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DIPLOMARBEIT

Existence of weak solutions to a degenerate reaction-cross-diffusion system desribing ion transport through confined geometries

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Univ.-Prof. Dr. Ansgar Jüngel

durch

Sebastian Ertel
Matrikelnummer: 1328851
Pfalzauerstraße 1a/3/16
3021 Pressbaum

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Kurzfassung

In dieser Diplomarbeit beweisen wir die Existenz schwacher Lösungen für ein degeneriertes Reaktions-Kreuz-Diffusionssystem, welches die Bewegung von Ionen durch räumlich beschränkte Geometrien beschreibt. Hierzu verwenden wir die sogenannte 'Boundedness-by-entropy' Methode, welche erstmals in [Jüngel2] zur Analysis von allgemeinen Kreuz-Diffusionssystemen eingesetzt wurde und welches auch in [GerstJün] verwendet wurde um die Existenz schwacher Lösungen für das von uns betrachtete System, aber hierbei ohne Reaktionsterm, zu zeigen.

Das wichtigste Ergebnis dieser Arbeit ist natürlich der Beweis das schwache Lösungen auch dann existieren, wenn ein Reaktionsterm die Evolution der Ionenkonzentrationen beeinflusst. Darüber hinaus wird in Lemma 3.8 bewiesen, das die Lösungen semilinearer Poisson Gleichungen Hölder-stetig, in der L^∞-Norm, von Parametern der Nichtlinearität abhängen, welches nach den Kenntnissen des Autors ein neues Resultat darstellt.

Der von uns geführte Existenzbeweis orientiert sich sehr stark an [GerstJün] and dementsprechend ähnelt auch der Aufbau dieser Arbeit dem des Papers von Gerstenmayer und Jüngel. Im ersten Kapitel führen wir das von uns betrachtete Gleichungssystem ein und definieren den Begriff der schwachen Lösung. Im zweiten Kapitel führen wir das Entropiefunktional ein, welches wir dann im dritten Kapitel dazu benutzen, zu beschreiben unter welchen Voraussetzungen schwache Lösungen existieren und deren Existenz dann auch zu beweisen. Dieser Existenzbeweis wird von uns in vier Schritte zerlegt. Im ersten Schritt zeigen wir zunächst die Existenz von Lösungen einer zeitdiskretisierten und im Raum regularisierten Gleichung.. Im zweiten Schritt beweisen wir eine Ungleichung für die zeitliche Evolution des Entropiefunktionals und benutzten diese Ungleichung um die in Schritt 1 gefundenen Approximationen in der Norm zu beschränken. Im dritten Schritt eliminieren wir die Regularisierung und zeigen die Existenz von zeitlich diskreten Approximationen, deren Konvergenz gegen eine schwache Lösung wir schlussendlich im vierten Schritt zeigen.

Abstract

In this thesis we prove the existence of weak solutions to a degenerate reaction-cross-diffusion system desribing the ion transport through confined geometries. Hereby we use the so called boundedness-by-entropy method, that was introduced in [Jüngel2] for the analysis of general cross-diffusion systems and was used in [GerstJün] to prove the existence of weak solutions for the same system but without the reaction term.

The main achievement of this thesis is of course the proof that weak solutions exist even when a reaction term is involved. Besides that, the thesis also includes a proof that the solutions of certain semilinear Poisson equations depend Hölder–continuously, in the L^∞-norm, on a paremeter that determines the nonlinearity, in Lemma 3.8 and, to the best knowledge of the author, is a novel result..

The structure of this thesis is similar to [GerstJün]. In chapter 1 we introduce the system, fix the notations and define the notion of weak solution that we will use. In chapter 2 we introduce the entropy functional associated to the system. This will later be used to prove the existence of weak solutions and is required to even state the existence theorem. Finally in chapter 3 we state under which assumptions weak solutions exist and prove the existence. The proof is conducted in 4 steps. First we prove the existence of solutions to an adapted system, that is discretized in time and regularized in space. Then we show a discrete entropy production inequality and use this inequality to derive bounds for the approximate solution constructed in step 1. In step 3 we construct an approximate solution that is time discretized but not space regularized via a compactness argument that uses the estimates derived in step 2. Finally, in step 4, we prove the existence of weak solutions by using a compactness argument for the approximate, solutions constructed in step 3.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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Sebastian Ertel

Sebastian Ertel

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1. Introduction

A classical model for describing the transport of ions through biological membranes and channels is given by the Poisson–Nernst-Planck equation, which can be derived in the mean-field limit from microscopic particle models. However when the finite size of the ions is taken into account, this model is no longer applicable. In [BurSchWol] the modifications of mobilites due to size exclusion effects lead to the following evolution equation

$$\partial_t u_i = \mathrm{div} J_i \text{ on } \Omega, \text{ for all times } t \in (0,T) \text{ and for all } i = 1, \cdots, n. \tag{1.1}$$

Hereby u_i is the concentration (volume fraction) of the i-th ion species and J_i the corresponding flux, which is given by

$$J_i := D_i \left(u_0 \nabla u_i - u_i \nabla u_0 + u_0 u_i (\beta z_i \nabla \Phi + \nabla W_i) \right), \text{ for } i = 1, \cdots, n. \tag{1.2}$$

The term $D_i > 0$ is the diffusivity of the i-th species and $u_0 := 1 - \Sigma_{j=1}^n u_j$ is the concentration (volume fraction) of the solvent. $\beta = \frac{q}{k_B \theta} > 0$ is the inverse thermal voltage with the elementary charge q and temperature θ. The Boltzmann constant is denoted by k_B. The valence of the i-th species, which is a measure for the combining power of its atoms, is denoted by z_i and W_i is an external potential. The term Φ is the electrical potential and is determined by the Poisson equation

$$-\lambda^2 \Delta \Phi = \sum_{j=1}^n z_j u_j + f, \text{ on } \Omega. \tag{1.3}$$

Hereby $\lambda > 0$ denotes the permittivity, $\sum_{j=1}^n z_j u_j$ is the total charge density and f is a permanent charge density, that only depends on space, i.e. $f = f(x)$.

We assume that the space $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, in which these ion species evolve, is a bounded domain. The boundary of the domain $\Gamma := \partial\Omega$ is decomposed into to parts Γ_D and Γ_N. On the insulating part Γ_N no flux boundary conditions are prescribed for both the electrical potential and the different ion species. On Γ_D the Dirichlet conditions prescribe the values of the ion concentrations and the electrical potential. Thus we obtain the set of mixed (Dirichlet and Neumann) boundary conditions

$$\begin{aligned} J_i \cdot \mathbf{n} = 0 \text{ on } \Gamma_N, u_i = u_i^D \text{ on } \Gamma_D, \text{ for } i = 1, \cdots, n \\ \frac{\partial \Phi}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_N, \Phi = \Phi^D \text{ on } \Gamma_D. \end{aligned} \tag{1.4}$$

Furthermore we prescribe the initial conditions

$$u_i(\cdot, 0) = u_i^0 \text{ on } \Omega, \text{ for all } i = 1, \cdots, n. \tag{1.5}$$

Combining the equations (1.1), (1.3),(1.4), (1.5) with a reaction terms $g_i(t, x, u)$, $i = 1, \cdots, n$, that depends on time t, space x and the concentration vector $u := (u_1, \cdots, u_n)^{\mathrm{T}}$, gives the system

$$\begin{aligned} \partial_t u_i &= \operatorname{div} J_i + g_i(t, x, u) \text{ on } \Omega \\ -\lambda^2 \Delta \Phi &= \sum_{j=1}^{n} z_j u_j + f \text{ on } \Omega \\ J_i \cdot \mathbf{n} &= 0 \text{ on } \Gamma_N, u_i = u_i^D \text{ on } \Gamma_D \\ \frac{\partial \Phi}{\partial \mathbf{n}} &= 0 \text{ on } \Gamma_N, \Phi = \Phi^D \text{ on } \Gamma_D. \\ u_i(\cdot, 0) &= u_i^0 \text{ on } \Omega, \end{aligned} \tag{1.6}$$

for $i = 1, \cdots, n$. In [GerstJün] the existence of weak solutions was shown for the case that there are no reactions, i.e. $g_i = 0$ for $i = 1, \cdots, n$. Herefore the boundedness-by-entropy method was used. This method also showed rather easily that $u_i \in [0, 1]$ for all $i = 0, \cdots, n$, which is a necessity for the model to make sense.
The goal of this thesis is to use similar methods as in [GerstJün] to prove the existence of (weak) solutions also for non-trivial reaction terms g_i, $i = 1, \cdots, n$.

1.1. Notations

Whenever we say in the following that a term depends on the parameters of system (1.6), we mean that this term depends on atleast one of

$$\Omega, \ \Gamma_D, \ D_i, \ z_i, \ W_i, \ \beta, \ f, \ \lambda, \ u_i^D, \ \Phi^D, \ g_i \text{ with } i = 1, \cdots, n.$$

For any set A its indicator function shall be denoted by $\mathbb{1}_A$ and for any vector $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$ we define

$$v_0 := 1 - \sum_{i=1}^{n} v_i.$$

With this notation we define the open domain $\mathcal{O} \subseteq \mathbb{R}^n$ by

$$\mathcal{O} := \{ \ v \in (0, 1)^n \mid v_0 \in (0, 1) \ \}.$$

For any multicoefficient $\alpha \in \mathbb{N}^n$ we define $|\alpha| := n$ and denote the corresponding derivative operator by D^α. The closure of a set O is denoted by $\bar{O}$ and the support of a function v is denoted by $\operatorname{supp}(v)$.

For any Banach space X we denote its norm by $\|\cdot\|_X$, its dual space by X' and the corresponding duality pairing by ${}_{X'}\langle \cdot, \cdot \rangle_X$. For a Hilbert space H its scalar product will be denoted by $\langle \cdot, \cdot \rangle_H$.

We denote the usual Lp-spaces on some space O with values in the Banach space X by $L^p(O;X)$ for any $p \in [1,+\infty]$.

The Sobolev space of order k and integrability $p \in [1,+\infty]$ on the domain O with values in the Banach space X will be denoted by

$$W^{k,p}(O;X) := \left\{ v \in L^p(O;X) \;\middle|\; \forall |\alpha| \le k : \; D^\alpha v \in L^p(O;X) \right\}.$$

As usual we denote $H^k(O;X) := W^{k,2}(O;X)$. The trace of $H^k(\Omega;\mathbb{R}^n)$ on the Dirichlet boundary Γ_D will be denoted by $\cdot|_{\Gamma_D}$. With this we define the two spaces

$$\begin{aligned} H_D^1(\Omega;\mathbb{R}^n) &:= \left\{ v \in H^1(\Omega;\mathbb{R}^n) \;\middle|\; v|_{\Gamma_D} = 0 \right\} \\ H_D^{-1}(\Omega;\mathbb{R}^n) &:= H_D^1(\Omega;\mathbb{R}^n)'. \end{aligned}$$

Finally we denote the k-times continuously differentiable, X-valued function on a domain O by $C^k(O;X)$ and the subset with compact support by

$$C_{00}^k(O;X) := \left\{ v \in C^k(O;X) \;\middle|\; \operatorname{supp}(v) \text{ is compact in } O \right\}.$$

1.2. Weak formulation

We can rewrite the first equation of (1.6) in vector form. To this end we make the following definitions.

Definition 1.1. *Let the matrix valued function $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be defined by*

$$A_{ij}(v) := \begin{cases} D_i v_i & , \text{ if } i \ne j \\ D_i(v_0 + v_i) & , \text{ if } i = j, \end{cases} \tag{1.7}$$

for $i,j = 1,\cdots,n$ and $v \in \mathbb{R}^n$.
Furthermore we define the reaction vector $g(t,x,u) := (g_1(t,x,u),\cdots,g_n(t,x,u))^{\mathrm{T}}$, the flux matrix $J := (J_1,\cdots,J_n)$ and the $\mathbb{R}^{n\times n}$-valued function

$$F(u,\Phi) := \big(F_1(u,\Phi),\cdots,F_n(u,\Phi)\big)$$

where

$$F_i(u,\Phi) := D_i u_0 u_i(\beta z_i \nabla\Phi + \nabla W_i),$$

for every $i = 1,\cdots,n$.

Remark 1.2. *In general the images of the matrix valued function A are neither symmetric, nor positive semidefinite on $\mathcal{O}$. More precisely it holds that for $v \in \mathcal{O}$:*

- *$A(v)$ is symmetric if and only if*

$$D_i(v_0 + v_i)A_{ij}(v) = A_{ji}(v) = D_j(v_0 + v_j) \text{ for all } i,j = 1,\cdots,n.$$

- *$A(v)$ is positive semidefinite, if and only if for all $x \in \mathbb{R}^n$, $x \neq 0$*

$$0 \leq x^{\mathrm{T}} A(v) x = \sum_{i,j=1}^{n} D_i v_i x_i x_j + v_0 \sum_{i=1}^{n} D_i x_i^2.$$

This is not always satisfied in the domain $\mathcal{O}$, as this inequality would imply by the continuity of A, that if we let v converge against $\mathbf{e}_1 = (1, 0, \cdots, 0)^{\mathrm{T}}$ within $\mathcal{O}$ (implying that also $v_0 \to 0$) and consider the vector $x = (1, -2, 0, \cdots, 0)^{\mathrm{T}}$, we would derive the contradiction

$$0 \leq \lim_{v \to \mathbf{e}_1} x^{\mathrm{T}} A(v) x = D_1 x_1^2 + D_1 x_1 x_2 = D_1 - 2D_1 = -D_1 < 0.$$

We can represent the ionflux using the matrix valued functions A and F, as

$$\begin{aligned} J_i &= D_i \left(u_0 \nabla u_i - u_i \nabla u_0 + u_0 u_i (\beta z_i \nabla \Phi + \nabla W_i) \right) = D_i u_0 \nabla u_i - D_i u_i \nabla u_0 + F_i(u, \Phi) \\ &= D_i u_0 \nabla u_i - D_i u_i \nabla \left(1 - \sum_{j=1}^{n} u_j \right) + F_i(u, \Phi) = D_i u_0 \nabla u_i + \sum_{j=1}^{n} D_i u_i \nabla u_j + F_i(u, \Phi) \\ &= \sum_{j=1}^{n} A_{ij}(u) \nabla u_j + F_i(u, \Phi). \end{aligned}$$

Thus (by the convention that the divergence of a matrix valued function is to be interpreted rowwise) we can rewrite the first equation in (1.6) as

$$\partial_t u = \operatorname{div} J + g(t, x, u) = \operatorname{div} \left(A(u) \nabla u + F(u, \Phi) \right) + g(t, x, u). \tag{1.8}$$

Remark 1.3. *The vectorized form (1.8) shows the main difficulty in the analysis of equation (1.6). Since A is neither symmetric, nor positive semidefinite, a Galerkin-approximation, which is the standard method for proving the existence of solutions to quasilinear equations, is not applicable to this problem.*

Using the vectorized form (1.8), the natural weak formulation of the system (1.6) up to a time horizon $T > 0$, would be to require that $u \in L^2\left((0, T); H^1(\Omega; \mathbb{R}^n)\right)$,
$\Phi \in L^2\left((0, T); H^1(\Omega)\right)$ and that

$$\begin{aligned} \int_0^T \langle \partial_t u(t), v(t) \rangle \, \mathrm{d}t &= \int_0^T \langle A(u(t)) \nabla u(t) + F(u(t), \Phi(t)), \nabla v(t) \rangle_{L^2(\Omega)} \, \mathrm{d}t + \cdots \\ &\quad \cdots + \int_0^T \langle g(t, \cdot, u(t)), v(t) \rangle_{L^2(\Omega)} \, \mathrm{d}t \\ \lambda^2 \int_0^T \langle \nabla \Phi(t), \nabla \theta(t) \rangle_{L^2(\Omega)} &= \int_0^T \left\langle \sum_{j=1}^{n} z_j u_j(t) + f, \theta(t) \right\rangle_{L^2(\Omega)} \mathrm{d}t, \end{aligned}$$

is satisfied for all $v \in L^2\left((0,T);H^1(\Omega;\mathbb{R}^n)\right)$, $\theta \in L^2\left((0,T);H^1(\Omega)\right)$. However we will not be able to prove existence for solutions u of regularity $u(t) \in H^1(\Omega,\mathbb{R}^n)$ for $t>0$. Thus we need a solution concept that requires less spatial regularity. To this end we use the identity

$$
\begin{aligned}
u_0\nabla u_i - u_i\nabla u_0 &= u_0\nabla u_i - 2u_i\frac{u_0^{-1/2}}{2}\nabla u_0 = u_0\nabla u_i - 2u_iu_0^{1/2}\nabla u_0^{1/2}\\
&= u_0^{1/2}\left(u_0^{1/2}\nabla u_i + u_i\nabla u_0^{1/2} - 3u_i\nabla u_0^{1/2}\right) = u_0^{1/2}\left(\nabla(u_0^{1/2}u_i) - 3u_i\nabla u_0^{1/2}\right).
\end{aligned}
$$

With this we can rewrite the i-th flux as

$$J_i = D_iu_0^{1/2}\left(\nabla(u_0^{1/2}u_i) - 3u_i\nabla u_0^{1/2}\right) + F_i(u,\Phi).$$

This identity leads to the following weak solution concept.

Definition 1.4. *We say that $u:[0,T]\times\Omega\to\bar{\mathcal{O}}$ and $\Phi:[0,T]\to\mathbb{R}$ form a weak solution to the system (1.6), up to time $T>0$, if for all $i=1,\cdots,n$*

$$
\begin{aligned}
u_i &\in L^\infty\left((0,T)\times\Omega\right)\\
\partial_t u_i &\in L^2\left((0,T);H_D^{-1}(\Omega)\right)\\
u_0^{1/2}u_i &\in L^2\left((0,T);H^1(\Omega)\right)\\
u_0^{1/2} &\in L^2\left((0,T);H^1(\Omega)\right)\\
\Phi &\in L^2\left((0,T);H^1(\Omega)\right)
\end{aligned}
$$

and the following identities hold

$$
\begin{aligned}
\int_0^T \langle\partial_t u_i(t),\xi_i(t)\rangle\,\mathrm{d}t &= \int_0^T \langle F_i(u(t),\Phi(t)),\nabla\xi_i(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t + \cdots\\
&\cdots + D_i\int_0^T\left\langle u_0^{1/2}(t)\left(\nabla(u_0^{1/2}(t)u_i(t)) - 3u_i(t)\nabla u_0^{1/2}(t)\right),\nabla\xi_i(t)\right\rangle_{L^2(\Omega)}\mathrm{d}t + \cdots \quad (1.9a)\\
&\cdots + \int_0^T + \langle g_i(t,x,u(t)),\xi_i(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t
\end{aligned}
$$

$$-\lambda^2\int_0^T\langle\nabla\Phi(t),\nabla\theta(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t = \int_0^T\left\langle\sum_{j=1}^n z_ju_j(t)+f,\theta(t)\right\rangle_{L^2(\Omega)}\mathrm{d}t \quad (1.9b)$$

for all $\xi \in L^2\left((0,T);H^1(\Omega;\mathbb{R}^n)\right)$, $\theta \in L^2\left((0,T);H^1(\Omega)\right)$. Furthermore the boundary conditions

$$\sqrt{u_0}|_{\Gamma_D} = \sqrt{u_0^D},\ \ u_i\sqrt{u_0}|_{\Gamma_D} = u_i^D\sqrt{u_0^D},\ \ \Phi|_{\Gamma_D} = \Phi^D. \quad (1.10)$$

have to be satisfied in the sense of H^1-traces on Γ_D and the initial conditions

$$u_i(0,\cdot) = u_i^0 \text{ on } \Omega \quad (1.11)$$

have to hold in the sense of equality in $H_D^{-1}(\Omega)$.

Before we can state the existence theorem, which we want to prove, we need to introduce a notion of entropy.

2. The boundedness-by-entropy method

In this chapter we apply the boundedness-by-entropy method to reformulate the system (1.6) into a formal gradient flow structure and derive bounds for the ion concentration. This method was introduced in [Jüngel2] for general cross-diffusion systems. A good summary of the method can be found in [Holz, Section 4.1]. We will follow [GerstJün, Section 1.2] to apply the method to our problem and also include all computations.

First we make the following assumption.

Assumption 1. *Throughout the rest of this thesis we assume that the boundary concentrations u^D are extended to the whole domain Ω, in such a way that $u_i^D \in (0,1)$ for $i = 0, \cdots, n$. Furthermore we assume that $\Phi^D \in L^\infty(\Gamma_D)$ and that it is extended to the whole domain Ω in such a way that it solves*

$$-\lambda^2 \Delta \Phi^D = \sum_{i=1}^{n} z_i u_i \text{ on } \Omega$$

$$\frac{\partial \Phi^D}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_N$$

in a weak fashion.

A key concept for the analysis of (1.6) is the notion of entropy/free energy.

Definition 2.1. *Let $v : \Omega \to \mathcal{O}$ be a measurable function. Define $\Phi(v)$ to be the weak solutions of the Poisson problem*

$$-\Delta \Phi(v) = \Sigma_{j=1}^{n} z_j v_j \text{ on } \Omega$$

$$\Phi(v) = 0 \text{ on } \Gamma_D, \frac{\partial \Phi(v)}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_N,$$

then the entropy density h of v is defined as

$$h(v) := \sum_{i=0}^{n} \int_{u_i^D}^{v_i} \log \frac{s}{u_i^D} \mathrm{d}s + \frac{\beta \lambda^2}{2} \left| \nabla (\Phi(v) - \Phi^D) \right|^2 + \sum_{i=1}^{n} v_i W_i.$$

The entropy/free energy of v is given by

$$H(v) := \int_\Omega h(v) \mathrm{d}x.$$

The following Lemma shows that our notion of entropy is bounded from below.

Lemma 2.2. *For any measurable $v : \Omega \to \mathcal{O}$ it holds that*

$$h(v) \geq -\sum_{i=1}^{n} |W_i|$$

and thus

$$H(v) \geq -\sum_{i=1}^{n} \|W_i\|_{L^1(\Omega)}.$$

Proof. Let $x \in \Omega$. If $v_i(x) > u_i^D(x)$, then $\log \frac{s}{u_i^D(x)} > \log \frac{u_i^D(x)}{u_i^D(x)} = 0$ for all $s \in \left(u_i^D(x), v_i(x)\right)$ and thus

$$\int_{u_i^D(x)}^{v_i(x)} \log \frac{s}{u_i^D(x)} \mathrm{d}s > 0.$$

On the other hand, if $v_i(x) \leq u_i^D(x)$, then $\log \frac{s}{u_i^D(x)} \leq \log \frac{u_i^D(x)}{u_i^D(x)} = 0$ for all $s \in \left[v_i(x), u_i^D(x)\right]$ and therefore

$$\int_{u_i^D(x)}^{v_i(x)} \log \frac{s}{u_i^D(x)} \mathrm{d}s = -\underbrace{\int_{v_i(x)}^{u_i^D(x)} \log \frac{s}{u_i^D(x)} \mathrm{d}s}_{\leq 0} \geq 0.$$

Using that $\|v\|_{L^\infty(\Omega)} \leq 1$, we easily can conclude that

$$h(u) \geq \sum_{i=1}^{n} u_i W_i \geq -\|u\|_{L^\infty(\Omega)} \sum_{i=1}^{n} |W_i|$$

and therefore

$$H(u) = \int_\Omega h(u) \mathrm{d}x \geq -\sum_{i=1}^{n} \int_\Omega |W_i| \, \mathrm{d}x = -\sum_{i=1}^{n} \|W_i\|_{L^1(\Omega)}.$$

□

Furthermore we define the entropy variables.

Definition 2.3. *Let $v : \Omega \to \mathcal{O}$ be a measurable function. For every $i = 1, \cdots, n$ we define the i-th entropy variable corresponding to v by*

$$w_i(v) := \log \frac{v_i}{v_0} + \beta z_i \Phi(v) + W_i,$$

We leave out the argument when it is a (weak) solution to (1.6), i.e. $w_i := w_i(u)$. Furthermore we define for every $i = 1, \cdots, n$ the i-th boundary entropy variable by

$$w_i^D := \log \frac{u_i^D}{u_0^D} + \beta z_i \Phi^D.$$

The name entropy variable becomes clear through the following Lemma.

Lemma 2.4. *Let $v : \Omega \to \mathcal{O}$ be a measurable function and $i = 1, \cdots, n$. Then*

$$\frac{\partial h(v)}{\partial v_i} = w_i(v) - w_i^D.$$

Hereby the derivative is to be understood in a Gateaux sense.

Proof. Since Φ^D suffices to $-\lambda^2 \Delta \Phi^D = f$ on Ω and $\frac{\partial \Phi^D}{\partial \mathbf{n}} = 0$ on Γ_N, it follows that the difference $\Phi(v) - \Phi^D$ solves

$$\begin{aligned} -\lambda^2 \Delta(\Phi(v) - \Phi^D) &= \sum_{i=0} z_i u_i \text{ on } \Omega \\ (\Phi(v) - \Phi^D) &= 0 \text{ on } \Gamma_D \\ \frac{\partial(\Phi(v) - \Phi^D)}{\partial \mathbf{n}} &= 0 \text{ on } \Gamma_N. \end{aligned}$$

Combining this fact with Lemma C.4, we get

$$\begin{aligned} \frac{\partial h(v)}{\partial v_i} &= \sum_{j=0}^{n} \frac{\partial}{\partial v_i} \int_{u_j^D}^{v_j} \log \frac{s}{u_j^D} \mathrm{d}s + \frac{\beta \lambda^2}{2} \underbrace{\frac{\partial}{\partial v_i} \left| \nabla(\Phi(v) - \Phi^D) \right|^2}_{= \frac{2z_i}{\lambda^2}(\Phi(v) - \Phi^D), \text{ by Lemma C.4}} + \sum_{j=1}^{n} \frac{\partial}{\partial v_i} v_j W_j \\ &= \log \frac{v_i}{u_i^D} + \frac{\partial}{\partial v_i} \int_{u_0^D}^{v_0} \log \frac{s}{u_0^D} \mathrm{d}s + \beta z_i (\Phi(v) - \Phi^D) + W_i. \end{aligned}$$

Using that $v_0 = 1 - \sum_{j=1}^{n} v_j$ we can conclude

$$\begin{aligned} \frac{\partial h(u)}{\partial u_i} &= \log \frac{v_i}{u_i^D} + \frac{\partial}{\partial v_i} \int_{u_0^D}^{v_0} \log \frac{s}{u_0^D} \mathrm{d}s + \beta z_i (\Phi(v) - \Phi^D) + W_i \\ &= \underbrace{\log \frac{v_i}{u_i^D} - \log \frac{v_0}{u_0^D}}_{= \log \frac{v_i u_0^D}{u_i^D v_0} = \log \frac{v_i}{v_0} - \log \frac{u_i^D}{u_0^D}} + \beta z_i (\Phi(v) - \Phi^D) + W_i \\ &= \log \frac{v_i}{v_0} - \log \frac{u_i^D}{u_0^D} + \beta z_i (\Phi(v) - \Phi^D) + W_i = w_i - w_i^D. \end{aligned}$$

□

Remark 2.5. *According to [Kardar, equation (1.27), page 15] the entropy h of a thermodynamical system suffices to*

$$\mathrm{d}h = \frac{\mathrm{d}E - \sum_{j=1}^{k} \psi_j \mathrm{d}x_j}{T},$$

where E is the energy of the system, T is the temperature, x_j is the j-th state function and ψ_j is its conjugate force. Therefore, if E and all states other than the i-th ion concentration are kept constant, one has

$$\mathring{w}_i := w_i - w_i^D = \frac{\partial h(u)}{\partial u_i} = \frac{\psi_i}{T},$$

where ψ_i is the conjugate force to the i-th ion concentration u_i.

Thus we could view the entropy variables as the quotient of conjugate forces and temperature. Indeed it was shown in [Jüngel2, Appendix A], that the entropy variables are strongly related to the chemical potential.

Note that one can easily obtain v_i from the corresponding entropy variable by a simple transformation

$$v_i = \frac{\exp(w_i(v) - \beta z_i \Phi(v) - W_i)}{1 + \sum_{j=1}^n \exp(w_j(v) - \beta z_j \Phi(v) - W_j)}. \tag{2.1}$$

This formula leads to the following definition

Definition 2.6. *For every vector $x \in \mathbb{R}^n$ and every scalar $y \in \mathbb{R}$ we define*

$$u_i(x, y) = \frac{\exp(x_i - \beta z_i y - W_i)}{1 + \sum_{j=1}^n \exp(x_j - \beta z_j y - W_j)}$$

for every $i = 1, \cdots, n$.

The entropy variables allow us to formally reformulate (1.6) in a (quasi) gradient flow structure as the following Lemma shows.

Lemma 2.7. *Let (u, Φ) be a strong solution to (1.6) and define the matrix valued function B by*

$$B_{ij} := \begin{cases} D_i u_0 u_i, & \text{if } i = j \\ 0, & \text{if} i \neq j, \end{cases}$$

for all $i, j = 1, \cdots, n$. Then the i-th ionflux can be written as

$$J_i = \sum_{j=1}^n B_{ij} \nabla w_j.$$

Proof. Due to the diagonal form of B we get for every $i = 1, \cdots, n$ that

$$\begin{aligned}
\sum_{j=1}^n B_{ij} \nabla w_j = B_{ii} \nabla w_i &= D_i u_0 u_i \nabla \left(\log \frac{u_i}{u_0} + \beta z_i \Phi + W_i \right) \\
&= D_i u_0 u_i \nabla \left(\log u_i - \log u_0 + \beta z_i \Phi + W_i \right) \\
&= D_i u_0 u_i \left(\frac{1}{u_i} \nabla u_i - \frac{1}{u_0} \nabla u_0 + \beta z_i \nabla \Phi + \nabla W_i \right) \\
&= D_i \left(u_0 \nabla u_i - u_i \nabla u_0 + \beta z_i u_0 u_i \nabla \Phi + u_0 u_i \nabla W_i \right) \\
&= J_i.
\end{aligned}$$

□

Thus we can write the first equation in (1.6) as a (quasi) gradient flow

$$\partial_t u_i = \operatorname{div}\left(\sum_{j=1}^{n} B_{ij}\nabla w_j\right) + g_i(t,x,u) \tag{2.2}$$

for every $i = 1, \cdots, n$. Or equivalently in matrix form

$$\partial_t u = \operatorname{div}(B\nabla w) + g(t,x,u). \tag{2.3}$$

We can further reformulate (2.3) entirely in terms of the entropy variables w. To do this we need the following definition.

Definition 2.8. *Similarly to the definitions in Lemma 2.7, we define for every vector $x \in \mathbb{R}^n$ and every scalar $y \in \mathbb{R}$ the diagonal matrix $B(x,y)$ by*

$$B_{ij}(x,y) := \begin{cases} D_i u_0(x,y)u_i(x,y), & \text{if } i = j \\ 0, & \text{if } i \neq j, \end{cases}$$

for all $i, j = 1, \cdots, n$.

Lemma 2.9. *For every $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}$ the matrix $B(x,y)$ is positive definite and is (uniformely) bounded in the maximum norm, i.e.*

$$\max_{i,j=1,\cdots,n} |B_{ij}(x,y)| \leq \max_{i=1,\cdots,n} D_i.$$

Proof. $B(x,y)$ is a diagonal matrix and all its diagonal entries are strictly positive, thus it is indeed positive definite. The upper bound of the maximum norm is a direct consequence of $0 < u_i(x,y) < 1$ for $i = 0, \cdots, n$.

□

Using the Definitions (2.6) and (2.8), the (quasi) gradient flow equation (2.3) can be written as a doubly nonlinear system without cross-diffusion

$$\partial_t u(w,\Phi) = \operatorname{div}(B(w,\Phi)\nabla w) + g(t,x,u(w,\Phi)). \tag{2.4}$$

The definition of the concentration vector u as a transformation of the entropy variable w and the potential Φ in 2.6 also lets us formally prove a physical property of the system, namely that the total charge density is monotonically decreasing with respect ot the potential. To prove this fact, we make the following definition.

Definition 2.10. *For any vector $x \in \mathbb{R}^n$ and any scalar $y \in \mathbb{R}$ we define*

$$\mathfrak{f}(x,y) := \sum_{i=1}^{n} z_i u_i(x,y),$$

The total charge density is thus given by $\mathfrak{f}(w,\Phi)$.

Remark 2.11. *Please note that, since the definition of $u(\cdot,\cdot)$ involves the external potential, $\mathfrak{f}$ is not a real valued function. For ever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, $\mathfrak{f}(x, y)$ is a function depending on space (just as the external potential).*

Lemma 2.12. *$\mathfrak{f}$ is monotonically decreasing with respect to the second argument(the variable y) and Lipschitz-continuous[1]. An upper bound for the optimal Lipschitz-constant for the second argument is given by*

$$L(z,\beta) := \beta \max\left\{\frac{(z_i - z_j)^2}{2}, z_i^2 \;:\; i,j = 1,\cdots n\right\}.$$

Proof. To make the formulas that appear in our proof less loaded, we denote $e_i := \exp(x_i - \beta z_i y - W_i)$ and $\mathfrak{e}_{ij} := e_i e_j$ for all $i, j = 1, \cdots, n$.

