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Generalized Steiner Point Maps

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Contents

Preface

In this thesis we study valuations. Let V be a Euclidean vector space and let A be an Abelian semigroup. A function μ defined on the non-empty convex bodies in V and taking values in A is called a valuation if

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B),$$

whenever A, B and $A \cap B$ are convex bodies. In [14] Hadwiger achieved a landmark result characterizing all scalar valued continuous Euclidean motion invariant valuations; this will be the first important result we will encounter. Alesker, Bernig and Schuster [8] accomplished a generalization of Hadwiger's characterization by characterizing all continuous translation invariant $\mathrm{SO}(n)$-equivariant valuations with values in an irreducible $\mathrm{SO}(n)$ representation Γ. Wannerer used this in [23] to determine the dimensions of the vector spaces of the continuous translation invariant unitary equivariant vector valued valuations. And from this he characterized the Steiner point map as the continuous unitary affine transformation equivariant valuation from the convex bodies in $\mathbb{C}^n$ to $\mathbb{C}^n$. Recently Böröczky, Domokas and Solanes [27] provided the dimension of the space of continuous translation invariant unitary equivariant tensor valuations, yielding Wannerer's result as special case. Additionally, utilizing the work of Wannerer [28], they provided a basis for the space of continuous translation invariant unitary equivariant vector valued valuations.

After a motivational problem from integral geometry in the first section we recall valuations and intrinsic volumes on parallelotopes. The second section transfers these definitions to the lattice of polyconvex sets, and we recall the volume theorem for polyconvex sets and Hadwiger's characterization theorem. In the third section we summarize some definitions from convex geometry and encounter Schneider's Steiner point characterization [2, 3].

In sections four to seven we compile the theory of compact Lie groups, that will be needed in the last two sections. Starting with basic definitions, we move on to representations, Schur's lemma and characters. Lie algebras provide a linearization of Lie groups, and Cartan subalgebras set up the theory of root space decompositions. This in turn will provide the highest weight classification of irreducible Lie group representations.

In section seven we gather more foundations for the generalized Hadwiger theorem: Frobenius reciprocity theorem and a branching theorem for $\mathrm{SO}(n)$. The normal cycle map provides a way to describe the smooth translation invariant valuations, and the Rumin operator enables us to fit the smooth translation invariant valuations into an exact sequence of $\mathrm{SO}(n)$-modules. With these prerequisites we present the statement and proof of the Hadwiger type theorem achieved by Alesker, Bernig and Schuster in [8].

Section eight starts with results from Klimyk [20] and Helgason [24]. We follow Schuster [22] and Wannerer [23] in order to calculate the dimension of the vector space of continuous translation invariant and $\mathrm{U}(n)$-equivariant valuations with values in $\mathbb{C}^n$. Finally the work of Böröczky, Domokas and Solanes [27] provides the dimension of the space of translation invariant unitary equivariant tensor valuations for $n \geq 2$ and a basis for the space vector valued case.

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Tobias Schadauer

1 Motivation

Starting out with a classical motivational problem from integral geometry, the Buffon needle problem, we move on to introducing valuations. We define intrinsic volumes on parallelotopes and see that they form a basis of the vector space of all continuous invariant valuations on the lattice of parallelotopes. This section is taken from [1].

1.1 The Buffon Needle Problem

We are first considering the Buffon needle problem, the classical example to introduce and motivate integral geometry: Given equidistant parallel straight lines across a plane. What is the probability of a needle of length L, randomly dropped on this plane, to hit one of the lines? Answering this question and going further to convex sets $A \subseteq B$, we find the conditional probability of a randomly drawn straight line meeting A, given that it meets B.

For the Buffon needle problem let's consider $\mathbb{R}^2$ with parallel straight lines, at a distance d from each other, drawn across it. If we drop a needle of length L at random on the plane, what is the probability of it to meet at least one of the straight lines?

For an instructive solution let's consider X to be the random variable counting the number of intersections of a randomly dropped needle of length L with any of the straight parallel lines. Let $L < d$ so, X takes values in $\{0, 1\}$. Denoting by p_n the probability that the needle meets exactly n straight lines, and by $E(X)$ the expectation of X, we get

$$E(X) = 0p_0 + 1p_1 = p_1,$$

making it sufficient to compute $E(X)$.

Adding another needle rigidly bound to the first one and using the additivity of the expectation as well as Cauchy's functional equation we get that

$$E(X) = rL,$$

where X is a polygonal line and $r \in \mathbb{R}$ is still to be determined.

Approximating a rigid wire C of length L by a polygonal line and passing to the limit we obtain a similar result. Plugging a circular wire of diameter d into this equation enables us to calculate r. In the end we get

$$E(X) = p_1 = \frac{2L}{\pi d},$$

for a needle with $L < d$.

We want to apply the methods just used to find answers to problems that are right at the center of interest of integral geometry. A subset K of $\mathbb{R}^2$ is called convex if for any two points x, y in K the complete line segment connecting x and y also lies within K. A closed curve C in $\mathbb{R}^2$ is called convex, given that C is enclosing a convex set.

We consider compact convex sets K_1, K_2 in $\mathbb{R}^2$ such that $K_1 \subseteq K_2$. Then the conditional probability that a random point belonging to K_2 also belongs to K_1 is given by

$$\frac{\text{area}(K_1)}{\text{area}(K_2)}.$$

Instead of points we will now look at lines, intersecting convex sets. Let $\text{AGr}(2,1)$ denote the set of all affine straight lines in $\mathbb{R}^2$, and let Z_1 be the random variable counting the number of intersections of a straight line taken at random with a line segment of length L_1. The integral

$$\int_{\text{AGr}(2,1)} Z_1 \; d\lambda_1^2,$$

where $d\lambda_1^2$ denotes up to normalization the unique rigid motion invariant measure on $\text{AGr}(2,1)$, depends only on L_1, therefore it can be expressed as a function $f(L_1)$. Because Z_1 takes values in $\{0,1\}$, the above integral is equal to the measure of all affine straight lines, that meet a straight line segment of length L_1. As before, we move from line segments to polygonal lines and to arbitrary curves. Given convex sets K_1, K_2 in $\mathbb{R}^2$ with $K_1 \subseteq K_2$, we denote the borders of K_1, K_2 by C_1, C_2 and the set of straight lines meeting C_i by D_i. Calculating the above integral for C_1, C_2, we get that the conditional probability that a random straight line meeting K_2 also meets K_1 is

$$\frac{\lambda_1^2(D_1)}{\lambda_1^2(D_2)} = \frac{L_1}{L_2} = \frac{\text{length}(\partial K_1)}{\text{length}(\partial K_2)}. \tag{1}$$

1.2 Valuations and Integrals

We are recalling the notion of a valuation, that is, a finitely additive set function with values in an abelian semigroup. We define an integral of simple functions with respect to a given valuation and state Groemer's integral theorem, which gives necessary conditions for this integral to be well defined.

Before we can define valuations we have to clarify their domains.

Definition. A partially ordered set L is called a *lattice* if for all $x, y \in L$ there exist a greatest lower bound $x \wedge y \in L$ and a least upper bound $x \vee y \in L$. A lattice is said to be *distributive* if for all $x, y, z \in L$ the following holds true:

$$x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z);$$
$$x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z).$$

Definition. A *valuation* on a lattice L of sets is a function $\mu : L \to \mathbb{R}$ satisfying the following conditions

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B); \tag{2}$$

$$\mu(\emptyset) = 0. \tag{3}$$

By iterating equation (2) we obtain

$$\mu(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) + \sum_{i<j<k} \mu(A_i \cap A_j \cap A_k) - \cdots ,$$

the *inclusion-exclusion principle* for a valuation μ on a lattice L.

We are interested in linear combinations of indicator functions, as we will be able to integrate them with respect to a valuation.

Definition. Let L be a lattice, $\alpha_i \in \mathbb{R}$, and $A_i \in L$. A finite linear combination of indicator functions $f = \alpha_1 I_{A_1} + \alpha_2 I_{A_2} + \cdots + \alpha_n I_{A_n}$ is said to be an *L-simple* function, or a *simple* function for short.

Because $I_{A\cap B} = I_A I_B$ holds for indicator functions, the set of all simple functions forms a ring under addition and multiplication.

Definition. A *generating set* of a lattice L is a subset G of L which is closed under finite intersections and where every element of L is a finite union of elements of G.

The following proposition is a consequence of the inclusion-exclusion principle for indicator functions and the equation $I_{A\cup B} = I_A + I_B - I_{A\cap B}$.

Proposition 1.1. *Given a generating set G of a lattice L, every L-simple function can be written as a finite linear combination $\sum_{i=1}^n \alpha_i I_{B_i}$ with $B_i \in G$.*

Definition. Given a lattice L with generating set G, we call $\nu : G \to \mathbb{R}$ a *valuation* on G if it satisfies (2) and (3) for all $A, B \in G$ with $A \cup B \in G$.

Because G is not required to be closed under unions, (2) might not make sense for all $A, B \in G$. For the same reason the inclusion-exclusion principle does not hold in general for ν if $n > 2$.

However, every element $B \in L$ can be expressed as a union $B = B_1 \cup B_2 \cup \cdots \cup B_n$. So the inclusion-exclusion principle suggests that we can attempt to extend ν to a valuation μ on all of L by setting

$$\mu(B) = \sum_i \nu(B_i) - \sum_{i<j} \nu(B_i \cap B_j) + \cdots . \tag{4}$$

It remains to be checked that $\mu(B)$ is well defined, that is, $\mu(B)$ does not depend on the possibly multiple ways B can be expressed as a union. This question is strongly related to the integrability of simple functions, so we postpone an answer until after the next definition.

Definition. Let L be a lattice with generating set G, and ν a valuation on G. For an L-simple function $f = \alpha_1 I_{A_1} + \alpha_2 I_{A_2} + \cdots + \alpha_n I_{A_n}$, we define the *integral* of f with respect to ν by

$$\int f d\nu = \sum_{i=1}^{n} \alpha_i \nu(A_i),$$

where $A_i \in G$, and $1 \leq i \leq n$.

In general there are infinetly many ways to express a simple function as a sum of indicator functions of sets in G. Consequently, it remains to be checked that the integral is well defined.

Theorem 1.2. *(Groemer's integral theorem) Let G be a generating set for a lattice L, and let μ be a valuation on G. The following statements are equivalent:*

(i) *μ extends uniquely to a valuation on L;*

(ii) *μ satisfies the inclusion-exclusion identities*

$$\mu(B_1 \cup B_2 \cup \cdots \cup B_n) = \sum_i \mu(B_i) - \sum_{i<j} \mu(B_i \cap B_j) + \cdots ;$$

if $B_i \in G$ and $B_1 \cup B_2 \cup \cdots \cup B_n \in G$ for all $n > 2$;

(iii) *μ defines an integral on the vector space of linear combinations of indicator functions of sets in L.*

1.3 The Intrinsic Volumes for Parallelotopes

As a toy case for the theory of invariant valuations on the lattice of finite unions of compact convex sets in $\mathbb{R}^n$, we consider the lattice of finite unions of orthogonal parallelotopes. In this setting many of the central results of integral geometry can be stated and proven with less effort. The intrinsic volumes for parallelotopes will not just be our first but also one of the most important examples of valuations, as they form a basis of the vector space of all continuous invariant valuations on the lattice of parallelotopes.

We first recall the definitions of the lattice of parallelotopes, continuity and invariance for valuations on parallelotopes and close with Groemer's extension theorem, providing a way to construct valuations. Throughout this chapter a Cartesian coordinate system in $\mathbb{R}^n$ is fixed.

Definition. $\mathrm{Par}(n)$ is defined to be the family of sets that are obtained by taking unions and intersections of orthogonal parallelotopes, that is, parallelotopes having edges parallel to a fixed basis of $\mathbb{R}^n$. Given $P \in \mathrm{Par}(n)$, we say that P is of dimension n or has *full dimension* if P is not contained in a finite union of hyperplanes of $\mathbb{R}^n$, that is, P has non-empty interior. Otherwise we shall say that P is of *lower dimension*. In general a set $P \in \mathrm{Par}(n)$ has dimension k if P is contained in a finite union of k-planes in $\mathbb{R}^n$, but is not in any finite union of $(k-1)$-planes.

Remark. Note that $\mathrm{Par}(n)$ is a distributive lattice.

By $\widetilde{T_n}$ we denote the group generated by translations and permutations of coordinates in $\mathbb{R}^n$. For $A \subseteq \mathbb{R}^n$ and $g \in \widetilde{T_n}$ we write $gA := g(A) = \{g(a) : a \in A\}$.

Definition. A valuation μ defined on $\mathrm{Par}(n)$ is said to be *invariant* if

$$\mu(gP) = \mu(P)$$

for all $g \in \widetilde{T_n}$ and $P \in \mathrm{Par}(n)$. If $\mu(gP) = \mu(P)$ only holds for translations g of $\mathbb{R}^n$, we say that μ is *translation invariant*.

We want to impose a continuity condition on the valuations defined on $\mathrm{Par}(n)$ to avoid pathological cases, when we determine all invariant valuations on $\mathrm{Par}(n)$.

Definition. For $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ the distance $d(x, A)$ from the point x to the set A is given by

$$d(x, A) := \inf_{a \in A} d(x, a),$$

where $d(a, x) = |x - a|$ is the usual distance between points in $\mathbb{R}^n$. For $K, L \subset \mathbb{R}^n$, the *Hausdorff distance* $\delta(K, L)$ is defined by

$$\delta(K, L) := \max\left(\sup_{a \in K} d(a, L), \sup_{b \in L} d(b, K)\right).$$

A sequence of compact subsets K_n of $\mathbb{R}^n$ *converges* to a set $K \subset \mathbb{R}^n$, or $K_n \to K$ if $\delta(K_n, K) \to 0$ as $n \to \infty$.

Definition. A valuation μ on $\mathrm{Par}(n)$ is said to be *continuous*, provided that $\mu(P_i) \to \mu(P)$, if P_i and P are parallelotopes and $P_i \to P$.

Another useful condition is that of monotonicity.

Definition. A valuation μ is said to be *increasing* on $\mathrm{Par}(n)$, provided that $\mu(P) \leq \mu(Q)$, if $P, Q \in \mathrm{Par}(n)$ and $P \subseteq Q$. Similarly one defines *decreasing* valuations. A valuation μ is said to be *monotone* on $\mathrm{Par}(n)$ if μ is either an increasing valuation or a decreasing valuation.

Theorem 1.3. *The distance δ defines a metric on the set of all compact subsets of $\mathbb{R}^n$.*

Theorem 1.4. *(Groemer's extension theorem for $\mathrm{Par}(n)$) A valuation μ defined on parallelotopes with edges parallel to the coordinate axes admits a unique extension to a valuation on the lattice $\mathrm{Par}(n)$.*

Our goal is the classification of invariant valuations on $\mathrm{Par}(n)$. To begin we consider the case of $\mathbb{R}^1$, where an element of $\mathrm{Par}(1)$ is a finite union of closed intervals.

Definition. For $A \in \text{Par}(1)$ set

$$\mu_0^1(A) := \text{number of connected components of } A;$$
$$\mu_1^1(A) := \text{lenght of } A.$$

Theorem 1.5. *Every continuous invariant valuation on* $\text{Par}(1)$ *is a linear combination of* μ_0^1 *and* μ_1^1.

Definition. The *k-th elementary symmetric functions* of $x_1, x_2, ..., x_n$ are the polynomials defined by

$$e_0(x_1, x_2, ..., x_n) = 1;$$
$$e_k(x_1, x_2, ..., x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n}^{n} x_{i_1} x_{i_2} \cdots x_{i_k}, 1 \le k \le n.$$

Theorem 1.6. *For* $0 \le k \le n$*, there exists a unique continuous valuation* μ_k *on* $\text{Par}(n)$*, invariant under translations and permutations of coordinates, such that*

$$\mu_k(P) = e_k(x_1, x_2, ..., x_n),$$

given that P *is a parallelotope with sides of lengths* $x_1, x_2, ..., x_n$.

Remark. The valuation μ_0 is called the *Euler characteristic*, and due to Theorem 1.6 it is the only valuation on $\text{Par}(n)$ having the value 1 on all nonempty parallelotopes. Furthermore $\mu_n(P)$ is the volume of $P \in \text{Par}(n)$, and $2\mu_{n-1}(P) = \text{surface area}(P)$.

Note that, if a parallelotope P has dimension $k < n$, the valuation $\mu_i(P)$ has ambiguous sense. It might indicate the value $\mu_i(P)$ in $\mathbb{R}^n$, but it might also denote the value of $\mu_i(P)$ in some $\mathbb{R}^m$ with $P \in \mathbb{R}^m$ and $k \le m < n$. Theorem 1.6 however implies that the two valuations coincide, that is, $\mu_i^n(P) = \mu_i^m(P)$. Therefore it is not necessary to indicate the dimension of the product space of copies of $\mathbb{R}$ we want calculate μ_i in. For this reason μ_k is called the *k-th intrinsic volume* for $0 \le k \le n$. We summarize this fact in the following corollary.

Corollary 1.7. *The valuations* μ_i *on* $\text{Par}(n)$ *are normalized independent of the dimension* n.

As the main result of this chapter we are now able to determine all continuous valuations that are invariant under translations and permutations of coordinates.

Theorem 1.8. *The valuations* $\mu_0, \mu_1, ..., \mu_n$ *form a basis of the vector space of all continuous invariant valuations defined on* $\text{Par}(n)$.

Remark. The last theorem is a special case of the famous Hadwiger characterization theorem, which will be discussed later in this chapter.

Definition. A valuation μ on $\mathrm{Par}(n)$ is said to be *homogeneous of degree* $k > 0$ if

$$\mu(\alpha P) = \alpha^k \mu(P)$$

for all $P \in \mathrm{Par}(n)$ and all $\alpha > 0$.

Corollary 1.9. *Let μ be a continuous invariant valuation defined on $\mathrm{Par}(n)$ that is homogeneous of degree k for some $0 \leq k \leq n$. Then there exists $c \in \mathbb{R}$ such that $\mu(P) = c\mu_k(P)$ for all $P \in \mathrm{Par}(n)$.*

2 The Theorem of Hadwiger

We extend the theory of valuations to the lattice of polyconvex sets, that is, the lattice of finite unions of compact convex subsets of $\mathbb{R}^n$. Defining an invariant measure on affine Grassmannians provides an extension of the intrinsic volumes to the lattice of polyconvex sets, that is consistent with the notion of the intrinsic volumes on the lattice of parallelotopes. Characterizing every continuous rigid motion invariant simple valuation on the lattice of polyconvex sets as the volume, provides a proof of Hadwiger's characterization theorem. This part is taken from [1].

2.1 The Lattice of Polyconvex Sets

After our observations about valuations on parallelotopes, we turn our attention to the lattice of polyconvex sets, which is a natural setting for the study of classical integral geometry. We extend the terminology from the previous chapter to this new setting, including Groemer's extension theorem.

Definition. Denote by $\mathcal{K}^n$ the collection of all non-empty compact convex subsets of $\mathbb{R}^n$. We call a finite union of compact convex sets a *polyconvex* set. For a polyconvex set A, we say that A is of dimension n if A is not contained in a finite union of hyperplanes in $\mathbb{R}^n$. Otherwise, we say that A is of *lower dimension*. By Polycon(n) we denote the distributive lattice of polyconvex sets in $\mathbb{R}^n$ together with the union and intersection of sets. Note that $\mathcal{K}^n$ is a generating set for Polycon(n).

Definition. For compact convex sets K and L the *Minkowski sum* $K + L$ is defined by

$$K + L = \{x + y : x \in K \text{ and } y \in L\}.$$

Note that $K + L$ in the above definition is convex and, due to $+$ being continuous as a map from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the compactness of $K \times L$, $K + L$ is also compact. So the Minkowski addition maps into $\mathcal{K}^n$.

In order to extend some of the definitions for valuations on parallelotopes to polyconvex sets, denote by E_n the Euclidean group on $\mathbb{R}^n$, that is, the group generated by translations and orthogonal transformations. If $A \subseteq \mathbb{R}^n$ and $g \in E_n$, we write $gA = g(A) = \{g(a) : a \in A\}$. The subgroup of translations shall be denoted by T_n.

Definition. A valuation $\mu : \text{Polycon}(n) \to \mathbb{R}$ is said to be *rigid motion invariant*, or simply *invariant* if

$$\mu(gA) = \mu(A) \tag{5}$$

for all $g \in E_n$ and all $A \in \text{Polycon}(n)$. If (5) only holds when $g \in T_n$, we say μ is *translation invariant*.

Definition. A valuation $\mu : \mathcal{K}^n \to \mathbb{R}$ is said to be *continuous*, provided that $\mu(A_n) \to \mu(A)$ if $A_n \to A$ with respect to the Hausdorff metric.

Due to the following result, the classification of all continuous valuations on $\text{Polycon}(n)$ can be reduced to the case of continuous valuations on $\mathcal{K}^n$.

Theorem 2.1. *(Groemer's extension theorem for* $\text{Polycon}(n)$*) A continuous valuation μ on $\mathcal{K}^n$ admits a unique extension to a valuation on the lattice* $\text{Polycon}(n)$*.*

We have seen that μ_0 is a functional on $\text{Par}(n)$ and that $\mu_0(P) = 1$ for a non-empty parallelotope. This motivates the following theorem, where we extend the valuation μ_0 to $\text{Polycon}(n)$.

Theorem 2.2. *There exists a unique continuous invariant valuation μ_0^n defined on* $\text{Polycon}(n)$ *such that $\mu_0^n(K) = 1$, given that K is a non-empty compact convex set.*

The valuation μ_0^n is called the Euler characteristic. It is *normalized* independent of the dimension of the space $\mathbb{R}^n$ and we write μ_0 instead of μ_0^n. This follows from the inclusion-exclusion principle, because $K \in \text{Polycon}(n)$ can be expressed as a union of elements of $\mathcal{K}^n$ and μ_0^n has the value 1 on all of $\mathcal{K}^n$.

2.2 Invariant Measures on Grassmannians

We recall invariant measures on linear subspaces of $\mathbb{R}^n$ and the flag coefficients. This will be a preparation for the next section, where we will introduce intrinsic volumes for polyconvex sets with the help of an invariant measure on affine linear subspaces on $\mathbb{R}^n$.

Definition. Let $\text{Mod}(n)$ denote the partially ordered set of linear subspaces of $\mathbb{R}^n$ together with the inclusion relation. For $x, y \in \text{Mod}(n)$ we define $x \vee y$ as the linear subspace spanned by x and y, and $x \wedge y$ as the intersection of x and y. This defines a lattice structure on $\text{Mod}(n)$.

Definition. An element $x \in \text{Mod}(n)$ has rank k, if the dimension of x is k. The set of elements of $\text{Mod}(n)$ of rank k, denoted $\text{Gr}(n, k)$, is called the *k-Grassmannian.*

If G with identity element e is a group and S is a set, a *left group action* of G on S is a function $\varphi : G \times S \to S$ satisfying the following properties

$$\varphi(e, x) = x, \text{ for all } x \in S;$$
$$\varphi(gh, x) = \varphi(g, \varphi(h, x)), \text{ for all } g, h \in G \text{ and } x \in S.$$

We say the group G acts on S.

The *orthogonal group $O(n)$* (that is, the group of rotations about the origin and reflections across hyperplanes through the origin in $\mathbb{R}^n$) acts naturally on $\text{Mod}(n)$.

It is known that there exists a rotation invariant Haar measure on $\mathrm{Gr}(n,k)$, which is unique up to a common factor. Our objective is to describe this Haar measure. We start out by considering the rotation invariant measure τ_n on $\mathrm{Gr}(n,1)$ and denote by

$$[n] := \tau_n(\mathrm{Gr}(n,1)) = \frac{n\kappa_n}{2\kappa_{n-1}},$$

where κ_n is the n-dimensional volume of the unit ball B_n.

Let σ_{n-1} be the invariant measure on the unit sphere S^{n-1}, and for any measurable subset A of $\mathrm{Gr}(n,1)$ let A' be the subset of S^{n-1} defined by

$$A' = \bigcup_{x\in A} x \cap S^{n-1}.$$

The measure τ_n then satisfies

$$\tau_n(A) = \frac{\sigma_{n-1}(A')}{2\kappa_{n-1}},$$

which is clearly invariant under rotations.

Definition. Let $(L,\le)$ be a partially ordered set. A *chain* in L is a linear ordered subset of L, that is, a subset in which for every pair x,y either $x\le y$ or $y\le x$. A *flag* is a maximal chain, i.e., a chain F such that if $G\supseteq F$ and G is a chain, then $F=G$.

Definition. Let $\mathrm{Flag}(n)$ be the set of all flags in $\mathrm{Mod}(n)$. For $x\in \mathrm{Mod}(n)$, denote by $\mathrm{Flag}(x)$ the set of all flags containing x.

Remark. $\mathrm{Flag}(x)$ is the set of all sequences $(x_0,x_1,...,x_n)$ with $x_i\in\mathrm{Mod}(n)$ where $\dim(x_i)=i$, $x_0\subseteq x_1\subseteq\cdots\subseteq x_n$, and $x_{i_0}=x$ for some $0\le i_0\le n$.

Definition. For $A\in\mathrm{Gr}(n,k)$ let $\mathrm{Flag}(A)$ be the set of all flags $(x_0,x_1,...,x_n)$ in $\mathrm{Flag}(n)$ such that $x_k\in A$.

We call a sequence of orthogonal straight lines a frame. For $(x_0,x_1,...,x_n)\in\mathrm{Flag}(n)$ we obtain a frame $(y_1,y_2,...,y_n)$ by setting

$$y_1=x_1, y_2=x_1^{\perp}\cap x_2, y_3=x_2^{\perp}\cap x_3, ..., y_n=x_{n-1}^{\perp}\cap x_n.$$

Conversely, given a frame $(y_1,y_2,...,y_n)$ we obtain a flag by setting

$$x_0=\{0\}, x_1=y_1, x_2=y_1\vee y_2, ..., x_n=y_1\vee\cdots\vee y_n.$$

Consequently, there exists a one-to-one correspondence between flags and frames, and for a real-valued measurable function $f(x_0,x_1,...,x_n)$ on flags we find a corresponding function $\bar f(y_1,y_2,...,y_n)$ on frames. Let ϕ_n be the invariant measure on $\mathrm{Flag}(n)$ defined by

$$\int f d\phi_n = \int\int\cdots\int \bar f(y_1,y_2,...,y_n)d\tau_1(y_n)d\tau_2(y_{n-1})\cdots d\tau_n(y_1).$$

The measure of Flag(n) can be calculated to be

$$\phi_n(\text{Flag}(n)) = [n][n-1]\cdots[2][1],$$

which is also denoted by $[n]!$, where $[1] = 1$. Note that due to the definition of $[n]$ we get

$$[n]! = \frac{n!\kappa_n\kappa_{n-1}\cdots\kappa_1}{2^n\kappa_{n-1}\kappa_{n-2}\cdots\kappa_0} = \frac{n!\kappa_n}{2^n}.$$

Definition. We define the rotation invariant measure ν_k^n on $\text{Gr}(n,k)$ by

$$\nu_k^n(A) = \frac{1}{[k]![n-k]!}\phi_n(\text{Flag}(A)).$$

2.3 The Intrinsic Volumes for Polyconvex Sets

Extending the invariant measure ν_k^n to affine subspaces of $\mathbb{R}^n$ yields the measure λ_k^n. This measure not only provides an alternative way of describing the intrinsic volumes for parallelotopes, but also opens the door to an extension of the intrinsic volumes to all polyconvex sets. Furthermore we will formulate Sylvester's theorem.

Definition. Let Aff(n) denote the partially ordered set of all affine subspaces of $\mathbb{R}^n$. The subset of elements of Aff(n) of rank k is denoted by $\text{AGr}(n,k)$ and is called the *affine k-Grassmannian*.

Remark. The minimal element of Aff(n) is the empty set. The Euclidean motion group E_n, that is, the group of translations and orthogonal transformations, acts naturally on Aff(n).

With the goal of constructing a measure on $\text{AGr}((n,k)$, that is invariant under the Euclidean group E_n, we parameterize $\text{AGr}(n,k)$ in the following way. For $V \in \text{AGr}((n,k)$ we find the maximal linear subspace $V^\perp$ of $\mathbb{R}^n$ that is orthogonal to V. Take note that as it is a linear subspace, $V^\perp$ contains the origin. Next we find the unique maximal linear subspace of $\mathbb{R}^n$ $\text{or}(V)$ that is orthogonal to $V^\perp$. $\text{or}(V)$ has dimension k and we say V and $\text{or}(V)$ are parallel. By $p(V)$ we denote the point $V \cap V^\perp$. We obtain a one-to-one correspondence between $\text{AGr}(n,k)$ and $\text{Gr}(n,k)\times\mathbb{R}^n$: For $V \in \text{AGr}(n,k)$ we get the unique pair $(or(V), p(V))$ in $\text{Gr}(n,k)\times\mathbb{R}^n$, where $p(V)$ lies in $\text{or}(V)^\perp = V^\perp$. Conversely for a pair $(V_0, p) \in \text{Gr}(n,k)\times\mathbb{R}^n$ we translate the subspace V by the vector p to get a unique element of $\text{AGr}(n,k)$.

