} F \vee G$ *with*

$$k = \begin{cases} min(m,n) & \text{if } \mathcal{I} \mid\!\sim_m^{CCL} F \text{ and } \mathcal{I} \mid\!\sim_n^{CCL} G \\ & \text{with } m \neq \infty \text{ or } n \neq \infty. \\ \infty & \text{otherwise} \end{cases}$$

5. $\mathcal{I} \mid\!\sim_k^{CCL} F \vec{\odot} G$ *with*

$$k = \begin{cases} m + npsd(G) & \text{if } \mathcal{I} \mid\!\sim_m^{CCL} F,\ m \neq 1, \text{ and } m \neq \infty \\ m & \text{if } \mathcal{I} \mid\!\sim_1^{CCL} F \text{ and } \mathcal{I} \mid\!\sim_m^{CCL} G \\ \infty & \text{otherwise} \end{cases}$$

We say that $\mathcal{I}$ is a model of F if we have that $\mathcal{I} \mid\!\sim_k^{\mathrm{CCL}} F$ with $k < \infty$. Preferred models in CCL are defined analogously to preferred models in QCL.

Definition 16. *Let $\mathcal{I}$ be an interpretation, and F be a CCL-formula. Then $\mathcal{I}$ is a preferred model of F if $\mathcal{I} \mid\!\sim_k^{CCL} F$ with $k < \infty$ and for all other interpretations $\mathcal{J}$ we have $\mathcal{J} \mid\!\sim_m^{CCL} F$ with $k \leq m$.*

It has to be mentioned that the above semantics does not capture the intended meaning of ordered conjunction. In [BB16], an example is given that the interpretation $\{x\}$ should ascribe a satisfaction degree of 2 to $(x \vec{\odot} y)$. However, with the semantics given in Definition 15, we have that $\{x\} \mid\!\sim_\infty^{\mathrm{CCL}} (x \vec{\odot} y)$. Therefore, we will describe an alternative semantics for CCL in Section 3.3.

2.4 Complexity Theory

In complexity theory, computational problems are categorized depending on how efficiently they can be solved. In this introduction we will briefly define the basic complexity classes P, NP, and coNP. We will then introduce the notion of oracles and the complexity classes Δ_2P and Θ_2P. The primary sources used in this section are [Pap94], [AB09] and [Sip12].

We assume familiarity with the concepts of deterministic Turing machines (TMs), decision problems, $\mathcal{O}$-notation, and reductions between decision problems. An example for a decision problem concerned with propositional logic is ModelChecking:

ModelChecking
Instance: A PL-formula F and an interpretation $\mathcal{I}$.
Question: $\mathcal{I} \models F$?

We say that $(F, \mathcal{I})$ is an instance of ModelChecking. More specifically, $(F, \mathcal{I})$ is called a yes-instance of ModelChecking if the question part of the problem can be answered with yes, i.e. if $\mathcal{I} \models F$. Otherwise, $(F, \mathcal{I})$ is called a no-instance. The efficiency of an algorithm is measured relative to input size, that is, to the size of the instance.

2.4.1 Basic Complexity Classes

The class P consists of exactly those decision problems that can be decided in polynomial time (with respect to input size) by a TM.

Definition 17. *A decision problem Q is in P if there is a deterministic Turing machine M and a constant $c \in \mathbb{N}$ such that, given an arbitrary instance I of Q, M decides whether I is a yes-instance of Q in $\mathcal{O}(n^c)$ time, where n is the length of the input string that encodes I.*

We have already seen a decision problem that is in P, namely ModelChecking. Before we prove this, let us fix some notation: We will use $|F|$ to denote the total number of variable occurrences in a propositional formula F. For example, if $G = (x \vee (\neg x \wedge y))$, then $|G| = 3$, since there are 2 occurrences of x and 1 occurrence of y in G. Regardless of the method by which formulas are encoded, $|F|$ will always be smaller than the size of F's encoding on the tape of a TM. Thus, an algorithm that runs in polynomial time with respect to $|F|$ also runs in polynomial time with respect to the input of ModelChecking. Conversely, there are encodings such that the size of F on the tape of a TM is polynomial in $|F|$. Since we can also assume that $\mathcal{I} \subseteq var(F)$ for any interpretation $\mathcal{I}$ we are dealing with (see Lemma 1), we can safely use $|F|$ as the input size of ModelChecking (or similar problems).

The strong form of the Church-Turing thesis states that "every physically realizable computation model can be simulated by a TM with polynomial overhead" [AB09, p. 26]. This statement is not uncontroversial, especially because of quantum computers. But since we do not deal with quantum algorithms here, we can resort to informal descriptions of algorithms instead of specifying Turing machines.

Proposition 5. ModelChecking *is in* P.

Proof. The truth value of F under $\mathcal{I}$ can be computed by applying v to $\mathcal{I}$ and F (see Definition 6). Clearly, every step in this recursion runs in polynomial time. The depth of the recursion does not exceed $|F|$. Neither does the width of the recursion exceed $|F|$, since every atom in F will be reached exactly once in the recursion. If $v(\mathcal{I}, F) = \mathit{true}$ we return "yes", and if $v(\mathcal{I}, F) = \mathit{false}$ we return "no". This shows that ModelChecking is in P. $\square$

Another important complexity class is NP, which is sometimes characterized by non-deterministic Turing machines. However, we will define it over so called certificate relations. Often, a certificate (or witness) is the corresponding solution to a yes-instance of the decision problem in question. For example, consider the problem of satisfiability:

Sat
Instance: A PL-formula F.
Question: Is there an interpretation $\mathcal{I}$ such that $\mathcal{I} \models F$?

Then $F = (x \lor y) \land (\neg x \lor z)$ is a yes-instance of Sat, and $\{x, z\}$ is a certificate for F. Before we can define NP using certificates, we have to introduce the following two concepts regarding binary relations:

Definition 18. *A binary relation R is polynomially decidable if there is an algorithm that decides whether $(x, y) \in R$ for any x and y in polynomial time.*

Definition 19. *A binary relation R is polynomially balanced if there is a constant c such that for every $(x, y) \in R$ we have that $|y| \leq |x|^c$.*

Now we can formally define the class NP.

Definition 20. *A decision problem Q is in NP if there exists a polynomially balanced and polynomially decidable relation R such that I is a yes-instance of Q if and only if there is a certificate C such that $(I, C) \in R$.*

With this definition, it is easy to show that Sat is in NP. Let

$$R = \{(F, \mathcal{I}) \mid \mathcal{I} \models F\}.$$

Clearly, an instance F is a yes-instance of Sat if and only if there is a certificate, i.e. an interpretation $\mathcal{I}$, such that $(F, \mathcal{I}) \in R$. R is polynomially balanced, since we can assume that $\mathcal{I} \subseteq \mathit{var}(F)$. Furthermore, R is polynomially decidable, since ModelChecking is in P. Thus, ModelChecking is in NP. Besides NP-membership, we are often also interested in NP-hardness:

Definition 21. *A decision problem Q is NP-hard if there exists a polynomial-time reduction from every decision problem in NP to Q.*

By the Cook-Levin Theorem, we know that SAT is NP-hard. Since we have both NP-membership and NP-hardness, SAT is NP-complete. Of course, the concepts of hardness and completeness can be extended to complexity classes other than NP.

The class coNP consists of the complimentary problems of the decision problems in NP. For example, the complement of SAT asks whether a given formula F is unsatisfiable.

UNSAT
Instance: A PL-formula F.
Question: Does $\mathcal{I} \not\models F$ hold for all interpretations $\mathcal{I}$?

Since UNSAT is the complementary problem of SAT, it is in coNP. Furthermore, we know that the problem TAUTOLOGY, which asks whether a given formula is satisfied by all interpretations, is coNP-complete. Since $\mathcal{I} \models F \iff \mathcal{I} \not\models \neg F$, it is easy to see that TAUTOLOGY can be reduced to UNSAT, and that therefore UNSAT is coNP-complete as well.

2.4.2 Oracle Complexity Classes

The classes P, NP and coNP can be generalized by the polynomial hierarchy, which is built out of infinitely many complexity classes. In this thesis, we only need two classes of the polynomial hierarchy other than P, NP and coNP, namely $\Delta_2\mathsf{P}$ and $\Theta_2\mathsf{P}$. To define these two classes, we first have to introduce the concept of oracles.

Definition 22. *A Q-oracle is a machine that is given an instance I of a decision problem Q as input, and decides whether I is a yes-instance of Q or not in constant time.*

If, for example, we could physically realize a SAT-oracle, then we could use it to solve SAT in constant time, which would also mean that SAT is in P. Since SAT is NP-complete, this in turn would imply that $\mathsf{P} = \mathsf{NP}$. Whether we can physically realize oracles or not, we can use them to define new complexity classes, such as $\mathsf{P}^{\textsc{Sat}}$:

Definition 23. *A decision problem Q is in $\mathsf{P}^{\textsc{Sat}}$ if it can be decided in polynomial time by an algorithm which is allowed an arbitrary number of calls to a SAT-oracle.*

In fact, because SAT is NP-hard, we could take any decision problem Q in NP, reduce it to SAT in polynomial time, and then use a SAT-oracle to solve Q in constant time. Thus, a SAT-oracle can function as an oracle for any problem in NP with only polynomial overhead. We can therefore also see $\mathsf{P}^{\textsc{Sat}}$ as consisting of exactly those decision problems that can be solved in polynomial time by an algorithm which is allowed an arbitrary number of calls to any NP-oracle, i.e. any Q-oracle where Q is a decision problem in NP. Consequently, $\mathsf{P}^{\textsc{Sat}}$ is often referred to as P^{NP}. Another name commonly used for $\mathsf{P}^{\textsc{Sat}}$ is that of $\Delta_2\mathsf{P}$, which we will use from now on.

An example for a $\Delta_2\mathsf{P}$-complete problem is that of LexMaxSat, which is called a prototypical $\Delta_2\mathsf{P}$-problem by Creignou, Pichler, and Woltran [CPW18].

> LexMaxSat
> *Instance:* A PL-formula F and an order $x_1 > \cdots > x_n$ on the variables in F.
> *Question:* Is x_n true in the lexicographically largest model of F?

We say that an interpretation $\mathcal{I}$ is lexicographically larger than another interpretation $\mathcal{J}$ with respect to the ordering $x_1 > \cdots > x_n$ if there is an index $k \in \{1, \ldots, n\}$ such that $x_i \in \mathcal{I} \iff x_i \in \mathcal{J}$ for all $i < k$, and $x_k \in \mathcal{I}$ but $x_k \notin \mathcal{J}$. For example, with respect to the ordering $x_1 > x_2 > x_3$, the lexicographically largest interpretation is $\{x_1, x_2, x_3\}$, the second largest is $\{x_1, x_2\}$, the third largest is $\{x_1, x_3\}$, and so on. Observe that $\emptyset$ is the lexicographically smallest interpretation with respect to any ordering.

Another complexity class closely related to $\Delta_2\mathsf{P}$ is that of $\Theta_2\mathsf{P}$. Instead of allowing an arbitrary number of calls to NP oracles, we restrict ourselves to only logarithmically many oracle calls.

Definition 24. *A decision problem Q is in $\Theta_2\mathsf{P}$ if it can be decided in polynomial time by an algorithm which is allowed $\mathcal{O}(\log n)$ number of calls to an NP-oracle, where n is the size of the input.*

$\Theta_2\mathsf{P}$ is sometimes also referred to as $\Delta_2\mathsf{P}[\log n]$ or $\mathsf{P}^{\mathsf{NP}[\log n]}$. An example for a $\Theta_2\mathsf{P}$-complete problem, which also appears in [CPW18], is that of LogLexMaxSat.

> LogLexMaxSat
> *Instance:* A PL-formula F and an order $x_1 > \cdots > x_n$ on some of the variables in F such that $n \leq \log(|F|)$.
> *Question:* Is x_n true in the lexicographically largest interpretation $\mathcal{J} \subseteq \{x_1, \ldots, x_n\}$ that can be extended to a model of F?

We say that an interpretation $\mathcal{J} \subseteq \{x_1, \ldots, x_n\}$ can be extended to a model of F if there is an interpretations $\mathcal{I}$ which is a model of F, and which behaves in the same way as $\mathcal{J}$ regarding the variables $\{x_1, \ldots, x_n\}$, i.e. $\mathcal{I} \cap \{x_1, \ldots, x_n\} = \mathcal{J}$.

The classes $\Delta_2\mathsf{P}$ and $\Theta_2\mathsf{P}$ have a connection to optimization problems, as was pointed out by Krentel [Kre88]. Indeed, if LexMaxSat could be solved in polynomial time, then one could also find the lexicographically largest model $\mathcal{I}$ of F in polynomial time as follows:

- Let $\mathcal{I} = \emptyset$.
- For each $x \in var(F)$, do the following:

- If x is contained in the lexicographically largest model of F, then let $\mathcal{I} = \mathcal{I} \cup \{x\}$ and modify F by replacing every occurrence of x by a tautology.
- Else, modify F by replacing every occurrence of x by a contradiction.

If we could solve LexMaxSat in polynomial time, then clearly this procedure would also run in polynomial time. Furthermore, by replacing a variable x with a tautology (or contradiction), we limit the search space to interpretations in which x is contained (or not contained). In this way, we can find the lexicographically largest model of F. LexMaxSat, and analogously LogLexMaxSat, can therefore be viewed as decision problems which capture the computational complexity of an optimization problem.

The complexity classes that we have encountered so far have a hierarchical relationship to each other. For example, P is a subset of NP, coNP, $\Delta_2\mathsf{P}$ and $\Theta_2\mathsf{P}$ respectively, which means that every problem that is contained in P is also contained in NP, coNP, $\Delta_2\mathsf{P}$ and $\Theta_2\mathsf{P}$. Whether P is a proper subset of any of these classes is not known, however. Similarly, NP and coNP are both subsets of $\Delta_2\mathsf{P}$ and $\Theta_2\mathsf{P}$, but it is unclear if this inclusion is strict or not. It is also easy to see that $\Theta_2\mathsf{P}$ is contained in $\Delta_2\mathsf{P}$, since a problem than can be solved in polynomial time with a logarithmic number of oracle calls can also be solved in polynomial time with an arbitrary number of oracle calls.

CHAPTER 3

Choice Logic Framework

In this chapter, we will formally define a framework for choice logics (CLs) and investigate some properties of the logics belonging to this framework. Additionally, we will show how QCL and CCL can be expressed in our framework, give examples for new choice logics, and show how two choice logics can be combined into a new choice logic.

3.1 Basic Concepts

When formally defining the framework of choice logics, there are some things that we would like to ensure: Most importantly, both QCL and CCL must be choice logics. Secondly, PL should be a CL as well. Furthermore, defining the semantics of a given CL ought to be, in a broad sense, similar to defining the semantic of QCL or CCL. More specifically, the notions of optionality and satisfaction degree will be present in every CL. In addition to capturing QCL, we also want to be able to represent PQCL and QCL+. Lastly, it should be possible to define new choice logics, based on different non-classical connectives.

3.1.1 Syntax

The formulas of a CL can be built with two types of connectives: Classical connectives (here we use $\neg$, $\wedge$, and $\vee$), and binary choice connectives with which a certain preference can be expressed.

Definition 25. *The set of choice connectives $\mathcal{C}_{\mathcal{L}}$ of a choice logic $\mathcal{L}$ is a finite set of binary connectives such that $\mathcal{C}_{\mathcal{L}} \cap \{\neg, \wedge, \vee\} = \emptyset$.*

For example, $\mathcal{C}_{\mathrm{QCL}} = \{\vec{\times}\}$, and $\mathcal{C}_{\mathrm{CCL}} = \{\vec{\odot}\}$. Note that the above definition does allow for CLs with multiple choice connectives.

Definition 26. *Let $\mathcal{F}_{PL}$ be the set of formulas of classical propositional logic, defined over the connectives $\wedge$, $\vee$, and $\neg$. Then the set $\mathcal{F}_{\mathcal{L}}$ of formulas of a choice logic $\mathcal{L}$ is defined inductively as follows:*

1. *if $F \in \mathcal{F}_{PL}$, then $F \in \mathcal{F}_{\mathcal{L}}$.*
2. *if $F \in \mathcal{F}_{\mathcal{L}}$, then $(\neg F) \in \mathcal{F}_{\mathcal{L}}$.*
3. *if $F, G \in \mathcal{F}_{\mathcal{L}}$, then $(F \circ G) \in \mathcal{F}_{\mathcal{L}}$ for $\circ \in \{\wedge, \vee\} \cup \mathcal{C}_{\mathcal{L}}$.*

If a formula F of some choice logic contains a choice connective, then we call it a choice formula. Otherwise, F is simply a PL-formula, and we refer to it as a classical formula. QCL and CCL both feature the notion of so called basic choice formulas. These are formulas in which classical connectives are applied only to classical formulas. If $F_1, \dots, F_n$ are PL-formulas, then $(F_1 \vec{\times} \cdots \vec{\times} F_n)$ is a basic QCL-formula. However, $((F \vec{\times} G) \wedge H)$ is not a basic QCL-formula, since $\wedge$ is applied to the non-classical formula $(F \vec{\times} G)$. Analogously for CCL. The concept of basic choice formulas can be defined for arbitrary choice logics.

Definition 27. *Let $\mathcal{L}$ be a choice logic. Then the set of basic $\mathcal{L}$-formulas is defined inductively as follows:*

1. *if F is a PL-formula, then F is a basic $\mathcal{L}$-formula.*
2. *if F and G are basic $\mathcal{L}$-formulas, then $(F \circ G)$ is a basic $\mathcal{L}$-formula, where $\circ \in \mathcal{C}_{\mathcal{L}}$.*

It is evident that for any choice logic $\mathcal{L}$, the set of basic $\mathcal{L}$-formulas is a proper subset of the set of all $\mathcal{L}$-formulas.

For the replacement of subformulas, we use the same notation as in PL (see Definition 8). Consequently, $F[A/B]_i$ stands for replacing the i-th occurrence of A in F by B, and $F[A/B]$ denotes replacing an arbitrary occurrence of A in F by B.

3.1.2 Semantics

First, we will discuss the semantics for the classical connectives. In order to easily capture both QCL and CCL within our framework, we could simply use the semantics of the classical connectives provided in both QCL and CCL (see Sections 2.2.2 and 2.3.2). Another possibility is to use one of the alternative semantics of QCL, i.e. PQCL or QCL+, for the classical connectives (see Section 2.2.3).

However, in PQCL, neither disjunction nor conjunction is commutative. This behavior is not satisfactory, as we want $\wedge$ and $\vee$ to function as similarly as possible to classical propositional logic in order to make specifications easy and intuitive. In fact, disjunction and conjunction in PQCL operate more like choice connectives, as they express a preference over which operand should be satisfied to a lower degree. The semantics for conjunction in QCL+ is problematic as well, as it is designed to deal with positive preferences only.

Negation in both PQCL and QCL+ is defined to distribute over non-atomic formulas. But the same effect can be achieved in QCL by restricting ourselves to formulas where $\neg$ occurs only in front of propositional variables. Indeed, there is an argument for not applying negation to choice formulas, since a sentence of the form "I do not prefer A over B" does not express a preference, but rather the absence of a preference. In some way, the negation featured in standard QCL can be seen as classical negation when applied to a classical formula, and as a tool to neutralize satisfaction degrees when applied to a choice formula.

For the above reasons, we will use the semantics of the classical connectives featured in QCL and CCL. A discussion about how PQCL and QCL+ fit into our framework can be found in Section 3.3. There, it will also become evident that if alternative semantics for the classical connectives are required, they can be implemented in our framework via choice connectives.

Now we will turn to the semantics of choice connectives. In QCL, there are two levels of preference for how the formula $(x \vec{\times} y)$ can be satisfied: It is best to satisfy x (level 1), and less optimal to satisfy only y (level 2). If neither x or y are satisfied, the formula is not satisfied. However, one can consider a choice connective $\circ$ such that $(x \circ y)$ has three or four levels of preference for how a formula can be satisfied. For example, it could be optimal to satisfy both x and y (level 1), less optimal to satisfy only x (level 2), and even less optimal to satisfy only y (level 3). If neither x nor y are satisfied, then the formula could be either not satisfied, or be satisfied even less preferably (level 4). Therefore, computing the optionality of $(F \circ G)$ by simply adding up the optionalities of F and G does not work for every choice connective. Instead, we could simply give an upper bound for a choice connective's optionality, e.g.

$$opt_{\mathcal{L}}(F) \leq 2^{N+1},$$

where N is the number of occurrences of choice connectives in F. The idea is that $(x \circ y)$ can induce at most $2^2 = 4$ different satisfaction degrees, and more generally $(x_1 \circ (x_2 \circ \cdots (x_{n-1} \circ x_n)))$ can induce at most 2^n different satisfaction degrees. Another possible upper bound is

$$opt_{\mathcal{L}}(F \circ G) \leq (opt_{\mathcal{L}}(F) + 1) \cdot (opt_{\mathcal{L}}(G) + 1).$$

F can have at most $opt_{\mathcal{L}}(F)$ many finite satisfaction degrees, plus the infinite degree ∞, i.e. there are at most $opt_{\mathcal{L}}(F) + 1$ degrees for F. Analogously for G. Thus, there are $(opt_{\mathcal{L}}(F) + 1) \cdot (opt_{\mathcal{L}}(G) + 1)$ possible combinations for how satisfaction degrees can be ascribed to $(F \circ G)$. The advantage of this upper bound is that the optionality of $F \circ G$ only depends on the optionalities of its immediate subformulas, i.e. F and G. It is not necessary to look inside F or G to count the occurrences of choice connectives.

In addition to giving an upper bound, we might also want to give a lower bound, e.g.

$$opt_{\mathcal{L}}(F \circ G) \geq max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G)).$$

The reasoning behind this is that a choice connective should introduce new ways to distinguish between interpretations, or at least not give less options for doing so.

Lastly, the optionality of $(F \circ G)$ should only depend on the optionality of F and G. No other factors, such as the structure of F or G, should have an influence in this matter.

Definition 28. *Let $\mathcal{L}$ be a choice logic. Then the optionality of an $\mathcal{L}$-formula is given by the function $opt_{\mathcal{L}} \colon \mathcal{F}_{\mathcal{L}} \to \mathbb{N}$ such that*

1. $opt_{\mathcal{L}}(x) = 1$, *for every* $x \in \mathcal{U}$.
2. $opt_{\mathcal{L}}(\neg F) = 1$.
3. $opt_{\mathcal{L}}(F \wedge G) = max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$.
4. $opt_{\mathcal{L}}(F \vee G) = max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$.
5. *for every* $\circ \in \mathcal{C}_{\mathcal{L}}$ *there is a function* $f \colon \mathbb{N}^2 \to \mathbb{N}$ *such that*

$$opt_{\mathcal{L}}(F \circ G) = f(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G))$$

with $max(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G)) \leq opt_{\mathcal{L}}(F \circ G) \leq (opt_{\mathcal{L}}(F) + 1) \cdot (opt_{\mathcal{L}}(G) + 1)$.

We are now ready to define the notion of satisfaction degrees for an arbitrary choice logic. As in CCL, we will use ∞ to signify unsatisfied formulas. The degrees of the classical connectives will be computed in the same way as in QCL and CCL. For the semantics of choice connectives, we impose two crucial restrictions.

Firstly, the satisfaction degree of a formula under any given interpretation should never be bigger than its optionality, unless the degree is ∞. After all, the purpose of optionality is to assert the number of possible satisfaction degrees that a formula can possibly have.

Secondly, the satisfaction degree of a formula $F \circ G$ under any given interpretation should only depend on the optionalities and satisfaction degrees of F and G. The structure of the formula or interpretation must not impact the satisfaction degree. For example, whether an interpretation consists of an even or uneven number of propositional variables should have no influence on satisfaction degree.

Definition 29. *Let $\mathcal{L}$ be a choice logic. Then the satisfaction degree of an $\mathcal{L}$-formula under any interpretation $\mathcal{I} \subseteq \mathcal{U}$ is given by the function $deg_{\mathcal{L}} \colon 2^{\mathcal{U}} \times \mathcal{F}_{\mathcal{L}} \to \mathbb{N} \cup \{\infty\}$ such that*

1. $deg_{\mathcal{L}}(\mathcal{I}, x) = \begin{cases} 1 & \text{if } x \in \mathcal{I} \\ \infty & \text{otherwise} \end{cases}$, *for every* $x \in \mathcal{U}$.

2. $deg_{\mathcal{L}}(\mathcal{I}, \neg F) = \begin{cases} 1 & \text{if } deg_{\mathcal{L}}(\mathcal{I}, F) = \infty \\ \infty & \text{otherwise} \end{cases}$

3. $deg_{\mathcal{L}}(\mathcal{I}, F \wedge G) = max(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$.

4. $deg_{\mathcal{L}}(\mathcal{I}, F \vee G) = min(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$.

5. *for every* $\circ \in \mathcal{C}_{\mathcal{L}}$ *there is a function* $g\colon (\mathbb{N} \cup \{\infty\})^4 \to (\mathbb{N} \cup \{\infty\})$ *such that*

$$deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = g(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G), deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$$

   with $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) \leq opt_{\mathcal{L}}(F \circ G)$ *or* $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = \infty$.