Using the quotient rule we get

$$\frac{\partial u_i}{\partial y}(x,y) = \frac{(-\beta z_i)e_i\left(1 + \sum\limits_{j=1}^{n} e_j\right) - e_i\left(\sum\limits_{j=1}^{n}(-\beta z_j)e_j\right)}{\left(1 + \sum\limits_{j=1}^{n} e_j\right)^2} = \frac{\beta\sum\limits_{j=1}^{n}(z_j - z_i)\mathfrak{e}_{ij} - \beta z_i e_i}{\left(1 + \sum\limits_{j=1}^{n} e_j\right)^2}.$$

Thus we derive

$$\frac{\partial \mathfrak{f}}{\partial y}(x,y) = \sum_{i=1}^{n} z_i \frac{\partial u_i}{\partial y}(x,y) = \frac{\beta\sum\limits_{i,j=1}^{n} z_i(z_j - z_i)\mathfrak{e}_{ij} - \beta\sum\limits_{i=1}^{n} z_i^2 e_i}{\left(1 + \sum\limits_{j=1}^{n} e_j\right)^2}.$$

Using the identity

$$\begin{aligned}\sum_{i,j=1}^{n} z_i(z_j - z_i)\mathfrak{e}_{ij} &= \sum_{i,j=1}^{n} z_i z_j \mathfrak{e}_{ij} - \sum_{i,j=1}^{n} z_i^2 \mathfrak{e}_{ij} \\ &= \frac{1}{2}\sum_{i,j=1}^{n} 2 z_i z_j \mathfrak{e}_{ij} - \frac{1}{2}\sum_{i,j=1}^{n} z_i^2 \mathfrak{e}_{ij} - \frac{1}{2}\sum_{i,j=1}^{n} z_j^2 \underbrace{\mathfrak{e}_{ji}}_{=\mathfrak{e}_{ij}} \\ &= \left(-\frac{1}{2}\right)\sum_{i,j=1}^{n}\left(z_j^2 - 2z_i z_j + z_i^2\right)\mathfrak{e}_{ij} = \left(-\frac{1}{2}\right)\sum_{i,j=1}^{n}(z_j - z_i)^2\,\mathfrak{e}_{ij},\end{aligned}$$

we obtain

$$\frac{\partial \mathfrak{f}}{\partial y}(x,y) = \frac{-\frac{\beta}{2}\sum\limits_{i,j=1}^{n}(z_j - z_i)^2\mathfrak{e}_{ij} - \beta\sum\limits_{i=1}^{n} z_i^2 e_i}{\left(1 + \sum\limits_{j=1}^{n} e_j\right)^2}$$

[1] As stated in Remark 2.11, $\mathfrak{f}$ is not a real valued function, as it involves the external potential. So the more accurate statement would be that the claimed properties hold pointwise for fixed spatial variable.

Since $\beta, e_i, \mathfrak{e}_{ij} > 0$ for all $i, j = 1, \cdots, n$, we get that $\frac{\partial \mathfrak{f}}{\partial y} \leq 0$ and thus $\mathfrak{f}$ is indeed monotonically decreasing.

By using that

$$\left(1 + \sum_{j=1}^{n} e_j\right)^2 = 1 + 2\sum_{j=1}^{n} e_j + \sum_{i,j=1}^{n} \mathfrak{e}_{ij},$$

we derive

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n, y\in\mathbb{R}} \left| \frac{\partial \mathfrak{f}}{\partial y}(x,y) \right| &= \sup_{x\in\mathbb{R}^n, y\in\mathbb{R}} \frac{\frac{\beta}{2}\sum\limits_{i,j=1}^{n}(z_j - z_i)^2 \mathfrak{e}_{ij} + \beta \sum\limits_{i=1}^{n} z_i^2 e_i}{\left(1 + \sum\limits_{j=1}^{n} e_j\right)^2} \\
&\leq \beta \max\left\{ \frac{(z_i - z_j)^2}{2}, z_i^2 \ : \ i, j = 1, \cdots n \right\} \sup_{x\in\mathbb{R}^n, y\in\mathbb{R}} \frac{\sum\limits_{i,j=1}^{n} \mathfrak{e}_{ij} + \sum\limits_{i=1}^{n} e_i}{\left(1 + \sum\limits_{j=1}^{n} e_j\right)^2} \\
&\leq \beta \max\left\{ \frac{(z_i - z_j)^2}{2}, z_i^2 \ : \ i, j = 1, \cdots n \right\} \leq L(z, \beta) < +\infty.
\end{aligned}$$

Similarly one can verify that $\frac{\partial \mathfrak{f}}{\partial x_k}$ can be uniformely bounded for all $k = 1, \cdots, n$. Thus $\mathfrak{f}$ is indeed Lipschitz-continuous.

□

Therefore the total charge density $\mathfrak{f}(w, \Phi)$ indeed decreases when the potential Φ is increased. This will be useful in the next chapter.

Another useful consequence of Definition 2.6 is the following.

Corollary 2.13. *For any $\xi \in L^\infty(\Omega; \mathbb{R}^n)$, any $y \in L^\infty(\Omega)$ and any $T > 0$, we define*

$$C^g(T) := \sup\left\{ \ |g(t,x,v)| \ \middle| \ t \leq T, \ x \in \bar{\Omega}, \ v \in \bar{\mathcal{O}} \ \right\}.$$

If $g \in C\left([0, +\infty) \times \bar{\Omega} \times \bar{\mathcal{O}}; \mathbb{R}^n\right)$, then it holds that

$$g\left(\cdot, \cdot, u(\xi, y)\right) \in L^\infty\left([0, T] \times \Omega\right),$$

with

$$\sup_{t\leq T, \ x\in\Omega} |g\left(t, x, u(\xi(x), y(x))\right)| \leq C^g(T) < +\infty. \tag{2.5}$$

Proof. The fact that $C^g(T)$ is bounded is a direct consequence of the continuity of g. Inequality 2.5 follows directly from Definition 2.6. □

3. Existence of weak solutions

Now we are in the position to state, under which assumptions we can prove the existence of weak solutions.

Assumption 2. *For the rest of this thesis the following assumptions shall hold.*

(H1) ***Domain:*** *$\Omega \subseteq \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, that is partitioned into two parts Γ_D and Γ_N, with*

$$\Gamma_D \cap \Gamma_N \neq \emptyset,\ \Gamma_D \cap \Gamma_N = \partial\Omega.$$

Additionally we require that Γ_N is relatively open with repsect to $\partial\Omega$ and that $\operatorname{meas}(\Gamma_D) > 0$, where meas *denotes the $(d-1)$-dimensional Hausdorff-measure.*

(H2) ***Parameters:*** *$D_i, \beta > 0$ and $z_i \in \mathbb{R}$ for every $i = 1, \cdots, n$.*

(H3) ***Initial data:*** *We assume that $u^0 \in L^\infty(\Omega;\mathbb{R}^n)$ with $0 < u_i^D < 1$ for every $i = 0, \cdots, n$.*

(H4) ***Given functions:*** *The permanent charge density $f \in L^\infty(\Omega)$ is bounded and the external potential $W \in H^1(\Omega;\mathbb{R}^n) \cap L^\infty(\Omega;\mathbb{R}^n)$ suffices to homogenous boundary conditions, i.e. $W_i = 0$ on Γ_D and $\nabla W_i \cdot \mathbf{n} = 0$ on Γ_N for every $i = 1, \cdots, n$.*

(H5) ***Reaction term:*** *We assume that*

$$g \in C\left([0,+\infty) \times \bar{\Omega} \times \bar{\mathcal{O}}; \mathbb{R}^n\right)$$

and that there exist an increasing function $\kappa^g : (0,+\infty) \to (0,+\infty)$ and a constant $\mathfrak{c} > 0$ such that

$$\sum_{i=1}^{n} g_i(t,\cdot,v)\frac{\partial h(v)}{\partial v_i} \leq \kappa^g(t)\left(\mathfrak{c} + h(v)\right)$$

for all times $t > 0$ and every measurable function $v : \Omega \to \mathcal{O}$.

The following simple Corollary follows directly from the Assumptions in 2.

Corollary 3.1. *Under the Assumptions 1 and 2 the extension of the boundary potential Φ^D is bounded, i.e. $\Phi^D \in L^\infty$.*

Proof. This is a direct consequence of the fact that by Assumption 2 $f \in L^\infty$ and from Theorem C.3. □

Now we state the main Theorem of this thesis.

Theorem 3.2. *Let $T > 0$ be arbitrary. Under the Assumption 1 and 2 there exists a weak solution (u, Φ) up to time T.*

We prove the existence of weak solutions by approximation via time-discretized and space-regularized problems. More precisely we split up the proof of Thm 3.2 into four steps. In the first step we prove the existence of weak solutions $(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k))_{k\in\mathbb{N}_0}$ to space-regularized implicit Euler discretizations of the doubly nonlinear equation (2.4). I.e. if we define the finite difference operator ∂^τ by

$$[\partial^\tau u\,(w^{\epsilon,\tau}, \Phi^{\epsilon,\tau})]\,(k) := \frac{u\,(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)) - u\,(w^{\epsilon,\tau}(k-1), \Phi^{\epsilon,\tau}(k-1))}{\tau}, \tag{3.1}$$

for every $k \in \mathbb{N}$, then for every $\epsilon, \tau > 0$ the sequence $(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k))_{k\in\mathbb{N}_0}$ satisfies

$$\begin{aligned} \partial^\tau u\,(w^{\epsilon,\tau}, \Phi^{\epsilon,\tau}) &= \operatorname{div}\left(B\,(w^{\epsilon,\tau}, \Phi^{\epsilon,\tau})\,\nabla w^{\epsilon,\tau}\right) + g\,(\tau\,\cdot\,, x, u\,(w^{\epsilon,\tau}, \Phi^{\epsilon,\tau})) + \epsilon\mathcal{L}^m \overset{\circ}{w}{}^{\epsilon,\tau} \\ \overset{\circ}{w}{}^{\epsilon,\tau} &= w^{\epsilon,\tau} - w^D \\ -\lambda^2\Delta\Phi^{\epsilon,\tau} &= \sum_{i=1}^{n} z_i u_i\,(w^{\epsilon,\tau}, \Phi^{\epsilon,\tau}) + f. \end{aligned} \tag{3.2}$$

for a fixed regularity parameter $m \in \mathbb{N}$, that is chosen such, that $H^m(\Omega;\mathbb{R}^n)$ is compactly embedded into $L^\infty(\Omega;\mathbb{R}^n)$. Here we define the operator $\mathcal{L}^m : H^m(\Omega;\mathbb{R}^n) \to H^m(\Omega;\mathbb{R}^n)'$ by

$$\langle \mathcal{L}^m \xi, \eta\rangle := \sum_{\alpha\in\mathbb{N}^m:\ |\alpha|=m} \langle D^\alpha \xi, D^\alpha \eta\rangle_{L^2(\Omega)} + \langle \xi, \eta\rangle_{L^2(\Omega)}\,, \text{ for all } \xi, \eta \in H^m(\Omega;\mathbb{R}^n). \tag{3.3}$$

In the second step we prove a discrete entropy production inequality for the sequence defined in the first step, which is then used to derive bounds for $\overset{\circ}{w}{}^{\epsilon,\tau}$, $u\,(w^{\epsilon,\tau}, \Phi^{\epsilon,\tau})$ and various derived quantities. With these bounds we then eliminate the space regularization in the third step and finally derive the existence of weak solutions in the fourth step, both via compactness arguments.

Remark 3.3. *We will not investigate the uniqueness in this thesis. Indeed [GerstJün] was only able to show the uniqueness for $H^1(\Omega;\mathbb{R}^n)$-valued concentrations and under additional restricing requirements regarding the parameters of system (1.6).*

Before we start with step 1, we investigate the semilinear elliptic equation (1.3).

3.1. Analysis of the semilinear elliptic equation

The following Lemma shows that weak solutions to (1.3) exist and are bounded for all potential entropy variables.

Theorem 3.4. *Let $w \in L^\infty(\Omega;\mathbb{R}^n)$. Then the non-linear Poisson-equation*

$$\begin{aligned} -\lambda^2\Delta\Phi &= \sum_{i=1}^n z_i u_i(w,\Phi) + f \ \text{ on } \Omega \\ \Phi = \Phi^D \ \text{ on } \Gamma_D,& \ \frac{\partial\Phi}{\partial\mathbf{n}} = 0 \ \text{ on } \Gamma_N \end{aligned} \tag{3.4}$$

has a unique weak solution $\Phi \in H^1(\Omega)$ and the following estimate holds

$$\|\Phi\|_{H^1(\Omega)} \le C(1 + \|\Phi^D\|_{H^1(\Omega)}). \tag{3.5}$$

Hereby the constant $C > 0$ depends only on $\Omega, \Gamma_D, \lambda, z$ and $\|f\|_{L^2(\Omega)}$.

Proof. We prove the existence of the solution by a standard fixed point argument that can, for example, be found in [Arnold, Satz 2.12, page 21].

Proof of existence: Let $\eta \in L^2(\Omega)$, then $u_i(w,\eta) \in L^\infty(\Omega) \subset L^2(\Omega)$ for all $i = 1, \cdots, n$ and thus

$$\mathfrak{f}(w,\eta) + f = \sum_{i=1}^n z_i u_i(w,\eta) + f \in L^2(\Omega).$$

By Theorem C.1 there exists a unique weak solution $\Phi^\eta \in H^1(\Omega)$ of the linear Poisson problem

$$\begin{aligned} -\lambda^2\Delta\Phi^\eta &= \mathfrak{f}(w,\eta) + f \text{ on } \Omega \\ \Phi^\eta &= \Phi^D \text{ on } \Gamma_D, \\ \frac{\partial\Phi}{\partial\mathbf{n}} &= 0 \text{ on } \Gamma_N, \end{aligned}$$

which satisfies

$$\|\Phi^\eta\|_{H^1(\Omega)} \le C(\Omega,\Gamma_D)\left(\|\mathfrak{f}(w,\eta) + f\|_{L^2(\Omega)} + \|\Phi^D\|_{H^1(\Omega)}\right),$$

for a constant $C(\Omega,\Gamma_D) > 0$, that only depends on Ω and Γ_D.

If we set $C := C(\Omega,\Gamma_D)\max\{\sum_{i=1}^n |z_i|\sqrt{|\Omega|} + \|f\|_{L^2(\Omega)}, 1\}$, then the triangle inequality gives us

$$\begin{aligned} \|\Phi^\eta\|_{H^1(\Omega)} &\le C(\Omega,\Gamma_D)\left(\sum_{i=1}^n |z_i| \|\underbrace{u_i(w,\eta)}_{\le 1}\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\Phi^D\|_{H^1(\Omega)}\right) \\ &\le C(\Omega,\Gamma_D)\left(\sum_{i=1}^n |z_i|\sqrt{|\Omega|} + \|f\|_{L^2(\Omega)} + \|\Phi^D\|_{H^1(\Omega)}\right) \\ &\le C\left(1 + \|\Phi^D\|_{H^1(\Omega)}\right) \end{aligned}$$

We define the map $S : L^2(\Omega) \to L^2(\Omega)$ by $S(\eta) := \Phi^\eta$. Then the image of S is a subset of

$$S(L^2(\Omega)) \subseteq \left\{ v \in H^1(\Omega) \ : \ \|v\|_{H^1(\Omega)} \leq C \left(1 + \left\|\Phi^D\right\|_{H^1(\Omega)}\right) \right\}, \tag{3.6}$$

and is thus, due to the compactness of the embedding of $H^1(\Omega)$ in $L^2(\Omega)$, precompact in $L^2(\Omega)$.

Let $\eta_1, \eta_2 \in L^2(\Omega)$, then $S(\eta_1) - S(\eta_2) \in H^1_D(\Omega)$ and therefore we derive by using the Cauchy–Schwarz inequality and the Lipschitz-continuity of the map $\mathfrak{f}$ in Lemma 2.12

$$\begin{aligned}
\|S(\eta_1) - S(\eta_2)\|^2_{H^1(\Omega)} &\leq \underbrace{(1 + C_P^2)}_{=:C_1} \langle \nabla(S(\eta_1) - S(\eta_2)), \nabla(S(\eta_1) - S(\eta_2))\rangle_{L^2(\Omega)} \\
&= C_1 \langle \mathfrak{f}(w, \eta_1) - \mathfrak{f}(w, \eta_2), S(\eta_1) - S(\eta_2)\rangle_{L^2(\Omega)} \\
&\leq C_1 \|\mathfrak{f}(w, \eta_1) - \mathfrak{f}(w, \eta_2)\|_{L^2(\Omega)} \|S(w, \eta_1) - S(w, \eta_2)\|_{L^2(\Omega)} \\
&\leq \underbrace{C_1 L(z, \beta)}_{=:C_2} \|\eta_1 - \eta_2\|_{L^2(\Omega)} \|S(w, \eta_1) - S(w, \eta_2)\|_{H^1(\Omega)}
\end{aligned}$$

Thus we get the inequality

$$\|S(\eta_1) - S(\eta_2)\|_{L^2(\Omega)} \leq \|S(\eta_1) - S(\eta_2)\|_{H^1(\Omega)} \leq C_2 \|\eta_1 - \eta_2\|_{L^2(\Omega)}.$$

Since S is a continuous map from $L^2(\Omega)$ into a precompact subset of $L^2(\Omega)$, we can apply the Schauder fixed point theorem, which tells us, that there is a (not necessarily unique) element $\Phi \in S(L^2(\Omega)) \subseteq H^1(\Omega)$, such that $\Phi = S(\Phi)$. Obviously this Φ is a weak solution of (3.4).

Proof of the bound (3.5): This is a trivial consequence of $\Phi = S(\Phi)$ and (3.6).

Proof of uniqueness: Let $\Phi_1, \Phi_2 \in H^1(\Omega)$ be two weak solutions, then $\Phi_1 - \Phi_2 \in H^1_D(\Omega)$ and we get by using the Poincare-inequality (Theorem B.5)

$$\begin{aligned}
\|\Phi_1 - \Phi_2\|^2_{H^1(\Omega)} &\leq (1 + C_P^2) \langle \nabla(\Phi_1 - \Phi_2), \nabla(\Phi_1) - \Phi_2)\rangle_{L^2(\Omega)} \\
&= (1 + C_P^2) \langle \mathfrak{f}(w, \Phi_1) - \mathfrak{f}(w, \Phi_2), \Phi_1 - \Phi_2\rangle_{L^2(\Omega)} \leq 0.
\end{aligned}$$

Hereby the last inequality is a direct consequence of the fact that $\mathfrak{f}$ is monotonically decreasing (as we proved in Lemma 2.12).
Thus $\Phi_1 = \Phi_2$ and therefore solutions to (3.4) are unique.

□

Remark 3.5. *In [GerstJün] the uniqueness of weak solutions to (3.4) is argued with the Lipschitz-continuity of $\mathfrak{f}$. This has the disadvantage that, at least when one uses a naive proof via test function, one would need to require that $L(z, \beta)C_P^2 < 1$. Using that the Poincare constant C_P is given by the measure of Ω scaled by some constant $C > 0$, that only depends on the form of Ω, and the definition $L(z, \beta)$, this is equivalent to*

$$\max \left\{ \frac{(z_i - z_j)^2}{2}, z_i^2 \;\middle|\; i, j = 1, \cdots, n \right\} \leq C^2 \frac{1/\beta}{|\Omega|^2}.$$

The first term can be viewed as a measure of the chemical reactiveness of the system[1] *and* $1/\beta$ *is the thermal voltage. Thus the reactiveness would have to be bounded by the thermal voltage per sqared area.*

Next we prove that weak solutions to (3.4) are bounded in the supremum norm.

Lemma 3.6. *Let* $w \in L^\infty(\Omega;\mathbb{R}^n)$. *Then the solution* Φ *to the non-linear Poisson-equation* (3.4) *suffices to* $\Phi \in L^\infty(\Omega)$ *and* $\|\Phi\|_{L^\infty(\Omega)} \leq C$ *form some positive constant, that only depends on the parameters of system* (1.6).

Proof. For the proof we employ a comparison principle. Let

$$M := \max_{i=1,\cdots,n} |z_i| + \|f\|_{L^\infty(\Omega)}\,, \quad l := \left\|\Phi^D\right\|_{L^\infty(\Gamma_D)}\,,$$

and $\bar{\Phi}$ be the weak solution to the linear Poisson-problem

$$\begin{aligned} -\lambda^2\Delta\bar{\Phi} &= M \text{ on } \Omega \\ \bar{\Phi} &= l \text{ on } \Gamma_D, \\ \frac{\partial\bar{\Phi}}{\partial\mathbf{n}} &= 0 \text{ on } \Gamma_N \end{aligned}$$

Due to Theorem C.3 it holds that $\bar{\Phi} \in L^\infty(\Omega)$.

By the Stampacchia Lemma B.14 we get $v := (\Phi - \bar{\Phi})^+ \in H^1_D(\Omega)$ and thus we derive, by using the Poincare-inequality (Theorem B.5) and again the Stampacchia Lemma B.14, that

$$\lambda^2\left\|(\Phi-\bar{\Phi})^+\right\|^2_{L^2(\Omega)} \leq \lambda^2 C_P\left\|\nabla(\Phi-\bar{\Phi})^+\right\|^2_{L^2(\Omega)} = C_P\lambda^2\left\langle\nabla(\Phi-\bar{\Phi}),\nabla v\right\rangle_{L^2(\Omega)}$$

The definition of weak solutions to linear and nonlinear elliptic equations gives us

$$\begin{aligned} \lambda^2\left\|(\Phi-\bar{\Phi})^+\right\|^2_{L^2(\Omega)} &\leq C_P\lambda^2\left\langle\nabla(\Phi-\bar{\Phi}),\nabla v\right\rangle_{L^2(\Omega)} \\ &= C_P\Big\langle\underbrace{\sum_{i=1}^n z_i u_i(w,\Phi) + f - M}_{\leq 0},\underbrace{v}_{\geq 0}\Big\rangle_{L^2(\Omega)} \leq 0 \end{aligned}$$

Thus we know that $\Phi \leq \bar{\Phi}$, which implies that Φ is bounded from above and the bound only depends on the parameters of system (1.6). Similarly one can derive that Φ is bounded from below and therefore the statement holds.

□

Similarly to the proof of uniqueness of weak solutions to (3.4), we prove the continuous dependence on the parameter w.

[1] The valence of an ion species is a measure of its combining power with other ions.

Lemma 3.7. *The solution to equation (3.4) depends Lipschitz-continuously, with respect to the $H^1(\Omega)$-norm, on $w \in L^\infty(\Omega;\mathbb{R}^n)$, i.e. there exists a constant $C > 0$, which only depends on the parameters of system (1.6), such that*

$$\left\| \Phi - \tilde{\Phi} \right\|_{H^1(\Omega)} \leq C \left\| w - \tilde{w} \right\|_{L^\infty(\Omega)}, \tag{3.7}$$

for any two weak solutions Φ, $\tilde{\Phi}$, corresponding to the functions $w, \tilde{w} \in L^\infty(\Omega;\mathbb{R}^n)$. A similar inequality also holds for the positive and the negative part of the difference, i.e.

$$\left\| \nabla \left(\Phi - \tilde{\Phi} \right)^{\pm} \right\|_{L^2(\Omega)} \leq \tilde{C} \left\| w - \tilde{w} \right\|_{L^\infty(\Omega)}, \tag{3.8}$$

for a constant $\tilde{C} > 0$ that again only depend on the parameters of system (1.6).

Proof. Let $\Phi, \tilde{\Phi} \in H^1(\Omega)$ be two weak solution to (3.4), corresponding to $w, \tilde{w} \in L^\infty(\Omega;\mathbb{R}^n)$. Since they suffice to the same Dirichlet conditions, $\Phi = \Phi^D = \tilde{\Phi}$ on Γ_D, it holds that $\Phi - \tilde{\Phi} \in H^1_D(\Omega)$.

Thus, due to the Stampacchia Lemma B.14, we can use $\left(\Phi - \tilde{\Phi}\right)^+$ as a test function and derive

$$\begin{aligned}
\left\| \nabla \left(\Phi - \tilde{\Phi} \right)^{+} \right\|^2_{L^2(\Omega)} &= \left\langle \nabla \left(\Phi - \tilde{\Phi} \right)^{+}, \nabla \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)} \\
&\overset{B.14}{=} \left\langle \mathbb{1}_{[\Phi > \tilde{\Phi}]} \nabla \left(\Phi - \tilde{\Phi} \right), \nabla \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)} = \frac{\lambda^2}{\lambda^2} \left\langle \nabla \left(\Phi - \tilde{\Phi} \right), \nabla \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)} \\
&\overset{(3.4)}{=} \frac{1}{\lambda^2} \left\langle \left(\sum_{i=1}^n z_i u_i(w,\Phi) + f \right) - \left(\sum_{i=1}^n z_i u_i(\tilde{w},\tilde{\Phi}) + f \right), \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)} \\
&= \frac{1}{\lambda^2} \left\langle \mathfrak{f}(w,\Phi) - \mathfrak{f}\left(\tilde{w},\tilde{\Phi}\right), \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)} \\
&= \frac{1}{\lambda^2} \left\langle \mathfrak{f}(w,\Phi) - \mathfrak{f}(\tilde{w},\Phi) + \mathfrak{f}(\tilde{w},\Phi) - \mathfrak{f}\left(\tilde{w},\tilde{\Phi}\right), \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)}.
\end{aligned}$$

Since $\mathfrak{f}$ is monotonically decreasing in the second argument, which was proven in Lemma 2.12, it holds that

$$\left\langle \mathfrak{f}(\tilde{w},\Phi) - \mathfrak{f}\left(\tilde{w},\tilde{\Phi}\right), \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)} \leq 0$$

and thus

$$\left\| \nabla \left(\Phi - \tilde{\Phi} \right)^{+} \right\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda^2} \left\langle \mathfrak{f}(w,\Phi) - \mathfrak{f}(\tilde{w},\Phi), \left(\Phi - \tilde{\Phi} \right)^{+} \right\rangle_{L^2(\Omega)}.$$

From the Cauchy–Schwarz inequality and the Lipschitz-continuity of $\mathfrak{f}$, proven in Lemma 2.12, we derive

$$\begin{aligned}\left\|\nabla\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)}^2 &\leq \frac{1}{\lambda^2}\left\|\mathfrak{f}(w,\Phi)-\mathfrak{f}(\tilde{w},\Phi)\right\|_{L^2(\Omega)}\left\|\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)}\\ &\leq \frac{C_{\text{Lip}}}{\lambda^2}\left\|w-\tilde{w}\right\|_{L^2(\Omega)}\left\|\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)}\end{aligned}$$

Since $\left(\Phi-\tilde{\Phi}\right)^{+}\in H_D^1(\Omega)$ we can apply the Poincare-inequality to get

$$\begin{aligned}\left\|\nabla\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)}^2 &\leq \frac{C_P C_{\text{Lip}}}{\lambda^2}\left\|w-\tilde{w}\right\|_{L^2(\Omega)}\left\|\nabla\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)}\\ &\leq \underbrace{\frac{C_P C_{\text{Lip}}\sqrt{|\Omega|}}{\lambda^2}}_{:=C_1}\left\|w-\tilde{w}\right\|_{L^\infty(\Omega)}\left\|\nabla\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)},\end{aligned}$$

which gives us

$$\left\|\nabla\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)} \leq C_1\left\|w-\tilde{w}\right\|_{L^\infty(\Omega)}.$$

Using that $\left(\Phi-\tilde{\Phi}\right)^{-}=\left(\tilde{\Phi}-\Phi\right)^{+}$, we can derive the same inequality for the negative part. Thus (3.8) indeed holds. Finally, by splitting up $\Phi-\tilde{\Phi}$ into its positive and negative part, we derive

$$\left\|\nabla\left(\Phi-\tilde{\Phi}\right)\right\|_{L^2(\Omega)} \leq \left\|\nabla\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)}+\left\|\nabla\left(\Phi-\tilde{\Phi}\right)^{+}\right\|_{L^2(\Omega)} \leq 2C_1\left\|w-\tilde{w}\right\|_{L^\infty(\Omega)}.$$

Using the Poincare inequality on $\Phi-\tilde{\Phi}$, lets us conclude the Lipschitz-continuous dependence on w.

□

Finally we prove that weak solutions to (3.4), depend (Hölder-)continuously in the $L^\infty(\Omega)$-norm on w. This will play a crucial role in the next section.

Lemma 3.8. *The solution to equation (3.4) depends Hölder-continuously, with respect to the $L^\infty(\Omega)$-norm, on $w\in L^\infty(\Omega;\mathbb{R}^n)$, i.e. there exist constants $C>0$ and $\gamma\in(0,1)$, which only depend on the parameters of system (1.6), such that*

$$\left\|\Phi-\tilde{\Phi}\right\|_{L^\infty(\Omega)} \leq C\left\|w-\tilde{w}\right\|_{L^\infty(\Omega)}^{\gamma},$$

for weak solutions Φ, $\tilde{\Phi}$ corresponding to the functions $w,\tilde{w}\in L^\infty(\Omega;\mathbb{R}^n)$.

Proof. Let $\Phi, \tilde{\Phi} \in H^1(\Omega)$ be two weak solution to (3.4) corresponding to $w, \tilde{w} \in L^\infty(\Omega; \mathbb{R}^n)$. We define the domain

$$\Omega^1 := \left[\Phi > \tilde{\Phi}\right] := \left\{ \; x \in \Omega \; : \; \Phi(x) > \tilde{\Phi}(x) \; \right\}.$$

Then it holds that

$$\begin{aligned}
\partial\Omega^1 &= \left(\left[\Phi = \tilde{\Phi}\right] \cap \bar{\Omega}\right) \bigcup \left(\left[\Phi > \tilde{\Phi}\right] \cap \partial\Omega\right) \\
&= \left(\left[\Phi = \tilde{\Phi}\right] \cap \bar{\Omega}\right) \bigcup \underbrace{\left(\left[\Phi > \tilde{\Phi}\right] \cap \Gamma_D\right)}_{=\emptyset, \text{ as } \Phi = \Phi^D = \tilde{\Phi} \text{ on } \Gamma_D} \bigcup \left(\left[\Phi > \tilde{\Phi}\right] \cap \Gamma_N\right) \\
&= \underbrace{\left(\left[\Phi = \tilde{\Phi}\right] \cap \bar{\Omega}\right)}_{:=\Gamma_D^1} \bigcup \underbrace{\left(\left[\Phi > \tilde{\Phi}\right] \cap \Gamma_N\right)}_{:=\Gamma_N^1}.
\end{aligned}$$

Therefore we know that $\left(\Phi - \tilde{\Phi}\right)^+ = 0$ on Γ_D^1. The Stampacchia Lemma B.14 implies that $\left(\Phi - \tilde{\Phi}\right)^+ \in H_D^1(\Omega)$ and that for any $v \in C^1(\Omega)$, with $\operatorname{supp}(v) \subseteq \Omega^1$ and $v \geq 0$, it holds that

$$\begin{aligned}
\lambda^2 \left\langle \nabla \left(\Phi - \tilde{\Phi}\right)^+, \nabla v \right\rangle_{L^2(\Omega^1)} &= \lambda^2 \left\langle \nabla \left(\Phi - \tilde{\Phi}\right)^+, \nabla v \right\rangle_{L^2(\Omega)} \\
= \lambda^2 \left\langle \mathbb{1}_{[\Phi > \tilde{\Phi}]} \nabla \left(\Phi - \tilde{\Phi}\right), \nabla v \right\rangle_{L^2(\Omega)} &\overset{\operatorname{supp}(v) \subseteq \Omega^1}{=} \lambda^2 \left\langle \nabla \left(\Phi - \tilde{\Phi}\right), \nabla v \right\rangle_{L^2(\Omega)}
\end{aligned}$$

Using the fact that both Φ and $\tilde{\Phi}$ are weak solutions, $\operatorname{supp}(v) \subseteq \Omega_1$ and the monotonicity of $\mathfrak{f}$, proven in Lemma 2.12, we derive

$$\begin{aligned}
&\lambda^2 \left\langle \nabla \left(\Phi - \tilde{\Phi}\right)^+, \nabla v \right\rangle_{L^2(\Omega^1)} = \lambda^2 \left\langle \nabla \left(\Phi - \tilde{\Phi}\right), \nabla v \right\rangle_{L^2(\Omega)} = \left\langle \mathfrak{f}(w, \Phi) - \mathfrak{f}(\tilde{w}, \tilde{\Phi}), v \right\rangle_{L^2(\Omega)} \\
&\quad = \left\langle \mathfrak{f}(w, \Phi) - \mathfrak{f}(\tilde{w}, \tilde{\Phi}), v \right\rangle_{L^2(\Omega_1)} = \left\langle \mathfrak{f}(w, \Phi) - \mathfrak{f}(\tilde{w}, \Phi) + \underbrace{\mathfrak{f}(\tilde{w}, \Phi) - \mathfrak{f}(\tilde{w}, \tilde{\Phi})}_{\leq 0 \text{ on } \Omega^1 \text{ due to Lemma 2.12}}, v \right\rangle_{L^2(\Omega^1)} \\
&\quad \overset{v \geq 0}{\leq} \left\langle \underbrace{\mathfrak{f}(w, \Phi) - \mathfrak{f}(\tilde{w}, \Phi)}_{=:\mathfrak{F}}, v \right\rangle_{L^2(\Omega^1)},
\end{aligned}$$

Together with the fact that $\left(\Phi - \tilde{\Phi}\right)^+ = 0$ on Γ_D^1, this implies that $\left(\Phi - \tilde{\Phi}\right)^+$ is a (weak) subsolution[2] of the system

$$\begin{aligned}
-\Delta\phi &= \mathfrak{F}/\lambda^2 \text{ on } \Omega^1 \\
\phi &= 0 \text{ on } \Gamma_D^1 \\
\frac{\partial\phi}{\partial\mathbf{n}} &= 0 \text{ on } \Gamma_N^1,
\end{aligned}$$

[2]See Lemma C.2 for the definition

We can therefore apply Lemma C.2, to derive the following bound

$$\left\| \left(\Phi - \tilde{\Phi} \right)^+ \right\|_{L^\infty(\Omega^1)} = \sup_{\Omega^1} \left(\Phi - \tilde{\Phi} \right)^+ \leq C_1 \left\| \mathfrak{F}/\lambda^2 \right\|^\alpha_{L^\infty(\Omega)} \left\| \nabla \left(\Phi - \tilde{\Phi} \right)^+ \right\|^\gamma_{L^2(\Omega^1)},$$

for some constants $C_1, \alpha > 0$ and some $\gamma \in (0,1)$, depending only on the parameters of system (1.6).

