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DIPLOMARBEIT

Matrix-Valued Hermite-Biehler Functions and Generalized Interlacing Property

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Abstract

The Hermite-Biehler Theorem, in its simplest form, characterizes complex polynomials whose zeros lie only in the lower half plane through the interesting property of its real and imaginary part to have only real and interlacing zeros. The aim of this thesis is to give an analogue of the Hermite-Biehler Theorem for matrix-valued entire functions. We will proceed by studying Herglotz functions, especially those that are meromorphic on $\mathbb{C}$ and satisfy $q(\overline{z}) = \overline{q(z)}$. It is known that the poles and zeros of scalar Herglotz functions that are meromorphic on $\mathbb{C}$ are all real, simple, and interlace, from which a product representation for functions of the aforementioned type can be deduced. Still, the more well-known representation for Herglotz functions is the Herglotz-Nevanlinna integral representation, which is better established mostly due to the fact that it holds not only for scalar meromorphic Herglotz functions, but rather for every scalar and matrix-valued (or even operator-valued) Herglotz function. In this thesis, we will introduce a generalized interlacing property which we use to prove a necessary and sufficient condition for a matrix-valued function which is real and meromorphic on $\mathbb{C}$ to be Herglotz. We can then apply this criterion to formulate a version of the Hermite-Biehler Theorem for matrix-valued entire functions.

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Chapter 1

Preliminaries

We start with some standard theory that serves to motivate the results of later chapters. Proofs for the theorems contained in this chapter can be found in [AD] and [L].

1.1 Reproducing kernel Hilbert spaces

First, we give a reminder on the topic of reproducing kernel Hilbert spaces. This is due to [AD, Chapter 5].

1.1.1 Definition. *A function* $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$*, where* Ω *is an arbitrary set, is called a* positive kernel *if for every* $m \in \mathbb{N}$*, points* $\omega_1, ..., \omega_m \in \Omega$*, and vectors* $u_1, ..., u_m \in \mathbb{C}^n$ *we have*

$$\sum_{j,k=1}^{m} u_j^* K(\omega_j, \omega_k) u_k \geq 0.$$

1.1.2 Remark. A positive kernel K always satisfies $K(w, z) = K(z, w)^*$. Observe that for fixed z_0 and arbitrary $x \in \mathbb{C}^n$, Definition 1.1.1 reads as $x^* K(z_0, z_0)x \geq 0$. In particular, $K(z_0, z_0) = K(z_0, z_0)^*$. Applying the same definition again for two points $w_1 = z, w_2 = w \in \Omega$ and $u_1 = x, u_2 = ix \in \mathbb{C}^n$, we find that

$$x^* \Big[K(z, z) + K(w, w) + i(K(z, w) - K(w, z))\Big] x \geq 0.$$

In particular, $K(z, z) + K(w, w) + i(K(z, w) - K(w, z))$ is self-adjoint, as is $i(K(z, w) - K(w, z))$. If, on the other hand, we take $u_1 = u_2 = x \in \mathbb{C}^n$, we conclude that

$$K(z, z) + K(w, w) + K(z, w) + K(w, z)$$

and $K(z, w) + K(w, z)$ are self-adjoint too. So,

$$\begin{aligned} 2K(z, w) &= [K(z, w) + K(w, z)] + [K(z, w) - K(w, z)] \\ &= [K(z, w)^* + K(w, z)^*] - [K(z, w)^* - K(w, z)^*] = 2K(w, z)^*. \end{aligned}$$

1.1.3 Lemma. *Let K be a positive kernel on Ω, and let $T : \Omega \to \mathbb{C}^{n\times n}$ be a function. Then the kernel*

$$L(z, w) := T(z)K(z, w)T(w)^*$$

is also positive on Ω.

Proof. This follows from

$$\sum_{j,k=1}^{m} u_j^* L(\omega_j, \omega_k) u_k = \sum_{j,k=1}^{m} \left(T(\omega_j)^* u_j\right)^* K(\omega_j, \omega_k)(T(\omega_k)^* u_k) \geq 0.$$

□

1.1.4 Definition. *Let $\mathcal{H}$ be a Hilbert space consisting of functions $f : \Omega \to \mathbb{C}^n$. Then $\mathcal{H}$ is called a reproducing kernel Hilbert space (RKHS for short) if there is an $n \times n$-positive kernel such that for every choice of $\omega \in \Omega$ and $u \in \mathbb{C}^n$,*

(1) *$K(\,\cdot\,, \omega)u$ belongs* [1] *to $\mathcal{H}$, and*

(2) $\left(f, K(\,\cdot\,, \omega)u\right)_{\mathcal{H}} = u^* f(\omega)$.

In this case, K is said to be the reproducing kernel of $\mathcal{H}$.

1.1.5 Remark. If, in the above definition, $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$ is a function that satisfies (1) and (2), then K is already a positive kernel. This follows from

$$\sum_{j,k=1}^{m} u_j^* K(\omega_j, \omega_k) u_k = \left(\sum_{j=1}^{m} K(\cdot, \omega_j) u_j, \sum_{j=1}^{m} K(\cdot, \omega_j) u_j \right)_{\mathcal{H}} \geq 0.$$

1.1.6 Lemma ([KK, Proposition 2.5.2])**.** *Let $\mathcal{H}$ be a RKHS with kernel $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$. Then the linear span $\mathcal{M}$ of all functions $K(\cdot, w)u$, where $w \in \Omega$ and $u \in \mathbb{C}^n$, is dense in $\mathcal{H}$.*

Proof. Suppose $\overline{\mathcal{M}} \neq \mathcal{H}$. Let $g \neq 0$ be orthogonal to $\mathcal{M}$. Then $u^* g(\omega) = \left(g, K(\cdot, \omega)u\right)_{\mathcal{H}} = 0$ for every $u \in \mathbb{C}^n$ yields $g = 0$, which contradicts our assumption. □

1.1.7 Theorem. *Let $K : \Omega \times \Omega \to \mathbb{C}^{n\times n}$ be a positive kernel. Then there exists exactly one RKHS of $\mathbb{C}^n$-valued functions on Ω with K as its reproducing kernel.*

Proof. This is found as Theorem 5.2 in [AD]. □

[1] $\cdot$ marks the argument of a function. $K(\,\cdot\,, \omega)u$ is to be understood as $z \mapsto K(z, \omega)u$.

1.1.8 Definition. *The* Hardy space H_2^n *is the space of vector-valued functions* $f : \mathbb{C}_+ \to \mathbb{C}^n$ *that are holomorphic in the upper half plane, such that*

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \|f(t+i\epsilon)\|^2 \, d\lambda(t) < +\infty.$$

H_2^n *becomes a Hilbert space when equipped with the scalar product*

$$(f,g)_{H_2^n} := \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} g(t+i\epsilon)^* f(t+i\epsilon) \, d\lambda(t).$$

Additionally, we define H_∞^n *to be the space of functions* $f : \mathbb{C}_+ \to \mathbb{C}^n$ *that are holomorphic in* $\mathbb{C}_+$*, and satisfy* $\lim_{\epsilon \searrow 0} \sup_{t \in \mathbb{R}} \|f(t+i\epsilon)\| < +\infty$*. Lastly, let* $H_\infty^{n \times n}$ *denote the space of* $n \times n$*-matrix-valued functions that belong to* $H_\infty := H_\infty^1$ *entrywise.*

1.1.9 Remark. Every function from H_2^n or from H_∞^n has boundary values almost everywhere on $\mathbb{R}$, or, put differently,

$$f(t) := \lim_{\epsilon \searrow 0} f(t+i\epsilon)$$

exists λ-a.e. on $\mathbb{R}$. So, instead of defining these spaces by a limit of integrals, we could, for example, also put

$$(f,g)_{H_2^n} := \int_{\mathbb{R}} g(t)^* f(t) \, d\lambda(t).$$

In this sense, H_2^n and H_∞^n become subspaces of L_2^n and L_∞^n, respectively. For details see Section 1 of Chapter 3 in [AD].

1.1.10 Remark. Not only is H_2^n a Hilbert space, it is even an RKHS. Its reproducing kernel is

$$K(z,w) = \frac{iI}{2\pi(z-\overline{w})}. \tag{1.1}$$

This is derived from the equation

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t-z} \, d\lambda(z) \tag{1.2}$$

which holds for every $f \in H_2^n$ and each $z \in \mathbb{C}_+$. See ([AD], Example 5.9) and ([D], Chapter 1, Theorem 12).

1.1.11 Lemma. *Let* P *be the orthogonal projection from* L_2^n *to* H_2^n*, and let* $f \in H_\infty^{n \times n}$ *as well as* $\xi \in \mathbb{C}^n$ *and* $w \in \mathbb{C}_+$*. Then*

$$P\Big[f^* K(\cdot,w)\xi\Big] = K(\cdot,w) f(w)^* \xi, \tag{1.3}$$

where $K(z,w) = \frac{iI}{2\pi(z-\overline{w})}$ *is the reproducing kernel of* H_2^n*.*

Proof. For $\alpha \in \mathbb{C}_+$, $\eta \in \mathbb{C}^n$ and using Definition 1.1.4, (2), we get

$$\Big(K(\cdot,\alpha)\eta, P\Big[f^*K(\cdot,w)\xi\Big]\Big)_{H_2^n} = \Big(K(\cdot,\alpha)\eta, f^*K(\cdot,w)\xi\Big)_{L_2^n}$$
$$= \Big(fK(\cdot,\alpha)\eta, K(\cdot,w)\xi\Big)_{L_2^n} = \xi^* f(w)K(w,\alpha)\eta = \Big(K(\cdot,\alpha)\eta, K(\cdot,w)f(w)^*\xi\Big)_{H_2^n}.$$

By Lemma 1.1.6, the set of functions of the form $K(\cdot,\alpha)\eta$, where $\alpha \in \mathbb{C}_+$ and $\eta \in \mathbb{C}^n$, is dense in H_2^n. This proves (1.3). ☐

1.2 The Schur class

1.2.1 Definition. *The* Schur class $\mathcal{S}_n$ *is the set of* $n \times n$*-matrix-valued functions* s *which are holomorphic on* $\mathbb{C}_+$ *and satisfy* [2]

$$s(z)^*s(z) \leq I, \quad z \in \mathbb{C}_+,$$

where I *denotes the* $n \times n$*-identity matrix.*

In the following, a few basic properties of Schur class functions are pointed out.

1.2.2 Lemma ([AD, Lemma 3.51])**.** *Let* $s \in \mathcal{S}_n$ *and* $\xi, \eta \in \mathbb{C}^n$ *such that* $\|\xi\| = \|\eta\|$. *If there is a point* $\omega \in \mathbb{C}_+$ *such that* $\xi = s(\omega)\eta$, *then* $\xi = s(z)\eta$ *and* $\eta = s(z)^*\xi$ *for every* $z \in \mathbb{C}_+$.

Proof. This is evident for $\eta = 0$, so w.l.o.g. we can assume $\|\xi\| = \|\eta\| = 1$. This leads to

$$|\xi^*s(z)\eta| \leq \|\xi\| \cdot \|s(z)\| \cdot \|\eta\| \leq 1 = \|\xi\|^2 = \xi^*s(\omega)\eta.$$

The maximum principle applied to the function $z \mapsto \xi^*s(z)\eta$ provides us with $\xi^*s(z)\eta = 1$, $z \in \mathbb{C}_+$. Now,

$$\|\xi - s(z)\eta\|^2 = \underbrace{\|\xi\|^2}_{=1} - \underbrace{\xi^*s(z)\eta}_{=1} - \underbrace{\eta^*s(z)^*\xi}_{=1} + \underbrace{\|s(z)\eta\|^2}_{\leq \|s(z)\|^2\|\eta\|^2 \leq 1} \leq 0,$$

so $\xi = s(z)\eta$. The other claim, $\eta = s(z)^*\xi$ is obtained in the same way from looking at $\|\eta - s(z)^*\xi\|^2$. ☐

1.2.3 Corollary ([AD, Corollary 3.52])**.** *Let* $s \in \mathcal{S}_n$. *Then* $s(z_0)^*s(z_0) < I$ *at one point* $z_0 \in \mathbb{C}_+$ *if and only if* $s(z)^*s(z) < I$ *at every point* $z \in \mathbb{C}_+$.

Proof. Suppose $s(z_0)^*s(z_0) < I$, but not $s(z)^*s(z) < I$ for every $z \in \mathbb{C}_+$. Then there is $z_1 \in \mathbb{C}_+$ and $\eta \in \mathbb{C}^n$ such that $\|s(z_1)\eta\| = \|\eta\|$. Let $\xi = s(z_1)\eta$, then from Lemma 1.2.2 we get $\xi = s(z)\eta$ for every $z \in \mathbb{C}_+$. But this would mean $\|s(z_0)\eta\| = \|\xi\| = \|\eta\|$, which is a contradiction. ☐

1.2.4 Corollary ([AD, Corollary 3.53])**.** *Let* $s \in \mathcal{S}_n$ *such that* $\det(I - s(z))$ *does not vanish identically on* $\mathbb{C}_+$. *Then* $\det(I - s(z)) \neq 0$ *for every* $z \in \mathbb{C}_+$.

[2] For $n \times n$-matrices C,D, by writing $C \leq D$ we mean that $D - C$ is positive semidefinite.

Proof. If there is $z_0 \in \mathbb{C}_+$ such that $\det(I - s(z_0)) = 0$, pick a nonzero $\xi \in \mathbb{C}^n$ with $\xi = s(z_0)\xi$. By Lemma 1.2.2, $\xi = s(z)\xi$ everywhere in the upper half plane, so $\det(I - s(z)) \equiv 0$. □

1.2.5 Theorem ([AD, Example 5.12])**.** *Let $s \in \mathcal{S}_n$. Then*

$$\Lambda(z,w) := i\frac{I - s(z)s(w)^*}{z - \overline{w}} \tag{1.4}$$

is a positive kernel on $\mathbb{C}_+$.

Proof. For any function f from $H_\infty^{n\times n}$, it is easy to see that multiplication of f with any H_2^n function gives again an H_2^n function, and that for the operator M_f on H_2^n of multiplication by f we have [3] $\|M_f\| \le \|f\|_{H_\infty^{n\times n}}$. Now, for $s \in \mathcal{S}_n \subseteq H_\infty^{n\times n}$ we have $\|M_s^*\| = \|M_s\| \le \|s\|_{H_\infty^{n\times n}} \le 1$ and

$$(M_s)^* K(\cdot, w)\xi = s(w)^* K(\cdot, w)\xi$$

which follows directly from (1.3). Then pick $m \in \mathbb{N}$, points $w_1, ..., w_m \in \mathbb{C}_+$, and vectors $\xi_1, ..., \xi_m \in \mathbb{C}^n$. The theorem is proven by

$$\begin{aligned}
&\frac{1}{2\pi}\sum_{j,k=1}^m i\frac{\xi_j^* s(w_j)s(w_k)^*\xi_k}{w_j - \overline{w_k}} = \sum_{j,k=1}^m [s(w_j)^*\xi_j]^* K(w_j, w_k)[s(w_k)^*\xi_k] \\
&= \sum_{j,k=1}^m \Big(K(\cdot, w_k)s(w_k)^*\xi_k, K(\cdot, w_j)s(w_j)^*\xi_j\Big)_{H_2^n} \\
&= \sum_{j,k=1}^m \Big((M_s)^*K(\cdot, w_k)\xi_k, (M_s)^*K(\cdot, w_j)\xi_j\Big)_{H_2^n} \\
&= \left\|(M_s)^*\sum_{j=1}^m K(\cdot, w_j)\xi_j\right\|^2 \le \left\|\sum_{j=1}^m K(\cdot, w_j)\xi_j\right\|^2 = \frac{1}{2\pi}\sum_{j,k=1}^m i\frac{\xi_j^*\xi_k}{w_j - \overline{w_k}}.
\end{aligned}$$

□

1.3 Asymptotic growth of holomorphic functions

We give a short overview of basic concepts regarding the growth of holomorphic functions. To gain a better understanding of this topic, see also [AD], [D] and [L].

1.3.1 Definition. *Let f be a $n \times n$-matrix-valued holomorphic function on $\mathbb{C}_+$. Then we say that f is a function of* bounded type *if there exist functions $g \in H_\infty^{n\times n}$ and $h \in H_\infty$ such that $f = g/h$. The set of all $n \times n$-matrix-valued functions of bounded type is also called $\mathcal{N}^{n\times n}$, the* Nevanlinna class*. For $n = 1$, we will write $\mathcal{N}$ instead of $\mathcal{N}^{1\times 1}$.*

1.3.2 Lemma ([D, Chapter 1, Problem 21])**.** *Whenever $f, g \in \mathcal{N}^{n\times n}$, then $f + g$ and fg belong to $\mathcal{N}^{n\times n}$ as well.*

[3] This turns out to be an equality, but that is not needed here.

1.3.3 Lemma. *For $f \in \mathcal{N}$, the limit*

$$h_f := \limsup_{\tau \to +\infty} \frac{\ln |f(i\tau)|}{\tau} \tag{1.5}$$

exists in $\mathbb{R}$.

Proof. This follows from Theorems 9 and 10 from Chapter 1 of [D]. □

1.3.4 Definition. *For $f \in \mathcal{N}$, the number h_f defined by (1.5) is called the* mean type *of f.*

The following two definitions are needed for a fundamental representation theorem for functions of bounded type.

1.3.5 Definition ([AD, Example 3.11])**.** *A* Blaschke product *is a function of the form*

$$b(z) = \prod_{j=1}^{n} \gamma_j \frac{z - \omega_j}{z - \overline{\omega_j}}, \quad n \in \mathbb{N} \cup \{\infty\}, \tag{1.6}$$

where all ω_j lie in the upper half plane and meet the Blaschke condition

$$\sum_{j=1}^{n} \frac{|\operatorname{Im} \omega_j|}{1 + |\omega_j|^2} < +\infty.$$

The constants γ_j are chosen in a way that ensures convergence of the Blaschke product: Usually, $\gamma_j = 1$ if $|\omega_j| \le 1$ and $\gamma_j = \frac{\overline{\omega_j}}{\omega_j}$ otherwise.

1.3.6 Definition. *A function $\phi \in H_\infty$ is called* outer *if there is a function $k \in L^1(\mathbb{R})$ and a constant γ with $|\gamma| = 1$, such that*

$$\phi(z) = \gamma \exp\left(\frac{i}{\pi} \int_{\mathbb{R}} \frac{1 + tz}{t - z} k(t)\, d\lambda(t) \right), \quad z \in \mathbb{C}_+. \tag{1.7}$$

In this case, $k(t) = -\lim_{\epsilon \searrow 0} \frac{\ln |\phi(t + i\epsilon)|}{1 + |t|^2}$.

1.3.7 Theorem ([D, Chapter 1, Theorem 9])**.** *Let f be holomorphic in $\mathbb{C}_+$ and suppose that 0 is not a limit point of zeros of f. Then $f \in \mathcal{N}$ if and only if there exist a Blaschke product b and an outer function ϕ such that $f(z) = e^{-ih_f z} b(z) \phi(z)$ everywhere in $\mathbb{C}_+$.*

We introduce another notion of growth for entire functions.

1.3.8 Definition ([R, Definition 6.16])**.** *An entire function f is* of exponential type *if there exist real constants m, M, where $M > 0$, such that*

$$|f(z)| \le M \exp(m|z|), \quad z \in \mathbb{C}. \tag{1.8}$$

The greatest lower bound τ_f of numbers m for which there exists $M > 0$ satisfying (1.8) is called the exponential type *of f.*

The exponential type of f can also be calculated by

$$\tau_f = \limsup_{|z|\to\infty} \frac{\ln|f(z)|}{|z|}. \tag{1.9}$$

The concepts of mean and exponential type are connected by the following theorem that was found by Kreĭn.

1.3.9 Theorem (Kreĭn 1947)**.** *For an entire function f, the following are equivalent:*

(i) f is of exponential type, and [4]

$$\int_{\mathbb{R}} \frac{\ln^+(f(t))}{1+t^2}\, d\lambda(t) < +\infty.$$

(ii) $f|_{\mathbb{C}_+}$ and $f^\#|_{\mathbb{C}_+}$ both belong to $\mathcal{N}$.

In this case, $\tau_f = \max\{h_f, h_{f^\#}\}$.

Proof. See [R, Theorems 6.17 and 6.18]. □

1.3.10 Theorem ([L, Chapter 1, Theorem 22 and Corollary])**.** *Let f be entire and of exponential type $\tau_f = 0$, and let $|f(z)|$ be bounded along some line. Then f is constant.*

1.3.11 Theorem ([L, Chapter I, Theorem 20])**.** *Let $f : \Omega \to \mathbb{C}$ be analytic on the domain $\Omega \subseteq \mathbb{C}$. Suppose that $f(z)$ has a limit at every point of the boundary $\partial\Omega$ of Ω, and that there exists $M > 0$ such that*

$$\lim_{z\to\zeta} |f(z)| \le M, \quad \zeta \in \partial\Omega.$$

If $f(z)$ is bounded in Ω, then $|f(z)| \le M$ for all $z \in \Omega$.

[4] $\ln^+$ is the positive part of ln, i.e., $\ln^+(t) := \max\{0, \ln(t)\}$.

Chapter 2

Matrix-valued Herglotz functions

This chapter contains the standard theory of (matrix valued) Herglotz functions. These being our central object of interest, proofs for the following theorems will be given. For readers not familiar with linear relations in Hilbert spaces, an alternative (and elegant) proof of the integral representation is provided in the appendix.

