Diplomarbeit

Zur Theorie der großen Abweichungen der Banachraum-wertigen Brown'schen Bewegung und Ciesielskis Isomorphismus in gewichteten Hölder-Räumen

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Univ.-Prof. Dipl.-Math. Dr. rer. nat. habil. Uwe Schmock

Durch

Aleksandar Arandjelović, BSc
Matrikelnummer 01425305

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Under the supervision of

Univ.-Prof. Dipl.-Math. Dr. rer. nat. habil. Uwe Schmock

By

Aleksandar Arandjelović, BSc
Matriculation number 01425305

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(Signature Author) (Signature Supervisor)

Zusammenfassung

Das Ziel der vorliegenden Diplomarbeit ist die Herleitung eines verschärften Resultats über das asymptotische Verhalten der Enden von Gauss'schen Wahrscheinlichkeitsmaßen. Wir zeigen, dass der Satz von Schilder, welcher ein gefeiertes Resultat aus der Theorie der großen Abweichungen darstellt, in einem Kontext gilt, wo skalierte, Banachraum-wertige Brown'sche Bewegung für unbeschränkte Zeiten und bezüglich einer Topologie betrachtet wird, welche durch eine Hölder-ähnliche Norm induziert wird. Wir beschäftigen uns somit mit einem Gebiet der Wahrscheinlichkeitstheorie, welches unter anderem in der Versicherungsmathematik eine große Tradition hat.

Nach einer kurzen Einführung im Kapitel 1 befassen wir uns in Kapitel 2 mit gewichteten Hölder-Räumen. Diese erlauben die Betrachtung von unbeschränkten Zeiten, verallgemeinerten Stetigkeitsmodulen sowie Bildräumen, welche von translationsinvarianten semi-Metriken induziert werden. Nachdem wir Wavelet-artige Reihendarstellungen und Approximationsresultate herleiten, zeigen wir unter welchen Bedingungen die betrachteten Räume separabel oder vollständig sind.

In Kapitel 3 zeigen wir eine verallgemeinerte Version von Ciesielskis Isomorphismus, welcher sich mit Abbildungen zwischen Funktionen- und Folgenräumen beschäftigt, wobei die betrachteten Folgen durch Differenzen zweiter Ordnung von Funktionswerten auf der Menge der dyadisch rationalen Zahlen gegeben sind. Ferner zeigen wir die Äquivalenz von verallgemeinerten Normen, welche durch Differenzen erster und zweiter Ordnung charakterisiert sind.

Schließlich beginnt Kapitel 4 mit grundlegenden Darstellungen zu Gaussmaßen auf lokalkonvexen topologischen Vektorräumen. In diesem Kontext betrachten wir das Konzept von Brownscher Bewegung, welche Werte in reellen separablen Banachräumen annimmt. Ein kurzer Ausflug über die Studie von Pfadeigenschaften dieser stochastischen Prozesse führt uns zuletzt zum angestrebten Satz von Schilder. Als Korollar erhalten wir außerdem eine Version des Satzes von Strassen. Zu guter Letzt skizzieren wir eine Methode zur Varianzreduktion von statistischen Schätzern als Anwendung im Risikomanagement.

Schlagworte: Abstrakter Wienerraum, Ciesielskis Isomorphismus, gewichteter Hölder-Raum, pseudo-quasi-Norm, Faber–Schauder System, Satz von Schilder.

Abstract

The aim of this thesis is to derive a strenghtened topological statement about asymptotic tail estimates of Gaussian probability measures. We show that Schilder's theorem, a celebrated result in the theory of large deviations, holds in the setting where scaled, Banach-space-valued Brownian motion runs on an unbounded time domain and with respect to a topology which is induced by a Hölder-like norm. We are thus studying an area of probability theory that has a long tradition especially in insurance mathematics.

After a short introduction in Chapter 1, we study the notion of weighted Hölder spaces in Chapter 2. These allow for unbounded time domains, generalized moduli of continuity, as well as image spaces whose topologies are induced by translation-invariant semi-metrics. After providing wavelet-like representation and approximation results, we show under which conditions the considered spaces are complete or separable.

In Chapter 3, we provide a generalization of Ciesieski's isomorphism, which deals with maps between function- and sequence spaces, where the sequences are essentially given by second order differences of functions evaluated on the set of dyadic rationals. Moreover, we establish equivalence between generalized norms that incorporate these second order differences, and those that encode first order differences.

Finally, Chapter 4 begins with a primer on Gaussian measures on locally convex topological vector spaces. In this context, we revisit the concept of Brownian motion which assumes values in real separable Banach spaces. A brief study of path properties of these stochastic processes finally leads to the generalized version of Schilder's theorem. As a corollary, we further obtain a variant of Strassen's theorem in Hölder norm. Finally, we outline a variance reduction method for statistical estimation problems as a potential application in risk management.

Keywords: Abstract Wiener Space, Ciesielski's isomorphism, weighted Hölder space, pseudo-quasi-norm, Faber–Schauder system, Schilder's theorem.

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Aleksandar Arandjelović

Statement of originality

I hereby declare that I have authored the present master thesis independently and did not use any sources other than those specified. I have not yet submitted the work to any other examining authority in the same or comparable form. It has not been published yet.

Vienna, September 18, 2020

Aleksandar Arandjelović

Contents

1. Introduction

The theory of large deviations is concerned with the asymptotic tail behavior of probability distributions. Its most classical usage is found within the discipline of insurance mathematics, where Harald Cramér and Filip Lund were the driving forces behind the development of ruin theory. Moreover, the first thorough formalization is due to S.R.S. Varadhan, see [Var66].

Let us consider an illustrative example. For a sequence $(X_i)_{i\in\mathbb{N}}$ of $\mathbb{R}$-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that are independent and identically distributed with mean μ and finite variance $\sigma^2 > 0$, we can study the asymptotic behavior of the sample mean $\overline{X}_n = \sum_{i=1}^n X_i/n$, as n grows large.

The law of large numbers states that the sample mean converges in probability and almost surely to μ. Moreover, according to the classical central limit theorem, the law of $\sqrt{n}(\overline{X}_n - \mu)$ converges in distribution to the normal distribution with mean 0 and variance σ^2. Therefore, we have our first asymptotic tail estimate of the following form: for each $x \in \mathbb{R}$, it holds that

$$\lim_{n\to\infty} \mathbb{P}\big(\sqrt{n}(\overline{X}_n - \mu) > x\big) = 1 - \lim_{n\to\infty} \mathbb{P}\big(\sqrt{n}(\overline{X}_n - \mu) \le x\big) = 1 - \Phi\Big(\frac{x}{\sigma}\Big), \tag{1.1}$$

where Φ denotes the cumulative distribution function of the standard normal distribution. As a special case, if $\mathcal{L}(X_1) = \mathcal{N}(\mu, \sigma^2)$, then $\mathcal{L}(\overline{X}_n) = \mathcal{N}(\mu, \sigma^2/n)$, and consequently $\mathcal{L}(\sqrt{n}(\overline{X}_n - \mu)) = \mathcal{N}(0, \sigma^2)$. However, the central limit theorem may fail to provide accurate tail estimates as long as n is not large enough, especially if the distribution of the random variables $(X_i)_{i\in\mathbb{N}}$ exhibits heavy tails. Moreover, the Equation (1.1) does not provide us with any information on the rate of convergence.

Cramér's theorem provides a solution to this issue. It states that, for any $x > \mathbb{E}[X_1]$, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(\overline{X}_n > x\big) = -\Lambda^*(x), \tag{1.2}$$

where Λ^* denotes the Legendre transform of the logarithmic moment generating function Λ, i.e.

$$\Lambda(t) = \log \mathbb{E}[\exp(tX_1)] \quad \text{and} \quad \Lambda^*(x) = \sup_{t\in\mathbb{R}}(tx - \Lambda(t)).$$

Equation (1.2) gives rise to a first tail estimate of the form

$$\mathbb{P}\big(\overline{X}_n > x\big) \approx \mathrm{e}^{-n\Lambda^*(x)}, \quad x > \mathbb{E}[X_1],$$

and we say that the distributions of the sequence $(\overline{X}_n)_{n\in\mathbb{N}}$ of sample means satisfies a large deviation principle with (good) rate function Λ^*.

In a more general context, one might replace the sample means $\overline{X}_n$ by $\sqrt{\varepsilon}$-scaled Brownian motion $(\sqrt{\varepsilon}B)_{\varepsilon>0}$ and look for an asymptotic tail estimate of the probability of these paths assuming values in some measurable set F of continuous functions, i.e. study the asymptotic behavior of $\mathbb{P}(\{\sqrt{\varepsilon}B \in F\})$ as $\varepsilon \searrow 0$. This is exactly the context of Schilder's theorem, and we will move on to extend this remarkable result, allowing for stronger topologies on path space as compared to the classical context. As it turns out, we will mostly rely on a pathwise characterization, which we will also motivate below.

Let us also remark that Brownian motion is a classical stochastic process that has a prominent role in financial- and actuarial mathematics. For example, it is being used for:

(a) Diffusion approximation of risk processes in actuarial mathematics;

(b) Modeling of the random dynamical evolution of the term structure of interest rates;

(c) Modeling of asset prices in financial markets and pricing of financial derivatives.

The study of stochastic processes allows for a great deal of flexibility, and depending on the context of application, many different settings are possible. To consider a simple case, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ denote a filtered probability probability space, $T = [0, 1]$ denote the unit time interval, and let $X\colon T \times \Omega \to \mathbb{R}$ be a stochastic process that is $\mathbb{F}$-adapted. In this context, many useful properties can be assumed. If the paths $X(\omega)\colon T \ni t \mapsto X_t(\omega)$, where $\omega \in \Omega$, are assumed to be continuous almost surely, then we can already conclude that they are bounded, because $X(\omega)$ would be almost surely a continuous function that maps the compact set T to $\mathbb{R}$.

A classical stochastic process that has been studied in this context is Brownian motion, which has already been mentioned above. We denote this process by $B\colon T \times \Omega \to \mathbb{R}$, with the usual defining properties. We will briefly revisit one way of constructing such a process, that has been attributed to Paul Lévy, see [Lév37], and Zbigniew Ciesielski, see [Cie61]. The Haar functions are given as the family $(H_n)_{n\in\mathbb{N}_0}$ of real-valued functions on the unit interval, such that $H_0 \equiv 1$ and

$$H_{2^k+l}(t) := \begin{cases} \sqrt{2^k} & \text{for } \frac{2l}{2^{k+1}}, \leq t < \frac{2l+1}{2^{k+1}}, \\ -\sqrt{2^k} & \text{for } \frac{2l+1}{2^{k+1}} \leq t < \frac{2l+2}{2^{k+1}}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.3}$$

where $n = 2^k + l$ and $k \in \mathbb{N}$ as well as $0 \leq l \leq 2^k - 1$. It is a classical result that $(H_n)_{n\in\mathbb{N}_0}$ constitutes a complete orthonormal system of the Hilbert space $L^2\big([0, 1]\big)$, see for instance [Aya19]. The Schauder functions $(K_n)_{n\in\mathbb{N}_0}$ are then defined as the primitives of the Haar functions, i.e

$$K_n(t) := \int_0^t H_n(s)\, \mathrm{d}s, \quad t \in [0, 1], n \in \mathbb{N}_0. \tag{1.4}$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(X_n)_{n\in\mathbb{N}_0}$ be an independent sequence of $\mathcal{N}(0, 1)$ distributed random elements on this space. Then the stochastic process $B\colon T \times \Omega \to \mathbb{R}$ given by

$$B(t) := \sum_{n=0}^{\infty} X_n K_n(t), \quad t \in [0, 1] \tag{1.5}$$

is continuous and most notable, a Brownian motion. Here, the series converges almost surely in the space $\mathcal{C} \coloneqq \mathcal{C}\big([0,1];\mathbb{R}\big)$ of continuous functions with respect to the uniform norm $\|f\|_\infty \coloneqq \sup_{t\in[0,1]} |f(t)|$, see [HIPP14, Theorem 2.5].

There is however a pitfall associated with this construction. That is, we can only treat a process where the time index runs in the unit interval, or by a slight generalization in compact subsets $K \subset \mathbb{R}_+$. Clearly, if we want to study dynamical systems for arbitrary times, this construction does not yield a fruitful approach.

On the other hand, one of the strengths is the pathwise characterization of stochastic processes. Moreover, due to the very specific structure of (1.5), we can actually describe continuous objects with discrete characterizations. Let us elaborate on this point. To this end, choose $\alpha \in (0,1)$ and define $\mathcal{C}^\alpha \subset \mathcal{C}$ as the space of real-valued Hölder continuous functions on the unit interval that vanish at zero, i.e. $f \in \mathcal{C}$ such that $f(0) = 0$ and

$$\|f\|_\alpha \coloneqq \sup_{0\le s<t\le 1} \frac{|f(t)-f(s)|}{(t-s)^\alpha} < \infty. \tag{1.6}$$

Further, let $\mathcal{C}_0^\alpha \subset \mathcal{C}^\alpha$ denote the so called little Hölder space of those $f \in \mathcal{C}^\alpha$ that further satisfy

$$\lim_{\delta\searrow 0} \sup_{\substack{s,t\in[0,1]\\ 0<t-s\le\delta}} \frac{|f(t)-f(s)|}{(t-s)^\alpha} = 0. \tag{1.7}$$

Let $l^\infty = l^\infty\big(\mathbb{N}_0,\mathbb{R}\big)$ denote the Banach space of real-valued, bounded sequences endowed with the norm $\|x\|_\infty \coloneqq \sup_{n\in\mathbb{N}_0} |x_n|$, and further $l_0^\infty \coloneqq l_0^\infty\big(\mathbb{N}_0,\mathbb{R}\big) \subset l^\infty$ denote the subspace of real-valued null sequences, i.e. those $x \in l^\infty$ that satisfy $\lim_{n\to\infty} x_n = 0$. A classical result attributed to Ciesielski, see [Cie60], shows that the pairs of spaces $\big(\mathcal{C}^\alpha, l^\infty\big)$ and $\big(\mathcal{C}_0^\alpha, l_0^\infty\big)$ are isomorphic, respectively. To see this, let $(c_n(\alpha))_{n\in\mathbb{N}_0}$ denote the sequence defined by $c_0(\alpha) = 1$ as well as $c_n(\alpha) = 2^{k(\alpha-1/2)+\alpha-1}$, for $n \ge 1$. The map $T^\alpha\colon \mathcal{C}^\alpha \to l^\infty$ given by

$$f \mapsto \Big(c_n(\alpha) \int_0^1 H_n(s)\, \mathrm{d}f(s)\Big)_{n\in\mathbb{N}_0} \tag{1.8}$$

is an isomorphism, bounded in operator norm, and the inverse $\big(T^\alpha\big)^{-1}$ is explicitly given by

$$x \mapsto \sum_{n=0}^\infty \frac{x_n}{c_n(\alpha)} K_n, \tag{1.9}$$

where an analogous result holds for the restriction $T_0^\alpha \coloneqq T^\alpha_{\restriction \mathcal{C}_0^\alpha}\colon \mathcal{C}_0^\alpha \to l_0^\infty$.

Given that paths of Brownian motion are almost surely elements of $\mathcal{C}_0^\alpha$, for $0 < \alpha < 1/2$, see [HIPP14, Theorem 2.6], we can study properties of the paths by inferring from the discrete sequence $T_0^\alpha\big(B(\omega)\big)$. This technique has been successfully used in many instances, some of which we want to revisit briefly.

(a) In [BBK92], the isomorphism T_0^α has been used to prove results in large deviations theory for Brownian motion on the unit interval, with respect to the Hölder seminorm.

(b) In [AIP13], the above result has been extended from the case of real-valued Brownian motion to the case of Brownian motion taking values in some separable Hilbert space.

(c) In [GIP16], the simple structure of the Schauder functions has been used to study pathwise stochastic integration that goes beyond Young integration.

However, as highlighted before, all of the above results restrict themselves to the case of the respective processes being defined on the unit time interval (or some compact subset $K \subset \mathbb{R}_+$), and it is not trivial to extend them to the case where time runs in unbounded domains, think of $T = \mathbb{R}_+$ or even $T = \mathbb{R}$ for example. Just consider that continuous paths on unbounded domains need not be bounded any more. More strikingly, the most important stochastic process, Brownian motion, when defined on $T = \mathbb{R}_+$, almost surely exceeds any given level $a \in \mathbb{R}$, see [Sch20, Example 4.97]. This begs the question whether it is possible to derive similar results as above for the case of unbounded domains. In [Str11, Chapter 8], a hint for this question has been provided. Since we know that real-valued Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ satisfies

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \quad \text{almost surely,} \tag{1.10}$$

we can see Brownian paths as elements of the separable Banach space $\mathcal{C}_0 := \mathcal{C}_0\big([0,\infty),\mathbb{R}\big)$ of continuous, real-valued functions f that satisfy $\lim_{t\to\infty} \frac{f(t)}{t} = 0$ as well as $f(0) = 0$, when endowed with the norm $\|f\| := \sup_{t\in\mathbb{R}_+} \frac{|f(t)|}{1+t}$. This approach extends the notion of the uniform norm. Topologies that are induced by Hölder norms are, however, stronger than those induced by uniform norms and thus allow for more delicate results.

It is therefore our aim to derive a characterization of paths that are defined on unbounded domains, which encodes information about Hölder continuity, and to study the application to the theory of large deviations. More precisely, we will consider Brownian motion that assumes values in real separable Banach spaces. In this context, we will study abstract Wiener spaces with Gaussian measures on path space of Hölder continuous maps. In these spaces, we will derive versions of Schilder's and Strassen's theorems.

That being said, let us briefly state our main findings. Definitions 2.47 and 2.58 provide us with the path spaces of interest, which we call (little) weighted Hölder spaces. These allow for the time domain $T = \mathbb{R}$, for generalized moduli of continuity as well as for image spaces that are endowed with pseudo-quasi-norms, see also Definition 2.10. In Theorem 2.61 we provide the essential approximation result of Hölder continuous functions with respect to wavelet-like series decompositions. Corollaries 2.67 and 2.70 establish sufficient conditions under which these path spaces are separable. In Theorems 3.6 and 3.12 we extend Ciesielski's isomorphism theorem to this setting, and show that we have another pseudo-quasi-norm equivalence in the sequence space that encodes first order differences of function values, see Proposition 3.17. After revisiting some path properties of Brownian motion and absolutely continuous functions, we collect all our findings in Theorem 4.22. Making use of the result [BBK92, Theorem 2.4], we finally obtain our main finding in the form of Theorem 4.27.

2. Generalized normed spaces of continuous functions on the real line

The theory of Schauder bases in Banach spaces is rich and diverse. It poses a generalization of the concept of Hamel bases, which are limited to finite linear combinations, thus allowing for a more involved treatment of general Banach spaces. The characterization is straightforward. Let $(X, \|\cdot\|)$ be a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A sequence $(e_n)_{n\in\mathbb{N}}$ of elements in X is then called a Schauder basis, if for every $x \in X$, there exists a unique sequence $(a_n(x))_{n\in\mathbb{N}}$ of scalars in $\mathbb{K}$, such that x can be written as

$$x = \sum_{n=1}^{\infty} a_n(x) e_n, \tag{2.1}$$

where the series on the right-hand side of Equation (2.1) converges with respect to the norm $\|\cdot\|$. Note that usually, we do need to specify the order of convergence, as Schauder bases in general are conditional bases, i.e. rearrangements can lead to different limits of the sequence of partial sums. We can thus lift the problem of representing an element $x \in X$ through a sequence $(e_n)_{n\in\mathbb{N}}$ by distinguishing between conditional and unconditional Schauder bases, depending on the type of convergence of the series stated in Equation (2.1).

Notably, all Banach spaces that contain a Schauder basis are necessarily separable, but not all separable Banach spaces admit a Schauder basis. This classical result is due to the Swedish mathematician Per Enflo, see [Enf73], and builds on previous works of Banach, Mazur and Grothendieck.

2.1. Faber–Schauder systems

In what follows, let X be a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For each $\delta \in (0, 1)$, let $\mathbb{Z} + \delta$ denote the set $\mathbb{Z} + \delta \coloneqq \{x + \delta \colon x \in \mathbb{Z}\}$. We moreover convene that $\mathbb{R}_+ = [0, \infty)$. Later on, X will be endowed with some topology $\mathcal{T}$ that we will use to characterize convergence of the objects in X, but for now we do not need to specify any. We will state a sequence of real-valued functions which will later be proven to structurally resemble a Schauder basis on a very specific space. In line with [Aya19, Chapter 3], let us define the Faber–Schauder system of the second kind, as well as the Faber–Schauder coefficients, both corresponding conceptually to the sequences $(e_n)_{n\in\mathbb{N}}$ and $(a_n(\cdot))_{n\in\mathbb{N}}$ in Equation (2.1).

Definition 2.1 (Faber–Schauder system)**.** Let $\theta, \tau \colon \mathbb{R} \to [0, 1]$ denote the hat functions given by $(t \in \mathbb{R})$

$$\theta(t) \coloneqq \max\left\{0, 1 - |t|\right\},$$
$$\tau(t) \coloneqq \max\left\{0, \frac{1}{2} - \left|t - \frac{1}{2}\right|\right\}.$$

The sequence of functions $\mathcal{M}_2^{\mathrm{FS}} := \{\theta_l(\cdot)\colon l \in \mathbb{Z}\} \cup \{\tau_{j,k}(\cdot)\colon (j,k) \in \mathbb{N}_0 \times \mathbb{Z}\}$ given by $(t \in \mathbb{R})$

$$\theta_l(t) := \theta(t-l), \tag{2.2}$$

$$\tau_{j,k}(t) := 2^{-j/2}\tau(2^j t - k), \tag{2.3}$$

is called the Faber–Schauder system of the second kind.

Definition 2.2 (Faber–Schauder coefficients)**.** For a map $f\colon \mathbb{R} \to X$, let the sequence $\mathcal{M}_2^{\mathrm{FSC}}(f) := \{a_l(f)\colon l \in \mathbb{Z}\} \cup \{b_{j,k}(f)\colon (j,k) \in \mathbb{N}_0 \times \mathbb{Z}\}$ of X-valued coefficients be given by

$$a_l(f) := f(l), \tag{2.4}$$

as well as

$$b_{j,k}(f) := 2^{j/2}\Big(2f\big(\tfrac{2k+1}{2^{j+1}}\big) - f\big(\tfrac{2k+2}{2^{j+1}}\big) - f\big(\tfrac{2k}{2^{j+1}}\big)\Big). \tag{2.5}$$

We call $\mathcal{M}_2^{\mathrm{FSC}}(f)$ the Faber–Schauder coefficients with respect to f.

Remark 2.3. Note that, for each $f\colon \mathbb{R} \to X$, the sequence $\mathcal{M}_2^{\mathrm{FSC}}(f)$ actually consists of elements of X, not $\mathbb{K}$. Moreover, the family $\mathcal{M}_2^{\mathrm{FS}}$ consists of $\mathbb{R}$-valued functions, regardless of the vector space X at hand. In short, the definitions above are not completely consistent with the definition that characterizes Schauder bases. However, since we will be considering more general spaces than just complete normed vector spaces, it does not pose a problem to us at this point.

Remark 2.4. From the defining Equations (2.4) and (2.5) is is clear that the Faber–Schauder coefficients linearly depend on the function f. Moreover, for each $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$, we have that $b_{j,k}(f) = 0$ if f is of affine form on the interval $[k/2^j, (k+1)/2^j]$.

Remark 2.5. Note how the maps and coefficients presented in the Definitions 2.1 and 2.2 encode only local behaviour of the respective function $f\colon \mathbb{R} \to X$. Most notably, the coefficients $b_{j,k}(f) \in \mathcal{M}_2^{\mathrm{FSC}}(f)$ defined in Equation (2.5) encode information about second order differences over intervals of length at most 1.

We could also fix an integer $J \in \mathbb{Z}$ and consider

$$\mathcal{M}_2^{\mathrm{FS}}(J) := \{\theta_{l,J}(\cdot)\colon l \in \mathbb{Z}\} \cup \{\tau_{j,k}(\cdot)\colon (j,k) \in \mathbb{Z}_{\geq J} \times \mathbb{Z}\},$$

where $\theta_{l,J}(t) = \theta_l(2^J t)$ for $t \in \mathbb{R}$, as well as

$$\mathcal{M}_2^{\mathrm{FSC}}(f,J) := \{a_l(f)\colon l \in \mathbb{Z}\} \cup \{b_{j,k}(f)\colon (j,k) \in \mathbb{Z}_{\geq J} \times \mathbb{Z}\}.$$

Then, the maps and coefficients presented in $\mathcal{M}_2^{\mathrm{FS}}(J)$ and $\mathcal{M}_2^{\mathrm{FSC}}(f,J)$ would encode local behavior about second order differences over intervals of length at most 2^J.

Clearly, the Definitions 2.1 and 2.2 are just special cases of this construction, where we assume that $J = 0$.

2.2. A dyadic decomposition

For $f\colon \mathbb{R} \to X$ and $m \in \mathbb{N}_0$, let $f_m, \tilde{f}_m\colon \mathbb{R} \to X$ be given by $(t \in \mathbb{R})$

$$f_m(t) := \sum_{l\in\mathbb{Z}} a_l(f)\theta_l(t) + \sum_{j=0}^{m}\sum_{k\in\mathbb{Z}} b_{j,k}(f)\tau_{j,k}(t), \tag{2.6}$$

$$\tilde{f}_m(t) := f_m(\gamma_m(t)), \tag{2.7}$$

where the time localization map $\gamma_m\colon \mathbb{R} \to \mathbb{R}$ is given by

$$\gamma_m(t) = \max\{-m, \min\{m, t\}\} = \begin{cases} m & \text{for } m \le t, \\ t & \text{for } -m \le t < m, \\ -m & \text{otherwise.} \end{cases} \tag{2.8}$$

Remark 2.6. Note that in Equation (2.6), for each $t \in \mathbb{R}$, we are actually looking at the sum of at most finitely many non-zero elements, because for each $t \in \mathbb{R}$ there are at most two integers $l_{1,2}$ such that $\theta_{l_{1,2}}(t) \neq 0$, and for each $j \in \mathbb{N}_0$ there is at most one integer k such that $\tau_{j,k}(t) \neq 0$.

The proofs of the following two Propositions 2.7 and 2.9 as well as Lemma 2.8 are very simple in nature, relying essentially on induction arguments.

Proposition 2.7. *For each $m \in \mathbb{N}_0$, the function $f_m\colon \mathbb{R} \to X$, defined in (2.6), is the piecewise linear interpolation of f on $\mathbb{D}^{(m+1)} \subset \mathbb{R}$, the set of dyadic rationals of order up to $m+1$, which is given by*

$$\mathbb{D}^{(m+1)} := \left\{ \tfrac{j}{2^{m+1}} : j \in \mathbb{Z} \right\}.$$

Proof. We prove the statement by induction.

Initial case. Let $m = 0$ and $t \in \mathbb{Z}$. Then, due to the simple structure of the elements of $\mathcal{M}_2^{\mathrm{FS}}$ we have $\tau_{0,k}(t) = 0$ for all $k \in \mathbb{Z}$, as well as $\theta_l(l) = 1$ for all $l \in \mathbb{Z}$ and $\theta_l(t) \neq 0$ if and only if $l = t$, which yields

$$f_0(t) = a_t(f)\theta_t(t) = a_t(f) = f(t).$$

For $\delta \in (0,1)$ and $t \in \mathbb{Z} + \delta$, there are exactly three integers $l_{1,2}, k \in \mathbb{Z}$ such that $\theta_{l_i}(t) \neq 0$ for $i \in \{1,2\}$, as well as $\tau_{0,k}(t) \neq 0$,

$$l_1 = \lfloor t \rfloor, \quad l_2 = \lceil t \rceil, \quad k = \lfloor t \rfloor.$$

If $t \in \mathbb{Z} + 1/2$, we have $\tau_{0,k}(t) = 1/2$ as well as $\theta_{l_i}(t) = 1/2$ for $i \in \{1,2\}$, hence

$$\begin{aligned} f_0(t) &= a_{l_1}(f)\theta_{l_1}(t) + a_{l_2}(f)\theta_{l_2}(t) + b_{0,k}(f)\tau_{0,k}(t) \\ &= \frac{1}{2}\big(f(\lfloor t \rfloor) + f(\lceil t \rceil) + 2f(\lfloor t \rfloor + 1/2) - f(\lceil t \rceil) - f(\lfloor t \rfloor)\big) \\ &= f(\lfloor t \rfloor + 1/2) = f(t). \end{aligned}$$

Let $r \in (0, 1/2)$ and $t \in \mathbb{Z} + r$. Then we have

$$\theta_{\lfloor t \rfloor}(t) = 1 - r, \quad \theta_{\lceil t \rceil}(t) = r, \quad \tau_{0,\lfloor t \rfloor}(t) = r,$$

hence

$$\begin{aligned} f_0(t) &= (1-r)f(\lfloor t \rfloor) + rf(\lceil t \rceil) + r(2f(\lfloor t \rfloor + 1/2) - f(\lceil t \rceil) - f(\lfloor t \rfloor)) \\ &= f(\lfloor t \rfloor) + \tfrac{t - \lfloor t \rfloor}{(\lfloor t \rfloor + 1/2) - \lfloor t \rfloor}\big(f(\lfloor t \rfloor + 1/2) - f(\lfloor t \rfloor)\big). \end{aligned}$$

Similarly, if $r \in (1/2, 1)$ and $t \in \mathbb{Z} + r$, then

$$\theta_{\lfloor t \rfloor}(t) = 1 - r, \quad \theta_{\lceil t \rceil}(t) = r, \quad \tau_{0,\lfloor t \rfloor}(t) = 1 - r,$$

hence

$$\begin{aligned} f_0(t) &= (1-r)f(\lfloor t \rfloor) + rf(\lceil t \rceil) + (1-r)(2f(\lfloor t \rfloor + 1/2) - f(\lceil t \rceil) - f(\lfloor t \rfloor)) \\ &= f(\lfloor t \rfloor + 1/2) + \tfrac{t - (\lfloor t \rfloor + 1/2)}{\lceil t \rceil - (\lfloor t \rfloor + 1/2)}\big(f(\lceil t \rceil) - f(\lfloor t \rfloor + 1/2)\big). \end{aligned}$$

This shows that f_0 is the linear interpolation of f on the set $\mathbb{D}^{(1)}$ of dyadic rationals up to order 1.

Induction step. Given $m \in \mathbb{N}$, we can write $f_{m+1}(\cdot)$ as

$$f_{m+1}(\cdot) = f_m(\cdot) + \sum_{k \in \mathbb{Z}} b_{m+1,k}(f)\tau_{m+1,k}(\cdot).$$

Let $t \in \mathbb{D}^{(m+2)} \setminus \mathbb{D}^{(m+1)}$ be a dyadic rational of the form

$$t = \tfrac{p}{2^{m+2}}, \quad p \in \mathbb{Z},$$

where we assume that $p/2 \notin \mathbb{Z}$, because otherwise we would have $f_{m+1}(t) = f_m(t)$ due to the fact that $\tau_{m+1,k}(t) = 0$ for all $t \in \mathbb{D}^{(m+1)}$ and $k \in \mathbb{Z}$. The two neighboring dyadic rationals of order $m+1$ are

$$t_l = \tfrac{\lfloor p/2 \rfloor}{2^{m+1}}, \quad t_u = \tfrac{\lceil p/2 \rceil}{2^{m+1}}.$$

Moreover, there is exactly one $k \in \mathbb{Z}$ such that $\tau_{m+1,k}(t) \neq 0$, $k = \lfloor p/2 \rfloor$. Moreover, we have

$$\tau_{m+1,k}(t) = 2^{-\frac{m+1}{2}-1}.$$

Given that f_m is the linear interpolation of f on the set $\mathbb{D}^{(m+1)}$ of dyadic rationals up to order $m+1$, we make use of the observation that

$$f_m(t) = f(t_l) + \tfrac{t - t_l}{t_u - t_l}(f(t_u) - f(t_l)),$$

which yields

$$\begin{aligned} f_{m+1}(t) &= f(t_l) + \tfrac{t - t_l}{t_u - t_l}(f(t_u) - f(t_l)) + b_{m+1,k}(f)2^{-\frac{m+1}{2}-1} \\ &= f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big) + \big(\tfrac{p}{2} - \lfloor \tfrac{p}{2} \rfloor\big)\Big(f\Big(\tfrac{\lceil p/2 \rceil}{2^{m+1}}\Big) - f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big) \\ &\quad + f\Big(\tfrac{\lfloor p/2 \rfloor + 1/2}{2^{m+1}}\Big) - \tfrac{1}{2}f\Big(\tfrac{\lceil p/2 \rceil}{2^{m+1}}\Big) - \tfrac{1}{2}f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big) \\ &= f\Big(\tfrac{\lfloor p/2 \rfloor + 1/2}{2^{m+1}}\Big) = f\big(\tfrac{p}{2^{m+2}}\big) = f(t). \end{aligned}$$

Let $t \in (t_l, 2^{-(m+2)}p)$, then we have

$$\tau_{m+1,\lfloor p/2 \rfloor}(t) = 2^m \Big(t - \tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big),$$

hence

$$\begin{aligned}
f_{m+1}(t) &= f_m(t) + b_{m+1,\lfloor p/2 \rfloor}(f)\tau_{m+1,\lfloor p/2 \rfloor}(t) \\
&= f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big) + 2^{m+1}\Big(t - \tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big(f\Big(\tfrac{\lceil p/2 \rceil}{2^{m+1}}\Big) - f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big) \\
&\quad + 2^m\Big(t - \tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big(2f\Big(\tfrac{\lfloor p/2 \rfloor + 1/2}{2^{m+1}}\Big) - f\Big(\tfrac{\lceil p/2 \rceil}{2^{m+1}}\Big) - f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big) \\
&= f\Big(\tfrac{p-1}{2^{m+2}}\Big) + \frac{t - \frac{p-1}{2^{m+2}}}{\frac{p}{2^{m+2}} - \frac{p-1}{2^{m+2}}}\Big(f\Big(\tfrac{p}{2^{m+2}}\Big) - f\Big(\tfrac{p-1}{2^{m+2}}\Big)\Big).
\end{aligned}$$

Similarly, given $t \in (2^{-(m+2)}p, t_u)$, we have

$$\tau_{m+1,\lfloor p/2 \rfloor}(t) = \Big(1 - 2^{m+2}\Big(t - \tfrac{p/2}{2^{m+1}}\Big)\Big),$$

hence

$$\begin{aligned}
f_{m+1}(t) &= f_m(t) + b_{m+1,\lfloor p/2 \rfloor}(f)\tau_{m+1,\lfloor p/2 \rfloor}(t) \\
&= f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big) + 2^{m+1}\Big(t - \tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big(f\Big(\tfrac{\lceil p/2 \rceil}{2^{m+1}}\Big) - f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big) \\
&\quad + \Big(1 - 2^{m+2}\Big(t - \tfrac{p/2}{2^{m+1}}\Big)\Big) \\
&\quad \cdot \Big(f\Big(\tfrac{\lfloor p/2 \rfloor + 1/2}{2^{m+1}}\Big) - \tfrac{1}{2}f\Big(\tfrac{\lceil p/2 \rceil}{2^{m+1}}\Big) - \tfrac{1}{2}f\Big(\tfrac{\lfloor p/2 \rfloor}{2^{m+1}}\Big)\Big) \\
&= f\Big(\tfrac{p}{2^{m+2}}\Big) + \frac{t - \frac{p}{2^{m+2}}}{\frac{p+1}{2^{m+2}} - \frac{p}{2^{m+2}}}\Big(f\Big(\tfrac{p+1}{2^{m+2}}\Big) - f\Big(\tfrac{p}{2^{m+2}}\Big)\Big),
\end{aligned}$$

which concludes our proof. □

Lemma 2.8. *Let $f\colon \mathbb{R} \to X$ and $m \in \mathbb{N}_0$ be given. Then the function $\tilde{f}_m\colon \mathbb{R} \to X$, given by Equation (2.7) satisfies*

$$\tilde{f}_m(t) = \begin{cases} f(m) & \text{for } t \geq m, \\ f_m(t) & \text{for } t \in (-m, m), \\ f(-m) & \text{for } t \leq -m. \end{cases}$$

Proof. We just have to observe that, for all $t \geq m$, we have

$$\tilde{f}_m(t) = f_m(\gamma_m(t)) = f_m(m) = f(m),$$

where the analogous argument holds for the case $t \leq -m$. If $t \in (-m, m)$, then $\gamma_m(t) = t$. □

Our goal is to prove a representation theorem, which allows us to identify a map $f\colon \mathbb{R} \to X$ with the series

$$f(\cdot) = \sum_{l \in \mathbb{Z}} a_l(f)\theta_l(\cdot) + \sum_{(j,k) \in \mathbb{N}_0 \times \mathbb{Z}} b_{j,k}(f)\tau_{j,k}(\cdot), \tag{2.9}$$

The exact type of convergence of the series will be investigated in detail later. Our next result deals with the uniqueness of the coefficients in series representations of functions f with respect to $\mathcal{M}_2^{\mathrm{FS}}$. As it turns out, any representation of the form as in Equation (2.9), that admits pointwise evaluation on the set $\mathbb{D}$ of dyadic rationals given by

$$\mathbb{D} := \bigcup_{m\in\mathbb{N}_0} \mathbb{D}^{(m)}$$

already has its coefficients uniquely determined.