From

$$\|\mathfrak{F}\|_{L^\infty(\Omega^1)} \leq \|\mathfrak{f}(w, \Phi)\|_{L^\infty(\Omega^1)} + \|\mathfrak{f}(\tilde{w}, \Phi)\|_{L^\infty(\Omega^1)} \leq 2 \max_{i=1,\cdots,n} |z_i|,$$

we get that

$$\left\| \left(\Phi - \tilde{\Phi} \right)^+ \right\|_{L^\infty(\Omega^1)} \leq C_2 \left\| \nabla \left(\Phi - \tilde{\Phi} \right)^+ \right\|^\gamma_{L^2(\Omega^1)}.$$

for $C_2 := C_1 \left(\frac{2 \max_{i=1,\cdots,n} |z_i|}{\lambda^2} \right)^\alpha$.

One can also dominate the negative part $\left(\Phi - \tilde{\Phi} \right)^-$ in the same manner. Together with inequality (3.8) in Lemma 3.7 this finally lets us conclude that

$$\left\| \Phi - \tilde{\Phi} \right\|_{L^\infty(\Omega)} \leq C \|w - \tilde{w}\|^\gamma_{L^\infty(\Omega)}.$$

□

Remark 3.9. *In the proof of Lemma 3.8 we did not use the explicit form of $\mathfrak{f}$. The Lemma can thus be easily extended to general $\mathfrak{f}$, that are Lipschitz-continuous and monotonically decreasing in the second argument. And which have the property that for every $w \in L^\infty$ and every $\phi \in H^1(\Omega) \cap L^\infty(\Omega)$*

$$\|\mathfrak{f}(w, \phi)\|_{L^p(\Omega)} \leq C,$$

for some constant $C > 0$ and some $p \in \left(\max\{\frac{d}{2}\}, +\infty\right]$. To the best knowledge of the author this represents a novel result.

3.2. Step 1: Analysis of the time discretized, space regularized equation

In this section our aim is to find a sequence

$$\left(\overset{\circ}{w}^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k) \right)_{k \in \mathbb{N}_0} \in \left(L^\infty(\Omega; \mathbb{R}^n) \cap H^1_D(\Omega; \mathbb{R}^n) \right) \times H^1(\Omega),$$

that satisfies (3.2) in a weak sense. This sequence will then be used in the following sections to approximate a weak solution to (1.6). We prove the existence of such a sequence by an induction argument. For $k = 0$ we use Lemma 2.4 to define

$$\overset{\circ}{w}^{\epsilon,\tau}(0) := \frac{\partial h(u^0)}{\partial u} = \left(\frac{\partial h(u^0)}{\partial u_1}, \cdots, \frac{\partial h(u^0)}{\partial u_n}\right)^{\mathrm{T}}$$

and furthermore we choose $\Phi^{\epsilon,\tau}(0) \in H^1(\Omega)$ arbitrarily.

Now assume that $\left(\overset{\circ}{w}^{\epsilon,\tau}(k-1), \Phi^{\epsilon,\tau}(k-1)\right)$ is given. Then we define

$$\overset{\circ}{w}^{\epsilon,\tau}(k) \in \left(H^m(\Omega;\mathbb{R}^n) \cap H^1_D(\Omega;\mathbb{R}^n)\right) \subseteq \left(L^\infty(\Omega;\mathbb{R}^n) \cap H^1_D(\Omega;\mathbb{R}^n)\right)$$

and $\Phi^{\epsilon,\tau}(k) \in H^1(\Omega)$, to satisfy

$$\begin{aligned}
\frac{1}{\tau}\left\langle u\left(\overset{\circ}{w}^{\epsilon,\tau}(k) + w^D, \Phi^{\epsilon,\tau}(k)\right) - u\left(\overset{\circ}{w}^{\epsilon,\tau}(k-1) + w^D, \Phi^{\epsilon,\tau}(k-1)\right), \xi\right\rangle_{L^2(\Omega)} + \cdots \\
\cdots + \left\langle B\left(\overset{\circ}{w}^{\epsilon,\tau}(k) + w^D, \Phi^{\epsilon,\tau}(k)\right)\nabla\left(\overset{\circ}{w}^{\epsilon,\tau}(k) + w^D\right), \nabla\xi\right\rangle_{L^2(\Omega)} + \cdots \\
\cdots + \epsilon\left(\sum_{|\alpha|=m}\left\langle D^\alpha\overset{\circ}{w}^{\epsilon,\tau}(k), D^\alpha\xi\right\rangle_{L^2(\Omega)} + \left\langle\overset{\circ}{w}^{\epsilon,\tau}(k), \xi\right\rangle_{L^2(\Omega)}\right) = \cdots \\
\cdots = \left\langle g\left(\tau k, x, u\left(\overset{\circ}{w}^{\epsilon,\tau}(k) + w^D, \Phi^{\epsilon,\tau}(k)\right)\right), \xi\right\rangle_{L^2(\Omega)} \\
\lambda^2\left\langle\nabla\Phi^{\epsilon,\tau}(k), \nabla\eta\right\rangle_{L^2(\Omega)} = \left\langle\sum_{i=1}^n z_i u_i\left(\overset{\circ}{w}^{\epsilon,\tau}(k) + w^D, \Phi^{\epsilon,\tau}(k)\right) + f, \eta\right\rangle_{L^2(\Omega)},
\end{aligned} \tag{3.9}$$

for every $\xi \in H^m(\Omega;\mathbb{R}^n) \cap H^1_D(\Omega;\mathbb{R}^n)$ and every $\eta \in H^1_D(\Omega)$, such that also $\Phi^{\epsilon,\tau}(k) = \Phi^D$ on Γ_D.

The following Lemma shows, that this is indeed plausible.

Theorem 3.10. *There exist*

$$\overset{\circ}{w}^{\epsilon,\tau}(k) \in H^m(\Omega;\mathbb{R}^n) \cap H^1_D(\Omega;\mathbb{R}^n),\ \ \Phi^{\epsilon,\tau}(k)H^1(\Omega)$$

with $\Phi^{\epsilon,\tau}(k) = \Phi^D$ *on* Γ_D*, that satisfy* (3.9) *for all* $\xi \in H^m(\Omega;\mathbb{R}^n) \bigcap H^1_D(\Omega;\mathbb{R}^n)$*,* $\eta \in H^1_D(\Omega)$*.*

Proof. Since (3.9) is the weak form of the semilinear[3] (elliptic) equation (3.2), we employ a standard fixed point argument to derive the existence of weak solutions.

We split the proof up into four parts. In the first part we define a bilinear and a linear form, that correspond to a linearization of (3.9) and which will later be used for the definition

[3]Note that the semilinearity is due to the added elliptic operator $\mathcal{L}^m$

of the fixed point problem. In the second part we show the solvability of the linearized problem via the Lax–Milgram Lemma. In the third part we show that the solution to the linearized problem depends continuously on the linearization parameter. In the fourth and final part we use the linearized problem to define a fixed point operator and show that it has a fixed point via the Schauder Theorem.

Part (i): *Definition of the linearized problem.*
For every $y \in L^\infty(\Omega;\mathbb{R}^n)$, we define $\Phi^y \in H^1(\Omega)$ to be the unique weak solution of

$$-\lambda^2 \Delta \Phi^y = \sum_{i=1}^{n} z_i u_i(y + w^D, \Phi^y) + f \text{ on } \Omega$$
$$\Phi^y = \Phi^D \text{ on } \Gamma_D, \quad \frac{\partial \Phi^y}{\partial \mathbf{n}} = 0 \text{ on } \Gamma_N.$$

Theorem 3.4 guarantees that this definition makes sense.

Now let $X := H^m(\Omega;\mathbb{R}^n) \cap H^1_D(\Omega;\mathbb{R}^n)$. From the continuity of the trace operator it follows that X is a closed subspace of $H^m(\Omega;\mathbb{R}^n)$ and therefore, when equipped with the scalar product $\langle \cdot,\cdot \rangle_{H^m(\Omega)}$, itself a Hilbert space.

For every $y \in L^\infty(\Omega;\mathbb{R}^n)$, we define the bilinear form $\mathcal{A}^y : X \times X \to \mathbb{R}$ by

$$\mathcal{A}^y(\xi,\eta) := \left\langle B\left(y + w^D, \Phi^y\right) \nabla \xi, \nabla \eta \right\rangle_{L^2(\Omega)} + \epsilon \left(\sum_{|\alpha|=m} \langle D^\alpha \xi, D^\alpha \eta \rangle_{L^2(\Omega)} + \langle \xi, \eta \rangle_{L^2(\Omega)} \right)$$

for all $\xi, \eta \in X$. Furthermore we define the linear form $\mathcal{F}^y : X \to \mathbb{R}$ by

$$\mathcal{F}^y(\xi) := \left\langle g\left(\tau k, \cdot, u\left(y + w^D, \Phi^y\right)\right), \xi \right\rangle_{L^2(\Omega)} - \left\langle B\left(y + w^D, \Phi^y\right) \nabla w^D, \nabla \xi \right\rangle_{L^2(\Omega)} \cdots$$
$$\cdots - \frac{1}{\tau} \left\langle u\left(y + w^D, \Phi^y\right) - u\left(\overset{\circ}{w}{}^{\epsilon,\tau}(k-1) + w^D, \Phi^{\epsilon,\tau}(k-1)\right), \xi \right\rangle_{L^2(\Omega)}$$

for every $\xi \in X$. Note that by Corollary 2.9, we have $B(y + w^D, \Phi^y) \in L^\infty(\Omega;\mathbb{R}^{n\times n})$ and thus both $\mathcal{A}^y$ and $\mathcal{F}^y$ are indeed well defined on X.

Part (ii): *Well posedness of the linearized problem.*
Using the Cauchy–Schwarz inequality we get that for all $\xi, \eta \in X$

$$|\mathcal{A}^y(\xi,\eta)| \le \left\| B(y + w^D, \Phi^y) \nabla \xi \right\|_{L^2(\Omega)} \|\nabla \eta\|_{L^2(\Omega)} + \cdots$$
$$\cdots + \epsilon \left(\sum_{|\alpha|=m} \|D^\alpha \xi\|_{L^2(\Omega)} \|D^\alpha \eta\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)} \|\eta\|_{L^2(\Omega)} \right).$$

Corollary 2.9 then gives us

$$\left\| B(y + w^D, \Phi^y) \nabla \xi \right\|_{L^2(\Omega)} \le \max_{i=1,\cdots,n} D_i \|\nabla \xi\|_{L^2(\Omega)}$$

and thus

$$|\mathcal{A}^y(\xi,\eta)| \le \left(\max_{i=1,\cdots,n} D_i + \epsilon\right) \|\xi\|_{H^m(\Omega)} \|\eta\|_{H^m(\Omega)} .$$

Therefore $\mathcal{A}^y$ is a continuous bilinear form. Similarly one can use the Cauchy–Schwarz inequality and Corollary 2.9 to derive

$$|\mathcal{F}^y(\xi)| \le \Bigg(\left\|g\left(\tau k, x, u\left(y+w^D,\Phi^y\right)\right)\right\|_{L^2(\Omega)} + \max_{i=1,\cdots,n} D_i \left\|\nabla w^D\right\|_{L^2(\Omega)} + \cdots$$
$$\cdots + \frac{\left\|u\left(y+w^D,\Phi^y\right)\right\|_{L^2(\Omega)}}{\tau} + \frac{\left\|u\left(\overset{\circ}{w}{}^{\epsilon,\tau}(k-1)+w^D,\Phi^{\epsilon,\tau}(k-1)\right)\right\|_{L^2(\Omega)}}{\tau} \Bigg) \|\xi\|_{H^m(\Omega)} .$$

Using inequality (2.5) to dominate the reaction term and the fact that $u(\cdot,\cdot)$ maps into $\mathcal{O}$, which implies that its euclidean norm is bounded by 1, this implies

$$|\mathcal{F}^y(\xi)| \le \left(|\Omega|^{1/2} C^g(\tau k) + \max_{i=1,\cdots,n} D_i \left\|\nabla w^D\right\|_{L^2(\Omega)} + \frac{2\,|\Omega|^{1/2}}{\tau} \right) \|\xi\|_{H^m(\Omega)} . \tag{3.10}$$

Thus $\mathcal{F}^y$ is continuous. Notice that the continuity constants for both $\mathcal{A}^y$ and $\mathcal{F}^y$ are bounded independently of y.

Using the positive definitness of $B(y+w^D,\Phi^y)$ we derive for any $\xi \in X$

$$\mathcal{A}^y(\xi,\xi) = \underbrace{\left\langle B\left(y+w^D,\Phi^y\right)\nabla\xi,\nabla\xi\right\rangle_{L^2(\Omega)}}_{\ge 0} + \epsilon\left(\sum_{|\alpha|=m} \langle D^\alpha\xi, D^\alpha\xi\rangle_{L^2(\Omega)} + \langle\xi,\xi\rangle_{L^2(\Omega)}\right)$$
$$\ge \left(\sum_{|\alpha|=m} \|D^\alpha\xi\|^2_{L^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}\right),$$

which by the generalized Poincare inequality implies

$$\mathcal{A}^y(\xi,\xi) \ge \epsilon C_{P,m} \|\xi\|^2_{H^m(\Omega)} . \tag{3.11}$$

Thus $\mathcal{A}^y$ is coercive and again its coercivity constant does not depend on y.

By the Lax–Milgram Lemma there exists a unique $v^y \in X$ such that

$$\mathcal{A}^y(v^y,\xi) = \mathcal{F}^y(\xi) \text{ for all } \xi \in X.$$

Note that the regularity parameter $m \in \mathbb{N}$ was chosen large enough, such that $H^m(\Omega;\mathbb{R}^n)$ is compactly embedded into $L^\infty(\Omega;\mathbb{R}^n)$ and thus $v^y \in X \subseteq L^\infty(\Omega;\mathbb{R}^n)$.

Part (iii): *Continuous parameter dependence of the linearized problem.*
We now want to show that the map $y \in L^\infty(\Omega;\mathbb{R}^n) \mapsto v^y \in X$ is continuous. Let therefore

$(y(j))_{j\in\mathbb{N}}$ be a sequence in $L^\infty(\Omega;\mathbb{R}^n)$ that converges with respect to the $L^\infty(\Omega;\mathbb{R}^n)$-norm against $y\in L^\infty(\Omega;\mathbb{R}^n)$.

We note that, according to Lemma 3.8, the function Φ^y depends (Hölder)-continuously, with respect to the $L^\infty(\Omega)$-norm, on y, i.e. the map $y\in L^\infty(\Omega;\mathbb{R}^n)\mapsto \Phi^y\in L^\infty(\Omega)$ is continuous. The Lipschitz-continuity of u, which was shown in Lemma 2.12, and the definition of B[4] gives us

$$\begin{aligned}\left\|u\left(y(j)+w^D,\Phi^{y(j)}\right)-u\left(y+w^D,\Phi^y\right)\right\|_{L^\infty(\Omega)}&\xrightarrow{j\to\infty}0.\\ \left\|B\left(y(j)+w^D,\Phi^{y(j)}\right)-B\left(y+w^D,\Phi^y\right)\right\|_{L^\infty(\Omega)}&\xrightarrow{j\to\infty}0\\ \left\|g\left(\tau k,\cdot,u\left(y(j)+w^D,\Phi^{y(j)}\right)\right)-g\left(\tau k,\cdot,u\left(y+w^D,\Phi^y\right)\right)\right\|_{L^\infty(\Omega)}&\xrightarrow{j\to\infty}0.\end{aligned}\tag{3.12}$$

Hereby the last convergence holds, since g is uniformely continuous on the compact set $\{\tau k\}\times\bar{\Omega}\times\bar{\mathcal{O}}$.

We define $\xi(j):=v^{y(j)}-v^y$. Then, using the generalized Poincare inequality and the definition of the bilinear forms $\mathcal{A}^{y(j)},\mathcal{A}^y$, we obtain

$$\begin{aligned}&\epsilon C_{P,m}\|\xi(j)\|^2_{H^m(\Omega)}\le\epsilon\left(\sum_{|\alpha|=m}\langle D^\alpha\xi(j),D^\alpha\xi(j)\rangle_{L^2(\Omega)}+\langle\xi(j),\xi(j)\rangle_{L^2(\Omega)}\right)\\ &=\mathcal{A}^{y(j)}\left(v^{y(j)},\xi(j)\right)-\mathcal{A}^y\left(v^y,\xi(j)\right)+\cdots\\ &\cdots+\langle B\left(y+w,\Phi^y\right)\nabla v^y,\nabla\xi(j)\rangle_{L^2(\Omega)}-\left\langle B\left(y(j)+w,\Phi^{y(j)}\right)\nabla v^{y(j)},\nabla\xi(j)\right\rangle_{L^2(\Omega)}\end{aligned}\tag{3.13}$$

First we estimate

$$\begin{aligned}&\langle B\left(y+w,\Phi^y\right)\nabla v^y,\nabla\xi(j)\rangle_{L^2(\Omega)}-\left\langle B\left(y(j)+w,\Phi^{y(j)}\right)\nabla v^{y(j)},\nabla\xi(j)\right\rangle_{L^2(\Omega)}\\ &=\left\langle\left(B\left(y+w,\Phi^y\right)-B\left(y(j)+w,\Phi^{y(j)}\right)\right)\nabla v^y,\nabla\xi(j)\right\rangle_{L^2(\Omega)}+\cdots\\ &\cdots+\left\langle B\left(y(j)+w,\Phi^{y(j)}\right)\nabla\underbrace{\left(v^y-v^{y(j)}\right)}_{=-\xi(j)},\nabla\xi(j)\right\rangle_{L^2(\Omega)}\\ &=\left\langle\left(B\left(y+w,\Phi^y\right)-B\left(y(j)+w,\Phi^{y(j)}\right)\right)\nabla v^y,\nabla\xi(j)\right\rangle_{L^2(\Omega)}-\cdots\\ &\cdots-\underbrace{\left\langle B\left(y(j)+w,\Phi^{y(j)}\right)\nabla\xi(j),\nabla\xi(j)\right\rangle_{L^2(\Omega)}}_{\ge0}\\ &\le\left\|B\left(y+w,\Phi^y\right)-B\left(y(j)+w,\Phi^{y(j)}\right)\right\|_{L^\infty(\Omega)}\|\nabla v^y\|_{L^2(\Omega)}\|\nabla\xi(j)\|_{L^2(\Omega)}\\ &\le\left\|B\left(y+w,\Phi^y\right)-B\left(y(j)+w,\Phi^{y(j)}\right)\right\|_{L^\infty(\Omega)}\|v^y\|_{H^m(\Omega)}\|\xi(j)\|_{H^m(\Omega)}\end{aligned}\tag{3.14}$$

[4] See Definition 2.8

Next we estimate $\mathcal{A}^{y(j)}\left(v^{y(j)},\xi(j)\right)-\mathcal{A}^{y}\left(v^{y},\xi(j)\right)$. By using the definition of $v^{y(j)}, v^y$ we derive

$$\begin{aligned}
&\mathcal{A}^{y(j)}\left(v^{y(j)},\xi(j)\right)-\mathcal{A}^{y}\left(v^{y},\xi(j)\right)=\mathcal{F}^{y(j)}\left(\xi(j)\right)-\mathcal{F}^{y}\left(\xi(j)\right)\\
&=\left\langle g\left(\tau k,x,u\left(y(j)+w^D,\Phi^{y(j)}\right)\right)-g\left(\tau k,x,u\left(y+w^D,\Phi^{y}\right)\right),\xi(j)\right\rangle_{L^2(\Omega)}-\cdots\\
&\cdots-\left\langle\left(B\left(y(j)+w^D,\Phi^{y(j)}\right)-B\left(y+w^D,\Phi^{y}\right)\right)\nabla w^D,\nabla\xi(j)\right\rangle_{L^2(\Omega)}-\cdots\\
&\cdots-\frac{1}{\tau}\left\langle u\left(y(j)+w^D,\Phi^{y(j)}\right)-u\left(y+w^D,\Phi^{y}\right),\xi(j)\right\rangle_{L^2(\Omega)}
\end{aligned}$$

Simply applying the Cauchy–Schwarz inequality gives us

$$\begin{aligned}
&\mathcal{A}^{y(j)}\left(v^{y(j)},\xi(j)\right)-\mathcal{A}^{y}\left(v^{y},\xi(j)\right)\\
&\leq\Bigg(\left\|g\left(\tau k,x,u\left(y(j)+w^D,\Phi^{y(j)}\right)\right)-g\left(\tau k,x,u\left(y+w^D,\Phi^{y}\right)\right)\right\|_{L^2(\Omega)}+\cdots\\
&\cdots+\left\|B\left(y(j)+w^D,\Phi^{y(j)}\right)-B\left(y+w^D,\Phi^{y}\right)\right\|_{L^\infty(\Omega)}\left\|\nabla w^D\right\|_{L^2(\Omega)}+\cdots\\
&\cdots+\frac{1}{\tau}\left\|u\left(y(j)+w^D,\Phi^{y(j)}\right)-u\left(y+w^D,\Phi^{y}\right)\right\|_{L^2(\Omega)}\Bigg)\left\|\xi(j)\right\|_{H^m(\Omega)}.
\end{aligned}\tag{3.15}$$

Combining the inequalities (3.13), (3.14) and (3.15), we derive

$$\begin{aligned}
\epsilon C_{P,m}\left\|\xi(j)\right\|_{H^m(\Omega)}&\leq\left\|B\left(y+w,\Phi^{y}\right)-B\left(y(j)+w,\Phi^{y(j)}\right)\right\|_{L^\infty(\Omega)}\left\|v^y\right\|_{H^m(\Omega)}\\
&\leq\left\|g\left(\tau k,x,u\left(y(j)+w^D,\Phi^{y(j)}\right)\right)-g\left(\tau k,x,u\left(y+w^D,\Phi^{y}\right)\right)\right\|_{L^2(\Omega)}\\
&\cdots+\left\|B\left(y(j)+w^D,\Phi^{y(j)}\right)-B\left(y+w^D,\Phi^{y}\right)\right\|_{L^\infty(\Omega)}\left\|\nabla w^D\right\|_{L^2(\Omega)}\\
&\cdots+\frac{1}{\tau}\left\|u\left(y(j)+w^D,\Phi^{y(j)}\right)-u\left(y+w^D,\Phi^{y}\right)\right\|_{L^2(\Omega)}.
\end{aligned}$$

Using (3.12), we see that the right side converges to zero for $j\to\infty$ and thus we get that $v^y\in H^m(\Omega,\mathbb{R}^n)$ indeed depends continuously on $y\in L^\infty(\Omega;\mathbb{R}^n)$.

Part (iv): *Defintion of the iteration map and existence of a fixed point.*

We define the constant

$$C:=\frac{|\Omega|^{1/2}C^g(\tau k)+\max_{i=1,\cdots,n}D_i\left\|\nabla w^D\right\|_{L^2(\Omega)}+\frac{2|\Omega|^{1/2}}{\tau}}{\epsilon C_{P,m}},$$

and the set $\mathfrak{K}\subseteq X\subseteq L^\infty(\Omega;\mathbb{R}^n)$ by

$$\mathfrak{K}:=\left\{\ v\in X\ :\ \|v\|_{H^m(\Omega)}\leq C\ \right\}$$

Note that since the regularity parameter m was chosen in such a way, that the embedding $H^m(\Omega;\mathbb{R}^n) \hookrightarrow L^\infty(\Omega;\mathbb{R}^n)$ is compact, the set $\mathfrak{K}$ is precompact in $L^\infty(\Omega;\mathbb{R}^n)$.

Now we define the iteration map $\mathcal{S} : L^\infty(\Omega;\mathbb{R}^n) \to L^\infty(\Omega;\mathbb{R}^n)$ by $\mathcal{S}(y) := v^y$ for every $y \in L^\infty(\Omega;\mathbb{R}^n)$.

Due to Part (iii) and the continuity of the embedding $H^m(\Omega;\mathbb{R}^n) \hookrightarrow L^\infty(\Omega;\mathbb{R}^n)$, we immediatley see that $\mathcal{S}$ is continuous. Combining the coercivity of $\mathcal{A}^y$ (see (3.11)) with the continuity estimate for $\mathcal{F}^y$ (see (3.10)) gives us

$$\|\mathcal{S}(y)\|^2_{H^m(\Omega)} \overset{(3.11)}{\leq} \frac{1}{\epsilon C_{P,m}} \mathcal{A}^y\left(\mathcal{S}(y), \mathcal{S}(y)\right) = \frac{1}{\epsilon C_{P,m}} \mathcal{F}^y\left(\mathcal{S}(y)\right) \overset{(3.10)}{\leq} C\,\|\mathcal{S}(y)\|_{H^m(\Omega)},$$

This means that the range of $\mathcal{S}$ is a subset of $\mathfrak{K}$ and thus, as a subset of a precompact set, itself precompact.

Since $\mathcal{S}$ is a continuous map from the Banach-space $L^\infty(\Omega;\mathbb{R}^n)$ into itself with precompact image, the Schauder theorem tells us that it has a fixed point, which we denote by

$$\mathring{w}^{\epsilon,\tau}(k) \in X, \text{ i.e. } \mathcal{S}\left(\mathring{w}^{\epsilon,\tau}(k)\right) = \mathring{w}^{\epsilon,\tau}(k).$$

Furthermore we define

$$\Phi^{\epsilon,\tau}(k) := \Phi^{\mathring{w}^{\epsilon,\tau}(k)} \in H^1(\Omega).$$

By definition of the iteration map $\mathcal{S}$ and the map $y \mapsto \Phi^y$, it immediatley follows that the pair $\left(\mathring{w}^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right)$ suffices (3.9).

□

3.3. Step 2: A discrete entropy production inequality and a priori estimate

Based on the sequence $\left(\mathring{w}^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right)_{k\in\mathbb{N}_0}$, defined in the previous section, we make the following definitions for every $k \in \mathbb{N}_0$.

$$\begin{aligned} w^{\epsilon,\tau}(k) &:= \mathring{w}^{\epsilon,\tau}(k) + w^D \\ u^{\epsilon,\tau}(k) &:= u\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \end{aligned}$$

Using this notation, we can rewrite (3.9) as

$$\begin{aligned}
&\frac{\langle u^{\epsilon,\tau}(k)-u^{\epsilon,\tau}(k-1),\xi\rangle_{L^2(\Omega)}}{\tau}+\langle B\left(w^{\epsilon,\tau}(k),\Phi^{\epsilon,\tau}(k)\right)\nabla w^{\epsilon,\tau}(k),\nabla\xi\rangle_{L^2(\Omega)}+\cdots\\
&\cdots+\epsilon\left(\sum_{|\alpha|=m}\left\langle D^\alpha \overset{\circ}{w}^{\epsilon,\tau}(k),D^\alpha\xi\right\rangle_{L^2(\Omega)}+\left\langle \overset{\circ}{w}^{\epsilon,\tau}(k),\xi\right\rangle_{L^2(\Omega)}\right)\\
&=\langle g\left(\tau k,\cdot,u^{\epsilon,\tau}(k)\right),\xi\rangle_{L^2(\Omega)}\\
&\lambda^2\langle\nabla\Phi^{\epsilon,\tau}(k),\nabla\eta\rangle_{L^2(\Omega)}=\left\langle\sum_{i=1}^n z_i u_i^{\epsilon,\tau}(k)+f,\eta\right\rangle_{L^2(\Omega)}
\end{aligned}\tag{3.16}$$

for every $\xi\in H^m(\Omega;\mathbb{R}^n)\cap H_D^1(\Omega;\mathbb{R}^n)$, $\eta\in H_D^1(\Omega)$ and every $k\in\mathbb{N}$.

Our goal in this section is to derive bounds for $u^{\epsilon,\tau}$ and $\overset{\circ}{w}^{\epsilon,\tau}$. To accomplish this we use a discrete entropy production inequality. But first we investigate the regularity and the boundary conditions of $u^{\epsilon,\tau}$.

Lemma 3.11. *For any $k\in\mathbb{N}$ we have*

$$\begin{aligned}
u^{\epsilon,\tau}(k)&\in H^1(\Omega;\mathbb{R}^n)\cap L^\infty(\Omega;\mathbb{R}^n)\\
\sqrt{u_i^{\epsilon,\tau}(k)}&\in H^1(\Omega)\cap L^\infty(\Omega)\ \text{for any } i=0,\cdots,n\\
u_i^{\epsilon,\tau}(k)\sqrt{u_0^{\epsilon,\tau}(k)}&\in H^1(\Omega)\cap L^\infty(\Omega)\ \text{for any } i=1,\cdots,n\\
\log u_i^{\epsilon,\tau}(k)&\in H^1(\Omega)\cap L^\infty(\Omega)\ \text{for any } i=0,\cdots,n
\end{aligned}$$

and the image of $u^{\epsilon,\tau}(k)$ is a subset of $\mathcal{O}$.

Proof. It is clear from the definition of $u^{\epsilon,\tau}(k)$ that it is a measurable function mapping Ω into $\mathcal{O}$. Thus it follows that $u_i^{\epsilon,\tau}(k),\sqrt{u_i^{\epsilon,\tau}(k)}\in L^\infty(\Omega)$ for every $i=0,\cdots,n$.

In the proof of Lemma 2.12 we argued that $u(\cdot,\cdot)\in C^1(\mathbb{R}^{n+1};\mathbb{R}^n)$ has a bounded first derivative and thus, by the chain rule (Lemma B.11), we have

$$u^{\epsilon,\tau}(k)=u\left(w^{\epsilon,\tau}(k),\Phi^{\epsilon,\tau}(k)\right)\in H^1(\Omega;\mathbb{R}^n).$$

Since $\overset{\circ}{w}^{\epsilon,\tau}(k),w^D\in L^\infty(\Omega;\mathbb{R}^n)$, we know that $w^{\epsilon,\tau}(k)=\overset{\circ}{w}^{\epsilon,\tau}(k)+w^D\in L^\infty(\Omega;\mathbb{R}^n)$. Furthermore we know from Lemma 3.6 that $\Phi^{\epsilon,\tau}(k)\in L^\infty(\Omega)$. Now let us define the compact set

$$\mathfrak{K}:=\left\{\ (x,y)\in\mathbb{R}^{n+1}\ \middle|\ |x|\le\|w^{\epsilon,\tau}(k)\|_{L^\infty(\Omega)},\ |y|\le\|\Phi^{\epsilon,\tau}(k)\|_{L^\infty(\Omega)}\ \right\},$$

and with this the two constants

$$\begin{aligned}
\underline{u}&:=\min_{i=0,\cdots,n}\underbrace{\min_{(x,y)\in\mathfrak{K}}u_i(x,y)}_{>0}>0\\
\bar{u}&:=\max_{i=0,\cdots,n}\underbrace{\max_{(x,y)\in\mathfrak{K}}u_i(x,y)}_{<1}<1,
\end{aligned}$$

as well as the function $\mathcal{S} : \mathbb{R} \to \mathbb{R}$ with

$$\mathcal{S}(y) := \begin{cases} \frac{1}{2\sqrt{\underline{u}}}\left(y - \underline{u}\right) + \sqrt{\underline{u}} & , \text{ if } y \leq \underline{u} \\ \sqrt{y} & , \text{ if } \underline{u} \leq y \leq \bar{u} \\ \frac{1}{2\sqrt{\bar{u}}}\left(y - \bar{u}\right) + \sqrt{\bar{u}} & , \text{ if } y \geq \bar{u}. \end{cases}$$

One easily verifies that $\mathcal{S} \in C^1(\mathbb{R})$ with $\mathcal{S}' \in L^\infty(\mathbb{R})$. Since $\underline{u} \leq u_i^{\epsilon,\tau}(k) \leq \bar{u}$ for every $i = 0, \cdots, n$, we can then use the chain rule (Lemma B.11) to derive

$$\sqrt{u_i^{\epsilon,\tau}(k)} = \mathcal{S}\left(u_i^{\epsilon,\tau}(k)\right) \in H^1(\Omega) \text{ for } i = 0, \cdots, n.$$

Similarly one can show that $\log u_i^{\epsilon,\tau}(k) \in H^1(\Omega)$ for all $i = 0, \cdots, n$. Finally, by using Lemma B.12, we conclude that $u_i^{\epsilon,\tau}(k)\sqrt{u_0^{\epsilon,\tau}(k)} \in H^1(\Omega)$ for every $i = 0, \cdots, n$. □

Lemma 3.12. *For any $k \in \mathbb{N}$ the following Dirichlet boundary conditions hold*

$$\begin{aligned} u^{\epsilon,\tau}(k)|_{\Gamma_D} &= u^D\big|_{\Gamma_D} \\ \sqrt{u_i^{\epsilon,\tau}(k)}\Big|_{\Gamma_D} &= \sqrt{u_i^D}\Big|_{\Gamma_D} \quad i = 0, \cdots, n \\ u_i^{\epsilon,\tau}(k)\sqrt{u_0^{\epsilon,\tau}(k)}\Big|_{\Gamma_D} &= u_i^D\sqrt{u_0^D}\Big|_{\Gamma_D} \quad i = 1, \cdots, n. \end{aligned}$$

Proof. As we have already proven in Lemma 3.11 that we can apply the chain rule, this is a trivial consequence of Corollary B.13.

□

The following Lemma will be key for deriving the discrete entropy production inequality.

Lemma 3.13. *The inequality*

$$\left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \overset{\circ}{w}^{\epsilon,\tau}(k)\right\rangle_{L^2(\Omega)} \geq H\left(u^{\epsilon,\tau}(k)\right) - H\left(u^{\epsilon,\tau}(k-1)\right). \tag{3.17}$$

holds for every $k \in \mathbb{N}$.