We also write $\mathcal{I} \models^{\mathcal{L}}_{k} F$ for $deg_{\mathcal{L}}(\mathcal{I}, F) = k$, and say that $\mathcal{I}$ satisfies F with a degree of k. If $\mathcal{I}$ satisfies F with a finite degree, then $\mathcal{I}$ is called a model of F. Observe that for classical formulas, we have that $\mathcal{I} \models F \iff \mathcal{I} \models^{\mathcal{L}}_{1} F$, and $\mathcal{I} \not\models F \iff \mathcal{I} \models^{\mathcal{L}}_{\infty} F$.

Now we will come to the central feature of choice logics, namely preferred models. Like in the specific cases of QCL and CCL, a model of a formula is an interpretation that ascribes to this formula a finite satisfaction degree. However, we are not only interested whether an interpretation is a model of some formula. Most often, we are interested in those interpretations that are the most preferable, i.e. that have the lowest satisfaction degree for a given formula.

Definition 30. *Let $\mathcal{I}$ be an interpretation, and F be a formula of some choice logic $\mathcal{L}$. Then $\mathcal{I}$ is a preferred model of F, written as $\mathcal{I} \in Mod_{\mathcal{L}}(F)$, if $deg_{\mathcal{L}}(\mathcal{I}, F) \neq \infty$ and for all other interpretations $\mathcal{J}$ we have $deg_{\mathcal{L}}(\mathcal{I}, F) \leq deg_{\mathcal{L}}(\mathcal{J}, F)$.*

Another important concept to consider is that of equivalence between formulas. We have already discussed equivalence in the context of classical propositional logic, see Definition 7. In choice logics, one could say that two formulas F and G are equivalent if they have the same preferred models, i.e. if $Mod_{\mathcal{L}}(F) = Mod_{\mathcal{L}}(G)$. This is the weakest notion of equivalence in choice logics. Consider the following, stronger definition.

Definition 31. *Let F and G be formulas of some choice logic $\mathcal{L}$. F and G are degree-equivalent, written as $F \equiv^{\mathcal{L}}_{d} G$, if for all interpretations $\mathcal{I}$ we have that $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I}, G)$.*

It is easy to see that degree-equivalence of two formulas implies that they also have the same preferred models.

Lemma 6. *If $F \equiv^{\mathcal{L}}_{d} G$, then $Mod_{\mathcal{L}}(F) = Mod_{\mathcal{L}}(G)$.*

Proof. Assume $F \equiv_d^{\mathcal{L}} G$. Then

$$\begin{aligned} \mathcal{I} \in Mod_{\mathcal{L}}(F) &\iff deg_{\mathcal{L}}(\mathcal{I}, F) \neq \infty \text{ and } deg_{\mathcal{L}}(\mathcal{I}, F) \leq deg_{\mathcal{L}}(\mathcal{J}, F) \text{ for all } \mathcal{J} \\ &\iff deg_{\mathcal{L}}(\mathcal{I}, G) \neq \infty \text{ and } deg_{\mathcal{L}}(\mathcal{I}, G) \leq deg_{\mathcal{L}}(\mathcal{J}, G) \text{ for all } \mathcal{J} \\ &\iff \mathcal{I} \in Mod_{\mathcal{L}}(G). \end{aligned}$$

□

The idea of degree-equivalence, while arguably natural, does not express that two formulas are completely semantically equivalent. For this, it is necessary that the formulas also have the same optionality.

Definition 32. *Let F and G be formulas of some choice logic $\mathcal{L}$. F and G are fully equivalent, written as $F \equiv_f^{\mathcal{L}} G$, if $F \equiv_d^{\mathcal{L}} G$ and $opt_{\mathcal{L}}(F) = opt_{\mathcal{L}}(G)$.*

The concept of full equivalence is called strong equivalence in [BBB04]. We will use the term strong equivalence differently, see Chapter 5. There, we will further investigate all notions of equivalence mentioned above.

3.2 Properties

From Item 5 in Definition 29 we can infer that, as intended, the satisfaction degree of $(F \circ G)$ only depends on the optionalities and satisfaction degrees of F and G. More formally, if $opt_{\mathcal{L}}(F) = opt_{\mathcal{L}}(F')$, $opt_{\mathcal{L}}(G) = opt_{\mathcal{L}}(G')$, $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I}, F')$, and $deg_{\mathcal{L}}(\mathcal{I}, G) = deg_{\mathcal{L}}(\mathcal{I}, G')$, then $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg_{\mathcal{L}}(\mathcal{I}, F' \circ G')$ for any choice connective $\circ$. From this it also follows that the satisfaction degree of a formula F only depends on those variables that actually occur in F, which is a generalization of Lemma 1.

Lemma 7. *Let $\mathcal{L}$ be a choice logic, and let $\mathcal{I}$ be an interpretation. If a propositional variable x does not occur in an $\mathcal{L}$-formula F, then $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F)$.*

Proof. By structural induction.

- Base case: If $F = y$ for some propositional variable y, and x does not occur in F, then $x \neq y$. Thus, $y \in \mathcal{I}$ if $y \in \mathcal{I} \setminus \{x\}$, i.e.

$$\begin{aligned} deg_{\mathcal{L}}(\mathcal{I}, y) &= \begin{cases} 1 & \text{if } y \in \mathcal{I} \\ \infty & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \in \mathcal{I} \setminus \{x\} \\ \infty & \text{otherwise} \end{cases} \\ &= deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, y). \end{aligned}$$

- Induction step: Assume x does not occur in F and G. Then, by the I.H., $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F)$ and $deg_{\mathcal{L}}(\mathcal{I}, G) = deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, G)$. It follows that

$$\begin{aligned} deg_{\mathcal{L}}(\mathcal{I}, \neg F) &= \begin{cases} 1 & \text{if } deg_{\mathcal{L}}(\mathcal{I}, F) = \infty \\ \infty & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F) = \infty \\ \infty & \text{otherwise} \end{cases} \\ &= deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, \neg F), \end{aligned}$$

$$\begin{aligned} deg_{\mathcal{L}}(\mathcal{I}, F \wedge G) &= max(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G)) \\ &= max(deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F), deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, G)) \\ &= deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F \wedge G), \end{aligned}$$

$$\begin{aligned} deg_{\mathcal{L}}(\mathcal{I}, F \vee G) &= min(deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G)) \\ &= min(deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F), deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, G)) \\ &= deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F \vee G), \end{aligned}$$

and for all $\circ \in \mathcal{C}_{\mathcal{L}}$

$$\begin{aligned} deg_{\mathcal{L}}(\mathcal{I}, F \circ G) &= g(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G), deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G)) \\ &= g(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G), deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F), deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, G)) \\ &= deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F \circ G) \end{aligned}$$

for some function g.

□

From Lemma 7, and from the fact that every formula F consists of only finitely many propositional variables, it follows that for every infinite interpretation $\mathcal{I}$ there exists a finite interpretation $\mathcal{J}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{J}, F)$. For this reason, interpretations will be assumed to be finite in the remaining text, unless explicitly stated otherwise.

The following lemma is concerned with the impact of renaming variables within a formula:

Lemma 8. *Let F be a $\mathcal{L}$-formula of some choice logic $\mathcal{L}$. Then there exists an $\mathcal{L}$-formula F' such that F' has no variables in common with F, and F' can be satisfied with degree k if and only if F can be satisfied with degree k.*

Proof. Let F be a $\mathcal{L}$-formula. Construct F' by replacing every variable x in F by a fresh variable x'. Now consider an arbitrary interpretation $\mathcal{I} \subseteq var(F)$. Then for $\mathcal{I}' = \{x' \mid x \in \mathcal{I}\}$ we have $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I}', F')$. □

3.3 Examples

This section gathers various choice logics. We will express already existing formalisms (PL, QCL, and CCL) in our framework of choice logics, as well as introduce entirely new CLs. Lastly, the possibility of combining CLs will be discussed.

3.3.1 Capturing Existing Choice Logics

First, we would like to ensure that our definition for choice logics given in Section 3.1 captures propositional logic.

Proposition 9. *PL is the choice logic such that* $\mathcal{C}_{PL} = \emptyset$*.*

Proof. We can observe that if a choice logic has no choice connective, then the only possible satisfaction degrees that can be ascribed to any formula of this choice logic are 1 and ∞. If we see 1 as another symbol for *true*, and ∞ as another symbol for *false*, then we can see that the semantics given to the classical connectives ($\neg$, $\wedge$, and $\vee$) in Definition 6 and Definition 29 are equivalent. □

Next, it will be shown that our framework encapsulates both QCL and CCL.

Proposition 10. *QCL and CCL are choice logics.*

Proof. Choose $\mathcal{C}_{\mathrm{QCL}} = \{\vec{\times}\}$ with

$$opt_{\mathrm{QCL}}(F \vec{\times} G) = opt_{\mathrm{QCL}}(F) + opt_{\mathrm{QCL}}(G),$$

and

$$deg_{\mathrm{QCL}}(\mathcal{I}, F \vec{\times} G) = \begin{cases} deg_{\mathrm{QCL}}(\mathcal{I}, F) & \text{if } deg_{\mathrm{QCL}}(\mathcal{I}, F) < \infty \\ deg_{\mathrm{QCL}}(\mathcal{I}, G) + opt_{\mathrm{QCL}}(F) & \text{if } deg_{\mathrm{QCL}}(\mathcal{I}, F) = \infty \\ & \text{and } deg_{\mathrm{QCL}}(\mathcal{I}, G) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

First, let us verify that the above definition indeed describes a CL. The requirements for the optionality of choice connectives are met: Since $opt_{\mathrm{QCL}}(F \vec{\times} G) = opt_{\mathrm{QCL}}(F) + opt_{\mathrm{QCL}}(G)$, we have that $opt_{\mathrm{QCL}}(F \vec{\times} G)$ only depends on the optionalities of F and G, and that

$$\begin{aligned} max(opt_{\mathrm{QCL}}(F), opt_{\mathrm{QCL}}(G)) &< opt_{\mathrm{QCL}}(F \vec{\times} G) \\ &< opt_{\mathrm{QCL}}(F) \cdot opt_{\mathrm{QCL}}(G) + opt_{\mathrm{QCL}}(F) + opt_{\mathrm{QCL}}(G) + 1 \\ &= (opt_{\mathrm{QCL}}(F) + 1) \cdot (opt_{\mathrm{QCL}}(G) + 1). \end{aligned}$$

The requirements for the satisfaction degree of choice connectives are met as well: $deg_{\mathrm{QCL}}(\mathcal{I}, F \vec{\times} G)$ only depends on the optionality of F and the satisfaction degrees of F

and G. Furthermore, the degree of $(F \vec{\times} G)$ never exceeds $opt_{\mathrm{QCL}}(F \vec{\times} G) = opt_{\mathrm{QCL}}(F) + opt_{\mathrm{QCL}}(G)$.

It remains to show that our choice logic is equivalent to QCL as it was defined in Section 2.2. Clearly, the above choice logic consists of exactly the same formulas as QCL. It is also evident that $opt_{\mathrm{QCL}}(F) = opt(F)$ holds for any QCL-formula F. Lastly, one can see that the semantics of QCL and the above choice logic are defined in such a way that $deg_{\mathrm{QCL}}(\mathcal{I}, F) = k$ with $k < \infty$ iff $\mathcal{I} \mid\!\sim_k^{\mathrm{QCL}} F$ and $deg_{\mathrm{QCL}}(\mathcal{I}, F) = \infty$ iff there is no k such that $\mathcal{I} \mid\!\sim_k^{\mathrm{QCL}} F$.

For CCL, choose $\mathcal{C}_{\mathrm{CCL}} = \{\vec{\odot}\}$ with

$$opt_{\mathrm{CCL}}(F \vec{\odot} G) = opt_{\mathrm{CCL}}(F) + opt_{\mathrm{CCL}}(G),$$

and

$$deg_{\mathrm{CCL}}(\mathcal{I}, F \vec{\odot} G) = \begin{cases} deg_{\mathrm{CCL}}(\mathcal{I}, F) + opt_{\mathrm{CCL}}(G) & \text{if } 1 < deg_{\mathrm{CCL}}(\mathcal{I}, F) < \infty \\ deg_{\mathrm{CCL}}(\mathcal{I}, G) & \text{if } deg_{\mathrm{CCL}}(\mathcal{I}, F) = 1 \\ \infty & \text{otherwise.} \end{cases}$$

The rest of the proof is analogous to the proof for QCL, except that is has to be shown that $deg_{\mathrm{CCL}}(\mathcal{I}, F) = k$ iff $\mathcal{I} \mid\!\sim_k^{\mathrm{CCL}} F$. $\square$

To encode PQCL as a choice logic, one could express the alternative semantics for negation, conjunction, and disjunction given in PQCL as choice connectives $\dot{\neg}$, $\dot{\wedge}$, and $\dot{\vee}$. Then one can consider the fragment of this choice logic that does not contain formulas with the classical connectives ($\neg$, $\wedge$, and $\vee$). Note that unary connectives, such as negation, can also be represented as binary connectives by simply ignoring one of the operands. As we will see, defining $\dot{\wedge}$ and $\dot{\vee}$ can be done quite easily. But expressing $\dot{\neg}$ is problematic, as the semantic of negation in PQCL depends on the structure of F. The case that $\dot{\neg} F = \dot{\neg}(G \dot{\wedge} H)$ would need to be handled differently than the case that $\dot{\neg} F = \dot{\neg}(G \dot{\vee} H)$, and therefore the negation of PQCL can not be encoded as a function over the optionality and satisfaction degree of its operand.

However, by the semantics of negation in PQCL, any PQCL-formula F can be transformed into a degree- and optionality-equivalent formula F^* such that negations only appear in front of atoms in F^*. But when applied to atoms, the negation of QCL and PQCL behave identically. Therefore, we will use the QCL semantics for negation in our choice logic.

We define the choice logic PQCL$'$ with $\mathcal{C}_{\mathrm{PQCL}'} = \{\vec{\times}, \dot{\wedge}, \dot{\vee}\}$,

$$\begin{aligned} opt_{\mathrm{PQCL}'}(F \vec{\times} G) &= opt_{\mathrm{PQCL}'}(F) + opt_{\mathrm{PQCL}'}(G), \\ opt_{\mathrm{PQCL}'}(F \dot{\wedge} G) &= opt_{\mathrm{PQCL}'}(F) \cdot opt_{\mathrm{PQCL}'}(G), \\ opt_{\mathrm{PQCL}'}(F \dot{\vee} G) &= opt_{\mathrm{PQCL}'}(F) \cdot opt_{\mathrm{PQCL}'}(G), \end{aligned}$$

and[1]

$$
\begin{aligned}
&deg_{\mathrm{PQCL}'}(\mathcal{I}, F\vec{\times}G) = deg_{\mathrm{QCL}}(\mathcal{I}, F\vec{\times}G),\\
&deg_{\mathrm{PQCL}'}(\mathcal{I}, F\dot{\wedge}G) = \begin{cases} (i-1)\cdot opt_{\mathrm{PQCL}'}(G) + j & \text{if } deg_{\mathrm{PQCL}'}(\mathcal{I}, F) = i,\\ & deg_{\mathrm{PQCL}'}(\mathcal{I}, G) = j,\\ & i < \infty,\ \text{and } j < \infty;\\ \infty & \text{otherwise.}\end{cases}\\
&deg_{\mathrm{PQCL}'}(\mathcal{I}, F\dot{\vee}G) = \begin{cases} 1 & \text{if } deg_{\mathrm{PQCL}'}(\mathcal{I}, F) = 1\\ & \text{or } deg_{\mathrm{PQCL}'}(\mathcal{I}, G) = 1;\\ (i-1)\cdot opt_{\mathrm{PQCL}'}(G) + 1 & \text{if } deg_{\mathrm{PQCL}'}(\mathcal{I}, F) = i,\\ & 1 < i < \infty,\\ & \text{and } deg_{\mathrm{PQCL}'}(\mathcal{I}, G) = \infty;\\ deg_{\mathrm{PQCL}'}(\mathcal{I}, G) & \text{if } deg_{\mathrm{PQCL}'}(\mathcal{I}, F) > 1\\ & \text{and } 1 < deg_{\mathrm{PQCL}'}(\mathcal{I}, G) < \infty;\\ \infty & \text{otherwise.}\end{cases}
\end{aligned}
$$

Take any PQCL formula F, and transform it into F^* such that negations appear only in front of atoms in F^* and

$$\mathcal{I} \mathrel{|\!\!\sim}^{\mathrm{PQCL}}_{k} F \iff \mathcal{I} \mathrel{|\!\!\sim}^{\mathrm{PQCL}}_{k} F^*.$$

This can be done by simply pushing the negation inwards according to the semantics of $\neg$ in PQCL, see Section 2.2.3. Furthermore, replace every occurrence of $\wedge$ and $\vee$ in F^* by $\dot{\wedge}$ and $\dot{\vee}$ respectively to obtain a PQCL$'$ formula F'. Then

$$\mathcal{I} \mathrel{|\!\!\sim}^{\mathrm{PQCL}}_{k} F^* \iff deg_{\mathrm{PQCL}'}(\mathcal{I}, F') = k$$

with $k < \infty$, and

$$\nexists\, k\colon \mathcal{I} \mathrel{|\!\!\sim}^{\mathrm{PQCL}}_{k} F^* \iff deg_{\mathrm{PQCL}'}(\mathcal{I}, F') = \infty.$$

This means that PQCL is semantically equivalent to the fragment of PQCL$'$ where $\neg$ only appears in front of atoms, and where $\wedge$ and $\vee$ do not appear at all.

QCL+ can be represented in our framework as well, similarly to PQCL. Define the choice logic QCL+$'$ with $\mathcal{C}_{\mathrm{QCL+}'} = \{\vec{\times}, \dot{\wedge}\}$,

$$
\begin{aligned}
opt_{\mathrm{QCL+}'}(F\vec{\times}G) &= opt_{\mathrm{QCL+}'}(F) + opt_{\mathrm{QCL+}'}(G),\\
opt_{\mathrm{QCL+}'}(F\dot{\wedge}G) &= max(opt_{\mathrm{QCL+}'}(F), opt_{\mathrm{QCL+}'}(G)),
\end{aligned}
$$

[1]By slight abuse of notation, we write $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg_{\mathcal{L}'}(\mathcal{I}, F \circ G)$ when the satisfaction degree of $\circ$ is defined by the same function in both $\mathcal{L}$ and $\mathcal{L}'$, see Definition 29. Analogously, we will sometimes write $opt_{\mathcal{L}}(F \circ G) = opt_{\mathcal{L}'}(F \circ G)$, see Definition 28.

and

$$deg_{\mathrm{QCL+}'}(\mathcal{I}, F\vec{\times}G) = deg_{\mathrm{QCL}}(\mathcal{I}, F\vec{\times}G),$$

$$deg_{\mathrm{QCL+}'}(\mathcal{I}, F\dot{\wedge}G) = \begin{cases} k & \text{if } deg_{\mathrm{QCL+}'}(\mathcal{I}, F) = deg_{\mathrm{QCL+}'}(\mathcal{I}, G) = k, \\ deg_{\mathrm{QCL+}'}(\mathcal{I}, F) & \text{if } deg_{\mathrm{QCL+}'}(\mathcal{I}, F) > opt_{\mathrm{QCL+}'}(G), \\ deg_{\mathrm{QCL+}'}(\mathcal{I}, G) & \text{if } deg_{\mathrm{QCL+}'}(\mathcal{I}, G) > opt_{\mathrm{QCL+}'}(F). \end{cases}$$

Analogously to our encoding of PQCL, we can conclude that QCL+ is semantically equivalent to the fragment of QCL+$'$ where $\neg$ only appears in front of atoms, and where $\dot{\wedge}$ does not appear at all.

3.3.2 Introducing New Choice Logics

Our definition of choice logics is not very restrictive. As a consequence, many different choice logics can be defined. Here we will look at some arguably sensible CLs.

An Alternative Semantics for Conjunctive Choice Logic (CCL)

As was already mentioned in Section 2.3, we want to give an alternative semantic for ordered conjunction. The following choice logic is what we will refer to as CCL from now on.

Definition 33. *CCL is the choice logic such that $\mathcal{C}_{CCL} = \{\vec{\odot}\}$,*

$$opt_{CCL}(F\vec{\odot}G) = opt_{CCL}(F) + opt_{CCL}(G),$$

and

$$deg_{CCL}(\mathcal{I}, F\vec{\odot}G) = \begin{cases} deg_{CCL}(\mathcal{I}, G) & \text{if } deg_{CCL}(\mathcal{I}, F) = 1 \\ & \text{and } deg_{CCL}(\mathcal{I}, G) < \infty \\ m + opt_{CCL}(G) & \text{if } deg_{CCL}(\mathcal{I}, F) = m, m < \infty \\ & \text{and } (m > 1 \text{ or } deg_{CCL}(\mathcal{I}, G) = \infty) \\ \infty & \text{otherwise.} \end{cases}$$

Let $F = (x\vec{\odot}(y\vec{\odot}z))$. Then we can see that, at least for F, this new definition of CCL captures the intuitive meaning of ordered conjunction: $\{x, y, z\} \models_1^{\mathrm{CCL}} F$, $\{x, y\} \models_2^{\mathrm{CCL}} F$, $\{x\} \models_3^{\mathrm{CCL}} F$, and $\emptyset \models_\infty^{\mathrm{CCL}} F$. Furthermore, we can show that $\vec{\odot}$ is associative under this new semantics, as was intended in [BB16].

Lemma 11. *The choice connective $\vec{\odot} \in \mathcal{C}_{CCL}$ is associative, i.e.*

$$((F\vec{\odot}G)\vec{\odot}H) \equiv_f^{CCL} (F\vec{\odot}(G\vec{\odot}H))$$

holds for arbitrary CCL-formulas F, G, and H.

Proof. By the definition of full equivalence (Definition 32), we need to show that $((F\vec{\odot}G)\vec{\odot}H)$ and $(F\vec{\odot}(G\vec{\odot}H))$ have the same optionality, and that they are degree-equivalent, i.e. that they are ascribed the same satisfaction degree by every interpretation. First, observe that

$$\begin{aligned} opt_{\mathrm{CCL}}((F\vec{\odot}G)\vec{\odot}H) &= opt_{\mathrm{CCL}}(F) + opt_{\mathrm{CCL}}(G) + opt_{\mathrm{CCL}}(H) \\ &= opt_{\mathrm{CCL}}(F\vec{\odot}(G\vec{\odot}H)). \end{aligned}$$

It remains to show that $((F\vec{\odot}G)\vec{\odot}H) \equiv_d^{\mathrm{CCL}} (F\vec{\odot}(G\vec{\odot}H))$. Let $\mathcal{I}$ be an arbitrary interpretation. We distinguish the following cases:

- $deg_{\mathrm{CCL}}(\mathcal{I}, F) = 1$, $deg_{\mathrm{CCL}}(\mathcal{I}, G) = 1$, and $deg_{\mathrm{CCL}}(\mathcal{I}, H) < \infty$. Then $deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}G) = 1$ and $deg_{\mathrm{CCL}}(\mathcal{I}, G\vec{\odot}H) = deg_{\mathrm{CCL}}(\mathcal{I}, H)$, which entails that
$$deg_{\mathrm{CCL}}(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = deg_{\mathrm{CCL}}(\mathcal{I}, H) = deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H)).$$
- $deg_{\mathrm{CCL}}(\mathcal{I}, F) = 1$, $deg_{\mathrm{CCL}}(\mathcal{I}, G) = 1$, and $deg_{\mathrm{CCL}}(\mathcal{I}, H) = \infty$. Then $deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}G) = 1$, and $deg_{\mathrm{CCL}}(\mathcal{I}, G\vec{\odot}H) = 1 + opt_{\mathrm{CCL}}(H)$, which entails that
$$deg_{\mathrm{CCL}}(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = 1 + opt_{\mathrm{CCL}}(H) = deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H)).$$
- $deg_{\mathrm{CCL}}(\mathcal{I}, F) = 1$ and $1 < deg_{\mathrm{CCL}}(\mathcal{I}, G) < \infty$. Then $deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}G) = deg_{\mathrm{CCL}}(\mathcal{I}, G)$ and $deg_{\mathrm{CCL}}(\mathcal{I}, G\vec{\odot}H) = deg_{\mathrm{CCL}}(\mathcal{I}, G) + opt_{\mathrm{CCL}}(H)$, which entails that
$$deg_{\mathrm{CCL}}(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = deg_{\mathrm{CCL}}(\mathcal{I}, G) + opt_{\mathrm{CCL}}(H) = deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H)).$$
- $deg_{\mathrm{CCL}}(\mathcal{I}, F) = 1$ and $deg_{\mathrm{CCL}}(\mathcal{I}, G) = \infty$. Then $deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}G) = 1 + opt_{\mathrm{CCL}}(G)$ and $deg_{\mathrm{CCL}}(\mathcal{I}, G\vec{\odot}H) = \infty$, which entails that
$$deg_{\mathrm{CCL}}(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = 1 + opt_{\mathrm{CCL}}(G) + opt_{\mathrm{CCL}}(H) = deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H)).$$
- $1 < deg_{\mathrm{CCL}}(\mathcal{I}, F) < \infty$. Then $deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}G) = deg_{\mathrm{CCL}}(\mathcal{I}, F) + opt_{\mathrm{CCL}}(G)$, which entails that
$$\begin{aligned} deg_{\mathrm{CCL}}(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) &= deg_{\mathrm{CCL}}(\mathcal{I}, F) + opt_{\mathrm{CCL}}(G) + opt_{\mathrm{CCL}}(H) \\ &= deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H)). \end{aligned}$$
- $deg_{\mathrm{CCL}}(\mathcal{I}, F) = \infty$. Then $deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}G) = \infty$, which entails that
$$deg_{\mathrm{CCL}}(\mathcal{I}, (F\vec{\odot}G)\vec{\odot}H) = \infty = deg_{\mathrm{CCL}}(\mathcal{I}, F\vec{\odot}(G\vec{\odot}H)).$$

$\square$

There are several reasons for why we characterize associativity by full equivalence. First of all, it is the characterization of associativity used by Brewka et al. for QCL (see Section 2.2.2). Secondly, as we will show in Section 5.2, full equivalence in CCL is the same as equivalence with respect to the replacement of subformulas. Thirdly, full equivalence is the strongest form of equivalence in choice logics (Section 5.3).

