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Asymptotic constraints on QCD correlators and holographic models

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Betreuer

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Kurzfassung

In dieser Arbeit betrachten wir den Bereich von hohen Energien von zwei holographischen Modellen der QCD, welche durch die AdS/CFT Korrespondenz motiviert sind. Wir berechnen speziell Übergangsformfaktoren der leichtesten pseudoskalaren Mesonen und unendlich vielen axialen Vektormesonen und deren Beiträge zum Licht-Licht-Streuungstensor. Die Resultate werden verglichen mit Berechnungen von der QCD, welche im Hochenergielimes durch die Operatorproduktentwicklung vereinfacht werden kann. Weiters werden die chirale Symmetriebrechung und dadurch auftretende Goldstone-Bosonen und Resultate über die chiralen Anomalien verwendet, um Ausdrücke aus der QCD zu gewinnen. Die mithilfe des holographischen Prinzips berechneten Observablen stimmen im Fall eines Modells (HW2) qualitativ mit den QCD Ausdrücken überein und im Fall des zweiten Modells (HW1) sogar quantitativ. Es wird auch gezeigt, dass die Zwangsbedingungen für den Licht-Licht-Streuungstensor nur dann erfüllt werden können, wenn unendlich viele axiale Vektormesonen berücksichtigt werden, während eine endliche Anzahl auf einen zu starken Abfall im Hochenergielimes führt.

Abstract

We study the high-energy asymptotics of two bottom-up holographic models of QCD motivated by the AdS/CFT correspondence. In particular we compute the transition form factors of the lowest lying pseudoscalar mesons and an infinite tower of axial vector mesons and their contributions to the hadronic light-by-light scattering tensor. The results obtained are compared to high-energy expressions of real QCD, which are obtained via the operator product expansion, results on Goldstone bosons, and the chiral anomaly that appears when the axial currents are coupled to external fields. The results from the holographic side qualitatively reproduce the QCD results for one model (HW2) and qualitatively in the case of another model (HW1). In particular it is shown that two constraints on the hadronic light-by-light scattering tensor can only be fulfilled when including an infinite number of axial vector resonances and a finite amount of resonances can not produce the right fall off behaviour.

Contents

1 Introduction

Low and intermediate energy regimes of correlators of gauge invariant operators in QCD have been under investigation for many years but the confining aspects of QCD are making it very difficult to obtain analytical and meaningful results. The main mathematical tool to explore quantum field theories is perturbation theory which does not work in low energy QCD. The effective running coupling constant $\alpha_s(Q)$ becomes larger and larger as one goes to lower energies and higher order diagrams give larger and larger contributions preventing the diagrammatic series from converging [1]. For very low momenta one may derive an effective Lagrangian describing low energy degrees of freedom such as the lowest lying pseudoscalar mesons, vector and axial vector mesons whose coefficients can be compared to experiment [2]. Another region where meaningful results can be derived are very high energies and momenta due to the asymptotic freedom of QCD. Results for correlation functions can be extracted using perturbation theory and more importantly the so called operator product expansion (OPE) of gauge invariant operators [3].

Recently another promising way of studying strongly interacting gauge theories has been found in applying the AdS/CFT correspondence [4] which in some formulations relates a strongly interacting gauge theory in 4 spacetime dimensions to a weakly interacting classical theory in 5 spacetime dimensions. This is called the holographic principle. There are so called top-down models which modify the underlying string quantum theory, which lives in an even higher dimensional space, in such a way as to make the dual theory more and more like real QCD [5]. There are also bottom-up models which take important ingredients from the AdS/CFT correspondence, but simply construct 5 dimensional Lagrangians by hand and try to reproduce certain aspects of QCD (see [6], [7]).

The purpose of this work is to first review various constraints coming from the OPE in the high-energy limit of QCD and then try to compare with the expressions coming from the holographic side. Top-down models have the known problem that their high-energy behavior differs qualitatively from real life QCD through their construction, so we will mainly focus on the bottom-up models created by hand.

In chapter 2 we will briefly motivate the correlators of interest and comment on the status of the OPE in particular in the presence of non-perturbative effects. In chapter 3 we will start with stating our conventions and then proceed to apply the OPE to lowest order to the light-by-light scattering tensor and to pseudoscalar meson transition form factors. In chapter 4 two bottom-up models will be introduced and matched to QCD. Then we proceed in calculating the hadronic observables in this holographic setting and compare with the high-energy expressions from chapter 3. The low energy predictions of these models will not be explored in detail here although we use decay constants and masses of the lowest lying resonances to fix some parameters.

2 QCD correlators and the Operator Product Expansion

In this section we motivate the strong interaction correlators of interest, recall some key properties and give examples of their applications. The most important ones are the hadronic light-by-light (HLbL) tensor $\Pi_{\mu\nu\alpha\beta}$ and the transition form factors (TFF) of the lowest lying pseudoscalar and pseudovector mesons. We then review the Operator Product Expansion (OPE) and its current status in QCD. We will discuss our conventions and the method to apply the OPE to the TFFs and the HLbL tensor in chapter 3.

2.1 QCD correlators

Before working with the relevant Green's functions it will be useful to motivate why we are interested in current correlators. In the calculation of the electromagnetic moment of the muon (or the electron) one comes across diagrams drawn in figure 2.1, where the square describes strong interactions which contribute to photon-photon scattering. Note that the external momenta of this square are off-shell in general. We denote this subdiagram by $\Pi_{\mu_1\mu_2\mu_3\mu_4}(q_1,q_2,q_3,q_4)$. In a theory with quarks coupling to gluons and photons we can write it as the sum of connected amputated diagrams with 4 external photons, where we omit the polarization vectors and do not restrict the amplitude to on-shell photons

$$\begin{aligned}
&\Pi_{\mu_1\mu_2\mu_3\mu_4}(q_1,q_2,q_3,q_4) = \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-i(q_1x_1+q_2x_2+q_3x_3+q_4x_4)} \\
&\int d^4y_1 d^4y_2 d^4y_3 d^4y_4 (G^{-1})^{\nu_1}_{\mu_1}(x_1-y_1)(G^{-1})^{\nu_2}_{\mu_2}(x_2-y_2)(G^{-1})^{\nu_3}_{\mu_3}(x_3-y_3)(G^{-1})^{\nu_4}_{\mu_4}(x_4-y_4) \\
&\langle\Omega\,|\,T\{A_{\nu_1}(y_1)A_{\nu_2}(y_2)A_{\nu_3}(y_3)A_{\nu_4}(y_4)\}\,|\,\Omega\rangle_{connected} = \\
&\frac{(\tilde{G}^{-1})^{\nu_1}_{\mu_1}(q_1)}{(q_1)^2}\frac{(\tilde{G}^{-1})^{\nu_2}_{\mu_2}(q_2)}{(q_2)^2}\frac{(\tilde{G}^{-1})^{\nu_3}_{\mu_3}(q_3)}{(q_3)^2}\frac{(\tilde{G}^{-1})^{\nu_4}_{\mu_4}(q_4)}{(q_4)^2} \\
&\int e^{-i(q_1x_1+q_2x_2+q_3x_3+q_4x_4)}\partial^2_{x_1}\partial^2_{x_2}\partial^2_{x_3}\partial^2_{x_4}\,\langle\Omega\,|\,T\{A_{\nu_1}(x_1)A_{\nu_2}(x_2)A_{\nu_3}(x_3)A_{\nu_4}(x_4)\}\,|\,\Omega\rangle_{connected}\,.
\end{aligned} \tag{2.1}$$

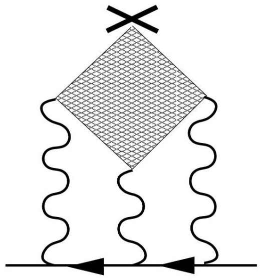


Figure 2.1: Light-by-light scattering diagram. Graphic taken from [8].

All fields in this equation are renormalized fields and $(G^{-1})^{\nu}_{\mu}(x-y)$ is the inverse operator of the 2-point function $\langle\Omega\,|\,T\{A_\mu(x)A_\nu(y)\}\,|\,\Omega\rangle$. In general this object includes both electroweak and strong interactions but in this work we focus on the lowest order in e contribution which is $\mathcal{O}(e^4)$. There will be no internal photon lines in diagrams contributing to this tensor and it can be evaluated purely within QCD, i.e. setting $e=0$. First however we would like to simplify the above equation. The connected 4-point function of the electro-magnetic field is gauge dependent by itself, but of course the gauge dependence cancels in any physically meaningful final result. We will choose the Feynman gauge to do computations. To continue we first write the connected part of the 4-point function as

$$\langle\Omega\,|\,T\{A_1A_2A_3A_4\}\,|\,\Omega\rangle_{connected} = \\ \langle\Omega\,|\,T\{A_1A_2A_3A_4\,|\,\Omega\rangle - (\langle\Omega\,|\,A_1A_2\,|\,\Omega\rangle\,\langle\Omega\,|\,A_3A_4\,|\,\Omega\rangle + crossings). \tag{2.2}$$

Within the path integral formalism we can derive a useful formula for the 4-point function when each leg gets upon by a massless Klein-Gordon operator ∂^2. After calculating this, we will throw away all terms with more than 2 delta functions in momentum space, since we only need the connected part of this correlator of gauge fields, i.e. terms with only one delta function in momentum space enforcing overall momentum conservation. The object to look at is (with abbreviated notation)

$$\partial_1^2\partial_2^2\partial_3^2\int\mathcal{D}A_0\mathcal{D}\varphi_0(A_0)_1(A_0)_2(A_0)_3e^{iS(A_0,\varphi_0)}, \tag{2.3}$$

where φ_0 denotes any other fields like quarks and gluons. The "0" subscript indicates that these are bare fields and the subscript outside of the bracket indicates the point at which it is evaluated $(A_0)_i := A_0(x_i)$. It is convenient to work with unrenormalized fields in the following since the Lagrangian has a nicer form. We will drop the "0" subscript in the upcoming calculation to make the computation more transparent. By renaming integration variables to $A'_\mu(x) = A_\mu(x) - \varepsilon_\mu(x)$ and using invariance of the measure wrt. this transformation we can make an expansion in ε. We also rename A' back to A . The term of order zero just cancels with the LHS and we get

$$\begin{aligned}\partial_1^2\partial_2^2\partial_3^2\int\mathcal{D}A\mathcal{D}(\varphi)A_1A_2A_3e^{iS(A,\varphi)}i\int d^4x\frac{\delta S}{\delta A(x)}\varepsilon(x)+\\ \partial_1^2\partial_2^2\partial_3^2\int\mathcal{D}A\mathcal{D}(\varphi)\varepsilon_1A_2A_3e^{iS(A,\varphi)}+\\ \partial_1^2\partial_2^2\partial_3^2\int\mathcal{D}A\mathcal{D}(\varphi)A_1\varepsilon_2A_3e^{iS(A,\varphi)}+\\ \partial_1^2\partial_2^2\partial_3^2\int\mathcal{D}A\mathcal{D}(\varphi)A_1A_2\varepsilon_3e^{iS(A,\varphi)}=0,\end{aligned}\tag{2.4}$$

which must hold for any ε that vanishes sufficiently quick near infinity. The variation of the action gives

$$\frac{\delta S}{\delta A_\mu(x)}=\partial^2A^\mu(x)+e_0\bar{\psi}\gamma^\mu\hat{Q}\psi=\partial^2A^\mu(x)+e_0J^\mu.\tag{2.5}$$

The matter fields and the coupling constant are all bare quantities here. The above equation arises since the action in the path integral is gauge fixed, i.e. not the classical action. To get the explicit form above one has to choose the Feynman gauge $\xi_0=1$. Picking $\varepsilon_\mu(x)=\delta^{(4)}(x-x_4)g_{\mu\mu_4}$ gives

$$\begin{aligned}\partial^2_{x_1}\partial^2_{x_2}\partial^2_{x_3}\partial^2_{x_4}\langle\Omega\,|\,T\{A_{\nu_1}(x_1)A_{\nu_2}(x_2)A_{\nu_3}(x_3)A_{\nu_4}(x_4)\}\,|\,\Omega\rangle=\\ -e_0\partial^2_{x_1}\partial^2_{x_2}\partial^2_{x_3}\langle\Omega\,|\,T\{A_{\nu_1}(x_1)A_{\nu_2}(x_2)A_{\nu_3}(x_3)J_{\nu_4}(x_4)\}\,|\,\Omega\rangle+\qquad(2.6)\\ (disc.)\qquad(2.7)\end{aligned}$$

with $(disc.)$ denoting terms that have more than one δ function in momentum space which we will discard later on. Continuing in the same manner we get

$$\begin{aligned}\partial^2_{x_1}\partial^2_{x_2}\partial^2_{x_3}\partial^2_{x_4}\langle\Omega\,|\,T\{A_{\nu_1}(x_1)A_{\nu_2}(x_2)A_{\nu_3}(x_3)A_{\nu_4}(x_4)\}\,|\,\Omega\rangle=\\ e_0^4\langle\Omega\,|\,T\{J_{\nu_1}(x_1)J_{\nu_2}(x_2)J_{\nu_3}(x_3)J_{\nu_4}(x_4)\}\,|\,\Omega\rangle+(disc.)\end{aligned}\tag{2.8}$$

Taking the connected part of this equation finally brings us to

$$\begin{aligned}\partial^2_{x_1}\partial^2_{x_2}\partial^2_{x_3}\partial^2_{x_4}\langle\Omega\,|\,T\{A_{\nu_1}(x_1)A_{\nu_2}(x_2)A_{\nu_3}(x_3)A_{\nu_4}(x_4)\}\,|\,\Omega\rangle_{connected}=\\ e_0^4\langle\Omega\,|\,T\{J_{\nu_1}(x_1)J_{\nu_2}(x_2)J_{\nu_3}(x_3).J_{\nu_4}(x_4)\}\,|\,\Omega\rangle_{connected}.\end{aligned}\tag{2.9}$$