Proof. Let $k \in \mathbb{N}$ be arbitrary. For the purpose of proving (3.17), let us define for every $x \in \Omega$ the function $\phi_x : \mathcal{O} \to \mathbb{R}$ by

$$\phi_x(v) := \sum_{i=0}^{n} \int_{u_i^D(x)}^{v_i} \log \frac{s}{u_i^D(x)} \mathrm{d}s$$

for every $v \in \mathcal{O}$. We denote its gradient by ϕ_x', in order to distinguish it from the gradient with respect to x. One sees immediately that, since $v_0 = 1 - \sum_{i=1}^n v_i$,

$$\phi_x'(v) = \begin{pmatrix} \log \frac{v_1}{u_1^D} - \log \frac{v_0}{u_0^D} \\ \vdots \\ \log \frac{v_n}{u_n^D} - \log \frac{v_0}{u_0^D} \end{pmatrix}.$$

By Lemma A.4 the function ϕ_x is convex and thus, by Lemma A.3, we derive the inequality

$$\phi_x\left(u^{\epsilon,\tau}(k)\right) - \phi_x\left(u^{\epsilon,\tau}(k-1)\right) \leq \phi_x'\left(u^{\epsilon,\tau}(k)\right) \cdot \left(u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1)\right). \tag{3.18}$$

Inverting $u^{\epsilon,\tau}(k) = u\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right)$ with respect to $w^{\epsilon,\tau}(k)$. we get

$$w_i^{\epsilon,\tau}(k) = \log \frac{u_i^{\epsilon,\tau}(k)}{u_0^{\epsilon,\tau}(k)} + \beta z_i \Phi^{\epsilon,\tau}(k) + W_i = w_i\left(u^{\epsilon,\tau}(k)\right). \tag{3.19}$$

Thus, by Lemma 2.4, we derive

$$\overset{\circ}{w}{}_i^{\epsilon,\tau}(k) = w_i^{\epsilon,\tau}(k) - w_i^D = \frac{\partial h\left(u^{\epsilon,\tau}(k)\right)}{\partial u_i}. \tag{3.20}$$

Combining this identity with the inequality (3.18) gives us

$$\begin{aligned}
&\left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \overset{\circ}{w}{}^{\epsilon,\tau}(k)\right\rangle_{L^2(\Omega)} = \int_\Omega \sum_{i=1}^n \left(u_i^{\epsilon,\tau}(k) - u_i^{\epsilon,\tau}(k-1)\right) \overset{\circ}{w}{}_i^{\epsilon,\tau}(k)\, \mathrm{d}x \\
&= \int_\Omega \sum_{i=1}^n \left(u_i^{\epsilon,\tau}(k) - u_i^{\epsilon,\tau}(k-1)\right) \Bigg(\underbrace{\log \frac{u_i^{\epsilon,\tau}(k)}{u_0^{\epsilon,\tau}(k)} - \log \frac{u_i^D}{u_0^D}}_{=\phi_x'(u^{\epsilon,\tau}(k))_i} + \beta z_i \left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right) + W_i \Bigg)\, \mathrm{d}x \\
&= \int_\Omega \left(u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1)\right) \cdot \phi_x'\left(u^{\epsilon,\tau}(k)\right) \mathrm{d}x + \cdots \\
&\cdots + \left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \beta z\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right) + W\right\rangle_{L^2(\Omega)} \\
&\overset{(3.18)}{\geq} \int_\Omega \phi_x\left(u^{\epsilon,\tau}(k)\right) - \phi_x\left(u^{\epsilon,\tau}(k-1)\right) \mathrm{d}x + \cdots \\
&\cdots + \left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \beta z\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right) + W\right\rangle_{L^2(\Omega)}
\end{aligned}$$

The term $\left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \beta z\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\rangle_{L^2(\Omega)}$ can be bounded from below by using the weak Poisson equation in recursion formula in (3.16)

$$\begin{aligned}
&\left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \beta z\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\rangle_{L^2(\Omega)} = \cdots \\
&= \beta \left\langle \sum_{i=1}^n z_i\left(u_i^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1)\right), \left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\rangle_{L^2(\Omega)} \\
&= \beta \left\langle \left(\sum_{i=1}^n z_i u_i^{\epsilon,\tau}(k) + f\right) - \left(\sum_{i=1}^n z_i u^{\epsilon,\tau}(k-1) + f\right), \underbrace{\Phi^{\epsilon,\tau}(k) - \Phi^D}_{\in H_D^1(\Omega)}\right\rangle_{L^2(\Omega)} \\
&\overset{(3.16)}{=} \lambda^2 \beta \left\langle \nabla\Phi^{\epsilon,\tau}(k) - \nabla\Phi^{\epsilon,\tau}(k-1), \nabla\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\rangle_{L^2(\Omega)} \\
&= \lambda^2 \beta \left\langle \nabla\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right) - \nabla\left(\Phi^{\epsilon,\tau}(k-1) - \Phi^D\right), \nabla\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\rangle_{L^2(\Omega)} \\
&= \lambda^2 \beta \left\|\nabla\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\|_{L^2(\Omega)}^2 - \lambda^2 \beta \underbrace{\left\langle \nabla\left(\Phi^{\epsilon,\tau}(k-1) - \Phi^D\right), \nabla\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\rangle_{L^2(\Omega)}}_{\leq \frac{\|\nabla(\Phi^{\epsilon,\tau}(k) - \Phi^D)\|_{L^2(\Omega)}^2 + \|\nabla(\Phi^{\epsilon,\tau}(k-1) - \Phi^D)\|_{L^2(\Omega)}^2}{2}} \\
&\geq \frac{\lambda^2 \beta}{2} \left(\left\|\nabla\left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right)\right\|_{L^2(\Omega)}^2 - \left\|\nabla\left(\Phi^{\epsilon,\tau}(k-1) - \Phi^D\right)\right\|_{L^2(\Omega)}^2 \right)
\end{aligned}$$

This lets us conclude

$$\left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} \geq \int_\Omega \phi_x \left(u^{\epsilon,\tau}(k)\right) - \phi_x \left(u^{\epsilon,\tau}(k-1)\right) \mathrm{d}x + \cdots$$
$$\cdots + \frac{\lambda^2\beta}{2} \left(\left\| \nabla \left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right) \right\|^2_{L^2(\Omega)} - \left\| \nabla \left(\Phi^{\epsilon,\tau}(k-1) - \Phi^D\right) \right\|^2_{L^2(\Omega)} \right) + \cdots$$
$$\cdots + \langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), W \rangle_{L^2(\Omega)}$$
$$= \underbrace{\int_\Omega \phi_x \left(u^{\epsilon,\tau}(k)\right) + \frac{\lambda^2\beta}{2} \left| \nabla \left(\Phi^{\epsilon,\tau}(k) - \Phi^D\right) \right|^2 + u^{\epsilon,\tau}(k) \cdot W \mathrm{d}x}_{H(u^{\epsilon,\tau}(k))} - \cdots$$
$$\cdots - \underbrace{\int_\Omega \phi_x \left(u^{\epsilon,\tau}(k-1)\right) + \frac{\lambda^2\beta}{2} \left| \nabla \left(\Phi^{\epsilon,\tau}(k-1) - \Phi^D\right) \right|^2 + u^{\epsilon,\tau}(k-1) \cdot W \mathrm{d}x}_{H(u^{\epsilon,\tau}(k-1))}$$
$$= H\left(u^{\epsilon,\tau}(k)\right) - H\left(u^{\epsilon,\tau}(k-1)\right).$$

□

Now we have the means to derive a discrete entropy production inequality for the sequence $(u^{\epsilon,\tau}(k))_{k\in\mathbb{N}_0}$.

Theorem 3.14. *For every $k \in \mathbb{N}$ the discrete entropy production inequality*

$$\begin{aligned} \frac{H\left(u^{\epsilon,\tau}(k)\right) - H\left(u^{\epsilon,\tau}(k-1)\right)}{\tau} &\leq \kappa^g(\tau k)\mathfrak{c}|\Omega| + \kappa^g(\tau k) H\left(u^{\epsilon,\tau}(k)\right) - \cdots \\ \cdots - \epsilon C_{P,m} \left\| \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|^2_{H^m(\Omega)} &- \left\langle B\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w^{\epsilon,\tau}(k), \nabla \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} \end{aligned} \tag{3.21}$$

holds.

Proof. Let $k \in \mathbb{N}$. By definition it holds that $\overset{\circ}{w}^{\epsilon,\tau}(k) \in H^m(\Omega) \cap H^1_D(\Omega)$ and thus $\overset{\circ}{w}^{\epsilon,\tau}(k)$ itself is an admissible test function in (3.16). Testing against $\xi := \overset{\circ}{w}^{\epsilon,\tau}(k)$ gives us

$$\frac{\left\langle u^{\epsilon,\tau}(k) - u^{\epsilon,\tau}(k-1), \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)}}{\tau} + \left\langle B\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w^{\epsilon,\tau}(k), \nabla \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} + \cdots$$
$$\cdots + \epsilon \left(\sum_{|\alpha|=m} \left\| D^\alpha \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|^2_{L^2(\Omega)} + \left\| \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|^2_{L^2(\Omega)} \right) = \left\langle g\left(\tau k, \cdot, u^{\epsilon,\tau}(k)\right), \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)}.$$

By (3.17) this implies

$$\begin{aligned} \frac{H\left(u^{\epsilon,\tau}(k)\right) - H\left(u^{\epsilon,\tau}(k-1)\right)}{\tau} &\leq \left\langle g\left(\tau k, \cdot, u^{\epsilon,\tau}(k)\right), \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} - \cdots \\ \cdots - \left\langle B\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w^{\epsilon,\tau}(k), \nabla \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} &- \cdots \\ \cdots - \epsilon \left(\sum_{|\alpha|=m} \left\| D^\alpha \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|^2_{L^2(\Omega)} + \left\| \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|^2_{L^2(\Omega)} \right)&. \end{aligned} \tag{3.22}$$

Now it only remains to find suitable bounds for the first and the last term on the right-hand side of (3.22), as the second term is already present in (3.21).

Using the generalized Poincare inequality, just as in (3.11) in the proof of Theorem 3.10, we can estimate the last term by

$$\left(\sum_{|\alpha|=m} \left\| D^\alpha \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|_{L^2(\Omega)}^2 + \left\| \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|_{L^2(\Omega)}^2 \right) \geq C_{P,m} \left\| \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|_{H^m(\Omega)}^2 . \tag{3.23}$$

From identity (3.20) and assumption **(H5)** one derives for the first term

$$\begin{aligned} \left\langle g\left(\tau k, \cdot, u^{\epsilon,\tau}(k)\right), \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} &= \int_\Omega \sum_{i=1}^n g_i\left(\tau k, \cdot, u^{\epsilon,\tau}(k)\right) \overset{\circ}{w}_i^{\epsilon,\tau}(k)\, \mathrm{d}x \\ &\overset{(3.20)}{=} \int_\Omega \sum_{i=1}^n g_i\left(\tau k, \cdot, u^{\epsilon,\tau}(k)\right) \frac{\partial h\left(u^{\epsilon,\tau}(k)\right)}{\partial u_i}\, \mathrm{d}x \\ &\overset{\mathbf{(H5)}}{\leq} \kappa^g(\tau k) \int_\Omega \left(\mathfrak{c} + h\left(u^{\epsilon,\tau}(k)\right)\right)\, \mathrm{d}x \\ &= \kappa^g(\tau k)\mathfrak{c}|\Omega| + \kappa^g(\tau k) H\left(u^{\epsilon,\tau}(k)\right). \end{aligned} \tag{3.24}$$

Combining the estimates (3.23) and (3.24) with (3.22) finally gives us the discrete entropy production inequality

$$\begin{aligned} \frac{H\left(u^{\epsilon,\tau}(k)\right) - H\left(u^{\epsilon,\tau}(k-1)\right)}{\tau} &\leq \kappa^g(\tau k)\mathfrak{c}|\Omega| + \kappa^g(\tau k) H\left(u^{\epsilon,\tau}(k)\right) - \epsilon C_{P,m} \left\| \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|_{H^m(\Omega)}^2 \\ &\quad \cdots - \left\langle B\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w^{\epsilon,\tau}(k), \nabla \overset{\circ}{w}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} . \end{aligned}$$

□

We can now use this entropy production inequality to derive bounds for $u^{\epsilon,\tau}$, that will later be used for deriving convergences via compactness arguments.

Theorem 3.15. *Let $T \in (0, +\infty)$ and $q' \in (0,1)$ be given. For any $\tau > 0$ with $\tau \kappa^g(T) \leq q'$, there exists constant $C > 0$, that only depends on $q*$, the time T and the parameters of system (1.6), such that*

$$\|u^{\epsilon,\tau}(k)\|_{L^\infty(\Omega)} + \epsilon\tau \sum_{j=1}^{\lfloor T/\tau \rfloor} \left\| \overset{\circ}{w}^{\epsilon,\tau}(j) \right\|_{H^m(\Omega)}^2 \leq C$$

for any $k = 0, \cdots, \lfloor \frac{T}{\tau} \rfloor$ and

$$\tau \sum_{j=1}^{\lfloor T/\tau \rfloor} \left(\left\| \sqrt{u_0^{\epsilon,\tau}(j)} \nabla \sqrt{u_i^{\epsilon,\tau}(j)} \right\|_{L^2(\Omega)}^2 + \|u_0^{\epsilon,\tau}(j)\|_{H^1(\Omega)}^2 + \left\| \sqrt{u_0^{\epsilon,\tau}(j)} \right\|_{H^1(\Omega)}^2 \right) \leq C.$$

Proof. Let $k \in \mathbb{N}$ with $\tau k \leq T$. By the discrete entropy production inequality (3.21) it holds that

$$\begin{aligned}(1-\tau\kappa^g(\tau k))\, H\left(u^{\epsilon,\tau}(k)\right) + \tau \left\langle B\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w^{\epsilon,\tau}(k), \nabla \overset{\circ}{w}{}^{\epsilon,\tau}(k) \right\rangle_{L^2(\Omega)} + \cdots \\ \cdots + \tau\epsilon C_{P,m} \left\| \overset{\circ}{w}{}^{\epsilon,\tau}(k) \right\|_{H^m(\Omega)}^2 \leq H\left(u^{\epsilon,\tau}(k-1)\right) + \tau\kappa^g(\tau k)\mathfrak{c}|\Omega|.\end{aligned} \tag{3.25}$$

From B being a diagonal matrix we obtain the inequality

$$\begin{aligned}\nabla \overset{\circ}{w}{}^{\epsilon,\tau}(k) : B\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w^{\epsilon,\tau}(k) &= \nabla\left(w^{\epsilon,\tau}(k) - w^D\right) : B\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w^{\epsilon,\tau}(k) \\ &= \sum_{i=1}^n D_i u_i\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) u_0\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla\left(w_i^{\epsilon,\tau}(k) - w_i^D\right) \cdot \nabla w_i^{\epsilon,\tau}(k) \\ &= \sum_{i=1}^n D_i u_i^{\epsilon,\tau}(k) u_0^{\epsilon,\tau}(k) \left|\nabla w_i^{\epsilon,\tau}(k)\right|^2 - \sum_{i=1}^n D_i u_i^{\epsilon,\tau}(k) u_0^{\epsilon,\tau}(k) \underbrace{w_i^D \cdot \nabla w_i^{\epsilon,\tau}(k)}_{\leq \frac{\left|\nabla w_i^{\epsilon,\tau}(k)\right|^2 + \left|\nabla w_i^D\right|^2}{2}} \\ &\geq \sum_{i=1}^n \frac{D_i}{2} u_i^{\epsilon,\tau}(k) u_0^{\epsilon,\tau}(k) \left|\nabla w_i^{\epsilon,\tau}(k)\right|^2 - \sum_{i=1}^n \frac{D_i}{2} u_i^{\epsilon,\tau}(k) u_0^{\epsilon,\tau}(k) \left|\nabla w_i^D\right|^2 \\ &\geq \frac{D_{\min}}{2} \sum_{i=1}^n u_i^{\epsilon,\tau}(k) u_0^{\epsilon,\tau}(k) \left|\nabla w_i^{\epsilon,\tau}(k)\right|^2 - \frac{D_{\max}}{2} \sum_{i=1}^n \left|\nabla w_i^D\right|^2,\end{aligned}$$

where $D_{\max/\min} := \max/\min\{D_i \ : \ 1, \cdots, n\}$.

Combining this inequality with (3.25) gives us

$$\begin{aligned}(1-\tau\kappa^g(\tau k)) H\left(u^{\epsilon,\tau}(k)\right) + \tau \frac{D_{\min}}{2} \int_\Omega \sum_{i=1}^n u_i^{\epsilon,\tau}(k) u_0^{\epsilon,\tau}(k) \left|\nabla w_i^{\epsilon,\tau}(k)\right|^2 \mathrm{d}x + \cdots \\ \cdots + \tau\epsilon C_{P,m} \left\| \overset{\circ}{w}{}^{\epsilon,\tau}(k) \right\|_{H^m(\Omega)}^2 \\ \leq H\left(u^{\epsilon,\tau}(k-1)\right) + \tau\kappa^g(\tau k)\mathfrak{c}|\Omega| + \tau \frac{D_{\max}}{2} \int_\Omega \sum_{i=1}^n \left|\nabla w_i^D\right|^2 \mathrm{d}x\end{aligned} \tag{3.26}$$

From identity (3.19) and inequality (A.1) we obtain

$$\begin{aligned}\left|\nabla w_i^{\epsilon,\tau}(k)\right|^2 &\overset{(3.19)}{=} \left|\nabla \log \frac{u_i^{\epsilon,\tau}(k)}{u_0^{\epsilon,\tau}(k)} + \beta z_i \nabla\left(\Phi^{\epsilon,\tau}(k) + W_i\right)\right|^2 \\ &\overset{(A.1)}{\geq} \frac{1}{2} \left|\nabla \log \frac{u_i^{\epsilon,\tau}(k)}{u_0^{\epsilon,\tau}(k)}\right|^2 - \left|\beta z_i \nabla\left(\Phi^{\epsilon,\tau}(k) + W_i\right)\right|^2.\end{aligned}$$

Together with (3.26) (and using that $u_i^{\epsilon,\tau}(k)u_0^{\epsilon,\tau}(k) \leq 1$) this gives us

$$\begin{aligned}&(1-\tau\kappa^g(\tau k))\,H\left(u^{\epsilon,\tau}(k)\right)+\tau\frac{D_{\min}}{4}\int_\Omega\sum_{i=1}^n u_i^{\epsilon,\tau}(k)u_0^{\epsilon,\tau}(k)\left|\nabla\log\frac{u_i^{\epsilon,\tau}(k)}{u_0^{\epsilon,\tau}(k)}\right|^2\mathrm{d}x+\cdots\\&\cdots+\tau\epsilon C_{P,m}\left\|\overset{\circ}{w}{}^{\epsilon,\tau}(k)\right\|^2_{H^m(\Omega)}\\&\leq H\left(u^{\epsilon,\tau}(k-1)\right)+\tau\kappa^g(\tau k)\mathfrak{c}|\Omega|+\tau\frac{D_{\max}}{2}\left\|\nabla w^D\right\|_{L^2(\Omega)}+\cdots\\&\cdots+\tau\int_\Omega\sum_{i=1}^n\frac{D_{\min}}{2}\left|\beta z_i\nabla\left(\Phi^{\epsilon,\tau}(k)+W_i\right)\right|^2\mathrm{d}x\end{aligned}\tag{3.27}$$

Since κ^g is monotone, we know that $\kappa^g(\tau k) \leq \kappa^g(T)$. Also we derive from (3.5) that there exists a constant $C_1 > 0$, that only depend on the parameters of system (1.6), such that

$$\begin{aligned}\int_\Omega\sum_{i=1}^n\frac{D_{\min}}{2}\left|\beta z_i\nabla\left(\Phi^{\epsilon,\tau}(k)+W_i\right)\right|^2\mathrm{d}x&\leq\sum_{i=1}^n D_{\min}\left(\beta z_i\left\|\nabla\Phi^{\epsilon,\tau}(k)\right\|^2_{L^2(\Omega)}+\left\|\nabla W_i\right\|^2_{L^2(\Omega)}\right)\\&\overset{(3.5)}{\leq}\sum_{i=1}^n D_{\min}\left(\beta z_i C_1\left(1+\left\|\Phi^D\right\|_{H^1(\Omega)}\right)^2+\left\|\nabla W_i\right\|^2_{L^2(\Omega)}\right)\end{aligned}$$

By defining the constant

$$C_2:=\kappa^g(T)\mathfrak{c}|\Omega|+\frac{D_{\max}}{2}\left\|\nabla w^D\right\|_{L^2(\Omega)}+\sum_{i=1}^n D_{\min}\left(\beta z_i C_1\left(1+\left\|\Phi^D\right\|_{H^1(\Omega)}\right)^2+\left\|\nabla W_i\right\|^2_{L^2(\Omega)}\right)$$

we can thus simplify (3.27) into

$$\begin{aligned}&(1-\tau\kappa^g(\tau k))\,H\left(u^{\epsilon,\tau}(k)\right)+\tau\frac{D_{\min}}{4}\int_\Omega\sum_{i=1}^n u_i^{\epsilon,\tau}(k)u_0^{\epsilon,\tau}(k)\left|\nabla\log\frac{u_i^{\epsilon,\tau}(k)}{u_0^{\epsilon,\tau}(k)}\right|^2\mathrm{d}x+\cdots\\&\cdots+\tau\epsilon C_{P,m}\left\|\overset{\circ}{w}{}^{\epsilon,\tau}(k)\right\|^2_{H^m(\Omega)}\leq H\left(u^{\epsilon,\tau}(k-1)\right)+C_2\tau.\end{aligned}\tag{3.28}$$

Define now for every $k = 0, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor$ the two terms terms

$$\begin{aligned}a(k)&:=H\left(u^{\epsilon,\tau}(k)\right)\\b(k)&:=\frac{D_{\min}}{4}\int_\Omega\sum_{i=1}^n u_i^{\epsilon,\tau}(k)u_0^{\epsilon,\tau}(k)\left|\nabla\log\frac{u_i^{\epsilon,\tau}(k)}{u_0^{\epsilon,\tau}(k)}\right|^2\mathrm{d}x+\epsilon C_{P,m}\left\|\overset{\circ}{w}{}^{\epsilon,\tau}(k)\right\|^2_{H^m(\Omega)}.\end{aligned}$$

We assume without loss of generality that all $a(k), b(k)$, $k = 0, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor$ are non-negative. This is justified, as we could simply replace each negative term with 0 and the recursive inequalities below would still hold.

Then (3.28) gives us the recursive inequality

$$(1-\tau\kappa^g(\tau k))\,a(k)\leq a(k-1)+C_2\tau-\tau b(k),$$

which can also be written as

$$(1-\tau\kappa^g(\tau k))\,a(k) \le (1-\tau\kappa^g(\tau(k-1)))\,a(k-1) + \tau\kappa^g(\tau(k-1))a(k-1) + C_2\tau - \tau b(k).$$

Now it is easy to solve this recursion and we obtain

$$(1-\tau\kappa^g(\tau k))\,a(k) + \tau\sum_{j=1}^{k} b(j) \le (1-\tau\kappa^g(0))\,a(0) + C_2\tau k + \tau\sum_{j=0}^{k-1}\kappa^g(\tau j)a(j).$$

We define the constants $q := \frac{1}{1-q'}$, $C_3 := q\left(\max\left\{H(u^0), 0\right\} + C_2T\right)$, which only depend on the parameters of system (1.6) and T. By using the monotonicity of κ^g and $\tau\kappa^g(T) \le q'$, we derive

$$\begin{aligned} a(k) + \tau\sum_{j=1}^{k} b(j) &\le a(k) + \frac{\tau}{1-\tau\kappa^g(\tau k)}\sum_{j=1}^{k} b(j) \\ &\le \underbrace{\frac{1}{1-\tau\kappa^g(\tau k)}}_{\le q}\left((1-\tau\kappa^g(0))\underbrace{a(0)}_{=H(u^0)} + C_2\underbrace{\tau k}_{\le T} + \sum_{j=0}^{k-1}\tau\kappa^g(\tau j)a(j)\right) \\ &\le C_3 + \sum_{j=0}^{k-1}\tau q\kappa^g(T)a(j). \end{aligned}$$

We furthermore define for every $k = 0, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor$ the term $c(k) := a(k) + \tau\sum_{j=1}^{k} b(j)$. Then the previous inequality gives us the recursion

$$c(k) \le C_3 + \sum_{j=0}^{k-1} q\kappa^g(T)\tau c(j) \text{ for all } k = 1, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor,$$

which, by the discrete Gronwall Lemma A.2, implies

$$c(k) \le C_3\left(1 + q\kappa^g(T)\tau\right)^k \text{ for every } k = 1, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor. \tag{3.29}$$

Due to the convergence

$$\left(1 + q\kappa^g(T)\tau\right)^{T/\tau} \xrightarrow{\tau\to 0} \exp\left(Tq\kappa^g(T)\right),$$

the constant

$$C_4 := C_3 \max_{\tau\in(0,1]} \left(1 + q\kappa^g(T)\tau\right)^{T/\tau} < +\infty$$

is well defined, positive and does only depend on the parameters of system (1.6) as well as on time T.

Thus we derive from (3.29) and Lemma 2.2 that

$$\tau \sum_{j=1}^{\lfloor T/\tau \rfloor} b(j) \le C_4 - a(\lfloor T/\tau \rfloor) = C_4 - \underbrace{H\left(u^{\epsilon,\tau}\left(\lfloor T/\tau \rfloor\right)\right)}_{\ge -\sum_{j=1}^{n} \|W_i\|_{L^1(\Omega)}} \le C_4 + \sum_{j=1}^{n} \|W_i\|_{L^1(\Omega)} := C_5. \tag{3.30}$$

Now it remains to use this inequality to derive the desired bounds. We note that by definition $\left\| \overset{\circ}{w}^{\epsilon,\tau}(j) \right\|_{H^m(\Omega)}^2 \le b(j)$ and thus

$$\underbrace{\|u^{\epsilon,\tau}(k)\|_{L^\infty(\Omega)}}_{\le 1} + \epsilon\tau \sum_{j=1}^{\lfloor T/\tau \rfloor} \left\| \overset{\circ}{w}^{\epsilon,\tau}(j) \right\|_{H^m(\Omega)}^2 \le 1 + \frac{\tau}{C_{P,m}} \sum_{j=1}^{\lfloor T/\tau \rfloor} \underbrace{\epsilon C_{P,m} \left\| \overset{\circ}{w}^{\epsilon,\tau}(j) \right\|_{H^m(\Omega)}^2}_{\le b(j)} \le 1 + \frac{C_5}{C_{P,m}} := C_6.$$

To show the second desired bound, we note that the chain rule (Lemma B.11), applied to[5] $\sqrt{u_i^{\epsilon,\tau}}$ and $\log u_i^{\epsilon,\tau}$, and some simple algebraic manipulations give us

$$\begin{aligned}
&\sum_{i=1}^{n} u_i^{\epsilon,\tau}(j) u_0^{\epsilon,\tau}(j) \left| \nabla \log \frac{u_i^{\epsilon,\tau}(j)}{u_0^{\epsilon,\tau}(j)} \right|^2 = \sum_{i=1}^{n} u_i^{\epsilon,\tau}(j) u_0^{\epsilon,\tau}(j) \left| \nabla \log u_i^{\epsilon,\tau}(j) - \nabla \log u_0^{\epsilon,\tau}(j) \right|^2 \\
&= \sum_{i=1}^{n} u_i^{\epsilon,\tau}(j) u_0^{\epsilon,\tau}(j) \left| \frac{\nabla u_i^{\epsilon,\tau}(j)}{u_i^{\epsilon,\tau}(j)} - \frac{\nabla u_0^{\epsilon,\tau}(j)}{u_0^{\epsilon,\tau}(j)} \right|^2 \\
&= \sum_{i=1}^{n} u_i^{\epsilon,\tau}(j) u_0^{\epsilon,\tau}(j) \left(\left| \frac{\nabla u_i^{\epsilon,\tau}(j)}{u_i^{\epsilon,\tau}(j)} \right|^2 - 2 \frac{\nabla u_i^{\epsilon,\tau}(j) \cdot \nabla u_0^{\epsilon,\tau}(j)}{u_i^{\epsilon,\tau}(j) u_0^{\epsilon,\tau}(j)} + \left| \frac{\nabla u_0^{\epsilon,\tau}(j)}{u_0^{\epsilon,\tau}(j)} \right|^2 \right) \\
&= u_0^{\epsilon,\tau}(j) \sum_{i=1}^{n} \underbrace{\frac{|\nabla u_i^{\epsilon,\tau}(j)|^2}{u_i^{\epsilon,\tau}(j)}}_{=4\left|\nabla \sqrt{u_i^{\epsilon,\tau}(j)}\right|^2} - 2 \sum_{i=1}^{n} \nabla u_i^{\epsilon,\tau}(j) \cdot \nabla u_0^{\epsilon,\tau}(j) + \left(\sum_{i=1}^{n} u_i^{\epsilon,\tau}(j) \right) \frac{|\nabla u_0^{\epsilon,\tau}(j)|^2}{u_0^{\epsilon,\tau}(j)} \\
&= 4 u_0^{\epsilon,\tau}(j) \sum_{i=1}^{n} \left| \nabla \sqrt{u_i^{\epsilon,\tau}(j)} \right|^2 - 2 \underbrace{\left(\sum_{i=1}^{n} \nabla u_i^{\epsilon,\tau}(j) \right)}_{=-\nabla u_0^{\epsilon,\tau}(j)} \cdot \nabla u_0^{\epsilon,\tau}(j) + \underbrace{\left(\sum_{i=1}^{n} u_i^{\epsilon,\tau}(j) \right)}_{=1-u_0^{\epsilon,\tau}(j)} \frac{|\nabla u_0^{\epsilon,\tau}(j)|^2}{u_0^{\epsilon,\tau}(j)} \\
&= 4 u_0^{\epsilon,\tau}(j) \sum_{i=1}^{n} \left| \nabla \sqrt{u_i^{\epsilon,\tau}(j)} \right|^2 + 2 \left| \nabla u_0^{\epsilon,\tau}(j) \right|^2 + \underbrace{\frac{|\nabla u_0^{\epsilon,\tau}(j)|^2}{u_0^{\epsilon,\tau}(j)}}_{=4\left|\nabla \sqrt{u_0^{\epsilon,\tau}(j)}\right|^2} - u_0^{\epsilon,\tau}(j) \frac{|\nabla u_0^{\epsilon,\tau}(j)|^2}{u_0^{\epsilon,\tau}(j)} \\
&= 4 \sum_{i=1}^{n} \left| \sqrt{u_0^{\epsilon,\tau}(j)} \nabla \sqrt{u_i^{\epsilon,\tau}(j)} \right|^2 + \left| \nabla u_0^{\epsilon,\tau}(j) \right|^2 + 4 \left| \nabla \sqrt{u_0^{\epsilon,\tau}(j)} \right|^2.
\end{aligned}$$

[5] Which can be justified just as in the proof of Lemma 3.11.

This identity lets us rewrite b as

$$\begin{aligned} b(j) &= \frac{D_{\min}}{4}\int_\Omega \sum_{i=1}^n u_i^{\epsilon,\tau}(j)u_0^{\epsilon,\tau}(j)\left|\nabla \log \frac{u_i^{\epsilon,\tau}(j)}{u_0^{\epsilon,\tau}(j)}\right|^2 \mathrm{d}x + \epsilon C_{P,m}\left\|\overset{\circ}{w}{}^{\epsilon,\tau}(j)\right\|^2_{H^m(\Omega)} \\ &= D_{\min}\left\|\sqrt{u_0^{\epsilon,\tau}(j)}\nabla\sqrt{u_i^{\epsilon,\tau}(j)}\right\|^2_{L^2(\Omega)} + \frac{D_{\min}}{4}\left\|\nabla u_0^{\epsilon,\tau}(j)\right\|^2_{L^2(\Omega)} + \cdots \\ &\cdots + D_{\min}\left\|\nabla\sqrt{u_0^{\epsilon,\tau}(j)}\right\|^2_{L^2(\Omega)} + \epsilon C_{P,m}\left\|\overset{\circ}{w}{}^{\epsilon,\tau}(j)\right\|^2_{H^m(\Omega)}. \end{aligned}$$

Which in turn lets us finally conclude, using $\|u^{\epsilon,\tau}(j)\|_{L^2(\Omega)} \leq \sqrt{|\Omega|}$, that

$$\begin{aligned} &\tau\sum_{j=1}^{\lfloor T/\tau\rfloor}\left(\left\|\sqrt{u_0^{\epsilon,\tau}(j)}\nabla\sqrt{u_i^{\epsilon,\tau}(j)}\right\|^2_{L^2(\Omega)} + \|u_0^{\epsilon,\tau}(j)\|^2_{H^1(\Omega)} + \left\|\sqrt{u_0^{\epsilon,\tau}(j)}\right\|^2_{H^1(\Omega)}\right) \\ &\leq \tau\sum_{j=1}^{\lfloor T/\tau\rfloor}\left(\left\|\sqrt{u_0^{\epsilon,\tau}(j)}\nabla\sqrt{u_i^{\epsilon,\tau}(j)}\right\|^2_{L^2(\Omega)} + \|\nabla u_0^{\epsilon,\tau}(j)\|^2_{L^2(\Omega)} + \left\|\nabla\sqrt{u_0^{\epsilon,\tau}(j)}\right\|^2_{L^2(\Omega)}\right) + \cdots \\ &\cdots + 2\underbrace{\tau\lfloor T/\tau\rfloor}_{\leq T}\sqrt{|\Omega|} \\ &\leq 2T\sqrt{|\Omega|} + \frac{4}{D_{\min}}\tau\sum_{j=1}^k b(j) \overset{(3.30)}{\leq} 2T\sqrt{|\Omega|} + \frac{4}{D_{\min}}C_5 := C_7. \end{aligned}$$

By defining $C := \max\{C_6, C_7\}$ we finish the proof.

□

3.4. Step 3: $\epsilon \to 0$

Our goal in this section is to eliminate the space regularization in (3.2). Therefore we first prove the existence of a limit (atleast for a subsequence) when the regularization parameter ϵ goes to zero.