Exclusive Disjunctive Choice Logic (XCL)

As another example for a new choice logic, we can consider a choice connective $\vec{\oplus}$ that is based on exclusive disjunction (also called XOR), similar to how $\vec{\times}$ is based on regular disjunction. We name this connective *ordered exclusive disjunction*, and call the corresponding choice logic Exclusive Disjunctive Choice Logic (XCL).

Definition 34. *XCL is the choice logic such that* $\mathcal{C}_{XCL} = \{\vec{\oplus}\}$,

$$opt_{XCL}(F\vec{\oplus}G) = opt_{XCL}(F) + opt_{XCL}(G),$$

and

$$deg_{XCL}(\mathcal{I}, F\vec{\oplus}G) = \begin{cases} deg_{XCL}(\mathcal{I}, F) & \text{if } deg_{XCL}(\mathcal{I}, F) < \infty \\ & \text{and } deg_{XCL}(\mathcal{I}, G) = \infty \\ deg_{XCL}(\mathcal{I}, G) + opt_{XCL}(F) & \text{if } deg_{XCL}(\mathcal{I}, F) = \infty \\ & \text{and } deg_{XCL}(\mathcal{I}, G) < \infty \\ \infty & \text{otherwise} \end{cases}$$

Intuitively, $F\vec{\oplus}G$ expresses that it is preferable to satisfy F. If this is not possible, then G should be satisfied. However, it is not acceptable to satisfy both F and G. As an example, consider $F^* = (x\vec{\oplus}(y\vec{\oplus}z))$. Then $\{x\} \models_1^{\text{XCL}} F^*$, $\{y\} \models_2^{\text{XCL}} F^*$, and $\{z\} \models_3^{\text{XCL}} F^*$. Furthermore, $\{x, y, z\} \models_1^{\text{XCL}} F^*$, since $(y\vec{\oplus}z)$ is satisfied to a degree of ∞ in this interpretation. In contrast to ordered disjunction, $\vec{\oplus}$ is not associative. Let $F' = ((x\vec{\oplus}y)\vec{\oplus}z)$. While $\{x\}$, $\{y\}$, and $\{z\}$ ascribe the same degrees to both F^* and F', we have that $\{x, y, z\} \models_3^{\text{XCL}} F'$.

As we will show in Lemma 12, $F\vec{\oplus}G$ can also be expressed in QCL. However, if exclusive ordered disjunction has to be expressed often in a given system, then a dedicated choice connective can simplify specifications. An example for such a system could be a calendar, in which preferences over time slots need to be considered. Since only one time slot may be assigned to each appointment, exclusive ordered disjunction might be used instead of exclusive disjunction.

Lemma 12. *Let H be an XCL-formula. Let H' be QCL-formula obtained by recursively replacing every occurrence of $F\vec{\oplus}G$ in H by $((F\vec{\times}G) \wedge \neg(F \wedge G))$. Then H and H' are fully equivalent.*

Proof. We show this by structural induction. The base case (H is a propositional variable) and the cases for the classical connectives are trivial. We will therefore only look at $F\vec{\oplus}G$, where F and G are XCL-formulas. Then, as our I.H., let F' and G' be QCL-formulas such that $opt_{\text{XCL}}(F) = opt_{\text{QCL}}(F')$, $opt_{\text{XCL}}(G) = opt_{\text{QCL}}(G')$, $deg_{\text{XCL}}(\mathcal{I}, F) = deg_{\text{QCL}}(\mathcal{I}, F')$, and $deg_{\text{XCL}}(\mathcal{I}, G) = deg_{\text{QCL}}(\mathcal{I}, G')$ for all interpretations $\mathcal{I}$.

Regarding optionality, we have that

$$\begin{aligned}
opt_{\mathrm{XCL}}(F \vec{\oplus} G) &= opt_{\mathrm{XCL}}(F) + opt_{\mathrm{XCL}}(G) \\
&= opt_{\mathrm{QCL}}(F') + opt_{\mathrm{QCL}}(G') \\
&= max(opt_{\mathrm{QCL}}(F') + opt_{\mathrm{QCL}}(G'), 1) \\
&= max(opt_{\mathrm{QCL}}(F' \vec{\times} G'), opt_{\mathrm{QCL}}(\neg(F' \wedge G'))) \\
&= opt_{\mathrm{QCL}}((F' \vec{\times} G') \wedge \neg(F' \wedge G')).
\end{aligned}$$

For satisfaction degree, we will look at four cases:

1. $deg_{\mathrm{XCL}}(\mathcal{I}, F) < \infty$ and $deg_{\mathrm{XCL}}(\mathcal{I}, G) < \infty$. Then

$$\begin{aligned}
deg_{\mathrm{XCL}}(\mathcal{I}, F \vec{\oplus} G) &= \infty \\
&= max(deg_{\mathrm{QCL}}(\mathcal{I}, F' \vec{\times} G'), \infty) \\
&= max(deg_{\mathrm{QCL}}(\mathcal{I}, F' \vec{\times} G'), deg_{\mathrm{QCL}}(\mathcal{I}, \neg(F' \wedge G'))) \\
&= deg_{\mathrm{QCL}}(\mathcal{I}, (F' \vec{\times} G') \wedge \neg(F' \wedge G')).
\end{aligned}$$

2. $deg_{\mathrm{XCL}}(\mathcal{I}, F) < \infty$ and $deg_{\mathrm{XCL}}(\mathcal{I}, G) = \infty$. Then

$$\begin{aligned}
deg_{\mathrm{XCL}}(\mathcal{I}, F \vec{\oplus} G) &= deg_{\mathrm{XCL}}(\mathcal{I}, F) \\
&= max(deg_{\mathrm{QCL}}(\mathcal{I}, F'), 1) \\
&= max(deg_{\mathrm{QCL}}(\mathcal{I}, F' \vec{\times} G'), deg_{\mathrm{QCL}}(\mathcal{I}, \neg(F' \wedge G'))) \\
&= deg_{\mathrm{QCL}}(\mathcal{I}, (F' \vec{\times} G') \wedge \neg(F' \wedge G')).
\end{aligned}$$

3. $deg_{\mathrm{XCL}}(\mathcal{I}, F) = \infty$ and $deg_{\mathrm{XCL}}(\mathcal{I}, G) < \infty$. Then

$$\begin{aligned}
deg_{\mathrm{XCL}}(\mathcal{I}, F \vec{\oplus} G) &= deg_{\mathrm{XCL}}(\mathcal{I}, G) + opt_{\mathrm{XCL}}(F) \\
&= max(deg_{\mathrm{QCL}}(\mathcal{I}, G') + opt_{\mathrm{QCL}}(F'), 1) \\
&= max(deg_{\mathrm{QCL}}(\mathcal{I}, F' \vec{\times} G'), deg_{\mathrm{QCL}}(\mathcal{I}, \neg(F' \wedge G'))) \\
&= deg_{\mathrm{QCL}}(\mathcal{I}, (F' \vec{\times} G') \wedge \neg(F' \wedge G')).
\end{aligned}$$

4. $deg_{\mathrm{XCL}}(\mathcal{I}, F) = \infty$ and $deg_{\mathrm{XCL}}(\mathcal{I}, G) = \infty$. Then

$$\begin{aligned}
deg_{\mathrm{XCL}}(\mathcal{I}, F \vec{\oplus} G) &= \infty \\
&= max(\infty, deg_{\mathrm{QCL}}(\mathcal{I}, \neg(F' \wedge G'))) \\
&= max(deg_{\mathrm{QCL}}(\mathcal{I}, F' \vec{\times} G'), deg_{\mathrm{QCL}}(\mathcal{I}, \neg(F' \wedge G'))) \\
&= deg_{\mathrm{QCL}}(\mathcal{I}, (F' \vec{\times} G') \wedge \neg(F' \wedge G')).
\end{aligned}$$

□

Simple Conjunctive Choice Logic (SCCL)

Now we will look at a CL based on a very simple choice connective:

Definition 35. *SCCL is the choice logic such that* $\mathcal{C}_{SCCL} = \{\circ\}$*,*

$$opt_{SCCL}(F \circ G) = opt_{SCCL}(F) + 1,$$

and

$$deg_{SCCL}(\mathcal{I}, F \circ G) = \begin{cases} deg_{SCCL}(\mathcal{I}, F) & \text{if } deg_{SCCL}(\mathcal{I}, F) < \infty \\ & \text{and } deg_{SCCL}(\mathcal{I}, G) < \infty \\ deg_{SCCL}(\mathcal{I}, F) + 1 & \text{if } deg_{SCCL}(\mathcal{I}, F) < \infty \\ & \text{and } deg_{SCCL}(\mathcal{I}, G) = \infty \\ \infty & \text{otherwise} \end{cases}$$

The idea behind $F \circ G$ in SCCL is that it is preferable to satisfy both F and G. If this is not possible, at least F should be satisfied. In this sense, the choice connective of SCCL fulfills the same purpose as ordered conjunction in CCL. However, SCCL does not use optionality to penalize less preferable interpretations. Instead, the degree of such interpretations is simply incremented by 1.

Consider $F = (x \circ (y \circ z))$. Then both $\{x, y\}$ and $\{x, y, z\}$ ascribe a degree of 1 to F. The fact that $(y \circ z)$ is not optimally satisfied by $\{x, y\}$ is irrelevant, as long as it is satisfied to some finite satisfaction degree. This is in contrast to CCL, where $(x \vec{\odot} (y \vec{\odot} z))$ would be satisfied to a degree of 2 by $\{x, y\}$. Also, $\circ \in \mathcal{C}_{\mathrm{SCCL}}$ is not associative, since $\{x, y\}$ ascribes a degree of 2 to $((x \circ y) \circ z)$.

Lexicographic Choice Logic (LCL)

Next, let us introduce a CL based on a choice connective $\circ$ that expresses more than two levels of satisfaction: For $F \circ G$, the best option will be the one that satisfies both F and G. If this is not possible, then at least F should be satisfied. But if this is also not possible, then it is still acceptable to satisfy only G. Satisfying neither F nor G is not acceptable.

Definition 36. *LCL is the choice logic such that* $\mathcal{C}_{LCL} = \{\circ\}$*,*

$$opt_{LCL}(F \circ G) = (opt_{LCL}(F) + 1) \cdot (opt_{LCL}(G) + 1) - 1,$$

and

$$deg_{LCL}(\mathcal{I}, F \circ G) = \begin{cases} (m-1) \cdot opt_{LCL}(G) + n & \textit{if } deg_{LCL}(\mathcal{I}, F) = m, \\ & deg_{LCL}(\mathcal{I}, G) = n, \\ & m < \infty, \;\; \textit{and } n < \infty; \\ opt_{LCL}(F) \cdot opt_{LCL}(G) + deg_{LCL}(\mathcal{I}, F) & \textit{if } deg_{LCL}(\mathcal{I}, F) < \infty, \\ & deg_{LCL}(\mathcal{I}, G) = \infty; \\ opt_{LCL}(F) \cdot opt_{LCL}(G) + opt_{LCL}(F) + n & \textit{if } deg_{LCL}(\mathcal{I}, F) = \infty, \\ & deg_{LCL}(\mathcal{I}, G) = n, \\ & \textit{and } n < \infty; \\ \infty & \textit{otherwise} \end{cases}$$

If both F and G are satisfied, then the degree of $F \circ G$ is determined in exactly the same way as for the regular conjunction ($\wedge$) of PQCL. This is also why, if G is not satisfied, we have to add $opt_{\text{LCL}}(F) \cdot opt_{\text{LCL}}(G)$ to the degree of F. LCL is based on classical disjunction, just as QCL is. However, their semantics differ greatly.

Let $F = (x \circ (y \circ z))$. The only interpretation that ascribes a degree of ∞ to F is $\emptyset$. The remaining 7 interpretations applicable to F each result in a different degree, ranging from 1 to 7. For example, $\{x, y, z\} \models_1^{\text{LCL}} F$, $\{x, y\} \models_2^{\text{LCL}} F$, $\{x, z\} \models_3^{\text{LCL}} F$, and $\{z\} \models_7^{\text{LCL}} F$.

In fact, LCL enables us to encode a lexicographic ordering over variables. Recall from Section 2.4.2 that an interpretation $\mathcal{I}$ is lexicographically larger than another interpretation $\mathcal{J}$ with respect to an ordering $x_1 > \cdots > x_n$ if there is an index $k \in \{1, \ldots, n\}$ such that $x_i \in \mathcal{I} \iff x_i \in \mathcal{J}$ for all $i < k$, and $x_k \in \mathcal{I}$ but $x_k \notin \mathcal{J}$.

Lemma 13. *Let $x_1 > \cdots > x_n$ be an ordering over n propositional variables. Let $\mathcal{I}_k \subseteq \{x_1, \ldots, x_n\}$ be the lexicographically k-th largest interpretation over this ordering, and let $F_n = (x_1 \circ (x_2 \circ (\cdots (x_{n-1} \circ x_n))))$ be an LCL-formula. Then*

$$deg_{LCL}(\mathcal{I}_k, F_n) = \begin{cases} k & \textit{if } k < 2^n \\ \infty & \textit{if } k = 2^n \end{cases}$$

Proof. First, we prove that $opt_{\text{LCL}}(F_n) = 2^n - 1$ by induction over n:

- Base case: $n = 1$. Then $F_n = x_1$, and therefore $opt_{\text{LCL}}(F_n) = opt_{\text{LCL}}(x_1) = 1 = 2^n - 1$.

- Step case: $n > 1$. Then $F_n = (x_1 \circ A)$, where $A = (x_2 \circ (\cdots (x_{n-1} \circ x_n)))$. By our

I.H., $opt_{\mathrm{LCL}}(A) = 2^{n-1} - 1$, and thus

$$\begin{aligned} opt_{\mathrm{LCL}}(F_n) &= opt_{\mathrm{LCL}}(x_1 \circ A) \\ &= (opt_{\mathrm{LCL}}(x_1) + 1) \cdot (opt_{\mathrm{LCL}}(A) + 1) - 1 \\ &= (1+1) \cdot ((2^{n-1} - 1) + 1) - 1 \\ &= (2 \cdot 2^{n-1}) - 1 \\ &= 2^n - 1. \end{aligned}$$

Now we proceed with the main proof, again by induction over n:

- Base case: $n = 1$. Then $F_n = x_1$, $\mathcal{I}_1 = \{x_1\}$, and $\mathcal{I}_2 = \emptyset$. Clearly, $deg_{\mathrm{LCL}}(\mathcal{I}_1, F_n) = 1$ and $deg_{\mathrm{LCL}}(\mathcal{I}_2, F_n) = \infty$, as required.
- Step case: $n > 1$. Then $F_n = (x_1 \circ A)$, where $A = (x_2 \circ (\cdots (x_{n-1} \circ x_n)))$. Let $\mathcal{J}_k$ be the lexicographically k-th largest interpretation over $x_2 > \cdots > x_n$. By our I.H., $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) = k$ if $k < 2^{n-1}$, and $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) = \infty$ if $k = 2^{n-1}$. We can obtain the k-th largest interpretation $\mathcal{I}_k$ over $x_1 > \cdots > x_n$ in one of two ways:
  1. If $k \leq 2^{n-1}$, then $\mathcal{I}_k = J_k \cup \{x_1\}$, and therefore $deg_{\mathrm{LCL}}(\mathcal{I}_k, x_1) = 1$. There are two cases:
     a) $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) < \infty$. Then $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = deg_{\mathcal{L}}(\mathcal{J}_k, A) = k$.
     b) $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) = \infty$. Then $k = 2^{n-1}$. Thus, $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = opt_{\mathrm{LCL}}(A) + 1 = (2^{n-1} - 1) + 1 = 2^{n-1} = k$.
  2. If $k > 2^{n-1}$, then $\mathcal{I}_k = \mathcal{J}_{(k-2^{n-1})}$, and therefore $deg_{\mathrm{LCL}}(\mathcal{I}_k, x_1) = \infty$. There are two cases:
     a) $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) < \infty$. Then $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = opt_{\mathrm{LCL}}(A) + 1 + deg_{\mathcal{L}}(\mathcal{J}_k, A) = (2^{n-1} - 1) + 1 + (k - 2^{n-1}) = k$.
     b) $deg_{\mathrm{LCL}}(\mathcal{J}_k, A) = \infty$. Then $k - 2^{n-1} = 2^{n-1}$, and therefore $k = 2^n$. Furthermore, $deg_{\mathcal{L}}(\mathcal{I}_k, F_n) = \infty$, as required.

□

The above result will be important when examining the computational complexity of LCL, as can be seen in Section 6.5.

3.3.3 Combining Choice Logics

So far, we have mostly dealt with CLs that consist of a single choice connective. But the CL-framework also allows for choice logics with multiple choice connectives. For example, we can define the choice logic that combines ordered disjunction from QCL and ordered conjunction from CCL.

Definition 37. *QCCL is the choice logic such that* $\mathcal{C}_{QCCL} = \{\vec{\times}, \vec{\odot}\}$,

$$\begin{aligned} opt_{QCCL}(F \vec{\times} G) &= opt_{QCL}(F \vec{\times} G), \\ opt_{QCCL}(F \vec{\odot} G) &= opt_{CCL}(F \vec{\odot} G), \\ deg_{QCCL}(\mathcal{I}, F \vec{\times} G) &= deg_{QCL}(\mathcal{I}, F \vec{\times} G), \text{ and} \\ deg_{QCCL}(\mathcal{I}, F \vec{\odot} G) &= deg_{CCL}(\mathcal{I}, F \vec{\odot} G). \end{aligned}$$

In QCCL, different types of preferences can be expressed in the same formula. An example for a QCCL formula is $(x \vec{\times} (y \vec{\odot} z))$, which expresses that it is most preferable to satisfy x, and less preferable, but still acceptable, to satisfy y and z. Satisfying only y is also acceptable, but it constitutes the least preferred option.

In general, a choice logic $\mathcal{L}$ consisting of multiple choice connectives $\mathcal{C}_\mathcal{L}$ can always be seen as the combination of other CLs, since one could take any choice connective from $\mathcal{C}_\mathcal{L}$ and built a choice logic solely from this connective. We will now define this notion of combining choice logics formally.

Definition 38. *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be choice logics such that* $\mathcal{C}_{\mathcal{L}_1} \cap \mathcal{C}_{\mathcal{L}_2} = \emptyset$. *Then the choice logic* $\mathcal{L}$ *is the combination of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *if* $\mathcal{C}_\mathcal{L} = \mathcal{C}_{\mathcal{L}_1} \cup \mathcal{C}_{\mathcal{L}_2}$, *and for every* $\circ \in \mathcal{C}_{\mathcal{L}_i}$, *where* $i \in \{1, 2\}$, *we have that*

$$\begin{aligned} opt_\mathcal{L}(F \circ G) &= opt_{\mathcal{L}_i}(F \circ G), \\ deg_\mathcal{L}(\mathcal{I}, F \circ G) &= deg_{\mathcal{L}_i}(\mathcal{I}, F \circ G). \end{aligned}$$

Note that we can always assume that $\mathcal{C}_{\mathcal{L}_1} \cap \mathcal{C}_{\mathcal{L}_2} = \emptyset$, since connectives can be renamed if there are conflicts. This assumption is necessary, as otherwise, if a choice connective appears in both $\mathcal{L}_1$ and $\mathcal{L}_2$, it would be unclear whether we should use the optionality and degree functions of $\mathcal{L}_1$ or $\mathcal{L}_2$. We will use the notation $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ if $\mathcal{L}$ is a combination of $\mathcal{L}_1$ and $\mathcal{L}_2$. For example, we have that $\text{QCCL} = \text{QCL} \cup \text{CCL}$.

CHAPTER 4

Classes of Choice Logics

Not all choice logics exhibit the same properties. For example, one could consider a CL in which optionality plays no role, or a CL where the satisfaction degree can never be higher than some fixed number. Such classes of choice logics will be defined and investigated in this chapter. We will also examine under which circumstances these classes are closed under combination.

4.1 Exhaustive Choice Logics

Our framework allows for CLs in which certain optionalities or satisfaction degrees can be skipped. For example, we can devise a CL in which no formula can have an even optionality, simply by specifying

$$opt_{\mathcal{L}}(F \circ G) = opt_{\mathcal{L}}(F) + opt_{\mathcal{L}}(G) + 1$$

for our only choice connective. Similarly, one can define a CL in which no formula can be ascribed an even satisfaction degree. It would also be possible to skip only a specific optionality or degree, e.g. there can be a CL in which no formula has an optionality of 3, while all other optionalities can be attained. As we will see in some of the coming sections, such behavior can be problematic, especially when we are combining two CLs. Therefore, we will now look at CLs in which skipping optionalities or satisfaction degrees is not possible. We will refer to such CLs as exhaustive choice logics.

Definition 39. *Let $\mathcal{L}$ be a choice logic. Then an $\mathcal{L}$-formula F is called exhaustive if for every $n \in \{1, \dots, opt_{\mathcal{L}}(F), \infty\}$ there exists an interpretation $\mathcal{I}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) = n$.*

For example, $F = (x_1 \vec{\times} x_2 \vec{\times} x_3)$ is an exhaustive QCL-formula, since $\{x_i\}$ ascribes a degree of i to F for every $i \in \{1, 2, 3\}$, and since $\emptyset$ ascribes a degree of ∞ to F. On the other hand, $((x \land \neg x) \vec{\times} y)$ is not exhaustive, since there is no way of obtaining a degree of 1. By definition, every satisfaction degree in an exhaustive formula can be obtained, given the right interpretation.

Definition 40. *A choice logic $\mathcal{L}$ is called exhaustive if for every $k \in \mathbb{N}$ there is an exhaustive $\mathcal{L}$-formula F with $opt_{\mathcal{L}}(F) = k$.*

In an exhaustive CL, for every pair $k \in \mathbb{N}$, $n \in \{1, \dots, k, \infty\}$, there exists a formula F and an interpretation $\mathcal{I}$ such that $opt_{\mathcal{L}}(F) = k$ and $deg_{\mathcal{L}}(\mathcal{I}, F) = n$. This means that, in principle, every possible combination of optionality and satisfaction degree can be obtained in an exhaustive CL.

Proposition 14. *QCL, CCL, XCL, and SCCL are exhaustive.*

Proof. We proof this by constructing appropriate formulas for each logic:

- QCL: Let $F_k = (x_1 \vec{\times} \cdots \vec{\times} x_k)$. It is easy to verify that $opt_{\text{QCL}}(F_k) = k$. For any $n \leq k$, we have that $deg_{\text{QCL}}(\{x_n\}, F_k) = n$. Furthermore, $deg_{\text{QCL}}(\emptyset, F_k) = \infty$.
- CCL: Let $F_k = (x_1 \vec{\odot} \cdots \vec{\odot} x_k)$. Again, $opt_{\text{CCL}}(F_k) = k$. For any $n \leq k$, we have that $deg_{\text{CCL}}(\{x_1, \dots, x_{k-n+1}\}, F_k) = n$. Furthermore, $deg_{\text{CCL}}(\emptyset, F_k) = \infty$.
- XCL: Let $F_k = (x_1 \circ (x_2 \circ \cdots (x_{k-1} \circ x_k)))$. Then, like in QCL, $opt_{\text{XCL}}(F_k) = k$, $deg_{\text{XCL}}(\{x_n\}, F_k) = n$ for all $n \leq k$, and $deg_{\text{XCL}}(\emptyset, F_k) = \infty$.
- SCCL: Let $F_k = (x_1 \circ (x_2 \circ \cdots (x_{k-1} \circ x_k)))$. Then, like in CCL, $opt_{\text{SCCL}}(F_k) = k$, $deg_{\text{SCCL}}(\{x_1, \dots, x_{k-n+1}\}, F_k) = n$ for all $n \leq k$, and $deg_{\text{SCCL}}(\emptyset, F_k) = \infty$.

□

Note that neither PL nor LCL are exhaustive, since it is not possible to obtain a formula with an optionality of 2 in either logic (compare Definition 36 for LCL).

It is easy to show that combining an exhaustive choice logics with any other CL yields another exhaustive choice logic.