2.1 Elementary properties

2.1.1 Definition. *Let $f : \Omega \to \mathbb{C}^{n\times n}$ be analytic on the open set $\Omega \subseteq \mathbb{C}$. The reflection of f is defined by $f^{\#}(z) := f(\overline{z})^*$ for $z \in \overline{\Omega} := \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.*
f is called real *if $f(z) = f^{\#}(z)$ for all $z \in \Omega \cap \overline{\Omega}$.*

2.1.2 Definition. *Let $Q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{n\times n}$ be holomorphic and real. Then Q is called a* Herglotz function *if $\operatorname{Im} Q(z) := \frac{Q(z)-Q(z)^*}{2i} \geq 0$ for all $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.*

2.1.3 Remark.

1. The restriction to real Q is purely technical, as for a given holomorphic function $\tilde{Q} : \Omega \to \mathbb{C}^{n\times n}$ with $\Omega \supseteq \mathbb{C}_+$, we can also look at the *real* function Q defined by

$$Q(z) = \begin{cases} \tilde{Q}(z), & z \in \mathbb{C}_+ \\ \tilde{Q}(\overline{z})^*, & z \in \mathbb{C}_- \end{cases}$$

   Note that, in general, $Q|_{\mathbb{C}_-}$ is *not* the analytic continuation of $Q|_{\mathbb{C}_+}$.

2. For Herglotz functions Q, R and scalar valued r, the functions $Q + R$ and $Q \circ r$ as well as $-Q^{-1}$ are Herglotz as well.

3. Elementary examples of Herglotz functions are:
   - ⇝ $Az + B$, where $A > 0$ and $B \in \mathbb{C}^{n\times n}$;
   - ⇝ $\log z$, if defined in a way that $0 < \arg z < \pi$ for $z \in \mathbb{C}_+$;
   - ⇝ $\sum_{j=1}^{n} \frac{A_j}{z_j - z}$ with real z_j and positive matrices A_j;
   - ⇝ The resolvent $(A - z)^{-1}$ of a self-adjoint matrix A.

2.1.4 Lemma. *Let q be a scalar Herglotz function and $z_0 \in \mathbb{C}_+$ such that $q(z_0) \in \mathbb{R}$. Then q is constant.*

Proof. Define $f(z) := \exp(iq(z))$, then

$$|f(z)| = \exp(-\operatorname{Im} q(z)) \leq 1 = |f(z_0)|,$$

which by the maximum principle means that f is constant, as is q. □

2.1.5 Lemma. *Let $Q : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{n\times n}$ be a Herglotz function. Then* $\ker(\operatorname{Im} Q(z))$ *is independent of $z \in \mathbb{C}_+$.*

Proof. Select $z_0 \in \mathbb{C}_+$ and $\xi_0 \in \ker(\operatorname{Im} Q(z))$. Hence, the scalar Herglotz function $g(z) := \xi_0^* Q(z)\xi_0$ takes on a real value at z_0, and by Lemma 2.1.4 is constant. This implies

$$0 = \xi_0^*(\operatorname{Im} Q(z))\xi_0 = \left\|(\operatorname{Im} Q(z))^{\frac{1}{2}}\xi_0\right\|^2,$$

which proves $\xi_0 \in \ker(\operatorname{Im} Q(z))$ for all $z \in \mathbb{C}_+$. □

2.1.6 Lemma. *A matrix M with $\operatorname{Im} M > 0$ is invertible.*

Proof. Suppose not, and choose $x \in \ker M$, $x \neq 0$. Then

$$0 \neq 2ix^*[\operatorname{Im} M]x = 2i \operatorname{Im}[x^* Mx] = x^* Mx - (Mx)^* x = 0,$$

which is a contradiction. □

2.1.7 Lemma. *Let Q be a Herglotz function such that* $\det Q$ *does not vanish identically. Then* $\det Q(z) \neq 0$ *for every $z \in \mathbb{C}\setminus\mathbb{R}$. In this case, $-Q(z)^{-1}$ is also a Herglotz function.*

Proof. At any point where $Q(z)$ is invertible, we have

$$\operatorname{Im}(-Q(z)^{-1}) = \frac{-Q(z)^{-1} + [Q(z)^*]^{-1}}{2i} = Q(z)^{-1} \operatorname{Im} Q(z)[Q(z)^{-1}]^* \geq 0.$$

Suppose that $\det Q(z)$ has a zero z_0 in the upper half plane. Then the meromorphic function $-Q(z)^{-1}$ has a pole at z_0. Let $\xi \in \mathbb{C}^n$ such that $z \mapsto -\xi^* Q(z)^{-1}\xi$ has a pole at z_0. This means that for a small enough circle γ around z_0,

$$\frac{1}{2\pi i}\int_\gamma \frac{[-\xi^* Q(z)^{-1}\xi]'}{-\xi^* Q(z)^{-1}\xi}\,dz \geq 1,$$

while at the same time $0 \leq \arg[-\xi^* Q(z)^{-1}\xi] \leq \pi$, $z \in \gamma$, which is a contradiction. □

2.1.8 Theorem. *Let $Q : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{n\times n}$ be a Herglotz function. Then*

$$K(z, w) := \frac{Q(z) - Q(w)^*}{z - \overline{w}} \tag{2.1}$$

is a positive kernel on $\Omega = \mathbb{C}_+$.

Proof. Consider the function $E(z) := [Q(z)+iI]|_{\mathbb{C}_+}$, which satisfies $\operatorname{Im} E(z) = \operatorname{Im} Q(z)+I > 0$. By Lemma 2.1.6, $E(z)$ is invertible for all $z \in \mathbb{C}_+$. Thus $s(z) := E(z)^{-1}E^{\#}(z)$ is a well-defined and analytic matrix-valued function on $\mathbb{C}_+$. Setting

$$\Lambda(z,w) = i\frac{I - s(z)s(w)^*}{z-\overline{w}},$$

we have

$$\begin{aligned}
E(z)\Lambda(z,w)E(w)^* &= E(z)\Big[i\frac{I - E(z)^{-1}(Q(z)-iI)(Q(w)^*+iI)\big[E(w)^*\big]^{-1}}{z-\overline{w}}\Big]E(w)^* \\
&= E(z)\Big[i\frac{I-(I-2iE(z)^{-1})(I+2i\big[E(w)^*\big]^{-1})}{z-\overline{w}}\Big]E(w)^* \\
&= i\frac{2i(E(w)^*-E(z))-4I}{z-\overline{w}} = i\frac{2i(Q(w)^*-Q(z)-2iI)-4I}{z-\overline{w}} = 2K(z,w).
\end{aligned} \tag{2.2}$$

Inserting $z = w$, we obtain

$$0 \le 2\frac{\operatorname{Im} Q(z)}{\operatorname{Im} z} = 2K(z,z) = 2E(z)\Lambda(z,z)E(z)^* = E(z)\frac{I-s(z)s(z)^*}{\operatorname{Im} z}E(z)^*.$$

Therefore, $I - s(z)s(z)^* \ge 0$ and further $\|s(z)\| = \|s(z)^*\| \le 1$, i.e., $s \in \mathcal{S}_n$. Applying Theorem 1.2.5 yields positivity of the kernel $\Lambda(z,w)$. By (2.2), $K(z,w)$ is a positive kernel, too. □

Much more involved is the theorem giving an integral representation for every Herglotz function. It reads as follows:
Any Herglotz function $Q : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{n\times n}$ admits the following integral representation:

$$Q(z) = C + Dz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\Sigma(t),$$

where C, D are self-adjoint with $D \ge 0$, and Σ is a matrix-valued positive measure such that $\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\Sigma(t) < +\infty$.
The standard proof is done without any additional theory and can be found in the appendix (though only for the scalar case). We will go a different direction and extract the integral representation from the operator model of a Herglotz function. This has the advantage of proving an important result about being able to continue partially defined "Herglotz functions" to all of $\mathbb{C}\setminus\mathbb{R}$. Note that this path of deriving the integral representation appears to be fundamentally different from the standard proof. However, Theorem 2.1.8 plays an important role in proving the operator representation of a Herglotz function. The proof of the latter, however, hinges on the theory of Hardy spaces, for which many of the same techniques as in the standard proof of the Herglotz integral representation are used. Still, the operator model gives a very interesting angle of perspective.

2.2 Defect families and Q-functions

This section is primarily taken from [KK]. Let $\mathcal{H}$ be a Hilbert space and T a linear relation on $\mathcal{H}$, i.e., a linear subspace $T \leq \mathcal{H} \times \mathcal{H}$. The following notation will be used:

2.2.1 Definition.

1. *The domain* $\operatorname{dom} T := \{x \in \mathcal{H} : \exists y \in \mathcal{H} \quad (x; y) \in T\}$*;*
2. *The range* $\operatorname{ran} T := \{y \in \mathcal{H} : \exists x \in \mathcal{H} \quad (x; y) \in T\}$*;*
3. *The kernel* $\ker T := \{x \in \mathcal{H} : (x; 0) \in T\}$*;*
4. *The multi-valued part* $\operatorname{mul} T := \{y \in \mathcal{H} : (0; y) \in T\}$*;*
5. $T_\infty := \{0\} \times \operatorname{mul} T$*;*
6. $T_s := T \ominus T_\infty = T \cap T_\infty^\perp$.

2.2.2 Lemma ([KK, Lemma 1.3.3])**.** *Let T be a closed linear relation on $\mathcal{H}$. Then*

(i) T_s *is an operator;*

(ii) $\operatorname{dom} T_s = \operatorname{dom} T$ *is dense in* $(\operatorname{mul} T^*)^\perp$*;*

(iii) $\operatorname{ran} T_s \subseteq (\operatorname{mul} T)^\perp$*;*

(iv) $\operatorname{ran} T = \operatorname{ran} T_s \oplus \operatorname{mul} T$*, where $\oplus$ denotes an orthogonal sum.*

Proof.
(i): Choose $x \in \operatorname{mul} T_s$. Then $(0; x) \in T_s \cap T_\infty = \{(0; 0)\}$, so $\operatorname{mul} T_s = \{0\}$.
(ii): Clearly, $\operatorname{dom} T_s \subseteq \operatorname{dom} T$. If, on the other hand, $x \in \operatorname{dom} T$, then there exists $y \in \mathcal{H}$ such that $(x; y) \in T$. T is closed, so $\operatorname{mul} T$ is closed, too, and therefore $\mathcal{H} = \operatorname{mul} T \oplus (\operatorname{mul} T)^\perp$. Write $y = y_1 + y_2$, where $y_1 \in \operatorname{mul} T$ and $y_2 \in (\operatorname{mul} T)^\perp$. Now, $(x; y_2) = (x; y) - (0; y_1) \in T$ is orthogonal to T_∞. This means that $(x; y_2) \in T_s$, which shows $x \in \operatorname{dom} T_s$.
Density of $\operatorname{dom} T$ in $(\operatorname{mul} T^*)^\perp$ follows from $(\operatorname{mul} T^*)^\perp = ((\operatorname{dom} T)^\perp)^\perp = \overline{\operatorname{dom} T}$.
(iii): Let $y \in \operatorname{ran} T_s$ and $u \in \operatorname{mul} T$. Taking $x \in \mathcal{H}$ such that $(x; y) \in T_s \perp T_\infty$ leads to $(y, u)_{\mathcal{H}} = \big((x; y), (0, u)\big)_{\mathcal{H} \times \mathcal{H}} = 0$.
(iv): The inclusion $\operatorname{ran} T_s \oplus \operatorname{mul} T \subseteq \operatorname{ran} T$ holds because of $\operatorname{ran} T_s, \operatorname{mul} T \subseteq \operatorname{ran} T$. If $y \in \operatorname{ran} T$, we find $x \in \mathcal{H}$ with $(x; y) \in T = T_s \oplus T_\infty$. Hence, $(x; y) = (x; y_1) + (0; y_2)$ leads to $y = y_1 + y_2 \in \operatorname{ran} T_s \oplus \operatorname{mul} T$. □

2.2.3 Corollary ([KK, Korollar 1.3.7])**.** *Let T be a self-adjoint linear relation on the Hilbert space $\mathcal{H}$. Then $T_s : (\operatorname{mul} T)^\perp \to (\operatorname{mul} T)^\perp$ is a densely defined and self-adjoint operator.*

Proof. We apply Lemma 2.2.2. By (i), T_s is an operator. By (ii), T_s is densely defined in $(\operatorname{mul} T^*)^\perp = (\operatorname{mul} T)^\perp$, and (iii) means that T_s has values in $(\operatorname{mul} T)^\perp$.
Let T_s^* be the adjoint of T_s in $(\operatorname{mul} T)^\perp$. We first show $T_s^* \subseteq T_s$. For $(x; y) \in T_s^*$, we claim

that $(x,y) \in T^* = T$, implying $(x;y) \in T_s$ because of $y \in (\operatorname{mul} T)^\perp$. Let $(u;v) \in T$, $(u;v) = (u;v_1) + (0;v2)$ with $(u;v_1) \in T_s$, $(0;v_2) \in T_\infty$. Now, $(x,v)_\mathcal{H} = (x,v_1)_\mathcal{H} = (y,u)_\mathcal{H}$ since $(x;y)$ belongs to T_s^*. So, indeed, $(x;y) \in T^*$.
The reverse inclusion holds because of $T_s \subseteq T = T^* \subseteq T_s^*$. □

2.2.4 Remark. Recall that for a linear relation T on a Hilbert space, the dimension of $[\operatorname{ran}(T-\lambda)]^\perp$ is locally constant on[1] $r(T) = \{\lambda \in \mathbb{C} \cup \infty : (T-\lambda)^{-1} \in L_b(\operatorname{ran}(T-\lambda), \mathcal{H})\}$. Since a symmetric linear relation S satisfies $r(S) \subseteq \mathbb{C} \setminus \mathbb{R}$, $\dim [\operatorname{ran}(T-\lambda)]^\perp$ can attain at most two different values, depending only on λ belonging to the upper or lower half-plane. We call those numbers the defect indices, i.e.,

$$n_\pm = \dim [\operatorname{ran}(T \pm i))]^\perp. \tag{2.3}$$

It is well known (see, for example, [B, Corollary 1.7.13]) that a closed and symmetric linear relation S on $\mathcal{H}$ has a self-adjoint extension on $\mathcal{H}$ if and only if $n_+ = n_-$. However, even in the case of inequality, a self-adjoint extension of S can be found in a bigger Hilbert space $\mathcal{H}' \supseteq \mathcal{H}$.

2.2.5 Definition. *Let T be a linear relation on $\mathcal{H}$. The subspaces*

$$\mathcal{N}_\lambda := [\operatorname{ran}(T - \overline{\lambda})]^\perp, \quad \lambda \in \mathbb{C}, \tag{2.4}$$

are called the defect spaces *of T. The notation $\mathcal{N}_\lambda(T)$ might be used in cases of ambiguity.*

In the context of the following lemma, $\mathcal{N}_\lambda$ always refers to $[\operatorname{ran}(S - \overline{\lambda})]^\perp$, while $T_{\mu,\lambda}$ will be defined with regard to a self-adjoint extension A of S.

2.2.6 Lemma ([KK, Proposition 2.1.5; Lemma 2.1.6])**.** *Let $S \subseteq S^*$ be a symmetric linear relation on a Hilbert space $\mathcal{H}$, and let $A = A^* \supseteq S$ be a self-adjoint extension of S in $\mathcal{H}$. For $\lambda, \mu \in \rho(A)$, define*

$$T_{\mu,\lambda} := I + (\lambda - \mu)(A - \lambda)^{-1}. \tag{2.5}$$

Then $T_{\mu,\lambda}$ is a bijective and continuous linear operator from $\mathcal{H}$ to $\mathcal{H}$. For $\lambda, \mu, \nu \in \rho(A)$, $T_{\nu,\lambda} T_{\mu,\nu} = T_{\mu,\lambda}$. Moreover, $T_{\mu,\lambda} \mathcal{N}_\mu = \mathcal{N}_\lambda$.

Proof. First, using the resolvent identity, the equality $T_{\nu,\lambda} T_{\mu,\nu} = T_{\mu,\lambda}$ is proven:

$$\begin{aligned} T_{\nu,\lambda} T_{\mu,\nu} &= \big(I + (\lambda-\nu)(A-\lambda)^{-1}\big)\big(I + (\nu-\mu)(A-\nu)^{-1}\big) \\ &= I + (\lambda-\nu)(A-\lambda)^{-1} + (\nu-\mu)(A-\nu)^{-1} + (\nu-\mu)\underbrace{(\lambda-\nu)(A-\lambda)^{-1}(A-\nu)^{-1}}_{=(A-\lambda)^{-1}-(A-\nu)^{-1}} \\ &= I + (\lambda-\mu)(A-\lambda)^{-1} = T_{\mu,\lambda}. \end{aligned}$$

[1] $L_b(X,Y)$ stands for the set of bounded linear operators $T : X \to Y$;
By $(T-\infty)^{-1}$ we understand T itself, so that $\infty \in r(T)$ if and only if T is a bounded linear operator on its domain.

In particular, $T_{\mu,\lambda}T_{\lambda,\mu} = T_{\mu,\mu} = I = T_{\lambda,\mu}T_{\mu,\lambda}$, so $T_{\mu,\lambda}$ is bijective.
This leaves us with the proof of $T_{\mu,\lambda}\mathcal{N}_\mu = \mathcal{N}_\lambda$. Taking $x \in \mathcal{N}_\mu$, we have to verify that, for every $y \in \operatorname{ran}(S - \overline{\lambda})$, $T_{\mu,\lambda}x$ is orthogonal to y. Start by writing $y = v - \overline{\lambda}u$ where $(u; v) \in S \subseteq A$. Hence, $(v - \overline{\lambda}u; u) \in (A - \overline{\lambda})^{-1}$, and further

$$(T_{\mu,\lambda}x, y) = (x, T^*_{\mu,\lambda}y) = (x, T^*(v - \overline{\lambda}u)) = (x, v - \overline{\lambda}u + (\overline{\lambda} - \overline{\mu})u) = (x, v - \overline{\mu}u) = 0.$$

So, $T_{\mu,\lambda}\mathcal{N}_\mu \subseteq \mathcal{N}_\lambda$. By symmetry, we also have $T_{\lambda,\mu}\mathcal{N}_\lambda \subseteq \mathcal{N}_\mu$. Applying $T_{\mu,\lambda}$ on both sides yields the missing inclusion. □

We are now on our way to establish the connection between Herglotz functions on one side and self-adjoint extensions of symmetric linear relations on the other side. The following definition is valid not only for finite defect indices (n, n). However, if the defect indices are infinite, the pair (S, A) does not correspond to a matrix-valued Herglotz function, but rather to an operator-valued.

2.2.7 Definition. *Let $S \subseteq S^*$ a closed and symmetric linear relation on a Hilbert space $\mathcal{H}$ with finite defect indices (n, n). Let $A = A^* \supseteq S$ be a selfadjoint extension of S. Pick $\mu_0 \in \rho(A)$, and let $\Gamma : \mathbb{C}^n \to \mathcal{N}_{\mu_0}$ be an isomorphism. Setting*

$$\Gamma_\lambda := T_{\mu_0,\lambda}\Gamma = \Big(I + (\lambda - \mu)(A - \lambda)^{-1}\Big)\Gamma, \quad \lambda \in \rho(A), \tag{2.6}$$

the family $(\Gamma_\lambda)_{\lambda \in \rho(A)}$ is called a defect family *of*[2] *(S, A).*

2.2.8 Remark. Setting $\lambda = \mu_0$, we get $\Gamma_{\mu_0} = \Gamma$.

2.2.9 Corollary ([KK, Korollar 2.1.8])**.** *Let $(\Gamma_\lambda)_{\lambda \in \rho(A)}$ be a defect family of (S, A), where S has defect indices (n, n). The following statements hold:*

(i) *For each $\lambda \in \rho(A)$, $\Gamma_\lambda : \mathbb{C}^n \to \mathcal{N}_\lambda$ is an isomorphism.*

(ii) *$\Gamma_\lambda = T_{\mu,\lambda}\Gamma_\mu$ for all $\lambda, \mu \in \rho(A)$.*

(iii) *Γ_λ depends analytically on $\lambda \in \rho(A)$.*

Proof.
(i): Clear from the definition of Γ_λ and Lemma 2.2.6.
(ii): $T_{\mu,\lambda}\Gamma_\mu = T_{\mu,\lambda}T_{\mu_0,\mu}\Gamma = T_{\mu_0,\lambda}\Gamma = \Gamma_\lambda$.
(iii): Because of *(ii)*, we have $\Gamma_{z+h} = T_{z,z+h}\Gamma_z$. Next,

$$\frac{\Gamma_{z+h} - \Gamma_z}{h} = \frac{1}{h}\Big((I + h(A - (z+h))^{-1})\Gamma_z - \Gamma_z\Big) = (A - (z+h))^{-1}\Gamma_z.$$

Thus, the limit for $h \to 0$ exists (and is equal to $(A - z)^{-1}\Gamma_z$). □

[2] The notation (S, A) will always refer to a self-adjoint extension A of a symmetric linear relation S.

