Cut-elimination for a Hypersequent Calculus for First-order Gödel Logic over $[0,1]$ with $\triangle$

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The family of Gödel logics has originally been introduced by Gödel [9] for the purpose of showing that intuitionistic logic cannot be characterized by finite truth tables. They were first studied in detail by Dummett [8]. Takeuti and Titani [10] based their "intuitionistic fuzzy set theory" on the first-order Gödel logic with truth values from real unit interval $[0,1]$. Nowadays Gödel logics are studied intensively in the context of mathematical fuzzy logic [4]. We will restrict attention to the version $\mathbf{G}^{\forall\triangle}_{[0,1]}$ of first-order Gödel logic over $[0,1]$, where the usual logical connectives are augmented by the projection operator $\triangle$ [1].

We work in a usual first-order language $\mathscr{L}$ with free $(a,\ b,\ \dots)$ and bound $(x,\ y,\ \dots)$ variables, predicate and function symbols, logical connectives $\vee$, $\wedge$, $\rightarrow$, a propositional constant $\bot$, quantifiers $\forall$, $\exists$, and a unary operator $\triangle$. Terms and formulas are defined in the usual way. We use $\neg$ as a defined connective; $\neg A \equiv A \rightarrow \bot$.

Definition 1 (Semantics of $\mathbf{G}^{\forall\triangle}_{[0,1]}$)**.** An *interpretation* $\mathfrak{I}$ into $[0,1]$ consists of

1. a nonempty set $|\mathfrak{I}|$, the 'universe' of $\mathfrak{I}$,
2. for each k-ary predicate symbol P, a function $P^{\mathfrak{I}} : |\mathfrak{I}| \rightarrow [0,1]$,
3. for each k-ary function symbol f, a function $f^{\mathfrak{I}} : |\mathfrak{I}| \rightarrow |\mathfrak{I}|$.
4. for each free variable a, a value $a^{\mathfrak{I}} \in [0,1]$.

Let $\mathscr{L}^{\mathfrak{I}}$ be the language $\mathscr{L}$ extended by constant symbols for the elements of $|\mathfrak{I}|$ (so that $d^{\mathfrak{I}} = d$).

Any interpretation $\mathfrak{I}$ extends to an evaluation function yielding a value $\mathfrak{I}(A)$ for any formula A of $\mathscr{L}^{\mathfrak{I}}$. For terms $t = f(u_1, \dots, u_k)$ we define $\mathfrak{I}(t) = f^{\mathfrak{I}}(\mathfrak{I}(u_1), \dots, \mathfrak{I}(u_k))$, for atomic formulas $A \equiv P(t_1, \dots, t_n)$, we define $\mathfrak{I}(A) = P^{\mathfrak{I}}(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n))$, and for composite formulas A we define $\mathfrak{I}(A)$ naturally by:

$$\mathfrak{I}(\bot) = 0 \tag{1}$$
$$\mathfrak{I}(A \wedge B) = \min(\mathfrak{I}(A), \mathfrak{I}(B)) \tag{2}$$
$$\mathfrak{I}(A \vee B) = \max(\mathfrak{I}(A), \mathfrak{I}(B)) \tag{3}$$
$$\mathfrak{I}(A \rightarrow B) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) \le \mathfrak{I}(B) \\ \mathfrak{I}(B) & \text{if } \mathfrak{I}(A) > \mathfrak{I}(B) \end{cases} \tag{4}$$
$$\mathfrak{I}(\triangle A) = \begin{cases} 1 & \text{if } \mathfrak{I}(A) = 1 \\ 0 & \text{if } \mathfrak{I}(A) < 1 \end{cases} \tag{5}$$
$$\mathfrak{I}(\forall x\, A(x)) = \inf\{\mathfrak{I}(A(u)) : u \in |\mathfrak{I}|\} \tag{6}$$
$$\mathfrak{I}(\exists x\, A(x)) = \sup\{\mathfrak{I}(A(u)) : u \in |\mathfrak{I}|\} \tag{7}$$

From a proof-theoretic perspective, several versions of hypersequent calculi for Gödel logics have been proposed, including systems for first-order logics [2, 3, 6] and systems with $\triangle$ [7]. In [5] the hypersequent calculus **HGIF** is shown to be complete for first-order $[0,1]$-based Gödel logic with $\triangle$. In this contribution we settle the problem of cut-elimination for **HGIF**.

Hypersequents are finite multisets of single-conclusion sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n.$$

The calculus **HGIF** is defined as follows.