Lemma 15. *Let $\mathcal{L}$ be an exhaustive choice logic, and let $\mathcal{L}'$ be any choice logic. Then $\mathcal{L} \cup \mathcal{L}'$ is an exhaustive choice logic.*

Proof. Because $\mathcal{L}$ is exhaustive, there is an exhaustive $\mathcal{L}$-formula F with $opt_{\mathcal{L}}(F) = k$ for every $k \in \mathbb{N}$. Since every $\mathcal{L}$-formula is also a $(\mathcal{L} \cup \mathcal{L}')$-formula, we know that there is an exhaustive $(\mathcal{L} \cup \mathcal{L}')$-formula F with $opt_{\mathcal{L} \cup \mathcal{L}'}(F) = k$ for every $k \in \mathbb{N}$, i.e. $\mathcal{L} \cup \mathcal{L}'$ is exhaustive. □

From Proposition 14 and Lemma 15 it directly follows that QCCL is exhaustive. In exhaustive CLs, it is also possible to take any interpretation, and find a formula that has a desired optionality and is satisfied by this interpretation to a desired degree.

Lemma 16. *Let $\mathcal{L}$ be an exhaustive choice logic. Let $k \in \mathbb{N}$ and $n \in \{1, \dots, k, \infty\}$. Then there is an $\mathcal{L}$-formula F such that $opt_{\mathcal{L}}(F) = k$ and $deg_{\mathcal{L}}(\mathcal{I}, F) = n$ for all interpretations $\mathcal{I}$.*

Proof. Let $\mathcal{L}$ be an exhaustive choice logic, and let $\mathcal{I}$ be an interpretation. Let $k \in \mathbb{N}$ and $n \in \{1, \dots, k, \infty\}$. Since $\mathcal{L}$ is exhaustive, there is a formula G such that $opt_{\mathcal{L}}(G) = k$, and $deg_{\mathcal{L}}(\mathcal{J}, G) = n$ for some interpretation $\mathcal{J}$. Let A and B be subformulas of G, and x be a propositional variable occuring in G. We obtain $F = T(G)$ by transforming G as follows:

1. $T(x) = \begin{cases} (x \vee \neg x) & \text{if } x \in \mathcal{J} \\ (x \wedge \neg x) & \text{otherwise} \end{cases}$

2. $T(\neg A) = \neg T(A)$

3. $T(A \circ B) = T(A) \circ T(B)$, where $\circ \in \{\wedge, \vee\} \cup \mathcal{C}_{\mathcal{L}}$.

The above transformation takes G, and replaces every x that is contained in $\mathcal{J}$ by $(x \vee \neg x)$. If x is not contained in $\mathcal{J}$, then it will be replaced by $(x \wedge \neg x)$. We can show that $opt_{\mathcal{L}}(F) = opt_{\mathcal{L}}(G)$ and $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{J}, G)$ by structural induction. Note that this means that $opt_{\mathcal{L}}(F) = k$ and $deg_{\mathcal{L}}(\mathcal{I}, F) = n$, as we intend to show.

- Base case: Let x be a propositional variable. Then either $T(x) = (x \vee \neg x)$ or $T(x) = (x \wedge \neg x)$. In both cases we have $opt_{\mathcal{L}}(T(x)) = 1 = opt_{\mathcal{L}}(x)$. Furthermore,

$$deg_{\mathcal{L}}(\mathcal{I}, T(x)) = \begin{cases} deg_{\mathcal{L}}(\mathcal{I}, x \vee \neg x) & \text{if } x \in \mathcal{J} \\ deg_{\mathcal{L}}(\mathcal{I}, x \wedge \neg x) & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in \mathcal{J} \\ \infty & \text{otherwise} \end{cases} = deg_{\mathcal{L}}(\mathcal{J}, x).$$

- Step case: As our I.H., assume that $opt_{\mathcal{L}}(T(A)) = opt_{\mathcal{L}}(A)$, $deg_{\mathcal{L}}(\mathcal{I}, T(A)) = deg_{\mathcal{L}}(\mathcal{J}, A)$, $opt_{\mathcal{L}}(T(B)) = opt_{\mathcal{L}}(B)$, and $deg_{\mathcal{L}}(\mathcal{I}, T(B)) = deg_{\mathcal{L}}(\mathcal{J}, B)$. Since the optionality and satisfaction degree of $\neg A$ is given by a function over the optionality and degree of A, we can conclude that

$$opt_{\mathcal{L}}(T(\neg A)) = opt_{\mathcal{L}}(\neg T(A)) = opt_{\mathcal{L}}(\neg A), \text{ and}$$
$$deg_{\mathcal{L}}(\mathcal{I}, T(\neg A)) = deg_{\mathcal{L}}(\mathcal{I}, \neg T(A)) = deg_{\mathcal{L}}(\mathcal{J}, \neg A).$$

The cases for the other connectives are analogous, since their optionalities and degrees are also given by functions over the optionalities and degrees of their operands.

□

With the above lemma, we can tackle the issue of synthesis in choice logics. In classical logic, synthesis describes the process of constructing a formula over a given set of variables such that this formula is satisfied by a certain set of interpretations. In choice logics, we will construct a formula over a set of variables such that this formula is satisfied to a given degree by certain interpretations.

Proposition 17. *Let $\mathcal{L}$ be an exhaustive choice logic. Let V be a finite set of propositional variables, and let s be a function $s\colon 2^V \to (\mathbb{N} \cup \infty)$. Then there is an $\mathcal{L}$-formula F such that for every $\mathcal{I} \subseteq V$ we have that $deg_{\mathcal{L}}(\mathcal{I}, F) = s(\mathcal{I})$.*

Proof. Let V be a finite set of propositional variables. Because of Lemma 7, we can assume that every interpretation we are dealing with is a subset of V. Let $G_{\mathcal{J}}$ be a classical formula that characterizes an interpretation $\mathcal{J}$, i.e.

$$G_{\mathcal{J}} = (\bigwedge_{x \in \mathcal{J}} x) \wedge (\bigwedge_{x \in (V \setminus \mathcal{J})} \neg x).$$

Observe that $\mathcal{J} \models G_{\mathcal{J}}$, but $\mathcal{J}' \not\models G_{\mathcal{J}}$ for all $\mathcal{J}' \neq \mathcal{J}$. Since $\mathcal{L}$ is exhaustive, and from Lemma 16, we know that for every $\mathcal{J}$, there is an $\mathcal{L}$-formula $S_{\mathcal{J}}$ such that $deg_{\mathcal{L}}(\mathcal{J}, S_{\mathcal{J}}) = s(\mathcal{J})$. Furthermore, let

$$F = \bigvee_{\mathcal{J} \subseteq V} (G_{\mathcal{J}} \wedge S_{\mathcal{J}}).$$

Let $\mathcal{I}$ be an arbitrary interpretation, and let C be an arbitrary clause in F, i.e. $C = (G_{\mathcal{J}} \wedge S_{\mathcal{J}})$ for some $\mathcal{J}$. We distinguish two cases:

1. $\mathcal{I} = \mathcal{J}$. Then $deg_{\mathcal{L}}(\mathcal{I}, G_{\mathcal{J}}) = 1$ and $deg_{\mathcal{L}}(\mathcal{I}, S_{\mathcal{J}}) = s(\mathcal{I})$, which implies that $deg_{\mathcal{L}}(\mathcal{I}, C) = s(\mathcal{I})$.

2. $\mathcal{I} \neq \mathcal{J}$. Then $deg_{\mathcal{L}}(\mathcal{I}, G_{\mathcal{J}}) = \infty$ and therefore $deg_{\mathcal{L}}(\mathcal{I}, C) = \infty$.

By construction, there is exactly one clause in F such that $\mathcal{I} = \mathcal{J}$. Since this clause is satisfied with degree $s(\mathcal{I})$ by $\mathcal{I}$, and all other clauses are ascribed a degree of ∞ by $\mathcal{I}$, we have that $deg_{\mathcal{L}}(\mathcal{I}, F) = s(\mathcal{I})$. □

4.2 Basic Exhaustive Choice Logics

Recall the definition of basic choice formulas (Definition 27). As we know from [BBB04], any QCL-formula F can be transformed into a basic QCL-formula F' such that $F \equiv_f^{\mathrm{QCL}} F'$. This transformation is realized with a normal form function. Such a function is also described in [BB16] for the original CCL. We will now examine under which circumstances a transformation to basic formulas is possible in general. For this, we need a variant of exhaustive choice logics.

Definition 41. *A choice logic $\mathcal{L}$ is called basic-exhaustive if for every $k \in \mathbb{N}$ there is a basic, exhaustive $\mathcal{L}$-formula F with $opt_{\mathcal{L}}(F) = k$.*

Of course, every basic-exhaustive choice logic is also exhaustive. As can be deduced from the proof of Proposition 14, QCL, CCL, XCL, and SCCL are basic-exhaustive. PL and LCL on the other hand are not, since they are not exhaustive. Combining a basic-exhaustive CL with another choice logic yields a basic-exhaustive CL. The proof is analogous to the proof of Lemma 15.

Proposition 18. *Let $\mathcal{L}$ be a basic-exhaustive choice logic. Then for every $\mathcal{L}$-formula there exists a basic $\mathcal{L}$-formula F' such that $F \equiv_f^{\mathcal{L}} F'$.*

Proof. Let F be an $\mathcal{L}$-formula. Let $M_k = \{\mathcal{I} \mid \mathcal{I} \models_k^{\mathcal{L}} F\}$. Without loss of generality, we can assume that M_k is a finite set, and that every interpretation we are dealing with is finite (see Lemma 7). Let G_k be the formula that characterizes the interpretations ascribing a degree of k to F, i.e.

$$G_k = \bigvee_{\mathcal{I} \in M_k} \Big((\bigwedge_{x \in \mathcal{I}} x) \land (\bigwedge_{x \in var(F) \setminus \mathcal{I}} \neg x) \Big).$$

We are assuming $\mathcal{L}$ to be basic-exhaustive, and therefore there is a basic, exhaustive $\mathcal{L}$-formula F^* such that $opt_{\mathcal{L}}(F^*) = opt_{\mathcal{L}}(F)$. Let $A_1, \dots, A_r$ be those classical formulas in F^* to which the choice connectives are applied, e.g. $F^* = A_1 \circ \dots \circ A_r$ if F^* is built with an associative choice connective. Of course, F^* could have any other structure, e.g. $F^* = ((A_1 \circ A_2) \circ (A_3 \circ A_4))$. To satisfy F^* with a degree of k, a certain subset of $\{A_1, \dots, A_r\}$ needs to be satisfied. This information will be contained in a set of indices, named S_k: For every $k \in \{1, \dots, opt_{\mathcal{L}}(F), \infty\}$, we pick exactly one interpretation $\mathcal{J}_k$ such that $\mathcal{J}_k \models_k^{\mathcal{L}} F^*$. This is possible, since F^* is exhaustive. Let $S_k = \{i \mid \mathcal{J}_k \models A_i\}$. Observe that $S_k \subseteq \{1, \dots, r\}$ and $S_k \neq S_l$ if $k \neq l$. Let $T_i = \{k \mid i \in S_k\}$ for $1 \leq i \leq r$, and consider

$$C_i = \bigvee_{k \in T_i} G_k.$$

We now construct F' by replacing A_i in F^* by C_i for every $1 \leq i \leq r$. Because $A_1, \dots A_r$ and $C_1, \dots C_r$ are classical formulas, we have that $opt_{\mathcal{L}}(F) = opt_{\mathcal{L}}(F^*) = opt_{\mathcal{L}}(F')$. It remains to show that $F \equiv_d^{\mathcal{L}} F'$:

- Assume $\mathcal{I} \models_k^{\mathcal{L}} F$. Then $\mathcal{I} \in M_k$, which entails that $\mathcal{I} \models G_k$ and $\mathcal{I} \not\models G_l$ for $k \neq l$. Thus, by construction, we have that $\mathcal{I} \models C_i$ iff $k \in T_i$ iff $i \in S_k$. Also observe that the basic structure of F^* is still intact in F', i.e. the choice connectives are still arranged in the same way as before, except that every A_i has been replaced with C_i. Therefore, if an interpretation satisfies exactly those C_i such that $i \in S_k$, then this interpretation satisfies F' with a degree of k. Thus, $\mathcal{I} \models_k^{\mathcal{L}} F'$.

- Assume $\mathcal{I} \models_k^{\mathcal{L}} F'$. Towards a contradiction, assume that $\mathcal{I} \models_s^{\mathcal{L}} F$ with $s \neq k$. By the same argument as above, we have that $\mathcal{I} \models_s^{\mathcal{L}} F'$. But this is impossible, since $deg_{\mathcal{L}}$ is defined by a function, and therefore it cannot be that $\mathcal{I} \models_k^{\mathcal{L}} F'$ and $\mathcal{I} \models_s^{\mathcal{L}} F'$ hold simultaneously. Contradiction.

□

As a small example for the above construction, we will transform the QCL-formula

$$F = (x \vec{\times} y) \wedge z.$$

into a fully equivalent basic formula, using the above semantic transformation. Observe that $opt_{\mathrm{QCL}}(F) = 2$, as well as $M_1 = \{\{x,z\},\{x,y,z\}\}$, $M_2 = \{\{y,z\}\}$, and $M_\infty = \{\emptyset,\{x\},\{y\},\{z\},\{x,y\}\}$. We now construct G_k for every $k \in \{1,2,\infty\}$, i.e.

$$\begin{aligned}
&G_1 = (x \wedge \neg y \wedge z) \vee (x \wedge y \wedge z),\\
&G_2 = (\neg x \wedge y \wedge z), \text{ and}\\
&G_3 = (\neg x \wedge \neg y \wedge \neg z) \vee (x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z) \vee (x \wedge y \wedge \neg z).
\end{aligned}$$

For our basic, exhaustive QCL-formula, we pick $F^* = a_1 \vec{\times} a_2$. Then we pick one interpretation $\mathcal{J}_k$ for every $k \in \{1,2,\infty\}$ such that $J_k \models F^*$: $\mathcal{J}_1 = \{a_1,a_2\}$, $\mathcal{J}_2 = \{a_2\}$, $\mathcal{J}_\infty = \emptyset$. Note that we could have also picked $\{a_1\}$ as our interpretation satisfying F^* with a degree of 1. However, it is necessary to pick exactly one interpretation for every satisfaction degree. Next, we have $S_1 = \{1,2\}$, $S_2 = \{2\}$, and $S_\infty = \emptyset$. Finally, $T_1 = \{1\}$, since $1 \in S_1$, and $T_2 = \{1,2\}$, since $2 \in S_1$ and $2 \in S_2$. This gives us $C_1 = G_1$ and $C_2 = G_1 \vee G_2$, i.e.

$$F' = G_1 \vec{\times} (G_1 \vee G_2).$$

We can verify that $opt_{\mathrm{QCL}}(F) = opt_{\mathrm{QCL}}(F')$ and $\mathcal{I} \models_k^{\mathrm{QCL}} F'$ for every $\mathcal{I} \in M_k$, which means that $F \equiv_f^{\mathrm{QCL}} F'$.

As another example, consider the CCL-formula

$$F = (x_1 \vec{\odot} x_2) \wedge (y_1 \vec{\odot} y_2 \vec{\odot} y_3).$$

Then $opt_{\mathrm{CCL}}(F) = 3$. The construction of G_k for every $k \in \{1,2,3,\infty\}$ is analogous to the QCL example above. We then pick $F^* = (a_1 \vec{\odot} a_2 \vec{\odot} a_3)$. We choose $S_1 = \{1,2,3\}$, since all of a_1, a_2, and a_3 would need to be satisfied in order for F^* to be ascribed a degree of 1. Likewise, we choose $S_2 = \{1,2\}$, $S_3 = \{1\}$, and $S_\infty = \emptyset$. Observe that, for example, we could have also picked $S_3 = \{1,3\}$ or $S_\infty = \{2,3\}$. We can now construct

$$F' = C_1 \vec{\odot} C_2 \vec{\odot} C_3$$

with the help of our index sets, i.e.

$$\begin{aligned} C_1 &= G_1 \vee G_2 \vee G_3, \\ C_2 &= G_1 \vee G_2, \\ C_3 &= G_1. \end{aligned}$$

Now, if all of C_1, C_2, and C_3 are satisfied, this must be because G_1 is satisfied. Then one of the interpretations ascribing a degree of 1 to F must be satisfied, i.e. both F and F' are ascribed a degree of 1. Similarly, if only C_1 and C_2 are satisfied, then G_2 must be satisfied, and therefore both F and F' are ascribed a degree of 2.

The transformation to basic choice formulas provided in Proposition 18 is semantic in nature. However, the transformations described in [BBB04] and [BB16] are syntactic in nature, i.e. the formulas are transformed purely on the basis of their structure. In both cases, a function N is given that transforms non-basic formulas into basic formulas as follows:

1. $N(\neg(A_1 \circ \cdots \circ A_m)) = C$
2. $N((A_1 \circ \cdots \circ A_m) \wedge (B_1 \circ \cdots \circ B_n)) = C_1 \circ \cdots \circ C_k$
3. $N((A_1 \circ \cdots \circ A_m) \vee (B_1 \circ \cdots \circ B_n)) = C'_1 \circ \cdots \circ C'_k$

In the above, $\circ$ is a choice connective, A_i, B_i, C, C_i, and C'_i are classical formulas, and $k = max(m, n)$. If N is defined in a way such that $F \equiv^{\mathcal{L}}_f N(F)$, then N can be applied repeatedly to any formula until a fully equivalent basic formula is obtained. We will not give all transformations for QCL or CCL here, but as an example, we provide the transformation for disjunction in QCL:

$N((A_1 \vec{\times} \cdots \vec{\times} A_m) \vee (B_1 \vec{\times} \cdots \vec{\times} B_n)) = C_1 \circ \cdots \circ C_k$, where $k = max(m, n)$ and

$$C_i = \begin{cases} (A_i \vee B_i) & \text{if } i \leq min(m, n) \\ A_i & \text{if } m < i \leq n \\ B_i & \text{if } n < i \leq m \end{cases}$$

Such a syntactic transformation depends highly on the semantics of the given choice connective. We will therefore not attempt to generalize this result to other choice logics at this point.

4.3 Optionality Ignoring Choice Logics

For any choice connective $\circ$, the satisfaction degree of $F \circ G$ can depend on the optionalities and satisfaction degrees of F and G. Of course, it does not *have to* depend on all of these. It could, for example, depend only on the degrees of F and G, but not on their optionalities. In such a CL, formulas with differing optionalities might exist, but this never has any effect on satisfaction degree.

Definition 42. *A choice logic $\mathcal{L}$ is called optionality-ignoring if for all $\circ \in \mathcal{C}_\mathcal{L}$ it holds that if $deg_\mathcal{L}(\mathcal{I}, F) = deg_\mathcal{L}(\mathcal{I}, F')$ and $deg_\mathcal{L}(\mathcal{I}, G) = deg_\mathcal{L}(\mathcal{I}, G')$, then $deg_\mathcal{L}(\mathcal{I}, F \circ G) = deg_\mathcal{L}(\mathcal{I}, F' \circ G')$.*

An example for an optionality-ignoring CL is SCCL (Definition 35). It is clear that the satisfaction degree of any formula in SCCL never depends on optionality in any way, and that therefore SCCL is optionality-ignoring. PL is optionality-ignoring as well. QCL, CCL, XCL, and LCL on the other hand are not. For example, in QCL, x and $(x \vec{\times} x)$ are degree-equivalent. However, we have $\{y\} \models_2^{\text{QCL}} (x \vec{\times} y)$ but also $\{y\} \models_3^{\text{QCL}} ((x \vec{\times} x) \vec{\times} y)$. The constructions for CCL and LCL are analogous. In XCL, $(x \land \neg x)$ and $(x \vec{\oplus} x)$ are degree-equivalent, but $((x \land \neg x) \vec{\oplus} y)$ and $((x \vec{\oplus} x) \vec{\oplus} y)$ are not.

Combining two optionality-ignoring choice logics $\mathcal{L}$ and $\mathcal{L}'$ does not necessarily yield another optionality-ignoring choice logic. Consider the following:

Definition 43. *L_1 is the choice logic such that $\mathcal{C}_{L_1} = \{\bullet\}$,*

$$opt_{L_1}(F \bullet G) = max(opt_{L_1}(F), opt_{L_1}(G)), \text{ and}$$
$$deg_{L_1}(\mathcal{I}, F \bullet G) = max(opt_{L_1}(F), opt_{L_1}(G)).$$

Observe that every formula in L_1 has an optionality of 1, and thus, $deg_{L_1}(\mathcal{I}, F \bullet G) = 1$ for any $\mathcal{I}$, F, and G. This means that L_1 is optionality-ignoring. If we now combine SCCL and L_1, we have $deg_{(\text{SCCL} \cup L_1)}(\emptyset, x) = \infty$ and $deg_{(\text{SCCL} \cup L_1)}(\emptyset, x \circ x) = \infty$, but $deg_{(\text{SCCL} \cup L_1)}(\emptyset, x \bullet y) = 1$ and $deg_{(\text{SCCL} \cup L_1)}(\emptyset, (x \circ x) \bullet y)) = 2$. Therefore, $(\text{SCCL} \cup L_1)$ is not optionality-ignoring. The underlying reason for this is that optionality does play a role in L_1, but only if there exist formulas with optionalities other than 1. As it turns out, optionality-ignoring choice logics can be combined and still remain optionality-ignoring, as long as they are also exhaustive. Note that L_1 is not exhaustive, since it is not possible to obtain a degree of ∞ for $F \bullet G$.

Lemma 19. *Let $\mathcal{L}$ and $\mathcal{L}'$ be exhaustive, optionality-ignoring choice logics. Then $\mathcal{L} \cup \mathcal{L}'$ is optionality-ignoring.*

Proof. Let $\mathcal{I}$ be an interpretation. Let F, F', G, and G' be $(\mathcal{L} \cup \mathcal{L}')$-formulas such that $deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, F) = deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, F')$ and $deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, G) = deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, G')$. Let $\circ$ be any choice connective of $\mathcal{L} \cup \mathcal{L}'$. There are two possibilities: Either $\circ \in \mathcal{C}_\mathcal{L}$, or $\circ \in \mathcal{C}_{\mathcal{L}'}$. Because of symmetry, we only need to consider the case that $\circ \in \mathcal{C}_\mathcal{L}$. Since $\mathcal{L}$ is exhaustive, and by Lemma 16, we know that there is an $\mathcal{L}$-formula A such that $opt_\mathcal{L}(A) = opt_{(\mathcal{L} \cup \mathcal{L}')}(F)$ and $deg_\mathcal{L}(\mathcal{I}, A) = deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, F)$. Analogously, there are $\mathcal{L}$-formulas A', B and B' that are equivalent in the same way to F', G, and G' respectively. Therefore, because the semantics of $\circ$ in $(\mathcal{L} \cup \mathcal{L}')$ is given by the same function as in $\mathcal{L}$, we have $deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, F \circ G) = deg_\mathcal{L}(\mathcal{I}, A \circ B)$ and $deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, F' \circ G') = deg_\mathcal{L}(\mathcal{I}, A' \circ B')$. Since $\mathcal{L}$ is optionality-ignoring, we also have $deg_\mathcal{L}(\mathcal{I}, A \circ B) = deg_\mathcal{L}(\mathcal{I}, A' \circ B')$, i.e.

$$deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, F \circ G) = deg_\mathcal{L}(\mathcal{I}, A \circ B) = deg_\mathcal{L}(\mathcal{I}, A' \circ B') = deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, F' \circ G').$$

□

4.4 Optionality Differentiating Choice Logics

If a choice logic is not optionality-ignoring, then the satisfaction degree of $F \circ G$ may depend on the optionality of F or G. Even if it does, it might depend on the optionality of only some F,G. For example, a CL could be defined in a way such that the optionality of G only plays a role if it is equal to some constant number, e.g. if $opt_{\mathcal{L}}(G) = 3$. Therefore, one could define a more restrictive family of CLs, in which as soon as F and G have different optionalities, this can have an impact on satisfaction degree.

Definition 44. *A choice logic $\mathcal{L}$ is called optionality-differentiating if for all $\mathcal{L}$-formulas A and B with $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$, there is an $\mathcal{L}$-formula F such that $F \not\equiv_d^{\mathcal{L}} F[A/B]$.*

Observe that QCL and CCL are optionality-differentiating choice logics. Let A and B be QCL (or CCL) formulas such that $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$. For QCL, choose $F = ((A \wedge x \wedge \neg x)\vec{\times}y)$. Since $(A \wedge x \wedge \neg x)$ can never be satisfied, and $opt_{\text{QCL}}(A \wedge x \wedge \neg x) = opt_{\text{QCL}}(A)$, we have that F is satisfied by $\{y\}$ with a degree of $opt_{\text{QCL}}(A) + 1$. On the other hand, $F[A/B]$ is satisfied by $\{y\}$ with a degree of $opt_{\text{QCL}}(B) + 1$. Similarly, for CCL we can choose $F = (y\vec{\odot}(A \wedge x \wedge \neg x))$. Interestingly, classical propositional logic is optionality-differentiating as well. Since there are no formulas A and B such that $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$, the statement in Definition 44 is vacuously true. Consequently, PL is both optionality-ignoring and optionality-differentiating.