For a measurable function $f : \text{AGr}(n,k) \to \mathbb{R}$ let $\bar{f} : \text{Gr}(n,k)\times\mathbb{R}^n \to \mathbb{R}$ be given by $\bar{f}(V_0,p) = f(V_0+p)$. We are now in a position to define an integral for a measurable function $f : \text{AGr}(n,k) \to \mathbb{R}$ via

$$\int f d\lambda_k^n = \int_{\text{Gr}(n,k)} \int_{V_0^\perp} \bar{f}(V_0,p)\; dp\; d\nu_k^n(V_0),$$

where dp denotes the Lebesgue measure on $V^\perp \cong \mathbb{R}^{n-k}$. The invariance of λ_k^n under E_n follows from the invariance of ν_k^n and the Lebesgue measure.

Definition. For $A \subseteq \mathbb{R}^n$, $\mathrm{AGr}(A;k)$ denotes the set of all $V \in \mathrm{AGr}(n,k)$ such that $A \cap V \neq \emptyset$.

For parallelotopes we can now interpret the intrinsic volumes μ_{n-k} with the help of the measure λ_k^n. For now, there would appear, a yet to be determined, factor C_k^n in Theorem 2.3. However, after calculating the intrinsic volumes of the unit ball with the help of Hadwiger's theorem, it can be shown that these factors C_k^n are actually all equal to 1. For the sake of brevity and simplicity, we will use this result already at this point, however note that of course all following results can be proven with the C_k^n still in place.

Theorem 2.3. *For a parallelotope P in* $\mathrm{Par}(n)$

$$\mu_{n-k}(P) = \lambda_k^n(\mathrm{AGr}(P;k)). \tag{6}$$

The last theorem motivates us to extend the intrinsic volumes in the following way.

Definition. For $K \in \mathcal{K}^n$ we define

$$\mu_{n-k}^n(K) := \lambda_k^n(\mathrm{AGr}(K;k)), \tag{7}$$

where $0 \leq k \leq n$.

It can be shown that μ_{n-k}^n is a continuous valuation on $\mathcal{K}^n$, and by Groemer's extension Theorem 2.1 it can be uniquely extended to a valuation μ_{n-k}^n on all polyconvex sets in $\mathbb{R}^n$. Note however that (7) does not hold for arbitrary polyconvex sets. Hadwiger's formula is however an explicit formulation of the extension of the intrinsic volumes to all of $\mathrm{Polycon}(n)$. We will later see that, like the intrinsic volumes on $\mathrm{Par}(n)$, $\mu_{n-k}^n(K)$ is not actually dependent of the dimension n of the embedding space.

Theorem 2.4. *(Hadwiger's formula) For an arbitrary $A \in$* $\mathrm{Polycon}(n)$*,*

$$\mu_{n-k}^n(K) = \int_{\mathrm{AGr}(n,k)} \mu_0(A \cap V)\; d\lambda_k^n(V).$$

For further details check [14].

We are now able to generalize (1). For non-empty $K, L \in \mathcal{K}^n$ with $K \subseteq L$ and L being of dimension n, we get $\mathrm{AGr}(K;k) \subseteq \mathrm{AGr}(L;k)$ and the conditional probability that $M \in \mathrm{AGr}(n,k)$ meets K, given that it meets L, is given by

$$\frac{\lambda_k^n(\mathrm{AGr}(K;k))}{\lambda_k^n(\mathrm{AGr}(L;k))}.$$

This leads us to the following theorem.

Theorem 2.5. *(Sylvester's theorem) For $K, L \in \mathcal{K}^n$ with $K \subseteq L$ and L being of dimension n the conditional probability that an affine subspace of $\mathbb{R}^n$ of dimension k meets K, assuming that it meets L, is given by*

$$\frac{\mu_{n-k}^n(K)}{\mu_{n-k}^n(L)}.$$

2.4 The Volume Theorem for Polyconvex Sets

In this short section we will see that every continuous rigid motion invariant simple valuation on the polyconvex sets is actually, up to a constant, the n-th intrinsic volume. This result will play a big part in the proof of Hadwiger's characterization theorem.

Definition. A valuation μ on $\mathcal{K}^n$ or $\text{Polycon}(n)$ is called *simple* if $\mu(K) = 0$, whenever K is not of dimension n (K is of lower dimension). It is called *even* if $\mu(-K) = \mu(K)$ for all $K \in \mathcal{K}^n$ and *odd* if $\mu(-K) = -\mu(K)$ for all $K \in \mathcal{K}^n$.

Remark. Note that every valuation $\mu : \mathcal{K}^n \to \mathbb{R}$ can be decomposed into an even and an odd part:

$$\mu = \mu_{even} + \mu_{odd}.$$

To see this, define $\mu_{even}(K) := \frac{1}{2}(\mu(K)+\mu(-K))$ and $\mu_{odd} := \frac{1}{2}(\mu(K)-\mu(-K))$.

The following theorem will be of utmost importance in the proof of Hadwiger's characterization theorem.

Theorem 2.6. *(The volume theorem for* $\text{Polycon}(n)$*) Let μ be a continuous rigid motion invariant simple valuation on $\mathcal{K}^n$ or* $\text{Polycon}(n)$*. Then there exists $c \in \mathbb{R}$, such that $\mu(K) = c\mu_n(K)$ for all K in $\mathcal{K}^n$ or* $\text{Polycon}(n)$*.*

For the intrinsic volumes on parallelotopes we saw that they are independent of the dimension of the embedding space. This result can be extended to the polyconvex sets.

Theorem 2.7. *The valuations μ_k^n are normalized independent of the dimension n of the embedding space.*

Due to the above theorem from now on we write μ_k for μ_k^n and call these valuations the *intrinsic volumes*.

2.5 Hadwiger's Characterization Theorem

We go on to the main result of this part - the famous Hadwiger characterization theorem and its direct consequences.

Theorem 2.8. *(Hadwiger's characterization theorem) The valuations $\mu_0, \mu_1, ..., \mu_n$ form a basis of the vector space of all continuous rigid motion invariant valuations defined on polyconvex sets in $\mathbb{R}^n$.*

Proof. Let μ be a continuous rigid motion invariant valuation and $H \subseteq \mathbb{R}^n$ a hyperplane. Because the restriction of μ to H is a continuous rigid motion invariant valuation on an affine space of dimension $n-1$, we can assume that

$$\mu(K) = \sum_{i=0}^{n-1} c_i \mu_i(A)$$

by induction on the dimension n. Note that the case $n = 0$ is trivial, considering $\mathbb{R}^0 = \{0\}$. As every convex set of lower dimension can be moved into H with a rigid motion, and because the valuations μ_i are invariant, we obtain that the valuation

$$\mu - \sum_{i=0}^{n-1} c_i \mu_i$$

is simple. Due to the volume theorem for polyconvex sets (Theorem 2.6)

$$\mu - \sum_{i=0}^{n-1} c_i \mu_i = c_n \mu_n.$$

So μ can be expressed as a linear combination of the intrinsic volumes. □

Definition. A valuation μ on Polycon(n) is said to be *homogeneous of degree* $k > 0$ if

$$\mu(\alpha P) = \alpha^k \mu(P)$$

for all $P \in \text{Polycon}(n)$ and all $\alpha > 0$.

Corollary 2.9. *For a continuous rigid motion invariant valuation $\mu : \text{Polycon}(n) \to \mathbb{R}$ of degree k there exists $c \in \mathbb{R}$ such that $\mu(K) = c\mu_k(K)$ for all $K \in \text{Polycon}(n)$.*

Theorem 2.10. *(The mean projection formula) Let $0 \leq k \leq n$ and $K \in \mathcal{K}^n$ then*

$$\mu_k(K) = \int_{\text{Gr}(n,k)} \mu_k(K|V_0)\ d\nu_k^n(V_0).$$

To put the last theorem into words: The k-th intrinsic volume of a compact convex subset of $\mathbb{R}^n$ is equal to the integral of the k-dimensional volumes of the projections of K onto all k-dimensional subspaces of $\mathbb{R}^n$.

3 The Classical Steiner Point Map

We have considered the Hadwiger theorem for scalar valued valuations, later on we will discuss a more general version for vector valued valuations. This will enable us to classify the Steiner point map with a different approach; But before that, we look at the characterization due to Schneider. Throughout this section we denote by $(\cdot,\cdot)$ the standard inner product on $\mathbb{R}^n$. We want to start out with some basic definitions and results from convex geometry and introduce the Steiner point map. This part is taken from [2] and [3].

3.1 Prerequisites

We begin with a few basic definitions from convex geometry.

Definition. For $\alpha \in \mathbb{R}$ and $u \in \mathbb{R}^n \backslash \{0\}$ the *hyperplane* $H_{u,\alpha}$ with *normal vector* u is defined by

$$H_{u,\alpha} := \{x \in \mathbb{R}^n : (x, u) = \alpha\}.$$

A hyperplane bounds two *closed halfspaces*

$$H^-_{u,\alpha} := \{x \in \mathbb{R}^n : (x, u) \leq \alpha\};$$
$$H^+_{u,\alpha} := \{x \in \mathbb{R}^n : (x, u) \geq \alpha\}.$$

Definition. We take $A \subseteq \mathbb{R}^n$, a hyperplane $H \subseteq \mathbb{R}^n$ and the two closed halfspaces bound by H and denoted by H^- and H^+. Then H is said to *support* A *at* x if $x \in A \cap H$ and either $A \subseteq H^-$ or $A \subseteq H^+$. H is called a *support plane* of A if H supports A at some point x. Given $H = H_{u,\alpha}$ supports A and $A \subseteq H^-_{u,\alpha}$, then $H^-_{u,\alpha}$ is called a *supporting halfspace* of A, and u is called an *exterior* or *outer normal vector* of H as well as $H^-_{u,\alpha}$.

The following theorem is the motivation for our particular interest in supporting planes and halfspaces.

Theorem 3.1. *Each $K \in \mathcal{K}^n$ is the intersection of its supporting halfspaces.*

We are not completely satisfied with this result and want to be more specific about the supporting halfspaces of a closed convex set. First however, we require some more definitions.

Definition. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\alpha \in \overline{\mathbb{R}}$ we define

$$\{f = \alpha\} := \{x \in \mathbb{R}^n : f(x) = \alpha\},$$

and the sets $\{f < \alpha\}$, $\{f \leq \alpha\}$,... are defined accordingly. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is called *convex*, given that $\{f = -\infty\} = \emptyset$, $\{f = \infty\} \neq \mathbb{R}^n$, and

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda y$$

for $x, y \in \mathbb{R}^n, 0 \leq \lambda \leq 1$. Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be convex, then $\text{dom} f := \{f < \infty\}$.

Now we are ready to introduce the support function of a closed convex set A in $\mathbb{R}^n$.

Definition. For $K \in \mathcal{K}^n$ we define the *support function* $h(K, \cdot) = h_K$ by

$$h(K, u) := \sup\{(x, u) : x \in K\}$$

for $u \in \mathbb{R}^n$. Furthermore for $u \in \text{dom } h(K, \cdot)$ we set

$$\begin{aligned} H(K, u) &:= \{x \in \mathbb{R}^n : (x, u) = h(K, u)\}; \\ H^-(K, u) &:= \{x \in \mathbb{R}^n : (x, u) \leq h(K, u)\}; \\ F(K, u) &:= H(K, u) \cap K. \end{aligned}$$

$H(K, u)$, $H^-(K, u)$, and $F(K, u)$ are called the *support plane* with exterior normal vector u, the *supporting halfspace* with exterior normal vector u, and the *support set* of K with exterior normal vector u, respectively. For bounded K the definitions of the support plane and the supporting halfspace coincide, however for unbounded K it might happen that $F(K, u) = \emptyset$.

In order to gain some intuition for the notion of the support function we consider $u \in S^{n-1} \cap \text{dom } h(K, \cdot)$. Due to the Cauchy-Schwarz inequality for inner products, $|(x, u)|$ is maximal for x being a scalar multiple of u. In this case $|(x, u)| = \|u\|$, so the support function is the signed distance between the support plane with exterior normal vector u and the origin. In the case that u is pointing into the open halfspace containing the origin, the signed distance is negative.

The support function has various properties that follow directly from the definition.

Proposition 3.2. *Given a non-empty closed convex set K in $\mathbb{R}^n$, then its support function $h(K, \cdot)$ has the following properties*

(*i*) $h(K + t, u) = h(K, u) + (t, u)$ *for* $u \in \mathbb{R}^n$;

(*ii*) $h(K, \lambda u) = \lambda h(K, u)$ *for* $\lambda \geq 0$ *and* $h(K, u + v) \leq h(K, u) + h(K, v)$.

For $K \in \mathcal{K}^n$, $h(K, \cdot)$ is sublinear and convex. Conversely the following statement holds.

Theorem 3.3. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there is a unique convex body K such that $f = h(K, \cdot)$.*

So a $K \in \mathcal{K}^n$ is completely determined by its support function.

3.2 Characterization of the Steiner Point Map

Our goal is to characterize the Steiner point map as the unique vector valued continuous rigid motion equivariant valuation.

By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$, and Γ denotes the gamma function. The surface area of the unit ball is $\omega_n = \mathcal{H}^{n-1}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$.

Definition. The *Steiner point map* $s : \mathcal{K}^n \to \mathbb{R}^n$ is defined by

$$s(K) := \frac{1}{\kappa_n} \int_{S^{n-1}} h(K,u)u \; d\mathcal{H}^{n-1}(u).$$

Definition. A function f on $\mathcal{K}^n$ with values in some abelian semigroup is called *Minkowski additive* if

$$f(K+L) = f(K) + f(L), \text{for } K, L \in \mathcal{K}^n.$$

A Minkowski additive function f with values in a real vector space or in $\mathcal{K}^n$ is called *Minkowski linear* if it satisfies

$$f(\lambda K) = \lambda f(K), \text{for } K \in \mathcal{K}^n \text{ and } \lambda \geq 0.$$

Theorem 3.4. *The support function $h : \mathcal{K}^n \to C(S^{n-1})$ is Minkowski additive, and so is the Steiner point map $s : \mathcal{K}^n \to \mathbb{R}^n$.*

Our special interest in the Steiner point map stems from its invariance properties.

Proposition 3.5. *The Steiner point map is equivariant under rigid motions, that is, given a rigid motion $g : \mathbb{R}^n \to \mathbb{R}^n$ and $K \in \mathcal{K}^n$, it satisfies $s(gK) = gs(K)$.*

Proof. Let g be a rotation, then $h(gK,u) = h(K, g^{-1}u)$. Considering the spherical Lebesgue measure is rotation invariant, we get $s(gK) = gs(K)$. Now let g be a translation, then, due to Proposition 3.2, we need to calculate

$$\frac{1}{\omega_n} \int_{S^{n-1}} (t,u)u \; d\mathcal{H}^{n-1}(u)$$

for $t \in \mathbb{R}^n$. As the integral is linear in t and invariant under rotations and reflections fixing t, we get

$$\frac{1}{\omega_n} \int_{S^{n-1}} (t,u)u \; d\mathcal{H}^{n-1}(u) = \alpha t$$

with $\alpha \in \mathbb{R}$ independent of t. Choosing $|t| = 1$ and an orthogonal basis $(e_1, .., e_n)$ of $\mathbb{R}^n$ we obtain

$$\begin{aligned}\alpha =& \int_{S^{n-1}} (t,u)^2 \; d\mathcal{H}^{n-1}(u) = \frac{1}{n} \sum_{i=1}^{n} \int_{S^{n-1}} (e_i,u)^2 \; d\mathcal{H}^{n-1}(u) \\ =& \frac{1}{n} \int_{S^{n-1}} |u|^2 \; d\mathcal{H}^{n-1}(u) = \omega_n.\end{aligned}$$

This results in

$$\frac{1}{\omega_n} \int_{S_{n-1}} (t,u)u \; d\mathcal{H}^{n-1}(u) = t$$

completing the proof. □

Theorem 3.6. *Let $n \geq 2$. If a map $\varphi : \mathcal{K}^n \to \mathbb{R}^n$ is Minkowski linear, equivariant under rigid motions and continuous at the unit ball B^n, then φ is the Steiner point map s.*

Proof. The group of rotations $\mathrm{SO}(n)$ is compact, therefore, given $\varepsilon > 0$, we can decompose $\mathrm{SO}(n)$ into finitely many nonempty Borel sets $\Delta_{1,\varepsilon}, ..., \Delta_{m(\varepsilon),\varepsilon}$ of diameter less then ε. We choose $\rho_{k,\varepsilon} \in \Delta_{k,\varepsilon}$ and denote $\nu_{k,\varepsilon} = \nu(\Delta_{k,\varepsilon})$, where ν denotes the Haar measure on $\mathrm{SO}(n)$. For any continuous real function f on $\mathrm{SO}(n)$ we have the usual estimate

$$\lim_{\varepsilon \to 0} \sum_{k=1}^{m(\varepsilon)} f(\rho_{k,\varepsilon})\nu_{k,\varepsilon} = \int_{\mathrm{SO}(n)} f d\nu.$$

Let $K \in \mathcal{K}^n$, if $K = \{x\}$, then $\varphi(K) = x = s(K)$, due to the rigid motion invariance of φ. Now let us assume that $\dim K > 0$. We choose $v \in \mathbb{R}^n, x \in S^{n-1}$ and $c > |v|$ and define a convex body K_ε by

$$K_\varepsilon := \sum_{k=1}^{m(\varepsilon)} [c + (v, \rho_{k,\varepsilon} x)]\nu_{k,\varepsilon}\rho_{k,\varepsilon}^{-1} K$$

for $\varepsilon > 0$. For an arbitrary $y \in S^{n-1}$ we get

$$(\varphi(K_\varepsilon), y) = \sum_{k=1}^{m(\varepsilon)} [c + (v, \rho_{k,\varepsilon} x)](\varphi(K), \rho_{k,\varepsilon} y)\nu_{k,\varepsilon}.$$

Hence

$$\begin{aligned}\lim_{\varepsilon \to 0}(\varphi(K_\varepsilon), y) &= \int_{\mathrm{SO}(n)} [c + (v, \rho x)](\varphi(K), \rho y)\ d\nu(\rho)\\ &= \int_{\mathrm{SO}(n)} (v, \rho x)(\varphi(K), \rho y)\ d\nu(\rho),\end{aligned}$$

considering $\int_{\mathrm{SO}(n)}(\varphi(K), \rho y)\ d\nu(\rho)$, as a function of y, is odd and rotation invariant and therefore zero.

The support function of K_ε is given by

$$h(K_\varepsilon, y) = \sum_{k=1}^{m(\varepsilon)} [c + (v, \rho_{k,\varepsilon} x)]h(K, \rho_{k,\varepsilon} y)\nu_{k,\varepsilon}$$

and hence satisfies

$$\lim_{\varepsilon \to 0} h(K_\varepsilon, y) = \int_{\mathrm{SO}(n)} [c + (v, \rho x)]h(K, \rho y)\ d\nu(\rho).$$

Now, as the integral is invariant under translations of K and positive if the origin $o \in \mathrm{relint}(K)$,

$$\int_{\mathrm{SO}(n)} ch(K, \rho y)\ d\nu(\rho) = r$$

is a positive real number that depends only on K and v. The integral

$$I(x,y) := \int_{\mathrm{SO}(n)} (v, \rho x) h(K, \rho y)\ d\nu(\rho)$$

satisfies $I(x,y) = I(y,x)$ if $n \geq 3$, as $I(\tau x, \tau y) = I(x,y)$ for each rotation $\tau \in \mathrm{SO}(n)$, and for $n \geq 3$ we can choose τ such that $\tau x = y$ and $\tau y = x$. Denoting

$$z := \int_{\mathrm{SO}(n)} \rho^{-1} v h(K, \rho x)\ d\nu(\rho)$$

for $n \geq 3$ we thus have $I(x,y) = (z,y)$. For $n = 2$, $I(x,y)$ is not symmetric in x and y, but in this case we can write

$$I(x,y) := \frac{1}{2\pi} \int_0^{2\pi} (v, u(\xi + \alpha)) h(K, u(\eta + \alpha))\ d\alpha$$

with $u(\alpha) := (\cos\alpha)e_1 + (\sin\alpha)e_2$, where $\{e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^2$, and $x = u(\xi), y = u(\eta)$. An elementary computation yields

$$I(x,y) = (A_1 \cos\xi + A_2 \sin\xi)\cos\eta + (A_1 \sin\xi - A_2 \cos\xi)\sin\eta$$

where the A_i only depend on K and v; therefore again we have $I(x,y) = (z,y)$ with some vector z only depending on K, v and x. Thus, in both cases we get

$$\lim_{\varepsilon \to 0} h(K_\varepsilon, y) = r + (z,y) = h(B(z,r), y),$$

where $B(z,r)$ is the ball with center z and radius r. This holds for each $y \in S^{n-1}$, hence $\lim_{\varepsilon\to 0} K_\varepsilon = B(z,r)$ in the Hausdorff metric. (Observe that point wise convergence of support functions implies uniform convergence on S^{n-1}.) From this we get $r^{-1}(K_\varepsilon - z) \to B^n$ for $\varepsilon \to 0$ and thus $\varphi(r^{-1}(K_\varepsilon - z)) \to \varphi(B^n)$ by the assumed continuity of φ at B^n. As $\varphi(B^n) = o$ by the rotation equivariance of φ, we arrive at $\varphi(K_\varepsilon) \to z$ and thus

$$\lim_{\varepsilon \to 0} (\varphi(K_\varepsilon), y) = (z,y)$$

for $y \in S^{n-1}$. We have proved that

$$\int_{\mathrm{SO}(n)} (v, \rho x)(\varphi(K), \rho y)\ d\nu(\rho) = (z,y),$$

which only depends on K, v, x, y and not on φ. Considering that the Steiner point map has all the properties of φ, this yields

$$\int_{\mathrm{SO}(n)} (v, \rho x)(\varphi(K) - s(K), \rho y)\ d\nu(\rho) = 0.$$

The choices $v := \varphi(K) - s(K)$ and $x = y$ now result in $\varphi(K) = s(K)$. $\square$

Minkowski additivity and continuity imply Minkowski linearity, so we get the following version of the characterization of the Steiner point map by Rolf Schneider as in [3].

Theorem 3.7. *For $n \geq 2$ let $\varphi : \mathcal{K}^n \to \mathbb{R}^n$ be a map with the following properties*

$$(i)\ \varphi(K_1 + K_2) = \phi(K_1) + \phi(K_2),\ \textit{for } K_1, K_2 \in \mathcal{K}^n;$$
$$(ii)\ \varphi \textit{ is equivariant under rigid motions;}$$
$$(iii)\ \varphi \textit{ is continuous,}$$

then φ is the Steiner point map s.

Proof. Given a Minkowski additive map φ from $\mathcal{K}^n$ into $\mathbb{R}^n$ or $\mathbb{R}$ we have $2K = K + K$ for $K \in \mathcal{K}^n$, and hence $\varphi(2K) = 2\varphi(K)$. By induction we get $\varphi(kK) = k\varphi(K)$ for $k \in \mathbb{N}$. For $k, m \in \mathbb{N}$ one obtains $k\varphi(K) = \varphi(kK) = \varphi(m(k/m)K) = m\varphi((k/m)K)$, and therefore $\varphi(qK) = q\varphi(K)$ for $q \in \mathbb{Q}$ with $q > 0$. The continuity now leads to $\varphi(\lambda K) = \lambda\varphi(K)$ for real $\lambda \geq 0$. □

Definition. Let $\mathcal{F}$ be a family of sets and $\mathcal{A}$ be an abelian semigroup, then a function $\varphi : \mathcal{F} \to \mathcal{A}$ is called a valuation on $\mathcal{F}$ if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

if $K \cup L$, $K \cap L$, K, $L \in \mathcal{F}$.

A *straight cylinder* is a convex body $K \in \mathcal{K}^n$ such that $K = K_1 + K_2$, where $\dim K_i \geq 1$ and the convex hulls of K_1 and K_2 are orthogonal. We need the following lemma from Hadwiger, see [3], to achieve a further characterization of the Steiner point map.

Lemma 3.8. *Let $\chi : \mathcal{K}^n \to \mathbb{R}$ be a functional with the following properties*

$$(i)\ \chi(K + a) = \chi(K) \textit{ for } a \in \mathbb{R}^n;$$
$$(ii)\ \chi(K_1 \cup K_2) = \chi(K_1) + \chi(K_2),\ \text{if } K_1, K_2, K_1 \cup K_2 \in \mathcal{K}^n \textit{ and } \dim(K_1 \cap K_2) < n;$$
$$(iii)\ \chi \textit{ is continuous;}$$
$$(vi)\ \chi(Z) = 0 \textit{ for straight cylinders } Z \in \mathcal{K}^n.$$

Then χ is Minkowski additive.

Theorem 3.9. *For $n \geq 2$ let $\varphi : \mathcal{K}^n \to \mathbb{R}^n$ be a map with the following properties*

$$(i)\ \varphi \textit{ is a valuation on } \mathcal{K}^n;$$
$$(ii)\ \varphi \textit{ is equivariant under rigid motions;}$$
$$(iii)\ \varphi \textit{ is continuous,}$$

then φ is the Steiner point map s.

Proof. We use induction on the dimension n. First let $n = 2$. For $K \in \mathcal{K}^2$ we define $\chi_b(K) := (\varphi(K) - s(K), b)$, where $b \in \mathbb{R}^2$ is arbitrary. Given that K is a line segment or a rectangle, there exists a rigid motion mapping K to K and fixing the point $s(K)$, and thus $\varphi(K) = s(K)$, as φ is rigid motion equivariant. It is easy to see that χ_b satisfies the other properties in Lemma 3.8, therefore χ_b is Minkowski additive. Considering that b was arbitrary and s is Minkowski additive, it follows that φ is Minkowski additive as well. So by Theorem 3.7 we get $\varphi = s$.

Now let $n > 2$ and assume the statement to be correct for $2 \leq m \leq n - 1$. Further let $E_p, E_q \subseteq \mathbb{R}^n$ be two orthogonal plains containing the origin of dimensions $p \geq 1$ and $q \geq 1$ respectively. Given convex bodies $K_1 \subseteq E_p$ and $K_2 \subseteq E_q$, then there exists a unique presentation

$$\varphi(K_1 + K_2) = \varphi_1(K_1, K_2) + \varphi_2(K_1, K_2),$$

where $\varphi_1(K_1, K_2) \in E_p$ and $\varphi_2(K_1, K_2) \in E_q$. We fix a body $Q \subseteq E_q$, and the above equation defines a functional $\varphi_1(\cdot, Q)$ mapping every convex body in E_p to a single point in E_p. Given convex bodies $K_1, K_2 \subseteq E_p$ such that $K_1 \cup K_2$ is a convex body in E_p, then φ being a valuation together with the trivial relations

$$(K_1 + Q) \cup (K_2 + Q) = (K_1 \cup K_2) + Q,$$
$$(K_1 + Q) \cap (K_2 + Q) = (K_1 \cap K_2) + Q$$

results in

$$\varphi_1((K_1 \cup K_2), Q) + \varphi_1((K_1 \cap K_2), Q) = \varphi_1(K_1, Q) + \varphi_1(K_2, Q).$$

Hence φ_1 is a valuation. If $A' : E_p \to E_p$ is a rigid motion fixing the origin, then there exists a rigid motion $A : \mathbb{R}^n \to \mathbb{R}^n$ which is the identity on E_q and such that $A|_{E_p} = A'$. Considering the rigid motion equivariance of φ, we get

$$\varphi_1(A'K, Q) = A'\varphi_1(K, Q).$$

If $A : \mathbb{R}^n \to \mathbb{R}^n$ is a translation mapping E_p to itself, we again have that φ is equivariant with respect to A, and thus φ_1 is rigid motion equivariant. As φ is continuous so is φ_1, and therefore φ_1 satisfies the same requirements on E_p as φ does on $\mathbb{R}^n$. In the case that $p \geq 2$ the induction hypothesis now results in $\varphi_1(K, Q) = s(K)$.

