DIPLOMARBEIT

Utility Indifference Pricing in Semi-Complete Markets: Large Deviations Effects

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durch

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Kurzfassung

Diese Diplomarbeit untersucht exponentielle Nutzenindifferenzpreise und das optimale Investitionsproblem für nicht replizierbare Eventualforderungen auf einem unvollständigen Markt, unter der Annahme, dass diese asymptotisch verschwindet. Eine tiefe Beziehung zwischen der Theorie großer Abweichungen und optimalen Abnahmemengen für einen Investor mit exponentiellem Nutzen wird vorgestellt. Um asymptotisches Verschwinden der nicht replizierbaren Komponenten zu betrachten, wird das Konzept der "halb-vollständigen" Märkte eingeführt und die erforderlichen Ergebnisse entsprechend diesem Rahmen formuliert. Darüber hinaus wird eine Folge von halb-vollständigen Märkten betrachtet, bei denen die unvollständigen Komponenten für den Markt im Grenzwert verschwinden. Dies führt zur Annahme eines Prinzips der großen Abweichungen für den nicht absicherbaren Teil. In diesem Rahmen werden der Grenzwert des Nutzenindifferenzpreises, sowie optimale Abnahmemengen ermittelt. Entgegen den Erwartungen, wird für den Grenzwert, dessen Markt vollständig ist, der Nutzenindifferenzpreis für unbeschränkte Positionen nicht mit dem Replikationskapital übereinstimmen, sowie es für beschränkte Handelsgrößen der Fall ist. Diese nicht triviale Differenz kann explizit mit Hilfe des Lemmas von Varadhan berechnet werden. Darüber hinaus wird gezeigt, wie große Positionen auf natürliche Weise entstehen, indem nach optimalen Mengen gefragt wird.
Diese Arbeit basiert weitgehend auf dem Artikel INDIFFERENCE PRICING FOR CONTINGENT CLAIMS: LARGE DEVIATIONS EFFECTS von Robertson und Spiliopoulos, 2018.

Abstract

This thesis studies exponential utility indifference prices and the optimal investment problem for non-replicable contingent claims in an incomplete market, which asymptotically vanish. A deep relationship between large deviations and optimal purchase quantities for an investor with exponential utility is presented. In order to consider asymptotic vanishing of the non-replicable components, the concept of semi-complete markets is introduced and the required results are formulated according to this framework. Furthermore, a sequence of semi-complete markets will be considered, therein the incomplete components vanishing for the limiting market, i.e. $n \to \infty$. This leads to an assumed large deviation principle (LDP) for the unhedgeable part. In this setting limiting utility indifference prices and the optimal purchase quantities will be determined. For unbounded positions the limiting utility indifference price will vary from the one for bounded position sizes. This non-trivial difference can be explicitly calculated by using Varadhan's integral lemma. In addition, there will be shown how large positions occur naturally by asking for optimal quantities.

This work is largely based on the paper *INDIFFERENCE PRICING FOR CONTINGENT CLAIMS: LARGE DEVIATIONS EFFECTS* by Robertson and Spiliopoulos, 2018.

Keywords: indifference pricing, large deviations, optimal investment, semi-complete markets, varadhan's integral lemma.

Danksagung

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 17. August 2020

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Contents

1. Introduction

This thesis deals with the connection of some considerable theories. In particular, large deviations theory will be applied to both, utility indifference pricing and optimal positions. Heuristically, the connections between these theories are made by demonstrating the deep relation between large deviations and the optimal investment problem, as it has been done originally by [Robertson and Spiliopoulos, 2018]. Therefore, the effects of large position sizes on the limiting indifference price as well as the natural arising of large positions for optimal purchase quantities in the presence of vanishing hedging errors will be pointed out. As in real financial markets, most instruments are not replicable by trading in the underlying market, the theory of utility indifference pricing became very popular. Moreover, the reason one should care about large investors is very simple - in the last decades there was an exponential growth in the amount of outstanding financial instruments, as can be seen in the annual report [Bank of International Settlements, 2020]. Furthermore, it is natural to assume vanishing hedging errors for unbounded positions. This will lead to the concept of semi-complete markets, where a part of the claim can be hedged perfectly, but the remaining components of it are completely unhedgeable, hence there is an additional source of risk [Becherer, 2003]. The considered instruments should have zero limiting hedging errors, i.e. should be asymptomatically replicable, but, as it will be shown, there is an additional term on the utility indifference price that might be non-trivial. As a result, unbounded purchasing positions will occur naturally. To see why this dependence holds, one may think of a complete market, where one is able to buy a claim for some price that does not equal the unique fair price, then the optimal position will be infinite, [Ílhan, Jonsson, and Sircar, 2005] . Finally, there will be an additional non-zero term in the limiting utility indifference price, added to the limiting replication capital, one would not expect naively and which does not occur for bounded positions.
This master thesis is structured as follows: First, Chapter 2 gives basic definitions and results of mathematical finance following the literature of [Delbaen and Schachermayer, 2006] and [Rheinländer and Sexton, 2011]. Furthermore, an overview on utility indifference pricing, leaning on [Carmona, 2009], and the duality results by [Delbaen, Grandits, et al., 2002] will be presented. Finally, this Chapter also provides the basic definitions and a few major results of large deviations theory. The content refers to the manuscripts of [Dembo and Zeitouni, 1998], [Hollander, 2008] and [Pham, 2010], which also offer a more profound view into theory.
In Chapter 3 the abstract semi-complete market setting will be developed. In particular, the aim is to precisely define the vanishing hedging in an incomplete market with underlying tradable assets. For an important example, one may think of a sequence of risky assets that are theoretically available to trade, but practically it is only possible to trade in the first n assets. As the claims depend on all sources of risk, the market is *semi-complete*. Furthermore, the main results from utility indifference pricing and optimal positions will be

applied to the semi-complete setting. Therefore, the optimal investment problem, the range of arbitrage free prices and an equation to obtain optimal purchase quantities are discussed therein. The chapter follows the associated section from [Robertson and Spiliopoulos, 2018], as well as [Becherer, 2003].

Chapter 4 deals with a detailed example of a large semi-complete market. Therein, the setting of Chapter 3 is embedded in a sequence of semi-complete markets, where the assets follow a geometric Brownian motion and the claim is described as sum of independent components. This model attracts a lot of authors as for example [Bouchard, Elie, and Moreau, 2012] and is consequently well known in literature.

After the mathematical foundations are set, the required examples and definitions are used in Chapter 5. Therein the mentioned connection of large deviations, optimal investment and utility indifference pricing by [Robertson and Spiliopoulos, 2018] is discussed in detail by assuming that in the sequence of semi-complete markets an LDP holds for the unhedgeable components of the claim. Herein, the non-trivial effects on the limiting indifference price for unbounded positions will be identified.

The final chapter includes some sufficient conditions that ensure an LDP for the unhedgeable part holds. Therein, two concrete examples will be given. For each, an LPD will be proven as a first step and then the results from Chapter 5 will be applied to explicitly calculate the quantities of interest. This happens in two various ways that are required for each setting.

The Appendix provides some technical lemmas, as well as various statements, which are used throughout this thesis.

2. Preliminary Definitions and Results

This chapter should give a brief introduction to the main definitions and results required later on in the thesis.
First of all, there are outlined some preliminaries that will be assumed to hold throughout the thesis unless explicitly indicated otherwise. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtrated probability space. Assume a finite time horizon T and $\mathcal{F} = \mathcal{F}_T$.

2.1. Mathematical Finance

If not explicit denoted, this section leans on definitions and results in [Rheinländer and Sexton, 2011] and [Delbaen and Schachermayer, 2006].
Let $S = \left(S^1, \ldots, S^d\right)$ denote a d-dimensional price process of a risky asset, where S_t^i describes the price of the i^{th}-asset at time t. S is imposed to be adapted and a locally bounded semi-martingale. As common, the interest rate is supposed to be zero throughout all of the thesis.

For readers convenience as well as consistent notation and understanding, the basic definitions and results are discussed in the following.

Definition 2.1.1. A probability measure $\mathbb{Q}$ absolutely continuous to $\mathbb{P}$ (written $\mathbb{Q} \ll \mathbb{P}$) is called a martingale measure for S, if S is a $\mathbb{Q}$-local martingale. If $\mathbb{Q}$ is equivalent to $\mathbb{P}$ (written $\mathbb{Q} \sim \mathbb{P}$), it is called an equivalent martingale measure. $\mathcal{M}(S)$ denotes the set of all martingale measures for S and $\mathcal{M}^e(S)$ the set of all equivalent martingale measures for S.

Definition 2.1.2. A strategy $\Delta = \left(\Delta^1, \ldots, \Delta^d\right)$ is a $\mathbb{R}^d$-valued S-integrable process. Δ_t^i denotes the units invested in S^i at time t. The associated value process $X^\Delta = X(x, \Delta)$, with initial capital x and strategy Δ, is given through the stochastic integral process

$$X_\cdot^\Delta = x + \int_0^\cdot \Delta_u dS_u.$$

Definition 2.1.3. A strategy Δ is called admissible if the associated value process X^Δ is a $\mathbb{Q}$-martingale for every equivalent martingale measure $\mathbb{Q}$.

Remark 2.1. The definition of an admissible strategy always depends on the required restrictions, e.g., different market models or optimization problems. In the ordinary definition a strategy is called admissible if the associated value process is a super-martingale. As the upcoming problems will need various restrictions, the definition will be adapted during this work. There will be an additional, less restrictive term of *allowable* strategies in section 2.3 (Definition 2.3.1).

Definition 2.1.4. A $\mathcal{F}_T$-measurable random variable B is named a claim. B is said to be replicable if there exists an admissible strategy Δ and a constant $c \in \mathbb{R}$, such that B could be written as sum of a stochastic integral process with respect to the underlying asset S_t and the constant c, i.e.

$$B = c + \int_0^T \Delta_t dS_t.$$

Definition 2.1.5. A filtrated probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ together with a price process S is called a market. It is said to be complete, if every claim is redundant, meaning that all claims are replicable according to Definition 2.1.4.

Complete markets are an idealization of reality and therefore are only used for theoretical market models. In a complete market every claim could be perfectly hedged by trading in the underlying assets. Theorem 2.1.2 below shows that in this case there is only one fair price for every claim.

Definition 2.1.6. A strategy Δ is an arbitrage opportunity if the associated value process X^Δ satisfies

(i) $X_0^\Delta \leq 0$

(ii) $X_T^\Delta \geq 0 \ \ \mathbb{P} - a.s.$

(iii) $\mathbb{P}\left(X_T^\Delta > 0\right) > 0.$

Theorem 2.1.1. *If $\mathcal{M}^e(S) \neq \emptyset$, then there are no arbitrage opportunities with admissible strategies in the $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, S)$-market .*

Theorem 2.1.1 is a more intuitive formulation of the well known first fundamental theorem of asset pricing [Theorem 8.2.1 Delbaen and Schachermayer, 2006]. To ensure there is no arbitrage in the selected market model, it is commonly assumed that $\mathcal{M}^e(S) \neq \emptyset$.

Definition 2.1.7. If there is at least one $\mathbb{Q} \in \mathcal{M}^e(S)$ the market is arbitrage free and for any contingent claim B

$$\pi(B) := \mathbb{E}^{\mathbb{Q}}[B]$$

is called an arbitrage free fair price of B for every $\mathbb{Q} \in \mathcal{M}^e(S)$.

It was shown by [Delbaen and Schachermayer, 2006, Theorem 2.4.1] that the set of all arbitrage free prices $\{\mathbb{E}^{\mathbb{Q}}[B] \,|\mathbb{Q} \in \mathcal{M}^e(S)\}$ is an open interval $(\underline{\pi}(B), \overline{\pi}(b))$ with

$$\begin{aligned} \underline{\pi}(B) &:= \inf\{\mathbb{E}^{\mathbb{Q}}[B] \,|\mathbb{Q} \in \mathcal{M}^e(S)\}; \\ \overline{\pi}(b) &:= \sup\{\mathbb{E}^{\mathbb{Q}}[B] \,|\mathbb{Q} \in \mathcal{M}^e(S)\}. \end{aligned} \tag{2.1}$$

To study the concept of complete or incomplete markets we need to formulate the second fundamental theorem of asset pricing by [Harrison and Pliska, 1983]

Theorem 2.1.2 (Second fundamental theorem of asset pricing)**.** *The following statements are equivalent:*

(i) The market is complete.

(ii) $\mathcal{M}^e(S)$ *consists of a singleton*

(iii) S *satisfies the predictable representation property from Definition A.2.2 with respect to* $(\mathbb{Q}, \mathbb{F})$*, where* $\mathbb{Q}$ *is the unique element of* $\mathcal{M}^e(S)$*.*

2.2. Modeling Stock Prices

The most popular stochastic model for stock prices is the geometric Brownian motion. Therefore, define

Definition 2.2.1. A stochastic process S_t follows a geometric Brownian motion, if S is a solution to the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW^t, \tag{2.2}$$

where W_t is a Brownian motion and μ and σ are constants.

The constant μ is called the percentage drift of return and σ the percentage volatility. μ models deterministic trends, while σ spreads the stochastic impact. If there is no uncertainty about the stock price, then $\sigma = 0$. Considering a short interval of time, Δt, this yields an expected increase in S of $\mu S \Delta t$. For this reason, μ is called the expected instantaneous rate of return and σ^2 the instantaneous variance of the rate of return [Hull, 2009]. Solving Equation (2.2) for an arbitrary initial value S_0 by applying Itô's Lemma, see [Protter, 2004], yields the solution

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right). \tag{2.3}$$

Hence, it follows that S_t is a log-normally distributed random variable for any value of t. In order to have no arbitrage opportunities, it is assumed that there exists a equivalent local martingale measure. Under this measure, the price of the underlying asset has to be a local martingale. Therefore, the price process S_t with drift should be transformed into such a local martingale under the risk neutral measure. The transformation can be performed effectively using Girsanov's Theorem, see [Protter, 2004, Theorem 35, III]. To remove the drift, the density of the risk neutral measure is defined by (remember we assumed $r \equiv 0$)

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \mathcal{E}(-\theta W)_T, \tag{2.4}$$

with market price of risk $\theta = \mu/\sigma$. Note that $\mathcal{E}(\cdot)$ denotes the stochastic exponential defined in A.2.1.

2.3. Exponential Utility Indifference Pricing

This section introduces the theory of utility indifference pricing, referring to [Carmona, 2009] and [Delbaen, Grandits, et al., 2002]. Theorem 2.1.2 implicates that, in a complete market, the price of every contingent claim B is uniquely determined by $\mathbb{E}^{\mathbb{Q}_0}[B]$, where $\mathbb{Q}_0$ is the sole element of $\mathcal{M}^e(S)$. This value represents the cost of replication for a claim. In an incomplete market, generally, the claims are not replicable and the set of all equivalent martingale measures $\mathcal{M}^e(S)$ is not a singleton. This leads to a range of prices, which do not allow arbitrage. Thus, it needs an approach to select a specific element of $\mathcal{M}^e(S)$ for exact claim pricing. This yields to the theory of indifference pricing.

First, note that a utility function $U(x)$ for the wealth x is defined as a twice continuously differentiable function, which is strictly increasing and strictly concave. Furthermore, to reflect the investors risk aversion, one defines by $r(x) = u''(x)/u'(x)$ the risk aversion parameter. Then, the basic concept of utility indifference pricing is that, for a given utility function U_a with absolute risk aversion $a > 0$ and payoff of a claim B, the utility indifference price p is the solution to the equation

$$U_a(x) = \mathbb{E}\left[U_a\left(x - p + B\right)\right],$$

where the constant x denotes the wealth level. Intuitively, the utility of the current wealth should coincide with the expected utility of buying a claim B for the premium p in addition to the wealth level x. This simple attempt, works in the absence of dynamical trading opportunities and was know for a long time, see [Bernoulli, 1738]. Nowadays, the consideration of dynamic financial markets is indispensable, which results in a more sophisticated problem formulation. To reduce the risk exposure to the terminal liability B, investors can dynamically trade in a risky asset with discounted price process S. Hence, the theory needs to involve trading strategies. Not any desired strategy is admissible and therefore, let $\mathcal{A}$ denote the space of all allowable trading strategies. In particular, define:

Definition 2.3.1. A trading strategy Δ is called allowable, if it is $\mathbb{F}$-predictable, $(\mathbb{P}, \mathbb{F}; S)$-integrable and the associated wealth process is a $(\mathbb{Q}, \mathbb{F})$-super martingale for all equivalent martingale measures for S.

Note that the definition of an allowable strategy is more restrictive than the one of an admissible, this restrictiveness will be needed in the optimization problem below.

Definition 2.3.2. Let B denote the payoff of a claim. For an initial capital $x \in \mathbb{R}$ and position size $q \in \mathbb{R}$ in B, the value function for the investor is given by

$$u(x, q) = \sup_{\Delta \in \mathcal{A}} \mathbb{E}\left[U_a\left(X_T^\Delta + qB\right)\right]; \qquad X_\cdot^\Delta = x + \int_0^\cdot \Delta_u dS_u. \tag{2.5}$$

Heuristically, the function $u(x, q)$ describes the optimal utility of trading in the underlying market, an investor with utility function U_a can achieve, starting with an initial capital x and q units of B.

Definition 2.3.3. For a given initial value $x \in \mathbb{R}$, a number $q \in \mathbb{R}$ denoting the units of B and a given utility function U_a, the indifference price of the claim B is defined as the unique solution $p(x,q)$ to the equation

$$u(x - qp(x,q), q) = u(x,0). \tag{2.6}$$

Since $p(x,q)$ typically does not admit an explicit formula, it is difficult to obtain properties of the indifference price just by using its definition. The famous *six authors paper* [Delbaen, Grandits, et al., 2002] first relates maximising the expected exponential utility from pure investment to minimising an entropy functional over all equivalent martingale measures for S. For this purpose define,

Definition 2.3.4. The relative entropy $H(\mathbb{Q},\mathbb{P})$ of a probability measure $\mathbb{Q}$ with respect to another probability measure $\mathbb{P}$ is defined as

$$H(\mathbb{Q},\mathbb{P}) = \begin{cases} \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] & \text{if } \mathbb{Q} \ll \mathbb{P} \\ +\infty & \text{otherwise.} \end{cases} \tag{2.7}$$

Intuitively, one can imagine the relative entropy as a measure of how close a probability distribution is to another one.
Recall the set $\mathcal{M}(S)$, which denotes all martingale measures for S. We are searching for a measure $\hat{\mathbb{Q}}$ that minimizes functional $H(\mathbb{Q},\mathbb{P})$ in (2.7) over all elements of $\mathcal{M}(S)$. If this certain $\hat{\mathbb{Q}}$ is also an element of $\mathcal{M}^e(S)$ the optimal martingale measure is obtained.

Definition 2.3.5. Denote by $\hat{\mathbb{Q}}$ the minimal entropy martingale measure, which satisfies

$$H(\hat{\mathbb{Q}},\mathbb{P}) = \min_{\mathbb{Q}\in\mathcal{M}(S)} H(\mathbb{Q},\mathbb{P}).$$

In the sequel, those martingale measures which have finite relative entropy are relevant and therefore set

$$\tilde{\mathcal{M}} = \{\mathbb{Q} \in \mathcal{M}(S) \,|H(\mathbb{Q},\mathbb{P}) < \infty\} \tag{2.8}$$

The following theorem ensures the existence and uniqueness of the minimal entropy measure, if there is at least one martingale measure for S absolute continuous to $\mathbb{P}$. For the original theorem and the proof see [Theorem 1, Frittelli, 2000].

Theorem 2.3.1 (Existence and uniqueness of the entropy measure)**.** *The minimal entropy martingale measure $\hat{\mathbb{Q}}$ exists and is unique if $\tilde{\mathcal{M}} \neq \emptyset$.*

Most of the relative entropy's elementary properties were shown by [Csiszar, 1975]. Some of them are reformulated hereafter according to [Frittelli, 2000] and [Grandits and Rheinländer, 2002]. For the related proofs see [Csiszar, 1975].

Lemma 2.3.1. *It holds that,*

(i) If one assumes a $\tilde{\mathbb{Q}} \in \tilde{\mathcal{M}}$, then $\tilde{\mathbb{Q}} = \hat{\mathbb{Q}}$, i.e. $\tilde{\mathbb{Q}}$ equals the minimal entropy martingale measure, if and only if

$$H(\mathbb{Q},\mathbb{P}) \geq H(\mathbb{Q},\tilde{\mathbb{Q}}) + H(\tilde{\mathbb{Q}},P) \qquad \textit{for all } \mathbb{Q} \in \tilde{\mathcal{M}} \tag{2.9}$$

(ii) *If $\mathcal{M}'$ denotes a convex subset of $\mathcal{M}(S)$ and for all $\mathbb{Q}_1 \in \mathcal{M}'$ there exists an $\alpha \in (0,1)$ and $\mathbb{Q}_2 \in \mathcal{M}'$ such that $\hat{\mathbb{Q}} = \alpha\mathbb{Q}_1 + (1-\alpha)\mathbb{Q}_2$ (i.e. $\tilde{\mathbb{Q}}$ is an algebraic inner point of $\mathcal{M}'$), then $\mathcal{M}' \subset \tilde{\mathcal{M}}$ and (2.9) holds with equality for all $\mathbb{Q} \in \mathcal{M}'$.*

As the optimal martingale measure to price a claim needs to be in $\mathcal{M}^e(S)$, one has to ensure that the minimal entropy measure, if it exists, is equivalent to $\mathbb{P}$. In fact, the next result by [Csiszar, 1975] shows this is true, as long as there is at least one equivalent martingale measure for S.

Theorem 2.3.2 (Equivalence of the minimal entropy martingale measure)**.** *Assuming that $\tilde{\mathcal{M}} \cap \mathcal{M}^e(S) \neq \emptyset$, then the minimal entropy martingale measure $\hat{\mathbb{Q}}$ is equivalent to $\mathbb{P}$.*

Proof. As the minimal entropy measure needs to satisfy $H(\hat{\mathbb{Q}}, \mathbb{P}) < \infty$, it follows by the definition of the relative entropy that $\hat{\mathbb{Q}} \ll \mathbb{P}$. At the same time, for some $\mathbb{Q} \in \tilde{\mathcal{M}} \cap \mathcal{M}^e(S)$ Equation (2.9) in Lemma 2.3.1 yields to

$$H(\mathbb{Q}, \hat{\mathbb{Q}}) \leq H(\mathbb{Q}, \mathbb{P}) - H(\hat{\mathbb{Q}}, \mathbb{P}).$$

Since both, $H(\mathbb{Q}, \mathbb{P})$ and $H(\hat{\mathbb{Q}}, \mathbb{P})$, are finite, it holds that $\mathbb{Q} \ll \hat{\mathbb{Q}}$. This immediately follows from the definition of the set $\tilde{\mathcal{M}}$ and the relative entropy. Furthermore, $\mathbb{Q} \sim \mathbb{P}$ and hence $\mathbb{P} \ll \hat{\mathbb{Q}}$. This implies $\mathbb{P} \sim \hat{\mathbb{Q}}$. $\square$

The next theorem is a direct consequence of *Csiszar's result*. Therein, the density of the minimal entropy martingale measure is given. It is an important structural result which states that the logarithm of the density of the minimal entropy martingale measure for S can always be written as sum of a constant and a stochastic integral with respect to S. The result itself and the proof are given in a more abstract setting, see [Grandits and Rheinländer, 2002], so only the conclusion for this thesis is given below.

Theorem 2.3.3 (Density structure for $\hat{\mathbb{Q}}$)**.** *Suppose that $\hat{\mathbb{Q}}$ is the minimal entropy martingale measure and $\tilde{\mathcal{M}} \cap \mathcal{M}^e(S) \neq \emptyset$. Then the probability density can be written as*

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \exp\left(c + \int_0^T \hat{\Delta}_t dS_t\right), \tag{2.10}$$

for a constant $c \in \mathbb{R}$ and a predictable process $\hat{\Delta}$ such that $\int \hat{\Delta} dS$ is a $\hat{\mathbb{Q}}$-martingale.

The reverse direction of Theorem 2.3.3 does not hold in general. This means, not every martingale measure with a density structure like (2.10) is a minimizer of the entropy functional. However, there is criterion for a martingale measure to coincide with the minimal entropy martingale measure. The following proposition is a characterisation of the minimal entropy measure and thus a partial converse to Theorem 2.3.3 originally in [Proposition 3.2, Grandits and Rheinländer, 2002].

Proposition 2.3.1. *Suppose there is a $\mathbb{Q}^* \in \tilde{\mathcal{M}} \cap \mathcal{M}^e(S)$, then $\mathbb{Q}^* = \hat{\mathbb{Q}}$ if and only if*

(i) $\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \exp\left(c + \int_0^T \Delta_t dS_t\right)$ *for some constant $c \in \mathbb{R}$ and an S-integrable Δ and*

(ii) $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \Delta_t dS_t\right] = 0$ *for $\mathbb{Q} = \mathbb{Q}^*, \hat{\mathbb{Q}}$.*

Proof. First, note that due to the condition $\mathbb{Q}^* \in \tilde{\mathcal{M}} \cap \mathcal{M}^e(S)$, it follows that $\tilde{\mathcal{M}} \neq \emptyset$ and hence, by Theorem 2.3.1, the minimal entropy martingale measure $\hat{\mathbb{Q}}$ exists and is unique. Conditions (i) and (ii) are necessary, since Theorem 2.3.3 holds. So, proving they are sufficient is enough. As a direct consequence of the demanded conditions one has

$$H(\mathbb{Q}^*, \mathbb{P}) = \mathbb{E}^{\mathbb{Q}^*}\left[\log \frac{d\mathbb{Q}^*}{d\mathbb{P}}\right] = c + \mathbb{E}^{\mathbb{Q}^*}\left[\int_0^T \Delta_t dS_t\right] = c$$

for the selected $\mathbb{Q}^*$. At the same time, as the relative entropy satisfies $H(\mathbb{Q}, \mathbb{P}) \geq 0$ for all probability measures, see [Ihara, 1993], it follows that

$$\begin{aligned}
H(\tilde{\mathbb{Q}}, \mathbb{P}) = \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\log \frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\right] &= \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\log\left(\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}^*}\frac{d\mathbb{Q}^*}{d\mathbb{P}}\right)\right] \\
&= \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\log \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}^*}\right] + \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\log \frac{d\mathbb{Q}^*}{d\mathbb{P}}\right] \\
&= H(\tilde{\mathbb{Q}}, \mathbb{Q}^*) + \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\log \frac{d\mathbb{Q}^*}{d\mathbb{P}}\right] \\
&\geq c + \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\int_0^T \Delta_t dS_t\right] = c.
\end{aligned}$$

The estimation holds due to condition (i) for the density of $\mathbb{Q}^*$. This indicates

$$H(\tilde{\mathbb{Q}}, \mathbb{P}) \geq H(\mathbb{Q}^*, \mathbb{P}),$$

which, by the uniqueness of $\tilde{\mathbb{Q}}$, sums up to $\tilde{\mathbb{Q}} = \mathbb{Q}^*$ □

2.4. Duality Results

The duality results by [Delbaen, Grandits, et al., 2002] are linking the exponential utility maximisation with minimisation of the entropy function over martingale measures. As mentioned before, these results are important to identify some qualitative properties of the utility indifference price. Therefore, recall the definition of an admissible strategy from 2.1.3. Define by Θ the set of all admissible strategies for S, i.e.,

$$\Theta = \left\{\Delta \in L(S) \,\middle|\, \int \Delta \; dS \text{ is a } \mathbb{Q}\text{-martingale for all } \mathbb{Q} \in \tilde{\mathcal{M}}\right\}. \tag{2.11}$$

The following theorem describes the reduced duality result without a claim B. For proof details of Theorem 2.4.1 and 2.4.2 see [Delbaen, Grandits, et al., 2002].

Theorem 2.4.1 (Entropic duality). *Let $a > 0$ denote the absolute risk aversion associated to the given utility function U_a, then*

$$\inf_{\Delta \in \Theta} \mathbb{E}\left[\exp\left(-a \int_0^T \Delta_t dS_t\right)\right] = \exp\left(-H(\hat{\mathbb{Q}}, \mathbb{P})\right) \tag{2.12}$$

holds and the infimum is attained by $\frac{1}{a}\hat{\Delta} \in \Theta$, where $\hat{\Delta}$ is determined by (2.10).

To make sure these duality also holds with claim B the definition of a new probability measure $\mathbb{P}^a$, which contains a normalizing constant, is required. To reduce the problem to the already approached case without a claim, the normalizing property of $\mathbb{P}^a$ will be necessary. Therefore, there has to exist a measure $\mathbb{Q}^a$ with a density of the form (2.10), that satisfies (2.12) with respect to $\mathbb{P}^a$. By assuming B to be bounded from below, this could be shown for any $\mathbb{P}$-equivalent probability measure of the form $d\mathbb{P}^a = \exp(c_a + qaB)d\mathbb{P}$ with normalising constant $\exp(c_a)$, see [Delbaen, Grandits, et al., 2002]. As in this work B is not assumed to be bounded, select a special measure $\mathbb{P}^a$, that performs in accordance with the problem formulation hereafter. By assuming e^{-qaB} to be $\mathbb{P}$-integrable one can define the measure $\mathbb{P}^a$ by

$$\frac{d\mathbb{P}^a}{d\mathbb{P}} = \frac{e^{-qaB}}{\mathbb{E}[e^{-qaB}]}. \tag{2.13}$$

Since $\mathbb{P}^a \sim \mathbb{P}$, the sets $\mathcal{M}(S)$ and $\mathcal{M}^e(S)$ remain equal by replacing $\mathbb{P}$ through $\mathbb{P}^a$, although the densities are different. To see that replacing $\mathbb{P}$ by $\mathbb{P}^a$ is also possible in the definition of $\tilde{\mathcal{M}}$ is less trivial. It was shown by [Becherer, 2003] that assuming

$$\mathbb{E}\left[e^{(-a+\epsilon)qB}\right] < \infty \quad \text{and} \quad \mathbb{E}\left[e^{-\epsilon qB}\right] < \infty \quad \text{for some } \epsilon > 0 \tag{2.14}$$

implies that B is in $L^1(\mathbb{Q})$ for every $\mathbb{Q} \in \tilde{\mathcal{M}}$ and that $\tilde{\mathcal{M}}$ coincides with respect to $\mathbb{P}$ and $\mathbb{P}^a$. Thus, with the convention that $\mathbb{P}$ corresponds to $a = 0$, the space Θ of admissible strategies does not depend on a. By assuming $\tilde{\mathcal{M}} \cap \mathcal{M}^e(S) \neq \emptyset$, Theorem 2.3.1 ensures the existence of an unique element $\mathbb{Q}^a$ of $\tilde{\mathcal{M}}$ that minimizes $H(\mathbb{Q}, \mathbb{P}^a)$ over all $\mathbb{Q} \in \tilde{\mathcal{M}}$. Moreover, Theorem 2.3.3 ensures that the density of $\mathbb{Q}^a$ can be written as

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}} = \exp\left(-a\left(c^B + \int_0^T \Delta_t^B \, dS_t + qB\right)\right)$$

for some constant $c^B \in \mathbb{R}$ and $\Delta^B \in \Theta$ such that $\int \Delta^B dS$ is a $\mathbb{Q}^a$-martingale. Thus, by changing the measure from $\mathbb{P}$ to $\mathbb{P}^a$ and the fact that (2.12) holds with respect to $\mathbb{P}^a$, one receives the entropic duality with claim.