Recall that the above fields are all bare fields. To connect this to (2.1) we have to divide by square roots of field renormalization constants Z_3 for each factor of A and insert the above equation into (2.1). As mentioned before we only want to know the $\mathcal{O}(e^4)$ term of (2.1), so we first express all the bare quantities in terms of a renormalized coupling e and then expand. Since e_0 is e to first order with no strong corrections, the rest of the terms can be taken with $e = 0$. The inverse propagator $(G^{-1})_{\mu\nu}(p)$ of the photon to zeroth order in e is then $\frac{p^2}{i} g_{\mu\nu}$ and the field renormalization constant is just 1 to this order. Our quantity of interest then becomes

$$\Pi_{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2, q_3, q_4) = e^4 \int \prod_i \left(d^4x_i e^{-i(q_i x_i)} \right)$$
$$\langle \Omega \,|\, T\{J_{\mu_1}(x_1) J_{\mu_2}(x_2) J_{\mu_3}(x_3) J_{\mu_4}(x_4)\} \,|\, \Omega \rangle_{connected} \tag{2.10}$$

where the current J is still built from bare quark fields and the correlator is calculated with e=0, that is in pure QCD. If we are interested in the full amplitude not just the $\mathcal{O}(e^4)$ approximation we get

$$\Pi_{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2, q_3, q_4) =$$
$$(\sqrt{Z_3})^4 e_0^4 \int \prod_i \left(d^4x_i \frac{G_b^{-1}(q_i)}{q_i^2} e^{-i(q_i x_i)} \right) \langle \Omega \,|\, T\{J_1 J_2 J_3 J_4\} \,|\, \Omega \rangle_{connected} =$$
$$e^4 \int \prod_i \left(d^4x_i \frac{G_b^{-1}(q_i)}{q_i^2} e^{-i(q_i x_i)} \right) \langle \Omega \,|\, T\{J_1 J_2 J_3 J_4\} \,|\, \Omega \rangle_{connected} \tag{2.11}$$

where G_b is the bare propagator and $e_0\sqrt{Z_3} = e$ was used. Now of course the current correlator is calculated with $e \neq 0$. The finiteness of all the above quantities is guaranteed by the renormalizability of QCD coupled to electromagnetism. The above tensor is gauge invariant with respect to all 4 indices on- and off-shell

$$\{q_1^{\mu_1}, q_2^{\mu_2}, q_3^{\mu_3}, q_4^{\mu_4}\} \Pi_{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2, q_3, q_4) = 0 \tag{2.12}$$

and is symmetric under any exchange of $(q_i, \mu_i) \longleftrightarrow (q_j, \mu_j)$. For many purposes of this work we do not need the most general version of the HLbL tensor, we need only the version where one photon has an infinitesimal momentum k which we take to be q_4 and is on-shell

$$\Pi_{\mu\nu\lambda}(q_1, q_2, q_3) = \int d^4x_1 d^4x_2 e^{-i(q_1 x_1 + q_2 x_2)} \langle \Omega \,|\, T\{J_\mu(x_1) J_\nu(x_2) J_\lambda(0)\} \,|\, \gamma(k, \epsilon_k) \rangle \, . \tag{2.13}$$

The momentum conserving delta function and e^4 have been extracted. In $N_f = 3$ QCD the current is

$$J_\mu := \bar{\psi} \gamma_\mu \hat{Q} \psi, \tag{2.14}$$

with the matrix $\hat{\mathrm{Q}} = \mathrm{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ acting on the flavour indices and the quark fields being bare fields. The off-shell behaviour of the above correlator is extremely

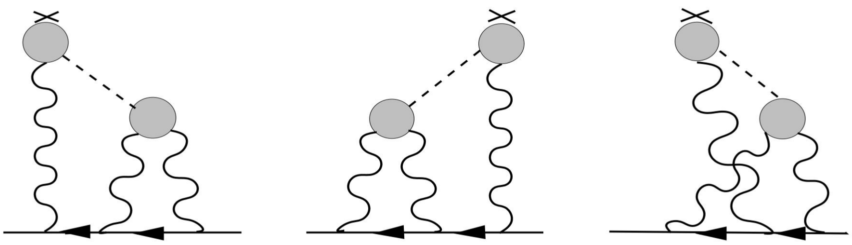


Figure 2.2: Three contributions to the polarization tensor arising from neutral hadron exchange. The blob represents the transition form factor, the wiggly lines represent photon propagators and the dashed line represents the hadron propagator. The cross marks the coupling to the external electro-magnetic field. Graphic taken from [8].

important, since this correlator contributes to the anomalous magnetic moment of the muon (and also the electron) via figure 2.1. It is then the part linear in k which contributes to the anomalous magnetic moment [8]. The polarization tensor can be decomposed in terms of a generalized tensor basis and the contribution to a_μ can then be expressed as a 3-dimensional integral as shown by [9].

The TFFs for various hadrons arise when one considers the decay of a neutral hadron into 2 photons. Taking the momenta of all 3 particles on-shell, the TFFs are proportional to the physical amplitude of 2 photon decay. In phenomenological models for the HLbL tensor, the off-shell values of these quantities contribute as shown in figure 2.2.

The pseudoscalar transition form factors $F_{P\gamma\gamma}(q_1^2, q_2^2)$ are defined by

$$i \int d^4x e^{-iq_1 x} \langle \Omega \,|\, T\{J_\mu(x) J_\nu(0) \,|\, P(p)\rangle = \varepsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F_{P\gamma\gamma}(q_1^2, q_2^2), \tag{2.15}$$

where $P = \pi, \eta, \eta'$ and the axial vector amplitudes $\mathcal{M}_{\mu\nu\alpha}$ are defined by

$$i \int d^4x e^{-iq_1 x} \langle \Omega \,|\, T\{J_\mu(x) J_\nu(0) \,|\, A\rangle = \mathcal{M}_{\mu\nu\alpha} \varepsilon_A^\alpha. \tag{2.16}$$

There are different tensor decompositions of $\mathcal{M}_{\mu\nu\alpha}$ so the definition of the scalar functions multiplying the tensor structures, i.e. the form factors, depend on the chosen tensor basis. We make our choice in chapter 4. To calculate for example asymptotic constraints on these quantities in QCD or in a holographic theory one computes vacuum expectation values of time-ordered products of two vector currents and one axial vector current. In QCD one can for example prove that pion states have nonzero overlap with these axial vector currents in the chiral limit $m_q \to 0$ via Goldstones theorem. The residue of such a vacuum expectation value at the pion mass, which is zero in the chiral limit, then represents the decay amplitude.

2.2 The BTT basis

In this section we would like to quickly summarize the decomposition of the HLbL tensor in terms of a certain tensor basis, the Bardeen-Tung-Tarach (BTT) basis, and coefficient functions, which was first presented in [9]. The basic quantities which appear in the Feynman rules are $\varepsilon_{\mu\nu\alpha\beta},\ g_{\mu\nu},\ q_i^\mu$. After using current conservation in each index and Bose symmetry, it can be shown that one needs at least 43 scalar functions and tensor structures for the decomposition of the HLbL tensor. This basis however has kinematic zeros and therefore another set of 54 tensor structures is chosen. This set of structures is redundant but free of kinematical singularities and zeros. Kinematic zeros are points or sets of points where a set of tensor structures are not linearly independent anymore. A trivial example would be the tensor structure $(q_1 \cdot q_2) q_1^\mu q_2^\nu$ which vanishes when q_1 is orthogonal to q_2. The scattering tensor can then be expressed as

$$\Pi^{\mu\nu\alpha\beta} = \sum_{i=1}^{54} T_i^{\mu\nu\alpha\beta} \Pi_i. \tag{2.17}$$

With this basis the contribution to the anomalous magnetic moment of the HLbL tensor can actually be greatly simplified to a three dimensional integral

$$a_\mu^{HLbL} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2} Q_1^3 Q_2^3 \times \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, Q_3), \tag{2.18}$$

where $\bar{\Pi}_i$ are linear combinations of Π_i and the T_i are known integration kernels, for example $\bar{\Pi}_1 = \Pi_1 + q_1 \cdot q_2 \Pi_{47}$. The tensor structure for which Π_1 is the coefficient will concern us the most here and is given by $T_1^{\mu\nu\lambda\sigma} = \varepsilon^{\mu\nu\alpha\beta} \varepsilon^{\lambda\sigma\gamma\delta} q_{1\alpha} q_{2\beta} q_{3\gamma} q_{4\delta}$. Also notice that in the above formula the momenta are Euclidean momenta with $Q_i^2 := -q_i^2 \geqslant 0$.

2.3 Operator product expansion

One version of the OPE states [3, 10] that given two renormalized local operators $A(x)$ and $B(y)$ their time-ordered product at nearby spacetime points can be written as a sum of products of singular coefficients $f_j(x-y)$ as $y \to x$ and renormalized operators $O_j(x)$

$$T\{A(x)B(y)\} = \sum_{j=1}^{m} f_j(x-y)O_j(x) + R(x, x-y). \tag{2.19}$$

The coefficients f_j depend on the direction of $\xi = x - y$ and its magnitude. We will however assume that they are polynomials in the direction times functions that only depend on the magnitude. The operator $R(x, \xi)$ vanishes as its second argument goes to zero. The f_j are ordered by decreasing singularity as its argument goes to zero

$$\lim_{\xi \to 0} \frac{f_{j+1}(\xi)}{f_j(\xi)} = 0. \tag{2.20}$$

More commonly the Fourier transformation of (2.19) is considered

$$\int d^4x e^{-ikx} T\{A(x)B(0)\} = \sum_{j=1}^{m} \tilde{f}_j(k)O_j(0) + \tilde{R}(0, k). \tag{2.21}$$

The canonical dimension of the operators $O_j(0)$ is assumed to increase with increasing j, i.e. they become more and more complicated. The coefficients $\tilde{f}_j(k)$ vanish now more and more rapidly with increasing j for $k \to \infty$ and the function $\tilde{R}(0, k)$ decays exponentially fast.

The OPE becomes most important in asymptotically free theories which are confining at low energies such as QCD. QCD is a renormalizeable theory but the perturbative series of most low energy processes does not appear to converge (exceptions are for example processes related to anomalies). In addition non-perturbative corrections which are completely missed by the Feynman diagram expansion are expected to give sizable contributions. The OPE allows one to calculate the coefficients $\tilde{f}_j(k)$ with perturbation theory since roughly speaking for high-momentum transfer the coupling constant decreases and the perturbative series gives meaningful answers while the low energy processes are encoded in the matrix elements of the $O_j(0)$ with external states. These can often be kinematically decomposed and the values of the coefficients or coefficient functions at specific momenta can be taken from experiment. One technical point is that the operators $O_j(0)$ are often renormalized versions of products of elementary fields at the same spacetime point. Therefore they depend on a normalization scale μ at which normalization conditions are specified. As the LHS of (2.21) is independent of this scale the coefficient functions $\tilde{f}_j(k)$ are also dependent on μ and cancel the μ dependence of the operators.

QCD also has non-perturbative contributions to observables which are completely missed by the standard Feynman diagram expansion. This has two important effects. First of all it gives some vacuum matrix elements nonzero values that would get no contribution from Feynman diagrams at all like $\langle\Omega\,|\,(\bar{q}q)_M(0)\,|\,\Omega\rangle$ (M is a normalization scale here). Secondly it can potentially violate the validity of the OPE since the standard proof of the OPE within QFT's relies on the perturbative expansion in terms of Feynman diagrams [10]. In [11] it is argued using approximate solutions for instantons that up until a critical dimension d_c the OPE is still valid, i.e. that a finite sum of local operators and perturbatively calculated coefficients as above is a good approximation to the high-energy behaviour of operator products until one gets to operators with canonical dimension $d \geq d_c$.

3 QCD Constraints

In this section we will review constraints to the HLbL tensor and the pseudoscalar TFFs coming from QCD. Some of these require the use of the OPE and others can be calculated in a straightforward manner from Feynman diagrams. As we will see, the chiral anomalies of QCD will play a big role in the determination of these constraints.

The first constraint we will look at concerns an asymmetric limit of the HLbL tensor first derived in [12] (2.13) namely

$$\lim_{Q_3\to\infty}\lim_{Q\to\infty}\Pi_{\mu\nu\lambda}(Q,Q,Q_3), \tag{3.1}$$

where the order of the limits matter.

To evaluate this we will use an OPE for the product of two currents

$$i\int d^4x_1 d^4x_2 e^{-i(q_1x_1+q_2x_2)}T\{J_\mu(x_1)J_\nu(x_2)\}, \tag{3.2}$$

in the limit when $|(q_1-q_2)^2| \ggg |q_3^2| = |(q_1+q_2)^2|$. We will start by describing our conventions for QCD in the following section.