2.2.10 Definition. *Let S be a closed and symmetric linear relation on $\mathcal{H}$ having defect indices (n,n). Let A be a self-adjoint extension of S and $(\Gamma_\lambda)_{\lambda\in\rho(A)}$ a defect family of (S,A). A function $Q:\rho(A)\to\mathbb{C}^{n\times n}$ is called a* Q-function *of $(S,A,(\Gamma_\lambda)_{\lambda\in\rho(A)})$ if*

$$\frac{Q(\lambda)-Q(\mu)^*}{\lambda-\overline{\mu}}=\Gamma_\mu^*\Gamma_\lambda, \quad \lambda,\mu\in\rho(A). \tag{2.7}$$

2.2.11 Proposition ([KK, Proposition 2.1.13])**.** *Under the assumptions of Definition 2.2.10, the function*

$$Q(\lambda):=(\lambda-\overline{\mu_0})\Gamma^*\Gamma_\lambda-\frac{1}{2}(\mu_0-\overline{\mu_0})\Gamma^*\Gamma$$

is a Q-function of $(S,A,(\Gamma_\lambda)_{\lambda\in\rho(A)})$. The Q-function is unique up to a self-adjoint constant $B=B^*\in\mathbb{C}^{n\times n}$.

Proof. We use Corollary 2.2.9, (ii), to calculate

$$Q(\lambda)=(\lambda-\overline{\mu_0})\Gamma^*T_{\mu_0,\lambda}\Gamma-\frac{1}{2}(\mu_0-\overline{\mu_0})\Gamma^*\Gamma.$$

Since $T_{\mu_0,\lambda}^*=T_{\overline{\mu_0},\overline{\lambda}}$, we obtain

$$Q(\mu)^*=(\overline{\mu}-\mu_0)\Gamma^*T_{\overline{\mu_0},\overline{\mu}}\Gamma-\frac{1}{2}(\overline{\mu_0}-\mu_0)\Gamma^*\Gamma.$$

Now,

$$\begin{aligned}Q(\lambda)-Q(\mu)^*&=\Gamma^*\Big[(\lambda-\overline{\mu_0})T_{\mu_0,\lambda}-(\overline{\mu}-\mu_0)T_{\overline{\mu_0},\overline{\mu}}-(\mu_0-\overline{\mu_0})I\Big]\Gamma\\&=\Gamma^*\Big[(\lambda-\overline{\mu})I+(\lambda-\overline{\mu_0})(\lambda-\mu_0)(A-\lambda)^{-1}-(\overline{\mu}-\mu_0)(\overline{\mu}-\overline{\mu_0})(A-\overline{\mu})^{-1}\Big]\Gamma.\end{aligned} \tag{2.8}$$

The expression within the squared brackets evaluates to $(\lambda-\overline{\mu})T_{\overline{\mu_0},\overline{\mu}}T_{\mu_0,\lambda}$:

$$\begin{aligned}T_{\overline{\mu_0},\overline{\mu}}T_{\mu_0,\lambda}&=\left[I+(\overline{\mu}-\overline{\mu_0})(A-\overline{\mu})^{-1}\right]\left[I+(\lambda-\mu_0)(A-\lambda)^{-1}\right]\\&=I+(\overline{\mu}-\overline{\mu_0})(A-\overline{\mu})^{-1}+(\lambda-\mu_0)(A-\lambda)^{-1}+(\overline{\mu}-\overline{\mu_0})(\lambda-\mu_0)(A-\overline{\mu})^{-1}(A-\lambda)^{-1}.\end{aligned}$$

After multiplicating the right side by $\lambda-\overline{\mu}$ and applying the resolvent identity, we have

$$\begin{aligned}&(\lambda-\overline{\mu})I+(\lambda-\overline{\mu})(\lambda-\mu_0)(A-\lambda)^{-1}+(\lambda-\overline{\mu})(\overline{\mu}-\overline{\mu_0})(A-\overline{\mu})^{-1}\\&\qquad+(\overline{\mu}-\overline{\mu_0})(\lambda-\mu_0)\left[(A-\lambda)^{-1}-(A-\overline{\mu})^{-1}\right]\\&=(\lambda-\overline{\mu})I+(\lambda-\overline{\mu_0})(\lambda-\mu_0)(A-\lambda)^{-1}-(\overline{\mu}-\mu_0)(\overline{\mu}-\overline{\mu_0})(A-\overline{\mu})^{-1}.\end{aligned}$$

This is exactly the expression within the squared brackets of (2.8), which implies

$$\begin{aligned}Q(\lambda)-Q(\mu)^*&=(\lambda-\overline{\mu})\Gamma^*T_{\overline{\mu_0},\overline{\mu}}T_{\mu_0,\lambda}\Gamma=(\lambda-\overline{\mu})\Gamma^*T_{\mu_0,\mu}^*T_{\mu_0,\lambda}\Gamma\\&=(\lambda-\overline{\mu})(T_{\mu_0,\mu}\Gamma)^*(T_{\mu_0,\lambda}\Gamma)=(\lambda-\overline{\mu})\Gamma_\mu^*\Gamma_\lambda.\end{aligned}$$

This shows that Q is a Q-function of $(S, A, (\Gamma_\lambda)_{\lambda\in\rho(A)})$.
Suppose that M is another Q-function of this triple. By definition,

$$\frac{M(\lambda) - M(\mu_0)^*}{\lambda - \overline{\mu_0}} = \Gamma^*\Gamma_\lambda,$$

which translates to[3]

$$\begin{aligned} M(\lambda) &= (\lambda - \overline{\mu_0})\Gamma^*\Gamma_\lambda + M(\mu)^* = (\lambda - \overline{\mu_0})\Gamma^*\Gamma_\lambda + \operatorname{Re} M(\mu_0) - i \operatorname{Im} M(\mu_0) \\ &= (\lambda - \overline{\mu_0})\Gamma^*\Gamma_\lambda + \operatorname{Re} M(\mu_0) - \frac{1}{2}(\mu_0 - \overline{\mu_0})\Gamma^*\Gamma = Q(\lambda) + \operatorname{Re} M(\mu_0). \end{aligned}$$

Hence, Q and M differ only by the self-adjoint constant $\operatorname{Re} M(\mu_0)$. □

2.3 Herglotz functions and extension theory

2.3.1 Lemma ([KK, Lemma 2.6.1]). *Let $\Omega \subseteq \mathbb{C}\setminus\mathbb{R}$, $n \in \mathbb{N}$, and $Q : \Omega \to \mathbb{C}^{n\times n}$ continuous and real, such that the kernel*

$$K(z, w) := \begin{cases} \frac{Q(z)-Q(w)^*}{z-\overline{w}}, & \text{if } z \neq \overline{w}, \\ \lim_{\zeta\to\overline{w}} \frac{Q(\zeta)-Q(w)^*}{\zeta-\overline{w}}, & \text{if } z = \overline{w} \text{ is a limit point of } \Omega, \\ I & \text{otherwise} \end{cases} \tag{2.9}$$

is well-defined and positive on Ω.
Let $\mathcal{H}$ be the unique RKHS having K as its kernel and consider the following linear relation on $\mathcal{H}$:

$$\begin{aligned} \tilde{S} := \Big\{ \Big(\sum_{i=1}^m K(\cdot, w_i)x_i; \sum_{i=1}^m \overline{w_i} K(\cdot, w_i)x_i \Big) \in \mathcal{H}\times\mathcal{H} : \\ m \in \mathbb{N}, w_1, ..., w_m \in \Omega, x_1, ..., x_m \in \mathbb{C}^n s.t. \sum_{i=1}^m x_i = 0 \Big\} \end{aligned} \tag{2.10}$$

Then the following statements hold:

(i) *$\tilde{S}$ is symmetric.*

(ii) *If Ω has a limit point in $\mathbb{C}_+$, then $\tilde{S}$ has defect index $n_+ = 0$. If Ω has a limit point in $\mathbb{C}_-$, then $n_- = 0$.*

(iii) *If Ω has a limit point both in $\mathbb{C}_+$ and in $\mathbb{C}_-$, then $A := \overline{\tilde{S}}$ is self-adjoint.*

Proof.
(i): It needs to be checked that whenever $(f; g), (u; v) \in \tilde{S}$, then $(f, v)_\mathcal{H} = (g, u)_\mathcal{H}$. Suppose that f has the form $\sum_{i=1}^m K(\cdot, w_i)x_i$, while $u = \sum_{j=1}^l K(\cdot, t_j)y_j$. We evaluate using Definition

[3]For a matrix B, we set $\operatorname{Re} B := \frac{1}{2}(B + B^*)$ and $\operatorname{Im} B := \frac{1}{2i}(B - B^*)$.

1.1.4, (2):

$$(f,v)_{\mathcal{H}} = \Big(\sum_{i=1}^{m} K(\cdot,w_i)x_i, \sum_{j=1}^{l} \overline{t_j}K(\cdot,t_j)y_j\Big)_{\mathcal{H}} = \sum_{i=1}^{m}\sum_{j=1}^{l}\Big(K(\cdot,w_i)x_i, \overline{t_j}K(\cdot,t_j)y_j\Big)_{\mathcal{H}}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{l}\overline{\overline{t_j}x_i^*K(w_i,t_j)y_j} = \sum_{i=1}^{m}\sum_{j=1}^{l} t_j y_j^* K(w_i,t_j)^* x_i = \sum_{i=1}^{m}\sum_{j=1}^{l} t_j y_j^* K(t_j,w_i)x_i.$$

The last equality holds because of Remark 1.1.2 (though it is also clear from the definition of K). On the other hand,

$$(g,u)_{\mathcal{H}} = \Big(\sum_{i=1}^{m} \overline{w_i}K(\cdot,w_i)x_i, \sum_{j=1}^{l} K(\cdot,t_j)y_j\Big)_{\mathcal{H}}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{l}\Big(\overline{w_i}(K(\cdot,w_i)x_i, K(\cdot,t_j)y_j\Big)_{\mathcal{H}} = \sum_{i=1}^{m}\sum_{j=1}^{l}\overline{w_i}y_j^*K(t_j,w_i)x_i.$$

Noticing that $(z-\overline{w})K(z,w) = Q(z) - Q(\overline{w})$ for all $z,w\in\Omega$, we finally obtain

$$(f,v)_{\mathcal{H}} - (g,u)_{\mathcal{H}} = \sum_{i=1}^{m}\sum_{j=1}^{l}(t_j-\overline{w_i})y_j^*K(t_j,w_i)x_i = \sum_{i=1}^{m}\sum_{j=1}^{l} y_j^*[Q(t_j) - Q(w_i)^*]x_i$$
$$= \sum_{j=1}^{l} y_j^* Q(t_j)\Big(\underbrace{\sum_{i=1}^{m} x_i}_{=0}\Big) - \sum_{i=1}^{m}\Big(\underbrace{\sum_{j=1}^{l} y_j}_{=0}\Big)^* Q(w_i)^* x_i = 0.$$

(ii): Let $z_0\in\Omega$. We start by showing

$$\operatorname{ran}(\tilde{S}-\overline{z_0}) = \Big\{g = \sum_{i=1}^{m} K(\cdot,w_i)x_i \in \mathcal{H} : w_1,...,w_m\in\Omega, x_1,...,x_m\in\mathbb{C}^n, x_i = 0 \text{ if } w_i = z_0\Big\}. \tag{2.11}$$

Firstly, we convince ourselves that every element of $\operatorname{ran}(\tilde{S}-\overline{z_0})$ is of the indicated form. Let $f = \sum_{i=1}^m K(\cdot,w_i)y_i \in \operatorname{dom}\tilde{S}$. Then $\Big(f;\sum_{i=1}^m K(\cdot,w_i)(\overline{w_i}-\overline{z_0})y_i\Big) \in \tilde{S}-\overline{z_0}$, and thus $x_i := (\overline{w_i}-\overline{z_0})y_i$ vanishes for $w_i = z_0$.
Secondly, let $g := \sum_{i=1}^m K(\cdot,w_i)x_i$, where $w_1,...,w_m\in\Omega, x_1,...,x_m\in\mathbb{C}^n$, and $x_i = 0$ if $w_i = z_0$. Set $M = \{i\in\{1,...,m\} : w_i \neq z_0\}$ and set $y_i := \frac{x_i}{\overline{w_i}-\overline{z_0}}$ for $i\in M$. Letting $y_0 := -\sum_{i\in M} y_i$, we have $\sum_{i\in M\cup\{0\}} y_i = 0$. With $w_0 := z_0$, we define

$$f := \sum_{i\in M\cup\{0\}} K(\cdot,w_i)y_i.$$

Evidently, $(f;g)$ belongs to $\tilde{S}-\overline{z_0}$, and therefore $g\in\operatorname{ran}(\tilde{S}-\overline{z_0})$.
For our next step, take a limit point z_0 of Ω. Since the linear span of all functions of the form $K(\cdot,w)u$ is dense in $\mathcal{H}$ is dense by Lemma 1.1.6, it suffices to show that every function of the

form $K(\cdot,w)u$ can be approximated with functions from $\operatorname{ran}(\tilde{S}-\overline{z_0})$. Since this is trivial for $w \neq z_0$, our task reduces to the approximation of $K(\cdot,z_0)u$ for every $u \in \mathbb{C}^n$.
Recall that, in Hilbert spaces, convergence of x_i to x is equivalent to convergence of $\|x_i\|$ to $\|x\|$ and (x_i,y) to (x,y) for every y from a dense subset. We continue by checking the latter with the net $K(\cdot,w)u$ defined on the directed set $\Omega \setminus \{z_0\}$.

$$(K(\cdot,w)u, K(\cdot,w)u)_{\mathcal{H}} = u^* K(w,w)u \to u^* K(z_0,z_0)u = (K(\cdot,z_0)u, K(\cdot,z_0)u)_{\mathcal{H}} \tag{2.12}$$

holds because Q was assumed to be continuous. What remains to be shown is that $(K(\cdot,w)u, K(\cdot,z)u)_{\mathcal{H}}$ converges to $(K(\cdot,z_0)u, K(\cdot,z)u)_{\mathcal{H}}$ for every $z \in \Omega$. If $z \neq \overline{z_0}$, then this follows from the continuity of Q in the same way as (2.12). If $z = \overline{z_0}$, then we have

$$(K(\cdot,w)u, K(\cdot,\overline{z_0})u)_{\mathcal{H}} = u^* K(\overline{z_0},w)u \to u^* K(\overline{z_0},z_0)u = (K(\cdot,z_0)u, K(\cdot,\overline{z_0})u)_{\mathcal{H}},$$

the limit existing because of the assumption that K is well-defined.
(iii): This follows immediately from (ii) because any closed and symmetric linear relation with defect indices $(0,0)$ is already self-adjoint. □

2.3.2 Theorem ([KK, Satz 2.7.1])**.** *Let $\emptyset \neq \Omega \subseteq \mathbb{C}\setminus\mathbb{R}$ be open, and let $Q : \Omega \to \mathbb{C}^{n\times n}$ such that $Q(\overline{z}) = Q(z)^*$ whenever $z, \overline{z} \in \Omega$. Then Q can be continued to a Herglotz function (on $\mathbb{C}\setminus\mathbb{R}$) if and only if the kernel*

$$K(z,w) = \begin{cases} \frac{Q(z)-Q(w)^*}{z-\overline{w}}, & z \neq \overline{w}, \\ Q'(z), & z = \overline{w} \end{cases} \tag{2.13}$$

is positive on Ω.

Proof. Suppose that $K(z,w)$ is positive. Lemma 2.3.1 gives us an RKHS $\mathcal{H}$ of functions on $\mathbb{C}\setminus\mathbb{R}$ with values in $\mathbb{C}^{n\times n}$, together with a symmetric relation $\tilde{S}$ on $\mathcal{H}$. As mentioned in Remark 2.2.4, we can construct a self-adjoint extension A' of $\tilde{S}$ in a bigger Hilbert space $\mathcal{H}' \supseteq \mathcal{H}$. For $\overline{\lambda} \in \Omega$, let

$$\Gamma_\lambda : \begin{cases} \mathbb{C}^n \to \mathcal{H} \\ x \mapsto K(\cdot,\overline{\lambda})x \end{cases} \tag{2.14}$$

If $\overline{\lambda}, \overline{\mu} \in \Omega$ and $x \in \mathbb{C}^n$, then by definition of $\tilde{S}$

$$\begin{aligned} &\Big(K(\cdot,\overline{\lambda})x - K(\cdot,\overline{\mu})x; \lambda K(\cdot,\overline{\lambda})x - \mu K(\cdot,\overline{\mu})x\Big) \in \tilde{S} \\ &\Rightarrow \Big(K(\cdot,\overline{\lambda})x - K(\cdot,\overline{\mu})x; (\lambda-\mu)K(\cdot,\overline{\mu})x\Big) \in \tilde{S} - \lambda \subseteq A' - \lambda \\ &\Rightarrow \Big(K(\cdot,\overline{\mu})x; K(\cdot,\overline{\lambda})x - K(\cdot,\overline{\mu})x\Big) \in (\lambda-\mu)(A'-\lambda)^{-1} \\ &\Rightarrow \Big(K(\cdot,\overline{\mu})x; K(\cdot,\overline{\lambda})x\Big) \in \big[I + (\lambda-\mu)(A'-\lambda)^{-1}\big]. \end{aligned} \tag{2.15}$$

Using the notation from Lemma 2.2.6, the last row takes the form

$$T_{\mu,\lambda}\Gamma_\mu = \Gamma_\lambda. \tag{2.16}$$

Let $\overline{\mu_0} \in \Omega$. Because of (2.16), we can define a continuation of $(\Gamma_\lambda)_{\overline{\lambda}\in\Omega}$ to $\mathbb{C}\setminus\mathbb{R}$ by

$$\Gamma_\lambda := T_{\mu_0,\lambda}\Gamma_{\mu_0}, \quad \lambda \in \mathbb{C}\setminus\mathbb{R}.$$

This also guarantees that the function $\lambda \mapsto \Gamma_\lambda$ is analytic on $\mathbb{C}\setminus\mathbb{R}$. Furthermore, equation (2.16) stays valid for all $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$ because of

$$T_{\mu,\lambda}\Gamma_\mu = T_{\mu,\lambda}T_{\mu_0,\mu}\Gamma_{\mu_0} = T_{\mu_0,\lambda}\Gamma_{\mu_0} = \Gamma_\lambda.$$

Next, we get $K(\overline{\mu},\overline{\lambda}) = \Gamma_\mu^*\Gamma_\lambda$ from

$$(\Gamma_\mu^*\Gamma_\lambda x, y)_{\mathbb{C}^n} = (K(\cdot,\overline{\lambda})x, K(\cdot,\overline{\mu})y)_{\mathcal{H}} = y^*K(\overline{\mu},\overline{\lambda})x = (K(\overline{\mu},\overline{\lambda})x, y)_{\mathbb{C}^n}.$$

Hence,

$$Q(\overline{\lambda})^* = (\lambda - \overline{\mu})\Gamma_\mu^*\Gamma_\lambda + Q(\overline{\mu}). \tag{2.17}$$

For $\lambda = \mu$, this reads as $Q(\overline{\mu})^* = (\mu - \overline{\mu})\Gamma_\mu^*\Gamma_\mu + Q(\overline{\mu})$, so

$$Q(\overline{\mu}) = (\overline{\mu} - \mu)\Gamma_\mu^*\Gamma_\mu + Q(\overline{\mu})^*.$$

We plug this into (2.17) to obtain

$$Q(\overline{\lambda})^* = (\lambda - \overline{\mu})\Gamma_\mu^*\Gamma_\lambda + (\overline{\mu} - \mu)\Gamma_\mu^*\Gamma_\mu + Q(\overline{\mu})^*.$$

We continue by taking the adjoint on both sides and calculating

$$\begin{aligned}
Q(\overline{\lambda}) &= (\overline{\lambda} - \mu)\Gamma_\lambda^*\Gamma_\mu + (\mu - \overline{\mu})\Gamma_\mu^*\Gamma_\mu + Q(\overline{\mu}) \\
&= \Gamma_\mu^*\Big[(\overline{\lambda} - \mu)T_{\mu,\lambda}^* + (\mu - \overline{\mu})I\Big]\Gamma_\mu + Q(\overline{\mu}) \\
&= \Gamma_\mu^*\Big[(\overline{\lambda} - \mu)(I + (\overline{\lambda} - \overline{\mu})(A' - \overline{\lambda})^{-1}) + (\mu - \overline{\mu})I\Big]\Gamma_\mu + Q(\overline{\mu}) \\
&= (\overline{\lambda} - \overline{\mu})\Gamma_\mu^*T_{\mu,\overline{\lambda}}\Gamma_\mu + Q(\overline{\mu}) = (\overline{\lambda} - \overline{\mu})\Gamma_\mu^*\Gamma_{\overline{\lambda}} + Q(\overline{\mu}).
\end{aligned}$$

Finally, Q can be continued analytically to $\mathbb{C}\setminus\mathbb{R}$ by picking $\overline{\mu_0} \in \Omega$ and setting

$$\hat{Q}(\lambda) := (\lambda - \overline{\mu_0})\Gamma_{\mu_0}^*\Gamma_\lambda + Q(\overline{\mu_0}).$$

It remains to be shown that the function we defined is Herglotz:

$$\hat{Q}(\lambda) - \hat{Q}(\mu)^* = (\lambda - \overline{\mu_0})\Gamma_{\mu_0}^*\Gamma_\lambda + Q(\overline{\mu_0}) - (\overline{\mu} - \mu_0)\Gamma_\mu^*\Gamma_{\mu_0} - Q(\overline{\mu_0})^*$$

Recall that

$$Q(\overline{\mu_0}) - Q(\overline{\mu_0})^* = (\overline{\mu_0} - \mu_0)K(\overline{\mu_0}, \overline{\mu_0}) = (\overline{\mu_0} - \mu_0)\Gamma_{\mu_0}^* \Gamma_{\mu_0}.$$