Now let us consider the case $p = 1$. We denote points on the straight line E_p by their oriented distance from the origin and the line segment with end points a and b by $\overline{ab}$. For a convex body $K \subseteq E_q$ we abbreviate $\varphi_1(\overline{ab}, K) = f(a, b)$. The translation equivariance and additivity of $\varphi_1(\cdot, K)$ yield the equations

$$f(a + c, b + c) = f(a, b) + c, \tag{8}$$

$$f(0, a + b) + f(a, a) = f(0, a) + f(a, a + b). \tag{9}$$

From (8) we derive $f(a,a) = \gamma + a$ where $\gamma = f(0,0)$ as well as $f(a, b+a) = f(0,b) + a$. With this (9) results in $f(0, a+b) + \gamma = f(0,a) + f(0,b)$, and thus the function $f(0,\cdot) - \gamma$ satisfies Cauchy's functional equation. With the continuity of $\varphi_1(\cdot, Q)$ we get $f(0,a) = \alpha a + \gamma$ with $\alpha = f(0,1) - \gamma$ and (8) yields $f(a,b) = f(0, b-a) + a = (1-\alpha)a + \alpha b + \gamma$. Thus we have

$$\varphi_1(\overline{ab}, K) = (1 - \alpha(K))a + \alpha(K)b + \gamma(K),$$

where $\alpha(K)$ and $\gamma(K)$ depend on the choice of K. In a similar way as above it can be shown that the functional $\varphi_1(\overline{ab}, \cdot)$ on E_q is additive, rigid motion invariant and continuous for a fixed line segment $\overline{ab}$. Considering a and b can be chosen at will, α and γ must have these properties as well. Due to the first functional theorem of Hadwiger, see [29] page 211, there exist real constants c_i, d_i such that for all convex bodies $K \subseteq E_q$ the equations

$$\alpha(K) = \sum_{i=0}^{q} c_i W_i(K), \qquad \gamma(K) = \sum_{i=0}^{q} d_i W_i(K)$$

hold, where W_i denote the quermassintergrals, see [2] page 209. For $0 \leq r \leq q$ we choose an r-dimensional convex body $K \subseteq E_q$ that is symmetric with respect to an $(n-2)$-dimensional plane E_{n-2} containing the origin. The reflection across the plane $E_{n-2} + \frac{1}{2}(a+b)$ is a rigid motion mapping the convex body $\overline{ab} + K$ to itself. Considering φ is rigid motion equivariant the point $\varphi(\overline{ab} + K)$ has to lie within the plane fixed by the reflection, thus $\varphi_1(\overline{ab} + K) = \frac{1}{2}(a+b)$. As a and b can still be varied, we get that $\alpha(K) = \frac{1}{2}$ and $\gamma(K) = 0$. Now we consider that $W_i(K) = 0$ for $0 \leq i \leq q-r-1$ and $W_i(K) \neq 0$ for $q - r \leq i \leq q$. Regarding $r = 0, 1, ..., q$ one after the other we find $c_q W_q = \frac{1}{2}, c_i = 0$ for $0 \leq i \leq q-1$ and $d_i = 0$ for $0 \leq i \leq q$. Thus $\alpha(K) = \frac{1}{2}$ and $\gamma(K) = 0$ for all convex bodies $K \subseteq E_q$, and therefore $\varphi_1(\overline{ab} + K) = \frac{1}{2}(a+b) = s(\overline{ab})$.

We have shown that $\varphi_1(K, Q) = s(K)$ holds for arbitrary convex bodies $Q \subseteq E_q$ and $K \subseteq E_p$. Similarly it can be shown that $\varphi_2(P, K) = s(K)$ holds for arbitrary convex bodies $K \subseteq E_q$ and $P \subseteq E_p$ and thus

$$\varphi(P + Q) = \varphi_1(P, Q) + \varphi_2(P, Q) = s(P) + s(Q) = s(P + Q)$$

for $P \in E_p$ and $Q \in E_q$. So we have proven $\varphi(Z) = s(Z)$ for an arbitrary straight cylinder Z.

Now we set

$$\chi_b(K) = (\varphi(K) - s(K), b)$$

for an arbitrary $b \in \mathbb{R}^n$ and $K \in \mathcal{K}^n$. Let E_{n-1} be a hyperplane containing the origin, then the restriction of φ to E_{n-1} is a valuation, rigid motion equivariant and continuous and thus $\varphi(K) = s(K)$ for $K \subseteq E_{n-1}$ due to the induction hypothesis and therefore $\chi_b(K) = 0$. Considering χ_b is translation invariant we get $\chi_b(K) = 0$ for all $K \in \mathcal{K}^n$ with $\dim(K) < n$. Due to φ being a valuation and I (38) in [3], we get $\chi_b(K_1 \cup K_2) = \chi_b(K_1) + \chi_b(K_2)$ for all $K_1, K_2 \in \mathcal{K}^n$

with $K_1 \cup K_2 \in \mathcal{K}^n$ and $\dim(K_1 \cap K_2) < n$. Furthermore χ_b is continuous and $\chi_b(Z) = 0$ for all straight cylinders Z, so due to Lemma 3.8, as b can be varied, it is Minkowski additive. Thus, due to Theorem 3.7 $\varphi(K) = s(K)$ for all $K \in \mathcal{K}^n$. □

We encountered Hadwiger's characterization theorem for continuous rigid motion invariant valuations with values in $\mathbb{R}$. The group of rigid motions in $\mathbb{R}^n$ consists of translations and rotations. The group of rotations $\mathrm{SO}(n)$ is a famous and important example of a so-called compact Lie group, that is, a compact topological space endowed with a group structure, such that some extra conditions are met. In this work we will encounter a more general version of Hadwiger's theorem proven by Alesker, Bernig and Schuster [8], characterizing continuous translation invariant $\mathrm{SO}(n)$-equivariant valuations with values in an irreducible $\mathrm{SO}(n)$ representation Γ. As well as the work of Wannerer [23], who characterized the Steiner point map (using said generalized Hadwiger theorem) as the continuous unitary affine transformation equivariant valuation from the convex bodies in $\mathbb{C}^n$ to $\mathbb{C}^n$. As background we need to recall some Lie group theory, in particular compact Lie groups. Introducing the definition of a Lie group and providing the compact classical Lie groups as examples, we move on to representations, which will prove to be integral in the understanding of Lie group structure and classification. The character of a Lie group representation determines a representation up to isomorphism, and the Peter-Weyl theorem will show us that every compact Lie group is indeed a closed subgroup of a unitary group. We move on to Lie algebras the linearizations of Lie groups. Cartan subalgebras will enable us to introduce the famous root space decomposition, and highest weights will provide a classification of irreducible representations. At last we will recall the second determinantal formula which we will directly use in the proof of the general version of Hadwiger's theorem. This part is mainly taken from [4]. For background on smooth manifolds and Lie algebras see [7], for a more detailed view of the exponential map see [6]; the root space decomposition of the special orthogonal group is taken from [9], see [10] for details on examples of highest weights and [19] for infinite dimensional representations.

4 Compact Lie Groups and Representations

We recall the definition of a Lie group and a notion for maps between Lie groups and give the compact classical Lie groups as examples. The later are of particular interest because they play an integral role in the structural theory of compact Lie groups. Furthermore the concept of a representation of a Lie group will be discussed, that is, a way of almost (up to the kernel of a Lie group homomorphism) viewing a Lie group as a subgroup of the general linear group of a vector space. We will have a look at the famous lemma of Schur, which has countless uses in representation theory due to its strong statement about irreducible representations. Finally we will see that every finite dimensional representation of a compact Lie group is a direct sum of smaller building blocks, namely irreducible representations. This section is taken from [4] and the subsection on infinite dimensional representations from [19].

4.1 Lie Groups

Definition. A *topological manifold* is a second countable Hausdorff topological space, which is locally Euclidean of dimension n.

Given a topological manifold we want to add a smooth structure to it. As it is locally Euclidean, we can move to $\mathbb{R}^n$ where smoothness is a familiar concept.

Definition. Let M be a topological manifold of dimension n. A *smooth atlas* is a set of charts $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ on M, such that $M = \bigcup_{\alpha\in A} U_\alpha$, and for all $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ the *transition map* $\varphi_{\alpha,\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap V) \to \varphi_\beta(U \cap V)$ is a smooth map on $\mathbb{R}^n$. A *smooth manifold* is a topological manifold with a maximal smooth atlas.

Remark. Every (smooth) atlas can be completed to a unique maximal (smooth) atlas.

We are now able to define the central object of this chapter: the Lie group.

Definition. A Lie group G is a group and a smooth manifold such that

(i) the *multiplication map* $\mu : G \times G \to G$ mapping $(g, h) \mapsto gh$ is smooth;

(ii) the *inversion* $\iota : G \to G$ mapping $g \mapsto g^{-1}$ is smooth.

Due to the manifold structure of Lie groups, finding subgroups is a bit more complicated.

Definition. Given a Lie group G and a subgroup H of G. H is a *Lie subgroup* of G given that it is provided with a topology and smooth structure making it a Lie group and an immersed submanifold of G. So the embedding $\iota : H \to G$ is a smooth immersion, that is, a smooth map with injective differential at every point of H.

Theorem 4.1. *Given a Lie group G with a subgroup H, then H is a regular Lie subgroup (a Lie subgroup the topology of which agrees with the relative topology) if and only if H is closed.*

Theorem 4.2. *A closed subgroup of a Lie group is a Lie group in its own right with respect to the relative topology.*

As in every theory, when introducing objects with a certain structure, we also want to talk about maps between these objects that are compatible with the given structure.

Definition. A *Lie group homomorphism* is a smooth (group) homomorphism between two Lie groups. A Lie group *isomorphism* f is a bijective Lie group homomorphism between two Lie groups.

Definition. A Lie group is called a *linear Lie group*, or a *matrix Lie group* if it is isomorphic to a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Later we will see that every compact Lie group is a linear Lie group.

4.2 The Compact Classical Lie Groups

We want to give some examples for Lie groups. Especially, due to their importance in the theory of Lie groups, we want to introduce the so called compact classical Lie groups $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{SU}(n)$, and $\mathrm{Sp}(n)$.

Definition. The *special linear group* is defined by

$$\mathrm{SL}(n, \mathbb{F}) := \{A \in \mathrm{GL}(n, \mathbb{F}) : \det(A) = 1\}$$

where $\mathbb{F}$ is a field.

As the determinant is continuous, $\mathrm{SL}(n, \mathbb{F})$ is a closed subgroup of the Lie group $\mathrm{GL}(n, \mathbb{F})$ and thus a Lie group (see [4] for example).

Definition. The *orthogonal group* is the closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ defined by

$$\mathrm{O}(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) : A^T A = I_n\}$$

where A^T denotes the transpose of the square matrix A.

The column vectors of an orthogonal matrix have length 1. So topologically speaking, $\mathrm{O}(n)$ may be viewed as a closed subset of the compact set $S^{n-1} \times S^{n-1} \times \cdots \times S^{n-1} \subseteq \mathbb{R}^{n^2}$. Therefore $\mathrm{O}(n)$ is a compact Lie group.

Definition. The *special orthogonal group* is the closed subgroup of $\mathrm{O}(n)$ defined by

$$\mathrm{SO}(n) := \{A \in \mathrm{O}(n) : \det(A) = 1\}.$$

As a closed subgroup of a compact Lie group SO(n) is a compact Lie group itself. Later we will see that the behavior of SO(n) depends heavily on the parity of n, and thus SO($2n$) and SO($2n+1$) are considered as two separate infinite families of Lie groups.

Definition. The *unitary group* is the closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ defined by

$$\mathrm{U}(n) := \{A \in \mathrm{GL}(n, \mathbb{C}) : A^*A = I_n\}$$

with A^* denoting the complex conjugate transpose of A.

For a unitary matrix each column vector has length 1. So U(n), topologically, is a closed subset of $S^{2n-1} \times S^{2n-1} \times \cdots \times S^{2n-1} \subseteq \mathbb{R}^{4n^2}$. As before it follows that U(n) is a compact Lie group.

Definition. The *special unitary group* is the closed subgroup of U(n) defined by

$$\mathrm{SU}(n) := \{A \in \mathrm{U}(n) : \det(A) = 1\}.$$

As a closed subgroup of a compact Lie group SU(n) is a compact Lie group itself.

Definition. The *symplectic group* is the subgroup of $\mathrm{GL}(n, \mathbb{H})$ defined by

$$\mathrm{Sp}(n) := \{A \in \mathrm{GL}(n, \mathbb{H}) : A^*A = I\}$$

with A^* denoting the quaternionic conjugate transpose of A.

Sp(n) is a compact Lie group (see [4] for example).

4.3 Representations

Representations are of utmost importance in the theory of Lie groups, as they provide a way to look at a general linear group from the point of view of a Lie group. There are two ways to introduce a representation of a Lie group, and considering both have their specific merit, we will discuss them both. The first way uses the concept of a Lie group homomorphism.

Definition. A (finite-dimensional) *representation* of a Lie group G on a finite dimensional complex vector space V is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$. The dimension of the representation is the dimension of V.

The second way to introduce Lie group representations, which is equivalent to the first one, uses group actions.

Definition. A (finite-dimensional) *representation* of a Lie group G on a finite dimensional complex vector space V is a map $\rho : G \times V \to V$ with the following properties:

$$(i)\ \rho(g) : V \to V,\ v \mapsto \rho(g, v) \text{ is linear;}$$
$$(ii)\ \rho(e, g) = g;$$
$$(iii)\ \rho(g_1, \rho(g_2, v)) = \rho(g_1 g_2, v),$$

where $g, g_1, g_2 \in G$, $v \in V$ and e is the identity element in G.

The fact that (ρ, V) is a representation might also be expressed by saying that V is a *G-module* or that *G acts on V*. Instead of $\rho(g, v)$ it is common to write $g \cdot v$ or gv if the representation is clear from the context.

Definition. Let (ρ, V) and (ρ', V') be finite dimensional G-modules, then $A \in \mathrm{Hom}(V, V')$ (the space of linear maps $V \to V'$) is called an *intertwining operator* or *G-map* if it is *equivariant*, i.e., it satisfies $A \circ \rho(g) = \rho'(g) \circ A$ for $g \in G$. The set of all G-maps is denoted by $\mathrm{Hom}_G(V, V')$. Two representations V and V' are *equivalent*, we write $V \cong V'$, if there exists a bijective G-map from V to V'. An element v of V is called *G-invariant* if $g \cdot v = v$ for all $g \in G$. The set of G-invariant elements of V is denoted by V^G.

From given representations and their vector spaces, one can look at natural ways to construct "new" vector spaces from given ones, i.e., the sum, tensor product, hom-spaces, and so on and find representations on these "new" vector spaces as well.

Theorem 4.3. *Let V and W be finite dimensional representations of a Lie group G then*

(i) *G acts on $V \oplus W$ by $g(v, w) = (gv, gw)$;*

(ii) *G acts on $V \otimes W$ by $g \sum v_i \otimes w_j = \sum gv_i \otimes gw_j$;*

(iii) *G acts on $\mathrm{Hom}(V, W)$ by $(gA)(v) = g(A(g^{-1}v))$;*

(iv) *G acts on $\bigotimes^k V$ by $g \sum v_{i_1} \otimes \cdots \otimes v_{i_k} = \sum gv_{i_1} \otimes \cdots \otimes gv_{i_k}$;*

(v) *G acts on $\bigwedge^k V$ by $g \sum v_{i_1} \wedge \cdots \wedge v_{i_k} = \sum gv_{i_1} \wedge \cdots \wedge gv_{i_k}$;*

(vi) *G acts on $S^k(V)$ by $g \sum v_{i_1} \cdots v_{i_k} = \sum (gv_{i_1}) \cdots (gv_{i_k})$;*

(vii) *G acts on V^* by $(gA)(v) = A(g^{-1}v)$;*

(iv) *G acts on $\overline{V}$ by the same action as it does on V.*

Here $\bigwedge^k$ denotes the k-fold *exterior product*, S^k the k-fold *symmetric product* and $\overline{V}$ the *conjugate* space where the scalar multiplication is given by multiplying with the conjugate.

We have seen that representations can be glued together to obtain representations on "larger" spaces. In order to build a classification of representations it makes sense to ask two questions: What are the smallest building blocks? Can every representation be built from these smallest building blocks? The first question leads directly to the next definition.

Definition. Let G be a Lie group acting on the finite dimensional complex vector space V. A subspace $U \subseteq V$ is *G-invariant* (we also say a *submodule* or a *subrepresentation*) if $gU \subseteq U$ for all $g \in G$. A nonzero (that is, not equal to the trivial representation $\{0\}$) representation is *irreducible* if the only G-invariant subspaces are trivial, i.e., $\{0\}$ or V. A representation is called *reducible* if it has a proper (non-trivial) G-invariant subspace.

Remark. A nonzero finite dimensional representation V is irreducible if and only if $V = \text{span}_{\mathbb{C}}\{gv : g \in G\}$ for each nonzero $v \in V$.

The next lemma has a variety of applications in the structural theory of representations.

Theorem 4.4. *(Schur's lemma) Let G be a Lie group and let V and W be finite dimensional G-modules. If V and W are irreducible, then*

$$\dim \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \ncong W. \end{cases}$$

Definition. Let G be a Lie group and V be a finite dimensional G-module. A bilinear form $(\cdot, \cdot) : V \times V \to \mathbb{C}$ is called *G-invariant* if $(gv, gw) = (v, w)$ for all $g \in G$ and for all $v, w \in V$. V is called *unitary* given that there exists a G-invariant Hermitian inner product on V.

Theorem 4.5. *Every representation of a compact Lie group is unitary.*

Another way to describe $\text{U}(n)$ is as $\{A \in \text{GL}(n, \mathbb{C}) : (Av, Aw) = (v, w)\}$, where $(\cdot, \cdot)$ denotes the standard Hermitian inner product on $\mathbb{C}^n$. This is equivalent to the way we introduced $\text{U}(n)$ considering $(Av, Aw) = (Av)^* I Aw = v^* A^* Aw$. As a consequence of the preceding theorem, we get that a finite dimensional representation $(\rho : G \to \text{GL}(V), V)$ of a compact Lie group G yields a Lie group homomorphism into a unitary group. We will see later that every compact Lie group is isomorphic to a subgroup of a unitary group. So every compact Lie group is a linear Lie group.

Definition. A finite dimensional representation of a Lie group is called *completely reducible* if it is a direct sum of irreducible submodules.

Now we can get back to the question we asked above: Can every representation be built from irreducible representations? For non-compact groups there are reducible but not completely reducible representations, for compact groups however we have the following result.

Corollary 4.6. *Every finite dimensional representation of a compact Lie group is completely reducible.*

We are now able to write every finite dimensional representation V of a compact Lie group G as $V \cong \bigoplus_{i=1}^n n_i V_i$ where the V_i are not equivalent irreducible G-modules and $n_i V_i = V_i \oplus \cdots \oplus V_i$ (n_i copies). The natural number n_i is called the *multiplicity* of V_i in V and is given by $n_i = \dim \text{Hom}_G(V_i, V)$. Given an irreducible G-module V and some arbitrary G-module W we denote the multiplicity of V in W by $m(V, W)$.

As a consequence of Schur's lemma we get:

Corollary 4.7. *Let V be a finite dimensional representation of a Lie group G then*

(i) *V is irreducible if and only if* $\dim \operatorname{Hom}_G(V, V) = 1$; *in particular for an irreducible V every G-map $f : V \to V$ is of the form $f = \lambda \operatorname{Id}$ for $\lambda \in \mathbb{C}$;*

(ii) *If V is irreducible, then the G-invariant inner product on V is unique up to multiplication by a positive real number.*

4.4 Characters and the Peter-Weyl Theorem

We consider the character of a Lie group representation, an important concept, as it determines a representation up to isomorphism. The Peter-Weyl theorem will provide the means to see that every compact Lie group is a closed subgroup of a unitary group.

Definition. Let G be a Lie group with finite dimensional unitary representation (ρ, V). The functions $f_{u,v}^V : G \to \mathbb{C}$ mapping $g \mapsto (gu, v)$ with $u, v \in V$, where $(\cdot,\cdot)$ denotes the G-invariant Hermitian inner product on V, are called *matrix coefficients* of G. The collection of all matrix coefficients is denoted by $\mathrm{MC}(G)$.

Definition. Let G be a Lie group with finite dimensional representation (ρ, V). The *character* $\chi_V : G \to \mathbb{C}$ of G is defined by $\chi_V(g) = \operatorname{tr}\, \rho(g)$ where $\operatorname{tr}\, \rho(g)$ denotes the trace of the matrix $\rho(g)$.

Theorem 4.8. *Let G be a compact Lie group and V_i and V be finite dimensional G-modules, then*

$$
\begin{aligned}
(i)\ & \chi_V \in \mathrm{MC}(G);\\
(ii)\ & \chi_V(e) = \dim V;\\
(iii)\ & \text{If } V_1 \cong V_2, \text{ then } \chi_{V_1} = \chi_{V_2};\\
(iv)\ & \chi_V(hgh^{-1}) = \chi_V(g) \text{ for } g, h \in G;\\
(v)\ & \chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2};\\
(vi)\ & \chi_{V_1 \otimes V_2} = \chi_{V_1}\chi_{V_2};\\
(vii)\ & \chi_{V^*}(g) = \chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1});\\
(viii)\ & \chi_{\mathbb{C}}(g) = 1 \text{ for the trivial representation } \mathbb{C}.
\end{aligned}
$$

Theorem 4.9. *1) Let V, V_i, W be finite dimensional representations of a compact Lie group G, then*

$$\int_G \chi_V(g)\overline{\chi_W(g)}\, dg = \dim \operatorname{Hom}_G(V, W).$$

In particular, $\int_G \chi_V(g)\, dg = \dim(V^G)$, where V^G denotes the set $\{v \in V : gv = v \text{ for } g \in G\}$, and if V and W are irreducible, then

$$\int_G \chi_V(g)\overline{\chi_W(g)}\, dg = \begin{cases} 0 & \text{if } V \ncong W \\ 1 & \text{if } V \cong W. \end{cases}$$

2) V is, up to equivalence, completely determined by its character, that is $V \cong W$ if and only if $\chi_V = \chi_W$. In particular, $V \cong \bigoplus_i n_i V_i$ if and only if $\chi_V = \sum_i n_i \chi_{V_i}$.
3) V is irreducible if and only if $\int_G |\chi_V(g)|^2 \, dg = 1$.

By $\mathrm{C}(G)$ we denote the set of continuous functions $f : G \to \mathbb{C}$ and by $\mathrm{L}^2(G)$ the set of square integrable functions $f : G \to \mathbb{C}$.

Theorem 4.10. *(Peter-Weyl theorem) Let G be a compact Lie group, then $\mathrm{MC}(G)$ is dense in $\mathrm{C}(G)$ and $\mathrm{L}^2(G)$.*

We say a representation (ρ, V) is *faithful given that ρ is injective.*

Theorem 4.11. *A compact Lie group possesses a finite dimensional faithful representation.*

As we have already hinted at in the last section, we obtain the following corollary.

Corollary 4.12. *Every compact Lie group is isomorphic to a closed subgroup of a unitary group $\mathrm{U}(n)$ for some $n \in \mathbb{N}$. Thus every compact Lie group is a linear Lie group.*

4.5 Infinite-dimensional Representations

We want to consider representations of compact Lie groups on infinite-dimensional vector spaces as well, because not every representation of interest is finite. There are some adaptions to be made to the infinite-dimensional case; adding some additional structure to our vector spaces, we work with topological vector spaces instead. We restrict ourselves to Banach spaces, but note that it is possible to develop infinite-dimensional representations for locally convex vector spaces. Most of the basic definitions for representations remain unchanged, however some need to be tweaked. In the end we see that every irreducible representation of a compact Lie group on a Banach space is finite-dimensional. As a reference for this subsection see [19] as well.

Given topological vector spaces V, V', let $\mathrm{Hom}(V, V')$ denote continuous linear maps from V to V' and let $\mathrm{GL}(V)$ denote the invertible elements of $\mathrm{Hom}(V, V)$.

Definition. A *representation* of a Lie group G on a topological vector space V is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$ such that the map $G \times V \to V$ defined by $(g, v) \mapsto \rho(g)v$ is continuous. Given two infinite-dimensional representations $(V, \rho), (V', \rho')$ on a topological vector space $f \in \mathrm{Hom}(V, V')$ is called an *intertwining* operator or *G-map* if it is *equivariant*, i.e., it satisfies $f \circ \rho(g) = \rho'(g) \circ f$ for $g \in G$. A closed G-invariant subspace is called a *submodule*. A representation of a Lie group G on a topological vector space V is called *irreducible* if the only submodules are $\{0\}$ and V. It is called reducible given that it has a non-trivial submodule.

Next up are some specific definitions needed to acquire a decomposition for infinite-dimensional representations.

Definition. Let V be an infinite-dimensional representation of a Lie group on a complex Banach space. A vector $v \in V$ is called *G-finite* if it is contained in a finite-dimensional G-invariant subspace of V. The linear subspace of V spanned by all G-finite vectors is denoted by V^f. For an infinite-dimensional complex irreducible representation W of G, we define the *W-isotopic component* $V_W \subseteq V$ as the space of all $v \in V$ for which there is a G-map $\varphi : W \to V$ with $v \in \operatorname{im}(\varphi)$.

Note that V_W is a linear subspace of V. Given $v \in V_W$ the image of the corresponding G-map φ is a finite-dimensional G-invariant subspace of V, so $V_W \subseteq V$. Now let us consider a compact Lie group G and a vector $v \in V^f$, so there exists a finite-dimensional G-invariant subspace $W \subseteq V$ containing v. We already know that finite-dimensional G-modules decompose into irreducible submodules, so $W \cong \bigoplus_{i=1}^n n_i W_i$ with irreducible W_i. Thus v can be written as a linear combination of vectors each contained in some W_i-isotopic component, and $V^f = \bigoplus_{U \in \hat{G}} V_U$ where $\hat{G}$ denotes the set of all isomorphism classes of finite-dimensional irreducible G-modules.

Theorem 4.13. *For any representation of a compact Lie group G on a complex Banach space V, the subspace V^f is dense in V.*

As a direct consequence we get:

Corollary 4.14. *Given a compact Lie group G, then any irreducible representation on a Banach space is finite-dimensional.*

5 Lie Algebras

Although in general not being linear objects, it is possible to approximate Lie groups by their tangent spaces at the identity. Such a tangent space is linear and we call it a Lie Algebra. As we did before with representations, we try to gain more insight into the theory of Lie groups by looking at simpler objects, in this case vector spaces, that are still closely related to Lie groups. After some basic definitions we will introduce the exponential map, a smooth map from the Lie algebra of a Lie group to the Lie group itself and compute the Lie algebras of the compact classical Lie groups. Besides [4] some results were taken from [6]. For more background on manifolds see [7] for example.

5.1 Basic Definitions

We introduce the very basic concepts: a Lie algebra, Lie subalgebras and Lie algebra homomorphisms.

Definition. A *Lie algebra* $\mathfrak{g}$ is a vector space over a field K with a bilinear product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket,* if the lie bracket additionally satisfies

$$(i)\ [X,Y] = -[Y,X];$$
$$(ii)\ [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

The second equation is called the *Jacobi identity.*

As with any new structures we want to define maps that preserve the given structure and its substructures.

Definition. A *Lie algebra homomorphism* is a linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ such that

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$$

for all $X, Y \in \mathfrak{g}$.

Definition. A *Lie subalgebra* $\mathfrak{h}$ of a given Lie algebra $\mathfrak{g}$ is a subspace that is closed under the Lie bracket. An ideal $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$. A Lie algebra $\mathfrak{g}$ is called *Abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

Remember that given a smooth manifold M the *tangent bundle* TM is the disjoint union of all tangent spaces to points in M. That is, $TM := \bigsqcup_{p\in M} T_pM = \bigcup_{p\in M}\{(p,v) : v \in T_pM\}$, and the projection π from TM onto M is given by $\pi(p,v) = p$. A *vector field* $X : M \to TM$ is a smooth section of the tangent bundle, i.e., a smooth right inverse to the projection π. Given a vector field X, we write the value of X at p as X_p. For a Lie group G the diffeomorphism $l_g : G \to G$ is defined by $l_g(h) = gh$. A vector field X is called left invariant given that $dl_gX = X$ or $d(l_g)_hX_h = X_{gh}$ at a point $h \in G$ where $d(l_g) : T_pM \to T_{gp}M$ denotes the differential of l_g. It can be shown (see [7] for example) that the set of left invariant vector fields together with the Lie bracket for vector fields is a Lie algebra.

Definition. The Lie algebra of smooth left invariant vector fields of a Lie group G is called the *Lie algebra of* G and is denoted by $\mathfrak{g}$.

For a Lie group G and $v \in T_eG$ we get a vector field that is left invariant by setting $v_g^l := d(l_g)_e v \in T_gG$. A fundamental fact is that this map gives an isomorphism between $\mathfrak{g}$ and T_eG.

Theorem 5.1. *Let G be a Lie group. The map $\varepsilon : \mathfrak{g} \to T_eG$ defined by $\varepsilon(X) = X_e$ is a vector space isomorphism with inverse $\tau : T_eG \to \mathfrak{g}$ defined by $\tau(v) = v^l$. Thus, $\mathfrak{g}$ is finite-dimensional with dimension equal to* $\dim G$.

For a compact Lie group G we already know that G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, so T_eG can be viewed as a subspace of $T_{I_n}(\mathrm{GL}(n, \mathbb{C}))$. We define $\mathfrak{gl}(n, \mathbb{F}) := M_{n,n}(\mathbb{F})$, the space of $n \times n$ matrices over the field $\mathbb{F}$. Remember that for a smooth manifold M tangent vectors in T_pM can also be understood as equivalence classes of curves γ in M such that $\gamma(0) = p$.

Theorem 5.2. *Let G be a Lie subgroup of $GL(n, \mathbb{C})$, then*

$$\mathfrak{g} \cong \{\gamma'(0) : \gamma(0) = I_n \textit{ and } \gamma : (-\varepsilon, \varepsilon) \to G, \varepsilon > 0, \textit{ is smooth}\} \subseteq \mathfrak{gl}(n, \mathbb{C})$$

where the Lie bracket is given by

$$[X, Y] = XY - YX.$$

For $X_i = \gamma_i'(0) \in \mathfrak{g}$ and $r \in \mathbb{R}$ consider the smooth curve $\gamma = \gamma_1(rt)\gamma_2(t)$ mapping some neighborhood of $0 \in \mathbb{R}$ to G. Then $\gamma'(0) = (r\gamma_1'(rt)\gamma_2(t) + \gamma_1(rt)\gamma_2'(t))|_{t=0} = rX_1 + X_2$. So $\mathfrak{g}$ is a real vector space, but not a complex one.