3.1 QCD

In this section we briefly describe the Lagrangian defining our theory. The classical Lagrangian is

$$\mathcal{L}_0 = \bar\psi_0(i\not{D})\psi_0 - \frac{1}{4}(F^a_{0\mu\nu})^2 - \bar\psi_0 M_0\psi_0, \tag{3.3}$$

where ψ describes fermions with N_f numbers of flavours and N_c numbers of colours, M is a diagonal matrix acting on the flavour degrees of freedom describing the masses of the fermions and $\not{D}$ is the usual Dirac operator $\gamma^\mu(\partial_\mu - ig_0A^a_{0\mu}t^a)$. The fermions are taken to be in the fundamental representation of $SU(N_c)\times SU(N_f)$, and the indices corresponding to these representations and the Dirac indices are suppressed in our notation. For the most part of this work we take $N_c = N_f = 3$. Correlation functions of gauge invariant combinations of the elementary fields for the quantum theory are formally given by

$$\langle\Omega\,|\,T\{F_1...F_n\}\,|\,\Omega\rangle = \frac{\int\mathcal{D}\bar\psi_0\mathcal{D}\psi_0\mathcal{D}A_0\; e^{i\int d^4x\mathcal{L}_0}F_1...F_n}{\int\mathcal{D}\bar\psi_0\mathcal{D}\psi_0\mathcal{D}A_0\; e^{i\int d^4x\mathcal{L}_0}}, \tag{3.4}$$

with $\mathcal{L}_0$ being given by (3.3) and $F_1, ..., F_n$ being gauge invariant functionals of the elementary fields. The right hand side contains infinities in the numerator and the denominator due to integrations over the gauge group at each point in spacetime. This makes it not suitable for a perturbative expansion. Furthermore it is not known whether this object even makes mathematical sense. A formally equivalent but more refined expression for correlation functions that gives finite diagrams after renormalization is obtained via the Faddeev-Popov method [13]. With this the correlation functions give

$$\langle \Omega \,|\, T\{F_1 ... F_n\} \,|\, \Omega \rangle = \frac{\int \mathcal{D}\bar{c}_0 \mathcal{D}c_0 \mathcal{D}\bar{\psi}_0 \mathcal{D}\psi_0 \mathcal{D}A_0 \; e^{i\int d^4x \mathcal{L}_{new}} F_1 ... F_n}{\int \mathcal{D}\bar{c}_0 \mathcal{D}c_0 \mathcal{D}\bar{\psi}_0 \mathcal{D}\psi_0 \mathcal{D}A_0 \; e^{i\int d^4x \mathcal{L}_{new}}}. \tag{3.5}$$

For a particlar choice of the Faddeev-Popov procedure (generalized ξ gauges) the new Lagrangian is

$$\mathcal{L}_{new} = \bar{\psi}_0(i\not{D})\psi_0 - \frac{1}{4}(F^a_{0\mu\nu})^2 - \bar{\psi}_0 M_0 \psi_0 - \frac{1}{2\xi_0}(\partial^\mu A^a_{0\mu})^2 + \bar{c}^a_0(-\partial^\mu D^{ab}_\mu)c^b_0. \tag{3.6}$$

In this formula c^a_0 is a Grassmann valued scalar field in the adjoint representation of $SU(N_c)$, $D^{ab} = \delta^{ab}\partial_\mu + g_0 f^{acb} A^c_{0\mu}$ and ξ_0 is a real parameter which fixes the gluon propagator in the Feynman diagrams. Correlators of gauge invariant operators are independent of ξ_0. The Lagrangian has no local gauge symmetry anymore but there is a new symmetry, the so called BRST symmetry [14]. A BRST transformation on the elementary matter and gauge fields looks like a gauge transformation with the Grassmann valued field c^a_0 as the parameter. Physical observables are now BRST invariant functionals of the fields. As a remark we would like to stress that with the Feynman rules that this Lagrangian gives it is easily possible to calculate non-gauge invariant correlation functions. These however in general do not have a physical meaning and must be part of a bigger gauge invariant diagram, i.e. these must always be subdiagrams. The new Lagrangian gives Feynman rules for diagrams involving quark and gluon operators. After multiplying one-particle irreducible (1PI) functions by appropriate factors $\sqrt{Z_i}$ and using renormalization conditions, the bare constants M_0, ξ_0 g_0 and Z_i can be expressed via finite mass and coupling parameters depending on some energy μ and a regularization parameter. This procedure then gives finite answers to 1PI diagrams for each order in the renormalized coupling in the limit when the cutoff constant is taken to infinity. Some of the field renormalization constants Z_i can be related to each other with the help of the Schwinger-Dyson equations. This is usually called bare perturbation theory. In QCD it is more difficult to connect diagrams to scattering amplitudes since no elementary operator is associated to a physical particle, which is an aspect of confinement. Renormalized versions of composite operators can describe physical degrees of freedom like mesons and baryons. In the section 2 we once made the choice $\xi_0 = 1$ for the computation involving the gauge invariant HLbL tensor. In most calculations it is however more convenient to choose the renormalized parameter $\xi = 1$.

For some applications it is better to use a different kind of Feynman diagram expansion called renormalized perturbation theory. Here we introduce new rescaled fields $\phi_{0i} = \sqrt{Z_i}\phi_i$ for each field and express the Lagrangian $\mathcal{L}_0$ in terms of them. In addition we split every bare mass and coupling to obtain a Lagrangian containing a part that looks like $\mathcal{L}_{new}$ but with finite renormalized couplings masses and fields and bunch of additional terms called counterterms. To obtain scattering amplitudes we compute the connected amputated Feynman diagrams with the Feynman rules obtained from this Lagrangian. In this procedure we do not need to multiply with $\sqrt{Z_i}$ for every external line. The diagramatic expansions of QCD amplitudes only converge in special situations where some momentum transfer is high. In addition to this problem there are contributions to the path integral which are completely missed by perturbation theory. In the case of the OPE coefficents, the diagramatic expression is valid however. It converges or appears to converge and non-perturbative terms are expected to be negligible.

3.2 OPE for correlation functions

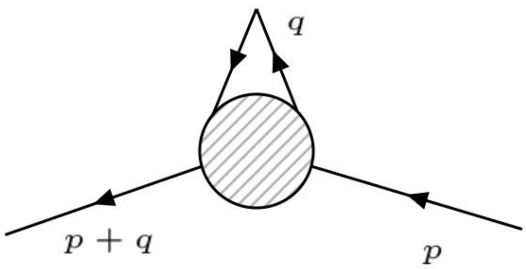
The first OPE we will perform is the one for $T\{J(x)J(0)\}$. Each one of the individual currents is BRST invariant and therefore the product expansion needs to be done in terms of BRST invariant operators. The external states that will sandwich these operators will annihilate any ghost or antighost operators so we really only need operators that are built out of the matter and gauge fields and are gauge invariant. We will first enumerate the relevant gauge invariant operators starting from the ones with the lowest canonical dimension d

$$\begin{aligned}
&\mathbb{I}, && (d=0)\\
&\bar{\psi}\Gamma F\psi, && (d=3)\\
&F^a_{\mu\nu}F^a_{\alpha\beta},\quad \bar{\psi}\Gamma F D_\mu\psi && (d=4)
\end{aligned}$$

We have not displayed the colour, flavour, and Dirac indices on the matter fields explicitly. Also the operators in this table that are composed of more than one elementary field have to be renormalized so the symbols in the table really represent renormalized versions of these operators. These renormalized versions have the same quantum numbers and transformation properties under various symmetries as the unrenormalized ones. In the above equation the Dirac matrix Γ is one of the 16 basis matrices and F is a matrix acting only on flavours. It is also important to say that even if one renormalizes composite operators in the usual way, i.e. making insertions of this operator into arbitrary Green's functions of elementary fields finite, it is not guaranteed that correlators of multiples of these operators are finite. An example is the correlator of two electro-magnetic currents and no elementary fields. It is possible to show that the contribution from the unit operator is divergent and therefore not defined. The contributions from all other operators are however well defined. A way to see this is to go back to (2.1) and try to do the derivation for 2 currents (not 4 as above). One can see that the connected amputated 2 point function $\Pi_{\mu\nu}$ is the correlator of two currents plus a counterterm.

It is important to stress that we intend to use the OPE for the current correlator to get an asymptotic formula for the HLbL tensor, i.e. the OPE will be sandwiched between two states that are orthogonal to each other. This means we do not need to worry about the unit operator but it also means that the operators that we use in the OPE need not be Lorentz scalars. For our purposes we are content with the first term in the OPE which uses dimension 3 operators. It might still be possible that the coefficients of these operators vanish or that they contain mass terms. In this case we would have to use the dimension 4 operators as well. We will see that this is not the case. In order to have the correct C-parity transformation properties the quark bilinear must be a scalar, a pseudoscalar or a pseudovector, which means the matrix Γ should be $\mathbb{I}$, γ_5 or $\gamma_5\gamma^\mu$.

Each of the operators with one of the above spinor matrices sandwiched between spinors $\bar{\psi}$ and ψ will only be multiplicatively renormalized (if at all) since each one is unique in the sense that there is no other gauge invariant operator of the same dimension with the same transformation behavior under Lorentz transformations and C and P. Even though we will only use the zeroth order result, we now give a short description how one obtains the renormalized versions of the operators above at one-loop level. Let us start with trying to find a renormalized version of $\bar{q}q$ which we call $(\bar{q}q)_M$ where M denotes the normalization point. The normalization condition for operator $(\bar{q}q)_M$ is



$= 1$

at spacelike momenta $p^2 = q^2 = (p + q^2) = -M^2$. The two lines meeting at "q" denote the insertion of $(\bar{q}q)_M(x)$ and the other two lines denote external fermions.

If we make the Ansatz $(\bar{q}q)_M(x) = (\bar{q}q)(x) + \delta_{\bar{q}q}(\bar{q}q)(x)$ then the zeroth order result always gives 1 and we have to find a counterterm $\delta_{\bar{q}q}$ which cancels the diagram in figure 3.1 with one gluon propagator exactly at the normalization point.

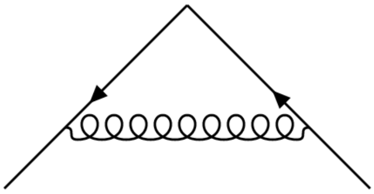


Figure 3.1: The first order diagram contributing to the counterterm.

A calculation using dimensional regularization reveals that

$$\delta_{\bar{q}q} = -4\frac{4}{3}\frac{g^2}{(4\pi)^2}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \tag{3.7}$$

The renormalized quark fields q also depend themselves on an energy scale M and the computation at one-loop level of the anomalous dimension of $(\bar{q}q)_M(x)$ gives

$$\gamma_{\bar{q}q} = M\frac{\partial}{\partial M}(-\delta_{\bar{q}q} + \delta_2) = -8\frac{g^2}{(4\pi)^2}, \tag{3.8}$$

where the first equality only holds at one-loop level and $\delta_2 = Z_2 - 1$ is the field renormalization constant for the quark field. The operator is now finite at one-loop level and depends on the energy scale M. If we define a running "mass" term $\bar{m}(Q)$ such that

$$Q\frac{d}{dQ}\bar{m}(Q) = \gamma_{\bar{q}q}(\bar{g}(Q))\bar{m}(Q), \tag{3.9}$$

where $\bar{g}(Q)$ is the running coupling constant, then one can show that $\bar{m}(Q)(\bar{q}q)_Q$ is independent of Q to all loop orders and is nothing but $m_0\bar{q}_0q_0$ where the subscript denotes bare quantities. This is useful since whenever there is a chance of $(\bar{q}q)_M(x)$ appearing in an OPE of scale independent quantities, its OPE coefficient must be proportional to a mass term to cancel the dependence on the normalization point. Using dimensional analysis this also means that the OPE coefficient in Fourier space will fall of faster for large momenta. The same analysis works for $\bar{q}\gamma_5 q$. At one-loop level the anomalous dimension is even the same as above. This is no coincidence as one can prove using Ward identities for conserved currents that. As an aside, it is also possible to show that in case we have different flavour structure $\bar{q}\tilde{q}$, with $q = u$ and $\tilde{q} = d$ for example, its renormalized version has the same anomalous dimension as for equal flavours.

$$\bar{m}(Q)(\bar{q}\gamma_5 q)_Q = m_0\bar{q}_0\gamma_5 q_0, \tag{3.10}$$

with the same $\bar{m}(Q)$ as above. The scalar density and the pseudoscalar density thereby have the same dependence on the normalization point.

Trying to find a renormalized version of a flavour current $J_5^\mu = \bar{\psi}\gamma_5\gamma^\mu F\psi$ at one-loop level involves the same computation as above but with a slightly different Lorentz index structure. The diagram 3.1 with one gluon inserted is again divergent and a counterterm needs to be introduced. However due to the different index structure, one can easily see that this counterterm is the same as δ_2, the field renormalization constant, and so at one-loop level

$$(J_5^\mu)_M = \bar{\psi}_0\gamma_5\gamma^\mu F\psi_0. \tag{3.11}$$

This is of course independent of M and finite at one-loop level for any F. If the flavour matrix F is a matrix in the algebra of $SU(N_f)$ then the associated bare currents are conserved. With the associated Ward identity one can prove that $\bar{\psi}_0\gamma_5\gamma^\mu F\psi_0$ is a finite operator to all loop orders and also of course independent of any scale since it is built from bare fields. We can take these to be the renormalized version of J_5^μ. The same line of arguments works of course for the $\bar{\psi}\gamma^\mu F\psi$ operators too and shows that their renormalized versions are $\bar{\psi}_0\gamma^\mu F\psi_0$. The slight difference is that for the vector currents we can take F to be any matrix. If the matrix F is proportional to the identity, then the above proof does not work anymore for the axial vector current due to anomalies. The renormalized operator $(J_5^\mu)_M$ can of course still be constructed order by order but this operator will not be independent of the scale M and in addition mixes with other operators of the same dimension when the scale is varied meaning

$$(J_5^\mu)_{M_1} = c(M_1, M_2)(J_5^\mu)_{M_2} + c^i(M_1, M_2)(O_i)_{M_2}. \tag{3.12}$$

The non-perturbative aspects of the $U(1)_A$ anomaly allows one to find the following equation

$$\begin{aligned}\partial_\mu (J_5^\mu)_M - \frac{\alpha_s(Q)}{8\pi} N_f (\varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma})_M = \\ \partial_\mu \bar{\psi}_0\gamma_5\gamma^\mu\psi_0 - \frac{\alpha_0}{8\pi} N_f \varepsilon^{\mu\nu\rho\sigma} F^a_{0\mu\nu} F^a_{0\rho\sigma}\end{aligned} \tag{3.13}$$

with $\alpha_s(Q) = \frac{\bar{g}^2(Q)}{4\pi}$. Since the second line is completely independent of M this can tell us about how the divergence of the current mixes with the field strength bilinear. As a last comment we mention that a renormalized version of $\varepsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$ requires different counterterms than a renormalized version of $F^a_{\mu\nu} F^{\mu\nu a}$.