Proposition 2.9. *Let $f\colon \mathbb{R}\to X$ be a map. Then for every representation of the form*

$$f(\cdot) = \sum_{l\in\mathbb{Z}} a_l\theta_l(\cdot) + \sum_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} b_{j,k}\tau_{j,k}(\cdot), \tag{2.10}$$

for some X-valued coefficients $(a_l)_{l\in\mathbb{Z}}$ and $(b_{j,k})_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}}$, which is valid with respect to pointwise evaluation on the set of dyadic rationals $\mathbb{D}\subset\mathbb{R}$, we have that

$$a_l = a_l(f), \quad b_{j,k} = b_{j,k}(f),$$

where the X-valued coefficients $a_l(f)$ and $b_{j,k}(f)$ are the Faber–Schauder coefficients with respect to f, given in Definition 2.2.

Proof. We prove the statement by successively evaluating the series in Equation (2.10) on dyadic rationals of increasing order.

Initial case. Let $t\in\mathbb{Z}$, then $\tau_{j,k}(t) = 0$ for all $(j,k)\in\mathbb{N}_0\times\mathbb{Z}$. Further, we have $\theta_l(t)\neq 0$ if and only if $l = t$, in which case $\theta_t(t) = 1$. Evaluating the right-hand side of Equation (2.10) at point t yields now

$$f(t) = a_l\theta_t(t) = a_t,$$

hence $a_t = f(t) = a_t(f)$ for arbitrary $t\in\mathbb{Z}$, where the last equality follows from the defining Equation (2.4). Next, assume that $t\in\mathbb{Z}+1/2$. Then there are exactly two integers $l_{1,2}\in\mathbb{Z}$ as well as one integer $k\in\mathbb{Z}$ such that $\theta_{l_i}(t)\neq 0$ for $i\in\{1,2\}$ and $\tau_{0,k}(t)\neq 0$, namely

$$l_1 = \lfloor t\rfloor, \quad l_2\lceil t\rceil, \quad k = \lfloor t\rfloor.$$

Evaluating again the right-hand side of Equation (2.10) at point t yields

$$\begin{aligned} f(t) &= a_{l_1}\theta_{l_1}(t) + a_{l_2}\theta_{l_2}(t) + b_{0,k}\tau_{0,k}(t) \\ &= a_{l_1}(f)\theta_{l_1}(t) + a_{l_2}(f)\theta_{l_2}(t) + b_{0,k}\tau_{0,k}(t). \end{aligned}$$

On the other hand, we know from the proof of Proposition 2.7 that for the chosen t we have

$$f(t) = f_0(t) = a_{l_1}(f)\theta_{l_1}(t) + a_{l_2}(f)\theta_{l_2}(t) + b_{0,k}(f)\tau_{0,k}(t),$$

hence $b_{0,k} = b_{0,k}(f)$ for all $k\in\mathbb{Z}$, as for each $k\in\mathbb{Z}$ we can find a $t\in\mathbb{Z}+1/2$ such that $k = \lfloor t\rfloor$ and repeat the above argument.

Induction step. Let $m \in \mathbb{N}$ and t be a dyadic rational of the form

$$t = \tfrac{p}{2^{m+2}}, \quad p \in \mathbb{Z},$$

where we assume that $p/2 \notin \mathbb{Z}$. Then there is at most one $k \in \mathbb{Z}$ such that $\tau_{m+1,k}(t) \neq 0$, namely $k = \lfloor p/2 \rfloor$. Evaluating once more the right-hand side of Equation (2.10) at t yields

$$\begin{aligned} f(t) &= a_{l_1}\theta_{l_1}(t) + a_{l_2}\theta_{l_2}(t) + \sum_{j=0}^{m+1} b_{j,\lfloor 2^j t\rfloor}\tau_{j,\lfloor 2^j t\rfloor}(t) \\ &= a_{l_1}(f)\theta_{l_1}(t) + a_{l_2}(f)\theta_{l_2}(t) + \sum_{j=0}^{m} b_{j,\lfloor 2^j t\rfloor}(f)\tau_{j,\lfloor 2^j t\rfloor}(t) \\ &\quad + b_{m+1,\lfloor 2^{m+1} t\rfloor}\tau_{m+1,\lfloor 2^{m+1} t\rfloor}(t). \end{aligned}$$

At the same time, again making use of the ideas displayed in the proof of Proposition 2.7, we have

$$f(t) = f_{m+1}(t) = a_{l_1}(f)\theta_{l_1}(t) + a_{l_2}(f)\theta_{l_2}(t) + \sum_{j=0}^{m+1} b_{j,\lfloor 2^j t\rfloor}(f)\tau_{j,\lfloor 2^j t\rfloor}(t),$$

hence $b_{m+1,k} = b_{m+1,k}(f)$ for all $k \in \mathbb{Z}$. □

2.3. Pseudo-quasi-normed spaces

Until now, we did not need to specify any topology on our vector space X. We did not treat the convergence of the series given in Equation (2.9), as pointwise evaluations of this series representation on the set $\mathbb{D}$ amounts to summing up finitely many non-zero elements. Now, we will specify exactly the notion of distance that is required to allow for an effective topological study of convergence of the series given in Equation (2.9), if we want to allow for evaluations on $\mathbb{R} \setminus \mathbb{D}$. As before, let X be a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We define the notion of a pseudo-quasi-norm. The following definition follows the exposition in [AM15, Section 8.1].

Definition 2.10 (Pseudo-quasi-norm)**.** Let $\|\cdot\|\colon X \to \mathbb{R}$ be a map that satisfies the following three conditions:

(a) (nondegeneracy) $\|x\| = 0$ if and only if $x = 0$, for all $x \in X$;

(b) (quasi-subadditivity) There exists a constant $C_{\mathrm{qs}} \in [1, \infty)$ such that

$$\Big\| \sum_{i=1}^{n} x_i \Big\| \leq C_{\mathrm{qs}} \sum_{i=1}^{n} \|x_i\|, \quad n \in \mathbb{N},\ x_1, \dots, x_n \in X; \tag{2.11}$$

(c) (pseudo-homogeneity) There exists a non-decreasing function

$$\phi\colon [0,\infty) \to [0,\infty)$$

that satisfies $\phi(\lambda) = 0 \iff \lambda = 0$, as well as

$$\|\lambda x\| \leq \phi(|\lambda|)\|x\|, \quad x \in X,\ \lambda \in \mathbb{K}, \tag{2.12}$$

and such that

$$\lim_{\lambda \searrow 0} \phi(\lambda) = 0. \tag{2.13}$$

We call the map $\|\cdot\|$ a pseudo-quasi-norm on the vector space X. Furthermore, we call each ϕ that satisfies the requirements of Definition 2.10(c) a pseudo-homogeneity function (in short, a pseudo-homogeneity) of $\|\cdot\|$. If $(X, \|\cdot\|)$ is complete, then we call it a pseudo-quasi-Banach space. If the quasi-subadditivity condition (2.11) holds with $C_{\mathrm{qs}} = 1$, we call $\|\cdot\|$ a pseudo-norm, and if the pseudo-homogeneity condition (2.12) is satisfied with equality and $\phi(\lambda) = \lambda$, we call $\|\cdot\|$ a quasi-norm.

Remark 2.11. In Definition 2.10, we mention the concept of completeness. This however assumes that the space is endowed with some topological structure, such that we can discuss convergence of Cauchy sequences. In fact, every pseudo-quasi-norm induces a topology on its space, see Proposition 2.25 below.

Remark 2.12. Every norm on X satisfies the requirements of Definition 2.10 with $C_{\mathrm{qs}} = 1$ and $\phi(\lambda) = \lambda$ for $\lambda \in \mathbb{K}$. However, it is important to note that a pseudo-quasi-norm that satisfies 2.10(b) with $C_{\mathrm{qs}} = 1$ as well as 2.10(c) with $\phi(\lambda) = \lambda$ for $\lambda \in \mathbb{K}$ in general does not need to be a norm. The reason for this is that in (2.12), equality does not need to hold.

Remark 2.13. From the defining property (2.12) we can deduce that $\phi(1) \geq 1$ in the case that X does not only contain the zero vector. This can be seen as follows. Let $x \in X$ such that $\|x\| \neq 0$. Then it holds that

$$\|x\| = \|1 \cdot x\| \leq \phi(1)\|x\|.$$

Dividing by $\|x\|$ verifies the claim. Moreover, we can deduce the non-negativity of $\|\cdot\|$ from the properties stated in Definition 2.10 as follows: For each $x \in X$ we have

$$0 = \|0\| = \|x - x\| = \|x + (-x)\| \leq C_{\mathrm{qs}}\big(\|x\| + \|-x\|\big) \leq C_{\mathrm{qs}}(1 + \phi(1))\|x\|,$$

which implies $\|x\| \geq 0$.

Remark 2.14. Let $C_1 \in (0, \infty)$ and $\theta \in (0, 1]$. Assume the pseudo-homogeneity ϕ to be of the form $\phi(\lambda) = C_1\lambda^\theta$. Then Definition 2.10 perfectly fits within the notion of a θ-pseudo-quasi-norm, as defined in [AM15, Definition 8.2]. Note however that the exposition in [AM15, Section 8.1] requires condition (2.11) to hold for $n = 2$, not for all $n \in \mathbb{N}$. Therefore, our concept of a pseudo-quasi-normed vector space can be considered a special case of [AM15, Definition 8.2] if $\phi(\lambda) = C_1\lambda^\theta$, and otherwise, one needs to compare both concepts with care.

Example 2.15 (L^p-space for $0 < p < 1$)**.** For further details and proofs of the statements below see [Kö83, Sections 15.9 & 15.10] as well as [Rud91, Section 1.47]. Let L^p denote the space of all λ-measurable $f\colon [0, 1] \to \mathbb{K}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, such that

$$\|f\|_p := \int_0^1 |f(t)|^p \ \lambda(\mathrm{d}t) < \infty,$$

where we identify functions that are equal λ-almost everywhere, and λ denotes the Lebesgue–Borel measure on $([0,1], \mathcal{B}([0,1]))$. This space is a complete topological vector space (see also Proposition 2.25 below) with respect to the pseudo-norm $\|\cdot\|_p$. It is metrizable and thus Hausdorff, and the invariant metric is given by

$$d(f,g) = \int_0^1 |f(t) - g(t)|^p \; \lambda(\mathrm{d}t).$$

Moreover, it holds that

$$\|\delta f\|_p = \phi(|\delta|)\|f\|_p, \quad \delta \in \mathbb{R},$$

where $\phi(\delta) = \delta^p$. On the other hand, the ε-balls $U_\varepsilon(0)$ of the form as given below in Equation (2.18), are not convex. In fact, the only convex neighborhood of the origin is the whole space L^p, and the only convex sets in L^p are the empty set and L^p itself.

Example 2.16. In line with the argument given right after [AM15, Definition 8.2], let $(X, \|\cdot\|)$ denote a pseudo-quasi-normed vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with a pseudo-homogeneity function ϕ according to Definition 2.10. Consider a map $\|\cdot\|_\star : X \to [0,\infty)$ that is equivalent to $\|\cdot\|$ in the sense that there exist some constants $K_1, K_2 > 0$ such that

$$K_1\|x\| \leq \|x\|_\star \leq K_2\|x\|, \quad x \in X.$$

One can easily verify that $\|\cdot\|_\star$ is non-degenerate. Moreover, we have for each $x \in X$ and $\lambda \in \mathbb{K}$:

$$\|\lambda x\|_\star \leq K_2\|\lambda x\| = K_2\phi(|\lambda|)\|x\| \leq \frac{K_2}{K_1}\phi(|\lambda|)\|x\|_\star.$$

Therefore, upon defining $\phi_\star(\lambda) = K_2\phi(\lambda)/K_1$, we can conclude that $\|\cdot\|_\star$ satisfies the pseudo-homogeneity property 2.10(c) with $\phi_\star$. Similarly, for each $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$, it holds that

$$\Big\|\sum_{i=1}^n x_i\Big\|_\star \leq K_2\Big\|\sum_{i=1}^n x_i\Big\| \leq K_2 C_{\mathrm{qs}} \sum_{i=1}^n \|x_i\| \leq \frac{K_2}{K_1} C_{\mathrm{qs}} \sum_{i=1}^n \|x_i\|_\star,$$

and so the quasi-subadditivity condition 2.10(b) is satisfied as well. All of the above now implies that $\|\cdot\|_\star$ is a pseudo-quasi-norm on X. Note that if $\|\cdot\|$ is a quasi-norm, i.e. $\phi(\lambda) = \lambda$, then $\phi_\star(\lambda) \neq \lambda$ in general. Consequently, this procedure shows how one can obtain a genuine pseudo-quasi-norm on a quasi-normed space.

In Chapter 3, it will become useful to deduce convergence of a series from its absolute convergence. This follows from the

Proposition 2.17. *Let $(X, \|\cdot\|)$ denote a pseudo-quasi-normed space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ in X. Then*

(a) If $(x_n)_{n\in\mathbb{N}}$ converges in X, then it is a Cauchy sequence.

(b) If X is complete and $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, i.e.

$$\sum_{n=1}^{\infty} \|x_n\| = \lim_{m\to\infty} \sum_{n=1}^{m} \|x_n\| < \infty,$$

then the series converges.

Proof. We will show both statements (a) and (b) separately.

(a) For a proof in the case that X is a metric space, see [KF75, Section 7.1]. Let $x \in X$ denote the limit of $(x_n)_{n\in\mathbb{N}}$, in other words,

$$\lim_{n\to\infty} \|x - x_n\| = 0.$$

For each $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that, for all $n \geq N$, it holds that

$$\|x - x_n\| \leq \frac{\varepsilon}{C_{\text{qs}}(1 + \phi(1))}.$$

This however implies that for each $m \geq N$, we have

$$\begin{aligned}\|x_m - x_n\| &\leq C_{\text{qs}}(\|x_m - x\| + \|x - x_n\|)\\ &\leq C_{\text{qs}}(\phi(1)\|x - x_m\| + \|x - x_n\|) \leq \varepsilon,\end{aligned}$$

and therefore, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

(b) For a proof in the case that $X = \mathbb{C}$, see [Rud76, Theorem 3.55]. Either by Part (a) or by the classical results that convergent sequences in metric spaces are indeed Cauchy sequences, we know that the sequence of partial sums of the absolutely convergent series, given by

$$\left(\sum_{n=1}^{m} \|x_n\|\right)_{m\in\mathbb{N}} \tag{2.14}$$

is a Cauchy sequence in $\mathbb{R}_+$. Consequently, for each $\varepsilon > 0$, we can find an $N \in \mathbb{N}$ such that, for $m, n \geq N$, where we assume without loss of generality that $n > m$, we have

$$\sum_{i=m+1}^{n} \|x_i\| \leq \frac{\varepsilon}{C_{\text{qs}}}.$$

Now it holds that

$$\left\|\sum_{i=m+1}^{n} x_i\right\| \leq C_{\text{qs}} \sum_{i=m+1}^{n} \|x_i\| \leq \varepsilon.$$

As we assume X to be complete, it follows that $\sum_{n=1}^{\infty} x_n$ converges.

This concludes our proof. □

Remark 2.18. Strictly speaking, the classical notion of an operator norm is defined for normed spaces, but we will extend this notion as follows: For two pseudo-quasi-normed spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ as well as a linear map $T\colon X_1 \to X_2$, we set for $X_1 \neq \{0\}$:

$$\|T\|_{\mathrm{op}} := \sup_{x \in X_1 \setminus \{0\}} \frac{\|T(x)\|_2}{\|x\|_1}, \tag{2.15}$$

and $\|T\|_{\mathrm{op}} = 0$ otherwise. Clearly, for two norms $\|\cdot\|_{1,2}$, this definition coincides with the classical one.

The following proposition should not come as a surprise.

Proposition 2.19. *Let V denote the set*

$$V := \{T\colon X_1 \to X_2 \text{ linear}\colon \|T\|_{\mathrm{op}} < \infty\}, \tag{2.16}$$

and $\|\cdot\|_{\mathrm{op}}$ be given by (2.15). Further, let $\mathbb{K}^{(2)}, C_{\mathrm{qs}}^{(2)}$ and $\phi^{(2)}$ denote the field, subadditivity constant and pseudo-homogeneity on $(X_2, \|\cdot\|_2)$. Then, $(V, \|\cdot\|_{\mathrm{op}})$ is a pseudo-quasi-normed vector space over $\mathbb{K}^{(2)}$ that inherits $C_{\mathrm{qs}}^{(2)}$ and $\phi^{(2)}$ from $\|\cdot\|_2$.

Proof. For a proof in the case that X_1 and X_2 are normed spaces, see [Rud91, Theorem 4.1]. We first show that $(V, \|\cdot\|_{\mathrm{op}})$ is actually a vector space. For all $T_1, T_2 \in V$ as well as $\lambda \in \mathbb{K}^{(2)}$, we have that $T := T_1 + \lambda T_2$ is a linear map from X_1 to X_2. Moreover, one can easily see that

$$\|T\|_{\mathrm{op}} \leq C_{\mathrm{qs}}^{(2)}(\|T_1\|_{\mathrm{op}} + \phi^{(2)}(|\lambda|)\|T_2\|_{\mathrm{op}}) < \infty.$$

Now it remains to verify the properties of Definition 2.10. Let $T \in V$. If $T \equiv 0$, then $T(x) = 0$ for each $x \in X_1$ and consequently, making use of the non-degeneracy of $\|\cdot\|_2$, it holds that $\|T(x)\|_2 = 0$ for each $x \in X_1 \setminus \{0\}$, which implies that $\|T\|_{\mathrm{op}} = 0$. On the other hand, $\|T\|_{\mathrm{op}} = 0$ implies that $\|T(x)\|_2 = 0$ for each $x \in X_1 \setminus \{0\}$, and again making use of the nondegeneracy of $\|\cdot\|_2$, we see that $T(x) = 0$ for each $x \in X_1 \setminus \{0\}$, which then implies that $T \equiv 0$. Therefore, $\|\cdot\|_{\mathrm{op}}$ is nondegenerate.

For $n \in \mathbb{N}$ as well as $T_1, \ldots, T_n \in V$, we have that

$$\begin{aligned}
\Big\|\sum_{i=1}^n T_i\Big\|_{\mathrm{op}} &= \sup_{x \in X_1 \setminus \{0\}} \frac{\big\|(\sum_{i=1}^n T_i)(x)\big\|_2}{\|x\|_1} = \sup_{x \in X_1 \setminus \{0\}} \frac{\big\|\sum_{i=1}^n T_i(x)\big\|_2}{\|x\|_1} \\
&\leq \sup_{x \in X_1 \setminus \{0\}} C_{\mathrm{qs}}^{(2)} \sum_{i=1}^n \frac{\|T_i(x)\|_2}{\|x\|_1} \leq C_{\mathrm{qs}}^{(2)} \sum_{i=1}^n \sup_{x \in X_1 \setminus \{0\}} \frac{\|T_i(x)\|_2}{\|x\|_1} \\
&= C_{\mathrm{qs}}^{(2)} \sum_{i=1}^n \|T_i\|_{\mathrm{op}}.
\end{aligned}$$

Therefore, $\|\cdot\|_{\mathrm{op}}$ satisfies the quasi-subadditivity property.

Finally, let $\lambda \in \mathbb{K}^{(2)}$ and $T \in V$. It follows that

$$\begin{aligned}
\|\lambda T\|_{\mathrm{op}} &= \sup_{x \in X_1 \setminus \{0\}} \frac{\|(\lambda T)(x)\|_2}{\|x\|_1} = \sup_{x \in X_1 \setminus \{0\}} \frac{\|\lambda T(x)\|_2}{\|x\|_1} \\
&\leq \sup_{x \in X_1 \setminus \{0\}} \phi^{(2)}(|\lambda|) \frac{\|T(x)\|_2}{\|x\|_1} = \phi^{(2)}(|\lambda|)\|T\|_{\mathrm{op}}.
\end{aligned}$$

Therefore, $\|\cdot\|_{\text{op}}$ also satisfies the pseudo-homogeneity property. This concludes our proof. □

Let us also investigate if the completeness can be preserved under certain mappings between pseudo-quasi-normed spaces.

Proposition 2.20. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote two pseudo-quasi-normed spaces over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Assume that there exists a map $T\colon X \to Y$ that is linear and bijective, such that both T and T^{-1} are bounded in operator pseudo-quasi-norm. Then, if one of the two spaces is complete, so is the other one.*

Proof. Without loss of generality, we assume that $(X, \|\cdot\|_X)$ is complete. Let $(y_n)_{n\in\mathbb{N}}$ denote a Cauchy sequence in Y, and set $x_n := T^{-1}(y_n)$ for $n \in \mathbb{N}$. For each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that, for all $m, n \geq N$, we have

$$\|T^{-1}\|_{\text{op}} \|y_m - y_n\|_Y \leq \varepsilon,$$

which yields that

$$\|x_m - x_n\|_X = \|T^{-1}(y_m) - T^{-1}(y_n)\|_X \leq \|T^{-1}\|_{\text{op}} \|y_m - y_n\|_Y \leq \varepsilon.$$

Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X, and since this vector space is assumed to be complete, we can deduce that there exists an $x \in X$ such that $\lim_{n\to\infty} \|x - x_n\|_X = 0$, and therefore, upon setting $y := T(x) \in Y$,

$$\begin{aligned}\lim_{n\to\infty} \|y - y_n\|_Y = \lim_{n\to\infty} \|T(x) - T(x_n)\|_Y = \lim_{n\to\infty} \|T(x - x_n)\|_Y \\ \leq \limsup_{n\to\infty} \|T\|_{\text{op}} \|x - x_n\|_X = 0.\end{aligned}$$

Therefore, we can conclude that $(y_n)_{n\in\mathbb{N}}$ converges to y in Y with respect to $\|\cdot\|_Y$, which concludes our proof. □

Before we can investigate continuity properties of functions that map into pseudo-quasi-normed spaces, let us first describe some basic topological properties. Let us begin with the

Definition 2.21. Let X denote a non-empty set, and consider $\mathcal{T} \subset \mathcal{P}(X)$. We then call $\mathcal{T}$ a topology on X, if the following holds:

(a) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;

(b) If $O_1 \in \mathcal{T}$ and $O_2 \in \mathcal{T}$, then $O_1 \cap O_2 \in \mathcal{T}$;

(c) For each index set I and $O_i \in \mathcal{T}, i \in I$ we have $\bigcup_{i\in I} O_i \in \mathcal{T}$.

Every pseudo-quasi-norm induces a topology on its space. In Definition 2.22, we will first provide the construction and then prove in Proposition 2.25 that the defined object is indeed a topology.

Definition 2.22 (Topology induced by $\|\cdot\|$)**.** Let $(X, \|\cdot\|)$ denote a pseudo-quasi-normed space. We set

$$\mathcal{T}^{\|\cdot\|} := \{O \subset X : O \text{ open}\}, \tag{2.17}$$

where we call a set $O \subset X$ open, if for each $x \in O$, there exists an $\varepsilon > 0$, such that $U_\varepsilon(x) \subset O$, where we set, for each $x \in X$ and $\varepsilon > 0$:

$$U_\varepsilon(x) := \{y \in X : \|x - y\| < \varepsilon\}. \tag{2.18}$$

Note that $y \in U_\varepsilon(x)$ implies $\|y\| \leq C_{\mathrm{qs}}(\phi(1)\varepsilon + \|x\|)$. Moreover, as $\|x - y\| \leq \phi(1)\|y - x\|$ we cannot expect the sets $U_\varepsilon(0)$ to be symmetric in the sense that $y \in U_\varepsilon(0)$ would imply $-y \in U_\varepsilon(0)$. However, it would be desirable for vector addition and scalar multiplication to be continuous operations. Therefore, let us state the

Definition 2.23. Let X denote a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ endowed with a topology $\mathcal{T}$. We then call $(X, \mathcal{T})$ a topological vector space, if the maps

(i) $+ : (X \times X, \mathcal{T} \otimes \mathcal{T}) \to (X, \mathcal{T})$, where $(x, y) \mapsto x + y$;

(ii) $\cdot : (\mathbb{K} \times X, \mathcal{T}^{\mathbb{K}} \otimes \mathcal{T}) \to (X, \mathcal{T})$, where $(\lambda, x) \mapsto \lambda \cdot x$;

are continuous. Here, $\mathcal{T}^{\mathbb{K}}$ denotes the Euclidean topology on $\mathbb{K}$ and we consider the respective product topologies on the product spaces $X \times X$ and $\mathbb{K} \times X$.

Remark 2.24. Note that Definition 2.23 does not require singletons $\{x\}$ to be closed, or equivalently, $\{x\}^{\mathrm{c}} = X \setminus \{x\}$ to be open. See [Rud91, Definition 1.6] for a classical definition that requires this additional assumption. Together with the properties stated in Definition 2.23, it would make $(X, \mathcal{T})$ Hausdorff, see [Rud91, Theorem 1.12]. This means that distinct points can be separated by disjoint neighborhoods. In many cases the space X will be Hausdorff, for instance when it is endowed with a metric.

We will now collect the crucial topological results in the

Proposition 2.25. *Let $(X, \|\cdot\|)$ denote a pseudo-quasi-normed vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Moreover, let $\mathcal{T}^{\|\cdot\|}$ denote the set defined in (2.17). Then $(X, \mathcal{T}^{\|\cdot\|})$ constitutes a topological vector space such that every singleton is a closed set.*

Proof. First, we argue that $\mathcal{T}^{\|\cdot\|}$ is indeed a topology on X. Clearly, we have that $\varnothing \in \mathcal{T}^{\|\cdot\|}$ as well as $X \in \mathcal{T}^{\|\cdot\|}$. Moreover, for $O_1 \in \mathcal{T}^{\|\cdot\|}$ and $O_2 \in \mathcal{T}^{\|\cdot\|}$, let us distinguish two cases. If $O_1 \cap O_2 = \varnothing$, then the intersection is clearly also contained in $\mathcal{T}^{\|\cdot\|}$. On the other hand, if the intersection is not the empty set, pick $x \in O_1 \cap O_2$. As O_1 and O_2 are open sets, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $U_{\varepsilon_1}(x) \subset O_1$ and $U_{\varepsilon_2}(x) \subset O_2$. If we now consider $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\} > 0$, then $U_\varepsilon(x) \subset U_{\varepsilon_1}(x) \cap U_{\varepsilon_2}(x) \subset O_1 \cap O_2$. Finally, if I denotes an index set and $O_i \in \mathcal{T}^{\|\cdot\|}$ for $i \in I$, let $x \in \bigcup_{i \in I} O_i$. Then there exists an $i \in I$ such that $x \in O_i$. As O_i is open, there exists an $\varepsilon > 0$ such that $U_\varepsilon(x) \subset O_i \subset \bigcup_{i \in I} O_i$. Therefore, we have shown that $\mathcal{T}^{\|\cdot\|}$ is indeed a topology on X.

Next, we will show that $(X, \mathcal{T}^{\|\cdot\|})$ is a topological vector space. Note that for all $(x_1, x_2), (y_1, y_2) \in X \times X$ and $(\lambda_1, x_1), (\lambda_2, x_2) \in \mathbb{K} \times X$, we have

$$\|(x_1 + x_2) - (y_1 + y_2)\| \leq C_{\mathrm{qs}}\big(\|x_1 - y_1\| + \|x_2 - y_2\|\big),$$

as well as

$$\|\lambda_1 x_1 - \lambda_2 x_2\| \le C_{\mathrm{qs}}\big(\phi(|\lambda_1|)\|x_1 - x_2\| + \phi(|\lambda_1 - \lambda_2|)\|x_2\|\big).$$

This implies for $\varepsilon > 0$ that $U_\varepsilon(x_1) + U_\varepsilon(x_2) \subset U_{2C_{\mathrm{qs}}\varepsilon}(x_1 + x_2)$ as well as $U_\varepsilon^{\mathbb{K}}(\lambda_1) \cdot U_\varepsilon(x_1) \subset U_{h(x_1,\lambda_1,\varepsilon)}(\lambda_1 \cdot x_1)$, where

$$h(x, \lambda, \varepsilon) = C_{\mathrm{qs}} \max\{\varepsilon, \phi(\varepsilon)\}(\phi(|\lambda|) + C_{\mathrm{qs}}(\phi(1)\varepsilon + \|x\|)).$$

Therefore, we can choose $\delta_1 = \varepsilon/(2C_{\mathrm{qs}})$ as well as δ_2 such that $h(x_1, \lambda_1, \delta_2) \le \varepsilon$ (note that we can always find such a δ_2 due to (2.13)) in order to argue continuity of vector addition and scalar multiplication.

Finally, for $x \in X$, we want to show that $\{x\}$ is closed. Equivalently, we show that $X \setminus \{x\}$ is open. Pick $y \in \{x\}^{\mathrm{c}}$, and set $\varepsilon := \|y - x\|$. Then $x \notin U_\varepsilon(y)$. Consequently, $U_\varepsilon(y) \subset \{x\}^{\mathrm{c}}$, and we conclude that every singleton is closed. □

Remark 2.26. Let us revisit now Remark 2.24 in the context of Proposition 2.25. Pseudo-quasi-normed spaces are topological vector spaces such that singletons are closed. Consequently, by [Rud91, Theorem 1.12], these spaces are Hausdorff, which means that distinct points can be separated by disjoint neighborhoods. Therefore, limits of convergent sequences are unique.

Moreover, by Proposition 2.25, we have that $(X, \mathcal{T}^{\|\cdot\|})$ is a first-countable space, which means that every point has a neighborhood basis that is countable. To see this, let $x \in X$, and consider the sets $U_{1/n}(x)$ for $n \in \mathbb{N}$. These are all elements of $\mathcal{T}^{\|\cdot\|}$ that contain x, thus they are neighborhoods of x. By Definition 2.22, for every neighborhood U of x, there exists an $n \in \mathbb{N}$, such that $x \in U_{1/m}(x) \subset U$ for every integer $m \ge n$.

Remark 2.27. As we have seen above, addition is a continuous operation. This however implies that translations on the space are continuous. Therefore, the topology is invariant under translation, and a linear map is continuous if and only if it is continuous at any given point, see [Rud91, Theorem 1.17]. Moreover, we can see any neighborhood of a point as a translated version of a neighborhood of the origin.

Remark 2.28. We can extend the quasi-subadditivity 2.10(b) to countable sums by the following argument: if $\sum_{i=1}^{\infty} x_i$ is a series that converges with respect to the topology induced by $\|\cdot\|$, then we have

$$\Big\|\sum_{i=1}^{\infty} x_i\Big\| = \lim_{n\to\infty}\Big\|\sum_{i=1}^{n} x_i\Big\| \le C_{\mathrm{qs}} \lim_{n\to\infty}\sum_{i=1}^{n}\|x_i\| = C_{\mathrm{qs}}\sum_{i=1}^{\infty}\|x_i\|,$$

where the right-most expression can be infinite.

In light of Proposition 2.25 and similarly as in Proposition 2.20, let us discuss mappings of dense sets between pseudo-quasi-normed spaces.

Proposition 2.29. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote two pseudo-quasi-normed spaces over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Assume that there exists a map $T\colon X \to Y$ that is bijective, such that both T and T^{-1} are continuous. Then, if one of the two spaces is separable, so is the other one.*

Proof. Without loss of generality, let us assume that $(X, \|\cdot\|_X)$ is separable. Let E_X denote a countable dense subset of X. We set $E_Y := \{y = T(x) \in Y : x \in E_X\}$ and argue that E_Y is dense in Y.

For each $y \in Y$ and neighborhood $V \in \mathfrak{U}(y)$ in the neighborhood filter of y, we set $x := T^{-1}(y)$ and $U := T^{-1}(V)$. Since T^{-1} is continuous by assumption, we have that $U \in \mathfrak{U}(x)$. As E_X is dense in X, there exists some $x^* \in E_X$ such that $x^* \in U$. Upon setting $y^* := T(x^*)$, it holds that $y^* \in V$. Consequently, E_Y is dense in Y, which concludes our proof. □

Proposition 2.30. *Let $(X, \|\cdot\|)$ denote a pseudo-quasi-normed vector space over a field $\mathbb{K}$. Then $\|\cdot\|$ induces a translation invariant and pseudo-homogeneous semimetric d on X. Moreover, d induces a topology $\mathcal{T}^d$ on X that is equivalent to the topology $\mathcal{T}^{\|\cdot\|}$. Consequently, $(X, \mathcal{T}^d)$ is a topological vector space.*

Proof. First, we have to show that $\|\cdot\|$ induces a map $d \colon X \times X \to \mathbb{R}$ that is nonnegative, nondegenerate, symmetric and satisfies a relaxed triangle inequality as well as a pseudo-homogeneous scaling property. To see this, set

$$d(x, y) := \frac{1}{2}(\|x - y\| + \|y - x\|), \quad x, y \in X. \tag{2.19}$$

For each pair (x, y) it immediately follows that $d(x, y) \geq 0$ because $\|\cdot\|$ is nonnegative. Similarly, one can easily show that d is nondegenerate because $\|\cdot\|$ is. The symmetry of d follows from the fact that the addition of real numbers is a commutative operation. Moreover, d satisfies the C_{qs}-relaxed triangle inequality: for all $x, y, z \in X$, we have

$$\begin{aligned} d(x, z) &= \frac{1}{2}(\|x - z\| + \|z - x\|) \\ &= \frac{1}{2}(\|x - y + y - z\| + \|z - y + y - x\|) \\ &\leq C_{\mathrm{qs}} \frac{1}{2}(\|x - y\| + \|y - z\| + \|z - y\| + \|y - x\|) \\ &= C_{\mathrm{qs}}(d(x, y) + d(y, z)). \end{aligned}$$

Next, d is translation invariant: for all $x, y, z \in X$, we have

$$\begin{aligned} d(x + z, y + z) &= \frac{1}{2}(\|x + z - y - z\| + \|y + z - x - z\|) \\ &= \frac{1}{2}(\|x - y\| + \|y - x\|) = d(x, y). \end{aligned}$$

Although d is not homogeneous, it is pseudo-homogeneous: for $x, y \in X$ and $\lambda \in \mathbb{K}$, we have

$$\begin{aligned} d(\lambda x, \lambda y) &= \frac{1}{2}(\|\lambda x - \lambda y\| + \|\lambda y - \lambda x\|) \\ &\leq \frac{1}{2}(\phi(|\lambda|)\|x - y\| + \phi(|\lambda|)\|y - x\|) = \phi(|\lambda|) d(x, y). \end{aligned}$$

This verifies our first claim.

The semimetric d induces a topology $\mathcal{T}^d$ on X as follows: For each $\varepsilon > 0$ and $x \in X$, we denote the ε-ball with center x and with respect to d to be the set

$$U_\varepsilon^d(x) := \{y \in X : d(x,y) < \varepsilon\}. \tag{2.20}$$

Similarly as in (2.17), we set

$$\mathcal{T}^d := \{O \subset X : O \text{ open}\}, \tag{2.21}$$

where we call a set $O \subset X$ open (with respect to d), if for each $x \in O$, there exists an $\varepsilon > 0$, such that $U_\varepsilon^d(x) \subset O$. Recall that Proposition 2.25 tells us that $\mathcal{T}^{\|\cdot\|}$ is a topology on X. Rather than arguing that $\mathcal{T}^d$ satisfies the defining properties of a topology as given by Definition 2.21, we will show that $\mathcal{T}^d = \mathcal{T}^{\|\cdot\|}$, which will then yield the stronger implication that, by Proposition 2.25, $(X, \mathcal{T}^d)$ is not just a vector space endowed with a topology, but even a topological vector space, see Definition 2.23.