Therefore, doing the same calculation as in (2.8),

$$\begin{aligned}\hat{Q}(\lambda) - \hat{Q}(\mu)^* &= \Gamma_{\mu_0}^* \Big[(\lambda - \overline{\mu_0})T_{\mu_0,\lambda} - (\overline{\mu} - \mu_0)T_{\overline{\mu_0},\overline{\mu}} + (\overline{\mu_0} - \mu_0)I\Big]\Gamma_{\mu_0} \\ &= (\lambda - \overline{\mu})\Gamma_{\mu_0}^* T_{\overline{\mu_0},\overline{\mu}} T_{\mu_0,\lambda}\Gamma_{\mu_0} = (\lambda - \overline{\mu})\Gamma_{\mu}^*\Gamma_{\lambda}. \end{aligned} \tag{2.18}$$

Setting $\lambda = \overline{\mu}$ yields $\hat{Q}(\overline{\mu}) = \hat{Q}(\mu)^*$, while from $\lambda = \mu$ we obtain

$$\operatorname{Im} \hat{Q}(\lambda) = (\operatorname{Im} \lambda)\Gamma_{\lambda}^*\Gamma_{\lambda}.$$

Thus, for $\lambda \in \mathbb{C}_+$ we get $\operatorname{Im} \hat{Q}(\lambda) \geq 0$.
We are left with the task of proving that for any Herglotz function Q, the kernel K as defined in (2.13) is positive on $\mathbb{C} \setminus \mathbb{R}$. From Theorem 2.1.8, we know that $K(z, w)$ is positive on $\mathbb{C}_+$. Using the same construction as in the first part of the proof, we know that $\hat{Q}$ is a continuation of Q to $\mathbb{C} \setminus \mathbb{R}$ that satisfies $\hat{Q}(\overline{z}) = \hat{Q}(z)^*$. Therefore $\hat{Q}$ and Q coincide, and (2.18) reads as $K(\lambda, \mu) = K(\overline{\mu}, \overline{\lambda}) = \Gamma_{\overline{\lambda}}^*\Gamma_{\overline{\mu}}$ for every choice of $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$. A straightforward calculation shows that $K(\lambda, \mu)$ is a positive kernel even on $\mathbb{C} \setminus \mathbb{R}$:

$$\sum_{j,k=1}^{m} u_j^* K(\omega_j, \omega_k) u_k = \sum_{j,k=1}^{m} u_j^* \Gamma_{\overline{\omega_j}}^* \Gamma_{\overline{\omega_k}} u_k = \sum_{j,k=1}^{m} (\Gamma_{\overline{\omega_k}} u_k, \Gamma_{\overline{\omega_j}} u_j)_{\mathcal{H}} = \Big(\sum_{k=1}^{m} \Gamma_{\overline{\omega_k}} u_k, \sum_{j=1}^{m} \Gamma_{\overline{\omega_j}} u_j\Big)_{\mathcal{H}} \geq 0.$$

□

2.3.3 Theorem ([KK, Satz 2.6.2])**.** *Let $Q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{n \times n}$ a Herglotz function such that $\operatorname{Im} Q(z_0) > 0$ for some $z_0 \in \mathbb{C}_+$. Then there exists a Hilbert space $\mathcal{H}$ together with*

- *a symmetric linear relation S in $\mathcal{H}$ with defect indices (n, n),*
- *a self-adjoint extension A of S in $\mathcal{H}$, and*
- *a defect family $(\Gamma_\lambda)_{\lambda \in \rho(A)}$,*

such that Q is a Q-function of the triple $(S, A, (\Gamma_\lambda)_{\lambda \in \rho(A)})$.

Proof. We know from Theorem 2.3.2 that the kernel

$$K(z, w) = \begin{cases} \frac{Q(z) - Q(w)^*}{z - \overline{w}}, & z \neq \overline{w}, \\ Q'(z), & z = \overline{w} \end{cases}$$

is positive on $\Omega = \mathbb{C} \setminus \mathbb{R}$. Lemma 2.3.1 gives us an RKHS $\mathcal{H}$ of functions on $\mathbb{C} \setminus \mathbb{R}$ with values in $\mathbb{C}^{n \times n}$, together with a symmetric relation $\tilde{S}$ on $\mathcal{H}$. In addition, $A := \overline{\tilde{S}}$ is a self-adjoint extension of $\tilde{S}$ in $\mathcal{H}$ because of Ω having limit points in both $\mathbb{C}_+$ and $\mathbb{C}_-$. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ set

$$\Gamma_\lambda : \begin{cases} \mathbb{C}^n \to \mathcal{H} \\ x \mapsto K(\cdot, \overline{\lambda})x \end{cases}$$

We show that, for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$, Γ_λ is injective. Because of Lemma 2.1.5 and $\ker(\operatorname{Im} Q(z_0)) = \{0\}$, even $\ker(\operatorname{Im} Q(z)) = \{0\}$ for every $z \in \mathbb{C}_+$. Pick $x \in \mathbb{C}^n$, $x \neq 0$, then

$$\|\Gamma_\lambda x\|_{\mathcal{H}}^2 = (K(\cdot, \overline{\lambda})x, K(\cdot, \overline{\lambda})x)_{\mathcal{H}} = x^* K(\overline{\lambda}, \overline{\lambda})x = x^* \frac{\operatorname{Im} Q(\lambda)}{\operatorname{Im} \lambda} x > 0.$$

Hence $\ker \Gamma_\lambda = \{0\}$, and $\Gamma_\lambda : \mathbb{C}^n \to \Gamma_\lambda(\mathbb{C}^n) \subseteq \mathcal{H}$ is bijective. Like in the previous theorem, formula (2.15), we have

$$\Big(K(\cdot, \overline{\mu})x; K(\cdot, \overline{\lambda})x\Big) \in [I + (\lambda - \mu)(\tilde{S} - \lambda)^{-1}] \subseteq [I + (\lambda - \mu)(A - \lambda)^{-1}],$$

more precisely, $T_{\mu,\lambda}\Gamma_\mu = \Gamma_\lambda$ for all $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$.
This yields a canonical continuation of Γ_λ to all $\lambda \in \rho(A)$: Take $\mu_0 \in \mathbb{C} \setminus \mathbb{R}$ and set

$$\Gamma_\lambda := T_{\mu_0,\lambda}\Gamma_{\mu_0}, \quad \lambda \in \mathbb{R} \cap \rho(A).$$

If we were to define this using λ_0 instead of μ_0, we would see that $T_{\lambda_0,\lambda}\Gamma_{\lambda_0} = T_{\lambda_0,\lambda}T_{\mu_0,\lambda_0}\Gamma_{\mu_0} = T_{\mu_0,\lambda}\Gamma_{\mu_0}$. The continuation of Γ_λ to $\lambda \in \mathbb{R} \cap \rho(A)$ is thus well-defined and satisfies

$$T_{\mu,\lambda}\Gamma_\mu = T_{\mu,\lambda}T_{\mu_0,\mu}\Gamma_{\mu_0} = T_{\mu_0,\lambda}\Gamma_{\mu_0} = \Gamma_\lambda.$$

We proceed by defining the linear relation

$$S := \{(f; g) \in A : g - \overline{\mu_0} f \perp \Gamma_{\mu_0}(\mathbb{C}^n)\} \tag{2.19}$$

which, as a restriction of A, is symmetric. In order for $(\Gamma_\lambda)_{\lambda \in \rho(A)}$ to be a defect family of (S, A), the identity $\Gamma_\lambda(\mathbb{C}^n) = \mathcal{N}_\lambda := [\operatorname{ran}(S - \overline{\lambda})]^\perp$ needs to hold. For $(f; g) \in S$ we have $(g - \overline{\mu_0} f; f) \in (A - \overline{\mu_0})^{-1}$; letting $x \in \mathbb{C}^n$ and $\lambda \in \rho(A)$,

$$\begin{aligned} 0 &= (g - \overline{\mu_0} f, \Gamma_{\mu_0} x)_{\mathcal{H}} = (g - \overline{\mu_0} f, T_{\lambda,\mu_0}\Gamma_\lambda x)_{\mathcal{H}} = (T_{\overline{\lambda},\overline{\mu_0}}(g - \overline{\mu_0} f), \Gamma_\lambda x)_{\mathcal{H}} \\ &= \Big(g - \overline{\mu_0} f + (\overline{\mu_0} - \lambda) f, \Gamma_\lambda x\Big)_{\mathcal{H}} = (g - \overline{\lambda} f, \Gamma_\lambda x)_{\mathcal{H}}. \end{aligned}$$

Hence $\operatorname{ran}(S - \overline{\lambda}) \subseteq [\Gamma_\lambda(\mathbb{C}^n)]^\perp$. If, on the other hand, $v \in [\Gamma_\lambda(\mathbb{C}^n)]^\perp$, take $f := (A - \overline{\lambda})^{-1} v$. Consequently, $(f; v + \overline{\lambda} f) \in A$. However, $(v + \overline{\lambda} f) - \overline{\lambda} f = v$ is orthogonal to $[\Gamma_\lambda(\mathbb{C}^n)]^\perp$, i.e., $(f; v + \overline{\lambda} f) \in S$. Therefore, $v \in \operatorname{ran}(S - \overline{\lambda})$. We conclude $\operatorname{ran}(S - \overline{\lambda}) = [\Gamma_\lambda(\mathbb{C}^n)]^\perp$. Note that $\Gamma_\lambda(\mathbb{C}^n)$ is of finite dimension and thus closed, so taking orthogonal complements yields $\mathcal{N}_\lambda = \Gamma_\lambda(\mathbb{C}^n)$. This proves that $(\Gamma_\lambda)_{\lambda \in \rho(A)}$ is a defect family of (S, A).

Lastly,

$$(\Gamma_\mu^*\Gamma_\lambda x, y)_{\mathbb{C}^n} = (K(\cdot,\overline{\lambda})x, K(\cdot,\overline{\mu})y)_{\mathcal{H}} = (K(\overline{\mu},\overline{\lambda})x, y)_{\mathbb{C}^n} = (K(\lambda,\mu)x, y)_{\mathbb{C}^n}.$$

$x, y \in \mathbb{C}^n$ being arbitrary, this implies $\Gamma_\mu^*\Gamma_\lambda = K(\lambda,\mu) = \frac{Q(\lambda)-Q(\mu)^*}{\lambda-\overline{\mu}}$, i.e., Q is a Q-function of $(S, A, (\Gamma_\lambda)_{\lambda\in\rho(A)})$. □

2.4 Integral representation

Let A be a self-adjoint linear relation in a Hilbert space $\mathcal{H}$. By Corollary 2.2.3, A can be decomposed into $A_\infty = \{0\} \times \operatorname{mul} A$ and a densely defined and self-adjoint operator $A_s : (\operatorname{mul} A)^\perp \to (\operatorname{mul} A)^\perp$. Let $E_s(\Delta)$ be the spectral measure of A_s, then we call

$$E(\Delta) := E_s(\Delta \cap \mathbb{R})P_{(\operatorname{mul} A)^\perp} + \mathbf{1}_\Delta(\infty)P_{\operatorname{mul} A}, \qquad \Delta \in \mathfrak{B}(\mathbb{R}\cup\{\infty\}) \tag{2.20}$$

the spectral measure of A.

2.4.1 Theorem ([KK, Satz 2.8.1])**.** *Let Q be the Q-function of the triple $(S, A, (\Gamma_\lambda)_{\lambda\in\rho(A)})$ and let E be the spectral measure of A. Then*

$$Q(z) = C + Dz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\Sigma(t), \tag{2.21}$$

where $D = \Gamma_i^* E(\{\infty\})\Gamma_i$ and Σ is the matrix-valued positive measure[4] *given by*

$$\Sigma(\Delta) = \int_\Delta (t^2+1)\, d[\Gamma_i^* E(t)\Gamma_i].$$

Proof. We prove this by showing that

$$M(z) := \Gamma_i^* E(\{\infty\})\Gamma_i z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)(t^2+1)\, d[\Gamma_i^* E(t)\Gamma_i]$$

is another Q-function of $(S, A, (\Gamma_\lambda)_{\lambda\in\rho(A)})$. Proposition 2.2.11 then states that the Q-functions Q and M differ only by a self-adjoint constant. Observe

$$\begin{aligned}
&\Big(\frac{M(z)-M(w)^*}{z-\overline{w}}x, y\Big)_{\mathbb{C}^n} \\
&= (\Gamma_i^* E(\{\infty\})\Gamma_i x, y)_{\mathbb{C}^n} + \frac{1}{z-\overline{w}} \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{1}{t-\overline{w}}\Big)(t^2+1)\, d(\Gamma_i^* E(t)\Gamma_i x, y)_{\mathbb{C}^n} \\
&= (E(\{\infty\})\Gamma_i x, E(\{\infty\})\Gamma_i y)_{\mathcal{H}} + \int_{\mathbb{R}} \frac{(t-i)(t+i)}{(t-z)(t-\overline{w})}\, d(E_s(t)\Gamma_i x, \Gamma_i y)_{\mathcal{H}}.
\end{aligned}$$

[4] i.e. for every $x \in \mathbb{C}^n$, $\Delta \mapsto x^*\Sigma(\Delta)x$ is a (scalar) positive measure. The integral being "$< \infty$" means that it should be finite for every choice of $x \in \mathbb{C}^n$.

Let P be the orthogonal projection onto $(\operatorname{mul} A)^\perp$ and I_s the identity on $(\operatorname{mul} A)^\perp$.

$$\int_{\mathbb{R}} \frac{(t-i)(t+i)}{(t-z)(t-\overline{w})}\, d(E_s(t)\Gamma_i x, \Gamma_i y)_{\mathcal{H}} = \int_{\mathbb{R}} \Big(1+\frac{\overline{w}-i}{t-\overline{w}}\Big)\Big(1+\frac{z-i}{t-z}\Big)\, d(E_s(t)P\Gamma_i x, P\Gamma_i y)_{\mathcal{H}}$$
$$= \Big([I_s + (\overline{w}-i)(A_s-\overline{w})^{-1}]\,[I_s+(z-i)(A_s-i)^{-1}]P\Gamma_i x, P\Gamma_i y\Big)_{\mathcal{H}}$$
$$= \Big([P+(z-i)(A_s-i)^{-1}]\Gamma_i x, [P+(w-i)(A_s-w)^{-1}]\Gamma_i y\Big)_{\mathcal{H}}.$$

Recall that $E(\{\infty\})$ is the orthogonal projection onto $\operatorname{mul} A$, and with the notation from above one can write $E(\{\infty\}) = I - P$.

$$\Big(\frac{M(z)-M(w)^*}{z-\overline{w}}x, y\Big)_{\mathbb{C}^n} = ((I-P)\Gamma_i x, (I-P)\Gamma_i y)_{\mathcal{H}}$$
$$+ \Big([P+(z-i)(A_s-i)^{-1}]\Gamma_i x, [P+(w-i)(A_s-w)^{-1}]\Gamma_i y\Big)_{\mathcal{H}}$$
$$= \Big([(I-P)+P+(z-i)(A_s-i)^{-1}]\Gamma_i x, [(I-P)+P+(w-i)(A_s-w)^{-1}]\Gamma_i y\Big)_{\mathcal{H}}$$
$$= (T_{i,z}\Gamma_i x, T_{i,w}\Gamma_i y)_{\mathcal{H}} = (\Gamma_w^*\Gamma_z x, y)_{\mathcal{H}}.$$

□

2.4.2 Theorem (Stieltjes inversion formula; [KK, Satz 2.3.6])**.** *Let $C = C^*, D = D^* \geq 0 \in \mathbb{C}^{n\times n}$ and let Σ be an $n\times n$-matrix-valued positive measure such that $\int_{\mathbb{R}} \frac{1}{1+t^2}\, d(\Sigma(t)x, x) < +\infty$ for every $x \in \mathbb{C}^n$. Let*

$$Q(z) = C + Dz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\Sigma(t). \tag{2.22}$$

Then, for every two real numbers $a < b$,

$$\frac{1}{\pi}\lim_{\tau\searrow 0}\int_{[a,b]} \operatorname{Im} Q(x+i\tau)\, d\lambda(x) = \Sigma((a,b)) + \frac{1}{2}\Big(\Sigma(\{a\}) + \Sigma(\{b\})\Big).$$

Proof. Choose $a, b \in \mathbb{R}$, $a < b$, and define

$$\chi(t) = \begin{cases} 1, & t \in (a,b), \\ \frac{1}{2}, & t \in \{a,b\}, \\ 0, & t \notin [a,b]. \end{cases}$$

Using this notation, we have

$$\Sigma((a,b)) + \frac{1}{2}\Big(\Sigma(\{a\}) + \Sigma(\{b\})\Big) = \int_{\mathbb{R}} \chi(t)\, d\Sigma(t).$$

Observing

$$\operatorname{Im} Q(z) = \operatorname{Im} z\Big(D + \int_{\mathbb{R}} \frac{1}{|t-z|^2}\, d\Sigma(t)\Big), \tag{2.23}$$

we calculate

$$\begin{aligned}
&\frac{1}{\pi}\int_{[a,b]} \operatorname{Im} Q(x+i\tau)\, d\lambda(x) - \int_{\mathbb{R}} \chi(t) d\Sigma(t) = \\
&= \frac{1}{\pi}\int_{[a,b]} \tau\Big(D + \int_{\mathbb{R}} \frac{1}{|t-x-i\tau|^2}\, d\Sigma(t)\Big) d\lambda(x) - \int_{\mathbb{R}} \chi(t)\, d\Sigma(t) = \\
&= \frac{1}{\pi} D(b-a)\tau + \frac{1}{\pi}\int_{\mathbb{R}} \Big(\int_{[a,b]} \frac{\tau}{(t-x)^2+\tau^2}\, d\lambda(x) - \pi\chi(t)\Big)\, d\Sigma(t) = \\
&= \frac{1}{\pi} D(b-a)\tau + \frac{1}{\pi}\int_{\mathbb{R}} \frac{R_\tau(t)}{1+t^2}\, d\Sigma(t),
\end{aligned}$$

where

$$R_\tau(t) := (1+t^2)\Big(\arctan\Big(\frac{b-t}{\tau}\Big) - \arctan\Big(\frac{a-t}{\tau}\Big) - \pi\chi(t)\Big).$$

We claim that there exists a constant C such that for every $t \in \mathbb{R}$ and $\tau \in (0,1)$ the inequality $|R_\tau(t)| \leq C$ holds. This is done by distinguishing the cases $t < a-1$, $a-1 < t < b+1$ and $t > b+1$.
For $t \in [a-1, b-1]$, there exists a constant $C_1 > 0$ satisfying $|R_\tau(t)| \leq C_1$ just by boundedness of arctan and continuity of $1+t^2$ on the compact set $[a-1, b+1]$.
For $t < a-1$, we apply the Mean Value Theorem and choose $\xi \in (\frac{a-t}{\tau}, \frac{b-t}{\tau})$ such that [5]

$$\begin{aligned}
R_\tau(t) &= \frac{1+t^2}{1+\xi^2}\Big(\frac{a-t}{\tau} - \frac{b-t}{\tau}\Big) \leq \frac{1+t^2}{\tau\Big(1+\frac{(a-t)^2}{\tau^2}\Big)}(b-a) = \\
&= (1+t^2)\frac{\tau(b-a)}{\tau^2+(a-t)^2} \leq \frac{1+t^2}{(a-t)^2}(b-a) \leq C_2.
\end{aligned}$$

Similar calculations provide a third constant C_3 for $t > b+1$, which proves that $R_\tau(t)$ is bounded for $t \in \mathbb{R}$, $0 < \tau < 1$. Using again the Dominated Convergence Theorem, we obtain

$$\lim_{\tau \searrow 0} \Big(\frac{1}{\pi} D(b-a)\tau + \frac{1}{\pi}\int_{\mathbb{R}} \frac{R_\tau(t)}{1+t^2}\, d\Sigma(t)\Big) = \frac{1}{\pi}\int_{\mathbb{R}} \lim_{\tau\searrow 0} \frac{R_\tau(t)}{1+t^2}\, d\Sigma(t) = 0,$$

completing the proof. □

2.4.3 Theorem ([KK, Satz 2.4.2]; [GT, Theorem 5.4]). *Let $Q : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{n\times n}$ be holomorphic and real. Then Q is Herglotz if and only if it admits a representation of the form*

$$Q(z) = C + Dz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\Sigma(t), \tag{2.24}$$

where C, D are self-adjoint with $D \geq 0$, and Σ is a matrix-valued positive measure such that $\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\Sigma(t) < +\infty$. Moreover,

(i) $C = \operatorname{Re} Q(i)$;

[5] Note that $\chi(t) = 0$ because of $t < c-1$.

(ii) $D = \lim_{\tau \to +\infty} \frac{\operatorname{Im} Q(i\tau)}{\tau}$;

(iii) *Let* $-\infty < c < d < +\infty$, *then*

$$\frac{1}{\pi} \lim_{\tau \searrow 0} \int_c^d \operatorname{Im} Q(x + i\tau)\, d\lambda(x) = \Sigma((c,d)) + \frac{1}{2}\Big(\Sigma(\{c\}) + \Sigma(\{d\})\Big);$$

(iv) *If all diagonal entries of* Q *vanish, then* $Q(z) \equiv C$.

