Diplomarbeit

Euler schemes and large deviations for stochastic Volterra equations

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Abstract

The main goal of this thesis is to show the Large Deviation Principle (LDP, see definition 4.2) for a family $\{X^\varepsilon, \varepsilon > 0\}$ where each X^ε is solving a stochastic Volterra integral equation of the form

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t b(t, s, X_s^\varepsilon)\ dt + \sqrt{\varepsilon} \int_0^t \sigma(t, s, X_s^\varepsilon)\ dW$$

on the same probability space where W is a Standard Brownian Motion. Chapter 2 contains the notations which will be used and in section 2.3 the assumptions under which the statements of this thesis hold are listed. The proof of the LDP will be done in chapter 5 by showing the Laplace Principle. The equivalence of these two principles and the conditions under which this equivalence holds true is stated in chapter 4 (see also [DE11]). In chapter 3 an Euler scheme for this type of integral equation is presented.

A large part of this thesis is dedicated to giving more detailed proofs of the statements from [Zha08], some of which are shown under stronger conditions than in the corresponding paper, since the proofs in [Zha08] for the weaker ones were not completely clear to me.

Contents

Chapter 1

Introduction

In this paper we will discuss stochastic Volterra integral equations

$$X_t = X_0 + \int_0^t b(t,s,X_s)dt + \int_0^t \sigma(t,s,X_s)dW_s$$

for which we will present an Euler scheme (chapter 3) and show the they satisfy the Large Deviation Principle (see definition 4.2). To be more precise about the LDP, in chapter 5 we will show this property for the family $\{X^\varepsilon, \varepsilon > 0\}$ where each X^ε solves the inteqgral equation

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t b(t,s,X_s^\varepsilon)dt + \int_0^t \sqrt{\varepsilon}\sigma(t,s,X_s^\varepsilon)dW_s$$

A large part of this thesis is dedicated to giving more detailed proofs of the statements from [Zha08] in which these two topics are covered. Regarding the Euler scheme we get the same results as in [Zha08, section 2]. For proving the latter (see theorem 5.9) we had to ask for stricter conditions. As is done in [RZ05, Theorem 3.12] we showed the Laplace Principle with respect to a certain good rate function I (given by (5.4)). As stated in theorem 4.4 this is equivalent to the LDP with respect to the same good rate function. The difference of the conditions needed arises from showing that I is indeed a good rate function.

In [Zha08, Lemma 3.8] and [RZ05, Lemma 3.5] it is claimed that I given as a mapping from $C([0,1] \times \mathbb{R}^d; \mathbb{R}^d)$ to $[0,\infty]$ satisfies the required condition (see definition 4.7) that $\{I \le a\}$ is compact for all $a < \infty$. But we were only able to show this property if we restrict ourselves to $C([0,1] \times D_R; \mathbb{R}^d)$ where $D_R := \{x \in \mathbb{R}^d : \|x\|_2 \le R\}$ for any $R > 0$ (see lemma 5.7). This is due to lemma 5.5 where we show that for the operator S which maps h to the solution of

$$S(h)(t) = S(h)(0) + \int_0^t b(t,s,S(h)(s))ds + \int_0^t \sigma(t,s,S(h)(s))h'(s)ds$$

it holds that $\{S(h) : h \in B_N\}$ is relatively compact. There, we had to ask for this stricter condition in comparison to [Zha08, Lemma 3.6] and [RZ05, Lemma 3.3]. The same goes for lemma 5.6 in opposition to [Zha08, Lemma 3.7] and [RZ05, Lemma 3.4] where the continuity of this mapping is shown.

We omitted the statements of [Zha08, Lemma 3.10] and [RZ05, Lemma 3.11], a technical lemma needed to show [RZ05, Theorem 3.12]. Instead we showed a weaker statement in lemma 5.8 which we needed in the proof of theorem 5.9.