Let $x, y \in X$. On the one hand, it holds that (recall $\phi(1) \geq 1$)

$$\begin{aligned} \|x-y\| &= \frac{1}{2}(\|x-y\| + \|x-y\|) \\ &\leq \frac{1}{2}(\|x-y\| + \phi(1)\|y-x\|) \leq \phi(1) d(x,y). \end{aligned}$$

On the other hand, in a similar manner we have that

$$\begin{aligned} d(x,y) &= \frac{1}{2}(\|x-y\| + \|y-x\|) \\ &\leq \frac{1}{2}(\|x-y\| + \phi(1)\|x-y\|) = (1+\phi(1))\|x-y\|. \end{aligned}$$

In order to see that $\mathcal{T}^d \subset \mathcal{T}^{\|\cdot\|}$, let $O \in \mathcal{T}^d$. This implies for each $x \in O$ the existence of an $\varepsilon > 0$ such that $U_\varepsilon^d(x) \subset O$. For $\tilde{\varepsilon} := \varepsilon/(1+\phi(1))$, it follows that $U_{\tilde{\varepsilon}}^{\|\cdot\|}(x) \subset U_\varepsilon^d(x)$, hence $O \in \mathcal{T}^{\|\cdot\|}$. Let $O \in \mathcal{T}^{\|\cdot\|}$ and $x \in O$. Then there exists an $\varepsilon > 0$ such that $U_\varepsilon^{\|\cdot\|}(x) \subset O$. For $\varepsilon := \varepsilon/\phi(1)$, it follows that $U_{\tilde{\varepsilon}}^d(x) \subset U_\varepsilon^{\|\cdot\|}(x)$, hence $O \in \mathcal{T}^d$. Consequently, we have $\mathcal{T}^d = \mathcal{T}^{\|\cdot\|}$, which concludes our proof. □

Remark 2.31. Let $C_1 \in (0,\infty)$ and $\theta \in (0,1]$. Set $\phi(\lambda) = C\lambda^\theta$. Assume ϕ to be a pseudo-homogeneity for some pseudo-quasi-norm $\|\cdot\|$ on a vector space X. Then, according to [AM15, Theorem 8.3] the space $(X, \mathcal{T}^{\|\cdot\|})$ is metrizable and therefore Hausdorff. However, it is in general not locally convex, as we would need a condition of the form $C_{\mathrm{qs}}(\phi(\lambda)+\phi(1-\lambda)) \leq 1$ for all $\lambda \in (0,1)$ to hold, see Example 2.15.

The semimetric $d\colon X \times X \to \mathbb{R}_+$ induces a uniform structure $\mathfrak{U}$ according to Definition 2.32 on $X \times X$, which is a remarkable fact that we want to show in Proposition 2.33. Below, we denote by $U \circ V$ the composition of two subsets of a set $X \times X$ which is given by $U \circ V = \{(x,z) : \exists\, y \in X : (x,y) \in U \wedge (y,z) \in V\}$. We follow the exposition in [Bou95, Chapter II].

Definition 2.32 (Uniform structure)**.** Let X denote a nonempty set. We then call a set $\mathfrak{U}$ of subsets of $X \times X$ a uniform structure or uniformity on X, if the following conditions are satisfied:

(a) Every set belonging to $\mathfrak{U}$ contains the diagonal $\Delta = \{(x,x) : x \in X\}$;

(b) Every subset of $X \times X$ which contains a set of $\mathfrak{U}$ belongs to $\mathfrak{U}$;

(c) Every finite intersection of sets of $\mathfrak{U}$ belongs to $\mathfrak{U}$;

(d) If $V \in \mathfrak{U}$ then $V^{-1} = \{(y,x) : (x,y) \in V\} \in \mathfrak{U}$;

(e) For each $V \in \mathfrak{U}$ there exists some $W \in \mathfrak{U}$ such that $W \circ W \subset V$.

We call the elements U of $\mathfrak{U}$ entourages of the uniformity $\mathfrak{U}$ on X and $(X, \mathfrak{U})$ a uniform space. Moreover, we call a set $\mathfrak{B} \subset \mathfrak{U}$ a fundamental system or base of the uniformity $\mathfrak{U}$, if every entourage of $\mathfrak{U}$ contains a set that belongs to $\mathfrak{B}$.

Proposition 2.33. *Let $(X, \|\cdot\|)$ denote a pseudo-quasi-normed vector space over a field $\mathbb{K}$. Then the semimetric d that is induced by $\|\cdot\|$ according to Proposition 2.30 induces a uniform structure on X.*

Proof. For each $\varepsilon > 0$, let $U_\varepsilon \subset X \times X$ denote the set

$$U_\varepsilon := \{(x,y) \in X \times X : d(x,y) < \varepsilon\}. \tag{2.22}$$

Moreover, we set

$$\mathfrak{U} := \{U \subset X \times X : \exists\, \varepsilon > 0 : U_\varepsilon \subset U\}. \tag{2.23}$$

We will now show that $\mathfrak{U}$ is a uniformity of X and that the set $\mathfrak{B} := \{U_\varepsilon : \varepsilon > 0\}$ is a fundamental system of $\mathfrak{U}$. Let us verify all defining properties of a uniformity according to Definition 2.32:

(a) Let $U \in \mathfrak{U}$. By (2.23) there exists an $\varepsilon > 0$ such that $U_\varepsilon \subset U$. By Definition 2.10(a), the pseudo-quasi-norm $\|\cdot\|$ is nondegenerate, which implies by (2.19) that $d(x,x) = 0$ for all $x \in X$. Consequently, the diagonal Δ is contained in U_ε and thus $\Delta \subset U_\varepsilon \subset U$.

(b) Let $V \subset X \times X$ such that $U \subset V$ for some $U \in \mathfrak{U}$. By (2.23) there exists an $\varepsilon > 0$ such that $U_\varepsilon \subset U$. Consequently, we have that $U_\varepsilon \subset V$ and thus $V \in \mathfrak{U}$.

(c) We will show the statement for the intersection of two sets, from which the general statement will follow directly. Let $U, V \in \mathfrak{U}$. By (2.23) there exist $\delta, \varepsilon > 0$ such that $U_\delta \subset U$ and $U_\varepsilon \subset V$. If we define $\eta = \min\{\delta, \varepsilon\}$, then $U_\eta \subset U \cap V$ and consequently $U \cap V \in \mathfrak{U}$.

(d) Let $U \in \mathfrak{U}$. By (2.23) there exists an $\varepsilon > 0$ such that $U_\varepsilon \subset U$. As the semimetric d is symmetric, we know that for any $(x,y) \in U_\varepsilon$ we also have $(y,x) \in U_\varepsilon$. Consequently, it holds that $U_\varepsilon \subset U^{-1}$ and thus $U^{-1} \in \mathfrak{U}$.

(e) Let $U \in \mathfrak{U}$. By (2.23) there exists an $\varepsilon > 0$ such that $U_\varepsilon \subset U$. Set $\delta := \varepsilon/(2C_{\mathrm{qs}})$. Then $U_\delta \in \mathfrak{U}$. By the C_{qs}-relaxed triangle inequality of the semimetric d, we know that for all $(x,z) \in U_\delta \circ U_\delta$, if we choose $y \in X$ such that $(x,y) \in U_\delta$ and $(y,z) \in U_\delta$, then it holds that $d(x,z) < \varepsilon$ and consequently $U_\delta \circ U_\delta \subset U_\varepsilon \subset U$.

Finally, the set $\mathfrak{B}$ is clearly a fundamental system of $\mathfrak{U}$ by (2.23) and Definition 2.32. $\square$

Let us finally point out that the study of bounded linear operators also allows us to derive topological properties of uniform structures. We will derive two version of this statement, one that based on direct estimates of pseudo-quasi-norms, and another one that takes the more general point of view of uniform structures.

Proposition 2.34. *Let $T\colon X_1 \to X_2$ denote a map between two pseudo-quasi-normed vector spaces $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ over fields $\mathbb{K}_1, \mathbb{K}_2 \in \{\mathbb{R}, \mathbb{C}\}$ with $\mathbb{K}_1 \subset \mathbb{K}_2$. Further, assume that T is $\mathbb{K}_1$-linear as well as bounded in operator pseudo-quasi-norm, see* (2.15)*. Then, T is in fact uniformly continuous as a map $T\colon (X_1, \mathcal{T}^{\|\cdot\|_1}) \to (X_2, \mathcal{T}^{\|\cdot\|_2})$.*

Proof. The proof is absolutely classical, and directly carries over from the setting of linear maps between normed spaces. The case $T \equiv 0$ is a trivial one. Therefore, assume that $T \not\equiv 0$. Let $\varepsilon > 0$, and set $\delta := \varepsilon / \|T\|_{\mathrm{op}}$. For each $x, y \in X_1$ such that $\|x - y\|_1 \leq \delta$ it then follows that

$$\|T(x) - T(y)\|_2 = \|T(x-y)\|_2 \leq \|T\|_{\mathrm{op}} \|x - y\|_1 \leq \varepsilon, \tag{2.24}$$

which implies that T is uniformly continuous. $\square$

The following definition originates from [Bou95, Section 2.1].

Definition 2.35 (Uniform continuity on uniform spaces)**.** Let T denote a map between two uniform spaces $(X_1, \mathfrak{U}_1)$ and $(X_2, \mathfrak{U}_2)$. We then call T uniformly continuous, if for every entourage $V \in \mathfrak{U}_2$, there exists an entourage $U \in \mathfrak{U}_1$, such that

$$(x, y) \in U \implies (T(x), T(y)) \in V.$$

Corollary 2.36. *In the context of Proposition 2.34 and Defintion 2.35, the map T is uniformly continuous as a map between the uniform spaces*

$$T\colon (X_1, \mathfrak{U}_1) \to (X_2, \mathfrak{U}_2).$$

Here, the uniformities $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are given as in (2.23) *and* (2.22) *by means of the respective semimetrics on X_1 and X_2 according to Proposition 2.30 and Proposition 2.33.*

Proof. Let $V \in \mathfrak{U}_2$. Then there exists an $\varepsilon > 0$ such that $U_\varepsilon^{(2)} \subset V$, where $U_\varepsilon^{(2)} = \{(x, y) \in X_2 \times X_2 \colon d_2(x, y) < \varepsilon\}$. Upon recalling the estimate (2.24) of Proposition 2.34, let us fix $\delta := \varepsilon / \|T\|_{\mathrm{op}}$. The set $U_\delta^{(1)}$ is contained in $\mathfrak{U}_1$ by definition. Furthermore, for all $(x, y) \in U_\delta^{(1)}$, it holds that

$$\begin{aligned} d_2(T(x), T(y)) &= \frac{1}{2}\big(\|T(x) - T(y)\|_2 + \|T(y) - T(x)\|_2\big) \\ &\leq \frac{\|T\|_{\mathrm{op}}}{2}\big(\|x - y\|_1 + \|y - x\|_1\big) \\ &= \|T\|_{\mathrm{op}}\, d_1(x, y) < \varepsilon. \end{aligned}$$

Consequently, it holds that $(T(x), T(y)) \in U_\varepsilon^{(2)} \subset V$. In line with Definition 2.35, we have thus shown that T is uniformly continuous. $\square$

2.4. Weighted Hölder spaces

We denote by $\mathcal{C} = \mathcal{C}(\mathbb{R}; X)$ the space of continuous maps $f\colon \mathbb{R} \to X$. It is our aim to further quantify the modulus of continuity of maps $f \in \mathcal{C}$. Therefore, we arrive at the

Definition 2.37 (Modulus of continuity)**.** Let $\varrho\colon (0,\infty) \to (0,\infty)$ denote a non-decreasing map that satisfies

$$\lim_{\delta \searrow 0} \varrho(\delta) = 0. \tag{2.25}$$

We then call ϱ a modulus of continuity.

Example 2.38. For $0 < \alpha < \infty$, consider the map $\varrho(\delta) = \delta^\alpha$ for $\delta > 0$. We then call ϱ the α-Hölder modulus of continuity. It will play a crucial role in Chapter 4.

Not all maps ϱ will be suitable for every setting that we are to consider below. The right choice of modulus essentially depends on the structure of the pseudo-quasi-normed space. More importantly, we need to impose very specific relations between ϱ and the pseudo-homogeneity ϕ that comes with $\|\cdot\|$. Therefore, let us further state the

Definition 2.39 (ϕ-admissible modulus of continuity)**.** In the context of Definition 2.37, let $(X, \|\cdot\|)$ denote a pseudo-quasi-normed space with a pseudo-homogeneity ϕ of $\|\cdot\|$. Assume that the following holds:

(a) There exists a positive real constant C_ϕ such that

$$\phi\Big(\frac{\delta}{\varepsilon}\Big)\frac{\varrho(\varepsilon)}{\varrho(\delta)} \leq C_\phi, \quad 0 < \delta \leq \varepsilon \leq 1. \tag{2.26}$$

(b) We have that ϱ dominates ϕ at 0, i.e.

$$\lim_{\delta \searrow 0} \frac{\phi(\delta)}{\varrho(\delta)} = 0. \tag{2.27}$$

We then call ϱ a ϕ-admissible modulus of continuity for the pseudo-quasi-norm $\|\cdot\|$ with pseudo-homogeneity ϕ. If the context allows, we will call ϱ an admissible modulus of continuity, and assume that (2.26) and (2.27) holds for a fixed pseudo-homogeneity ϕ.

Remark 2.40. Condition (2.26) will first be needed in Proposition 2.54, whereas condition (2.27) will make its first appearance in Corollary 2.67, see also Remark 2.69. In Chapter 3, we will make the additional assumptions (3.1), (3.2) and (3.3). These are not needed at this point though.

Example 2.41. Let $0 < \alpha < \beta \leq 1$, and set $\phi(\lambda) := \lambda^\beta$ for $\lambda \geq 0$ as well as $\varrho(\delta) := \delta^\alpha$ for $\delta > 0$. Then ϱ is a ϕ-admissible modulus of continuity for every pseudo-quasi-norm $\|\cdot\|$ endowed with the pseudo-homogeneity ϕ, and (2.26) holds with $C_\phi = 1$.

Since the domain of functions $f \in \mathcal{C}$, being the whole of $\mathbb{R}$, is not bounded, it is possible that these become unbounded even under moderate growth conditions. Therefore, we restrict to subspaces of $\mathcal{C}$ where we can control the behaviour of the functions for large times. To motivate this thought, let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$ or $\mathbb{C}$. Note that we neither assume the space to be complete (hence a Banach space) nor separable (hence containing a countable, dense subset). Since the domain is not bounded, we cannot expect f to be of finite uniform norm, i.e. describing f in terms of

$$\|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\|$$

would restrict the space $\{f \in \mathcal{C} \colon \|f\|_\infty < \infty\}$ to bounded functions. On the other hand, describing f through the family $\{\|\cdot\|_n \colon n \in \mathbb{N}\}$ of semi-norms given by

$$\|f\|_n := \sup_{\substack{t\in\mathbb{R}\\ |t|\le n}} \|f(t)\|$$

might be an interesting approach, but the topology generated by this family would make the space $\{f \in \mathcal{C} \colon \|f\|_n < \infty \ \forall\, n \in \mathbb{N}\}$ potentially Fréchet, not Banach. However, if we control the growth of f by some weight function w, see Definition 2.42, and assume that the weighted term

$$\frac{\|f(t)\|}{w(t)}, \quad t \in \mathbb{R},$$

is uniformly bounded from above or even vanishes, as $|t| \to \infty$, we obtain reasonable normed spaces that allow for moderate growth of the functions. For this, we need to specify the weight function w.

Definition 2.42 (Weight function)**.** A continuous function $w\colon \mathbb{R} \to (0,\infty)$ is called a weight function, if there exists a real constant $C \ge 1$ such that

$$\sup_{\substack{s,t\in\mathbb{R}\\ |t-s|\le 1}} \frac{w(t)}{w(s)} \le C. \tag{2.28}$$

We call a weight function w non-decreasing, if both maps $\mathbb{R}_+ \ni t \mapsto w(t)$ and $\mathbb{R}_+ \ni t \mapsto w(-t)$ are non-decreasing.

Remark 2.43. Condition (2.28) implies that w does not change too much on small time intervals, i.e. for $s, t \in \mathbb{R}$ such that $|t-s| \le 1$, we always have the estimate $w(t) \le Cw(s)$.

Example 2.44. Let $\beta \in \mathbb{R}$ and $w_1, w_2\colon \mathbb{R} \to (0,\infty)$ be given by

$$w_1(t) := \mathrm{e}^{\beta|t|}, \qquad w_2(t) := (1+|t|)^\beta, \quad t \in \mathbb{R}. \tag{2.29}$$

Let us show that they are indeed weight functions.

1. First of all, upon recalling the reverse triangle inequality

$$-|t-s| \leq |t| - |s| \leq |t-s|, \tag{2.30}$$

   note that for all s, t in $\mathbb{R}$, we have

$$\frac{w_1(t)}{w_1(s)} = \mathrm{e}^{\beta(|t|-|s|)} \leq \mathrm{e}^{|\beta||t-s|},$$

   Hence, an admissible constant for w_1 such that (2.28) is satisfied is given by $C_1 = C(w_1) := \mathrm{e}^{|\beta|}$.

2. Moreover, note that inequality (2.30) together with $|t-s| \leq 1$ implies $|s| \leq 1 + |t|$ and $|t| \leq 1 + |s|$, which yields the estimate

$$\Big(\frac{1+|t|}{1+|s|}\Big)^{\beta} \leq \Big(\frac{2+|s|}{1+|s|}\Big)^{\beta} \leq 2^{\beta},$$

   in the case that β is positive, and

$$\Big(\frac{1+|t|}{1+|s|}\Big)^{\beta} = \Big(\frac{1+|s|}{1+|t|}\Big)^{-\beta} \leq \Big(\frac{2+|t|}{1+|t|}\Big)^{-\beta} \leq 2^{-\beta},$$

   in the case that β is negative. Hence an admissible constant for w_2 such that (2.28) is satisfied is given by $C_2 = C(w_2) := 2^{|\beta|}$.

Remark 2.45. The idea of incorporating time-dependent weights w in this thesis comes from [DS89, Section 1.3]. There, the authors endow the norm on path space of Brownian motion with the weight $w(t) = 1+t$, for $t \in \mathbb{R}_+$. Inspired by the exposition in [HL15, Chapter 2], we chose to move for a more general formulation in Defintion 2.42 that also allows us to consider – amongst others – weight functions of the form $w(t) = (1+|t|)^{\beta}$ for $t \in \mathbb{R}$ and $0 < \beta < 1$.

In what follows, we will need to be able to compare the growth behaviour of different weight functions. Therefore, let us state the

Definition 2.46. Let w_1 and w_2 denote two weight functions according to Definition 2.42. We then say that w_1 dominates w_2, if there exists a positive real constant C^* such that

$$\sup_{t\in\mathbb{R}} \frac{w_2(t)}{w_1(t)} \leq C^*. \tag{2.31}$$

We further say that w_1 and w_2 are equivalent, $w_1 \sim w_2$, if w_1 and w_2 dominate each other.

The weighted Hölder space that we are about to define will be the prime object of our interest. We will be treating paths that are elements of this space. These exhibit certain local continuity and global growth properties, that are controlled by admissible moduli of continuity ϱ as well as weight functions w.

Definition 2.47 (Weighted Hölder space)**.** Let X be a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, endowed with a pseudo-quasi-norm $\|\cdot\|$ and a pseudo-homogeneity ϕ according to Definition 2.10. Further, let two weight functions w_1 and w_2 according to Definition 2.42 as well as a ϕ-admissible modulus of continuity ϱ according to Definition 2.39 be given. Let $\|\cdot\|_{\varrho,w_{1,2}} \colon \mathcal{C} \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ denote the map

$$f \mapsto \|f\|_{\varrho,w_{1,2}} \coloneqq \sup_{t\in\mathbb{R}} \frac{\|f(t)\|}{w_1(t)} + \sup_{\substack{s,t\in\mathbb{R}\\ 0<|t-s|\leq 1}} \frac{\|f(t)-f(s)\|}{\varrho(|t-s|)\, w_2(t)}. \tag{2.32}$$

We call $\mathcal{C}_{\varrho,w_{1,2}} \coloneqq \{f \in \mathcal{C} \colon \|f\|_{\varrho,w_{1,2}} < \infty\}$ endowed with $\|\cdot\|_{\varrho,w_{1,2}}$ weighted Hölder space. If $w_1 = w_2 = w$, then we simply write $(\mathcal{C}_{\varrho,w}, \|\cdot\|_{\varrho,w})$.

Remark 2.48. The weighted Hölder space is a pseudo-quasi-normed space. To see this, choose $f_1, f_2 \in \mathcal{C}_{\varrho,w_{1,2}}$ and $\lambda \in \mathbb{K}$. Upon defining $g = f_1 + \lambda f_2$, we immediately see that $g \in \mathcal{C}$, as linear combinations preserve continuity. Moreover, we have

$$\|g\|_{\varrho,w_{1,2}} \leq C_{\mathrm{qs}} \max\{1, \phi(|\lambda|)\}\big(\|f_1\|_{\varrho,w_{1,2}} + \|f_2\|_{\varrho,w_{1,2}}\big) < \infty.$$

Therefore, $\mathcal{C}_{\varrho,w_{1,2}}$ is a vector space over the field $\mathbb{K}$. The defining properties of a pseudo-quasi-norm, as stated in Definition 2.10, can be verified by hand, where the constant C_{qs} in 2.10(b) and the function ϕ in 2.10(c) are directly inherited from $\|\cdot\|$ on the image space X. Moreover, one can also show that the weighted Hölder space is a normed vector space, as long as $\|\cdot\|$ is a norm on the vector space X.

Remark 2.49. In Equation (2.32), we have chosen to evaluate w_2 at the point t. This might seem somewhat arbitrary, as we might have chosen to evaluate w_2 at s, $|s| \vee |t|$ or at $|s| \wedge |t|$. However, condition (2.28) guarantees that all of these choices will give us the same function space, as the pseudo-quasi-norms $\|\cdot\|_{\varrho,w_{1,2}}$, where we alter the evaluation of w_2 to be on t, s, $|s| \vee |t|$ or $|s| \wedge |t|$, are all equivalent in the sense of classical norm equivalence, see also Example 2.16.

Next, we state another technical result. The proof is very classical and can be used for many different settings, where one is having uniform convergence of (Hölder) continuous functions.

Lemma 2.50. *In the context of Definition 2.47, assume the image space X of the functions $f \in \mathcal{C}_{\varrho,w_{1,2}}$ to be a pseudo-quasi-Banach space. Then so is $(\mathcal{C}_{\varrho,w_{1,2}}, \|\cdot\|_{\varrho,w_{1,2}})$.*

Proof. According to Remark 2.48, we are only left to argue completeness of the space. Let $(f_n)_{n\in\mathbb{N}}$ denote a Cauchy sequence in $\mathcal{C}_{\varrho,w_{1,2}}$, i.e. for each $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that, for all $m, n \geq M$, we have that

$$\|f_m - f_n\|_{\varrho,w_{1,2}} \leq \varepsilon.$$

For each $t \in \mathbb{R}$, this implies in particular

$$\|f_m(t) - f_n(t)\| \leq \varepsilon w_1(t),$$

and as w_1 takes values in $(0,\infty)$, we conclude that the X-valued sequence $(f_n(t))_{n\in\mathbb{N}}$ is in fact a Cauchy sequence. As X is assumed to be complete, the sequence converges to some element in X with respect to $\|\cdot\|$. Define now the map $f\colon \mathbb{R}\to X$ by

$$f(t) := \lim_{n\to\infty} f_n(t),$$

which by the above argumentation is pointwise well defined.

We need to verify two properties, i.e. $\lim_{n\to\infty}\|f_n - f\|_{\varrho,w_{1,2}} = 0$ and $f\in\mathcal{C}_{\varrho,w_{1,2}}$. For ε and M as above, let $n\geq M$. We then have, for each $t\in\mathbb{R}$:

$$\frac{\|f(t)-f_n(t)\|}{w_1(t)} = \lim_{m\to\infty}\frac{\|f_m(t)-f_n(t)\|}{w_1(t)} \leq \limsup_{m\to\infty}\|f_m - f_n\|_{\varrho,w_{1,2}} \leq 2\varepsilon.$$

Similarly, for $s,t\in\mathbb{R}$ such that $0<|t-s|\leq 1$ we have

$$\begin{aligned}&\frac{\|(f(t)-f_n(t))-(f(s)-f_n(s))\|}{\varrho(|t-s|)\ w_2(t)}\\ &\quad = \lim_{m\to\infty}\frac{\|(f_m(t)-f_n(t))-(f_m(s)-f_n(s))\|}{\varrho(|t-s|)\ w_2(t)}\\ &\quad \leq \limsup_{m\to\infty}\|f_m - f_n\|_{\varrho,w_{1,2}}\leq\varepsilon,\end{aligned}$$

and therefore $\|f-f_n\|_{\varrho,w_{1,2}}\leq\varepsilon$, hence $\lim_{n\to\infty}\|f-f_n\|_{\varrho,w_{1,2}}=0$. Upon setting $n=M$ we also get $\|f-f_M\|_{\varrho,w_{1,2}}\leq\varepsilon$ and thus we have

$$\|f\|_{\varrho,w_{1,2}} \leq C_{\mathrm{qs}}(\|f-f_M\|_{\varrho,w_{1,2}} + \|f_M\|_{\varrho,w_{1,2}}) < \infty.$$

This implies that $f\in\mathcal{C}_{\varrho,w_{1,2}}$, as $\|f\|_{\varrho,w_{1,2}}<\infty$ and f is continuous, being the limit of continuous functions, uniformly over compact sets. Here, we implicitly use the fact that the continuous weight functions are bounded over compact sets. This concludes our proof. □

Next, we will be taking a closer look at the approximating functions f_m and $\tilde{f}_m$ defined in (2.6) and (2.7). Our goal is to understand in what sense these converge to function series in a reasonable way, as $m\to\infty$. Apart from the pseudo-quasi-norm $\|\cdot\|_{\varrho,w_{1,2}}$, let us define two other specific notions of generalized norms. Let $\|\cdot\|_{w_1}\colon\mathcal{C}\to\overline{\mathbb{R}}_+$ be given by

$$\|f\|_{w_1} := \sup_{t\in\mathbb{R}}\frac{\|f(t)\|}{w_1(t)}. \tag{2.33}$$

For each $q\in(0,1]$, let $\|\cdot\|_{w_2|\varrho;q}\colon\mathcal{C}\to\overline{\mathbb{R}}_+$ be given by

$$\|f\|_{\varrho|w_2;q} := \sup_{\substack{s,t\in\mathbb{R}\\ 0<|t-s|\leq q}}\frac{\|f(t)-f(s)\|}{\varrho(|t-s|)\ w_2(t)}. \tag{2.34}$$

Moreover, set

$$\|\cdot\|_{\varrho|w_2} := \|\cdot\|_{\varrho|w_2;1}.$$

The notation for these different notions of generalized norms is intentionally chosen,

$$\|\cdot\|_{w_1}, \quad \|\cdot\|_{\varrho|w_2;q}, \quad \|\cdot\|_{\varrho|w_2}, \quad \|\cdot\|_{\varrho,w_{1,2}},$$

where the usage of the vertical bar "|" is to indicate that we are looking at the second (Hölder) part of $\|\cdot\|_{\varrho,w_{1,2}}$, and the sole usage of w_1 indicates that we are looking at the first weighted part of $\|\cdot\|_{\varrho,w_{1,2}}$.

Remark 2.51. The map $\|\cdot\|_{w_1}$ again constitutes a pseudo-quasi-norm on the vector space $\mathcal{C}_{\varrho,w_{1,2}}$. On the other hand, the maps $\|\cdot\|_{\varrho|w_2;q}$ for $q \in (0,1]$ lack the nondegeneracy property 2.10(a), because constant functions are mapped to 0. One can think of $\|\cdot\|_{\varrho|w_2;q}$ as being a generalized semi-norm. If we restricted the space $\mathcal{C}_{\varrho,w_{1,2}}$ to those functions that satisfy $f(0) = 0$, then $\|\cdot\|_{\varrho|w_2;q}$ would satisfy all the defining properties of a pseudo-quasi-norm.

We will now state our first approximation result.

Proposition 2.52. *Let $f \in \mathcal{C}$ satisfy $\|f\|_{\varrho|w_2} < \infty$, for an admissible modulus of continuity ϱ according to Definition 2.39 and a weight function w_2. Further, let $(f_m)_{m\in\mathbb{N}_0}$ be the sequence defined by (2.6), and w_1 be a weight function that dominates w_2. Then we have*

$$\|f - f_m\|_{w_1} = \mathcal{O}(\varrho(2^{-(m+1)})), \quad m \in \mathbb{N}_0.$$

Proof. Let $m \in \mathbb{N}_0$ and, without loss of generality, choose $t \in \mathbb{R} \setminus \mathbb{D}^{m+1}$, because otherwise $f_m(t) = f(t)$, hence $\|f(t) - f_m(t)\| = 0$. Let t_l, t_u denote the neighboring dyadic rationals of order $m+1$ such that $t_l < t < t_u$. Then we have by Proposition 2.7

$$\begin{aligned}
\|f(t) - f_m(t)\| &= \Big\|f(t) - \big(f(t_l) + \tfrac{t-t_l}{t_u-t_l}(f(t_u) - f(t_l))\big)\Big\| \\
&= \Big\|(f(t) - f(t_l)) + \big(-\tfrac{t-t_l}{t_u-t_l}(f(t_u) - f(t_l))\big)\Big\| \\
&\le C_{\mathrm{qs}}\big(\|f(t) - f(t_l)\| + \phi\big(\tfrac{t-t_l}{t_u-t_l}\big)\|f(t_u) - f(t_l)\|\big) \\
&\le C_{\mathrm{qs}}\big(\|f(t) - f(t_l)\| + \phi(1)\|f(t_u) - f(t_l)\|\big).
\end{aligned} \tag{2.35}$$

Making use of (2.28), (2.31) as well as the fact that $\phi(1) \ge 1$, a division by $w_1(t)$ yields the upper bound

$$\begin{aligned}
\frac{\|f(t) - f_m(t)\|}{w_1(t)} &\le C_{\mathrm{qs}}C^*\varrho(|t-t_l|)\frac{\|f(t) - f(t_l)\|}{\varrho(|t-t_l|)\ w_2(t)} \\
&\quad + C_{\mathrm{qs}}CC^*\phi(1)\varrho(|t_u-t_l|)\frac{\|f(t_u) - f(t_l)\|}{\varrho(|t_u-t_l|)\ w_2(t_l)} \\
&\le 2C_{\mathrm{qs}}CC^*\phi(1)\|f\|_{\varrho|w_2;2^{-(m+1)}}\varrho(2^{-(m+1)}) \\
&\le 2C_{\mathrm{qs}}CC^*\phi(1)\|f\|_{\varrho|w_2}\varrho(2^{-(m+1)}),
\end{aligned} \tag{2.36}$$

which concludes our proof. □

Remark 2.53. Note that we are very generous in (2.36), giving away the fact that the increments are very small, in estimating

$$\|f\|_{\varrho|w_2;2^{-(m+1)}} \le \|f\|_{\varrho|w_2}.$$

This sharper bound will be used in the proofs of Proposition 2.54 and Corollary 2.55.

Let $\mathcal{C}_{\varrho|w,0}$ denote the set of all $f \in \mathcal{C}$ that satisfy

$$\|f\|_{\varrho|w} < \infty, \quad \lim_{q\searrow 0} \|f\|_{\varrho|w,q} = 0. \tag{2.37}$$

The next proposition deals with an approximation of elements $f \in \mathcal{C}_{\varrho|w,0}$.

Proposition 2.54. *Let $f \in \mathcal{C}$ satisfy $\|f\|_{\varrho|w} < \infty$, for an admissible modulus of continuity ϱ according to Definition 2.39 and a weight function w. Further, let $(f_m)_{m\in\mathbb{N}_0}$ be the sequence defined by (2.6). Then we have*

$$\|f - f_m\|_{\varrho|w} = \mathcal{O}(\|f\|_{\varrho|w;2^{-(m+1)}}), \quad m \in \mathbb{N}_0.$$

Proof. Let $m \in \mathbb{N}_0$ and choose $s, t \in \mathbb{R} \setminus \mathbb{D}^{(m+1)}$. Assume $2^{-(m+1)} < |t-s| \le 1$ and let $t_l < t < t_u$ as well as $s_l < s < s_u$ be the neighboring dyadic rationals of order $m+1$. First of all, we have

$$\|(f(t) - f_m(t)) - (f(s) - f_m(s))\| \le C_{\mathrm{qs}}(\|f(t) - f_m(t)\| + \phi(1)\|f(s) - f_m(s)\|),$$

hence, making use of (2.36), we obtain

$$\begin{aligned}\frac{\|(f(t) - f_m(t)) - (f(s) - f_m(s))\|}{\varrho(|t-s|)\; w(t)}&\\ &\le C_{\mathrm{qs}}\Big\{\frac{\|f(t) - f_m(t)\|}{\varrho(|t-s|)\; w(t)} + C\phi(1)\frac{\|f(s) - f_m(s)\|}{\varrho(|t-s|)\; w(s)}\Big\}\\ &\le 4C_{\mathrm{qs}}^2 C^2 \phi(1)^2 \|f\|_{\varrho|w;2^{-(m+1)}}.\end{aligned}$$

Next, let $m \in \mathbb{N}_0$ and choose $s, t \in \mathbb{R} \setminus \mathbb{D}^{(m+1)}$. Assume $0 < |t-s| \le 2^{-(m+1)}$ and that s and t belong to the same dyadic interval of order $m+1$, i.e. $d_l < s, t < d_u$ for the neighboring dyadic rationals d_l, d_u of order $m+1$. We then estimate

$$\|(f(t) - f_m(t)) - (f(s) - f_m(s))\| \le C_{\mathrm{qs}}(\|f(t) - f(s)\| + \phi(1)\|f_m(t) - f_m(s)\|).$$

The first summand allows the direct estimate

$$\frac{\|f(t) - f(s)\|}{\varrho(|t-s|)\; w(t)} \le \|f\|_{\varrho|w;2^{-(m+1)}},$$

whereas estimating the latter part yields by Proposition 2.7 up to the constant $\phi(1)$

$$\|f_m(t) - f_m(s)\| = \|\tfrac{t-s}{d_u - d_l}\big(f(d_u) - f(d_l)\big)\| \le \phi\big(\tfrac{|t-s|}{d_u - d_l}\big)\|f(d_u) - f(d_l)\|,$$

hence

$$\begin{aligned}\frac{\|f_m(t) - f_m(s)\|}{\varrho(|t-s|)\; w(t)} &\le C\phi\big(\tfrac{|t-s|}{d_u - d_l}\big)\frac{\varrho(d_u - d_l)}{\varrho(|t-s|)}\|f\|_{\varrho|w;2^{-(m+1)}}\\ &\le CC_\phi\|f\|_{\varrho|w;2^{-(m+1)}},\end{aligned}$$

making use of (2.26). In the case that s and t (assuming w.l.o.g. that $s < t$) do not belong to the same dyadic interval of order $m+1$, i.e. $d_l < s < d < t < d_u$ for the neighboring dyadic rationals d_l, d, d_u of order $m+1$, we again estimate

$$\|(f(t) - f_m(t)) - (f(s) - f_m(s))\| \leq C_{\mathrm{qs}}(\|f(t) - f(s)\| + \|f_m(t) - f_m(s)\|).$$

The first part allows the same estimate as above, whereas in the second part we add the term $f(d)$:

$$\begin{aligned}\|f_m(t) - f_m(s)\| &= \|(f_m(t) - f(d)) + (f(d) - f_m(s))\| \\ &\leq C_{\mathrm{qs}}(\|f_m(t) - f(d)\| + \|f(d) - f_m(s)\|).\end{aligned}$$

In the same manner as above, we have

$$\|f_m(t) - f(d)\| \leq \phi\big(\tfrac{t-d}{d_u-d}\big)\|f(d_u) - f(d)\|,$$

hence

$$\begin{aligned}\frac{\|f_m(t) - f(d)\|}{\varrho(|t-s|)\; w(t)} &\leq C\phi\big(\tfrac{t-d}{d_u-d}\big)\frac{\varrho(d_u - d)}{\varrho(t-d)}\|f\|_{\varrho|w;2^{-(m+1)}} \\ &\leq CC_\phi\|f\|_{\varrho|w;2^{-(m+1)}}.\end{aligned}$$

For the last term, we apply the same line of reasoning, using a neat trick of rewriting the linear interpolation with respect to the lower boundary point in terms of the upper boundary point, i.e.

$$f_m(s) = f(d) - \tfrac{d-s}{d-d_l}(f(d) - f(d_l)).$$

This yields

$$\|f(d) - f_m(s)\| \leq \phi\big(\tfrac{d-s}{d-d_l}\big)\|f(d) - f(d_l)\|,$$

hence

$$\begin{aligned}\frac{\|f(d) - f_m(s)\|}{\varrho(|t-s|)\; w(t)} &\leq C\phi\Big(\tfrac{d-s}{d-d_l}\Big)\frac{\varrho(d-d_l)}{\varrho(d-s)}\|f\|_{\varrho|w;2^{-(m+1)}} \\ &\leq CC_\phi\|f\|_{\varrho|w;2^{-(m+1)}}.\end{aligned}$$

Combining the above yields

$$\sup_{\substack{s,t\in\mathbb{R}\setminus\mathbb{D}^{(m+1)} \\ 0<|t-s|\leq 1}} \frac{\|(f(t) - f_m(t)) - (f(s) - f_m(s))\|}{\varrho(|t-s|)\; w(t)} = \mathcal{O}(\|f\|_{\varrho|w;2^{-(m+1)}}).$$

If $s, t \in \mathbb{D}^{(m+1)}$, then we have $f_m(t) = f(t)$ as well as $f_m(s) = f(s)$, hence this case is of no interest for the convergence.