Theorem 2.4.2 (Entropic duality with claim)**.** *For a claim B, that satisfies (2.14), we receive*

$$\begin{aligned} &\inf_{\Delta \in \Theta} \mathbb{E}\left[\exp\left(-a\left(\int_0^T \Delta_t \, dS_t + qB\right)\right)\right] \\ &= \exp\left(-\inf_{\mathbb{Q} \in \tilde{\mathcal{M}}}\left(H(\mathbb{Q}, \mathbb{P}) + a\mathbb{E}^{\mathbb{Q}}[qB]\right)\right). \end{aligned} \tag{2.15}$$

The infima are attained at $\mathbb{Q}^a \in \tilde{\mathcal{M}} \cap \mathcal{M}^e$ and $-\frac{1}{a}\Delta^B \in \Theta$, determined by the density of $\mathbb{Q}^a$.

This duality result yields to a closed form of the utility indifference price $p(x, q)$ depending on the selected utility function U_a. Thus, set $U_a(x) = -\frac{1}{a}e^{-ax}$. Now, recall the value function $u(x,q)$ from Definition 2.3.2 and the balance equation (2.6) that defines $p(x,q)$. One can rewrite $u(x,q)$ with the results from above and obtains

$$u(x,q) = -\frac{1}{a}\exp(-ax)\exp\left(-\inf_{\mathbb{Q}\in\tilde{\mathcal{M}}}\left(H(\mathbb{Q},\mathbb{P}) + a\mathbb{E}^{\mathbb{Q}}[qB]\right)\right). \tag{2.16}$$

In view of that, it follows

$$\begin{aligned} u(0,0) &= u(-qp(q),q) \\ -\frac{1}{a}\exp(-H(\hat{\mathbb{Q}},P)) &= \sup_{\Delta\in\Theta}\mathbb{E}\left[\exp\left(-a\left(X_T^\Delta + q\left(B-p(q)\right)\right)\right)\right]; \\ &= -\frac{1}{a}\exp\left(-\inf_{\mathbb{Q}\in\tilde{\mathcal{M}}}\left(H(\mathbb{Q},\mathbb{P}) + a\mathbb{E}^{\mathbb{Q}}\left[q\left(B-p(q)\right)\right]\right)\right) \end{aligned}$$

and hence

$$p(q) = \inf_{\mathbb{Q}\in\tilde{\mathcal{M}}}\mathbb{E}^{\mathbb{Q}}[B] + \frac{1}{aq}\left(H(\mathbb{Q},\mathbb{P}) - H(\hat{\mathbb{Q}},\mathbb{P})\right). \tag{2.17}$$

This closed form allows a more specific analysis of the utility indifference prices' properties. The most obvious one is that for a replicable claim B with $B = c + \int_0^T \Delta_t dS_t$ it follows that, if Δ_t is an allowable strategy, the utility indifference price coincides with the fair price of the claim, in that case c. Furthermore, there were many properties shown by [Frittelli, 2000] and [Becherer, 2003]. The one mentioned in the next lemma will be useful in the upcoming proofs.

Lemma 2.4.1. *The utility indifference price $p(q)$ for q units of the claim B is non-increasing in q, i.e. $q \mapsto p(q)$ is non-increasing.*

Proof. For $\epsilon > 0$ the closed form of $p(q)$ (2.17) for the quantity $q+\epsilon$ yields

$$p(q+\epsilon) = \inf_{\mathbb{Q}\in\tilde{\mathcal{M}}}\mathbb{E}^{\mathbb{Q}}[B] + \frac{1}{a(q+\epsilon)}\left(H(\mathbb{Q},\mathbb{P}) - H(\hat{\mathbb{Q}},\mathbb{P})\right)$$

Now, denote by $\tilde{\mathbb{Q}}$ the specific element of $\tilde{\mathcal{M}}$ achieving the infimum of the problem (2.17) for $p(q)$. Then the indifference price for $q+\epsilon$ units could be estimated by

$$\begin{aligned} p(q+\epsilon) &\le \mathbb{E}^{\tilde{\mathbb{Q}}}[B] + \frac{1}{a(q+\epsilon)}\left(H(\tilde{\mathbb{Q}},\mathbb{P}) - H(\hat{\mathbb{Q}},\mathbb{P})\right) \\ &= \mathbb{E}^{\tilde{\mathbb{Q}}}[B] + \frac{1}{aq}\left(H(\tilde{\mathbb{Q}},\mathbb{P}) - H(\hat{\hat{\mathbb{Q}}},\mathbb{P})\right) - \frac{1}{aq}\left(H(\tilde{\mathbb{Q}},\mathbb{P}) - H(\hat{\hat{\mathbb{Q}}},\mathbb{P})\right) \\ &\quad + \frac{1}{a(q+\epsilon)}\left(H(\tilde{\mathbb{Q}},\mathbb{P}) - H(\hat{\mathbb{Q}},\mathbb{P})\right) \\ &= p(q) - \left(\frac{1}{aq} - \frac{1}{a(q+\epsilon)}\right)\left(H(\tilde{\mathbb{Q}},\mathbb{P}) - H(\hat{\mathbb{Q}},\mathbb{P})\right) \\ &\le p(q), \end{aligned}$$

where the last inequality follows observing that $\hat{\mathbb{Q}}$ is the minimal entropy measure, hence $H(\tilde{\mathbb{Q}},\mathbb{P}) - H(\hat{\mathbb{Q}},\mathbb{P}) > 0$. □

2.5. Large Deviations Theory

The theory of large deviations is a sector of probability theory which deals with the asymptotic estimates of the probabilities of very rare events. In many stochastic models, the law of large numbers that describes the "typical" behaviour, holds, meaning that the mean values of certain random variables converging against a deterministic limit value. The probabilities for large deviations from this typical behaviour are often very small and decrease exponentially. The precise estimation of the probabilities for such large deviations is one of the most important modern tools of probability theory [Dembo and Zeitouni, 1998]. In this process, connections between stochastic questions and variation problems arise naturally. The idea is to change the probability measure to a measure under which the concerned rare event is no longer rare. The mentioned definitions and results, that will be used in the sequel mainly follow the ones from [Dembo and Zeitouni, 1998], [Hollander, 2008] and [Pham, 2010].

At first, a concrete definition of a large deviation principle (LDP) is given below. The LDP characterizes the limiting behaviour for $\epsilon \to 0$ of a family of probability measures on a topological space in terms of a rate function. Throughout this section, let S be a Polish space with Borel σ-algebra $\mathcal{B}(S)$.

Definition 2.5.1. A rate function $I : S \to [0, \infty]$ is lower semicontinuous, in the sense that for all $s \geq 0$ the level set $\Phi_I(s) = \{x \in S : I(x) \leq s\}$ is a closed subset of S. The rate function is said to be good if all of its level sets $\Phi_I(s)$ are compact subsets of S. The effective domain of I, denoted by $\mathcal{D}_I$, is the set of points in S of finite rate, i.e., $\mathcal{D}_I = \{s \in S : I(s) < \infty\}$.

Remark 2.2. A mapping I on the Polish space S is called lower semicontinuous if and only if, for every $x \in S$ and a sequence $x_n \to x$, $\lim_{x_n \to x} I(x_n) \geq I(x)$ holds. Hence, a good rate function achieves its infimum over every closed set.

In the common literature, the LDP is usually defined as property of a sequence of probability measures. As the work in this thesis will need a definition for a sequence of random variables, this one will be given.

Definition 2.5.2. For the Polish space S with Borel σ-algebra $\mathcal{B}(S)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a collection $(\xi_n)_{n\in\mathbb{N}}$ from Ω to S satisfies a large deviation principle (LDP) with good rate function $I : S \to [0, \infty]$ and scaling r_n if $r_n \to \infty$ and

(i) For every $s \geq 0$, the set $\Phi(s) = \{x \in S : I(x) \leq s\}$ is a compact subset of S, hence I is lower semicontinuous.

(ii) For every open subset $G \subset S$ it holds that

$$\liminf_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[\xi_n \in G\right]\right) \geq -\inf_{s\in G} I(s) \tag{2.18}$$

(iii) For every closed subset $F \subset S$ it holds that

$$\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[\xi_n \in G\right]\right) \leq -\inf_{s\in F} I(s) \tag{2.19}$$

Note, that a collection of random variables is said to satisfy a LDP if its laws meet the conditions of definition 2.5.2.

In the following an alternative characterization of a large deviation principle by [Dembo and Zeitouni, 1998] is given. Denote by $\Phi_I(s)$ the level set of the rate function I. Since (ii) trivially holds if $\inf_{s\in G} I(s) = \infty$ and (iii) if $\inf_{s\in F} I(s) = 0$, it is easy to see, that these two conditions are equivalent to the following

(ii*) For every $x \in \mathcal{D}_I$ and each measurable set Γ with $x \in \Gamma^o$

$$\liminf_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[\xi_n \in \Gamma\right]\right) \geq -I(x) \tag{2.20}$$

(iii*) For every $s \in (0, \infty)$ and each measurable set Γ with $\bar{\Gamma} \subset \Phi_I(s)^c$

$$\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[\xi_n \in \Gamma\right]\right) \leq -s. \tag{2.21}$$

As proving the upper bound (iii) often becomes a challenge, it is common to first proof it for compact sets only and then proof the *exponential tightness*, as defined below. First, define the weak variant of a principle of large deviations, which drops the compactness of the level sets, (but not the closure) and requires the upper bound in (iii) only for compact sets.

Definition 2.5.3. Assume that all compact subsets of S are also in $\mathcal{B}(S)$. If for a collection of random variables $(\xi_n)_{n\in\mathbb{N}}$, condition (2.21) holds for every $s \in (0, \infty)$ and all compact subsets of $\Phi_I(s)^c$, and further, (2.20) holds for all measurable sets, then $(\xi_n)_{n\in\mathbb{N}}$ is said to satisfy a weak LDP with rate function I and rate r_n.

To make a connection to the stronger definition of 2.5.2, the following definition will be indispensable.

Definition 2.5.4. Assume that all compact subsets of S are also in $\mathcal{B}(S)$. A collection of random variables $(\xi_n)_{n\in\mathbb{N}}$ is called exponentially tight on the scale r_n if for each $s < \infty$ there is a compact subset $K_s \subset S$ such that

$$\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[\xi_n \in K_s^c\right]\right) < -s.$$

Having set these two definitions, makes it sometimes easier to prove an LDP, as the following lemma by [Section 1.2, Dembo and Zeitouni, 1998] shows.

Lemma 2.5.1. *Suppose $(\xi_n)_{n\in\mathbb{N}}$ is exponentially tight. Then it follows, that*

- *If (2.21) holds for some $s < \infty$ and all the compact subsets of $\Phi_I(s)^c$, then it also holds for all measurable sets Γ with $\bar{\Gamma} \subset \Phi_I(s)^c$.*

- *If (2.20) holds for all measurable sets, i.e. all open sets, then the function $I(\cdot)$ is a good rate function as in Definition 2.5.1.*

Hence, if a collection of random variables is exponentially tight and satisfies a weak LDP, then the rate function I is good and the LDP according to Definition 2.5.2 holds. Note, [Dembo and Zeitouni, 1998] provides a slightly more general version of Lemma 2.5.1 as well as the proof.

An additional tool to prove the upper bound (2.19) is the following lemma, again by [Section 1.2, Dembo and Zeitouni, 1998].

Lemma 2.5.2. *For a fixed integer M and every $\alpha^i_{r_n} \geq 0$ it follows that*

$$\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\sum_{i=1}^{M} \alpha^i_{r_n}\right) = \max_{i=1,\dots,M} \left\{\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\alpha^i_{r_n}\right)\right\}.$$

The next theorem provides the limiting behaviour of a specific exception. It is an extension of the famous Laplace method, that evaluates the asymptotic of specific integrals on $\mathbb{R}$. This theorem will be a key tool to obtain the results from [Robertson and Spiliopoulos, 2018]. For more details and the proof of Theorem 2.5.1 see [Section 4.3, Dembo and Zeitouni, 1998].

Theorem 2.5.1 (Varadhan's integral lemma)**.** *Let $(\xi_n)_{n\in\mathbb{N}}$ be a collection of random variables satisfying a LDP with good rate function $I : S \to [0, \infty]$. Then for any continuous function $\phi : S \to \mathbb{R}$ fulfilling the tail condition*

$$\lim_{\hat{M}\to\infty} \limsup_{\epsilon\to 0} \epsilon \log\left(\mathbb{E}\left[e^{\phi(\xi_n)/\epsilon} \mathbb{1}_{\{\phi(\xi_n)>\hat{M}\}}\right]\right) = -\infty,$$

or the following moment condition for $\gamma > 1$

$$\limsup_{\epsilon\to 0} \epsilon \log\left(\mathbb{E}\left[e^{\gamma\phi(\xi_n)/\epsilon}\right]\right) < \infty,$$

it holds that

$$\lim_{\epsilon\to 0} \epsilon \log\left(\mathbb{E}\left[e^{\phi(\xi_n)/\epsilon}\right]\right) = \sup_{x\in\mathbb{R}} \left(\phi(x) - I(x)\right).$$

In Varadhan's integral lemma above an LDP for $(\xi_n)_{n\in\mathbb{N}}$ was supposed to hold, in the sequel some conditions for an LDP to hold will be given. A well known theorem with such a statement is Cramér's theorem about the large deviations associated with the empirical mean of independent and identical distributed random variables taking values in a finite set. As this thesis will need a more general result, the expanded version of Cramér's theorem, namely the Gärtner-Ellis theorem, for random sequences that have some moderate dependence, is given below. It was originally proofed by [Gärtner, 1977] and [Ellis, 1984]. The version given below leans on [Hollander, 2008].

First, some preliminaries are required. Consider the collection of random variables $(\xi_n)_{n\in\mathbb{N}}$ on the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$ and set $\Lambda_n(\lambda) = \log(\mathbb{E}[e^{\lambda\xi_n}])$ for every $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. Furthermore, denote the limit of Λ_n by Λ, i.e. $\lim_{n\to\infty} \Lambda_n =: \Lambda$. Then define

Definition 2.5.5. Denote by Λ^* the Legendre transform of Λ, i.e.

$$\Lambda^*(y) = \sup_{\lambda\in\mathbb{R}}(\lambda y - \Lambda(\lambda)), \qquad y \in \mathbb{R}.$$

Further, a point $x \in \mathbb{R}$ is called exposed for the function Λ^* if there is a point $t \in \mathbb{R}$ satisfying

$$\Lambda^*(y) - \Lambda^*(x) > (y-x)t \qquad \text{for each } y \neq x.$$

Such t is called (the normal to) an exposing hyperplane for x.

As it is also done in Cramér's theorem, it will be assumed that $0 \in \mathcal{D}_\Lambda^o$. This assumption ensures that under certain assumptions $\Lambda > -\infty$. Additionally, observe that the map $y \mapsto \Lambda^*(y)$ is convex as supremum of a convex function. Furthermore, the function Λ^* will work as a rate function in the Gärtner-Ellis theorem. The following lemma by [Hollander, 2008] covers these claims.

Lemma 2.5.3. *Assume that $\Lambda(y)$ exists in $[-\infty,\infty]$ for all $y \in \mathbb{R}$ and $0 \in \mathcal{D}_\Lambda^o$, then*

(i) Λ is convex and $\Lambda(y) > -\infty$ for each $y \in \mathbb{R}$.

(ii) Λ^* is a rate function according to Definition 2.5.1.

The proof of (i) is a simply conclusion of the convexity of the Λ_n and the following convexity of Λ^*. For the proof of (ii) and more detail see [Lemma V.4, Hollander, 2008].

Theorem 2.5.2 (Gärtner-Ellis)**.** *Let $(\xi_n)_{n\in\mathbb{N}}$ be a collection of random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Further, suppose that $\Lambda(y)$ exists in $[-\infty,\infty]$ for all $y \in \mathbb{R}$ and $0 \in \mathcal{D}_\Lambda^o$, then*

(i) For every closed set $C \subset \mathbb{R}$

$$\limsup_{n\to\infty} \frac{1}{n} \log\left(\mathbb{P}\left[\xi_n \in C\right]\right) \leq -\inf_{y\in C} \Lambda^*(y)$$

(ii) For every open set $O \subset \mathbb{R}$

$$\liminf_{n\to\infty} \frac{1}{n} \log\left(\mathbb{P}\left[\xi_n \in O\right]\right) \geq -\inf_{y\in O\cap E} \Lambda^*(y)$$

where E is the set of those exposed points for Λ^* which have an exposing hyperplane in $\mathcal{D}_\Lambda^o$.

If, in addition, Λ is lower semicontinuous on $\mathbb{R}$, differentiable on $\mathcal{D}_\Lambda^o$ and either $\mathcal{D}_\Lambda = \mathbb{R}$ or Λ is steep at $\partial\mathcal{D}_\Lambda$, i.e.

$$\lim_{t\to\partial\Lambda:\ t\in\mathcal{D}_\Lambda} |\dot{\Lambda}(t)| = \infty$$

then $O \cap E$ may be replaced by O in (ii). As a result, the LDP holds on $\mathbb{R}$ with rate n and rate function Λ^*.

This theorem is rather general but still it does not capture all the cases in which a sequence of random variables on $\mathbb{R}^d$ satisfies a LDP, as will be shown in Chapter 6 by a concrete examples.

3. The Semi-Complete Market Framework

3.1. Definition

In this part of the thesis, the semi-complete market for a fixed model will be introduced. The range of arbitrage free prices, the utility indifference price and optimal purchase quantity for exponential investors will be identified herein. This chapter mainly follows the results in [Robertson and Spiliopoulos, 2018], as well as [Becherer, 2003].
To specify the semi-complete market model and the technical assumptions, let B be a contingent claim. A market model is called semi-complete if every claim B allows the decomposition

$$B = D + Y \tag{3.1}$$

where D is replicable according to Definition 2.1.4. Y will denote the *unhedgeable* part of the claim. Therefore, Y is assumed to be independent of the underlying assets.
To precisely define the semi-complete market one has to impose the structure of the claims to the quintuple $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}; S)$ at the beginning. Therefore, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The filtration $\mathbb{F}$, the probability space will be equipped with, needs to allow the decomposition

$$\mathbb{F} = \mathbb{G} \vee \mathbb{H} = (\mathcal{F}_t \vee \mathcal{H}_t)_{0 \leq t \leq T}\,. \tag{3.2}$$

Then the following theorem by [Wu and Gang, 1982] ensures that the usual conditions for filtrations also hold for the composed filtration.

Lemma 3.1.1. *$\mathbb{F}$ from (3.2) satisfies the usual conditions, if $\mathbb{G}$ and $\mathbb{H}$ do and $\mathcal{G}_T, \mathcal{H}_T \subset \mathcal{F}$ are $\mathbb{P}$ independent.*

Proof. It has to be shown that $\mathbb{F}$ is right continuous and contains all $\mathbb{P}$-null sets. Because by assumption $\mathbb{G}$ and $\mathbb{H}$ contain all $\mathbb{P}$-null sets and the decomposition (3.2) holds, $\mathbb{F}$ does the same.
Consider $\eta \in \mathcal{G}_T$ and $\zeta \in \mathcal{H}_T$ some integrable random variables. $\mathcal{G}_T, \mathcal{H}_T$ are assumed to be $\mathbb{P}$ independent and hence

$$\mathbb{E}[\eta\zeta|\mathcal{F}_t] = \mathbb{E}[\eta|\mathcal{G}_t]\ \mathbb{E}[\zeta|\mathcal{H}_t] \tag{3.3}$$

holds for all $t \geq 0$. For an arbitrary $t_0 \geq 0$, let $t \searrow t_0$ in (3.3). The right continuity of $\mathbb{G}$ and $\mathbb{H}$, as well as Levy's continuity theorem, see [17.38, Kusolitsch, 2014], yield to

$$\begin{aligned}
\mathbb{E}[\eta\zeta|\mathcal{F}_{t_0}] &= \mathbb{E}[\eta|\mathcal{G}_{t_0}]\mathbb{E}[\zeta|\mathcal{H}_{t_0}] \\
&= \mathbb{E}[\eta|\mathcal{G}_{t_0}^{+}]\ \mathbb{E}[\zeta|\mathcal{H}_{t_0}^{+}] \\
&= \mathbb{E}[\eta\zeta|\mathcal{F}_{t_0}^{+}]
\end{aligned}$$

Furthermore, linear combinations of integrable random variables, as η and ζ are dense in $L^1(\mathcal{F}_T)$. Hence, for all $\xi \in L^1(\mathcal{F}_T)$

$$\mathbb{E}[\xi|\mathcal{F}_{t_0}^+] = \mathbb{E}[\eta\zeta|\mathcal{F}_{t_0}]$$

holds and therefore $\mathcal{F}_t^+ = \mathcal{F}_t$. □

To gain more specific results and build a semi-complete market framework, now, some standing assumptions are supposed to hold.

Assumption 3.1. *The filtration $\mathbb{F}$ admits the decomposition (3.2). $\mathcal{G}_T, \mathcal{H}_T \subset \mathcal{F}$ are assumed to be $\mathbb{P}$ independent and $\mathbb{G}, \mathbb{H}$ fulfil the usual conditions.*

Assumption 3.2. *The price process $S = (S^1, \ldots, S^d)$ is a d-dimensional, locally bounded $(\mathbb{P}, \mathbb{G})$-semi martingale. The market $(\Omega, \mathcal{F}, \mathbb{G}, \mathbb{P}; S)$ is supposed to be complete and arbitrage free.*

Since Assumption 3.2 holds, it follows from Theorem 2.1.2 and Definition 2.1.5 that there exists a unique probability measure, denoted by $\mathbb{Q}_0$, equivalent to $\mathbb{P}$ on $\mathcal{G}_T$, in order that S is a $(\mathbb{Q}_0, \mathbb{G})$-local martingale. Because of the equivalence of $\mathbb{P}$ and $\mathbb{Q}_0$ and Definition 2.3.4

$$H(\mathbb{Q}_0, \mathbb{P}|_{\mathcal{G}_T}) < \infty$$

holds. Furthermore, every claim ξ, which is $\mathcal{G}_T$-measurable by definition, is perfectly replicable in the complete market. For all claims, that additionally satisfy $\mathbb{E}^{\mathbb{Q}_0}[|\xi|] < \infty$, there exists a unique $x \in \mathbb{R}$ and a $(\mathbb{P}, \mathbb{G}; S)$- and, as $\mathbb{P} \sim \mathbb{Q}_0$, also $(\mathbb{Q}^0, \mathbb{G}; S)$-integrable, d-dimensional trading strategy Δ, so that the associated value process

$$X_\cdot^\Delta = x + \int_0^\cdot \Delta_u dS_u \tag{3.4}$$

is a $(\mathbb{Q}_0, \mathbb{G})$-martingale. This immediately follows from (iii) in Theorem 2.1.2 and Definition 2.1.5. Moreover, $X_T^\Delta = \xi, \mathbb{P} - a.s.$ holds for the finite maturity T.
Since $\mathbb{G}$ fulfils the usual conditions by Assumption 3.1, [Corollary 2.11 Ethier and Kurtz, 2005] guarantees the existence of a càdlàg modification of the value process X^Δ. Hence, without loss of generality X^Δ is assumed to be càdlàg.

Assumption 3.3. *The contingent claim B admits the decomposition (3.1). Assume D to be $\mathcal{G}_T$-measurable and Y to be $\mathcal{H}_T$-measurable.*

Equipped with these assumptions, the main results for the semi-complete setting can be formulated. As a first consequence of Assumption 3.1 and Assumption 3.2, more precisely the $\mathbb{P}$-independence of $\mathcal{G}_T$ and $\mathcal{H}_T$, it follows from equation (3.3) that every $(\mathbb{P}, \mathbb{G})$-martingale is a $(\mathbb{P}, \mathbb{F})$-martingale as well. Moreover, S is even a special $(\mathbb{P}, \mathbb{F})$-semi martingale. In fact, S is supposed to be a $(\mathbb{P}, \mathbb{G})$-semi martingale and therefore a $(\mathbb{P}, \mathbb{F})$-semi martingale by Theorem A.2.1. Further, S is locally bounded, hence special. Therefore, the stochastic integral with respect to S of a $\mathbb{F}$-predictable and $(\mathbb{P}, \mathbb{F}; S)$-integrable process exists. To make sure the stochastic integral process $\int_0^\cdot \Delta_u dS_u$ coincides under $\mathbb{G}$ and $\mathbb{F}$, one may observe that it is well defined on $\mathbb{G} \subset \mathbb{F}$ for a $\mathbb{G}$-predictable process and a $(\mathbb{P}, \mathbb{F})$(hence

$(\mathbb{P}, \mathbb{G})$)-special semi martingale. Further note, [Proposition 8, Jacod, 1980] and Theorem A.2.2 ensure, that under these conditions the stochastic integrals coincide and

$$L(S, \mathbb{G}) \subset L(S, \mathbb{F}).$$

Thus, all $\mathbb{G}$-predictable and $(\mathbb{P}, \mathbb{G}; S)$-integrable processes are also $\mathbb{F}$-predictable and $(\mathbb{P}, \mathbb{F}; S)$ -integrable.
Recall the martingale measure $\mathbb{Q}_0$ on $\mathcal{G}_T$, which by Assumption 3.2 is unique. This one has to be expanded to the enlarged filtration $\mathcal{F}_T$ and hence, on the semi-complete market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}; S)$. Obviously, the extended measure should have the martingale property for the enlarged market. To this end, define $\mathbb{Q}_0$ on $\mathcal{F}_T$ by

$$\mathbb{Q}_0[A] := \mathbb{E}\left[\left.\frac{d\mathbb{Q}_0}{d\mathbb{P}}\right|_{\mathcal{G}_T} \mathbb{1}_A\right]; \qquad A \in \mathcal{F}_T. \tag{3.5}$$

This kind of extension is call *"martingale preserving probability measure"* by [Amendinger, Imkeller, and Schweizer, 1998]. It can be shown that under some assumptions the extended measure retains the martingale property of all processes. For the results in this thesis, it is only necessary to prove that S is a local martingale with respect to the extended measure. Thus, define

$$Z_t^0 := \left.\frac{d\mathbb{Q}_0}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \left.\frac{d\mathbb{Q}_0}{d\mathbb{P}}\right|_{\mathcal{F}_t}; \qquad t \leq T, \tag{3.6}$$

the density process for $\mathbb{Q}_0$, and for an arbitrary $\mathbb{P}$-equivalent martingale measure $\mathbb{Q}$

$$Z_t^{\mathbb{Q}} := \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}; \qquad t \leq T. \tag{3.7}$$

According to Definition 2.1.1 further define

$$\mathcal{M} = \{\mathbb{Q} \sim \mathbb{P} \text{ on } \mathcal{F}_T : S \text{ is a } (\mathbb{Q}, \mathbb{F}) - \text{local martingale}\}. \tag{3.8}$$

In the following the elements of $\mathcal{M}$ are stated more precisely. It will be shown, that $\mathbb{Q}_0$ is a martingale measure for S with respect to $\mathbb{F}$, hence an element of $\mathcal{M}$. Therefore, some structural results of the equivalent martingale measures in the semi-complete market are given below. To this end we recall the results from [Becherer, 2003].

Lemma 3.1.2. *For $\mathbb{Q} \in \mathcal{M}$, define the process R implicit through*

$$Z_t^{\mathbb{Q}} = Z_t^0 R_t; \qquad t \leq T. \tag{3.9}$$

In that case it follows that $\mathbb{Q} = \mathbb{Q}_0$ on $\mathcal{G}_T$ and $\mathbb{E}[R_t|\mathcal{G}_t] = 1$ for every $t \leq T$.