Lemma 3.16. *Let $T > 0$ be given and let $\tau > 0$ be so small that $\tau\kappa^g(T) < \frac{1}{2}$. Then there exists a sequence ϵ_j, $j \in \mathbb{N}$, which we simply denote by (ϵ) and a finite sequence of functions $u^\tau(k) \in H^1(\Omega;\mathbb{R}^n)$, $\Phi^\tau(k) \in H^1(\Omega)$ for $k = 1, \cdots, \lfloor\frac{T}{\tau}\rfloor$, such that*

$$u^{(\epsilon),\tau}(k) \xrightharpoonup[(\epsilon)\to 0]{*} u^\tau(k) \text{ in } L^\infty(\Omega;\mathbb{R}^n) \tag{3.31a}$$

$$\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightharpoonup{(\epsilon)\to 0} \sqrt{u_0^\tau(k)},\ \Phi^{(\epsilon),\tau}(k) \xrightharpoonup{(\epsilon)\to 0} \Phi^\tau(k) \quad \text{in } H^1(\Omega) \tag{3.31b}$$

$$u_0^{(\epsilon),\tau}(k) \xrightarrow{(\epsilon)\to 0} u_0^\tau(k),\ \sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} \sqrt{u_0^\tau(k)},\ \Phi^{(\epsilon),\tau}(k) \xrightarrow{(\epsilon)\to 0} \Phi^\tau(k) \quad \textit{in } L^2(\Omega) \tag{3.31c}$$

$$u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightharpoonup{(\epsilon)\to 0} u_i^\tau(k)\sqrt{u_0^\tau(k)} \ \textit{in } H^1(\Omega) \tag{3.31d}$$

$$u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} u_i^\tau(k)\sqrt{u_0^\tau(k)} \ \textit{in } L^2(\Omega) \tag{3.31e}$$

for every $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$ *and every* $i = 1, \cdots, n$*. Furthermore the following property holds*

$$0 \le u_i^\tau(k) \le 1 \ \textit{for all } i = 0, \cdots, n. \tag{3.32}$$

Proof. The estimates derived in Theorem 3.15, give us uniform bounds for the sequences

$$\|u^{\epsilon,\tau}(k)\|_{L^\infty(\Omega)}, \|u_0^{\epsilon,\tau}(k)\|_{H^1(\Omega)}, \left\|\sqrt{u_0^{\epsilon,\tau}(k)}\right\|_{H^1(\Omega)} \quad \text{for } k = 1, \cdots, \left\lfloor \frac{T}{\tau} \right\rfloor,$$

where we vary in $\epsilon > 0$ (τ stays fixed). Furthermore, by inequality (3.5), we know that the sequence $\|\Phi^{\epsilon,\tau}(k)\|_{H^1(\Omega)}$, $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$, where we again vary in ϵ, is uniformly bounded.

Thus we derive by the Banach-Alaoglu theorem that there exists a subsequence in ϵ, that is denoted by (ϵ) and functions $u^\tau(k) \in L^\infty(\Omega; \mathbb{R}^n)$, $\Phi^\tau(k) \in H^1(\Omega)$, $v(k), w(k) \in H^1(\Omega)$ for every $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$, such that

$$\begin{aligned} u^{(\epsilon),\tau}(k) &\xrightharpoonup[(\epsilon)\to 0]{*} u^\tau(k) \text{ in } L^\infty(\Omega; \mathbb{R}^n) \\ \Phi^{(\epsilon),\tau}(k) &\xrightharpoonup{(\epsilon)\to 0} \Phi^\tau(k) \text{ in } H^1(\Omega) \\ u_0^{(\epsilon),\tau}(k) &\xrightharpoonup{(\epsilon)\to 0} v(k) \text{ in } H^1(\Omega) \\ \sqrt{u_0^{(\epsilon),\tau}(k)} &\xrightharpoonup{(\epsilon)\to 0} w(k) \text{ in } H^1(\Omega) \end{aligned}$$

First we show that these limits are consistent, i.e. that $v(k) = u_0^\tau(k)$ and $w(k) = \sqrt{u_0^\tau(k)}$. We note that by the linearity of the (weak-star) limit, we get that

$$u_0^{(\epsilon),\tau}(k) = 1 - \sum_{i=1}^n u_i^{(\epsilon),\tau}(k) \xrightharpoonup[(\epsilon)\to 0]{*} 1 - \sum_{i=1}^n u_i^\tau(k) = u_0^\tau(k) \text{ in } L^\infty(\Omega),$$

which, by Corollary B.9, implies that $u_0^{(\epsilon),\tau}(k) \xrightharpoonup{(\epsilon)\to 0} u_0^\tau(k)$ in $L^2(\Omega)$. On the other hand the compactness of the embedding from $H^1(\Omega)$ into $L^2(\Omega)$ (see Theorem (B.7)) implies that $u_0^{(\epsilon),\tau}(k) \xrightarrow{(\epsilon)\to 0} v(k)$ in $L^2(\Omega)$. Since weak and strong limits must coincide we therefore derive

$$v(k) = u_0^\tau(k).$$

We note that strong convergence in $L^2(\Omega)$ implies the convergence in measure (with respect to the Lebesgue-measure) and therefore we obtain

$$\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} w(k) \text{ and } u_0^{(\epsilon),\tau}(k) \xrightarrow{(\epsilon)\to 0} v(k) \text{ in measure.}$$

But since the square root is continuous, with respect to the convergence in measure, this implies that $\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} \sqrt{v(k)}$ in measure and thus

$$w(k) = \sqrt{v(k)} = \sqrt{u_0^\tau(k)}.$$

So far we have shown the identities (3.31a), (3.31b) and (3.31c).

Now we look at the term $u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)}$ for arbitrary, but fixed $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$ and $i = 1, \cdots, n$. First we note that, since

$$\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} \sqrt{u_0^\tau(k)} \text{ in } L^2(\Omega) \text{ and } u_i^{(\epsilon),\tau}(k) \xrightarrow[(\epsilon)\to 0]{*} u_i^\tau(k) \text{ in } L^\infty(\Omega),$$

it holds that

$$\left\langle u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)}, \xi \right\rangle_{L^2(\Omega)} = \int_\Omega u_i^{(\epsilon),\tau}(k) \underbrace{\sqrt{u_0^\tau(k)}\xi}_{\in L^1(\Omega)} \, \mathrm{d}x + \cdots$$

$$\cdots + \Big\langle \underbrace{u_i^{(\epsilon),\tau}(k)}_{\in[0,1]} \underbrace{\left(\sqrt{u_0^{(\epsilon),\tau}(k)} - \sqrt{u_0^\tau(k)}\right)}_{\to 0 \text{ in } L^2(\Omega)}, \xi \Big\rangle_{L^2(\Omega)} \xrightarrow{(\epsilon)\to 0} \int_\Omega u_i^\tau(k)\sqrt{u_0^\tau(k)}\xi \, \mathrm{d}x$$

for any $\xi \in L^2(\Omega)$. Thus we deduce that

$$u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} u_i^\tau(k)\sqrt{u_0^\tau(k)} \text{ in } L^2(\Omega). \tag{3.33}$$

On the other hand we know from Lemma 3.11 that $u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \in H^1(\Omega)$. Using Corollary B.12 (which gives a product rule for derivatives in Sobolev spaces) we get

$$\begin{aligned}
&\left\| \nabla \left(u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \right) \right\|_{L^2(\Omega)} \overset{\text{Cor. } B.12}{=} \left\| u_i^{(\epsilon),\tau}(k)\nabla\sqrt{u_0^{(\epsilon),\tau}(k)} + \sqrt{u_0^{(\epsilon),\tau}(k)}\nabla u_i^{(\epsilon),\tau}(k) \right\|_{L^2(\Omega)} \\
&\le \left\| u_i^{(\epsilon),\tau}(k) \right\|_{L^\infty(\Omega)} \left\| \nabla\sqrt{u_0^{(\epsilon),\tau}(k)} \right\|_{L^2(\Omega)} + \left\| \sqrt{u_0^{(\epsilon),\tau}(k)}\nabla u_i^{(\epsilon),\tau}(k) \right\|_{L^2(\Omega)} \\
&= \left\| u_i^{(\epsilon),\tau}(k) \right\|_{L^\infty(\Omega)} \left\| \nabla\sqrt{u_0^{(\epsilon),\tau}(k)} \right\|_{L^2(\Omega)} + \left\| 2\sqrt{u_i^{(\epsilon),\tau}(k)}\sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)} \right\|_{L^2(\Omega)} \\
&\le \left\| u_i^{(\epsilon),\tau}(k) \right\|_{L^\infty(\Omega)} \left\| \sqrt{u_0^{(\epsilon),\tau}(k)} \right\|_{H^1(\Omega)} + 2 \underbrace{\left\| \sqrt{u_i^{(\epsilon),\tau}(k)} \right\|_{L^\infty(\Omega)}}_{\le 1} \left\| \sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)} \right\|_{L^2(\Omega)} \\
&\le \left\| u_i^{(\epsilon),\tau}(k) \right\|_{L^\infty(\Omega)} \left\| \sqrt{u_0^{(\epsilon),\tau}(k)} \right\|_{H^1(\Omega)} + 2 \left\| \sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)} \right\|_{L^2(\Omega)}
\end{aligned}$$

Using Theorem 3.15, we see that all three terms on the right hand side can be uniformely bounded and thus $\left\| \nabla \left(u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \right) \right\|_{L^2(\Omega)}$ is bounded. Additionally we know from the weak $L^2(\Omega)$-convergence (see equation (3.33)), that $\left\| u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \right\|_{L^2(\Omega)}$ must be bounded too and thus $\left\| u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \right\|_{H^1(\Omega)}$ is bounded. This in turn implies by the Banach–Alaoglu Theorem that a subsequence, which we assume to be the original sequence, will converge weakly in $H^1(\Omega)$. Due to (3.33) this limit must coincide with $u_i^\tau(k)\sqrt{u_0^\tau(k)}$, i.e.

$$u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} u_i^\tau(k)\sqrt{u_0^\tau(k)} \text{ in } H^1(\Omega).$$

Thus (3.31d) is proven and by the Rellich–Kondrachov Theorem this in turn implies (3.31e).

Finally we note that from (3.31c) we can conclude that (3.32) holds, as it holds for $u^{(\epsilon),\tau}$ and because $L^2(\Omega)$-convergence implies convergence in measure.

□

Now we show, by using (3.31), that $(u^\tau(k), \Phi^\tau(k))$, $k = 0, \cdots, \lfloor \frac{T}{\tau} \rfloor$ also suffices to a recursion formula.

Theorem 3.17. *Let $T > 0$ be given and $\tau > 0$ be so small that $\tau\kappa^g(T) < \frac{1}{2}$. The finite sequence $(u^\tau(k), \Phi^\tau(k))$, $k = 0, \cdots, \lfloor \frac{T}{\tau} \rfloor$ defined in Lemma 3.16 suffices to*

$$\begin{aligned} &\frac{\langle u_i^\tau(k) - u_i^\tau(k-1), v_i\rangle_{L^2(\Omega)}}{\tau} + D_i \left\langle \sqrt{u_0^\tau(k)}\nabla\left(u_i^\tau(k)\sqrt{u_0^\tau(k)}\right), \nabla v_i \right\rangle_{L^2(\Omega)} \\ &\cdots - 3D_i \left\langle \sqrt{u_0^\tau(k)} u_i^\tau(k) \nabla \sqrt{u_0^\tau(k)}, \nabla v_i \right\rangle_{L^2(\Omega)} + \langle F_i\left(u^\tau(k), \Phi^\tau(k)\right), \nabla v_i\rangle_{L^2(\Omega)} \\ &= \langle g_i\left(\tau k, \cdot, u^\tau(k)\right), v_i\rangle_{L^2(\Omega)} \end{aligned} \tag{3.34}$$

for any $v \in H_D^1(\Omega;\mathbb{R}^n)$, $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$, as well as to

$$\lambda^2 \langle \nabla\Phi^\tau(k), \nabla\eta\rangle_{L^2(\Omega)} = \left\langle \sum_{i=1}^n z_i u_i^\tau(k) + f, \eta \right\rangle_{L^2(\Omega)} \tag{3.35}$$

for any $\eta \in H_D^1(\Omega)$, $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$. Furthermore the following boundary conditions hold for any $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$

$$\left.\sqrt{u_0^\tau(k)}\right|_{\Gamma_D} = \left.\sqrt{u_0^D}\right|_{\Gamma_D}, \quad \left.u_i^\tau(k)\sqrt{u_0^\tau(k)}\right|_{\Gamma_D} = \left.u_i^D\sqrt{u_0^D}\right|_{\Gamma_D}, \quad \Phi^\tau(k)|_{\Gamma_D} = \Phi^D|_{\Gamma_D}. \tag{3.36}$$

Proof. Let $k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$ be arbitrary. First we note that by Theorem 3.15, the term

$$\left\| \sqrt{\epsilon}\overset{\circ}{w}^{\epsilon,\tau}(k) \right\|_{H^m(\Omega)}^2 = \epsilon \left\| \overset{\circ}{w}^{\epsilon,\tau}(k) \right\|_{H^m(\Omega)}^2$$

is bounded and thus

$$\epsilon \overset{\circ}{w}^{\epsilon,\tau}(k) = \sqrt{\epsilon} \underbrace{\sqrt{\epsilon} \overset{\circ}{w}^{\epsilon,\tau}(k)}_{\text{bounded norm}} \xrightarrow{\epsilon \to 0} 0 \text{ in } H^m(\Omega). \tag{3.37}$$

Furthermore it holds that

$$g(\tau k, \cdot, u^{(\epsilon),\tau}) \xrightarrow{(\epsilon)\to 0} g(\tau k, \cdot, u^\tau) \text{ in } L^2(\Omega; \mathbb{R}^n). \tag{3.38}$$

To see this we use the criteria for $L^2(\Omega;\mathbb{R}^n)$-convergence, stated in Lemma B.2. From (3.31c) it follows that $u^{(\epsilon),\tau} \xrightarrow{(\epsilon)\to 0} u^\tau$ in measure, and thus, by the continuity of the reaction term g, we also get $g(\tau k, \cdot, u^{(\epsilon),\tau}) \xrightarrow{(\epsilon)\to 0} g(\tau k, \cdot, u^\tau)$ in measure.
The convergence of the norms $\left\|g(\tau k, \cdot, u^{(\epsilon),\tau})\right\|_{L^2(\Omega)} \xrightarrow{(\epsilon)\to 0} \left\|g(\tau k, \cdot, u^\tau)\right\|_{L^2(\Omega)}$ follows from a dominated convergence argument, as $\left|g(\tau k, \cdot, u^{(\epsilon),\tau})\right| \leq C^g(T)$. Thus (3.38) indeed holds.

By Corollary B.4 we derive from (3.31d) and (3.31c), that

$$\sqrt{u_0^{(\epsilon),\tau}(k)} \nabla \left(u_i^{(\epsilon),\tau}(k) \sqrt{u_0^{(\epsilon),\tau}(k)} \right) \xrightarrow{(\epsilon)\to 0} \sqrt{u_0^\tau(k)} \nabla \left(u_i^\tau(k) \sqrt{u_0^\tau(k)} \right), \tag{3.39}$$

and from (3.31e) and (3.31b), that

$$\sqrt{u_0^{(\epsilon),\tau}(k)} u_i^{(\epsilon),\tau}(k) \nabla \sqrt{u_0^{(\epsilon),\tau}(k)} \xrightarrow{(\epsilon)\to 0} \sqrt{u_0^\tau(k)} u_i^\tau(k) \nabla \sqrt{u_0^\tau(k)}. \tag{3.40}$$

Now we want to prove

$$F_i \left(u^{(\epsilon),\tau}(k), \Phi^{(\epsilon),\tau}(k) \right) \xrightarrow{(\epsilon)\to 0} F_i \left(u^\tau(k), \Phi^\tau(k) \right) \text{ in } L^2(\Omega). \tag{3.41}$$

Looking at the definition of F, we see that

$$F_i \left(u^{(\epsilon),\tau}(k), \Phi^{\epsilon,\tau}(k) \right) = D_i u_0^{(\epsilon),\tau}(k) u_i^{(\epsilon),\tau}(k) \left(\beta z_i \nabla \Phi^{\epsilon,\tau}(k) + W_i \right)$$

and thus we aim to apply Corollary B.4 again. Therefore we note that, since

$$0 \leq u_i^{(\epsilon),\tau}(k), u_i^\tau(k) \leq 1 \text{ for every } i = 0, \cdots, n,$$

it holds that

$$\begin{aligned}
&\left\| u_0^{(\epsilon),\tau}(k) u_i^{(\epsilon),\tau}(k) - u_0^\tau(k) u_i^\tau(k) \right\|_{L^2(\Omega)} \leq \left\| \left(\sqrt{u_0^{(\epsilon),\tau}(k)} - \sqrt{u_0^\tau(k)} \right) u_i^\tau(k) \sqrt{u_0^\tau(k)} \right\|_{L^2(\Omega)} + \cdots \\
&\cdots + \left\| \sqrt{u_0^{(\epsilon),\tau}(k)} \left(u_i^{(\epsilon),\tau}(k) \sqrt{u_0^{(\epsilon),\tau}(k)} - u_i^\tau(k) \sqrt{u_0^\tau(k)} \right) \right\|_{L^2(\Omega)} \\
&\leq \underbrace{\left\| \sqrt{u_0^{(\epsilon),\tau}(k)} - \sqrt{u_0^\tau(k)} \right\|_{L^2(\Omega)}}_{\to 0 \text{ by } (3.31c)} + \underbrace{\left\| u_i^{(\epsilon),\tau}(k) \sqrt{u_0^{(\epsilon),\tau}(k)} - u_i^\tau(k) \sqrt{u_0^\tau(k)} \right\|_{L^2(\Omega)}}_{\to 0 \text{ by } (3.31e)} \xrightarrow{(\epsilon)\to 0} 0
\end{aligned}$$

Together with $\Phi^{(\epsilon),\tau}(k) \xrightarrow{(\epsilon)\to 0} \Phi^{\tau}(k)$ (see (3.31b)) this shows that we can indeed employ Corollary B.4, which then gives us (3.41).

Similar to the computations of section 1.2, we see that for every $i = 1, \cdots, n$

$$\begin{aligned}\sum_{j=1}^{n} B_{ij}\left(w^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right) \nabla w_j^{\epsilon,\tau}(k) &\overset{\text{Lemma 2.7}}{=} J_i\left(u^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k), W\right) \\ = D_i \sqrt{u_0^{\epsilon,\tau}(k)} \nabla \left(u_i^{\epsilon,\tau}(k) \sqrt{u_0^{\epsilon,\tau}(k)} \right) &- 3D_i \sqrt{u_0^{\epsilon,\tau}(k)} u_i^{\epsilon,\tau}(k) \nabla \sqrt{u_0^{\epsilon,\tau}(k)} + \cdots \\ \cdots + F_i\left(u^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right).\end{aligned} \tag{3.42}$$

Combining this with the space regularized, time discretized equation (3.16) gives us

$$\begin{aligned}&\underbrace{\left\langle g\left(\tau k, \cdot, u^{(\epsilon),\tau}(k)\right), v\right\rangle_{L^2(\Omega)}}_{\to \langle g(\tau k,\cdot,u^{\tau}(k)),v\rangle_{L^2(\Omega)} \text{ by } (3.38)} - \underbrace{\epsilon\left(\sum_{|\alpha|=m} \left\langle D^\alpha \mathring{w}^{(\epsilon),\tau}(k), D^\alpha v \right\rangle_{L^2(\Omega)} + \left\langle \mathring{w}^{(\epsilon),\tau}(k), v \right\rangle_{L^2(\Omega)} \right)}_{\to 0, \text{ due to } (3.37)} \\
&\overset{(3.16)}{=} \frac{\left\langle u_i^{(\epsilon),\tau}(k) - u_i^{(\epsilon),\tau}(k-1), v\right\rangle_{L^2(\Omega)}}{\tau} + \left\langle \sum_{j=1}^{n} B_{ij}\left(w^{(\epsilon),\tau}(k), \Phi^{(\epsilon),\tau}(k)\right) \nabla w_j^{(\epsilon),\tau}(k), \nabla v \right\rangle_{L^2(\Omega)} \\
&\overset{(3.42)}{=} \frac{\left\langle u_i^{(\epsilon),\tau}(k) - u_i^{(\epsilon),\tau}(k-1), v\right\rangle_{L^2(\Omega)}}{\tau} + D_i \underbrace{\left\langle \sqrt{u_0^{(\epsilon),\tau}(k)} \nabla \left(u_i^{(\epsilon),\tau}(k) \sqrt{u_0^{(\epsilon),\tau}(k)} \right), \nabla v \right\rangle_{L^2(\Omega)}}_{\to \left\langle \sqrt{u_0^{\tau}(k)} \nabla \left(u_i^{\tau}(k) \sqrt{u_0^{\tau}(k)} \right), \nabla v \right\rangle_{L^2(\Omega)} \text{ by } (3.39)} \\
&\cdots - 3D_i \underbrace{\left\langle \sqrt{u_0^{(\epsilon),\tau}(k)} u_i^{(\epsilon),\tau}(k) \nabla \sqrt{u_0^{(\epsilon),\tau}(k)}, \nabla v \right\rangle_{L^2(\Omega)}}_{\to \left\langle \sqrt{u_0^{\tau}(k)} u_i^{\tau}(k) \nabla \sqrt{u_0^{\tau}(k)}, \nabla v \right\rangle_{L^2(\Omega)} \text{ by } (3.40)} + \underbrace{\left\langle F_i\left(u^{\epsilon,\tau}(k), \Phi^{\epsilon,\tau}(k)\right), \nabla v \right\rangle_{L^2(\Omega)}}_{\to \langle F_i(u^{\tau}(k),\Phi^{\tau}(k)),\nabla v\rangle_{L^2(\Omega)} \text{ by } (3.41)}\end{aligned}$$

for any $i = 1, \cdots, n$ and any $v \in H^m(\Omega) \cap H^1_D(\Omega)$.

Taking the limits on both side of the equation lets us conclude

$$\begin{aligned}&\frac{\langle u_i^{\tau}(k) - u_i^{\tau}(k-1), v\rangle_{L^2(\Omega)}}{\tau} + D_i \left\langle \sqrt{u_0^{\tau}(k)} \nabla \left(u_i^{\tau}(k) \sqrt{u_0^{\tau}(k)} \right), \nabla v \right\rangle_{L^2(\Omega)} - \cdots \\
&\cdots - 3D_i \left\langle \sqrt{u_0^{\tau}(k)} u_i^{\tau}(k) \nabla \sqrt{u_0^{\tau}(k)}, \nabla v \right\rangle_{L^2(\Omega)} + \langle F_i\left(u^{\tau}(k), \Phi^{\tau}(k)\right), \nabla v\rangle_{L^2(\Omega)} \\
&= \langle g\left(\tau k, \cdot, u^{\tau}(k)\right), v\rangle_{L^2(\Omega)}\end{aligned}$$

and thus the scheme (3.34) indeed holds.

From (3.16) we derive by using (3.31) that

$$\begin{aligned}\lambda^2 \left\langle \nabla\Phi^\tau(k), \nabla\eta \right\rangle_{L^2(\Omega)} &\overset{(3.31b)}{=} \lim_{(\epsilon)\to 0} \lambda^2 \left\langle \nabla\Phi^{(\epsilon),\tau}(k), \nabla\eta \right\rangle_{L^2(\Omega)} \\ &\overset{(3.16)}{=} \lim_{(\epsilon)\to 0} \left\langle \sum_{i=1}^n z_i u_i^{(\epsilon),\tau}(k) + f, \eta \right\rangle_{L^2(\Omega)} \\ &\overset{(3.31c)}{=} \left\langle \sum_{i=1}^n z_i u_i^\tau(k) + f, \eta \right\rangle_{L^2(\Omega)}\end{aligned}$$

for every $\eta \in H^1_D(\Omega)$.

To prove that the boundary conditions (3.36) are satisfied, we note that by Corollary B.13, the fact that $u_i^{(\epsilon),\tau}(k)\big|_{\Gamma_D} = u_i^D(k)\big|_{\Gamma_D}$ holds, implies that the same is true for the transformed variables

$$\sqrt{(\epsilon), u_0^\tau(k)}\Big|_{\Gamma_D} = \sqrt{u_0^D}\Big|_{\Gamma_D}, \quad u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)}\Big|_{\Gamma_D} = u_i^D\sqrt{u_0^D}\Big|_{\Gamma_D} \quad \text{for every } i = 1, \cdots, n.$$

Since the trace operator $\cdot|_{\Gamma_D}$ is continuous, it translates weak convergence into weak convergence, according to Lemma A.8. As $u_i^{(\epsilon),\tau}(k), \Phi_i^{(\epsilon),\tau}(k)$ satisfy the boundary conditions (3.36), the same is thus true for the limit $u_i^\tau(k), \Phi_i^\tau(k)$.

□

The final task in this section is to derive bounds for u^τ, that are used in the final step of the existence proof for a compactness argument.

Lemma 3.18. *Let $T > 0$ be given and $\tau > 0$ be so small that $\tau\kappa^g(T) < \frac{1}{2}$. Let $u^\tau(k)$, $k = 0, \cdots, \lfloor \frac{T}{\tau} \rfloor$ be the finite sequence defined in Lemma 3.16. There exists a constant $C > 0$, that only depends on the parameters of system (1.6) and time T, such that for any $i = 1, \cdots, n$*

$$\tau \sum_{j=1}^{\lfloor T/\tau \rfloor} \left(\left\| \sqrt{u_0^\tau(j)} \nabla \sqrt{u_i^\tau(j)} \right\|^2_{L^2(\Omega)} + \|u_0^\tau(j)\|^2_{H^1(\Omega)} + \left\| \sqrt{u_0^\tau(j)} \right\|^2_{H^1(\Omega)} \right) \le C.$$

Proof. Let $i = 1, \cdots, n$ be arbitrary but fixed. For any $j = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor$ it holds that $0 \le u_i^{(\epsilon),\tau}(j) \le 1$ and therefore

$$\left\| u_i^{(\epsilon),\tau}(j) \sqrt{u_0^{(\epsilon),\tau}(j)} \right\|_{L^2(\Omega)} \le \left\| \sqrt{u_0^{(\epsilon),\tau}(j)} \right\|_{L^2(\Omega)}. \tag{3.43}$$

By the product rule for Sobolev-spaces (Lemma B.12) we get

$$
\begin{aligned}
&\left\|\nabla\left(u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)}\right)\right\|_{L^2(\Omega)} = \left\|\nabla\left(\sqrt{u_i^{(\epsilon),\tau}(k)}\sqrt{u_i^{(\epsilon),\tau}(k)u_0^{(\epsilon),\tau}(k)}\right)\right\|_{L^2(\Omega)}\\
&\leq \left\|\sqrt{u_i^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)u_0^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)} + \left\|\sqrt{u_i^{(\epsilon),\tau}(k)u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)}\\
&\leq \left\|\nabla\sqrt{u_i^{(\epsilon),\tau}(k)u_0^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)} + \left\|\sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)}\\
&\leq \left\|\sqrt{u_i^{(\epsilon),\tau}(k)}\nabla\sqrt{u_0^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)} + 2\left\|\sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)}\\
&\leq \left\|\nabla\sqrt{u_0^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)} + 2\left\|\sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)}.
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\left\|\nabla\left(u_i^{(\epsilon),\tau}(k)\sqrt{u_0^{(\epsilon),\tau}(k)}\right)\right\|^2_{L^2(\Omega)}\\
&\leq 2\left\|\nabla\sqrt{u_0^{(\epsilon),\tau}(k)}\right\|^2_{L^2(\Omega)} + 8\left\|\sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)}\right\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{3.44}
$$

The two inequalities (3.43) and (3.44) clearly impliy

$$
\begin{aligned}
\left\|u_i^{(\epsilon),\tau}(j)\sqrt{u_0^{(\epsilon),\tau}(j)}\right\|^2_{H^1(\Omega)} &\leq \left\|u_i^{(\epsilon),\tau}(j)\sqrt{u_0^{(\epsilon),\tau}(j)}\right\|^2_{L^2(\Omega)} + \left\|\nabla\left(u_i^{(\epsilon),\tau}(j)\sqrt{u_0^{(\epsilon),\tau}(j)}\right)\right\|^2_{L^2(\Omega)}\\
&\leq 2\left\|\sqrt{u_0^{(\epsilon),\tau}(t)}\right\|^2_{H^1(\Omega)} + 8\left\|\sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)}\right\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Now let C' denote the constant in Theorem 3.15. If we define $C := 8C'$ then this inequality lets us derive from Theorem 3.15 that

$$
\begin{aligned}
&\tau\sum_{j=1}^{\lfloor T/\tau\rfloor}\left(\left\|u_i^{(\epsilon),\tau}(j)\sqrt{u_0^{(\epsilon),\tau}(j)}\right\|^2_{H^1(\Omega)} + \left\|u_0^{(\epsilon),\tau}(j)\right\|^2_{H^1(\Omega)} + \left\|\sqrt{u_0^{(\epsilon),\tau}(j)}\right\|^2_{H^1(\Omega)}\right)\\
&\leq \tau\sum_{j=1}^{\lfloor T/\tau\rfloor}\left(8\left\|\sqrt{u_0^{(\epsilon),\tau}(k)}\nabla\sqrt{u_i^{(\epsilon),\tau}(k)}\right\|_{L^2(\Omega)} + \left\|u_0^{(\epsilon),\tau}(j)\right\|^2_{H^1(\Omega)} + 2\left\|\sqrt{u_0^{(\epsilon),\tau}(j)}\right\|^2_{H^1(\Omega)}\right)\\
&\overset{\text{Thm 3.15}}{\leq} 8C' = C
\end{aligned}
$$

Finally the fact that Hilbert-space norms are weakly subcontinuous and the convergences (3.31) let us conclude

$$\tau \sum_{j=1}^{\lfloor T/\tau \rfloor} \left(\left\| u_i^\tau(j) \sqrt{u_0^\tau(j)} \right\|_{H^1(\Omega)}^2 + \|u_0^\tau(j)\|_{H^1(\Omega)}^2 + \left\| \sqrt{u_0^\tau(j)} \right\|_{H^1(\Omega)}^2 \right)$$
$$\overset{(3.31)}{\leq} \liminf_{(\epsilon)\to 0} \tau \sum_{j=1}^{\lfloor T/\tau \rfloor} \left(\left\| u_i^{(\epsilon),\tau}(j) \sqrt{u_0^{(\epsilon),\tau}(j)} \right\|_{H^1(\Omega)}^2 + \left\| u_0^{(\epsilon),\tau}(j) \right\|_{H^1(\Omega)}^2 + \left\| \sqrt{u_0^{(\epsilon),\tau}(j)} \right\|_{H^1(\Omega)}^2 \right)$$
$$\leq C.$$

□

3.5. Step 4: $\tau \to 0$

Before we can derive the existence of a weak solution, we extend the discrete sequence $(u^\tau(k), \Phi^\tau(k))$, $k = 0, \cdots, \lfloor \frac{T}{\tau} \rfloor$ to the whole time interval $[0, +\infty)$. Throughout the whole section we assume that $T > 0$ is fixed and that $\tau > 0$ is so small that $\tau < T$ and $\tau \kappa^g(T) < \frac{1}{2}$.

Definition 3.19. *We define the function $u^\tau : [0, +\infty) \to L^\infty(\Omega; \mathbb{R}^n)$ by*

$$u^\tau(t) := \begin{cases} u^0 & , \text{ if } t = 0 \\ u^\tau(k) & , \text{ if } t \in ((k-1)\tau, k\tau] \text{ for some } k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor \\ u^\tau\left(\lfloor \frac{T}{\tau} \rfloor\right) & , \text{ if } t > \lfloor \frac{T}{\tau} \rfloor \end{cases}.$$

Similarly we define the function $\Phi^\tau : [0, +\infty) \to H^1(\Omega) \cap L^\infty(\Omega)$ by

$$\Phi^\tau(t) := \begin{cases} \Phi^0 & , \text{ if } t = 0 \\ \Phi^\tau(k) & , \text{ if } t \in ((k-1)\tau, k\tau] \text{ for some } k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor \\ \Phi^\tau\left(\lfloor \frac{T}{\tau} \rfloor\right) & , \text{ if } t > \lfloor \frac{T}{\tau} \rfloor \end{cases}.$$

These two functions have the following properties.

Lemma 3.20. *Both functions are L^2-integrable in time, i.e.*

$$u^\tau \in L^2\left((0,T); L^2(\Omega; \mathbb{R}^n)\right), \quad \Phi^\tau \in L^2\left((0,T); H^1(\Omega)\right), \tag{3.45}$$

u^τ is uniformely bounded as

$$u_i^\tau, \Phi^\tau \in L^\infty\left([0,T] \times \bar{\Omega}\right), \text{ with } 0 \leq u_i^\tau \leq 1, \text{ for every } i = 0, \cdots, n \tag{3.46}$$

and it holds that

$$u_i^\tau \sqrt{u_0^\tau}, \sqrt{u_0^\tau} \in L^2\left((0,T); H^1(\Omega)\right) \tag{3.47}$$

Furthermore the following boundary conditions hold for every $t \in (0, +\infty)$

$$\sqrt{u_0^\tau(t)}\Big|_{\Gamma_D} = \sqrt{u_0^D}\Big|_{\Gamma_D}, \quad u_i^\tau(t)\sqrt{u_0^\tau(t)}\Big|_{\Gamma_D} = u_i^D\sqrt{u_0^D}\Big|_{\Gamma_D}, \quad \Phi^\tau(t)|_{\Gamma_D} = \Phi^D\big|_{\Gamma_D}. \tag{3.48}$$

for every $i = 1, \cdots, n$.

Proof. (3.45),(3.46) and (3.47) follow directly from the fact that both u^τ and Φ^τ are by definition simple functions, i.e. piecewise constant with respect to the time variable. Equation (3.48) on the other hand is a trivial consequence of (3.36).