This leaves us with finding an estimate for the case that either t or s is a dyadic rational of order up to $m+1$. To this end, assume without loss of generality that $s \in \mathbb{D}^{(m+1)}$

and $t \in \mathbb{R} \setminus \mathbb{D}^{(m+1)}$ as well as $0 < |t-s| \leq 1$ and $s < t$. Let $t_l < t < t_u$ denote the two neighboring dyadic rationals of order $m+1$. Then we have that

$$s \leq t_l < t < t_u,$$

which is the crucial observation. The case $t_l < s < t$ is not possible, because $s \in \mathbb{D}^{(m+1)}$, and when t_l and t_u are the neighboring dyadic rationals of order $m+1$, then $t_l < s < t_u$ fails to hold, since in this case s and t_u would be neighboring t; the case $t_l = s$ is possible and hence not excluded. Now we can write

$$\begin{aligned}\|(f(t) - f_m(t)) - (f(s) - f_m(s))\| &= \|(f(t) - f_m(t)) - (f(s) - f(s))\| \\ &= \|f(t) - f_m(t)\| \leq C_{\mathrm{qs}}\big(\|f(t) - f(t_l)\| + \phi\big(\tfrac{t-t_l}{t_u-t_l}\big)\|f(t_u) - f(t_l)\|\big).\end{aligned}$$

The first part allows the weighted estimate

$$\frac{\|f(t) - f(t_l)\|}{\varrho(|t-s|)\, w(t)} \leq C \frac{\|f(t) - f(t_l)\|}{\varrho(t-t_l)\, w(t_l)} \leq C\|f\|_{\varrho|w;2^{-(m+1)}},$$

whereas the second part allows the weighted estimate

$$\begin{aligned}\phi\big(\tfrac{t-t_l}{t_u-t_l}\big)\frac{\|f(t_u) - f(t_l)\|}{\varrho(|t-s|)\, w(t)} &\leq C\phi\big(\tfrac{t-t_l}{t_u-t_l}\big)\frac{\varrho(t_u-t_l)}{\varrho(t-t_l)}\frac{\|f(t_u) - f(t_l)\|}{\varrho(t_u-t_l)\, w(t_l)} \\ &\leq CC_\phi\|f\|_{\varrho|w;2^{-(m+1)}}.\end{aligned}$$

Collecting all of the above estimates, we have

$$\|f - f_m\|_{\varrho|w} = \mathcal{O}(\|f\|_{\varrho|w;2^{-(m+1)}}), \tag{2.38}$$

which concludes the proof. □

As a consequence of Propositions 2.52 and 2.54 as well as Remark 2.53 we obtain the

Corollary 2.55. *Let $f \in \mathcal{C}$ satisfy $\|f\|_{\varrho|w_2} < \infty$, for an admissible modulus of continuity ϱ and a weight function w_2. Further, let $(f_m)_{m\in\mathbb{N}_0}$ be the sequence defined by (2.6), and w_1 be a weight function that dominates w_2. We then have*

$$\|f - f_m\|_{\varrho,w_{1,2}} = \mathcal{O}(\|f\|_{\varrho|w_2;2^{-(m+1)}}), \quad m \in \mathbb{N}_0. \tag{2.39}$$

Proof. First, note that in line with Remark 2.53 the last line in the estimate (2.36) can be replaced by another upper bound, which is

$$\|f - f_m\|_{w_1} \leq 2C_{\mathrm{qs}}CC^*\phi(1)\varrho(1)\|f\|_{\varrho|w_2;2^{-(m+1)}}.$$

But together with Proposition 2.54, this already implies the assertion. □

Remark 2.56. Note that the statement (2.39) implies the first strong approximation property: In the context of Corollary 2.55, if f additionally satisfies

$$\lim_{m\to\infty} \|f\|_{\varrho|w_2;2^{-(m+1)}} = 0,$$

then Corollary 2.55 yields the convergence statement

$$\lim_{m\to\infty} \|f - f_m\|_{\varrho,w_{1,2}} = 0.$$

Remark 2.57. For the sake of clarity, let us note that Corollary 2.55 implies that as long as $f \in \mathcal{C}_{\varrho,w_{1,2}}$, we immediately have that $f_m \in \mathcal{C}_{\varrho,w_{1,2}}$ for all $m \in \mathbb{N}_0$. This can be directly seen through (2.39) and the fact that $\|\cdot\|_{\varrho,w_{1,2}}$ satisfies the quasi-subadditivity property (2.11), yielding

$$\begin{aligned}\|f_m\|_{\varrho,w_{1,2}} &\le C_{\mathrm{qs}}(\|f\|_{\varrho,w_{1,2}} + \|f - f_m\|_{\varrho,w_{1,2}}) \\ &= C_{\mathrm{qs}}\|f\|_{\varrho,w_{1,2}} + \mathcal{O}(\|f\|_{\varrho|w_2;2^{-(m+1)}}) < \infty.\end{aligned}$$

2.5. Little weighted Hölder spaces

We are finally ready to prove the most important mode of the convergence, which will be the content of Theorem 2.61. The motivation behind this is that we want to consider spaces which are separable. In order to accomplish this, we will need to pass to a subspace of the weighted Hölder space, which will be defined as follows.

Definition 2.58 (Little weighted Hölder space)**.** In the context of Definition 2.47, let $\mathcal{C}_{\varrho,w_{1,2},0} \subset \mathcal{C}_{\varrho,w_{1,2}}$ denote the subset of all $f \in \mathcal{C}_{\varrho,w_{1,2}}$ that further satisfy the following conditions:

(a) f grows slower than w_1, i.e.

$$\lim_{|t|\to\infty} \frac{\|f(t)\|}{w_1(t)} = 0; \tag{2.40}$$

(b) weighted Hölder constants vanish uniformly for small time increments, i.e.

$$\lim_{q\searrow 0} \|f\|_{\varrho|w_2,q} = 0; \tag{2.41}$$

(c) weighted Hölder constants vanish uniformly for distant times, i.e.

$$\lim_{n\to\infty} \sup_{\substack{s,t\in\mathbb{R}\setminus[-n,n]\\ 0<|t-s|\le 1}} \frac{\|f(t)-f(s)\|}{\varrho(|t-s|)\; w_2(t)} = 0. \tag{2.42}$$

We call $(\mathcal{C}_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$ little weighted Hölder space.

Lemma 2.59. *In the context of Definition 2.58, assume the image space X of the functions $f \in \mathcal{C}_{\varrho,w_{1,2},0}$ to be a pseudo-quasi-Banach space. Then so is $(\mathcal{C}_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$.*

Proof. One can easily verify that $(\mathcal{C}_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$ is indeed a pseudo-quasi-normed vector space. Therefore, we will only argue that it is also complete. First of all, observe that $\mathcal{C}_{\varrho,w_{1,2},0} \subset \mathcal{C}_{\varrho,w_{1,2}}$ and that, by Lemma 2.50, $(\mathcal{C}_{\varrho,w_{1,2}}, \|\cdot\|_{\varrho,w_{1,2}})$ is indeed complete as long as X is.

Let $(f_n)_{n\in\mathbb{N}} \in \mathcal{C}_{\varrho,w_{1,2},0}$ denote a Cauchy sequence. Then it is obviously also a Cauchy sequence in $(\mathcal{C}_{\varrho,w_{1,2}}, \|\cdot\|_{\varrho,w_{1,2}})$. By Lemma 2.50, there exists an $f \in \mathcal{C}_{\varrho,w_{1,2}}$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{\varrho,w_{1,2}} = 0.$$

If we can show that actually $f \in \mathcal{C}_{\varrho,w_{1,2},0}$, we are done.

Let $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that, for all $n \geq N$ we have

$$\|f - f_n\|_{\varrho,w_{1,2}} \leq \frac{\varepsilon}{2C_{\text{qs}}}.$$

As f_N satisfies (2.40), there exists some $m > 0$ such that, for all $t \in \mathbb{R} \setminus [-m, m]$ it holds that

$$\frac{\|f_N(t)\|}{w_1(t)} \leq \frac{\varepsilon}{2C_{\text{qs}}}.$$

This however implies already that f satisfies (2.40), because

$$\frac{\|f(t)\|}{w_1(t)} \leq \frac{C_{\text{qs}}}{w_1(t)}\big(\|f_N(t)\| + \|f(t) - f_N(t)\|\big) \leq C_{\text{qs}}\Big(\frac{\|f_N(t)\|}{w_1(t)} + \|f - f_N\|_{\varrho,w_{1,2}}\Big) \leq \varepsilon.$$

Next, f_N also satisfies (2.41), so there exists some $q^* \in (0,1)$ such that for all $q \in (0, q^*)$ we have

$$\sup_{\substack{s,t\in\mathbb{R}\\ 0<t-s\leq q}} \frac{\|f_N(t) - f_N(s)\|}{\varrho(|t-s|)\, w_2(t)} \leq \frac{\varepsilon}{2C_{\text{qs}}}.$$

This implies that f satisfies (2.41), because

$$\sup_{\substack{s,t\in\mathbb{R}\\ 0<t-s\leq q}} \frac{\|f(t) - f(s)\|}{\varrho(|t-s|)\, w_2(t)} \leq C_{\text{qs}}\Big(\sup_{\substack{s,t\in\mathbb{R}\\ 0<t-s\leq q}} \frac{\|f_N(t) - f_N(s)\|}{\varrho(|t-s|)\, w_2(t)} + \|f - f_N\|_{\varrho,w_{1,2}}\Big) \leq \varepsilon.$$

Finally, f_N also satisfies (2.42), so there exists some $M > 0$ such that, for all $m \geq M$, we have

$$\sup_{\substack{s,t\in\mathbb{R}\setminus[-m,m]\\ 0<t-s\leq 1}} \frac{\|f_N(t) - f_N(s)\|}{\varrho(|t-s|)\, w_2(t)} \leq \frac{\varepsilon}{2C_{\text{qs}}}.$$

Consequently, it holds that

$$\sup_{\substack{s,t\in\mathbb{R}\setminus[-m,m]\\ 0<t-s\leq 1}} \frac{\|f(t) - f(s)\|}{\varrho(|t-s|)\, w_2(t)} \leq C_{\text{qs}}\Big(\sup_{\substack{s,t\in\mathbb{R}\setminus[-m,m]\\ 0<t-s\leq 1}} \frac{\|f_N(t) - f_N(s)\|}{\varrho(|t-s|)\, w_2(t)} + \|f - f_N\|_{\varrho,w_{1,2}}\Big) \leq \varepsilon.$$

Therefore, $f \in \mathcal{C}_{\varrho,w_{1,2},0}$ and we are done. □

Remark 2.60. Let us briefly comment on the terminology that we have chosen in Definition 2.58. Let $\alpha \in (0,1)$ and M_α be the vector space of all $f\colon [0,1] \to \mathbb{R}$ such that

$$\|f\|_\alpha := \sup_{0\leq s<t\leq 1} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty.$$

The semi-normed space $(M_\alpha, \|\cdot\|_\alpha)$ is classically termed Hölder space, but it lacks the important property of being separable. Therefore, the little Hölder space is introduced as the subset $M_{\alpha,0} \subset M_\alpha$ of functions that additionally satisfy

$$\lim_{\delta \searrow 0} \sup_{\substack{0 \le s < t \le 1 \\ 0 < t-s \le \delta}} \frac{|f(t) - f(s)|}{(t-s)^\alpha} = 0. \tag{2.43}$$

This space is classically shown to be separable by Ciesielski's isomorphism, see [Cie60]. Alternatively, one can show that polynomials with rational coefficients are dense. The generalization of the arguments behind these statements will be provided below. Note that condition (2.43) corresponds to the property (2.41). However, since we are dealing with unbounded domains, we have to consider weight functions as introduced in Definition 2.42 and additionally require that apart from (2.41), the conditions (2.40) and (2.42) also hold.

We are finally able to formulate and prove the

Theorem 2.61. *Let $f \in \mathcal{C}_{\varrho,w_{1,2},0}$. Assume that w_1 is non-decreasing and dominating w_2 in the sense of Definition 2.46. Further, let $(\tilde f_m)_{m\in\mathbb{N}_0}$ be the sequence defined in (2.7). Then*

$$\lim_{m\to\infty} \|f - \tilde f_m\|_{\varrho,w_{1,2}} = 0,$$

where the order of convergence depends on (2.40), (2.41) and (2.42), see the estimates (2.44) as well as (2.45).

Proof. Let us first recall that

$$\|f - \tilde f_m\|_{\varrho,w_{1,2}} = \|f - \tilde f_m\|_{w_1} + \|f - \tilde f_m\|_{\varrho|w_2}.$$

For the first part, $\|f - \tilde f_m\|_{w_1}$, let $m \in \mathbb{N}$ and $t \in (-m, m)$, then $\tilde f_m(t) = f_m(t)$, and by Proposition 2.52 as well as Remark 2.53, we have

$$\frac{\|f(t) - \tilde f_m(t)\|}{w_1(t)} = \frac{\|f(t) - f_m(t)\|}{w_1(t)} \le 2C_{\mathrm{qs}} C C^* \phi(1) \varrho(1) \|f\|_{\varrho|w_2;2^{-(m+1)}}.$$

If $t \ge m$, then $\tilde f_m(t) = f(m)$, hence

$$\begin{aligned}
\frac{\|f(t) - \tilde f_m(t)\|}{w_1(t)} &\le C_{\mathrm{qs}} \frac{\|f(t)\|}{w_1(t)} + \phi(1) C_{\mathrm{qs}} \frac{\|f(m)\|}{w_1(t)} \\
&\le C_{\mathrm{qs}} \frac{\|f(t)\|}{w_1(t)} + \phi(1) C_{\mathrm{qs}} \frac{\|f(m)\|}{w_1(m)} \le 2C_{\mathrm{qs}} \phi(1) \sup_{t \ge m} \frac{\|f(t)\|}{w_1(t)}.
\end{aligned}$$

The case $t \le -m$ can be argued in a similar way. Pulling all of these estimates together, we have, up to a constant

$$\|f - \tilde f_m\|_{w_1} \lesssim \max\Big\{ \|f\|_{\varrho|w_2;2^{-(m+1)}}, \sup_{\substack{t\in\mathbb{R} \\ |t| \ge m}} \frac{\|f(t)\|}{w_1(t)} \Big\}, \tag{2.44}$$

which vanishes, as $m \to \infty$.

Now we will estimate $\|f - f_m\|_{\varrho|w_2}$. We will treat the cases $-m < s < t < m$, as well as $-m < s < m \leq t$ and $m \leq s < t$ separately. Note that we always implicitly assume, without loss of generality, that $0 < t - s \leq 1$. First of all, let $-m < s < t < m$, then $\tilde{f}_m(t) = f_m(t)$ and $\tilde{f}_m(s) = f_m(s)$, hence

$$\frac{\|(f(t) - \tilde{f}_m(t)) - (f(s) - \tilde{f}_m(s))\|}{\varrho(t-s)\ w_2(t)} = \frac{\|(f(t) - f_m(t)) - (f(s) - f_m(s))\|}{\varrho(t-s)\ w_2(t)} \leq \|f - f_m\|_{\varrho|w_2} = \mathcal{O}(\|f\|_{\varrho|w_2;2^{-(m+1)}}),$$

where the last equality readily follows from (2.38), because the assumptions of Proposition 2.54 are met. For $m \leq s < t$, we can use the fact that $\tilde{f}_m(t) = \tilde{f}_m(s) = f(m)$ to obtain

$$\frac{\|(f(t) - \tilde{f}_m(t)) - (f(s) - \tilde{f}_m(s))\|}{\varrho(t-s)\ w_2(t)} = \frac{\|f(t) - f(s)\|}{\varrho(t-s)\ w_2(t)} \leq \sup_{\substack{m \leq s < t < \infty \\ 0 < t-s \leq 1}} \frac{\|f(t) - f(s)\|}{\varrho(t-s)\ w_2(t)},$$

which, by (2.42), vanishes as $m \to \infty$. Next, assume that $s < m \leq t$, then we have to distinguish two cases. Let $2^{-(m+1)} < t - s \leq 1$. Then it follows that, for the two dyadic rationals s_l, s_u of order $m+1$ neighboring s, the inequality $s_l \leq s \leq s_u < t$ holds. We first make the observation that

$$\|(f(t) - \tilde{f}_m(t)) - (f(s) - \tilde{f}_m(s))\| \leq C_{\mathrm{qs}}\|f(t) - f(m)\| + C_{\mathrm{qs}}\phi(1)\|f(s) - f_m(s)\|.$$

The first summand, upon being weighted by $\varrho(t-s)w_2(t)$, satisfies the upper bound

$$C_{\mathrm{qs}}\frac{\|f(t) - f(m)\|}{\varrho(t-s)\ w_2(t)} \leq C_{\mathrm{qs}} \sup_{\substack{m \leq s < t < \infty \\ 0 < t-s \leq 1}} \frac{\|f(t) - f(s)\|}{\varrho(t-s)\ w_2(t)}.$$

The second part can be rewritten as follows:

$$\begin{aligned}\|f(s) - f_m(s)\| &\leq C_{\mathrm{qs}}\Big(\|f(s) - f(s_l)\| + \phi\Big(\tfrac{s-s_l}{s_u-s_l}\Big)\|f(s_u) - f(s_l)\|\Big) \\ &\leq C_{\mathrm{qs}}\|f(s) - f(s_l)\| + C_{\mathrm{qs}}\phi(1)\|f(s_u) - f(s_l)\|.\end{aligned}$$

Making use of the fact that $s - s_l \leq s_u - s_l < t - s$, we obtain the upper bound

$$\frac{\|f(s) - f_m(s)\|}{\varrho(t-s)\ w_2(t)} \leq C_{\mathrm{qs}}C(1 + \phi(1))\|f\|_{\varrho|w_2,2^{-(m+1)}}.$$

Now we are left to treat the case when $0 < t - s \leq 2^{-(m+1)}$. We first make the observation that

$$\|(f(t) - \tilde{f}_m(t)) - (f(s) - \tilde{f}_m(s))\| \leq C_{\mathrm{qs}}\|f(t) - f(s)\| + C_{\mathrm{qs}}\phi(1)\|f(m) - f_m(s)\|.$$

The first part allows a straightforward weighted upper bound

$$\frac{\|f(t)-f(s)\|}{\varrho(t-s)\ w_2(t)} \le C_{\mathrm{qs}} \|f\|_{\varrho|w_2, 2^{-(m+1)}},$$

whereas for the second part we argue differently. If $s \in \mathbb{D}^{(m+1)}$, then $f_m(s) = f(s)$ and consequently $\|f(m) - f_m(s)\| = \|f(m) - f(s)\|$. This yields

$$\frac{\|f(m)-f_m(s)\|}{\varrho(t-s)\ w_2(t)} \le C \frac{\|f(m)-f(s)\|}{\varrho(m-s)\ w_2(m)} \le C\|f\|_{\varrho|w_2, 2^{-(m+1)}}.$$

If $s \notin \mathbb{D}^{(m+1)}$, we consider the neighboring dyadic rationals s_l, s_u of order $m+1$. Because $t - s \le 2^{-(m+1)}$ we thus have $s_l < s < s_u = m \le t$, the case $s_u < m$ would yield a contradiction. Recall that we can write

$$f_m(s) = f(m) - \tfrac{m-s}{m-s_l}\big(f(m) - f(s_l)\big).$$

But then

$$\|f(m) - f_m(s)\| \le \phi\big(\tfrac{m-s}{m-s_l}\big)\|f(m) - f(s_l)\|,$$

and we finally arrive at

$$\frac{\|f(m)-f_m(s)\|}{\varrho(t-s)\ w_2(t)} \le \phi\big(\tfrac{m-s}{m-s_l}\big)\frac{\varrho(m-s_l)}{\varrho(m-s)}\frac{\|f(m)-f(s_l)\|}{\varrho(m-s_l)\ w_2(m)} \le C_\phi\|f\|_{\varrho|w_2, 2^{-(m+1)}}.$$

In pulling all of these estimates together, we obtain, up to a constant,

$$\|f - \tilde{f}_m\|_{\varrho|w_2} \lesssim \max\left\{\|f\|_{\varrho|w_2, 2^{-(m+1)}}, \sup_{\substack{s,t\in\mathbb{R}\setminus[-m,m]\\ 0<|t-s|\le 1}} \frac{\|f(t)-f(s)\|}{\varrho(|t-s|)w_2(t)}\right\}, \tag{2.45}$$

which vanishes, as $m \to \infty$. □

Remark 2.62. For the sake of clarity, let us note that Theorem 2.61 implies that as long as $f \in \mathcal{C}_{\varrho,w_{1,2}}$, we immediately have that $\tilde{f}_m \in \mathcal{C}_{\varrho,w_{1,2}}$ for all $m \in \mathbb{N}_0$. This can be directly seen through the estimates (2.44) and (2.45) as well as the fact that $\|\cdot\|_{\varrho,w_{1,2}}$ satisfies the quasi-subadditivity property (2.11), yielding

$$\|\tilde{f}_m\|_{\varrho,w_{1,2}} \le C_{\mathrm{qs}}(\|f\|_{\varrho,w_{1,2}} + \|f - \tilde{f}_m\|_{\varrho,w_{1,2}}) < \infty.$$

Remark 2.63. Note that Theorem 2.61 essentially relies on the assumption that w_1 is non-decreasing. In Corollary 2.67 below, we will also require that $\lim_{|t|\to\infty} w_1(t) = \infty$. These assumptions are necessary due to the form of the functions $\tilde{f}_m$. One could try to choose different functions, in order to loosen these two requirements. An interesting approach might be to consider $\tilde{g}_m$ which equals f_m on $[-m, m]$, vanishes outside of $[-m-1, m+1]$, and is smooth between the points $(m, f_m(m)), (m+1, 0)$ and $(-m, f_m(-m)), (-m-1, 0)$ respectively. This might be accomplished by means of applying an appropriate mollifier to f_m. However, it would require a more involved treatment of the Theorem 2.61, as the current setup essentially relies on linearity arguments, not integral estimates.

Remark 2.64. As already hinted, the family $\mathcal{M}_2^{\mathrm{FS}}$ given in Definition 2.1 does not constitute a Schauder basis in $\mathcal{C}_{\varrho,w_{1,2},0}$. However, if the image space X of the functions in $\mathcal{C}_{\varrho,w_{1,2},0}$ is separable, then we will conclude separability of the respective little weighted Hölder space, see Corollaries 2.67 and 2.70. This result will come in handy when dealing with Brownian motion on a real separable Banach space.

Before we state and prove Corollary 2.67, let us first provide the

Lemma 2.65. *For $s,t \in \mathbb{R}$ and $l \in \mathbb{Z}$ as well as $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$, we have*

$$|\theta_l(t) - \theta_l(s)| \leq |t-s| \wedge 1,$$
$$|\tau_{j,k}(t) - \tau_{j,k}(s)| \leq 2^{j/2}|t-s| \wedge 2^{-j/2-1},$$

where θ_l and $\tau_{j,k}$ are the elements of the Faber–Schauder system $\mathcal{M}_2^{\mathrm{FS}}$, as given in Definition 2.1.

Proof. We will show the claim by means of integral estimates.

Recall the definition of θ_l, see (2.2). From this, we immediately see that, if $|t-s| > 1$, the best upper bound is

$$|\theta_l(t) - \theta_l(s)| \leq |\theta_l(l) - \theta_l(l-1)| = |\theta_l(l)| = 1.$$

Note that for each $l \in \mathbb{Z}$ and $t \in [l-1, l+1]$, we can write

$$\theta_l(t) = \int_{l-1}^{t} \mathcal{K}_l(s) \, \mathrm{d}s,$$

where $\mathcal{K}_l$ denotes the step function

$$\mathcal{K}_l = 1_{[l-1,l)} - 1_{[l,l+1)}.$$

But this implies that, for $0 < |t-s| \leq 1$, we actually have

$$|\theta_l(t) - \theta_l(s)| \leq \int_{|s|\wedge|t|}^{|s|\vee|t|} |\mathcal{K}_l(s)| \, \mathrm{d}s \leq |t-s|.$$

In a similar manner, recall the definition of $\tau_{j,k}$, see (2.3). If $|t-s| > 2^{-j-1}$, then

$$|\tau_{j,k}(t) - \tau_{j,k}(s)| \leq \tau_{j,k}(\tfrac{2k+1}{2^{j+1}}) - \tau_{j,k}(\tfrac{k}{2^j})| = |\tau_{j,k}(\tfrac{2k+1}{2^{j+1}})| = 2^{-j/2-1}.$$

Note that for each $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$ and $t \in \operatorname{supp}(\tau_{j,k})$ we can write

$$\tau_{j,k}(t) = \int_{\frac{k}{2^j}}^{t} \mathcal{H}_{j,k}(s) \, \mathrm{d}s,$$

where $\mathcal{H}_{j,k}$ denotes the step function

$$\mathcal{H}_{j,k}(s) = 2^{j/2}\mathcal{H}(2^j s - k),$$

and $\mathcal{H}$ denotes the Haar mother wavelet

$$\mathcal{H} = 1_{[0,1/2)} - 1_{[1/2,1)}.$$

But this implies that, for $0 < |t-s| \leq 2^{-j-1}$, we actually have

$$|\tau_{j,k}(t) - \tau_{j,k}(s)| \leq \int_{|s|\wedge|t|}^{|s|\vee|t|} |\mathcal{H}_{j,k}(s)| \, \mathrm{d}s \leq 2^{j/2}|t-s|,$$

which concludes our proof. □

Remark 2.66. Lemma 2.65 shows that the Faber–Schauder system of the second kind $\mathcal{M}_2^{\mathrm{FS}}$ consists of Lipschitz-continuous maps, which will be of importance in Chapter 4.

Corollary 2.67. *In the setting of Theorem 2.61, assume that the image space X of the functions $f \in \mathcal{C}_{\varrho,w_{1,2},0}$ is a separable pseudo-quasi-normed space, that $\lim_{|t|\to\infty} w_1(t) = \infty$ holds and that ϱ is ϕ-admissible according to Definition 2.39. Then $(\mathcal{C}_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$ is also separable, i.e. there exists a countable dense subset $E \subset \mathcal{C}_{\varrho,w_{1,2},0}$.*

Proof. Let E^X be a countable dense subset in $(X, \|\cdot\|)$. Further, let $\varepsilon > 0$ and $f \in \mathcal{C}_{\varrho,w_{1,2},0}$. According to Theorem 2.61, we can find an $M \in \mathbb{N}_0$, such that for all $m \in \mathbb{N}_0$ that satisfy $m \geq M$, we have

$$\|f - \tilde{f}_m\|_{\varrho,w_{1,2}} \leq \frac{\varepsilon}{2C_{\mathrm{qs}}}.$$

Let $m \geq M$ be a fixed integer, and set, for some X-valued coefficients a_l and $b_{j,k}$, which we will specify below, the function $g_m\colon \mathbb{R} \to X$ by

$$g_m(t) := \sum_{l\in\mathbb{Z}} a_l \theta_l(\gamma_m(t)) + \sum_{j=0}^{m} \sum_{k\in\mathbb{Z}} b_{j,k}\tau_{j,k}(\gamma_m(t)), \quad t \in \mathbb{R}.$$

Note the similarities between g_m and $\tilde{f}_m$. Moreover, almost all $\theta_l(\gamma_m(\cdot))$ and $\tau_{j,k}(\gamma_m(\cdot))$ are zero, hence, only finitely many coefficients a_l and $b_{j,k}$ need to be specified. Further, if we choose the coefficients of g_m close enough to those of $\tilde{f}_m$ such that $\|\tilde{f}_m - g_m\|_{\varrho,w_{1,2}} \leq \varepsilon/(2C_{\mathrm{qs}})$ holds, we obtain that

$$\begin{aligned}\|f - g_m\|_{\varrho,w_{1,2}} &\leq C_{\mathrm{qs}}\Big(\|f - \tilde{f}_m\|_{\varrho,w_{1,2}} + \|\tilde{f}_m - g_m\|_{\varrho,w_{1,2}}\Big) \\ &\leq C_{\mathrm{qs}}\Big(\frac{\varepsilon}{2C_{\mathrm{qs}}} + \|\tilde{f}_m - g_m\|_{\varrho,w_{1,2}}\Big) \leq \varepsilon.\end{aligned}$$

Note that the distance between $\tilde{f}_m$ and g_m can be estimated pointwise:

$$\begin{aligned}&\|\tilde{f}_m(t) - g_m(t)\| \\ &\quad = \Big\|\sum_{l\in\mathbb{Z}}(a_l(f) - a_l)\theta_l(\gamma_m(t)) + \sum_{j=0}^{m}\sum_{k\in\mathbb{Z}}(b_{j,k}(f) - b_{j,k})\tau_{j,k}(\gamma_m(t))\Big\| \\ &\quad \leq C_{\mathrm{qs}}\Big(\phi(1)\sum_{l=-m}^{m}\|a_l(f) - a_l\| + \sum_{j=0}^{m}\phi\Big(2^{-\frac{j}{2}-1}\Big)\sum_{k=-2^j m}^{2^j m-1}\|b_{j,k}(f) - b_{j,k}\|\Big),\end{aligned}$$

and further

$$
\begin{aligned}
&\|(\tilde{f}_m(t) - g_m(t)) - (\tilde{f}_m(s) - g_m(s))\| \\
&\quad = \Big\| \sum_{l\in\mathbb{Z}} (a_l(f) - a_l)(\theta_l(\gamma_m(t)) - \theta_l(\gamma_m(s))) \\
&\qquad + \sum_{j=0}^{m} \sum_{k\in\mathbb{Z}} (b_{j,k}(f) - b_{j,k})(\tau_{j,k}(\gamma_m(t)) - \tau_{j,k}(\gamma_m(s)))\Big\| \\
&\quad \le \phi(|t-s|) C_{\mathrm{qs}} \Big(\sum_{l=-m}^{m} \|a_l(f) - a_l\| + \sum_{j=0}^{m} \sum_{k=-2^j m}^{2^j m-1} \|2^{\frac{j}{2}}(b_{j,k}(f) - b_{j,k})\| \Big).
\end{aligned}
$$

Because E^X is dense in X, we can choose finitely many coefficients $\{a_l : l \in \{-m, \dots, m\}\}$ as well as $\{b_{j,k} : j \in \{0, \dots, m\}, k \in \{-2^j m, \dots 2^j m - 1\}\}$ in E^X and close enough to the respective coefficients $a_l(f)$ and $b_{j,k}(f)$ with respect to $\|\cdot\|$ such that, upon defining the two upper bounds

$$
c_1 := \frac{1}{4C_{\mathrm{qs}}^2 \phi(1) c(m)} \inf_{\substack{t\in\mathbb{R} \\ |t|\le m}} w_1(t), \quad c_2 := \frac{1}{4C_{\mathrm{qs}}^2 c(m)} \frac{\varrho(1)}{C_\phi} \inf_{\substack{t\in\mathbb{R} \\ |t|\le m}} w_2(t),
$$

where

$$
c(m) := (2m+1) + 4m(2^m - 1),
$$

we get the estimates

$$
\begin{aligned}
\max\Big\{ \max_{\substack{l\in\mathbb{Z} \\ |l|\le m}} \|a_l(f) - a_l\|, \max_{\substack{j=0,\dots,m \\ k=-2^j m,\dots,2^j m-1}} \|b_{j,k}(f) - b_{j,k}\| \Big\} &\le c_1\varepsilon, \\
\max\Big\{ \max_{\substack{l\in\mathbb{Z} \\ |l|\le m}} \|a_l(f) - a_l\|, \max_{\substack{j=0,\dots,m \\ k=-2^j m,\dots,2^j m-1}} \|2^{\frac{j}{2}}(b_{j,k}(f) - b_{j,k})\| \Big\} &\le c_2\varepsilon,
\end{aligned}
$$

which finally yields $\|\tilde{f}_m - g_m\|_{\varrho,w_{1,2}} \le \varepsilon/(2C_{\mathrm{qs}})$, hence $\|f - g_m\|_{\varrho,w_{1,2}} \le \varepsilon$. This implies that the countable set of all functions g_m of the form

$$
g_m(t) := \sum_{l\in\mathbb{Z}} a_l \theta_l(\gamma_m(t)) + \sum_{j=0}^{m} \sum_{k\in\mathbb{Z}} b_{j,k} \tau_{j,k}(\gamma_m(t)), \quad t \in \mathbb{R}, \tag{2.46}
$$

where $m \in \mathbb{N}_0$, and with coefficients a_l and $b_{j,k}$ in E^X, is dense in $(\mathcal{C}_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$. $\square$

Remark 2.68. In the above proof, we have chosen the coefficients a_l and $b_{j,k}$ to be elements of the countable, dense subset E^X of X. However, we then argued that we can push the expressions

$$
\|a_l(f) - a_l\|, \quad \|2^{j/2}(b_{j,k}(f) - b_{j,k})\|
$$

arbitrarily close to 0 by choosing appropriate elements of E^X. As the given pseudo-quasi-norm is not absolutely homogeneous, we will briefly elaborate on this point. Due to the pseudo-homogeneity property 2.10(c) of $\|\cdot\|$ we have

$$\|2^{j/2}(b_{j,k}(f) - b_{j,k})\| \leq \phi(2^{j/2})\|b_{j,k}(f) - b_{j,k}\|.$$

Therefore, we have to choose $b_{j,k}$ closer to $b_{j,k}(f)$ such that the scalar factor $\phi(2^{j/2})$ is accounted for. But this we can do, as E^X is dense in X.

Remark 2.69. To be precise, we need to provide justification why the functions g_m defined in (2.46) are actually elements of $\mathcal{C}_{\varrho,w_{1,2},0}$. Fix some $m \in \mathbb{N}$. Then g_m is constant on $(-\infty, -m]$ and $[m, \infty)$, equalling $g_m(-m)$ and $g_m(m)$ respectively. Hence

$$\lim_{|t|\to\infty} \frac{\|g_m(t)\|}{w_1(t)} = \lim_{|t|\to\infty} \frac{\|g_m(\pm m)\|}{w_1(t)} = 0,$$

as we require w_1 to explode asymptotically. Moreover, differences $g_m(t) - g_m(s)$ are zero if $|t| \geq m$, $|s| \geq m$ and $0 < |t - s| \leq 1$, therefore condition (2.42) holds automatically. In the case that $-m < s, t < m$, we recall the formula

$$g_m(t) - g_m(s) = \sum_{l=-m}^{m} a_l(\theta_l(t) - \theta_l(s)) + \sum_{j=0}^{m} \sum_{k=-2^j m}^{2^j m-1} b_{j,k}(\tau_{j,k}(t) - \tau_{j,k}(s)). \tag{2.47}$$

As we are dealing with the sum of finitely many elements here, and $\|\cdot\|$ satisfies the properties (2.11) and (2.12), we can make use of the statement of Lemma 2.65 to conclude that the condition

$$\lim_{\delta\searrow 0} \sup_{\substack{s,t\in[-m,m] \\ 0<t-s\leq\delta}} \frac{\|g_m(t) - g_m(s)\|}{\varrho(t-s)\ w_2(t)} = 0,$$

follows, as soon as we can show that actually

$$\lim_{\delta\searrow 0} \frac{\phi(\delta)}{\varrho(\delta)} = 0.$$

But this is precisely the condition (2.27). On the other hand, if $s < m \leq t$, then we have to replace $\theta_l(t)$ by $\theta_l(m)$ and $\tau_{j,k}(t)$ by $\tau_{j,k}(m)$ in Equation (2.47). By a similar line of reasoning, the problem boils down to showing that indeed

$$\lim_{\delta\searrow 0} \sup_{\substack{s<m\leq t \\ 0<t-s\leq\delta}} \frac{\phi(m-s)}{\varrho(t-s)} = 0.$$

But this is trivially satisfied, as ϕ is assumed to be non-decreasing. The case $s \leq -m < t$ can be treated in the same way. Finally, we can also show that $\|g_m\|_{\varrho,w_{1,2}} < \infty$ by additionally making use of the bound (2.26).

In the next corollary, we will further expand on the approximation property of the functions g_m defined in Equation (2.46). The following statement will be of particular interest when studying abstract Wiener spaces.