Proof. First, let $A \in \mathcal{G}_t$ and consider the claim $\mathbb{1}_A$. The completeness of the $(\mathbb{P}, \mathbb{G}; S)$-market ensures the existence of a $\mathbb{G}$-predictable and $(\mathbb{P}, \mathbb{G}; S)$-integrable strategy Δ such that for some unique $x \in \mathbb{R}$

$$\mathbb{1}_A = x + \int_0^T \Delta_t dS_t = X_T^{\Delta} \tag{3.10}$$

holds, see (3.4). As mentioned above this is a $(\mathbb{Q}_0, \mathbb{G})$-martingale. Furthermore, due to $|X^\Delta| \leq 1$, it is a bounded martingale. By assumption $\mathbb{Q} \in \mathcal{M}$ and as the stochastic integral coincides, it holds that Δ is $(\mathbb{P}, \mathbb{F}; S)$-integrable. From Lemma A.2.1 it follows that X^Δ is a $(\mathbb{Q}, \mathbb{F})$-local martingale and hence even a $(\mathbb{Q}, \mathbb{F})$-martingale. Now, consider for $A \in \mathcal{G}_t$

$$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_A] = \mathbb{E}^{\mathbb{Q}}\left[X_T^\Delta\right] = x = \mathbb{E}^{\mathbb{Q}_0}\left[X_T^\Delta\right] = \mathbb{Q}_0[A].$$

Therefore, the densities coincide on $\mathcal{G}_t$, i.e., $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t} = \frac{d\mathbb{Q}_0}{d\mathbb{P}}|_{\mathcal{G}_t} = Z_t^0$. On the other hand $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{G}_t} = Z_t^0 \mathbb{E}[R_t|\mathcal{G}_t]$ holds by definition. This yields to the result $\mathbb{E}[R_t|\mathcal{G}_t] = 1$ and $\mathbb{Q} = \mathbb{Q}_0$ on $\mathcal{G}_T$. □

Lemma 3.1.3. *Define $\mathbb{Q}$ on $\mathcal{F}_T$ via*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^0 R_T,$$

where R_T is $\mathcal{H}_T$-measurable, strictly positive and satisfies $\mathbb{E}[R_T] = 1$. Then $\mathbb{Q}$ is an equivalent martingale measure for S with respect to $\mathbb{F}$, i.e., $\mathbb{Q} \in \mathcal{M}$

Proof. By assuming S to be a locally bounded $(\mathbb{Q}_0, \mathbb{G})$-local martingale, it follows that there exists a sequence of $\mathbb{G}$-stopping times $(\tau_m)_{m\in\mathbb{N}}$ such that $S^m_\cdot = S_{\tau_m \wedge \cdot}$ is bounded and hence a $(\mathbb{Q}_0, \mathbb{G})$-martingale. Set $R_u = \mathbb{E}[R_T|\mathcal{H}_u]$ for $u \leq T$, which apparently is a $(\mathbb{P}, \mathbb{G})$-martingale. For some $A_s \in \mathcal{G}_s$, $B_s \in \mathcal{H}_s$ and fixed $0 \leq s \leq t \leq T$ one obtains

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{A_s}\mathbb{1}_{B_s}S_{\tau_m\wedge t}\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}\right] &= \mathbb{E}\left[\mathbb{1}_{A_s}\mathbb{1}_{B_s}S_{\tau_m\wedge t}Z_t^0 R_t\right] \overset{(3.3)}{=} \mathbb{E}\left[\mathbb{1}_{A_s}S_{\tau_m\wedge t}Z_t^0\right]\mathbb{E}\left[\mathbb{1}_{B_s}R_t\right] \\ &= \mathbb{E}\left[\mathbb{1}_{A_s}S_{\tau_m\wedge s}Z_s^0\right]\mathbb{E}\left[\mathbb{1}_{B_s}R_s\right] = \mathbb{E}\left[\mathbb{1}_{A_s}\mathbb{1}_{B_s}S_{\tau_m\wedge s}\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_s}\right],\end{aligned}$$

where the third equality follows from the fact that $\mathcal{G}_t$ and $\mathcal{H}_t$ are assumed to be $\mathbb{P}$-independent. Thus, the stopped process S^m is a bounded $(\mathbb{Q}, \mathbb{F})$-martingale. By observing that $(\tau_m)_{m\in\mathbb{N}}$ is also a $\mathbb{F}$-stopping time, it follows that S is a local $(\mathbb{Q}, \mathbb{F})$-martingale, which yields to the result. □

Assumption 3.1, 3.2 together with Lemma 3.1.3 imply that $\mathbb{Q}_0$ extended on $\mathcal{F}_T$, as defined in (3.5) is in $\mathcal{M}$. Thus, one could consider the optimal investment problem on the semi-complete market.

3.2. Optimal Investment Problem

Consider a utility function $U(x) = -\frac{1}{a}e^{-ax}$, $x \in \mathbb{R}$ of an exponential investor with $a > 0$, the absolute risk aversion. Exponential utility and indifference pricing is always linked to the theory of relative entropy, see Definition 2.3.4. Especially the set of measures having finite relative entropy with respect to $\mathbb{P}$, a subset of $\mathcal{M}$, is relevant. This subset is given by

$$\tilde{\mathcal{M}} = \{\mathbb{Q} \in \mathcal{M} : H(\mathbb{Q}|\mathbb{P}) < \infty\}. \tag{3.11}$$

As Lemma 3.1.3 shows, the unique measure $\mathbb{Q}_0$ extended on $\mathcal{F}_T$ is in $\mathcal{M}$. It follows immediately from the definition of $\tilde{\mathcal{M}}$ (3.11) and the relative entropy (2.3.4), that $\mathbb{Q}_0 \in \tilde{\mathcal{M}}$. As a result $\tilde{\mathcal{M}} \neq \emptyset$, which is related to the lack of arbitrage in the $(\mathbb{P}, \mathbb{F}; S)$-market by the fundamental theorem of asset pricing by [Delbaen and Schachermayer, 2006]. Now, recall the set of allowable strategies $\mathcal{A}$. As defined in 2.3.1, a strategy Δ is called allowable with respect to $\mathbb{P}$ and $\mathbb{F}$ if it is $\mathbb{F}$-predictable, $(\mathbb{P}, \mathbb{F}; S)$-integrable and the resultant value process is a $(\mathbb{Q}, \mathbb{F})$-super martingale for all $\mathbb{P}$-equivalent martingale measures $\mathbb{Q}$ for S, here $\mathbb{Q} \in \tilde{\mathcal{M}}$. Furthermore, recall the value function $u\left(x,q\right)$ from Definition 2.3.2. The goal will be to show that there exists an $\mathbb{F}$-optimal strategy $\hat{\Delta}$ in $\mathcal{A}$, solving the optimal investment problem

$$u\left(x,q\right) = \sup_{\Delta \in \mathcal{A}} \mathbb{E}\left[U\left(X_T^{\Delta} + qB\right)\right]; \qquad X_\cdot^{\Delta} = x + \int_0^{\cdot} \Delta_u dS_u, \tag{3.12}$$

for an initial value x and position size q in B. Therefore, first determine the value function without the contingent claim $u\left(x,0\right)$. The exponential utility yields to the following property of the value function.

$$\begin{aligned} u\left(x,0\right) = \sup_{\Delta \in \mathcal{A}} \mathbb{E}\left[U\left(X_T^{\Delta}\right)\right] =& \sup_{\Delta \in \mathcal{A}} \mathbb{E}\left[-\frac{1}{a}\exp\left(-ax - a\int_0^T \Delta_t dS_t\right)\right] \\ =& e^{ax} \sup_{\Delta \in \mathcal{A}} \mathbb{E}\left[-\frac{1}{a}\exp\left(-a\int_0^T \Delta_t dS_t\right)\right] \\ =& e^{-ax} u\left(0,0\right) \end{aligned} \tag{3.13}$$

Hence, consider $x = 0$. The following proposition identifies the value function and the associated optimization problem (3.12) for $x = q = 0$, i.e. without a claim.

Proposition 3.2.1. *Let Assumption 3.1 and 3.2 hold. It follows that there exists an optimal strategy $\psi \in \mathcal{A}$ to the optimal investment problem (3.12) for $q = x = 0$. Precisely, ψ is $\mathbb{G}$-predictable, $(\mathbb{P}, \mathbb{G}; S)$-integrable and satisfies the first order condition*

$$\left.\frac{d\mathbb{Q}_0}{d\mathbb{P}}\right|_{\mathcal{G}_T} = \frac{e^{-aX_T^{\psi}}}{\mathbb{E}\left[e^{-aX_T^{\psi}}\right]}. \tag{3.14}$$

Further, X^{ψ} is a $\mathbb{Q}$-uniformly integrable $(\mathbb{Q}, \mathbb{F})$-martingale for all $\mathbb{Q} \in \tilde{\mathcal{M}}$.

Sketch of proof. Before diving into the full proof, we would first like to give a brief overview of the steps which are necessary to obtain the result. First note that, as $\mathbb{Q}_0$ is the unique equivalent martingale measure in the complete $(\mathbb{P}, \mathbb{G}; S)$-market, it also has to be the unique minimal entropy martingale measure. Therefore, the density takes the form from 2.3.3 and by taking a closer look on the entropy $H(\mathbb{Q}_0, \mathbb{P}|_{\mathcal{G}_T})$, perceive by the duality results, that there has to be an optimal strategy ψ solving the maximization problem (3.12) for $x = q = 0$. In a second step, identify the extended $\mathbb{Q}_0$ on $\mathcal{F}_T$ to be the minimal entropy martingale measure relative to $(\mathbb{P}, \mathbb{F})$. This implies ψ to be the optimal trading strategy in the $(\mathbb{P}, \mathbb{F}; S)$-market as well, if it can be shown that ψ is allowable, hence in $\mathcal{A}$. As mentioned above, the $(\mathbb{P}, \mathbb{F})$-integrability, as well as the fact that ψ is $\mathbb{F}$-predictable, hold as ψ has this properties relative to $\mathbb{G}$. Thus, in the last step, it has to be proven that the associated value process X^ψ is a $(\mathbb{Q},\mathbb{F})$-super-martingale for every $\mathbb{Q} \in \tilde{\mathcal{M}}$. To this end, we show it is a $(\mathbb{Q},\mathbb{F})$-local martingale for every $\mathbb{Q} \in \tilde{\mathcal{M}}$. Together with the results for the density processes $Z^{\mathbb{Q}}$ for some $\mathbb{Q} \in \tilde{\mathcal{M}}$ by [Kabanov and Strickler, 2002, Chapter 4] and [Delbaen, Grandits, et al., 2002], one may observe that the family of the stopped value processes is $\mathbb{Q}$ uniformly integrable and hence X^ψ is a $\mathbb{Q}$ uniformly integrable $(\mathbb{Q},\mathbb{F})$-martingale for every $\mathbb{Q} \in \tilde{\mathcal{M}}$. Therefore, X^ψ is particularly a $(\mathbb{Q},\mathbb{F})$-super-martingale. This finishes the proof.

Proof. Consider the complete market $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P}, S)$. Hereunder, the optimal investment problem (3.12) needs to be solved for trading strategies Δ which are $\mathbb{G}$-predictable, $(\mathbb{P},\mathbb{G};S)$-integrable and the resultant value or wealth process X^Δ is a $(\mathbb{Q}_0,\mathbb{G})$-super martingale. Remember, that S is assumed to be locally bounded by Assumption 3.2 and that $\mathbb{Q}_0$ is the unique equivalent local martingale measure for S on $\mathcal{G}_T$. Clearly, $H\left(\mathbb{Q}_0|\mathbb{P}|_{\mathcal{G}_T}\right) < \infty$ holds. Since the $(\mathbb{P},\mathbb{G};S)$-market is complete and hence $\mathbb{Q}_0$ is unique, it follows from Theorem 2.3.1, that $\mathbb{Q}_0$ needs to be the minimal martingale measure, as defined in 2.3.5. By the characterisation of the minimal entropy martingale measure 2.3.1 and Theorem 2.3.1, one may conclude that (3.14) holds for some $(\mathbb{P},\mathbb{G};S)$-integrable trading strategy ψ such that the value process X^ψ is a $(\mathbb{Q}_0,\mathbb{G})$-martingale. Thus,

$$\begin{aligned} H(\mathbb{Q}_0,\mathbb{P}|_{\mathcal{G}_T}) &= \mathbb{E}^{\mathbb{Q}_0}\left[\log \frac{d\mathbb{Q}_0}{d\mathbb{P}}|_{\mathcal{G}_T}\right] = \mathbb{E}^{\mathbb{Q}_0}\left[-aX_T^\psi - \log\left(\mathbb{E}\left[e^{-aX_T^\psi}\right]\right)\right] \\ &= -\log\left(\mathbb{E}\left[e^{-aX_T^\psi}\right]\right), \end{aligned} \tag{3.15}$$

which yields

$$\mathbb{E}\left[U(X_T^\psi)\right] = -\frac{1}{a}\mathbb{E}\left[e^{-aX_T^\psi}\right] = -\frac{1}{a}e^{-H(\mathbb{Q}_0,\mathbb{P}|_{\mathcal{G}_T})}. \tag{3.16}$$

Hence by Theorem 2.4.1, ψ is the optimal trading strategy in the $(\mathbb{P},\mathbb{G};S)$-market. It remains to prove ψ is optimal among the larger class of trading strategies $\mathcal{A}$ in the $(\mathbb{P},\mathbb{F};S)$-market. Therefore, recall the extended $\mathbb{Q}_0$ from (3.5). Assumptions 3.1 and 3.2 and Lemma 3.1.3 implicate that the extended $\mathbb{Q}_0$ is in $\mathcal{M}$. For any $\mathbb{Q} \in \mathcal{M}$, Lemma 3.1.2 implies

$$\begin{aligned} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] &= \mathbb{E}\left[Z_T^0 R_T\left(\log\left(Z_T^0\right) + \log\left(R_T\right)\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[Z_T^0 R_T\left(\log\left(Z_T^0\right) + \log\left(R_T\right)\right)|\mathcal{G}_T\right]\right] \\ &= \mathbb{E}\left[Z_T^0 \log\left(Z_T^0\right)\right] + \mathbb{E}\left[Z_T^0 \mathbb{E}\left[R_T \log\left(R_T\right)|\mathcal{G}_T\right]\right] \\ &\geq \mathbb{E}\left[Z_T^0 \log\left(Z_T^0\right)\right] + \mathbb{E}\left[Z_T^0\right] = \mathbb{E}\left[\frac{d\mathbb{Q}_0}{d\mathbb{P}}\log\left(\frac{d\mathbb{Q}_0}{d\mathbb{P}}\right)\right], \end{aligned}$$

where R is determined by Lemma 3.1.2. The third equality follows by the fact, that the density process Z_T^0 is a $\mathbb{P}$-martingale and $\mathbb{E}[R_T|\mathcal{G}_T] = 1$ holds again by Lemma 3.1.2. The conditional Jensen inequality yields to the estimation. Since, this estimation holds for every $\mathbb{Q} \in \mathcal{M}$, $\mathbb{Q}_0$ has to be the $(\mathbb{P},\mathbb{F})$-minimal entropy martingale measure. As equation

(3.16) still holds, it remains to prove that $\psi \in \mathcal{A}$. This yields that ψ is the optimal trading strategy, i.e., the solution of the optimal investment problem (3.12) for $q = x = 0$ in the semi-complete market. To this end, it has to be proven that X^ψ is a $(\mathbb{Q}, \mathbb{F})$-super-martingale for all $\mathbb{Q} \in \tilde{\mathcal{M}}$. As discussed above, ψ is $\mathbb{F}$-predictable and $(\mathbb{P}, \mathbb{F}; S)$-integrable, due to these properties hold for the filtration $\mathbb{G}$.
First, make sure that X^ψ is a $(\mathbb{Q}, \mathbb{F})$-local martingale for all $\mathbb{Q} \in \mathcal{M}$. It was already shown that X^ψ is a $(\mathbb{Q}_0, \mathbb{G})$-martingale and hence a $(\mathbb{Q}_0, \mathbb{G})$-special semi-martingale. From Proposition A.2.1 it follows that, since $x = X_0^\psi = 0$, $Y_t = \sup_{s \leq t} |X_0^\psi|$ is $(\mathbb{Q}_0, \mathbb{G})$-locally integrable for all $t \leq T$. Therefore, there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \nearrow \infty$ for $n \to \infty$ and that $\mathbb{E}^{\mathbb{Q}_0}[\sup_{s \leq T\wedge\tau_n} |X_s^\psi|] < \infty$. Now, fix this sequence $(\tau_n)_{n\in\mathbb{N}}$ and let $\mathbb{Q} \in \mathcal{M}$. As a consequence, this gives

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\sup_{s\leq T\wedge\tau_n} |X_s^\psi|\right] &= \mathbb{E}\left[Z_T^0 R_T \sup_{s\leq T\wedge\tau_n} |X_s^\psi|\right] = \mathbb{E}\left[\mathbb{E}\left[Z_T^0 R_T \sup_{s\leq T\wedge\tau_n} |X_s^\psi| \middle| \mathcal{G}_T\right]\right] \\
&= \mathbb{E}^{\mathbb{Q}_0}\left[\sup_{s\leq T\wedge\tau_n} |X_s^\psi|\right] < \infty,
\end{aligned}
$$

by the fact that $\mathbb{E}[R_T|\mathcal{G}_T] = 1$ and all τ_n are $\mathbb{G}$-stopping times. Hence, $\left(X^\psi\right)^-$ is $(\mathbb{Q}, \mathbb{G})$-integrable, further implying it is also $(\mathbb{Q}, \mathbb{F})$-locally integrable. Together with the fact that S is a $(\mathbb{Q}, \mathbb{F})$-local martingale by imposing $\mathbb{Q} \in \mathcal{M}$, Lemma A.2.1 ensures that X^ψ is a $(\mathbb{Q}, \mathbb{F})$-local martingale.
It remains to show that X^ψ is a $(\mathbb{Q}, \mathbb{F})$-super-martingale for all $\mathbb{Q} \in \tilde{\mathcal{M}}$. To this end, first conclude that X^ψ is a $(\mathbb{Q}_0, \mathbb{F})$-martingale. Therefore, fix $0 \leq s \leq t \leq T$ and let $A_s \in \mathcal{G}_s, B_s \in \mathcal{H}_s$. Further, consider

$$
\mathbb{E}[\mathbb{1}_{A_s}\mathbb{1}_{B_s} X_t^\psi Z_t^0] = \mathbb{P}[B_s]\mathbb{E}[\mathbb{1}_{A_s} X_t^\psi Z_t^0] = \mathbb{P}[B_s]\mathbb{E}[\mathbb{1}_{A_s} X_s^\psi Z_s^0] = \mathbb{E}[\mathbb{1}_{A_s}\mathbb{1}_{B_s} X_s^\psi Z_s^0].
$$

where the first equality follows by the $\mathbb{P}$-independence of $\mathbb{G}$ and $\mathbb{H}$, the second one holds as it is already known that X^ψ is a $(\mathbb{Q}_0, \mathbb{G})$-martingale and Z_t^0 is the density process of $\mathbb{Q}_0$.
In the following, the results from [Kabanov and Strickler, 2002] are applied. Therefore set

$$
\begin{aligned}
\mathcal{D} &:= \left\{Z^{\mathbb{Q}} : \mathbb{Q} \in \tilde{\mathcal{M}}\right\}; \\
\mathcal{T}_T &:= \{\tau : \tau \text{ is } \mathbb{F}\text{-stopping time such that } \tau \leq T\}; \\
\bar{Z}_t &:= \exp\left(\mathbb{E}^{\mathbb{Q}_0}[\log(Z_T^0)|\mathcal{F}_t]\right); \quad t \leq T,
\end{aligned}
\tag{3.17}
$$

as defined in A.2.3 and Lemma A.2.2. Together with the density of $\mathbb{Q}_0$ with respect to $\mathbb{P}$ (3.14) one receives

$$
\begin{aligned}
\log(\bar{Z}_t) = \mathbb{E}^{\mathbb{Q}_0}\left[\log(Z_T^0)|\mathcal{F}_t\right] &\overset{(3.14)}{=} \mathbb{E}^{\mathbb{Q}_0}\left[-aX_T^\psi - \log\left(\mathbb{E}\left[e^{-aX_T^\psi}\right]\right)|\mathcal{F}_t\right]; \\
&= -aX_t^\psi - \log\left(\mathbb{E}\left[e^{-aX_T^\psi}\right]\right); \quad t \leq T,
\end{aligned}
$$

To complete the proof, one has to show that X^ψ is of class DL, i.e. $\{X_\tau : \tau \leq t\}$ is $\mathbb{Q}$ uniformly integrable for each $t \leq T$. To this end, apply Theorem A.2.2, which implies that

$\{\log(\bar{Z}_{\tau\wedge T})\}_{\tau\in\mathcal{T}_T}$ is $\mathbb{Q}$ uniformly integrable. Recall $\tilde{Z}$ from Definition A.2.3,

$$\tilde{Z} = Z^{\mathbb{Q}_1}\mathbb{1}_{[0,\tau)} + \frac{Z_\tau^{\mathbb{Q}_1}}{Z_\tau^{\mathbb{Q}_2}} Z^{\mathbb{Q}_2}\mathbb{1}_{[\tau,T]}.$$

One may observe easily that $\mathbb{E}[\tilde{Z}_T\log(\tilde{Z}_T)] < \infty$, $\tilde{Z}_T > 0$ by definition of $\tilde{Z}_t$. By the optimal sampling theorem, one has $\mathbb{E}[\tilde{Z}_T] = 1$. To make sure $\mathcal{D}$ is stable under concatenation, note that as S is locally bounded by assumption and by again using the optimal sampling theorem, it follows that $\tilde{Z}_t$ is the density process with respect to $\mathbb{P}$ of some $\tilde{\mathbb{Q}} \in \tilde{\mathcal{M}}$, see [Delbaen, Grandits, et al., 2002, Lemma 4]. Thus, $\mathcal{D}$ is stable under concatenation. Therefore, by Lemma A.2.2 one receives that $\{\log(\bar{Z}_{\tau\wedge T})\}_{\tau\in\mathcal{T}_T}$ $\mathbb{Q}$ uniformly integrable, and so is $\{X^\psi_{\tau\wedge T}\}_{\tau\in\mathcal{T}_T}$. Therefore, X^ψ is of class DL and since it is a $(\mathbb{Q},\mathbb{F})$-local martingale for every $\mathbb{Q}\in\tilde{\mathcal{M}}$, it is even a $\mathbb{Q}$ uniformly integrable $(\mathbb{Q},\mathbb{F})$-martingale for every $\mathbb{Q}\in\tilde{\mathcal{M}}$. This completes the proof. □

3.3. The Indifference Price in the Semi-Complete Market

As this work considers an investor with exponential utility, it follows immediately from the balance equation of the utility indifference price (2.6) that $p(x,q)$ does not depend on the initial capital x, as seen in (3.13). Thus, one may write $p(q)$ instead and assume $x = 0$ throughout. To identify $p(q)$ consider

$$u(-qp(q),q) = \sup_{\Delta\in\mathcal{A}}\mathbb{E}\left[-\frac{1}{a}\exp\left(-a\left(-qp(q)+\int_0^T\Delta_t dS_t + qB\right)\right)\right] = e^{aqp(q)}u(0,q)$$

and hence

$$\begin{aligned} u(-qp(q),q) = e^{aqp(q)}u(0,q) &= u(0,0)\\ aqp(q) &= -\log\left(\frac{u(0,q)}{u(0,0)}\right)\\ p(q) &= -\frac{1}{aq}\log\left(\frac{u(0,q)}{u(0,0)}\right). \end{aligned} \tag{3.18}$$

To solve the optimal investment problem in the semi-complete market, the value function $u(0,q)$, together with the optimal trading strategy, which is denoted by $\hat{\Delta}$, as well as the optimal local marginal measure $\hat{\mathbb{Q}}$ has to be determined. The idea is that, as D is perfectly replicable by some initial capital denoted by d, the utility indifference price $p(q)$ should be decomposable to the form $p(q) = d + p(q;Y)$, where $p(q;Y)$ is the indifference price for q units of the "*unhedgeable*" part Y of the claim B. By continuing this heuristic consideration, a consequence of Y being independent of S, and therefore trading in S should not affect $p(q;Y)$, is that $p(q;Y)$ should coincide with the average certainty equivalent. The equation

$$U(0) = \mathbb{E}\left[U\left(qY - q\hat{p}(q;Y)\right)\right]$$

is the implicit definition of the average certainty equivalent, denoted by $\hat{p}(q;Y)$. In the case of an exponential investor with utility function $U(x) = -\frac{1}{a}e^{-ax}$, $\hat{p}(q;Y)$ can be expressed by

$$\hat{p}(q;Y) = -\frac{1}{qa}\mathbb{E}\left[e^{-qaY}\right]. \tag{3.19}$$

By specifying this thought, one should conclude that the allowable strategies in $\mathcal{A}$ have to be $\mathbb{F}$-predictable and hence are not independent of Y. As Proposition 3.3.1 below shows, this has no effect on the decomposition of $p(q)$.
In the sequel, the cumulant generating function of Y, as defined in A.1.1, will be used to express the utility indifference price $p(q)$. Therefore, set

$$\Lambda(\lambda) := \log\left(\mathbb{E}\left[e^{\lambda Y}\right]\right); \quad \lambda \in \mathbb{R} \tag{3.20}$$

As already discussed as a direct consequences of Assumption 3.2 and the formula for $\hat{p}(q;Y)$, it makes sense to consider integrable claims and hence the following conditions for D and Y are imposed.

Assumption 3.4. *For the claim $B = D + Y$, as in Assumption 3.3, it holds that for some $\epsilon > 0$, $\mathbb{E}^{\mathbb{Q}_0}[|D|^{1+\epsilon}] < \infty$ and for all $\lambda \in \mathbb{R}$ that $\Lambda(\lambda) < \infty$.*

Remark 3.1. As it can be seen in the proof of Proposition 3.2.1, if the duality results should be used to solve the optimal investment problem, the integrability is a main property to ensure that every $(\mathbb{Q}_0, \mathbb{G})$-martingale is also a $(\mathbb{Q}, \mathbb{F})$-martingale as for any $\mathbb{Q} \in \tilde{\mathcal{M}}$. Thus, it is essential for finding an optimal strategy in $\mathcal{A}$. Therefore, the stronger condition upon D, opposed to the one we needed in the consequences of Assumption 3.2 above is imposed. In view of the formula for $\hat{p}(q;Y)$ and consequently the one of the utility indifference price, we assume Y to have exponential moments of all order. Otherwise one would have to matter about position size. Despite all that, B is not required to be bounded and therefore Assumption 3.4 will be sufficient.

Together with Assumption 3.4 the following proposition, that completes the optimal investment problem and determines the utility indifference price in the semi-complete market, holds.

Proposition 3.3.1. *Let Assumption 3.1, 3.2, 3.3 and 3.4 hold. Then for each $q \in \mathbb{R}$ it follows that*

$$\frac{u(0,q)}{u(0,0)} = e^{-qad}\mathbb{E}\left[e^{-qaY}\right], \tag{3.21}$$

where d is the initial capital required to replicate D. Therefore, the indifference price p can be written in the form

$$p(q) = d - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right) = d - \frac{1}{qa}\Lambda(-qa). \tag{3.22}$$

The $\mathbb{F}$-optimal (i.e. $\mathbb{G}$-predictable, $(\mathbb{P}, \mathbb{G}; S)$-integrable) trading strategy $\hat{\Delta} \in \mathcal{A}$ is given by $\hat{\Delta} = -q\Delta^1 + \psi$, where Δ^1 is the replication strategy for D and ψ is the optimal strategy from Proposition 3.2.1. The associated value process $X^{\hat{\Delta}}$ is a $(\mathbb{Q}, \mathbb{F})$-martingale for every $\mathbb{Q} \in \tilde{\mathcal{M}}$. Furthermore, the optimal local martingale measure $\hat{\mathbb{Q}} \in \tilde{\mathcal{M}}$ for solving the optimal investment problem (3.12) with claim has the form

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \frac{d\mathbb{Q}_0}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} \frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]}. \tag{3.23}$$

Sketch of proof. The proof will be split up in three parts. First, one has to ensure that the measure $\hat{\mathbb{Q}}$ as defined by (3.23) is in $\tilde{\mathcal{M}}$ by using the independence of $d\mathbb{Q}_0/d\mathbb{P}|_{\mathcal{G}_T}$ and Y, as well as the existence of all exponential moments of Y from Assumption 3.4.
As a second step, it will be shown that the trading strategy $\hat{\Delta}$ is in $\mathcal{A}$, where the structure of $\hat{\Delta}$ is taken into account. Proposition 3.2.1 implies that $\psi \in \mathcal{A}$, hence it suffices to show $\Delta^1 \in \mathcal{A}$. The third and main step, it to prove that the duality result 2.4.2 holds in this setting and as a result $\hat{\mathbb{Q}}$ is the optimal local martingale measure and $\hat{\Delta}$ the optimal trading strategy.

Proof. 1.Step. To make sure $\hat{\mathbb{Q}} \in \tilde{\mathcal{M}}$, firstly conclude that $\hat{\mathbb{Q}} \in \mathcal{M}$ by Lemma 3.1.3. This is an immediate consequence of the fact that the normalizing factor $e^{-qaY}/\mathbb{E}[e^{-qaY}]$ of the measure $\hat{\mathbb{Q}}$ has exception 1 and is $\mathcal{H}_T$-measurable. Therefore, it is sufficient to show that $H(\hat{\mathbb{Q}}, P) < \infty$. Consider

$$
\begin{aligned}
H(\hat{\mathbb{Q}}, \mathbb{P}) &= \mathbb{E}\left[Z_T^0 \frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]} \left(\log\left(Z_T^0\right) - qaY - \log\left(\mathbb{E}\left[e^{-qaY}\right]\right)\right)\right] \\
&= H(\mathbb{Q}_0, \mathbb{P}|_{\mathcal{G}_T}) - \mathbb{E}\left[Z_T^0 \frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]}\, qaY\right] - \mathbb{E}\left[Z_T^0 \frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]} \log\left(\mathbb{E}\left[e^{-qaY}\right]\right)\right] \\
&= H(\mathbb{Q}_0, \mathbb{P}|_{\mathcal{G}_T}) - qa\frac{\mathbb{E}[Ye^{-qaY}]}{\mathbb{E}[e^{-qaY}]} - \log\left(\mathbb{E}\left[e^{-qaY}\right]\right) < \infty.
\end{aligned}
$$

The last equality follows from the fact that Z^0, the density process of $\mathbb{Q}_0$ on $\mathcal{G}_T$, and Y, a $\mathcal{H}_T$-measurable process, are independent. The inequality follows as there exist all exponential moments for Y by Assumption 3.4, i.e. $\mathbb{E}[e^{\lambda Y}] < \infty$ for all $\lambda \in \mathbb{R}$. Thus, $\hat{\mathbb{Q}} \in \tilde{\mathcal{M}}$.