Proof. Suppose that Q admits the representation (2.24). We rewrite the integral in the following way:

$$\int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\Sigma(t) = \int_{\mathbb{R}} \frac{1+tz}{(t-z)(1+t^2)}\, d\Sigma(t) = \int_{\mathbb{R}} \frac{1+tz}{t-z}\, d\Big(\frac{\Sigma(t)}{1+t^2}\Big). \tag{2.25}$$

The integrand on the right is bounded in t for fixed nonreal z and analytic in $z \in \mathbb{C} \setminus \mathbb{R}$ for fixed t, while the measure is finite by assumption. Therefore, Q is analytic on $\mathbb{C} \setminus \mathbb{R}$. Clearly, $Q(\overline{z}) = Q(z)^*$. Lastly, for $z \in \mathbb{C}_+$,

$$\begin{aligned}\operatorname{Im} Q(z) &= \frac{Q(z) - Q(z)^*}{2i} = D \operatorname{Im} z + \frac{1}{2i} \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{1}{t-\overline{z}}\Big)\, d\Sigma(t) \\ &= D \operatorname{Im} z + \operatorname{Im} z \int_{\mathbb{R}} \frac{1}{|t-z|^2}\, d\Sigma(t) \geq 0,\end{aligned}$$

so Q is Herglotz.

On the other hand, if Q is Herglotz, then $\tilde{Q}(z) := Q(z) + Iz$ is Herglotz as well and satisfies $\operatorname{Im} \tilde{Q}(z) = \operatorname{Im} Q(z) + I \operatorname{Im} z > 0$. By Theorem 2.3.3, there exists a Hilbert space $\mathcal{H}$ as well as a triple $(S, A, (\Gamma_\lambda)_{\lambda \in \rho(A)})$, such that $\tilde{Q}$ is a Q-function of this triple. Theorem 2.4.1 then gives the integral representation of $\tilde{Q}$:

$$\tilde{Q}(z) = \tilde{C} + \tilde{D} z + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\tilde{\Sigma}(t).$$

Therefore, letting $C := \tilde{C}$, $D := \tilde{D} - I$, and $\Sigma := \tilde{\Sigma}$, a representation for Q is obtained. We check that $D \geq 0$ by first proving (ii):

$$\frac{\operatorname{Im} Q(i\tau)}{\tau} = D + \int_{\mathbb{R}} \frac{1}{|t - i\tau|^2}\, d\Sigma(t) \xrightarrow{\tau \to +\infty} D$$

holds because of the Dominated Convergence Theorem. In particular, $D \geq 0$ because it is the limit of positive semidefinite matrices.

Checking (i) is left to the reader, while (iii) is precisely the statement of Theorem 2.4.2. To show (iv), observe that if all diagonal entries of Q vanish, we obtain from (ii) that diagonal entries of the positive definite matrix D also vanish, and thus $D = 0$. In a similar way, we see that Σ vanishes by looking at (iii). □

2.4.4 Corollary. *Let* Q *be an* $n \times n$*-matrix-valued real function that is meromorphic on* $\mathbb{C}$ *and* *holomorphic on $\mathbb{C}\backslash\mathbb{R}$. Then Q is Herglotz if and only if Q allows for the follwing representation,*

$$Q(z) = C + Dz + \sum_{j\in\mathbb{Z}} A_j\Big(\frac{1}{z_j - z} - \frac{z_j}{1+z_j^2}\Big), \tag{2.26}$$

where C, D, A_j are self-adjoint matrices, $D, A_j \geq 0$, and $z_j \in \mathbb{R}$. In this case, the sum $\sum_{j\in\mathbb{Z}} \frac{A_j}{1+z_j^2}$ converges.

Proof. From the assumptions on Q it follows that $\operatorname{Im} Q(x) = 0$ for every real x. From (iii) in the last theorem we conclude that for any finite interval (a, b) which does not contain a pole of Q, we have

$$0 \leq \Sigma\big((a,b)\big) \leq \Sigma\big((a,b)\big) + \frac{1}{2}\Big(\Sigma(\{a\}) + \Sigma(\{b\})\Big) = \frac{1}{\pi}\int_{[a,b]} \operatorname{Im} Q(x)\, d\lambda(x) = 0.$$

This implies that Σ is discrete, which means that Q can be written in the form (2.26) with $A_j = \Sigma(\{z_j\})$. In this case, aforementioned sum converges because of

$$\sum_{j=-N}^{N} \frac{A_j}{1+z_j^2} = \int_{[z_{-N}, z_N]} \frac{1}{1+t^2}\, d\Sigma(t) \xrightarrow{N\to\infty} \int_{\mathbb{R}} \frac{1}{1+t^2}\, d\Sigma(t) < +\infty.$$

□

2.4.5 Lemma. *If Q is an $n \times n$-matrix-valued Herglotz function, then $Q \in \mathcal{N}^{n\times n}$. Moreover, each entry Q_{kl} of Q has mean type $h_{Q_{kl}} = 0$.*

Proof. Let $x \in \mathbb{C}^n$, then $q_x(z) := x^*Q(z)x$ is a scalar Herglotz function. The Möbius transform $\psi(z) := \frac{z-i}{z+i}$ maps $\mathbb{C}_+$ onto the unit disk. In particular, $|(\psi \circ q_x)(z)| \leq 1$. Now, the following representation of q_x holds:

$$q_x(z) = \psi^{-1}(\psi \circ q_x)(z) = \frac{i(1 + (\psi \circ q_x)(z))}{1 - (\psi \circ q_x)(z)}.$$

Both numerator and denominator are bounded, i.e., $q_x \in \mathcal{N}$. Since x is arbitrary, we can easily check that each entry of Q belongs to $\mathcal{N}$ to conclude $Q \in \mathcal{N}^{n\times n}$.
For the second assertion, look at the integral representation (2.24) to obtain that $|Q_{kl}(i\tau)|$ grows at most linearly for $\tau \to +\infty$. □

Chapter 3

The classical Hermite-Biehler Theorem

Scalar Herglotz functions that can be continued to a meromorphic function on all of $\mathbb{C}$ exhibit some intriguing properties, aside from the integral representation which holds for every Herglotz function. More precisely, the zeros and poles of a scalar meromorphic Herglotz function[1] interlace. This fact is the key in the proof of the Hermite-Biehler theorem.
The following is largely due to the book of Levin, ([L], Chapter VII).

3.1 Product representation for scalar Herglotz functions

3.1.1 Definition. *We say that the (real) zeros and poles of a function q are interlacing, if between two successing real zeros of q, there is a pole of q, and between two successing real poles of q, there is a zero of q.*

3.1.2 Lemma. *Let $q \not\equiv 0$ be Herglotz, and assume that q is meromorphic on $\mathbb{C}$ and real. Then the zeros and poles of q are real, simple, and interlace.*

Proof. We know from Lemma 2.1.4 that all zeros of q have to be real. The same goes for poles of q, by definition. For the zeros and poles of q to be simple and interlacing, we show that for any finite interval (u, v), the number of zeros[2] of q in (u, v) differs from the number of poles of q in (u, v) by at most 1. So, let $u, v \in \mathbb{R}$, $u < v$, and assume that neither u nor v is a zero or pole of q. Setting $m := \frac{u+v}{2}$, $r := \frac{u-v}{2}$ and $\gamma(t) := m + r\exp(it)$, $t \in [0, 2\pi]$, the difference between the numbers of zeros and poles of q in (u, v) will be counted by the logarithmic integral

$$\frac{1}{2\pi i}\int_\gamma \frac{q'(z)}{q(z)}\,dz = \frac{1}{2\pi i}\int_{\gamma|_{[0,\pi]}} \frac{q'(z)}{q(z)}\,dz + \frac{1}{2\pi i}\int_{\gamma|_{[\pi,2\pi]}} \frac{q'(z)}{q(z)}\,dz.$$

[1]"Meromorphic Herglotz function" always refers to a Herglotz function that has a meromorphic (and real) continuation to all of $\mathbb{C}$. This is equivalent to the measure in the integral representation being discrete, see also 2.4.4.

[2]By multiplicity.

Since $\operatorname{Im} q(z) \geq 0$ for $\operatorname{Im} z \geq 0$, and $\operatorname{Im} q(z) \leq 0$ for $\operatorname{Im} z \leq 0$, both of the summands can be calculated by a suitable logarithm. Additionally, we have $0 \leq \arg q(z) \leq \pi$ for $\operatorname{Im} z \geq 0$ and $-\pi \leq \operatorname{Im} z \leq 0$ for $\operatorname{Im} z \leq 0$, and this guarantees that the imaginary parts of both summands have an absolute value of at most π. In total, $|\frac{1}{2\pi i}\int_\gamma \frac{q'(z)}{q(z)}\,dz| \leq 1$, which means that zeros and poles of q are simple and interlacing. □

The following two theorems are very similar and are based on ([L, Chapter VII, Theorem 1]).

3.1.3 Theorem. *Let q be rational and real, but not constant. Then q is Herglotz if and only if it can be written in the form*

$$q(z) = c(b_1 - z)^{\alpha_1}\Big(\prod_{j=1}^{k} \frac{a_j - z}{b_j - z}\Big)(a_k - z)^{-\alpha_2}, \quad z \in \mathbb{C}, \tag{3.1}$$

where $b_1 < a_1 < b_2 < a_2 < ... < b_k < a_k$, $c \in \mathbb{R}$, and $\alpha_1, \alpha_2 \in \{0,1\}$ such that $(\alpha_1 = 0) \Leftrightarrow (c > 0)$.

Proof. Let q be Herglotz. Then we know from the last lemma that zeros and poles of q are real, simple, and interlace. By rationality of q, this already gives the representation (3.1), and we only need to check that $(\alpha_1 = 1) \Leftrightarrow (c > 0)$. This is obtained by looking at representation (2.26):

$$q(z) = a + bz + \sum_{j=1}^{k} \frac{q_j}{b_j - z}, \tag{3.2}$$

which means that residues at poles are negative. So, if q has at least one pole, then either $\alpha_1 = 0$ and $c > 0$ as q does not change sign left of the pole b_1; or $\alpha_1 = 1$, which means that left of b_2, q changes its sign exactly one time, so $c < 0$. If there is no pole, then $q(z) = c(a_1 - z)$, which leads to $c < 0$ as well as $\alpha_1 = 1$.
The sufficiency of (3.1) for q to be Herglotz is easily obtained from looking at the partial fraction decomposition

$$q(z) = a + bz + \frac{q_1}{b_1 - z} + \sum_{j=2}^{k} \frac{q_j}{b_j - z},$$

for which we have to show that $b \geq 0$ and $q_1 \geq 0$ and [3] $q_2, ..., q_k > 0$. If $b \neq 0$, then $\alpha_1 = 1$ and $\alpha_2 = 0$ for degree reasons. So, $c < 0$ and therefore

$$b = \lim_{\tau \to +\infty} \frac{\operatorname{Im} q(i\tau)}{\tau} = -c > 0,$$

while from the condition $(\alpha_1 = 0) \Leftrightarrow (c > 0)$ it follows that $q_1 = \lim_{z \to b_1}(b_1 - z)q(z) \geq 0$, and

[3] This is because b_1 is not a pole if $\alpha_1 = 1$

if $q_1 = 0$, then $\alpha_1 = 1$, such that at least $q_2 > 0$. Also, for each $j \in \{1, ..., k\}$, the function

$$r_j(z) := \frac{b_j - z}{a_{j+1} - z}(b_{j+1} - z)q(z)$$

has no zeros or poles in $[b_j, b_{j+1}]$, which under the assumption $q_j > 0$ gives

$$\operatorname{sgn} q_{j+1} = \operatorname{sgn}\left(\lim_{z\to b_{j+1}} (b_{j+1} - z)q(z)\right) = \operatorname{sgn}\left(\frac{a_{j+1} - b_{j+1}}{b_j - b_{j+1}} r_j(b_{j+1})\right) = \operatorname{sgn} q_j = 1.$$

□

3.1.4 Theorem ([L, Chapter VII, Theorem 1])**.** *Let q be meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{R}$ such that the set of (real) poles of q is bounded neither from below nor from above. Then q is Herglotz if and only if*

$$q(z) = c\frac{a_0 - z}{b_0 - z} \prod_{j\in\mathbb{Z}\setminus\{0\}} \frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}}, \tag{3.3}$$

where $b_j < a_j < b_{j+1}$, $j \in \mathbb{Z}$, $a_{-1} < 0 < b_1$, and $c > 0$.

Proof. We start by proving the necessity of representation (3.3). Let

$$\psi(z) := \frac{a_0 - z}{b_0 - z} \prod_{j\in\mathbb{Z}\setminus\{0\}} \frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}},$$

where a_j are all the zeros of q and b_j are all of its poles, which by Lemma 3.1.2 are all simple, and interlace. Now, we show that $\psi(z)$ converges uniformly on any compact subset of $\mathbb{C}$ not containing any poles of ψ. First, from interlacing of a_j and b_j, it is easily seen that

$$\sum_{j\in\mathbb{Z}\setminus\{0\}} \left(\frac{1}{b_j} - \frac{1}{a_j}\right)$$

converges. From this follows the convergence of

$$\sum_{j\in\mathbb{Z}} \left(\frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}} - 1\right) = z\sum_{j\in\mathbb{Z}} \left(\frac{1}{b_j} - \frac{1}{a_j}\right)\left(1 - \frac{z}{b_j}\right)^{-1},$$

which confirms that ψ converges uniformly on every compact subset of $\mathbb{C} \setminus \{b_j : j \in \mathbb{Z}\}$. Now, observe that if $\operatorname{sgn} a_j = \operatorname{sgn} b_j$,

$$\arg\left(\frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}}\right) = \arg\left(\frac{z - a_j}{z - b_j}\right)$$

and

$$0 < \sum_{j\in\mathbb{Z}} \Big[\arg(z - a_j) - \arg(z - b_j)\Big] = \arg\psi(z) < \sum_{j\in\mathbb{Z}} \Big[\arg(z - b_{j+1}) - \arg(z - b_j)\Big] < \pi.$$

So, $g := \frac{q}{\psi}$ is entire, and $-\pi < \arg g < \pi$. We can now define a logarithm of g, which is entire and maps $\mathbb{C}$ into the strip $-\pi < \operatorname{Im} z < \pi$, and is therefore constant. This means that $g(z) \equiv c$, and q being real and Herglotz implies $c > 0$.
The proof of sufficiency was done implicitly in the previous part of the proof. □

Similar theorems apply for all meromorphic real functions. The remaining two cases shall be listed here, but proofs will be left out.

3.1.5 Corollary. *Let q be meromorphic on $\mathbb{C}$ and real, but not constant. Then q is Herglotz if and only if it admits a representation of one of the following forms:*

(A)

$$q(z) = c(b_1 - z)^{\alpha_1}\Big(\prod_{j=1}^{k} \frac{a_j - z}{b_j - z}\Big)(a_k - z)^{-\alpha_2}, \quad z \in \mathbb{C},$$

where $k \in \mathbb{N}$, $b_1 < a_1 < b_2 < a_2 < \ldots < b_k < a_k$, $c \in \mathbb{R}$, and $\alpha_1, \alpha_2 \in \{0, 1\}$ such that $(\alpha_1 = 0) \Leftrightarrow (c > 0)$.

(B)

$$q(z) = c\frac{a_0 - z}{b_0 - z} \prod_{j \in \mathbb{Z}\setminus\{0\}} \frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}},$$

where $b_j < a_j < b_{j+1}$, $j \in \mathbb{Z}$, $a_{-1} < 0 < b_1$, and $c > 0$.

(C)

$$q(z) = c(b_1 - z)^{\alpha_1}\Big(\prod_{j=1}^{k} \frac{a_j - z}{b_j - z}\Big)\Big(\prod_{j=k+1}^{\infty} \frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}}\Big)$$

where $b_1 < a_1 < b_2 < a_2 < \ldots$, $c \in \mathbb{R}$, and $\alpha_1 \in \{0, 1\}$ such that $(\alpha_1 = 0) \Leftrightarrow (c > 0)$.

(D)

$$q(z) = c\Big(\prod_{j=k+1}^{\infty} \frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}}\Big)\Big(\prod_{j=1}^{k} \frac{a_j - z}{b_j - z}\Big)(a_1 - z)^{-\alpha_2}$$

where $a_1 > b_1 > a_2 > b_2 > \ldots$, $c \in \mathbb{R}$, and $\alpha_2 \in \{0, 1\}$ such that $(\alpha_2 = 0) \Leftrightarrow (c > 0)$.

□

Note that for (C) and (D), the infinite product has to be split into two parts - the infinite one, containing factors of the form $(1 - \frac{z}{a_j})/(1 - \frac{z}{b_j})$ to ensure convergence; and the finite ones, which makes sure that if one of the zeros or poles happens to be zero, it will not be divided through.

3.2 The Hermite-Biehler Theorem for polynomials

For any entire function f, we use the notation

$$(Rf)(z) := \frac{f(z) + f^{\#}(z)}{2}$$

and

$$(If)(z) := \frac{f(z) - f^{\#}(z)}{2i}.$$

We will call those functions the real and imaginary part of f, even though $(Rf)(z)$ does not coincide with $\operatorname{Re} f(z)$, and $(If)(z)$ does not coincide with $\operatorname{Im} f(z)$

3.2.1 Theorem. *Let E be a polynomial, and $A := RE$, $B := IE$ its real and imaginary part. Suppose that neither A nor B vanish identically nor are they constant multiples of one another. Then the following statements are equivalent:*

(i) $E(z) \neq 0$ for all z in the upper half plane $\mathbb{C}_+$.

(ii) The zeros and poles of $\frac{A}{B}$ are all real, simple, and interlace. In addition, there is $x_0 \in \mathbb{R}$ such that $(\frac{A}{B})'(x_0) > 0$.

Proof.
$(i) \Rightarrow (ii)$: Let $E(z) = \alpha \prod_{j=1}^{m}(z - z_j)$, where $\operatorname{Im} z_j \leq 0$, $j = 1, ..., m$. Set

$$\omega(z) := \frac{E^{\#}(z)}{E(z)} = \frac{\overline{\alpha}}{\alpha} \prod_{j=1}^{m} \frac{(z - \overline{z_j})}{(z - z_j)}. \tag{3.4}$$

Thus, if $z \in \mathbb{C}_+$, we have $|z - \overline{z_j}| \leq |z - z_j|$, $j = 1, ..., m$ and therefore $|\omega(z)| \leq 1$. Note that $|\omega(z)| = 1$ does not occur, because that would imply that all zeros of E are real, contradicting the condition that A is not a constant multiple of B. The Möbius transform $i\frac{1+z}{1-z}$ takes the unit disk to the upper half plane, which implies that

$$q(z) := \frac{A(z)}{B(z)} = i\frac{1 + \omega(z)}{1 - \omega(z)} \in \mathbb{C}_+, \quad z \in \mathbb{C}_+. \tag{3.5}$$

So, q is a meromorphic Herglotz function, and thus its zeros and poles interlace. But, since real zeros of A and B do not coincide, it follows that zeros of q are precisely the zeros of A, while poles of q are exactly the zeros of B. The second condition follows from

$$q'(x) = \operatorname{Re} q'(x) = \operatorname{Re} \lim_{\epsilon \searrow 0} \frac{q(x + i\epsilon) - q(x)}{i\epsilon} = \lim_{\epsilon \searrow 0} 2 \operatorname{Im} \frac{q(x + i\epsilon)}{\epsilon} \geq 0, \quad x \in \mathbb{R}. \tag{3.6}$$

$(ii) \Rightarrow (i)$: Again, let

$$q(z) = \frac{A(z)}{B(z)} = c(b_1 - z)^{e_1} \Big[\prod_{j=1}^{k} \frac{(a_j - z)}{(b_j - z)} \Big] (a_k - z)^{-e_2}$$

with $b_1 < a_1 < b_2 < a_2 < ...$ and $e_1, e_2 \in \{0,1\}$ depending on whether the series of zeros and poles of q "starts" and "ends" by a zero and/or by a pole. So, either q or $-q$ is a Herglotz function, which by (3.6) means that $q'(x)$ has the same sign for each real x, which by assumption is positive. This guarantees that $-q$ is not Herglotz, so q has to be Herglotz. Therefore, $\mathrm{Im}(q(z)+i) > 0$, $z \in \mathbb{C}_+$, from which we obtain that $E(z) = B(z)(q(z)+i) \neq 0$, $z \in \mathbb{C}_+$. □

3.3 The class HB

It was already seen in the last proof that the core part of the Hermite-Biehler theorem is a simple Möbius transform and the fact that[4] $\omega(z) = \frac{E^{\#}(z)}{E(z)} \in \mathbb{D}$, $z \in \mathbb{C}_+$, if E is a polynomial whose zeros lie only in the lower half plane. However, this does not work anymore if E is an arbitrary entire function. Then, even if E has no zeros in the upper half plane, it is far from guaranteed that $|\omega(z)| < 1$ for every $z \in \mathbb{C}_+$. In order to generalize the Hermite-Biehler Theorem to a wider range of entire functions, this condition has to be treated separately. This section was also taken directly from ([L, Chapter VII]).