Corollary 2.70. *In the context of Theorem 2.61, assume that the image space* $(X, \|\cdot\|)$ *of the functions* $f \in \mathcal{C}_{\varrho,w_{1,2},0}$ *is a pseudo-quasi-normed space, and that* $\lim_{|t|\to\infty} w_1(t) = \infty$ *holds. Further, let* $(Y, \mathcal{T})$ *denote a separable topological space, such that there exists a continuous map* $i\colon Y \to X$ *with the property that* $i(Y) := \{x = i(y) \in X : y \in Y\}$ *is dense in* X*. Then* $(\mathcal{C}_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$ *is also separable.*

Proof. Let E^Y denote a countable dense subset in Y. For each $x \in X$ and $\varepsilon > 0$, since $i(Y)$ is dense in X, there exists some $y \in Y$ such that

$$\|x - i(y)\| \leq \frac{\varepsilon}{2C_{\mathrm{qs}}}.$$

Since i is continuous in y, there exists a neighborhood $U \in \mathfrak{U}(y)$ in the neighborhood filter of y such that

$$i(U) \subset U_{\frac{\varepsilon}{2C_{\mathrm{qs}}}}(i(y)).$$

As U is a neighborhood of y, there exists an open set O in the topology $\mathcal{T}$ such that $y \in O \subset U$ and consequently

$$i(y) \in i(O) \subset i(U) \subset U_{\frac{\varepsilon}{2C_{\mathrm{qs}}}}(i(y)).$$

Since E^Y is a dense subset in Y, we can find a $y^* \in E^Y$ such that $y^* \in O$, hence

$$i(y^*) \in i(O) \subset i(U) \subset U_{\frac{\varepsilon}{2C_{\mathrm{qs}}}}(i(y)),$$

and consequently $\|i(y) - i(y^*)\| \leq \frac{\varepsilon}{2C_{\mathrm{qs}}}$. All of the above now yields

$$\|x - i(y^*)\| \leq C_{\mathrm{qs}}\big(\|x - i(y)\| + \|i(y) - i(y^*)\|\big) \leq \varepsilon.$$

Therefore, $E^X := i(E^Y)$ is a countable dense subset in X. The claim now follows by Corollary 2.67. □

Remark 2.71. In the context of Corollary 2.70, if $(Y, \mathcal{T})$ would be a topological vector space, and i linear as well as continuous, then we know how a countable dense subset of $(\mathcal{C}_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$ looks like. For each $m \in \mathbb{N}_0$, consider maps $g_m^Y\colon \mathbb{R} \to Y$ given by

$$g_m^Y(t) = \sum_{l\in\mathbb{Z}} a_l^Y \theta_l(\gamma_m(t)) + \sum_{j=0}^{m} \sum_{k\in\mathbb{Z}} b_{j,k}^Y \tau_{j,k}(\gamma_m(t)),$$

with coefficients a_l^Y and $b_{j,k}^Y$ in E^Y. The corresponding countable dense subset is given by functions $g_m\colon \mathbb{R} \to X$ of the form

$$g_m(t) = i(g_m^Y(t)) = \sum_{l\in\mathbb{Z}} i(a_l^Y)\theta_l(\gamma_m(t)) + \sum_{j=0}^{m} \sum_{k\in\mathbb{Z}} i(b_{j,k}^Y)\tau_{j,k}(\gamma_m(t)).$$

3. An extension of Ciesielski's isomorphism

For an n-dimensional vector space over a field $\mathbb{K}$, the choice of a basis induces a $\mathbb{K}$-linear isomorphism onto $\mathbb{K}^n$, and the study of linear maps between finite-dimensional vector spaces is reduced to the study of matrices with entries from $\mathbb{K}$. If the n-dimensional vector space is over the real or complex numbers and has an inner product, then the choice of an orthonormal basis leads to an isomorphism that preserves the inner product, provided we choose the standard Euclidean inner product on $\mathbb{K}^n$. All these subjects are studied in linear algebra.

For infinite-dimensional separable Hilbert spaces over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, the choice of a complete orthonormal system induces a $\mathbb{K}$-linear isomorphism onto the sequence space $l^2_{\mathbb{K}}$, consisting of all sequences $x = (x_n)_{n \in \mathbb{N}}$ in $\mathbb{K}$, that means elements of $\mathbb{K}^{\mathbb{N}}$, such that the norm $\|x\|_2$, given by the square root of the series $\sum_{n \in \mathbb{N}} |x_n|^2$, is finite. Note that $(l^2_{\mathbb{K}}, \|\cdot\|_2)$ is a separable Hilbert space itself and that the isomorphism preserves the inner product and, therefore, the corresponding norm.

The notion of Ciesielski's isomorphism refers to closely related $\mathbb{K}$-linear isomorphisms, defined on (generalized) Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ of certain continuous functions, defined on (subintervals of) the real line. Such an isomorphism arises from Faber–Schauder systems and identifies a function with a (double-indexed) sequence of numbers in $\mathbb{K}$, the possible sequences form a suitable (generalized) Banach space, too. Since the notion of orthogonality is missing, one can't expect to preserve the norm, but the isomorphism should still be a homeomorphism, i.e. continuous with continuous inverse.

Homeomorphic maps induce one-to-one correspondences between neighborhoods as well as open and closed sets. Consequently, one is free to choose either one of the homeomorphic spaces as domain of topological study. In view of the previous paragraph, convergence in sequence space implies convergence in function space, and vice versa. Let us also highlight that homeomorphisms preserve the whole topological structure, meaning all topological invariants. These include, but are not limited to:

(a) Separability as well as the axioms of first and second countability;

(b) The separation axioms T_0 (Kolmogorov), T_1 (Fréchet) and T_2 (Hausdorff);

(c) Connectedness;

(d) Compactness and sequential compactness;

(e) Metrizability and local metrizability.

A very classical homeomorphism can be constructed on open subsets of n-dimensional Euclidean space. For an open subset $U \subset \mathbb{R}^n$ and a map $f\colon U \to \mathbb{R}^n$ that is injective and continuous, it can be shown that U and $V := f(U)$ are homeomorphic under f. The proof goes back to L.E.J. Brouwer, see [Bro11].

The goal of this chapter now is to further investigate the representation

$$f(\cdot) = \sum_{l\in\mathbb{Z}} a_l(f)\theta_l(\cdot) + \sum_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} b_{j,k}(f)\tau_{j,k}(\cdot).$$

More precisely, we will be looking at the map, that assigns each $f \in \mathcal{C}_{\varrho,w_{1,2}}$ to the (double indexed) sequence of coefficients $\mathcal{M}_2^{\mathrm{FSC}}$ as given in Definition 2.2. To this end, let $(X, \|\cdot\|)$ be a pseudo-quasi-normed vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $f\colon \mathbb{R} \to X$. Set $\mathbb{N}^* := \{-1\} \cup \mathbb{N}_0$, and consider the linear map T defined on the space $\mathcal{C}$ of continuous functions and given by

$$T\colon \mathcal{C} \to X^{\mathbb{N}^*\times\mathbb{Z}},$$
$$f \mapsto (x^f_{j,k})_{(j,k)\in\mathbb{N}^*\times\mathbb{Z}},$$

where $x^f_{-1,l} := a_l(f)$ for $l \in \mathbb{Z}$, and $x^f_{j,k} := b_{j,k}(f)$ for $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$.

In this section, we will need to require additional relational properties for the pseudo-homogeneity ϕ as defined in 2.10(c) and the ϕ-admissible modulus of continuity ϱ as given by Definition 2.39. Therefore, let us make the

Assumption 3.1. For Chapter 3, we assume that there exists a map

$$\chi\colon (0,\infty) \to \mathbb{R}_+$$

that is non-decreasing, such that the following conditions hold:

(a) The map ϱ is χ-submultiplicative on the set of powers of $1/2$, i.e.,

$$\varrho(2^{-m}) \le \chi(2^{n-m})\varrho(2^{-n}), \quad m,n \in \mathbb{N}_0; \tag{3.1}$$

(b) Dyadic summability of χ:

$$\sum_{j=0}^{\infty} \chi(2^{-j}) < \infty; \tag{3.2}$$

(c) The map χ exhibits a ϕ-controlled growth condition outside of the unit interval, i.e.

$$\sum_{j=1}^{\infty} \phi(2^{-j})\chi(2^j) < \infty. \tag{3.3}$$

Example 3.2. Let us provide an example for when (3.1), (3.2) and (3.3) are satisfied. Fix some $0 < \alpha < \beta < \infty$ and set $\phi(\lambda) := \lambda^\beta$ for $\lambda \ge 0$ as well as $\varrho(\delta) := \delta^\alpha$ for $\delta > 0$. Upon defining $\chi(\delta) = \delta^\alpha$ for $\delta > 0$, we immediately see that (3.1) holds, even with equality. The next trivial observation to be made is that

$$\sum_{j=0}^{\infty} \chi(2^{-j}) = \sum_{j=0}^{\infty} 2^{-\alpha j} = \frac{2^\alpha}{2^\alpha - 1} < \infty,$$

and therefore (3.2) holds. On the other hand, we see that

$$\sum_{j=1}^{\infty} \phi(2^{-j})\chi(2^j) = \sum_{j=1}^{\infty} 2^{-(\beta-\alpha)j} = \frac{1}{2^{\beta-\alpha}-1} < \infty,$$

which yields (3.3).

3.1. Isomorphism in weighted Hölder spaces

In this section, we want to show that for each $f \in \mathcal{C}_{\varrho,w_{1,2}}$, the sequence $x^f = T(f)$ also has some very specific properties. We will now define the right subspace of $X^{\mathbb{N}^*\times\mathbb{Z}}$, such that the map T becomes an isomorphism.

Definition 3.3 (Weighted Hölder sequence space)**.** Let X be a vector space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, endowed with a pseudo-quasi-norm $\|\cdot\|$ and pseudo-homogeneity ϕ according to Definition 2.10. Further, let two weight functions w_1 and w_2 according to Definition 2.42 as well as a ϕ-admissible modulus of continuity ϱ according to Definition 2.39 be given. Let $\|\cdot\|_{\varrho,w_{1,2}}\colon X^{\mathbb{N}^*\times\mathbb{Z}} \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ denote the map

$$x \mapsto \|x\|_{\varrho,w_{1,2}} := \sup_{l\in\mathbb{Z}} \frac{\|x_{-1,l}\|}{w_1(l)} + \sup_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} \frac{\|2^{-\frac{j}{2}} x_{j,k}\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k+1}{2^{j+1}}\big)}. \tag{3.4}$$

We call $\ell_{\varrho,w_{1,2}} := \{x \in X^{\mathbb{N}^*\times\mathbb{Z}}\colon \|x\|_{\varrho,w_{1,2}} < \infty\}$ endowed with $\|\cdot\|_{\varrho,w_{1,2}}$ weighted Hölder sequence space.

Remark 3.4. Note that we used the notation $\|\cdot\|_{\varrho,w_{1,2}}$ on both the function space $\mathcal{C}_{\varrho,w_{1,2}}$ as well as the sequence space $\ell_{\varrho,w_{1,2}}$. This should not lead to any confusions, as the argument that we plug in will be either a function or a sequence, hence it should be clear from the context whether we refer to (2.32) or (3.4).

Remark 3.5. Let us show that the weighted Hölder sequence space is a pseudo-quasi-normed space. In line with Remark 2.48, choose $x_1, x_2 \in \ell_{\varrho,w_{1,2}}$ and $\lambda \in \mathbb{K}$. Upon defining $y = x_1 + \lambda x_2$, we have

$$\|y\|_{\varrho,w_{1,2}} \le C_{\mathrm{qs}} \max\{1, \phi(|\lambda|)\}\big(\|x_1\|_{\varrho,w_{1,2}} + \|x_2\|_{\varrho,w_{1,2}}\big) < \infty.$$

Therefore, $\ell_{\varrho,w_{1,2}}$ is a vector space over the field $\mathbb{K}$. The defining properties of a pseudo-quasi-norm, as stated in Definition 2.10, can be verified by hand, where the constant C_{qs} in 2.10(b) and the function ϕ in 2.10(c) are directly inherited from $\|\cdot\|$ on the image space X. Moreover, one can also show that the weighted Hölder sequence space is a normed vector space, as long as $\|\cdot\|$ is a norm on the vector space X.

The main result of this section is the

Theorem 3.6. *In the context of Definition 3.3, recalling Assumption 3.1, let w_1, w_2 be two weight functions that are equivalent in the sense of Definition 2.46.*

(a) For each $f \in \mathcal{C}_{\varrho,w_{1,2}}$ we have that $x^f \in \ell_{\varrho,w_{1,2}}$. In other words

$$T\colon \mathcal{C}_{\varrho,w_{1,2}} \to \ell_{\varrho,w_{1,2}}.$$

Furthermore, T is an isomorphism, and the inverse map $T^{-1}\colon \ell_{\varrho,w_{1,2}} \to \mathcal{C}_{\varrho,w_{1,2}}$ is explicitly given by

$$x \mapsto f^x\colon \Big(\mathbb{R} \ni t \mapsto \sum_{l\in\mathbb{Z}} x_{-1,l}\theta_l(t) + \sum_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} x_{j,k}\theta_{j,k}(t)\Big), \tag{3.5}$$

where the series converges pointwise with respect to $\|\cdot\|$ and is thus well defined.

(b) The operator pseudo-quasi-norms of T and T^{-1} as given in Remark 2.18 satisfy the upper bounds:

$$\|T\|_{\mathrm{op}} \le C_{\mathrm{qs}}(1 + C\phi(1)) \quad \text{and} \quad \|T^{-1}\|_{\mathrm{op}} \le \Psi,$$

where the constant Ψ is given by

$$\Psi := C_{\mathrm{qs}} C\big(2\phi(1) + C^*\phi(\tfrac{1}{2})\varrho(1)C_\varrho + 4C^*\frac{1}{\varrho(1)}C_\phi + 2C_{\phi,\varrho}\big), \tag{3.6}$$

with

$$C_\varrho := \sum_{j=0}^{\infty} \chi(2^{-j-1}) \quad \text{and} \quad C_{\phi,\varrho} := \sup_{j_0\in\mathbb{N}_0} \sum_{j=0}^{\infty} \phi(2^j 2^{-j_0} \wedge 2^{-1}) \frac{\varrho(2^{-j-1})}{\varrho(2^{-j_0-1})}.$$

The constants C_ϱ, $C_{\phi,\varrho}$ and consequently also Φ are real valued due to Assumption 3.1.

(c) The pseudo-quasi-norms $\|\cdot\|_{\varrho,w_{1,2}}$ on the spaces $\mathcal{C}_{\varrho,w_{1,2}}$ and $\ell_{\varrho,w_{1,2}}$ are T-equivalent, i.e. there exist real constants $K, K' > 0$, such that, for all $f \in \mathcal{C}_{\varrho,w_{1,2}}$, we have

$$K\|T(f)\|_{\varrho,w_{1,2}} \le \|f\|_{\varrho,w_{1,2}} \le K'\|T(f)\|_{\varrho,w_{1,2}}. \tag{3.7}$$

(d) $\ell_{\varrho,w_{1,2}}$ is complete.

(e) The linear isomorphisms T and T^{-1} are homeomorphisms as maps between the vector spaces $\mathcal{C}_{\varrho,w_{1,2}}$ and $\ell_{\varrho,w_{1,2}}$ endowed with the topologies induced by the respective pseudo-quasi-norms according to Proposition 2.25.

(f) The linear isomorphisms T and T^{-1} are uniformly continuous as maps between the vector spaces $\mathcal{C}_{\varrho,w_{1,2}}$ and $\ell_{\varrho,w_{1,2}}$ endowed with the uniformities induced by the respective semimetrics according to Proposition 2.30 and Proposition 2.33.

Remark 3.7. Below, we will revisit the statement of Theorem 3.6(c) and further expand it to a setting where the image space X of the functions $f \in \mathcal{C}_{\varrho,w_{1,2}}$ can be a metric space without a vector space structure, see Proposition 3.17 and Remark 3.18. The basic idea behind the proofs of 3.6(a) and 3.6(b) originates from [HIPP14]. See [HIPP14, Theorem 2.2] for the proof in the case that $X = \mathbb{R}$ and the domain of the functions is the unit interval.

We will first collect some preliminary results in the form of Proposition 3.8 and Proposition 3.9 before formulating the proof of Theorem 3.6 on page 49.

Proposition 3.8. *In the setting of Theorem 3.6, let $f \in \mathcal{C}_{\varrho,w_{1,2}}$. Then we have $x^f \in \ell_{\varrho,w_{1,2}}$.*

Proof. Upon recalling Definition 2.2 as well as (2.32) and (3.4), this easily follows from the definitions of the spaces $\mathcal{C}_{\varrho,w_{1,2}}$ and $\ell_{\varrho,w_{1,2}}$, as for $l \in \mathbb{Z}$, we have

$$\frac{\|x^f_{-1,l}\|}{w_1(l)} = \frac{\|a_l(f)\|}{w_1(l)} = \frac{\|f(l)\|}{w_1(l)} \le \|f\|_{w_1}.$$

Moreover, for each $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$, we have

$$\begin{aligned}\|2^{-\frac{j}{2}} x^f_{j,k}\| = \|2^{-\frac{j}{2}} b_{j,k}(f)\| &= \left\|2f\left(\tfrac{2k+1}{2^{j+1}}\right) - f\left(\tfrac{2k+2}{2^{j+1}}\right) - f\left(\tfrac{2k}{2^{j+1}}\right)\right\| \\ &\le C_{\mathrm{qs}}\Big(\left\|f\left(\tfrac{2k+1}{2^{j+1}}\right) - f\left(\tfrac{2k}{2^{j+1}}\right)\right\| + \phi(1)\left\|f\left(\tfrac{2k+2}{2^{j+1}}\right) - f\left(\tfrac{2k+1}{2^{j+1}}\right)\right\|\Big),\end{aligned}$$

hence

$$\frac{\|2^{-\frac{j}{2}} x^f_{j,k}\|}{\varrho\left(2^{-j-1}\right) w_2\left(\frac{2k+1}{2^{j+1}}\right)} \le C_{\mathrm{qs}}(1 + C\phi(1))\|f\|_{\varrho|w_2;2^{-j-1}},$$

which proves our claim. □

Proposition 3.9. *In the setting of Theorem 3.6, let $x \in \ell_{\varrho,w_{1,2}}$. Then $f^x = T^{-1}(x) \in \mathcal{C}_{\varrho,w_{1,2}}$.*

Proof. First of all, let $t \in \mathbb{R}$. For each $j \in \mathbb{N}_0$, we denote by $k(j,t)$ the unique integer, such that $\tau_{j,k(j,t)}(t) \neq 0$, if it exists, and ignore the respective element in the series below otherwise. The first observation to be made is that in line with Definition 2.1, (2.12), (2.28), (3.1) and (3.4), we actually have

$$\begin{aligned}\sum_{j=0}^{\infty} \|x_{j,k(j,t)}\tau_{j,k(j,t)}(t)\| &\le \phi\left(\tfrac{1}{2}\right) \sum_{j=0}^{\infty} \|2^{-j/2} x_{j,k(j,t)}\| \\ &\le C\phi\left(\tfrac{1}{2}\right) w_2(t)\|x\|_{\varrho,w_{1,2}} \sum_{j=0}^{\infty} \varrho\left(2^{-j-1}\right) \\ &\le C\phi\left(\tfrac{1}{2}\right) w_2(t)\|x\|_{\varrho,w_{1,2}}\,\varrho(1) \sum_{j=0}^{\infty} \chi\left(2^{-j-1}\right),\end{aligned} \tag{3.8}$$

which by (3.2) is finite. This implies in particular, that the sequence of partial sums

$$(f^x(t))_m = x_{-1,\lfloor t\rfloor}\theta_{\lfloor t\rfloor}(t) + x_{-1,\lceil t\rceil}\theta_{\lceil t\rceil}(t) + \sum_{j=0}^{m} x_{j,k(j,t)}\tau_{j,k(j,t)}(t)$$

converges absolutely, as $m \to \infty$. Since X is assumed to be complete, we can deduce convergence of the X-valued series $f^x(t)$, due to Proposition 2.17(b), and therefore the following chain of inequalities is justified by (3.8) as well as Definition 2.46:

$$\begin{aligned}\|f^x(t)\| &= \Big\| x_{-1,\lfloor t\rfloor}\theta_{\lfloor t\rfloor}(t) + x_{-1,\lceil t\rceil}\theta_{\lceil t\rceil}(t) + \sum_{j=0}^{\infty} x_{j,k(j,t)}\tau_{j,k(j,t)}(t)\Big\| \\ &\leq C_{\mathrm{qs}}\Big(\phi(1)\|x_{-1,\lfloor t\rfloor}\| + \phi(1)\|x_{-1,\lceil t\rceil}\| + \sum_{j=0}^{\infty}\|x_{j,k(j,t)}\tau_{j,k(j,t)}(t)\|\Big) \\ &\leq C_{\mathrm{qs}} C\|x\|_{\varrho,w_{1,2}}\Big(2\phi(1) + C^*\phi(\tfrac{1}{2})\varrho(1)\sum_{j=0}^{\infty}\chi(2^{-j-1})\Big)w_1(t).\end{aligned} \tag{3.9}$$

But this implies that

$$\frac{\|f^x(t)\|}{w_1(t)} \leq C_{\mathrm{qs}} C\|x\|_{\varrho,w_{1,2}}\Big(2\phi(1) + C^*\phi(\tfrac{1}{2})\varrho(1)\sum_{j=0}^{\infty}\chi(2^{-j-1})\Big) < \infty. \tag{3.10}$$

Next on, let $s,t \in \mathbb{R}$ such that $0 < |t-s| \leq 1$, and define for each $j \in \mathbb{N}_0$ the unique integers $k(j,s)$ and $k(j,t)$, such that $\tau_{j,k(j,s)}(s) \neq 0$ and $\tau_{j,k(j,t)}(t) \neq 0$, if they exist. Moreover, there exist at most 4 integers $l_i : i \in \{1,2,3,4\}$ such that $\theta_{l_i}(s)$ and $\theta_{l_i}(t)$ do not vanish. We then observe that, making use of Lemma 2.65, and similar as above:

$$\begin{aligned}&\sum_{\substack{j=0 \\ k\in\{k(j,t),k(j,s)\}}}^{\infty} \|x_{j,k}(\tau_{j,k}(t) - \tau_{j,k}(s))\| \\ &\qquad\leq \sum_{\substack{j=0 \\ k\in\{k(j,t),k(j,s)\}}}^{\infty} \phi(2^j|t-s| \wedge 2^{-1})\|2^{-j/2}x_{j,k}\| \\ &\qquad\leq 2C\|x\|_{\varrho,w_{1,2}} w_2(t)\varrho(|t-s|)\sum_{j=0}^{\infty}\phi(2^j|t-s| \wedge 2^{-1})\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)}.\end{aligned}$$

Choose j_0 to be the unique integer such that

$$2^{-j_0-1} < |t-s| \leq 2^{-j_0}.$$

Then, making use of (3.1) and (3.2), we have

$$\begin{aligned}\sum_{j=j_0}^{\infty}\phi(2^j|t-s| \wedge 2^{-1})\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)} &\leq \phi(2^{-1})\sum_{j=j_0}^{\infty}\frac{\varrho(2^{-j-1})}{\varrho(2^{-j_0-1})} \\ &\leq \phi(2^{-1})\sum_{j=j_0}^{\infty}\chi(2^{j_0-j}) = \phi(2^{-1})\sum_{j=0}^{\infty}\chi(2^{-j}) < \infty.\end{aligned}$$

In a similar manner, making use of (3.1) and (3.3), we have

$$\begin{aligned}\sum_{j=0}^{j_0-1}\phi(2^j|t-s|\wedge 2^{-1})\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)} &\leq \sum_{j=0}^{j_0-1}\phi(2^{j-j_0})\frac{\varrho(2^{-j-1})}{\varrho(2^{-j_0-1})}\\ &\leq \sum_{j=0}^{j_0-1}\phi(2^{j-j_0})\chi(2^{j_0-j}) = \sum_{j=0}^{j_0-1}\phi(2^{-j-1})\chi(2^{j+1})\\ &\leq \sum_{j=1}^{\infty}\phi(2^{-j})\chi(2^j) < \infty.\end{aligned}$$

As above, this implies the absolute convergence and thus convergence of the sequence of partial sums

$$\begin{aligned}(f^x(t)-f^x(s))_m = &\sum_{l\in\{l_1,\dots,l_4\}} x_{-1,l}(\theta_l(t)-\theta_l(s))\\ &+\sum_{\substack{j=0\\ k\in\{k(j,t),k(j,s)\}}}^{m} x_{j,k}(\tau_{j,k}(t)-\tau_{j,k}(s)),\end{aligned}$$

as $m\to\infty$. Therefore the following chain of inequalities is justified:

$$\begin{aligned}\|f^x(t)-f^x(s)\| &= \Bigg\|\sum_{l\in\{l_1,\dots,l_4\}} x_{-1,l}(\theta_l(t)-\theta_l(s))\\ &\quad+\sum_{\substack{j=0\\ k\in\{k(j,t),k(j,s)\}}}^{\infty} x_{j,k}(\tau_{j,k}(t)-\tau_{j,k}(s))\Bigg\|\\ &\leq C_{\mathrm{qs}}\Bigg(\sum_{l\in\{l_1,\dots,l_4\}}\phi\big(|t-s|\big)\|x_{-1,l}\|\\ &\quad+\sum_{\substack{j=0\\ k\in\{k(j,t),k(j,s)\}}}^{\infty}\phi\big(2^j|t-s|\wedge 2^{-1}\big)\|2^{-\frac{j}{2}}x_{j,k}\|\Bigg)\\ &\leq \Bigg(4C^*\frac{1}{\varrho(1)}C_\phi + 2\sum_{j=0}^{\infty}\phi(2^j|t-s|\wedge 2^{-1})\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)}\Bigg)\\ &\quad\times C_{\mathrm{qs}}C\|x\|_{\varrho,w_{1,2}}\varrho(|t-s|)w_2(t).\end{aligned}\tag{3.11}$$

Moreover, we have the estimate

$$\begin{aligned}\frac{\|f^x(t)-f^x(s)\|}{\varrho(|t-s|)\,w_2(t)} &\leq C_{\mathrm{qs}}C\Bigg(4C^*\frac{1}{\varrho(1)}C_\phi\\ &\quad+2\sum_{j=0}^{\infty}\phi\big(2^j|t-s|\wedge 2^{-1}\big)\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)}\Bigg)\|x\|_{\varrho,w_{1,2}} < \infty,\end{aligned}\tag{3.12}$$

where the right-hand side is uniformly bounded over all $0<|t-s|\leq 1$ by the above argumentation. Finally, the inequalities (3.10) and (3.12) prove our claim. □

Proof of Theorem 3.6.

(a) By Proposition 3.8 and 3.9, both $T\colon f \mapsto x^f$ and $G\colon x \mapsto f^x$ are well defined maps between $\mathcal{C}_{\varrho,w_{1,2}}$ and $\ell_{\varrho,w_{1,2}}$. The convergence of the series (3.5) follows from the estimate (3.9). Both T and G are linear by construction. Moreover, given f and g such that $T(f) = T(g)$, we immediately conclude that $f = g$ due to the uniqueness of coefficients in the respective expansions of f and g, see Proposition 2.9. For any $x \in \ell_{\varrho,w_{1,2}}$, we have $f^x = G(x) \in \mathcal{C}_{\varrho,w_{1,2}}$, hence $T(f^x) = x$, again by the uniqueness of the coefficients. We conclude that T is linear and bijective, hence an isomorphism. Moreover, given that $T(G(x)) = x$ as well as $G(T(f)) = f$, we can conclude that $G = T^{-1}$.

(b) The estimates in Proposition 3.8 imply that, for each $f \in \mathcal{C}_{\varrho,w_{1,2}}$,

$$\begin{aligned}\|x^f\|_{\varrho,w_{1,2}} &\leq \|f\|_{w_1} + C_{\text{qs}}(1 + C\phi(1))\|f\|_{\varrho|w_2} \\ &\leq C_{\text{qs}}(1 + C\phi(1))\|f\|_{\varrho,w_{1,2}},\end{aligned}$$

because all three constants C, C_{qs} and $\phi(1)$ are larger or equal to 1. But then (recall that $T(f) = x^f$)

$$\|T\|_{\text{op}} = \sup_{f \in \mathcal{C}_{\varrho,w_{1,2}} \setminus \{0\}} \frac{\|T(f)\|_{\varrho,w_{1,2}}}{\|f\|_{\varrho,w_{1,2}}} \leq C_{\text{qs}}(1 + C\phi(1)).$$

The statement for $\|T^{-1}\|_{\text{op}}$ is a direct consequence of the estimates (3.10) and (3.12).

(c) This direct follows from 3.6(b). We have that $K = 1/(C_{\text{qs}}(1 + C\phi(1)))$, and $K' = \Psi$, which is defined in (3.6).

(d) By Lemma 2.50, we know that $\mathcal{C}_{\varrho,w_{1,2}}$ is complete. According to 3.6(a) and 3.6(b), the assumptions of Proposition 2.20 are satisfied, which yields the assertion.

(e) By 3.6(b), both T and T^{-1} are bounded with respect to $\|\cdot\|_{\text{op}}$ and bounded by construction. By Proposition 2.34 they are uniformly continuous and thus continuous.

(f) This is a direct consequence of 3.6(b) and Corollary 2.36.

This concludes our proof. □

Remark 3.10. Let us briefly comment on the statement that $T(f) = T(g)$ implies $f = g$, as it is not trivial. Consider $f, g \in \mathcal{C}_{\varrho,w_{1,2}}$ such that they both allow the expansion

$$f(\cdot) = g(\cdot) = \sum_{l \in \mathbb{Z}} a_l \theta_l(\cdot) + \sum_{(j,k) \in \mathbb{N}_0 \times \mathbb{Z}} b_{j,k} \tau_{j,k}(\cdot),$$

with convergence either in the sense of Propositions 2.52, 2.54, Corollary 2.55 or Theorem 2.61. According to Proposition 2.7, we know that the continuous functions f and g are equal on the set $\mathbb{D}$ of dyadic rationals. This set is dense in $\mathbb{R}$. Therefore, for each $t \in \mathbb{R}$, we can find a sequence $(d_n)_{n\in\mathbb{N}} \in \mathbb{D}$ of dyadic rationals such that $\lim_{n\to\infty} d_n = t$, and consequently

$$f(t) = \lim_{n\to\infty} f(d_n) = \lim_{n\to\infty} g(d_n) = g(t),$$

which implies $f = g$ on all of $\mathbb{R}$.

3.2. Isomorphism in little weighted Hölder spaces

Our next goal is to derive similar results as in Section 3.1, but we will be focused on the little weighted Hölder function space $\mathcal{C}_{\varrho,w_{1,2},0}$. To this end, we first need to define the analogue of the little weighted Hölder space for sequence spaces, see also Definition 2.58.

Definition 3.11 (Little weighted Hölder sequence space)**.** In the context of Definition 3.3, let $\ell_{\varrho,w_{1,2},0} \subset \ell_{\varrho,w_{1,2}}$ denote the subset of all $x \in \ell_{\varrho,w_{1,2}}$ that further satisfy the following conditions:

(a) The sequence $(x_{-1,l})_{l\in\mathbb{Z}}$ is a null sequence upon being weighted by $(w_1(l))_{l\in\mathbb{Z}}$ componentwise, i.e.

$$\lim_{|l|\to\infty} \frac{\|x_{-1,l}\|}{w_1(l)} = 0; \tag{3.13}$$

(b) Upon being weighted and scaled, the sequence x vanishes uniformly in $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$ as $j \to \infty$, i.e.

$$\lim_{J\to\infty} \sup_{(j,k)\in\mathbb{N}_{\geq J}\times\mathbb{Z}} \frac{\|2^{-\frac{j}{2}} x_{j,k}\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k+1}{2^{j+1}}\big)} = 0; \tag{3.14}$$

(c) Upon being weighted and scaled, the sequence x vanishes uniformly in $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$ as $k \to \infty$, i.e.

$$\lim_{K\to\infty} \sup_{\substack{(j,k)\in\mathbb{N}_0\times\mathbb{Z} \\ |k|\geq 2^j K}} \frac{\|2^{-\frac{j}{2}} x_{j,k}\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k+1}{2^{j+1}}\big)} = 0. \tag{3.15}$$

We call $(\ell_{\varrho,w_{1,2},0}, \|\cdot\|_{\varrho,w_{1,2}})$ little weighted Hölder sequence space.

The main result of this section is the

Theorem 3.12. *In the context of Theorem 3.6, let T_0 denote the restriction of T to $\mathcal{C}_{\varrho,w_{1,2},0}$. Then*

(a) For each $f \in \mathcal{C}_{\varrho,w_{1,2},0}$ we have that $T_0(f) = x^f \in \ell_{\varrho,w_{1,2},0}$. In other words

$$T_0\colon \mathcal{C}_{\varrho,w_{1,2},0} \to \ell_{\varrho,w_{1,2},0}.$$

Furthermore, T_0 is an isomorphism, and the inverse $T_0^{-1}\colon \ell_{\varrho,w_{1,2},0} \to \mathcal{C}_{\varrho,w_{1,2},0}$ is given by (3.5), where the series converges pointwise with respect to $\|\cdot\|$ and is thus well defined.

(b) The operator pseudo-quasi-norms of T_0 and T_0^{-1} satisfy the upper bounds:

$$\|T_0\|_{\mathrm{op}} \leq C_{\mathrm{qs}}(1 + C\phi(1)), \quad \|T_0^{-1}\|_{\mathrm{op}} \leq \Psi,$$

where the constant Ψ is defined in (3.6).

(c) *The pseudo-quasi-norms* $\|\cdot\|_{\varrho,w_{1,2}}$ *on the spaces* $\mathcal{C}_{\varrho,w_{1,2},0}$ *and* $\ell_{\varrho,w_{1,2},0}$ *are equivalent, i.e. there exist real constants* $K, K' > 0$*, such that, for all* $f \in \mathcal{C}_{\varrho,w_{1,2},0}$*, we have*

$$K\|T_0(f)\|_{\varrho,w_{1,2}} \leq \|f\|_{\varrho,w_{1,2}} \leq K'\|T_0(f)\|_{\varrho,w_{1,2}}. \tag{3.16}$$

(d) $\ell_{\varrho,w_{1,2},0}$ *is complete.*

(e) *The linear isomorphisms* T_0 *and* T_0^{-1} *are homeomorphisms as maps between the vector spaces* $\mathcal{C}_{\varrho,w_{1,2},0}$ *and* $\ell_{\varrho,w_{1,2},0}$ *endowed with the topologies induced by the respective pseudo-quasi-norms according to Proposition 2.25.*

(f) *The linear isomorphisms* T_0 *and* T_0^{-1} *are uniformly continuous as maps between the vector spaces* $\mathcal{C}_{\varrho,w_{1,2},0}$ *and* $\ell_{\varrho,w_{1,2},0}$ *endowed with the uniformities induced by the respective semimetrics according to Proposition 2.30 and Proposition 2.33.*

(g) *If additionally, we have that* X *is separable and* w_1 *is nondecreasing such that* $w_1(t) \to \infty$*, as* $|t| \to \infty$*, then* $\ell_{\varrho,w_{1,2},0}$ *is separable as well.*

Remark 3.13. The basic idea behind the proof of 3.12(a) and 3.12(b) originates from [HIPP14]. See [HIPP14, Theorem 2.3] and [AIP13, Theorem 1] for the proof in the case that $X = \mathbb{R}$ or X is a real Hilbert space, and the domain of the functions is the unit interval. Let us also remark that the assumptions of Theorem 3.12(g) are actually too restrictive. However, in the chosen setting, they allow to draw a direct connection to separability properties of the little weighted Hölder spaces $\mathcal{C}_{\varrho,w_{1,2},0}$ in line with Corollary 2.67. We will also provide a much simpler and much more direct proof under milder conditions below in Proposition 3.16.

We will first collect some preliminary results in the form of Proposition 3.14 and Proposition 3.15 before formulating the proof of Theorem 3.12 on page 55.