Remark. In the same way T_pG can be identified with $\{\gamma'(0) : \gamma(0) = g$ and $\gamma : (-\varepsilon, \varepsilon) \to G, \varepsilon > 0,$ is smooth$\} \subseteq \mathfrak{gl}(n, \mathbb{C})$.

5.2 The Exponential Map

The exponential map is a smooth map from the Lie algebra of a Lie group to the Lie group itself with interesting properties; it commutes with the differential of a Lie group homomorphism and provides a way to compute the Lie algebra of a Lie group.

Remember that for $v \in T_eG$ we get a smooth vector field v^l and the *integral curve* $\gamma : J \to G, J \subseteq \mathbb{R}$ of v^l through e, that is, the unique maximally defined smooth curve in G satisfying $\gamma(0) = e$ and $\gamma'(t) = v^l_{\gamma(t)}$.

Theorem 5.3. *Let G be a linear Lie group, $X \in \mathfrak{g}$,and γ be the integral curve of X through e. Then*

$$\gamma(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n.$$

Moreover γ is defined for all $t \in \mathbb{R}$ and a homomorphism.

Definition. Let G be a linear Lie group, $X \in \mathfrak{g}$ and γ be the integral curve of X through e. The *exponential map of* G $\exp : \mathfrak{g} \to G$ is defined by $\exp(X) = \gamma(1)$.

Theorem 5.4. *Let G be a linear Lie group, then:*

- (i) *The exponential map is a smooth map from $\mathfrak{g}$ to G;*
- (ii) *The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to some neighborhood of e in G;*
- (iii) $\mathfrak{g} = \{X \in \mathfrak{gl}(n, \mathbb{C}) : e^{tX} \in G \text{ for } t \in \mathbb{R}\}$;
- (iv) *When G is connected,* $\exp \mathfrak{g}$ *generates G.*

Definition. Let $\varphi : H \to G$ be a Lie group homomorphism between linear Lie groups H and G. The *differential of φ, $d\varphi : \mathfrak{h} \to \mathfrak{g}$* is defined by

$$d\varphi(X) = \frac{d}{dt}\varphi(e^{tX})|_{t=0}.$$

Theorem 5.5. *Suppose $\varphi, \varphi_i : H \to G$ are Lie group homomorphisms between linear Lie groups H and G, then:*

- (i) *The following diagram is commutative*

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{d\varphi} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ H & \xrightarrow{\varphi} & G \end{array}$$

  that is, $e^{d\varphi(X)} = \varphi(e^X)$ for $X \in \mathfrak{g}$;
- (ii) *The differential $d\varphi$ is a Lie algebra homomorphism;*
- (iii) *If H is connected and $d\varphi_1 = d\varphi_2$, then $\varphi_1 = \varphi_2$.*

Definition. Let G be a linear Lie group. For $g \in G$ the *conjugation* $c_g : G \to G$ is the Lie group homomorphism given by $c_g(h) = ghg^{-1}$. The *adjoint representation* of G on $\mathfrak{g}$, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, is defined by $\mathrm{Ad}(g) = dc_g$. The *adjoint representation* of $\mathfrak{g}$ on $\mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, is defined by $\mathrm{ad}(g) = d\mathrm{Ad}$, that is, $\mathrm{ad}(X)Y = \frac{d}{dt}(\mathrm{Ad}(e^{tX})Y)|_{t=0}$ for $X, Y \in \mathfrak{g}$.

Remark. Note that

$$\mathrm{Ad}(g)X = dc_g(X) = \frac{d}{dt}(ge^{tX}g^{-1})|_{t=0} = gXg^{-1},$$

and

$$\mathrm{ad}(X)Y = \frac{d}{dt}(\mathrm{Ad}(e^{tX})Y)|_{t=0} = \frac{d}{dt}(e^{tX}Ye^{-tX})|_{t=0} = XY - YX = [X, Y].$$

We calculate the Lie algebra of $\mathrm{U}(n)$ using Theorem 5.4. Let X be an element of the Lie algebra of $\mathrm{U}(n)$, then $I_n = e^{tX}(e^{tX})^* = e^{tX}e^{tX^*}$ holds for

$t \in \mathbb{R}$. Differentiating with respect to t at 0, implies $X + X^* = 0$. Conversely, $X = -X^*$ implies $e^{tX}e^{tX^*} = e^{tX}e^{-tX} = I_n$. So the Lie algebra of $\mathrm{U}(n)$, denoted by $\mathfrak{u}(n)$, turns out to be

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}(\mathfrak{n}, \mathbb{C}) : X^* = -X\}.$$

For the special unitary group $\mathrm{SU}(n)$ we get an additional condition: $1 = \det(e^{tX}) = e^{t\mathrm{tr}X}$ for $t \in \mathbb{R}$, implying, after differentiation, that $\mathrm{tr}X = 0$. Considering $\mathrm{tr}X = 0$ implies $\det(e^{tX}) = 1$, we get that

$$\mathfrak{su}(n) = \{X \in \mathfrak{gl}(\mathfrak{n}, \mathbb{C}) : X^* = -X, \mathrm{tr}X = 0\}.$$

In a similar way the Lie algebras of $\mathrm{O}(n)$, $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$ can be calculated to be

$$\begin{aligned}
\mathfrak{o}(n) &= \{X \in \mathfrak{gl}(\mathfrak{n}, \mathbb{R}) : X^t = -X\};\\
\mathfrak{so}(n) &= \{X \in \mathfrak{gl}(\mathfrak{n}, \mathbb{R}) : X^t = -X, \mathrm{tr}X = 0\} = \mathfrak{o}(n);\\
\mathfrak{sp}(n) &= \{X \in \mathfrak{gl}(\mathfrak{n}, \mathbb{H}) : X^* = -X\}.
\end{aligned}$$

5.3 Abelian Lie Subgroups and Lie Algebra Structure

The idea of a Cartan subalgebra is introduced, which we will later use to decompose a Lie algebra into root spaces. We have seen that every compact Lie group G is isomorphic to a Lie subgroup of $\mathrm{U}(n)$. Thus every element in G can be diagonalized via conjugation in $\mathrm{U}(n)$. However, it is even possible to diagonalize every $g \in G$ using conjugation in G. Furthermore we will see that every Lie algebra of a compact Lie group can be decomposed into a semisimple and an Abelian part.

Definition. Let G be a compact Lie group. A *maximal torus* of G is a maximal connected Abelian Lie subgroup of G. A *Cartan subalgebra of* $\mathfrak{g}$ is a maximal Abelian Lie subalgebra of $\mathfrak{g}$.

Theorem 5.6. *Given a compact Lie group G and a connected Lie subgroup T of G, then T is a maximal torus if and only if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. In particular, maximal tori and Cartan subalgebras exist.*

Theorem 5.7. *(Maximal torus theorem) For a compact connected Lie group G with maximal torus T and $h \in G$,*

(i) *there exists $g \in G$ such that $ghg^{-1} \in T$;*

(ii) *the exponential map is surjective, i.e., $G = \exp\mathfrak{g}$.*

Example. We want to examine the Lie group $\mathrm{SO}(n)$ which behaves differently in odd and even dimensions. For $\mathrm{SO}(2l)$ a maximal torus is given by

$$T = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -\sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_l & \sin\theta_l \\ & & & -\sin\theta_l & \cos\theta_l \end{pmatrix} : \theta_i \in \mathbb{R} \right\},$$

and the corresponding Cartan subalgebra is given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & & 0 & \theta_l \\ & & & & -\theta_l & 0 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}.$$

For odd dimension we get a maximal torus of $\mathrm{SO}(2l+1)$ by

$$T = \left\{ \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\theta_l & \sin\theta_l & \\ & & & -\sin\theta_l & \cos\theta_l & \\ & & & & & 1 \end{pmatrix} : \theta_i \in \mathbb{R} \right\},$$

and the corresponding Cartan subalgebra is given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta_1 & & & & \\ -\theta_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \theta_l & \\ & & & -\theta_l & 0 & \\ & & & & & 0 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}.$$

Definition. Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group. Then $\mathfrak{g}$ is called *simple* if it has no proper ideals and $\dim \mathfrak{g} > 1$, that is, $\mathfrak{g}$ has no ideals besides $\{0\}$ and $\mathfrak{g}$ and it is not Abelian. $\mathfrak{g}$ is called *semisimple* if it is a direct sum of simple Lie algebras. $\mathfrak{g}$ is called *reductive* if it is the direct sum of a semisimple Lie algebra and an Abelian Lie Algebra. By $\mathfrak{g}'$ we denote the ideal of $\mathfrak{g}$ spanned by $[\mathfrak{g}, \mathfrak{g}]$ and by $\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} : [X, \mathfrak{g}] = 0\}$ the center of $\mathfrak{g}$.

Theorem 5.8. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group, then*

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g}),$$

so $\mathfrak{g}$ is reductive, considering $\mathfrak{g}'$ is semisimple and $\mathfrak{z}(\mathfrak{g})$ is Abelian.

6 Roots, Highest Weights and Highest Weight Classification

After introducing the concept of a representation of a Lie algebra, we define weight spaces as a way to decompose a given Lie algebra representation. The well known root space decomposition is a weight space decomposition with respect to the adjoint representation, and it provides tremendous insight in the structural theory of Lie algebras. The Killing form, a symmetric complex bilinear form on the complexification of a Lie algebra, provides, inter alia, an identification of roots with elements of the Cartan subalgebra. We see that the root space decomposition decomposes a representing vector space into triples of one dimensional subspaces isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. And the lattice of analytically integral weights will play an important role in the classification of highest weights of irreducible representations. This part was taken from [4] and the part on the second determinantal formula from [8].

6.1 Representations and Complexification of Lie Algebras

Similarly to Lie groups we want to define a morphism from a Lie Algebra to a vector space of endomorphisms. This will turn out to be a useful tool in the study of Lie algebras.

Definition. Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group, V a finite-dimensional complex vector space and $\rho : \mathfrak{g} \to \mathrm{End}(V)$ linear. The pair (ρ, V) is a *representation* of $\mathfrak{g}$ if ρ is compatible with the Lie bracket, that is, $\rho([X,Y]) = \rho(X) \circ \rho(Y) - \rho(Y) \circ \rho(X)$ for $X, Y \in \mathfrak{g}$. A representation is called *irreducible* if there are no proper $\rho(\mathfrak{g})$-invariant subspaces. Otherwise it is said to be *reducible*.

Remark. Again, depending on the context, we simply write V or ρ for a representation and $X \cdot v$ or Xv instead of $\rho(X)(v)$ for $v \in V$. For a finite dimensional vector space V of dimension n, we can view a representation as a map $\rho : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$.

Definition. Let (ρ, V) and (ρ', V') be finite-dimensional representations of a Lie algebra $\mathfrak{g}$, then $\phi \in \mathrm{Hom}(V, V')$ is called an *intertwining operator* if it is *equivariant*, i.e., it satisfies $\phi \circ \rho(X) = \rho'(X) \circ \phi$ for $X \in \mathfrak{g}$. Two representations V and V' are *equivalent*, we write $V \cong V'$, if there exists a bijective intertwining operator from V to V'.

In the next theorem we see that Lie group and Lie algebra representations are strongly interconnected, and this connection is compatible with the exponential map.

Theorem 6.1. *Let G be a linear Lie group with a finite-dimensional representation (ρ, V), then $(d\rho, V)$ is a representation of $\mathfrak{g}$ such that $e^{d\rho X} = \rho(e^X)$. If G is connected, ρ is completely determined by $d\rho$, and a subspace $W \subseteq V$ is $\rho(G)$-invariant if and only if it is $d\rho(\mathfrak{g})$-invariant. In particular, for connected G, V*

is irreducible under G if and only if it is irreducible under $\mathfrak{g}$. For a connected compact G, V is irreducible if the only endomorphisms of V commuting with all the operators $d\rho(\mathfrak{g})$ are scalar multiples of the identity map.

In the previous chapter we saw that $\mathfrak{g}$ is a real vector space. However, our representations map into endomorphism spaces of complex vector spaces. So we need to expand $\mathfrak{g}$ to become a complex vector space in order to get the vector space of a representation.

Definition. For a real vector space V the *complexification* of V, $V_\mathbb{C}$, is defined by $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ with the scalar multiplication $z'(v \otimes z) := v \otimes (z'z)$.

The complexification of a real vector space is a complex vector space, and it is isomorphic to $V \oplus iV$. Given a Lie algebra $\mathfrak{g}$ we can extend the Lie bracket to $\mathfrak{g}_\mathbb{C}$ by $\mathbb{C}$-linearity and get a Lie algebra $\mathfrak{g}_\mathbb{C}$. A representation (ρ, V) of $\mathfrak{g}$ can be extended, again by $\mathbb{C}$-linearity, to a representation of $\mathfrak{g}_\mathbb{C}$. So given a Lie algebra $\mathfrak{g}$ of a compact Lie group, we can identify $\mathfrak{g}_\mathbb{C}$ with $\mathfrak{g} \oplus i\mathfrak{g}$ where the Lie bracket is the $\mathbb{C}$-linear extension of the Lie bracket in $\mathfrak{gl}(n, \mathbb{C})$.

Example. The complexification of $\mathfrak{so}(n)$ can be realized by $\mathfrak{so}(n)_\mathbb{C} = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X^t = -X\}$ and will be denoted by $\mathfrak{so}(n, \mathbb{C})$.

Lemma 6.2. *Let $\mathfrak{g}$ be the Lie algebra of a linear Lie algebra, then a representation V of $\mathfrak{g}$ is irreducible if and only if it is an irreducible representation of $\mathfrak{g}_\mathbb{C}$.*

6.2 Weights

The weights of a representation provide a decomposition of a representing vector space indexed by elements of the dual space of a Cartan subalgebra.

Let (ρ, V) be a finite-dimensional representation of a compact Lie group G. Every representation of a compact Lie group is unitary, so there exists a G-invariant inner product $(\cdot, \cdot)$ on V. Because $\frac{d}{dt}|_{t=0}$ applied to $(\rho(e^{tX})Y_1, \rho(e^{tX})Y_2) = (Y_1, Y_2)$ results in $(d\rho(X)Y_1, Y_2) + (Y_1, d\rho(X)Y_2) = 0$, we see that $d\rho$ is skew-Hermitian on $\mathfrak{g}$ and Hermitian on $i\mathfrak{g}$. Now let us consider a Cartan subalgebra $\mathfrak{t}$ and its complexification $\mathfrak{t}_\mathbb{C}$. Then $\mathfrak{t}_\mathbb{C}$ acts as a family of commuting normal operators on V. Due to the spectral theorem each of these operators $d\rho(H), H \in \mathfrak{t}_\mathbb{C}$ is diagonalizable, and as they all commute they are simultaneously diagonalizable. So the following is well defined.

Definition. Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G with finite-dimensional representation (ρ, V) and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. There is a finite set $\Delta(V) = \Delta(V, \mathfrak{t}_\mathbb{C}) \subseteq \mathfrak{t}_\mathbb{C}^*$ called the *weights* of V such that

$$V = \bigoplus_{\alpha \in \Delta(V)} V_\alpha,$$

where

$$V_\alpha = \{v \in V : d\rho(H)v = \alpha(H)v, H \in \mathfrak{t}_\mathbb{C}\}$$

is nonzero. This decomposition of V is called the *weight space decomposition* of V with respect to $\mathfrak{t}_{\mathbb{C}}$.

Let (ρ, V) and (ρ', V') be two equivalent finite-dimensional Lie algebra representations of a Lie algebra $\mathfrak{g}$ with $\phi : V \to V'$ being the corresponding bijective intertwining operator and $v \in V_\alpha$ for some $\alpha \in \Delta(V)$. Then $\rho'(H)\phi(v) = \phi(\rho(H)(v)) = \phi(\alpha(H)v) = \alpha(H)\phi(v)$, so the weights of (ρ, V) are weights of (ρ', V') as well. The same argument can be made for ϕ^{-1} and therefore equivalent representations of a Lie algebra have the same weights. Given a Lie algebra isomorphism $\phi : \mathfrak{a} \to \mathfrak{b}$ and a finite-dimensional representation (ρ, V) of $\mathfrak{a}$, then $\rho \circ \phi^{-1}$ is a representation of $\mathfrak{b}$. So isomorphic Lie algebras have the same finite-dimensional representing vector spaces.

Example. The trivial representation of a Lie group G is given by $\rho : G \to \mathrm{GL}(1, \mathbb{C}), \rho(g) = 1$. The corresponding Lie algebra representation $d\rho(H) = \frac{d}{dt}\rho(e^{Ht})|_{t=0} = 0$ is the zero map on $\mathbb{C}$. So we get $\mathbb{C}$ as the weight space of the trivial map $0 \in \mathfrak{t}_{\mathbb{C}}^*$.

Example. We have found a Cartan subalgebra for $\mathfrak{so}(n)_{\mathbb{C}}$, however, in the realization we utilized, there is no Cartan subalgebra consisting of diagonal matrices. Considering it will be easier to work with diagonal matrices, we introduce an isomorphic realization of $\mathrm{SO}(n)$ that gives rise to a Cartan subalgebra consisting of diagonal matrices. Define

$$T_{2l} := \frac{1}{\sqrt{2}} \begin{pmatrix} I_l & I_l \\ iI_l & -iI_l \end{pmatrix} \text{ and } E_{2l} := \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$$

for even dimensions $n = 2l$ and

$$T_{2l+1} := \begin{pmatrix} T_{2l} & 0 \\ 0 & 1 \end{pmatrix} \text{ and } E_{2l+1} := \begin{pmatrix} E_{2l} & 0 \\ 0 & 1 \end{pmatrix}$$

for odd dimensions $n = 2l+1$. Then our new realization of the special orthogonal group is defined by

$$\mathrm{SO}(E_n) = \{g \in \mathrm{SL}(n, \mathbb{C}) : \overline{g} = E_n g E_n, g^t E_n g = E_n\}$$

with the corresponding Lie algebra

$$\mathfrak{so}(E_n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : \overline{X} = E_n X E_n, X^t E_n + E_n X = 0\}$$

and complexification

$$\mathfrak{so}(E_n)_{\mathbb{C}} = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X^t E_n + E_n X = 0\}.$$

Note that $E_n = T_n^t T_n$ and $\overline{T_n}^t = T_n^{-1}$. It can be shown that $\mathrm{SO}(E_n)$ is a compact subgroup of $\mathrm{SU}(n)$, the map $g \to T_n^{-1} g T_n$ is a Lie group isomorphism from $\mathrm{SO}(n)$ to $\mathrm{SO}(E_n)$, and the map $X \to T_n^{-1} X T_n$ is a Lie algebra isomorphism from $\mathfrak{so}(n)$ to $\mathfrak{so}(E_n)$ and from $\mathfrak{so}(n)_{\mathbb{C}}$ to $\mathfrak{so}(E_n)_{\mathbb{C}}$.

For even dimensions a maximal torus of $\mathrm{SO}(2l)$ is given by

$$T = \{\mathrm{diag}(e^{i\theta_1}, ..., e^{i\theta_l}, e^{-i\theta_1}, ..., e^{-i\theta_l}), \theta_i \in \mathbb{R}\}$$

with the corresponding Cartan subalgebra

$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, ..., i\theta_l, -i\theta_1, ..., -i\theta_l), \theta_i \in \mathbb{R}\}$$

and complexification

$$\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(a_1, a_2, ..., a_l, -a_1, -a_2, ..., -a_l) : a_i \in \mathbb{C}\}.$$

For odd dimensions $n = 2l + 1$ we get a maximal torus

$$T = \{\mathrm{diag}(e^{i\theta_1}, ..., e^{i\theta_l}, e^{-i\theta_1}, ..., e^{-i\theta_l}, 1), \theta_i \in \mathbb{R}\}$$

with the corresponding Cartan subalgebra

$$\mathfrak{t} = \{\mathrm{diag}(i\theta_1, ..., i\theta_l, -i\theta_1, ..., -i\theta_l, 0), \theta_i \in \mathbb{R}\}$$

and complexification

$$\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(a_1, a_2, ..., a_l, -a_1, -a_2, ..., -a_l, 0) : a_i \in \mathbb{C}\}.$$

The standard representation of $\mathrm{SO}(E_n)$ on $\mathbb{C}^n$ is given by the matrix multiplication from the left by $X \in \mathrm{SO}(E_n)$, $\rho(X) = X$ and $d\rho(H) = \frac{d}{dt} e^{Ht}|_{t=0} = H$ because $H \in \mathfrak{t}_{\mathbb{C}}$ is a diagonal matrix. Define $\varepsilon_i \in \mathfrak{t}_{\mathbb{C}}^*$ via $\varepsilon_i(H)$ being the i-th entry of the diagonal of $H \in \mathfrak{t}_{\mathbb{C}}$, then the weights of the standard representation of $\mathrm{SO}(E_n)$ on $\mathbb{C}^n$ for even n are given by $\{\pm\varepsilon_i : 1 \le i \le \lfloor n/2 \rfloor\}$ and the weight spaces by $V_{\varepsilon_i} = \mathrm{span}_{\mathbb{C}}\{e_i\}$ and $V_{-\varepsilon_i} = \mathrm{span}_{\mathbb{C}}\{e_{i+\lfloor n/2 \rfloor}\}$. In the case that n is odd we have to add the weight 0 with weight space $V_0 = \mathrm{span}_{\mathbb{C}}\{e_n\}$ to the previously obtained.

Theorem 6.3. *Let G be a compact Lie group, with maximal torus T and a finite-dimensional representation (ρ, V), and let $V = \bigoplus_{\alpha \in \Delta(V)} V_\alpha$ be the weight space decomposition with respect to $\mathfrak{t}_{\mathbb{C}}$. Then every $\alpha \in \Delta(V)$ is imaginary valued on $\mathfrak{t}$ and real valued on $i\mathfrak{t}$. For $t \in T$ choose $H \in \mathfrak{t}$ such that $e^H = t$, then $tv_\alpha = e^{\alpha(H)} v_\alpha$ for $v_\alpha \in V_\alpha$.*

Remark. Note that because $\alpha \in \Delta(V)$ is completely determined by its values on either $\mathfrak{t}$ or $\mathfrak{t}_{\mathbb{C}}$ by $\mathbb{C}$-linearity, we can view α as an element of $\mathfrak{t}^*$ or $(i\mathfrak{t})^*$ as well.

6.3 Roots

The weight space decomposition applied to the adjoint representation of a Lie algebra results in the root space decomposition, an integral part of the structural theory of Lie algebras.

Let us consider a compact Lie group G, then the domain of $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$ can be extended to $\mathfrak{g}_{\mathbb{C}}$ via $\mathbb{C}$-linearity. This provides a representation $(\mathrm{Ad}, \mathfrak{g}_{\mathbb{C}})$ with the differential $\mathrm{ad} : \mathfrak{g}_{\mathbb{C}} \to \mathrm{End}(\mathfrak{g}_{\mathbb{C}})$ (extended by $\mathbb{C}$-linearity) being a representation of $\mathfrak{g}_{\mathbb{C}}$. This leads to a weight space decomposition that is important enough to get its own name.

Definition. Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group and $\mathfrak{t}$ a Cartan subalgebra. The elements of the finite set $\Delta(\mathfrak{g}_\mathbb{C}) = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C}) \subseteq \mathfrak{t}_\mathbb{C}^*$, that satisfies

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{Z \in \mathfrak{g}_\mathbb{C} : [H, Z] = \alpha(H)Z, H \in \mathfrak{t}_\mathbb{C}\}$$

is nonzero, are called *roots* of $\mathfrak{g}_\mathbb{C}$. This decomposition of $\mathfrak{g}_\mathbb{C}$ is called the *root space decomposition* of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{t}_\mathbb{C}$.

Note that $\mathfrak{t}_\mathbb{C} = \mathfrak{g}_0 = \{Z \in \mathfrak{g}_\mathbb{C} : [H, Z] = 0, H \in \mathfrak{t}_\mathbb{C}\}$ as $\mathfrak{t}$ is a maximal abelian subspace in $\mathfrak{g}$.

Example. For $\mathfrak{so}(E_{2l+1})_\mathbb{C}$ the condition $X^t E_n + E_n X = 0$ leads to a block form for its elements

$$\mathfrak{so}(E_{2l+1})_\mathbb{C} = \left\{ \begin{pmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{pmatrix} : p = -p^t, q = -q^t \right\},$$

where m, p, q are complex $l \times l$ matrices and $b, c \in \mathbb{C}^l$. A basis for the elements of $\mathfrak{so}(E_{2l+1})_\mathbb{C}$ where the only nonzero matrix entries are contained in b and c is given by $b_i := e_{i,0} - e_{0,l+i}$ and $c_i := e_{0,i} - e_{l+i,0}$ for $0 \leq i \leq l$, where $e_{i,j}$ is the matrix having all zeros except a single 1 in i-th row and j-th column. To calculate the corresponding roots, we look at

$$[H, b_i] = a_i b_i, \quad [H, c_i] = -a_i c_i$$

for $H \in \mathfrak{t}_\mathbb{C}$. We extend to a basis of all of $\mathfrak{so}(E_{2l+1})_\mathbb{C}$ via

$$\begin{aligned} m_{i,j} :=& e_{i,j} - e_{l+j,l+i} \text{ for } 1 \leq i \neq j \leq l; \\ p_{i,j} :=& e_{i,l+j} - e_{j,l+i} \text{ for } 1 \leq i < j \leq l; \\ q_{i,j} :=& e_{l+j,i} - e_{l+i,j} \text{ for } 1 \leq i < j \leq l, \end{aligned}$$

and again we see that our basis elements are simultaneous eigenvectors under the $\mathrm{ad}(H)$ for $H \in \mathfrak{t}_\mathbb{C}$ as

$$\begin{aligned} [H, m_{i,j}] =& (a_i - a_j) m_{i,j}; \\ [H, p_{i,j}] =& (a_i + a_j) p_{i,j}; \\ [H, q_{i,j}] =& -(a_i + a_j) q_{i,j}. \end{aligned}$$

So we obtain the roots of $\mathfrak{so}(E_{2l+1})_\mathbb{C}$ given by

$$\Delta(\mathfrak{so}(E_{2l+1})_\mathbb{C}) = \{\pm\varepsilon_i : 1 \leq i \leq l\} \cup \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i \leq j \leq l\}.$$

For even dimensions the situation is just a simplification and the roots are given by

$$\Delta(\mathfrak{so}(E_{2l})_\mathbb{C}) = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i \leq j \leq l\}$$

(see [30] and [9] for details).

6.4 The Killing Form, the Standard $\mathfrak{sl}(2,\mathbb{C})$ Triple and Lattices

The Lie algebra of a linear Lie group can be equipped with a symmetric bilinear form: the Killing form. Given that the Lie algebra is semisimple, the Killing form yields an inner product. We will see that the Lie algebra of a compact Lie group consists of copies of $\mathfrak{sl}(2,\mathbb{C})$, and the lattice of analytically integral weights will be of particular interest in the highest weight theory.

Definition. Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group. The *Cartan involution* $\theta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ is given by $\theta(X \otimes z) = X \otimes \overline{z}, X \in \mathfrak{g}, z \in \mathbb{C}$. Equivalently if $Z \in \mathfrak{g}_{\mathbb{C}}$ is given by $X + iY$ for $X, Y \in \mathfrak{g} \otimes 1$, then $\theta(Z) = X - iY$.

Lemma 6.4. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group. If $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$, then $-\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}})$ and $\mathfrak{g}_{-\alpha} = \theta\mathfrak{g}_{\alpha}$.*

Definition. Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group. The *Killing form* is the symmetric complex bilinear form $B : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$ defined by $B(X,Y) = \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y))$ for $X, Y \in \mathfrak{g}_{\mathbb{C}}$.

Theorem 6.5. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group G, then*

(i) For $X, Y \in \mathfrak{g}, B(X,Y) = \operatorname{tr}(\operatorname{ad}(X) \circ \operatorname{ad}(Y))$ on $\mathfrak{g}$;

(ii) B is Ad*-invariant, i.e., $B(\operatorname{Ad}(g)X, \operatorname{Ad}(g)Y) = B(X,Y)$ for $g \in G$ and $X, Y \in \mathfrak{g}_{\mathbb{C}}$;*

(iii) B is skew ad*-invariant, i.e., $B(\operatorname{ad}(Z)X, Y) = -B(X, \operatorname{ad}(Z)Y)$ for $X, Y, Z \in \mathfrak{g}_{\mathbb{C}}$;*

(iv) B restricted to $\mathfrak{g}' \times \mathfrak{g}'$ is negative definite;

(v) B restricted to $\mathfrak{g}_{\alpha} \times \mathfrak{g}_{\beta}$ is zero when $\alpha + \beta \neq 0$ for $\alpha, \beta \in \Delta(\mathfrak{g}_{\mathbb{C}}) \cup \{0\}$;

(vi) B is nondegenerate on $\mathfrak{g}_{\alpha} \times \mathfrak{g}_{-\alpha}$. If $\mathfrak{g}$ is semisimple with a Cartan subalgebra $\mathfrak{t}$, then B is also nondegenerate on $\mathfrak{t}_{\mathbb{C}} \times \mathfrak{t}_{\mathbb{C}}$;

(vii) The radical of B, $\operatorname{rad}B = \{X \in \mathfrak{g}_{\mathbb{C}} : B(X, \mathfrak{g}_{\mathbb{C}}) = 0\}$, is the center of $\mathfrak{g}_{\mathbb{C}}$;

(viii) If $\mathfrak{g}$ is semisimple, the form $(X,Y) = -B(X, \theta Y)$ is an Ad*-invariant inner product on $\mathfrak{g}_{\mathbb{C}}$;*

(ix) Let G be simple and choose a linear realization of G such that $\mathfrak{g} \subseteq \mathfrak{u}(n)$. Then there exists a positive $c \in \mathbb{R}$ such that $B(X,Y) = c\operatorname{tr}(XY)$ for $X, Y \in \mathfrak{g}_{\mathbb{C}}$.