Furthermore, we give a heuristic as to how to deduce the good rate function of choice (see section 4.2). As is shown in [Aze80] the family $\{\varepsilon W, \varepsilon > 0\}$ satisfies the LDP with the mapping λ given by (4.3). For a certain family of SDEs driven by the same Brownian motion (see remark

4.11) it is then shown that its solutions satisfy the LDP with a mapping $\tilde{\lambda}$ given by (4.4) which looks similar to our good rate function I.

In chapter 3 we give a specific bound for the paramter p for which the L^p-estimate can be shown. In [Zha08] in the corresponding section 2 it only says for p sufficiently large. Furthermore, we highlight which parameters the constants in these estimates depend on. For lemma 5.5 we give a slightly different proof than in [Zha08, Lemma 3.6] using the same techniques as in the proofs of chapter 3.

There were a few typos in [Zha08]:

- Theorem 1.2 : It should say "$A \in \mathcal{B}(\mathbb{C})$" instead of "$A \in \mathbb{C}$".
- In the definition of $\mathbb{H}$ at the beginning of section 3.1 the condition $h(0) = 0$ is missing.
- In section 3.3 it should say "$X^{\varepsilon,h^{\varepsilon}}$" instead of "$X^{\varepsilon}$" in the definition of the SDE at the beginning of the section and the lemmata 3.9 and 3.10.

Chapter 2

Setting and Notation

2.1 General Setting

Definition 2.1. (probability space)
We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ given by

$$\begin{aligned}
\Omega &:= \{\omega \in C([0,1];\mathbb{R}^m) \mid \omega(0) = 0\} \\
\mathbb{P} &:= \text{Wiener measure on } \Omega \\
& \qquad \mathcal{N} := \{A \subset \Omega \mid \exists B \in \mathcal{B}(\Omega), A \subset B : \mathbb{P}(B) = 0\} \\
\mathcal{F} &:= \sigma(\mathcal{N} \cup \mathcal{B}(\Omega))
\end{aligned} \tag{2.1}$$

where the σ-Algebra $\mathcal{F}$ is the $\mathbb{P}$-completion of $\mathcal{B}(\Omega)$, the Borel field of Ω endowed with the topology induced by the uniform norm, $\mathcal{B}(\Omega) = \sigma(\mathcal{T}_{\|\cdot\|_\infty})$. For simplicity we will identify $\mathbb{P}$ with the augmented probability measure defined in the completion.

Remark 2.2.
Endowed with the uniform norm the space $(\Omega, \|\cdot\|_\infty)$ is a separable Banach space. Separability follows from Stone-Weierstrass. To show completeness let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Since f_n is continuous for all $n \in \mathbb{N}$, $[0,1]$ compact and $(\mathbb{R}^m, \|\cdot\|_2)$ complete, one can define a function pointwise by $f(x) = \lim_{n\to\infty} f_n(x)$. From there it is easy to show that $f_n \to f$ uniformly and f is continuous.

Definition 2.3. (Brownian Motion)
On this space a m-dimensional Brownian Motion $W = (W_t)_{t\in[0,1]}$ is defined

$$\begin{aligned}
W_t(\omega) &:= \omega(t) \\
\mathcal{F}_t &:= \sigma(W_s, s \le t) \\
\mathbb{F} &:= (\mathcal{F}_t)_{t\in[0,1]}
\end{aligned} \tag{2.2}$$

We thereby obtain a filtrated probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Definition 2.4. ($\mathbb{H}$)
If existing we denote for $h \in \Omega$ the vector of the derivatives of the components by h'. We define

$$\mathbb{H} := \{h \in \Omega : \ \|h\|_{\mathbb{H}}^2 < \infty\} \tag{2.3}$$

where

$$\|h\|_{\mathbb{H}}^2 := \int_0^1 \|h'(s)\|_2^2 ds \tag{2.4}$$

Remark 2.5.
Endowed with the scalar product

$$\langle f,g\rangle_{\mathbb{H}} := \int_0^1 \langle f'(s),g'(s)\rangle_2 ds, \quad f,g\in\mathbb{H} \tag{2.5}$$

the space $\mathbb{H}$ becomes a separable Hilbert space. Because for every $h\in\mathbb{H}$ it follows that $h'\in L^2([0,1];\mathbb{R}^d)$ and conversely for all $f\in L^2([0,1];\mathbb{R}^d)$ its "antiderivative" starting in zero defined componentwise by $F_i(t):=\int_0^t f_i(s)ds$ is an Element of $\mathbb{H}$. Hence, it exists a bijection between $\mathbb{H}$ and L^2, so we have $\mathbb{H}\cong L^2$. From there it follows that the space $(\mathbb{H},\langle\cdot,\cdot\rangle_{\mathbb{H}})$ is separable and complete.