Proposition 3.14. *In the context of Theorem 3.12, let* $f \in \mathcal{C}_{\varrho,w_{1,2},0}$*. Then* $T_0(f) = x^f \in \ell_{\varrho,w_{1,2},0}$*.*

Proof. Similar as in Proposition 3.8, we have for each $l \in \mathbb{Z}$

$$\frac{\|x^f_{-1,l}\|}{w_1(l)} = \frac{\|a_l(f)\|}{w_1(l)} = \frac{\|f(l)\|}{w_1(l)},$$

therefore, by making use of (2.40), it follows that

$$\lim_{|l|\to\infty} \frac{\|x^f_{-1,l}\|}{w_1(l)} = \lim_{|t|\to\infty} \frac{\|f(t)\|}{w_1(t)} = 0.$$

Next, given the estimate

$$\begin{aligned}\|2^{-\frac{j}{2}} x^f_{j,k}\| &= \|2^{-\frac{j}{2}} b_{j,k}(f)\| = \|2f\left(\tfrac{2k+1}{2^{j+1}}\right) - f\left(\tfrac{2k+2}{2^{j+1}}\right) - f\left(\tfrac{2k}{2^{j+1}}\right)\| \\ &\leq C_{\mathrm{qs}}\Big(\|f\left(\tfrac{2k+1}{2^{j+1}}\right) - f\left(\tfrac{2k}{2^{j+1}}\right)\| + \phi(1)\|f\left(\tfrac{2k+2}{2^{j+1}}\right) - f\left(\tfrac{2k+1}{2^{j+1}}\right)\|\Big),\end{aligned}$$

which for $(j,k) \in \mathbb{N}_{\geq J} \times \mathbb{Z}$ for $J \in \mathbb{N}$, implies on the one hand that

$$\frac{\|2^{-\frac{j}{2}} x^f_{j,k}\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k+1}{2^{j+1}}\big)} \leq C_{\mathrm{qs}}(1 + C\phi(1))\|f\|_{\varrho|w_2;2^{-J-1}},$$

and for $(j,k) \in \mathbb{N}_0 \times \mathbb{Z}$ such that $|k| \geq 2^j K$ for some $K \in \mathbb{N}$ on the other hand

$$\frac{\|2^{-\frac{j}{2}} x^f_{j,k}\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k+1}{2^{j+1}}\big)} \leq C_{\mathrm{qs}}(1 + C\phi(1)) \sup_{\substack{s,t\in\mathbb{R}\setminus[-K+1,K-1] \\ 0<|t-s|\leq 1}} \frac{\|f(t) - f(s)\|}{\varrho(|t-s|)\, w_2(t)}.$$

By (2.41) and (2.42), we can conclude that both (3.14) and (3.15) hold. □

Proposition 3.15. *In the setting of Theorem 3.12, let $x \in \ell_{\varrho,w_{1,2},0}$. Then $f^x = T_0^{-1}(x) \in \mathcal{C}_{\varrho,w_{1,2},0}$.*

Proof. We start with the estimate (3.9) from Proposition 3.9, in particular

$$\begin{aligned}\|f^x(t)\| &= \Big\|x_{-1,\lfloor t\rfloor}\theta_{\lfloor t\rfloor}(t) + x_{-1,\lceil t\rceil}\theta_{\lceil t\rceil}(t) + \sum_{j=0}^{\infty} x_{j,k(j,t)}\tau_{j,k(j,t)}(t)\Big\| \\ &\leq C_{\mathrm{qs}}\Big(\phi(1)\|x_{-1,\lfloor t\rfloor}\| + \phi(1)\|x_{-1,\lceil t\rceil}\| + \phi(\tfrac{1}{2})\sum_{j=0}^{\infty}\|2^{-j/2}x_{j,k(j,t)}\|\Big).\end{aligned}$$

The first two summands inside the bracket satisfy, upon division by $w_1(t)$, the upper bounds

$$C\frac{\|x_{-1,m}\|}{w_1(m)}, \qquad m \in \big\{\lfloor t\rfloor, \lceil t\rceil\big\}.$$

The last element of the sum, when weighted by $w_1(t)$, satisfies the upper bound

$$\begin{aligned}&CC^*\sum_{j=0}^{\infty}\varrho\big(2^{-j-1}\big)\frac{\|2^{-j/2}x_{j,k(j,t)}\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k(j,t)+1}{2^{j+1}}\big)} \\ &\qquad\leq CC^*\varrho(1)\sum_{j=0}^{\infty}\chi\big(2^{-j-1}\big) \sup_{\substack{(j,k)\in\mathbb{N}_0\times\mathbb{Z} \\ |k|\geq\lfloor 2^j|t|\rfloor-1}} \frac{\|2^{-j/2}x_{j,k}\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k+1}{2^{j+1}}\big)}.\end{aligned}$$

By (3.13) and (3.15) as well as (3.2), these upper bounds converge to 0, as $|t| \to \infty$.

In order to show (2.41), we make use of the estimate (3.11), in particular:

$$\begin{aligned}\|f^x(t) - f^x(s)\| &= \Big\| \sum_{l\in\{l_1,\dots,l_4\}} x_{-1,l}(\theta_l(t) - \theta_l(s)) \\ &\quad + \sum_{\substack{j=0 \\ k\in\{k(j,t),k(j,s)\}}}^{\infty} x_{j,k}(\tau_{j,k}(t) - \tau_{j,k}(s))\Big\| \\ &\le C_{\mathrm{qs}}\Big(\sum_{l\in\{l_1,\dots,l_4\}} \phi\big(|t-s|\big)\|x_{-1,l}\| \\ &\quad + \sum_{\substack{j=0 \\ k\in\{k(j,t),k(j,s)\}}}^{\infty} \phi\big(2^j|t-s| \wedge 2^{-1}\big)\|2^{-\frac{j}{2}}x_{j,k}\|\Big)\end{aligned}$$

The first four summands, upon dividing by $\varrho(|t-s|)w_2(t)$, allow for the upper bound

$$CC^* \frac{\phi(|t-s|)}{\varrho(|t-s|)} \sup_{l\in\mathbb{Z}} \frac{\|x_{-1,l}\|}{w_1(l)}.$$

The last term, being an infinite series, allows the upper bound, when divided by $\varrho(|t-s|)w_2(t)$:

$$2C\sum_{j=0}^{\infty}\phi\big(2^j|t-s| \wedge 2^{-1}\big)\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)} \sup_{\substack{u\ge j \\ k\in\mathbb{Z}}} \frac{\|2^{-u/2}x_{u,k}\|}{\varrho(2^{-u-1})\,w_2(\frac{2k+1}{2^{u+1}})}.$$

For the sake of notational simplicity, let us define the monotonically decreasing sequence $(h(j))_{j\in\mathbb{N}_0}$ by

$$h(j) = \sup_{\substack{u\ge j \\ k\in\mathbb{Z}}} \frac{\|2^{-u/2}x_{u,k}\|}{\varrho(2^{-u-1})\,w_2(\frac{2k+1}{2^{u+1}})}$$

and note that, by (3.14), $h(j) \to 0$, as $j \to \infty$. Let us split the series into two parts. Given $s, t \in \mathbb{R}$ such that $0 < |t-s| \le 1$, choose j_0 to be the unique integer such that

$$2^{-j_0-1} < |t-s| \le 2^{-j_0}.$$

Then, making use of (3.1) and (3.3), we have

$$\begin{aligned}\sum_{j=j_0}^{\infty}\phi(2^j|t-s| \wedge 2^{-1})\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)}h(j) &\le \phi(\tfrac{1}{2})\sum_{j=j_0}^{\infty}\frac{\varrho(2^{-j-1})}{\varrho(2^{-j_0-1})}h(j_0) \\ &\le \phi(\tfrac{1}{2})\sum_{j=j_0}^{\infty}\chi(2^{j_0-j})h(j_0) = \phi(\tfrac{1}{2})\sum_{j=0}^{\infty}\chi(2^{-j})h(j_0).\end{aligned}$$

Making use of (3.2) as well as the fact that h is a null sequence, we conclude that the last expression above vanishes, as $j_0 \to \infty$. In a similar manner, making use of (3.1) and (3.3), we have

$$\begin{aligned}\sum_{j=0}^{j_0-1}\phi(2^j|t-s|\wedge 2^{-1})\frac{\varrho(2^{-j-1})}{\varrho(|t-s|)}h(j) &\le \sum_{j=0}^{j_0-1}\phi(2^{j-j_0})\frac{\varrho(2^{-j-1})}{\varrho(2^{-j_0-1})}h(j)\\ &\le \sum_{j=0}^{j_0-1}\phi(2^{j-j_0})\chi(2^{j_0-j})h(j) = \sum_{j=0}^{j_0-1}\phi(2^{-j-1})\chi(2^{j+1})h(j_0-j-1)\\ &= \sum_{j=1}^{\infty}\phi(2^{-j})\chi(2^j)h(j_0-j)1_{j\le j_0}.\end{aligned}$$

In order to show that the last expression above vanishes, as $j_0 \to \infty$, let us take a closer look at

$$\sum_{j=1}^{\infty}\phi(2^{-j})\chi(2^j)h(j_0-j)1_{j\le j_0}. \tag{3.17}$$

On the one hand, we know that (3.3) holds. Therefore, the sequence

$$\big(\phi(2^{-j})\chi(2^j)\big)_{j\in\mathbb{N}}$$

induces a finite measure on $\mathbb{N}$ that we denote μ. Moreover, let us consider a sequence of functions $(f_{j_0})_{j_0\in\mathbb{N}}$ given by

$$f_{j_0}\colon \mathbb{N}\to\mathbb{R}_+ : j \mapsto h(j_0-j)1_{j\le j_0}.$$

It follows that $(f_{j_0})_{j_0\in\mathbb{N}}$ is uniformly bounded from above by $h(0)$ and pointwise it holds that

$$\lim_{j_0\to\infty} f_{j_0}(j) = \lim_{j_0\to\infty} h(j_0-j)1_{j\le j_0} = \lim_{l\to\infty} h(l) = 0.$$

Upon observing that (3.17) can be written as

$$\int_{\mathbb{N}} f_{j_0}(j)\ \mu(\mathrm{d}j), \tag{3.18}$$

we can apply the dominated convergence theorem to conclude that (3.17) vanishes, as $j_0 \to \infty$.

This was the most involved line of arguments. We are finally left with showing (2.42), which goes as follows: Let $n \in \mathbb{N}_0$ and $s, t \in \mathbb{R}\setminus[-n,n]$ such that $0 < |t-s| \le 1$ be given. Let us recycle some of the above ideas. First, we have the same estimate

$$\begin{aligned}\|f^x(t)-f^x(s)\| \le C_{\mathrm{qs}}\Big(&\sum_{l\in\{l_1,\dots,l_4\}}\phi\big(|t-s|\big)\|x_{-1,l}\|\\ &+\sum_{\substack{j=0\\ k\in\{k(j,t),k(j,s)\}}}^{\infty}\phi\big(2^j|t-s|\wedge 2^{-1}\big)\|2^{-\frac{j}{2}}x_{j,k}\|\Big).\end{aligned}$$

The first sum admits, similarly as above, the weighted upper bound

$$CC^* \sup_{\substack{l\in\mathbb{Z}\\ |l|\geq n-1}} \frac{\|x_{-1,l}\|}{w_1(l)} \frac{\phi(|t-s|)}{\varrho(|t-s|)} \leq CC^* \frac{C_\phi}{\varrho(1)} \sup_{\substack{l\in\mathbb{Z}\\ |l|\geq n-1}} \frac{\|x_{-1,l}\|}{w_1(l)}.$$

Clearly, by (3.13), the right-hand side goes to 0, as $n \to \infty$. The last term now admits the weighted upper bound

$$2C \sum_{j=0}^{\infty} \phi(2^j|t-s| \wedge 2^{-1}) \frac{\varrho(2^{-j-1})}{\varrho(|t-s|)} \sup_{\substack{u\in\mathbb{N}_0\\ |k|\geq 2^u n-1}} \frac{\|2^{-u/2} x_{u,k}\|}{\varrho(2^{-u-1})\, w_2(\frac{2k+1}{2^{u+1}})},$$

which consists of a series that is dominated from above by (3.2) and (3.3), uniformly over all $0 < |t-s| \leq 1$, and a sequence that by (3.15) converges to 0, as $n \to \infty$.

This concludes our proof. □

Proof of Theorem 3.12. This can be shown exactly as in the proof of Theorem 3.12 on page 49, upon noting that by Proposition 3.14 and Proposition 3.15, both $T_0 \colon f \mapsto x^f$ and $G \colon x \mapsto f^x$ are well defined maps between $\mathcal{C}_{\varrho,w_{1,2},0}$ and $\ell_{\varrho,w_{1,2},0}$. Point (g) follows from 3.12(e), Corollary 2.67 and Proposition 2.29. □

Proposition 3.16. *In the context of Definition 3.11, assume the pseudo-quasi-normed space X to be separable. Then so is the little weighted Hölder sequence space $\ell_{\varrho,w_{1,2},0}$.*

Proof. Let E_X denote a countable dense subset in X, and M be the set of double indexed sequences $x \in E_X^{\mathbb{N}^*\times\mathbb{Z}}$ with at most finitely many nonzero entries. The first observation to be made is that M is a countable set.

Next, for each $x \in \ell_{\varrho,w_{1,2},0}$ and $\varepsilon > 0$, according to (3.13), (3.14) and (3.15), we can find a pair $(J,K) \in \mathbb{N}\times\mathbb{N}$ such that, for all $(j,k) \in \mathbb{N}^*\times\mathbb{Z}$ with $j \geq J$ or $|k| \geq K$, the respective scaled entries in the sequence norm (3.4) are bounded from above by ε. As the set E_X is dense in X, we can find for each $(j,k) \in \mathbb{N}^*\times\mathbb{Z}$ with $j < J$ and $|k| < K$ an element $x^*_{j,k} \in E_X$, such that it approximates the elements $x_{j,k}$ arbitrarily close in the respective scaled norm parts of (3.4). To be precise, we choose for each $(j,k) \in \mathbb{N}^*\times\mathbb{Z}$ with $j < J$ and $|k| < K$ an element $x^*_{j,k} \in E_X$ such that, for all $j = -1$ and $|k| < K$:

$$\frac{\|x_{-1,l} - x^*_{-1,l}\|}{w_1(l)} \leq \varepsilon,$$

and for all $j \in \{0, 1, \dots J-1\}$ and $|k| < K$:

$$\frac{\|2^{-\frac{j}{2}}(x_{j,k} - x^*_{j,k})\|}{\varrho\big(2^{-j-1}\big)\, w_2\big(\frac{2k+1}{2^{j+1}}\big)} \leq \varepsilon.$$

Consequently, if we consider the sequence x^* that is composed of the finitely many $x^*_{j,k} \in E_X$ chosen above, and is set to zero otherwise, we have that $x^* \in M$ and $\|x - x^*\|_{\varrho,w_{1,2}} \leq \varepsilon$, which concludes our proof. □

3.3. First order differences and metric space setting

The ideas of the following line of reasoning are not new. In fact, they have been applied in the context of Besov spaces in [Ros09] as well as [LPT20]. The idea goes as follows: In the context of Theorem 3.6, according to point (c), we have for each $f \in \mathcal{C}_{\varrho,w_{1,2}}$ equivalence of the norms $\|f\|_{\varrho,w_{1,2}} \sim \|f\|_{\varrho,w_{1,2};(2)}$, given by

$$\|f\|_{\varrho,w_{1,2}} := \sup_{t\in\mathbb{R}} \frac{\|f(t)\|}{w_1(t)} + \sup_{\substack{s,t\in\mathbb{R} \\ 0<t-s\leq 1}} \frac{\|f(t)-f(s)\|}{\varrho(t-s)\, w_2(t)}, \tag{3.19}$$

and

$$\|f\|_{\varrho,w_{1,2};(2)} := \sup_{l\in\mathbb{Z}} \frac{\|f(l)\|}{w_1(l)} + \sup_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} \frac{\left\|2f\left(\frac{2k+1}{2^{j+1}}\right) - f\left(\frac{2k+2}{2^{j+1}}\right) - f\left(\frac{2k}{2^{j+1}}\right)\right\|}{\varrho\left(2^{-(j+1)}\right) w_2\left(\frac{2k+1}{2^{j+1}}\right)}. \tag{3.20}$$

It is important to note that there is a canonical counterpart of (3.19) for the metric space valued setting, by identifying $\|f(t)\| = \|f(t)-f(0)\|$ with $d(f(t), f(0))$ and $\|f(t)-f(s)\|$ with $d(f(t), f(s))$. However, for (3.20) this is not the case, because the second order differences appearing do not have a canonical generalization to metric spaces, due to the fact that these need not be endowed with a vector space structure. Luckily, we can draw from the ideas of [Ros09, Theorem 1], and try to show that we actually have the equivalence of three norms $\|f\|_{\varrho,w_{1,2}} \sim \|f\|_{\varrho,w_{1,2};(1)} \sim \|f\|_{\varrho,w_{1,2};(2)}$, where

$$\|f\|_{\varrho,w_{1,2};(1)} := \sup_{l\in\mathbb{Z}} \frac{\|f(l)\|}{w_1(l)} + \sup_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} \frac{\left\|f\left(\frac{k+1}{2^j}\right) - f\left(\frac{k}{2^j}\right)\right\|}{\varrho\left(2^{-j}\right) w_2\left(\frac{2k+1}{2^{j+1}}\right)}. \tag{3.21}$$

We will now try to verify that indeed, for any $f \in \mathcal{C}_{\varrho,w_{1,2}}$, the equivalence $\|f\|_{\varrho,w_{1,2};(1)} \sim \|f\|_{\varrho,w_{1,2};(2)}$ holds. Then, we will draw from the observation that the first order differences appearing in (3.21) do in fact have a canonical generalization to the metric space setting. Finally, by means of a suitably chosen result in the spirit of Kuratowski's embedding, see [Kur35], we will extend the equivalence $\|f\|_{\varrho,w_{1,2}} \sim \|f\|_{\varrho,w_{1,2};(1)}$ to the setting of functions on metric spaces.

Recall that each $f \in \mathcal{C}_{\varrho,w_{1,2}}$ admits the expansion

$$f(t) = \sum_{l\in\mathbb{Z}} a_l(f)\theta_l(t) + \sum_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} b_{j,k}(f)\tau_{j,k}(t),$$

and that the estimates of Lemma 2.65 as well as Assumption (3.3) hold. Pick some $(n,r) \in \mathbb{N}_0 \times \mathbb{Z}$. Then, by making use of the quasi-subadditivity property (2.11), we have

$$\begin{aligned}\left\|f\left(\tfrac{r+1}{2^n}\right) - f\left(\tfrac{r}{2^n}\right)\right\| \leq C_{\mathrm{qs}}\Big(&\sum_{l\in\mathbb{Z}} \left\|a_l(f)\left(\theta_l\left(\tfrac{r+1}{2^n}\right) - \theta_l\left(\tfrac{r}{2^n}\right)\right)\right\| \\ &+ \sum_{j=0}^{n-1}\sum_{k\in\mathbb{Z}} \left\|b_{j,k}(f)\left(\tau_{j,k}\left(\tfrac{r+1}{2^n}\right) - \tau_{j,k}\left(\tfrac{r}{2^n}\right)\right)\right\|\Big).\end{aligned}$$

Note that we are summing over $j \in \{0, 1, \dots, n-1\}$, as evaluations of $\tau_{j,k}$ for $j \geq n$ just equal 0. In particular, we are applying the quasi-subadditivity property to a finite sum, therefore we do not need to worry about convergence issues.

We will first focus our attention on the second part, the sum over j. Note that for each $(j, n, r) \in \mathbb{N}_0^2 \times \mathbb{Z}$, there exists at most one $k = k_{(j,n,r)} \in \mathbb{Z}$, such that

$$0 \neq \left|\tau_{j,k_{(j,n,r)}}\left(\tfrac{r+1}{2^n}\right) - \tau_{j,k_{(j,n,r)}}\left(\tfrac{r}{2^n}\right)\right| \leq 2^{j/2-n},$$

in which case, we actually can make the crucial observation that

$$\left|\tfrac{2r+1}{2^{n+1}} - \tfrac{2k_{(j,n,r)}+1}{2^{j+1}}\right| \leq 1.$$

The idea behind the following estimates originates from [Ros09]:

$$\begin{aligned}
&\sup_{r\in\mathbb{Z}} \frac{\sum_{j=0}^{n-1} \|2^{-j/2} b_{j,k_{(j,n,r)}}\| \phi(2^{j-n})}{\varrho\big(2^{-n}\big)\, w_2\big(\frac{2r+1}{2^{n+1}}\big)} = \sum_{j=0}^{n-1} \frac{\phi(2^{j-n})}{\varrho(2^{-n})} \sup_{r\in\mathbb{Z}} \frac{\|2^{-j/2} b_{j,k_{(j,n,r)}}\|}{w_2\big(\frac{2r+1}{2^{n+1}}\big)} \\
&\leq C \sum_{j=0}^{n-1} \frac{\phi(2^{j-n})}{\varrho(2^{-n})} \sup_{r\in\mathbb{Z}} \frac{\|2^{-j/2} b_{j,k_{(j,n,r)}}\|}{w_2\big(\frac{2k_{(j,n,r)}+1}{2^{j+1}}\big)} \leq C \sum_{j=0}^{n-1} \frac{\phi(2^{j-n})}{\varrho(2^{-n})} \sup_{k\in\mathbb{Z}} \frac{\|2^{-j/2} b_{j,k}\|}{w_2\big(\frac{2k+1}{2^{j+1}}\big)} \\
&\leq C\|f\|_{\varrho,w_{1,2};(2)} \sum_{j=0}^{n-1} \phi(2^{j-n}) \frac{\varrho(2^{-j-1})}{\varrho(2^{-n})} \leq C\|f\|_{\varrho,w_{1,2};(2)} \sum_{j=0}^{n-1} \phi(2^{j-n}) \chi(2^{n-j-1}) \\
&\leq C\|f\|_{\varrho,w_{1,2};(2)} \sum_{j=0}^{n-1} \phi(2^{-j-1}) \chi(2^{j}) \leq C\|f\|_{\varrho,w_{1,2};(2)} \sum_{j=1}^{\infty} \phi(2^{-j}) \chi(2^{j}).
\end{aligned}$$

Recall that, by (3.3), we have

$$\sum_{j=1}^{\infty} \phi(2^{-j}) \chi(2^{j}) < \infty.$$

All of the above now yields

$$\begin{aligned}
\sup_{(n,r)\in\mathbb{N}_0\times\mathbb{Z}} \frac{\sum_{j=0}^{n} \|2^{-j/2} b_{j,k_{(j,n,r)}}\| \phi(2^{j-n})}{\varrho\big(2^{-n}\big)\, w_2\big(\frac{2r+1}{2^{n+1}}\big)} &\leq C\|f\|_{\varrho,w_{1,2};(2)} \sup_{n\in\mathbb{N}_0} \sum_{j=0}^{n} \phi(2^{-j-1}) \chi(2^{j}) \\
&\leq C\|f\|_{\varrho,w_{1,2};(2)} \sum_{j=1}^{\infty} \phi(2^{-j}) \chi(2^{j}) \lesssim \|f\|_{\varrho,w_{1,2};(2)}.
\end{aligned}$$

Now we are still left with finding an estimate for

$$\sup_{r\in\mathbb{Z}} \frac{\sum_{l\in\mathbb{Z}} \|a_l(f)(\theta_l\big(\frac{r+1}{2^n}\big) - \theta_l\big(\frac{r}{2^n}\big))\|}{\varrho\big(2^{-n}\big)\, w_2\big(\frac{2r+1}{2^{n+1}}\big)}.$$

Similarly as above, for each $(n, r) \in \mathbb{N}_0 \times \mathbb{Z}$, there exist at most two nonnegative integers $l_1 < l_2$ such that

$$0 \neq \left|\theta_{l_i}\big(\tfrac{r+1}{2^n}\big) - \theta_{l_i}\big(\tfrac{r}{2^n}\big)\right| \leq 2^{-n}, \quad i \in \{1, 2\},$$

in which case we actually also have the crucial observation that

$$l_1 \le \tfrac{2r+1}{2^{n+1}} \le l_2.$$

Making use of (2.26), we have

$$\sup_{(n,r)\in\mathbb{N}_0\times\mathbb{Z}} \frac{\sum_{l\in\mathbb{Z}} \|a_l(f)(\theta_l(\frac{r+1}{2^n}) - \theta_l(\frac{r}{2^n}))\|}{\varrho(2^{-n})\, w_2(\frac{2r+1}{2^{n+1}})} \le \sup_{n\in\mathbb{N}_0} 2\frac{\phi(2^{-n})}{\varrho(2^{-n})} CC^* \sup_{l\in\mathbb{Z}} \frac{\|a_l(f)\|}{w_1(l)}$$
$$\le \frac{2CC^*}{\varrho(1)} \|f\|_{\varrho,w_{1,2};(2)} \sup_{n\in\mathbb{N}_0} \phi\big(\tfrac{2^{-n}}{1}\big) \frac{\varrho(1)}{\varrho(2^{-n})} \le \frac{2CC^*C_\phi}{\varrho(1)} \|f\|_{\varrho,w_{1,2};(2)}.$$

To sum up, in pulling all of the above estimates together we have shown that for $f \in \mathcal{C}_{\varrho,w_{1,2}}$ it holds that $\|f\|_{\varrho,w_{1,2};(1)} \lesssim \|f\|_{\varrho,w_{1,2};(2)}$. But the other direction

$$\|f\|_{\varrho,w_{1,2};(2)} \le C_{\mathrm{qs}} C(1+\phi(1)) \|f\|_{\varrho,w_{1,2};(1)}$$

readily follows from making use of the quasi-subadditivity (2.11) as well as the bounded growth condition (2.28) of the weights. Therefore, we have proven the

Proposition 3.17. *In the context of Theorem 3.6 each $f \in \mathcal{C}_{\varrho,w_{1,2}}$ satisfies the equivalence property*

$$\|f\|_{\varrho,w_{1,2}} \sim \|f\|_{\varrho,w_{1,2};(1)} \sim \|f\|_{\varrho,w_{1,2};(2)}. \tag{3.22}$$

Remark 3.18. How can we use Proposition 3.17 for an extension of the norm equivalence to metric spaces? We have to note that the setting of Proposition 3.17 contains the case where f takes values in a Banach space. Assume now that our image space is actually a metric space (Y,d). According to Kuratowski's embedding, see [Kur35], there exists a map

$$\Phi\colon (Y,d) \to (C_b(Y), \|\cdot\|_\infty)$$

into the space of all bounded, continuous and real-valued maps, endowed with the uniform norm, such that Φ is an isometry. Since $(C_b(Y), \|\cdot\|_\infty)$ is a Banach space, we can use the machinery of Proposition 3.17 to conclude that, for each $f \in \mathcal{C}_{\varrho,w_{1,2},d}$, where $\mathcal{C}_{\varrho,w_{1,2},d}$ denotes the metric space valued version of the space $\mathcal{C}_{\varrho,w_{1,2}}$, we have

$$\sup_{t\in\mathbb{R}} \frac{d(0,f(t))}{w_1(t)} + \sup_{\substack{s,t\in\mathbb{R}\\ 0<t-s\le 1}} \frac{d(f(t),f(s))}{\varrho(t-s)\, w_2(t)}$$
$$= \sup_{t\in\mathbb{R}} \frac{\|\Phi(f(t))\|_\infty}{w_1(t)} + \sup_{\substack{s,t\in\mathbb{R}\\ 0<t-s\le 1}} \frac{\|\Phi(f(t)) - \Phi(f(s))\|_\infty}{\varrho(t-s)\, w_2(t)}$$
$$\sim \sup_{l\in\mathbb{Z}} \frac{\|\Phi(f(l))\|_\infty}{w_1(l)} + \sup_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} \frac{\big\|\Phi\big(f\big(\frac{k+1}{2^j}\big)\big) - \Phi\big(f\big(\frac{k}{2^j}\big)\big)\big\|_\infty}{\varrho(2^{-j})\, w_2\big(\frac{2k+1}{2^{j+1}}\big)}$$
$$= \sup_{l\in\mathbb{Z}} \frac{d(0,f(l))}{w_1(l)} + \sup_{(j,k)\in\mathbb{N}_0\times\mathbb{Z}} \frac{d\big(f\big(\frac{k+1}{2^j}\big), f\big(\frac{k}{2^j}\big)\big)}{\varrho(2^{-j})\, w_2\big(\frac{2k+1}{2^{j+1}}\big)}.$$

This concludes our section on metric space valued functions.

4. Large deviations in weighted Hölder spaces for Banach-space-valued Brownian motion on the half-line

4.1. Infinite-dimensional Gaussian distributions

The aim of this section is to establish some basic ideas about Gaussian measures on topological vector spaces. We will revisit some concepts from [Bog98, Chapter 2 and Appendix A] with the intention to familiarize ourselves to a more general notion of Brownian motion, which will complement the setting of the following sections, where we will consider stochastic processes with values in real separable Banach spaces.

Let X denote a vector space over $\mathbb{R}$. We begin with some basic terminology.

(a) We call a map $p\colon X \to \mathbb{R}_+$ a seminorm, if for all $\lambda \in \mathbb{R}$ and $x, y \in X$ it holds that $p(\lambda x) = |\lambda| p(x)$ (absolute homogeneity) as well as $p(x+y) \leq p(x) + p(y)$ (subadditivity).

(b) Let A denote a non-empty set. We call the family $P = (p_\alpha)_{\alpha \in A}$ of seminorms on X point-separating, if for each $x \in X \setminus \{0\}$, there exists an $\alpha \in A$ such that $p_\alpha(x) > 0$.

(c) We call X a locally convex topological vector space, if the topology on X is generated by a family P of point-separating seminorms. In this case, a neighborhood basis of any point $x \in X$ is given by sets of the form

$$\big\{y \in X \,:\, p_{\alpha_i}(x-y) < \varepsilon,\ i = 1, \dots, n\big\},$$

where $\varepsilon > 0$ and $\alpha_1, \dots, \alpha_n \in A$ with $n \in \mathbb{N}$.

(d) Let $(X, \mathcal{T})$ denote a locally convex topological vector space. We then denote X^* the topological dual, i.e. the space of continuous linear functionals on X. Similarly, we denote X' the algebraic dual, i.e. the space on linear functionals on X.

(e) Let $\mathcal{E}(X)$ denote the smallest σ-algebra that makes all elements of X^* measurable.

(f) Let F denote a set of functions on X. We neither assume them to be linear, nor continuous. We then denote $\mathcal{E}(X, F)$ the smallest σ-algebra that makes all elements of F measurable. In general, we only have $\mathcal{E}(X) \subset \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel σ-algebra. However, in many cases we actually have equality, especially if X is a separable Fréchet space.

(g) By the previous point, it clearly holds that $\mathcal{E}(X) = \mathcal{E}(X, X^*)$.

(h) Let μ denote a measure on $(X, \mathcal{E}(X))$, and choose $h \in X$. We denote μ_h to be the pushforward measure under the map $X \ni x \mapsto x + h$. This implies for $A \in \mathcal{E}(X)$ that $\mu_h(A) = \mu(A - h)$. Note that μ does not need to be a probability measure.

(i) Let μ denote a measure on $(X, \mathcal{E}(X))$. Then $\mathcal{E}(X)_\mu$ denotes the Lebesgue completion of $\mathcal{E}(X)$. This means that $A \in \mathcal{E}(X)_\mu$ if there exist $B_1, B_2 \in \mathcal{E}(X)$ such that $B_1 \subset A \subset B_2$ and

$$\mu(B_2 \setminus B_1) = 0.$$

Definition 4.1 (Gaussian measure)**.** Let X denote a locally convex topological vector space over $\mathbb{R}$. We call a probability measure γ on $(X, \mathcal{E}(X))$ Gaussian, if one of the following two equivalent conditions holds:

(a) For each $f \in X^*$, there exists an $a \in \mathbb{R}$ and $\sigma \geq 0$, such that for all $F \in \mathcal{B}(\mathbb{R})$, we have in the case $\sigma > 0$:

$$\gamma(\{f \in F\}) = \int_F \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(x-a)^2}{2\sigma^2}} \ \lambda(\mathrm{d}x),$$

and in the case $\sigma = 0$:

$$\gamma(\{f \in F\}) = \delta_a(F),$$

where δ_a denotes the Dirac point mass at a, and λ denote the Lebesgue–Borel measure.

(b) There exist a linear functional $L \in (X^*)'$ and a positive semidefinite symmetric bilinear form q on X^*, such that for each $f \in X^*$, we have

$$\int_X \mathrm{e}^{\mathrm{i}f(x)} \ \gamma(\mathrm{d}x) = \mathrm{e}^{\mathrm{i}L(f) - \frac{1}{2}q(f,f)}.$$

If we have for each $f \in X^*$ that $a = 0$, or equivalently, $L \equiv 0$, we call γ a centered Gaussian measure. Moreover, if $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space, and ξ is a random vector, i.e. a measurable map $\xi \colon (\Omega, \mathcal{F}) \to (X, \mathcal{E}(X))$, then we call ξ Gaussian, if it induces a Gaussian measure on $(X, \mathcal{E}(X))$. We call ξ a Gaussian random variable, if it is a Gaussian random vector with image space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Remark 4.2. Note that condition 4.1(a) is a statement about the pushforward-measures $\gamma \circ f^{-1}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, while the equivalent condition 4.1(b) is a statement about the Fourier transform of the measure γ on $(X, \mathcal{E}(X))$. The fact that these two conditions are equivalent is not a triviality, for a proof see [Bog98, Theorem 2.2.4]. Moreover, the special case of a centered Gaussian measure originates from the defining requirement that for all $F \in \mathcal{E}(X)$, we have $\gamma(F) = \gamma(-F)$, see [Bog98, Corollary 2.2.5]. In 4.1(a), we call γ degenerate if there exists an $f \in X^*$ that is not the zero functional, such that the corresponding σ satisfies $\sigma = 0$, and nondegenerate otherwise.

Definition 4.3 (Mean and covariance operator)**.** Let X denote a locally convex topological vector space over $\mathbb{R}$. Consider a probability measure γ on $(X, \mathcal{E}(X))$ and assume that $X^* \subset L^2(\gamma)$ holds. Then, the mean of γ is given as the element a_γ in the algebraic dual of X^*, i.e. $a_\gamma \in (X^*)'$, such that

$$a_\gamma(f) = \int_X f(x)\ \gamma(\mathrm{d}x), \quad f \in X^*.$$

Moreover, the covariance operator $R_\gamma\colon X^* \to (X^*)'$ of γ is given by

$$R_\gamma(f)(g) = \int_X \big(f(x) - a_\gamma(f)\big)\big(g(x) - a_\gamma(g)\big)\ \gamma(\mathrm{d}x), \quad f, g \in X^*,$$

where we call the quadratic form $f \mapsto R_\gamma(f)(f)$ the covariance of γ.

Definition 4.4 (Abstract Cameron–Martin space)**.** Let X denote a locally convex topological vector space over $\mathbb{R}$ and consider a Gaussian measure γ on $(X, \mathcal{E}(X))$. The abstract Cameron–Martin space $H(\gamma) \subset X$ is then given by

$$H(\gamma) \coloneqq \big\{h \in X\colon \|h\|_{H(\gamma)} < \infty\big\},$$

where

$$\|h\|_{H(\gamma)} \coloneqq \sup\big\{f(h)\colon f \in X^*, R_\gamma(f)(f) \le 1\big\}.$$

Now we move on to processes.

Definition 4.5 (Gaussian random process)**.** Let T be a nonempty set. Consider a family $(\xi_t)_{t\in T}$ of random vectors. It is then called a Gaussian random process, if for each $n \in \mathbb{N}$ and $t_1, \dots, t_n \in T$ as well as $f_1, \dots, f_n \in X^*$, we have that $(f_1(\xi_{t_1}), \dots, f_n(\xi_{t_n}))$ induces a Gaussian measure on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ in the sense of Definition 4.1.

The Gaussian random process ξ induces a Gaussian measure γ^ξ on $\mathbb{R}^T$, the space of functions $f\colon T \to \mathbb{R}$, endowed with the topology of pointwise convergence, see [Bog98, Proposition 2.3.9]. This topology is induced by the family $(\delta_t)_{t\in T}$ of seminorms mapping $\mathbb{R}^T \ni x \mapsto \delta_t(x) \coloneqq |x(t)|$. The function q for this measure γ^ξ from Definition 4.1(b) is uniquely determined by the covariance function $K\colon T \times T \to \mathbb{R}$ given by

$$K(s, t) = \mathbb{E}\big[(\xi_s - \mathbb{E}[\xi_s])(\xi_t - \mathbb{E}[\xi_t])\big].$$

It then holds that $K(s, t) = q(\delta_s, \delta_t)$.

Example 4.6. Let T denote a nonempty set, and consider $X = \mathbb{R}^T$. Let L and q denote a linear and positive semidefinite bilinear form on $(\mathbb{R}^T)^*$, respectively, and consider the map $F\colon (\mathbb{R}^T)^* \to \mathbb{C}$ given by

$$f \mapsto \mathrm{e}^{\mathrm{i}L(f) - \frac{1}{2}q(f,f)},$$

see also Definition 4.1(b). By [Bog98, Proposition 2.3.9], it holds that F is the Fourier transform of some Gaussian measure γ on X. We will now consider some specific examples of processes that can be constructed from these measures on suitably chosen subsets of X of full measure, by choosing $T = \mathbb{R}_+$, $L \equiv 0$ as well as different covariance functions K, see also [Bog98, Examples 2.3.11 and 2.3.15].