In the following we will omit the dependence of the renormalized operators on M from the notation and we will only be concerned with the zeroth order contributions from the operators. Via symmetries and dimensional analysis we find that for small euclidean x (omitting the identity operator)

$$
\begin{aligned}
&T\{J_\mu(x)J_\nu(0)\}\\
&= A_{\mu\nu}{}^{\beta}(x)(\bar\psi\gamma_5\gamma_\beta F_A\psi)(0) + B_{\mu\nu}(x)(\bar\psi F_B\psi)(0) + C_{\mu\nu}(x)(\bar\psi\gamma_5 F_C\psi)(0)\\
&\approx A_{\mu\nu}{}^{\beta}(x)(\bar\psi\gamma_5\gamma_\beta F_A\psi)(\frac{x}{2}) + B_{\mu\nu}(x)(\bar\psi F_B\psi)(\frac{x}{2}) + C_{\mu\nu}(x)(\bar\psi\gamma_5 F_C\psi)(\frac{x}{2}) \quad (3.14)
\end{aligned}
$$

with $F_{A,B,C}$ acting on flavour indices and the coefficients having a divergence no worse than $\frac{1}{x^3}$. In the second line we evaluate the operators at the point between 0 and x which makes no difference to leading order. We want to use this short distance expansion in the Fourier transformation of $T\{J_\mu(x)J_\nu(y)\}$. Using coordinates $z = \frac{x+y}{2}$, $\eta = x - y$ and writing $\xi = \frac{q_1-q_2}{2}$ we have

$$
\begin{aligned}
&\int d^4x d^4y e^{-i(q_1x+q_2y)} T\{J_\mu(x)J_\nu(y)\} =\\
&\int d^4z d^4\eta \det(J) e^{i\eta\xi} e^{-iz(q_1+q_2)} U(z) T\{J_\mu(\frac{\eta}{2})J_\nu(-\frac{\eta}{2})\} U^{-1}(z) =\\
&\int d^4z e^{-iz(q_1+q_2)} (\bar\psi\gamma_5\gamma_\beta F_A\psi)(z) \tilde A_{\mu\nu}{}^{\beta}(\xi) +\\
&\int d^4z e^{-iz(q_1+q_2)} (\bar\psi F_B\psi)(z) \tilde B_{\mu\nu}(\xi) +\\
&\int d^4z e^{-iz(q_1+q_2)} (\bar\psi\gamma_5 F_C\psi)(z) \tilde C_{\mu\nu}(\xi), \quad (3.15)
\end{aligned}
$$

with J the Jacobian of the coordinate-transformation which has been absorbed into the Fourier transformed coefficients of the operators and $U(z)$ a translation by the vector z. In the second line above we can perform the Fourier transform of $T\{J_\mu(\frac{\eta}{2})J_\nu(-\frac{\eta}{2})\}$ separately. If this function was real analytic in η, then the Fourier transform would decay exponentially fast for large $\frac{q_1-q_2}{2} = \xi$. Only the singular parts yield non-exponential behaviour so we can replace the time-ordered product in the integral with the short-distance expansion. In other words the short distance behaviour dominates the integral in the limit of high-momentum ξ. The coefficients $\tilde B_{\mu\nu}(\xi)$ and $\tilde C_{\mu\nu}(\xi)$ by Lorentz invariance can be written as linear combinations of $g_{\mu\nu}$ and $\xi_\mu\xi_\nu$. The operators multiplying these coefficients have however different parity transformation behavior. A calculation reveals that $\tilde C_{\mu\nu}(\xi)$ has to vanish. Also the time-ordered product of two currents has a symmetry, namely interchanging of μ and ν with a simultaneous exchange of q_1 and q_2 leaves the object invariant. The coefficient $\tilde A_{\mu\nu}{}^{\beta}(\xi)$ can then be written as a scalar function of ξ times $\varepsilon_{\mu\nu\alpha}{}^{\beta}\xi^\alpha$. The time reversal symmetry doesn't lead to any new constraints on the coefficients since by the CPT theorem we can relate T to CP. If the coefficient functions in x-space diverge no worse than $\frac{1}{x^3}$, then the Fourier coefficients are expected to go like $\frac{1}{q}$ as q goes to infinity. Less divergent coefficients in x space means faster fall off behavior in q space. With this knowledge we write

$$\tilde{A}_{\mu\nu}{}^{\beta}(\xi) = \frac{a}{\xi^2}\varepsilon_{\mu\nu\alpha}{}^{\beta}\xi^{\alpha},$$
$$\tilde{B}_{\mu\nu}(\xi) = \frac{b_1}{\sqrt{\xi^2}}g_{\mu\nu} + \frac{b_2}{\xi^3}\xi_\mu\xi_\nu. \tag{3.16}$$

The coefficients a, b_1, b_2 will in general depend on the normalization scale at which the operators multiplying them are defined and could in priniple also contain logarithms of the momentum ξ. We want to stress also that the above OPE in momentum space is valid for large $\xi = \frac{q_1-q_2}{2}$ independent of what value the sum $q_1 + q_2$ has. To calculate the coefficients we will sandwich (3.15) between an outgoing vacuum state and an incoming state that contains an antiquark and a quark. Since states with external quarks might not be directly related to Feynman diagrams because the Kählen-Lehman decomposition of the quark propagator in QCD does not reveal the pole structure at low momenta it is more correct to say that we consider a connected diagram with one amputated quark line and one amputated antiquark line and also choose operator normalization points using these diagrams instead of external states. We also ignored the unit operator on the RHS of the OPE in our discussion since we will want to take matrix elements between two orthogonal states. The first 2 diagrams that arise from sandwiching the LHS with the 2 states mentioned before are shown in figure 3.2.

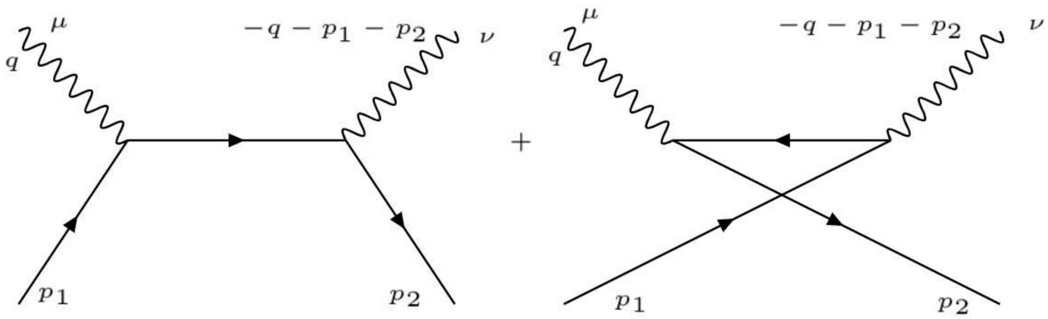


Figure 3.2: Two diagrams contributing to the coefficients. All momenta are incoming in this diagram.

The incoming quark and antiquark have momenta p_1 and p_2 respectively. By sandwiching (3.15) with the aforementioned states we get

$$\left\langle \Omega \middle| i\int d^4x T\{J_\mu(x)J_\nu(0)\}e^{-iqx} \middle| q_f(p_1), \bar{q}_f(p_2)\right\rangle$$
$$= c\varepsilon_{\mu\nu\alpha\beta}\frac{q^\alpha}{q^2}\bar{u}_{f'}(p_2)\gamma^5\gamma^\beta F_{f'f}u_f(p_1) + (OT), \tag{3.17}$$

where OT stands for the terms coming from the other operator in the OPE. The first contribution to the first term is then $-i$ times the values of the two diagrams in figure 3.2 which give

$$\bar{u}(p_2)\Big\{\hat{Q}\gamma_\nu \frac{i}{\not{q}+\not{p}_1-m}\hat{Q}\gamma_\mu + \hat{Q}\gamma_\mu \frac{i}{-\not{q}-\not{p}_2-m}\hat{Q}\gamma_\nu\Big\}u(p_1)$$
$$\approx \frac{iq^\alpha}{q^2}\bar{u}(p_2)\hat{Q}^2[\gamma_\nu\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\gamma_\nu]u(p_1). \tag{3.18}$$

We have already extracted the high-momentum behaviour in the second line above. To further simplify we use the identity

$$\gamma_\nu\gamma_\alpha\gamma_\mu - \gamma_\mu\gamma_\alpha\gamma_\nu = -i\varepsilon_{\nu\mu\alpha\beta}\gamma_5\gamma^\beta, \tag{3.19}$$

with $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The first thing we see is that these diagrams only contribute to the a coefficient and to this order the scalar operator with coefficients b_1 and b_2 doesn't contribute. We can then compare with (3.17) and find $cF = 2i\hat{Q}^2$. In summary we have

$$i\int d^4x T\{J_\mu(x)J_\nu(0)\}e^{-iqx} = 2i\varepsilon_{\mu\nu\alpha\beta}\frac{q^\alpha}{q^2+i\epsilon}J_5^\beta(0), \tag{3.20}$$

with J_5^β being $\bar{\psi}_0\hat{Q}^2\gamma_5\gamma^\beta\psi_0$. The flavour singlet part of the RHS is a finite operator up to one-loop order and independent of the normalization scale M and the flavour octet part is finite to all loop orders and independent of the normalization scale to all orders.

3.3 The MV constraint

A soft photon with vanishing momentum k in the light-by-light scattering amplitude can be modeled by an external state so we only need to look at the time-ordered product of three currents. We now insert the operator expansion into the HLbL tensor

$$\Pi_{\mu\nu\lambda}(q_1,q_2,q_3) = \int d^4x_1 d^4x_2 e^{-i(q_1x_1+q_2x_2)}\left\langle\Omega\,|\,T\{J_\mu(x_1)J_\nu(x_2)J_\lambda(0)\}\,|\,\gamma(k,\epsilon_k)\right\rangle. \tag{3.21}$$

We change the integration to an integration over the midpoint of x and y called z and an integration over relative coordinates called η. For high relative momentum $\xi = \frac{q_1-q_2}{2}$ we only need to integrate over a small interval of relative coordinates to get the asymptotics right. We then approximate the above integral as

$$\int_{z^0>\frac{\eta_b^0}{2}} d^4z e^{-i(q_1+q_2)z} \int d^4\eta e^{-i\xi\eta} \langle\Omega\,|\,T\{J_\mu(z+\eta/2)J_\nu(z-\eta/2)\}J_\lambda(0)\,|\,\gamma(k,\epsilon_k)\rangle$$
$$+\int_{z^0<-\frac{\eta_b^0}{2}} d^4z e^{-i(q_1+q_2)z} \int d^4\eta e^{-i\xi\eta} \langle\Omega\,|\,J_\lambda(0)T\{J_\mu(z+\eta/2)J_\nu(z-\eta/2)\}\,|\,\gamma(k,\epsilon_k)\rangle$$
$$+(OT), \tag{3.22}$$

where the η integration is constrained to a small region around 0 with the temporal boundary of integration having time coordinates η_b^0 and $-\eta_b^0$. The terms labeled by (OT) are terms where the J_α current is in between the other two currents, i.e. where 0 is in the interval $(z^0-\frac{\eta_b^0}{2}, z^0+\frac{\eta_b^0}{2})$. These other terms produce singularities in z and η too since two or three currents come close to each other. But when Fourier transforming they do not contribute to the high ξ asymptotics but to different limits. For example the region of integration where x_1 is very close to 0 determines the asymptotics when $2q_1+q_2$ becomes large. So by taking the $\xi\to\infty$ limit before any other limit, we can neglect these terms and also approximate $\eta_b^0=0$ in the above equation. The asymmetric limit (3.1) is used for this very reason of being able to neglect such terms at this stage. In summary we get

$$\int_{z^0>0} d^4z e^{-i(q_1+q_2)z} \int d^4\eta e^{-i\xi\eta} \langle\Omega\,|\,T\{J_\mu(z+\eta/2)J_\nu(z-\eta/2)\}J_\lambda(0)\,|\,\gamma(k,\epsilon_k)\rangle$$
$$+\int_{z^0<0} d^4z e^{-i(q_1+q_2)z} \int d^4\eta e^{-i\xi\eta} \langle\Omega\,|\,J_\lambda(0)T\{J_\mu(z+\eta/2)J_\nu(z-\eta/2)\}\,|\,\gamma(k,\epsilon_k)\rangle \tag{3.23}$$

as our limit for the HLbL tensor. We can now pull the η integration inside the matrix element, insert the OPE and make use of the time ordering symbol for $J_5^\beta(z)$ and $J^\alpha(0)$.

To shorten notation we write $q_3=-q_1-q_2$. We have now completely separated the ξ dependence of the amplitude from the q_3 dependence. The q_3 dependence is contained in the quantities

$$T_{\alpha\beta}(k,q_3)=i\left\langle\Omega\left|\int d^4z e^{iq_3z}T\{J_{5\beta}(z)J_\alpha(0)\}\right|\gamma(k,\epsilon_k)\right\rangle. \tag{3.24}$$

We can decompose the current J_5^α in terms of fundamental $SU(3_f)$ currents and a singlet current

$$J_{5\alpha}=\sum_{a=0,3,8}\frac{Tr[\lambda_a\hat{Q}^2]}{Tr[\lambda_a^2]}J_{5\alpha}^a, \tag{3.25}$$

where λ^a are the standard Gell-Mann matrices for $a=3,8$ and λ^0 being the appropriately normalized identity matrix. We denote by $T_{\alpha\beta}^{(a)}(k,q_3)$ the quantity on the RHS of (3.24) but with $J^a_{5\beta} = \bar{\psi}_0\gamma_\beta\gamma_5\lambda^a\psi_0$ replacing $J_{5\beta}$. We now want to make a kinematical decomposition of (3.24). The amplitude is linear in the soft photon polarization ε^μ and must be invariant under the replacement $\varepsilon^\mu \to \varepsilon^\mu + k^\mu$ by current conservation. The amplitude also obeys $(q_3 - k)^\alpha T^{(a)}_{\alpha\beta}(k, q_3) = 0$ by current conservation. As mentioned before, the part that is interesting for the anomalous magnetic moment is the part linear in k. The linear part then obeys

$$q_3^\alpha T^{(a)}_{\alpha\beta}(k, q_3) = 0, \tag{3.26}$$

since the amplitude vanishes for $k = 0$. This can be seen by trying to do a kinematical decomposition of the $\mathcal{O}(k^0)$ part with the tensors $\varepsilon^\mu, q_3^\mu, \eta_{\mu\nu}$ and $\varepsilon_{\mu\nu\alpha\beta}$ linear in ε^μ and obeying the zeroth order current conservation conditions. Before enforcing (3.26) the part linear in k can then be written in terms of 3 different tensors and invariant functions

$$\begin{aligned} T_{\alpha\beta}(k, q) = {} & f_1(q^2) q_\alpha \varepsilon_{\beta\mu_1\mu_2\mu_3} q^{\mu_1} k^{\mu_2} \varepsilon^{\mu_3} + f_2(q^2) q_\beta \varepsilon_{\alpha\mu_1\mu_2\mu_3} q^{\mu_1} k^{\mu_2} \varepsilon^{\mu_3} \\ & + f_3(q^2) \varepsilon_{\alpha\beta\mu_1\mu_2} k^{\mu_1} \varepsilon^{\mu_2} \end{aligned} \tag{3.27}$$

By enforcing (3.26) we can express f_3 via f_1. We then order the parts that do not vanish when contracting $T_{\alpha\beta}$ with q^α and we get (see [12])

$$\begin{aligned} T^{(a)}_{\alpha\beta}(q_3^2) = \frac{-ieN_c Tr[\lambda_a \hat{Q}^2]}{4\pi^2} \Big\{ & \omega_L^{(a)}(q_3^2) q_{3\beta} q_3^\sigma \tilde{f}_{\sigma\alpha} \\ & + \omega_T^{(a)}(q_3^2)(-q_3^2 \tilde{f}_{\alpha\beta} + q_{3\alpha} q_3^\sigma \tilde{f}_{\sigma\beta} - q_{3\beta} q_3^\sigma \tilde{f}_{\sigma\alpha}) \Big\}, \end{aligned} \tag{3.28}$$

where $\tilde{f}$ is the Hodge dual of $f^{\alpha\beta} = q_3^\alpha \varepsilon^\beta - q_3^\beta \varepsilon^\alpha$. Also an overall factor has been extracted from the coefficient functions $\omega_{L,T}$ which are linear combinations of $f_{1,2,3}$. The factor of N_c naturally arises when computing the diagrams associated to the invariant functions. We still cannot use this to extract a constraint, we need to calculate the functions $\omega^{(a)}_{L,T}$.