SCCL is, however, not optionality-differentiating: While $opt_{\text{SCCL}}(x) \neq opt_{\text{SCCL}}(x \circ x)$ holds, it is also true that $x \equiv_d^{\text{SCCL}} (x \circ x)$. More generally, since the degree of $F \circ G$ depends only on the satisfaction degrees of F and G, and not on their optionalities, we have that $F \equiv_d^{\text{SCCL}} F[x/(x \circ x)]$ for all $F \in \mathcal{F}_{\text{SCCL}}$.

There also exist choice logics that are neither optionality-ignoring, nor optionality-differentiating. Consider the following:

Definition 45. *L_2 is the choice logic such that $\mathcal{C}_{L_2} = \{\circ\}$,*

$$opt_{L_2}(F \circ G) = opt_{L_2}(F) + opt_{L_2}(G),$$

and

$$deg_{L_2}(\mathcal{I}, F \circ G) = \begin{cases} deg_{L_2}(\mathcal{I}, F) & \text{if } deg_{L_2}(\mathcal{I}, F) < \infty \text{ and } opt_{L_2}(G) \neq 3 \\ deg_{L_2}(\mathcal{I}, F) + opt_{L_2}(\mathcal{I}, G) & \text{if } deg_{L_2}(\mathcal{I}, F) < \infty \text{ and } opt_{L_2}(G) = 3 \\ \infty & \text{otherwise} \end{cases}$$

In L_2, when the optionality of G is equal to 3, the satisfaction degree of $F \circ G$ depends on the optionality of G. Otherwise, the degree of $F \circ G$ does not depend on optionality in any way. To concretely show that L_2 is not optionality-ignoring, we will look at the formulas x and $(x \circ (x \circ x))$. Both are satisfied to a degree of 1 under the interpretation $\{x, y\}$. However, $\{x, y\}$ ascribes a degree of 1 to $(y \circ x)$ and, because of optionality, a degree of

4 to $(y \circ (x \circ (x \circ x)))$. To show that L_2 is not optionality-differentiating, consider the formulas $A = (x \circ (x \circ (x \circ x)))$ and $B = (x \circ (x \circ (x \circ (x \circ x))))$. Observe that $opt_{L_2}(A) = 4$ and $opt_{L_2}(B) = 5$. This means that a formula G has an optionality of 3 if and only if $G[A/B]$ has an optionality of 3 as well. Also observe that $A \equiv_d^{L_2} B$. From these two facts, we can infer that for any formula F, it holds that $F \equiv_d^{L_2} F[A/B]$. Therefore, since also $opt_{L_2}(A) \neq opt_{L_2}(B)$, we know that L_2 is not optionality-differentiating. One could prove this more formally by structural induction.

As for combining optionality-differentiating choice logics, we have essentially the same problem as the one we encountered for optionality-ignoring CLs: A choice logic may be optionality-differentiating, but when combined with another choice logic, new formulas with new optionality values could be created, and the combined choice logic may no longer be optionality-differentiating. We therefore introduce the following:

Definition 46. *Let $\mathcal{L}$ be a choice logic, and let $\circ \in \mathcal{C}_{\mathcal{L}}$. Let g be the function over which the satisfaction degree of $\circ$ is defined, i.e.*

$$deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = g(opt_{\mathcal{L}}(F), opt_{\mathcal{L}}(G), deg_{\mathcal{L}}(\mathcal{I}, F), deg_{\mathcal{L}}(\mathcal{I}, G))$$

for all $\mathcal{L}$-formulas F and G. Then $\circ$ is called naturally optionality-differentiating if either $g(k, 1, \infty, 1) \neq g(k', 1, \infty, 1)$ or $g(1, k, 1, \infty) \neq g(1, k', 1, \infty)$ holds for all $k, k' \in \mathbb{N}$ such that $k \neq k'$.

Choice logics that have a naturally optionality-differentiating choice connective will themselves be called naturally optionality-differentiating. By their semantics, QCL, CCL, XCL, and LCL are naturally optionality-differentiating.

Lemma 20. *If a choice logic is naturally optionality-differentiating, then it is also optionality-differentiating.*

Proof. Let $\mathcal{L}$ be a naturally optionality-differentiating choice logic. Let A and B be $\mathcal{L}$-formulas such that $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$. Then, since $\mathcal{L}$ is naturally optionality-differentiating, there are two cases:

1. $g(opt_{\mathcal{L}}(A), 1, \infty, 1) \neq g(opt_{\mathcal{L}}(B), 1, \infty, 1)$. Then choose $F = ((A \wedge x \wedge \neg x) \vec{\times} y)$. Since $(A \wedge x \wedge \neg x)$ can never be satisfied, and $opt_{\mathrm{QCL}}(A \wedge x \wedge \neg x) = opt_{\mathrm{QCL}}(A)$, we have that F is satisfied by $\{y\}$ with a degree of $g(opt_{\mathcal{L}}(A), 1, \infty, 1)$. On the other hand, $F[A/B]$ is satisfied by $\{y\}$ with a degree of $g(opt_{\mathcal{L}}(B), 1, \infty, 1)$. Thus, $deg_{\mathcal{L}}(\{y\}, F) \neq deg_{\mathcal{L}}(\{y\}, F[A/B])$, i.e. $F \not\equiv_d^{\mathcal{L}} F[A/B]$.

2. $g(1, opt_{\mathcal{L}}(A), 1, \infty) \neq g(1, opt_{\mathcal{L}}(B), 1, \infty)$. Analogous to the first case, except that we choose $F = (y \vec{\times} (A \wedge x \wedge \neg x))$.

$\square$

By definition, combining a naturally optionality-differentiating CL with any other CL yields a naturally optionality-differentiating CL.

Lemma 21. *Let $\mathcal{L}$ be a naturally optionality-differentiating choice logic, and let $\mathcal{L}'$ be any choice logic. Then $\mathcal{L} \cup \mathcal{L}'$ is a naturally optionality-differentiating choice logic.*

Proof. This follows directly from the fact that if $\circ \in \mathcal{C}_{\mathcal{L}}$, then $\circ \in \mathcal{C}_{(\mathcal{L} \cup \mathcal{L}')}$, and from Definition 46. □

From Lemma 21, and from the fact that QCL is naturally optionality-differentiating, we can infer that QCCL is naturally optionality-differentiating as well.

4.5 Reasonable Choice Logics

If a formula has a finite satisfaction degree under some interpretation, we say that it is *satisfied* to this degree. If it has a satisfaction degree of ∞ however, we say that the formula is *not satisfied* to any (finite) degree. We will now examine how this concept of satisfaction with respect to degrees relates to the notion of satisfaction in PL. Choice logics in which the two concepts are related in a natural way will be called reasonable. We will first attempt the following:

Definition 47. *A choice logic $\mathcal{L}$ is called seemingly reasonable if for every $\mathcal{L}$-formula F there exists a PL-formula F' such that for all interpretations $\mathcal{I}$ it holds that $deg_{\mathcal{L}}(\mathcal{I}, F) < \infty$ iff $\mathcal{I} \models F'$.*

This is actually the case for all choice logics. Because of Lemma 7, we only need to consider interpretations such that $\mathcal{I} \subseteq var(F)$, i.e. we only need to consider a finite number of interpretations. For any formula F, there is a finite set S of these interpretations such that $deg_{\mathcal{L}}(\mathcal{I}, F) < \infty$. If $S = \emptyset$, then construct $F' = (x \wedge \neg x)$. Otherwise, construct

$$F' = \bigvee_{\mathcal{I} \in S} \Big(\Big(\bigwedge_{x \in \mathcal{I}} x \Big) \wedge \Big(\bigwedge_{x \in var(F) \setminus \mathcal{I}} \neg x \Big) \Big).$$

Then for all $\mathcal{I}$ it holds that $deg_{\mathcal{L}}(\mathcal{I}, F) < \infty$ iff $\mathcal{I} \models F'$. However, this is hardly satisfying. The above transformation from F into F' is not natural in the sense that the structure of F is not preserved in F' in any shape or form.

A possible remedy is to require that every choice connective of a CL has a designated classical binary connective as a counterpart, as is the case for QCL and CCL. The choice connective in QCL is called ordered disjunction, and can indeed be (partly) characterized by classical disjunction: For any interpretation $\mathcal{I}$, it holds that $x \vec{\times} y$ has a finite satisfaction degree under some interpretation $\mathcal{I}$ exactly when $x \vee y$ is classically satisfied by $\mathcal{I}$. The ordered conjunction in CCL can not be characterized by classical conjunction in quite the same way. It can, however, be characterized by another classical binary connective, namely by the first projection: $x \vec{\odot} y$ has a finite satisfaction degree under some interpretation $\mathcal{I}$ exactly when x is classically satisfied by $\mathcal{I}$. For now, we will only consider connectives applied to atoms:

Definition 48. *Let $\mathcal{L}$ be a choice logic, let $\circ \in \mathcal{C}_{\mathcal{L}}$, and let $\circledast$ be one of the 16 classical binary connectives. Then $\circledast$ is a classical counterpart for $\circ$ if*

$$deg_{\mathcal{L}}(\mathcal{I}, x \circ y) < \infty \text{ iff } \mathcal{I} \models (x \circledast y).$$

holds for all interpretations $\mathcal{I}$ and all propositional variables x and y.

Looking, for example, at XCL, one can easily see that exclusive disjunction is a classical counterpart for ordered exclusive disjunction. Should we therefore call those choice logics reasonable for which every choice connective possesses a classical counterpart? No, since again, this is the case for every CL.

Proposition 22. *Let $\mathcal{L}$ be a choice logic and let $\circ \in \mathcal{C}_{\mathcal{L}}$. Then $\circ$ has exactly one classical counterpart.*

Proof. There are four interpretations relevant to $(x \circ y)$, namely $\emptyset$, $\{x\}$, $\{y\}$, and $\{x, y\}$. For any of them, we will either have $deg_{\mathcal{L}}(\mathcal{I}, x \circ y) < \infty$ or $deg_{\mathcal{L}}(\mathcal{I}, x \circ y) = \infty$. We can simply pick the classical binary connective that is *true* under $\mathcal{I}$ when $deg_{\mathcal{L}}(\mathcal{I}, x \circ y) < \infty$ and *false* under $\mathcal{I}$ when $deg_{\mathcal{L}}(\mathcal{I}, x \circ y) = \infty$. There is exactly one such classical binary connective. $\square$

The point is not that $\vec{\times}$ has a classical counterpart, but that one can take any formula F of QCL, replace $\vec{\times}$ by its counterpart, and obtain a formula F' of PL such that F has a finite degree under $\mathcal{I}$ if and only if F' is classically satisfied by $\mathcal{I}$. This fact is contained as Proposition 1 in [BBB04]. We therefore define the following:

Definition 49. *Let $\mathcal{L}$ be a choice logic. Let x be a propositional variable, and let F and G be $\mathcal{L}$-formulas. Then the classical counterpart of an $\mathcal{L}$-formula is given by the function $cp_{\mathcal{L}} \colon \mathcal{F}_{\mathcal{L}} \to \mathcal{F}_{PL}$ such that*

1. $cp_{\mathcal{L}}(x) = x$;
2. $cp_{\mathcal{L}}(\neg F) = \neg(cp_{\mathcal{L}}(F))$;
3. $cp_{\mathcal{L}}(F \wedge G) = (cp_{\mathcal{L}}(F) \wedge cp_{\mathcal{L}}(G))$;
4. $cp_{\mathcal{L}}(F \vee G) = (cp_{\mathcal{L}}(F) \vee cp_{\mathcal{L}}(G))$;
5. $cp_{\mathcal{L}}(F \circ G) = (cp_{\mathcal{L}}(F) \circledast cp_{\mathcal{L}}(G))$ *for every* $\circ \in \mathcal{C}_{\mathcal{L}}$, *where* $\circledast$ *is the classical counterpart of* $\circ$.

The above transformation from $\mathcal{L}$-formula to PL-formula simply replaces every choice connective by its classical counterpart. Note that we distinguish between the classical counterpart of a choice connective (e.g. $\lor$ for $\vec{\times}$) and the classical counterpart of a formula (e.g. $(x \lor (y \land z))$ for $(x \vec{\times} (y \land z))$).

Definition 50. *A choice logic $\mathcal{L}$ is called reasonable if*

$$deg_{\mathcal{L}}(\mathcal{I}, F) < \infty \text{ iff } \mathcal{I} \models cp_{\mathcal{L}}(F).$$

holds for all interpretations $\mathcal{I}$ and all $\mathcal{L}$-formulas F.

PL is trivially reasonable in this sense. We will now show that some of the other choice logics we have encountered so far are reasonable as well.

Proposition 23. *QCL, CCL, XCL, SCCL, LCL, and L_2 are reasonable choice logics.*

Proof. We will prove that XCL is reasonable by structural induction. The proofs for the remaining choice logics are analogous.

- Base case: $F = x$. Then $cp_{\mathrm{XCL}}(x) = x$. For any interpretation $\mathcal{I}$ we have that $deg_{\mathrm{XCL}}(\mathcal{I}, x) = 1$ if $x \in \mathcal{I}$, and $deg_{\mathrm{XCL}}(\mathcal{I}, x) = \infty$ if $x \notin \mathcal{I}$. Thus, $deg_{\mathrm{XCL}}(\mathcal{I}, x) < \infty$ iff $\mathcal{I} \models cp_{\mathcal{L}}(F)$.
- Step case: Assume that $deg_{\mathcal{L}}(\mathcal{I}, G) < \infty$ iff $\mathcal{I} \models cp_{\mathcal{L}}(G)$ and $deg_{\mathcal{L}}(\mathcal{I}, H) < \infty$ iff $\mathcal{I} \models cp_{\mathcal{L}}(H)$ holds for all interpretations $\mathcal{I}$.

  1. $F = \neg G$. Then

  $$\begin{aligned} deg_{\mathrm{XCL}}(\mathcal{I}, F) < \infty &\iff deg_{\mathrm{XCL}}(\mathcal{I}, G) = \infty \\ &\iff \mathcal{I} \not\models cp_{\mathrm{XCL}}(G) \\ &\iff \mathcal{I} \models \neg(cp_{\mathrm{XCL}}(G)) \\ &\iff \mathcal{I} \models cp_{\mathrm{XCL}}(F). \end{aligned}$$

  2. $F = G \land H$. Then

  $$\begin{aligned} deg_{\mathrm{XCL}}(\mathcal{I}, F) < \infty &\iff max(deg_{\mathrm{XCL}}(\mathcal{I}, G), deg_{\mathrm{XCL}}(\mathcal{I}, H)) < \infty \\ &\iff deg_{\mathrm{XCL}}(\mathcal{I}, G) < \infty \text{ and } deg_{\mathrm{XCL}}(\mathcal{I}, H) < \infty \\ &\iff \mathcal{I} \models cp_{\mathrm{XCL}}(G) \text{ and } \mathcal{I} \models cp_{\mathrm{XCL}}(H) \\ &\iff \mathcal{I} \models (cp_{\mathrm{XCL}}(G) \land cp_{\mathrm{XCL}}(H)) \\ &\iff \mathcal{I} \models cp_{\mathrm{XCL}}(F). \end{aligned}$$

  3. $F = G \lor H$. Analogous to the case that $F = G \land H$.

4. $F = G \vec{\oplus} H$. Then

$$\begin{aligned}
deg_{\mathrm{XCL}}(\mathcal{I}, F) < \infty \iff & (deg_{\mathrm{XCL}}(\mathcal{I}, G) < \infty \text{ and } deg_{\mathrm{XCL}}(\mathcal{I}, H) = \infty) \text{ or} \\
& (deg_{\mathrm{XCL}}(\mathcal{I}, G) = \infty \text{ and } deg_{\mathrm{XCL}}(\mathcal{I}, H) < \infty) \\
\iff & (\mathcal{I} \models cp_{\mathrm{XCL}}(G) \text{ and } \mathcal{I} \not\models cp_{\mathrm{XCL}}(H)) \text{ or} \\
& (\mathcal{I} \not\models cp_{\mathrm{XCL}}(G) \text{ and } \mathcal{I} \models cp_{\mathrm{XCL}}(H)) \\
\iff & \mathcal{I} \models (cp_{\mathrm{XCL}}(G) \oplus cp_{\mathrm{XCL}}(H)) \\
\iff & \mathcal{I} \models cp_{\mathrm{XCL}}(F).
\end{aligned}$$

□

We still need to ensure that we have not again accidentally captured every choice logic with our definition of reasonable. We therefore introduce the following CL, which is based on SCCL.

Definition 51. *L_3 is the choice logic such that $\mathcal{C}_{L_3} = \{\circ\}$,*

$$opt_{L_3}(F \circ G) = opt_{L_3}(F) + 1,$$

and

$$deg_{L_3}(\mathcal{I}, F \circ G) = \begin{cases} deg_{L_3}(\mathcal{I}, F) & \text{if } deg_{L_3}(\mathcal{I}, F) < 3 \text{ and } deg_{L_3}(\mathcal{I}, G) < 3 \\ deg_{L_3}(\mathcal{I}, F) + 1 & \text{if } deg_{L_3}(\mathcal{I}, F) < 3 \text{ and } deg_{L_3}(\mathcal{I}, G) \geq 3 \\ \infty & \text{otherwise} \end{cases}$$

The classical counterpart for the choice connective of L_3 is the first projection, just as in SCCL. This means that $cp_{L_3}(x \circ y) = x$. Consider the L_3-formula $F = (((x \circ y) \circ y) \circ y)$. Since $\{x\} \models_{\infty}^{L_3} y$, we have that$\{x\} \models_{2}^{L_3} (x \circ y)$. Consequently, $\{x\} \models_{3}^{L_3} ((x \circ y) \circ y)$. It follows that $\{x\} \models_{\infty}^{L_3} F$. But $cp_{L_3}(F) = x$, and therefore $\{x\} \models cp_{L_3}(F)$. We can conclude that L_3 is not a reasonable CL.

Reasonable choice logics can be combined and will remain reasonable, provided they are exhaustive.

Lemma 24. *Let $\mathcal{L}$ and $\mathcal{L}'$ be exhaustive, reasonable choice logics. Then $\mathcal{L} \cup \mathcal{L}'$ is reasonable.*

Proof. We will show this by structural induction. The base case and the case for the classical connectives are already contained in the proof of Proposition 23. Therefore, we only need to consider the case that $F = G \circ H$, where $\circ \in \mathcal{L}$. Because of symmetry, the case that $\circ \in \mathcal{L}'$ is analogous. Let $\mathcal{I}$ be any interpretation. As our I.H., assume that $deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, G) < \infty$ iff $\mathcal{I} \models cp_{(\mathcal{L} \cup \mathcal{L}')}(G)$ and $deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, H) < \infty$ iff $\mathcal{I} \models cp_{(\mathcal{L} \cup \mathcal{L}')}(H)$.

Because $\mathcal{L}$ is exhaustive, and by Lemma 16, there are $\mathcal{L}$-formulas G' and H' such that $opt_{\mathcal{L}}(G') = opt_{(\mathcal{L} \cup \mathcal{L}')}(G)$, $deg_{\mathcal{L}}(\mathcal{I}, G') = deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, G)$, $opt_{\mathcal{L}}(H') = opt_{(\mathcal{L} \cup \mathcal{L}')}(H)$, and $deg_{\mathcal{L}}(\mathcal{I}, H') = deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, H)$. Since the semantics of $\circ$ depends only on the optionalities and satisfaction degrees of its operands, we also have that $deg_{\mathcal{L}}(\mathcal{I}, G' \circ H') = deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, G \circ H)$. By the I.H. and the fact that $\mathcal{L}$ is reasonable we have that

$$\mathcal{I} \models cp_{(\mathcal{L} \cup \mathcal{L}')}(G) \iff deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, G) < \infty \iff deg_{\mathcal{L}}(\mathcal{I}, G') < \infty \iff \mathcal{I} \models cp_{\mathcal{L}}(G').$$

Analogously, $\mathcal{I} \models cp_{(\mathcal{L} \cup \mathcal{L}')}(H)$ iff $\mathcal{I} \models cp_{\mathcal{L}}(H')$. Let $\circledast$ be the classical counterpart of $\circ$. Then, since $\mathcal{L}$ is reasonable,

$$\begin{aligned}
deg_{(\mathcal{L} \cup \mathcal{L}')}(\mathcal{I}, G \circ H) < \infty &\iff deg_{\mathcal{L}}(\mathcal{I}, G' \circ H') < \infty \\
&\iff \mathcal{I} \models cp_{\mathcal{L}}(G' \circ H') \\
&\iff \mathcal{I} \models cp_{\mathcal{L}}(G') \circledast cp_{\mathcal{L}}(H') \\
&\iff \mathcal{I} \models cp_{(\mathcal{L} \cup \mathcal{L}')}(G) \circledast cp_{(\mathcal{L} \cup \mathcal{L}')}(H) \\
&\iff \mathcal{I} \models cp_{(\mathcal{L} \cup \mathcal{L}')}(G \circ H).
\end{aligned}$$

$\square$

CHAPTER 5

Strong Equivalence

We have already introduced the notions of degree-equivalence and full equivalence in Definitions 31 and 32. This section aims to investigate what we will call strong equivalence.

Definition 52. *Let A and B be formulas of some choice logic $\mathcal{L}$. A and B are strongly equivalent, written as $A \equiv_s^{\mathcal{L}} B$, if $Mod_{\mathcal{L}}(F) = Mod_{\mathcal{L}}(F[A/B])$ for all $\mathcal{L}$-formulas F.*

This definition is based on the concept of strong equivalence described by Faber et al. in [FTW13a]. In Sections 5.1 and 5.2 we will show that strong equivalence is the same as full equivalence in QCL and CCL. Section 5.3 will relate strong equivalence to degree- and full equivalence for choice logics in general, as well as specifically for optionality-ignoring and optionality-differentiating choice logics.

5.1 Qualitative Choice Logic

Before examining strong equivalence for arbitrary choice logics, we will investigate it in the context of QCL. The following lemma is analogous to Lemma 2, which is concerned with propositional logic.

Lemma 25. *$A \equiv_f^{QCL} B$ if and only if $F \equiv_f^{QCL} F[A/B]$ for all QCL-formulas F.*

Proof. The only-if-direction is already contained as a lemma in [BBB04]. The proof is not given explicitly there, but it is stated that it can be done by structural induction. The validity of the if-direction is not mentioned in [BBB04], but can be easily shown by contrapositive: Assume that $A \not\equiv_f^{\mathrm{QCL}} B$. Choose $F = A$. Then $F[A/B] = B$, and therefore $F \not\equiv_f^{\mathrm{QCL}} F[A/B]$.

For the sake of completeness, we will now explicitly prove the only-if-direction by structural induction. Assume $A \equiv_f^{\mathrm{QCL}} B$. Note that if A is not contained in F, then trivially $F[A/B] = F$, and therefore $F \equiv_f^{\mathrm{QCL}} F[A/B]$.

- Base case: If $F = x$, where x is a propositional variable, and if A is contained in F, then $A = x$. Thus, $F = A$ and $F[A/B] = B$. Since $A \equiv_f^{\mathrm{QCL}} B$, we have that $F \equiv_f^{\mathrm{QCL}} F[A/B]$.

- Induction step: As the I.H., assume that $G \equiv_f^{\mathrm{QCL}} G[A/B]$ and $H \equiv_f^{\mathrm{QCL}} H[A/B]$ for QCL-formulas G and H.