3.3.1 Definition. *An entire function E belongs to the class HB if it has no real zeros, and $|E^{\#}(z)| < |E(z)|$ for all $z \in \mathbb{C}_+$.*

3.3.2 Theorem. *Let E be entire and let $A = \mathcal{R}E$ and $B = \mathcal{I}E$ be its real and imaginary part. Let A and B be represented in the following way:*

$$A(z) = C\exp(u(z))(a_0 - z)^{r_1} \prod_{j\in\mathbb{Z}\setminus\{0\}} (1 - \frac{z}{a_j}) \exp(P_j(\frac{z}{a_j})), \tag{3.7}$$

$$B(z) = D\exp(u(z))(b_0 - z)^{r_2} \prod_{j\in\mathbb{Z}\setminus\{0\}} (1 - \frac{z}{b_j}) \exp(P_j(\frac{z}{b_j})), \tag{3.8}$$

where $C, D \in \mathbb{R}$, $u(z)$ and $v(z)$ are real entire functions, $u(0) = v(0) = 0$, and a_j as well as b_j are real with $a_j \leq a_{j+1}$, $b_j \leq b_{j+1}$, $j \in \mathbb{Z}$. Moreover, $a_j \neq 0 \neq b_j$ for $j \neq 0$, and let P_j be the j-th Taylor polynomial of $\ln \frac{1}{1-z}$, ensuring convergence. Then E belongs to HB if and only if the following conditions are fulfilled:

(i) *The zeros of A and B are simple and interlace, such that $b_j < a_j < b_{j+1}$;*

(ii) $w(z) := u(z) - v(z) + \sum_{j\in\mathbb{Z}\setminus\{0\}} \left[P_j(\frac{z}{a_j}) - P_j(\frac{z}{b_j})\right] = 0$, $z \in \mathbb{C}$;

(iii) $\operatorname{sgn} C = \operatorname{sgn} D$.

Proof. If E belongs to HB, we have that

$$\omega(z) = \frac{E^{\#}(z)}{E(z)}$$

maps the upper half plane to the unit disk, and by the Möbius transform $i\frac{1+z}{1-z}$, just as in Theorem 3.2.1, we get that $q(z) := \frac{A(z)}{B(z)}$ is a Herglotz function. Now, A and B cannot have

[4] $\mathbb{D}$ is the open unit disk.

common real roots, because E has no real roots. So, roots of A and B are simple and interlace, proving (i). Now, recall the (Herglotz) function

$$\psi(z) := \frac{a_0 - z}{b_0 - z} \prod_{j \in \mathbb{Z} \setminus \{0\}} \frac{1 - \frac{z}{a_j}}{1 - \frac{z}{b_j}}$$

from the proof of Theorem 3.1.4. Seeing that

$$q(z) = \frac{C}{D} \exp(w(z)) \psi(z),$$

just as in the aforementioned theorem, $\arg \frac{q(z)}{\psi(z)} \in (-\pi, \pi)$ where $\frac{q(z)}{\psi(z)} = \frac{C}{D} \exp(w(z))$ is an entire function. So, $w(z)$ is constant, i.e. $w(z) \equiv 0$, so (ii) holds. Now, Theorem 3.1.4 also implies $\frac{C}{D} > 0$, which means that (iii) holds. The other direction of the proof is done by the same arguments. □

3.3.3 Lemma ([L, Chapter V, Lemma 4])**.** *Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of complex numbers satisfying $\sum_{k=1}^{\infty} \left| \operatorname{Im} \frac{1}{a_k} \right| < +\infty$. Then the product*

$$\chi(z) := \prod_{k=1}^{\infty} \Big(1 - \frac{z}{a_k}\Big) \Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1}$$

converges uniformly on every compact set not containing any of the points $\overline{a_k}$, $k \in \mathbb{N}$.

Proof. Writing

$$\Big(1 - \frac{z}{a_k}\Big)\Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1} = \Big(1 - \frac{z}{\overline{a_k}} + \frac{z}{\overline{a_k}} - \frac{z}{a_k}\Big)\Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1} = 1 + z\Big(\frac{1}{\overline{a_k}} - \frac{1}{a_k}\Big)\Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1},$$

the product converges compactly if and only if the series

$$\sum_{k=1}^{\infty} z \Big(\frac{1}{\overline{a_k}} - \frac{1}{a_k}\Big)\Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1} = \frac{i}{2} \sum_{k=1}^{\infty} z \Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1} \Big(\operatorname{Im} \frac{1}{a_k}\Big)$$

converges absolutely. Since $z\Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1}$ is bounded on every compact set that does not contain any of the points a_k, this is the case if and only if $\sum_{k=1}^{\infty} \left| \operatorname{Im} \frac{1}{a_k} \right| < +\infty$. □

The proof of the following lemma is omitted.

3.3.4 Lemma ([L, Chapter V, Remark to Theorem 2])**.** *Let Ω be an open set containing $\mathbb{C}_+$, and let $f : \Omega \to \mathbb{C}$ be an analytic function of exponential type in $\mathbb{C}_+$ that is bounded on the real axis. Let a_k, $k \in \mathbb{N}$, be the zeros of f in $\mathbb{C}_+$ (by multiplicity). Then $\sum_{k=1}^{\infty} \left| \operatorname{Im} \frac{1}{a_k} \right| < +\infty$.*

3.3.5 Theorem ([L, Chapter VII, Theorem 6])**.** *Let E be a function of class HB. Then E can be represented as*

$$E(z) = cz^m e^{u(z) - i\nu z} \prod_{k=1}^{\infty} \Big(1 - \frac{z}{a_k}\Big) e^{(RP_k)(\frac{z}{a_k})}, \tag{3.9}$$

where $c \in \mathbb{C}$, u is a real entire function, $\nu \geq 0$, and P_k is the k-th Taylor polynomial of $\ln \frac{1}{1-z}$. If, on the other hand, E is the entire function given by formula (3.9), then E belongs to HB.

Proof. We start by decomposing E into the product

$$E(z) = z^m e^{g(z)} \prod_{k=1}^{\infty} \Big(1 - \frac{z}{a_k}\Big) e^{P_k(\frac{z}{a_k})}.$$

Let $\omega(z) := \frac{E^\#(z)}{E(z)}$, which is bounded by 1 in the upper half plane since E belongs to HB. By Lemma 3.3.4, the zeros a_k, $k \in \mathbb{N}$, of E satisfy

$$\sum_{k=1}^{\infty} \Big| \operatorname{Im} \frac{1}{a_k} \Big| = \sum_{k=1}^{\infty} \Big| \operatorname{Im} \frac{1}{\overline{a_k}} \Big| < +\infty,$$

since the zeros of ω coincide with the zeros of $E^\#$, which are given by $\overline{a_k}$, $k \in \mathbb{N}$. Lemma 3.3.3 gives convergence of

$$\chi(z) := \prod_{k=1}^{\infty} \Big(1 - \frac{z}{a_k}\Big)\Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1}.$$

Setting

$$\chi_n(z) := \prod_{k=1}^{n} \Big(1 - \frac{z}{a_k}\Big)\Big(1 - \frac{z}{\overline{a_k}}\Big)^{-1}$$

and

$$E_n(z) = z^m e^{g(z)} \prod_{k=1}^{n} \Big(1 - \frac{z}{a_k}\Big) e^{P_k(\frac{z}{a_k})}$$

as well as $\omega_n := \frac{E_n^\#}{E_n}$, we obtain

$$\exp\Big(-2i\Big[(Ig)(z) + \sum_{k=1}^{n} (IP_k)(\frac{z}{a_k})\Big]\Big) = \omega_n(z)\chi_n(z).$$

Since the right side converges uniformly on any compact subset of $\mathbb{C}$, the same goes for the series $\sum_{k=1}^{\infty} (IP_k)(\frac{z}{a_k})$. Therefore,

$$v(z) := (Ig)(z) + \sum_{k=1}^{\infty} (IP_k)(\frac{z}{a_k})$$

is well defined and satisfies $e^{-2iv(z)} = \lim_{n\to\infty}[\omega_n(z)\chi_n(z)] = \omega(z)\chi(z)$. Observe that

$$|\omega(x)\chi_n(x)| = |\omega(x)| \leq 1, \quad x \in \mathbb{R}.$$

Moreover,

$$\Big(1-\frac{z}{a_k}\Big)\Big(1-\frac{z}{\overline{a_k}}\Big)^{-1} = \frac{\overline{a_k}}{a_k}\cdot\frac{a_k-z}{\overline{a_k}-z} \xrightarrow{|z|\to\infty} \frac{\overline{a_k}}{a_k}.$$

We conclude that χ_n is bounded in $\mathbb{C}_+$. Hence, it follows from Theorem 1.3.11 that $|\omega(z)\chi_n(z)| \leq 1$ everywhere in $\mathbb{C}_+$. Passing to the limit, we have

$$|e^{-2iv(z)}| = |\omega(z)\chi(z)| \leq 1, \quad z \in \mathbb{C}_+.$$

Therefore, $\operatorname{Im} v(z) \leq 0$ for every $z \in \mathbb{C}_+$, i.e., $-v$ is Herglotz. Since v is entire and real, Theorem 2.4.3 implies that $v(z) = -(\nu z + \delta)$, where $\nu \geq 0$ and $\delta \in \mathbb{R}$. Thus,

$$E(z) = z^m e^{u(z)-i(\nu z+\delta)} \prod_{k=1}^{\infty} \Big(1-\frac{z}{a_k}\Big) e^{(RP_k)(\frac{z}{a_k})}.$$

This proves representation (3.9) if we put $c := e^{-i\delta}$.

For the other implication of the proof, consider a function E with a representation of the form (3.9). Then

$$\omega(z) = \frac{E^{\#}(z)}{E(z)} = \frac{\bar{c}}{c\chi(z)} e^{2i\nu z}$$

and thus E belongs to HB. $\square$

Chapter 4

Generalized interlacing property

The integral representation for a scalar Herglotz function given by Theorem A.0.1 has the straightforward matrix version in Theorem 2.4.3. In this chapter we give a matrix analogue of the interlacing property that was studied for the scalar case in Section 3.1.

4.1 The n-interlacing property

In general, matrix-valued Herglotz functions that are meromorphic on $\mathbb{C}$ and real do not satisfy the interlacing property from Section 3.1 (see Definition 3.1.1). Look, for example, at $Q(z) := \operatorname{diag}(q_{11}(z), ..., q_{nn}(z))$, where q_{jj} are scalar meromorphic Herglotz functions. Then the pattern of zeros and poles of[1] $\det Q = q_{11} \cdots q_{nn}$ is simply an overlay of the patterns of zeros and poles of q_{jj}, $j = 1, ..., n$. Hence, the classical interlacing property is lost, but the distribution of zeros and poles of $\det Q$ is still far from arbitrary. We start with a reformulation of the classical interlacing property that allows to be taken to higher dimensions.

4.1.1 Definition. *Let $f \not\equiv 0$ be meromorphic on $\Omega \subseteq \mathbb{C}$ and let $f(z) = \sum_{j=N}^{\infty} a_j(z - z_0)^j$ be its unique Laurent series at $z_0 \in \Omega$, i.e. $N \in \mathbb{Z}$ and $a_n \neq 0$. We define $\theta_f(z_0) := N$. The function*

$$\theta_f : \begin{cases} \Omega & \to & \mathbb{Z} \\ z & \mapsto & \theta_f(z) \end{cases}$$

will be called the divisor function *of f.*

If z is a zero of f, the number $\theta_f(z)$ indicates its multiplicity, and if z is a pole of f, its multiplicity is given by $-\theta_f(z)$. If f has neither zero nor pole at z, then $\theta_f(z) = 0$. Note that for $f \not\equiv 0$, the set of all z with $\theta_f(z) \neq 0$ is discrete because of the Identity Theorem.

4.1.2 Lemma. *Let f be meromorphic on $\Omega \supseteq \mathbb{R}$. Then the real zeros and poles of f are all*

[1] While it is not immediately clear what should be understood by zeros and poles of Q, for a diagonal Herglotz function it is quite reasonable to look at $\det Q$.

simple and interlace, if and only if for every choice of $-\infty < a < b < \infty$ *the inequality*

$$\Big| \sum_{x \in (a,b)} \theta_f(x) \Big| \leq 1. \tag{4.1}$$

holds.

□

This motivates the following definition.

4.1.3 Definition. *Let* $\theta : \mathbb{R} \to \mathbb{Z}$ *have discrete support, and let* $n \in \mathbb{N}$*. Then* θ *is called* n-interlacing*, if for every* $-\infty < a < b < \infty$ *we have*

$$\Big| \sum_{x \in (a,b)} \theta(x) \Big| \leq n. \tag{4.2}$$

If f *is a meromorphic function on some open set* $\Omega \supseteq \mathbb{R}$*, we say that* f *satisfies the* n-interlacing condition *if all zeros and poles of* f *are real and the function* $\theta_f|_{\mathbb{R}}$ *is* n*-interlacing, that is, (4.2) holds.*

Observe that, if $\theta_1, ..., \theta_n$ are 1-interlacing functions, $\theta := \sum_{j=1}^n \theta_j$ is n-interlacing:

$$\Big| \sum_{x \in (a,b)} \theta(x) \Big| = \Big| \sum_{x \in (a,b)} \sum_{j=1}^{n} \theta_j(x) \Big| \leq \sum_{j=1}^{n} \Big| \sum_{x \in (a,b)} \theta_j(x) \Big| \leq n. \tag{4.3}$$

In fact, every n-interlacing function can be written as the sum of n functions that are 1-interlacing, as will be seen in the following theorem, which is elementary but appears to be new.

4.1.4 Theorem. *Let* $\theta : \mathbb{R} \to \mathbb{Z}$ *have discrete support and let* $n \in \mathbb{N}$*. Then* θ *is* n*-interlacing if and only if there exist* 1*-interlacing functions* $\theta_1, ..., \theta_n$ *such that* $\theta = \sum_{j=1}^n \theta_j$*.*

Proof. If $\theta = \sum_{j=1}^n \theta_j$, where θ_j are 1-interlacing, then θ is n-interlacing by (4.3). We show that if θ is n-interlacing, it can be written as the sum of n functions that are 1-interlacing.
Let

$$\Theta(x) = \begin{cases} \sum_{t \in [0,x)} \theta(t), & x \geq 0, \\ -\sum_{t \in (x,0)} \theta(t), & x < 0 \end{cases}$$

Θ is a well-defined step function because of θ having discrete support. We use the notation $\Theta(x-)$ for $\lim_{t \nearrow x} \Theta(t)$ and $\Theta(x+)$ for $\lim_{t \searrow x} \Theta(t)$. For $j \in \mathbb{Z}$, we define

$$\theta_j(x) := \begin{cases} 1 & \text{if } \Theta(x+) > j \geq \Theta(x-), \\ -1 & \text{if } \Theta(x-) > j \geq \Theta(x+), \\ 0 & \text{otherwise} \end{cases}$$

We show that each $\theta_j(x)$ is already 1-interlacing.
Firstly, suppose we are given $x < y$ such that $\theta_j(x) = \theta_j(y) = 1$. Then $\Theta(x-) \leq j < \Theta(x+)$ and $\Theta(y-) \leq j < \Theta(y+)$. In particular,

$$\Theta(x+) > j \geq \Theta(y-).$$

Hence, $t_0 := \inf\{t > x : \Theta(t+) \leq j\}$ is well-defined, and $t_0 \in (x, y)$. Moreover, $\Theta(t_0-) > j \geq \Theta(t_0+)$ can be derived from the fact that Θ is a step function. Therefore, $\theta_j(t_0) = -1$. Analogously, we see that between points $\tilde{x} < \tilde{y}$ with $\theta_j(\tilde{x}) = \theta_j(\tilde{y}) = -1$, there is $\tilde{t_0}$ satisfying $\theta_j(\tilde{t_0}) = 1$. We conclude that θ_j is 1-interlacing.
In the second step, we utilize the fact that θ is n-interlacing. Letting $x < y$, observe that

$$|\Theta(y) - \Theta(x)| = \Big| \sum_{t \in (x,y)} \theta(t) \Big| \leq n.$$

Writing $j_- := \min\{\Theta(x) : x \in \mathbb{R}\}$ and $j_+ := \max\{\Theta(x) : x \in \mathbb{R}\}$, it follows that $j_+ - j_- \leq n$. However, from our definition of θ_j we can see that $\theta_j \equiv 0$ if $j \geq j_+$ or $j < j_-$, which means that all but at most n of the functions θ_j vanish.
Thirdly, the definition of Θ yields

$$\theta(x) = \Theta(x+) - \Theta(x-) = \begin{cases} |\{j \in \mathbb{Z} : \Theta(x+) > j \geq \Theta(x-)\}|, & \Theta(x+) > \Theta(x-), \\ -|\{j \in \mathbb{Z} : \Theta(x-) > j \geq \Theta(x+)\}|, & \Theta(x+) < \Theta(x-), \\ 0, & \Theta(x+) = \Theta(x-) \end{cases}$$
$$= \sum_{j \in \mathbb{Z}} \theta_j(x).$$

We already know that the number of non-vanishing θ_j is at most n. The proof is complete. □

4.2 Necessity of the interlacing property

4.2.1 Lemma. *Let Q be a $n \times n$-matrix-valued Herglotz function. Then there exist scalar Herglotz functions $q_1, ..., q_n$ such that* $\det Q(z) = q_1(z) \cdots q_n(z)$*. If Q is meromorphic on $\mathbb{C}$, so are $q_1, ..., q_n$.*

Proof. The proof proceeds by induction on n. Since the assertion is evident for $n = 1$, only the induction step is to be done. Suppose that the assertion holds for every $n \times n$-matrix-valued Herglotz function, and let Q be a $(n+1) \times (n+1)$-matrix-valued Herglotz function. If $Q(z) \equiv C$ (with self-adjoint C), then there is nothing to be proven. The same goes for the case that $\det Q \equiv 0$. Otherwise, by Lemma 2.1.7, $\det Q(z) \neq 0$ for all $z \in \mathbb{C}_+$. Letting $Q_{(j)}(z)$, $j = 1, ..., n$, be the submatrix of $Q(z)$ obtained by deleting the j-th row and column from Q, we can write

$$-Q(z)^{-1} = \begin{pmatrix} -\frac{\det Q_{(1)}(z)}{\det Q(z)} & * & * & * \\ * & -\frac{\det Q_{(2)}(z)}{\det Q(z)} & * & * \\ * & * & \ddots & * \\ * & * & * & -\frac{\det Q_{(n)}(z)}{\det Q(z)} \end{pmatrix}$$

Because $-Q(z)^{-1}$ is Herglotz, all of its diagonal entries $-\frac{\det Q_{(j)}(z)}{\det Q(z)}$ are scalar Herglotz functions. Because we assumed $Q(z)$ to not be constant, and because of Theorem 2.4.3, (iv), there exists j_0 such that $q(z) := -\frac{\det Q_{(j_0)}(z)}{\det Q(z)}$ is not identically zero. By Lemma 2.1.7, $-q(z)^{-1}$ is also a scalar Herglotz function, which implies the assertion by

$$\det Q(z) = \det Q_{(j_0)}(z)[-q(z)^{-1}] = q_1(z) \cdots q_n(z)[-q(z)^{-1}]. \tag{4.4}$$

□

What does this result mean for zeros and poles of $\det Q$? Of course, a result about interlacing of zeros and poles can only exist in the case where Q is meromorphic on $\mathbb{C}$, and therefore $\det Q(z)$ can be written as a product of scalar Herglotz functions that are meromorphic on $\mathbb{C}$. By Lemma 3.1.2, every scalar Herglotz function satisfies the 1-interlacing condition as in Definition 4.1.3. In analogy to (4.3), this implies that $\det Q$ satisfies the n-interlacing condition.
However, $\det Q$ satisfying the n-interlacing condition is not enough for Q to be Herglotz. Think, for example, of a 2×2 diagonal matrix with diagonal entries r_1, r_2 where $-r_1$ and $-r_2$ are scalar Herglotz functions. Still, it turns out to be sharp in the sense that for every pattern of poles and zeros that satisfy this generalized interlacing pattern, there is a Herglotz function having said pattern of zeros and poles.

4.2.2 Example. *Let θ be n-interlacing. Then there exists a matrix-valued Herglotz function Q such that $\theta(x) = \theta_{\det Q}(x)$, $x \in \mathbb{R}$. Set, for example,*

$$r_j(z) = z^{\theta_j(0)} \prod_{x \in \mathbb{R}\setminus\{0\}} (1 - \frac{z}{x})^{\theta_j(x)}, \quad j = 1, ..., n,$$

where all θ_j's are 1-interlacing and $\theta = \sum_{j=1}^{n} \theta_j$. Let $q_j(z) := \pm r_j(z)$, choosing for every j the sign that makes q_j a scalar Herglotz function. Then $Q(z) := \operatorname{diag}(q_1(z), ..., q_n(z))$ is an $n \times n$-matrix-valued Herglotz function with $\det Q = q_1 \cdots q_n$.

4.2.3 Theorem. *Let $n \in \mathbb{N}$ and let f be meromorphic on $\mathbb{C}$. Then f satisfies the n-interlacing condition if and only if there exist scalar Herglotz functions $q_1, ..., q_n$ and an entire function g such that*

$$f(z) = \exp(g(z)) \prod_{j=1}^{n} q_j(z), \quad z \in \mathbb{C}. \tag{4.5}$$

If, in addition, $f|_{\mathbb{C}_+} \in \mathcal{N}$ and $f^{\#}(z)|_{\mathbb{C}_+} \in \mathcal{N}$, then $g(z) = c - ih_f z$, $z \in \mathbb{C}$, with some constant $c \in \mathbb{C}$.