3.4 The invariant functions

It is well known that if we compute $q^\beta T^{(a)}_{\alpha\beta}$ the result will be given by famous anomaly relations. These give values to the longitudinal functions for $a = 3, 8$ which receive no further corrections either perturbatively or non-perturbatively in the chiral limit $m_q = 0$. The $a = 0$ function does receive corrections. To reliably calculate the invariant functions we again have to make use of asymptotic freedom and pick a high external momentum q_3. Then the first diagram, a triangle diagram dominates and gives

$$\omega_L^{(a)}(q^2) = \frac{-2}{q^2}, \quad \omega_T^{(a)}(q^2) = \frac{-1}{q^2}. \tag{3.29}$$

If we now insert all the relevant functions we get for the asymmetric limit the following expression

$$\Pi_{\mu_1\mu_2\mu_3}(\hat{q}, q_3^2, k)_{\text{a. lim.}} = \alpha^2 N_c \frac{8}{\hat{q}^2}\varepsilon_{\mu_1\mu_2\delta\rho}\hat{q}^\delta \sum_{a=0,3,8} \frac{(Tr[\lambda_a \hat{Q}^2])^2}{Tr[\lambda_a^2]} \Big\{ \omega_L^{(a)}(q_3^2) q_3^\rho q_3^\sigma \tilde{f}_{\sigma\mu_3} + \omega_T^{(a)}(q_3^2)(-q_3^2 \tilde{f}^\rho_{\mu_3} + q_{3\mu_3} q_3^\sigma \tilde{f}^\rho_\sigma - q_3^\rho q_3^\sigma \tilde{f}_{\sigma\mu_3}) \Big\}. \tag{3.30}$$

This is the final form of the MV constraint and once a model for the HLbL tensor is given we can perform the asymmetric limit and check if it satisfies the above expression. By extracting the ε tensor contained in $\tilde{f}_{\mu\nu}$ we see that in terms of the BTT basis the above expression contributes to $\bar{\Pi}_1$

$$\lim_{Q_3\to\infty} \lim_{Q\to\infty} Q^2 Q_3^2 \bar{\Pi}_1(Q, Q, Q_3) = -\frac{2}{3\pi^2}, \tag{3.31}$$

where again we mention that the order of the limits matters. We also want to mention that there exists a theorem [15] on the perturbative contributions to the invariant functions $\omega_{T,L}^{(a)}$ which states that $2\omega_T^{(a)}(q^2) = \omega_L^{(a)}(q^2)$ for all q. This holds only for the additional contributions from higher order Feynman diagrams and is violated by perturbative corrections. For constraints on the HLbL tensor the limit of high q has to be taken to make the non-perturbative corrections vanish and we arrive again at the asymmetric limit. Non-perturbative contributions to $T_{\alpha\beta}^{(a)}$ have been calculated in [15] using an OPE of $J_\alpha J_{5\beta}^{(a)}$ in a weak external electro-magnetic field. The next-to-leading order term involves a condensate of the renormalized version of $\bar{\psi}[\gamma_\mu, \gamma_\nu]\psi$ in an external field. The holographic models in their simplest form will not be able to account for these terms.

3.5 The symmetric momenta constraint

The second constraint on the HLbL tensor comes from the momentum region where all Euclidean momenta $q_i^2,\ i = 1, 2, 3$ are large and comparable in magnitude [16]. This regime is not available for ordinary perturbation theory since one of the momenta q_4 is infinitesimal. We will model this external photon by adding a new term $\bar{\psi}_0(x)\hat{Q}\gamma^\mu\psi_0(x)A_{\mu(x)}$ to the Lagrangian which models the interaction with an external classical electro-magnetic field. Our polarization tensor is then the same as the correlator of three currents under the influence of an external field. The term of zeroth order in the external field vanishes by Furry's theorem (or charge conjugation invariance) and we are left with the first-order term. The momenta of the other 3 currents are all hard. In an OPE the external photon field has two effects. In a Feynman diagram the line for the external photon can be inserted on a hard quark line and secondly operators whose vacuum expectation value usually vanishes by Lorentz invariance can now obtain nonzero values. This procedure incorporates the large logarithms that we would have in the standard perturbative procedure into new expectation values. The most important contribution is then the diagram where the external line is inserted into a quark loop diagram (see figure 3.3). The contribution to the HLbL tensor in this limit of this diagram is

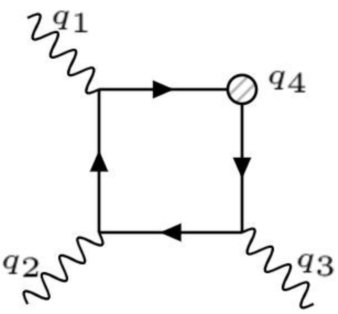


Figure 3.3: The quark loop current with the external photon inserted. The blob represents the interaction with the external photon.

proportional to

$$\int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\{\gamma^\mu S(p)\gamma^\nu S(p+q_2)\gamma^\lambda S(p+q_2+q_3)\gamma^\sigma S(p+q_2+q_3+q_4)\}\varepsilon_\sigma(q_4),$$

$$S(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\varepsilon}.$$

The limit in which the three photon momenta go to infinity involves a lot of computation so we will just quote the result. Just as in the case of the MV constraint this contributes mainly to the coefficient of the $T_1^{\mu\nu\lambda\sigma}$ tensor structure in the BTT basis

$$\lim_{Q\to\infty} Q^4 \bar{\Pi}_1(Q,Q,Q) = -\frac{4}{9\pi^2}. \tag{3.32}$$

It is possible to calculate subleading terms of the OPE in this particular limit which would give additional constraints. For the models considered later on we will see that the above constraint is at least qualitatively obeyed but all subleading terms vanish exponentially so in this work only the leading term will be of value.

3.6 Constraints on the pion transition form factor

In this section we will look at the pion transition form factor defined in (2.15). The first constraint can be obtained using the OPE (3.20) that we already obtained.

To use it we have to have q_1 and q_2 very large in magnitude and space-like, but its sum $-p$ has to be a light-like vector or one with a very small invariant mass.

$$i\int d^4x e^{-iq_1x}\left\langle\Omega\,\middle|\,T\{J_\mu(x)J_\nu(0)\,\middle|\,\pi^0\right\rangle$$
$$= 2i\varepsilon_{\mu\nu\alpha\beta}\frac{q_1^\alpha}{q_1^2}\sum_{a=0,3,8}\frac{Tr[\lambda_a\hat{Q}^2]}{Tr[\lambda_a^2]}\left\langle\Omega\,\middle|\,J^a_{5\beta}(0)\,\middle|\,\pi^0\right\rangle \tag{3.33}$$

The pion state is a (pseudo) Goldstone boson of QCD and therefore it can be proven that the pion state has overlap with the current of the associated spontaneously broken symmetry, which is $\bar{\psi}_0\gamma_5\gamma^\mu\lambda_3\bar{\psi}_0$. The currents with $a = 0, 8$ do not have the right quantum numbers and therefore they do not contribute. Lorentz invariance allows to parametrize the matrix element as follows

$$\left\langle\Omega\,\middle|\,J_5^{3\mu}(0)\,\middle|\,\pi^0\right\rangle = 2iF_\pi p^\mu. \tag{3.34}$$

Usually the factor 2 is missing in the above equation in the literature but there the flavour matrix contained in the current built out of bare fields has an additional $\frac{1}{2}$ factor. F_π is known as the pion decay constant and can be independently measured using the weak decays of the π^+, π^- particles. Its value is about 93 MeV. Inserting this and writing $q_1^2 = q_2^2 = -Q^2$ we get for the transition form factor

$$F_{\pi\gamma\gamma}(-Q^2, -Q^2) = \frac{2F_\pi}{3}\frac{1}{Q^2}. \tag{3.35}$$

In the chiral limit where up, down and strange masses vanish, the η particle which is also a pseudoscalar boson can be interpreted as the Goldstone boson associated to the current $\bar{\psi}_0\gamma_5\gamma^\mu\lambda_8\bar{\psi}$ and the derivation before works exactly in the same way. The only thing that is different is that $Tr[\lambda_a\hat{Q}^2]$ is not 1 like for the pion (colour gets traced too) but $\frac{1}{\sqrt{3}}$. So if we write the result as above we would obtain $F_\eta = \frac{F_\pi}{\sqrt{3}}$. Since chiral symmetry of the Lagrangian is only approximate, the actual η decay constant might differ from that exact value so we will just denote it as F_η. Note that the way we have defined the η decay constant is different from the way we have defined the π decay constant, i.e. as a factor in the overlap of the pion state and the Goldstone current.

The η' particle can not be identified with a broken symmetry. First of all in the chiral limit we only have 8 broken $SU(3)_A$ symmetry currents which describe the lowest pseudoscalar meson octet among which the η' is not included. It can also not be the the Goldstone boson associated to the $U(1)_A$ symmetry with the current $\bar{\psi}_0\gamma_5\gamma^\mu\bar{\psi}$ even though it has the same quantum numbers, since this current is not conserved. We can find a conserved current from this one since its divergence can be written as $\partial_\mu K^\mu$ and we just subtract K^μ from the symmetry current.

The resulting charge Q is however not gauge invariant due to instanton effects. The instantons have the effect that when we perform a gauge transformation with nonzero winding number, our charge Q would change by an integer times a well defined prefactor. Hence this charge can not be used to prove the existence of another Goldstone boson.

If we denote by $\mathbf{3}$ the fundamental representation of $SU(3_f)_V$ and by $\bar{\mathbf{3}}$ its anti-fundamental one, the particles can be characterized by members of irreducible sub-representations of tensor representations of $SU(3)$, i.e. elements in invariant subspaces of $\mathbf{3} \otimes ... \otimes \mathbf{3} \otimes \bar{\mathbf{3}} \otimes ... \otimes \bar{\mathbf{3}}$. The number of fundamental representations need not be the same as the number of anti-fundamental representations. The representation space $\mathbf{3} \otimes \bar{\mathbf{3}}$ can be decomposed into a direct sum of an 8-dimensional and a 1-dimensional space $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$. The 1-dimensional subspace is spanned by the vector $\delta_{i\bar{j}} e_i \otimes \bar{e}_{\bar{j}} \in \mathbf{3} \otimes \bar{\mathbf{3}}$, where $e_i \in \mathbf{3}$ and $\bar{e}_{\bar{j}} \in \bar{\mathbf{3}}$ are basis in their respective spaces. The η' is then identified with this one dimensional subspace. This is the best we can do and it is not possible from these considerations to find a constraint on the η' transition functions. One can however show that as $N_c \to \infty$ with fixed t'Hooft coupling the $U(1)_A$ anomaly vanishes and the η' becomes a Goldstone boson. It then joins the GB octet to form a nonet.

The states that describe the η and η' particles strictly speaking do not have definite flavour quantum numbers. Experiments show that both are linear combinations of states η_8 and η_0 which have the same quantum numbers as $\bar{\psi}_0 \gamma_5 \gamma^\mu \lambda_8 \bar{\psi}$ and $\bar{\psi}_0 \gamma_5 \gamma^\mu \bar{\psi}$. The mixing is however quite small and to a good approximation we can take $\eta = \eta_8$ and $\eta' = \eta_0$.

The next constraint we want to look at regards the zero momentum limit of the transition form factors $F_{P\gamma\gamma}(0,0)$. This region of momenta can not be treated in the asymptotic freedom framework, but two useful results help us there. The first one is the fact that the pion state has nonzero overlap with $J_5^{\mu,(3)} |\Omega\rangle$ and the second one is the exact computations one can do thanks to the axial anomaly.