  1. $F = \neg G$. Then
  $$opt_{\mathrm{QCL}}(\neg G) = 1 = opt_{\mathrm{QCL}}(\neg G[A/B])$$
  and
  $$\begin{aligned} deg_{\mathrm{QCL}}(\mathcal{I}, \neg G) &= \begin{cases} 1 & \text{if } \mathcal{I} \models_{\infty}^{\mathrm{QCL}} G \\ \infty & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } \mathcal{I} \models_{\infty}^{\mathrm{QCL}} G[A/B] \\ \infty & \text{otherwise} \end{cases} \\ &= deg_{\mathrm{QCL}}(\mathcal{I}, \neg G[A/B]) \end{aligned}$$
  for any interpretation $\mathcal{I}$. Thus, $\neg G \equiv_f^{\mathrm{QCL}} \neg G[A/B]$, i.e. $F \equiv_f^{\mathrm{QCL}} F[A/B]$.
  2. $F = G \wedge H$. Then the replacement of A by B in F could occur in either G or H, i.e. $F[A/B] = (G[A/B] \wedge H)$ or $F[A/B] = (G \wedge H[A/B])$. Because of symmetry, we only need to consider the case that $F[A/B] = (G[A/B] \wedge H)$:
  $$\begin{aligned} opt_{\mathrm{QCL}}(G \wedge H) &= max(opt_{\mathrm{QCL}}(G), opt_{\mathrm{QCL}}(H)) \\ &= max(opt_{\mathrm{QCL}}(G[A/B]), opt_{\mathrm{QCL}}(H)) \\ &= opt_{\mathrm{QCL}}(G[A/B] \wedge H) \end{aligned}$$
  and
  $$\begin{aligned} deg_{\mathrm{QCL}}(\mathcal{I}, G \wedge H) &= max(deg_{\mathrm{QCL}}(\mathcal{I}, G), deg_{\mathrm{QCL}}(\mathcal{I}, H)) \\ &= max(deg_{\mathrm{QCL}}(\mathcal{I}, G[A/B]), deg_{\mathrm{QCL}}(\mathcal{I}, H)) \\ &= deg_{\mathrm{QCL}}(\mathcal{I}, G[A/B] \wedge H) \end{aligned}$$
  for any interpretation $\mathcal{I}$. Thus, $(G \wedge H) \equiv_f^{\mathrm{QCL}} (G[A/B] \wedge H)$, i.e. $F \equiv_f^{\mathrm{QCL}} F[A/B]$.
  3. $F = G \vee H$. Identical to $F = G \wedge H$, except for using *min* instead of *max* for $deg_{\mathrm{QCL}}(\mathcal{I}, G \vee H)$.
  4. $F = G \vec{\times} H$. Consider the case that $F[A/B] = (G[A/B] \vec{\times} H)$. Then
  $$\begin{aligned} opt_{\mathrm{QCL}}(G \vec{\times} H) &= opt_{\mathrm{QCL}}(G) + opt_{\mathrm{QCL}}(H) \\ &= opt_{\mathrm{QCL}}(G[A/B]) + opt_{\mathrm{QCL}}(H) \\ &= opt_{\mathrm{QCL}}(G[A/B] \vec{\times} H) \end{aligned}$$

and

$$deg_{\text{QCL}}(\mathcal{I}, G \vec{\times} H) = \begin{cases} m & \text{if } \mathcal{I} \models_m^{\text{QCL}} G \text{ and } m \neq \infty \\ n + opt_{\text{QCL}}(G) & \text{if } \mathcal{I} \models_\infty^{\text{QCL}} G,\ \mathcal{I} \models_n^{\text{QCL}} H, \text{ and } n \neq \infty \\ \infty & \text{otherwise} \end{cases}$$

$$= \begin{cases} m & \text{if } \mathcal{I} \models_m^{\text{QCL}} G[A/B] \text{ and } m \neq \infty \\ n + opt_{\text{QCL}}(G[A/B]) & \text{if } \mathcal{I} \models_\infty^{\text{QCL}} G[A/B],\ \mathcal{I} \models_n^{\text{QCL}} H, \\ & \text{and } n \neq \infty \\ \infty & \text{otherwise} \end{cases}$$

for any interpretation $\mathcal{I}$. Thus, $(G \vec{\times} H) \equiv_f^{\text{QCL}} (G[A/B] \vec{\times} H)$, i.e. $F \equiv_f^{\text{QCL}} F[A/B]$. The case that $F[A/B] = (G \vec{\times} H[A/B])$ is analogous.

$\square$

With the help of Lemma 25, we can show that strong equivalence can be characterized by full equivalence in QCL.

Proposition 26. *$A \equiv_s^{QCL} B$ if and only if $A \equiv_f^{QCL} B$.*

Proof. The if-direction of Proposition 26 is quite straight forward: If $A \equiv_f^{\text{QCL}} B$, then, by Lemma 25, $F \equiv_f^{\text{QCL}} F[A/B]$ holds for all QCL-formulas F. Consequently, by Lemma 6, $Mod_{\text{QCL}}(F) = Mod_{\text{QCL}}(F[A/B])$ holds for all QCL-formulas F, i.e. $A \equiv_s^{\text{QCL}} B$.

As for the only-if-direction, we proceed by contrapositive: Assume $A \not\equiv_f^{\text{QCL}} B$. It remains to show that $A \not\equiv_s^{\text{QCL}} B$, i.e. that there is a formula F such that $Mod_{\text{QCL}}(F) \neq Mod_{\text{QCL}}(F[A/B])$. If $A \not\equiv_f^{\text{QCL}} B$, there are two cases:

1. $A \not\equiv_d^{\text{QCL}} B$. Then there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models_m^{\text{QCL}} A$ and $\mathcal{I} \models_n^{\text{QCL}} B$ with $m \neq n$. Let $k = min(m, n)$, and let x_i and y_i with $1 \leq i \leq k$ be fresh variables that do not appear in $\mathcal{I}$, A, or B. We then construct $F = ((A \wedge x_1) \vee y_1)$ if $k = 1$, and

   $$F = (A \wedge (x_1 \vec{\times} \cdots \vec{\times} x_k) \wedge \bigwedge_{i=1}^{k-1} \neg x_i) \vee ((y_1 \vec{\times} \cdots \vec{\times} y_k) \wedge \bigwedge_{i=1}^{k-1} \neg y_i)$$

   if $k > 1$. Observe that the minimal degree with which F (or $F[A/B]$) can possibly be satisfied is k, as either x_k or y_k need to be satisfied. Furthermore, $\{y_k\} \models_k^{\text{QCL}} F$ and $\{y_k\} \models_k^{\text{QCL}} F[A/B]$. This means that any preferred model of F must satisfy F with a degree of k. The same is true for preferred models of $F[A/B]$.

   We can again distinguish two cases:

a) $k = m$. Then, $\mathcal{I} \models_k^{\text{QCL}} A$, and therefore also $\mathcal{I} \cup \{x_k\} \models_k^{\text{QCL}} A$. Thus, $\mathcal{I} \cup \{x_k\} \models_k^{\text{QCL}} F$, i.e. $\mathcal{I} \cup \{x_k\} \in Mod_{\text{QCL}}(F)$. Analogously, since $\mathcal{I} \models_n^{\text{QCL}} B$, we have that $\mathcal{I} \cup \{x_k\} \models_n^{\text{QCL}} B$ and $\mathcal{I} \cup \{x_k\} \models_n^{\text{QCL}} F[A/B]$. Since $n > k$, $\mathcal{I} \cup \{x_k\} \notin Mod_{\text{QCL}}(F[A/B])$.

b) $k = n$. Analogous to (a), but with $\mathcal{I} \cup \{x_k\} \notin Mod_{\text{QCL}}(F)$ and $\mathcal{I} \cup \{x_k\} \in Mod_{\text{QCL}}(F[A/B])$.

2. $opt_{\text{QCL}}(A) \neq opt_{\text{QCL}}(B)$. Let $k = min(opt_{\text{QCL}}(A), opt_{\text{QCL}}(B))$, and let x, y and z_i with $1 \leq i \leq k+1$ be fresh variables that do not appear in A or B. We then construct

$$F = ((A \wedge x \wedge \neg x) \vec{\times} y) \vee ((z_1 \vec{\times} \cdots \vec{\times} z_{k+1}) \wedge \bigwedge_{i=1}^{k} \neg z_i).$$

Observe that $(A \wedge x \wedge \neg x)$ can not be satisfied, and that $opt_{\text{QCL}}(A \wedge x \wedge \neg x) = opt_{\text{QCL}}(A)$. Analogously, $(B \wedge x \wedge \neg x)$ can not be satisfied, and $opt_{\text{QCL}}(B \wedge x \wedge \neg x) = opt_{\text{QCL}}(B)$. Thus, the the minimal degree with which F (or $F[A/B]$) can possibly be satisfied is $k+1$, as either y or z_{k+1} need to be satisfied. Furthermore, $\{z_{k+1}\} \models_{k+1}^{\text{QCL}} F$ and $\{z_{k+1}\} \models_{k+1}^{\text{QCL}} F[A/B]$. This means that any preferred model of F must satisfy F with a degree of $k+1$. The same is true for preferred models of $F[A/B]$.

We can again distinguish two cases:

a) $k = opt_{\text{QCL}}(A)$. Then $\{y\} \models_{k+1}^{\text{QCL}} F$, and therefore $\{y\} \in Mod_{\text{QCL}}(F)$. But $\{y\} \models_{n+1}^{\text{QCL}} F[A/B]$ with $n = opt_{\text{QCL}}(B)$. Thus, since $n > k$, $\{y\} \notin Mod_{\text{QCL}}(F[A/B])$

b) $k = opt_{\text{QCL}}(B)$. Analogous to (a), but with $\{y\} \notin Mod_{\text{QCL}}(F)$ and $\{y\} \in Mod_{\text{QCL}}(F[A/B])$.

□

5.2 Conjunctive Choice Logic

Similar results to those regarding strong equivalence in QCL can be found for CCL.

Lemma 27. *$A \equiv_f^{CCL} B$ if and only if $F \equiv_f^{CCL} F[A/B]$ for all CCL-formulas F.*

Proof. This proof is analogous to the proof for Lemma 25. The structural induction of the only-if-direction can be altered to fit the definition of CCL rather than QCL: Instead of the case that $F = G \vec{\times} H$, we have the case that $F = G \vec{\odot} H$. If $F[A/B] = G[A/B] \vec{\odot} H$, then

$$\begin{aligned} opt_{\text{CCL}}(G \vec{\odot} H) &= opt_{\text{CCL}}(G) + opt_{\text{CCL}}(H) \\ &= opt_{\text{CCL}}(G[A/B]) + opt_{\text{CCL}}(H) \\ &= opt_{\text{CCL}}(G[A/B] \vec{\odot} H) \end{aligned}$$

and

$$\begin{aligned} deg_{\text{CCL}}(\mathcal{I}, G \vec{\odot} H) &= \begin{cases} n & \text{if } \mathcal{I} \models_1^{\text{CCL}} G \text{ and } \mathcal{I} \models_n^{\text{CCL}} H \text{ with } n \neq \infty \\ m + opt_{\text{CCL}}(H) & \text{if } \mathcal{I} \models_m^{\text{CCL}} G,\ m \neq \infty, \\ & \text{and } (m > 1 \text{ or } \mathcal{I} \models_\infty^{\text{CCL}} H) \\ \infty & \text{otherwise} \end{cases} \\ &= \begin{cases} n & \text{if } \mathcal{I} \models_1^{\text{CCL}} G[A/B] \text{ and } \mathcal{I} \models_n^{\text{CCL}} H \text{ with } n \neq \infty \\ m + opt_{\text{CCL}}(H) & \text{if } \mathcal{I} \models_m^{\text{CCL}} G[A/B],\ m \neq \infty, \\ & \text{and } (m > 1 \text{ or } \mathcal{I} \models_\infty^{\text{CCL}} H) \\ \infty & \text{otherwise} \end{cases} \\ &= deg_{\text{CCL}}(\mathcal{I}, G[A/B] \vec{\odot} H) \end{aligned}$$

for any interpretation $\mathcal{I}$. Thus, $(G \vec{\odot} H) \equiv_f^{\text{CCL}} (G[A/B] \vec{\odot} H)$, i.e. $F \equiv_f^{\text{CCL}} F[A/B]$. The case that $F[A/B] = (G \vec{\odot} H[A/B])$ is analogous. □

Proposition 28. *$A \equiv_s^{CCL} B$ if and only if $A \equiv_f^{CCL} B$.*

Proof. The if-direction of this proof is analogous to the proof of Proposition 26. The only-if-direction is also similar to Proposition 26, but requires a slightly different construction of F. We again argue by contrapositive: Assume $A \not\equiv_f^{\text{CCL}} B$. Now there are two cases:

1. $A \not\equiv_d^{\text{CCL}} B$. Then there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models_m^{\text{CCL}} A$ and $\mathcal{I} \models_n^{\text{CCL}} B$ with $m \neq n$. Let $k = min(m, n)$, and let x_i and y_i with $1 \leq i \leq k$ be fresh variables that do not appear in $\mathcal{I}$, A, or B. We then construct $F = ((A \wedge x_1) \vee y_1)$ if $k = 1$, and

   $$F = (A \wedge (x_1 \vec{\odot} \cdots \vec{\odot} x_k) \wedge \bigwedge_{i=2}^{k} \neg x_i) \vee ((y_1 \vec{\odot} \cdots \vec{\odot} y_k) \wedge \bigwedge_{i=2}^{k} \neg y_i)$$

   if $k > 1$. Observe that the minimal degree with which F (or $F[A/B]$) can possibly be satisfied is k, as either x_1 or y_1 need to be satisfied, but x_i or y_i with $2 \leq i \leq k$ can not be satisfied. Furthermore, $\{y_1\} \models_k^{\text{CCL}} F$ and $\{y_1\} \models_k^{\text{CCL}} F[A/B]$. This means that any preferred model of F must satisfy F with a degree of k. The same is true for preferred models of $F[A/B]$.

   We can again distinguish two cases:

a) $k = m$. Then, $\mathcal{I} \models_k^{\text{CCL}} A$, and therefore also $\mathcal{I} \cup \{x_1\} \models_k^{\text{CCL}} A$. Thus, $\mathcal{I} \cup \{x_1\} \models_k^{\text{CCL}} F$, i.e. $\mathcal{I} \cup \{x_1\} \in \mathit{Mod}_{\text{CCL}}(F)$. Analogously, since $\mathcal{I} \models_n^{\text{CCL}} B$, we have that $\mathcal{I} \cup \{x_1\} \models_n^{\text{CCL}} B$ and $\mathcal{I} \cup \{x_1\} \models_n^{\text{CCL}} F[A/B]$. Since $n > k$, $\mathcal{I} \cup \{x_1\} \notin \mathit{Mod}_{\text{CCL}}(F[A/B])$.

b) $k = n$. Analogous to (a), but with $\mathcal{I} \cup \{x_1\} \notin \mathit{Mod}_{\text{CCL}}(F)$ and $\mathcal{I} \cup \{x_1\} \in \mathit{Mod}_{\text{CCL}}(F[A/B])$.

2. $\mathit{opt}_{\text{CCL}}(A) \neq \mathit{opt}_{\text{CCL}}(B)$. Let $k = \mathit{min}(\mathit{opt}_{\text{CCL}}(A), \mathit{opt}_{\text{CCL}}(B))$, and let x, y and z_i with $1 \leq i \leq k+1$ be fresh variables that do not appear in A or B. We then construct

$$F = (y \vec{\odot} (A \wedge x \wedge \neg x)) \vee ((z_1 \vec{\odot} \cdots \vec{\odot} z_{k+1}) \wedge \bigwedge_{i=2}^{k+1} \neg z_i).$$

Observe that $(A \wedge x \wedge \neg x)$ can not be satisfied, and that $\mathit{opt}_{\text{CCL}}(A \wedge x \wedge \neg x) = \mathit{opt}_{\text{CCL}}(A)$. Analogously, $(B \wedge x \wedge \neg x)$ can not be satisfied, and $\mathit{opt}_{\text{CCL}}(B \wedge x \wedge \neg x) = \mathit{opt}_{\text{CCL}}(B)$. Thus, the the minimal degree with which F (or $F[A/B]$) can possibly be satisfied is $k+1$, as either y or z_1 need to be satisfied. Furthermore, $\{z_1\} \models_{k+1}^{\text{CCL}} F$ and $\{z_1\} \models_{k+1}^{\text{CCL}} F[A/B]$. This means that any preferred model of F must satisfy F with a degree of $k+1$. The same is true for preferred models of $F[A/B]$.

We can again distinguish two cases:

a) $k = \mathit{opt}_{\text{CCL}}(A)$. Then $\{y\} \models_{k+1}^{\text{CCL}} F$, and therefore $\{y\} \in \mathit{Mod}_{\text{CCL}}(F)$. But $\{y\} \models_{n+1}^{\text{CCL}} F[A/B]$ with $n = \mathit{opt}_{\text{CCL}}(B)$. Thus, since $n > k$, $\{y\} \notin \mathit{Mod}_{\text{CCL}}(F[A/B])$

b) $k = \mathit{opt}_{\text{CCL}}(B)$. Analogous to (a), but with $\{y\} \notin \mathit{Mod}_{\text{CCL}}(F)$ and $\{y\} \in \mathit{Mod}_{\text{CCL}}(F[A/B])$.

□

5.3 Choice Logics in General

We have already examined QCL and CCL with regard to some properties of strong equivalence. In both QCL and CCL, strong equivalence can be characterized by full equivalence. We will now examine whether this also holds for choice logics in general.

Lemma 29. *Let $\mathcal{L}$ be a choice logic. Then $A \equiv_f^{\mathcal{L}} B$ if and only if $F \equiv_f^{\mathcal{L}} F[A/B]$ for all $\mathcal{L}$-formulas F.*

Proof. The if-direction is analogous to the proof for QCL: Assume that $A \not\equiv_f^{\mathcal{L}} B$. Choose $F = A$. Then $F[A/B] = B$, and therefore $F \not\equiv_f^{\mathcal{L}} F[A/B]$. The only-if-direction can be carried out by structural induction, similar to the proof of Lemma 25. The cases for the classical connectives remain unchanged. For any choice connective $\circ \in \mathcal{C}_{\mathcal{L}}$, we know that $opt_{\mathcal{L}}(F \circ G)$ depends only on $opt_{\mathcal{L}}(F)$ and $opt_{\mathcal{L}}(G)$. Furthermore, $deg_{\mathcal{L}}(\mathcal{I}, F \circ G)$ depends only on $opt_{\mathcal{L}}(F)$, $opt_{\mathcal{L}}(G)$, $deg_{\mathcal{L}}(\mathcal{I}, F)$, and $deg_{\mathcal{L}}(\mathcal{I}, G)$. By I.H., we also know that $F \equiv_f^{\mathcal{L}} F[A/B]$ and $G \equiv_f^{\mathcal{L}} G[A/B]$. Therefore we can conclude that $opt_{\mathcal{L}}(F \circ G) = opt_{\mathcal{L}}((F \circ G)[A/B])$ and $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg_{\mathcal{L}}(\mathcal{I}, (F \circ G)[A/B])$, i.e. $F \equiv_f^{\mathcal{L}} F[A/B]$. □

With the help of the above lemma, we can compare strong equivalence with full equivalence for arbitrary choice logics.

Proposition 30. *Let $\mathcal{L}$ be a choice logic. If $A \equiv_f^{\mathcal{L}} B$, then $A \equiv_s^{\mathcal{L}} B$.*

Proof. Assume $A \equiv_f^{\mathcal{L}} B$. Then, by Lemma 29, $F \equiv_f^{\mathcal{L}} F[A/B]$ holds for all formulas F. By Lemma 6, this entails that any F and $F[A/B]$ have the same preferred models, i.e. $A \equiv_s^{\mathcal{L}} B$. □

However, unlike for the analogous QCL- and CCL-statements (see Propositions 26 and 28), the converse of Proposition 30 does not hold: Consider a choice logic $\mathcal{L}$ that is not optionality-differentiating. Then there are $\mathcal{L}$-formulas A and B such that $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$, and for all $F \in \mathcal{F}_{\mathcal{L}}$ we have that $F \equiv_d^{\mathcal{L}} F[A/B]$. Therefore, also $Mod_{\mathcal{L}}(F) = Mod_{\mathcal{L}}(F[A/B])$ for all $F \in \mathcal{F}_{\mathcal{L}}$, i.e. $A \equiv_s^{\mathcal{L}} B$. But $A \not\equiv_f^{\mathcal{L}} B$, since $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$.

To examine how strong equivalence and degree-equivalence are related, we first introduce the following lemma:

Lemma 31. *If $\mathcal{I} \models_k^{\mathcal{L}} F$ for some interpretation $\mathcal{I}$, then there exists an $\mathcal{L}$-formula G such that the minimum degree that satisfies G is k.*

Proof. Let $\mathcal{I}$ be an interpretation and F be a formula such that $\mathcal{I} \models_k^{\mathcal{L}} F$. Due to Lemma 7 we can assume that $\mathcal{I} \subseteq var(F)$. Consider

$$G = F \wedge \Big(\bigwedge_{x \in \mathcal{I}} x \Big) \wedge \Big(\bigwedge_{x \in var(F) \setminus \mathcal{I}} \neg x \Big).$$

Then $\mathcal{I} \models_k^{\mathcal{L}} G$. Also, $\mathcal{I} \cup \mathcal{J} \models_k^{\mathcal{L}} G$, for any $\mathcal{J}$ with $\mathcal{J} \cap var(F) = \emptyset$. All other interpretations ascribe an infinite satisfaction degree to G. □

Proposition 32. *Let $\mathcal{L}$ be a choice logic. If $A \equiv_s^{\mathcal{L}} B$, then $A \equiv_d^{\mathcal{L}} B$.*

Proof. We proceed by contrapositive. Assume $A \not\equiv_d^{\mathcal{L}} B$. We want to show that $A \not\equiv_s^{\mathcal{L}} B$, i.e. that there is a formula F such that $Mod_{\mathcal{L}}(F) \neq Mod_{\mathcal{L}}(F[A/B])$.

Since $A \not\equiv_d^{\mathcal{L}} B$, there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models_m^{\mathcal{L}} A$ and $\mathcal{I} \models_n^{\mathcal{L}} B$ with $m \neq n$. Let $k = min(m, n)$. Due to Lemma 31, we know that there are formulas G and H such that the minimum degree that satisfies G or H is k. Thus, there are interpretations $\mathcal{I}_G$, $\mathcal{I}_H$ such that $\mathcal{I}_G \models_k^{\mathcal{L}} G$ and $\mathcal{I}_H \models_k^{\mathcal{L}} H$. Because of Lemma 8 we can assume that G

and H are variable disjoint from each other as well as from A and B. By Lemma 7, this implies that $\mathcal{I} \cap \mathcal{I}_G = \emptyset$, $\mathcal{I} \cap \mathcal{I}_H = \emptyset$, and $\mathcal{I}_G \cap \mathcal{I}_H = \emptyset$. We now construct

$$F = (A \wedge G) \vee (x \wedge H),$$

where x is a fresh variable that does not occur in A, B, G or H. We can therefore also assume that x is not contained in $\mathcal{I}$, $\mathcal{I}_G$, or $\mathcal{I}_H$.

Observe that the minimal degree with which F (or $F[A/B]$) can possibly be satisfied is k, as either G or H need to be satisfied. Furthermore, $\mathcal{I}_H \cup \{x\} \models_k^{\mathcal{L}} F$ and $\mathcal{I}_H \cup \{x\} \models_k^{\mathcal{L}} F[A/B]$. This means that any preferred model of F must satisfy F with a degree of k. The same is true for preferred models of $F[A/B]$. Also observe that since x is not contained in $\mathcal{I}$ or $\mathcal{I}_G$, $\mathcal{I} \cup \mathcal{I}_G \models_\infty^{\mathcal{L}} (x \wedge H)$.

We can distinguish two cases:

1. $k = m$. Then $\mathcal{I} \models_k^{\mathcal{L}} A$, and therefore $\mathcal{I} \cup \mathcal{I}_G \models_k^{\mathcal{L}} (A \wedge G)$. Thus, $\mathcal{I} \cup \mathcal{I}_G \models_k^{\mathcal{L}} F$, i.e. $\mathcal{I} \cup \mathcal{I}_G \in Mod_{\mathcal{L}}(F)$. Analogously, since $\mathcal{I} \models_n^{\mathcal{L}} B$, we have that $\mathcal{I} \cup \mathcal{I}_G \models_n^{\mathcal{L}} (B \wedge G)$. Therefore $\mathcal{I} \cup \mathcal{I}_G \models_n^{\mathcal{L}} F[A/B]$. Since $n > k$, we have $\mathcal{I} \cup \mathcal{I}_G \notin Mod_{\mathcal{L}}(F[A/B])$.
2. $k = n$. Analogous to (a), but with $\mathcal{I} \cup \mathcal{I}_G \notin Mod_{\mathcal{L}}(F)$ and $\mathcal{I} \cup \mathcal{I}_G \in Mod_{\mathcal{L}}(F[A/B])$.

□

The converse of Proposition 32 does not hold. In the proof of Proposition 26 we have already seen how to construct a formula F in QCL such that $A \equiv_d^{\mathrm{QCL}} B$ but $Mod_{\mathrm{QCL}}(F) \neq Mod_{\mathrm{QCL}}(F[A/B])$.

In summary, the different notions of equivalence for choice logics are related as follows:

$$A \equiv_f^{\mathcal{L}} B \implies A \equiv_s^{\mathcal{L}} B \implies A \equiv_d^{\mathcal{L}} B \implies Mod_{\mathcal{L}}(A) = Mod_{\mathcal{L}}(B).$$

For all of the implications above, the inverse direction does not hold in general. But when dealing with choice logics where optionality plays no role, it can be shown that strong equivalence is interchangeable with degree-equivalence.

Proposition 33. *Let $\mathcal{L}$ be an optionality-ignoring choice logic. Then $A \equiv_s^{\mathcal{L}} B$ if and only if $A \equiv_d^{\mathcal{L}} B$.*

Proof. The only-if-direction follows directly from Proposition 32. For the if-direction, we prove that if $A \equiv_d^{\mathcal{L}} B$, then $F \equiv_d^{\mathcal{L}} F[A/B]$ for all $\mathcal{L}$-formulas F, which implies that $Mod_{\mathcal{L}}(F) = Mod_{\mathcal{L}}(F[A/B])$ for all $\mathcal{L}$-formulas F. This can be done by structural induction, analogous to the proof of Lemma 29. However, in this case, since $\mathcal{L}$ is optionality-ignoring, an I.H. of $F \equiv_d^{\mathcal{L}} F[A/B]$ and $G \equiv_d^{\mathcal{L}} G[A/B]$ is enough to conclude that $deg_{\mathcal{L}}(\mathcal{I}, F \circ G) = deg_{\mathcal{L}}(\mathcal{I}, (F \circ G)[A/B])$ for any $\circ \in \mathcal{C}_{\mathcal{L}}$. □

Similar to how strong equivalence can by characterized by degree-equivalence in optionality-ignoring choice logics, we can characterize strong equivalence by full equivalence in optionality-differentiating choice logics.