Due to Theorem 6.5, given a semisimple Lie algebra of a compact Lie group B is a nondegenerate form on $\mathfrak{t}_{\mathbb{C}} \cong \mathfrak{t} \oplus i\mathfrak{t}$, and as such it induces an isomorphism between $i\mathfrak{t}$ and $(i\mathfrak{t})^*$ in the following way.

Definition. Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\alpha \in (i\mathfrak{t})^*$. Assume that $\mathfrak{g}$ is semisimple and let $u_{\alpha} \in i\mathfrak{t}$ be uniquely determined by the equation

$$\alpha(H) = B(H, u_{\alpha})$$

for all $H \in i\mathfrak{t}$, and given that $\alpha \neq 0$, let

$$h_\alpha = \frac{2u_\alpha}{B(u_\alpha, u_\alpha)}.$$

If $\mathfrak{g}$ is not semisimple, that is, $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ with a nonzero center, define $u_\alpha \in i\mathfrak{t}' \subseteq i\mathfrak{t}$ by restricting B to $i\mathfrak{t}'$. Given $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, as discussed before, α can be viewed as an element of $(i\mathfrak{t})^*$. Now we can find u_α and h_α as above, and by $\mathbb{C}$-linear extension $\alpha(H) = B(H, u_\alpha)$ holds for $H \in \mathfrak{t}_\mathbb{C}$.

The Lie algebra of a compact Lie group contains many copies of $\mathfrak{sl}(2,\mathbb{C})$. In fact root spaces come in pairs $(\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha})$, and together with $h_\alpha \in \mathfrak{t}$, where h_α is given by the Lie bracket of the basis vectors of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, they span an isomorphic copy of $\mathfrak{sl}(2,\mathbb{C})$.

Theorem 6.6. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$. For a nonzero $E_\alpha \in \mathfrak{g}_\alpha$ let $F_\alpha = -\theta E_\alpha \in \mathfrak{g}_{-\alpha}$. Then E_α and F_α can be rescaled such that $[E_\alpha, F_\alpha] = h_\alpha$ and $\mathfrak{sl}(2,\mathbb{C}) \cong \operatorname{span}_\mathbb{C}\{E_\alpha, F_\alpha, h_\alpha\}$ with $\{E_\alpha, F_\alpha, h_\alpha\}$ corresponding to the standard basis*

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of $\mathfrak{sl}(2,\mathbb{C})$.

Corollary 6.7. *Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group, $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, then:*

(*i*) *The only multiple of α is $\pm\alpha$;*

(*ii*) $\dim \mathfrak{g}_\alpha = 1$;

(*iii*) *If $\beta \in \Delta(\mathfrak{g}_\mathbb{C})$, then $\alpha(h_\beta) = \pm\{0, 1, 2, 3\}$;*

(*iv*) *If (ρ, V) is a finite-dimensional representation of G and $\lambda \in \Delta(V)$, then $\lambda(h_\alpha) \in \mathbb{Z}$.*

So the root space decomposition decomposes V into $\mathfrak{t}_\mathbb{C}$ and the one-dimensional subspaces $\mathfrak{g}_\alpha$, where a triple $E_\alpha \in \mathfrak{g}_\alpha$, $F_\alpha = \theta E_\alpha \in \mathfrak{g}_{-\alpha}$ and $[E_\alpha, F_\alpha] \in \mathfrak{t}_\mathbb{C}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

It is possible to transport the Killing form to $(i\mathfrak{t})^*$ by setting

$$B(\lambda_1, \lambda_2) = B(u_{\lambda_1}, u_{\lambda_2})$$

for $\lambda_1, \lambda_2 \in (i\mathfrak{t})^*$.

In the following paragraphs we define lattices of a Lie algebra and the lattice of analytically integral weights will be of particular importance in the highest weight theory.

Definition. Let $\mathfrak{g}$ be the Lie algebra of a compact Lie group with maximal torus T and $\mathfrak{t}$ the corresponding Cartan subalgebra of $\mathfrak{g}$. The *root lattice,* $R = R(\mathfrak{t})$ is the lattice in $(i\mathfrak{t})^*$ defined by

$$R = \operatorname{span}_\mathbb{Z}\{\alpha : \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}.$$

The set of *algebraically integral weights*, $P = P(\mathfrak{t})$, is the lattice given by

$$P = \{\lambda \in (i\mathfrak{t})^* : \lambda(h_\alpha) \in \mathbb{Z} \text{ for } \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\},$$

where $\lambda \in (i\mathfrak{t})^*$ is extended to an element of $(\mathfrak{t}_\mathbb{C})^*$. The set of *analytically integral weights*, $A = A(T)$, is the lattice given by

$$A = \{\lambda \in (i\mathfrak{t})^* : \lambda(H) \in 2\pi i\mathbb{Z} \text{ whenever } \exp(H) = I \text{ for } H \in \mathfrak{t}\}.$$

Example. The root lattice of $\mathfrak{so}(E_{2l})_\mathbb{C}$ is given by

$$\begin{aligned} R(\mathfrak{so}(E_{2l})_\mathbb{C}) &= \operatorname{span}_\mathbb{Z}\{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i \leq j \leq l\} \\ &= \left\{ \sum_{i=i}^{l} \lambda_i \varepsilon_i : \lambda_i \in \mathbb{Z}, \sum_{i=1}^{l} \lambda_i \in 2\mathbb{Z} \right\}, \end{aligned}$$

where the second equality holds because the roots are given by pairs of ε_is.

Let E_i denote the $2l \times 2l$ matrix with all zeroes except a single 1 in the i-th diagonal position. Then h_α for $\alpha \in \Delta(\mathfrak{so}(E_{2l})_\mathbb{C})$ is given by $h_{\varepsilon_i - \varepsilon_j} = (E_i - E_j) - (E_{i+l} - E_{j+l})$, $h_{\varepsilon_i + \varepsilon_j} = (E_i + E_j) - (E_{i+l} + E_{j+l})$ and $h_{-\alpha} = -h_\alpha$. The algebraically integral weights can be calculated as

$$P(\mathfrak{so}(E_{2l})_\mathbb{C}) = \left\{ \sum_{i=i}^{l} \left(\lambda_i + \frac{\lambda_0}{2} \varepsilon_i \right) : \lambda_i \in \mathbb{Z} \right\}.$$

Recall that a Cartan subalgebra of $\mathfrak{so}(E_{2l})$ is given by $\mathfrak{t} = \{\operatorname{diag}(i\theta_1, ..., i\theta_l, -i\theta_1, ..., -i\theta_l) : \theta_i \in \mathbb{R}\}$ so the condition $\exp(H) = I$ for $H \in \mathfrak{t}$ boils down to $i\theta_i \in 2\pi i\mathbb{Z}$. Therefore the set of analytically integral weights for $\mathfrak{so}(E_{2l})_\mathbb{C}$ is given by

$$A(\mathfrak{so}(E_{2l})_\mathbb{C}) = \left\{ \sum_{i=i}^{l} \lambda_i \varepsilon_i : \lambda_i \in \mathbb{Z}, \lambda_i \in \mathbb{Z} \right\}.$$

For odd dimensions, we get in a similar way that

$$R(\mathfrak{so}(E_{2l+1})_\mathbb{C}) = A(\mathfrak{so}(E_{2l+1})_\mathbb{C}) = \left\{ \sum_{i=i}^{l+1} \lambda_i \varepsilon_i : \lambda_i \in \mathbb{Z}, \lambda_i \in \mathbb{Z} \right\}$$

and

$$P(\mathfrak{so}(E_{2l+1})_\mathbb{C}) = \left\{ \sum_{i=i}^{l+1} \left(\lambda_i + \frac{\lambda_0}{2} \right) \varepsilon_i : \lambda_i \in \mathbb{Z} \right\},$$

(see [30]).

Definition. Given a compact Lie group G with maximal torus T the *character group on* T, $\chi(T)$, *is the group of all Lie homomorphisms* $\xi : T \to \mathbb{C} \setminus \{0\}$.

Theorem 6.8. *Given a compact Lie group G with maximal torus T, then*

(i) $R \subseteq A \subseteq P$;

(ii) *Given $\lambda \in (i\mathfrak{t})^*, \lambda$ is in A if and only if there exists $\xi_\lambda \in \chi(T)$ satisfying $\xi_\lambda(exp(H)) = e^{\lambda(H)}$ for $H \in \mathfrak{t}$, where $\lambda \in (i\mathfrak{t})^*$ is extended to an element of $(\mathfrak{t}_\mathbb{C})^*$ by $\mathbb{C}-linearity$. The map $\lambda \to \xi_\lambda$ establishes a bijection $A \longleftrightarrow \chi(T)$.*

6.5 The Weyl Group, Simple Roots and Weyl Chambers

The Weyl group is introduced in an algebraic and in a geometric way. Furthermore we introduce systems of simple roots, a concept that in some way can be compared to that of the basis of a vector space.

Definition. Given a compact Lie group G with maximal torus T. The *normalizer* of T in G is defined by $N(T) := \{g \in G : gTg^{-1} = T\}$. The *Weyl group* of G is defined by $W(G,T) := N/T$.

It can be shown, that this definition is up to isomorphism independent of the choice of a maximal torus.

For $g \in N, H \in \mathfrak{t}$ and $\lambda \in \mathfrak{t}^*$ we can define an action of N on $\mathfrak{t}$ and $\mathfrak{t}^*$ by

$$g(H) = \mathrm{Ad}(g)H;$$
$$g(\lambda)(H) = \lambda(g^{-1}(H)) = \lambda(\mathrm{Ad}(g^{-1})H).$$

As before this action can be extended to an action on $\mathfrak{t}_\mathbb{C}, i\mathfrak{t}, (\mathfrak{t}_\mathbb{C})^*$ and $(i\mathfrak{t})^*$ by $\mathbb{C}$-linearity.

We want to consider the realization of the Weyl group as a reflection group as well.

Definition. Given the Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{t}$ to a compact Lie group for $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$, $r_\alpha : (i\mathfrak{t})^* \to (i\mathfrak{t})^*$ is defined by

$$r_\alpha(\lambda) := \lambda - 2\frac{B(\lambda,\alpha)}{B(\alpha,\alpha)}\alpha = \lambda - \lambda(h_\alpha)\alpha.$$

By $W(\Delta(\mathfrak{g}_\mathbb{C}))$ we denote the group generated by $\{r_\alpha : \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}$.

As usual the action of $W(\Delta(\mathfrak{g}_\mathbb{C}))$ on $(i\mathfrak{t})^*$ is extended to an action on $\mathfrak{t}^*$ by $\mathbb{C}$-linear extension. r_α acts on $(i\mathfrak{t}')^*$ as the reflection across the hyperplane perpendicular to α.

Definition. Given a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$ of a compact Lie group, let $\mathfrak{t}' := \mathfrak{t} \cap \mathfrak{g}'$. A *system of simple roots,* $\Pi = \Pi(\mathfrak{g}_\mathbb{C})$, is a subset of the set of roots $\Delta(\mathfrak{g}_\mathbb{C})$ that is a basis of $(i\mathfrak{t}')^*$ and furthermore satisfies the property that any root $\beta \in \Delta(\mathfrak{g}_\mathbb{C})$ can be written as

$$\beta = \sum_{\alpha \in \Pi} k_\alpha \alpha,$$

with the set $\{k_\alpha : \alpha \in \Pi\}$ being completely contained either in $\mathbb{Z}_{\geq 0} := \{k \in \mathbb{Z} : k \geq 0\}$ or in $\mathbb{Z}_{\leq 0} := \{k \in \mathbb{Z} : k \leq 0\}$. The elements of Π are called *simple roots*. Given a system of simple roots Π, the set of *positive roots* with respect to Π is

$$\Delta^+(\mathfrak{g}_\mathbb{C}) := \left\{ \beta \in \Delta(\mathfrak{g}_\mathbb{C}) : \beta = \sum_{\alpha \in \Pi} k_\alpha \alpha \text{ such that } k_\alpha \in \mathbb{Z}_{\geq 0} \forall \alpha \in \Pi \right\}.$$

The set of *negative roots* with respect to Π is

$$\Delta^-(\mathfrak{g}_\mathbb{C}) := \left\{ \beta \in \Delta(\mathfrak{g}_\mathbb{C}) : \beta = \sum_{\alpha \in \Pi} k_\alpha \alpha \text{ such that } k_\alpha \in \mathbb{Z}_{\leq 0} \forall \alpha \in \Pi \right\}.$$

So $\Delta(\mathfrak{g}_\mathbb{C})$ is the disjoint union of $\Delta^+(\mathfrak{g}_\mathbb{C})$ and $\Delta^-(\mathfrak{g}_\mathbb{C})$, and $\Delta^-(\mathfrak{g}_\mathbb{C}) = -\Delta^+(\mathfrak{g}_\mathbb{C})$.

Example. For $\mathfrak{so}(E_{2l})_\mathbb{C}$ a system of simple roots is given by $\{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq l-1\} \cup \{\varepsilon_{l-1} + \varepsilon_l\}$, and for odd dimensions, we get a system of simple roots for $\mathfrak{so}(E_{2l+1})_\mathbb{C}$ as $\{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq l-1\} \cup \{\varepsilon_l\}$.

In order to see that systems of simple roots exist we need the definition of a Weyl chamber and the next theorem.

Definition. Given a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$ of a compact Lie group the connected components of $(i\mathfrak{t}')^* \setminus \bigcup_{\alpha \in \Delta(\mathfrak{g}_\mathbb{C})} \alpha^\perp$ are called *Weyl chambers* of $(i\mathfrak{t})^*$. Let C be a Weyl chamber, then $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$ is called *C-positive* if $B(C, \alpha) > 0$. Furthermore $\alpha \in \Delta(\mathfrak{g}_\mathbb{C})$ is called *indecomposable* with respect to C if α cannot be expressed as $\alpha = \beta + \gamma$ with C-positive $\beta, \gamma \in \Delta(\mathfrak{g}_\mathbb{C})$.

Given a Weyl chamber C of $(i\mathfrak{t})^*$ we define a system of roots as

$$\Pi(C) := \{\alpha \in \Delta(\mathfrak{g}_\mathbb{C}) : \alpha \text{ is indecomposable and } C\text{-positive}\}.$$

Conversely, given a system of simple roots Π the *associated Weyl chamber of* $(i\mathfrak{t})^*$ is defined by

$$C(\Pi) := \{\lambda \in (i\mathfrak{t})^* : B(\lambda, \alpha) > 0 \text{ for } \alpha \in \Pi\}.$$

This correspondence establishes a bijection between Weyl chambers and systems of simple roots, thus guaranteeing the existence of the latter.

Theorem 6.9. *Given a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$ of a compact Lie group G there is a one-to-one correspondence between*

$$\{\textit{systems of simple roots}\} \longleftrightarrow \{\textit{Weyl chambers of } (i\mathfrak{t})^*\}$$

mapping a system of simple roots Π to the associated Weyl chamber $C(\Pi)$ and a Weyl chamber C to the system of simple roots $\Pi(C)$. Furthermore $W(G) \cong W(\Delta(\mathfrak{g}_\mathbb{C}))$ and $W(G)$ acts simply transitively on the set of Weyl chambers.

6.6 Highest Weights

We recall highest weights, which will lead to a classification of irreducible representations.

Given a compact Lie group G with maximal torus T and a system of simple roots $\Pi(\mathfrak{g}_\mathbb{C})$, the decomposition of the roots of $\Delta(\mathfrak{g}_\mathbb{C})$ into positive and negative roots leads to

$$\mathfrak{g}_\mathbb{C} = \mathfrak{n}^- \oplus \mathfrak{t}_\mathbb{C} \oplus \mathfrak{n}^+,$$

where

$$\mathfrak{n}^\pm := \bigoplus_{\alpha \in \Delta^\pm(\mathfrak{g}_\mathbb{C})} \mathfrak{g}_\alpha.$$

With this in mind we are prepared to define highest weights.

Definition. Given a representation of a compact Lie group G with Lie algebra $\mathfrak{g}$, a system of positive roots $\Delta^+(\mathfrak{g}_\mathbb{C})$ and weight space decomposition $V = \bigoplus_{\lambda \in \Delta(V)} V_\lambda$, a nonzero $v \in V_{\lambda_0}$ is called a *highest weight vector* of weight λ_0 with respect to $\Delta^+(\mathfrak{g}_\mathbb{C})$ if $Xv = 0$ for all $X \in \mathfrak{n}^+$. We then call λ_0 a *highest weight* of V. A weight $\lambda \in \Delta(\mathfrak{g}_\mathbb{C})$ is called *dominant* given that $B(\lambda, \alpha) \geq 0$ for all $\alpha \in \Pi(\mathfrak{g}_\mathbb{C})$. That is, λ lies within the closed Weyl chamber associated to $\Pi(\mathfrak{g}_\mathbb{C})$.

Highest weights are of particular interest considering they determine irreducible representations up to isomorphism. As we have seen in Theorem 6.1, for a compact connected Lie group G a representation of its Lie algebra determines a representation on G.

Theorem 6.10. *Given a connected compact Lie group G and an irreducible representation V of G the following statements hold true:*

(*i*) *V has a unique highest weight λ_0;*

(*ii*) *The highest weight is dominant and analytically integral;*

(*iii*) *Up to scalar multiplication there is a unique highest weight vector;*

(*iv*) *Any weight $\lambda \in \Delta(V)$ is of the form*

$$\lambda = \lambda_0 - \sum_{\alpha_i \in \Pi(\mathfrak{g}_\mathbb{C})} n_i \alpha_i$$

with $n_i \in \mathbb{Z}_{\geq 0}$;

(*v*) *For $w \in W(G), wV_\lambda = V_{w\lambda}$ and therefore $\dim V_\lambda = \dim V_{w\lambda}$;*

(*vi*) *Using the norm induced by the Killing form, $\|\lambda\| \leq \|\lambda_0\|$ with equality if and only if $\lambda = w\lambda_0$ for a $w \in W(G)$;*

(*vii*) *V is uniquely determined by λ_0 up to isomorphism.*

Because for a compact connected Lie group G an irreducible representation V is uniquely determined by its highest weight λ, we write V_λ for V and χ_λ for its character. With the help of Theorem 6.15, which states that we can compute highest weights as analytically integral weights, we want to give some examples of highest weights.

Example. For $\mathfrak{so}(E_{2l})_{\mathbb{C}}$ a Cartan subalgebra is given by $\mathfrak{t}_{\mathbb{C}} = \{\mathrm{diag}(a_1, a_2, ..., a_l, -a_1, -a_2, ..., -a_l) : a_i \in \mathbb{C}\}$. The corresponding roots are given by $\Delta(\mathfrak{so}(E_{2l})_{\mathbb{C}}) = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i \neq j \leq l\}$ and a system of simple roots by $\Pi(\mathfrak{so}(E_{2l})_{\mathbb{C}}) = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq l-1\} \cup \{\alpha_l = \varepsilon_{l-1} + \varepsilon_l\}$. As the highest weight λ_0 is also analytically integral we get that $\lambda_0 = \sum_{i=1}^{l} \lambda_i \varepsilon_i$ with $\lambda_i \in \mathbb{Z}$, so we can also think of λ_0 as an l-tuple of integers. Because λ_0 is also dominant and $\mathfrak{so}(E_{2l})_{\mathbb{C}}$ is simple ([25] page 94)

$$B(\lambda_0, \alpha) = c\ \mathrm{tr}(\lambda\alpha) = c\ \mathrm{tr}(u_\lambda u_\alpha) \geq 0$$

for $\alpha \in \Pi(\mathfrak{so}(E_{2l})_{\mathbb{C}})$. In order to calculate $u_{\alpha_i} \in i\mathfrak{t}$, we take a look at its defining equation $\alpha_i(H) = B(H, u_{\alpha_i})$ for $H \in i\mathfrak{t}$. For the right side we get

$$\begin{aligned} B(H, u_{\alpha_i}) &= c\ \mathrm{tr}(\mathrm{diag}(a_1, ..., a_l, -a_1, ..., -a_l)\ \mathrm{diag}(u_1, ..., u_l, -u_1, ..., -u_l)) \\ &= 2c \sum_{i=1}^{l} a_i u_i. \end{aligned}$$

For $1 \leq i \leq l-1$ this leads to $a_i - a_{i+1} = 2c \sum_{i=1}^{l} a_i u_i$ and therefore

$$u_{\alpha_i} = \frac{1}{2c}(e_{i,i} - e_{i+1,i+1} - e_{l+i,l+i} + e_{l+i+1,l+i+1}).$$

For $i = l$ we obtain

$$u_{\alpha_l} = \frac{1}{2c}(e_{l-1,l-1} + e_{l,l} - e_{2l-1,2l-1} - e_{2l,2l}).$$

For u_{λ_0} we get

$$\lambda_0(H) = \sum_{i=1}^{l} \lambda_i a_i = c\mathrm{tr}(H, u_{\lambda_0}) = 2c \sum_{i=1}^{l} a_i u_i$$

and, thus,

$$u_{\lambda_0} = \frac{1}{2c}\mathrm{diag}(\lambda_1, ..., \lambda_l, -\lambda_1, ..., -\lambda_l).$$

Now we are able to calculate

$$B(\lambda_0, \alpha_i) = c\mathrm{tr}(u_{\lambda_0} u_\alpha) = \frac{c}{4c^2}(\lambda_i - \lambda_{i+i} + \lambda_i - \lambda_{i+1}) = \frac{1}{2c}(\lambda_i - \lambda_{i+1})$$

for $1 \leq i \leq l-1$ and

$$B(\lambda_0, \alpha_l) = \frac{1}{2c}(\lambda_i + \lambda_{i+1}).$$

Considering λ_0 is dominant these values of the Killing form are greater or equal to 0 and therefore we obtain

$$\lambda_1 \geq \cdots \geq \lambda_{l-1} \geq |\lambda_l|$$

as a condition for the highest weight of an irreducible representation of $\mathrm{SO}(E_{2l})$. For odd dimensions it can be calculated in the same fashion that for the highest weight of an irreducible representation of $\mathrm{SO}(E_{2l+1})$,

$$\lambda_1 \geq \cdots \geq \lambda_l \geq 0.$$

To sum this up, the highest weights of irreducible representation of $\mathrm{SO}(E_n)$ are given by tuples of integers such that

$$\begin{cases} \lambda_1 \geq ... \geq \lambda_l \geq 0 & \text{for odd } n, \\ \lambda_1 \geq ... \geq \lambda_{l-1} \geq |\lambda_l| & \text{for even } n. \end{cases} \tag{10}$$

Example. For the trivial representation of the Lie group $\mathrm{SO}(n)$, $\rho : \mathrm{SO}(n) \to \mathrm{GL}(1, \mathbb{C}), \rho(g) = 1$, the only weight 0 is also the highest weight. As an $\lfloor n/2 \rfloor$-tuple it is given by $\lambda_0 = (0, ..., 0)$.

Example. Let's consider the case of the standard representation of $\mathrm{SO}(E_n)$ on $\mathbb{R}^n$, then the highest weight is given by $\lambda_0 = (1, 0, ..., 0)$.

Example. Let Γ be the standard representation $\mathrm{SO}(n)$ on $\mathbb{R}^n$, then $\Lambda^k \Gamma_{\mathbb{C}}$ is an irreducible representation with the highest weight $(1, ..., 1, 0, ..., 0)$ where 1 appears k times for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor - 1$. For odd n this is also true for $k = \lfloor \frac{n}{2} \rfloor$. However, in case that n is even $\Lambda^{n/2}\Gamma_{\mathbb{C}}$ is the direct sum of two irreducible representations $\Lambda^{n/2}\Gamma_{\mathbb{C}} = \Gamma_{(1,...1)} \oplus \Gamma_{(1,...,1,-1)}$ (See [10] for more details). From the identification of exterior powers we get a natural isomorphism $\Lambda^k \Gamma_{\mathbb{C}} \cong \Lambda^{n-k}\Gamma_{\mathbb{C}}$.

6.7 The Weyl Integration and Character Formulas and the Highest Weight Classification

Our goal in this subsection is to establish a highest weight classification for irreducible representations of a compact connected Lie group. We come across the Weyl integration formula and the Weyl character formula, both important results for the proof of this classification. Furthermore we get to know another consequence of the Weyl character formula, the so called second determinantal formula which will be used in the proof of the generalized Hadwiger theorem. For more details see [4] and for more details on the second determinantal formula see [8, 10].

Definition. Let G be a compact Lie group with maximal torus T and Lie algebra $\mathfrak{g}$. $X \in \mathfrak{g}$ is called a regular element of $\mathfrak{g}$ if $\mathfrak{z}(X) := \{Y \in \mathfrak{g} : [X, Y] = 0\}$ is a Cartan subalgebra. Let $Z_G(g) := \{h \in G : gh = hg\}$ be the *centralizer* of $g \in G$, and given a subgroup H of G denote by H^0 the connected component of H containing e. An element $g \in G$ is called regular if $Z_G(g)^0$ is a maximal torus. For the sets of regular elements of G and $\mathfrak{g}$ we write G^{reg} and $\mathfrak{g}^{reg}$, respectively.

The sets of regular elements have a couple of useful properties, in particular they are dense in a connected Lie group and its Lie algebra.

Theorem 6.11. *Given a compact connected Lie group G, then $\mathfrak{g}^{reg}$ is open dense in $\mathfrak{g}$, and G^{reg} is open dense in G.*

For the following theorem, recall Theorem 6.8 and the bijection between analytically integral weights A and the character group mapping $\lambda \in A$ to the character ξ_λ.

Theorem 6.12. *(Weyl integration formula) Let G be a compact connected Lie group with maximal torus T and $f \in C(G)$. Then*

$$\int_G f(g)\ dg = \frac{1}{|W(G)|} \int_T d(t) \int_{G/T} f(gtg^{-1})\ dgT\ dt,$$

where $d(t) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} |1 - \xi_{-\alpha}(t)|^2$ for $t \in T$.

Definition. Given a compact Lie group G with maximal torus T let $f : \mathfrak{t} \to \mathbb{C}$ be a function. We say f descends to T if $f(H + Z) = f(H)$ for $H, Z \in \mathfrak{t}$ with $Z \in \ker(\exp)$. In that case we write $F : T \to \mathbb{C}$ for the function given by $F(e^H) := f(H)$. $F : T \to \mathbb{C}$ is called *W-invariant* if $F(c_w t) = F(t)$ for $w \in N(T)$.

Definition. Let G be a compact Lie group with maximal torus T. Then $\Delta : \mathfrak{t} \to \mathbb{C}$ is defined by

$$\Delta(H) = \prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} (e^{\alpha(H)/2} - e^{-\alpha(H)/2})$$

for $H \in \mathfrak{t}$.

Definition. Let G be a compact Lie group with maximal torus T and λ be an analytically integral weight. Let $\Xi := \{H \in \mathfrak{t} : \alpha(H) \notin 2\pi i\mathbb{Z} \text{ for all } \alpha \in \Delta(\mathfrak{g}_\mathbb{C})\}$ (this is open dense in $\mathfrak{t}$) and $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} \alpha$, then we define $\Theta_\lambda : \Xi \to \mathbb{C}$ by

$$\begin{aligned} \Theta_\lambda(H) &= \frac{\sum_{w \in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w) e^{[w(\lambda+\rho)](H)}}{\Delta(H)} \\ &= \frac{\sum_{w \in W(\Delta(\mathfrak{g}_\mathbb{C}))} \det(w) e^{[w(\lambda+\rho)-\rho](H)}}{\prod_{\alpha \in \Delta^+(\mathfrak{g}_\mathbb{C})} (1 - e^{-\alpha(H)})} \end{aligned}$$

for $H \in \Xi$.

Lemma 6.13. *For a compact Lie group G with maximal torus T and an analytically integral weight λ the function Θ_λ descends to a smooth W-invariant function on T^{reg}, and this function uniquely extends to a smooth class function on G^{reg}.*

Theorem 6.14. *(Weyl character formula) Given a compact connected Lie group G with maximal torus T and V_λ, an irreducible representation of G with highest weight λ, then χ_λ, the character of V_λ, satisfies*

$$\chi_\lambda(g) = \Theta_\lambda(g)$$

for $g \in G^{reg}$.

With the help of Weyl's integration and character formulas the highest weight classification can be proven. Recall that we have already established the well-definedness and injectivity of the correspondence between irreducible representations and highest weights in Theorem 6.10.