□

Furthermore we define the time-shifted approximation.

Definition 3.21. *The time-shifted approximation* $\sigma_\tau u^\tau : (0, +\infty) \to L^\infty(\Omega; \mathbb{R}^n)$ *is defined by*

$$\sigma_\tau u^\tau(t) := \begin{cases} u^\tau(k-1) & , \text{ if } t \in ((k-1)\tau, k\tau] \text{ for some } k = 1, \cdots, \lfloor \frac{T}{\tau} \rfloor \\ u^\tau\left(\lfloor \frac{T}{\tau} \rfloor - 1\right) & , \text{ if } t > \lfloor \frac{T}{\tau} \rfloor. \end{cases}$$

With this definition we can reformulate (3.34) and (3.35).

Lemma 3.22. *For any* $\xi \in L^2\left((0,T); H_D^1(\Omega; \mathbb{R}^n)\right)$ *and any* $\eta \in L^2\left((0,T); H_D^1(\Omega)\right)$ *it holds that for any* $i = 1, \cdots, n$

$$\int_0^T \frac{\langle u_i^\tau(t) - \sigma_\tau u_i^\tau(t), \xi_i(t)\rangle_{L^2(\Omega)}}{\tau} + D_i \left\langle \sqrt{u_0^\tau(t)}\nabla\left(u_i^\tau(t)\sqrt{u_0^\tau(t)}\right), \nabla\xi_i(t)\right\rangle_{L^2(\Omega)}$$
$$\cdots + \langle F_i(u^\tau(t), \Phi^\tau(t)), \nabla\xi_i(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t \tag{3.49a}$$
$$= \int_0^T \langle g(t, \cdot, u^\tau(t)), \xi_i(t)\rangle_{L^2(\Omega)} + 3D_i \left\langle \sqrt{u_0^\tau(t)} u_i^\tau(t) \nabla\sqrt{u_0^\tau(t)}, \nabla\xi_i(t)\right\rangle_{L^2(\Omega)} \mathrm{d}t$$

$$\lambda^2 \int_0^T \langle \nabla\Phi^\tau(t), \nabla\eta(t)\rangle_{L^2(\Omega)} \,\mathrm{d}t = \int_0^T \left\langle \sum_{i=1}^n z_i u_i^\tau(t) + f, \eta(t)\right\rangle_{L^2(\Omega)} \mathrm{d}t \tag{3.49b}$$

Proof. Let $\xi \in L^2\left((0,T); H^1(\Omega; \mathbb{R}^n)\right)$ and $\eta \in L^2\left((0,T); H^1(\Omega)\right)$. From Lemma 3.20 one directly sees that for any $i = 1, \cdots, n$

$$u_i^\tau, \; \sigma_\tau u_i^\tau, \; \Phi^\tau, \; \sum_{i=1}^n z_i u_i^\tau(t) + f \in L^2\left((0,T); L^2(\Omega)\right)$$

and

$$\sqrt{u_0^\tau}\nabla\left(u_i^\tau\sqrt{u_0^\tau}\right), \; \sqrt{u_0^\tau}u_i^\tau\nabla\sqrt{u_0^\tau}, \; g(\cdot, \cdot, u^\tau) \in L^2\left((0,T); L^2(\Omega; \mathbb{R}^n)\right),$$

and thus the equations (3.49) make sense. Theorem 3.17 tells us, that the integrands on the left and on the right hand side in (3.49a) and (3.49b) coincide. Thus (3.49) indeed holds.

□

Now we want to derive estimates, which will be crucial for applying the usual compactness arguments, when constructing the weak solution. First we estimate Φ^τ.

Lemma 3.23. *There exists a constant $C > 0$, that only depends on the parameters of system (1.6), such that*

$$\|\Phi^\tau\|^2_{L^2((0,T);H^1(\Omega))} \leq CT.$$

Proof. Using the fact that for every $t > 0$ the function $\Phi^\tau(t)$ is a solution to a Poisson problem of the form (3.4) and the bound (3.5) applies, we derive

$$\|\Phi^\tau\|^2_{L^2((0,T);H^1(\Omega))} = \int_0^T \underbrace{\|\Phi^\tau(t)\|^2_{H^1(\Omega)}}_{\leq C\left(1+\|\Phi^D\|_{H^1(\Omega)}\right)^2} \,\mathrm{d}t \leq C\left(1 + \|\Phi^D\|_{H^1(\Omega)}\right)^2 T,$$

which proves the claim.

□

Next we derive bounds for u^τ.

Lemma 3.24. *There exists a constant $C > 0$, that only depends on the parameters of system (1.6) and time T, such that for any $i = 1, \cdots, n$*

$$\left\|u_i^\tau\sqrt{u_0^\tau}\right\|^2_{L^2((0,T);H^1(\Omega))} + \|u_0^\tau\|^2_{L^2((0,T);H^1(\Omega))} + \left\|\sqrt{u_0^\tau}\right\|^2_{L^2((0,T);H^1(\Omega))} \leq C.$$

Proof. Define, for the sake of brevity in formulas, the integer $K := \lfloor \frac{T}{\tau} \rfloor$. Since u^τ is a simple function we have

$$\begin{aligned}
&\left\|u_i^\tau\sqrt{u_0^\tau}\right\|^2_{L^2((0,T);H^1(\Omega))} + \|u_0^\tau\|^2_{L^2((0,T);H^1(\Omega))} + \left\|\sqrt{u_0^\tau}\right\|^2_{L^2((0,T);H^1(\Omega))} \\
&= \int_0^T \left\|u_i^\tau(t)\sqrt{u_0^\tau(t)}\right\|^2_{H^1(\Omega)} + \|u_0^\tau(t)\|^2_{H^1(\Omega)} + \left\|\sqrt{u_0^\tau(t)}\right\|^2_{H^1(\Omega)} \,\mathrm{d}t \\
&= \tau\sum_{j=1}^{K}\left(\left\|u_i^\tau(j)\sqrt{u_0^\tau(j)}\right\|^2_{H^1(\Omega)} + \|u_0^\tau(j)\|^2_{H^1(\Omega)} + \left\|\sqrt{u_0^\tau(j)}\right\|^2_{H^1(\Omega)}\right) + \cdots \\
&\cdots + \underbrace{\left(T - \tau\left\lfloor\frac{T}{\tau}\right\rfloor\right)}_{\leq\tau}\left(\left\|u_i^\tau(K)\sqrt{u_0^\tau(K)}\right\|^2_{H^1(\Omega)} + \|u_0^\tau(K)\|^2_{H^1(\Omega)} + \left\|\sqrt{u_0^\tau(K)}\right\|^2_{H^1(\Omega)}\right) \\
&\leq \underbrace{2\,\tau\sum_{j=1}^{K}\left(\left\|u_i^\tau(j)\sqrt{u_0^\tau(j)}\right\|^2_{H^1(\Omega)} + \|u_0^\tau(j)\|^2_{H^1(\Omega)} + \left\|\sqrt{u_0^\tau(j)}\right\|^2_{H^1(\Omega)}\right)}_{\leq C \text{ by Lemma 3.18}}
\end{aligned}$$

By Lemma 3.18 the right hand side can be bounded by some constant, that only depends on the parameters of system (1.6) and time T, and thus the claim holds.

□

Finally we need to estimate the difference quotient $\frac{u_i^\tau - \sigma_\tau u_i^\tau}{\tau}$, $i = 1, \cdots, n$.

Lemma 3.25. *There exists a constant $C > 0$, that only depends on the parameters of system (1.6) and time T, such that for every $i = 1, \cdots, n$*

$$\left\| \frac{u_i^\tau - \sigma_\tau u_i^\tau}{\tau} \right\|_{L^2\left((0,T);H_D^{-1}(\Omega)\right)} \leq C$$

Proof. Let $i = 1, \cdots, n$ and $\xi \in L^2\left((0,T); H_D^1(\Omega)\right)$ be arbitrary but fixed. From formula (3.49a) we derive

$$\left| \int_0^T \frac{\langle u_i^\tau(t) - \sigma_\tau u_i^\tau(t), \xi(t)\rangle_{L^2(\Omega)}}{\tau} \mathrm{d}t \right| \overset{(3.49a)}{\leq} \int_0^T D_i \left| \left\langle \sqrt{u_0^\tau(t)} \nabla \left(u^\tau(t) \sqrt{u_0^\tau(t)} \right), \nabla \xi(t) \right\rangle_{L^2(\Omega)} \right|$$
$$\cdots + \left| \langle F_i \left(u^\tau(t), \Phi^\tau(t) \right), \nabla \xi(t) \rangle_{L^2(\Omega)} \right| + \left| \langle g\left(t, \cdot, u^\tau(t)\right), \xi(t) \rangle_{L^2(\Omega)} \right| + \cdots$$
$$\cdots + 3D_i \left| \left\langle \sqrt{u_0^\tau(t)} u_i^\tau(t) \nabla \sqrt{u_0^\tau(t)}, \nabla \xi(t) \right\rangle_{L^2(\Omega)} \right| \mathrm{d}t$$

Using the Cauchy–Schwarz inequality we can dominate this further by

$$\left| \int_0^T \frac{\langle u_i^\tau(t) - \sigma_\tau u_i^\tau(t), \xi(t)\rangle_{L^2(\Omega)}}{\tau} \mathrm{d}t \right| \leq \left(D_i \left\| \sqrt{u_0^\tau} \nabla \left(u^\tau \sqrt{u_0^\tau} \right) \right\|_{L^2((0,T);L^2(\Omega))} + \cdots \right.$$
$$\cdots + \| F_i \left(u^\tau, \Phi^\tau \right) \|_{L^2((0,T);L^2(\Omega))} + \| g\left(\cdot, \cdot, u^\tau\right) \|_{L^2((0,T);L^2(\Omega))} + \cdots$$
$$\left. \cdots + 3D_i \left\| \sqrt{u_0^\tau} u_i^\tau \nabla \sqrt{u_0^\tau} \right\|_{L^2((0,T);L^2(\Omega))} \right) \|\xi\|_{L^2((0,T);H^1(\Omega))} .$$

Since $0 \leq u_j^\tau \leq 1$ for every $j = 0, \cdots, n$, this can further be estimated by

$$\left| \int_0^T \frac{\langle u_i^\tau(t) - \sigma_\tau u_i^\tau(t), \xi(t)\rangle_{L^2(\Omega)}}{\tau} \mathrm{d}t \right| \leq \left(D_i \left\| u^\tau \sqrt{u_0^\tau} \right\|_{L^2((0,T);H^1(\Omega))} + \cdots \right.$$
$$\cdots + \| F_i \left(u^\tau, \Phi^\tau \right) \|_{L^2((0,T)\times\Omega)} + \| g\left(\cdot, \cdot, u^\tau\right) \|_{L^2((0,T)\times\Omega)}$$
$$\left. \cdots + 3D_i \left\| \sqrt{u_0^\tau} \right\|_{L^2((0,T);H^1(\Omega))} \right) \|\xi\|_{L^2((0,T);H^1(\Omega))} .$$

The first and the last term on the right hand side, namely $\left\| u^\tau(t) \sqrt{u_0^\tau(t)} \right\|_{L^2((0,T);H^1(\Omega))}$ and $\left\| \sqrt{u_0^\tau(t)} \right\|_{L^2((0,T);H^1(\Omega))}$, can be dominated with Lemma 3.24.

The norm of the reaction term can simply be bounded by

$$\| g\left(\cdot, \cdot, u^\tau(t)\right) \|_{L^2((0,T)\times\Omega)} \leq \sqrt{|\Omega| T} \sup \left\{ \, |g(t,x,v)| \; \middle| \; t \in [0,T], x \in \bar{\Omega}, v \in \bar{\mathcal{O}} \, \right\} .$$

Finally we dominate the term $\|F_i(u^\tau(t), \Phi^\tau(t))\|_{L^2((0,T)\times\Omega)}$ by

$$\begin{aligned}\|F_i(u^\tau(t), \Phi^\tau(t))\|_{L^2((0,T)\times\Omega)} &= \|D_i u_i^\tau u_0^\tau(\beta z_i \nabla\Phi^\tau + \nabla W_i)\|_{L^2((0,T)\times\Omega)} \\ &\leq D_i\beta|z_i|\,\|\nabla\Phi^\tau\|_{L^2((0,T)\times\Omega)} + D_i\|\nabla W_i\|_{L^2((0,T)\times\Omega)} \\ &\leq D_i\beta|z_i|\,\|\Phi^\tau\|_{L^2((0,T);H^1(\Omega))} + D_i\|\nabla W_i\|_{L^2((0,T);L^2(\Omega))}.\end{aligned}$$

By Lemma 3.23 there exists an upper bound for the right hand side, that only depends on the parameters of system (1.6), and therefore we can conclude that there exists a constant $C > 0$, such that

$$\left|\int_0^T \frac{\langle u_i^\tau(t) - \sigma_\tau u_i^\tau(t), \xi(t)\rangle_{L^2(\Omega)}}{\tau}\mathrm{d}t\right| \leq C\,\|\xi\|_{L^2((0,T);H^1(\Omega))}.$$

Since $L^2\left((0,T); H_D^{-1}(\Omega)\right)$ is the dual space of $L^2\left((0,T); H_D^1(\Omega)\right)$ and ξ was chosen arbitrarily this indeed concludes the proof.

□

Now we are in the position to construct the weak solution.

Theorem 3.26. *There exist functions $u \in L^\infty((0,T)\times\Omega;\mathbb{R}^n)$ and $\Phi \in L^2\left((0,T);H^1(\Omega)\right)$, such that for a subsequence of $(u^\tau,\Phi^\tau)_{\tau>0}$, which we denote by (τ), it holds that*

$$u^{(\tau)} \xrightharpoonup[(\tau)\to 0]{*} u \text{ in } L^\infty\left([0,T]\times\bar{\Omega};\mathbb{R}^n\right) \tag{3.50a}$$

$$u_0^{(\tau)} \xrightarrow{(\tau)\to 0} u_0 \text{ in } L^2(\Omega_T;\mathbb{R}^n) \text{ and } \sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} \sqrt{u_0} \text{ in } L^2(\Omega_T) \tag{3.50b}$$

$$\Phi^{(\tau)} \xrightharpoonup{(\tau)\to 0} \Phi \text{ in } L^2\left((0,T);H^1(\Omega)\right) \tag{3.50c}$$

$$\sqrt{u_0^{(\tau)}} \xrightharpoonup{(\tau)\to 0} \sqrt{u_0} \text{ in } L^2\left((0,T);H^1(\Omega)\right) \tag{3.50d}$$

$$u_i^{(\tau)}\sqrt{u_0^{(\tau)}} \xrightharpoonup{(\tau)\to 0} u_i\sqrt{u_0} \text{ in } L^2\left((0,T);H^1(\Omega)\right), \text{ for every } i = 1,\cdots,n \tag{3.50e}$$

$$\frac{u_i^\tau - \sigma_\tau u_i^\tau}{\tau} \xrightharpoonup{(\tau)\to 0} \partial_t u_i \text{ in } L^2\left((0,T);H_D^{-1}(\Omega)\right), \text{ for every } i = 1,\cdots,n \tag{3.50f}$$

Furthermore it holds that $u \in \bar{\mathcal{O}}$ almost everywhere[6]

[6]With respect to the Lebesgue measure on Ω_T.

Proof. From the bounds in equation (3.46), Lemma 3.23, Lemma 3.24 and Lemma 3.25 we immediatley derive by the Banach–Alaoglu Theorem, that there exists a subsequence with respect to the parameter τ, which we denote by (τ), such that

$$\begin{aligned} u^{(\tau)} &\xrightarrow[(\tau)\to 0]{*} u \text{ in } L^\infty\left((0,T)\times\Omega;\mathbb{R}^n\right) \\ \Phi^{(\tau)} &\xrightharpoonup{(\tau)\to 0} \Phi \text{ in } L^2\left((0,T);H^1(\Omega)\right) \\ \sqrt{u_0^{(\tau)}} &\xrightharpoonup{(\tau)\to 0} w \text{ in } L^2\left((0,T);H^1(\Omega)\right) \\ u_i^{(\tau)}\sqrt{u_0^{(\tau)}} &\xrightharpoonup{(\tau)\to 0} \psi_i \text{ in } L^2\left((0,T);H^1(\Omega)\right), \text{ for every } i=1,\cdots,n \\ \frac{u_i^\tau-\sigma_\tau u_i^\tau}{\tau} &\xrightharpoonup{(\tau)\to 0} v_i \text{ in } L^2\left((0,T);H_D^{-1}(\Omega)\right), \text{ for every } i=1,\cdots,n \end{aligned}$$

for some $u\in L^\infty\left((0,T)\times\Omega;\mathbb{R}^n\right)$, $\Phi,w,\psi_i\in L^2\left((0,T);H^1(\Omega)\right)$ and $v_i\in L^2\left((0,T);H_D^{-1}(\Omega)\right)$, $i=1,\cdots,n$.

From Lemma B.10 we know that weak star convergence preserves positivity and thus we have $u_j\geq 0$ for every $j=1,\cdots,n$. By the linearity of convergence we derive that

$$u_0^{(\tau)} = 1-\sum_{j=1}^n u_j^{(\tau)} \xrightarrow[(\tau)\to 0]{*} 1-\sum_{j=1}^n u_j = u_0 \text{ in } L^\infty\left((0,T)\times\Omega;\mathbb{R}^n\right).$$

Using again Lemma B.10 we see that $u_0\geq 0$ and therefore indeed $u\in\bar{\mathcal{O}}$.

First let us show by discrete partial integration that $v_i=\partial_t u_i$ for every $i=1,\cdots,n$. For this let $\xi\in C^1_{00}\left((0,T);H^1(\Omega)\right)$. The definition of $u^{(\tau)}$ as a piecewise constant function and the definition of the shift operator $\sigma_{(\tau)}$ give us by a simple index change

$$\begin{aligned} &\int_0^T \frac{\left\langle u_i^{(\tau)}(t)-\sigma_{(\tau)}u_i^{(\tau)}(t),\xi(t)\right\rangle_{L^2(\Omega)}}{(\tau)}\mathrm{d}t = \sum_{j=1}^{\lfloor T/(\tau)\rfloor}\int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(t)-\sigma_{(\tau)}u_i^{(\tau)}(t),\xi(t)\right\rangle_{L^2(\Omega)}}{(\tau)}\mathrm{d}t \\ &= \sum_{j=1}^{\lfloor T/(\tau)\rfloor}\int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(j)-u_i^{(\tau)}(j-1),\xi(t)\right\rangle_{L^2(\Omega)}}{(\tau)}\mathrm{d}t \\ &= \sum_{j=1}^{\lfloor T/(\tau)\rfloor}\int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(j),\xi(t)\right\rangle_{L^2(\Omega)}}{(\tau)}\mathrm{d}t - \sum_{j=0}^{\lfloor T/(\tau)\rfloor-1}\int_{j(\tau)}^{(j+1)(\tau)} \frac{\left\langle u_i^{(\tau)}(j),\xi(t)\right\rangle_{L^2(\Omega)}}{(\tau)}\mathrm{d}t \\ &= \sum_{j=1}^{\lfloor T/(\tau)\rfloor}\int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(j),\xi(t)\right\rangle_{L^2(\Omega)}}{(\tau)}\mathrm{d}t - \sum_{j=0}^{\lfloor T/(\tau)\rfloor-1}\int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(j),\xi(t+(\tau))\right\rangle_{L^2(\Omega)}}{(\tau)}\mathrm{d}t. \end{aligned}$$

When (τ) is so small that

$$\operatorname{supp}(\xi)\subseteq\left((\tau),(\tau)\left\lfloor\frac{T}{(\tau)}\right\rfloor\right), \tag{3.51}$$

this can be rewritten as

$$\begin{aligned}
&\int_0^T \frac{\left\langle u_i^{(\tau)}(t) - \sigma_{(\tau)} u_i^{(\tau)}(t), \xi(t) \right\rangle_{L^2(\Omega)}}{(\tau)} \mathrm{d}t \\
&\overset{(3.51)}{=} \sum_{j=1}^{\lfloor T/(\tau) \rfloor} \int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(j), \xi(t) \right\rangle_{L^2(\Omega)}}{(\tau)} \mathrm{d}t - \sum_{j=1}^{\lfloor T/(\tau) \rfloor} \int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(j), \xi(t+(\tau)) \right\rangle_{L^2(\Omega)}}{(\tau)} \mathrm{d}t \\
&= - \sum_{j=1}^{\lfloor T/(\tau) \rfloor} \int_{(j-1)(\tau)}^{j(\tau)} \frac{\left\langle u_i^{(\tau)}(t), \xi(t+(\tau)) - \xi(t) \right\rangle_{L^2(\Omega)}}{(\tau)} \mathrm{d}t \\
&= - \int_0^T \frac{\left\langle u_i^{(\tau)}(t), \xi(t+(\tau)) - \xi(t) \right\rangle_{L^2(\Omega)}}{(\tau)} \mathrm{d}t.
\end{aligned}$$

By taking the limit on the left and on the right hand side of this identity, we derive

$$\int_0^T \langle v_i(t), \xi(t) \rangle_{L^2(\Omega)} \, \mathrm{d}t = - \int_0^T \langle u_i(t), \partial_t \xi(t) \rangle_{L^2(\Omega)} \, \mathrm{d}t.$$

Since ξ was an arbitrary element of $C^1_{00}\left((0,T); H^1(\Omega)\right)$, we can conclude that $v_i = \partial_t u_i$.

By equation (3.46) and by Lemma 3.25 the term

$$\begin{aligned}
&\left\| \frac{u_0^{(\tau)} - \sigma_{(\tau)} u_0^{(\tau)}}{\tau} \right\|_{L^1\left((0,T); H_D^{-1}(\Omega)\right)} + \left\| u_0^{(\tau)} \right\|_{L^2((0,T);H^1(\Omega))} \\
&\leq \sqrt{T} \left\| \frac{u_0^{(\tau)} - \sigma_{(\tau)} u_0^{(\tau)}}{\tau} \right\|_{L^2\left((0,T); H_D^{-1}(\Omega)\right)} + \left\| u_0^{(\tau)} \right\|_{L^2((0,T);H^1(\Omega))}
\end{aligned}$$

is uniformely[7] bounded and therefore we can apply Lemma B.15 to deduce the strong convergence

$$u_0^{(\tau)} \xrightarrow{(\tau)\to 0} u_0 \text{ in } L^2(\Omega_T) \tag{3.52}$$

Thus the first part of (3.50b) holds. This in turn implies the convergence in measure. By the continuity of the square root we thus see that

$$\sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} \sqrt{u_0} \text{ in measure.} \tag{3.53}$$

Using the strong convergence of $u_0^{(\tau)}$ with respect to the $L^2(\Omega_T)$-norm, we get

$$\left\| \sqrt{u_0^{(\tau)}} \right\|^2_{L^2(\Omega_T)} = \left\langle u_0^{(\tau)}, 1 \right\rangle_{L^2(\Omega_T)} \xrightarrow{(\tau)\to 0} \langle u_0, 1 \rangle_{L^2(\Omega_T)} = \|\sqrt{u_0}\|^2_{L^2(\Omega_T)}. \tag{3.54}$$

[7]W.r.t. (τ)

By Lemma B.2 the combination of (3.53) and (3.54) implies the $L^2(\Omega_T)$-convergence

$$\sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} \sqrt{u_0} \text{ in } L^2(\Omega_T), \tag{3.55}$$

and thus the second part of (3.50b) holds.

The continuity of the embedding $L^2\left((0,T);H^1(\Omega)\right) \hookrightarrow L^2(\Omega_T)$ and the fact that continuous (linear) operators translate weak convergence into weak convergence (see Lemma A.8), gives us that $\sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} w$ in $L^2\left((0,T);H^1(\Omega)\right)$ also implies $\sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} w$ in $L^2(\Omega_T)$. Thus, by the uniqueness of weak limits, we derive $w = \sqrt{u_0}$, which proves (3.50d).

By (3.55) the sequence $\left(\sqrt{u_0^{(\tau)}}\right)_{\tau>0}$ is convergent and thus relatively compact in $L^2\left(\Omega_T\right)$. By (3.46) we know that $0 \leq u_i^\tau \leq 1$ and by Lemma 3.24 and 3.25 there exists a constant $C > 0$ such that

$$\left\|\sqrt{u_0^{(\tau)}}\right\|_{L^2((0,T);H^1(\Omega))} + \left\|\sqrt{u_0^{(\tau)}}u_i^{(\tau)}\right\|_{L^2((0,T);H^1(\Omega))} + \frac{\left\|u_i^{(\tau)} - \sigma_{(\tau)}u_i^{(\tau)}\right\|_{L^2\left((0,T);H_D^{-1}(\Omega)\right)}}{(\tau)} \leq C.$$

Thus we can apply Lemma (B.16) to derive

$$u_i^{(\tau)}\sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} u_i\sqrt{u_0} \text{ in } L^2(\Omega_T), \text{ for every } i = 1,\cdots,n.$$

Just as before, we can deduce $u_i^{(\tau)}\sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} \psi_i$ in $L^2(\Omega_T)$ from $u_i^{(\tau)}\sqrt{u_0^{(\tau)}} \xrightarrow{(\tau)\to 0} \psi_i$ in $L^2((0,T);H^1(\Omega))$ and thus the uniqueness of weak limits lets us conclude $\psi_i = u_i\sqrt{u_0}$ for every $i = 1,\cdots,n$. Therefore (3.50e) indeed holds, which concludes the proof.

□

Next we show that (u,Φ) satisfy the initial and boundary conditions.

Lemma 3.27. *For any $i = 1,\cdots,n$ and almost any $t \in (0,T)$ (in the sense of the Lebesgue-measure) u and Φ satisfy the boundary conditions (1.10). Additionally it holds that*

$$u \in C\left([0,T];H_D^{-1}(\Omega;\mathbb{R}^n)\right), \tag{3.56}$$

and the intial condition (1.11) is satisfied.

Proof. Since the trace operator $\cdot|_{\Gamma_D} : H^1(\Omega) \to L^2(\Gamma_D)$ is a continuous linear operator, it is clear that its pointwise extension $\cdot|_{\Gamma_D} : L^2\left((0,T);H^1(\Omega)\right) \to L^2\left((0,T);L^2(\Gamma_D)\right)$, defined by

$$\left(v|_{\Gamma_D}\right)(t) := v(t)|_{\Gamma_D}(t) \text{ for every } v \in L^2\left((0,T);H^1(\Omega)\right),$$

is also a continuous linear operator. From Lemma A.8 we know that such an operator translates weak convergence into weak convergence and thus (3.50c) gives us

$$\Phi^{(\tau)}\Big|_{\Gamma_D} \xrightarrow{(\tau)\to 0} \Phi|_{\Gamma_D} \text{ in } L^2\left((0,T);L^2(\Gamma_D)\right).$$

On the other hand we know from (3.36) that $\Phi^{(\tau)}\big|_{\Gamma_D} = \Phi^D\big|_{\Gamma_D}$. As weak limits are unique, we can thus conclude that $\Phi(t)|_{\Gamma_D} = \Phi^D(t)\big|_{\Gamma_D}$ for almost every $t \in (0,T)$. One can prove the other identities in (1.10) in a similar fashion.

Since $u \in L^2\left((0,T); H^1(\Omega;\mathbb{R}^n)\right)$ and $\partial_t u \in L^2\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right)$ we know that

$$u \in H^1\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right)$$

and thus, by Sobolev embedding[8], (3.56) indeed holds.

To derive the initial condition for u, we define another approximation $\tilde{u} \in L^2(\Omega_T)$, which is the piecewise linear interpolation of $u^{(\tau)}(k), k = 1, \cdots, \left\lfloor \frac{T}{(\tau)} \right\rfloor$, by

$$\tilde{u}^{(\tau)}(t) := \begin{cases} u^{(\tau)}(k) + \frac{((\tau)k-t)}{(\tau)}\left(u^{(\tau)}(k-1) - u^{(\tau)}(k)\right) & , \text{ for } t \in \left[(\tau)(k-1), (\tau)k\right) \\ & \quad \text{and } k = 1, \cdots, \left\lfloor \frac{T}{(\tau)} \right\rfloor \\ u^{(\tau)}(\lfloor T/(\tau) \rfloor) & , \text{ for } t \geq (\tau)\left\lfloor \frac{T}{(\tau)} \right\rfloor. \end{cases}$$

Since $\tilde{u}^{(\tau)}$ is piecewise linear and globally continuous, it is easy to see that

$$\tilde{u}^{(\tau)} \in H^1\left((0,T); L^2(\Omega;\mathbb{R}^n)\right) \subseteq H^1\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right).$$

Due to Lemma 3.25 we can dominate the time derivative of $\tilde{u}$ by

$$\begin{aligned} \left\|\partial_t \tilde{u}^{(\tau)}\right\|^2_{L^2((0,T);H_D^{-1})} &= \int_0^{(\tau)\left\lfloor \frac{T}{(\tau)} \right\rfloor} \left\|\partial_t \tilde{u}^{(\tau)}(t)\right\|^2_{H_D^{-1}} \mathrm{d}t = \sum_{k=1}^{\lfloor T/(\tau) \rfloor} (\tau) \left\| \frac{u^{(\tau)}(k) - u^{(\tau)}(k-1)}{(\tau)} \right\|^2_{H_D^{-1}} \\ &= \sum_{k=1}^{\lfloor T/(\tau) \rfloor} (\tau) \int_{(\tau)(k-1)}^{(\tau)k} \left\| \frac{u^{(\tau)}(t) - \sigma_{(\tau)} u^{(\tau)}(t)}{(\tau)} \right\|^2_{H_D^{-1}} \\ &\leq \left\| \frac{u^{(\tau)}(t) - \sigma_{(\tau)} u^{(\tau)}(t)}{(\tau)} \right\|^2_{L^2((0,T);H_D^{-1})} \leq C, \end{aligned}$$

for some constant $C > 0$, that is independent of (τ). This, together with the fact that $0 \leq \tilde{u}_i^{(\tau)} \leq 1$ for every $i = 0, \cdots, n$, implies that we can uniformely bound $\tilde{u}^{(\tau)}$ in the $H^1\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right)$-norm. By Banach–Alaoglu there exists a subsequence with respect to the parameter (τ) (without loss of generality we assume that this is the original sequence), that converges weakly against some $w \in H^1\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right)$, i.e.

$$\tilde{u}^{(\tau)} \xrightarrow{(\tau)\to 0} w \text{ in } H^1\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right).$$

As $H^1\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right)$ is compactly embedded into $C\left([0,T]; H_D^{-1}(\Omega;\mathbb{R}^n)\right)$, this implies $w(0) = \lim_{(\tau)\to 0} \tilde{u}^{(\tau)}(0) = \lim_{(\tau)\to 0} u^{(\tau)}(0) = u^D$.

[8] Which also holds for Sobolev-valued functions. See [Arnold, Satz 3.4, page 42]

It only remains to show that $w = u$. From the compactness of the embedding of $H^1\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right)$ into $L^2\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right)$ we know that

$$\tilde{u}^{(\tau)} \xrightarrow{(\tau)\to 0} w \text{ in } L^2\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right), \tag{3.57}$$

Furthermore Theorem 3.26 implies

$$u^{(\tau)} \xrightarrow{(\tau)\to 0} u \text{ in } L^2\left((0,T); H_D^{-1}(\Omega;\mathbb{R}^n)\right). \tag{3.58}$$

From (3.57) and (3.58) we derive

$$\begin{aligned}
\|u-w\|^2_{L^2((0,T);H_D^{-1})} &= \lim_{(\tau)\to 0} \left\| u^{(\tau)} - \tilde{u}^{(\tau)} \right\|^2_{L^2((0,T);H_D^{-1})} \\
&= \lim_{(\tau)\to 0} \int_0^{(\tau)\left\lfloor \frac{T}{(\tau)} \right\rfloor} \left\| u^{(\tau)}(t) - \tilde{u}^{(\tau)}(t) \right\|^2_{H_D^{-1}} \mathrm{d}t \\
&= \lim_{(\tau)\to 0} \sum_{k=1}^{\lfloor T/(\tau) \rfloor} \int_{(\tau)(k-1)}^{(\tau)k} \underbrace{\left(\frac{(\tau)k - t}{(\tau)} \right)^2}_{\le 1} \left\| u^{(\tau)}(k) - u^{(\tau)}(k-1) \right\|^2_{H_D^{-1}} \mathrm{d}t \\
&\le \limsup_{(\tau)\to 0} \int_0^{(\tau)\left\lfloor \frac{T}{(\tau)} \right\rfloor} \left\| u^{(\tau)}(t) - \sigma_{(\tau)} u^{(\tau)}(t) \right\|^2_{H_D^{-1}} \mathrm{d}t \\
&= \limsup_{(\tau)\to 0} (\tau) \left\| \frac{u^{(\tau)}(t) - \sigma_{(\tau)} u^{(\tau)}(t)}{(\tau)} \right\|^2_{L^2((0,T);H_D^{-1})} = 0,
\end{aligned}$$

and thus we can conclude that indeed $u(0) = w(0) = u^0$.

□

Finally we are in the position to prove Theorem 3.2.

Proof of Theorem 3.2. We have already constructed a candidate for a weak solution in (u, Φ) and seen that it satisfies the boundary and initial conditions. Thus it only remains to show that (u, Φ) satisfy the equations (1.9).

Let therefore $\xi \in L^2\left((0,T); H_D^1(\Omega;\mathbb{R}^n)\right)$ and $\eta \in L^2\left((0,T); H_D^1(\Omega)\right)$.