Proof. By assumption, θ_f is n-interlacing as in Definition 4.1.3. Theorem 4.1.4 provides us with 1-interlacing functions $\theta_1, ..., \theta_n$ such that $\theta_f = \sum_{j=1}^n \theta_n$. Then, for every $j \in \{1, ..., n\}$, zeros and poles of the function

$$r_j(z) = z^{\theta_j(0)} \prod_{x \in \mathbb{R}\setminus\{0\}} (1 - \frac{z}{x})^{\theta_j(x)}$$

are simple and interlace. Note that the product converges just because of this interlacing (compare to the proof of Theorem 3.1.4). So, $\frac{f}{r_1 \cdots r_n}$ is entire and does not have any zeros, and therefore can be written as $\exp \circ \tilde{g}$, with some entire function $\tilde{g}$. Observe that, for each j, either r_j or $-r_j$ is a scalar Herglotz function. Letting $q_j := \pm r_j$ such that q_j is Herglotz for every j, we get that $f(z) = \exp(\tilde{g}(z) + s\pi i) q_1(z) \cdots q_n(z)$, where s is either 0 or 1. Representation (4.5) then holds for $g(z) := \tilde{g}(z) + s\pi i$.
Suppose that, additionally, $f|_{\mathbb{C}_+}$ and $f^{\#}|_{\mathbb{C}_+}$ both belong to $\mathcal{N}$. Then, by Theorem 1.3.7, $\hat{f} := \exp \circ g = \frac{f}{q_1 \cdots q_n}$ can be represented by $\hat{f}(z) = ce^{-ih_f z}\phi(z)$ with some complex constant c. The Blaschke product does not occur because of $\hat{f}$ having no zeros. Similarly, $\hat{f}^{\#}(z) = \bar{c}e^{ih_f z}\phi^{\#}(z)$. Therefore, both ϕ and $\phi^{\#}$ are outer in the upper half plane. In particular, they belong to H_∞, making ϕ a bounded entire function which is therefore constant. □

4.2.4 Remark. Let $n \in \mathbb{N}$ and let f be meromorphic on $\mathbb{C}$ and real. In addition, let $f|_{\mathbb{C}_+} \in \mathcal{N}$ with $h_f = 0$, and suppose that f satisfies the n-interlacing property. Applying Theorem 4.2.3 yields the representation

$$f(z) = c \prod_{j=1}^{n} q_j(z), \quad z \in \mathbb{C}, \tag{4.6}$$

with some real constant c and scalar Herglotz functions q_j that are meromorphic on $\mathbb{C}$ and real. Depending only on whether c is positive or negative, we can assume w.l.o.g. that $f = \prod_{j=1}^n q_j$ or $f = -\prod_{j=1}^n q_j$. However, in regard of Lemma 4.2.1, we are only interested in the first case, i.e. $f = \prod_{j=1}^n q_j$, or rather $c > 0$. Unfortunately, there seems to be no straightforward condition that determines the sign of c.

The relevancy of the next statement is that the functions $q_1, ..., q_n$ can be chosen to be meromorphic.

4.2.5 Theorem. *Let f be meromorphic on $\mathbb{C}$ and real. Assume that f is not constant. For $n \in \mathbb{N}$, the following conditions are equivalent:*

(*i*) *There exist scalar Herglotz functions $q_1, ..., q_n$ that are meromorphic on $\mathbb{C}$ and real, such that $f = \prod_{j=1}^n q_j$;*

(*ii*) *There exists a holomorphic logarithm u of $f|_{\mathbb{C}_+}$ with $\operatorname{Im} u(z) \in (0, n\pi)$ for every $z \in \mathbb{C}_+$.*

Proof. If $f = \prod_{j=1}^n q_j$, then $u := \sum_{j=1}^n \log q_j$ is a logarithm of f with $\operatorname{Im} u(z) \in (0, n\pi)$, $z \in \mathbb{C}_+$. We show that (i) follows from (ii). Observe that f has no zeros or poles in $\mathbb{C}_+$ because of $f|_{\mathbb{C}_+}$ having a holomorphic logarithm. Consider now any finite interval (a, b) and assume that neither a nor b is a pole of f. Set $x_0 := \frac{a+b}{2}$ and $r := \frac{a-b}{2}$. For $\epsilon > 0$ define $\gamma_+^\epsilon(z) := x_0 + re^{it}$ for $t \in [\epsilon, \pi - \epsilon]$ and $\gamma_-^\epsilon(z) := x_0 + re^{it}$ for $t \in [\pi + \epsilon, 2\pi - \epsilon]$. Then

$$\sum_{x \in (a,b)} \theta_f(x) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \left(\int_{\gamma_+^\epsilon} \frac{f'(z)}{f(z)} \, dz + \int_{\gamma_-^\epsilon} \frac{f'(z)}{f(z)} \, dz \right). \tag{4.7}$$

Observe that $f = f^\#$ implies that $u^\#$ is a holomorphic logarithm of $f^\#$ satisfying $\operatorname{Im} g^\#(z) \in (-n\pi, 0)$, $z \in \mathbb{C}_-$. Thus, the modulus of the imaginary parts of both integrals in (4.7) is bounded by $n\pi$. So, $|\sum_{x \in (a,b)} \theta_f(x)| \leq n$. Applying Theorem 4.2.3 yields scalar meromorphic Herglotz function $q_1, ..., q_n$ and an entire function g such that $f(z) = \exp(g(z)) \prod_{j=1}^n q_j(z), \quad z \in \mathbb{C}$. Again, each q_j has a holomorphic logarithm in the upper half plane, which leads to $g(z) = 2\pi i k + u(z) - \sum_{j=1}^n \log q_j(z)$ with some $k \in \mathbb{Z}$. Therefore, $|\operatorname{Im} g(z)|$ is bounded in $\mathbb{C}_+$. g being real implies that $|\operatorname{Im} g(z)|$ is also bounded in $\mathbb{C}_-$ and thus in all of $\mathbb{C}$. By the Liouville Theorem, $g(z) \equiv c$, where c is a real constant. If $c \geq 0$, then $f = (cq_1) \prod_{j=2}^n q_j$, which is the desired representation. This leaves the case where $c < 0$. First, we write $f = -(-cq_1) \prod_{j=2}^n q_j$, where $-cq_1$ is Herglotz. Then the function $\tilde{u}(z) := i\pi + \log(-cq_1) + \sum_{j=2}^n \log q_j$ is another holomorphic logarithm of f and satisfies $\operatorname{Im} \tilde{u}(z) \in (\pi, (n+1)\pi)$ for z in the upper half plane. Observing that $u(z) = 2r\pi i + \tilde{u}(z)$ for some $r \in \mathbb{Z}$, we get that $(2r+1)\pi < \operatorname{Im} u(z) < (2r+1+n)\pi$. At the same time, we know that $0 < \operatorname{Im} u(z) < n\pi$. In total, we have that either $\operatorname{Im} u(z) \in (0, (n-1)\pi)$ (if $r < 0$) or $\operatorname{Im} u(z) \in (\pi, n\pi)$. In either case, from what was proven so far, we conclude that there exist scalar Herglotz functions $r_1, ..., r_{n-1}$ such that either $f = \prod_{j=1}^{n-1} r_j$ or $f = -\prod_{j=1}^{n-1} r_j$. Putting either $\delta(z) \equiv 1$ or $\delta(z) \equiv -1$ leads to $f = \delta(z) \prod_{j=1}^{n-1} r_j$, which proves the theorem because δ is, in both cases, a Herglotz function. □

4.2.6 Corollary. *Let f be meromorphic on $\mathbb{C}$ and real. The following conditions are equivalent:*

(i) *There exist scalar Herglotz functions q_1, q_2 that are meromorphic on $\mathbb{C}$ and real, such that $f = q_1 q_2$;*

(ii) *If f is not constant, then $f(z) \notin [0, +\infty)$ for every $z \in \mathbb{C}_+$.*

□

4.3 Sufficiency of the interlacing property

Corollary 4.2.3 gives some hope in characterizing meromorphic Herglotz functions through the pattern of zeros and poles of its determinant. First, we give a short reminder on some topics of linear algebra.

4.3.1 Definition. *Let $M = (m_{ij}) \in \mathbb{C}^{n\times n}$, and let $k \in \{1, ..., n\}$ together with $1 \le i_1 < ... < i_k \le n$. We define the matrix $M_{(i_1,...,i_k)}$ by*

$$M_{(i_1,...,i_k)} := \begin{pmatrix} m_{i_1 i_1} & m_{i_1 i_2} & \cdots & m_{i_1 i_k} \\ m_{i_2 i_1} & m_{i_2 i_2} & \cdots & m_{i_2 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ m_{i_k i_1} & m_{i_k i_2} & \cdots & m_{i_k i_k} \end{pmatrix}.$$

A matrix X is called a principal submatrix *of M if there exist indices $i_1, ..., i_k$ such that $X = M_{(i_1,...,i_k)}$.*

The following criterion of Sylvester is well-known, but it should not be confused with the principal minor criterion, which can only be used to check definiteness, but not semidefiniteness.

4.3.2 Theorem (Sylvester's criterion)**.** *Let M be a Hermitian matrix. Then M is positive semidefinite, if and only if for each principal submatrix[2] $\tilde{M}$ of M we have $\det \tilde{M} \ge 0$.*

Proof. See ([M, Equations (7.6.9)-(7.6.12)]). □

We are now ready for the main result of this chapter.

4.3.3 Theorem. *Let Q be an $n \times n$-matrix-valued function that is holomorphic on $\mathbb{C} \setminus \mathbb{R}$, meromorphic on $\mathbb{C}$, and real. Suppose that Q is of bounded type, and that for each entry Q_{kl} of Q we have $\limsup_{\tau\to+\infty} \frac{|Q_{kl}(i\tau)|}{\tau} < +\infty$. Then the following statements are equivalent:*

- *(i) Q is Herglotz;*
- *(ii) All residues of Q as well as $\lim_{\tau\to+\infty}\left(-\frac{\operatorname{Im} Q(i\tau)}{\tau}\right)$ are nonpositive matrices, and for every choice of $1 \le i_1 < i_2 \le n$, $f(z) := \det[Q_{(i_1,i_2)}(z)]$ is either constant, or it does not take on nonnegative real values.*
- *(iii) For every $m \in \{1, ..., n\}$ and $1 \le i_1 < ... < i_m \le n$, $f(z) := \det[Q_{(i_1,...,i_m)}(z)]$ satisfies at least one of the following properties:*
  - *a. f can be represented as the product of m scalar Herglotz functions that are meromorphic on $\mathbb{C}$ and real;*
  - *b. If f is not constant, it has a holomorphic logarithm u satisfying $\operatorname{Im} u(z) \in (0, m\pi)$;*
  - *c. f satisfies the m-interlacing property, and if the set of poles of multiplicity m is nonempty, then $\lim_{z\to z_0} \left[(z_0 - z)^m f(z)\right] > 0$ for at least one of those poles z_0,. In addition, $\lim_{\tau\to+\infty} \frac{f(i\tau)}{(i\tau)^m} \ge 0$.*

If (i) – (iii) hold, then in (iii), all of a. – c. hold.

[2]This includes M itself.

Proof. Every principal submatrix of Q is itself a Herglotz function. Applying Lemma 4.2.1 implies that each principal submatrix can be represented as the product of scalar Herglotz functions that are meromorphic on $\mathbb{C}$. Also, residues at poles of Q as well as the limit $\lim_{\tau\to+\infty}\left(-\frac{\operatorname{Im} Q(i\tau)}{\tau}\right)$ are nonpositive because of Corollary 2.4.4. This proves implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$.

$(ii) \Rightarrow (i)$:
It suffices to show that Q can be represented in the form

$$Q(z) = C + Dz + \sum_{j\in\mathbb{Z}} A_j\left(\frac{1}{z_j - z} - \frac{z_j}{1+z_j^2}\right). \tag{4.8}$$

with real z_j. The assertion on the non-positivity of residues and $-D = \lim_{\tau\to+\infty}\left(-\frac{\operatorname{Im} Q(i\tau)}{\tau}\right)$ then proves the implication. This is done by showing that for any $k, l \in \{1, ..., n\}$,

$$q_{kl}(z) = c_{kl} + d_{kl}z + \sum_{j\in\mathbb{Z}} A_j^{kl}\left(\frac{1}{z_j - z} - \frac{z_j}{1+z_j^2}\right).$$

Of course, for $k = l$, we obtain this representation from the fact that every diagonal entry of Q is a 1×1-principal submatrix of Q, and thus is a scalar Herglotz function. Consider the function

$$R_{kl}(z) := \begin{pmatrix} Q_{kk}(z) & Q_{kl}(z) \\ Q_{lk}(z) & Q_{ll}(z) \end{pmatrix}$$

Since R_{kl} is a 2×2-principal submatrix of Q, Corollary 4.2.6 provides scalar Herglotz functions r_1, r_2 satisfying $\det R_{kl}(z) = r_1(z)r_2(z)$. Let $z_{j_0} \in \mathbb{R}$ be a pole of Q. Using the notation from Definition 4.1.1, $2\theta_{Q_{kl}}(z_{j_0}) = \theta_{Q_{kl}Qlk}(z_{j_0})$ because $Q_{lk} = Q_{kl}^{\#}$, and therefore poles and zeros of those functions have the same multiplicities. This leads to

$$\begin{aligned}\theta_{Q_{kl}}(z_{j_0}) &= \frac{1}{2}\theta_{Q_{kl}Q_{lk}}(z_{j_0}) = \frac{1}{2}\theta_{Q_{kk}Q_{ll}-r_1r_2}(z_{j_0}) \leq \\ &\leq \frac{1}{2}\max\{\theta_{Q_{kk}}(z_{j_0}) + \theta_{Q_{ll}}(z_{j_0}), \theta_{r_1}(z_{j_0}) + \theta_{r_2}(z_{j_0})\} \leq 1\end{aligned}$$

because poles of the scalar Herglotz functions Q_{kk}, Q_{ll}, r_1, r_2 are all simple. Since k, l are arbitrary, this shows that all poles of Q are simple.
It is also easily seen that for the residue $A_{j_0}^{(k,l)}$ of R_{kl} at z_{j_0}, we have

$$\begin{aligned}\det A_{j_0}^{(k,l)} &= \lim_{z\to z_0}[(z - z_{j_0})^2 \det R_{kl}(z)] = \\ &= \left(\lim_{z\to z_0}\Big[(z - z_{j_0})r_1(z)\Big]\right)\left(\lim_{z\to z_0}\Big[(z - z_{j_0})r_2(z)\Big]\right) \geq 0\end{aligned}$$

since residues of scalar Herglotz functions are always nonpositive. Writing

$$A_{j_0}^{(k,l)} = \begin{pmatrix} A_{j_0}^{kk} & A_{j_0}^{kl} \\ A_{j_0}^{lk} & A_{j_0}^{ll} \end{pmatrix},$$

we have

$$|A_{j_0}^{kl}| = |A_{j_0}^{kl} A_{j_0}^{lk}|^{\frac{1}{2}} \leq (A_{j_0}^{kk} A_{j_0}^{ll})^{\frac{1}{2}}.$$

Now Corollary 2.4.4 implies

$$\sum_{j\in\mathbb{Z}} \frac{|A_j^{kl}|}{1+z_j^2} \leq \sum_{j\in\mathbb{Z}} \left(\frac{A_j^{kk}}{1+z_j^2}\right)^{\frac{1}{2}} \left(\frac{A_j^{ll}}{1+z_j^2}\right)^{\frac{1}{2}} \leq \left(\sum_{j\in\mathbb{Z}} \frac{A_j^{kk}}{1+z_j^2}\right)^{\frac{1}{2}} \left(\sum_{j\in\mathbb{Z}} \frac{A_j^{ll}}{1+z_j^2}\right)^{\frac{1}{2}} < +\infty.$$

This ensures the convergence of

$$\psi_{kl}(z) := \sum_{j\in\mathbb{Z}} A_j^{kl} \left(\frac{1}{z_j - z} - \frac{z_j}{1+z_j^2}\right) = \sum_{j\in\mathbb{Z}} A_j^{kl} \frac{1+z_j z}{(z_j - z)(1+z_j^2)}$$

on every compact subset of $\mathbb{C}$ not containing any of the points z_j.
Therefore, $g_{kl}(z) := Q_{kl}(z) - \psi_{kl}(z)$ is entire. From the assumption on Q we obtain that g_{kl} is of bounded type, as is $g_{lk} = g_{kl}^{\#}$. Moreover, $h_{g_{kl}} = 0$ because of $\limsup_{\tau\to+\infty} \frac{|Q_{kl}(i\tau)|}{\tau} < +\infty$. We apply Kreĭn's Theorem (Theorem 1.3.9) to conclude that g_{kl} is of exponential type $\tau_{g_{kl}} = 0$. Hence, the entire function $\tilde{g}_{kl}(z) := \frac{g_{kl}(z) - g_{kl}(0)}{z}$ is of exponential type 0 as well. In addition to that, it is bounded on the imaginary axis. By Theorem 1.3.10, $\tilde{g}_{kl}$ is constant. Therefore, $g_{kl}(z) = c_{kl} + d_{kl} z$ with some constants c_{kl}, d_{kl}, which proves representation (4.8).

$(iii) \Rightarrow (i)$:
Step 1. For a meromorphic $m \times m$-matrix-valued function $X(z)$ that has a pole of multiplicity one at z_0, the residue of X at z_0 can be calculated by

$$\operatorname{Res}(X, z_0) = \lim_{z\to z_0} (z - z_0) X(z).$$

In particular,

$$\det \operatorname{Res}(X, z_0) = \lim_{z\to z_0} \det\left((z - z_0)X(z)\right) = \lim_{z\to z_0} (z - z_0)^m \det X(z). \tag{4.9}$$

Step 2. By Theorem 4.2.5, *a.* is equivalent to *b.* If one of those conditions holds, then there exist scalar meromorphic Herglotz functions $r_1, ..., r_m$ such that $f(z) = \det Q_{(i_1,...,i_m)}(z) = r_1(z) \cdots r_m(z)$. At any pole z_0 of f we have

$$\lim_{z\to z_j} \left[(z_j - z)^m f(z)\right] = \left[\lim_{z\to z_0} (z_0 - z) r_1(z)\right] \cdots \left[\lim_{z\to z_0} (z_0 - z) r_m(z)\right] \geq 0. \tag{4.10}$$

Condition *c.* is, by Theorem 4.2.3, only equivalent to f or $-f$ being a product of m Herglotz functions. However, if f does not have a pole of multiplicity m at z_0, then $\lim_{z\to z_j}\left[(z_j - z)^m f(z)\right] = 0$. Otherwise, if z_0 is a pole of multiplicity m of f, the assertion guarantees that $-f$ is not a product of m Herglotz functions, which means that f is. So, just as in (4.10), $\lim_{z\to z_j}\left[(z_j - z)^m f(z)\right] > 0$. In both cases, this limit is nonnegative if condition *c.* holds. Hence, representation 4.8 is obtained in the same way as for the implication $(ii) \Rightarrow (i)$. Only negativity of the residues has to be checked.

Step 3. We prove that $A_j \geq 0$ as well as $D \geq 0$, which suffices for Q to be Herglotz. Let $-A_j$ be the residue of Q at the pole z_j. By the Sylvester Criterion, Theorem 4.3.2, in order for A_j to be positive semidefinite, we only need to show that all of its principal subdeterminants (principal minors) are nonnegative. Let $m \in \{1, ..., n\}$ and $1 \leq i_1 < i_2 < ... < i_m \leq n$, and $(A_j)_{(i_1,...,i_m)}$ be the corresponding principal submatrix. So, by (4.9) and (4.10),

$$\det[(A_j)_{(i_1,...,i_m)}] = \lim_{z\to z_j}\Big[(z_j - z)^m \det Q(z)_{(i_1,...,i_m)}\Big] = \quad lim_{z\to z_j}\Big[(z_j - z)^m f(z)\Big] \geq 0.$$

So, indeed, $A_j \geq 0$. It remains to verify that $D \geq 0$ as well. Choose $1 \leq i_1 < ... < i_m \leq n$. Then f satisfies one of $a. - c.$ If it satisfies *c.*, then $\det D_{(i_1,...,i_m)} = \lim_{\tau\to+\infty} \frac{f(i\tau)}{(i\tau)^m} \geq 0$. Otherwise, *a.* is satisfied[3]. Therefore, $\tilde{f}(z) := f(-\frac{1}{z})$ still satisfies *a.*, which leads to $\det D_{(i_1,...,i_m)} = \lim_{\tau\to+\infty} \frac{f(i\tau)}{(i\tau)^m} = \lim_{z\to 0}\Big((-z)^m \tilde{f}(z)\Big) \geq 0$.

□

4.3.4 Example. *In the previous theorem, we considered matrix-valued functions that are meromorphic on $\mathbb{C}$ and real. The theorem does not hold when dropping this assumption. This is shown by the simple counterexample*

$$Q(z) = \begin{cases} \begin{pmatrix} z & 2 \\ z & 1 \end{pmatrix}, & z \in \mathbb{C}_+, \\ \begin{pmatrix} z & z \\ 2 & 1 \end{pmatrix}, & z \in \mathbb{C}_-. \end{cases}$$

Q is real, but not Herglotz, since $\det(\operatorname{Im} Q(z)) = -|\frac{2-\bar{z}}{2i}| < 0$ *for $z \in \mathbb{C}_+$. Nevertheless,* $\det Q(z) = -z$ *is the product of the scalar Herglotz functions $q_1(z) := -1$ and $q_2(z) := z$ which are meromorphic on $\mathbb{C}$ and real. Moreover, diagonal entries of Q are scalar meromorphic Herglotz functions.*
So, in a sense, the subdeterminants of a non-meromorphic Q do not contain enough information about non-diagonal entries of Q.

[3] Remember that *a.* and *b.* are equivalent, as was seen in Theorem 4.2.5.

Chapter 5

Theorem of Hermite-Biehler for matrix-valued entire functions

5.1 The classes LHB_n and RHB_n

The theory layed out in Section 1.2 allows us to prove a matrix-valued version of the Hermite-Biehler theorem. Our first step is to define a matrix analogue to the class HB.