Theorem 6.15. *(Highest weight classification) Given a compact connected Lie group G with maximal torus T, then there is a one-to-one correspondence between irreducible representations and dominant analytically integral weights given by the mapping $V_\lambda \to \lambda$.*

We will close this section with the second determinantal formula, which will be used in the proof of the general version of Hadwiger's theorem.

Given a tuple of non-negative integers $\lambda = (\lambda_1, ..., \lambda_{\lfloor n/2 \rfloor})$ satisfying (10) we define an $\mathrm{SO}(n)$ module $\bar{\Gamma}_\lambda$ by

$$\bar{\Gamma}_\lambda := \begin{cases} \Gamma_\lambda \oplus \Gamma_{\lambda'} & \text{for even } n \text{ and } \lambda_{n/2} \neq 0, \\ \Gamma_\lambda & \text{otherwise,} \end{cases} \tag{11}$$

where $\lambda' = (\lambda_1, ..., -\lambda_{\lfloor n/2 \rfloor})$. The second determinantal formula expresses the character of $\bar{\Gamma}_\lambda$ as a polynomial of the characters F_i of the fundamental representations $\Lambda^i \Gamma_{\mathbb{C}}$ for $i \in \mathbb{Z}$. Note that $F_0 = F_n = 1$, and we set $F_i = 0$ for $i < 0$ and $i > n$. Given a highest weight λ the *conjugate* μ of λ is given by the tuple $\mu = (\mu_1, ..., \mu_s)$ where $s = \lambda_1$, and μ_j is the number of λ_is in λ such that $\lambda_i \geq j$.

Theorem 6.16. *(Second determinantal formula) Let $\lambda = (\lambda_1, ..., \lambda_{\lfloor n/2 \rfloor})$ be a tuple of non-negative integers satisfying (10) and let $\mu = (\mu_1, ..., \mu_s)$ be the conjugate of λ. The character of $\bar{\Gamma}_\lambda$ equals the determinant of the $s \times s$ matrix the i-th row of which is given by*

$$\begin{pmatrix} F_{\mu_i - i + 1} & F_{\mu_i - i + 2} + F_{\mu_i - i} & F_{\mu_i - i + 3} + F_{\mu_i - i - 1} & \dots & F_{\mu_i - i + s} + F_{\mu_i - i - s + 2} \end{pmatrix}.$$

Sometimes we allow s to be greater than λ_1, but this only adds more zeros to the end of the conjugate not changing the determinant of the matrix defined above. Let $\#(\lambda, j)$ be the number of λ_is that are equal to j. We can formulate a corollary which we will need in the proof of the generalized Hadwiger theorem.

Corollary 6.17. *If $i, j \in \mathbb{N}$ are such that $n/2 \leq i \leq n$ and $i + j \leq n$, then*

$$F_i F_j - F_{i-1} F_{j-1} = \sum_\lambda \mathrm{char}(\bar{\Gamma}_\lambda),$$

where the sum ranges over all $\lfloor n/2 \rfloor$-tuples of non-negative integers $\lambda = (\lambda_1, ..., \lambda_{\lfloor n/2 \rfloor})$ satisfying (10) and

$$\lambda_1 \leq 2, \quad \#(\lambda, 1) = n - i - j, \quad \#(\lambda, 2) \leq j.$$

7 The Generalized Hadwiger Theorem by Alesker, Bernig and Schuster

We recall the Frobenius reciprocity theorem and a branching theorem for $\mathrm{SO}(n)$. The normal cycle map provides a way to describe the smooth translation invariant valuations, and the Rumin operator enables us to fit the smooth translation invariant valuations into an exact sequence of $\mathrm{SO}(n)$-modules. With these prerequisites we are able to generalize Hadwiger's theorem for valuations with values in a finite-dimensional irreducible $\mathrm{SO}(n)$-modules. This section is largely based on [8].

7.1 Prerequisites

We give an overview of the foundations needed in the proof and formulation of the generalized Hadwiger theorem. Given a Lie group G and a representation of a closed Lie subgroup H, there exists an induced representation on G itself, and the Frobenius reciprocity theorem gives a connection between those two representations. For an arbitrary Lie subgroup H the branching theorem is about achieving a decomposition of a G-module into irreducible H-modules. The natural action on the space of smooth translation invariant valuations $\mathbf{Val}^\infty$ turns it into an $\mathrm{SO}(n)$-module. The Rumin operator enables us to describe the kernel of the normal cycle map and to fit $\mathbf{Val}^\infty$ into an exact sequence of $\mathrm{SO}(n)$-modules. This subsection is based on [8] and [11].

7.1.1 Induced Representations and Frobenius Reciprocity Theorem

For this subsubsection we want to give [4] and [8] as references. Let G be a Lie group with closed subgroup H. Given a representation of G or H it only is natural to wonder if this induces a representation of the other. The first part of this question can be answered effortlessly: Given a representation Θ of G, a representation $\mathrm{Res}_H^G\Theta$ of H is obtained by restriction. For the converse, let Γ be a finite dimensional complex vector space and $C^\infty(G;\Gamma)$ be the space of all smooth functions from G to Γ. Furthermore let Γ be a representation of H, then $\mathrm{Ind}_G^H\Gamma \subseteq C^\infty(G;\Gamma)$ is defined by

$$\mathrm{Ind}_G^H\Gamma := \{f \in C^\infty(G;\Gamma) : f(gh) = h^{-1}f(g) \text{ for all } g \in G, h \in H\},$$

and the action of G on $\mathrm{Ind}_G^H\Gamma$ is given by left translation

$$(gf)(u) = f(g^{-1}u)$$

for $g, u \in G$. The Frobenius reciprocity theorem gives a connection between these two induced representations.

Theorem 7.1. *Let G be a Lie group and H a closed subgroup of G. If Θ is a representation of G and Γ is a representation of H, then*

$$\mathrm{Hom}_G(\Theta, \mathrm{Ind}_G^H\Gamma) \cong \mathrm{Hom}_H(\mathrm{Res}_H^G\Theta, \Gamma)$$

as vector spaces.

Due to Schur's lemma the multiplicity of an irreducible representation V of G in an arbitrary representation of G is given by $m(V,W) = \dim \operatorname{Hom}_G(V,W) = \dim \operatorname{Hom}_G(W,V)$. So for irreducible Γ and Θ the Frobenius reciprocity theorem results in

$$m(\Theta, \operatorname{Ind}_G^H \Gamma) = m(\operatorname{Res}_H^G \Theta, \Gamma). \tag{12}$$

7.1.2 Branching Theorem for SO(n)

Let G be a Lie group, V a G-module that decomposes into irreducible submodules $V \cong \bigoplus_{i=1}^n n_i V_i$ and H be a subgroup of G. In general an irreducible G-module V_i is not an irreducible H-module. However, in some settings such as G being compact and V being finite dimensional, V_i can be decomposed into irreducible H-modules. So V itself can be decomposed into a sum of irreducible H-modules. There exist various formulas describing the multiplicities of those subgroup modules for classical groups and subgroups, and they are called branching theorems or formulas. Here we are interested in a branching theorem in the case of an $\mathrm{SO}(n)$ representation restricted to $\mathrm{SO}(n-1)$. See [10] page 426 for further details.

Theorem 7.2. *Let Γ_λ be an irreducible $\mathrm{SO}(n)$-module with highest weight $\lambda = (\lambda_1, ..., \lambda_{\lfloor n/2 \rfloor})$, so λ satisfies (10). Then*

$$\operatorname{Res}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \Gamma_\lambda = \bigoplus_\mu \Gamma_\mu,$$

where the sum ranges over all $\mu = (\mu_1, ..., \mu_k)$ with $k := \lfloor (n-1)/2 \rfloor$ and

$$\begin{cases} \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \mu_{k-1} \geq \lambda_{\lfloor n/2 \rfloor} \geq |\mu_k| & \text{for odd } n, \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \mu_k \geq |\lambda_{n/2}| & \text{for even } n. \end{cases} \tag{13}$$

7.1.3 Valuations and Normal Cycles

We will define a norm on the space of continuous translation invariant complex valued valuations turning it into a Banach space. This Banach space becomes a $\mathrm{GL}(n)$-module under the natural action. $\mathrm{O}(n)$ finite and smooth valuations are defined, and the normal cycle map establishes an $\mathrm{SO}(n)$-module isomorphism between the space of smooth translation invariant valuations of degree i and those of degree $n-i$.

First we make some slight adjustments and generalizations to the setting in which we consider valuations. Let A be an abelian semigroup and V be an n-dimensional Euclidean vector space. Denote by $\mathcal{K}(V)$ the set of all nonempty convex compact subsets of V.

Definition. A function $\phi : \mathcal{K}(V) \to A$ is called a *valuation* if it satisfies

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$$

for $K, L, K \cup L \in \mathcal{K}(V)$.

We have already come across the following notions in the first part.

Definition. A valuation ϕ is called *translation invariant* given that $\phi(K+v) = \phi(K)$, it has *degree* i, or is *homogeneous of degree* i, if $\phi(tK) = t^i\phi(K)$, and it is called *even* if $\phi(-K) = \phi(K)$ and *odd* if $\phi(-K) = -\phi(K)$ for all $v \in V$, $K \in \mathcal{K}(V)$ and $t > 0$. ϕ is called *continuous* given that it is continuous with respect to the Hausdorff metric on $\mathcal{K}(V)$.

By **Val** we denote the vector space of all continuous translation invariant complex valued valuations and by $\mathbf{Val}_i^{\pm}$ its subspace of valuations of degree i of even/odd parity. The dimensions of $\mathbf{Val}_0$ and $\mathbf{Val}_n$ are given by

$$\dim \mathbf{Val}_0 = \dim \mathbf{Val}_n = 1, \tag{14}$$

where the first claim is easy to check, the second one was shown by Hadwiger in [14].

Due to an important result by McMullen, see [12], **Val** can be decomposed:

$$\mathbf{Val} = \bigoplus_{i=0}^{n}(\mathbf{Val}_i^{+} \oplus \mathbf{Val}_i^{-}). \tag{15}$$

A well known consequence of McMullen's decomposition is the following corollary.

Corollary 7.3. *Let $C \in \mathcal{K}(V)$ be a fixed convex body with non-empty interior. Then under the norm*

$$\|\phi\| = \sup\{|\phi(K)| : K \subseteq C\}$$

the space **Val** *becomes a Banach space. Moreover, a different choice of C yields an equivalent norm.*

There is s natural continuous action of $\mathrm{SO}(n)$ on **Val** defined by

$$A \cdot \phi(K) = \phi(A^{-1}K), A \in \mathrm{SO}(n), K \in \mathcal{K}(V),$$

turning **Val** into an $\mathrm{SO}(n)$-module.

The following theorem, known as the irreducibility theorem, was shown by Alesker [13].

Theorem 7.4. *The natural action of* $\mathrm{GL}(n)$ on $\mathbf{Val}_i^{\pm}$ is irreducible for every $i \in \{1, ..., n\}$.

We want to introduce two subsets of **Val** which will turn out to be dense.

Definition. A valuation $\phi \in \mathbf{Val}$ is called $\mathrm{O}(n)$ *finite* if the $\mathrm{O}(n)$ orbit of ϕ, i.e., the subspace $\mathrm{span}\{A\phi : A \in \mathrm{O}(n)\}$, is finite-dimensional. $\phi \in \mathbf{Val}$ is called *smooth* if the map $\mathrm{GL}(n) \to \mathbf{Val}$ defined by $A \mapsto A\phi$ is infinitely differentiable.

The space of continuous translation invariant $\mathrm{O}(n)$ finite valuations is denoted by $\mathbf{Val}^f$, the space of smooth translation invariant valuations by $\mathbf{Val}^\infty$, and the subspaces of given parity and degree are denoted by $\mathbf{Val}_i^{\pm,f}$ and $\mathbf{Val}_i^{\pm,\infty}$. In [6] on page 141 it is shown that $\mathbf{Val}_i^{\pm,f}$ is a dense $\mathrm{O}(n)$-invariant subspace of $\mathbf{Val}_i^{\pm}$ and that $\mathbf{Val}_i^{\pm,\infty}$ is a dense $\mathrm{GL}(n)$-invariant subspace of $\mathbf{Val}_i^{\pm}$. Furthermore, $\mathbf{Val}^f \subseteq \mathbf{Val}^\infty$ and from (15) a decomposition of the spaces $\mathbf{Val}^f$ and $\mathbf{Val}^\infty$ can be deduced:

$$\mathbf{Val}^f = \bigoplus_{i=0}^{n} (\mathbf{Val}_i^{+,f} \oplus \mathbf{Val}_i^{-,f});$$
$$\mathbf{Val}^\infty = \bigoplus_{i=0}^{n} (\mathbf{Val}_i^{+,\infty} \oplus \mathbf{Val}_i^{-,\infty}).$$

The smooth translation invariant valuations can be equivalently described by the normal cycle map. Let $SV = V \times S^{n-1}$ denote the unit sphere bundle on V. The product structure of SV induces a bigrading on $\Omega^*(SV)$, that is, the space of all smooth differential forms on SV. $\omega \in \Omega^*$ is translation invariant given that

$$t_y^* \omega = \omega$$

for $y \in V$, where t_y^* is the pullback of the map $t_y : SV \to SV$ given by

$$t_y(x,u) = (x+y,u).$$

Definition. Given $K \in \mathcal{K}(V)$, the *tangent cone* to K at x is the set $T(K,x) := \mathrm{cl}\{y \in V : \exists \varepsilon > 0 \ x + \varepsilon y \in K\}$, where cl denotes the closure. The *normal cone to k at x* is defined by $N(K,x) = \{f \in V^* : f(y) \geq 0 \text{ for all } y \in T(K,x)\}$ (see [17]). The *normal cycle* is the Lipschitz submanifold of SV defined by

$$nc(K) = \{(x,u) \in SV : x \in \partial K, u \in N(K,x)\}$$

(see [8]). Let $\Omega^{k,l}$ denote the space of smooth translation invariant differential forms on SV with bidegree (k,l). Considering a special case of Theorem 5.2.1 in [17] we get the following lemma.

Lemma 7.5. *For $0 \leq i \leq n-1$ the map $\nu : \Omega^{i,n-i-1} \to \mathbf{Val}_i^\infty$ defined by*

$$\nu(\omega)(K) = \int_{nc(K)} \omega,$$

is surjective.

In the next subsection we will discuss the kernel of this map.

Lemma 7.5 is the main tool used in [16] to establish a Hard Lefschetz theorem for translation invariant valuations. A direct consequence of this result is the following theorem.

Theorem 7.6. *For $0 \leq i \leq n$ the spaces $\mathbf{Val}_i^\infty$ and $\mathbf{Val}_{n-i}^\infty$ are isomorphic as* $\mathrm{SO}(n)$*-modules.*

7.1.4 The Rumin-de Rham Complex

Describing the unit sphere bundle as a contact manifold yields the Rumin operator. This operator enables us to describe the kernel of the normal cycle map and fit the space $\mathbf{Val}_i^\infty$ into an exact sequence of SO(n)-modules.

We start by introducing the notion of a contact manifold (see [18]). Let M be a differentiable manifold of dimension n, TM its tangent bundle and $\xi \subset TM$ be a smooth field of hyperplanes , that is, at $p \in M$ $\xi_p \subset T_pM$ is of codimension 1. Locally ξ induces a 1-form up to multiplication by a smooth non-vanishing function $f : M \to \mathbb{R}$ via $\xi_p = \ker \alpha_p$ with $\alpha_p \in T_p^*M \setminus \{0\}$. Note that for α to be globally defined in this way, some extra condition has to be met, namely ξ has to be coorientable.

Definition. Let M be a differentiable manifold of dimension $2n+1$. A *contact structure* is a maximally non-integrable hyperplane field $\xi = \ker \alpha \subset TM$, that is, the (locally) defining differential 1–form α is required to satisfy $\alpha \wedge (d\alpha)^n \neq 0$. Such an α is called a *contact form,* and the pair (M, ξ) is called *contact manifold.*

Note that the condition $\alpha \wedge (d\alpha)^n \neq 0$ is independent of the choice of α, as $(f\alpha) \wedge (df\alpha)^n = f^{n+1}\alpha \wedge (d\alpha)^n$.

Returning to our setting, we have the $(2n-1)$-dimensional manifold SV, which becomes a contact manifold with the canonical contact form

$$\alpha_{(x,u)}(w) = \langle u, d_{(x,u)}\pi(w)\rangle$$

for $w \in T_{(x,u)}SV$, where $\pi : SV \to V$ is the canonical projection (see [8], [11] and [16]).

Lemma 7.7. *Let M be a smooth $(2n-1)$-dimensional contact manifold with global contact form α. If $\omega \in \Omega^{n-1}(M)$, then there exists a unique differential form $D\omega \in \Omega^n(M)$ such that $D\omega$ annihilates the contact distribution and such that there exists $\zeta \in \Omega^{n-2}(M)$ with $D\omega = d(\omega + \alpha \wedge \zeta)$ and $D\omega \wedge \alpha = 0$. D is called the Rumin operator.*

With the Rumin operator we are able to describe the kernel of the normal cycle map. As stated in [15], the following theorem is a special case of Theorem 1 in [16].

Theorem 7.8. *Given the normal cycle map $\nu : \Omega^{i,n-i-1} \to \mathbf{Val}_i^\infty$, then for $0 \leq i \leq n-1$, $\omega \in \ker \nu$ if and only if $D\omega = 0$ and $\pi_*\omega = 0$.*

Our goal is to fit $\mathbf{Val}_i^\infty$ into an exact sequence of SO(n)-modules. The product structure of SV induces a bigrading on $\Omega^*(SV)$, the space of all complex valued smooth differential forms on SV. By $\Omega^{k,l}(SV)$ we denote the space of smooth differential forms on SV of bidegree (k,l). Therefore we get the decomposition

$$\Omega^*(SV) = \bigoplus \Omega^{k,l}(SV).$$

Recall that $\Omega^{k,l} \subseteq \Omega^{k,l}(SV)$ is the subspace of translation invariant forms. Furthermore we introduce to following subspaces:

The ideal generated by α and $d\alpha$ where α is the contact form of SV

$$\mathcal{I}^{i,j} := \{\omega \in \Omega^{i,j} : \omega = \alpha \wedge \xi + d\alpha \wedge \phi, \xi \in \Omega^{i-1,j}, \phi \in \Omega^{i-1,j-1}\};$$

the subspace of *vertical* forms

$$\Omega_v^{i,j} := \{\omega \in \Omega^{i,j} : \alpha \wedge \omega = 0\};$$

and the subspace of *horizontal* forms

$$\Omega_h^{i,j} := \Omega^{i,j} / \Omega_v^{i,j}.$$

We fix a point $u_0 \in S^{n-1}$ and let $\mathrm{SO}(n-1)$ be realized as the *stabilizer* of $\mathrm{SO}(n)$ at u_0, that is, $\mathrm{SO}(n-1) \cong \mathrm{SO}(n)_{u_0} := \{A \in \mathrm{SO}(n) : Au_0 = u_0\}$. Let $W_0 = T_{u_0}S^{n-1} \otimes \mathbb{C}$ be the complexification of the tangent space of S^{n-1} at u_0.

Lemma 7.9. *For $i, j \in \mathbb{N}$ there is an isomorphism of $\mathrm{SO}(n)$-modules*

$$\Omega_h^{i,j} \cong \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^i W_0^* \otimes \Lambda^j W_0^*).$$

Corollary 7.10. *If $i, j \in \mathbb{N}$ are such that $i + j \leq n - 1$, then there is an isomorphism of $\mathrm{SO}(n)$-modules*

$$\Omega_p^{i,j} \oplus \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^{i-1} W_0^* \otimes \Lambda^{j-1} W_0^*) \cong \mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^i W_0^* \otimes \Lambda^j W_0^*).$$

The subspace of *primitive* forms is defined by

$$\Omega_p^{i,j} := \Omega^{i,j} / \mathcal{I}^{i,j}.$$

The primitive forms are of particular interest, considering the space $\mathbf{Val}_i$ fits into an exact sequence of spaces of primitive forms, as shown in [15]. Note that $d\mathcal{I}^{i,j} \subseteq \mathcal{I}^{i,j+1}$, thus the exterior derivative induces a linear operator

$$d_Q : \Omega_p^{i,j} \to \Omega_p^{i,j+1}.$$

Theorem 7.11. *For $0 \leq i \leq n$ there exists an exact sequence*

$$0 \to \Lambda^i V_{\mathbb{C}}^* \hookrightarrow \Omega_p^{i,0} \xrightarrow{d_Q} \Omega_p^{i,1} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega_p^{i,n-i-1} \xrightarrow{\nu} \mathbf{Val}_i^\infty \to 0.$$

The natural smooth action of $\mathrm{SO}(n)$ on SV is given by

$$l_\vartheta(x, u) = (\vartheta x, \vartheta u)$$

with $\vartheta \in \mathrm{SO}(n)$. The vector spaces $\Omega^{k,l}$ become $\mathrm{SO}(n)$-modules under the continuous action

$$\vartheta \cdot \omega = l_{\vartheta^{-1}}^* \omega$$

for $\vartheta \in \mathrm{SO}(n)$ and $\omega \in \Omega^{k,l}$. As d_Q and ν are both $\mathrm{SO}(n)$-equivariant, we get the following corollary:

Corollary 7.12. *For $0 \leq i \leq n$ there is an exact $\mathrm{SO}(n)$-equivariant sequence of $\mathrm{SO}(n)$-modules*

$$0 \to \Lambda^i V_{\mathbb{C}}^* \hookrightarrow \Omega_p^{i,0} \xrightarrow{d_Q} \Omega_p^{i,1} \xrightarrow{d_Q} \cdots \xrightarrow{d_Q} \Omega_p^{i,n-i-1} \xrightarrow{\nu} \mathbf{Val}_i^\infty \to 0.$$

7.2 Statement and Proof

Following Alesker, Bernig and Schuster (see [8]) we formulate and proof the generalized Hadwiger characterization theorem and see that the Hadwiger theorem is indeed a special case of this result.

Before we can prove the generalized Hadwiger characterization theorem we prove an equivalent result, that describes the decomposition of $\mathbf{Val}_i$ into irreducible $\mathrm{SO}(n)$-modules.

Theorem 7.13. *Let $0 \leq i \leq n$. The vector space $\mathbf{Val}_i$ is the direct sum of the irreducible representations of $\mathrm{SO}(n)$ with highest weights $(\lambda_1, \lambda_2, ..., \lambda_{\lfloor n/2 \rfloor})$ precisely satisfying the following conditions:*

$$\begin{aligned} (i)&\ \lambda_j = 0 \text{ for } j > \min\{i, n-i\}; \\ (ii)&\ |\lambda_j| \neq 1 \text{ for } 1 \leq j \leq \lfloor n/2 \rfloor; \\ (iii)&\ |\lambda_2| \leq 2. \end{aligned} \tag{16}$$

In particular, under the action of $\mathrm{SO}(n)$ the space $\mathbf{Val}_i$ is multiplicity free.

Proof. We have seen that any representation of a compact group decomposes into irreducible summands, so $\mathbf{Val}_i \cong \bigoplus m_\lambda \Gamma_\lambda$ where m_λ is the multiplicity of the irreducible $\mathrm{SO}(n)$-module Γ_λ in $\mathbf{Val}_i$ and the direct sum ranges over all highest weights. Denote by S the set of highest weights satisfying the conditions (16), then we need to show that $m(\mathbf{Val}_i, \lambda) = 1$ for $\lambda \in S$ and $m(\mathbf{Val}_i, \lambda) = 0$ otherwise. The cases $i = 0$ and $i = n$ are trivial (see equation (14)). Due to Theorem 7.6 we only need to deal with the cases where $n/2 \leq i < n$.

Let Γ_λ be an arbitrary $\mathrm{SO}(n)$-module with highest weight $\lambda = (\lambda_1, ..., \lambda_{\lfloor n/2 \rfloor})$. $\Lambda^i W_0^* \otimes \Lambda^j W_0^*$ is finite-dimensional, so the multiplicity $m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)}(\Lambda^i W_0^* \otimes \Lambda^j W_0^*), \lambda)$ is finite-dimensional due to (12), and therefore the multiplicity of Γ_λ in the modules $\Omega^{i,j}$ is finite as well. Because the spaces $\mathbf{Val}_i^\infty$ are quotients of $\Omega_p^{i,n-i-1}$ according to Corollary 7.12 the multiplicity of Γ_λ in $\mathbf{Val}_i^\infty$ is finite too. According to the exact sequence in Corollary 7.12, $\mathbf{Val}_i^\infty \cong \Omega_p^{i,n-i-1}/\ker(\nu) = \Omega_p^{i,n-i-1}/d_Q(\Omega_p^{i,n-i-2})$. For an arbitrary G-module A with submodule B and an irreducible G-module Γ we have $\mathrm{Hom}_G(A/B, \Gamma) = \mathrm{Hom}_G(A, \Gamma)/\mathrm{Hom}_G(B, \Gamma)$ and therefore

$$\dim(\mathrm{Hom}_G(A/B, \Gamma)) = \dim(\mathrm{Hom}_G(A, \Gamma)) - \dim(\mathrm{Hom}_G(B, \Gamma)).$$

So we get

$$m(\mathbf{Val}_i, \lambda) = m(\Omega_p^{i,n-i-1}, \lambda)) - m(d_Q(\Omega_p^{i,n-i-2}, \lambda)$$

and repeating this procedure for

$$d_Q(\Omega_p^{i,n-i-k}) \cong \Omega_p^{i,n-i-2}/\ker(d_Q) = \Omega_p^{i,n-i-k}/d_Q(\Omega_p^{i,n-i-k-1})$$

results in

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} m(\Lambda^i V_{\mathbb{C}}, \lambda) + \sum_{j=0}^{n-i-1} (-1)^{n-1-i-j} m(\Omega_p^{i,j}, \lambda). \tag{17}$$

Let $W \cong W^*$ be the complex standard representation of $\mathrm{SO}(n-1)$, then by Corollary 7.10 and $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$ we get

$$\begin{aligned} m(\Omega_p^{i,j}, \lambda) = & \; m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} (\Lambda^i W \otimes \Lambda^j W), \lambda) \\ & - m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} (\Lambda^{i-1} W \otimes \Lambda^{j-1} W), \lambda). \end{aligned}$$

An application of Corollary 6.17, with n replaced by $n-1$ and $0 \leq j \leq n-i-1$, results in

$$m(\Omega_p^{i,j}, \lambda) = \sum m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda), \tag{18}$$

where $\bar{\Gamma}_\sigma$ is defined by (11) and the sum ranges over all $k := \lfloor (n-1)/2 \rfloor$-tuples of non-negative highest weights $\sigma = (\sigma_1, ..., \sigma_k)$ of $\mathrm{SO}(n-1)$-modules such that

$$\sigma_1 \leq 2 \quad \#(\sigma, 1) = n - 1 - i - j \quad \#(\sigma, 2) \leq j. \tag{19}$$

Denoting by $\mathcal{P}_i$ the set containing the k-tuples satisfying the conditions (19) and combining (17) and (18) we get

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} m(\Lambda^i V_{\mathbb{C}}, \lambda) + \sum_{\sigma \in \mathcal{P}_i} (-1)^{|\sigma|} m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda), \tag{20}$$

where $|\sigma|$ stands for the sum over all integers of the integer tuple σ.