Definition 2.6.
The following notations will be used in this thesis.

$$\mathcal{A}_b := \{H\ \mathbb{F}\text{-progressive} \mid H(\omega)\in\mathbb{H}, \forall\omega\in\Omega \wedge \exists C>0 : \|H\|_{\mathbb{H}}^2 \le C\} \tag{2.6}$$
$$D_R := \{x\in\mathbb{R}^d : \|x\|_2 \le R\}, \quad R>0 \tag{2.7}$$

Definition 2.7. ($\mathbb{F}$-progressive)
Let $T\subset\overline{\mathbb{R}}$ be an interval, $T_t := \{s\in T \mid s\le t\}$, $(S,\mathcal{S})$ a measurable space. A process $X : T\times\Omega\to S$ is called $\mathbb{F}$-progressive if and only if

$$X|_{T_t\times\Omega}\ \ \mathcal{B}(T_t)\otimes\mathcal{F}_t\text{-measureable}, \quad t\in T \tag{2.8}$$

Remark 2.8.
We define
$$\Sigma_{pm}(T,\mathbb{F}) := \{A\subset T\times\Omega \mid A\cap(T_t\times\Omega)\in\mathcal{B}(T_t)\otimes\mathcal{F}_t,\ \forall t\in T\} \tag{2.9}$$
This set is a σ-algebra and fulfills: a process X is $\mathbb{F}$-progressive if and only if X is $\Sigma_{pm}(T,\mathbb{F})$ measurable. Furthermore it holds that $\Sigma_{pm}(T,\mathbb{F})\subset\mathcal{B}(T)\otimes\mathcal{F}_{t^*}$ where $t^* := \sup T$ (see also [Sch20, Def. 3.33 and Ex. 3.34].

Definition 2.9. ($\mathbb{F}$-predictable)
Let $T\subset\mathbb{R}$ be an interval, $\mathcal{V} := \{X : T\times\Omega\to\mathbb{R} \mid X\text{adapted, left-continuous}\}$. We define

$$\Sigma_{pre}(T,\mathbb{F}) := \sigma(\bigcup_{X\in\mathcal{V}} X^{-1}(\mathcal{B}(\mathbb{R}))) \tag{2.10}$$

Then a process Y is $\mathbb{F}$-predictable if and only if it is $\Sigma_{pre}(T,\mathbb{F})$ measurable.

Remark 2.10.
In the setting of (2.1) and (2.2) it holds that a stochastic process X is $\mathbb{F}$-progressive if and only if it is $\mathbb{F}$-predictable (see also [DM78, Ch. IV. 94-97]).

2.2 Stochastic Volterra Integral Equation

Definition 2.11. (Volterra Integral Equation)
Let $b : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ measurable, $x \in \mathbb{R}^d$. We consider the integral equation

$$\begin{aligned} X_t &= X_0 + \int_0^t b(t,s,X_s)ds + \int_0^t \sigma(t,s,X_s)dW_s \\ X_0 &= x \end{aligned} \tag{2.11}$$

For $n \in \mathbb{N}$ we define

$$\begin{aligned} \tilde{t}_n &:= t\mathbb{1}_{\{t\geq 2^{-n}\}} + 2^{-n}\mathbb{1}_{\{t<2^{-n}\}} \\ s_n &:= \lfloor 2^n s \rfloor 2^{-n} \\ \tilde{s}_n &:= s_n\mathbb{1}_{\{s\geq 2^{-n}\}} + 2^{-n-1}\mathbb{1}_{\{s<2^{-n}\}} \end{aligned} \tag{2.12}$$