The quantity to look at is

$$\int d^4x e^{-ipx} \left\langle \gamma(q_1), \gamma(q_2) \,\middle|\, J_5^{\alpha,(3)}(x) \,\middle|\, \Omega \right\rangle = (2\pi)^4 \delta^{(4)}(p + q_1 + q_2) \mathscr{M}^\alpha_{\mu\nu} \varepsilon^{*\mu} \varepsilon^{*\nu} \quad (3.36)$$

in QCD coupled to electromagnetism. Standard polology results tell us that if the momentum p comes close to the pion mass shell $p^2 = 0$ then the LHS can be written as

$$\begin{aligned} & - 2iF_\pi p^\alpha \frac{i}{p^2} \left\langle \gamma(q_1), \gamma(q_2) \,\middle|\, \pi^0(p) \right\rangle \\ & = -(2\pi)^4 \delta^{(4)}(p + q_1 + q_2) 2iF_\pi p^\alpha \frac{i}{p^2} \frac{e^2}{i} F_{\pi\gamma\gamma}(q_1^2, q_2^2) \varepsilon_{\mu\nu\lambda\sigma} \varepsilon^{*\mu} \varepsilon^{*\nu} q_1^\lambda q_2^\sigma, \end{aligned} \quad (3.37)$$

where we only kept terms of order e^2 and $F_{\pi\gamma\gamma}$ is calculated within pure QCD giving

$$\mathscr{M}^{\alpha}_{\mu\nu} = -2F_\pi p^\alpha e^2 \frac{i}{p^2} F_{\pi\gamma\gamma}(q_1^2, q_2^2)\varepsilon_{\mu\nu\lambda\sigma} q_1^\lambda q_2^\sigma, \tag{3.38}$$

Contracting with ip_α and inserting on-shell values for all momenta we get

$$ip_\alpha \mathscr{M}^{\alpha}_{\mu\nu} = 2F_\pi e^2 F_{\pi\gamma\gamma}(0,0)\varepsilon_{\mu\nu\lambda\sigma} q_1^\lambda q_2^\sigma. \tag{3.39}$$

The value of $ip_\alpha \mathscr{M}^{\alpha}_{\mu\nu}$ can be calculated in a second way, namely via perturbative diagrams. It turns out that there are only two diagrams contributing to it and this is the famous result of the non-perturbative nature of quantum anomalies. We express $\mathscr{M}^{\alpha}_{\mu\nu}$ as

$$\mathscr{M}^{\alpha}_{\mu\nu} = e_0^2 \int d^4x d^4y e^{-iq_1\cdot x} e^{-iq_2\cdot y} \left\langle \Omega \middle| T\{J_\mu(x)J_\nu(y)J_5^{\alpha,(3)}(0)\} \middle| \Omega \right\rangle. \tag{3.40}$$

If one now dots ip_α into this expression the so called contact terms will not contribute and we can insert the divergence of the axial current into the correlator, which is

$$\frac{1}{16\pi^2}\varepsilon_{\alpha\beta\mu\nu} F_0^{\alpha\beta} F_0^{\mu\nu} \mathrm{tr}\{\lambda_3 Q^2\}, \tag{3.41}$$

where the trace is just 1 for our case of normalization (remember that the trace here runs over color indices too). The external photon lines require wave function renormalization, i.e multiplication by $\sqrt{Z_3}$ for each external line. We are content with the order e^2 terms and the matrix element can be computed straightforwardly to give

$$\frac{1}{2\pi^2}\varepsilon_{\alpha\beta\mu\nu}\varepsilon^{*\mu}\varepsilon^{*\nu} q_1^\lambda q_2^\sigma. \tag{3.42}$$

This should be compared now to (3.39) to give

$$F_{\pi\gamma\gamma}(0,0) = \frac{1}{4\pi^2 F_\pi}. \tag{3.43}$$

The above analysis can be repeated for the η current $J_5^{\mu,(8)}$ in almost the same way, giving

$$F_{\eta\gamma\gamma}(0,0) = \frac{1}{4\pi^2 F_\eta}, \tag{3.44}$$

where again $F_\eta = \frac{1}{\sqrt{3}} F_\pi$ for exact chiral symmetry.

The last constraint that we will look at gives values to the meson transition form factors for one on-shell photon and one highly virtual photon [17]. The derivation involves techniques such as light cone perturbation theory, hadronic wave functions and is beyond the scope of this work, so we simply state the result

$$\lim_{Q^2\to\infty} Q^2 F_{\pi\gamma^*\gamma}(-Q^2, 0) = 2F_\pi \tag{3.45}$$

for the pion, and the result for the η transition form factor has the same form with F_π replaced by F_η which is $\frac{F_\pi}{\sqrt{3}}$ for exact chiral symmetry.

4 Holographic models and their asymptotic behaviour

In this chapter we will discuss models of the low energy degrees of freedom inspired by the AdS/CFT correspondence [4]. Once we have obtained some of them we will calculate contributions to the HLbL tensor and the meson transition form factors and check if the constraints derived above are satisfied.

The holographic principle provides a way to relate two apparently very different theories. Two theories describing the same phenomena are called dual. A situation that occurs very often with the sort of dualities that we will discuss is that the strong coupling regime of one theory is described by the small coupling regime of the dual theory, which makes perturbation theory applicable. Frequently such dualities are fundamentally relations between two quantum theories at least one of which is a string theory. However in certain limits these string theories are well described by an effective action in 5 dimensions at low energy computed from tree processes. So we will have a theory in 5 spacetime dimensions whose dual is a 4 dimensional gauge theory similar to QCD. In many formulations of holographic dualities the dual theory will be quite different from real QCD. There have been many creative ideas which aim to uphold a holographic duality but which change both theories in such a way to make the 4 dimensional theory more and more similar to QCD. Here it is of importance to break any inital supersymmetry and conformal symmetry in the 4 dimensional theory.

Motivated by these dualities originating from string theory, models have also been created by hand, i.e. specifying a set of fields in a 5 dimensional spacetime, an effective action and a background geometry. These models are called "bottom-up" models. Predictions of these models include "static" observables such as masses of mesons and baryons and decay widths as well as dynamical quantities such as scattering amplitudes and correlation functions also involving photons. We will use these models to calculate the HLbL tensor and some meson transition form factors and check whether they satisfy the constraints derived above.

In each of these models we will have a 5-dimensional bulk space and fields ϕ with an action $S[\phi]$, where the fields are supposed to obey the equations of motion and have a fixed value on the UV boundary $\phi(0, x) := \phi_0(x)$. Here (z, x) could be Poincaré coordinates on AdS. The message from the holographic principle [18] is now that each of these fields ϕ_0 is a source for a corresponding operator in QCD living on the $z = 0$ boundary, i.e.

$$\exp(iS[\phi])|_{\phi(0,x)=\phi_0(x)} = \left\langle \exp\left[i \int d^4x \phi_0(x) O(x)\right] \right\rangle_{QCD}. \tag{4.1}$$

In our case we will have field theories of 5-dimensional $U(N_f)$ flavour gauge fields V^a and A^a whose values at the UV boundary are to be interpreted as sources for the QCD currents $\bar{\psi}\lambda^a\gamma_\mu\psi$ and $\bar{\psi}\lambda^a\gamma_\mu\gamma_5\psi$ with λ^a representing a generator of the algebra of $U(N_f)$. This is another recipe from the holographic principle: Global symmetries in the boundary theory are gauged in the bulk theory. For $N_f = 2$ their generators can be taken to be the 3 Pauli matrices and the identity matrix and for $N_f = 3$ these can be taken to be the 8 Gell-Mann matrices and the identity matrix or more correctly appropriately normalized versions of the above. It is quite reasonable to believe that a highly complicated quantum theory like QCD which is supposed to describe so many phenomena cannot be accurately described by just one smooth functional $S[\phi]$ in one dimension higher. We will however see that the agreement is in many cases quite good qualitatively and sometimes also quantitatively. This is quite remarkable for a classical theory with only a handful of coupling constants and boundary conditions. The above equation (4.1) is only the weak form of the holographic correspondence. In its strong form it connects the boundary theory, which is supposed to model the strong interactions with a string QT. In the case where stringy corrections can be neglected the partition function will still include loops. Approximating the path integral by the value of the integrand at the saddle point we arrive at the LHS of (4.1).

In the following we will look at various bottom-up holographic models of QCD. The main connection to QCD is that the holographic partition function with fixed boundary conditions will be the generating functional of Green's functions of left and right current operators in QCD. In all of these models we will have two $U(N_f)$ gauge fields $L_M(x, z)$ and $R_M(x, z)$ whose values of the x components at the boundary at $z = 0$ are defined as $l_\mu(x)$ and $r_\mu(x)$. Depending on which model one uses, there will be additional fields denoted by X here, whose boundary values will describe sources of different operators.

With

$$Z_{QCD}[l_\mu, r_\mu] = e^{iW[l_\mu, r_\mu]} = \int DA_0 D\bar{\psi}_0 D\psi_0 e^{i\int \mathscr{L}_{QCD} + l^a_\mu J^{a\mu}_L + r^a_\mu J^{a\mu}_R} \tag{4.2}$$

the precise correspondence reads

$$Z_{QCD}[l_\mu, r_\mu] = Z[l_\mu, r_\mu] \tag{4.3}$$

where

$$Z[l_\mu, r_\mu] = \int_{L_\mu(z=0)=l_\mu; R_\mu(z=0)=r_\mu} DLDRDX e^{iS_{hol}[L,R,X]} \tag{4.4}$$

By demanding boundary conditions for X one can have additional sources for other operators. It turns out that the above is just the exponential of the effective action $\Gamma[L,R]$ with above mentioned boundary conditions and where L and R are extrema of $\Gamma[L,R]$. So we can abbreviate the above by

$$W_{QCD}[l_\mu, r_\mu] = \Gamma[L,R]|_{b.c}, \tag{4.5}$$

i.e. the connected correlators of currents in QCD can be obtained by functional differentiation of the effective action of the holographic theory evaluated on solutions to the quantum equations of motion subject to boundary conditions with respect to the boundary fields.

One can in principle differentiate $\Gamma[L,R]|_{b.c}$ wrt. fields L or R evaluated inside the bulk $z > 0$ and then insert the solutions to the equations of motion. The resulting objects have no direct interpretation in terms of QCD observables, but they will be of computational value. These functions will encode information about QCD bound states. In some models it is also possible to decompose the 5D gauge fields with boundary conditions and the additional fields X into 2 sets depending on only the x coordinates where the fields in the first set can be interpreted as meson, vector and axial vector meson fields, and fields in the second set are purely the boundary fields l and r. In this formulation one sees that the holographic actions try to describe QCD observables in terms of observable dynamical degrees of freedom, i.e. hadrons. In all of the following models, baryons are excluded and these correspondences hold only in the large N_c limit of QCD with fixed but large t'Hooft coupling. Colour degrees of freedom are not encoded in the holographic models directly, but instead free parameters of the theory will be proportional to N_c. The action S_{hol} will always include a Yang-Mills term

$$S_{YM} = -\frac{1}{4g_5^2} tr \int d^4x dz \sqrt{|\text{detg}|} g^{MN} g^{RS} (L_{MR} L_{NS} + R_{MR} R_{NS}), \tag{4.6}$$

with $R_{MN} = \nabla_M R_N - \nabla_N R_M - i[R_M, R_N]$. The integral over the z coordinate will be cut off at $z = z_0$. As an aside, Poincaré coordinates do not cover all of Lorentzian AdS, but, if it is cut off at a finite z value, then we may use them without hesitation [19]. Other terms which are added to the action include a Chern-Simons term and often terms containing the additional fields X. After rescaling the gauge fields so as to have the usual propagator, the coupling constants of various interactions will have negative mass dimension (g_5^2 has mass dimension -1) so renormalization appears to not be possible. We can however compute the effective action at tree level and compare its predictions to experiments and constraints coming from QCD. The qualitative agreement will be quite impressive. In the tree approximation we solve the equations of motion for the quadratic part of the action for our boundary conditions and insert them back into the action. This will yield all the n-point functions up to $n = 3$, then one has to build the respective terms out of the given vertices and propagators. In all models we will work with the fields $V = L + R$ and $A = L - R$.

4.1 HW1 model

In the HW1 model of [7] an additional field X with two flavour indices in the fundamental representation is introduced which is minimally coupled to the gauge fields. In addition to the usual kinetic term it also comes with a potential which breaks the flavour gauge symmetry of the Lagrangian to the diagonal subgroup $g_R = g_L$. The contribution to the total action is

$$S_X = tr \int d^4xdz\sqrt{|\mathrm{detg}|}\{|DX|^2 + 3|X|^2\}, \tag{4.7}$$

with $D_AX = \nabla_AX - iL_AX + iXR$. The coefficient in front of the mass term "3" (in units of the AdS radius) is determined by the conformal dimension of the dual operator which is the quark condensate. This is how the chiral symmetry breaking of QCD is implemented in this model. A broken local symmetry in 4 dimensions leads via the Higgs mechanism to masses for gauge bosons. In our 5D theory the axial gauge fields which correspond to the broken generators get an additional z dependent term in the action that acts like a mass and lifts the degeneracy of vector and axial vector fields. The field $\frac{2}{z}X$ on the boundary acts as the source for the quark condensate which is already incorporated into the QCD Lagrangian. The source X at the boundary is therefore related to the renormalized current quark mass matrix via $\frac{2}{\varepsilon}X(\varepsilon) = M = m_q\mathbb{I}$. Solving the classical equations of motion for X gives rise to another free parameter σ. Writing the scalar field then as $X = \frac{v(z)}{2}e^{i2\pi(x,z)}$ where $v(z) = zm_q + z^3\sigma$ contains the symmetry breaking parameters and $\pi(x,z)$ will later be associated to the physical pion we can start solving the linearized EOM for the vector and the axial vector field. As boundary conditions on the IR brane we pick the gauge invariant conditions $L_{z\mu} = R_{z\mu} = 0$. We may also choose the axial gauge $L_z = R_z = 0$ to simplify computations. If the above action $S = S_{YM} + S_X$ is to be interpreted as an action in a path integral with boundary conditions, then the effective action at tree level is comprised of S and of gauge fixing and ghost terms for L and R coming from the Faddeev-Popov procedure. Using as gauge fixing functions simple delta functions of the fifth components R_5, L_5 one can remove the fifth component from the equations of motion [20] without introducing ghosts or other new terms to the Lagrangian. The longitudinal part is defined via the Fourier transform as the projection of $\tilde{V}^\mu(q,z)$ with $\frac{q^\mu q_\nu}{q^2}$ for spacelike and timelike momenta. The equation of motion for $V^\mu = V^\mu_\perp$ then reads

$$-\partial^2 V^\mu + z\partial_z(\frac{1}{z}\partial_z V^\mu) = 0, \tag{4.8}$$

while the transverse part is just a constant. Since the above equation treats every spacetime component and flavour component in the same way, we could introduce a scalar function f which solves the above equation and which has $f(x,0) = \delta^{(d)}(x)$ and $\partial_z f(x, z_0) = 0$ and then $\int dy f(x-y)v^\mu(y)$ also solves the equation and has $v^\mu(x)$ as boundary condition. The more standard way is to go to Fourier space and write $\tilde{V}^\mu(p,z) = \tilde{v}^\mu(p)\mathscr{V}(p,z)$ with $\mathscr{V}(p,0) = 1$ and $\partial_z\mathscr{V}(p,z_0) = 0$. The term $\tilde{v}^\mu(p)$ is then the Fourier transform of the source $l(x)+r(x)$. $\mathscr{V}$ is the Fourier transform and called the bulk-to-boundary propagator. It makes explicit the sources in the effective action after reinsertion of the above solution. The version of $\mathscr{V}$ with Euclidean momenta inserted is usually denoted by $\mathscr{J}$.