2.Step. Now, consider $\hat{\Delta} = -q\Delta^1 + \psi$, where Δ^1 is the replication strategy for D and ψ is the optimal strategy from Proposition 3.2.1. To see $\hat{\Delta} \in \mathcal{A}$, let $\mathbb{Q} \in \tilde{\mathcal{M}}$. As the $(\mathbb{P}, \mathbb{G}; S)$-market was supposed to be complete in Assumption 3.2 and D is an $\mathcal{G}_T$-measurable claim, it follows that the replication strategy Δ^1 is $\mathbb{G}$-, hence $\mathbb{F}$-predictable and both, $(\mathbb{P}, \mathbb{G}; S)$ and $(\mathbb{P}, \mathbb{F}; S)$-integrable. The same holds for ψ due to Proposition 3.2.1 and hence also for $\hat{\Delta}$. Furthermore, the process $X^{\hat{\Delta}}$ coincides under $\mathbb{G}$ and $\mathbb{F}$ with respect to $\mathbb{P}$ and thus, for every $\mathbb{P}$-equivalent measure on $\mathcal{F}_T$. Therefore, it is left to show that $X^{\hat{\Delta}}_\cdot$, which can be written as

$$
X^{\hat{\Delta}}_\cdot = \int_0^\cdot \hat{\Delta}_u dS_u = -q\int_0^\cdot \Delta_u^1 dS_u + \int_0^\cdot \psi_u dS_u = -q\left(X^{\Delta^1}_\cdot - d\right) + X^{\psi}_\cdot, \tag{3.24}
$$

is a $(\mathbb{Q}, \mathbb{F})$-super martingale. Since X^ψ is a $\mathbb{Q}$ uniformly integrable $(\mathbb{Q}, \mathbb{F})$-martingale by Proposition 3.2.1, it has to be shown X^{Δ^1} is a $(\mathbb{Q}, \mathbb{F})$-martingale as well. As $\mathbb{Q}_0$-integrability for D was imposed in Assumption 3.4, Assumption 3.2 implies that $(\mathbb{Q}_0, \mathbb{G})$-martingale. Note that X^{Δ^1} is $\mathbb{G}$-adapted and that $\mathbb{Q}_0 = \mathbb{Q}$ on $\mathcal{G}_T$ ($\mathbb{Q}_0$ is the unique local martingale measure on $\mathcal{G}_T$). As there is a càdlàg modification of X^{Δ^1}, we obtain

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\leq T}|X_t^{\Delta^1}|\right] &= \mathbb{E}^{\mathbb{Q}_0}\left[\sup_{t\leq T}|X_t^{\Delta^1}|\right] = \mathbb{E}^{\mathbb{Q}_0}\left[\sup_{t\leq T}\left|\mathbb{E}^{\mathbb{Q}_0}[\underbrace{X_T^{\Delta^1}}_{D}|\mathcal{G}_t]\right|\right]\\ &\leq \mathbb{E}^{\mathbb{Q}_0}\left[\left(\sup_{t\leq T}\left|\mathbb{E}^{\mathbb{Q}_0}\left[D|\mathcal{G}_t\right]\right|\right)^{1+\epsilon}\right]^{\frac{1}{1+\epsilon}}\\ &\leq \left(\frac{1+\epsilon}{\epsilon}\right)\mathbb{E}^{\mathbb{Q}_0}\left[|D|^{1+\epsilon}\right]^{\frac{1}{1+\epsilon}} \overset{3.4}{<} \infty,\end{aligned}$$

where the first inequality follows by Hölder's inequality and the second from Doob's maximal inequality. Lemma A.2.1 then implies that X^{Δ^1} is a $(\mathbb{Q},\mathbb{F})$-local martingale. Further, let τ be an arbitrary $\mathbb{F}$-stopping time and $\lambda > 0$. Under these premises the following inequality holds.

$$\mathbb{E}^{\mathbb{Q}}\left[|X_{t\wedge\tau}^{\Delta^1}|\mathbb{1}_{|X_{t\wedge\tau}^{\Delta^1}|\geq\lambda}\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\sup_{t\leq T}|X_t^{\Delta^1}|\mathbb{1}_{\sup\limits_{t\leq T}|X_t^{\Delta^1}|\geq\lambda}\right] \leq \mathbb{E}^{\mathbb{Q}}\left[\sup_{t\leq T}|X_t^{\Delta^1}|\right].$$

Therefore, X^{Δ^1} is of class DL with respect to $(\mathbb{Q},\mathbb{F})$, consequently a $(\mathbb{Q},\mathbb{F})$-martingale and $\hat{\Delta} \in \mathcal{A}$.

3.Step. For the last step, recall the equation (3.15) in Proposition 3.2.1, which leads to

$$H(\mathbb{Q}_0, \mathbb{P}|_{\mathcal{G}_T}) = -\log\left(\mathbb{E}\left[e^{-aX_T^{\psi}}\right]\right) = -\log\left(-au(0,0)\right). \tag{3.25}$$

The last equality holds since ψ is the optimal strategy for $q = x = 0$ and thus, the supremum is attained at ψ. Conclude from above that $\hat{\Delta}$ is $\mathbb{G}$-predictable and so $X^{\hat{\Delta}}$, as integral process, is $\mathbb{G}$-adapted. Due to the $\mathbb{P}$-independence of $\mathbb{G}$ and $\mathbb{H}$, it follows that $X^{\hat{\Delta}}$ is independent of $\mathbb{H}$. Moreover, $\hat{\Delta} \in \mathcal{A}$ and as a consequence $X^{\hat{\Delta}}$ is a $(\mathbb{Q},\mathbb{F})$-martingale for every $\mathbb{Q} \in \tilde{\mathcal{M}}$. Concerning the structure of $X^{\hat{\Delta}}$ from (3.24), one arrives at

$$a(X_T^{\hat{\Delta}} + qB) = -aX_T^{\psi} + qaX_T^{\Delta^1} - qad - qaD - qaY = -aX_T^{\psi} - qad - qaY,$$

since $X_T^{\Delta^1} = D$ $\mathbb{P}$-a.s. due to Assumption 3.2. Thus

$$\begin{aligned}\mathbb{E}\left[U(X_T^{\hat{\Delta}} + qB)\right] &= -\frac{1}{a}\mathbb{E}\left[e^{-a(X_T^{\hat{\Delta}}+qB)}\right] = -\frac{1}{a}e^{-qad}\mathbb{E}\left[e^{-aX_T^{\psi}+qaY}\right]\\ &= -\frac{1}{a}e^{-qad}\mathbb{E}\left[e^{-aX_T^{\psi}}\right]\mathbb{E}\left[e^{-qaY}\right]\\ &= u(0,0)e^{-qad}\mathbb{E}\left[e^{-qaY}\right].\end{aligned} \tag{3.26}$$

To complete the proof, the duality result 2.4.2 has to hold for $\hat{\mathbb{Q}}$ and $\hat{\Delta}$. Hence

$$\begin{aligned}-\frac{1}{a}\exp\left(-H(\hat{\mathbb{Q}},\mathbb{P}) - a\mathbb{E}^{\hat{\mathbb{Q}}}\left[qB\right]\right) &= -\frac{1}{a}\exp\left(-\mathbb{E}\left[Z_T^{\hat{\mathbb{Q}}}\left(\log(Z_T^{\hat{\mathbb{Q}}}) + qaB\right)\right]\right);\\ &= -\frac{1}{a}\exp\left(-qa\mathbb{E}\left[Z_T^{\hat{\mathbb{Q}}}\left(B + \frac{1}{qa}\log(Z_T^{\hat{\mathbb{Q}}})\right)\right]\right)\end{aligned} \tag{3.27}$$

in turn has to coincide with (3.26), i.e. $\mathbb{E}[U(X_T^{\hat{\Delta}} + qB)]$. First, consider the term $B + \frac{1}{qa}\log(Z_T^{\hat{\mathbb{Q}}})$ and note that by recalling the structure of $\hat{\mathbb{Q}}$ on $\mathcal{F}_T$ in (3.23) it follows that

$$\begin{aligned} B + \frac{1}{qa}\log(Z_T^{\hat{\mathbb{Q}}}) &= D + Y + \frac{1}{qa}\log\left(Z_T^0 \frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]}\right); \\ &= D + Y + \frac{1}{qa}\log\left(Z_T^0\right) - Y - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right); \\ &= X_T^{\Delta^1} + \frac{1}{qa}\log\left(Z_T^0\right) - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right). \end{aligned}$$

Identifying that $\mathbb{E}[Z_T^{\hat{\mathbb{Q}}}] = 1$ and $\mathbb{E}\left[e^{-qaY}/\mathbb{E}[e^{-qaY}]\right] = 1$ results in

$$\begin{aligned} &\mathbb{E}\left[Z_T^{\hat{\mathbb{Q}}}\left(B + \frac{1}{qa}\log(Z_T^{\hat{\mathbb{Q}}})\right)\right] \\ &= \mathbb{E}\left[X_T^{\Delta^1} Z_T^0 \frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]}\right] + \frac{1}{qa}\mathbb{E}\left[\log\left(Z_T^0\right) Z_T^0 \frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]}\right] - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right) \\ &= \mathbb{E}\left[X_T^{\Delta^1} Z_T^0\right]\mathbb{E}\left[\frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]}\right] + \frac{1}{qa}\mathbb{E}\left[\log\left(Z_T^0\right) Z_T^0\right]\mathbb{E}\left[\frac{e^{-qaY}}{\mathbb{E}[e^{-qaY}]}\right] - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right) \\ &= d + \frac{1}{qa}H(\mathbb{Q}_0, \mathbb{P}|_{\mathcal{G}_T}) - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right) \\ &= d - \frac{1}{qa}\log\left(-au(0,0)\right) - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right). \end{aligned}$$

In the equation above, the second equality follows as a result of the independence of Y and $X_T^{\Delta^1} Z_T^0$, the third one holds as $X_T^{\Delta^1}$ is a $(\mathbb{Q}_0, \mathbb{G})$-martingale with expected value d regarding to $\mathbb{Q}_0$ and the last one follows from (3.25). Now, inserting these results to (3.27) gives

$$\begin{aligned} -\frac{1}{a}e^{-qa\mathbb{E}\left[Z_T^{\hat{\mathbb{Q}}}\left(B+\frac{1}{qa}\log(Z_T^{\hat{\mathbb{Q}}})\right)\right]} &= -\frac{1}{a}e^{-qad+\log(-au(0,0))+\log\left(\mathbb{E}\left[e^{-qaY}\right]\right)} = u(0,0)e^{-qad}\mathbb{E}\left[e^{-qaY}\right] \\ &= -\frac{1}{a}\mathbb{E}[e^{-a(X_T^{\hat{\Delta}}+qB)}]. \end{aligned}$$

Thus, it was shown that (3.26) coincides with (3.27), consequently Theorem 2.4.2 ensures that $\hat{\Delta}$ is the optimal strategy, $\hat{\mathbb{Q}}$ is the optimal local martingale measure. Therefore, $\mathbb{E}[U(X_T^{\hat{\Delta}} + qB)] = u(q, 0)$, in fact (3.21) holds and the indifference price $p(q)$ equals (3.22). □

3.4. Arbitrage Free Prices

As an incomplete market is considered, it is well known that there is no unique arbitrage free price, actually there is a range of possible prices that do not admit arbitrage. For the consideration of optimal purchase quantities subsequently, it is necessary to identify this range for the claim B precisely. Recall that the range of arbitrage free prices for $Y \neq 0$, which is defined by equation (2.1). It is given as the open interval $I = (\underline{b}, \overline{b})$, where

$$\underline{b} = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[B] \ ; \qquad \overline{b} = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[B] \,. \tag{3.28}$$

The constructed semi-complete market setting permits an explicit characterization for the boundaries $\underline{b}$ and $\overline{b}$ of I.

Lemma 3.4.1. *If Assumptions 3.1, 3.2, 3.3 and 3.4 hold, it follows that*

$$\underline{b} = d + essinf_{\mathbb{P}}(Y) \ ; \qquad \overline{b} = d + esssup_{\mathbb{P}}(Y) \tag{3.29}$$

where d is the initial capital required to replicate D. Both, $\underline{b}$ and $\overline{b}$, do not need to be finite, since Y is not necessarily bounded.

Proof. First, fix a $\mathbb{Q} \in \mathcal{M}$, a $\mathbb{P}$-equivalent local martingale measure for S. Then Lemma 3.1.2 implies that in this case $Z_T^{\mathbb{Q}} = Z_T^0 R_T$ for a $\mathcal{H}_T$-measurable R_T, satisfying $\mathbb{E}[R_T|\mathcal{G}_T] = 1$. Now, consider

$$\mathbb{E}^{\mathbb{Q}}[B] = \mathbb{E}\left[Z_T^0 R_T (X_T^{\Delta^1} + Y)\right] = \mathbb{E}\left[Z_T^0 R_T X_T^{\Delta^1}\right] + E\left[Z_T^0 R_T Y\right], \tag{3.30}$$

wherein the fact, that $D = X_T^{\Delta^1}$ almost surely, was used. Furthermore, one may observe that $X_T^{\Delta^1}$ is a $(\mathbb{Q}_0, \mathbb{G})$-martingale with initial value d. Thus, continue the consideration as follows,

$$\begin{aligned}(3.30) &= \mathbb{E}\left[\mathbb{E}\left[Z_T^0 R_T X_T^{\Delta^1} | \mathcal{G}_T\right]\right] + E\left[Z_T^0 R_T Y\right] = \mathbb{E}\left[Z_T^0 X_T^{\Delta^1} \mathbb{E}\left[R_T | \mathcal{G}_T\right]\right] + E\left[Z_T^0 R_T Y\right] \\ &= \mathbb{E}^{\mathbb{Q}_0}\left[X_T^{\Delta^1}\right] + \mathbb{E}\left[Z_T^0 R_T Y\right] = d + \mathbb{E}\left[Z_T^0 R_T Y\right].\end{aligned}$$

The fact that $\mathbb{E}\left[Z_T^0 R_T Y\right] \geq essinf_{\mathbb{P}}(Y)$ together with the equation above immediately yields to

$$\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[B] \geq d + essinf_{\mathbb{P}}(Y). \tag{3.31}$$

The reverse direction needs a little more effort. First, consider the class of all strictly positive, $\mathcal{H}_T$-measurable random variables R_T, such that $\mathbb{E}[R_T] = 1$ and denote it by $\mathbf{M}_T$. Lemma 3.1.3 implies that for any $R_T \in \mathbf{M}_T$ the through $d\mathbb{Q}/d\mathbb{P} = Z_T^0 R_T$ defined probability measure $\mathbb{Q}$ is in $\mathcal{M}$. By additionally using the independence of $\mathbb{G}$ and $\mathbb{H}$ at the identification of $\mathbb{E}^{\mathbb{Q}}[B]$, it is easy to see that $\mathbb{E}^{\mathbb{Q}}[B] = d + \mathbb{E}[R_T Y]$. Therefore, by the triangle inequality

$$\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[B] \leq \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[D] + \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[Y] = d + \inf_{R_T\in\mathbf{M}_T} \mathbb{E}[R_T Y]\,. \tag{3.32}$$

As a next step, fix an $m \in \mathbb{R}$ such that $\mathbb{P}[Y < m] > 0$ and set $A_m = \{Y < m\} \in \mathcal{H}_T$. Further, define for $0 < \delta < 1$

$$R_T^{m,\delta} = \frac{(1-\delta)\mathbb{1}_{A_m} + \delta\mathbb{1}_{A_m^c}}{(1-\delta)\mathbb{P}[A_m] + \delta\mathbb{P}[A_m^c]}.$$

As $R_T^{m,\delta}$ is strictly positive, $\mathcal{H}_T$-measurable and $\mathbb{E}[R_T^{m,\delta}] = 1$, one can conclude that $R_T^{m,\delta} \in \mathbf{M}_T$. Therefore,

$$\inf_{R_T \in \mathbf{M}_T} \mathbb{E}\left[R_T Y\right] \le \mathbb{E}\left[R_T^{m,\delta} Y\right] = \frac{(1-\delta)\mathbb{E}[Y \mathbb{1}_{\{Y<m\}}] + \delta \mathbb{E}[Y \mathbb{1}_{\{Y \ge m\}}]}{(1-\delta)\mathbb{P}[Y<m] + \delta \mathbb{P}[Y \ge m]}$$
$$\le \frac{m(1-\delta)\mathbb{P}[Y<m] + \delta \mathbb{E}[Y \mathbb{1}_{\{Y \ge m\}}]}{(1-\delta)\mathbb{P}[Y<m] + \delta \mathbb{P}[Y \ge m]},$$

where the last inequity follows as Y was estimated by m. Since Y has all exponential moments by Assumption 3.4, $\mathbb{E}[|Y|] < \infty$ or rather $\mathbb{E}[|R_T^{m,\delta} Y|] < \infty$ holds. So by taking $\delta \searrow 0$ it follows that

$$\inf_{R_T \in \mathbf{M}_T} \mathbb{E}\left[R_T Y\right] \le m,$$

and therefore taking $m \searrow essinf_{\mathbb{P}}(Y)$ yields to $\inf_{R_T \in \mathbf{M}_T} \mathbb{E}\left[R_T Y\right] \le essinf_{\mathbb{P}}(Y)$. Combining this with (3.32) proofs the reverse direction of (3.31) and thus

$$\underline{b} = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}\left[B\right] = d + essinf_{\mathbb{P}}(Y). \tag{3.33}$$

The upper bound $\overline{b} = d + esssup_{\mathbb{P}}(Y)$ can be calculated analogical. □

3.5. Optimal Quantities

As the range of arbitrage free prices was identified above, the optimal purchase quantities shall be determined next. So, let $\tilde{p} \in (\underline{b}, \overline{b})$. If an investor could buy an arbitrary amount of B for a unit prices $\tilde{p}$, one may ask what the optimal purchase number $\hat{q}$ would be. Therefore, $u(-q\tilde{p}, q)$ shall be maximized over all $q \in \mathbb{R}$, i.e.

$$\begin{aligned}\sup_{q \in \mathbb{R}} u(-q\tilde{p}, q) &= \sup_{q \in \mathbb{R}} e^{aq\tilde{p}} u(0,q)\\ &= \sup_{q \in \mathbb{R}} -\frac{1}{a} e^{aq\tilde{p}} \exp\left(-aqp(q) + \inf_{\mathbb{Q} \in \tilde{\mathcal{M}}} H(\mathbb{Q}, \mathbb{P})\right)\\ &= \sup_{q \in \mathbb{R}} e^{a(q\tilde{p} - qp(q))} u(0,0),\end{aligned}$$

where the second equality follows by the dual formulation (2.16) of $u(0,q)$ and the closed form of $p(q)$ from (2.17), whereas the last one by (3.18). Although the set of arbitrage free prices was defined in terms of all equivalent martingale measures $\mathbb{Q} \in \mathcal{M}$, [Theorem 7.1, İlhan, Jonsson, and Sircar, 2005] implies, that it is sufficient to consider the measures $\mathbb{Q} \in \tilde{\mathcal{M}}$ to obtain optimal purchase quantities. Since $u(0,0) < 0$ a value $\hat{q}$ is required that minimizes $q\tilde{p} - qp(q)$ or more precisely $\hat{q} \in \arg\min_{q \in \mathbb{R}}(q\tilde{p} - qp(q))$. Inserting the received formula for $p(q)$ from (3.22), results in

$$\begin{aligned}\inf_{q \in \mathbb{R}} \left(q\tilde{p} - q\left(d - \frac{1}{qa}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right)\right)\right) &= \inf_{q \in \mathbb{R}} \left(q(\tilde{p} - d) + \frac{1}{a}\log\left(\mathbb{E}\left[e^{-qaY}\right]\right)\right)\\ &= \inf_{q \in \mathbb{R}} \left(q(\tilde{p} - d) + \frac{1}{a}\Lambda(-qa)\right).\end{aligned} \tag{3.34}$$

This equation can be solved in order to obtain the optimal $\hat{q}$. The following proposition shows that the optimal purchase number $\hat{q}$ exists and is unique in $\mathbb{R}$.

Proposition 3.5.1. *If Assumption 3.1, 3.2, 3.3 and 3.4 hold and $\tilde{p} \in (\underline{b}, \overline{b})$, where $\underline{b}, \overline{b}$ are defined as in (3.28), then there exists a unique purchase quantity $\hat{p} \in \mathbb{R}$ that solves the optimization problem (3.34). Moreover, $\hat{q}$ is the unique element in $\mathbb{R}$ satisfying the first order condition for the optimization*

$$\tilde{p} - d = \dot{\Lambda}(-qa), \tag{3.35}$$

where $\dot{\Lambda}(\cdot)$ denotes the derivative of $\Lambda(\cdot)$.

Proof. First, rewrite the problem (3.34) in a more convenient form by taking $q = -\lambda/a$, i.e.,

$$\frac{1}{a} \inf_{\lambda \in \mathbb{R}} \left(\Lambda(\lambda) - \lambda(\tilde{p} - d)\right).$$

Now, define the function $f(\lambda) := \Lambda(\lambda) - \lambda(\tilde{p} - d)$. By taking a closer look to the derivative of $f(\lambda)$, one may realize that, since $\Lambda(\lambda)$ is strictly convex, $\dot{\Lambda}(\lambda)$ exists and is finite for all $\lambda \in \mathbb{R}$ by Lemma A.1.1 (iii). The same is true for $f(\lambda)$. To see $f(\cdot)$ is coercive too, consider $f(\lambda)/\lambda$. For $\lambda \neq 0$ one arrives at

$$\frac{f(\lambda)}{\lambda} = \frac{\Lambda(\lambda)}{\lambda} - (\tilde{p} - d).$$

As $\tilde{p} \in (\underline{b}, \overline{b})$, it follows from Lemma 3.4.1 $essinf_{\mathbb{P}}(Y) < \tilde{p} - d < esssup_{\mathbb{P}}(Y)$. Parts (i) and (ii) of Lemma A.1.1 imply that there exists an $\epsilon > 0$ such that $\liminf_{|\lambda| \nearrow \infty} f(\lambda)/|\lambda| \geq \epsilon$. Therefore, $\liminf_{|\lambda| \nearrow \infty} f(\lambda)/|\lambda| \cdot \lambda = \infty$. Consequently f is strictly convex and coercive. Thus there exists a unique minimizer $\hat{\lambda} \in \mathbb{R}$ for the function f and by the standard results of minimizer's differentiable function, it follows that the minimizer $\hat{\lambda}$ must fulfil the first order condition (3.35). To make this plausible, first note that, since $\hat{\lambda}$ minimizes $f(\lambda)$, for every $\lambda \in \mathbb{R}$, $f(\lambda) - f(\hat{\lambda}) \geq 0$ holds, hence

$$\Lambda(\lambda) - \Lambda(\hat{\lambda}) \geq (\lambda - \hat{\lambda})(\tilde{p} - d).$$

Now, assume $\lambda > \hat{\lambda}$. From the inequality above, one has

$$\tilde{p} - d \leq \frac{1}{\lambda - \hat{\lambda}} \int_{\hat{\lambda}}^{\lambda} \dot{\Lambda}(\tau) d\tau.$$

Taking $\lambda \nearrow \hat{\lambda}$, together with the smoothness of $\Lambda(\cdot)$, which is a consequence of Assumption 3.4, gives $\tilde{p} - d \leq \dot{\Lambda}(\hat{\lambda})$. For the reverse direction, i.e., $\hat{\lambda} > \lambda$, a similar calculation provides the opposite inequality. □

4. Large Embedding of Semi-Complete Markets

4.1. The Sequence of Markets

In this section the semi-complete market setting is embedded in a sequence of markets. The below constructed model motivates the consideration of semi-complete markets with asymptotically vanishing hedging errors. For a fixed $n \in \mathbb{N}$ this setting describes a market in which a sequence of risky assets are theoretically available to trade. In practice one is only able to trade in the first n assets and since any contingent claim always depends on all the sources of uncertainty the market is incomplete for every finite value of n. The considered and fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is required to admit a sequence of independent Brownian motions $(W^j)_{j\in\mathbb{N}}$. Further, the right-continuous, $\mathbb{P}$-augmented enlargement of the induced filtration of $(W^j)_{j\in\mathbb{N}}$ on $[0, T]$ shall be denoted by $\mathbb{F}$.

Assumption 4.1. *For a given $\mu = (\mu_i)_{i\in\mathbb{N}}$ and $\Sigma = (\Sigma_{ij})_{i,j\in\mathbb{N}}$ assume $\sum_{i=1}^{\infty} \mu_i^2 < \infty$ and Σ is symmetric, meaning that $\Sigma_{ij} = \Sigma_{ji}$. Further, Σ is supposed to be uniformly elliptic, i.e. there exists a $\tilde{\lambda} > 0$, such that for every $\xi = (\xi_i)_{i\in\mathbb{N}}$ with $|\xi|^2 = \sum_{i=1}^{\infty} \xi_i^2 < \infty$, it holds that $\xi^T \Sigma \xi = \sum_{i,j=1}^{\infty} \xi_i \Sigma_{ij} \xi_j \geq \tilde{\lambda} |\xi|^2$.*

In the following, let σ denote the lower triangular matrix uniquely determined by $\sigma\sigma^T = \Sigma$. Since Σ is symmetric and positive definite, as a consequence of being uniformly elliptic, σ can be calculated by applying the recursive Cholesky factorization, see [Meister, 2008]. The sequence of risky assets $(S^i)_{i\in\mathbb{N}}$ develops according to the dynamic

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \sum_{j=1}^{i} \sigma_{ij} dW_t^j; \quad i \in \mathbb{N}, \tag{4.1}$$

which can be summarized to $dS^t/S_t = \mu dt + \sigma dW^t$. Hence, the assets follow a geometric Brownian motion, see Definition 2.2.1. Therefore, for every $i, j \in \mathbb{N}$, the instantaneous rate of return of S^i is μ_i and the instantaneous return covariance of S^i, S^j is described through Σ_{ij}.
For a claim B in the sequence of markets, i.e. the $\mathcal{F}_T$-measurable non-traded asset, it holds that

$$B = \sum_{i=1}^{\infty} B_i, \tag{4.2}$$

with B_i denoting a measurable random variable with respect to the σ-algebra generated by W_T^i. As all the Brownian motions were assumed to be independent under the measure $\mathbb{P}$, so are $(B_i)_{i\in\mathbb{N}}$. Since the cumulant generating function of B_i, as before, will be part of the explicit representation of the indifference price below, the notation will be simplified by denoting it by Γ_i, so

$$\Gamma_i(\lambda) = \log \mathbb{E}\left[e^{\lambda B_i}\right]; \quad \lambda \in \mathbb{R}. \tag{4.3}$$

To ensure that the assumptions of Section 3 hold and that B is well defined, some further assumptions are required in the sequel.

Assumption 4.2. *For every $i \in \mathbb{N}$, the cumulant generating function of B_i, $\Gamma_i(\lambda)$ is finite for all $\lambda \in \mathbb{R}$, i.e. $\Gamma_i(\lambda) < \infty$.*

Assumption 4.3. *The limit*

$$\sum_{i=1}^{\infty} \Gamma_i(\lambda) = \lim_{N\nearrow\infty} \sum_{i=1}^{N} \Gamma_i(\lambda), \tag{4.4}$$

exists and is finite for every $\lambda \in \mathbb{R}$.

As a direct consequence of Assumption 4.2, it follows that all moments of B_i exist, especially $\mathbb{E}\left[B_i^2\right] < \infty$. Therefore, $\dot{\Gamma}_i(0) = \mathbb{E}[B_i]$, with $\dot{\Gamma}_i(\cdot)$ denoting the first derivative of $\Gamma_i(\cdot)$. Furthermore, $\lim_{N\nearrow\infty}\sum_{i=1}^N \Gamma_i(\lambda)$ may depend on the order of summation, as $\sum_{i=1}^{\infty} |\Gamma_i(\lambda)| < \infty$ was not assumed. The following lemma implies that by Assumption 4.2 and Assumption 4.3, the claim B is well defined.

Lemma 4.1.1. *Under Assumption 4.2 and 4.3 the sum $\sum_{i=1}^N B_i$ converges for $N \to \infty$ $\mathbb{P}$-almost surely, as well as in $L^2(\mathbb{P})$ to a random variable B. Particularly, the limits $\lim_{N\nearrow\infty}\sum_{i=1}^N \mathbb{E}[B_i]$ and $\lim_{N\nearrow\infty}\sum_{i=1}^N Var(B_i)$ exists and are finite.*

Proof. To ensure $\sum_{i=1}^N B_i$ converges to an element of $L^2(\mathbb{P})$, it has to be shown that $\lim_{N\nearrow\infty}\sum_{i=1}^N \mathbb{E}[B_i]$ and $\lim_{N\nearrow\infty}\sum_{i=1}^N \mathrm{Var}(B_i)$ exist and are finite. Then

$$\lim_{N\nearrow\infty}\sum_{i=1}^{N}\mathrm{Var}(B_i) = \sum_{i=1}^{\infty}\mathrm{Var}(B_i) = \sum_{i=1}^{\infty}\mathbb{E}\left[(B_i - \mathbb{E}[B_i])^2\right]$$

holds and implies $L^2(\mathbb{P})$-convergence. Further, if the limits exist and are finite, Lemma A.2.3 guarantees the $\mathbb{P}$-almost sure convergence. Thus, consider $\lim_{N\nearrow\infty}\sum_{i=1}^N \mathbb{E}[B_i]$. Due to the convexity of each Γ_i, it follows that $\Gamma_i(-\lambda) \geq \Gamma_i(0) - \lambda\dot{\Gamma}_i(0)$ for every $\lambda > 0$ and consequently

$$-\frac{1}{\lambda}\Gamma_i(-\lambda) \leq \dot{\Gamma}_i(0) = \mathbb{E}[B_i] = \log\left(\exp\left(\mathbb{E}\left[\frac{1}{\lambda}\lambda B_i\right]\right)\right) \leq \frac{1}{\lambda}\Gamma_i(\lambda),$$

where the last inequality follows immediately by Jensen's inequality. As a result, one obtains for every $M \in \mathbb{N}$ with $M > N$

$$-\frac{1}{\lambda}\sum_{i=N+1}^{M}\Gamma_i(-\lambda) \leq \sum_{i=N+1}^{M}\mathbb{E}[B_i] \leq \frac{1}{\lambda}\sum_{i=N+1}^{M}\Gamma_i(\lambda).$$

Since, $-\frac{1}{\lambda}\sum_{i=1}^{\infty}\Gamma_i(-\lambda)$ and $\frac{1}{\lambda}\sum_{i=1}^{\infty}\Gamma_i(\lambda)$ exist and are finite for every $\lambda > 0$ by Assumption 4.3, conclude that for every $\epsilon > 0$ there is a $N_\epsilon \in \mathbb{N}$, so that for all $M, N > N_\epsilon$, the sum $\left|\sum_{i=N+1}^{M}\mathbb{E}[B_i]\right| \le \epsilon$. Therefore, $\sum_{i=1}^{N}\mathbb{E}[B_i]$ is a Cauchy sequence and by the completeness of $L^2(\mathbb{P})$, the limit exists and is finite. It remains to show $\sum_{i=1}^{\infty}\mathrm{Var}(B_i) < \infty$. To this end, one may apply the equation

$$x^2 \le \frac{2}{\lambda^2}\left(e^{\lambda x} + e^{-\lambda x}\right); \quad x \in \mathbb{R}, \lambda > 0, \tag{4.5}$$

to $x = \sum_{i=1}^{N}(B_i - \mathbb{E}[B_i])$. Therefore, consider

$$\left(\sum_{i=1}^{N}(B_i - \mathbb{E}[B_i])\right)^2 = \sum_{i=1}^{N}(B_i - \mathbb{E}[B_i])^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{N}(B_i - \mathbb{E}[B_i])(B_j - \mathbb{E}[B_j])$$

and observe, that by taking the exception yields

$$\mathbb{E}\left[\left(\sum_{i=1}^{N}(B_i - \mathbb{E}[B_i])\right)^2\right] = \sum_{i=1}^{N}\mathbb{E}\left[(B_i - \mathbb{E}[B_i])^2\right] + \sum_{\substack{i,j=1\\ i\neq j}}^{N}\mathbb{E}[B_iB_j] - \mathbb{E}[B_i]\,\mathbb{E}[B_j]\,.$$

Using the independence of B_i and B_j for $i \neq j$ yields

$$\mathbb{E}\left[\left(\sum_{i=1}^{N}(B_i - \mathbb{E}[B_i])\right)^2\right] = \sum_{i=1}^{N}\mathbb{E}\left[(B_i - \mathbb{E}[B_i])^2\right] = \sum_{i=1}^{N}\mathrm{Var}\left(B_i\right).$$

Due to inequality (4.5), one obtains

$$\sum_{i=1}^{N}\mathrm{Var}\left(B_i\right) \le \frac{2}{\lambda^2}\left(\mathbb{E}\left[e^{\lambda\sum_{i=1}^{N}B_i}\right]e^{-\lambda\sum_{i=1}^{N}\mathbb{E}[B_i]} + \mathbb{E}\left[e^{-\lambda\sum_{i=1}^{N}B_i}\right]e^{\lambda\sum_{i=1}^{N}\mathbb{E}[B_i]}\right)$$

As $N \to \infty$, note that $e^{\pm\lambda\sum_{i=1}^{\infty}\mathbb{E}[B_i]} < \infty$, since $\sum_{i=1}^{\infty}\mathbb{E}[B_i]$ exists and is finite. By Assumption 4.3 $\sum_{i=1}^{\infty}\Gamma_i(\lambda) = \log(\mathbb{E}[e^{\lambda\sum_{i=1}^{\infty}B_i}]) < \infty$ and consequently $\mathbb{E}[e^{\lambda\sum_{i=1}^{\infty}B_i}] < \infty$. □

In order to verify the assumptions from Chapter 3, especially to identify the equivalent martingale measure for every market, the market price of risk θ has to be defined. To this end, one may firstly notice that σ is invertible, as it is the unique non singular lower triangular satisfying $\Sigma = \sigma^T\sigma$, computed by the Cholesky factorization. Thus, the inverse σ^{-1} again is a lower triangular with diagonal elements $\sigma_{ii}^{-1} = 1/\sigma_{ii}$. Forward substitution yields

$$\sigma_{ij}^{-1} = \frac{1}{\sigma_{ii}}\left(\delta_{ij} - \sum_{k=1}^{i-1}\sigma_{ik}\sigma_{kj}^{-1}\right),$$

thus, the market price of risk θ can be defined as in the one dimensional case, see Section 2.2, hence

$$\theta = \sigma^{-1}\mu.$$

As by Assumption 4.1 $\sum_{i=1}^{\infty} \mu_i^2 < \infty$, θ can also be defined iteratively. Therefore, set $\theta_1 = \mu/\sigma_{11}$ and observe that

$$\begin{aligned}\theta_2 &= \sigma_{21}^{-1}\mu_1 + \frac{1}{\sigma_{22}}\mu_2 \\ &= \frac{1}{\sigma_{22}}\left(-\sigma_{21}\frac{1}{\sigma_{11}}\right)\mu_1 + \frac{1}{\sigma_{22}}\mu_2 \\ &= \frac{1}{\sigma_{22}}\left(\mu_2 - \sigma_{21}\theta_1\right).\end{aligned}$$

Precisely, by induction it follows that $\theta = \sigma^{-1}\mu$ has an inductive representation

$$\theta_i = \frac{1}{\sigma_{ii}}\left(\mu_i - \sum_{j=1}^{i-1}\sigma_{ij}\theta_j\right), \qquad \text{for } i \geq 2. \tag{4.6}$$

Assumption 4.1 also implies that $\sum_{i=1}^{\infty}\theta_i^2 = \theta^T\theta = \mu^T\Sigma^{-1}\mu \leq (1/\tilde{\lambda})\mu^T\mu < \infty$, which allows the definition of an equivalent measure $\tilde{\mathbb{Q}} \sim \mathbb{P}$ on $\mathcal{F}_T$ by

$$\left.\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \mathcal{E}\left(\sum_{i=1}^{\infty} -\theta_i W_\cdot^i\right)_T, \tag{4.7}$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential as defined in A.2.1. After the definition of all the necessary parts, the described market model is semi-complete for each $n \in \mathbb{N}$. The next lemma guarantees that the market fulfils all the assumptions from Chapter 3.