(1.9b) follows directly from passing to the limit in (3.49b) and applying both (3.50c) and (3.50b), as

$$\begin{aligned}
\lambda^2 \int_0^T \langle \nabla\Phi(t), \nabla\eta(t) \rangle_{L^2(\Omega)} \mathrm{d}t &\overset{(3.50c)}{=} \lim_{(\tau)\to 0} \lambda^2 \int_0^T \left\langle \nabla\Phi^{(\tau)}(t), \nabla\eta(t) \right\rangle_{L^2(\Omega)} \mathrm{d}t \\
&\overset{(3.49b)}{=} \lim_{(\tau)\to 0} \int_0^T \left\langle \sum_{j=1}^n z_j u_j^{(\tau)}(t) + f, \eta(t) \right\rangle_{L^2(\Omega)} \mathrm{d}t \\
&\overset{(3.50b)}{=} \int_0^T \left\langle \sum_{j=1}^n z_j u_j(t) + f, \eta(t) \right\rangle_{L^2(\Omega)} \mathrm{d}t.
\end{aligned}$$

Let now $i = 1, \cdots, n$ be arbitrary. The identity (1.9a) also follows from passing to the limit in (3.49a). Let us treat the convergence of each part of (3.49a) individually. First we get from (3.50f) that

$$\int_0^T \frac{\left\langle u_i^{(\tau)}(t) - \sigma_{(\tau)} u_i^{(\tau)}(t), \xi_i(t)\right\rangle_{L^2(\Omega)}}{(\tau)} \mathrm{d}t \overset{(\tau)\to 0}{\longrightarrow} \int_0^T \langle \partial_t u_i(t), \xi(t)\rangle \,\mathrm{d}t. \tag{3.59}$$

Next we derive from the boundedness of $u^{(\tau)}$ and from (3.50b) that

$$\begin{aligned}\left\|u_i^{(\tau)}u_0^{(\tau)} - u_i u_0\right\|_{L^2(\Omega_T)} &\le \left\|u_i^{(\tau)}\left(u_0^{(\tau)} - u_0\right)\right\|_{L^2(\Omega_T)} + \left\|u_0\left(u_i^{(\tau)} - u_i\right)\right\|_{L^2(\Omega_T)} \\ &\le \left\|u_0^{(\tau)} - u_0\right\|_{L^2(\Omega_T)} + \left\|u_i^{(\tau)} - u_i\right\|_{L^2(\Omega_T)} \overset{(\tau)\to 0}{\longrightarrow} 0.\end{aligned}$$

By (3.50c) it also holds that $\Phi^{(\tau)} \overset{(\tau)\to 0}{\rightharpoonup} \Phi$ in $L^2\left((0,T); H^1(\Omega)\right)$ and thus we can apply Lemma B.4 to derive

$$u_0^{(\tau)}u_i^{(\tau)}\nabla\Phi^{(\tau)} \overset{(\tau)\to 0}{\rightharpoonup} u_0 u_i \nabla\Phi \text{ in } L^2\left(\Omega_T; \mathbb{R}^d\right)$$

and therefore

$$F_i\left(u^{(\tau)}, \Phi^{(\tau)}\right) \overset{(\tau)\to 0}{\rightharpoonup} F_i\left(u, \Phi\right) \text{ in } L^2\left(\Omega_T; \mathbb{R}^d\right). \tag{3.60}$$

Similarly we derive from the combination of (3.50e) and (3.50b) the convergence

$$\sqrt{u_0^{(\tau)}}\nabla\left(\sqrt{u_0^{(\tau)}}u_i^{(\tau)}\right) \overset{(\tau)\to 0}{\rightharpoonup} \sqrt{u_0}\,\nabla\left(\sqrt{u_0}u_i\right) \text{ in } L^2\left(\Omega_T; \mathbb{R}^d\right), \tag{3.61}$$

and from the combination of (3.50d) and

$$\begin{aligned}\left\|\sqrt{u_0^{(\tau)}}u_i^{(\tau)} - \sqrt{u_0}u_i\right\|_{L^2(\Omega_T)} &\le \left\|\sqrt{u_0^{(\tau)}}\left(u_i^{(\tau)} - u_i\right)\right\|_{L^2(\Omega_T)} + \left\|u_i\left(\sqrt{u_0^{(\tau)}} - \sqrt{u_0}\right)\right\|_{L^2(\Omega_T)} \\ &\le \left\|u_i^{(\tau)} - u_i\right\|_{L^2(\Omega_T)} + \left\|\sqrt{u_0^{(\tau)}} - \sqrt{u_0}\right\|_{L^2(\Omega_T)} \overset{(\tau)\to 0}{\longrightarrow} 0,\end{aligned}$$

the convergence

$$\sqrt{u_0^{(\tau)}}u_i^{(\tau)}\nabla\sqrt{u_0^{(\tau)}} \overset{(\tau)\to 0}{\rightharpoonup} \sqrt{u_0}u_i\nabla\sqrt{u_0} \text{ in } L^2\left(\Omega_T; \mathbb{R}^d\right). \tag{3.62}$$

Finally the convergence of the reaction term

$$g(\cdot,\cdot,u^{(\tau)}) \overset{(\tau)\to 0}{\longrightarrow} g(\cdot,\cdot,u) \text{ in } L^2(\Omega_T). \tag{3.63}$$

can be shown just as in the proof of Theorem 3.17.

Due to (3.59), (3.60), (3.61), (3.62) and (3.63), taking the limit in (3.49a), indeed shows that (u, Φ), constructed in Theorem 3.26, satisfies (1.9a) and thus Theorem 3.2 indeed holds.

□

4. Conclusion and outlook

We have seen that under assumption **(H5)** we can introduce a (time and space dependent) reaction term g to the evolution equation (1.1) and still obtain the existence of weak solutions of system (1.6). This may be used to model the evolution of ion concentrations in confined spaces under the influence of noise, as we may set g to be a (regularized) noise term.

For example we could set $g(t,x,v) := \sigma(x,v)\xi^\epsilon(t)$, where σ is a matrix valued function and $\xi^\epsilon := \xi * \rho^\epsilon$. Hereby ξ denotes temporal white noise, ρ^ϵ denotes a molifier and $*$ denotes the convolution. Note that the convolution against the molfier turns the white noise from a distribution into a differentiable function. If σ satisfies

$$\left| \sum_{i=1}^{n} \sigma_{ij}(x,v) \frac{\partial h(v)}{\partial v_i} \right| \leq C_j \left(\mathfrak{c} + h(v)\right)$$

for positive constants $C_j, \mathfrak{c} > 0$ for every $j = 1, \cdots, n$, then we have

$$\begin{aligned}
\sum_{i=1}^{n} g_i(t,x,v) \frac{\partial h(v)}{\partial v_i} &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}(x,v) \xi_j^\epsilon(t) \frac{\partial h(v)}{\partial v_i} = \sum_{j=1}^{n} \xi_j^\epsilon(t) \sum_{i=1}^{n} \sigma_{ij}(x,v) \frac{\partial h(v)}{\partial v_i} \\
&\leq \sum_{j=1}^{n} \left|\xi_j^\epsilon(s)\right| \left| \sum_{i=1}^{n} \sigma_{ij}(x,v) \frac{\partial h(v)}{\partial v_i} \right| \leq \sum_{j=1}^{n} \left|\xi_j^\epsilon(s)\right| C_j \left(\mathfrak{c} + h(v)\right) \\
&\leq \kappa^g(t) \left(\mathfrak{c} + h(v)\right)
\end{aligned}$$

for the monotone function $\kappa^g(t) := \max_{s\in[0,t]} \sum_{j=1}^{n} C_j \left|\xi_i^\epsilon(s)\right|$.

Thus we see that by Theorem 3.2 weak solutions exist pathwise for this form of regularized noise. An interesting next step would now be to eliminate the molifier, i.e. set $g(t,x,v) := \sigma(x,v)\xi(t)$. In this case the definition of weak solutions is trivial using stochastic integrals. Indeed this kind of noise was investigated in [DhJüZa] for general cross-diffusion systems. Unfortunately their analysis required the diffusion matrix A to be componentwise quadratic, which is not the case for our system. Thus a the incorporation of non-regularized noise may from an interesting subject for future investigations.

Due to the assumed boundedness of the external potentials W_i, $i = 1, \cdots, n$, it is easy to see that a trivial reaction term $g_i = 0$ for $i = 1, \cdots, n$ also suffices to **(H5)**. Thus our existence Theorem 3.2 is a true generalization of the one found in [GerstJün].

Remark 4.1. *As we have argued, solutions to the problem with regularized noise exist pathwise. However such a pathwise existence theorem can not show that such a solution is*

indeed measurable with respect to the filtration of the noise. Since there is no result available showing the continuous dependence of the solution with respect to the reaction term g[1], *one can not use standard arguments to conclude measurability and one has thus to resort to using the constructed approximations.*
This is indeed possible, as under suitable assumptions on the parameters of system (1.6) *one can prove, similarly to Part (iii) of the proof of Theorem 3.10, that $u^{\epsilon,\tau}$ and $\Phi^{\epsilon,\tau}$ depend continuously on the reaction term g. Then one would argue that both u and Φ are measurable due to the strong convergence of the approximations in $L^2(\Omega;\mathbb{R}^n)$ and $L^2(\Omega_T;\mathbb{R}^n)$. If one would also like to obtain the measurability of Φ, $u_i\sqrt{u_0}$, $\sqrt{u_0}$ as elements of $L^2\left((0,T);H^1(\Omega)\right)$, one would have to use variational arguments in combination with the weak convergences of the approximations, as the measurability of the limit is only guaranteed in metrizable topologies.*

[1] In [GerstJün] uniqueness could only be shown for $H^1(\Omega)$-valued and thus more regular solutions. Since continuous dependence with respect to the parameters of the equations is a generalization of uniqueness, obtaining such a result might not be trivial.

Appendices

A. Some results in analysis and linear algebra

Lemma A.1. *For all* $a, b \in \mathbb{R}^n$ *the inequality*

$$|a+b|^2 \geq \frac{|a|^2}{2} - |b|^2 \tag{A.1}$$

holds.

Proof. Let $a, b \in \mathbb{R}^n$ be arbitrary. By using that

$$2a \cdot b = a \cdot (2b) \leq \frac{|a|^2}{2} + \frac{|2b|^2}{2} = \frac{|a|^2}{2} + 2|b|^2,$$

we easily derive that

$$|a+b|^2 + |b|^2 = |a|^2 + 2a \cdot b + 2|b|^2 \geq |a|^2 - \left(\frac{|a|^2}{2} + 2|b|^2\right) + 2|b|^2 = \frac{|a|^2}{2},$$

and thus (A.1) indeed holds.

□

Lemma A.2 (discrete Gronwall Lemma, [Clark])**.** *Let* $x(k), a(k), b(k) \in \mathbb{R},\ k \in \mathbb{N}_0$ *be three sequences of real numbers, with* $b(k) \geq 0$ *and* $a(k) \leq \bar{a}$ *for some* $\bar{a} \in \mathbb{R}$. *If the recursive inequality*

$$x(k) \leq a(k) + \sum_{j=0}^{k-1} b(j)x(j)$$

holds for all $k \in \mathbb{N}_0$, *then we know that*

$$x(k) \leq \bar{a} \prod_{j=0}^{k-1} (1 + b(j))$$

is satisfied for any $k \in \mathbb{N}$.

□

Lemma A.3. *Let* $O \subseteq \mathbb{R}^n$ *be an open, convex domain. A function* $\phi \in C^1(O)$, *is convex if and only if*

$$\phi(y) - \phi(x) \geq \nabla\phi(x) \cdot (y - x)$$

holds for all $x, y \in O$.

Proof. Assume that ϕ is convex, then for any $x, y \in O$ we get

$$\nabla\phi(x)\cdot(y-x) = \lim_{\lambda\searrow 0}\frac{\phi\left(x+\lambda(y-x)\right)-\phi(x)}{\lambda} = \lim_{\lambda\searrow 0}\frac{\phi\left((1-\lambda)x+\lambda y\right)-\phi(x)}{\lambda}$$
$$\overset{\text{convexity}}{\leq} \lim_{\lambda\searrow 0}\frac{\lambda\phi(y)+(1-\lambda)\phi(x)-\phi(x)}{\lambda} = \phi(y)-\phi(x).$$

Conversely, if the inequality holds for all elements of O, we obtain that for all $x, y \in O$ the function

$$s\in(0,1)\mapsto\frac{\phi(y)-\phi(x+s(y-x))}{1-s}$$

is monotonically increasing, as

$$\frac{\partial}{\partial s}\frac{\phi(y)-\phi(x+s(y-x))}{1-s} = \frac{(\phi(y)-\phi(x+s(y-x)))-(1-s)\nabla\phi\left(x+s(y-x)\right)\cdot(y-x)}{(1-s)^2}$$
$$= \frac{(\phi(y)-\phi(x+s(y-x)))-\nabla\phi\left(x+s(y-x)\right)\cdot(y-(x+s(y-x)))}{(1-s)^2} \geq 0.$$

Thus we get for every $\lambda\in(0,1)$

$$\phi(x+\lambda(y-x)) = \phi(y)-(1-\lambda)\frac{\phi(y)-\phi(x+\lambda(y-x))}{1-\lambda} \leq \phi(y)-(1-\lambda)\left(\phi(y)-\phi(x)\right)$$
$$= \phi(x)+\lambda\left(\phi(y)-\phi(x)\right).$$

This means that ϕ is indeed convex.

□

Lemma A.4. *For any fixed $a\in\mathcal{O}$ the map $\phi:\mathcal{O}\to\mathbb{R}$, defined by*

$$\phi(y) := \sum_{i=0}^{n}\int_{a_i}^{y_i}\log\frac{s}{a_i}\mathrm{d}s \textit{ for every } y\in\mathcal{O},$$

is convex.

Proof. It is easy to prove that the set $\mathcal{O}$ is convex, as for every $y^1, y^2 \in \mathcal{O}$ and every $\lambda\in(0,1)$ it holds that

$$\left(\lambda y^1+(1-\lambda)y^2\right)_0 = 1-\lambda\sum_{i=1}^{n}y_i^1-(1-\lambda)\sum_{i=1}^{n}y_i^2 = \lambda\left(1-\sum_{i=1}^{n}y^1\right)+(1-\lambda)\left(1-\sum_{i=1}^{n}y_i^2\right)$$
$$= \lambda y_0^1+(1-\lambda)y_0^2.$$

Now define for every $c\in(0,+\infty)$ the function $\phi_c:(0,+\infty)\to\mathbb{R}$ by

$$\phi_c(x) := \int_c^x\log\frac{s}{c}\mathrm{d}s \text{ for every } x\in(0,+\infty).$$

The Fundamental Theorem of Calculus tells us that the derivative of ϕ_c is given by $\phi_c'(x) = \log\frac{x}{c}$ and is thus a monotonically increasing function. Thus ϕ_c is convex. The convexity of ϕ now follows easily from the convexity of all $\phi_c, c > 0$, as

$$\begin{aligned}\phi\left(\lambda y^1 + (1-\lambda)y^2\right) &= \sum_{i=0}^{n} \phi_{a_i}\left(\lambda y_i^1 + (1-\lambda)y_i^2\right) \leq \sum_{i=0}^{n} \left(\lambda\ \phi_{a_i}\left(y_i^1\right) + (1-\lambda)\ \phi_{a_i}\left(y_i^2\right)\right) \\ &= \lambda\ \phi\left(y^1\right) + (1-\lambda)\ \phi\left(y^2\right)\end{aligned}$$

holds for every $y^1, y^2 \in \mathcal{O}$ and every $\lambda \in (0,1)$.

□

Lemma A.5. *Let $N \in \mathbb{N}$ and $\beta > 1$. The function $H_{N,\beta} : \mathbb{R} \to \mathbb{R}$, defined by*

$$H_{N,\beta}(z) := \begin{cases} 0, & \textit{for } z < 0 \\ z^\beta, & \textit{for } z \in [0, N] \\ \beta N^{\beta-1}(z-N) + N^\beta, & \textit{for } z > N \end{cases}$$

is continuously differentiable and monotonically increasing. The derivative is bounded, i.e. it holds that $H'_{N,\beta} \in L^\infty(\mathbb{R})$. $\left|H'_{N,\beta}\right|^2$ is also a monotonically increasing function. Furthermore for every $z \in \mathbb{R}$, both $H_{N,\beta}(z)$ and $\left|H'_{N,\beta}(z)\right|^2$ are monotonically increasing in the parameter N.

Proof. Since $H_{N,\beta}$ is a piecewise smooth function, its derivative is given by

$$H'_{N,\beta}(z) := \beta \mathbb{1}_{(0,+\infty)}(z) \min\{z, N\}^{\beta-1}.$$

We immediately see that both $H_{N,\beta}$ and $H'_{N,\beta}$ are continuous, which lets us conclude that $H_{N,\beta} \in C^1(\mathbb{R})$, that $H'_{N,\beta} \geq 0$, which shows that $H_{N,\beta}$ is a monotonically increasing function, and that $H'_{N,\beta} \in L^\infty(\mathbb{R})$.

Furthermore one can verify immediatley that $\left|H'_{N,\beta}\right|^2$ is a monotonically increasing function too. Thus it only remains to show that both $H_{N,\beta}(z)$ and $\left|H'_{N,\beta}(z)\right|^2$ are monotonically increasing in $N \in \mathbb{N}$ for all fixed but arbitrary $z \in \mathbb{R}$.

It is obvious by just looking at the definition of $H'_{N,\beta}$, that for all $z \in \mathbb{R}$ the series $\left(H'_{N,\beta}(z)\right)_{N\in\mathbb{N}}$ is monotonically increasing. Since $H'_{N,\beta}$ is positive, this also implies that $\left|H'_{N,\beta}(z)\right|^2$ increases in the parameter N. As for $z \leq 0$ it holds that $H_{N,\beta}(z) = 0 = H_{N+1,\beta}(z)$, the monotonicity of $H'_{N,\beta}$ with respect to N lets us also conclude that

$$H_{N,\beta}(z) = \int_0^z H'_{N,\beta}(z)\mathrm{d}z \leq \int_0^z H'_{N+1,\beta}(z)\mathrm{d}z = H_{N+1,\beta}(z)$$

for all $z \in \mathbb{R}$, which finally shows that $H_{N,\beta}(z)$ is also monotonically increasing in N.

□

Lemma A.6. *Let $\chi \in (1, +\infty)$ and $a \in (0,1)$ be given. The infinite product*

$$(a; 1/\chi)_\infty := \prod_{k=1}^{\infty} \left(1 - a\chi^{-k}\right)$$

converges and $(a; 1/\chi)_\infty \in (0,1)$

Proof. From the fact that $a\chi^{-k} \in (0,1)$ for all $k \in \mathbb{N}$ we follow $1 - a\chi^{-k} \in (0,1)$ and thus we can derive from convergence by monotonicity that $(a; 1/\chi)_\infty \in [0,1)$.

As there exists a $k_0 \in \mathbb{N}$ such that $k^2 \leq a\chi^k$ for all $k \geq k_0$, we get

$$\begin{aligned}(a; 1/\chi)_\infty &= \prod_{k=1}^{k_0} \left(1 - a\chi^{-k}\right) \prod_{k=k_0+1}^{\infty} \left(1 - a\chi^{-k}\right) \geq \prod_{k=1}^{k_0} \left(1 - a\chi^{-k}\right) \prod_{k=k_0+1}^{\infty} \left(1 - \frac{1}{k^2}\right) \\ &= \frac{\prod_{k=1}^{k_0} \left(1 - a\chi^{-k}\right)}{\prod_{k=2}^{k_0} \left(1 - \frac{1}{k^2}\right)} \prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right).\end{aligned}$$

Due to

$$\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \prod_{k=2}^{\infty} \frac{k^2-1}{k^2} = \prod_{k=2}^{\infty} \frac{(k-1)(k+1)}{k^2} = \underbrace{\prod_{k=2}^{\infty} \frac{k-1}{k}}_{=1} \underbrace{\prod_{k=2}^{\infty} \frac{k+1}{k}}_{=\frac{1}{2}} = \frac{1}{2},$$

this lets us conclude that

$$(a; 1/\chi)_\infty \geq \frac{1}{2} \frac{\prod_{k=1}^{k_0} \left(1 - a\chi^{-k}\right)}{\prod_{k=2}^{k_0} \left(1 - \frac{1}{k^2}\right)} > 0.$$

□

Remark A.7. *For any $k \in \mathbb{N} \cup \{+\infty\}$ the term $(a;q)_k$ denotes the so called q-Pochhammmer symbol, see [Andrews], and is formally defined by a the q-series*

$$(a;q)_k := \prod_{j=0}^{k-1} (1 - aq^j).$$

q-series arise in different applications in combinatorics and even mathematical physics.

Lemma A.8. *Let X, Y be two Banach-spaces and let $\mathcal{S} : X \to Y$ be a continuous linear operator. Then $\mathcal{S}$ translates weak convergence into weak convergence, i.e. for any series $x^k \in X,\ k \in \mathbb{N}$ with $x^k \xrightharpoonup{k\to\infty} x$ in X, one has $\mathcal{S}(x^k) \xrightharpoonup{k\to\infty} \mathcal{S}(x)$ in Y,*

Proof. Let $y' \in Y'$ be arbitrary and denote the conjugate operator of $\mathcal{S}$ by $\mathcal{S}'$. Note that the existence of $\mathcal{S}'$ is guaranteed by the continuity of $\mathcal{S}$. Then we know that

$${}_{Y'}\langle y', \mathcal{S}(x^k)\rangle_Y = {}_{X'}\langle \mathcal{S}'(y'), x^k\rangle_X \xrightarrow{k\to\infty} {}_{X'}\langle \mathcal{S}'(y'), x^k\rangle_X = {}_{Y'}\langle y', \mathcal{S}(x)\rangle_Y.$$

Since y' was arbitrary, this indeed shows the claim.

□

B. Sobolev spaces

Theorem B.1 ([Kuso, Satz 13.22]). *For any measurable function $v : \Omega \to \mathbb{R}$ it holds that*

$$\lim_{p \to +\infty} \|v\|_{L^p(\Omega)} = \|v\|_{L^\infty(\Omega)} .$$

□

Lemma B.2 ([Kuso, Satz 13.25]). *Let $p \in [1, +\infty)$ and $v^k, v \in L^p(\Omega; \mathbb{R}^d)$, $k \in \mathbb{N}$. Then $v^k \xrightarrow{k\to\infty} v$ in $L^p(\Omega; \mathbb{R}^d)$ holds, if and only if $v^k \xrightarrow{k\to\infty} v$ in measure and $\lim_{k\to\infty} \left\|v^k\right\|_{L^p(\Omega)} = \|v\|_{L^p(\Omega)}$.*

□

Lemma B.3. *Weak convergence in $H^1(\Omega)$ implies weak convergence of the gradient in $L^2(\Omega; \mathbb{R}^d)$, i.e. for any sequence $v^k, v \in H^1(\Omega)$, $k \in \mathbb{N}$*

$$v^k \xrightharpoonup{k\to\infty} v \text{ in } H^1(\Omega) \text{ implies } \nabla v^k \xrightharpoonup{k\to\infty} \nabla v \text{ in } L^2(\Omega; \mathbb{R}^d)$$

Proof. Let $v^k, v \in H^1(\Omega)$, $k \in \mathbb{N}$. By the definition of the $H^1(\Omega)$-norm, the gradient is a bounded linear operator from $H^1(\Omega)$ to $L^2(\Omega; \mathbb{R}^d)$. Denote by $\nabla^* : L^2(\Omega; \mathbb{R}^d) \to H^1(\Omega)$ its adjoint. Then we have due to $v^k \xrightharpoonup{k\to\infty} v$ in $H^1(\Omega)$, that for every $\xi \in L^2(\Omega; \mathbb{R}^d)$

$$\left\langle \nabla v^k, \xi \right\rangle_{L^2(\Omega)} = \left\langle v^k, \nabla^* \xi \right\rangle_{H^1(\Omega)} \xrightarrow{k\to\infty} \langle v, \nabla^* \xi \rangle_{H^1(\Omega)} = \langle \nabla v, \xi \rangle_{L^2(\Omega)} ,$$

which concludes the proof.

□

Corollary B.4. *Let $w^k, w \in H^1(\Omega)$, $k \in \mathbb{N}$ with $w^k \xrightharpoonup{k\to\infty} w$ in $H^1(\Omega)$.*
Let $v^k, v \in L^\infty(\Omega)$, $k \in \mathbb{N}$ with $v^k \xrightarrow{k\to\infty} v$ in $L^2(\Omega)$ and such that there exists a constant $C > 0$ with $\left\|v^k\right\|_{L^\infty(\Omega)} \leq C$ for all $k \in \mathbb{N}$.
Then we have that

$$v^k \nabla w^k \rightharpoonup v \nabla w \text{ in } L^2(\Omega; \mathbb{R}^d)$$

Proof. Let $\xi \in L^2(\Omega; \mathbb{R}^d)$ be arbitrary. First we note that $L^2(\Omega; \mathbb{R}^d)$-convergence implies the convergence in measure and second we deduce from $\left\|v^k\right\|_{L^\infty(\Omega)} \leq C$, by using convergence by domination, that

$$\lim_{k\to\infty} \left\|v^k \xi\right\|_{L^2(\Omega)} = \|v\xi\|_{L^2(\Omega)}$$

Thus we derive from Lemma B.2 that $v^k\xi \to v\xi$ in $L^2(\Omega;\mathbb{R}^d)$. From Lemma B.3 we also know that $\nabla w^k \rightharpoonup \nabla w$ in $L^2(\Omega;\mathbb{R}^d)$. Since a scalar product between a weak and a strongly convergent sequence converges against the product of the limits, we finally derive that

$$\left\langle v^k \nabla w^k, \xi \right\rangle_{L^2(\Omega)} = \left\langle \nabla w^k, v^k\xi \right\rangle_{L^2(\Omega)} \xrightarrow{k\to\infty} \langle \nabla w, v\xi\rangle_{L^2(\Omega)} = \langle v\nabla w, \xi\rangle_{L^2(\Omega)}.$$

□

Theorem B.5 (Poincare inequality)**.** *There exists a constant $C_P > 0$ that only depends on Ω and Γ_D such that*

$$\|v\|_{L^2(\Omega)} \le C_P \|\nabla v\|_{L^2(\Omega)} \quad \text{for all } v \in H^1_D(\Omega). \tag{B.1}$$

The constant C_P is called the Poincare constant.

□

Theorem B.6 (generalized Poincare inequality)**.** *Let $m \in \mathbb{N}$ be arbitrary. There exists a constant $C_{P,m} > 0$ that only depends on Ω and m such that*

$$\|v\|^2_{H^m(\Omega)} \le C_{P,m} \left(\sum_{\alpha\in\mathbb{N}^m:\ |\alpha|=m} \|D^\alpha v\|^2_{L^2(\Omega)} \|\nabla v\|^2_{L^2(\Omega)} \right) \quad \text{for all } v \in H^1_D(\Omega). \tag{B.2}$$

We call $C_{P,m}$ the generalized Poincare constant[1].

□

Theorem B.7 (Sobolev embedding, [GilbTrud, Theorem 7.26])**.** *Let $k \in \mathbb{N}_0$ and $p \in [1,+\infty)$. If $kp < d$, the space $W^{k,p}(\Omega)$ is continuously embedded into $L^r(\Omega)$ for all $r \in \left[1, \frac{dp}{d-kp}\right]$. The embedding is even compact when $r < \frac{dp}{d-kp}$.*
If $d \in \{1,2\}$, then $H^1(\Omega)$ is compactly embedded into $L^r(\Omega)$ for all $r \in [1,+\infty)$. For $d = 1$ this is even the case for $r = +\infty$.

□

Lemma B.8. *Let $p, r \in [1,+\infty]$ with $p \ge r$. Then $L^p(\Omega)$ is continuously embedded into $L^r(\Omega)$.*

Proof. This is a direct consequence of the Hölder-inequality, as we know that for $q := \frac{p}{r} \ge 1$ and $q' \in [1,+\infty]$ with $\frac{1}{q} + \frac{1}{q'} = 1$

$$\|v\|_{L^r(\Omega)} = \left(\int_\Omega |v|^r \ \mathrm{d}x\right)^{1/r} \le \left(\|1\|_{L^{q'}(\Omega)} \||v|^r\|_{L^q(\Omega)}\right)^{1/r} = |\Omega|^{1/(q'r)} \|v\|_{L^p(\Omega)}.$$

□

[1] The name generalized Poincare constant is used, because Theorem B.5 is based on a generalization of the Poincare inequality. Admittedly this is not the best naming convention.

Corollary B.9. *For any $r \in [1,+\infty)$ the embedding from $L^\infty(\Omega)$ into $L^r(\Omega)$ translates weak star convergence into weak convergence. In other words for any $v_k, v \in L^\infty(\Omega),\ k \in \mathbb{N}$ with $v_k \xrightarrow[k\to\infty]{*} v$ in $L^\infty(\Omega)$, meaning that*

$$\lim_{k\to\infty} \int_\Omega v_k\ w\ \mathrm{d}x = \int_\Omega v\ w\ \mathrm{d}x \text{ for every } w \in L^1(\Omega),$$

it holds that $v_k \xrightharpoonup{k\to\infty} v$ in $L^r(\Omega)$.

Proof. Denote by $r' \in (1,+\infty]$ the Hölder conjugate of r, meaning that $\frac{1}{r} + \frac{1}{r'} = 1$. For every $w \in L^{r'}(\Omega)$ we know form Lemma B.8 that $w \in L^1(\Omega)$, and thus

$$\lim_{k\to\infty} \int_\Omega v_k\ w\ \mathrm{d}x = \int_\Omega v\ w\ \mathrm{d}x.$$

Since w was arbitrary we can therefore conclude $v_k \xrightharpoonup{k\to\infty} v$ in $L^r(\Omega)$.

□

Lemma B.10. *Let $(v_k)_{k\in\mathbb{N}}$ be a sequence of non-negative $L^\infty(\Omega)$-functions. Assume that $v_k \xrightarrow[k\to\infty]{*} v$ in $L^\infty(\Omega)$. Then v is also non-negative, i.e. $v \geq 0$ almost everywhere on Ω*[2].

Proof. Since Ω is a bounded domain, the indicator $\mathbb{1}_{[v<0]}$ is in $L^1(\Omega)$ and is thus an admissible test function. From the non-negativity of v_k and the weak star convergence $v_k \xrightarrow[k\to\infty]{*} v$ we thus derive

$$0 \leq \lim_{k\to\infty} \int_\Omega v_k\ \mathbb{1}_{[v<0]}\ \mathrm{d}x = \int_{[v<0]} v\ \mathrm{d}x.$$

This can only hold if $v \geq 0$ almost everywhere.

□

Theorem B.11 (Chain rule)**.** *Let $n, m \in \mathbb{N}$ and $F \in C^1(\mathbb{R}^n;\mathbb{R}^m)$ with $DF \in L^\infty(\mathbb{R}^n;\mathbb{R}^m)$. Then for $p \in [1,+\infty)$ and every $v \in W^{1,p}(\Omega;\mathbb{R}^n)$ it holds that*

$$F(v) \in W^{1,p}(\Omega;\mathbb{R}^m) \text{ with } DF(v) = [DF](v)Dv. \tag{B.3}$$

Proof. Let $v \in W^{1,k}(\Omega;\mathbb{R}^n)$ and $v_k \in C^1(\bar{\Omega};\mathbb{R}^n),\ k \in \mathbb{N}$ be an approximating sequence, i.e. $\|v - v_k\|_{W^{1,p}} \xrightarrow{k\to\infty} 0$. Clearly (B.3) holds for all $v_k,\ k \in \mathbb{N}$.

First we note that

$$\begin{aligned}\|F(v)\|_{L^p(\Omega)} &\overset{\text{Taylor}}{=} \left\| F(0) + \int_0^1 [DF](tv)v\mathrm{d}t \right\|_{L^p(\Omega)} \\ &\leq \|F(0)\|_{L^p(\Omega)} + \int_0^1 \underbrace{\|[DF](tv)\|_{L^\infty}}_{\leq \|DF\|_{L^\infty(\Omega)}} \|v\|_{L^p(\Omega)}\,\mathrm{d}t < +\infty,\end{aligned}$$

[2]With respect to the Lebesgue measure.

and thus $F(v) \in L^p(\Omega;\mathbb{R}^m)$.

Similarly we get

$$\begin{aligned}\|F(v)-F(v_k)\|_{L^p(\Omega)} &= \left\|\int_0^1 [DF]\left((1-t)v+tv_k\right)(v_k-v)\mathrm{d}t\right\|_{L^p(\Omega)} \\ &\leq \|DF\|_{L^\infty(\Omega)}\|v-v_k\|_{L^p(\Omega)},\end{aligned}$$

and thus $F(v_k)\xrightarrow{k\to\infty}F(v)$ in $L^p(\Omega;\mathbb{R}^m)$.