We want to compute $\sum_{\sigma \in \mathcal{P}_i} (-1)^{|\sigma|} m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda)$. Due to the Frobenius reciprocity Theorem 7.1

$$m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda) = m(\mathrm{Res}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \Gamma_\lambda, \bar{\Gamma}_\sigma).$$

Now we use the branching theorem for $\mathrm{SO}(n)$, Theorem 7.2, and get

$$m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda) = m(\bigoplus_\mu \Gamma_\mu, \bar{\Gamma}_\sigma),$$

where μ satisfies the conditions (13). In order to account for the definition of $\bar{\Gamma}_\sigma$ and the conditions we have derived in (19) for σ let $\lambda^* = (\lambda_1^*, ..., \lambda_{\lfloor n/2 \rfloor}^*)$, where $\lambda_1^* := \min(\lambda_1, 2)$ and $\lambda_j^* := |\lambda_j|$ for $0 < j \leq \lfloor n/2 \rfloor$ and we achieve that

$$\sum_{\sigma \in \mathcal{P}_i} (-1)^{|\sigma|} m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda) = \sum_\mu (-1)^{|\mu|},$$

where the sum on the right ranges over all sequences $\mu = (\mu_1, ..., \mu_k)$ with $\mu_{n-i} = 0$ and

$$\begin{cases} \lambda_1^* \geq \mu_1 \geq \lambda_2^* \geq \mu_2 \geq ... \geq \mu_{k-1} \geq \lambda_{\lfloor n/2 \rfloor}^* \geq |\mu_k| & \textit{for odd } n, \\ \lambda_1^* \geq \mu_1 \geq \lambda_2^* \geq \mu_2 \geq ... \geq \mu_k \geq |\lambda_{n/2}^*| & \textit{for even } n. \end{cases}$$

If $\lambda^*_{n-i+1} > 0$ there is no such sequence (as $\mu_{n-i} = 0$), so we will continue with $\lambda^*_{n-i+1} = 0$ and obtain

$$\sum_\mu (-1)^{|\mu|} = \prod_{j=1}^{n-i-1} \sum_{\mu_j=\lambda^*_{j+1}}^{\lambda^*_j} (-1)^{\mu_j}.$$

These sums and the resulting product are zero, given that the λ^*_j for $0 \leq j \leq n-1$ have different parity. So we just consider the cases where the λ^*_j have the same parity, which leads to further simplification

$$\prod_{j=1}^{n-i-1} \sum_{\mu_j=\lambda^*_{j+1}}^{\lambda^*_j} (-1)^{\mu_j} = (-1)^{(n-i-1)\lambda^*_1},$$

as $\sum_{\mu_j=\lambda^*_{j+1}}^{\lambda^*_j} (-1)^{\mu_j} = (-1)^{\lambda^*_j} = (-1)^{\lambda^*_1}$. Recall that $m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda) = m(\mathrm{Res}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \Gamma_\lambda, \bar{\Gamma}_\sigma)$, so we obtain for $i > n/2$

$$\sum_{\sigma \in \mathcal{P}_i} (-1)^{|\sigma|} m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda) = \begin{cases} (-1)^{n-i-1} & \text{if } \Gamma_\lambda \cong \Lambda^{n-i} V_\mathbb{C}, \\ 1 & \text{if } \lambda \in S, \\ 0 & \text{otherwise.} \end{cases}$$

In case $i = n/2$, so n has to be even, we get

$$\sum_{\sigma \in \mathcal{P}_i} (-1)^{|\sigma|} m(\mathrm{Ind}_{\mathrm{SO}(n-1)}^{\mathrm{SO}(n)} \bar{\Gamma}_\sigma, \lambda) = \begin{cases} (-1)^{i-1} & \text{if } \lambda = (1, ..., 1, \pm 1), \\ 1 & \text{if } \lambda \in S, \\ 0 & \text{otherwise.} \end{cases}$$

We are now able to calculate $m(\mathbf{Val}_i, \lambda)$, while differentiating between the cases above. First we consider $i > n/2$:

$\lambda = (1, ..., 1, 0, ..., 0)$, that is, $\Gamma_\lambda \cong \Lambda^i V_\mathbb{C} \cong \Lambda^{n-i} V_\mathbb{C}$:

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} \underbrace{m(\Lambda^i V_\mathbb{C}, \lambda)}_{=1} + (-1)^{n-i-1} = 0.$$

$\lambda \in S$:

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} \underbrace{m(\Lambda^i V_\mathbb{C}, \lambda)}_{=0} + 1 = 1.$$

for all other λs:

$$m(\mathbf{Val}_i, \lambda) = (-1)^{n-i} \underbrace{m(\Lambda^i V_\mathbb{C}, \lambda)}_{=0} + 0 = 0.$$

If $i = n/2$, we obtain:

$\lambda = (1, ..., 1, \pm 1)$, recall that $\Lambda^{n/2} V_{\mathbb{C}} \cong \Gamma_{(1,...,1)} \oplus \Gamma_{(1,...,1,-1)}$:

$$m(\mathbf{Val}_{n/2}, \lambda) = (-1)^{n/2} \underbrace{m(\Lambda^{n/2} V_{\mathbb{C}}, \lambda)}_{=0} + (-1)^{n/2-1} = 0.$$

$\lambda \in S$:

$$m(\mathbf{Val}_{n/2}, \lambda) = (-1)^{n/2} \underbrace{m(\Lambda^{n/2} V_{\mathbb{C}}, \lambda)}_{=0} + 1 = 1.$$

for all other λs:

$$m(\mathbf{Val}_{n/2}, \lambda) = (-1)^{n/2} \underbrace{m(\Lambda^{n/2} V_{\mathbb{C}}, \lambda)}_{=0} + 0 = 0.$$

So we have shown that $m(\mathbf{Val}_i, \lambda) = 1$ if and only if $\lambda \in S$ for $1 \leq i \leq n$, that is, λ satisfies the conditions (16). ☐

Our next goal is to reformulate this theorem in terms of continuous translation invariant $\mathrm{SO}(n)$-equivariant i-homogeneous valuations.

Definition. The vector space of maps $f : X \to Y$ between two G-sets X, Y becomes a G-module with the natural action

$$g \cdot f(x) := g \cdot \phi(g^{-1} \cdot x), g \in G, x \in X.$$

Denote by $\Gamma\mathbf{Val}$ the space of non-trivial continuous translation invariant valuations with values in Γ. This space becomes an $\mathrm{SO}(n)$-module with the natural action from above. We recall what equivariance means in our setting of valuations: Given a Lie group G and a finite-dimensional G-module Γ, $\phi \in \Gamma\mathbf{Val}$ is called G-equivariant if

$$\phi(gK) = g \cdot \phi(K), g \in G, K \in \mathcal{K}(V).$$

By $\Gamma\mathbf{Val}^G$ we denote the subspace of G-equivariant elements of the module $\Gamma\mathbf{Val}$. The above notation coincides with the notation for G-invariant elements of a G-module, however the following lemma clarifies.

Proposition 7.14. *A map $f : X \to Y$ between two G-sets X, Y is equivariant if and only if it is invariant under the natural G-action on the space of maps between G-modules.*

Proof. Considering f is G-equivariant and evaluating at $g^{-1}K$ we get $\phi(K) = g \cdot \phi(g^{-1}K) = g \cdot \phi(K)$, where the last action is the natural action on the space of functions between G-modules, therefore f is G-invariant. The other direction is similar. ☐

Lemma 7.15. *Given a finite-dimensional $SO(n)$-module Γ, then $\iota : \mathbf{Val} \otimes \Gamma \to \Gamma\mathbf{Val}$ defined by $\iota(\phi \otimes v) = \phi \cdot v$ is an isomorphism of vector spaces. This statement holds true if we restrict to valuations of degree i.*

Proof. Let $\iota : \mathbf{Val} \otimes \Gamma \to \Gamma\mathbf{Val}$ be defined by $\iota(\phi \otimes v) = \phi v$. Then ι is injective because its kernel consists only of the zero vector. To show that ι is surjective, take $\phi \in \Gamma\mathbf{Val}$ a basis $(b_i)_{i=1}^k$ of Γ and let $b_i^*(\phi)$ be defined via $b_i^*(\phi)(K) := b_i^*(\phi(K)), K \in \mathcal{K}(\Gamma)$ where b_i^* is the dual element to b_i. It is easy to see that the $b_i^*(\phi)$ are continuous translation invariant valuations, so an arbitrary $\phi \in \Gamma\mathbf{Val}$ can be expressed as $\phi = \sum_{i=1}^k b_i^*(\phi)b_i$, and thus ι is surjective. Given valuations of degree i, the $b_i^*(\phi)$ are of degree i as well, completing the proof. $\square$

Given a representation Γ the representation on the dual space Γ^* is given by the action

$$(A \cdot f)(u) = f(A^{-1}u)$$

for $A \in \mathrm{SO}(n), f \in \Gamma^*, u \in \Gamma$. A representation is called *self-dual* given that Γ and Γ^* are isomorphic. The following lemma (see Lemma 3.2 in [8]) shows that if a representation Γ satisfies the conditions (16), then Γ^* satisfies these conditions as well.

Lemma 7.16. *Let $\lambda = (\lambda_1, ..., \lambda_{\lfloor n/2 \rfloor})$ be a tuple of integers satisfying the conditions (16). Given that $n \not\equiv 2 \mod 4$, then all irreducible representations Γ_λ of $\mathrm{SO}(n)$ are self-dual. If $n \equiv 2 \mod 4$, then the dual of an irreducible representation Γ_λ is given by $\Gamma_{(\lambda_1,...,\lambda_{n/2-1},-\lambda_{n/2})}$.*

Now we can formulate the main theorem of this part: a Hadwiger-type characterization theorem for continuous translation invariant $\mathrm{SO}(n)$-equivariant i-homogeneous valuations with values in a $\mathrm{SO}(n)$-irreducible space Γ. It is a consequence of Theorem 7.13 - in fact it is even equivalent.

Theorem 7.17. *Let Γ be a finite-dimensional irreducible $\mathrm{SO}(n)$ representation and let $0 \leq i \leq n$. There exists a non-trivial continuous translation invariant $\mathrm{SO}(n)$-equivariant valuation of degree i with values in Γ if and only if the highest weight of Γ satisfies the conditions (16). This valuation is unique up to scaling.*

Proof. Let Γ_μ be a finite-dimensional irreducible $\mathrm{SO}(n)$-module of dimension k and $\Gamma_\mu\mathbf{Val}_i$ be the space of non-trivial continuous translation invariant valuations of degree i with values in Γ_μ. Theorem 7.13 states that $\mathbf{Val}_i = \bigoplus_{\lambda \in S} \Gamma_\lambda$ where S is the set containing all highest weights satisfying the conditions (16). We want to calculate the dimension of the vector space of the $\mathrm{SO}(n)$-equivariant maps of $\Gamma_\lambda\mathbf{Val}_i$. Due to Lemma 7.15 we get

$$\begin{aligned}\dim(\Gamma_\mu\mathbf{Val}_i)^{\mathrm{SO}(n)} &= \dim(\mathbf{Val}_i \otimes \Gamma_\mu)^{\mathrm{SO}(n)} = \dim(\bigoplus_{\lambda \in S} \Gamma_\lambda \otimes \Gamma_\mu)^{\mathrm{SO}(n)} \\ &= \dim(\bigoplus_{\lambda \in S} \Gamma_\lambda^{\mathrm{SO}(n)} \otimes \Gamma_\mu^{\mathrm{SO}(n)}) = \sum_{\lambda \in S} \dim(\bigoplus_{\lambda \in S} \Gamma_\lambda \otimes \Gamma_\mu)^{\mathrm{SO}(n)},\end{aligned}$$

where the second to last equation holds, as $\mathrm{SO}(n)$ acts component wise on tensor products and sums. Because $\phi : V^* \otimes W \to \mathrm{Hom}(V, W)$ given by $f \otimes w \mapsto f(\cdot)w$

is an isomorphism which translates SO(n)-invariance into SO(n)-equivariance, we get that

$$\sum_{\lambda \in S} \dim(\bigoplus_{\lambda \in S} \Gamma_\lambda \otimes \Gamma_\mu)^{\mathrm{SO}(n)} = \sum_{\lambda \in S} \dim \ \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda^*, \Gamma_\mu)$$
$$= \sum_{\lambda \in S} \dim \ \mathrm{Hom}_{\mathrm{SO}(n)}(\Gamma_\lambda, \Gamma_\mu) = \begin{cases} 1 & \text{if } \mu \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The second equation above holds due to Lemma 7.16 and the last one is a consequence of Schur's lemma, so the proof is complete. □

Example. Consider the trivial representation $\Gamma_{(0,\dots,0)}$ of SO(n), then λ satisfies the conditions (16), thus by Theorem 7.17 there exists a non-trivial continuous rigid motion invariant valuation of degree i which is unique up to scaling. So the Hadwiger characterization Theorem 2.8 is a special case of Theorem 7.17.

Example. Given the standard representation $\Gamma_{(1,\dots,0)} \cong V_{\mathbb{C}}$ of SO(n), we see that there is no non-trivial continuous translation invariant valuation of degree i, because λ does not satisfy the conditions (16).

8 Unitary Steiner Points

Klimyk [20] introduced a formula for decomposing the tensor product of irreducible representations Γ_1, Γ_2, that is easier to calculate than the formula of Steinberg [32], however requires the knowledge of all weights of Γ_1 or Γ_2. Given a compact symmetric pair (G, K) a theorem of Helgason [24] provides a bijection from the spherical representations of (G, K) to $\mathbb{Z}$-linear combinations of specific fundamental weights. With these two results and the general Hadwiger characterization theorem, Wannerer [23] calculated the dimensions of the spaces of continuous translation invariant $\mathrm{U}(n)$-equivariant valuations of degree i with values in $\mathbb{C}^n$ for $n \geq 3$. Furthermore he characterized the Steiner point map as the continuous Minkowski additive unitary affine transformation-equivariant map from $\mathcal{K}(\mathbb{C}^n)$ to $\mathbb{C}^n$. Recently Böröczky, Domokas and Solanes [27] calculated the dimensions of the space of translation invariant unitary equivariant tensor valuations for $n \geq 2$ using branching, in particular a theorem of King, and the generalized Hadwiger theorem we encountered above. Additionally, with the work of Wannerer [28], they were able to provide a basis for the vector valued case.

8.1 Decomposition of Tensor Products of Irreducible Representations - Klimyk's Formula

In this subsection we want to decompose a tensor product of irreducible representations into a direct sum of irreducible representations. We use a formula taken from Klimyk's Paper [20] and apply it to the $\mathrm{SO}(2n)$ representation $\mathbb{C}^{2n} \otimes \Gamma_\lambda$.

Let us consider a semisimple complex Lie algebra $\mathfrak{g}$ with irreducible representations Γ_{λ_1} and Γ_{λ_2}. Then the tensor product of these representations is fully reducible, that is, it is decomposable into a direct sum of irreducible representations:

$$\Gamma_{\lambda_1} \otimes \Gamma_{\lambda_2} = \sum_{\mu} m_\mu \Gamma_\mu \tag{21}$$

where the summation is over all highest weights μ of $\mathfrak{g}$ and m_μ is the multiplicity of Γ_μ in $\Gamma_{\lambda_1} \otimes \Gamma_{\lambda_2}$. There exists an explicit formula from Steinberg [32] to calculate m_μ, however given a large Weyl group, it is not easy to apply (see [21] for more detail). Given that the weights of one of the representations are known, the formula can be significantly simplified.

Let P be the set of algebraically integral weights and P_+ be the set of dominant weights in P. To see that our definition of algebraically integral weights matches the one in Klimyk's Paper see [5] Proposition 4.62. Given the Weyl Group $W = W(\Delta(\mathfrak{g}))$ we say, elements of P are equivalent if one can be obtained by the action of the Weyl group on the other. Denote by $\{\nu\}$ the dominant element equivalent to $\nu \in P$ and by $r := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g})} \alpha$ the half-sum of positive roots.

Theorem 8.1. *If Γ_{λ_1} and Γ_{λ_2} are irreducible representations of a semisimple Lie algebra $\mathfrak{g}$ and (21) holds, then*

$$m_\mu = \sum_{\substack{\nu \\ \{\nu+\lambda_2+r\}=\mu+r}} m_\nu \; \beta_{\nu+\lambda_2+r},$$

where the summation is over all weights ν of Γ_{λ_1} such that $\{\nu+\lambda_2+r\} = \mu+r$ and

$$\beta_{\nu+\lambda_2+r} = \begin{cases} 0 & \text{if there exists } s \in W, s \neq e \text{ such that } s(\nu+\lambda_2+r) = \\ & \nu+\lambda_2+r, \\ \det t & t \in W \text{ if such } s \text{ does not exist and } t(\nu+\lambda_2+r) = \\ & \{\nu+\lambda_2+r\}. \end{cases}$$

We want to apply this formula to the $\mathrm{SO}(2n)$ representation $\mathbb{C}^{2n} \otimes \Gamma_\lambda$ where $\mathbb{C}^{2n}$ is the standard representation of $\mathrm{SO}(2n)$ with the highest weight $\lambda_1 = (1, 0, ..., 0)$ and Γ_λ is some irreducible $\mathrm{SO}(2n)$-module. We have seen that the algebraically integral weights are given by

$$P = \left\{ \sum_{i=i}^{l} \left(\lambda_i + \frac{\lambda_0}{2}\varepsilon_i \right) : \lambda_i \in \mathbb{Z} \right\}.$$

Recall that for a simple root $\alpha \in \Pi(\mathfrak{so}(E_{2n})_\mathbb{C})$ the h_α are given by $h_{\varepsilon_i-\varepsilon_{i+1}} = (E_i - E_{i+1}) - (E_{i+n} - E_{i+n+1})$ and $h_{\varepsilon_i+\varepsilon_{i+1}} = (E_i + E_{i+1}) - (E_{i+n} + E_{i+n+1})$. So for $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ the condition $B(\lambda, \alpha) = B(h_\lambda, h_\alpha) = \lambda(h_\alpha) \geq 0$ is equivalent to the system of equations

$$\lambda_i - \lambda_{i+1} \geq 0, \qquad \text{for} \qquad 1 \leq i \leq n-1;$$

$$\lambda_{n-1} + \lambda_n \geq 0,$$

resulting in the condition

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n| \,.$$

Therefore the dominant algebraically integral weights are given by

$$P_+ = \{\lambda \in P : \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|\}.$$

We take a maximal torus $T = \{\mathrm{diag}(e^{i\theta_1}, ..., e^{i\theta_l}, e^{-i\theta_1}, ..., e^{-i\theta_l}), \theta_i \in \mathbb{R}\}$ of $\mathrm{SO}(E_{2n})$. The Weyl group $W = W(\mathrm{SO}(E_{2n}), T)$ of $\mathrm{SO}(E_{2n})$ is given by the semidirect product

$$W \cong S_n \ltimes (\mathbb{Z}_2)^{n-1}$$

and it acts on $(\theta_1, ..., \theta_n) \in i\mathfrak{t}$ and $(\lambda_1, ..., \lambda_n) \in (i\mathfrak{t})^*$ by all permutations and even sign changes of the coordinates. Check [4] page 138 for further details.

The positive roots $\Delta^+(\mathfrak{so}(E_{2n})_\mathbb{C})$ are given by $\{\varepsilon_i - \varepsilon_j : 1 \le i < j \le n\} \cup \{\varepsilon_i + \varepsilon_j : 1 \le i < j \le n\}$ and we calculate

$$r = \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{so}(E_{2n})_\mathbb{C})} \alpha = \sum_{i=0}^{n} \varepsilon_i.$$

We are now able to calculate the multiplicities m_μ in the decomposition of $\mathbb{C}^{2n} \otimes \Gamma_\lambda = \sum_\mu m_\mu \Gamma_\mu$ into a direct sum of irreducible representations using Theorem 8.1:

$$m_\mu = \sum_{\substack{\nu \\ \{\nu+\lambda+(1,\dots,1)\}=\mu+(1,\dots,1)}} m_\nu \ \beta_{\nu+\lambda+(1,\dots,1)},$$

where the summation is over all weights of $\mathbb{C}^{2n}$, that is, $\{\pm\varepsilon_i : 1 \le i \le n\}$, and m_ν is the multiplicity of ν in $\mathbb{C}^{2n}$. Using that m_ν is 1 it follows that

$$\mathbb{C}^{2n} \otimes \Gamma_\lambda = \sum_\mu \Gamma_\mu,$$

where the summation ranges over all highest weights μ of $\mathrm{SO}(2n)$ that can be expressed as $\mu = \lambda \pm \varepsilon_i$ for some $1 \le i \le n$.

8.2 Theorem of Helgason

We have recalled the theory for the classification of highest weights of a compact Lie group G. In this section, we want to extend this theory to a compact Lie group with a compact subgroup K and an involutive automorphism θ on G. θ decomposes the Lie algebra $\mathfrak{g}$ into a direct sum of two vector spaces, one corresponding to the Lie algebra $\mathfrak{k}$ of K. A theory of weights and roots, similar to the one we have already encountered, relative to this decomposition is described. Our goal is the classification of spherical representations via $D(G, K)$, which is the equivalent to the dominant analytically integral weights relative to the decomposition of $\mathfrak{g}$. This provides an additional condition for highest weights of $\mathrm{SO}(2n)$ that have $\mathrm{U}(n)$ invariant elements. We will follow Takeuchi's "Modern Spherical Functions" [24] and give a rough sketch of the underlying theory.

Definition. Let G be a connected Lie group and K be a compact subgroup of G. An irreducible representation of G with a K-invariant element is called a spherical representation, and by $\mathcal{D}(G, K)$ we denote the set of all equivalence classes of spherical representations.

Definition. Let G be a connected Lie group, and let K be a compact subgroup of G. The pair (G, K) is called a *Riemannian symmetric pair* if there exists an involutive C^∞ automorphism θ of G such that $G_\theta^0 \subseteq K \subseteq G_\theta$ where $G_\theta = \{x \in G : \theta(x) = x\}$ and G_θ^0 is the identity component of G_θ. Given that G is compact, a Riemannian symmetric pair (G, K) is called a *compact symmetric pair.*

Denoting the differential of θ by θ as well and the Lie algebra of G by $\mathfrak{g}$, the Lie algebra $\mathfrak{k}$ of the subgroup K is given by

$$\mathfrak{k} = \{X \in \mathfrak{g} : \theta(X) = X\}.$$

Setting

$$\mathfrak{m} = \{X \in \mathfrak{g} : \theta(X) = -X\},$$

and considering θ is involutive, we get a vector space decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is called the canonical complement of the pair (G, K) or the pair $(\mathfrak{g}, \mathfrak{k})$. Because θ is an automorphism of $\mathfrak{g}$, we have $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{k}$.

Definition. Let (G, K) be a Riemannian symmetric pair. A maximal Abelian subalgebra which is contained in the canonical complement $\mathfrak{m}$ is called a Cartan subalgebra of the pair (G, K).

Let $\mathfrak{a}$ be a Cartan subalgebra of the pair (G, K) and $\mathfrak{t}$ a Cartan subalgebra of G containing $\mathfrak{a}$, then $\mathfrak{t}$ decomposes into $\mathfrak{t} = \mathfrak{a} \oplus \mathfrak{b}$ where $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$. Now take an inner product$(\cdot, \cdot)$ on $\mathfrak{g}$ which is invariant under G and θ, and fix it. Notice that as θ is an involution, there always exists such an inner product. Let us denote by $O(\mathfrak{t})$ the group of orthogonal transformations of $\mathfrak{t}$ with respect to the inner product $(\cdot, \cdot)$. We define $\sigma \in O(\mathfrak{t})$ by

$$\sigma(H) = \sigma(H_1 + H_2) = -H_1 + H_2, H_1 \in \mathfrak{b}, H_2 \in \mathfrak{a}.$$

For $0 \neq \alpha \in \mathfrak{t}$ set $\alpha^* := \frac{2\alpha}{(\alpha,\alpha)}$ which is called the *inversion* of α. The following definition of the roots of $\mathfrak{g}$ relative to the Cartan subalgebra $\mathfrak{t}$ is, up to a constant, equivalent to the one we used before.

Definition. For $\alpha \in \mathfrak{t}$ set

$$\tilde{\mathfrak{g}}_\alpha = \{X \in \mathfrak{g}_\mathbb{C} : [H, X] = 2\pi i(\alpha, H)X, H \in \mathfrak{t}\}$$

the *root subspace* associated with α and

$$\Sigma(G) = \{\alpha \in \mathfrak{t} : \alpha \neq 0, \tilde{\mathfrak{g}}_\alpha \neq \{0\}\}$$

the set of *roots* of G relative to $\mathfrak{t}$. This leads to a *decomposition of* $\mathfrak{g}_\mathbb{C}$ *into the root subspaces relative to* $\mathfrak{t}$:

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} + \sum_{\alpha \in \Sigma(G)} \tilde{\mathfrak{g}}_\alpha.$$

By definition $(\alpha, \mathfrak{z}(\mathfrak{g})) = \{0\}$ and therefore, as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}'$, $\Sigma(G) \subseteq \mathfrak{t}' = \mathfrak{t} \cap \mathfrak{g}'$.

In a similar fashion we can define roots relative to the maximal abelian subalgebra $\mathfrak{a}$ contained in $\mathfrak{m}$.

Definition. For $\gamma \in \mathfrak{a}$ set

$$\tilde{\mathfrak{g}}_\gamma = \{X \in \mathfrak{g}_\mathbb{C} : [H, X] = 2\pi i(\gamma, H)X, H \in \mathfrak{a}\}$$

the *root subspace* associated with γ and

$$\Sigma(G, K) = \{\gamma \in \mathfrak{t} : \gamma \neq 0, \tilde{\mathfrak{g}}_\gamma \neq \{0\}\}$$

the set of *roots* of G relative to $\mathfrak{a}$. This leads to a *decomposition of* $\mathfrak{g}_\mathbb{C}$ *into the root subspaces relative to* $\mathfrak{a}$:

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} + \sum_{\alpha \in \Sigma(G,K)} \tilde{\mathfrak{g}}_\alpha.$$

We set $\Sigma_0(G) = \Sigma(G) \cap \mathfrak{b}$, then $\Sigma(G, K) = \Sigma(G) \setminus \Sigma_0(G)$. There is a another approach (compare to Subsection 6.5) to the concepts of positive and simple roots: given a linear order on $\mathfrak{t}$, $\lambda \in \mathfrak{t}$ is called *positive* (*negative*) if $\lambda > 0$ ($\lambda < 0$). A positive root α is called *simple* if $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in \Sigma(G), \beta, \gamma > 0$. The *fundamental system* is the set of all simple roots of $\Sigma(G)$ with respect to the order $>$.

Definition. A linear order on $\mathfrak{t}$ is said to be a σ-order if for $\alpha \in \Sigma(G, K), \alpha > 0$ $\sigma(\alpha) > 0$ holds.

There always exists a σ-order on $\mathfrak{t}$. To see that, take a basis $\{H_1, ..., H_l\}$ of $\mathfrak{a}$ and a basis $\{H_{l+1}, ..., H_m\}$ of $\mathfrak{b}$ and define $\lambda > \mu$ if

$$(\lambda, H_1) = (\mu, H_1), ..., (\lambda, H_r) = (\mu, H_r), (\lambda, H_{r+1}) > (\mu, H_{r+1})$$

for $\lambda, \mu \in \mathfrak{t}$ and some $1 \leq r \leq m$. Then this order is a σ-order.

Definition. A fundamental system with respect to a σ-order $>$ is called a *σ-fundamental system*.

Let $m' = \dim(\mathfrak{t}')$ and $\Pi(G) = \{\alpha_1, ..., \alpha_{m'}\}$ be a fundamental system with respect to the σ-order $>$, then we denote $\Pi_0(G) = \Pi(G) \cap \Sigma_0(G)$.

Theorem 8.2. *Let* (G, K) *be a compact symmetric pair. Then* $\Sigma(G, K)$ *is a root system in* $\mathfrak{a}'$*, that is:*

- (i) $\Sigma(G, K)$ is a subset of $\mathfrak{a}'$ consisting of finite nonzero elements and spans $\mathfrak{a}'$ over $\mathbb{R}$;
- (ii) $\Sigma(G, K)$ is invariant under the reflection s_γ for every $\gamma \in \Sigma(G, K)$;
- (iii) $\dfrac{2(\delta, \gamma)}{(\gamma, \gamma)} \in \mathbb{Z}$ for every $\gamma, \delta \in \Sigma(G, K)$.

Furthermore if $\Pi(G)$ *is a* σ*-fundamental system, then there exists a permutation* p *of* $\Pi(G) \setminus \Pi_0(G)$ *of order 2 such that*

$$\sigma\alpha_i \equiv p\alpha_i \mod \{\Pi_0(G)\}_\mathbb{Z}, \quad 1 \leq i \leq m' - |\Pi_0(G)|,$$

where $\{\Pi_0(G)\}_\mathbb{Z}$ *denotes the subgroup of* $\mathfrak{t}$ *generated by* $\Pi_0(G)$*. The permutation* p *is called the Satake involution of the* σ*-fundamental system* $\Pi(G)$*.*

Again the positive roots are interesting enough to get their own notation

$$\Sigma^+(G) = \{\alpha \in \Sigma(G) : \alpha > 0\};$$
$$\Sigma^+(G,K) = \{\gamma \in \Sigma(G,K) : \gamma > 0\}.$$

Let $\{a_1^*, ..., a_{m'}^*\}$ be the inversions of the fundamental system $\{a_1, ..., a_{m'}\}$ of $\Sigma(G)$ with respect to the order $>$, and let $\{\Lambda_1, ..., \Lambda_{m'}\} \subseteq \mathfrak{t}'$ be the basis of $\mathfrak{t}'$ dual to the inversions, that is,

$$(\Lambda_i, \alpha_j^*) = \frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}, \quad 1 \leq i, j \leq m'.$$

$\Lambda_1, ...\Lambda_{m'}$ are called the *fundamental weights* of $\mathfrak{g}'$ with respect to the order $>$. We set $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$, $l' = \dim \mathfrak{a}'$, and considering the Satake involution p leaves $\Pi(G)$ invariant, we define $M_i, ..., M_{l'} \in \mathfrak{t}'$ by

$$M_i = \begin{cases} 2\Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_0(G)) = \{0\}, \\ \Lambda_i & \text{if } p\alpha_i = \alpha_i \text{ and } (\alpha_i, \Pi_0(G)) \neq \{0\}, \\ \Lambda_i + \Lambda_{i'} & \text{if } p\alpha_i = \alpha_{i'} \text{ and } \alpha_i \neq \alpha_{i'}. \end{cases}$$

We call $M_i, ..., M_{l'}$ the *fundamental weights* for the pair $(\mathfrak{g}', \mathfrak{t}')$ with respect to the σ-order $>$. These fundamental weights for the pair $(\mathfrak{g}', \mathfrak{t}')$ with respect to the σ-order $>$ will help us to describe the weight of a spherical representation in the case that G/K is simply connected.