Lemma 4.1.2. *If Assumption 4.1, 4.2 and 4.3 hold, then for every $n \in \mathbb{N}$ consider the market model with tradable assets $S = (S^1, \ldots, S^n)$, a claim B splitting in $D = \sum_{i=1}^{n} B^i$ and $Y = \sum_{i=n+1}^{\infty} B^i$, the right-continuous, $\mathbb{P}$-augmented filtration generated by $W^1, \ldots, W^n$ denoted by $\mathbb{G}$ and right-continuous, $\mathbb{P}$-augmented filtration generated by $W^{n+1}, W^{n+2}, \ldots$ denoted by $\mathbb{H}$. It follows, that this market is semi-complete for each $n \in \mathbb{N}$. Indeed, Assumption 3.1, 3.2, 3.3 and 3.4 hold, hence with*

$$d^n = \sum_{i=1}^{n} \mathbb{E}^{\tilde{\mathbb{Q}}}\left[B^i\right] \tag{4.8}$$

the utility indifference price $p^n(q)$ for q units of the claim B has the form

$$p^n(q) = d^n - \frac{1}{qa}\sum_{i=n+1}^{\infty}\Gamma_i(-qa); \qquad \textit{for } q \in \mathbb{R}. \tag{4.9}$$

The range of arbitrage free prices from Lemma 3.4.1 is given by $(\underline{b}_n, \overline{b}_n)$*, where*

$$\underline{b}_n = d^n + \sum_{i=n+1}^{\infty} essinf_{\mathbb{P}}\left(B_i\right); \qquad \overline{b}_n = d^n + \sum_{i=n+1}^{\infty} esssup_{\mathbb{P}}\left(B_i\right). \tag{4.10}$$

For any price $\tilde{p}^n \in (\underline{b}_n, \overline{b}_n)$ *the optimal purchase quantity* $\hat{q}_n$ *fulfils the first order condition according to Proposition 3.5.1*

$$\tilde{p}^n - d^n = \sum_{i=n+1}^{\infty} \dot{\Gamma}_i\left(-\hat{q}_n a\right). \tag{4.11}$$

Proof. First, note that, as the Brownian motions W^i are independent, so are the generated filtrations $\mathbb{G}$ and $\mathbb{H}$. Further, as both satisfy the usual conditions, Assumption 3.1 holds. The claim B from (4.2) can be written as the sum of D and Y, hence Assumption 3.3 is true. To make sure that Assumption 3.2 holds, firstly observe that σ is lower triangular and square root of Σ, so S, i.e. the first n assets, only depend on $W^1, \ldots, W^n$. Therefore, S is $\mathbb{G}$ adapted. By the martingale representation theorem every contingent claim in the $(\mathbb{P}, \mathbb{G}; S)$-market can be exactly replicated by trading in the underlying stock. Indeed, as $Z^0 S$ (recall (3.6)) has to be a local martingale with respect to the generated Brownian filtration $\mathbb{G}$, the martingale representation theorem implies that there is an almost everywhere unique process eliminating the drift term in the dynamics of $Z^0 S$. As this process is unique, the associated local martingale measure $\mathbb{Q}^0$ is unique too and therefore, the $(\mathbb{P}, \mathbb{G}; S)$-market is complete by the second fundamental theorem of asset pricing 2.1.2. Since S is given in terms of the Brownian motions $W^1, \ldots, W^n$, the process can be calculated by using Itô's formula. Hence, the unique equivalent local martingale measure is

$$\left.\frac{d\mathbb{Q}_0}{d\mathbb{P}}\right|_{\mathcal{G}_T} = \left.\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{G}_T} = \mathcal{E}\left(\sum_{i=1}^{n} -\theta_i W^i_\cdot\right)_T,$$

with relative entropy

$$\begin{aligned} H\left(\mathbb{Q}_0 | \, \mathbb{P}|_{\mathcal{G}_T}\right) &= \mathbb{E}^{\mathbb{Q}_0}\left[\log\left(\mathcal{E}\left(\sum_{i=1}^{n} -\theta_i W^i_\cdot\right)_T\right)\right] \\ &= \frac{1}{2}\sum_{i=1}^{n} \theta_i^2 < \infty. \end{aligned}$$

As a direct consequence, Assumption 3.2 is satisfied as well. Thus, Assumption 3.4 remains to be verified. To prove the integrability assumptions on D and Y, notice that by the assumed existence of all cumulant generating functions of B_i, it follows that all moments exist, in particular $\mathbb{E}\left[B_i\right] < \infty$ for every $i \in \mathbb{N}$. Furthermore, for every $0 < \epsilon < 1$ applying Hölder's inequality gives

$$\mathbb{E}^{\mathbb{Q}_0}\left[|D|^{1+\epsilon}\right] = \mathbb{E}\left[Z_T^0\left|\sum_{i=1}^n B_i\right|^{1+\epsilon}\right] \le \mathbb{E}\left[(Z_T^0)^{\frac{2}{1-\epsilon}}\right]^{\frac{1-\epsilon}{2}} \mathbb{E}\left[\left(\sum_{i=1}^n B_i\right)^2\right]^{1+\epsilon};$$

$$\le e^{\frac{T}{2}\frac{1+\epsilon}{1-\epsilon}\sum_{i=1}^n \theta_i^2}\left(n\sum_{i=1}^n [B_i^2]\right)^{\frac{1+\epsilon}{2}} < \infty,$$

with Hölder exponents $p = 2/(1-\epsilon)$ and $q = 2/(1+\epsilon)$. The second inequality holds due to Cauchy-Schwarz inequality and

$$\begin{aligned}\mathbb{E}\left[(Z_T^0)^{\frac{2}{1-\epsilon}}\right]^{\frac{1-\epsilon}{2}} &= \mathbb{E}\left[\exp\left(-\sum_{i=1}^n \theta_i W_T^i - \frac{1}{2}\sum_{i=1}^n \theta_i^2 T\right)^{\frac{2}{1-\epsilon}}\right]^{\frac{1-\epsilon}{2}} \\ &= \exp\left(\frac{2}{1-\epsilon}\frac{T}{2}\sum_{i=1}^n \theta_i^2\right)\exp\left(-\frac{T}{2}\sum_{i=1}^n \theta_i^2\right) \\ &= \exp\left(\frac{T}{2}\frac{1+\epsilon}{1-\epsilon}\sum_{i=1}^n \theta_i^2\right).\end{aligned}$$

Concerning the integrability assumption for Y, the independence of the $(B_i)_{i\in\mathbb{N}}$ is taken into account. In fact,

$$\begin{aligned}\mathbb{E}\left[e^{\lambda Y}\right] = \mathbb{E}\left[e^{\lambda\sum_{i=n+1}^\infty B_i}\right] &= e^{\log\left(\mathbb{E}\left[e^{\lambda\sum_{i=n+1}^\infty B_i}\right]\right)} \\ &= e^{\sum_{i=n+1}^\infty \Gamma_i(\lambda)} < \infty\end{aligned}$$

by Assumption 4.3. As the required assumptions for Proposition 3.3.1 hold, it follows that for each n, the indifference price for q units of the claim B in the n^{th}-market equals $p(q) = d - (1/(qa))\log(\mathbb{E}[e^{-qaY}])$. Since, by definition $\mathbb{Q}_0$ and $\tilde{\mathbb{Q}}$ agree on the $\mathcal{G}_T$, together with the definition of d and d^n in Section 3.3 and (4.8) it follows that

$$d = \mathbb{E}^{\mathbb{Q}_0}[D] = \sum_{i=1}^n \mathbb{E}^{\tilde{\mathbb{Q}}}[B_i] = d^n$$

$$\log\left(\mathbb{E}\left[e^{-qaY}\right]\right) = \log\left[e^{-qa\sum_{i=n+1}^\infty B_i}\right] = \sum_{i=n+1}^\infty \Gamma_i(-qa).$$

The next thing to verify is the range of arbitrage free prices. By proving $\operatorname{essinf}_{\mathbb{P}}(\sum_{i=n+1}^\infty B_i) = \sum_{i=n+1}^\infty \operatorname{essinf}_{\mathbb{P}}(B_i)$ and $\operatorname{esssup}_{\mathbb{P}}(\sum_{i=n+1}^\infty B_i) = \sum_{i=n+1}^\infty \operatorname{esssup}_{\mathbb{P}}(B_i)$, Equation (4.10) follows instantaneously. To show $\operatorname{essinf}_{\mathbb{P}}(\sum_{i=n+1}^\infty B_i) = \sum_{i=n+1}^\infty \operatorname{essinf}_{\mathbb{P}}(B_i)$ and $\operatorname{esssup}_{\mathbb{P}}(\sum_{i=n+1}^\infty B_i) = \sum_{i=n+1}^\infty \operatorname{esssup}_{\mathbb{P}}(B_i)$ firstly remember that $\sum_{i=1}^\infty \mathbb{E}[B_i]$ exists due to Lemma 4.1.1 and the B_i are independent for every $i \in \mathbb{N}$. Without loss of generality, assume that $\mathbb{E}[B_i] = 0$ and, as every $\mathbb{E}[B_i^2] < \infty$, $\sum_{i=1}^\infty \mathbb{E}[B_i^2] < \infty$. Now, define

$M_\infty = \sum_{i=n+1}^{\infty} B_i$ and observe $M_\infty < \infty$ almost surely. Let $c < \sum_{i=n+1}^{\infty} \text{essinf}_{\mathbb{P}}(B_i)$. Then it follows by the independence of the B_i, that $\mathbb{P}[M_\infty < c] = 0$ and consequently, $\text{essinf}_{\mathbb{P}}(\sum_{i=n+1}^{\infty} B_i) \geq \sum_{i=n+1}^{\infty} \text{essinf}_{\mathbb{P}}(B_i)$. The reverse direction requires the definition of $M = (M_m)_{m\in\mathbb{N}}$ with $M_m = \sum_{i=n+1}^{m+n} B_i$, which is a $L^2(\mathbb{P})$-bounded martingale with respect to the filtration $\mathcal{F}_m = \sigma(B_i, i = n+1, \ldots, n+m)$. Namely, as $\sum_{i=1}^{\infty} \mathbb{E}[B_i^2] < \infty$ was assumed, the L^2 bound follows immediately and for $m < u$ the martingale property is verified by

$$\begin{aligned}\mathbb{E}\left[M_u|\mathcal{F}_m\right] &= \mathbb{E}\left[\sum_{i=n+1}^{u+n} B_i\middle|\mathcal{F}_m\right] = \mathbb{E}\left[\sum_{i=n+1}^{m+n} B_i\middle|\mathcal{F}_m\right] + \mathbb{E}\left[\sum_{i=n+m+1}^{u+n} B_i\middle|\mathcal{F}_m\right] \\ &= \sum_{i=n+1}^{m+n} B_i = M_m.\end{aligned}$$

Thus, $M_m = \mathbb{E}[M_\infty|\mathcal{F}_m]$ and therefore for $m \in \mathbb{N}$

$$\begin{aligned}\sum_{i=n+1}^{m+n} \text{essinf}_{\mathbb{P}}\left(B_i\right) &= \text{essinf}_{\mathbb{P}}\left(\sum_{i=n+1}^{m+n} B_i\right) = \text{essinf}_{\mathbb{P}}\left(M_m\right) = \text{essinf}_{\mathbb{P}}\left(\mathbb{E}\left[M_\infty|\mathcal{F}_m\right]\right) \\ &\geq \text{essinf}_{\mathbb{P}}\left(M_\infty\right) = \text{essinf}_{\mathbb{P}}\left(\sum_{i=n+1}^{\infty} B_i\right)\end{aligned}$$

where the first equality holds as a finite sum of the independent B_i is considered. By taking m to ∞ the reverse direction is proved. The first order condition (4.11) follows from Proposition 3.5.1 as one observes that $\Lambda(\lambda)$ from (3.20) results in $\Lambda(\lambda) = \sum_{i=n+1}^{\infty} \Gamma_i(\lambda)$ and that $\frac{d}{d\lambda} \sum_{i=1}^{\infty} \Gamma_i(\lambda) = \sum_{i=1}^{\infty} \dot{\Gamma}_i(\lambda)$. Thus, notice that as the cumulant generating function $\Gamma_i(\lambda)$ is convex the derivative of Γ_i increases. This yields to

$$\sum_{i=k}^{n}(\Gamma_i(\lambda) - \Gamma_i(\lambda-1)) \leq \sum_{i=k}^{n} \dot{\Gamma}_i(\lambda) \leq \sum_{i=k}^{n}(\Gamma_i(\lambda+1) - \Gamma_i(\lambda)),$$

which means that $\sum_{i=0}^{\cdot} \dot{\Gamma}_i(\lambda)$ is a Cauchy sequence, since $\lim_{n\to\infty} \sum_{i=1}^{n} \Gamma_i(\lambda)$ is assumed to exist in Assumption 4.3. As a result $\sum_{i=0}^{\cdot} \dot{\Gamma}_i(\lambda)$ converges to a finite limit, which completes the lemma's proof. □

By Lemma 4.1.2 it follows that for every $n \in \mathbb{N}$ the n^{th}-market is semi-complete and fulfils the requirements from Section 3.3. To follow the aim of considering the market for $n = \infty$ it requires a proof for the unique existence of the limit of the indifference price p^n. By Assumption 4.3 the second term of p^n exists for $n \to \infty$. Concerning d^n from (4.8) define

$$d = \lim_{n\to\infty} d^n. \tag{4.12}$$

To make sure d is well defined, i.e. the limit exists, recall Lemma 4.1.1 which proves that $\lim_{N\nearrow\infty} \sum_{i=1}^{N} \mathbb{E}[B_i]$ and $\lim_{N\nearrow\infty} \sum_{i=1}^{N} \text{Var}(B_i)$ both exist and are finite. Further, consider for any $n \leq N$ in $\mathbb{N}$

$$d^N - d^n = \sum_{i=n+1}^{N} \mathbb{E}^{\tilde{\mathbb{Q}}}[B_i] = \sum_{i=n+1}^{N} \mathbb{E}^{\mathbb{Q}}[B_i] + \sum_{i=n+1}^{N} (\mathbb{E}^{\tilde{\mathbb{Q}}}[B_i] - \mathbb{E}[B_i])$$

and

$$\begin{aligned}\left| \sum_{i=n+1}^{N} (\mathbb{E}^{\tilde{\mathbb{Q}}}[B_i] - \mathbb{E}[B_i]) \right|^2 &= \left| \mathbb{E}\left[\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \left(\sum_{i=n+1}^{N} (B_i - \mathbb{E}[B_i]) \right) \right] \right|^2 \\ &\leq \mathbb{E}\left[\left| \frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \right|^2 \right] \mathbb{E}\left[\left| \left(\sum_{i=n+1}^{N} (B_i - \mathbb{E}[B_i]) \right) \right|^2 \right] \\ &\leq e^{T \sum_{i=1}^{\infty} \theta_i^2} \sum_{i=n+1}^{N} \mathrm{Var}(B_i)\end{aligned}$$

wherein the first inequality follows by the Hölder inequality with Hölder exponents $p = q = 2$. The second inequality follows by the fact that

$$\mathbb{E}\left[\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}}^2 \right] = \mathbb{E}\left[\exp\left(-2 \sum_{i=1}^{\infty} \theta_i W_T^i - \sum_{i=1}^{\infty} \theta_i^2 T \right) \right] = \exp\left(T \sum_{i=1}^{\infty} \theta_i^2 \right)$$

and the independence of $(B_i)_{i\in\mathbb{N}}$. Consequently, the replicating capitals d^n are converging to an unique replicating capital d. Therefore, in the $n = \infty$ market model, where all the assets $(S^i)_{i\in\mathbb{N}}$ are available to trade, it follows from Lemma 4.1.1 that $B \in L^2(\mathbb{P})$ and that the unique arbitrage free price for B is given by $d = \mathbb{E}^{\tilde{\mathbb{Q}}}[B]$. Hence the market is complete.

In this large market example there occur large position sizes when the hedging errors for the claim B are getting small. Due to a simple consideration, the connection of hedging errors and position size shall be pointed out. In a complete market there are no hedging errors, since every claim is perfectly replicable. Moreover, there is only one arbitrage free price d for a given claim, so if there is a possibility to purchase claims for a price $\tilde{p} \neq d$ it would be optimal to buy an infinite amount of claims. This simple idea also appears in the following examples, where optimal purchase quantities can be calculated explicitly.

4.2. Examples

Example 4.1. Assume $B_i \overset{\mathbb{P}}{\sim} \mathcal{N}(\gamma_i, \delta_i^2)$ is normally distributed for every $i \in \mathbb{N}$. Due to the properties of the normal distribution it immediately follows that the cumulant generating functions exist for every B_i, so Assumption 4.2 is satisfied. Calculating $\Gamma_i(\lambda)$ by (4.3) gives $\Gamma_i(\lambda) = \lambda\gamma_i + (1/2)\lambda^2\delta_i^2$, and therefore, if $\sum_{i=1}^{\infty} |\gamma_i| < \infty$ and $\sum_{i=1}^{\infty} \delta_i^2 < \infty$ hold additionally, Assumption 4.3 is satisfied as well. Therefore, it follows by Lemma 4.1.2 that the range of arbitrage free prices goes from $\underline{b}_n = -\infty$ to $\overline{b}_n = \infty$ for each $n \in \mathbb{N}$. Hence, by (4.11) it follows that the optimal purchase quantity $\hat{q}_n$ for every $\tilde{p}^n \in \mathbb{R}$ satisfies

$$\sum_{i=n+1}^{\infty} \dot{\Gamma}_i(\hat{q}_n a) = \sum_{i=n+1}^{\infty} \gamma_i - \hat{q}_n a \sum_{i=n+1}^{\infty} \delta_i^2 = \tilde{p}^n - d^n,$$

thus

$$\hat{q}_n = \frac{d^n - \tilde{p}^n + \sum_{i=n+1}^{\infty} \gamma_i}{a \sum_{i=n+1}^{\infty} \delta_i^2}.$$

As $\sum_{i=n+1}^{\infty} \delta_i^2 \to 0$ for $n \to \infty$, one may observe that if $\liminf_{n\to\infty} |\tilde{p}^n - d| > 0$ holds, this yields to $|\hat{q}_n| \to \infty$ at a rate proportional to $(\sum_{i=n+1}^{\infty} \delta_i^2)^{-1}$.

Example 4.2. Another example for optimal positions becoming large is a claim B where each $B_i \overset{\mathbb{P}}{\sim} Poi(\beta_i)$ is Poisson for every $i \in \mathbb{N}$. Since the cumulant generating function of the Poisson distribution always exists, Assumption 4.2 is satisfied. Calculating $\Gamma_i(\lambda)$ by (4.3) gives $\Gamma_i(\lambda) = (e^{\lambda} - 1)\beta_i$, and therefore, if $\sum_{i=1}^{\infty} |\beta_i| < \infty$ holds additionally, Assumption 4.3 is satisfied as well. Therefore, it follows by Lemma 4.1.2 that the range of arbitrage free prices goes from $\underline{b}_n = d^n$ to $\overline{b}_n = \infty$ for each $n \in \mathbb{N}$. Hence, by (4.11) it follows that the optimal purchase quantity $\hat{q}_n$ for every $\tilde{p}^n \in \mathbb{R}$ satisfies

$$\sum_{i=n+1}^{\infty} \dot{\Gamma}_i(\hat{q}_n a) = e^{-qa} \sum_{i=n+1}^{\infty} \beta_i = \tilde{p}^n - d^n.$$

Since by construction of the range of arbitrage free prices $\tilde{p}^n > \underline{b}_n$ should always hold and further $d^n \nearrow d$, conclude that $\underline{b}_n = d_n < d \leq \tilde{p}^n$. Therefore, for any $\tilde{p}^n \geq d > d_n$ the optimal purchase quantity $\hat{q}_n$ for every $\tilde{p}^n \in \mathbb{R}$ takes the form

$$\hat{q}_n = -\frac{1}{a} \log\left(\frac{\tilde{p}^n - d_n}{\sum_{i=n+1}^{\infty} \beta_i}\right).$$

As $\sum_{i=n+1}^{\infty} \beta_i^2 \to 0$ for $n \to \infty$, one may observe that if $\liminf_{n\to\infty} \tilde{p}^n - d > 0$ holds, this yields to $|\hat{q}_n| \to \infty$ at a rate proportional to $-\log(\sum_{i=n+1}^{\infty} \beta_i^2)$.

5. Relation of Limiting Indifference Price and Optimal Quantities with Large Deviations

5.1. Introduction

This chapter deals with the main results of [Robertson and Spiliopoulos, 2018] concerning the limiting indifference price and optimal purchase quantities in the sequence of semi-complete markets. The vanishing unhedgeable component Y_n of B_n in the n^{th}-market, combines the theory of large deviations from Section 2.5 with the one of utility indifference pricing from Section 2.2. In the sequence of semi-complete markets, the limit of indifference price of B_n shall be determined for $n \to \infty$ while assuming that $(Y_n)_{n\in\mathbb{N}}$ has an LDP according to Definition 2.5.2. Naturally, there will occur large positions naturally while dealing with optimal quantities and, as demonstrated below, there will be non-trivial terms in the limiting indifference price. To correctly define a general embedding of semi-complete markets the following assumption is supposed to hold.

Assumption 5.1. *There is a complete filtered probability* $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n)$ *for each* $n \in \mathbb{N}$*, with associated subfiltrations* $\mathbb{G}^n$ *and* $\mathbb{H}^n$*, assets* S^n*, probability measures* $\mathbb{Q}_0^n$ *and claims* B^n *so that Assumptions 3.1, 3.2, 3.3 and 3.4 are satisfied.*

As shown in Chapter 4, there is an embedding that satisfies Assumption 5.1. Indeed, the sequence of semi-complete markets, where one is only able to trade in the first n assets $S^1, \dots, S^n$ in the n^{th}-market, fulfils all the required conditions from Assumption 5.1. Next, recall the definition of a large deviation principle 2.5.2. In the required setting for this section, the Polish space S is imposed to be $\mathbb{R}$, while for each $n \in \mathbb{N}$ the collection of random variables ξ_n shall be Y_n in the filtrated probability space $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbb{P}^n)$. In view of the conditions for some important theorems, like Theorem 2.5.1 and 2.5.2, an additional convergence condition for Y_n is required. Due to $\mathbb{P}^n[Y_n \in \mathbb{R}] = 1$ for every $n \in \mathbb{N}$, it is necessary that $\inf_{y\in\mathbb{R}} I(y) = 0$ for the upper bound to hold [Section 1.2 Dembo and Zeitouni, 1998]. Since I is a good rate function, defined in 2.5.1, meaning that the level sets are compact, there has to be at least one element $y \in \mathbb{R}$ with $I(y) = 0$. Therefore, by assuming $I(y) = 0$ if and only if $y = 0$, it follows by the lower semicontinuity of the rate function I, that for every $\epsilon > 0$ the probability $\mathbb{P}[|Y_n| \geq \epsilon] \to 0$ for $n \to \infty$. Indeed, due to the upper bound from (2.19), one receives

$$\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[|Y_n| \geq \epsilon\right]\right) \leq -\inf_{s\in[\epsilon,\infty)} I(s),$$

which by rearrangement of the terms implies that $\mathbb{P}[|Y_n| \geq \epsilon] \to 0$ for $n \to \infty$. Hence, the laws of Y_n weakly converge to the Dirac mass at 0, which means that for $n \to \infty$ the

unhedgeable component Y_n vanishes. Assuming this is reasonable, as one may think of Lemma 4.1.1 where $Y_n = \sum_{i=n+1}^{\infty} B_i$ converges to 0 in $L^2(\mathbb{P})$, hence in probability and moreover, as will be shown below, it is very supportive by determining the limiting utility indifference price and the optimal position size for an exponential investor.
Recall Proposition 3.3.1, Assumption 5.1 guarantees that the required conditions hold and therefore,

$$p^n(q_n) = d^n - \frac{1}{q_n a} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-q_n a Y_n}\right]\right), \tag{5.1}$$

with $d^n = \mathbb{E}^{\mathbb{Q}_0^n}[D_n]$ denoting the replication capital for the hedgeable part D_n of the claim B_n for each $n \in \mathbb{N}$. Furthermore, assume that the sequence of unhedgeable parts $(Y_n)_{n\in\mathbb{N}}$ of $(B_n)_{n\in\mathbb{N}}$ satisfies an LDP with $I(y) = 0$ if and only if $y = 0$. As shown above, this results in vanishing of the unhedgeable components according to the LDP. One may expect that as a consequence the limiting indifference price would be $d = \lim_{n\to\infty} d^n$. As will be shown in 5.2.1, this is true for finite position sizes, i.e. $\sup_{n\in\mathbb{N}} |q_n| < \infty$. As large optimal purchase quantities will arise naturally, for unbounded ones with $\lim_{n\to\infty} q_n/r_n = l \neq 0$ Varadhan's integral lemma 2.5.1 together with some adapted integrability assumptions, stated in Assumption 5.2, implicates that

$$\lim_{n\to\infty} p^n(q_n) - d^n = -\frac{1}{la} \sup_{y\in\mathbb{R}} (-lay - I(y)).$$

Further, note that commonly the rate function is assumed to be convex, which leads to $-1/(la)\sup_{y\in\mathbb{R}}(-lay - I(y)) \neq 0$ for all $l \neq 0$. Therefore, there is a non-trivial large deviations effect on the limiting utility indifference price. These heuristic observations give rise of some major questions which are stated in [Robertson and Spiliopoulos, 2018]. The questions that shall be discussed are given in the following.

1. Does the LDP hold for some $(Y_n)_{n\in\mathbb{N}}$? What are sufficient conditions for an LDP to hold?
2. If a LDP is assumed to hold for $(Y_n)_{n\in\mathbb{N}}$, what would be the limiting indifference prices? Which impact has the limiting ratio of $|q_n|$ and $|r_n|$ on the asymptotic behaviour of $p^n(q_n)$?
3. Which circumstances yield $q_n/r_n \to l$ for a value $0 < |l| < \infty$? What is the asymptotic ratio of $\hat{q}_n$, the optimal purchase quantity from Proposition 3.5.1, and the scaling r_n?