Axioms:

$$A \Rightarrow A \qquad \bot \Rightarrow$$

Internal structural rules:

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ iw\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow A}\ \Rightarrow iw \qquad \frac{G \mid A, A, \Gamma \Rightarrow \Delta}{G \mid A, \Gamma \Rightarrow \Delta}\ ic\Rightarrow$$

External structural rules:

$$\frac{G}{G \mid \Gamma \Rightarrow \Delta}\ ew \qquad \frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ ec$$

Logical rules:

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \neg A, \Gamma \Rightarrow}\ \neg\Rightarrow \qquad \frac{G \mid A, \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \neg A}\ \Rightarrow\neg$$

$$\frac{G \mid A, \Gamma \Rightarrow \Delta \quad G \mid B, \Gamma \Rightarrow \Delta}{G \mid A \vee B, \Gamma \Rightarrow \Delta}\ \vee\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A \quad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B}\ \Rightarrow\wedge$$

$$\frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma \Rightarrow A \vee B}\ \Rightarrow\vee_1 \qquad \frac{G \mid A, \Gamma \Rightarrow \Delta}{G \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge\Rightarrow_1$$

$$\frac{G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \vee B}\ \Rightarrow\vee_2 \qquad \frac{G \mid B, \Gamma \Rightarrow \Delta}{G \mid A \wedge B, \Gamma \Rightarrow \Delta}\ \wedge\Rightarrow_2$$

$$\frac{G \mid \Gamma_1 \Rightarrow A \quad G \mid B, \Gamma_2 \Rightarrow \Delta}{G \mid A \rightarrow B, \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ \rightarrow\Rightarrow \qquad \frac{G \mid A, \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \rightarrow B}\ \Rightarrow\rightarrow$$

$$\frac{G \mid A(t), \Gamma \Rightarrow \Delta}{G \mid (\forall x)A(x), \Gamma \Rightarrow \Delta}\ \forall\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow (\forall x)A(x)}\ \Rightarrow\forall$$

$$\frac{G \mid A(a), \Gamma \Rightarrow \Delta}{G \mid (\exists x)A(x), \Gamma \Rightarrow \Delta}\ \exists\Rightarrow \qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow (\exists x)A(x)}\ \Rightarrow\exists$$

The rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ are subject to eigenvariable conditions: the free variable a must not occur in the lower hypersequent.

Rules for $\triangle$:

$$\frac{G \mid \quad A, \Gamma \Rightarrow \Delta}{G \mid \triangle A, \Gamma \Rightarrow \Delta}\ \triangle\Rightarrow \qquad \frac{G \mid \triangle\Gamma \Rightarrow \quad A}{G \mid \triangle\Gamma \Rightarrow \triangle A}\ \Rightarrow\triangle$$

$$\frac{G \mid \triangle\Gamma, \Gamma' \Rightarrow \Delta}{G \mid \triangle\Gamma \Rightarrow \mid \Gamma' \Rightarrow \Delta}\ \triangle cl$$

Communication:

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta \quad G \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta'}{G \mid \Gamma_1 \Rightarrow \Delta \mid \Gamma_2 \Rightarrow \Delta'}\ cm$$

Cut:

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid A, \Pi \Rightarrow \Lambda}{G \mid \Gamma, \Pi \Rightarrow \Lambda}\ cut$$

Our main result is the following:

Theorem 2 (Cut-Elimination)**.** *Every proof in* **HGIF** *of some hypersequent* σ *can be transformed into a proof of* σ *that does not contain applications of* (*cut*)*.*

The problem of cut-elimination in hypersequent calculi is that Gentzen's original method is not suitable due to the lack of a definable mix-rule. This implies that induction on the height and size of the cut-formula does not lead to the desired result, as the contraction rule appears as obstacle. We therefore adopt the so-called Schütte-Tait procedure, where cut-elimination proceeds by iteratively removing the maximal cuts; i.e., applications of *cut*, where the cut-formula is of maximal size. This method of cut-elimination is based on the reduction of one side of the cut without moving the cut-formula. Our adaption of this procedure is that not the highest maximal cut is reduced, but the highest cut with a specific cut-formula is reduced top-down with possibly multiplying the occurrences, but not the number of other maximal cut formulas.

Spelling out details of the cut-elimination procedure requires quite a few technical preparations. Obviously this abstract is not the right place to do so. However we formulate a few interesting corollaries that follow straightforwardly from the proof of Theorem 2.

Corollary 3 (Mid-Hypersequent Theorem)**.** *Let the end-hypersequent of a cut-free proof* π *contain prenex formulas only. There is a hypersequent* σ *in* π *such that, besides structural inferences, all inferences in* π *ocurring above* σ *are propositional and all inferences below* σ *are quantificational.*

Corollary 4. *The prenex fragment of* $\mathbf{G}^{\forall\triangle}_{[0,1]}$ *admits Skolemization and interpolation.*

The following corollary in essence entails a version of Herbrand's Theorem:

Corollary 5. *Let* A *be quantifier-free, then the following rule is admissible in* **HGIF***:*

$$\frac{\Rightarrow \triangle \exists x A(x)}{\Rightarrow \exists x \triangle\, A(x)}$$

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