Expressing the covariant derivative of the scalar field X with the V and A fields we see that there is a term proportional to A with no pion fields multiplying it but no corresponding term for V. We will denote the longitudinal part of the axial vector field as $A^\mu_\parallel(x,z) = \partial^\mu\psi(x,z)$. The equations of motion for A_M split into 3 equations

$$\begin{aligned}
&-\partial^2 A^\mu_\perp + z\partial_z(\frac{1}{z}\partial_z A^\mu_\perp) - \frac{g_5^2 v^2}{2z^3}A^\mu_\perp = 0,\\
&\partial_z\big(\frac{1}{z}\partial_z\psi^a\big) + \frac{g_5^2 v^2}{z^2}(\pi^a - \psi^a) = 0\\
&\partial^2\partial_z\psi^a + \frac{g_5^2 v^2}{z^2}\partial_z\pi^a = 0.
\end{aligned} \tag{4.9}$$

This defines the transverse axial vector bulk-to-boundary propagator $\mathscr{A}(q,z)$ in the same way as above for the vector case and the last two equations are coupled now. Inserting the solution for the vector field back into the action and differentiating twice wrt. the Fourier representation of the source $\tilde{v}^\mu(p)$ we obtain the vector current 2-point function

$$\Pi_V(-q^2) = -\frac{1}{2g_5^2 Q^2}\frac{\partial_z\mathscr{V}(q,z)}{z}|_{z=0}. \tag{4.10}$$

One way to fix a parameter of the holographic model would be to compute the high-momentum transfer limit and compare with the OPE of these two currents obtained within QCD. The first term in the OPE is divergent, independent of the QCD coupling and is obtained from the quark loop graph. After manually subtracting the divergent term at $Q^2 = \mu^2$ the OPE gives $\Pi_V(-q^2) = -\frac{1}{4\pi}\ln\frac{Q^2}{\mu^2}$ and this may be compared to (4.10). One should be cautious however since the holographic model does not model the running coupling of QCD and thereby asymptotic freedom correctly and is only expected to agree well with QCD for low and intermediate energies. In this model we will usally set $m_q = 0$ so there will be 3 free parameters z_0, g_5, σ which need to be fitted. We will choose the above constraint coming from perturbative QCD as one constraint. Two further constraints will come from the low energy parameters of the lowest lying mesons. By inserting the identity operator in the form of a sum over projection operators onto the various states describing physical particles, polology tells us that the current correlator has poles at the masses of these states and the residue is proportional to their decay constants. Indeed the correlator computed from holography has such poles. For this we introduce the vector bulk-to-bulk propagator $G(z, z', q)$. Consider solutions ψ_n of (4.8) for an arbitrary component of V and with $q^2 = m_n^2$ with boundary conditions $\psi_n(0) = 0$ and $\partial_z \psi_n(z_0) = 0$. Normalizable modes exist only for discrete m_n and the $\psi_n(z)$ are called holographic wave functions. The Green's function $G(z, z', q)$ associated to (4.8) for an arbitrary component can then be shown to be

$$G(z, z', q) = \sum_{n=1}^{\infty} \frac{\psi_n(z)\psi_n(z')}{q^2 - m_n^2 + i\varepsilon} \tag{4.11}$$

and

$$\mathscr{V}(q, z') = \lim_{\epsilon \to 0} -\frac{1}{\epsilon} G(\epsilon, z', q). \tag{4.12}$$

This allows us to express the current correlator in terms of sums of poles

$$\Pi_V(-q^2) = -\frac{1}{g_5^2} \sum_n \frac{[\psi_n'(\epsilon)/\epsilon^2]}{(q^2 - m_n^2 + i\varepsilon) m_n^2} \tag{4.13}$$

and identify m_n as masses of vector meson resonances and the derivatives of the wave functions at the UV boundary with decay constants F_n. The same thing can be done for the axial current correlator $\mathscr{A}(q, z)$. We first define the holographic wave functions of the transverse compontents of A as the solution to the equations of motion for discrete momenta and with the same boundary conditions. In the axial sector we also have an equation for the longitudinal component of A which is encoded in $\psi(q,z)$. The equation for $\psi(q,z)$ at discrete momenta defines the so called pion wave function. The current correlator of axial currents then gets contributions from the pion sector and the axial vector meson sector. The three free parameters will be fitted to the asymptotic constraint, the ρ meson mass and the pion decay constant F_π.

We are now in a position to calculate the pion transition form factor. The transition form factor can be extracted from the current correlator of two electro-magnetic currents and one axial current with the same quantum numbers as the neutral pion, i.e $J_5^{(3)}$ from the residue of the pole at $p = 0$ where p is brought in by the axial current. The pion state (in QCD) has no overlap with the transverse part of the axial current so we need only the longitudinal one, which in holography is described by $\tilde{\psi}(p,z) = \frac{1}{i} p_\alpha \tilde{a}^\alpha(p)\psi(z)$. $\psi(z)$ can be computed approximately from the equations of motion in the axial sector as $1 - \mathscr{A}(0,z)$ [7].

In the upcoming calculations we will choose $N_f = 2$ which is sufficient for this observable and mainly follow [21]. The pion is described by a vector in the $I = 1$ representation which has only the 3-component $\neq 0$, where I labels the value of the Casimir operator of the $SU(2)$ subgroup of $U(2)$. Since states in different irreducible representations of a symmetry group are orthogonal to each other, the only terms surviving from the electro-magnetic current, which has $I = 0$ and $I = 1$ parts, are $J^{I=1;3}J^{I=0}$ and $J^{I=0}J^{I=1;3}$.

The terms contributing to the current 3-point function will come from AVV parts of the Chern-Simons term. The trilinear parts in the Yang Mills action do not contribute. The Chern-Simons term for our model in terms of L and R gauge fields has the following form

$$\begin{aligned} S_{CS}[L,R] &= \tilde{S}_{CS}[L] - \tilde{S}_{CS}[R] \\ \tilde{S}_{CS}[B] &= \frac{N_c}{24\pi^2}\int tr\Big(BF^2 + \frac{i}{2}B^3F - \frac{1}{10}B^5\Big) \end{aligned} \tag{4.14}$$

The fact that the prefactor of this term is quantized follows from certain considerations of Chern-Simons terms on manifolds with boundaries [22] which we will come to later. When a gauge transformation is performed, the Chern-Simons term is not invariant and gets an additional boundary term $S_{CS} \to S_{CS}$+boundary term. To obtain the equations of motion in the bulk one always varies the the action in such a way that the variations of the fields at the boundary is zero, so a gauge transformed solution will still obey the equations of motion, but when reinserted into the action there might appear additional terms if the gauge transformation does not vanish on the boundary. This would mean that current correlators could be gauge dependent which is not desirable. In 5 dimensional gauge theories with a Chern-Simons term one therefore has to introduce chiral fermions living on the boundary and being coupled to the boundary values of the gauge field. Integrating out these fermions in the path integral which can be done if they have high mass will introduce another term in the effective action

Alternatively one can determine the prefactor by requiring it to be consistent with the QCD anomaly result on $\pi^0 \to \gamma\gamma$ decay (3.43). We decompose each gauge field in terms of the generators

$$B_\mu = B_\mu^a t^a + \hat{B}_\mu \frac{\mathbb{I}}{2}. \tag{4.15}$$

Each one of the generators is normalized in a standard way and the coefficient function of the generator of the $U(1)$ subgroup is distinguished by a hat symbol.

The only terms contributing from the Chern-Simons action are the terms trilinear in the fields. After inserting $L, R = V \pm A$ we keep the terms linear in the axial field. Only the longitudinal part of A is needed for this computation and because of the 5-dimensional ε-tensor we need only terms in which no derivative acts on A. We can then use $A_5 = V_5 = 0$ which means that always one derivative has to be wrt. the coordinate z and we only keep terms with one $\hat{V}$. We can then easily compute the trace over the flavour indices since there are always only two $SU(N_f)$ generators multiplying each other. Using the antisymmetry of the 4 dimensional ε-tensor and shifting derivatives we arrive at

$$S_{CS} = 2\frac{N_c}{24\pi^2}\varepsilon^{\mu\nu\rho\sigma}\int d^4x\left\{[-\psi^a(\partial_\rho V_\mu^a)(\partial_\sigma \hat{V}_\nu)]|_{z=z_0} + 3\int dz \partial_z \psi^a(\partial_\rho V_\mu^a)(\partial_\sigma \hat{V}_\nu)\right\}. \tag{4.16}$$

The boundary term is problematic and would not reproduce the correct QCD anomaly result. The aforementioned boundary term resulting from integrating out the chiral fermions on the boundary will actually cancel this term exactly leaving us only with the second one. Writing the vector fields in terms of their bulk-to-boundary propagators and sources and the $\psi(q,z)$ in terms of $\psi(z)$ and the axial source and then differentiating wrt. to these, the 3-point correlator is given by

$$\langle J_{5\,\alpha}(-p) J_\mu^{EM}(q_1) J_\mu^{EM}(q_2)\rangle = i(2\pi)^4\delta^{(4)}(q_1+q_2-p)\frac{N_c}{12\pi^2}\frac{p_\alpha}{p^2}\varepsilon_{\mu\nu\rho\sigma}q_1^\rho q_2^\sigma K_b(Q_1^2, Q_2^2), \tag{4.17}$$

with

$$K_b(Q_1^2, Q_2^2) = -\int dz\, \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\partial_z \psi(z). \tag{4.18}$$

A surface term at $z = z_0$ also needs to be subtracted by hand from the above equation in order to reproduce the QCD anomaly giving

$$K(Q_1^2, Q_2^2) = -\int dz\, \mathcal{J}(Q_1, z)\mathcal{J}(Q_2, z)\partial_z \Psi(z) + \mathcal{J}(Q_1, z_0)\mathcal{J}(Q_2, z_0)\Psi(z_0), \tag{4.19}$$

with $\Psi(z) = 1 - \psi(z)$.

In the HW1 model the bulk-to-boundary propagator and the function $\Psi(z)$ can be represented in terms of Bessel functions

$$\begin{aligned} \mathcal{J}(Q, z) &= Qz\big[K_1(Qz) + I_1(Qz)\frac{K_0(Qz_0)}{I_0(Qz_0)}\big], \\ \Psi(z) &= z\Gamma(2/3)(\frac{\xi}{2})^{1/3}\big[I_{-1/3}(\xi z^3) - I_{1/3}(\xi z^3)\frac{I_{2/3}(\xi z_0^3)}{I_{-2/3}(\xi z_0^3)}\big], \end{aligned} \tag{4.20}$$

with $\xi = \frac{g_5 \sigma}{3}$. For the function $K(Q_1^2, Q_2^2)$ there is no closed formula for generic momenta. By construction it satisfies the anomaly QCD constraint. Using $\mathcal{J}(Q, \xi/Q) = \xi K_1(\xi) + \mathcal{O}(e^{-Q})$ and $\partial_z \Psi \approx -2\frac{z}{z_0^2}$ for small z we can find that for high momenta Q we get

$$\begin{aligned} K(Q^2, Q^2) &= \frac{4}{3z_0^2}\frac{1}{Q^2}, \\ K(0, Q^2) &= \frac{4}{z_0^2}\frac{1}{Q^2}. \end{aligned} \tag{4.21}$$

This shows that the Brodsky-Lepage constraint (3.45) where one photon is real and the constraint coming from the OPE (3.35) where both photons are highly virtual are both satisfied. Considering that these holographic models are built to describe low to intermediate energies this is a remarkable achievment. In the case where the two photons are highly virtual, one can compute subleading terms in the $\frac{1}{Q^2}$ expansion coming from the OPE. The holographic models considered here can not reproduce any of these subleading terms. Before we check if the 2 constraints on the HLbL tensor, the MV constraint (3.31) and the symmetric momenta constraint (3.32), are fulfilled in this model, we will look at 2 more holographic models, the HW2 model and the Sakai-Sugimoto model.

4.2 HW2 model

The HW2 model of [6] is very similar to the HW1 model, it only differs essentially in the way chiral symmetry is broken spontaneously and how the pion fields arise during this process. The HW2 model has only 5D gauge fields L and R and no scalar field which breaks the symmetry. Instead different boundary conditions are used for the vector and axial vector combinations at the IR brane $z = z_0$. This model will therefore have one less parameter to fix and fixing them via low energy resonance properties will cause various observables to not obey perturbative QCD expressions quantitatively. The boundary conditions in the path integral on the IR brane are

$$\begin{aligned} R_\mu(x, z_0) - L_\mu(x, z_0) &= 0 \\ F_{R,5\mu}(x, z_0) + F_{L,5\mu}(x, z_0) &= 0 \end{aligned} \tag{4.22}$$

The theory, i.e. the action together with the above boundary condition does not possess the full gauge symmetry anymore. At $z = z_0$ the above boundary conditions demand that $g_L(x, z_0) = g_R(x, z_0)$ so the symmetry is broken down to the diagonal vector subgroup at the IR boundary. One proceeds now by defining a Wilson line which starts at the IR brane [23] by

$$\xi_L(x, z) = P\exp\left\{ -i \int_z^{z_0} dz' L_5(x, z') \right\}, \tag{4.23}$$

and analogously for R. An action with a Chern-Simons term is not gauge invariant anymore, so it is generally not possible to set $L_5 = R_5 = 0$ without acquiring boundary terms. To proceed we simply do a field redefinition

$$\begin{aligned} L^\xi_M(x, z) &= \xi^\dagger_L(x, z)[L_M(x, z) + i\partial_M]\xi_L(x, z) \\ R^\xi_M(x, z) &= \xi^\dagger_R(x, z)[R_M(x, z) + i\partial_M]\xi_R(x, z) \end{aligned} \tag{4.24}$$

such that $L^\xi_5 = R^\xi_5 = 0$, and insert it back into the action. The Chern-Simons term changes and via the definition of

$$U(x) = \xi_L(x, 0)\xi^\dagger_R(x, 0) =: e^{\frac{i2\pi(x)}{F_\pi}}, \tag{4.25}$$

one can show that the resulting action describes an effective action of pions, vector and axial vector fields. We rename the fields L^ξ, R^ξ into L, R again and consider their vector and axial vector combinations. Analogously to the HW1 model we decompose them according to

$$V_\mu(q, z) = v(q, z)\hat{v}^\perp_\mu(q) \tag{4.26}$$

$$A_\mu(q, z) = a(q, z)\hat{a}^\perp_\mu(q) + a(\bar{q}, z)\hat{a}^\parallel_\mu(q) + \frac{\alpha(z)}{F_\pi}\partial_\mu\pi(x),$$

where $\hat{v}$ and $\hat{a}$ are identified with the sources of the QCD currents and the unhatted version represent transverse and longitudinal bulk-to-boundary propagators. The $\pi(x)$ appears since the above fields are the redefined versions of the original fields whose boundary values are sources for the currents. The boundary values of the new fields get contributions from the redefinition.