The remaining work of this thesis is to fully answer those question. First, in Section 5.2 a LDP for $(Y_n)_{n\in\mathbb{N}}$ is supposed to hold. This will lead to the answer of the second and third question. In Proposition 5.2.1 the limiting utility indifference prices depending on the asymptotic value of q_n/r_n will be identified. Furthermore, in Section 5.3 by still assuming an LDP to hold, Proposition 5.3.1 considers the relations between $\hat{q}_n$, the optimal purchase quantity, and the scaling r_n of the large deviation. It will be shown that the possibility to buy claims at price $\tilde{p} \neq d$ for all *reasonable* prices $\tilde{p}$, as specified later, leads to $0 < \liminf_{n\to\infty} |\hat{q}_n|/r_n \leq \limsup_{n\to\infty} |\hat{q}_n|/r_n < \infty$. In other words, one is typically within the large deviations regime where non-zero effects on the limiting indifference price occur by purchasing optimal quantities. To finally answer the first question, Chapter 6 identifies sufficient conditions for a LDP to hold and precisely proves the LDP of $(Y_n)_{n\in\mathbb{N}}$ for two concrete examples.

5.2. Analysis of Large Claims satisfying LDP

For each $n \in \mathbb{N}$

$$\Lambda_n(\lambda) = \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\lambda Y_n}\right]\right) \tag{5.2}$$

denotes the cumulant generating function of Y_n. Therefore, (5.1) takes the form

$$p^n(q_n) = d^n - \frac{1}{q_n a}\Lambda_n\left(-q_n a\right). \tag{5.3}$$

Due to Assumption 5.1, Assumption 3.4 holds for every $n \in \mathbb{N}$, hence $\Lambda_n(\lambda) < \infty$ for every $\lambda \in \mathbb{R}$. Further, by Lemma 2.4.1, $q \mapsto p^n(q)$ is non-increasing.
Next, a LDP for $(Y_n)_{n\in\mathbb{N}}$ with some suitable integrability conditions will be imposed to hold.

Assumption 5.2. *The sequence of random variables $(Y_n)_{n\in\mathbb{N}}$ satisfies a LDP according to Definition 2.5.2 with scaling r_n and good rate function $I(y)$. Furthermore, $I(y) = 0$ is supposed to hold if and only if $y = 0$ and there exists a constant $\delta > 0$ such that for $\epsilon = \pm\delta$*

$$\limsup_{n\to\infty}\frac{1}{r_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\epsilon r_n Y_n}\right]\right) = \limsup_{n\to\infty}\frac{1}{r_n}\Lambda_n(\epsilon r_n) < \infty \tag{5.4}$$

Remark 5.1. In this setting, the convexity of $I(y)$ is not assumed and hence, $-1/(la)$ $\sup_{y\in\mathbb{R}}(-lay - I(y))$ may be zero for an $l \neq 0$. In Section 5.3 the definition of *reasonable prices* will replace this.

Due to (5.4), the moment condition of Theorem 2.5.1 is satisfied. Hence, one may ask for the maximal ϵ fulfilling the bound (5.4). Therefore, define

$$\begin{aligned}\bar{M} &:= \sup\left\{M \in \mathbb{R}: \limsup_{n\to\infty}\frac{1}{r_n}\Lambda_n(Mr_n) < \infty\right\};\\ \underline{M} &:= \inf\left\{M \in \mathbb{R}: \limsup_{n\to\infty}\frac{1}{r_n}\Lambda_n(Mr_n) < \infty\right\}\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}M^* &:= \sup\left\{M \in \mathbb{R}: \sup_{y\in\mathbb{R}}(My - I(y)) < \infty\right\},\\ M_* &:= \inf\left\{M \in \mathbb{R}: \sup_{y\in\mathbb{R}}(My - I(y)) < \infty\right\}.\end{aligned} \tag{5.6}$$

By the adapted integrability conditions in Assumption 5.2 it follows that $\underline{M} \leq -\delta < \delta \leq \bar{M}$. Recall the moment condition of Varadhan's integral lemma 2.5.1, then one may notice that it is satisfied by Assumption 5.2. Indeed, for $\tilde{\epsilon} > 0$ choose $\gamma = (M + \tilde{\epsilon})/M > 1$ with $M + \tilde{\epsilon} < \bar{M}$, the $\limsup_{n\to\infty}(1/r_n)\Lambda_n(\gamma M r_n) = \limsup_{n\to\infty}(1/r_n)\Lambda_n((M + \tilde{\epsilon})r_n) < \infty$. Thus, it follows for an affine function with $M \in (\underline{M}, \bar{M})$

$$\lim_{n\to\infty} \frac{1}{r_n}\Lambda_n(Mr_n) = \sup_{y\in\mathbb{R}}(My - I(y)) < \infty \tag{5.7}$$

Since $\underline{M}$ and $\bar{M}$ are working as bounds for the moment condition of Varadhan's integral lemma 2.5.1 and consequently ensure its validity, M_* and M^* may be different from $\underline{M}$ and $\bar{M}$. Thus

$$-\infty \leq M_* \leq \underline{M} \leq -\delta < 0 < \delta \leq \bar{M} \leq M^* \leq \infty. \tag{5.8}$$

As mentioned in the introduction, the limiting ratio of the sequence of purchase quantities $(q_n)_{n\in\mathbb{N}}$ and the scaling r_n of the LDP plays an important role in the evaluation of the limiting indifference price. For the further analysis suppose Assumption 5.2 holds. One then has to study the limiting indifference price $p^n(q_n)$ in three different regimes for every arbitrary sequence $(q_n)_{n\in\mathbb{N}}$ with $|q_n| \to \infty$ up to oscillations. Define the regimes as follows

$$\lim_{n\to\infty} \frac{|q_n|}{r_n} \begin{cases} = 0 & \text{Regime 1,} \\ \in (0,\infty) & \text{Regime 2,} \\ = \infty & \text{Regime 3.} \end{cases} \tag{5.9}$$

The following proposition shows the different forms of the limiting indifference price depending on both the regimes and the different values of $M_*, \underline{M}, \bar{M}$ and M^*. The result is given for all limits $l \in [-\infty, \infty]$ with $q_n/r_n \to l$ such that the moment condition for Varadhan's integral lemma 2.5.1 and (5.4) are satisfied. Thus, apart from $l \in [-\underline{M}/a, -M_*/a]$ and $l \in [-M^*/a, -\bar{M}/a]$ all $l \in \mathbb{R}$ are treated below.

Proposition 5.2.1. *Under Assumption 5.1 and 5.2 it holds that,*

1. *Regime 1: If* $\lim_{n\to\infty} |q_n|/r_n = 0$, *then* $\lim_{n\to\infty}(p^n(q_n) - d^n) = 0$. *This holds especially if* $\sup_{n\in\mathbb{N}} |q_n| < \infty$.
2. *Regime 2:*
   - *If* $\lim_{n\to\infty} q_n/r_n = l \in (0, \infty)$
     - *with* $l \in \left(0, -\frac{\underline{M}}{a}\right)$ *it follows that*

$$\lim_{n\to\infty}(p^n(q_n) - d^n) = \inf_{y\in\mathbb{R}}\left(y + \frac{1}{al}I(y)\right) \in (-\infty, 0], \tag{5.10}$$

- *with* $M_* > -\infty$ *and* $l > -\frac{M_*}{a}$ *it follows that*

$$\lim_{n\to\infty}(p^n(q_n) - d^n) = -\infty. \tag{5.11}$$

- *If* $\lim_{n\to\infty} q_n/r_n = l \in (-\infty, 0)$
  - *with* $l \in \left(-\frac{\bar{M}}{a}, 0\right)$ *it follows that*

$$\lim_{n\to\infty}(p^n(q_n) - d^n) = \sup_{y\in\mathbb{R}}\left(y + \frac{1}{al}I(y)\right) \in [0,\infty), \tag{5.12}$$

  - *with* $M^* < \infty$ *and* $l < -\frac{M^*}{a}$ *it follows that*

$$\lim_{n\to\infty}(p^n(q_n) - d^n) = \infty. \tag{5.13}$$

3. *Regime 3:*
   - *If* $\lim_{n\to\infty} q_n/r_n = \infty$*, then*
     - *if* $M_* > -\infty$ *it follows that*

$$\lim_{n\to\infty}(p^n(q_n) - d^n) = -\infty, \tag{5.14}$$

     - *if* $\underline{M} = -\infty$ *it follows that*

$$\limsup_{n\to\infty}(p^n(q_n) - d^n) \leq \inf\{y \in \mathbb{R} : I(y) < \infty\}. \tag{5.15}$$

   - *If* $\lim_{n\to\infty} q_n/r_n = -\infty$*, then*
     - *if* $M^* < \infty$ *it follows that*

$$\lim_{n\to\infty}(p^n(q_n) - d^n) = \infty, \tag{5.16}$$

     - *if* $\bar{M} = \infty$ *it follows that*

$$\liminf_{n\to\infty}(p^n(q_n) - d^n) \geq \sup\{y \in \mathbb{R} : I(y) < \infty\}. \tag{5.17}$$

Proposition 5.2.1 covers all cases for the different values of the bounds $M_*, \underline{M}, \bar{M}$ and M^* in every regime. It is the most general case and therefore not easily readable. For reader's convenience the following proposition handles a special case of Proposition 5.2.1 pointing out the main results by assuming $\underline{M} = -\infty$ and $\bar{M} = \infty$. This implicates that Varadhan's integral lemma 2.5.1 holds for all $M \in \mathbb{R}$ and moreover, it implies that $M_* = -\infty$ and $M^* = \infty$.

Proposition 5.2.2. *Under Assumption 5.1 and 5.2 it holds that,*

1. *Regime 1: If* $\lim_{n\to\infty} |q_n|/r_n = 0$*, then* $\lim_{n\to\infty}(p^n(q_n) - d^n) = 0$*. This holds especially if* $\sup_{n\in\mathbb{N}} |q_n| < \infty$*.*

2. *Regime 2: If* $\lim_{n\to\infty} |q_n|/r_n = l \in (0,\infty)$ *it follows that*

$$\lim_{n\to\infty}(p^n(q_n) - d^n) = -\frac{1}{la}\sup_{y\in\mathbb{R}}(-lay - I(y)) \in \mathbb{R} \tag{5.18}$$

3. *Regime 3:*
   - *If* $\lim_{n\to\infty} q_n/r_n = \infty$*, then* $\limsup_{n\to\infty}(p^n(q_n) - d^n) \le \inf\{y \in \mathbb{R} : I(y) < \infty\}$.
   - *If* $\lim_{n\to\infty} q_n/r_n = -\infty$*, then* $\liminf_{n\to\infty}(p^n(q_n) - d^n) \ge \sup\{y \in \mathbb{R} : I(y) < \infty\}$.

As Proposition 5.2.2 is a special case of Proposition 5.2.1, the proof follows immediately from the one of 5.2.1 below.

Proof of Proposition 5.2.1. Regime 1: In order to guarantee the moment condition (5.4), recall δ from Assumption 5.2 and fix an $\epsilon > 0$ such that $\epsilon a < \delta$. Further, observe that, as $\lim_{n\to\infty}|q_n|/r_n = 0$, there exists an $n \in \mathbb{N}$ such that $|q_n| < \epsilon r_n$ for n large enough. As $p^n(q_n)$ is non-increasing due to Lemma 2.4.1 and $-\epsilon r_n \le q_n \le \epsilon r_n$ the following inequality holds,

$$-\frac{1}{a\epsilon r_n}\Lambda_n(-a\epsilon r_n) = p^n(\epsilon r_n) - d^n \le p^n(q_n) - d^n \le p^n(-\epsilon r_n) - d^n = \frac{1}{a\epsilon r_n}\Lambda_n(a\epsilon r_n).$$

Together with Varadhan's integral lemma 2.5.1 this implicates

$$\begin{aligned}
\liminf_{n\to\infty}\ p^n(q_n) - d^n &\ge \liminf_{n\to\infty}\ p^n(\epsilon r_n) - d^n = -\frac{1}{a\epsilon}\sup_{y\in\mathbb{R}}(-a\epsilon y - I(y)) \\
\limsup_{n\to\infty}\ p^n(q_n) - d^n &\le \limsup_{n\to\infty}\ p^n(-\epsilon r_n) - d^n = \frac{1}{a\epsilon}\sup_{y\in\mathbb{R}}(a\epsilon y - I(y)).
\end{aligned}$$

Note, that for some $y_{\pm\epsilon} \in [l^{\pm\epsilon a}, u^{\pm\epsilon a}]$, with ϵ small enough as required above and l^ϵ and u^ϵ as defined in Lemma A.1.3, the maximum of the function $(\pm a\epsilon y - I(y))$ over all $y \in \mathbb{R}$ is attained, hence

$$\begin{aligned}
\liminf_{n\to\infty}\ p^n(q_n) - d^n &\ge y_{-\epsilon} + \frac{1}{a\epsilon}I(y_{-\epsilon}) \ge y_{-\epsilon} \\
\limsup_{n\to\infty}\ p^n(q_n) - d^n &\le y_{+\epsilon} - \frac{1}{a\epsilon}I(y_{+\epsilon}) \le y_{+\epsilon}.
\end{aligned} \tag{5.19}$$

Additionally, Lemma A.1.3 states that $l^{\pm\epsilon} \to 0$ as well as $u^{\pm\epsilon} \to 0$ for $\epsilon \to 0$ and therefore $y_{\pm\epsilon} \to 0$. As (5.19) holds for all ϵ small enough, the inequality also holds for the limiting value for $\epsilon \to 0$. This concludes the proof for Regime 1, as $\lim_{n\to\infty}\ p^n(q_n) - d^n = 0$.

Regime 2: Suppose $\lim_{n\to\infty}|q_n|/r_n = l \in (0,\infty)$. To prove (5.10) assume $0 < l < -\underline{M}/a$. By the limit requirement, it follows that there exists an $n \in \mathbb{N}$ large enough such that $(l-\gamma)r_n \le q_n \le (l+\gamma)r_n$ for a constant $\gamma > 0$. By choosing γ small enough it follows that

$\underline{M} < -a(l+\gamma) < -a(l-\gamma) < 0$ which by the fact that $q_n \mapsto p^n(q_n)$ is non-increasing, see Lemma 2.4.1, yields

$$p^n(q_n) - d^n \leq p^n((l-\gamma)r_n) - d^n = -\frac{1}{a(l-\gamma)r_n}\Lambda_n(-a(l-\gamma)r_n)$$

$$p^n(q_n) - d^n \geq p^n((l+\gamma)r_n) - d^n = -\frac{1}{a(l+\gamma)r_n}\Lambda_n(-a(l+\gamma)r_n).$$

Further, note that as $I(y) \geq 0$ for every $y \in \mathbb{R}$ and $I(y) = 0$ if and only if $y = 0$, $\sup_{y\in\mathbb{R}}(-(l-\gamma)ay - I(y)) \geq 0$. As $-a(l+\gamma) > \underline{M} \geq M_*$ and by the definition of M_* it follows that $\sup_{y\in\mathbb{R}}(-(l+\gamma)ay - I(y)) \leq \infty$. Then, Varadhan's integral lemma 2.5.1 for $\epsilon = r_n$ implies

$$\begin{aligned}\limsup_{n\to\infty} p^n(q_n) - d^n &\leq \limsup_{n\to\infty} p^n((l-\gamma)r_n) - d^n \overset{2.5.1}{=} -\frac{1}{a(l-\gamma)}\sup_{y\in\mathbb{R}}(-a(l-\gamma)ay - I(y)) \\ &= \inf_{y\in\mathbb{R}}\left(y + \frac{I(y)}{(l-\gamma)a}\right) \leq 0; \\ \liminf_{n\to\infty} p^n(q_n) - d^n &\geq \liminf_{n\to\infty} p^n((l+\gamma)r_n) - d^n \overset{2.5.1}{=} -\frac{1}{a(l+\gamma)}\sup_{y\in\mathbb{R}}(-a(l+\gamma)ay - I(y)) \\ &= \inf_{y\in\mathbb{R}}\left(y + \frac{I(y)}{(l+\gamma)a}\right) > -\infty.\end{aligned}$$

As affine functions are convex and concave and the infimum of a concave function again is concave, one can draw the conclusion that the function $\tau \mapsto \inf_{y\in\mathbb{R}}(y + \tau I(y))$ is continuous on the interior of its effective domain. Thus, (5.10) follows by taking $\gamma \to 0$.

To prove (5.11) assume that $M_* > -\infty$ and $l > M_*/a$. As $M_* \leq \underline{M}$, $l > -\underline{M}/a$ follows immediately from (5.8). Consequently, there is a $\gamma > 0$ such that for an $n \in \mathbb{N}$ large enough it is reasonable to assume that $q_n \geq (l-\gamma)r_n$ and $l - \gamma > -M_*/a \geq -\underline{M}/a$. Again, due to the fact that $q_n \mapsto p^n(q_n)$ is non-increasing, it follows that

$$p^n(q_n) - d^n \leq p^n((l-\gamma)r_n) - d^n = -\frac{1}{a(l-\gamma)r_n}\Lambda_n(-a(l-\gamma)r_n). \tag{5.20}$$

Recall the definition of $\underline{M}$ in (5.5) and note that this together with $-a(l-\gamma) < \underline{M}$ implies that $\limsup_{n\to\infty}(1/r_n)\Lambda_n(-a(l-\gamma)r_n) = \infty$. To show that even $\lim_{n\to\infty}(1/r_n)\Lambda_n(-a(l-\gamma)r_n) = \infty$ suppose there exists a subsequence with $\limsup_{k\to\infty}(1/r_k)\Lambda_k(-a(l-\gamma)r_k) < \infty$. Therefore, for the subsequence for which the LDP still holds, the conditions for Varadhan's integral lemma 2.5.1 are satisfied. Thus, for γ sufficiently small, it results in

$$\lim_{k\to\infty}(1/r_k)\Lambda_k(-a(l-2\gamma)r_k) = \sup_{y\in\mathbb{R}}(-a(l-2\gamma)y - I(y)) < \infty.$$

Due to γ being such that $-a(l-2\gamma) < M_*$ there is a contradiction to the definition of M_*, as according to this definition $\sup_{y\in\mathbb{R}}(-a(l-2\gamma)y - I(y))$ has to be ∞. Hence, as

$-1/(a(l-\gamma)r_n) < 0$ and equation (5.20) holds, it follows that $\lim_{n\to\infty} p^n(q_n) - d^n = -\infty$.

Consider $\lim_{n\to\infty} q_n/r_n = l$ for an $l \in (-\infty, 0)$. To receive the result in this case the approach is analogue to the one for $l > 0$. As a first step, suppose $\bar{M}/a < l < 0$. By the limit requirement, it follows that there exists an $n \in \mathbb{N}$ large enough such that $(l-\gamma)r_n \leq q_n \leq (l+\gamma)r_n$ for a constant $\gamma > 0$. By choosing γ small enough it follows that $0 < -a(l+\gamma) < -a(l-\gamma) < \bar{M} < 0$ which due to the fact that $q_n \mapsto p^n(q_n)$ is non-increasing, see Lemma 2.4.1, yields

$$p^n(q_n) - d^n \leq p^n((l-\gamma)r_n) - d^n = -\frac{1}{a(l-\gamma)r_n}\Lambda_n(-a(l-\gamma)r_n)$$
$$p^n(q_n) - d^n \geq p^n((l+\gamma)r_n) - d^n = -\frac{1}{a(l+\gamma)r_n}\Lambda_n(-a(l+\gamma)r_n).$$

Further, note that as $I(y) \geq 0$ for every $y \in \mathbb{R}$ and $I(y) = 0$ if and only if $y = 0$, $\sup_{y\in\mathbb{R}}(-(l+\gamma)ay - I(y)) \geq 0$. As $-a(l-\gamma) < \bar{M}$ and by the definition of $\bar{M}$ it follows that $\sup_{y\in\mathbb{R}}(-(l-\gamma)ay - I(y)) < \infty$. Varadhan's integral lemma 2.5.1 for $\epsilon = r_n$ implies

$$\begin{aligned}
\limsup_{n\to\infty} p^n(q_n) - d^n &\leq \limsup_{n\to\infty} p^n((l-\gamma)r_n) - d^n \overset{2.5.1}{=} -\frac{1}{a(l-\gamma)}\sup_{y\in\mathbb{R}}(-a(l-\gamma)ay - I(y)) \\
&= \sup_{y\in\mathbb{R}}\left(y + \frac{I(y)}{(l-\gamma)a}\right) < \infty; \\
\liminf_{n\to\infty} p^n(q_n) - d^n &\geq \liminf_{n\to\infty} p^n((l+\gamma)r_n) - d^n \overset{2.5.1}{=} -\frac{1}{a(l+\gamma)}\sup_{y\in\mathbb{R}}(-a(l+\gamma)ay - I(y)) \\
&= \sup_{y\in\mathbb{R}}\left(y + \frac{I(y)}{(l+\gamma)a}\right) \geq 0.
\end{aligned}$$

As affine functions are convex and concave and the supremum of a convex function again is convex, one may conclude that the function $\tau \mapsto \inf_{y\in\mathbb{R}}(y + \tau I(y))$ is continuous on the interior of its effective domain. Thus, (5.12) follows by taking $\gamma \to 0$.

To prove (5.13) assume that $M^* < \infty$ and $l < M^*/a$. As $M^* \geq \bar{M}$, $l < -\bar{M}/a$ follows immediately from (5.8). Consequently, there is a $\gamma > 0$ such that for a $n \in \mathbb{N}$ large enough it can be assumed that $q_n \geq (l+\gamma)r_n$ and $l + \gamma < -M^*/a \leq -\bar{M}/a$. Again, due to the fact that $q_n \mapsto p^n(q_n)$ is non-increasing it follows that

$$p^n(q_n) - d^n \geq p^n((l+\gamma)r_n) - d^n = -\frac{1}{a(l+\gamma)r_n}\Lambda_n(-a(l+\gamma)r_n). \tag{5.21}$$

Recall the definition of $\bar{M}$ in (5.5) and note that it, together with $-a(l+\gamma) > \bar{M}$, implies that $\limsup_{n\to\infty}(1/r_n)\Lambda_n(-a(l+\gamma)r_n) = \infty$. An analogous argumentation as presented above yields to $\lim_{n\to\infty} p^n(q_n) - d^n = \infty$.

Regime 3: First, assume that $q_n/r_n \to \infty$. This implies that for each $M > 0$ there exists a $N \in \mathbb{N}$ such that $q_n \geq Mr_n$ for every $n > N$. As $p^n(q_n)$ is non-increasing in q_n, one receives

$$p^n(q_n) - d^n \leq p^n(Mr_n) - d^n = -\frac{1}{Mr_n}\Lambda_n(-Mr_n). \tag{5.22}$$

Then, in case $M_* > -\infty$, it follows by (5.8) that $\underline{M} > -\infty$. By choosing M sufficiently large such that $-aM < M_* \leq \underline{M}$ the definition of M_* yields $\limsup_{n\to\infty}(1/r_n)\Lambda_n(-aMr_n) = \infty$. To see that even $\lim_{n\to\infty}(1/r_n)\Lambda_n(-aMr_n) = \infty$ suppose there exists a subsequence with $\limsup_{k\to\infty}(1/r_k)\Lambda_k(-a(l-\gamma)r_k) < \infty$. Therefore, for the subsequence, for which the LDP still holds, the conditions for Varadhan's integral lemma 2.5.1 are satisfied. Thus, for γ small enough that $-a(M-\gamma) < M_*$, it results in

$$\lim_{k\to\infty}(1/r_k)\Lambda_k(-a(M-\gamma)r_k) = \sup_{y\in\mathbb{R}}(-a(M-\gamma)y - I(y)) < \infty.$$

This is a contradiction to the definition of M_*, as according to this $\sup_{y\in\mathbb{R}}(-a(M-\gamma)y - I(y))$ has to be ∞. Hence, as $-1/(a(M-\gamma)r_n) < 0$ and equation (5.22) holds, it follows that $\lim_{n\to\infty} p^n(q_n) - d^n = -\infty$, proving (5.14).
To show (5.15), note that assuming $\underline{M} = -\infty$ forces $M_* = -\infty$. Then, by equation (5.22) and Varadhan's integral lemma 2.5.1 for every $M > 0$, it follows that

$$\begin{aligned}\limsup_{n\to\infty}\, p^n(q_n) - d^n \leq \limsup_{n\to\infty}\, p^n(Mr_n) - d^n &= -\frac{1}{aM}\sup_{y\in\mathbb{R}}(-aMy - I(y))\\ &= \inf_{y\in\mathbb{R}}\left(y + \frac{I(y)}{aM}\right).\end{aligned}$$

Moreover, $\inf_{y\in\mathbb{R}}(y + I(y)/(aM))$ is decreasing in M. Note that

$$\limsup_{n\to\infty}\, p^n(q_n) - d^n \leq \lim_{M\to\infty}\inf_{y\in\mathbb{R}}\left(y + \frac{I(y)}{aM}\right) \tag{5.23}$$

and observe that as $I(y) \geq 0$ for any $y \in \mathbb{R}$

$$\lim_{M\to\infty}\inf_{y\in\mathbb{R}}\left(y + \frac{I(y)}{aM}\right) = \lim_{M\to\infty}\inf_{\substack{y\in\mathbb{R}\\ I(y)<\infty}}\left(y + \frac{I(y)}{aM}\right) \geq \inf\left\{y \in \mathbb{R} : I(y) < \infty\right\}.$$

Then fix a y with $I(y) < \infty$. Since $\inf_{y\in\mathbb{R}}(y + I(y)/(aM)) \leq y + I(y)/(aM)$, by taking $M \to \infty$ one receives

$$\lim_{M\to\infty}\inf_{y\in\mathbb{R}}\left(y + \frac{I(y)}{aM}\right) \leq \lim_{M\to\infty} y + I(y)/(aM) = y.$$

Taking $y \to \inf\{y \in \mathbb{R} : I(y) < \infty\}$ results in

$$\lim_{M\to\infty}\inf_{y\in\mathbb{R}}\left(y+\frac{I(y)}{aM}\right)=\inf\left\{y\in\mathbb{R}:I(y)<\infty\right\},$$

which by (5.23) provides the result in (5.15).

Proving (5.16) for $q_n/r_n\to-\infty$ requires a similar argumentation as for $q_n/r_n\to\infty$. For $\bar{M}=\infty$, note that $M^*=\infty$ follows instantaneously. This implies that for each $M<0$ there exists an $N\in\mathbb{N}$ such that $q_n\le Mr_n$ for every $n>N$ as $r_n\to\infty$. As $p^n(q_n)$ is non-increasing in q_n one receives

$$p^n(q_n)-d^n\ge p^n(Mr_n)-d^n=-\frac{1}{Mr_n}\Lambda_n(-Mr_n). \tag{5.24}$$

Together with Varadhan's integral lemma 2.5.1, this implicates for every $M<0$

$$\begin{aligned}\liminf_{n\to\infty}p^n(q_n)-d^n\ge\liminf_{n\to\infty}p^n(Mr_n)-d^n&=-\frac{1}{aM}\sup_{y\in\mathbb{R}}(-aMy-I(y))\\&=\sup_{y\in\mathbb{R}}\left(y+\frac{I(y)}{aM}\right).\end{aligned}$$

Observe that $\sup_{y\in\mathbb{R}}(y+I(y)/(aM))$ is decreasing in M and that

$$\liminf_{n\to\infty}p^n(q_n)-d^n\ge\lim_{M\to\infty}\sup_{y\in\mathbb{R}}\left(y+\frac{I(y)}{aM}\right). \tag{5.25}$$

Further, note that as $I(y)\ge0$ for any $y\in\mathbb{R}$ and $M<0$

$$\lim_{M\to\infty}\sup_{y\in\mathbb{R}}\left(y+\frac{I(y)}{aM}\right)=\lim_{M\to\infty}\sup_{\substack{y\in\mathbb{R}\\I(y)<\infty}}\left(y+\frac{I(y)}{aM}\right)\le\sup\left\{y\in\mathbb{R}:I(y)<\infty\right\}.$$

Then fix a y with $I(y)<\infty$. Since $\sup_{y\in\mathbb{R}}(y+I(y)/(aM))\ge y+I(y)/(aM)$, by taking $M\to-\infty$ one receives

$$\lim_{M\to-\infty}\sup_{y\in\mathbb{R}}\left(y+\frac{I(y)}{aM}\right)\ge\lim_{M\to-\infty}y+I(y)/(aM)=y.$$

Taking $y\to\sup\{y\in\mathbb{R}:I(y)<\infty\}$ yields

$$\lim_{M\to-\infty}\sup_{y\in\mathbb{R}}\left(y+\frac{I(y)}{aM}\right)=\sup\left\{y\in\mathbb{R}:I(y)<\infty\right\},$$

which by (5.25) provides the result in (5.17).

□

5.3. Relations of Optimal Quantities and Large Deviations Scaling

In accordance to the appropriate section of [Robertson and Spiliopoulos, 2018], this section is about the specification of the limiting value l when q_n is the sequence of optimal purchasing quantities. Section 5.2 deals with the evaluation of all possible limiting indifference prices for each limit $l = \lim_{n\to\infty} q_n/r_n$. The major achievement is proving that by purchasing optimal quantities the limiting values$l \in \{0, \pm\infty\}$ does not occur for all *reasonable prices* as defined below. First, recall Proposition 3.5.1, more precisely the unique value $\hat{q}_n(\tilde{p}^n)$ satisfying the first order condition (3.35) for every $\tilde{p}^n \in (\underline{b}_n, \bar{b}_n)$ with $\underline{b}_n$ and $\bar{b}_n$ as in Lemma 3.4.1. This gives

$$\underline{b}_n = d^n + \text{essinf}_{\mathbb{P}^n}(Y_n), \qquad \bar{b}_n = d^n + \text{esssup}_{\mathbb{P}^n}(Y_n).$$

In order to motivate the *reasonable prices*, two special cases should be excluded in the following considerations. A reasonable price $\tilde{p}^n \in (\underline{b}_n, \bar{b}_n)$ should satisfy $\liminf_{n\to\infty} |\tilde{p}^n - d^n| > 0$. The first case violates this condition is a trivial one. In fact, if there exists some subsequence $(Y_k)_{k\in\mathbb{N}}$ of $(Y_n)_{n\in\mathbb{N}}$ and some limit l^y denoting

$$l^y = \lim_{k\to\infty} \text{esssup}_{\mathbb{P}^k}(Y_k) = \lim_{k\to\infty} \text{essinf}_{\mathbb{P}^k}(Y_k),$$

by the definition of the essential supremum and infimum, more specifically as $\mathbb{P}^n[Y_n \geq l^y + \epsilon] \to 0$ and $\mathbb{P}^n[Y_n \leq l^y - \epsilon] \to 0$, it follows that $(1/r_n)\log(\mathbb{P}^n[Y_n \geq l^y + \epsilon]) \to -\infty$ and $(1/r_n)\log(\mathbb{P}^n[Y_n \leq l^y - \epsilon]) \to -\infty$ for any $\epsilon > 0$. As $I(y) = 0$ if and only if $y = 0$ by Assumption 5.2 and therefore the laws of Y_n converge to the Dirac measure at 0, this forces $l^y = 0$ and I takes the form $I(0) = 0, I(y) = \infty$ for $y \neq 0$. Moreover, one may observe that the set of arbitrage free prices converges to the singleton $\{d^n\}$ and hence, for all arbitrage free prices $\tilde{p}^n(q_n)$ of every sequence $(q_n)_{n\in\mathbb{N}}$, one obtains $\lim_{n\to\infty} \tilde{p}^n(q_n) - d^n = 0$. This result is completely unaffected by the limiting relation between q_n and r_n. This trivial case, together with one special case, where there is some oscillation in the range of arbitrage free prices, meaning that the prices may be arbitrage free for some subsequence $(n_k)_{k\in\mathbb{N}}$ of q_n converging to infinity, but not for another $(n_j)_{j\in\mathbb{N}}$, were excluded in the considerations of [Robertson and Spiliopoulos, 2018] by imposing the following assumption.