(a) Brownian motion: Choose

$$K(s,t) = \min\{s,t\}, \quad 0 \le s \le t < \infty.$$

(b) Fractional Brownian motion: Choose $\alpha \in (0,1]$ and set

$$K(s,t) = \tfrac{1}{2}\big(s^{2\alpha} + t^{2\alpha} - (t-s)^{2\alpha}\big), \quad 0 \le s \le t < \infty.$$

(c) Stationary Ornstein–Uhlenbeck process: Choose

$$K(s,t) = \mathrm{e}^{-(t-s)}, \quad 0 \le s \le t < \infty.$$

(d) Fractional Ornstein–Uhlenbeck process: Choose $\alpha \in (0,1]$ and set

$$K(s,t) = \mathrm{e}^{-(t-s)^{2\alpha}}, \quad 0 \le s \le t < \infty.$$

One of the overarching concepts that we will be dealing with is the notion of abstract Wiener spaces. For the remainder of this chapter, we denote by $\mathcal{M}_1(E)$ the set of probability measures on $(E, \mathcal{B}(E))$, for a given real separable Banach space E. We follow the expositions in [BBK92] as well as [Str11, Chapter 8]. Let us state the

Definition 4.7 (Abstract Wiener space)**.** Let H denote a real separable Hilbert space, E denote a real separable Banach space and $\mathcal{W}$ be a probability measure on $(E, \mathcal{B}(E))$. The triple $(H, E, \mathcal{W})$ is called an abstract Wiener space, if

- H is continuously embedded as a dense subspace of E;
- For every w^* in the topological dual E^*, we have

$$\int_E \mathrm{e}^{\mathrm{i}\langle w^*, w\rangle}\ \mathcal{W}(\mathrm{d}w) = \mathrm{e}^{-\frac{1}{2}\|h_{w^*}\|_H^2}, \tag{4.1}$$

  where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of (E^*, E) and $h_{w^*} \in H$ is the unique element such that $(h_{w^*}, h)_H = \langle w^*, h\rangle$ for all $h \in H$.

Note that the Equation (4.1) is a statement about the Fourier transform of the probability measure $\mathcal{W}$. The existence and uniqueness of h_{w^*} is obtained by means of the Fréchet–Riesz representation theorem, see [Sch20, Theorem 14.16]. Moreover, Equation (4.1) implies that $\mathcal{W}$ is a centered and nondegenerate Gaussian measure on E. In contrast to Definition 4.1, we write $\mathcal{W}$ instead of γ, because Gaussian measures on abstract Wiener spaces are also referred to as Wiener measures, in honor of the mathematician Norbert Wiener, who made significant contributions to the theory.

If $\mathcal{W}$ denotes a centered and nondegenerate Gaussian measure on E in the sense of Definition 4.1, where E denotes a real separable Banach space, then, according to [Str11, Theorem 8.2.5], there exists a unique real, separable Hilbert space H, such that $(H, E, \mathcal{W})$ becomes an abstract Wiener space. We call H the corresponding Cameron–Martin space, see also Definition 4.4.

In what follows, let H denote a real, separable Hilbert space.

Definition 4.8 (absolute continuity)**.** A map $h\colon \mathbb{R}_+ \to H$ is called absolutely continuous, if for each $\varepsilon > 0$, there exists a $\delta > 0$, such that, for all $n \in \mathbb{N}$ and $0 \le a_1 \le b_1 \le a_2 \le b_2 \le \ldots \le a_n \le b_n < \infty$ with $\sum_{i=1}^n (b_i - a_i) < \delta$, it follows that

$$\sum_{i=1}^{n} \|h(b_i) - h(a_i)\|_H < \varepsilon.$$

Every Lipschitz-continuous map is absolutely continuous. Moreover, every absolutely continuous map is uniformly continuous and thus continuous. But more strikingly, it is differentiable λ-almost everywhere. In fact, we have the following theorem, which also holds if the image space is more generally a reflexive Banach space (see [AGS08, Remark 1.1.3]):

Theorem 4.9. *Let $h\colon \mathbb{R}_+ \to H$ be an absolutely continuous map, and λ denote the Lebesgue–Borel measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Then h is differentiable λ-almost everywhere with H-valued density $\dot{h}$. Moreover, for any two points $a < b$ in $\mathbb{R}_+$, we have*

$$h(b) - h(a) = \int_a^b \dot{h}(s)\ \lambda(\mathrm{d}s).$$

Note that Theorem 4.9 can actually be generalized to the case where the image space of $h\colon \mathbb{R}_+ \to X$ is a metric space, see [AGS08, Section 1.1].

Upon having treated the basic framework, we are ready to state the

Definition 4.10 (Classical Cameron–Martin space)**.** See [Str11, Section 8.6.1]. Let $H^1(H)$ denote the vector space of maps $h\colon \mathbb{R}_+ \to H$ that satisfy $h(0) = 0 \in H$, that are absolutely continuous, and admit a square-integrable density $\dot{h}$. Endowed with the norm

$$\|h\|_{H^1(H)} := \|\dot{h}\|_{L^2(\mathbb{R}_+,H)} = \Big(\int_{\mathbb{R}_+} \|\dot{h}(s)\|_H^2\ \lambda(\mathrm{d}s) \Big)^{1/2},$$

$H^1(H)$ actually constitutes a Hilbert space. We call $(H^1(H), \|\cdot\|_{H^1(H)})$ the classical Cameron–Martin space.

Remark 4.11. The connection between the Definitions 4.4 and 4.10 is established in [Bog98, Lemma 2.3.14] for the case of real-valued Gaussian random processes on the unit time interval.

Lemma 4.12. *The Cameron–Martin space as given in Definition 4.10 is a real separable Hilbert space.*

Proof. According to [Rud91, Definition 12.1], we first have to show that $H^1(H)$ is a vector space over $\mathbb{R}$ that carries an inner product, i.e. a positive definite and symmetric map $\langle \cdot, \cdot \rangle_{H^1(H)}\colon H^1(H) \times H^1(H) \to \mathbb{R}$ that is linear in its first argument, which induces a norm that makes the space complete.

The vector space structure is easy to see. For $g, h \in H^1(H)$, we set

$$\langle g, h \rangle_{H^1(H)} := \int_{\mathbb{R}_+} \langle \dot{g}(s), \dot{h}(s) \rangle_H\ \lambda(\mathrm{d}s),$$

where $\langle\cdot,\cdot\rangle_H\colon H\times H\to\mathbb{R}$ denotes the inner product on the Hilbert space H. It is straightforward to verify the symmetry and linearity in the first argument of $\langle\cdot,\cdot\rangle_{H^1(H)}$. Moreover, we have that $\langle\cdot,\cdot\rangle_{H^1(H)}$ is well defined, which can be seen by means of applying the Cauchy–Schwarz inequality twice.

We focus on showing positive definiteness. We have to show that $\langle h,h\rangle_{H^1(H)}=0$ if $h=0$ and $\langle h,h\rangle_{H^1(H)}>0$ if $h\neq 0$. If $h=0$, then $\dot h\equiv 0$, and therefore $\langle h,h\rangle_{H^1(H)}=0$ due to the fact that the inner product $\langle\cdot,\cdot\rangle_H$ is indeed positive definite. On the other hand, if we assume that $h\neq 0$, then there exists at least one $t\in\mathbb{R}_+$ such that $h(t)\neq 0$. Let us now write

$$\begin{aligned}\langle h,h\rangle_{H^1(H)}&=\int_{\mathbb{R}_+}\langle\dot h(s),\dot h(s)\rangle_H\ \lambda(\mathrm{d}s)\\&=\int_{\{\dot h=0\}}\langle\dot h(s),\dot h(s)\rangle_H\ \lambda(\mathrm{d}s)+\int_{\{\dot h\neq0\}}\langle\dot h(s),\dot h(s)\rangle_H\ \lambda(\mathrm{d}s)\\&=\int_{\{\dot h\neq0\}}\langle\dot h(s),\dot h(s)\rangle_H\ \lambda(\mathrm{d}s).\end{aligned}$$

Since on the set $\{\dot h\neq 0\}$ it holds that $\langle\dot h(s),\dot h(s)\rangle_H>0$, we just have to verify that $\lambda(\{\dot h\neq 0\})>0$. Let us assume that this is not the case, i.e. we assume that $\lambda(\{\dot h\neq 0\})=0$. But then, upon recalling that $h(0)=0$ holds, we have for the above $t\in\mathbb{R}_+$ where $h(t)\neq 0$:

$$h(t)=\int_0^t\dot h(s)\ \lambda(\mathrm{d}s)=0,$$

which yields a contradiction to the assumption $h\neq 0$ and implies that, $\langle h,h\rangle_{H^1(H)}>0$.

In order to show completeness of the space $H^1(H)$, note that $L^2(\mathbb{R}_+;H)$ is complete. If $(h_n)_{n\in\mathbb{N}}$ denotes a Cauchy sequence in $H^1(H)$, then $(\dot h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}_+;H)$, and thus converges to some $g\in L^2(\mathbb{R}_+;H)$. If we now consider $h:=\mathbb{R}_+\ni t\mapsto\int_0^t g(s)\ \lambda(\mathrm{d}s)$, then $h\in H^1(H)$ with $\|h_n-h\|_{H^1(H)}=\|\dot h_n-g\|_{L^2(\mathbb{R}_+;H)}\to 0$, as $n\to\infty$.

Finally, we show that $H^1(H)$ is separable. According to [Str11, Section 8.3.2], the family $\{\dot h_{j,k}\colon (j,k)\in\mathbb{N}_0^2\}$ of functions given by $\dot h_{0,k}=\mathbb{1}_{[k,k+1)}$ and

$$\dot h_{j,k}=2^{(j-1)/2}(\mathbb{1}_{[k2^{1-j},(2k+1)2^{-j})}-\mathbb{1}_{[(2k+1)2^{-j},(k+1)2^{1-j})})$$

is an orthonormal basis in $L^2(\mathbb{R}_+;\mathbb{R})$. As we assume H to be separable, there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}_0}$ in H. Upon defining $h_{j,k}\colon\mathbb{R}_+\ni t\mapsto\int_0^t\dot h_{j,k}(s)\ \lambda(\mathrm{d}s)$, it follows that the family $\{h_{j,k}e_i\colon (i,j,k)\in\mathbb{N}_0^3\}$ is an orthonormal basis in $H^1(H)$. Consequently, $H^1(H)$ is separable. □

Let $\Theta(E)$ denote the space of all continuous $\theta\colon\mathbb{R}_+\to E$ that satisfy $\theta(0)=0$ as well as

$$\lim_{t\to\infty}\frac{\|\theta(t)\|_E}{t}=0.$$

Endowed with the norm $\|\theta\|_{\Theta(E)}:=\sup_{t\in\mathbb{R}_+}\|\theta(t)\|_E/(1+t)$ the space $\Theta(E)$ constitutes a separable Banach space, and it can be further shown that $H^1(H)$ is continuously embedded into $\Theta(E)$ as a dense subspace, which can be shown similarly as in [Str11, Section 8.1].

According to [Str11, Theorem 8.6.1], there exists a unique probability measure $\mathcal{W}^{(E)} \in \mathcal{M}_1(\Theta(E))$ such that $(H^1(H), \Theta(E), \mathcal{W}^{(E)})$ constitutes an abstract Wiener space.

Let us revisit in this general context the notion of a Brownian motion.

Definition 4.13 ($\mathcal{W}$-Brownian motion)**.** Let $(H, E, \mathcal{W})$ denote an abstract Wiener space as given in Definition 4.7. Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and a measurable map $B\colon \mathbb{R}_+ \times \Omega \to E$. We then call B a $\mathcal{W}$-Brownian motion, if

1. B is $\mathbb{F}$-progressively measurable, i.e. for all $T \in \mathbb{R}_+$, the restricted map $B_{\restriction_{[0,T]\times\Omega}}$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}_T$-measurable;
2. $B_0(\omega) = 0$ and $B_\cdot(\omega) \in C(\mathbb{R}_+; E)$, $\mathbb{P}$-almost surely;
3. $\mathcal{L}(B_1) = \mathcal{W}$, and for all $0 \le s < t < \infty$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and has distribution $\mathcal{L}(B_t - B_s) = \mathcal{L}(\sqrt{t-s}B_1)$.

The existence of $\mathcal{W}$-Brownian motion B is established in [Str11, Theorem 8.6.6], and it is moreover shown that we can see $\mathcal{W}^{(E)}$ as the pushforward measure of $\mathbb{P}$ under B on $\Theta(E)$. Furthermore, one can employ a generalized version of Fernique's theorem in Banach spaces, see [Bog98, Theorem 2.8.5] or [Str11, Theorem 8.2.1], together with Kolmogorov–Chentsov's continuity theorem (see [Sch20, Theorem 2.98] for a version for stochastic processes with values in Polish spaces), in order to show that there exists a modification $\tilde{B}$ of $\mathcal{W}$-Brownian motion B such that all paths are locally uniformly Hölder continuous for every exponent $\alpha \in (0, 1/2)$. We will write B instead of $\tilde{B}$ and consider this modification from now on.

The Brownian time inversion invariance property is given in [Str11, Exercise 8.6.8 (ii)]. This implies that, if we consider the map $I\colon \Theta(E) \to C(\mathbb{R}_+; E)$ given by

$$I(\theta)(t) = \begin{cases} 0 & \text{for } t = 0, \\ t\theta(\frac{1}{t}) & \text{otherwise,} \end{cases}$$

then I is actually an involuntory isometry from $\Theta(E)$ onto itself, and the pushforward measure of $\mathcal{W}^{(E)}$ under I coincides with $\mathcal{W}^{(E)}$. Let us set $B(\omega) \equiv 0$, if $I(B(\omega))$ is not locally uniformly Hölder continuous for all exponents $\alpha \in (0, 1/2)$, and further $X \coloneqq I(B)$.

4.2. The classical Cameron–Martin space and the weighted Hölder spaces

We will now define the appropriate weight functions and admissible moduli of continuity, which will allow us to apply our findings from Chapters 2 and 3 to the study of path space of Brownian motion. For $\alpha > 0$ and $\beta \ge 0$, consider the modulus of continuity

$$\varrho_\alpha\colon (0, \infty) \to (0, \infty)\colon \delta \mapsto \delta^\alpha,$$

as well as the weight function

$$w_\beta\colon \mathbb{R}_+ \to (0, \infty)\colon t \mapsto \max\{1, t^\beta\}.$$

Recall the definition of little weighted Hölder spaces in the context of this chapter:

Definition 4.14 (Little weighted Hölder space)**.** Let $\alpha > 0$ and $\beta, \gamma \geq 0$ be given. We denote by $\mathcal{M}^{\alpha,\beta,\gamma}$ the vector space of all maps $f\colon \mathbb{R}_+ \to E$ such that $f(0) = 0$ and

$$\|f\|_{\alpha,\beta,\gamma} := \sup_{t\in\mathbb{R}_+} \frac{\|f(t)\|_E}{w_\beta(t)} + \sup_{\substack{s,t\in\mathbb{R}_+\\ 0<t-s\leq 1}} \frac{\|f(t)-f(s)\|_E}{\varrho_\alpha(t-s)\, w_\gamma(t)} < \infty,$$

where $\|\cdot\|_E$ denotes the norm on the separable Banach space E over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Further, we denote by $\mathcal{M}_0^{\alpha,\beta,\gamma} \subset \mathcal{M}^{\alpha,\beta,\gamma}$ the subset of all maps $f \in \mathcal{M}^{\alpha,\beta,\gamma}$ that satisfy the following conditions:

$$\lim_{t\to\infty} \frac{\|f(t)\|_E}{w_\beta(t)} = 0,$$

$$\lim_{\delta\searrow 0} \sup_{\substack{s,t\in\mathbb{R}_+\\ 0<t-s\leq \delta}} \frac{\|f(t)-f(s)\|_E}{\varrho_\alpha(t-s)\, w_\gamma(t)} = 0,$$

$$\lim_{n\to\infty} \sup_{\substack{s,t\in\mathbb{R}_+\setminus[0,n]\\ 0<t-s\leq 1}} \frac{\|f(t)-f(s)\|_E}{\varrho_\alpha(t-s)\, w_\gamma(t)} = 0.$$

We call the space $(\mathcal{M}_0^{\alpha,\beta,\gamma}, \|\cdot\|_{\alpha,\beta,\gamma})$ little weighted Hölder space.

Remark 4.15. According to Remark 2.48, Lemma 2.50 as well as Lemma 2.59, it follows that both $\mathcal{M}^{\alpha,\beta,\gamma}$ and $\mathcal{M}_0^{\alpha,\beta,\gamma}$ are Banach spaces over $\mathbb{K}$, because E is. Moreover, in line with Corollary 2.67, we have that $\mathcal{M}_0^{\alpha,\beta,\gamma}$ is separable, as long as $\beta \geq \gamma$ and $\beta > 0$.

Next, we will show that under suitable assumptions, the classical Cameron–Martin space $H^1(H)$ as defined in 4.10 is continuously embedded into the spaces $\mathcal{M}_0^{\alpha,\beta,\gamma}$ as a dense subset.

Proposition 4.16. *Let E denote a separable Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and H be a separable Hilbert space over $\mathbb{K}$ that is continuously embedded into E as a dense subset.*

(a) If $\alpha \in (0, 1/2]$, $\beta \geq 1/2$ and $\gamma \geq 0$, then for each $h \in H^1(H)$ as well as $\delta > 0$, we have

$$\sup_{t\in\mathbb{R}_+} \frac{\|h(t)\|_H}{w_\beta(t)} + \sup_{\substack{s,t\in\mathbb{R}_+\\ 0<t-s\leq \delta}} \frac{\|h(t)-h(s)\|_H}{\varrho_\alpha(t-s)\, w_\gamma(t)} \leq (1+\delta^{1/2-\alpha})\|h\|_{H^1(H)} < \infty, \tag{4.2}$$

and consequently, $H^1(H)$ is continuously embedded into $\mathcal{M}^{\alpha,\beta,\gamma}$ as a subset.

(b) If $\alpha \in (0, 1/2)$, $\beta > 1/2$ and $\gamma > 0$, then $H^1(H)$ is continuously embedded into $\mathcal{M}_0^{\alpha,\beta,\gamma}$ as a subset.

(c) In the context of part (b), if in addition $\gamma \leq \beta$ holds, then $H^1(H)$ is dense in $\mathcal{M}_0^{\alpha,\beta,\gamma}$.

Remark 4.17. Note that, in contrast to Theorem 4.18 below, part (a) of Proposition 4.16 allows for $\alpha = 1/2$, $\beta = 1/2$ and $\gamma = 0$.

Proof. We begin the proof with some observations. For $h \in H^1(H)$, upon applying the Cauchy–Schwarz inequality, we get, for each $s, t \in \mathbb{R}_+$ such that $s \leq t$,

$$\begin{aligned}\|h(t) - h(s)\|_H &\leq \int_s^t \|\dot{h}(s)\|_H \ \lambda(\mathrm{d}s) \\ &\leq \Big(\int_s^t |1|^2 \ \lambda(\mathrm{d}s)\Big)^{1/2}\Big(\int_s^t \|\dot{h}(s)\|_H^2 \ \lambda(\mathrm{d}s)\Big)^{1/2} \\ &\leq \sqrt{t-s}\Big(\int_{\mathbb{R}_+} \|\dot{h}(s)\|_H^2 \ \lambda(\mathrm{d}s)\Big)^{1/2} = \sqrt{t-s}\|h\|_{H^1(H)},\end{aligned}$$

see also [Str11, Section 8.1.2]. Moreover, recall that we assume H to be continuously embedded into E as a subset. Therefore, the canonical embedding, which is a linear map between the Banach spaces H and E, is bounded in operator norm, hence

$$\|h\|_E \leq C_{\mathrm{e}}\|h\|_H, \quad h \in H, \tag{4.3}$$

where C_{e} denotes the operator norm of the canonical embedding.

(a) For each $t \in \mathbb{R}_+$ (set $s = 0$), we have

$$\frac{\|h(t)\|_H}{w_\beta(t) \ \|h\|_{H^1(H)}} = \frac{\|h(t) - h(0)\|_H}{w_\beta(t) \ \|h\|_{H^1(H)}} \leq \begin{cases} t^{1/2} \leq 1 & \text{for } t \in [0,1], \\ t^{1/2-\beta} \leq 1 & \text{for } t > 1. \end{cases} \tag{4.4}$$

On the other hand, we get for each $\delta > 0$ and $s, t \in \mathbb{R}_+$ such that $0 < t - s \leq \delta$

$$\frac{\|h(t) - h(s)\|_H}{\varrho_\alpha(t-s) \ w_\gamma(t) \ \|h\|_{H^1(H)}} \leq \begin{cases} (t-s)^{1/2-\alpha} \leq \delta^{1/2-\alpha} & \text{for } t \in (0,1], \\ (t-s)^{1/2-\alpha} t^{-\gamma} \leq \delta^{1/2-\alpha} & \text{for } t > 1. \end{cases} \tag{4.5}$$

The estimates (4.4) and (4.5) now imply (4.2), and the continuity of the canonical embedding of $H^1(H)$ into $\mathcal{M}^{\alpha,\beta,\gamma}$ follows from (4.2) and (4.3), as we now have

$$\sup_{t \in \mathbb{R}_+} \frac{\|h(t)\|_E}{w_\beta(t)} + \sup_{\substack{s,t \in \mathbb{R}_+ \\ 0 < t-s \leq \delta}} \frac{\|h(t) - h(s)\|_E}{\varrho_\alpha(t-s) \ w_\gamma(t)} \leq C_{\mathrm{e}}(1 + \delta^{1/2-\alpha})\|h\|_{H^1(H)} < \infty.$$

(b) The inequalities (4.4) and (4.5) actually point towards some further properties, which we will collect now. For each $h \in H^1(H)$, we have

$$\lim_{t \to \infty} \frac{\|h(t)\|_H}{w_\beta(t)} \leq \lim_{t \to \infty} t^{1/2-\beta}\|h\|_{H^1(H)} = 0,$$

hence the weighted expression vanishes as time goes to infinity. Moreover,

$$\begin{aligned}\lim_{\delta \searrow 0} \sup_{\substack{s,t \in \mathbb{R}_+ \\ 0 < t-s \leq \delta}} \frac{\|h(t) - h(s)\|_H}{\varrho_\alpha(t-s) \ w_\gamma(t)} &\leq \lim_{\delta \searrow 0} \sup_{\substack{s,t \in \mathbb{R}_+ \\ 0 < t-s \leq \delta}} (t-s)^{1/2-\alpha}\|h\|_{H^1(H)} \\ &\leq \lim_{\delta \searrow 0} \delta^{1/2-\alpha}\|h\|_{H^1(H)} = 0,\end{aligned}$$

hence the Hölder constants vanish uniformly for small time increments. Now fix a parameter $\delta > 0$ and conclude that

$$\begin{aligned}\lim_{n\to\infty} \sup_{\substack{s,t\in\mathbb{R}_+\setminus[0,n]\\ 0<t-s\le\delta}} \frac{\|h(t)-h(s)\|_H}{\varrho_\alpha(t-s)\; w_\gamma(t)} &\le \lim_{n\to\infty} \sup_{\substack{s,t\in\mathbb{R}_+\setminus[0,n]\\ 0<t-s\le\delta}} (t-s)^{1/2-\alpha} t^{-\gamma} \|h\|_{H^1(H)} \\ &\le \lim_{n\to\infty} \delta^{1/2-\alpha} n^{-\gamma} \|h\|_{H^1(H)} = 0,\end{aligned}$$

hence the Hölder constants vanish uniformly for distant times (just set $\delta = 1$). Together with the estimate (4.3) and part (a), we obtain that $H^1(H)$ is continuously embedded into $\mathcal{M}_0^{\alpha,\beta,\gamma}$.

(c) Let us state some very specific functions that are elements of the Cameron–Martin space $H^1(H)$. Recall the definition of the time change $\gamma_m\colon \mathbb{R}_+ \to \mathbb{R}_+$ for $m \in \mathbb{N}$ given by $t \mapsto \min\{t, m\}$. Further, recall the sequence $\{\theta_l : l \in \mathbb{N}\}$ defined in Equation (2.2). Starting from the map $\theta\colon \mathbb{R} \ni t \to \max\{0, 1-|t|\}$, we set, for each $l \in \mathbb{N}$, $\theta_l(t) := \theta(t-l)$, and for each $m \in \mathbb{N}$, $\theta_{l,m}(t) := \theta_l(\gamma_m(t))$. But according to Lemma 2.65, θ_l is Lipschitz-continuous for each $l \in \mathbb{N}$ and thus also $\theta_{l,m}$ for each $(l, m) \in \mathbb{N}^2$. Hence, these maps are absolutely continuous, and admissible densities are given by

$$\dot{\theta}_{l,m} = \begin{cases} 1_{(l-1,l)} - 1_{(l,l+1)} & \text{for } l < m, \\ 1_{(l-1,l)} & \text{for } l = m, \\ 0 & \text{otherwise,} \end{cases}$$

which are clearly elements of $\mathrm{L}^2(\mathbb{R}_+, \mathbb{R})$. Consequently, the functions $\theta_{l,i,m} := \theta_{l,m} e_i$, where $(l, m, i) \in \mathbb{N}^3$ and $(e_i)_{i\in\mathbb{N}}$ denotes an orthonormal basis in the real separable Hilbert space H, are all elements of $(H^1(H), \|\cdot\|_{H^1(H)})$.

Next, recall the sequence $\{\tau_{j,k} : (j, k) \in \mathbb{N}_0^2\}$ defined in (2.3). Starting from the map $\tau\colon \mathbb{R} \ni t \to \max\{0, 1/2-|t-1/2|\}$, we set, for each $(j, k) \in \mathbb{N}_0^2$, $\tau_{j,k}(t) := 2^{-j/2}\tau(2^j t-k)$, and for each $m \in \mathbb{N}$, $\tau_{j,k,m}(t) := \tau_{j,k}(\gamma_m(t))$. According to Lemma 2.65, we know that $\tau_{j,k}$ is Lipschitz-continuous, for each $(j, k) \in \mathbb{N}_0^2$ and thus also $\tau_{j,k,m}$ for each $(j, k, m) \in \mathbb{N}_0^2 \times \mathbb{N}$, and admissible densities are given by

$$\dot{\tau}_{j,k,m} = \begin{cases} 2^{-j/2}\big(1_{(\frac{2k}{2^{j+1}},\frac{2k+1}{2^{j+1}})} - 1_{(\frac{2k+1}{2^{j+1}},\frac{2k+2}{2^{j+1}})}\big) & \text{for } (j,k,m) \in \mathbb{N}_0^2 \times \mathbb{N}, k \le 2^j m - 1, \\ 0 & \text{for } (j,k,m) \in \mathbb{N}_0^2 \times \mathbb{N}, k \ge 2^j m, \end{cases}$$

which are clearly elements of $\mathrm{L}^2(\mathbb{R}_+, \mathbb{R})$. Again, as before, the functions $\tau_{j,k,i,m} := \tau_{j,k,m} e_i$, where $(j, k, m, i) \in \mathbb{N}_0^2 \times \mathbb{N}^2$ and $(e_i)_{i\in\mathbb{N}}$ denotes an orthonormal basis in the separable Hilbert space H, are all elements of $(H^1(H), \|\cdot\|_{H^1(H)})$.

Due to the fact that the Cameron–Martin space $H^1(H)$ is a vector space, all finite linear combinations of the maps discussed above are also contained in it. Most notably, if we fix $(m, N) \in \mathbb{N}_0 \times \mathbb{N}$ and consider $\mathbb{Q}$-valued sequences $\{a_{l,i} : (l, i) \in \{1, 2, \dots, m\} \times \{1, 2, \dots, N\}\}$ and $\{b_{j,k,i} : (j, k, i) \in \{0, 1, \dots, m\} \times \mathbb{N}_0 \times \{1, 2, \dots, N\}, k \le 2^j m - 1\}$, the the maps $f_{m,N}\colon \mathbb{R}_+ \to H$ given by

$$\begin{aligned} f_{m,N} &= \sum_{l=1}^{m}\sum_{i=1}^{N} a_{l,i}\theta_{l,i,m} + \sum_{j=0}^{m}\sum_{k=0}^{2^j m-1}\sum_{i=1}^{N} b_{j,k,i}\tau_{j,k,i,m} \\ &= \sum_{l=1}^{m}\Big(\sum_{i=1}^{N} a_{l,i}e_i\Big)\theta_{l,m} + \sum_{j=0}^{m}\sum_{k=0}^{2^j m-1}\Big(\sum_{i=1}^{N} b_{j,k,i}e_i\Big)\tau_{j,k,m} \end{aligned} \tag{4.6}$$

are elements of $(H^1(H), \|\cdot\|_{H^1(H)})$. Moreover, if we set

$$E^X := \Big\{\sum_{i=1}^{N} a_i e_i \colon a_i \in \mathbb{Q}, N \in \mathbb{N}\Big\},$$

then E^X is a countable and dense subset of X, and according to the proof of Corollary 2.67, maps of the form (4.6) with coefficients in E^X are dense in the little weighted Hölder space $\mathcal{M}_0^{\alpha,\beta,\gamma}$, if $\gamma \le \beta$. This fact, combined with part (b), yields the density of $H^1(H)$ in $\mathcal{M}_0^{\alpha,\beta,\gamma}$.

This concludes our proof. □

4.3. Path properties of Banach-space-valued Brownian motion

Theorem 4.18. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ denote a filtered probability space. Further, let $(H, E, \mathcal{W})$ denote an abstract Wiener space and $B = (B_t)_{t\in\mathbb{R}_+}$ a $\mathcal{W}$-Brownian motion. Let $(H^1(H), \|\cdot\|_{H^1(H)})$ denote the Cameron–Martin space given in Definition 4.10. Then, for each $\alpha \in (0, 1/2)$, $\beta > 1/2$ and $\gamma > 0$, it holds that*

(a) Paths of B satisfy almost surely

$$\sup_{t\in\mathbb{R}_+} \frac{\|B_t\|_E}{w_\beta(t)} + \sup_{\substack{s,t\in\mathbb{R}_+\\ 0<t-s\le 1}} \frac{\|B_t - B_s\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)} < \infty. \tag{4.7}$$

(b) Paths of B are almost surely in $\mathcal{M}_0^{\alpha,\beta,\gamma}$.

Proof. Let us begin the proof with some initial remarks.

For a map $f\colon \mathbb{R}_+ \to E$, we set

$$\|f\|_{\alpha,\gamma} := \sup_{\substack{s,t\in\mathbb{R}_+\\ 0<t-s\le 1}} \frac{\|f(t) - f(s)\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)}, \quad \|f\|_\beta := \sup_{t\in\mathbb{R}_+} \frac{\|f(t)\|_E}{w_\beta(t)}.$$

Clearly, it follows that $\|f\|_{\alpha,\beta,\gamma} = \|f\|_\beta + \|f\|_{\alpha,\gamma}$. Note that, for each $f\colon \mathbb{R}_+ \to E$, we can actually write

$$\|f\|_{\alpha,\gamma} \le 2\max\Big\{\sup_{0\le s<t\le 1} \frac{\|f(t) - f(s)\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)}, \sup_{\substack{1\le s<t<\infty\\ 0<t-s\le 1}} \frac{\|f(t) - f(s)\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)}\Big\},$$

because, due to the subadditivity property of the norm $\|\cdot\|_E$, where it suffices to consider the case $s < 2^n < t$, with $0 < t-s \leq 1$, for some $n \in \mathbb{N}_0$, we have the estimate

$$\frac{\|f(t)-f(s)\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)} \leq \frac{\|f(t)-f(2^n)\|_E}{\varrho_\alpha(t-2^n)\ w_\gamma(t)} + \frac{\|f(2^n)-f(s)\|_E}{\varrho_\alpha(2^n-s)\ w_\gamma(2^n)}.$$

Moreover, upon applying the exact same logic, we actually obtain the estimate $\|f\|_{\alpha,\gamma} \leq 2\|f\|^*$, where we set

$$\|f\|^* := \max\Big\{\sup_{0\leq s<t\leq 1} \frac{\|f(t)-f(s)\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)}, \sup_{n\in\mathbb{N}_0} \sup_{\substack{2^n\leq s<t\leq 2^{n+1}\\ 0<t-s\leq 1}} \frac{\|f(t)-f(s)\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)}\Big\},$$

see also [Pic11, Remark 2.1].

(a) The restriction $\alpha \in (0, 1/2)$ is motivated by [Str11, Corollary 4.3.3] as well as [Sch20, Theorem 2.98].

For each $\alpha \in (0, 1/2)$, let $C_{\alpha;i}\colon \Omega \to \mathbb{R}_+$, where $i \in \{1, 2\}$, denote the random constants from the uniform α-Hölder continuity of B and $X = I(B)$ on the unit interval, respectively. For simplicity, we set $C_\alpha := \max\{C_{\alpha;1}, C_{\alpha;2}\}$. This yields

$$\max\Big\{\sup_{0\leq s<t\leq 1} \frac{\|B_t - B_s\|_E}{\varrho_\alpha(t-s)}, \sup_{0\leq s<t\leq 1} \frac{\|X_t - X_s\|_E}{\varrho_\alpha(t-s)}\Big\} \leq C_\alpha.$$

Let $n \in \mathbb{N}_0$, and $2^n \leq s < t \leq 2^{n+1}$ such that $0 < t-s \leq 1$. Choose further $\alpha' = \max\{\alpha, 1/2(1-\gamma)\}$, and let $\tilde{C} = \max\{C_{\alpha;2}, C_{\alpha';2}\}$ denote the maximum of the two random constants that come from the uniform α- and α'-Hölder continuity of X on $[0,1]$. In line with the arguments found in [Pic11], we can then write almost surely

$$\begin{aligned}\|B_t - B_s\|_E &= \|t\ X_{1/t} - s\ X_{1/s}\|_E \\ &\leq t\ \|X_{1/t} - X_{1/s}\|_E + (t-s)\ \|X_{1/s}\|_E \\ &= t\ \|X_{1/t} - X_{1/s}\|_E + (t-s)\ \|X_{1/s} - X_0\|_E \\ &\leq 2^{n+1} C_{\alpha';2} \big|\tfrac{1}{t} - \tfrac{1}{s}\big|^{\alpha'} + (t-s) C_{\alpha;2} \big|\tfrac{1}{s} - 0\big|^{\alpha} \\ &\leq \tilde{C}\big(2^{n+1}\big(\tfrac{t-s}{ts}\big)^{\alpha'} + (t-s)\big(\tfrac{1}{s}\big)^{\alpha}\big).\end{aligned} \tag{4.8}$$

A division by $\varrho_\alpha(t-s)\ w_\gamma(t)$ yields the almost sure estimate

$$\begin{aligned}\frac{\|B_t - B_s\|_E}{\varrho_\alpha(t-s)\ w_\gamma(t)} &\leq \tilde{C}\big(2^{n+1}(t-s)^{\alpha'-\alpha}\tfrac{1}{t^{\alpha'+\gamma}s^{\alpha'}} + (t-s)^{1-\alpha}\tfrac{1}{t^\gamma s^\alpha}\big) \\ &\leq \tilde{C}\big(2^{-n(2\alpha'+\gamma-1)+1} + 2^{-n(\alpha+\gamma)}\big) \\ &\leq 3\tilde{C}.\end{aligned}$$

All of the above estimates finally imply the almost sure upper bound

$$\|B\|_{\alpha,\gamma} \leq 2\|B\|^* \leq 2\max\big\{C_{\alpha;1}, 3\tilde{C}\big\}.$$

Next, we want to show that $\|B\|_\beta < \infty$ holds pathwise. Since $t \mapsto \|B_t\|_E / w_\beta(t)$ is continuous, we just have to verify that

$$\limsup_{t \nearrow \infty} \frac{\|B_t\|_E}{w_\beta(t)} < \infty$$

holds for every path. To this end, let us note that for all $t \in \mathbb{R}_+$ such that $t \geq 1$ and $\tilde{\alpha} \in [1-\beta, 1/2)$ if $\beta < 1$ or $\tilde{\alpha} \in (0, 1/2)$ otherwise, we have

$$\frac{\|B_t\|_E}{w_\beta(t)} = \frac{\|t\, X_{1/t}\|_E}{w_\beta(t)} = t^{1-\beta-\tilde{\alpha}} \frac{\|X_{1/t} - X_0\|_E}{\left(\frac{1}{t} - 0\right)^{\tilde{\alpha}}} \leq C_{\tilde{\alpha};2}.$$

(b) Choose $\tilde{\beta} \in (1/2, \beta)$. By part (a), we know that

$$\sup_{t \in \mathbb{R}_+} \frac{\|B_t\|_E}{w_{\tilde{\beta}}(t)} = C_1 < \infty,$$

for some random variable $C_1 \colon \Omega \to \mathbb{R}_+$. For $t \geq 1$, we estimate

$$\frac{\|B_t\|_E}{w_\beta(t)} = t^{\tilde{\beta}-\beta} \frac{\|B_t\|_E}{w_{\tilde{\beta}}(t)} \leq t^{\tilde{\beta}-\beta}\, C_1.$$

But due to $\tilde{\beta} < \beta$, the right-hand side vanishes, as $t \to \infty$.