5.1.1 Definition. *Let $E : \mathbb{C} \to \mathbb{C}^{n\times n}$ be entire, and let* $\det E(z) \neq 0$ *for every $z \in \mathbb{C}_+$. Then E belongs to the* left Hermite-Biehler class (LHB_n) *if and only if $s_E(z) := E(z)^{-1}E^{\#}(z)$ belongs to the Schur class $\mathcal{S}_n$. Analogously, E belongs to the* right Hermite-Biehler class (RHB_n) *if and only if $t_E(z) := E^{\#}(z)E(z)^{-1}$ lies in the Schur class $\mathcal{S}_n$.*

Evidently, $\mathrm{LHB}_1 = \mathrm{RHB}_1 = \mathrm{HB}$. Surprisingly, this still holds for every $n > 1$.

5.1.2 Lemma. *Let $n \in \mathbb{N}$. Then E belongs to LHB_n if and only if it belongs to RHB_n. In this case $s_E = t_E$.*

Proof. The function t_E belongs to $\mathcal{S}_n$ if and only if, for every $z \in \mathbb{C}_+$, we have $0 \leq I - t_E(z)^* t_E(z) = I - (E(z)^{-1})^* E(\overline{z}) E(\overline{z})^* E(z)^{-1}$. Multiplying with $E(z)^*$ from the left and with $E(z)$ from the right, this is equivalent to $E(z)^*E(z) - E(\overline{z})E(\overline{z})^* \geq 0$. This inequality also holds for real x, i.e., $R(x) := E(x)^*E(x) - E(x)E(x)^*$ is a positive semidefinite matrix for every real x. At the same time,

$$\operatorname{tr} R(x) = \operatorname{tr}(E(x)^*E(x) - E(x)E(x)^*) = \operatorname{tr}(E(x)^*E(x)) - \operatorname{tr}(E(x)E(x)^*) = 0. \qquad (5.1)$$

Hence, $R(x) = 0$. Now consider the entire function $S(z) := E(z)E^{\#}(z) - E^{\#}(z)E(z)$. For $x \in \mathbb{R}$ we have $S(x) = R(x) = 0$, and the Identity Theorem shows that $E(z)E^{\#}(z) = E^{\#}(z)E(z)$ for every $z \in \mathbb{C}$. This leads to $s_E = t_E$.

If, on the other hand, E belongs to LHB_n, then $\|s_E(z)\| \leq 1$ for all $z \in \mathbb{C}_+$. Thus, $\|s_E(z)^*\| \leq 1$, or, put differently, $I - s_E(z)s_E(z)^* \geq 0$. By multiplication with $E(z)$ from the left and $E(z)^*$ from the right, we obtain $E(z)E(z)^* - E(\overline{z})^*E(\overline{z}) \geq 0$. From this point, we can use an argument analogous to (5.1) to verify that $E(z)$ and $E^{\#}(z)$ commute. □

The latter gives rise to the following definition.

5.1.3 Definition. *The* Hermite-Biehler class (HB_n) *is defined as the set of all entire functions* $E : \mathbb{C} \to \mathbb{C}^{n\times n}$ *such that* $\det E(z) \neq 0$ *for every* $z \in \mathbb{C}_+$*, and* $s(z) := E(z)^{-1}E^{\#}(z) = E^{\#}(z)E(z)^{-1}$ *belongs to the Schur class* $\mathcal{S}_n$*.*

5.2 The Hermite-Biehler Theorem for matrix-valued functions

5.2.1 Theorem. *Let* $E : \mathbb{C} \to \mathbb{C}^{n\times n}$ *be entire, and let* $A(z) := (RE)(z) = \frac{E(z)+E^{\#}(z)}{2}$, $B(z) := (IE)(z) = \frac{E(z)-E^{\#}(z)}{2i}$ *be its real and imaginary part. Then the following conditions are equivalent:*

- *(i)* $E \in HB_n$*, and* $\det B(z) \not\equiv 0$*;*
- *(ii) There is an open, nonempty set* $\Omega \subseteq \mathbb{C}_+$ *such that* $\det E(z) \neq 0$ *for all* $z \in \Omega$*, and*

$$\Lambda(z,w) := i\frac{I - s_E(z)s_E(w)^*}{z-\overline{w}} \tag{5.2}$$

is a positive kernel on Ω*. Additionally,* $\det B(z) \neq 0$ *for all* $z \in \mathbb{C}_+$*;*

- *(iii)* $Q(z) := B(z)^{-1}A(z)$*,* $z \in \mathbb{C}$*, is Herglotz and meromorphic on* $\mathbb{C}$*, and* $\det B(z) \neq 0$ *for all* $z \in \mathbb{C}_+$*.*

Proof.
$(i) \Rightarrow (ii)$:
Except for invertibility of B, this was done in Theorem 1.2.5. However, Corollary 1.2.4 provides us with the fact that $2iE(z)^{-1}B(z) = I - s(z)$ is invertible at every point. Hence, $B(z)$ is also invertible everywhere in the upper half plane.

$(ii) \Rightarrow (iii)$:
We put

$$K(z,w) := \frac{Q(z) - Q(w)^*}{z - \overline{w}}$$

and show that K is also a positive kernel on Ω. Indeed, looking at

$$[B(z)^{-1}E(z)]\Lambda(z,w)[B(w)^{-1}E(w)]^* = B(z)^{-1}i\frac{E(z)E(w)^* - E^{\#}(z)\left(E^{\#}(w)\right)^*}{z-\overline{w}}\left(B(w)^{-1}\right)^*$$
$$= 2\frac{B(z)^{-1}A(z) - (B(z)^{-1}A(z))^*}{z-\overline{w}} = 2\frac{Q(z) - Q(w)^*}{z-\overline{w}} = 2K(z,w).$$

By Lemma 1.1.3, K is positive. Moreover, Lemma 5.1.2 states that, for every z, $E(z)$ and $E^{\#}(z)$ commute. In particular, $A(z)$ and $B(z)$ commute. Therefore,

$$Q(\overline{z}) = B(\overline{z})^{-1}A(\overline{z}) = A(\overline{z})B(\overline{z})^{-1} = A(z)^*\left(B(z)^{-1}\right)^* = Q(z)^*.$$

Hence, Q is real and meromorphic on $\mathbb{C}$. The assertion now follows from Theorem 2.3.2 and the Identity Theorem.

$(iii) \Rightarrow (i)$:
Invertibility of $E(z)$ follows from $E(z) = B(z)(Q(z) + iI)$. Repeating the calculation from the last step in the inverse order, we get

$$I - s(z)s(z)^* = \frac{1}{4}[E(z)^{-1}B(z)] \operatorname{Im} Q(z)[E(z)^{-1}B(z)]^* \geq 0.$$

So, $\|s(z)\| = \|s(z)^*\| \leq 1$ for every $z \in \mathbb{C}_+$, which proves $s \in \mathcal{S}_n$. □

Theorem 5.2.1 can be combined nicely with Theorem 4.3.3 to obtain the following matrix-valued version of the Hermite-Biehler Theorem.

5.2.2 Corollary (Hermite-Biehler, Version 1)**.** *Let $E : \mathbb{C} \to \mathbb{C}^{n\times n}$ be entire, and let $E|_{\mathbb{C}_+}$ belong to $\mathcal{N}^{n\times n}$. Let $A := RE$, $B := IE$ be the real and imaginary part of E. Set $Q(z) := B(z)^{-1}A(z)$. Then the following statements are equivalent:*

(i) *$E \in HB_n$, and $\det B(z) \not\equiv 0$;*

(ii) *There is an open, nonempty set $\Omega \subseteq \mathbb{C}_+$ such that $\det E(z) \neq 0$ for all $z \in \Omega$, and*

$$\Lambda(z, w) := i\frac{I - s_E(z)s_E(w)^*}{z - \overline{w}} \tag{5.3}$$

is a positive kernel on Ω. Moreover, $\det B(z) \neq 0$ for all $z \in \mathbb{C}_+$;

(iii) *All of the following conditions hold:*

- $\det B(z) \neq 0$, $z \in \mathbb{C}_+$;
- *for each entry Q_{kl} of Q we have $\limsup_{\tau\to+\infty} \frac{|Q_{kl}(i\tau)|}{\tau} < +\infty$;*
- *for every $m \in \{1, ..., n\}$ and $1 \leq i_1 < ... < i_m \leq n$, $f(z) := \det[Q_{(i_1,...,i_m)}(z)]$ satisfies the m-interlacing condition. If the set of poles of multiplicity m is nonempty, then $\lim_{z\to z_0} [(z_0 - z)^m f(z)] > 0$ for at least one of those poles z_0;*
- $\lim_{\tau\to+\infty} \frac{f(i\tau)}{(i\tau)^m} \geq 0$.

□

5.2.3 Remark. One might ask whether it is really necessary to impose some kind of invertibility for B in each of the equivalent statements. The answer seems to be that it depends on the formulation that you want to achieve - for the latter, invertibility of B is indeed important. However, in some cases, it follows automatically from the properties of E. Notably, we know from Corollary 1.2.3 that for $E \in \mathrm{HB}_n$ we have $I - s_E(z_0)^*s_E(z_0) > 0$ at one point z_0 if and only if $I - s_E(z)^*s_E(z) > 0$ at every $z \in \mathbb{C}_+$. Because of invertibility of $E(z)$, it follows that

$E(z)E(z)^* - E^\#(z)\big(E^\#(z)\big)^* > 0$ at every z in the upper half plane. Now,

$$\begin{aligned} 0 &< E(z)E(z)^* - E^\#(z)\big(E^\#(z)\big)^* \\ &= \Big(A(z) + iB(z)\Big)\Big(A(z)^* - iB(z)^*\Big) - \Big(A(z) - iB(z)\Big)\Big(A(z)^* + iB(z)^*\Big) \\ &= 2i\Big(B(z)A(z)^* - A(z)B(z)^*\Big). \end{aligned} \tag{5.4}$$

Assume that there is $z_0 \in \mathbb{C}_+$ and $\xi_0 \in \mathbb{C}^n$ such that $B(z_0)^*\xi_0 = 0$. Then we run into the contradiction

$$0 \neq \xi_0^* B(z)A(z)^*\xi_0 - \xi_0^* A(z)B(z)^*\xi_0 = 0,$$

which means that $B(z)$ is invertible for every $z \in \mathbb{C}_+$.

5.2.4 Theorem (Hermite-Biehler, Version 2)**.** *Let $E : \mathbb{C} \to \mathbb{C}^{n\times n}$ be entire such that $E|_{\mathbb{C}_+} \in \mathcal{N}^{n\times n}$, and let $A := RE$, $B := IE$ be its real and imaginary part. Assume* $\det B(z) \not\equiv 0$, *and set $Q(z) := B(z)^{-1}A(z)$. Then the following statements are equivalent:*

(*i*) *There is an open and nonempty set $\Omega \subseteq \mathbb{C}_+$ such that $L(z,w) := i\frac{E(z)E(w)^* - E^\#(z)\left(E^\#(w)\right)^*}{z-\overline{w}}$ is a positive kernel on Ω;*

(*ii*) *The conditions below are all satisfied:*

- *Q has no poles in $\mathbb{C}\setminus\mathbb{R}$;*
- *for each entry Q_{kl} of Q we have $\limsup_{\tau\to+\infty}\frac{|Q_{kl}(i\tau)|}{\tau} < +\infty$;*
- *for every $m \in \{1,...,n\}$ and $1 \le i_1 < ... < i_m \le n$, $f(z) := \det[Q_{(i_1,...,i_m)}(z)]$ satisfies the m-interlacing condition. If the set of poles of multiplicity m is nonempty, then $\lim_{z\to z_0}\left[(z_0-z)^m f(z)\right] > 0$ for at least one of those poles z_0;*
- $\lim_{\tau\to+\infty}\frac{f(i\tau)}{(i\tau)^m} \ge 0$.

Proof. Set $M_B := \{z \in \mathbb{C} : \det B(z) \neq 0$. By the identity theorem, $\mathbb{C}\setminus M_B$ is discrete, and therefore $\Omega_B := \Omega \cap M_B$ is open and nonempty. (5.4) leads to

$$K(z,w) := \frac{B(z)^{-1}A(z) - \Big(B(w)^{-1}A(w)\Big)^*}{z-\overline{w}} = \frac{1}{2}B(z)^{-1}L(z,w)(B(w)^{-1})^*,$$

implying that K is a positive kernel on Ω_B. By Theorem 2.3.2, the function $B(z)^{-1}A(z)$ defined on Ω_B can be continued to a Herglotz function Q. The Identity Theorem implies $Q(z) = B(z)^{-1}A(z)$ for every $z \in M_B$. Analogous as in Lemma 5.1.2, one can show that $E(z)$ always commutes with $E^\#(z)$, and therefore $A(z)$ commutes with $B(z)$ at every z. By defining $Q(x) := B(x)^{-1}A(x) = A(x)B(x)^{-1}$ for real $x \in M_B$, Q becomes meromorphic on $\mathbb{C}$. We can now apply Theorem 4.3.3, proving the assertion. The other direction follows from inverting the previous steps. Note that in Theorem 4.3.3, Q has to be holomorphic on $\mathbb{C}\setminus\mathbb{R}$. This is ensured by the requirement that Q has no nonreal poles (at any point where $B(z)$ is not invertible). □

Appendix A

Direct proof of the Herglotz integral representation

In order to derive Theorem 2.4.3, we branched out to the theory of linear relations. This was necessary to prove Theorem 2.3.2. For readers that are interested only in the integral representation, we present the standard proof of the integral representation for scalar Herglotz functions. Once this representation is established for scalar Herglotz functions, it can be generalized to matrix-valued Herglotz functions $Q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{n\times n}$ using the fact that, for every $x \in \mathbb{C}^n$, the function $x^*Q(z)x$ is a scalar Herglotz function. See also [B, Theorem A.4.2].

A.1 Theorem. ([KA], Theorem 5.1); ([KK], Satz 2.3.6)

A function $q : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ is Herglotz if and only if it admits a representation of the form

$$q(z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{1+t^2}\Big)\, d\sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{A.1}$$

where $a, b \in \mathbb{R}$, $b \geq 0$, and μ is a positive Borel measure with $\int_{\mathbb{R}} \frac{1}{1+t^2}\, d\sigma(t) < +\infty$. This representation is unique.

Proof. Checking that formula (A.1) always gives a Herglotz function for all admissible a, b and σ will be left as an exercise to the reader. In the following three steps, we prove that any Herglotz function has a representation of the form (A.1).

Step 1. Let f be holomorphic on some open domain containing the unit disk. For every z in the unit disk, using the Cauchy Integral Theorem and the Cauchy Integral Formula, we find that

$$f(z) = \frac{1}{2\pi i} \int_\gamma \left(\frac{f(\zeta}{\zeta - z} - \frac{f(\zeta)}{\zeta - \frac{1}{z}} \right) d\zeta, \tag{A.2}$$

where $\gamma(t) := e^{it}$, $t \in [0, 2\pi]$. A straightforward calculation shows that, for $|\zeta| = 1$,

$$\frac{1}{\zeta}\,\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right) = \frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{1}{z}}.$$

Inserting this into (A.2) yields

$$f(z) = \frac{1}{2\pi i}\int_\gamma \operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right)\frac{f(z)}{\zeta}\,d\zeta = \frac{1}{2\pi}\int_{[0,2\pi]} \operatorname{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) f(e^{it})\,d\lambda(t).$$

Now look at the function

$$k(z) := \frac{1}{2\pi}\int_{[0,2\pi]} \frac{e^{it}+z}{e^{it}-z}\operatorname{Re} f(e^{it})\,d\lambda(t),$$

which is holomorphic on the unit disk and satisfies $\operatorname{Re} k(z) = \operatorname{Re} f(z)$. Hence, $k(z) = f(z)+ci$ for every z in the unit disk, where $c \in \mathbb{R}$. We observe that $k(0) = \operatorname{Re} f(0)$ and thus $c = -\operatorname{Im} f(0)$. Therefore,

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi}\int_{[0,2\pi]} \frac{e^{it}+z}{e^{it}-z}\operatorname{Re} f(e^{it})\,d\lambda(t). \tag{A.3}$$

Step 2. Let f be holomorphic in the unit disk such that $\operatorname{Re} f(z) \geq 0$ for every z. Define for each $r \in [0,1)$ the positive measure ν_r on $[0,2\pi)$ given by the density

$$\frac{d\nu_r}{d\lambda}(t) := \frac{1}{2\pi}\operatorname{Re} f(re^{it}).$$

Letting $\gamma(t) := e^{it}$ for $t \in [0,2\pi)$, we further define the measure $\mu_r := \nu_r \circ \gamma^{-1}$ on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Using (A.3) for $z \mapsto f(rz)$ yields

$$\int_\mathbb{T} \frac{\zeta+z}{\zeta-z}\,d\mu_r(\zeta) = \frac{1}{2\pi}\int_{[0,2\pi)} \frac{e^{it}+z}{e^{it}-z}\operatorname{Re} f(re^{it})\,dt = f(rz) - i\operatorname{Im} f(0). \tag{A.4}$$

Putting $z = 0$ implies $\|\mu_r\| = \operatorname{Re} f(0)$, and thus we can apply the theorem of Banach-Alaoglu for $r \to 1$. We obtain a subnet μ_{r_j}, $j \in J$, of μ_r which converges to some measure μ with respect to the weak-$*$ topology. In particular, it follows from (A.4) that

$$f(z) = i\operatorname{Im} f(0) + \int_\mathbb{T} \frac{\zeta+z}{\zeta-z}\,d\mu(\zeta) \tag{A.5}$$

for every z in the unit disk.

Step 3. Let $\beta(z) := \frac{z-i}{z+i}$ which takes the upper half plane to the unit disk and the real line to $\mathbb{T}\setminus\{1\}$. Let g be holomorphic on $\mathbb{C}\setminus\mathbb{R}$ such that $\operatorname{Re} g(z) \geq 0$ for every $z \in \mathbb{C}_+$. Setting $f(z) := g(\beta^{-1}(z))$ allows us to use (A.5), leading to

$$f(z) = i\operatorname{Im} f(0) + \int_\mathbb{T} \frac{\zeta+z}{\zeta-z}\,d\mu(\zeta) = i\operatorname{Im} f(0) + \mu(\{1\})\frac{1+z}{1-z} + \int_\mathbb{R} \frac{\beta(t)+z}{\beta(t)-z}\,d(\mu\circ\beta^{-1})(t).$$

Some straightforward calculations yield

$$g(z) = f(\beta(z)) = i\,\mathrm{Im}\, f(0) + \mu(\{1\})\frac{1+\beta(z)}{1-\beta(z)} + \int_{\mathbb{R}} \frac{\beta(t)+\beta(z)}{\beta(t)-\beta(z)}\, d(\mu\circ\beta^{-1})(t) =$$

$$= \frac{1}{i}\Bigg(-\mathrm{Im}\, f(0) + \mu(\{1\})z + \int_{\mathbb{R}} \frac{1+tz}{t-z}\, d(\mu\circ\beta^{-1})(z)\Bigg). \tag{A.6}$$

We now define the measure σ given by

$$\frac{d\sigma}{d(\mu\circ\beta^{-1})}(t) := 1+t^2.$$

Because of $\int_{\mathbb{T}} d\mu < +\infty$, we have $\int_{\mathbb{R}} d(\mu\circ\beta^{-1}) < +\infty$ and therefore $\int_{\mathbb{R}} (1+t^2)^{-1}\, d\sigma(t) < +\infty$. Finally, by (A.6),

$$g(z) = \frac{1}{i}\Bigg(-\mathrm{Im}\, f(0) + \mu(\{1\})z + \int_{\mathbb{R}} \frac{1+tz}{(t-z)(1+t^2)}\, d\sigma(t)\Bigg) =$$

$$= \frac{1}{i}\Bigg(-\mathrm{Im}\, f(0) + \mu(\{1\})z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) d\sigma(t)\Bigg).$$

The integral representation for a given Herglotz function q follows from putting $g(z) := \frac{1}{i}q(z)$, $z \in \mathbb{C}_+$.

Uniqueness of the parameters (a, b, σ) is proven in the same way as in Theorem 2.4.3. $\square$

Bibliography

[AD] Damir Z. Arov, Harry Dym: *J-Contractive Matrix Valued Functions and Related Topics*
Cambridge University Press, 2008.

[B] Jussi Behrndt, Seppo Hassi, Henk de Snoo: *Boundary Value Problems, Weyl Functions, and Differential Operators*
Birkhäuser, 2020.

[D] Louis de Branges: *Hilbert Spaces of Entire Function*
Prentice Hall, Inc., 1968.

[GT] Fritz Gesztesy, Eduard Tsekanovskii: *On Matrix-Valued Herglotz Functions*
Mathematische Nachrichten, 2000, Vol.218, pp.61-138.

[KA] Johannes Kaiser: *Canonical Systems*
Master Thesis. TU Wien, 2018.

[KK] Maximilian Kleinert, Florian Kren: *Ausgewählte Kapitel aus der Operatortheorie*
Lecture notes for a lecture by Michael Kaltenbäck, 2007.

[L] Boris Y. Levin: *Distribution of Zeros of Entire Functions*
American Mathematical Society, 1964. Translation of
Raspredelenie kornei zelykh funktsii, 1956.

[M] Carl D. Meyer: *Matrix Analysis and Applied Linear Algebra*
Society for Industrial and Applied Mathematics, 2000.

[R] Marvin Rosenblum, James Rovnyak: *Topics in Hardy Classes and Univalent Functions*
Birkhäuser, 1994.