Proposition 34. *Let $\mathcal{L}$ be an optionality-differentiating choice logic. Then $A \equiv_s^{\mathcal{L}} B$ if and only if $A \equiv_f^{\mathcal{L}} B$.*

Proof. The if-direction follows directly from Proposition 30. For the only-if-direction, assume that $A \equiv_s^{\mathcal{L}} B$. Then, from Proposition 32, we know that $A \equiv_d^{\mathcal{L}} B$. It remains to show that $opt_{\mathcal{L}}(A) = opt_{\mathcal{L}}(B)$.

Towards a contradiction, assume that $opt_{\mathcal{L}}(A) \neq opt_{\mathcal{L}}(B)$. This means that, since $\mathcal{L}$ is optionality-differentiating, there exists a formula $A' \in \mathcal{F}_{\mathcal{L}}$ such that $A' \not\equiv_d^{\mathcal{L}} A'[A/B]$. Observe that therefore A must occur in A'. Let $B' = A'[A/B]$. Then $A' \not\equiv_d^{\mathcal{L}} B'$. By the contrapositive of Proposition 32, there exists a formula F such that $Mod_{\mathcal{L}}(F) \neq Mod_{\mathcal{L}}(F[A'/B'])$. By the construction of F in the proof for Proposition 32, we can assume that A' occurs only once in F, and that A only occurs in A'. Therefore, replacing A' by $A'[A/B]$ in F is the same as simply replacing A by B in F, i.e. $F[A'/B'] = F[A'/A'[A/B]] = F[A/B]$. Thus, $Mod_{\mathcal{L}}(F) \neq Mod_{\mathcal{L}}(F[A/B])$. But then $A \not\equiv_s^{\mathcal{L}} B$. Contradiction. $\square$

In fact, as we have already argued, if a choice logic is not optionality-differentiating, then strong equivalence and full equivalence are not interchangable. We can therefore conclude that $A \equiv_s^{\mathcal{L}} B \iff A \equiv_f^{\mathcal{L}} B$ holds only when $\mathcal{L}$ is optionality-differentiating.

CHAPTER 6

Computational Complexity

This chapter contains results for the complexity of certain decision problems pertaining to choice logics. We will not investigate the complexity of all choice logics, but rather of those in which the satisfaction degree of a formula given an interpretation can be computed in polynomial time. Such a choice logic will be referred to as *tractable*.

In total, we will introduce four new decision problems: $\mathcal{L}$-ModelChecking and $\mathcal{L}$-Sat are similar to the classical decision problems of ModelChecking and Sat (see Section 2.4), but instead of asking whether classical formulas evaluate to *true* or *false*, they ask questions regarding the satisfaction degree of $\mathcal{L}$-formulas. $\mathcal{L}$-PrefModelChecking asks if a given interpretation is a preferred model of a given $\mathcal{L}$-formula, and $\mathcal{L}$-PrefModelSat asks whether a given $\mathcal{L}$-formula has a preferred model containing a given variable.

In the following, by Lemma 7, we can assume that if F is the input to a decision problem, then $\mathcal{I} \subseteq var(F)$ for any interpretation $\mathcal{I}$ we are dealing with. Just as for propositional logic, we denote the input size of our decision problems by $|F|$, i.e. the total number of variable occurrences in F (compare Section 2.4).

6.1 Tractable Choice Logics

Since the degree- and optionality-functions of choice connectives can be given by arbitrary functions over the natural numbers, they can also be given by non-computable functions. For example, consider the following choice logic, where M_i is the i-th Turing machine:

Definition 53. *L_4 is the choice logic such that $\mathcal{C}_{L_4} = \{\circ\}$,*

$$opt_{L_4}(F \circ G) = opt_{L_4}(F) + opt_{L_4}(G),$$

and

$$deg_{L_4}(\mathcal{I}, F \circ G) = \begin{cases} 1 & \textit{if } M_{opt_{L_4}(F)} \textit{ halts on input } opt_{L_4}(G). \\ \infty & \textit{otherwise} \end{cases}$$

Then the simple decision problem of asking whether a formula is satisfied to a degree of 1 by some interpretation is not computable for L_4. By the definition of opt_{L_4}, we can construct L_4-formulas F and G with any optionality we require. Also note that $deg_{L_4}(\mathcal{I}, F \circ G)$ does not depend on $\mathcal{I}$ at all, but only on the optionalities of F and G. Thus, asking whether $deg_{L_4}(\mathcal{I}, F \circ G)$ is equal to 1 is the same as asking whether the Turing machine with the index $opt_{L_4}(F)$ halts on the input $opt_{L_4}(G)$. If there was an algorithm that would decide our problem, then this algorithm would also decide the halting problem, which is known to be undecidable [AB09]. We can therefore conclude that our decision problem is not computable. But when analyzing problems with respect to their complexity, we are only interested in computable problems. Thus, we need to exclude at least some of the choice logics that can be defined in our framework.

For all of the choice logics we have seen so far, except L_4, the degree- and optionality-functions over which the choice connectives are defined are polynomial time computable. This is convenient when analyzing complexity, as we do not need to worry about how the complexity of a certain decision problem might change if e.g. the degree-function of some choice connective is not polynomial time computable.

Definition 54. *A choice logic $\mathcal{L}$ is called tractable if for every $\circ \in \mathcal{C}_{\mathcal{L}}$, the optionality- and degree functions which define the semantics of $\circ$ are polynomial-time computable.*

We will only consider tractable choice logics in this chapter. Alternatively, one could also analyze the complexity of choice logics with respect to oracles. But since (almost) all choice logics we encountered so far are tractable, and since a non-tractable choice logic can arguably be seen as impractical, we will not do so.

Note that the term *tractable* only refers to the complexity of the degree- and optionality functions, and does not imply that all decision problems concerned with tractable choice logics are solvable in polynomial time.

6.2 Model Checking for Choice Logics

In ModelChecking, which is concerned with classical propositional logic, we ask whether a given formula F is satisfied by a given interpretation $\mathcal{I}$. A similar problem can be defined for a choice logic $\mathcal{L}$.

$\mathcal{L}$-ModelChecking *Instance:* An $\mathcal{L}$-formula F, an interpretation $\mathcal{I}$, and a satisfaction degree $k \in \mathbb{N} \cup \{\infty\}$. *Question:* $deg_{\mathcal{L}}(\mathcal{I}, F) \leq k$?

$\mathcal{L}$-ModelChecking is a generalization of ModelChecking: If F is a classical formula, and $k = 1$, then the two problems are equivalent. Since we are dealing with tractable choice logics only, we can show that $\mathcal{L}$-ModelChecking is decidable in polynomial time, just like the standard ModelChecking problem.

Proposition 35. *Let $\mathcal{L}$ be a tractable choice logic. Then $\mathcal{L}$-ModelChecking is in* P.

Proof. We can compute the degree of F under $\mathcal{I}$ by applying $deg_{\mathcal{L}}$ to $\mathcal{I}$ and F (see Definition 29). Since $\mathcal{L}$ is tractable, we know that every step in this recursion runs in polynomial time. The depth of the recursion can not exceed the length of F. Neither can the width of the recursion exceed the length of F, since every atom in F will be reached exactly once in the recursion. After we computed $deg_{\mathcal{L}}(\mathcal{I}, F)$, we can simply compare the result with k, and return either "yes" or "no" accordingly. □

Note that, in $\mathcal{L}$-ModelChecking, we ask whether F is satisfied to a degree less or equal to k by $\mathcal{I}$, instead of whether F is satisfied to a degree of *exactly* k by $\mathcal{I}$. But, as a matter of fact, there is no significant difference between these two questions: Since $\mathcal{L}$-ModelChecking is in P, and since P is closed under complement [HMU07], the complementary problem of $\mathcal{L}$-ModelChecking is also in P. By solving $\mathcal{L}$-ModelChecking we can find out if $deg_{\mathcal{L}}(\mathcal{I}, F) \leq k$, and by solving the complimentary problem, we can find out if $deg_{\mathcal{L}}(\mathcal{I}, F) \geq k$. This procedure clearly constitutes a polynomial time algorithm, and it is enough to solve the question of whether $deg_{\mathcal{L}}(\mathcal{I}, F) = k$.

6.3 Satisfiability for Choice Logics

In Sat, the satisfiability problem for classical propositional logic, we ask whether a formula F is satisfiable, i.e. if there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models F$. Similarly, we can ask if a formula of some choice logic $\mathcal{L}$ can be satisfied to a given degree.

> $\mathcal{L}$-Sat
> *Instance:* An $\mathcal{L}$-formula F and a satisfaction degree $k \in \mathbb{N} \cup \{\infty\}$.
> *Question:* Is there an interpretation $\mathcal{I}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) \leq k$?

$\mathcal{L}$-Sat is a generalization of the standard Sat problem, just as $\mathcal{L}$-ModelChecking is a generalization of ModelChecking. Just like Sat, $\mathcal{L}$-Sat is in NP for tractable choice logics.

Proposition 36. *Let $\mathcal{L}$ be a tractable choice logic. Then $\mathcal{L}$-Sat is in* NP.

Proof. We prove this by providing a polynomially balanced and polynomially decidable certificate relation for $\mathcal{L}$-Sat. Let

$$R = \{((F, k), \mathcal{I}) \mid deg_{\mathcal{L}}(\mathcal{I}, F) \leq k\}.$$

Clearly, (F, k) is a yes-instance of $\mathcal{L}$-Sat iff there is an interpretation $\mathcal{I}$ such that $((F, k), \mathcal{I}) \in R$. R is polynomially balanced, since we can assume that $\mathcal{I} \subseteq var(F)$. Furthermore, R is polynomially decidable, since $\mathcal{L}$-ModelChecking is in P. □

In addition to being contained in NP, $\mathcal{L}$-Sat is also NP-hard. This is easy to show, since Sat is NP-hard, and because $\mathcal{L}$-Sat is a generalization of Sat.

Proposition 37. *Let $\mathcal{L}$ be a tractable choice logic. Then $\mathcal{L}$-Sat is NP-hard.*

Proof. We prove this by providing a reduction from Sat to $\mathcal{L}$-Sat. Let F be an arbitrary instance of Sat. Then we construct an instance $(F, 1)$ of $\mathcal{L}$-Sat. Since F is a classical formula, for any interpretation $\mathcal{I}$ it holds that $\mathcal{I} \models F \iff \mathcal{I} \models_1^{\mathcal{L}} F \iff deg_{\mathcal{L}}(\mathcal{I}, F) \leq 1$. Thus,

$$\begin{aligned}
F \text{ is a yes-instance of Sat} &\iff \text{there is an interpretation } \mathcal{I} \text{ such that } \mathcal{I} \models F \\
&\iff \text{there is an interpretation } \mathcal{I} \text{ such that } deg_{\mathcal{L}}(\mathcal{I}, F) \leq 1 \\
&\iff (F, 1) \text{ is a yes-instance of } \mathcal{L}\text{-Sat}.
\end{aligned}$$

□

In conclusion, $\mathcal{L}$-Sat is NP-complete for a tractable $\mathcal{L}$, as it is both contained in NP and NP-hard.

6.4 Preferred Model Checking

The main question regarding a choice formula and an interpretation is most often not to what degree the formula is satisfied by the interpretation, but rather whether the interpretation is a preferred model of the formula.

$\mathcal{L}$-PrefModelChecking
Instance: An $\mathcal{L}$-formula F and an interpretation $\mathcal{I}$.
Question: $\mathcal{I} \in Mod_{\mathcal{L}}(F)$?

To solve $\mathcal{L}$-PrefModelChecking, we essentially need to ensure that no interpretation has a smaller degree than $\mathcal{I}$. Intuitively, this places $\mathcal{L}$-PrefModelChecking in coNP, and in fact we can prove this:

Proposition 38. *Let $\mathcal{L}$ be a tractable choice logic. Then $\mathcal{L}$-PrefModelChecking is in coNP.*

Proof. We will show that the complementary problem is in NP by providing a polynomially balanced, polynomially decidable certificate relation. First of all, observe that $(F, \mathcal{I})$ is a yes-instance of co-$\mathcal{L}$-PrefModelChecking iff $\mathcal{I} \notin Mod_{\mathcal{L}}(F)$. Let

$$R = \{((F,\mathcal{I}),\mathcal{J}) \mid deg_{\mathcal{L}}(\mathcal{J},F) < deg_{\mathcal{L}}(\mathcal{I},F) \text{ or } deg_{\mathcal{L}}(\mathcal{I},F) = \infty\}.$$

Then $((F,\mathcal{I}),\mathcal{J}) \in R$ iff there is an interpretation $\mathcal{J}$ such that $deg_{\mathcal{L}}(\mathcal{J},F) < deg_{\mathcal{L}}(\mathcal{I},F)$ or $deg_{\mathcal{L}}(\mathcal{I},F) = \infty$, which is the case exactly when $\mathcal{I} \notin Mod_{\mathcal{L}}(F)$. R is polynomially balanced, since we can assume that $\mathcal{J} \subseteq var(F)$. Furthermore, R is polynomially decidable, since we can compute $deg_{\mathcal{L}}(\mathcal{I},F)$ and $deg_{\mathcal{L}}(\mathcal{J},F)$ in polynomial time. □

As it turns out, we can also show coNP-hardness of $\mathcal{L}$-ModelChecking for certain choice logics. However, we have to exclude at least PL from this proof, since $\mathcal{I} \in Mod_{\text{PL}}(F) \iff \mathcal{I} \models_1^{\text{PL}} F$. Therefore, PL-ModelChecking and PL-PrefModelChecking are identical. If we showed that PL-PrefModelChecking is coNP-hard, then PL-ModelChecking would be both coNP-hard and contained in P, which would mean that coNP is equal to P. Because P is closed under complement, this in turn would imply that P is equal to NP, which is widely believed to not be the case [Aar17]. Therefore, we will only consider choice logics where degrees other than 1 and ∞ can be obtained.

Definition 55. *A choice logic $\mathcal{L}$ is called non-binary if there is an $\mathcal{L}$-formula G and an interpretation $\mathcal{I}$ such that $\mathcal{I} \models_k^{\mathcal{L}} G$ with $k \notin \{1,\infty\}$.*

With this restriction, we can show coNP hardness. Note that most choice logics mentioned in this thesis (including QCL, CCL, XCL, SCCL, and LCL) are non-binary. Also observe that a binary choice logic must not necessarily amount to PL. If this were the case, then L_4 would be equivalent to PL, and would therefore be decidable.

Proposition 39. *Let $\mathcal{L}$ be a tractable, non-binary choice logic. Then $\mathcal{L}$-PrefModelChecking is coNP-hard.*

Proof. We prove this by providing a reduction from Unsat to $\mathcal{L}$-PrefModelChecking. Let F be an arbitrary instance of Unsat. Since $\mathcal{L}$ is non-binary, there exists an $\mathcal{L}$-formula G and an interpretation $\mathcal{I}$ such that $\mathcal{I} \models_k^{\mathcal{L}} G$ with $k \notin \{1,\infty\}$. Then, by Lemma 31, there must be a formula H such that the minimum degree that satisfies H is k, i.e. there is an interpretation $\mathcal{I}'$ such that $\mathcal{I}' \models_k^{\mathcal{L}} H$ and $deg_{\mathcal{L}}(\mathcal{I}',H) \leq deg_{\mathcal{L}}(\mathcal{J},H)$ for all other interpretations $\mathcal{J}$. Crucially, by the construction in the proof of Lemma 31, we can also assume that there is an interpretation $\mathcal{I}^*$ such that $\mathcal{I}^* \models_\infty^{\mathcal{L}} H$. By Lemma 8, we can assume F and H to be variable disjoint. Observe that the size of H is constant with respect to the size of F. We construct an instance $(F',\mathcal{I}')$ of $\mathcal{L}$-PrefModelChecking, where

$$F' = (F \vee H) \wedge \neg(F \wedge H).$$

We will show that F is a yes-instance of Unsat if and only if $(F',\mathcal{I}')$ is a yes-instance of $\mathcal{L}$-PrefModelChecking:

" $\Longrightarrow$ ": Assume F is a yes-instance of Unsat. Then there is no interpretation $\mathcal{J}$ such that $\mathcal{J} \models F$, i.e. $deg_{\mathcal{L}}(\mathcal{J}, F) = \infty$ for all $\mathcal{J}$. Since $deg_{\mathcal{L}}(\mathcal{I}', H) = k$, we have that $deg_{\mathcal{L}}(\mathcal{I}', F') = k$. Indeed, F' can not be satisfied to a degree lower than k since F is unsatisfiable and since k is the lowest degree with which H can be satisfied. Thus, $\mathcal{I}' \in Mod_{\mathcal{L}}(F')$, which means that $(F', \mathcal{I}')$ is a yes-instance of $\mathcal{L}$-PrefModelChecking.

" $\Longleftarrow$ ": We proceed by contrapositive. Assume F is a no-instance of Unsat. Then there is an interpretation $\mathcal{J}$ such that $\mathcal{J} \models F$. Because F and H are variable disjoint, we can assume that $\mathcal{I}^* \cap \mathcal{J} = \emptyset$. Since $\mathcal{I}^* \models^{\mathcal{L}}_{\infty} H$, we have that $(\mathcal{I}^* \cup \mathcal{J}) \models^{\mathcal{L}}_{1} (F \vee H)$ and $(\mathcal{I}^* \cup \mathcal{J}) \models^{\mathcal{L}}_{1} \neg(F \wedge H)$, which implies that $(\mathcal{I}^* \cup \mathcal{J}) \models^{\mathcal{L}}_{1} F'$. Recall that $\mathcal{I}' \models^{\mathcal{L}}_{k} H$. We distinguish two cases:

1. $\mathcal{I}' \models F$. Then $\mathcal{I}' \models^{\mathcal{L}}_{1} (F \vee H)$ and $\mathcal{I}' \models^{\mathcal{L}}_{\infty} \neg(F \wedge H)$, and therefore $\mathcal{I}' \models^{\mathcal{L}}_{\infty} F'$.
2. $\mathcal{I}' \not\models F$. Then $\mathcal{I}' \models^{\mathcal{L}}_{k} (F \vee H)$ and $\mathcal{I}' \models^{\mathcal{L}}_{1} \neg(F \wedge H)$, and therefore $\mathcal{I}' \models^{\mathcal{L}}_{k} F'$.

In both cases, it holds that $deg_{\mathcal{L}}(\mathcal{I}', F') > deg_{\mathcal{L}}(\mathcal{I}^* \cup \mathcal{J}, F')$. This means that $\mathcal{I}'$ is not a preferred model of F'. Thus, $\mathcal{I}' \in Mod_{\mathcal{L}}(F')$, which implies that $(F', \mathcal{I}')$ is a no-instance of $\mathcal{L}$-PrefModelChecking. □

In conclusion, $\mathcal{L}$-PrefModelChecking is coNP-complete for a tractable and non-binary $\mathcal{L}$, as it is both contained in coNP and coNP-hard.

6.5 Preferred Model Satisfiability

The last decision problem regarding choice logics that we will investigate is that of preferred model satisfiability. Instead of simply asking whether a formula F has a preferred model, we will ask if F has a preferred model containing a given variable $x \in var(F)$. In fact, asking whether F has a preferred model is the same as asking if there is an interpretation $\mathcal{I}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) \leq opt_{\mathcal{L}}(F)$. But this question can be answered by solving $\mathcal{L}$-Sat, and is therefore in NP, as we have already shown in Proposition 36. In preferred model satisfiability however, we are interested in how difficult it is to *find* a preferred model. As we have already discussed in Section 2.4.2, an optimization problem of this form can be represented as a decision problem by asking whether there is a preferred model containing a variable x.

$\mathcal{L}$-PrefModelSat
Instance: An $\mathcal{L}$-formula F and a variable x such that $x \in var(F)$.
Question: Is there an interpretation $\mathcal{I}$ such that $x \in \mathcal{I}$ and $\mathcal{I} \in Mod_{\mathcal{L}}(F)$?

Before analyzing the complexity of $\mathcal{L}$-PrefModelSat, we will give an upper bound for the optionality of a choice formula relative to its size:

Lemma 40. *Let $\mathcal{L}$ be a choice logic. Then, for every $\mathcal{L}$-formula F it holds that $opt_{\mathcal{L}}(F) < 2^{(|F|^2)}$.*

Proof. We show this by structural induction over the formulas of $\mathcal{L}$.

- Base case. $F = x$, where x is a propositional variable. Then $|F| = 1$ and $opt_{\mathcal{L}}(F) = 1 < 2^{(|F|^2)}$.
- Step case. As our I.H., let G and H be $\mathcal{L}$-formulas such that $opt_{\mathcal{L}}(G) < 2^{(|G|^2)}$ and $opt_{\mathcal{L}}(H) < 2^{(|H|^2)}$. We distinguish the following cases:
  1. $F = (\neg G)$. Then $|F| = |G| \geq 1$ and $opt_{\mathcal{L}}(F) = 1 < 2^{(|F|^2)}$.
  2. $F = (G \land H)$ or $F = (G \lor H)$. Then $|F| = |G| + |H|$ and
  $$opt_{\mathcal{L}}(F) = max(opt_{\mathcal{L}}(G), opt_{\mathcal{L}}(H)) < max(2^{(|G|^2)}, 2^{(|H|^2)}) < 2^{(|F|^2)}.$$
  3. $F = (G \circ H)$, where $\circ \in \mathcal{C}_{\mathcal{L}}$. Then $|F| = |G| + |H|$. Observe that $opt_{\mathcal{L}}(G) < 2^{(|G|^2)}$ is the same as $opt_{\mathcal{L}}(G) \leq 2^{(|G|^2)} - 1$. Likewise for H. Thus,
  $$\begin{aligned} opt_{\mathcal{L}}(F) &\leq (opt_{\mathcal{L}}(G) + 1) \cdot (opt_{\mathcal{L}}(H) + 1) \\ &\leq ((2^{(|G|^2)} - 1) + 1) \cdot ((2^{(|H|^2)} - 1) + 1) \\ &= 2^{(|G|^2)} \cdot 2^{(|H|^2)} \\ &= 2^{(|G|^2) + (|H|^2)} \\ &< 2^{((|G|+|H|)^2)} \\ &= 2^{(|F|^2)}. \end{aligned}$$

∎

The above upper bound for the optionality of a formula is likely not a tight bound, but it is enough to show the Δ_2P-membership of $\mathcal{L}$-PrefModelSat. Because we will also show that there is a choice logic for which $\mathcal{L}$-PrefModelSat is Δ_2P-hard, finding a tighter bound would not reduce the complexity of $\mathcal{L}$-PrefModelSat in general.

Proposition 41. *Let $\mathcal{L}$ be a tractable choice logic. Then $\mathcal{L}$-PrefModelSat is in Δ_2P.*

Proof. Let (F, x) be an arbitrary instance of $\mathcal{L}$-PrefModelSat. We provide a decision procedure which runs in polynomial time with respect to $|F|$, makes $\mathcal{O}(|F|^2)$ calls to an NP-oracle, and determines whether (F, x) is a yes-instance of $\mathcal{L}$-PrefModelSat:

1. Construct the formula F' by replacing every occurrence of x in F by the tautology $(x \lor \neg x)$. This means that for any $\mathcal{I}$ with $x \in \mathcal{I}$, we have that $deg_{\mathcal{L}}(\mathcal{I}, F) = deg_{\mathcal{L}}(\mathcal{I} \setminus \{x\}, F')$. In fact, F can be satisfied to a degree of k by some interpretation containing x if and only if F' can be satisfied to the degree of k by any interpretation. Also note that $|F'| \leq 2 \cdot |F|$, and that $opt_{\mathcal{L}}(F') = opt_{\mathcal{L}}(F)$.

2. Conduct a binary search over $(1, \ldots, opt_{\mathcal{L}}(F), \infty)$. In each step of the binary search, we call an NP-oracle that decides $\mathcal{L}$-Sat to check whether there is an interpretation $\mathcal{I}$ such that $deg_{\mathcal{L}}(\mathcal{I}, F) \leq k$, where k is the mid-point of the current step in the binary search. In the end, we will find the minimum k such that $deg_{\mathcal{L}}(\mathcal{J}, F) = k$ for some $\mathcal{J}$. Since binary search runs in logarithmic time, we require $\mathcal{O}(\log(opt_{\mathcal{L}}(F)))$ calls to our NP-oracle. By Lemma 40 we know that $opt_{\mathcal{L}}(F) < 2^{(|F|^2)}$, and therefore we need at most $\mathcal{O}(\log(2^{(|F|^2)})) = \mathcal{O}(|F|^2)$ oracle calls.

3. Conduct a binary search over $(1, \ldots, opt_{\mathcal{L}}(F'), \infty)$ to find the minimum k' such that $deg_{\mathcal{L}}(\mathcal{J}, F') = k'$ for some $\mathcal{J}$. As before, this requires $\mathcal{O}(|F'|^2) = \mathcal{O}(|F|^2)$ NP-oracle calls.

4. If $k < k'$, then F' can not be satisfied to a degree of k. This means that F can not be satisfied to a degree of k by an interpretation containing x. But F can be satisfied to a degree of k in general. Thus, there is no preferred model of F containing x, and we can return "no". If $k = k'$, then F' can be satisfied to a degree of k, and therefore F can be satisfied to a degree of k by an interpretation containing x. Thus, we can return "yes". Note that it can not be that $k > k'$.