Assumption 5.3. *For $l^y = \lim_{n\to\infty} essinf_{\mathbb{P}^n}(Y_n)$ and $u^y = \lim_{n\to\infty} esssup_{\mathbb{P}^n}(Y_n)$ assume that $l^y < u^y$. Therefore, for some $p \in (l^y, u^y)$, it follows that $p + d^n \in (\underline{b}_n, \bar{b}_n)$ for a n large enough, hence, it is an arbitrage free price.*

Remark 5.2. Recall the large market setting of Section 4, where $\underline{b}_n = \text{essinf}_{\mathbb{P}}(Y_n) = \sum_{i=n+1}^{\infty} \text{essinf}_{\mathbb{P}}(B_i)$ and $\bar{b}_n = \text{esssup}_{\mathbb{P}}(Y_n) = \sum_{i=n+1}^{\infty} \text{esssup}_{\mathbb{P}}(B_i)$. Then, Assumption 5.3 forces $l^y = 0$ or $l^y = -\infty$ and $u^y = 0$ or $u^y = \infty$. To state this argumentation more precisely, remember that Lemma 4.1.1 guarantees that $\sum_{i=1}^{\infty} \mathbb{E}[B_i]$ exists and is finite, which leads to

$$\sum_{i=n+1}^{\infty} \text{esssup}_{\mathbb{P}}(B_i) = \sum_{i=n+1}^{\infty} \left(\text{esssup}_{\mathbb{P}}(B_i) - \mathbb{E}\left[B_i\right]\right) + \sum_{i=n+1}^{\infty} \mathbb{E}\left[B_i\right].$$

Consequently, in case $\sum_{i=1}^{\infty}(\mathrm{esssup}_{\mathbb{P}}(B_i) - \mathbb{E}[B_i]) < \infty$, the sequence of the $\mathrm{esssup}_{\mathbb{P}}(B_i)$ has to be a null sequence, hence $u^y = 0$. Then again, if the sum is infinite, one receives $u^y = \infty$ since $\sum_{i=1}^{\infty}\mathbb{E}[B_i]$ is finite. Analogue arguments yield $l^y = 0$ or $l^y = -\infty$.
In general, since it was assumed that $I(y) = 0$ if and only if $y = 0$ and therefore Y_n converges to zero, one must have $l^y \leq 0 \leq u^y$.

Before formulating the main result of this section, the idea of reasonable prices should be briefly discussed. As mentioned above, for such a reasonable price $\tilde{p}^n$ it should hold that $\liminf_{n\to\infty}|\tilde{p}^n - d^n| > 0$. This means, the probability of $|Y_n|$ being greater than $|\tilde{p}^n - d^n|$ should not be zero. To state this in the large deviations setting, it will be assumed that there is at least one $y < \tilde{p}^n - d^n$ and one $y > \tilde{p}^n - d^n$, such that the rate function I from Assumption 5.2 is finite at these y.

In the following, note that there exists an $N \in \mathbb{N}$ such that $\tilde{p}^n - d^n = p^n \in (l^y, u^y)$ for each $n \geq N$. Further, denote the limit $\lim_{n\to\infty} p^n =: p$. Without loss of generality, it can be assumed that $p^n \equiv p$, as the results in the following proposition do not change for $\tilde{p}^n - d^n = p^n \to p$. After all this preliminaries, the proposition below gives the desired result.

Proposition 5.3.1. *Suppose that Assumption 5.1, 5.2 and 5.3 hold. Given a $\tilde{p}^n \in (\underline{b}_n, \bar{b}_n)$ let $\hat{q}_n = \hat{q}_n(\tilde{p}^n)$ denote the optimal purchase quantity for each arbitrage free price $\tilde{p}^n$ as in Proposition 3.5.1. Further, let (l^y, u^y) be as defined in Assumption 5.3. Then, it follows:*

1. *If $l^y < 0$ and $\tilde{p}^n = d^n + p$ for $l^y < p < 0$, then*
   - $\liminf_{n\to\infty} \hat{q}_n/r_n > 0$.
   - *If $I(y) < \infty$ for at least one $y < p$, then* $\limsup_{n\to\infty} \hat{q}_n/r_n < \infty$.
2. *If $u^y > 0$ and $\tilde{p}^n = d^n + p$ for $0 < p < u^y$, then*
   - $\limsup_{n\to\infty} \hat{q}_n/r_n < 0$.
   - *If $I(y) < \infty$ for at least one $y > p$, then* $\liminf_{n\to\infty} \hat{q}_n/r_n > -\infty$.

Proof. First note that (3.35) for Y_n writes as

$$p = \tilde{p}^n - d_n = \dot{\Lambda}_n(-a\hat{q}_n) = \frac{\mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{-a\hat{q}_n Y_n}\right]}{\mathbb{E}^{\mathbb{P}^n}\left[e^{-a\hat{q}_n Y_n}\right]}, \tag{5.26}$$

where $\dot{\Lambda}_n$ is the first derivative of Λ_n as in (5.2). By Lemma A.1.1 it is clear that Λ_n is convex and hence the function $q \mapsto \mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{qY_n}\right]/\mathbb{E}^{\mathbb{P}^n}\left[e^{qY_n}\right]$ is increasing in q.

Statement 1.: By way of contradiction, suppose that $\liminf_{n\to\infty}\hat{q}_n/r_n \leq 0$. This allows to assume that for some $\epsilon > 0$, there is a subsequence, which is still labeled by n, that satisfies $a\hat{q}_n \leq \epsilon r_n$ for sufficiently large n. As this yields $-a\hat{q}_n \geq -\epsilon r_n$ and $\dot{\Lambda}_n$ is increasing, one arrives at

$$p = \dot{\Lambda}_n(-a\hat{q}_n) \geq \dot{\Lambda}_n(-\epsilon r_n) = \frac{\mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{-\epsilon r_n Y_n}\right]}{\mathbb{E}^{\mathbb{P}^n}\left[e^{-\epsilon r_n Y_n}\right]} \tag{5.27}$$

This gives as $n \to \infty$ and $\epsilon \to 0$

$$p \geq \liminf_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{-\epsilon r_n Y_n}\right]}{\mathbb{E}^{\mathbb{P}^n}\left[e^{-\epsilon r_n Y_n}\right]} = 0$$

The equality holds due to Appendix Lemma A.1.4. Therein, it was shown that the desired limit is 0, if one takes $-\epsilon$ instead of ϵ. Observe that this leads to $p \geq 0$, which is a contradiction to condition $p < 0$. Thus, under the required conditions, $\liminf_{n\to\infty} \hat{q}_n/r_n > 0$ has to hold.
In the sequel, suppose $p < 0$ in order to guarantee the existence of a $y < p$ such that $I(y) < \infty$. Assume $\limsup_{n\to\infty} \hat{q}_n/r_n = \infty$. Then consider the subsequence, still labeled with n, satisfying $\lim_{n\to\infty} \hat{q}_n/r_n = \infty$. By Proposition 3.5.1 it follows that for $\hat{q}_n$ the infimum of (3.34) is uniquely attained, and hence minimizes

$$qp + \frac{1}{a}\Lambda_n(-qa) = qp + \frac{1}{a}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-aqY_n}\right]\right)$$

over all $q \in \mathbb{R}$. Therefore,

$$\hat{q}_n p + \frac{1}{a}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-a\hat{q}_n Y_n}\right]\right) \leq qp + \frac{1}{a}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-aqY_n}\right]\right)\bigg|_{q=0} = 0.$$

Furthermore, $\hat{q}_n/r_n \to \infty$ as $n \to \infty$ yields $\hat{q}_n > 0$ for some n sufficiently large. Dividing by $\hat{q}_n$ and subtracting p then gives

$$\frac{1}{a\hat{q}_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-a\hat{q}_n Y_n}\right]\right) \leq -p.$$

Apart from that, note that $q \mapsto (1/q)\log(\mathbb{E}^{\mathbb{P}_n}[e^{-qY_n}])$ is increasing for $q > 0$. Indeed, for some $\tilde{q} > q$ Hölder's inequality with Hölder exponents $r = \tilde{q}/q$ and $s = r/(r-1)$ gives

$$\frac{1}{q}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-qY_n}\right]\right) \leq \frac{1}{q}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-\tilde{q}Y_n}\right]^{\frac{q}{\tilde{q}}}\right) = \frac{1}{\tilde{q}}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-\tilde{q}Y_n}\right]\right). \tag{5.28}$$

For the next stage, note that $\hat{q}_n/r_n \to \infty$ as $n \to \infty$ implies that for some $M > 0$ there is an index N, such that for n greater than N the condition $0 \leq Mr_n \leq \hat{q}_n$ is met. This observation yields

$$\frac{1}{Mr_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-Mr_nY_n}\right]\right) \leq \frac{1}{a\hat{q}_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-a\hat{q}_n Y_n}\right]\right) \leq -p.$$

By Assumption 5.2 $(Y_n)_{n\in\mathbb{N}}$ has an LDP with rate function I and scaling $(r_n)_{n\in\mathbb{N}}$. Further, observe that since $-p < \infty$, it follows for each $M' < M \leq \bar{M}$ by Varadhan's integral lemma 2.5.1 and by (5.28) that

$$\lim_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-M' r_n Y_n}\right]\right) = \sup_{y\in\mathbb{R}}(-M'y - I(y)) \leq -M'p.$$

This immediately gives for each $y \in \mathbb{R}$

$$-M'y - I(y) \leq -M'p \Leftrightarrow -y - \frac{I(y)}{M'} \leq -p.$$

Due to assumption, there is at least one y with $I(y) < \infty$. Further, remember that $M > 0$ was arbitrary, meaning that all the conclusions above also hold for any other $M > 0$. Therefore, one should take a look on the result for taking $M' \to \infty$, which leads $-y \leq -p$. One may conclude that therefore $I(y) < \infty$ implies $y \geq p$, which yields $I(y) = \infty$ for $y < p$. This is a contradiction, as p should be such that there is at least one $y < p$ which satisfies $I(y) < \infty$. Thus, the assumption that $\limsup_{n\to\infty} \hat{q}_n/r_n = \infty$ has to be rejected and as a result $\limsup_{n\to\infty} \hat{q}_n/r_n < \infty$.

Statement 2.: To prove the second statement, i.e. the result for $p > 0$, one can follow the argumentation of Statement 1. Indeed, assume $\limsup_{n\to\infty} \hat{q}_n/r_n \geq 0$, which implicates that for some $\epsilon > 0$, there is a subsequence, which is also labeled by n, that satisfies $a\hat{q}_n \leq -\epsilon r_n$ for sufficiently large n. As above in Equation (5.26) one has

$$p = \dot{\Lambda}_n(-a\hat{q}_n) \leq \dot{\Lambda}_n(\epsilon r_n) = \frac{\mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{\epsilon r_n Y_n}\right]}{\mathbb{E}^{\mathbb{P}^n}\left[e^{\epsilon r_n Y_n}\right]},$$

which yields

$$p \leq \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{\epsilon r_n Y_n}\right]}{\mathbb{E}^{\mathbb{P}^n}\left[e^{\epsilon r_n Y_n}\right]} = 0,$$

as $n \to \infty$ and $\epsilon \to 0$. As the desired limit is 0 due to Appendix Lemma A.1.4 the equation above holds. However, this is a contradiction to condition $p > 0$. As a result, under the required conditions $\limsup_{n\to\infty} \hat{q}_n/r_n < 0$ has to hold.
Further, suppose $p > 0$ in order to guarantee the existence a $y > p$ such that $I(y) < \infty$ and assume $\liminf_{n\to\infty} \hat{q}_n/r_n = -\infty$. Then consider the subsequence, again labeled with n, satisfying $\lim_{n\to\infty} \hat{q}_n/r_n = -\infty$. As $\hat{q}_n$ minimizes

$$qp + \frac{1}{a}\Lambda_n(-qa) = qp + \frac{1}{a}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-aqY_n}\right]\right)$$

over all $q \in \mathbb{R}$, taking into account that $\hat{q}_n < 0$ gives

$$\hat{q}_n p + \frac{1}{a}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-a\hat{q}_n Y_n}\right]\right) \leq qp + \frac{1}{a}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-aqY_n}\right]\right)\Big|_{q=0} = 0.$$

Observe that this is equivalent to

$$-\frac{1}{a\hat{q}_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-a\hat{q}_n Y_n}\right]\right) \leq p.$$

Apart from that, note that $q \mapsto (1/q)\log(\mathbb{E}^{\mathbb{P}_n}[e^{-qY_n}])$ is increasing for $q > 0$.
For the next stage, note that $\hat{q}_n/r_n \to -\infty$ as $n \to \infty$ implies that for some $M > 0$ it follows that there is an index N, such that for every n larger that N it holds that $0 \leq Mr_n \leq -\hat{q}_n$. This observation yields

$$\frac{1}{Mr_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{Mr_n Y_n}\right]\right) \leq \frac{1}{a\hat{q}_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-a\hat{q}_n Y_n}\right]\right) \leq p.$$

By Assumption 5.2 $(Y_n)_{n\in\mathbb{N}}$ has an LDP with rate function I and scaling $(r_n)_{n\in\mathbb{N}}$. Further, observe that since $p < \infty$, it follows for each $M' < M \leq \bar{M}$ by Varadhan's integral lemma 2.5.1 and as $q \mapsto (1/q)\log(\mathbb{E}^{\mathbb{P}_n}[e^{-qY_n}])$ is increasing, that

$$\lim_{n\to\infty}\frac{1}{r_n}\log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{M' r_n Y_n}\right]\right) = \sup_{y\in\mathbb{R}}(M'y - I(y)) \leq M'p.$$

This immediately yields that $M'y - I(y) \leq M'p \Leftrightarrow y - I(y)/M' \leq p$ for each $y \in \mathbb{R}$. Note that by assumption, there is at least one y with $I(y) < \infty$ and $M > 0$ was arbitrarily chosen, consequently all the considerations also hold for any other $M > 0$. Therefore, one should take a look on the result for $M' \to \infty$, which leads to $y \leq p$. One may conclude that $I(y) < \infty$ implies $y \leq p$, which yields $I(y) = \infty$ for $y > p$. This is a contradiction to the assumption that $\liminf_{n\to\infty} \hat{q}_n/r_n = \infty$, as p should be such that there is at least one $y > p$ such that $I(y) < \infty$. Thus, the assumption has to be rejected and consequently $\liminf_{n\to\infty} \hat{q}_n/r_n > -\infty$.

□

6. Existence of an LDP

In this chapter, the validity of Assumption 5.2 shall be pointed out. As a first step it is argued below that the Gärtner-Ellis theorem 2.5.2, which provides condition for an LDP to hold, implicates Assumption 5.2. As mentioned in Section 2.5, Gärtner-Ellis does not cover all the sequences of random variables satisfying an LDP, as defined in 2.5.2, hence it is only sufficient but not necessary. To demonstrate this, two concrete examples will be given according to [Robertson and Spiliopoulos, 2018]. To this end, Example 4.1 and 4.2 will be expanded by firstly proving the LDP holds and then applying the herein build theory to it. For the first one the Gärtner-Ellis theorem holds and the LDP is given via it, so limiting indifference price and optimal quantities can be calculated according to the appropriate results. For the second example Gärnter-Ellis can not be applied, but despite this even for the considered example the LDP holds and the desired quantities can be calculated explicitly.

Recall Gärtner-Ellis theorem 2.5.2 and suppose it holds for $\xi_n = Y_n$, some sequence r_n with $r_n \to \infty$ and $\Lambda(\lambda) = \lim_{n\to\infty}(1/r_n)\Lambda_n(\lambda r_n)$, with Λ_n from (5.2). Then, since $I(y) = \sup_{y\in\mathbb{R}}(\lambda y - \Lambda(\lambda))$, the Legendre transform of $\Lambda(\lambda)$, performs as a (good) rate function and the condition $0 \in \mathcal{D}_\Lambda^o$ implies the moment condition (5.4), it follows that Assumption 5.2 holds.

6.1. Gaussian Case

Herein Example 4.1 should be extended, hence assume $B_i \overset{\mathbb{P}}{\sim} \mathcal{N}(\gamma_i, \delta_i^2)$ and

$$Y_n = \sum_{i=n+1}^{\infty} B_i \overset{\mathbb{P}}{\sim} \mathcal{N}\left(\sum_{i=n+1}^{\infty} \gamma_i, \sum_{i=n+1}^{\infty} \delta_i^2\right).$$

Again by Example 4.1, let $r_n = (\sum_{i=n+1}^{\infty} \delta_i^2)^{-1}$, which obviously satisfies $\lim_{n\to\infty} r_n = \infty$. Furthermore, one has for every $\lambda \in \mathbb{R}$

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\lambda r_n Y_n}\right]\right) = \lim_{n\to\infty}\left(\frac{1}{2}\lambda^2 + \lambda \sum_{i=n+1}^{\infty} \gamma_i\right) = \frac{1}{2}\lambda^2, \tag{6.1}$$

which is continuous and differentiable on $\mathbb{R}$. Therefore, one may use Gärtner-Ellis theorem to see that $(Y_n)_{n\in\mathbb{N}}$ satisfies an LDP with rate r_n and good rate function $I(y) = \sup_{\lambda\in\mathbb{R}}(\lambda y - \lambda^2/2) = y^2/2$. This implicates that Assumption 5.2 holds.

6.1.1 (**The Limiting Indifference Price**). For any large market example as introduced in Section 4, it was shown that the limit $d = \lim_{n\to\infty} d^n$ exists. Further, recall the utility indifference prices in the sequence of semi-complete markets from (4.9) and that Γ_i can be explicitly calculated by $\Gamma_i(\lambda) = \lambda\gamma_i + (1/2)\lambda^2\delta_i^2$, as done in Example 4.1. This gives

$$\begin{aligned} p^n(q_n) - d^n &= -\frac{1}{q_n a}\sum_{i=n+1}^{\infty}\Gamma_i(-q_n a) = -\frac{1}{q_n a}\sum_{i=n+1}^{\infty}\left(\frac{1}{2}q_n^2 a^2\delta_i^2 - q_n\gamma_i\right) \\ &= \frac{1}{2}q_n a\sum_{i=n+1}^{\infty}\delta_i^2 + \sum_{i=n+1}^{\infty}\gamma_i = -\frac{aq_n}{2r_n} + \sum_{i=n+1}^{\infty}\gamma_i, \end{aligned}$$

where the definition of r_n was taken into account, i.e. $r_n^{-1} = \sum_{i=n+1}^{\infty}\delta_i^2$. As a consequence, it follows that $\lim_{n\to\infty}(p^n(q_n) - d^n + aq_n/(2r_n)) = 0$, since $\sum_{i=n+1}^{\infty}\gamma_i \to 0$ for $n \to \infty$. Thus, one can explicitly calculate the values of $\lim_{n\to\infty} p^n(q_n)$ depending on the limit ratio of q_n/r_n by just using this result. This gives for any subsequence such that $\lim_{n\to\infty}|q_n|/r_n$ exists

- *(Regime 1)*: If $\lim_{n\to\infty}|q_n|/r_n = 0$ then $\lim_{n\to\infty} p^n(q_n) = d$.
- *(Regime 2)*: If $\lim_{n\to\infty}|q_n|/r_n = l \neq 0$ then $\lim_{n\to\infty}(p^n(q_n)) = d - (1/2)al$.
- *(Regime 3)*: If $\lim_{n\to\infty}|q_n|/r_n = \infty$ then $\lim_{n\to\infty}(p^n(q_n) - d^n) = \pm\infty$ for $q_n/r_n \to \mp\infty$.

Now, recall the definition of $\underline{M}$ and $\bar{M}$ from (5.5). It follows immediately by (6.1) and as $I(y) = y^2/2$, that $\underline{M} = M^* = -\infty$ and $M = \bar{M}_* = \infty$. Therefore, one may also use the results from Proposition 5.2.2, thus for any subsequence such that $\lim_{n\to\infty}|q_n|/r_n$ exists

- *(Regime 1)*: If $\lim_{n\to\infty}|q_n|/r_n = 0$ then $\lim_{n\to\infty} p^n(q_n) - d^n) = 0$ or equivalent $\lim_{n\to\infty}(p^n(q_n) = d$.
- *(Regime 2)*: If $\lim_{n\to\infty}|q_n|/r_n = l \neq 0$ then $\lim_{n\to\infty}(p^n(q_n) - d^n) = -\frac{1}{la}\sup_{y\in\mathbb{R}}(-lay - y^2/2) = -(1/2)al$, as the supremum is attained by $y = al$.
- *(Regime 3)*: If $\lim_{n\to\infty} q_n/r_n = \infty$, then $\limsup_{n\to\infty}(p^n(q_n) - d^n) \leq \inf\{y \in \mathbb{R} : y^2/2 < \infty\} = -\infty$. Else, if $\lim_{n\to\infty} q_n/r_n = -\infty$, then $\liminf_{n\to\infty}(p^n(q_n) - d^n) \geq \sup\{y \in \mathbb{R} : y^2/2 < \infty\} = \infty$.

Therefore, the results perfectly coincide.

6.1.2 (**Optimal Quantities**). From Example 4.1, the optimal purchase quantity $\hat{q}_n$ for some $\tilde{p}^n \in (\underline{b}_n, \bar{b}_n)$ takes the form

$$\hat{q}_n = \frac{d^n - \tilde{p}^n + \sum_{i=n+1}^{\infty}\gamma_i}{a\sum_{i=n+1}^{\infty}\delta_i^2} = \frac{r_n}{a}\left(d^n - \tilde{p}^n + \sum_{i=n+1}^{\infty}\gamma_i\right), \tag{6.2}$$

where the second equality follows by the definition of r_n. For a reasonable price $\tilde{p}^n$ with $\tilde{p}^n - d^n = p \neq 0$ the equation above yields $\lim_{n\to\infty} \hat{q}_n/r_n = -p/a$. If $\tilde{p}^n = d^n$ then taking $n \to \infty$ results in $\lim_{n\to\infty} \hat{q}_n/r_n = 0$, although (6.2) states that $|q_n| \to \infty$ is possible. To sum it up, optimal quantities for some reasonable price with $\tilde{p}^n - d^n = p$ yield $\hat{q}_n/r_n \to l = -p/a$. Thus, together with the results from 6.1.1 it follows that $\lim_{n\to\infty} p^n(q_n) = d + p/2$, with some non-trivial large deviation effect of $p/2$.

6.2. Poisson Case

Herein Example 4.2 shall be extended, hence assume $B_i \overset{\mathbb{P}}{\sim} \mathrm{Poi}(\beta_i)$ and

$$Y_n = \sum_{i=n+1}^{\infty} B_i \overset{\mathbb{P}}{\sim} \mathrm{Poi}\left(\sum_{i=n+1}^{\infty} \beta_i\right) \tag{6.3}$$

Due to the summation property of the Poisson distribution (6.3) holds, i.e. for each $\lambda \in \mathbb{R}$ the moment generating function of Y_n satisfies $\mathbb{E}^{\mathbb{P}}[e^{\lambda Y_n}] = e^{(e^{\lambda}-1)\sum_{i=n+1}^{\infty}\beta_i}$. Again, by Example 4.2, let $r_n = -\log(\sum_{i=n+1}^{\infty} \beta_i)$, which obviously satisfies $\lim_{n\to\infty} r_n = \infty$. Note that the definition of r_n yields $Y_n \overset{\mathbb{P}}{\sim} \mathrm{Poi}(e^{-r_n})$ and for each $\lambda \in \mathbb{R}$ it follows

$$\frac{1}{r_n} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\lambda r_n Y_n}\right]\right) = \frac{1}{r_n}\left(e^{\lambda r_n} - 1\right) e^{-r_n} = \frac{e^{(\lambda-1)r_n} - e^{-r_n}}{r_n}. \tag{6.4}$$

As a consequence, this together with L'Hôpital's rule gives

$$\lim_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\lambda r_n Y_n}\right]\right) = \begin{cases} \infty & \lambda > 1 \\ 0 & \lambda \leq 1. \end{cases}$$

Thus, in this case one is not able to use Gärtner-Ellis theorem 2.5.2 to prove the existence of an LDP for Y_n, as the additional assumptions are violated in this case. Nevertheless, explicit calculations in the following proposition will show the LDP for Y_n still holds.

Proposition 6.2.1. *Suppose $(\beta_i)_{i\in\mathbb{N}}$ is $\mathbb{P}$-independent such that $B_i \overset{\mathbb{P}}{\sim}$ Poi(β_i) for each $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \beta_i < \infty$. Further, set $r_n = -\log(\sum_{i=n+1}^{\infty} \beta_i)$. Then Y_n, as defined in* (6.3) *has an LDP with rate $(r_n)_{n\in\mathbb{N}}$ and good rate function*

$$I(y) = \begin{cases} \infty & y \notin \{0,1,2,3,\dots\} \\ 0 & y \in \{0,1,2,3,\dots\}. \end{cases}$$

Proof. First, consider for any $y \in \{0,1,2,3,\dots\}$

$$\frac{1}{r_n} \log\left(\mathbb{P}\left[Y_n = y\right]\right) = \frac{1}{r_n} \log\left(\frac{e^{-r_n y}}{y!} e^{-e^{-r_n}}\right) = -\frac{1}{r_n} e^{-r_n} - y - \frac{1}{r_n}\log(y!), \tag{6.5}$$

which is a consequence of $Y_n \overset{\mathbb{P}}{\sim} \mathrm{Poi}(e^{-r_n})$. For $y \in \{0,1,2,3,\dots\}$ assume $A \subset \mathbb{R}$ with $y \in A$ is an open set. Then, (6.5) together with the monotony of the probability measure yields

$$\liminf_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[Y_n \in A\right]\right) \geq \liminf_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[Y_n = y\right]\right) = -y = -I(y)$$

which implicates (2.20). As a next step suppose $A \subset \mathbb{R}$ is compact and in the first place assume $A \cap \{0,1,2,3,\dots\} = \emptyset$. Then one receives that

$$\lim_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[Y_n \in A\right]\right) = -\infty = -\inf_{y\in A} I(y),$$

where the first equality follows as $\mathbb{P}\left[Y_n \in A\right] = 0$ and the second one as I satisfies $I(y) = \infty$ for $y \in A$ by construction. Now, assume $A \cap \{0,1,2,3,\dots\} \neq \emptyset$ and denote by $\{y_1,\dots,y_M\}$ the finite set of non-negative integers in A. Then again, by (6.5) it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[Y_n \in A\right]\right) &= \limsup_{n\to\infty} \frac{1}{r_n} \log\left(\sum_{m=1}^{M} \mathbb{P}\left[Y_m = y_m\right]\right) \\
&= \max_{m=1,\dots,M} \left\{\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[Y_m = y_m\right]\right)\right\} \\
&= \max_{m=1,\dots,M} \{-y_m\} = -\min_{m=1,\dots,M} \{I(y_m)\} = -\inf_{y\in A} I(y).
\end{aligned}$$

Therein the second equality follows by Lemma 2.5.2 and the last as A is supposed to be compact. Hence, it was shown that $(Y_n)_{n\in\mathbb{N}}$ satisfies a weak LDP as defined in 2.5.3. By Lemma 2.5.1, it remains to prove that $(Y_n)_{n\in\mathbb{N}}$ is exponentially tight according to Definition 2.5.4. Thus, fix an arbitrary $K > 0$. Then for every $\lambda > 0$ it follows by applying the Markov inequality to $e^{\lambda Y_n}$, that

$$\begin{aligned}
\mathbb{P}\left[Y_n \geq K\right] &= \mathbb{P}\left[e^{\lambda Y_n} \geq e^{\lambda K}\right] \leq \frac{\mathbb{E}^{\mathbb{P}^n}\left[e^{\lambda Y_n}\right]}{e^{\lambda K}} \\
&= e^{-\lambda K + \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\lambda Y_n}\right]\right)} = e^{-\lambda K + e^{-r_n}(e^{\lambda}-1)}.
\end{aligned}$$

As this equation holds for each $\lambda \in \mathbb{R}$, one may take the minimizing value $\hat{\lambda} = r_n + \log(K)$ for the right hand side as well as the limits on both sides and obtains

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{P}\left[Y_n \geq K\right]\right) &\leq \limsup_{n\to\infty} \frac{1}{r_n} \left((-r_n + \log(K))K + e^{-r_n}\left(e^{r_n + \log(K)} - 1\right)\right) \\
&= \limsup_{n\to\infty} \frac{e^{\log(K)}}{r_n} - K\frac{\log(K)}{r_n} - \frac{e^{-r_n}}{r_n} = -K.
\end{aligned}$$

This provides the result, as due to this estimations $(Y_n)_{n\in\mathbb{N}}$ is exponentially tight and thus satisfies an LDP by Lemma 2.5.1. □

The proposition, together with the fact that I is lower semicontinuous and satisfies $I(y) = 0$ if and only if $y = 0$, implies that I fulfils the conditions for the rate function from Assumption 5.2. Together with (6.4) it follows that $M_* = \underline{M} = -\infty$ and $\bar{M} = M_* = 1$, thus Assumption 5.2 holds for $(Y_n)_{n\in\mathbb{N}}$ with $\delta = 1$.