Theorem 8.3. *Let (G, K) be a compact symmetric pair and let $\rho : G \to \mathrm{GL}(V)$ be a spherical representation of G relative to K, then the multiplicity m_ρ of ρ is identical to 1.*

Next up we develop something comparable to the dominant analytically integral weights we have already encountered, but this time we also have to take the subgroup K into account. The set

$$\Gamma(G,K) = \{H \in \mathfrak{a} : \exp H \in K\}$$

has the structure of a geometric lattice, that is, it is a discrete subgroup (isomorphic to $\mathbb{Z}^{\dim \mathfrak{a}}$) of the commutative additive group $\mathfrak{a}$ which contains a basis of $\mathfrak{a}$. Let $Z(G,K)$ be the lattice given by

$$Z(G,K) = \{\lambda \in \mathfrak{a} : (\lambda, H) \in \mathbb{Z}, H \in \Gamma(G,K)\}$$

and $D(G,K)$ be the semigroup given by

$$D(G,K) = \{\lambda \in Z(G,K) : (\lambda, \gamma) \geq 0, \gamma \in \Sigma^+(G,K)\}.$$

Similarly to the role of dominant analytically integral weights in the description of irreducible representation, we will see $D(G,K)$ playing its part in the description of spherical representations.

Theorem 8.4. *Let (G, K) be a compact symmetric pair such that G/K is simply connected, then*

$$D(G,K) = \left\{ \sum_{i=1}^{l'} m_i M_i, m_i \in \mathbb{Z}, m_i \geq 0, 1 \leq i \leq l' \right\}.$$

Theorem 8.5. *Let (G, K) be a compact symmetric pair, then the mapping*

$$\mathcal{D}(G,K) \to D(G,K)$$

sending ρ to its highest weight $\lambda(\rho)$ is a bijection.

Corollary 8.6. *Let (G, K) be a compact symmetric pair such that G/K is simply connected, then the mapping that sends $\rho \in \mathcal{D}(G, K)$ to its highest weight $\lambda(\rho) \in \mathfrak{a}$ is a bijection into the semigroup*

$$\left\{ \sum_{i=1}^{l'} m_i M_i, m_i \in \mathbb{Z}, m_i \geq 0, 1 \leq i \leq l' \right\}.$$

We want to apply what we have achieved to our specific case of interest: $\mathrm{SO}(2n)$ with the subgroup $\mathrm{U}(n)$. Recall that

$$\begin{aligned} \mathrm{SO}(2n) &= \{A \in \mathrm{gl}(2n, \mathbb{R}) : A^T A = I, \det A = 1\}; \\ \mathrm{U}(n) &= \{A \in \mathrm{gl}(n, \mathbb{C}) : A^* A = I\}. \end{aligned}$$

The map

$$\begin{aligned} i : \mathrm{U}(n) &\to \mathrm{gl}(2n, \mathbb{R}) \\ X + iY &\mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \end{aligned}$$

is an injective Lie group homomorphism that maps into $\mathrm{SO}(2n)$, as

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}^T = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \begin{pmatrix} X^T & Y^T \\ -Y^T & X^T \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

is equivalent to $(X+iY)(X+iY)^* = I_n$. As $X+iY$ is unitary $\det(X+iY) = \pm 1$. Considering the connected $\mathrm{U}(n)$ maps onto the connected component of $\mathrm{O}(2n)$ containing the identity, $\det(X + iY) = 1$.Thus we will identify $\mathrm{U}(n)$ with the subgroup of $\mathrm{SO}(2n)$ given by

$$\left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{SO}(2n) : (X+iY)(X+iY)^* = I_n \right\}.$$

Define

$$E = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

then

$$E^T = E^{-1} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and

$$\begin{aligned} \theta : \mathrm{SO}(2n) &\to \mathrm{SO}(2n) \\ G &\mapsto EGE^{-1} \end{aligned}$$

is a C^∞ involutive inner automorphism of $\mathrm{SO}(2n)$. Because

$$\theta(G) = \begin{pmatrix} G_4 & -G_3 \\ -G_2 & G_1 \end{pmatrix}$$

the fixed points of θ are given by

$$\mathrm{SO}(2n)_\theta = \{G \in \mathrm{SO}(2n) : \theta(G) = G\} = \mathrm{U}(n)$$

and $(\mathrm{SO}(2n), \mathrm{U}(n))$ is a compact symmetric pair. The Lie algebra corresponding to $\mathrm{SO}(2n)$ is given by

$$\mathfrak{so}(2n) = \{X \in \mathfrak{gl}(2n, \mathbb{R}) : X^T = -X\}.$$

Then $\mathfrak{so}(2n)$ decomposes into

$$\begin{aligned} \mathfrak{k} &= \{X \in \mathfrak{so}(2n) : \theta(X) = X\} \\ &= \left\{ \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathfrak{so}(2n) : X^T = -X, Y^T = -Y \right\} \end{aligned}$$

and

$$\begin{aligned} \mathfrak{m} &= \{X \in \mathfrak{so}(2n) : \theta(X) = -X\} \\ &= \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \in \mathfrak{so}(2n) : X^T = -X, Y^T = -Y \right\}. \end{aligned}$$

We recall that a Cartan subalgebra for $\mathfrak{so}(2n)$ was given by

$$\mathfrak{t} = \left\{ \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \theta_n \\ & & & -\theta_n & 0 \end{pmatrix} : \theta_i \in \mathbb{R} \right\}.$$

Set

$$H_{\theta_i} = 2\pi \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}$$

for $\theta_i \in \mathbb{R}$. For even n we can decompose $\mathfrak{t}$ into a maximal abelian subalgebra

$$\mathfrak{a} = \left\{ \begin{pmatrix} H_{\theta_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{\theta_{n/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -H_{\theta_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & -H_{\theta_{n/2}} \end{pmatrix} : \theta_i \in \mathbb{R}, 1 \leq i \leq n/2 \right\}$$

contained in $\mathfrak{m}$ and

$$\mathfrak{b} = \left\{ \begin{pmatrix} H_{\theta_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{\theta_{n/2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{\theta_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & H_{\theta_{n/2}} \end{pmatrix} : \theta_i \in \mathbb{R}, 1 \leq i \leq n/2 \right\}.$$

The inner product on $\mathfrak{so}(2n)$ given by $(X, Y) = -\mathrm{tr}(X, Y)$ is invariant under θ and $\mathrm{SO}(2n)$ (see Theorem 6.5). We write $H(\theta_1, ..., \theta_n)$ for the matrix $\mathrm{diag}(H_{\theta_1}, ..., H_{\theta_n})$. Then the functionals $\vartheta_i \in \mathfrak{t}^*$ are defined by

$$\vartheta_i(H(\theta_1, ..., \theta_n)) = \theta_i$$

and the roots relative to $\mathfrak{t}$ are given by

$$\Sigma(\mathrm{SO}(2n)) = \{\pm\vartheta_i \pm \vartheta_j : 1 \leq i < j \leq n\}$$

(check to [6] page 219). A basis of $\mathfrak{a}$ and $\mathfrak{b}$ is given by

$$\begin{aligned} a_1 &= H(1, 0, ..., 0, -1, 0, ..., 0), \\ a_2 &= H(0, 1, 0, ..., 0, 0, -1, 0, ..., 0), \\ &\vdots \\ a_{n/2} &= H(0, ..., 0, 1, 0, ...0, -1), \end{aligned}$$

and

$$\begin{aligned} b_1 &= H(1, 0, ..., 0, 1, 0, ..., 0), \\ b_2 &= H(0, 1, 0, ..., 0, 0, 1, 0, ..., 0), \\ &\vdots \\ b_{n/2} &= H(0, ..., 0, 1, 0, ...0, 1), \end{aligned}$$

respectively. We combine it to a basis $\{t_1, ..., t_n\}$ of $\mathfrak{t}$, where $t_1 = a_1, .., t_{n/2} = a_{n/2}, t_{n/2+1} = b_1, ..., t_n = b_{n/2}$. With this basis we define a σ-order on $\mathfrak{t}$: Let $\lambda, \mu \in \mathfrak{t}$, then $\lambda > \mu$ if and only if

$$\lambda(t_1) = \mu(t_1), ..., \lambda(t_r) = \mu(t_r), \lambda(t_{r+1}) > \mu(t_{r+1})$$

for some $1 \leq r \leq n$. With respect to this σ-order the positive roots are given by

$$\Sigma^+(\mathrm{SO}(2n)) = \{\vartheta_i \pm \vartheta_j : 1 \leq i \leq n/2, i < j \leq n\} \cup \\ \{-\vartheta_i \pm \vartheta_j : n/2 < i \leq n-1, i < j \leq n\}.$$

It can be shown that the application of Corollary 8.6 results in the following conditions for the highest weight $\lambda(\rho) = (\lambda_1, ..., \lambda_n)$ of a spherical representation ρ of the compact symmetric pair $(\mathrm{SO}(2n), \mathrm{U}(n))$:

$$\lambda(\rho) = \begin{cases} \lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq ... \geq \lambda_{n-1} = \lambda_n & \text{if } n \text{ is even,} \\ \lambda_1 = \lambda_2 \geq \lambda_3 = \lambda_4 \geq ... \geq \lambda_{n-2} = \lambda_{n-1} \geq \lambda_n = 0 & \text{if } n \text{ is odd.} \end{cases}$$

8.3 Unitary Vector Valued Valuations

We follow Section 6.5.3 by Schuster in [22] and Wannerer [23] and use Theorem 7.13 and the results of Klimyk [20] and Helgason [24] in order to calculate the dimension of the vector space of continuous translation invariant and $\mathrm{U}(n)$-equivariant valuations with values in $\mathbb{C}^n$. The last part follows the work of Böröczky, Domokas and Solanes [27] calculating the dimensions of the space of translation invariant unitary equivariant tensor valuations for $n \geq 2$ and providing a basis for the vector valued case.

Let $n \geq 3$, as in the previous subsection we consider $\mathrm{U}(n)$ as a subgroup of $\mathrm{SO}(n)$. Let $\mathcal{K}^{2n}$ be the space of convex bodies in $\mathbb{R}^{2n}$ and $\mathbf{Val}$ ($\mathbf{Val}_i$) be the space of continuous translation invariant valuations $\Phi : \mathcal{K}^{2n} \to \mathbb{R}$ (of degree i). By $\mathbb{C}^n\mathbf{Val}$ we denote the real vector space of continuous translation invariant valuations $\Phi : \mathcal{K}^{2n} \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ and by $\mathbb{C}^n\mathbf{Val}^{\mathrm{U}(n)}$ its subspace of $\mathrm{U}(n)$-equivariant valuations. For any of those vector spaces a subscript i denotes the subspace of valuations of the given space of degree i. From McMullen's decomposition [12] we get a decomposition of $\mathbb{C}^n\mathbf{Val}$ into the subspaces of valuations of degree i by

$$\mathbb{C}^n\mathbf{Val} = \bigoplus_{0 \leq i \leq 2n} \mathbb{C}^n\mathbf{Val}_i.$$

Keeping this in mind the following theorem is due to Wannerer [23].

Theorem 8.7. *Suppose that $0 \leq i \leq 2n$, then*

$$\dim_{\mathbb{R}} \mathbb{C}^n\mathbf{Val}_i^{\mathrm{U}(n)} = 2 \min\left\{ \left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{2n-i}{2} \right\rfloor \right\}.$$

Proof. We abbreviate $V := \mathbb{R}^{2n}$ and denote by $V_{\mathbb{C}}$ the complexification of V. The dimension of $V\mathbf{Val}_i^{\mathrm{U}(n)}$ over $\mathbb{R}$ is the same as the dimension of it's complexification over the complex numbers, and Lemma 7.15 together with Proposition 7.14 imply

$$\dim_{\mathbb{R}} V\mathbf{Val}_i^{\mathrm{U}(n)} = \dim_{\mathbb{C}}(V\mathbf{Val}_i \otimes \mathbb{C})^{\mathrm{U}(n)} = \dim_{\mathbb{C}}(\mathbf{Val}_i \otimes V_{\mathbb{C}})^{\mathrm{U}(n)}.$$

Notice that the first two terms in the last equation are a statement about the space of unitary equivariant valuations where the last term is about a unitary invariant subspace. With the decomposition from Theorem 7.13 this leads to

$$\begin{aligned}\dim_{\mathbb{C}}(\mathbf{Val}_i \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} &= \dim_{\mathbb{C}}(\bigoplus_{\lambda} \Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} \\ &= \dim_{\mathbb{C}}(\bigoplus_{\lambda} (\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)}) = \sum_{\lambda} \dim_{\mathbb{C}}(\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)},\end{aligned}$$

where the sum ranges over all highest weights $\lambda = (\lambda_1, ..., \lambda_n)$ of $\mathrm{SO}(2n)$ satisfying

$$\begin{aligned}&(i)\ \lambda_j = 0 \textit{ for } j > \min\{i, n-i\}; \\ &(ii)\ |\lambda_j| \neq 1 \textit{ for } 1 \leq j \leq \lfloor n/2 \rfloor; \\ &(iii)\ |\lambda_2| \leq 2.\end{aligned} \tag{22}$$

The application of Klimyk's Formula further decomposes

$$\Gamma_\lambda \otimes V_{\mathbb{C}} = \bigoplus_{\mu} \Gamma_\mu,$$

where the μs are highest $\mathrm{SO}(2n)$ weights that can be expressed as $\mu = \lambda \pm \varepsilon_k$ for some $1 \leq k \leq n$. Let us consider Γ_μ containing a $\mathrm{U}(n)$ invariant element, i.e., it is a spherical representation of $\mathrm{SO}(2n)$ with respect to the compact symmetric pair $(\mathrm{SO}(2n), \mathrm{U}(n))$. In this case, due to Corollary 8.6, the following additional conditions apply to $\mu = (\mu_1, ..., \mu_n)$

$$\begin{cases} \mu_1 = \mu_2 \geq \mu_3 = \mu_4 \geq ... \geq \mu_{n-1} = \mu_n & \text{if } n \text{ is even,} \\ \mu_1 = \mu_2 \geq \mu_3 = \mu_4 \geq ... \geq \mu_{n-2} = \mu_{n-1} \geq \mu_n = 0 & \text{if } n \text{ is odd.} \end{cases} \tag{23}$$

Therefore we have

$$\dim_{\mathbb{C}} \Gamma_\mu^{\mathrm{U}(n)} = \begin{cases} 1 & \text{if } \mu \text{ satisfies (23),} \\ 0 & \text{otherwise.} \end{cases}$$

Now we show that $\dim_{\mathbb{C}}(\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} = 2$ if λ satisfies

$$\lambda_1 = 3, \qquad \lambda_2 = ... = \lambda_{2m} = 2, \qquad \text{and } \lambda_j = 0 \text{ for } j < 2m$$

for some integer $1 \leq m \leq \min\{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{2n-i}{2} \rfloor\}$ and $\dim_{\mathbb{C}}(\Gamma_\lambda \otimes V_{\mathbb{C}})^{\mathrm{U}(n)} = 0$ otherwise. The conditions (22) result in λ_2 being either 2 or 0, and thus the components $\lambda_3, \lambda_4, ...$ of λ are restricted to these two values as well. Therefore, $\mu = \lambda \pm \varepsilon_k$ under the condition (23) needs to be equal to $\lambda + \varepsilon_2$ or $\lambda - \varepsilon_1$. This results in $\lambda_1 = 3, \lambda_2 = 2$, $\lambda_3 = ... = \lambda_{2m} = 2$ and $\lambda_j = 0$ for $j > 2m$. Conclusively there are exactly $\min\{\lfloor \frac{i}{2} \rfloor, \lfloor \frac{2n-i}{2} \rfloor\}$ choices for m resulting in the formula we wanted to prove. □

Denote by $\overline{\mathrm{U}(n)} := \mathrm{U}(n) \ltimes \mathbb{C}^n$ the group of unitary affine transformations of $\mathbb{C}^n$. Now recall Theorem 3.7 characterizing the Steiner point map as a continuous Minkowski additive rigid motion equivariant map from $\mathcal{K}^n$ to $\mathbb{R}^n$. For a complex vector space, due to Theorem 8.7, we can weaken the requirements in Theorem 3.7 to $\overline{\mathrm{U}(n)}$-equivariance and get the following corollary taken from Wannerer [23].

Corollary 8.8. *Let $f : \mathcal{K}(\mathbb{C}^n) \to \mathbb{C}^n$ be a continuous map which satisfies*

$$(i)\ f(K + L) = f(K) + f(L) \text{ for } K, L \in \mathcal{K}(V);$$
$$(ii)\ f \circ g = g \circ f \text{ for } g \in \overline{\mathrm{U}(n)},$$

then $f = s$.

Proof. Let $K, L \in \mathcal{K}(\mathbb{C}^n)$. As $K \cup L + K \cap L = K + L$ if $K \cup L$ is convex, we see that f is a valuation. (i) and the continuity of f can be used to show it is of degree 1. Because s is in particular $\overline{\mathrm{U}(n)}$-equivariant, $f - s$ is unitary equivariant and translation invariant, thus $f - s \in \mathbf{Val}_1^{\mathrm{U}(n)} = \{0\}$. ☐

Notice that the $\overline{\mathrm{U}(n)}$-equivariant valuations do not form a vector space, considering they are not closed under the addition of functions. However, if φ is continuous translation invariant $\mathrm{U}(n)$-equivariant valuation then $\varphi + s$, where s is the Steiner point map, is continuous and $\overline{\mathrm{U}(n)}$-equivariant. Thus, translating the vector space of continuous translation invariant $\mathrm{U}(n)$-equivariant valuations by the Steiner point map s results in the affine space of continuous $\overline{\mathrm{U}(n)}$-equivariant valuations.

Corollary 8.9. *The continuous $\overline{\mathrm{U}(n)}$-equivariant valuations $\phi : \mathcal{K}(\mathbb{C}^n) \to \mathbb{C}^n$ constitute a complex affine subspace of dimension*

$$\begin{cases} 2k^2 - k & \text{for } n = 2k, \\ 2k^2 + k & \text{for } n = 2k + 1. \end{cases}$$

Proof. Due to Theorem 8.7 and Corollary 8.8 the complex dimension of the affine space of continuous $\overline{\mathrm{U}(n)}$-equivariant valuations is given by

$$\begin{aligned} \sum_{i=0}^{2n} \min\left\{ \left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{2n-i}{2} \right\rfloor \right\} &= \sum_{i=0}^{n} \left\lfloor \frac{i}{2} \right\rfloor + \sum_{i=n+1}^{2n} \left\lfloor \frac{2n-i}{2} \right\rfloor \\ &= \sum_{i=0}^{n} \left\lfloor \frac{i}{2} \right\rfloor + \sum_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor = 2\sum_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor. \end{aligned}$$

Considering

$$\sum_{i=0}^{n-1} \left\lfloor \frac{i}{2} \right\rfloor = \begin{cases} 2\sum_{i=0}^{k-1} i & \text{for } n = 2k \\ 2\sum_{i=0}^{k-1} i + k & \text{for } n = 2k + 1 \end{cases}$$

we get that

$$\sum_{i=0}^{2n} \min\left\{\left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{2n-i}{2} \right\rfloor\right\} = \begin{cases} 2k^2 - k & \text{for } n = 2k, \\ 2k^2 + k & \text{for } n = 2k+1. \end{cases}$$

□

In contrast to the last corollary, we reformulate the Steiner point map characterization (Theorem 3.9).

Theorem 8.10. *Let* $n \geq 2$*. The continuous translation equivariant* $\mathrm{SO}(n)$*-equivariant valuations* $\phi : \mathcal{K}^n \to \mathbb{R}^n$ *constitute a real affine subspace of dimension* 0 *consisting of the Steiner point map* s*.*

So the Steiner point map from $\mathcal{K}^n$ to $\mathbb{R}^n$ is the unique continuous translation equivariant $\mathrm{SO}(n)$-equivariant valuation. However, if we restrict equivariance to the subgroup $\overline{\mathrm{U}(n)}$ the unitary Steiner point maps from $\mathcal{K}(\mathbb{C}^n)$ to $\mathbb{C}^n$ constitute a complex affine subspace of dimension

$$\begin{cases} 2k^2 - k & \text{for } n = 2k, \\ 2k^2 + k & \text{for } n = 2k+1. \end{cases}$$

Due to the recent work of Böröczky, Domokas and Solanes, see [27], and Wannerer [28] a basis for the $\mathbb{C}$-vector space of continuous translation invariant unitary equivariant vector valued valuations was found. Let φ be an even valuation of degree k taking values in a finite dimensional real vector space V. Then the Klain function, see [30], is a function Kl_φ on the k-Grassmanian $\mathrm{Gr}(\dim V, k)$ given by $\varphi(A) = \mathrm{Kl}_\varphi(E)\mathrm{vol}_k(A)$ for $A \subseteq E \in \mathrm{Gr}(\dim V, k)$. In the scalar case Bernig and Fu [29] constructed a basis of $\mathbf{Val}^{\mathrm{U}(n)}$ consisting of the so-called hermitian intrinsic volumes $\mu_{k,q}$ defined, considering they are even, by their Klain functions $\mu_{k,q}$ for $0 \leq k \leq 2n$ and $0, k-n \leq q \leq \frac{k}{2}$. For $\max(0, k-n) \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor$ the valuations $\mu_{k,q}$ comprise a basis for the vector space $\mathbf{Val}_k^{\mathrm{U}(n)}$. For $k \leq m$ the Klain function $\mu_{k,q}$ is given by

$$\mathrm{Kl}_{\mu_{k,q}}(E) = \sum_{i=q}^{\left\lfloor \frac{k}{2} \right\rfloor} (-1)^{i+q} \binom{i}{q} \sigma_i(\cos^2(\theta_1), ..., \cos^2(\theta_{\lfloor k/2 \rfloor})),$$

where σ_i is the i-th elementary symmetric function, and $\theta_1, ..., \theta_{\lfloor k/2 \rfloor}$ are the Kähler angles of the k-dimensional linear subspace E. Given the endomorphism ψ_E of E mapping $u \in E$ to the orthogonal projection of $\sqrt{-1}u$ to E, then ψ_E has eigenvalues $\pm\sqrt{-1}\cos\theta_1, ..., \pm\sqrt{-1}\cos\theta_{\lfloor k/2 \rfloor}$, plus a zero eigenvalue for odd k. These eigenvalues characterize the Kähler angles. For $k \geq n$

$$\mathrm{Kl}_{\mu_{k,q}}(E) = \mathrm{Kl}_{\mu_{2n-k,n-k+q}}(E^\perp).$$

Wannerer [28] introduced the space $\mathrm{Area}(V)$ of smooth area measures on an Euclidean vector space V as certain translation invariant valuations on V taking values in the space of signed measures (i.e., measures that can take values in all of $\mathbb{R}$) of the unit sphere $S(V)$. Therefore, given $\Phi \in \mathrm{Area}(V)$ and a convex body $A \subseteq V$, $\Phi(A, \cdot)$ is a signed measure on $S(V)$.

Definition. [23] (Smooth area measures) The vector space $\mathrm{Area}(V)$ of smooth area measures on V is given by all expressions of the form

$$\Phi(K, A) = \int_{N(K)\cap \pi_2^{-1}(A)} \omega,$$

where $K \in \mathcal{K}(V)$, $\omega \in \Omega^{n-1}(SV)$ a translation invariant smooth $(n-1)$-form, $A \subseteq S(V)$ is a Borel set and $\pi_2 : SV \to S(V)$ the canonical projection.

Denote by $\mathbf{Val}(V)$ the space of continuous translation invariant valuations with values in $\mathbb{R}$ and by $V\mathbf{Val}(V)$ the valuations in $\mathbf{Val}(V)$ with values in V. The globalization map $\mathrm{glob} : \mathrm{Area}(V) \to \mathbf{Val}(V)$ is defined by

$$\mathrm{glob}(\Phi)(A) = \Phi(A, S(V)),$$

and the centroid map by $C : \mathrm{Area}(V) \to V\mathbf{Val}(V)$ is defined by

$$C(\Phi)(A) = \int_{S(V)} u \; d\Phi(A, u),$$

where we integrate with respect to the measure $\Phi(A, \cdot)$. Given a linear subspace $E \subseteq V$, we characterize a restriction map $r : \mathrm{Area}(V) \to \mathrm{Area}(E)$ as follows. For a Borel set $U \subseteq S(V)$ denote $\overline{U} = (U + E^{\perp}) \cap S(V)$. Then the restriction of $\Phi \in \mathrm{Area}(V)$ to E is given by

$$r(\Phi)(A, U) = \Phi(A, \overline{U})$$

for $A \in \mathcal{K}(E)$ and $U \subseteq S(E)$.

Proposition 8.11. *[28] Given $0 \leq k < 2n$, there exists a family $\Delta_{k,q} \in \mathrm{Area}(\mathbb{V})_k^{\mathrm{U}(n)}$ with $0, k-n \leq q \leq \frac{k}{2}$ such that*

(i) $\mathrm{glob}(\Delta_{k,q}) = \mu_{k,q}$;

(ii) for every polytope P and every Borel set $U \subseteq S(\mathbb{C}^n)$

$$\Delta_{k,q}(P, U) = \sum_{F \in \mathcal{F}_k} \mathrm{Kl}_{\mu_{k,q}}(\vec{F}) \frac{\mathrm{vol}_{2n-k-1}(N(P,F) \cap U)}{\mathrm{vol}_{2n-k-1}(S^{2n-k-1})} \mathrm{vol}_k(F), \tag{24}$$

where $\mathcal{F}_k$ is the set of k-dimensional faces, $N(P,F)$ is the set of outer unit normal vectors to P at points of F, and $\vec{F}$ is the k-dimensional linear subspace parallel to F;

(iii) The restriction $r : \mathrm{Area}(\mathbb{C}^{n+l}) \to \mathrm{Area}(\mathbb{C}^n)$ corresponding to the inclusion $\mathbb{C}^n \to \mathbb{C}^{n+l}$ satisfies $r(\Delta_{k,q}) = \Delta_{k,q}$ if $q \geq k - m$.

Given a p-dimensional real subspace $E \subseteq \mathbb{C}^n$ and the corresponding restriction map r, it follows from (24) that

$$C(r(\Delta_{k,q}))(A) = c_{n,p,k} C(\Delta_{k,q})(A)$$

for a convex body A in E and $c_{n,p,k} \neq 0$ only depending on n, p, k. In [28] it was shown that the family $C(\Delta_{k,q})$ with $0, k-n < q \leq \frac{k}{2}$ is $\mathbb{R}$-linearly independent. Due to Theorem 8.7 $\dim_{\mathbb{R}} \mathbb{C}^n \mathbf{Val}_i^{\mathrm{U}(n)} = 2 \min \left\{ \left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{2n-i}{2} \right\rfloor \right\}$. In [27] it was shown that the family $C(\Delta_{k,q})$ with $0, k-n < q \leq \frac{k}{2}$ is in fact $\mathbb{C}$-linearly independent, thus the following theorem holds.

Theorem 8.12. [27] *For $n \geq 2$ a $\mathbb{C}$-vector space basis of the complex vector space $(\mathbf{Val}_k \otimes \mathbb{C}^n)^{\mathrm{U}(n)}$ is given by the family $C(\Delta_{k,q})$ where $0 \leq k \leq 2n$ and* $\max(0, k-n) < q \leq \frac{k}{2}$.

Corollary 8.13. [27] *An $\mathbb{R}$-vector space basis of $(\mathbf{Val}_k \otimes \mathbb{R}^{2n})^{\mathrm{U}(n)}$ is given by*

$$\left\{ C(\Delta_{k,q}), \sqrt{-1} \cdot C(\Delta_{k,q}) | 0, k-n < q \leq \frac{k}{2} \right\}.$$

Let $\mathbb{S}^d(\mathbb{R}^{2n})$ denote the space of symmetric rank d tensors of $\mathbb{R}^{2n}$. Using branching, in particular a theorem of King [31] and the generalized Hadwiger theorem (Theorem 7.17)) Böröczky, Domokas and Solanes achieved the following theorem.

Theorem 8.14. *[27] For $n \geq 2, k = 0, ..., 2n$ and $d \geq 0$, using the notation $f := \left\lfloor \frac{d}{2} \right\rfloor$ and $l := \min\{k, 2n-k\}$, the dimensions of $W := (\mathbb{S}^d(\mathbb{R}^{2n})\mathbf{Val}_k)^{\mathrm{U}(n)}$ is as follows:*

$$\dim(W) = \begin{cases} 1 + \left\lfloor \frac{l}{2} \right\rfloor & \text{for } d = 0, \\ 1 & \text{for } d = 2f > 0, l = 0, \\ 3lf^2 + 2\left\lfloor \frac{l}{2} \right\rfloor - 2f^2 + 2f + 1 & \text{for } d = 2f > 0, 1 \leq l \leq n, \\ 3nf^2 + 2\left\lfloor \frac{n}{2} \right\rfloor - 3f^2 + 2f + 1 & \text{for } d = 2f > 0, l = n, \\ 0 & \text{for } d = 2f+1, l = 0, \\ 3lf^2 + 3lf + 2\left\lfloor \frac{l}{2} \right\rfloor - 2f^2 & \text{for } d = 2f+1, 1 \leq l \leq n, \\ 3nf^2 + 3mf + 2\left\lfloor \frac{n}{2} \right\rfloor - 3f^2 - f & \text{for } d = 2f+1, l = n. \end{cases}$$

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