The corresponding inteqral equation is given by

$$\begin{aligned} X_n(t) &= X_n(0) + \int_0^t b(\tilde{t}_n, \tilde{s}_n, X_n(s_n))ds + \int_0^t \sigma(\tilde{t}_n, \tilde{s}_n, X(s_n))dW_s \\ X_n(0) &= x \end{aligned} \tag{2.13}$$

To highlight that a solution X_n of (2.13) is starting at $X_n(0) = x$ we use the notation $X_n(\cdot, x)$. The same goes for X solving equation (2.11) starting in $X_0 = x$ where we write $X(\cdot, x)$.

Definition 2.12. ($\mathbb{H}$ Integral Equation)
For $H \in \mathcal{A}_b$ and $\varepsilon > 0$ the process $X^{\varepsilon,H}$ describes the solution of

$$\begin{aligned} X_t^{\varepsilon,H} &= X_0^{\varepsilon,H} + \int_0^t b(t,s,X_s^{\varepsilon,H})ds + \int_0^t \sigma(t,s,X_s^{\varepsilon,H})H'(s)ds + \int_0^t \sqrt{\varepsilon}\sigma(t,s,X_s^{\varepsilon,H})dW_s \\ X_0^{\varepsilon,H} &= x \end{aligned} \tag{2.14}$$

For the special case $H \equiv 0$ we write $X^\varepsilon := X^{\varepsilon,0}$. Existence and Uniqueness of a solution of equation (2.11) and (2.14) is discussed in section 2.4.

2.3 Assumptions

Assumption 1

$$\exists \alpha, \beta > 1,\ K_1, K_2 : [0,1]^2 \to (0,\infty),\ C_1 > 0\ \forall t,s \in [0,1]\ \forall x,y \in \mathbb{R}^d : \tag{2.15}$$

$$\begin{aligned} &(i) && \|b(t,s,x) - b(t,s,y)\|_2 \leq K_1(t,s)\|x-y\|_2 \\ &(ii) && \|\sigma(t,s,x) - \sigma(t,s,y)\|_F^2 \leq K_2(t,s)\|x-y\|_2^2 \\ &(iii) && \int_0^t (\ \|b(t,s,0)\|_2^\beta + \|\sigma(t,s,0)\|_F^{2\beta}\ )ds \leq C_1 \\ &(iv) && \int_0^t (\ (K_1(t,s))^\alpha + (K_2(t,s))^\alpha\ )ds \leq C_2 \end{aligned}$$

In particular, in (iii) and (iv) the integrals of each summand of the integrands are bounded as well by C since they are all nonnegative

Assumption 2

$$\begin{aligned} & \exists \gamma > 0,\ F_1, F_2 : [0,1]^3 \to (0,\infty),\ C_2 > 0\ \forall t', t, s \in [0,1]\ \forall x \in \mathbb{R}^d : \\ (i)\quad & \|b(t',s,x) - b(t,s,x)\|_2 \le F_1(t',t,s)(1 + \|x\|_2) \\ (ii)\quad & \|\sigma(t',s,x) - \sigma(t,s,x)\|_F^2 \le F_2(t',t,s)(1 + \|x\|_2^2) \\ (iii)\quad & \int_0^{t'\wedge t} (\ F_1(t',t,s) + F_2(t',t,s)\ )ds \le C_2|t-t'|^\gamma \end{aligned} \tag{2.16}$$

where $t' \wedge t := \min\{t', t\}$. Again as $F_1, F_2 > 0$ it holds that the integrals of each summand of the integrands in (iii) are bounded as well by $C|t-t'|^\gamma$.