The vector fields $A_\mu(x,z)$ and $V_\mu(x,z)$ now obey the same differential equations in our approximation. They however have to obey different boundary conditions, which models spontaneous symmetry breaking. The vector bulk-to-boundary propagator $v(q,z)$ therefore coincides with $\mathscr{V}(q,z)$ from the HW1 model.

To compute correlators of currents we insert the solution to the linearized equations of motion into the action again and using a pole decomposition of the bulk-to-boundary propagator we fit the 2 parameters of the model g_5, z_0 to the pion decay constant and the ρ meson mass. The asymptotic constraint on the QCD current correlator is now not fulfilled. The dependence on the momentum Q is reproduced but the precise coefficient is not.

The computations for the pion transition form factor are mostly analogous with the key differences that no boundary term needs to be added to the action like above and also that no term must be added by hand to the function $K_b(Q_1, Q_2)$ because of the boundary conditions. This gives us

$$K(Q_1^2, Q_2^2) = -\int dz\, \mathscr{J}(Q_1, z)\mathscr{J}(Q_2, z)\partial_z\alpha(z) \tag{4.27}$$

where $\alpha(z)$ solves

$$\partial_z(\frac{1}{z}\partial_z\alpha) = 0 \tag{4.28}$$

with boundary conditions $\alpha(0) = 1$. Setting $\alpha(z_0) = 0$ gives

$$\alpha(z) = 1 - \frac{z^2}{z_0^2}. \tag{4.29}$$

The vector bulk-to-boundary propagator obeys the same differential equation as in the HW1 model with the same boundary conditions, so when expressed in terms of z_0 and g_5 they are the same. Fitting to the ρ meson mass implies $z_0 = 3.103\ \mathrm{GeV}^{-1}$ for both models. If in the HW1 model g_5 and σ are fitted with the pion decay constant and the constraint from asymptotic QCD then one gets $g_5^2 = 12\pi^2/N_c$ and $\xi = (0.424\ \mathrm{GeV})^3$. The HW2 model being fitted to the pion decay constant gives $g_5^2 = \frac{2}{F_\pi^2 z_0^2}$, which differs from the HW1 model. The HW2 TFF already satisfies the QCD anomaly constraint $K(0,0) = 1$ which can be checked by using the expansions of the Bessel functions near $Q = 0$. For very high momenta the integration over the holographic coordinate is dominated by the region near $z = 0$ where the derivatives of the functions $\Psi(z)$ and $\alpha(z)$ coincide. The HW2 TFF therefore has the same functional form as the HW1 TFF in the high Q regime but with differently fitted parameters. The coefficients in the HW2 for both constraints (3.35) and (3.45) only reach about 62 % of the desired values [24].

The most interesting model of low energy QCD is the Sakai-Sugimoto model [5] which is a top-down model constructed from string theory. It would be beyond the scope of this paper to describe the process by which this model is obtained, but in the end it can be put into a form which is very similar to HW2. The action is formulated on a space in which the holographic coordinate Z goes from $-\infty$ to ∞ and describes one $U(N_f)$ flavour gauge field B_μ. Its even and odd parts in the holographic coordinate are identified with the fields V_μ and A_μ. Using the decomposition into even and odd parts allows one to restrict the interval of Z to $Z \geq 0$ and with certain coordinate transformations we may identify the UV and IR branes in this theory. We will again be able to find a bulk-to-boundary propagator where the axial one satisfies different boundary conditions than the vector one just as in the HW2 model. From its derivation from string theory it is clear from the start that the dual theory will approximate QCD only in the low energy regime and will certainly not fulfill asymptotic constraints from QCD. Explicit computations also show this, for example the TFF in the double virtual case scales like $\frac{1}{Q^4}$, see [24]. For this reason we leave this model out of our discussion regarding asymptotic constraints. We now turn to the holographic computation of the HLbL tensor and check if the results obey the MV constraint and the symmetric momenta constraint.

4.3 Holographic computation of the light-by-light scattering tensor

To compute the HLbL tensor from holography we first want to introduce the axial vector transition form factor. It describes the decay of an axial vector meson into two in general virtual photons. When both photons are on-shell, the resulting amplitude has to be zero by the Landau Yang theorem [25, 26]. Axial vector mesons contribute to the Π tensor just as the neutral pseudoscalar mesons do and we will see that they are absolutely necessary for the implementation of the constraints. Even though their on-shell value vanishes, the axial TFF's also contribute to the anomalous magnetic of the muon a_μ. We will work with $N_f = 3$ from now on. We parametrize the amplitude $\gamma^*(q_1)\gamma^*(q_2) \to A^a$ as (see [27])

$$i\frac{N_c}{4\pi^2}\mathrm{tr}(Q^2 t^a)\varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_A^{*\rho}\varepsilon_{\mu\nu\rho\sigma}[q_2^\sigma Q_1^2 A(Q_1^2, Q_2^2) - q_1^\sigma Q_2^2 A(Q_2^2, Q_1^2)]. \tag{4.30}$$

To compute this within holography we compute the current correlator of one axial current and two vector currents from the Chern-Simons terms in the action and then look at the residue of the pole at the desired axial vector meson mass. This brings in the holographic wave function ψ^A of that resonance and in the two HW models it leads to

$$A(Q_1^2, Q_2^2) = \frac{2}{Q_1^2}\int_0^{z_0} dz[\frac{d}{dz}\mathscr{J}(Q_1, z)]\mathscr{J}(Q_2, z)\psi^A/[g_5^{-2}\int_0^{z_0}\frac{dz}{z}(\psi^A)^2]^{1/2}. \tag{4.31}$$

The integral in the denominator becomes 1 if the wavefunctions are normalized correctly. There is experimental data on the decay of the f_1 axial vector meson into one real and one virtual photon which can be compared to the holographic results above. The Sakai-Sugimoto which we leave out on our discussion on asymptotic constraints and the HW2 model agree very well with the experimental data [27]. To get the terms that are relevant for the MV constraint, we only have to look at diagrams in which the q_1 and q_2 photon legs are connected by an axial form factor (see figure 4.1), the other two diagrams fall off faster.

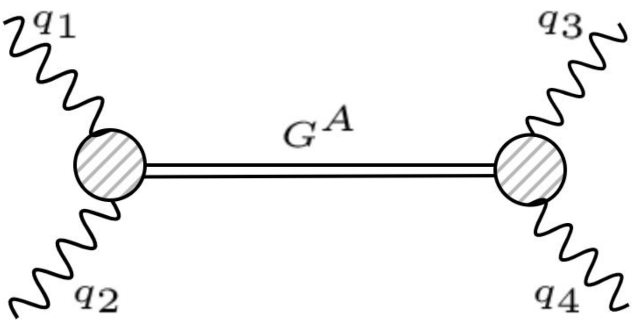


Figure 4.1: One of the diagrams contributing to the HLbL tensor. The double lines represent the axial vector propagator and the blob represents the axial vector form factor. There are two additional diagrams related to this one via crossing symmetry. In these two diagrams different pairs of photon lines are connected via an axial vector form factor.

It is also only the longitudinal part of the axial vector propagator $q_3^\mu q_3^\nu/(M_n^A Q_3)^2$ that contributes to $\bar{\Pi}_1$ thereby giving

$$\bar{\Pi}_1 = -\frac{g_5^2}{2\pi^4}\sum_{n=1}^{\infty}\int_0^{z_0} dz[\frac{d}{dz}\mathscr{J}(Q,z)]\mathscr{J}(Q,z)\psi_n^A(z)\frac{1}{(M_n^A Q_3)^2}$$
$$\times\int_0^{z_0} dz'[\frac{d}{dz'}\mathscr{J}(Q_3,z')]\psi_n^A(z'), \quad (4.32)$$

The vector bulk-to-boundary propagator associated to the soft photon does not appear explicitly above since $\mathscr{J}(Q,0) = 1$. Analysis of the asymptotics of any truncated version of the above infinite sum shows that the MV constraint is completely missed and the only chance to fulfill it is to consider the whole infinite sum. The axial bulk-to-bulk propagator

$$G^A(Q;z,z') = \sum_{n=1}^{\infty}\frac{\psi_n(z)\psi_n(z')}{Q^2+M_n^2} \quad (4.33)$$

is the appropriate tool since the sum above reduces to the zero momentum limit of this propagator for which a closed form can be derived at least within the HW2 model where one can express it in terms of Bessel functions. The zero momentum limit in the HW2 model is particularly simple

$$G^A(0;z,z') = \frac{\min(z,z')(z_0^2 - \max(z,z'))}{2z_0^2}. \quad (4.34)$$

Changing integration variables to $Qz = \xi$ and $Q_3 z' = \xi'$ and restricting to $Q^2 \gg Q_3^2$ we get

$$-\frac{g_5^2}{2\pi^4}\frac{1}{2Q_3^2}\int_0^{\infty} d\xi\int_0^{\infty} d\xi'\xi K_1\xi\frac{d}{d\xi}[\xi K_1(\xi)]\frac{d}{d\xi'}[\xi K_1(\xi)]\xi^2/Q^2. \quad (4.35)$$

The $\xi K_1(\xi)$ term of $\mathscr{J}(Q,\xi/Q)$ is the only part that survives the high-momentum limit since the second part falls of exponentially with Q. The $\xi' = 0$ integrand is a total derivative and gives contributions at the ξ' boundary, since K_1 has a pole of order 1 there. Performing the last integral we arrive at

$$\bar{\Pi}_1 = -\frac{g_5^2}{4\pi^2}\frac{2}{3\pi^2 Q^2 Q_3^2}. \quad (4.36)$$

For the HW2 model fitted to low lying resonance parameters this matches the constraint quantitatively only up to 62 %.

For the HW1 no closed form for the double integral has been found but the HW1 and the HW2 should yield the same predictions for high-energy observables and if for a moment we pretend that the above result holds also for HW1 then we see that due to the different fitting of HW1, $g_5^2 = 4\pi^2$ for $N_c = 3$, the constraint is fulfilled exactly. For a long time it was not known how to naturally incorporate the MV constraint (3.31) into hadronic models with a finite number of resonances since one could never get the asymptotic matching naturally. This has occasionally led to rather ad hoc modifications of models such as in [12] but the above results show that an infinite number of resonances can at least qualitatively reproduce the asymptotic behaviour and no unnatural modifications have to be made.

To perform the symmetric limit we write

$$\bar{\Pi}_1 = \frac{-g_5^2}{2\pi^4 Q_3^2} \int dz \int dz' \mathscr{J}'(Q_1, z)\mathscr{J}(Q_2, z)\mathscr{J}'(Q_3, z')G^A(0; z, z') \tag{4.37}$$

as before where the prime on $\mathscr{J}$ denotes differentiation wrt. z and we shift the derivatives from the bulk-to-boundary propagators to G^A via partial integration. This works since G^A vanishes when either z or z' is on one of the two boundaries and the above function is symmetric wrt. Q_1 and Q_2 so we can replace $2\mathscr{J}'(Q_1, z)\mathscr{J}(Q_2, z)$ by $\mathscr{J}'(Q_1, z)\mathscr{J}(Q_2, z) + \mathscr{J}'(Q_2, z)\mathscr{J}(Q_1, z)$. Inserting then (see for example [23])

$$\partial_z \partial_{z'} G^A(0; z, z') = \frac{z_0^2}{2}[\alpha'(z)\alpha'(z') + \alpha'(z)\delta(z - z')] \tag{4.38}$$

we can simplify and discard terms coming from the first term above since they decay too fast to contribute in the limit. Using then the explicit form of the $\alpha(z)$ wave function, making a variable transformation $zQ = \xi$ and inserting the high-momentum limit of $\mathscr{J}(Q, \xi/Q)$ we find

$$\bar{\Pi}(Q, Q, Q) = -\frac{g_5^2}{4\pi^2}\frac{1}{\pi^2 Q^4}\int_0^\infty d\xi \xi^4 K_1^3(\xi). \tag{4.39}$$

The value of the integral is roughly 0.36 whereas the coefficient from (3.32) is $\frac{4}{9} = 0.44$. For the IR fixed HW2 model the factor $\frac{g_5^2}{4\pi^2}$ is about 0.62 so the axial vector sector is able to account for 51 % of the constraint, the UV fitted one for 81 %. Like before in the MV constraint, the pions and any finite number of axial vector resonances cannot obtain the right asymptotic behaviour, infinitely many axial vector resonances are needed. It is hoped that inclusion of other resonances will lead to a better matching to the two asymptotic constraint on the HLbL tensor [27].

4.4 Conclusion

We first reviewed some key asymptotic constraints on mesonic transition form factors and the HLbL tensor coming from QCD. The main tools that were used to derive these constraints was the OPE and the asymptotic freedom of QCD. We also briefly commented on the validity of the OPE in QCD and derived the explicit form of the HLbL tensor as a 4-point current correlator of the electro-magnetic current. We also saw that the chiral flavour anomaly and spontaneous symmetry breaking play an important roles in deriving these constraints. We then focused on two models of QCD coming from the holographic prinicple. These models took inspiration from the AdS/CFT correspondence to build a 5D gauge theory which describes current correlators of QCD. They incorporate pions, vector and axial vector resonances and have no colour degrees of freedom. We were able to show that both models satisfy the asymptotic constraints for the pseudoscalar meson TFF's at least qualitatively where we had to make subtractions by hand in the HW1 model. The fall off behaviour of the HLbL tensor in holography was shown to be consistent with QCD results in the HW2 model but the exact coefficient was not obtained. In the case of the HW1 model we argued that the constraints are even better fulfilled but were not able to show it rigorously.

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