Let $\xi \in C_{00}^\infty(\Omega;\mathbb{R}^m)$ be a test function. Due to the continuity of DF and since $v_k \xrightarrow{k\to\infty} v$ in $L^p(\Omega;\mathbb{R}^n)$, we get that a subsequence of $[DF](v_k)$ converges to $[DF](v)$ almost everywhere. Without loss of generality we assume that this actually holds for the whole sequence and thus we get, by convergence by domination, that for any $i=1,\cdots,d$

$$\int_\Omega \xi\cdot[DF](v_k)\frac{\partial v}{\partial x_i}\mathrm{d}x \xrightarrow{k\to\infty} \int_\Omega \xi\cdot[DF](v)\frac{\partial v}{\partial x_i}\mathrm{d}x.$$

From this we derive that by definition of the weak derivative

$$\begin{aligned}&\left|\left\langle\frac{\partial F(v)}{\partial x_i}-[DF](v)\frac{\partial v}{\partial x_i},\xi\right\rangle\right| = \left|\int_\Omega \xi\cdot\frac{\partial F(v)}{\partial x_i}\mathrm{d}x-\int_\Omega\xi\cdot[DF](v)\frac{\partial v}{\partial x_i}\mathrm{d}x\right| \\ &= \left|-\int_\Omega\frac{\partial\xi}{\partial x_i}\cdot F(v)\mathrm{d}x-\int_\Omega\xi\cdot[DF](v)\frac{\partial v}{\partial x_i}\mathrm{d}x\right| \\ &= \lim_{k\to\infty}\left|-\int_\Omega\frac{\partial\xi}{\partial x_i}\cdot F(v_k)\mathrm{d}x-\int_\Omega\xi\cdot[DF](v)\frac{\partial v}{\partial x_i}\mathrm{d}x\right| \\ &= \lim_{k\to\infty}\left|\int_\Omega\xi\cdot\frac{\partial F(v_k)}{\partial x_i}\mathrm{d}x-\int_\Omega\xi\cdot[DF](v)\frac{\partial v}{\partial x_i}\mathrm{d}x\right| \\ &= \lim_{k\to\infty}\left|\int_\Omega\xi\cdot[DF](v_k)\frac{\partial v_k}{\partial x_i}\mathrm{d}x-\int_\Omega\xi\cdot[DF](v)\frac{\partial v}{\partial x_i}\mathrm{d}x\right| \\ &\leq \lim_{k\to\infty}\left|\int_\Omega\xi\cdot[DF](v_k)\left(\frac{\partial v_k}{\partial x_i}-\frac{\partial v_k}{\partial x_i}\right)\mathrm{d}x\right|+\left|\int_\Omega\xi\cdot\left([DF](v_k)-[DF](v)\right)\frac{\partial v}{\partial x_i}\mathrm{d}x\right| \\ &\leq \lim_{k\to\infty}\|DF\|_{L^\infty(\mathbb{R}^n)}\|\xi\|_{L^q(\Omega)}\underbrace{\|v-v_k\|_{W^{1,p}(\Omega)}}_{\to 0}+\left|\int_\Omega\xi\cdot\left([DF](v_k)-[DF](v)\right)\frac{\partial v}{\partial x_i}\mathrm{d}x\right| = 0\end{aligned}$$

Therefore we can indeed conclude that (B.3) holds.

□

Corollary B.12. *Let $p\in[1,+\infty)$. For all $v_1,v_2\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ it holds that*

$$v_1v_2\in W^{1,p}(\Omega) \text{ with } \nabla(v_1v_2)=v_2\nabla v_1+v_1\nabla v_2.$$

Proof. Let $C:=\max\left\{\|v_1\|_{L^\infty(\Omega)},\|v_1\|_{L^\infty(\Omega)}\right\}$. Then let $\phi\in C_{00}^\infty(\mathbb{R};\mathbb{R})$ be such, that

$$\phi(x)=x \text{ for } x\in[-C,C].$$

With this we define the function prod : $\mathbb{R}^2 \to \mathbb{R}$ by

$$\text{prod}(x,y) := \phi(x)\phi(y) \text{ for } x, y \in \mathbb{R}.$$

We note that

$$\nabla\text{prod}(x,y) = \begin{pmatrix} \phi'(x)\phi(y) \\ \phi(x)\phi'(y) \end{pmatrix} \text{ for any } x, y \in \mathbb{R},$$

and therefore $\nabla\text{prod} \in L^\infty(\mathbb{R}^2, \mathbb{R}^2)$.

Since $\phi(v_i) = v_i$ and $\phi'(v_i) = 1$ for $i = 1, 2$ we can use Theorem B.11 to conclude

$$v_1 v_2 = \text{prod}(v_1, v_2) \in W^{1,p}(\Omega)$$
$$\nabla(v_1 v_2) = \nabla\text{prod}(v_1, v_2) = v_2 \nabla v_1 + v_1 \nabla v_2.$$

□

Corollary B.13. *Let $n, m \in \mathbb{N}$ and $\iota \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ with $D\iota \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Then for any $v \in H^1(\Omega; \mathbb{R}^n)$ it holds that*

$$\iota(v)|_{\Gamma_D} = \iota\left(v|_{\Gamma_D}\right)$$

Proof. Let $v^k \in C^\infty(\bar{\Omega}; \mathbb{R}^n)$, $k \in \mathbb{N}$ be chosen in such a way that $\|v - v^k\|_{H^1(\Omega)} \overset{k\to\infty}{\longrightarrow} 0$. From the proof of theorem B.11 we know that

$$\iota(v^k) \xrightarrow{k\to\infty} \iota(v) \text{ in } H^1(\Omega; \mathbb{R}^m).$$

By definition, the trace operator $\cdot|_{\Gamma_D}$ is a bounded and linear operator from $H^1(\Omega; \mathbb{R}^m)$ to $L^2(\Gamma_D; \mathbb{R}^m)$. Denote by $\mathcal{S}$ its adjoint. Then we have for any $\xi \in L^2(\Gamma_D; \mathbb{R}^m)$

$$\left\langle \iota\left(v^k\right)\Big|_{\Gamma_D}, \xi \right\rangle_{L^2(\Gamma_D)} = \left\langle v^k, \mathcal{S}(\xi)\right\rangle_{L^2(\Gamma_D)} \overset{k\to\infty}{\longrightarrow} \langle v, \mathcal{S}(\xi)\rangle_{L^2(\Gamma_D)} = \left\langle \iota(v)|_{\Gamma_D}, \xi \right\rangle_{L^2(\Gamma_D)}$$

and thus $\iota\left(v^k\right)\big|_{\Gamma_D} \xrightarrow{k\to\infty} \iota(v)|_{\Gamma_D}$ in $L^2(\Gamma_D; \mathbb{R}^m)$.

Due to the boundedness of the trace operator there exists a constant $C > 0$, such that

$$\begin{aligned} \left\| \iota\left(v|_{\Gamma_D}\right) - \iota\left(v^k\Big|_{\Gamma_D}\right) \right\|_{L^2(\Gamma_D)} &\leq \|D\iota\|_{L^\infty(\mathbb{R}^n)} \left\| v|_{\Gamma_D} - v^k\Big|_{\Gamma_D} \right\|_{L^2(\Gamma_D)} \\ &\leq C \|D\iota\|_{L^\infty(\mathbb{R}^n)} \left\| v - v^k \right\|_{H^1(\Omega)} \overset{k\to\infty}{\longrightarrow} 0 \end{aligned}$$

Thus we derive that for every $\xi \in L^2(\Gamma_D; \mathbb{R}^m)$

$$\begin{aligned} \left\langle \iota(v)|_{\Gamma_D}, \xi \right\rangle_{L^2(\Gamma_D)} &= \lim_{k\to\infty} \left\langle \iota\left(v^k\right)\Big|_{\Gamma_D}, \xi \right\rangle_{L^2(\Gamma_D)} = \lim_{k\to\infty} \left\langle \iota\left(v^k\Big|_{\Gamma_D}\right), \xi \right\rangle_{L^2(\Gamma_D)} \\ &= \left\langle \iota\left(v|_{\Gamma_D}\right), \xi \right\rangle_{L^2(\Gamma_D)}, \end{aligned}$$

which implies $F(v)|_{\Gamma_D} = F\left(v|_{\Gamma_D}\right)$. □

Lemma B.14 (Stampacchia, [Jüngel1, Lemma 2.14]). *For $p \in [1,+\infty)$ and every $v \in W^{1,p}(\Omega)$ it holds that and*

$$v^+ \in W^{1,p}(\Omega) \text{ with } \nabla v^+ = \mathbb{1}_{[v>0]} \nabla v.$$

□

Lemma B.15 (piecewise constant Aubin–Lions Lemma, [DrehJün, Theorem 1]). *Let X, Y, Z be three nested Banach spaces, such that the embedding $X \hookrightarrow Y$ is compact and the embedding $Y \hookrightarrow Z$ is continuous, and let $p \in [1,+\infty)$.*
Any sequence $(v^\tau)_{\tau>0}$ of functions in $L^p((0,T);X)$, such that for each τ the function v^τ is piecewise constant on the uniform grid of stepsize τ, that suffice to the bound

$$\frac{\|v^\tau(\cdot+\tau) - v^\tau\|_{L^1((0,T-\tau);Z)}}{\tau} + \|v^\tau\|_{L^p((0,T);X)} \le C,$$

for some constant $C > 0$, independent of τ, is relatively compact in $L^p((0,T);Y)$.

□

Lemma B.16 (degenerate Aubin–Lions Lemma, [Jüngel2, Lemma 13]).
Let for every $\tau > 0$ the terms $a^\tau, b^\tau \in L^\infty((0,T);L^\infty(\Omega))$ be two piecewise constant functions on a uniform grid of stepsize τ. Assume that both sequences $(a^\tau)_{\tau>0}$, $(b^\tau)_{\tau>0}$ are bounded[3], which implies by Banach–Alaoglu that there exists some $b \in L^\infty((0,T);L^\infty(\Omega))$ such that, for a subsequence that is not relabled, $b^\tau \xrightarrow[\tau\to 0]{*} b$.
Assume furthermore that the sequence $(a^\tau)_{\tau>0}$ is relatively compact in $L^2\big((0,T);L^2(\Omega)\big)$, i.e. up to subsequences, which are not relabled, $a^\tau \xrightarrow{\tau\to 0} a$ in $L^2\big((0,T);L^2(\Omega)\big)$. Finally we assume that there exists a constant $C > 0$ such that

$$\|a^\tau\|_{L^2((0,T);H^1(\Omega))} + \|a^\tau b^\tau\|_{L^2((0,T);H^1(\Omega))} + \frac{\|b^\tau(\cdot+\tau) - b^\tau\|_{L^2\left((0,T-\tau);H_D^{-1}(\Omega)\right)}}{\tau} \le C$$

for every $\tau > 0$. Then there exists a subsequence, which we do not relable, such that

$$a^\tau b^\tau \xrightarrow{\tau\to 0} ab \text{ in } L^p((0,T);L^p(\Omega)) \text{ for every } p \in [1,+\infty).$$

□

[3] Bounded with respect to the $L^\infty((0,T);L^\infty(\Omega))$-norm

C. Linear elliptic PDEs

Theorem C.1 ([Praet, Proposition 2.18])**.** *Let $f \in L^2(\Omega), g \in L^2(\Gamma_N), \phi^D \in H^1(\Omega)$ be given. Then there exists a unique weak solution $\phi \in H^1(\Omega)$ of the Poisson-problem*

$$\begin{aligned} -\Delta\phi &= f \ \text{ on } \Omega \\ \phi &= \phi^D \ \text{ on } \Gamma_D, \\ \frac{\partial\phi}{\partial\mathbf{n}} &= 0 \ \text{ on } \Gamma_N. \end{aligned} \tag{C.1}$$

This means there exists a unique $\phi \in H^1(\Omega)$, such that for all $v \in H^1_D(\Omega)$

$$\langle \nabla\phi, \nabla v\rangle_{L^2(\Omega)} = \langle f, v\rangle_{L^2(\Omega)}.$$

Additionally there exists a constant $C(\Omega, \Gamma_D) > 0$, that only depends on Ω and Γ_D, such that

$$\|\phi\|_{H^1(\Omega)} \leq C(\Omega, \Gamma_D)\left(\|f\|_{L^2(\Omega)} + \left\|\phi^D\right\|_{H^1(\Omega)}\right)$$

□

Lemma C.2. *Let $p \in \left(\max\left\{\frac{d}{2}, 2\right\}, +\infty\right]$, $f \in L^p(\Omega)$ and $\phi^D \in H^1(\Omega) \cap L^\infty(\Gamma_D)$. Let $\phi \in H^1(\Omega)$ be a weak subsolution (supersolution) of (C.1), meaning that*

$$\langle \nabla\phi, \nabla v\rangle_{L^2(\Omega)} \leq (\geq) \langle f, v\rangle_{L^2(\Omega)} \ \text{ for all } v \in H^1_D(\Omega) \text{ with } v \geq 0$$

and

$$\phi \leq (\geq)\phi^D \ \text{ on } \Gamma_D,$$

then there exist constants $C > 0$, $\alpha > 0$, $\gamma \in (0,1)$, that only depends ond d, Ω, Γ_D and p, such that

$$\sup_\Omega \phi(-\phi) \leq C\,\|f\|^\alpha_{L^p(\Omega)}\,\|\nabla\phi\|^\gamma_{L^2(\Omega)} + \left\|\phi^D\right\|_{L^\infty(\Omega)}$$

Proof. For the proof we employ techniques found in [GilbTrud, Theorem 8.15]. We assume that ϕ is a subsolution (the bound for the supersolution then directly follows from this case) and split the proof up into five steps. In the first step we find a suitable $r \in [1, +\infty)$, that will be needed in the fourth step. The second step serves to introduce two test functions that are used in the third step to derive a recursive inequality. This inequality is then used in the fourth and final step to derive the wanted inequality. Hereby we employ an argument that was first introduced by Jürgen Moser.

Step 1: Some calculations for required Sobolev embeddings.

Let $q \in [1, +\infty]$ be the Hölder conjugate of p, i.e. $q = \frac{p-1}{p}$, which implies $\frac{1}{p} + \frac{1}{q} = 1$. We show that there exists a $r \in [1, +\infty)$ with $r > 2q$, such that $H^1(\Omega)$ can be continuously embedded into $L^r(\Omega)$.

In the cases $d = 1$ and $d = 2$ this is clear, as $p > 1 \geq \frac{d}{2}$ implies $q \in [1, +\infty)$ and the Sobolev embedding Theorem B.7 tells us that $H^1(\Omega)$ can be continuously embedded into all $L^r(\Omega)$ with $r \in [1, +\infty)$.

In the case $d \geq 3$ the inequality $p > \frac{d}{2} = \frac{d-2}{2} + 1$ implies $p - 1 > \frac{d-2}{2}$ and thus

$$2q = 2\frac{p}{p-1} = 2\left(1 + \frac{1}{p-1}\right) < 2\left(1 + \frac{2}{d-2}\right) = \frac{2d}{d-2}.$$

By the Sobolev embedding Theorem B.7 there exists an $r > 2q$ such that $H^1(\Omega)$ can be continuously embedded into $L^r(\Omega)$.

Step 2: Definition of suitable test functions.

Let $l := \|\phi^D\|_{L^\infty(\Gamma_D)}$ and define $w := (\phi - l)^+$. Due to $\phi = \phi^D$ on Γ_D and the Stampacchia Lemma B.14 we get $w \in H^1_D(\Omega)$.

For every $N \in \mathbb{N}$ and every $\beta > 1$ we denote by $H_{N,\beta}$ the function studied in Lemma A.5 and define $G_{N,\beta} : \mathbb{R} \to \mathbb{R}$ by

$$G_{N,\beta}(z) := \int_0^z |H'(s)|^2 \, ds \text{ for all } z \in \mathbb{R}$$

It is easy to see that since $H_{N,\beta} \in C^1(\mathbb{R})$ with $H'_{N,\beta} \in L^\infty(\mathbb{R})$ (see Lemma A.5) it also holds that $G_{N,\beta} \in C^1(\mathbb{R})$, $G'_{N,\beta} \in L^\infty(\mathbb{R})$ and thus the chain rule (Theorem B.11) gives us $H_{N,\beta}(w), G_{N,\beta}(w) \in H^1(\Omega)$. Since $H_{N,\beta}(0) = G_{N,\beta}(0) = 0$, it actually holds that

$$H_{N,\beta}(w), G_{N,\beta}(w) \in H^1_D(\Omega). \tag{C.2}$$

Since $|H'_{N,\beta}|^2$ is a monotonically increasing function (see Lemma A.5) we can derive directly from the definition of $G_{N,\beta}$, that

$$0 \leq G_{N,\beta}(w) = \int_0^w |H'(s)|^2 \, ds \leq |H'_{N,\beta}(w)|^2 w. \tag{C.3}$$

Step 3: Deriving the recursive inequality.

Let $r \in [1, +\infty)$ be chosen as in step 1. Due to (C.2), the Sobolev embedding Theorem B.7 and the Poincare inequality (Theorem B.5) there exists a positive constant $C_1 > 0$, that only depend on d, Ω and Γ_D, such that

$$\|H_{N,\beta}(w)\|^2_{L^r(\Omega)} \leq C_1 \|\nabla H_{N,\beta}(w)\|^2_{L^2(\Omega)}.$$

Using the chain rule (Theorem B.11) we get

$$\begin{aligned}\|H_{N,\beta}(w)\|_{L^r(\Omega)}^2 &\le C_1 \|\nabla H_{N,\beta}(w)\|_{L^2(\Omega)}^2 = C_1 \left\|H'_{N,\beta}(w)\nabla w\right\|_{L^2(\Omega)}^2 . \\ &= C_1 \int_\Omega |\nabla w|^2 \underbrace{\left|H'(w)\right|^2}_{=G'_{N,\beta}(w)} \mathrm{d}x = C_1 \int_\Omega \nabla w \cdot \underbrace{\left(G'_{N,\beta}(w)\nabla w\right)}_{=\nabla G_{N,\beta}(w)} \mathrm{d}x \\ &= C_1 \left\langle \nabla w, \nabla G_{N,\beta}(w)\right\rangle_{L^2(\Omega)} .\end{aligned}$$

Since $w = (\phi - l)^+$ we can use the chain rule (Theorem B.11) and the Stampacchia Lemma B.14 to derive

$$\mathbb{1}_{[\phi>l]}\nabla G_{N,\beta}(w) = G'_{N,\beta}(w)\mathbb{1}_{[\phi>l]}\nabla w = G'_{N,\beta}(w)\nabla w = \nabla G_{N,\beta}(w),$$

which implies

$$\begin{aligned}\|H_{N,\beta}(w)\|_{L^r(\Omega)}^2 &\le C_1 \left\langle \nabla w, \nabla G_{N,\beta}(w)\right\rangle_{L^2(\Omega)} = C_1 \langle \underbrace{\nabla(\phi-l)^+}_{=\mathbb{1}_{[\phi>l]}\nabla\phi}, \nabla G_{N,\beta}(w)\rangle_{L^2(\Omega)} \\ &= C_1 \left\langle \nabla\phi, \mathbb{1}_{[\phi>l]}\nabla G_{N,\beta}(w)\right\rangle_{L^2(\Omega)} = C_1 \left\langle \nabla\phi, \nabla G_{N,\beta}(w)\right\rangle_{L^2(\Omega)}\end{aligned}$$

As ϕ is a weak subsolution of (C.1) and since (C.2) implies that $G_{N,\beta}(w) \ge 0$ is an admissible test function we derive

$$\|H_{N,\beta}(w)\|_{L^r(\Omega)}^2 \le C_1 \left\langle \nabla\phi, \nabla G_{N,\beta}(w)\right\rangle_{L^2(\Omega)} \le C_1 \left\langle f, G_{N,\beta}(w)\right\rangle_{L^2(\Omega)} .$$

Using the Cauchy–Schwarz inequality and inequality (C.3) gives us

$$\begin{aligned}\|H_{N,\beta}(w)\|_{L^r(\Omega)} &\le \sqrt{C_1 \left\langle f, G_{N,\beta}(w)\right\rangle_{L^2(\Omega)}} \le \underbrace{\sqrt{C_1 \|f\|_{L^p(\Omega)}}}_{:=C_2} \|G_{N,\beta}(w)\|_{L^q(\Omega)}^{\frac{1}{2}} . \\ &\overset{(C.3)}{\le} C_2 \left\||H'_{N,\beta}(w)|^2 w\right\|_{L^q(\Omega)}^{\frac{1}{2}} .\end{aligned} \tag{C.4}$$

As both $H_{N,\beta}$ and $|H'_{N,\beta}|^2$ are monotonically increasing in the parameter N (see Lemma A.5), we can use the convergence by monotonicity to take the limit $N \to +\infty$ in inequality (C.4) and by that obtain

$$\begin{aligned}\left\|w^\beta\right\|_{L^r(\Omega)} &= \lim_{N\to+\infty} \|H_{N,\beta}(w)\|_{L^r(\Omega)} \le C_2 \lim_{N\to+\infty} \left\||H'_{N,\beta}(w)|^2 w\right\|_{L^q(\Omega)}^{\frac{1}{2}} \\ &= C_2 \left\|\beta^2 w^{2(\beta-1)} w\right\|_{L^q(\Omega)}^{\frac{1}{2}} = C_2\beta \left\|w^{2\beta-1}\right\|_{L^q(\Omega)}^{\frac{1}{2}} .\end{aligned}$$

This gives us the inequality

$$\|w\|_{L^{\beta r}(\Omega)} = \left\|w^\beta\right\|_{L^r(\Omega)}^{\frac{1}{\beta}} \le (C_2\beta)^{\frac{1}{\beta}} \left\|w^{2\beta-1}\right\|_{L^q(\Omega)}^{\frac{1}{2\beta}} = (C_2\beta)^{\frac{1}{\beta}} \|w\|_{L^{(2\beta-1)q}(\Omega)}^{\frac{2\beta-1}{2\beta}}$$

As $\frac{1}{2\beta} + \frac{2\beta-1}{2\beta} = 1$, one can use Hölder inequality with exponents 2β and $\frac{2\beta}{2\beta-1}$ to derive

$$\|w\|_{L^{(2\beta-1)q}(\Omega)} = \left(\int_\Omega w^{(2\beta-1)q} \mathrm{d}x\right)^{\frac{1}{(2\beta-1)q}} \le \|1\|_{2\beta}^{\frac{1}{(2\beta-1)q}} \left(\left\|w^{(2\beta-1)q}\right\|_{\frac{2\beta}{2\beta-1}}\right)^{\frac{1}{(2\beta-1)q}}$$
$$= |\Omega|^{\frac{1}{2\beta(2\beta-1)q}} \left(\int_\Omega w^{2\beta q} \mathrm{d}x\right)^{\frac{2\beta-1}{2\beta(2\beta-1)q}} = |\Omega|^{\frac{1}{2\beta(2\beta-1)q}} \|w\|_{L^{2\beta q}(\Omega)},$$

which, applied to the previous inequality, gives us

$$\|w\|_{L^{\beta r}(\Omega)} \le (C_2\beta)^{\frac{1}{\beta}} \left(|\Omega|^{\frac{1}{2\beta(2\beta-1)q}} \|w\|_{L^{2\beta q}(\Omega)}\right)^{\frac{2\beta-1}{2\beta}} = (C_2\beta)^{\frac{1}{\beta}} |\Omega|^{\frac{1}{4\beta^2 q}} \|w\|_{L^{2\beta q}(\Omega)}^{\frac{2\beta-1}{2\beta}}. \tag{C.5}$$

Step 4: Mosers iteration argument.
If we set $\hat{q} := 2q, \chi := \frac{r}{2q} > 1$ and $\beta := \chi^m$ for $m \in \mathbb{N}$. Then inequality (C.5) implies recursion formula

$$\|w\|_{L^{\chi^{m+1}\hat{q}}(\Omega)} \le C_2^{\chi^{-m}} \chi^{m\chi^{-m}} |\Omega|^{\frac{1}{4\chi^{2m}}} \|w\|_{L^{\chi^m \hat{q}}(\Omega)}^{1-\chi^{-m}/2}.$$

Thus we obtain by recursion that for all $m \in \mathbb{N}$

$$\|w\|_{L^{\chi^{m+1}\hat{q}}(\Omega)} \le C_2^{\sigma_1(m)} \chi^{\sigma_2(m)} |\Omega|^{\sigma_3(m)} \|w\|_{L^{\hat{q}}(\Omega)}^{\Pi_{k=1}^m \left(1-\frac{1}{2\chi^k}\right)}. \tag{C.6}$$

with

$$\sigma_1(m) = \sum_{k=1}^m \frac{1}{\chi^k} \prod_{j=k+1}^m \left(1 - \frac{\chi^{-j}}{2}\right)$$
$$\sigma_2(m) = \sum_{k=1}^m \frac{k}{\chi^k} \prod_{j=k+1}^m \left(1 - \frac{\chi^{-j}}{2}\right)$$
$$\sigma_3(m) = \sum_{k=1}^m \frac{1}{4\chi^{2k}} \prod_{j=k+1}^m \left(1 - \frac{\chi^{-j}}{2}\right).$$

By comparing $\sigma_i(m),\ i = 1, 2, 3$ to the geometric series, one can easily see that due to $\chi > 1$, they converge to finite values and we denote their limits by $\sigma_i := \lim_{m\to+\infty} \sigma_i(m) < +\infty$ for $i = 1, 2, 3$.

By taking the limit in inequality (C.6) and using Theorem B.1 and Lemma A.6 we derive

$$\|w\|_{L^\infty(\Omega)} = \lim_{m\to+\infty} \|w\|_{L^{\chi^{m+1}\hat{q}}(\Omega)} \le C_2^{\sigma_1} \chi^{\sigma_2} |\Omega|^{\sigma_3} \|w\|_{L^{\hat{q}}(\Omega)}^{\left(\frac{1}{2};\frac{1}{\chi}\right)_\infty}. \tag{C.7}$$

Based on inequality (C.7) and the definition of C_2 (found in inequality (C.4)), we set

$$C_3 := \frac{C_2^{\sigma_1} \chi^{\sigma_2} |\Omega|^{\sigma_3}}{\|f\|_{L^p(\Omega)}^{\sigma_1/2}} \overset{(C.4)}{=} C_1^{\sigma_1/2} \chi^{\sigma_2} |\Omega|^{\sigma_3} < +\infty.$$

Finally inequality (C.7) together with the Sobolev embedding Theorem B.7, the Poincare inequality B.5 and the definition of w, l and C_3 gives us the desired inequality

$$\begin{aligned}\sup_\Omega \phi = \sup_\Omega(\phi - l) + l \leq \left\|(\phi - l)^+\right\|_{L^\infty(\Omega)} + l &\overset{\text{by definition}}{=} \|w\|_{L^\infty(\Omega)} + \left\|\phi^D\right\|_{L^\infty(\Gamma_D)} \\ &\overset{(C.7)}{\leq} C_3 \|f\|_{L^p(\Omega)}^{\sigma_1/2} \|w\|_{L^{\hat{q}}(\Omega)}^{\left(\frac{1}{2};\frac{1}{\chi}\right)_\infty} + \left\|\phi^D\right\|_{L^\infty(\Gamma_D)} \\ &\overset{\text{Sobolev embedding}}{\leq} C_3 C_4 \|f\|_{L^p(\Omega)}^{\sigma_1/2} \|w\|_{H^1(\Omega)}^{\left(\frac{1}{2};\frac{1}{\chi}\right)_\infty} + \left\|\phi^D\right\|_{L^\infty(\Gamma_D)} \\ &\overset{\text{Poincare inequality}}{\leq} \underbrace{C_3 C_4 C_5}_{:=C} \|f\|_{L^p(\Omega)}^{\sigma_1/2} \|\nabla w\|_{L^2(\Omega)}^{\left(\frac{1}{2};\frac{1}{\chi}\right)_\infty} + \left\|\phi^D\right\|_{L^\infty(\Gamma_D)} \\ &\overset{\text{Stampacchia}}{\leq} C \|f\|_{L^p(\Omega)}^{\sigma_1/2} \|\nabla \phi\|_{L^2(\Omega)}^{\left(\frac{1}{2};\frac{1}{\chi}\right)_\infty} + \left\|\phi^D\right\|_{L^\infty(\Gamma_D)}.\end{aligned}$$

□

Theorem C.3. *Let $p \in \left(\max\left\{\frac{d}{2}, 2\right\}, +\infty\right]$. If $f \in L^p(\Omega)$ and $\phi^D \in H^1(\Omega) \cap L^\infty(\Gamma_D)$ then the weak solution of (C.1) suffices to $\phi \in L^\infty(\Omega)$.*

Proof. We follow the proof of [GerstJün, Lemma 7, page 541]. If $d = 1$ this is a trivial consequence of Theorem (B.7). For proving the case $n \geq 2$ we use Lemma C.2. Since ϕ is a weak solution, it is both a sub- and a supersolution, thus we get

$$\|\phi\|_{L^\infty(\Omega)} = \max\left\{\sup_\Omega \phi, \sup_\Omega(-\phi)\right\} \leq C \|f\|_{L^p(\Omega)}^{\alpha} \|\nabla\phi\|_{L^2(\Omega)}^{\gamma} + \left\|\phi^D\right\|_{L^\infty(\Omega)} < +\infty.$$

□

Lemma C.4. *Let $n \in \mathbb{N}$, $z_1, \cdots, z_n \in \mathbb{R}$ and $v \in L^2(\Omega;\mathbb{R}^n)$. Define $\phi(v) \in H^1_D(\Omega)$ to be the unique weak solution of*

$$-\Delta\phi(v) = \Sigma_{j=1}^n z_j v_j \ \text{ on } \Omega$$
$$\phi(u) = 0 \ \text{ on } \Gamma_D, \frac{\partial\phi(v)}{\partial\mathbf{n}} = 0 \ \text{ on } \Gamma_N.$$

Then it holds that

$$\frac{\partial|\nabla\phi(v)|^2}{\partial v_i} = 2z_i\phi(v), \ \text{for all } i = 1, \cdots, n$$

Proof. First we note that for any $\xi \in L^2(\Omega;\mathbb{R}^n)$ it holds that

$$\langle\Sigma_{j=1}^n z_j v_j, \phi(\xi)\rangle_{L^2(\Omega)} = \langle\nabla\phi(v), \nabla\phi(\xi)\rangle_{L^2(\Omega)} = \langle\phi(v), \Sigma_{j=1}^n z_j \xi_j\rangle_{L^2(\Omega)} \tag{C.8}$$

Let $i = 1, \cdots, n$ and $\mathbf{e}_i$ be the i-th unit vector of $\mathbb{R}^n$. By the linearity of ϕ (in v) and by equation (C.8) we derive that

$$\begin{aligned}
\left\langle \frac{\partial |\nabla \phi(v)|^2}{\partial v_i}, \xi_i \right\rangle_{L^2(\Omega)} &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega |\nabla \phi(u + \epsilon \xi_i \mathbf{e}_i)|^2 - |\nabla \phi(v)|^2 \mathrm{d}x \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega |\nabla \phi(v) + \epsilon \nabla \phi(\xi_i \mathbf{e}_i)|^2 - |\nabla \phi(v)|^2 \mathrm{d}x \\
&= \int_\Omega 2 \nabla \phi(v) \cdot \nabla \phi(\xi_i \mathbf{e}_i) \mathrm{d}x \\
&= 2 \langle \nabla \phi(v), \nabla \phi(\xi_i \mathbf{e}_i) \rangle_{L^2(\Omega)} \overset{(C.8)}{=} 2 \langle \phi(v), z_i \xi_i \rangle_{L^2(\Omega)} \\
&= \langle 2 z_i \phi(v), \xi_i \rangle_{L^2(\Omega)}.
\end{aligned}$$

Thus the fundamental theorem of calculus lets us conclude $\frac{\partial |\nabla \phi(v)|^2}{\partial v_i} = 2 z_i \phi(v)$ for any $i = 1, \cdots, n$. □

List of Symbols and Notations

A	matrix valued function describing the cross-diffusion of the system, see Definition 1.1 on page 3.
$B(\cdot,\cdot)$	matrix valued function, see Definition 2.8 on page 10.
β	inverse thermal voltage, given by $\beta = q/(k_B\theta)$, see page 1.
$C^g(T)$	positive constant depending on time $T > 0$, see Lemma 2.13 on page 12.
$\mathfrak{c}$	positive constant, see assumption **(H5)** on page 13.
C_P	Poincare constant, see Theorem B.5 on page 64.
$C_{P,m}$	generalized Poincare constant, see Theorem B.6 on page 64.
D_i	diffusivity of the i-th ion species, see page 1.
D^α	(distributional) derivative corresponding to the multiindex α, see page 2.
∂^τ	finite difference operator of step size τ, see (3.1) on page 14.
(ϵ)	denotes a special sequence $\epsilon_j,\ j \in \mathbb{N}$ of positive numbers, see Lemma 3.16 on page 37.
F	matrix valued function describing the interaction between potential and the ion concentrations, see Definition 1.1 on page 3.
f	permanent charge density, see page 1.
$\mathfrak{f}$	total charge density function, see Definition 2.10 on page 10.
g	reaction term, see page 2.
Γ_D	part of the boundary on which Dirichlet conditions are prescribed, see page 1.
Γ_N	part of the boundary on which no-flux conditions are prescribed, see page 1.
h	entropy density, see Definition 2.1 on page 6.
H	entropy functional, see Definition 2.1 on page 6.
$\mathbb{1}_A$	indicator function of the set A, see page 2.
$J = (J_1, \cdots, J_n)^{\mathrm{T}}$	matrix of the ionfluxes , see equation (1.2), see page 1.
κ^g	positive monotonicall increasing function, see Assumption **(H5)** on page 13.
$\mathcal{L}^m$	elliptic operator on $H^m(\Omega)$, see (3.3) on page 14.
$L(z,\beta)$	Lipschitz constant of the second argument of $\mathfrak{f}$, see Lemma 2.12 on page 11.
m	natural number so large that $H^m(\Omega)$ is compactly embedded into $L^\infty(\Omega)$, see page 14.
n	number of ion species, see page 1.
$\mathbf{n}$	outer normal vector of Ω, see page 1.
$\mathcal{O}$	the domain given by $\mathcal{O} := \{\ v \in (0,1)^n \mid v_0 \in (0,1)\ \}$, see page 2.
Φ	potential, see page 1.

(τ)	denotes a special sequence $\tau_j,\ j \in \mathbb{N}$ of positive numbers, see Lemma 3.26 on page 49.
$u = (u_1, \cdots, u_n)^{\mathrm{T}}$	concentration vector of the different ion species, see page 1.
$u(\cdot, \cdot)$	function relating ion concentrations and entropy variables, see Definition 2.6 on page 9.
u^D	boundary concentration vector, extended to the whole domain in Assumption 1 on page 6.
u_0	concentration of the solvent, $u_0 = 1 - \sum_{i=1}^{n} u_i$, see page 1.
u^0	initial ion concentration vector
W_i	external potential influencing the i-th ion species, see page 1.
w	entropy variables, see Definition 2.3 on page 7.
w^D	boundary entropy variables, see Definition 2.3 on page 7.
$\mathring{w}$	entropy variables withouth boundary condititions defined by $\mathring{w} := w - w^D$, see (3.2) on page 14.
$z = (z_1, \cdots, z_n)^{\mathrm{T}}$	vector of the valences of the different ion species, see page 1.

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