6.2.1 (**The Limiting Indifference Price**)**.** As mentioned before, for any large market example as introduced in Section 4, it was shown that the limit $d = \lim_{n\to\infty} d^n$ exists. As Example 4.2 satisfies all required conditions from Assumption 5.1 and it was argued above that Assumption 5.2 holds, one may use Proposition 5.2.1 to identify the limiting indifference prices. To this end, recall the form of I from Proposition 6.2.1 and suppose a subsequence of $(q_n)_{n\in\mathbb{N}}$, still indexed by n, so that $\lim_{n\to\infty} |q_n|/r_n$ exists, then

- *(Regime 1)*: If $\lim_{n\to\infty} |q_n|/r_n = 0$ then $\lim_{n\to\infty} p^n(q_n) - d_n = 0$.
- *(Regime 2)*: If $\lim_{n\to\infty} |q_n|/r_n = l \neq 0$ then

$$\lim_{n\to\infty} (p^n(q_n) - d^n) = \begin{cases} \inf_{y\in\mathbb{R}} \left(y + \frac{1}{al} I(y)\right) = 0 & l > 0 \\ \sup_{y\in\mathbb{R}} \left(y + \frac{1}{al} I(y)\right) = 0 & -\dfrac{1}{a} < l < 0 \\ \infty & l < -\dfrac{1}{a} \end{cases}$$

- *(Regime 3)*: If $\lim_{n\to\infty} q_n/r_n = \infty$, then $\limsup_{n\to\infty} (p^n(q_n) - d^n) \leq \inf\{y \in \mathbb{R} : I(y) < \infty\} = 0$. Else, if $\lim_{n\to\infty} q_n/r_n = -\infty$, then as $M^* < \infty$ $\lim_{n\to\infty}(p^n(q_n) - d^n) = \infty$.

Furthermore, as the explicit formula of $p^n(q_n)$ is available, one can precise $\lim_{n\to\infty}(p^n(q_n) - d^n)$ for $\lim_{n\to\infty} q_n/r_n = \infty$. Indeed, consider

$$\lim_{n\to\infty} (p^n(q_n) - d^n) = -\lim_{n\to\infty} \frac{1}{q_n a} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-q_n a Y_n}\right]\right) = -\lim_{n\to\infty} \frac{1}{q_n a} e^{-r_n}(e^{-q_n a} - 1) = 0,$$

which strengthens the statement for Regime 3.

6.2.2 (**Optimal Quantities**)**.** From Example 4.2 the optimal purchase quantity $\hat{q}_n$ for some $\tilde{p}^n \in (\underline{b}_n, \bar{b}_n)$ takes the form

$$\hat{q}_n = -\frac{1}{a} \log\left(\frac{\tilde{p}^n - d_n}{\sum_{i=n+1}^{\infty} \beta_i}\right) = -\frac{r_n}{a} - \frac{1}{a}\log(\tilde{p}^n - d_n).$$

For $\tilde{p}^n - d_n = p > 0$ this yields

$$\lim_{n\to\infty} \hat{q}_n/r_n = \lim_{n\to\infty} -\frac{1}{a} - \frac{\log(p)}{ar_n} = -\frac{1}{a}.$$

Note that the limit does not depend on p and hence optimal purchase quantities for every reasonable price $d^n + p \in (\underline{b}_n, \bar{b}_n)$ will result in the same limit ratio of $\hat{q}_n/r_n$. As for this limiting value Proposition 5.2.1 yields no result, one has to calculate the limiting indifference price $p^n(\hat{q}_n)$ for optimal purchase quantities $\hat{q}_n$ explicitly. Therefore, recall

$$\lim_{n\to\infty} p^n(\hat{q}_n) - d^n = -\frac{1}{\hat{q}_n a} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-\hat{q}_n a Y_n}\right]\right), \tag{6.6}$$

which by plugging in the value of $\hat{q}_n$ and, as $\mathbb{E}^{\mathbb{P}}[e^{\lambda Y_n}] = e^{(e^\lambda - 1)e^{-r_n}}$ for every $\lambda \in \mathbb{R}$, yields

$$\begin{aligned}(6.6) &= \lim_{n\to\infty} \frac{1}{r_n + \log(\tilde{p}^n - d^n)} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{-(r_n + \log(\tilde{p}^n - d^n))Y_n}\right]\right) \\ &= \lim_{n\to\infty} \frac{(\tilde{p}^n - d^n) - e^{-r_n}}{r_n + \log(\tilde{p}^n - d^n)} = 0.\end{aligned}$$

Hence, the limiting indifference price $\lim_{n\to\infty} p^n(\hat{q}_n) = \lim_{n\to\infty} d^n = d$.

A. Appendix

A.1. Supporting Lemmas

Definition A.1.1. The cumulant generating function for a random variable X is defined by

$$\kappa(\lambda) = \log\left(\mathbb{E}\left[e^{\lambda X}\right]\right); \quad \lambda \in \mathbb{R}. \tag{A.1}$$

If the cumulant generating function is finite for all $\lambda \in \mathbb{R}$ one sustains some useful properties for $\kappa(\cdot)$.

Lemma A.1.1. *If the cumulant generating function $\kappa(\lambda)$ is finite for all $\lambda \in \mathbb{R}$ one obtains the following results:*

(i) $\lim_{\lambda \nearrow \infty} \frac{1}{\lambda}\kappa(\lambda) = \mathit{esssup}_{\mathbb{P}}(X)$

(ii) $\lim_{\lambda \searrow -\infty} \frac{1}{\lambda}\kappa(\lambda) = \mathit{essinf}_{\mathbb{P}}(X)$

(iii) For every $\lambda \in \mathbb{R}$ it holds that

$$\dot{\kappa}(\lambda) = \frac{\mathbb{E}\left[Xe^{\lambda X}\right]}{\mathbb{E}\left[e^{\lambda X}\right]}. \tag{A.2}$$

Furthermore, the function $\kappa(\lambda)$ is strictly convex, consequently it follows that the map $\lambda \mapsto \dot{\kappa}(\lambda)$ is increasing in λ.

Proof. First, notice that for $\lambda > 0$

$$\frac{1}{\lambda}\kappa(\lambda) = \frac{1}{\lambda}\log\left(\mathbb{E}^{\mathbb{P}}\left[e^{\lambda X}\right]\right) \leq \frac{1}{\lambda}\log\left(\mathbb{E}^{\mathbb{P}}\left[e^{\lambda\, \mathrm{esssup}_{\mathbb{P}}(X)}\right]\right) = \mathrm{esssup}_{\mathbb{P}}(X) \tag{A.3}$$

Then select $m > 0$ such that $\mathbb{P}[X > m] > 0$, or rather $m < \mathrm{esssup}_{\mathbb{P}}(X)$. By observing that $X \geq m\mathbb{1}_{\{X>m\}}$, it follows that

$$\begin{aligned}\frac{1}{\lambda}\kappa(\lambda) &\geq \frac{1}{\lambda}\log\left(\mathbb{E}^{\mathbb{P}}\left[e^{\lambda m \mathbb{1}_{\{X>m\}}}\right]\right)\\ &\geq \frac{1}{\lambda}\log\left(\mathbb{E}^{\mathbb{P}}\left[e^{\lambda m}\mathbb{1}_{\{X>m\}}\right]\right) = m + \frac{1}{\lambda}\log\left(\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\{X>m\}}\right]\right)\end{aligned}$$

For $\lambda \nearrow \infty$ one receives that $\liminf_{\lambda \nearrow \infty} \frac{1}{\lambda}\kappa(\lambda) \geq m$ and by taking $m \nearrow \mathrm{esssup}_{\mathbb{P}}(X)$ together with (A.3), $\lim_{\lambda \nearrow \infty} \frac{1}{\lambda}\kappa(\lambda) = \mathrm{esssup}_{\mathbb{P}}(X)$ holds. An analogous consideration yields $\lim_{\lambda \searrow -\infty} \frac{1}{\lambda}\kappa(\lambda) = \mathrm{essinf}_{\mathbb{P}}(X)$, which turn in proves (i) and (ii) from above. To prove (iii) use the inequality

$$|x|e^{\lambda x} \leq C(\lambda)\left(e^{2\lambda x} + e^{-2\lambda x}\right), \qquad x \in \mathbb{R} \tag{A.4}$$

that holds for some constant $C(\lambda) < \infty$. It immediately follows by the product rule that

$$\dot{\kappa}(\lambda) = \frac{1}{\mathbb{E}\left[e^{\lambda X}\right]} \cdot \frac{d}{d\lambda}\mathbb{E}\left[e^{\lambda X}\right] = \frac{\mathbb{E}\left[Xe^{\lambda X}\right]}{\mathbb{E}\left[e^{\lambda X}\right]}.$$

The last equality holds due to the dominated convergence theorem and as inequality (A.4), that holds for some $C(\lambda) < \infty$. If $\dot{\kappa}(\lambda)$ exists for all $\lambda \in \mathbb{R}$, (iii) follows by the observation that $\dot{\kappa}(\lambda)$ is increasing. To this end, consider $E\left[e^{\lambda X}\right]$ and note that by Jensen's inequality and as $\kappa(\lambda)$ was supposed to be finite for all $\lambda \in \mathbb{R}$, particularly as $\mathbb{E}\left[|X|\right] < \infty$

$$E\left[e^{\lambda X}\right] \geq e^{\lambda \mathbb{E}[X]} \geq e^{-|\lambda|\mathbb{E}[|X|]} > 0.$$

Moreover, as inequality (A.4) holds and all exponential moments of X exist, one obtains

$$|\mathbb{E}\left[Xe^{\lambda X}\right]| \leq |C(\lambda)|\left(|\mathbb{E}\left[e^{2\lambda X}\right]| + |\mathbb{E}\left[e^{-2\lambda X}\right]|\right) < \infty,$$

and therefore $\dot{\kappa}(\lambda)$ exists for all $\lambda \in \mathbb{R}$ and one may conclude (iii). □

For the following three lemmas, suppose that Assumption 5.1 and 5.2 hold.

Lemma A.1.2. *Let δ be the constant and I the good rate function from Assumption 5.2, then $\liminf_{|y|\to\infty} I(y)/|y| \geq \delta$.*

Proof. Recall the moment condition (5.4), which implies that Varadhan's integral lemma 2.5.1 holds for all $\epsilon \in (-\delta, \delta)$ and hence

$$\Gamma(\epsilon) := \lim_{n\to\infty} \frac{1}{r_n} \log\left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\epsilon r_n Y_n}\right]\right) = \sup_{y\in\mathbb{R}}(\epsilon y - I(y)) < \infty$$

Further, observe that for $\epsilon > 0$ and $y > 0$ the equation above yields $\Gamma(\epsilon) \geq \epsilon y - I(y)$ and thus

$$\frac{I(y)}{y} \geq \epsilon - \frac{\Gamma(\epsilon)}{y},$$

which by taking $y \nearrow \infty$ and $\epsilon \nearrow \delta$ provides the result.
Moreover, for $\epsilon < 0$ and $y < 0$ the inequality above yields $\Gamma(\epsilon) \geq \epsilon y - I(y)$ and thus

$$\frac{I(y)}{(-y)} \geq -\epsilon - \frac{\Gamma(\epsilon)}{(-y)},$$

which by taking $y \searrow -\infty$ and $\epsilon \searrow -\delta$ provides the result.

□

Lemma A.1.3. *For δ as in Assumption 5.2 and $\epsilon \in (-\delta, \delta)$ define*

$$\begin{aligned} l^\epsilon &:= \inf\left\{y \in \mathbb{R} : y \in argmax_{y\in\mathbb{R}}(\epsilon y - I(y))\right\}, \\ u^\epsilon &:= \sup\left\{y \in \mathbb{R} : y \in argmax_{y\in\mathbb{R}}(\epsilon y - I(y))\right\}. \end{aligned} \tag{A.5}$$

Then it follows, that $\lim_{\epsilon\to 0} l^\epsilon = 0 = \lim_{\epsilon\to 0} u^\epsilon$.

Proof. First, note that since $I(y) \geq 0$ for each $y \in \mathbb{R}$ and $I(y) = 0$ if and only if $y = 0$, it follows that $u^\epsilon \leq 0$ for $\epsilon < 0$, $l^\epsilon \geq 0$ for $\epsilon < 0$ and that for $\epsilon = 0$ $l^\epsilon = u^\epsilon = 0$. Moreover, as $\lim_{n\to\infty} \frac{1}{r_n} \log \left(\mathbb{E}^{\mathbb{P}^n}\left[e^{\epsilon r_n Y_n}\right]\right) = \sup_{y\in\mathbb{R}}(\epsilon y - I(y)) < \infty$, it follows by Lemma A.1.2 that for all $|\epsilon| < \delta/2$ there exists a constant $K > 0$ independent of ϵ such that $-K \leq l^\epsilon \leq u^\epsilon \leq K$. In the sequel, consider the case that $\epsilon < 0$. Let $\epsilon \to 0$ and assume by way of contradiction that l^ϵ converges to $-l < 0$ for a fixed $l > 0$. Then, as l^ϵ is defined as infimum of all $y \in \mathbb{R}$ maximizing $(\epsilon y - I(y))$, it follows that there exists a sequence $y^\epsilon \to y < -l/2$ with $y \geq -l$ and for each ϵ, $y_\epsilon \in argmax_{y\in\mathbb{R}}(\epsilon y - I(y))$. Then by Lemma A.1.2

$$0 \leq \liminf_{\epsilon\to 0} \left(\epsilon y_\epsilon - I(y_\epsilon)\right) \leq -I(y),$$

where the last inequality holds as $I(y)$ is lower semicontinuous by Assumption 5.2, see Remark 2.2. According to the inequality above, $I(y) \leq 0$ holds, which contradicts the fact $I(y) \geq 0$ for all $y \in \mathbb{R}$ by Assumption 5.2. As $y < -l/2 < 0$, but $I(y) = 0$ holds if and only if $y = 0$, there is a contradiction. Thus, it follows that $l^\epsilon \to 0$ for $\epsilon \to 0$, which leads to the result, as it was already shown that $u^\epsilon \leq 0$. By an analogous argumentation for $\epsilon > 0$ and $\epsilon \to 0$, the lemma is proven. □

Lemma A.1.4. *For δ as in Assumption 5.2 and $\epsilon \in (-\delta, \delta)$ define*

$$p_n^\epsilon = \frac{\mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{\epsilon r_n Y_n}\right]}{\mathbb{E}^{\mathbb{P}^n}\left[e^{\epsilon r_n Y_n}\right]}.$$

Then it follows that

$$\liminf_{\epsilon\to 0} \liminf_{n\to\infty} p_n^\epsilon = \limsup_{\epsilon\to 0} \limsup_{n\to\infty} p_n^\epsilon = 0.$$

Proof. First, observe that $\Lambda_n(\lambda) = \log(\mathbb{E}^{\mathbb{P}^n}[e^{\lambda Y_n}])$ as defined in (5.2) satisfies $\Lambda_n(\lambda) < \infty$ by Assumption 5.1. Thus, it is strictly convex and by Lemma A.1.1 it follows that $\dot{\Lambda}_n(\lambda) = \mathbb{E}^{\mathbb{P}^n}\left[Y_n e^{\lambda Y_n}\right] / \mathbb{E}^{\mathbb{P}^n}\left[e^{\lambda Y_n}\right]$. In addition, set $\Lambda(\lambda) := \sup_{y\in\mathbb{R}}(\lambda y - I(y))$, which is convex as supremum of an affine function. Varadhan's integral lemma or rather (5.7) yields that $(1/r_n)\Lambda_n(\epsilon r_n) \to \Lambda(\epsilon)$ for $n \to \infty$ due to the assumption $\epsilon \in (-\delta, \delta)$. Further, note that Proposition 3.5.1 states

$$\tilde{p}^n - d_n = \dot{\Lambda}_n(-q_n a)$$

with $\tilde{p}_n \in (\underline{b}_n, \bar{b}_n)$, only holds for a unique value q_n that also minimizes the function $q_n(\tilde{p} - d_n) + (1/a)\Lambda_n(-aq_n)$. Hence, $\lambda = \epsilon r_n$ is the unique value minimizing the function $\lambda \mapsto -\lambda p_n^\epsilon + \Lambda_n(\lambda)$. This leads to the following inequality for any arbitrarily chosen $\gamma \in \mathbb{R}$

$$-\epsilon r_n p_n^\epsilon + \Lambda_n(\epsilon r_n) \leq -\gamma p_n^\epsilon + \Lambda_n(\gamma). \tag{A.6}$$

Now, fix $\lambda > 0$ in such a way that $\lambda + \epsilon < \delta$ and insert $\lambda + \epsilon$ instead of γ. Then due to the inequality (A.6) one has

$$-\epsilon r_n p_n^\epsilon + \Lambda_n(\epsilon r_n) \leq -(\lambda+\epsilon) r_n p_n^\epsilon + \Lambda_n((\lambda+\epsilon) r_n).$$

After some equivalent transformation, i.e. cancel out the $-\epsilon p_n^\epsilon$ and divide both sides through λr_n, this inequality yields

$$p_n^\epsilon \leq \frac{1}{\lambda}\left(\frac{1}{r_n}\Lambda_n((\lambda+\epsilon)r_n) - \frac{1}{r_n}\Lambda_n(\epsilon r_n)\right).$$

As aforementioned, for $n \to \infty$ Equation (5.7) yields

$$\liminf_{n\to\infty} p_n^\epsilon \leq \limsup_{n\to\infty} p_n^\epsilon \leq \frac{1}{\lambda}(\Lambda(\lambda+\epsilon) - \Lambda(\epsilon)).$$

By taking $\lambda \to 0$, this implies

$$\liminf_{n\to\infty} p_n^\epsilon \leq \limsup_{n\to\infty} p_n^\epsilon \leq \lim_{\lambda\to 0}\frac{\Lambda(\lambda+\epsilon)-\Lambda(\epsilon)}{\lambda} = \dot{\Lambda}_+(\epsilon),$$

with $\dot{\Lambda}_+(\epsilon)$ being the right derivative of Λ at ϵ. On the other hand, by (A.6) and using $\gamma = (\epsilon - \lambda)r_n$ for with $\lambda > 0$ such that $-\delta < -\epsilon - \lambda$ one obtains

$$-\limsup_{n\to\infty} p_n^\epsilon \leq -\liminf_{n\to\infty} p_n^\epsilon \leq \lim_{\lambda\to 0}\frac{\Lambda(\epsilon-\lambda)-\Lambda(\epsilon)}{\lambda} = \dot{\Lambda}_-(\epsilon),$$

with $\dot{\Lambda}_-(\epsilon)$ being the left derivative of Λ at ϵ. These two inequalities together with Theorem 23.2 in [Rockafellar, 1997] imply, as Λ is convex by construction, that $\limsup_{n\to\infty} p_\epsilon^n = \liminf_{n\to\infty} p_\epsilon^n = \partial\Lambda(\epsilon)$. Note that $\partial\Lambda(\epsilon)$ denotes the subdifferential, i.e. the set of all subgradients g^ϵ of Λ at ϵ defined through the inequality $\Lambda(\lambda) \geq \Lambda(\epsilon) + g^\epsilon(\lambda - \epsilon)$. In the sequel, assume an arbitrary $g^\epsilon \in \partial\Lambda(\epsilon)$. Proving $\lim_{\epsilon\to 0}|g^\epsilon| = 0$ provides the desired result. To this end, it is assumed by way of contradiction, that for some subsequence $\epsilon_k \to 0$ and $\tau > 0$, it holds that $g^{\epsilon_k} \geq \tau$. Fix $0 < \lambda < \delta$ in such a way that $\epsilon_k < \lambda$ as of some k large enough. As $g^{\epsilon_k} \in \partial\Lambda(\epsilon_k)$ for every k, one finds that

$$\Lambda(\lambda) \geq \Lambda(\epsilon_k) + g^{\epsilon_k}(\lambda - \epsilon_k) \geq \Lambda(\epsilon_k) + \tau(\lambda - \epsilon_k).$$

Further, note that by Lemma A.1.3, there exists an $y^\lambda \in \operatorname{argmax}_{y\in\mathbb{R}}(\lambda y - I(y))$ for λ sufficiently small that still satisfies $\lambda > \epsilon_k$, such that $\Lambda(\lambda) = \lambda y^\lambda - I(y^\lambda)$, which immediately yields

$$\Lambda(\lambda) = \lambda y^\lambda - I(y^\lambda) \geq \Lambda(\epsilon_k) + \tau(\lambda - \epsilon_k).$$

Due to the convexity of Λ and as its finiteness for all $\lambda \in (-\delta, \delta)$, it follows that the function Λ is continuous. Furthermore, since $\Lambda(0) = 0$ the limit for $k \to \infty$ gives $\lambda y^\lambda - I(y^\lambda) \geq \tau\lambda$, which is equivalent to $y^\lambda - I(y^\lambda)/\lambda \geq \tau$ and therefore implies $y^\lambda \geq \tau$. Lemma A.1.3 states $y^\lambda \to 0$ for $\lambda \to 0$, which is a contradiction, as this means $0 \geq \tau$. The analogous argumentation yields a contradiction for assuming a subsequence $g^{\epsilon_k} \leq -\tau$ for $\tau > 0$ and

$\epsilon_k \to 0$. Indeed, for $0 > \lambda > -\delta$ in such a way that $\epsilon_k > \lambda$ as of some k large enough, again for some $g^{\epsilon_k} \in \partial\Lambda(\epsilon_k)$ for every $k \in \mathbb{N}$ one has

$$\Lambda(\lambda) \geq \Lambda(\epsilon_k) + g^{\epsilon_k}(\lambda - \epsilon_k) \geq \Lambda(\epsilon_k) - \tau(\lambda - \epsilon_k).$$

Moreover, by Lemma A.1.3, there exists a $y^\lambda \in \operatorname{argmax}_{y\in\mathbb{R}}(\lambda y - I(y))$ for λ sufficiently small, that still satisfies $\lambda < \epsilon_k$, such that $\Lambda(\lambda) = \lambda y^\lambda - I(y^\lambda)$, which immediately yields

$$\Lambda(\lambda) = \lambda y^\lambda - I(y^\lambda) \geq \Lambda(\epsilon_k) - \tau(\lambda - \epsilon_k).$$

Similar to the argumentation above, the limit for $k \to \infty$ gives $\lambda y^\lambda - I(y^\lambda) \geq -\tau\lambda$, which is equivalent to $-y^\lambda + I(y^\lambda)/\lambda \geq \tau$ and hence $-y^\lambda \geq \tau$. Lemma A.1.3 states $y^\lambda \to 0$ for $\lambda \to 0$, which is a contradiction, as this means $0 \geq \tau$. All together, this yields $|g^\epsilon| \to 0$ by taking $\epsilon \to 0$ for each $g^\epsilon \in \partial\Lambda(\epsilon) = \lim_{n\to\infty} p_n^\epsilon$, which finishes the proof. □

A.2. Additional Theory

Definition A.2.1. For a continuous $(\mathbb{P}, \mathbb{F})$-semi-martingale X, the corresponding stochastic exponential is defined as

$$\mathcal{E}(X) = \exp(X - \frac{1}{2}[X]) \tag{A.7}$$

Definition A.2.2. A $(\mathbb{P}, \mathbb{F})$-local martingale S has the predictable representation property if for any $\mathbb{F}^S$-local martingale X there is a predictable process θ in $L^2(S)$ such that

$$X_t = X_0 + \int_0^t \theta_u dS_u.$$

Here $\mathbb{F}^S$ denotes the natural filtration of S.

Lemma A.2.1 ([Delbaen and Schachermayer, 2006])**.** *If the following holds:*

(i) S is a local martingale,

(ii) h_t is S-integrable process

(iii) $\left(\int_0^\cdot h_t dS_t\right)^-$ is locally integrable, i.e., there exists a sequence $(\tau_n)_{n\in\mathbb{N}}$ of increasing stopping times with $\tau_n \nearrow \infty$ for $n \to \infty$, such that $\mathbb{E}\left[-\inf_{0\leq t\leq\tau_n} \int_0^t h_u \, dS_u\right] < \infty$,

then the stochastic integral process $\int_0^\cdot h_t dS_t$ is a local martingale.

Proof. First, note that the notation $(h \cdot S)_t$ is used for the stochastic integral process in the sense of semi-martingale integration. $(h, \Delta S)_t$ denotes the process of jumps of the stochastic integral. According to the lemma of Ansel and Stricker, see [Theorem 7.3.7, Delbaen and Schachermayer, 2006], one has to verify the existence of a sequence of stopping time $T_n \nearrow \infty$, as well as the existence of a sequence of integrable functions $\zeta_n \geq 0$, such that the stopped process of jumps is bounded from below. Hence, proving that $(h, \Delta S)^{T_n} \geq \zeta_n$ for all n, provides the required result. Therefore, set $R_n := \inf\{t | (h \cdot S)_t \geq n\}$ for each $n \in \mathbb{N}$, which are also stopping times. Consider a sequence $(\tau_n)_{n\in\mathbb{N}}$ such that the stopped process satisfies $(h \cdot S)_{\tau_n} \geq \zeta_n$ for a sequence of any integrable functions ζ_n, $n \in \mathbb{N}$. Further, set $T_n = \min(R_n, \tau_n)$, which again is a stopping time, as it is the minimum of two stopping times. As $R_n \nearrow \infty$ and $\tau_n \nearrow \infty$ for $n \to \infty$, the same holds true for T_n. One may easily observe that this yields to $(h, \Delta S)^{T_n} \geq -n + \zeta_n$, which in turn yields the result. □

Theorem A.2.1 ([Protter, 2004])**.** *Let X be a $(\mathbb{P}, \mathbb{G})$semi-martingale with decomposition $X = M + A$ on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$. $\mathcal{H}$ is a σ-algebra $\mathbb{P}$-independent of the local martingale term M of X. Further, $\mathbb{F}$ shall denote the expanded filtration obtained by the extension of $\mathbb{G}$ with the σ-algebra $\mathcal{H}$, i.e., $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$ for each $t \geq 0$. Then X is a $(\mathbb{P}, \mathbb{F})$-semi-martingale with the same decomposition.*

Proof. As the local martingale term M is independent of $\mathcal{G}$, it remains a local martingale under $\mathbb{F}$. Therefore, the result follows immediately. □

Theorem A.2.2 ([Protter, 2004])**.** *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ denote two filtrations satisfying the usual conditions, such that $\mathcal{G}_t \subset \mathcal{F}_t$ for each $t \geq 0$. X is a semi-martingale for $\mathbb{F}$ and $\mathbb{G}$. Let h_t be a locally bounded and predictable process for $\mathbb{G}$. Then the stochastic integral processes exist and coincide for both $\mathbb{F}$ and $\mathbb{G}$.*

Proof. Since all $\mathbb{G}$-stopping times are $\mathbb{F}$-stopping times as well, the process h is also locally bounded and predictable for $\mathbb{F}$. Without loss of generality h is supposed to be bounded since stopping yields a bounded process. Set

$$H = \left\{h_t \in L(X) \,|h_t \text{ bounded, } \mathcal{G}\text{-predictable such that } (h \cdot X)_t^{\mathbb{G}} = (h \cdot X)_t^{\mathbb{F}},\ \text{ for all } t \geq 0\right\},$$

where $(h \cdot X)_\cdot^{\mathbb{G}} = (h \cdot X)_\cdot^{\mathbb{F}}$ denotes the integral processes computed with respect to the filtrations $\mathbb{F}$ and $\mathbb{G}$, respectively. Clearly, H is a vector space, all constant functions are in H and for every sequence of processes converging to a bounded limit, the limiting process is also in H . Hence, H is a monotone vector space. Further, H contains the multiplicative class of all left continuous and $\mathbb{F}$-adapted processes. Thus, the dominated convergence theorem, together with the monotone class theorem prove the result. □

The following proposition brings up a main property of semi-martingales, which is included in some proofs of this thesis. For more detail and the proofs see [Protter, 2004, Therorem 33, III] or [Jacod and Shiryaev, 2003, Proposition 4.23]

Proposition A.2.1. *Let X be a semi-martingale. Then X is a special semi-martingale, if and only if the associated process $X_t^* = \sup_{s\leq t} |X_s - X_0|$ is locally integrable.*

Definition A.2.3. Define

$$\mathcal{D} := \{Z^{\mathbb{Q}} : \mathbb{Q} \in \tilde{\mathcal{M}}\} \text{ and } \mathcal{T}_T := \{\tau : \tau \text{ is } \mathbb{F}\text{-stopping time such that } \tau \leq T\}\,.$$

The set $\mathcal{D}$ is said to be stable under concatenation for $\mathbb{F}$ if for all $\tau \in \mathcal{T}_T$, one has that $Z^{\mathbb{Q}_1}, Z^{\mathbb{Q}_2} \in \mathcal{D}$ implies that

$$\tilde{Z} = Z^{\mathbb{Q}_1}\mathbb{1}_{[0,\tau)} + \frac{Z^{\mathbb{Q}_1}_\tau}{Z^{\mathbb{Q}_2}_\tau} Z^{\mathbb{Q}_2}\mathbb{1}_{[\tau,T]} \in \mathcal{D}.$$

Lemma A.2.2 ([Kabanov and Strickler, 2002])**.** *Let $\mathbb{Q} \in \tilde{\mathcal{M}}$ such that $Z_t d\mathbb{P} = d\mathbb{Q}$ and $\mathbb{E}[Z_T \log(Z_T)] < \infty$. If $\mathcal{D}$ is stable under concatenation it follows that $\{\log(\bar{Z}_\tau)\}_{\tau \in \mathcal{T}_T}$ with $\bar{Z}_t = \exp\left(\mathbb{E}^{\mathbb{Q}_0}[\log(Z^0_T)|\mathcal{F}_t]\right)$ is $\mathbb{Q}$ uniformly integrable.*

Lemma A.2.3 ([Stroock, 2011])**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables. If every X_n is in $L^2(\mathbb{P})$ and if*

$$\sum_{i=1}^{\infty} Var(X_n) < \infty,$$

then

$$\sum_{i=1}^{\infty} (X_n - \mathbb{E}[X_n])$$

converges $\mathbb{P}$-almost surely.

Proof. First, assume without loss of generality that for all $n \in \mathbb{N}$, $\mathbb{E}[X_n] = 0$. Then, by applying Kolmogorov's inequality to $\{X_{N+n} : n \in \mathbb{N}\}$, it follows that

$$\mathbb{P}\left[\sup_{n>N} |\sum_{i=1}^{n} X_i - \sum_{i=1}^{N} X_i| \geq \epsilon\right] \leq \frac{1}{\epsilon^2} \sum_{i=N+1}^{\infty} \mathbb{E}\left[X_i^2\right] \to 0 \quad \text{as } N \to \infty$$

for any $\epsilon > 0$. Hence, it follows immediately that the sequence of partial sums $(\sum_{i=1}^{n} X_i)_{n\in\mathbb{N}}$ is $\mathbb{P}$-almost sure a Cauchy sequence $\mathbb{P}$-almost sure, which completes the proof. □

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