Next, choose $\tilde{\alpha} \in (\alpha, 1/2)$. By part (a), we know that, almost surely,

$$\sup_{\substack{s,t \in \mathbb{R}_+ \\ 0 < t-s \leq 1}} \frac{\|B_t - B_s\|_E}{\varrho_{\tilde{\alpha}}(t-s)\; w_\gamma(t)} = C_2 < \infty,$$

for some random variable $C_2 \colon \Omega \to \mathbb{R}_+ \cup \{+\infty\}$ that almost surely takes values in $\mathbb{R}_+$. For $\delta \in (0, 1]$ as well as $s, t \in \mathbb{R}_+$ such that $0 < t - s \leq \delta$, we estimate, almost surely

$$\frac{\|B_t - B_s\|_E}{\varrho_\alpha(t-s)\; w_\gamma(t)} = (t-s)^{\tilde{\alpha}-\alpha} \frac{\|B_t - B_s\|_E}{\varrho_{\tilde{\alpha}}(t-s)\; w_\gamma(t)} \leq (t-s)^{\tilde{\alpha}-\alpha}\, C_2.$$

But due to $\tilde{\alpha} > \alpha$, the right-hand side vanishes almost surely, as $\delta \searrow 0$.

Finally, choose $\tilde{\gamma} \in (0, \gamma)$. By part (a), we know that, almost surely,

$$\sup_{\substack{s,t \in \mathbb{R}_+ \\ 0 < t-s \leq 1}} \frac{\|B_t - B_s\|_E}{\varrho_\alpha(t-s)\; w_{\tilde{\gamma}}(t)} = C_3 < \infty,$$

for some random variable $C_3 \colon \Omega \to \mathbb{R}_+ \cup \{+\infty\}$ that almost surely takes values in $\mathbb{R}_+$. For $n \geq 1$ as well as $s, t \in \mathbb{R}_+ \setminus [0, n]$ such that $0 < t - s \leq 1$, we estimate, almost surely

$$\frac{\|B_t - B_s\|_E}{\varrho_\alpha(t-s)\; w_\gamma(t)} = t^{\tilde{\gamma}-\gamma} \frac{\|B_t - B_s\|_E}{\varrho_\alpha(t-s)\; w_{\tilde{\gamma}}(t)} \leq t^{\tilde{\gamma}-\gamma}\, C_3 \leq n^{\tilde{\gamma}-\gamma}\, C_3,$$

where the last inequality follows due to $\tilde{\gamma} < \gamma$. Now the right-hand side vanishes almost surely, as $n \to \infty$.

This concludes our proof. □

Remark 4.19. We did not use any arguments that involved the filtration on the underlying filtered probability space so far, only path properties which hold almost surely. Therefore, all of the above results in this chapter can be extended by a symmetry argument to the setting of two-sided Brownian motion.

Remark 4.20. For each $\alpha \in (0, 1/2)$, $\beta \in (1/2, 1]$ and $\gamma \in (0, \beta]$, we have that both $\mathcal{M}^{\alpha,\beta,\gamma}$ and $\mathcal{M}_0^{\alpha,\beta,\gamma}$ are Borel measurable subsets of $\Theta(E)$. To see this, note that due to the continuity of the functions in the (little) weighted Hölder spaces $\mathcal{M}^{\alpha,\beta,\gamma}$ and $\mathcal{M}_0^{\alpha,\beta,\gamma}$, we have that the suprema which appear in Definition 4.14 can all be reduced to suprema where s and t are elements of the countable set $\mathbb{Q}_+$ of nonnegative rationals. Consequently, $\|\cdot\|_{\alpha,\beta,\gamma}$ is a measurable map, and thus $\mathcal{M}^{\alpha,\beta,\gamma}$ is a measurable subset of $\Theta(E)$. Finally, the defining limits in Definition 4.14 which describe the little weighted Hölder space do also preserve measurability, and thus $\mathcal{M}_0^{\alpha,\beta,\gamma}$ is also a Borel set.

4.4. Hölderian Wiener spaces as abstract Wiener spaces

In this section, it is our aim to identify the little weighted Hölder spaces as the appropriate real separable Banach spaces, with which we can construct abstract Wiener spaces. To this end, we will make use of a result found in [BBK92, Theorem 2.4]:

Theorem 4.21. *Let W_1 and W_2 denote real separable Banach spaces, and H be a real separable Hilbert space such that $H \hookrightarrow W_1 \hookrightarrow W_2$, both embeddings being continuous. Let μ denote a probability measure on W_2 such that (H, W_2, μ) is an abstract Wiener space. Further, assume that $\mu^*(W_1) = 1$, where μ^* denotes the outer measure on $(W_2, \mathcal{B}(W_2))$, and let H be densely embedded in W_1. If ν denotes the trace measure of μ on W_1, then (H, W_1, ν) is an abstract Wiener space.*

Now we are ready to formulate and prove the

Theorem 4.22. *Let $(H, E, \mathcal{W})$ denote an abstract Wiener space and $(H^1(H), \|\cdot\|_{H^1(H)})$ be the Cameron–Martin space given in Definition 4.10. Then, for each $\alpha \in (0, 1/2)$, $\beta \in (1/2, 1]$ and $\gamma \in (0, \beta]$, the trace of $\mathcal{W}^{(E)}$ on the little weighted Hölder space $\mathcal{M}_0^{\alpha,\beta,\gamma} \subset \Theta(E)$ is a probability measure, called $\mathcal{W}^{\alpha,\beta,\gamma}$, such that the triple*

$$(H^1(H), \mathcal{M}_0^{\alpha,\beta,\gamma}, \mathcal{W}^{\alpha,\beta,\gamma})$$

constitutes an abstract Wiener space according to Definition 4.7.

Proof. Let us begin the proof by stating a classical example of an abstract Wiener space. As already mentioned above, there exists an unique probability measure

$$\mathcal{W}^{(E)} \in \mathcal{M}_1(\Theta(E))$$

such that $(H^1(H), \Theta(E), \mathcal{W}^{(E)})$ constitutes an abstract Wiener space. Moreover, we can identify $\mathcal{W}^{(E)}$ with the Borel measure induced by $\mathcal{W}$-Brownian motion B on $\Theta(E)$.

In the context and terminology of Theorem 4.21, let (H, W_2, μ) be given by the abstract Wiener space $(H^1(H), \Theta(E), \mathcal{W}^{(E)})$. Moreover, let W_1 denote the real separable Banach space $\mathcal{M}_0^{\alpha,\beta,\gamma}$. Under the restriction $\beta \leq 1$, the continuity of the canonical embedding $\mathcal{M}_0^{\alpha,\beta,\gamma} \hookrightarrow \Theta(E)$ simply follows from the observation

$$\frac{1}{1+t} \leq \frac{1}{w_\beta(t)}, \quad t \geq 0.$$

By Proposition 4.16, we also know that $H^1(H)$ is densely and continuously embedded into $\mathcal{M}_0^{\alpha,\beta,\gamma}$. Moreover, we have that $\mathcal{W}^{(E)}(\mathcal{M}_0^{\alpha,\beta,\gamma}) = 1$ by Theorem 4.18 and Remark 4.20, because $\mathcal{W}$-Brownian paths are almost surely elements of the Borel set $\mathcal{M}_0^{\alpha,\beta,\gamma}$. Therefore, we can conclude by Theorem 4.21 that the triple

$$(H^1(H), \mathcal{M}_0^{\alpha,\beta,\gamma}, \mathcal{W}^{(E)})$$

constitutes an abstract Wiener space, which concludes our proof. □

Remark 4.23. Due to Remark 4.15, Proposition 4.16 and Theorem 4.18 we can deduce that for each $\alpha \in (0, 1/2)$, $\beta \in (1/2, 1]$ and $\gamma \in (0, \beta]$, the little weighted Hölder space $\mathcal{M}_0^{\alpha,\beta,\gamma}$ is a complete and separable normed space, hence it is a Polish space. Moreover, the classical Cameron–Martin space $H^1(H)$ is continuously embedded as a dense subspace, and almost all $\mathcal{W}$-Brownian paths are elements therein. Consequently, if we consider the space $\mathcal{M}_0^\infty$ given by

$$\mathcal{M}_0^\infty := \bigcap_{\alpha \in (0,1/2)} \bigcap_{\beta \in (1/2,1]} \bigcap_{\gamma \in (0,\beta]} \mathcal{M}_0^{\alpha,\beta,\gamma}, \tag{4.9}$$

almost all $\mathcal{W}$-Brownian paths and all Cameron–Martin paths are actually elements of $\mathcal{M}_0^\infty$. The spaces $\mathcal{M}_0^{\alpha,\beta,\gamma}$ also carry a monotonic structure in the following sense: for fixed $\beta \in (1/2, 1]$ and $\gamma \in (0, \beta]$, if $0 < \alpha_1 < \alpha_2 < 1/2$, then $\mathcal{M}_0^{\alpha_2,\beta,\gamma} \subset \mathcal{M}_0^{\alpha_1,\beta,\gamma}$. Moreover, $\alpha \in (0, 1/2)$, $1/2 < \beta_1 < \beta_2 \leq 1$ and $\gamma \in (0, \beta_1]$ implies $\mathcal{M}_0^{\alpha,\beta_1,\gamma} \subset \mathcal{M}_0^{\alpha,\beta_2,\gamma}$. Similarly, $\alpha \in (0, 1/2)$, $\beta \in (1/2, 1]$ and $0 < \gamma_1 < \gamma_2 \leq \beta$ implies $\mathcal{M}_0^{\alpha,\beta,\gamma_1} \subset \mathcal{M}_0^{\alpha,\beta,\gamma_2}$. Consequently, we can write (4.9) as an intersection over countably many parameters (α, β, γ), all for which the corresponding Borel sets $\mathcal{M}_0^{\alpha,\beta,\gamma}$ are of full $\mathcal{W}^{(E)}$-measure. Therefore, we have that $\mathcal{M}_0^\infty$ is a Borel set with $\mathcal{W}^{(E)}(\mathcal{M}_0^\infty) = 1$.

4.5. Schilder's and Strassen's theorem in weighted Hölder norms

For a topological space $(Y, \mathcal{T})$ and $F \subset Y$, we denote by F° and $\overline{F}$ the interior and closure of F, respectively, i.e.

$$F^\circ = \bigcup_{\substack{G \subset F \\ G \in \mathcal{T}}} G, \quad \overline{F} = \bigcap_{\substack{F \subset C \\ Y \setminus C \in \mathcal{T}}} C.$$

Moreover, we convene that the infimum over the empty set is $+\infty$.

Let us state some initial definitions.

Definition 4.24 (Rate function)**.** Let Y denote a Hausdorff topological space. We then call the map $I\colon Y \to \overline{\mathbb{R}}_+$ a rate function, if it is lower semicontinuous, i.e. if for all $a \geq 0$, the level sets $\Psi_I(a) := \{y \in Y : I(y) \leq a\}$ are closed subsets of Y. We call I a good rate function, if for all $a \geq 0$, the level sets $\Psi_I(a)$ are compact subsets of Y.

Definition 4.25 (Large deviation principle)**.** A family $(\mu_\varepsilon)_{\varepsilon>0}$ of probability measures on $(Y, \mathcal{B}(Y))$, where $\mathcal{B}(Y)$ denotes the Borel σ-field over Y, satisfies the large deviation principle with (good) rate function I, if for all $F \in \mathcal{B}(Y)$, we have

$$-\inf_{x\in F^\circ} I(x) \leq \liminf_{\varepsilon\searrow 0} \varepsilon \log \mu_\varepsilon(F) \leq \limsup_{\varepsilon\searrow 0} \varepsilon \log \mu_\varepsilon(F) \leq -\inf_{x\in \overline{F}} I(x). \tag{4.10}$$

Let us state a first abstract large deviations theorem, which can be found in [Str11, Theorem 8.4.1], [BBK92, Theorem 2.3] as well as [DS89, Theorem 3.4.5]. Below, the property of the rate function being a good rate function follows from [DS89, Theorem 3.4.12].

Theorem 4.26. *Let $(H, E, \mathcal{W})$ denote an abstract Wiener space. Further, for $\varepsilon > 0$, let $\mathcal{W}_\varepsilon$ denote the pushforward measure of $\mathcal{W}$ under the map $E \ni f \mapsto \sqrt{\varepsilon} f$. Then $(\mathcal{W}_\varepsilon)_{\varepsilon>0}$ satisfies the large deviation principle with good rate function $I_{\mathcal{W}}\colon E \to \overline{\mathbb{R}}_+$ given by*

$$I_{\mathcal{W}}(f) = \begin{cases} \frac{1}{2}\|f\|_H^2 & \text{for } f \in H, \\ \infty & \text{otherwise.} \end{cases}$$

By Theorem 4.22 and Theorem 4.26, we finally arrive at the

Theorem 4.27. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ denote a filtered probability space. Further, let $(H, E, \mathcal{W})$ denote an abstract Wiener space and $B = (B_t)_{t\in\mathbb{R}_+}$ a $\mathcal{W}$-Brownian motion. Let $(H^1(H), \|\cdot\|_{H^1(H)})$ denote the Cameron–Martin space given in Definition 4.10 and $\alpha \in (0, 1/2)$, $\beta \in (1/2, 1]$ and $\gamma \in (0, \beta]$ be given. If $\mathcal{W}^{\alpha,\beta,\gamma}$ denotes the Borel measure induced by B on $\mathcal{M}_0^{\alpha,\beta,\gamma}$ according to Theorem 4.22 and, for each $\varepsilon > 0$, $\mathcal{W}_\varepsilon^{\alpha,\beta,\gamma}$ is the pushforward measure of $\mathcal{W}^{\alpha,\beta,\gamma}$ under the map $\mathcal{M}_0^{\alpha,\beta,\gamma} \ni f \mapsto \sqrt{\varepsilon} f$, then $(\mathcal{W}_\varepsilon^{\alpha,\beta,\gamma})_{\varepsilon>0}$ satisfies the large deviation principle with good rate function $I_{\mathcal{W}^{\alpha,\beta,\gamma}}\colon \mathcal{M}_0^{\alpha,\beta,\gamma} \to \overline{\mathbb{R}}_+$ given by*

$$I_{\mathcal{W}^{\alpha,\beta,\gamma}}(f) = \begin{cases} \frac{1}{2}\|f\|_{H^1(H)}^2 & \text{for } f \in H^1(H), \\ \infty & \text{otherwise.} \end{cases}$$

As a corollary to Theorem 4.27 that directly follows from [Str11, Corollary 8.4.3], we also get the result

Corollary 4.28. *In the context of Theorem 4.27, assume that $E \neq \{0\}$ and let C denote the positive real constant implicitly given by $C^{-1} = \inf\{\|h\|_{H^1(H)} : \|h\|_{\alpha,\beta,\gamma} = 1\}$. Then*

$$\lim_{R\to\infty} \frac{1}{R^2} \log \mathbb{P}(\|B\|_{\alpha,\beta,\gamma} \geq R) = -\frac{1}{2C^2}. \tag{4.11}$$

Moreover, this yields

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\alpha^2 \|B\|_{\alpha,\beta,\gamma}^2\Big)\Big] < \infty \quad \text{if and only if} \quad \alpha < C^{-1}. \tag{4.12}$$

Another useful Lemma that comes from Fernique's theorem, see [Str11, Theorem 8.2.1] and [Str11, Exercise 8.2.16] yields that, for any $n \in \mathbb{N}$,

$$\mathbb{E}\big[\|B\|_{\alpha,\beta,\gamma}^{2n}\big] \leq (72)^n n!\ \mathbb{E}\big[\|B\|_{\alpha,\beta,\gamma}^{2}\big]^n \left(\mathrm{e}^{\frac{1}{2}} + \sum_{m=0}^{\infty} \left(\frac{\mathrm{e}}{3}\right)^{2^m}\right).$$

Another notable result that follows from Theorem 4.27 for our setting is Strassen's law of the iterated logarithm. The following formulation follows from [Str11, Theorem 8.4.4]. For another classical treatment that also weakens the assumption of Gaussianity, see [Che94]. Below, we call as subset G of a topological Hausdorff space Y to be relatively compact, if its closure is compact. We further call a sequence $(x_n)_{n\in\mathbb{N}}$ in Y relatively compact, if the set $G = \{x_n : n \in \mathbb{N}\}$ is.

Theorem 4.29. *In the context of Theorem 4.27, let $(X_m)_{m\in\mathbb{N}}$ denote a sequence of independent $\mathcal{W}$-Brownian motions on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. For each $n \in \mathbb{N}$, let S_n denote the sequence of partial sums given by $S_n = \sum_{m=1}^{n} X_m$, and let Λ_n denote the constant $\Lambda_n = \sqrt{2n \log(\log(n \vee 3))}$. We rescale S_n to $\tilde{S}_n := S_n/\Lambda_n$. Then, $\mathbb{P}$-almost surely, the sequence $(\tilde{S}_n)_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{M}_0^{\alpha,\beta,\gamma}$, and the closed unit ball $\overline{B_1(0)}$ in $H^1(H)$ coincides with the set of accumulation points of $(\tilde{S}_n)_{n\in\mathbb{N}}$. Equivalently, $\mathbb{P}$-almost surely, we have*

$$\lim_{n\to\infty} \|\tilde{S}_n - \overline{B_1(0)}\|_{\alpha,\beta,\gamma} = 0, \tag{4.13}$$

$$\forall\ h \in \overline{B_1(0)} \colon \liminf_{n\to\infty} \|\tilde{S}_n - h\|_{\alpha,\beta,\gamma} = 0. \tag{4.14}$$

Remark 4.30. Note that the topology on $\mathcal{M}_0^{\alpha,\beta,\gamma}$ is stronger than the one on $\Theta(E)$. Therefore, the bounds stated in (4.10) can be sharper than in the classical formulation of Schilder's theorem on $\Theta(E)$. Moreover, recall the definition of continuity in the context of topological spaces: a map $f\colon (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is continuous, if for each $O \in \mathcal{T}^Y$, we have that $f^{-1}(O) \in \mathcal{T}^X$. Consequently, there are more continuous maps ϕ on $\mathcal{M}_0^{\alpha,\beta,\gamma}$ than there are on $\Theta(E)$. As a corollary to this remark, we state a well known theorem due to S.R.S. Varadhan, see also [DZ10, Theorem 4.3.1].

Theorem 4.31 (Varadhan's lemma)**.** *Consider a family $(Z_\varepsilon)_{\varepsilon>0}$ of Y-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Y denotes a topological Hausdorff space. Assume that the induced family of probability measures $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies the large deviation principle with good rate function $I\colon Y \to \overline{\mathbb{R}}_+$, and let $\phi\colon Y \to \mathbb{R}$ be a continuous function. Assume further that either of the following conditions holds:*

(a) Tail condition:

$$\lim_{M\to\infty} \limsup_{\varepsilon\searrow 0} \varepsilon \log \mathbb{E}\big[\mathrm{e}^{\phi(Z_\varepsilon)/\varepsilon} 1_{\{\phi(Z_\varepsilon)\geq M\}}\big] = -\infty; \tag{4.15}$$

(b) Exponential moment condition: there exists some $\gamma > 1$ such that

$$\limsup_{\varepsilon\searrow 0} \varepsilon \log \mathbb{E}\big[\mathrm{e}^{\gamma\phi(Z_\varepsilon)/\varepsilon}\big] < \infty. \tag{4.16}$$

Then we have

$$\lim_{\varepsilon \searrow 0} \varepsilon \log \mathbb{E}\big[\mathrm{e}^{\phi(Z_\varepsilon)/\varepsilon}\big] = \sup_{x \in E} \big\{\phi(x) - I(x)\big\}. \tag{4.17}$$

The proof of the following result can be found in [DZ10, Theorem 4.2.1].

Theorem 4.32 (Contraction principle)**.** *Consider two topological Hausdorff spaces X and Y. Assume further that there is a family $(Z_\varepsilon)_{\varepsilon>0}$ of X-valued random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that the corresponding induced family of probability measures $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies the large deviation principle with good rate function $I\colon X \to \overline{\mathbb{R}}_+$. Let $f\colon X \to Y$ denote a continuous function. For each $y \in Y$, set*

$$I'(y) := \inf_{x \in f^{-1}(\{y\})} I(x),$$

where we convene that the infimum over the empty set is $+\infty$. Then, the following holds:

(a) $I'\colon Y \to [0, \infty]$ is a good rate function on Y;

(b) The family $(\mu_\varepsilon \circ f^{-1})_{\varepsilon>0}$ of probability measures on Y that are induced by f satisfies the large deviation principle with good rate function I'.

Remark 4.33. Coming back to the setting of Theorem 4.27, consider the spaces $\mathcal{M}^\beta$ and $\mathcal{M}^{\alpha,\gamma}$ of continuous functions $f\colon \mathbb{R}_+ \to E$ such that $f(0) = 0$ and $\|f\|_\beta < \infty$ as well as $\|f\|_{\alpha,\gamma} < \infty$ holds, respectively. Here, we set

$$\|f\|_\beta := \sup_{t \in \mathbb{R}_+} \frac{\|f(t)\|_E}{w_\beta(t)},$$
$$\|f\|_{\alpha,\gamma} := \sup_{\substack{s,t \in \mathbb{R}_+ \\ 0<t-s\leq 1}} \frac{\|f(t) - f(s)\|_E}{\varrho_\alpha(t-s)\; w_\gamma(t)},$$

where $\|\cdot\|_E$ denotes the norm on the real separable Banach space E. We immediately obtain that $\|f\|_\beta \leq \|f\|_{\alpha,\beta,\gamma}$ as well as $\|f\|_{\alpha,\gamma} \leq \|f\|_{\alpha,\beta,\gamma}$. Moreover, under the assumption that $\beta \leq 1$, which is part of Theorem 4.27 we further have that $\|f\|_{\Theta(E)} \leq \|f\|_\beta$. Consequently, the canonical embeddings

$$\mathcal{M}_0^{\alpha,\beta,\gamma} \hookrightarrow \mathcal{M}^\beta, \quad \mathcal{M}_0^{\alpha,\beta,\gamma} \hookrightarrow \mathcal{M}^{\alpha,\gamma}, \quad \mathcal{M}_0^{\alpha,\beta,\gamma} \hookrightarrow \Theta(E), \quad \mathcal{M}^\beta \hookrightarrow \Theta(E)$$

are bounded linear operators between these Banach spaces, which implies that they are continuous. An application of the contraction principle as stated in Theorem 4.32 allows us to transfer large deviation principles onto these spaces by making use of Theorem 4.27.

4.6. An application to importance sampling

The following ideas originate from [GR08], [GHS99] as well as [Pha07].

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Let $B = (B_t)_{t\in\mathbb{R}_+}$ denote a real-valued $(\mathbb{F}, \mathbb{P})$-Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let us mention right from the start that we could also consider more generally d-dimensional Brownian motion, which would allow the treatment of Hölder-path dependent basket options below. In the context of Theorem 4.18(b), let $G\colon \mathcal{M}_0^{\alpha,\beta,\gamma} \to \mathbb{R}_+$ denote a continuous function. We also call $G(B)$ a random payoff. We are interested in computing an estimator of the expected payoff of Brownian motion, i.e.

$$\mathbb{E}[G(B)] = \int_\Omega G(B(\omega))\ \mathrm{d}\mathbb{P}(\omega). \tag{4.18}$$

For each $h \in H^1(\mathbb{R})$, we perform a change of measure as follows: Let Z^h denote the stochastic exponential

$$Z^h := \mathcal{E}\Big(\int_0^\cdot \dot h(t)\ \mathrm{d}B_t\Big) = \exp\Big(\int_0^\cdot \dot h(t)\ \mathrm{d}B_t - \frac{1}{2}\int_0^\cdot \dot h^2(t)\ \mathrm{d}t\Big).$$

By [Sch20, Theorem 6.46], we know that the stochastic exponential Z^h is a strictly positive continuous local martingale. Moreover, as elements of the Cameron–Martin space have square-integrable densities, we have

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\Big[\int_0^\cdot \dot h(t)\ \mathrm{d}B_t\Big]_\infty\Big)\Big] = \exp\Big(\frac{1}{2}\int_0^\infty \dot h^2(t)\ \mathrm{d}t\Big) < \infty,$$

we can apply Novikov's criterion (see [Sch20, Theorem 7.84]) in order to conclude that Z^h is a uniformly integrable martingale. By Doobs L^1-convergence theorem (see [Sch20, Theorem 7.48]), we have that Z^h_∞ is the limit of $(Z^h_t)_{t\in\mathbb{R}_+}$ in L^1 and Z^h_∞ closes the martingale Z^h. Upon setting $\mathrm{d}\mathbb{Q}^h/\mathrm{d}\mathbb{P} = Z^h_\infty$, we can apply Girsanov's theorem, see [Sch20, Theorem 7.33] in order to show that $B^h := B - h$ is indeed an $(\mathbb{F}, \mathbb{Q}^h)$-Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q}^h)$. Moreover, the problem (4.18) can now be rewritten as

$$\mathbb{E}_\mathbb{P}[G(B)] = \mathbb{E}_{\mathbb{Q}^h}[G(B)(Z^h_\infty)^{-1}] = \int_\Omega G(B(\omega))\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^h}(\omega)\ \mathrm{d}\mathbb{Q}^h(\omega).$$

If we now consider the modified payoff $H(B) = G(B)(Z^h_\infty)^{-1}$, then we have that the $\mathbb{P}$-expectation of $G(B)$ and the $\mathbb{Q}^h$-expectation of $H(B)$ are identical. Therefore, we can tackle the computation of the expectation given in Equation 4.18 by means of obtaining an estimator for

$$\mathbb{E}_{\mathbb{Q}^h}[H(B)] = \int_\Omega H(B(\omega))\ \mathrm{d}\mathbb{Q}^h(\omega).$$

only that now we can make use of the additional degree of freedom in the form of the function h in order to choose the optimal such function that minimizes the variance of $H(B)$ under $\mathbb{Q}^h$, which is given by

$$\mathbb{E}_\mathbb{P}[G^2(B)(Z^h_\infty)^{-1}] - \mathbb{E}_\mathbb{P}[G(B)]^2.$$

As the second term in the difference does not depend on h, we will solely focus on the first term, which is the second moment and can be written as

$$\mathbb{E}_\mathbb{P}\Big[\exp\Big(2F(B) - \int_0^\infty \dot h(t)\ \mathrm{d}B_t + \frac{1}{2}\int_0^\infty \dot h^2(t)\ \mathrm{d}t\Big)\Big], \tag{4.19}$$

where we set $F = \log G$. Note that the image space of F is $\{-\infty\} \cup \mathbb{R}$. Finding a minimizer $h \in H^1(\mathbb{R})$ of the expression (4.19) is in general intractable if one employs Monte-Carlo methods to estimate the expectation. Therefore, we consider a small-noise approximation for each $h \in H^1(\mathbb{R})$ of the form

$$L(h) := \lim_{\varepsilon \searrow 0} \varepsilon \log \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(\frac{1}{\varepsilon}\Big(2F(\sqrt{\varepsilon}B) - \int_0^\infty \sqrt{\varepsilon}\dot{h}(t)\,\mathrm{d}B_t + \frac{1}{2}\int_0^\infty \dot{h}^2(t)\,\mathrm{d}t\Big)\Big)\Big], \tag{4.20}$$

which corresponds to approximating (4.19) by $\exp(L(h))$. Moreover, we can recover (4.19) from $\exp(L(h))$ by simply fixing $\varepsilon = 1$ in (4.20). Finally, we consider the optimization problem

$$\inf_{h \in H^1(\mathbb{R})} L(h). \tag{4.21}$$

In order to find the optimal h, we can employ the machinery of Varadhan's lemma, see Theorem 4.31. Consider for each $h \in H^1_{\mathrm{bv}}(\mathbb{R})$, where $H^1_{\mathrm{bv}}(\mathbb{R})$ denotes the set of all $h \in H^1(\mathbb{R})$ such that the density $\dot{h}$ is of bounded variation, the function F_h which is given on $\mathcal{M}_0^{\alpha,\beta,\gamma}$ by

$$f \mapsto F_h(f) := 2F(f) - \int_0^\infty \dot{h}(t)\,\mathrm{d}f(t) + \frac{1}{2}\int_0^\infty \dot{h}^2(t)\,\mathrm{d}t.$$

Here the restrictive assumption of h having a density of bounded variation comes into play, as we now can identify the first integral as a Stieltjes integral. As F can assume $-\infty$, we cannot apply Varadhan's lemma directly. However, we can employ a modified version that allows for this case, see [GR08, Lemma 7.5]. Let us assume now that F satisfies some sufficient properties such that F_h is pointwise well defined and we can apply the discussed modified version of Theorem 4.31. This yields:

$$L(h) = \limsup_{\varepsilon \searrow 0} \varepsilon \log \mathbb{E}_{\mathbb{P}}\Big[\exp\Big(\frac{1}{\varepsilon}F_h(\sqrt{\varepsilon}B)\Big)\Big] = \sup_{f \in \mathcal{M}_0^{\alpha,\beta,\gamma}} \Big(F_h(f) - I_{\mathcal{W}^{\alpha,\beta,\gamma}}(f)\Big). \tag{4.22}$$

Since F_h is well defined, and we know that for the good rate function $I_{\mathcal{W}^{\alpha,\beta,\gamma}}$ it holds that $I_{\mathcal{W}^{\alpha,\beta,\gamma}}(f) = +\infty$, if $f \in \mathcal{M}_0^{\alpha,\beta,\gamma} \setminus H^1(\mathbb{R})$, we further have

$$\begin{aligned}
\sup_{f \in \mathcal{M}_0^{\alpha,\beta,\gamma}} \Big(F_h(f) - I_{\mathcal{W}^{\alpha,\beta,\gamma}}(f)\Big) &= \sup_{f \in H^1(\mathbb{R})} \Big(F_h(f) - I_{\mathcal{W}^{\alpha,\beta,\gamma}}(f)\Big) \\
&= \sup_{f \in H^1(\mathbb{R})} \Big(2F(f) + \frac{1}{2}\int_0^\infty \big(\dot{f}(t) - \dot{h}(t)\big)^2\,\mathrm{d}t - \int_0^\infty \dot{f}^2(t)\,\mathrm{d}t\Big) \\
&\geq \sup_{f \in H^1(\mathbb{R})} \Big(2F(f) - \int_0^\infty \dot{f}^2(t)\,\mathrm{d}t\Big).
\end{aligned}$$

We can now try to find an optimal shift $h \in H^1_{\mathrm{bv}}(\mathbb{R})$ as follows:

(a) Assume that the following optimization problem has a (unique) solution:

$$\sup_{f \in H^1(\mathbb{R})} \Big(2F(f) - \int_0^\infty \dot{f}^2(t)\,\mathrm{d}t\Big),$$

which we denote by h^*. Note that this optimization problem is of Euler–Lagrange type.

(b) If the density of h^* is of bounded variation, then (4.22) yields

$$L(h^*) = \sup_{f \in H^1(\mathbb{R})} \Big(2F(f) + \frac{1}{2} \int_0^\infty \big(\dot{f}(t) - \dot{h}^*(t)\big)^2 \,\mathrm{d}t - \int_0^\infty \dot{f}^2(t) \,\mathrm{d}t \Big).$$

(c) Finally, if we can verify that indeed $h \in H^1_{\mathrm{bv}}$, and that the following condition holds:

$$L(h^*) = 2F(h^*) - \int_0^\infty (\dot{h}^*(t))^2 \,\mathrm{d}t,$$

then we have found a solution to the optimization problem (4.21). The reason for this is the following: for each $h \in H^1_{\mathrm{bv}}(\mathbb{R})$, we have

$$\begin{aligned} L(h) &= \sup_{f \in H^1(\mathbb{R})} \Big(2F(f) + \frac{1}{2} \int_0^\infty \big(\dot{f}(t) - \dot{h}(t)\big)^2 \,\mathrm{d}t - \int_0^\infty \dot{f}^2(t) \,\mathrm{d}t \Big) \\ &\geq \sup_{f \in H^1(\mathbb{R})} \Big(2F(f) - \int_0^\infty \dot{f}^2(t) \,\mathrm{d}t \Big). \end{aligned}$$

Consequently, it holds that

$$\inf_{h \in H^1_{\mathrm{bv}}(\mathbb{R})} L(h) \geq 2F(h^*) - \int_0^\infty \big(\dot{h}^*(t)\big)^2 \,\mathrm{d}t,$$

and since the last expression equals $L(h^*)$ by the previous point, we have found a solution.

A. Open questions and ideas

Many of the concepts that we have touched upon could be pursued further. Below, we will collect some unfinished tasks and ideas for the future.

(A) Constructing fractional Brownian motion on locally convex topological vector spaces: The classical idea is the Lévy–Ciesielski construction, see [Lév37] and [Cie61]. A nice explanation of this construction can further be found in [McK69, Section 1.2]. For a generalization to Brownian motion on a Hilbert space, see [AIP13, Proposition 1 and Proposition 2]. Another very fine paper that might be studied in this context is given by [LLQ02].

The ideas of this thesis should be applicable to the study of fractional Brownian motion with Hurst parameter $H \in (0,1)$. The results from Chapter 2 and Chapter 3 in principle allow the treatment of locally convex topological vector spaces, where the topology is induced by a family of seminorms. We could then extend the notion of weighted Hölder spaces to this setting, because every seminorm satisfies the subadditivity property (2.11) with $C_{\text{qs}} = 1$ as well as the pseudo-homogeneity property 2.10(c) with $\phi(\lambda) = \lambda$. The concept of admissible moduli of continuity as stated in Definition 2.39, as well as the assumptions of Chapter 3 are then naturally satisfied for $\varrho(\delta) = \delta^\alpha$, where $0 < \alpha \leq 1$. A thorough treatment of Gaussian measures on locally convex vector spaces, very much in the spirit of the introductory section of Chapter 4 might then lead to all the necessary requirements for us to be able to write fractional Brownian motion as a decomposition with respect to the Faber–Schauder system of the second kind, where now the coefficients of the series would actually be elements of a locally convex topological vector space X. Some relevant theory for this can be found for instance in [Kue73].

(B) Extension to Besov spaces: The original motivation behind this thesis was to study Hölder regularity of Brownian motion. However, Hölder norms are actually a special case of Besov norms. In [LPT20], the more involved Besov-version of Proposition 3.17 was proven by means of atomic decompositions for functions on the unit interval that assume values in Banach spaces. However, it might be possible to prove the Besov-version of Proposition 3.17 in a more direct way, only making use of the fine properties of the Faber–Schauder system of the second kind as given in Definition 2.1. This might then lead to a better understanding of the construction of stochastic processes whose paths do not exhibit Hölder regularity, but Besov regularity, very much in the spirit of the previous point.

(C) Finding norms in $\mathcal{M}_0^\infty$: For an $\mathbb{R}$-valued $(\mathbb{F}, \mathbb{P})$-Brownian motion $B = (B_t)_{t\in[0,1]}$ on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, we can argue by means of the Kolmogorov–

Chentsov continuity theorem that, for each $\alpha \in (0, 1/2)$, almost surely

$$\sup_{0\leq s<t\leq 1} \frac{|B_t - B_s|}{(t-s)^\alpha} = C_\alpha < \infty. \tag{A.1}$$

The problem is that the random positive constant C_α explicitly depends on the index α itself. If we were to look for a bound of (A.1) uniformly over all $\alpha \in (0, 1/2)$ – see also the space $\mathcal{M}_0^\infty$ defined in (4.9) – this would require a deeper understanding of the random map $\alpha \mapsto C_\alpha$.

One way to approach this problem in a pathwise manner is by making use of the fact that, according to Lévy's modulus of continuity theorem, almost surely,

$$\sup_{0\leq s<t\leq 1} \frac{|B_t - B_s|}{h(t-s)} = C < \infty, \tag{A.2}$$

for some random positive constant C, where $h(\delta) = \sqrt{\delta \log(1/\delta)}$ for $\delta > 0$. Take some $\varepsilon \in (0, 1/2)$ close to 0, then, for each $\alpha \in [\varepsilon, 1/2)$, almost surely, for each $\delta \in (0, 1]$,

$$\frac{h(\delta)}{\delta^\alpha} = \delta^{1/2-\alpha}\sqrt{\log(1/\delta)} \leq \delta^{1/2-\varepsilon}\sqrt{\log(1/\delta)} \leq C_{(\varepsilon)} < \infty,$$

where the upper bound $C_{(\varepsilon)}$ does not depend on δ. The fact that the above expression and therefore the deterministic positive constant $C_{(\varepsilon)}$ do not explode for small enough δ can be verified by means of applying L'Hôpital's rule. Finally, we can conclude that, again almost surely

$$\sup_{\alpha\in[\varepsilon,1/2)} \sup_{0\leq s<t\leq 1} \frac{|B_t - B_s|}{(t-s)^\alpha} \leq C_{(\varepsilon)} \sup_{0\leq s<t\leq 1} \frac{|B_t - B_s|}{h(t-s)} \leq C_{(\varepsilon)} C < \infty.$$

This approach however does not work well for considering all $\alpha \in (0, 1/2)$ because it is not clear whether $h(\delta)\delta^{-\alpha}$ is uniformly bounded over all $(\delta, \alpha) \in (0, 1] \times (0, 1/2)$, especially for small α. Most notably, the statement (A.2) is mostly concerned with the behavior for $\delta = t - s \approx 0$, which is where $\sqrt{\log(1/\delta)}$ explodes.

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