□

Regarding hardness, we can prove that $\mathcal{L}$-PrefModelSat is NP-hard for all tractable choice logics. This is to be expected, since classical propositional logic is contained as a fragment in every choice logic.

Proposition 42. *Let $\mathcal{L}$ be a tractable choice logic. Then $\mathcal{L}$-PrefModelSat is NP-hard.*

Proof. We prove this by providing a reduction from Sat to $\mathcal{L}$-PrefModelSat. Let F be an arbitrary instance of Sat. We then construct an instance (F', x) of $\mathcal{L}$-PrefModelSat, where x does not occur in F, and

$$F' = F \wedge x.$$

Since F is a classical formula, and since x does not occur in F, we have that

$$\begin{aligned}
F \text{ is a yes-instance of } \textsc{Sat} &\iff \text{there is an interpretation } \mathcal{I} \text{ such that } \mathcal{I} \models F \\
&\iff \text{there is an interpretation } \mathcal{I} \text{ such that } \mathcal{I} \cup \{x\} \models F' \\
&\iff \text{there is an interpretation } \mathcal{I} \text{ such that } \mathcal{I} \cup \{x\} \models_1^{\mathcal{L}} F' \\
&\iff \mathcal{I} \cup \{x\} \in Mod_{\mathcal{L}}(F') \\
&\iff (F', x) \text{ is a yes-instance of } \mathcal{L}\text{-}\textsc{PrefModelSat}.
\end{aligned}$$

□

The $\Delta_2\mathsf{P}$-membership gives us an upper bound for the complexity of $\mathcal{L}$-PrefModelSat, while the NP-hardness constitutes a lower bound. But, as already mentioned, we can also show $\Delta_2\mathsf{P}$-hardness of $\mathcal{L}$-PrefModelSat for a specific choice logic, namely LCL (see Definition 36). We choose LCL for this purpose because it enables us to represent a lexicographic ordering over variables as an LCL-formula, and it will therefore make a reduction from LexMaxSat quite straightforward.

Proposition 43. *LCL-*PrefModelSat *is* $\Delta_2\mathsf{P}$*-hard.*

Proof. We prove this by providing a reduction from LexMaxSat. Let $(F, (x_1, \dots, x_n))$ be an arbitrary instance of LexMaxSat. We then construct an instance (F', x_n) of LCL-PrefModelSat, where

$$F' = F \wedge (x_1 \circ (x_2 \circ (\cdots (x_{n-1} \circ x_n)))).$$

It remains to show that $(F, (x_1, \dots, x_n))$ is a yes-instance of LexMaxSat if and only if (F', x_n) is a yes-instance of LCL-PrefModelSat.

" $\Longrightarrow$ ": Let $(F, (x_1, \dots, x_n))$ be a yes-instance of LexMaxSat. Then there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models F$, $x_n \in \mathcal{I}$, and $\mathcal{I}$ is the lexicographically largest model of F with respect to the ordering $x_1 > \cdots > x_n$. Let $\mathcal{J}$ be any interpretation other than $\mathcal{I}$. If $\mathcal{J} \not\models F$, then $deg_{\mathrm{LCL}}(\mathcal{J}, F') = \infty$, and $\mathcal{J}$ is not a preferred model of F'. If $\mathcal{J} \models F$, then $\mathcal{J}$ must be lexicographically smaller than $\mathcal{I}$. By Lemma 13, we can directly infer that $deg_{\mathrm{LCL}}(\mathcal{I}, x_1 \circ (x_2 \circ (\cdots (x_{n-1} \circ x_n)))) < deg_{\mathrm{LCL}}(\mathcal{J}, x_1 \circ (x_2 \circ (\cdots (x_{n-1} \circ x_n))))$, and therefore $deg_{\mathrm{LCL}}(\mathcal{I}, F') < deg_{\mathrm{LCL}}(\mathcal{J}, F')$. This means that $\mathcal{I} \in Mod_{\mathrm{LCL}}(F')$. Since also $x_n \in \mathcal{I}$, we have that (F', x_n) is a yes-instance of LCL-PrefModelSat.

" $\Longleftarrow$ ": Let (F', x_n) be a yes-instance of LCL-PrefModelSat. Then there is an interpretation $\mathcal{I}$ such that $x_n \in \mathcal{I}$ and $\mathcal{I} \in Mod_{\mathrm{LCL}}(F')$. Towards a contradiction, assume that there is an interpretation $\mathcal{J}$ such that $\mathcal{J} \models F$, and such that $\mathcal{J}$ is lexicographically larger than $\mathcal{I}$ with respect to $x_1 > \cdots > x_n$. But, by Lemma 13, this means that $deg_{\mathrm{LCL}}(\mathcal{J}, F') < deg_{\mathrm{LCL}}(\mathcal{I}, F')$, which means that $\mathcal{I} \notin Mod_{\mathrm{LCL}}(F')$. Contradiction. Thus, $\mathcal{I}$ is the lexicographically largest model of F. Since $x_n \in \mathcal{I}$, we have that $(F, (x_1, \dots, x_n))$ is a yes-instance of LexMaxSat. □

In summary, LCL-PrefModelSat is $\Delta_2\mathsf{P}$-complete. But still, $\mathcal{L}$-PrefModelSat could be easier for some other $\mathcal{L}$. The complexity of LCL-PrefModelSat is owed in part to the fact that the optionality of a formula F in LCL can be exponential in the size of F, and that we therefore require a polynomial number of NP oracle calls when conducting the binary search. But for choice logics where the optionality of a formula F is only polynomial in the size of F, we only need a logarithmic number of oracle calls.

Proposition 44. *Let $\mathcal{L}$ be a tractable choice logic such that $opt_{\mathcal{L}}(F) \in \mathcal{O}(|F|^c)$ holds for all $\mathcal{L}$-formulas F, where c is a constant. Then $\mathcal{L}$-*PrefModelSat *is in* $\Theta_2\mathsf{P}$.

Proof. The proof is analogous to the proof of Proposition 41, except for the crucial difference that, since $opt_{\mathcal{L}}(F) \in \mathcal{O}(|F|^c)$, the binary search can be executed in $\mathcal{O}(\log(|F|))$ time, which means that we only need $\mathcal{O}(\log(|F|))$ oracle calls. □

Note that QCL, CCL, XCL, and SCCL all fulfill the condition necessary for Proposition 44. Thus, $\mathcal{L}$-PREFMODELSAT is in $\Theta_2\mathsf{P}$ for $\mathcal{L} \in \{\mathrm{QCL}, \mathrm{CCL}, \mathrm{XCL}, \mathrm{SCCL}\}$. We will now show $\Theta_2\mathsf{P}$-hardness for QCL and CCL.

Proposition 45. *QCL-*PREFMODELSAT *is* $\Theta_2\mathsf{P}$*-hard.*

Proof. We provide a reduction from LOGLEXMAXSAT to QCL-PREFMODELSAT. Let $(F, (x_1, \dots, x_n))$ be an arbitrary instance of LOGLEXMAXSAT. We then construct an instance (F', x_n) of QCL-PREFMODELSAT as follows: Let $\mathcal{J}_i$ be the lexicographically i-th largest interpretation over $x_1 > \cdots > x_n$. For example, $\mathcal{J}_1 = \{x_1, \dots, x_n\}$, $\mathcal{J}_2 = \{x_1, \dots, x_{n-1}\}$, and $\mathcal{J}_{(2^n)} = \emptyset$. We characterize each of those interpretations by a formula, namely

$$A_i = \Big(\bigwedge_{x \in \mathcal{J}_i} x\Big) \wedge \Big(\bigwedge_{x \in \{x_1,\dots,x_n\} \setminus \mathcal{J}_i} \neg x\Big).$$

Then, for any interpretation $\mathcal{I}$, we have that $\mathcal{I} \models A_i \iff \mathcal{I} \cap \{x_1, \dots, x_n\} = \mathcal{J}_i$. Now let

$$F' = F \wedge (A_1 \vec{\times} A_2 \vec{\times} \cdots \vec{\times} A_{(2^n)}).$$

Observe that this construction is polynomial in $|F|$, as $n \leq \log(|F|)$, and therefore $2^n \leq |F|$. It remains to show that $(F, (x_1, \dots, x_n))$ is a yes-instance of LOGLEXMAXSAT if and only if (F', x_n) is a yes-instance of QCL-PREFMODELSAT.

" $\Longrightarrow$ ": Let $(F, (x_1, \dots, x_n))$ be a yes-instance of LOGLEXMAXSAT. Then there exists an interpretation $\mathcal{I}$ such that $x_n \in \mathcal{I}$, $\mathcal{I} \models F$, and such that $\mathcal{J}_k = \mathcal{I} \cap \{x_1, \dots, x_n\}$ is the lexicographically largest interpretation over $x_1 > \cdots > x_n$ that can be extended to a model of F. Observe that $\mathcal{I} \models A_k$, but $\mathcal{I} \not\models A_r$ for any $r \neq k$. Therefore, by the semantics of ordered disjunction in QCL, $deg_{\mathrm{QCL}}(\mathcal{I}, F') = k$. Let $\mathcal{I}'$ be any interpretation. If $\mathcal{I}' \not\models F$, then $deg_{\mathrm{QCL}}(\mathcal{I}', F') = \infty$. If $\mathcal{I}' \models F$, then it can not be that $\mathcal{J}_{k'} = \mathcal{I}' \cap \{x_1, \dots, x_n\}$ is lexicographically larger than $\mathcal{J}_k$ with respect to $x_1 > \cdots > x_n$. Thus, $k \leq k'$. But by the same reasoning as above, we have that $deg_{\mathrm{QCL}}(\mathcal{I}', F') = k'$. This means that there is no interpretation that satisfies F' to a smaller degree than $\mathcal{I}$, i.e. $\mathcal{I} \in Mod_{\mathrm{QCL}}(F')$. Since also $x_n \in \mathcal{I}$, we can conclude that (F', x_n) is a yes-instance of QCL-PREFMODELSAT.

" $\Longleftarrow$ ": Let (F', x_n) be a yes-instance of QCL-PREFMODELSAT. Then there is an interpretation $\mathcal{I}$ such that $x_n \in \mathcal{I}$ and $\mathcal{I} \in Mod_{\mathrm{QCL}}(F')$. By the construction of F', we have that $\mathcal{I} \models F$. Towards a contradiction, assume there is an interpretation $\mathcal{I}'$ such that $\mathcal{I}' \models F$, and such that $\mathcal{J}_{k'} = \mathcal{I}' \cap \{x_1, \dots, x_n\}$ is lexicographically larger than $\mathcal{J}_k = \mathcal{I} \cap \{x_1, \dots, x_n\}$ with respect to $x_1 > \cdots > x_n$. Then $k' < k$. But by the same argument as in the only-if-direction, we can conclude that $deg_{\mathrm{QCL}}(\mathcal{I}, F') = k$

and $deg_{\text{QCL}}(\mathcal{I}', F') = k'$, i.e. $deg_{\text{QCL}}(\mathcal{I}, F') < deg_{\text{QCL}}(\mathcal{I}', F')$. But then $\mathcal{I}$ is not a preferred model of F'. Contradiction. This means that $\mathcal{J}_k$ is the lexicographically largest interpretation with respect to $(x_1, \ldots, x_n)$ that can be extended to a model of F. Therefore, $(F, (x_1, \ldots, x_n))$ is a yes-instance of LogLexMaxSat. □

Proposition 46. *CCL-*PrefModelSat *is* $\Theta_2\mathsf{P}$*-hard.*

Proof. This proof is similar to the proof of Proposition 45. Again, we provide a reduction from LogLexMaxSat. Let $(F, (x_1, \ldots, x_n))$ be an arbitrary instance of LogLex-MaxSat. We then construct an instance (F', x_n) of CCL-PrefModelSat. Analogously to the proof of Proposition 45, we characterize the lexicographically i-th largest interpretation $\mathcal{J}_i$ over $x_1 > \cdots > x_n$ by a formula A_i, i.e.

$$A_i = \Big(\bigwedge_{x \in \mathcal{J}_i} x\Big) \wedge \Big(\bigwedge_{x \in \{x_1, \ldots, x_n\} \setminus \mathcal{J}_i} \neg x\Big).$$

We now further construct

$$C_i = \bigvee_{j=1}^{2^n-(i-1)} A_j.$$

for every $1 \leq i \leq 2^n$. Then $\mathcal{J}_i \models C_j$ for $1 \leq j \leq 2^n - (i-1)$, and $\mathcal{J}_i \not\models C_j$ for $j > 2^n - (i-1)$. For example, $\mathcal{J}_1$ satisfies $C_1, \ldots, C_{(2^n)}$, $\mathcal{J}_2$ satisfies $C_1, \ldots, C_{(2^n-1)}$ but not $C_{(2^n)}$, and $\mathcal{J}_{(2^n)}$ satisfies only C_1. Let

$$F' = F \wedge (C_1 \overrightarrow{\times} C_2 \overrightarrow{\times} \cdots \overrightarrow{\times} C_{(2^n)}).$$

This construction is still polynomial in $|F|$: Recall that $n \leq \log(|F|)$, and therefore $2^n \leq |F|$. For every $1 \leq i \leq 2^n$ we have that $|A_i| \leq \log(|F|)$, and thus $|C_i| \leq \log(|F|) \cdot |F|$. This means that $|F'| \leq |F| + \log(|F|) \cdot |F|^2$. Also note that, by the semantics of ordered conjunction in CCL, we have that $deg_{\mathcal{L}}(\mathcal{J}_i, C_1 \overrightarrow{\times} C_2 \overrightarrow{\times} \cdots \overrightarrow{\times} C_{(2^n)}) = i$ for all $1 \leq i \leq 2^n$. Thus, by the same argument as in the proof of Proposition 45, we can conclude that $(F, (x_1, \ldots, x_n))$ is a yes-instance of LogLexMaxSat if and only if (F', x_n) is a yes-instance of CCL-PrefModelSat. □

In summary, QCL-PrefModelSat and CCL-PrefModelSat are $\Theta_2\mathsf{P}$-complete. Since LCL-PrefModelSat is $\Delta_2\mathsf{P}$-complete, we know that $\mathcal{L}$-PrefModelSat is not equally hard for all $\mathcal{L}$, unless of course $\Delta_2\mathsf{P} = \Theta_2\mathsf{P}$. In fact, we can show NP-completeness for $\mathcal{L} = \text{PL}$.

Proposition 47. *PL-*PrefModelSat *is in* NP*.*

Proof. We prove this by providing a polynomially balanced and polynomially decidable certificate relation for PL-PrefModelSat. Let

$$R = \{((F, x), \mathcal{I}) \mid \mathcal{I} \models F \text{ and } x \in \mathcal{I}\}.$$

Clearly, (F, x) is a yes-instance of PL-PrefModelSat iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \in Mod_{\mathrm{PL}}(F)$ and $x \in \mathcal{I}$ iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models F$ and $x \in \mathcal{I}$ iff $((F, x), \mathcal{I}) \in R$. R is polynomially balanced, since we can assume that $\mathcal{I} \subseteq var(F)$. Furthermore, R is polynomially decidable, since PL-ModelChecking is in P, and because we can verify whether $x \in \mathcal{I}$ in linear time. □

By the above result, and by Proposition 42, PL-PrefModelSat is NP-complete.

6.6 Summary of Complexity Results

The results of our complexity analysis are summarized in Table 6.1. As we can see, $\mathcal{L}$-ModelChecking is in P for all tractable choice logics. Similarly, we showed the NP-completeness of $\mathcal{L}$-Sat for every tractable $\mathcal{L}$.

Regarding $\mathcal{L}$-PrefModelChecking, we proved coNP-membership for all tractable choice logics, and coNP-completeness for tractable, non-binary choice logics. Limiting ourselves to non-binary choice logics here is not as restrictive as it may seem, since the purpose of choice logics is to rank interpretations based on more than just truth and falsity.

For $\mathcal{L}$-PrefModelSat, we showed Δ_2P-membership and NP-hardness for all tractable choice logics. These results can be seen as upper and lower bounds for the complexity of $\mathcal{L}$-PrefModelSat. For tractable choice logics where the optionality of a formula F is polynomially bounded by $|F|$, $\mathcal{L}$-PrefModelSat is contained in Θ_2P. In fact, we obtained different completeness results for PL (NP-complete), QCL/CCL (Θ_2P-complete), and LCL (Δ_2P-complete).

	$\mathcal{L} = \mathrm{PL}$	$\mathcal{L} \in \{\mathrm{QCL}, \mathrm{CCL}\}$	$\mathcal{L} = \mathrm{LCL}$	tractable $\mathcal{L}$
$\mathcal{L}$-ModelChecking	in P	in P	in P	in P
$\mathcal{L}$-Sat	NP-c	NP-c	NP-c	NP-c
$\mathcal{L}$-PrefModelChecking	in P	coNP-c	coNP-c	in coNP
$\mathcal{L}$-PrefModelSat	NP-c	Θ_2P-c	Δ_2P-c	in Δ_2P/NP-h

Table 6.1: Summary of complexity results

CHAPTER 7

Conclusion

In this chapter, a summary of the results of this thesis will be given, followed by a discussion on work related to choice logics. Finally, several possible directions for future work on the choice logic framework will be proposed.

7.1 Summary

The main contribution of this thesis is the generalization of Qualitative Choice Logic (QCL) and Conjunctive Choice Logic (CCL), realized by the introduction of a formal framework. A logic of this framework, i.e. a choice logic, is an extension of classical propositional (PL) logic by so-called choice connectives, with which preferences can be expressed. The semantics of a choice logic is given by two functions:

1. The satisfaction degree of a formula given an interpretation indicates how preferable this interpretation is: The lower this satisfaction degree, the more preferable the interpretation.

2. The optionality of a formula represents the number of possible satisfaction degrees that this formula can be ascribed. In this sense, optionality is an upper bound for satisfaction degrees.

It was shown that the original definitions of QCL and CCL, but also PL and the two alternative satisfaction relations for QCL proposed by Benferhat and Sedki [BS08b], can be expressed within our framework. Additionally, we provided an alternative satisfaction relation for CCL (see Definition 33), which captures the intended meaning of ordered conjunction. The logic defined by this alternative semantics is what we refer to as CCL in the rest of this chapter. We also introduced completely new choice logics, among which the following are featured prominently in this thesis:

- Exclusive Disjunctive Choice Logic (XCL). Based on exclusive disjunction similarly to how QCL is based on regular disjunction.
- Simple Conjunctive Choice Logic (SCCL). A choice logic in which optionality has no impact on the satisfaction degree of a formula.
- Lexicographic Choice Logic (LCL). Features a choice connective with more than two levels of satisfaction. Capable of encoding a lexicographic ordering over variables.

Note that any two choice logics can be combined into a new choice logic, simply by extending propositional logic by both of their choice connectives. For example, QCCL is the choice logic that has ordered disjunction from QCL and ordered conjunction from CCL as choice connectives.

Since our framework is not very restrictive, and a lot of different choice logics can be defined within it, we introduced several classes of choice logics and examined their properties. Table 7.1 shows which classes some of the main choice logics belong to.

- Exhaustive choice logics: Every possible combination of satisfaction degree and optionality can be obtained. In such a logic, the problem of formula synthesis can be solved, i.e. it is possible to construct a formula that satisfies any given interpretation to a desired degree.
- Basic exhaustive choice logics: A basic choice formula is a formula in which classical connectives are applied only to classical formulas. In basic exhaustive choice logics, for every formula there is a fully equivalent basic choice formula.
- Optionality-ignoring choice logics: The satisfaction degree of a formula does not depend on optionality.
- Optionality-differentiating choice logics: Replacing a subformula by any other formula with a different optionality value can have an impact on the satisfaction degree.
- Reasonable choice logics: Choice connectives are related to classical binary connectives in a natural way.

	PL	QCL	CCL	XCL	SCCL	LCL
exhaustive		×	×	×	×	
basic-exhaustive		×	×	×	×	
optionality-ignoring	×				×	
optionality-differentiating	×	×	×	×		×
reasonable	×	×	×	×	×	×

Table 7.1: Classes of choice logics

The concept of strong equivalence (two formulas can be replaced without affecting preferred models) was compared with the notions of degree equivalence (two formulas have the same satisfaction degrees under all interpretations) and full equivalence (two formulas are degree equivalent and have the same optionality). For arbitrary choice logics, we showed that full equivalence implies strong equivalence, which in turn implies degree equivalence. Furthermore, strong equivalence was found to be interchangeable with degree equivalence for optionality-ignoring choice logics, and with full equivalence for optionality-differentiating choice logics, which includes QCL and CCL.

Finally, we investigated the computational complexity of so-called tractable choice logics, in which the satisfaction degree and optionality of a formula must be computable in polynomial time. Note that PL, QCL, CCL, XCL, SCCL, and LCL are all tractable in this sense. Four new decision problems were defined and analyzed:

- $\mathcal{L}$-ModelChecking. Asks whether a given formula is satisfied to a certain degree by a given interpretation. This problem is in P for tractable choice logics.
- $\mathcal{L}$-Sat. Asks whether a given formula can be satisfied to a certain degree or lower. This problem is NP-complete for tractable choice logics.
- $\mathcal{L}$-PrefModelChecking. Asks whether a given interpretation is a preferred model of a given formula. This problem is in P for PL, but it is coNP-complete for tractable choice logics in which more than two satisfaction degrees can be expressed, which includes QCL, CCL, XCL, SCCL, and LCL.
- $\mathcal{L}$-PrefModelSat. Asks whether a given formula has a preferred model in which a given variable is contained. This problem captures the complexity of finding a preferred model, and we showed that it is contained in $\Delta_2\mathsf{P}$ and NP-hard for all tractable choice logics. Furthermore, we proved that $\mathcal{L}$-PrefModelSat is NP-complete for PL, $\Theta_2\mathsf{P}$-complete for QCL and CCL, and $\Delta_2\mathsf{P}$-complete for LCL.

7.2 Related Work

Choice logics are a tool to express preferences, but there are also other systems that are concerned with preference handling. Surveys on this topic include [BLW10], [DHKP11], and the more recent [PTV16]. Some formalisms represent preferences quite differently than choice logics. For example, CP-nets [BBD+11] and lexicographic preference trees [BCL+10] do so in a graphical manner.

Closely related to our work are of course systems that specify preferences via logic. This includes nonmonotonic logics, which are inherently related to preferences [Sho87, BNT08]. An example for such a logic is propositional circumscription, which can be expressed by QCL and vice versa [BBB04, p. 220]. Logics that are more focused on reasoning about preferences rather than expressing them include a formalism introduced by von Wright [vW63, Liu10], and a modal logic described by van Benthem, Girard, and Roy [vBGR09].

Some other logic formalisms are also related to our work, even if they were not designed with preferences in mind. Many-valued logics, for example, are similar to choice logics in the sense that they deal with more than two truth values [Got01]. Another relevant formalism is possibilistic logic, whose connection to QCL has been studied by Brewka et al. [BBB04, p. 221].

There are also some systems that are directly inspired by the concept of QCL's ordered disjunction. For example, Jiang et al. [JZPZ15] introduced a modal logic that contains a binary connective with a similar meaning to ordering disjunction, while Zhang and Thielscher [ZT15] stated that they took inspiration from QCL for their prioritized disjunction, which is used to reason about game strategies. It has also been demonstrated that Answer Set Programming (ASP) can be extended by ordered disjunction [BNS04, Bre05]. This extension is called Logic Programming with Ordered Disjunction (LPOD).

In this thesis, the notion of strong equivalence was defined in the sense of replaceability with respect to preferred models. This notion is based on the concept of strong equivalence for preference systems outlined by Faber et al. [FTW13a], which in turn builds on the notion of strong equivalence between logic programs described by Lifschitz, Pearce, and Valverde [LPV01]. Strong equivalence was also investigated for LPOD [FTW08], and other ASP formalisms concerned with preferences [FK05, FTW13b]. For a discussion on strong equivalence in knowledge representation formalisms, refer for example to [BS16].

7.3 Future Work

Regarding future work, choice logics other than those defined in this thesis could be defined and examined. One possibility is to introduce a choice logic based on material implication, which might be useful for expressing conditional preferences.

Another open issue is that of associativity. Although we know that ordered disjunction in QCL and ordered conjunction in CCL are associative, it is not clear which conditions are necessary in general for a choice connective to be associative. This, and the properties of choice logics with associative choice connectives, could be investigated.

We have shown that for every formula in a basic exhaustive choice logic, there is a fully equivalent basic choice formula. But we have only shown this via a semantic transformation, i.e. the structure of the original formula is not preserved in any shape or form. From the original QCL and CCL papers [BBB04, BB16] we know that there are syntactic transformations to basic choice formulas in QCL and CCL, in which the structure of the original formula is preserved to a certain degree. Thus, one might want to investigate for which choice logics such syntactic transformations to basic choice formulas are possible.

Furthermore, one could examine in detail how certain choice logics relate to other logics, such as circumscription logic or multi-valued logics. Of course it is also possible to investigate how choice logics are related to each other, similarly to how we have shown that XCL can be expressed in QCL.

Lastly, regarding computational complexity, other decision problems than those defined in this thesis may be considered. For example, an interesting problem would be the complexity of testing for full equivalence (or another notion of equivalence) between two formulas. One could also investigate choice logics with respect to Fixed-Parameter Tractability [CFK^{+}15] by using, e.g., the optionality of a formula as a parameter.

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