Assumption 3

$$\begin{aligned} & \exists \delta > 0,\ F_3, F_4 : [0,1]^3 \to (0,\infty),\ C_3 > 0\ \forall t, s, s' \in [0,1]\ \forall x \in \mathbb{R}^d : \\ (i)\quad & \|b(t,s,x) - b(t,s',x)\|_2 \le F_3(t,s,s')(1 + \|x\|_2) \\ (ii)\quad & \|\sigma(t,s,x) - \sigma(t,s',x)\|_F^2 \le F_4(t,s,s')(1 + \|x\|_2^2) \\ (iii)\quad & \int_0^{t'\wedge t} (\ F_3(\tilde{t}_n, s, \tilde{s}_n) + F_4(\tilde{t}_n, s, \tilde{s}_n)\ )ds \le C_3 2^{-n\delta} \end{aligned} \tag{2.17}$$

For (iii) the same holds as before.

Assumption 4

$$\begin{aligned} & \forall x \in \mathbb{R}^d\ \forall t \in [0,1] : \\ (i)\quad & \text{The mapping } b(t,\cdot,x) : (0,t) \to \mathbb{R}^d : s \mapsto b(t,s,x) \text{ is continuous} \\ (ii)\quad & \text{The mapping } \sigma(t,\cdot,x) : (0,t) \to \mathbb{R}^d : s \mapsto \sigma(t,s,x) \text{ is continuous} \end{aligned}$$

2.4 Existence and Uniqueness of the SVIE

With these assumptions we can state the existance and uniqueness of a continuous solution.

Theorem 2.13.
There exist progressive solutions of the integral equations (2.11) and (2.14). In particular they are unique.

Proof: Existence and uniqueness of a solution of the integral equation (2.11) already follows from assumption 1 as is shown in [Wan08, Theorem 1.1]. There the assumptions are, it exists $p > 2$, a concave function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and functions K_1, K_2 as in assumption 1, such that for all $T > 0$ there is a $C_T > 0$ such that

$$\begin{aligned} (a)\quad & \|b(t,s,x) - b(t,s,y)\|_2 \le C_T K_1(t,s)\rho^{1/p}(\|x-y\|_2^p) \\ (b)\quad & \|\sigma(t,s,x) - \sigma(t,s,y)\|_F^2 \le C_T K_2(t,s)\rho^{2/p}(\|x-y\|_2^p) \\ (c)\quad & \int_0^t (\ \|b(t,s,0)\|_2 + \|\sigma(t,s,0)\|_F^2\ )ds \le C_T \\ (d)\quad & \int_0^t (\ (K_1(t,s))^{p/(p-1)} + (K_2(t,s))^{p/(p-2)}\ )ds \le C_T \end{aligned}$$

We only consider $T = 1$. Obviously (i) and (ii) in assumption 1 are the same as (a) und (b) mit $C_T = 1$ und $\rho(u) = u$ for all $u \in \mathbb{R}_+$. Condition (c) is weaker than (iii) and the exponents in (d) are necessary to proof the statement there, which can be generalised to (iv) due to the special choice of ρ. Therefore, all the assumptions for [21, Theorem 1.1] are met and we get the existence and uniqueness for equation (2.11). From Girsanov the same follows for equation (2.14).

Lemma 2.14.
There exists a $\theta > 0$ such that the solutions of the equations (2.11) and (2.14) λ-hoelder continuous for all $\lambda \in (0, \theta)$. In particular solutions are continuous.

Proof: Assumption 2 and assumption 1 together with the remarks in the proof of theorem 2.4.13 guarantee that the assumptions of [Wan08, Theorem 1.3] are met. Hence, the statement of this lemma holds true. $\square$

Chapter 3

Euler Scheme

In this section we will prove that the solutions X_n of (2.13) converge to the solution X of (2.11) in the sense that

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2^p] = 0$$

for all $R > 0$. This will be done in theorem 3.5. First, we need to prove some lemmata. All constants arising from estimates will be denoted by C and we will highlight which parameters it is depending on by writing by subscripting those to it.

Lemma 3.1.
Under assumption 1 let $X_n(\cdot, x)$ be a solution of (2.13). Then for all $p \geq \frac{2\alpha}{\alpha-1}$ there exists a constant $C_{p,d} > 0$ such that

$$\begin{aligned} (i) \quad & \mathbb{E}[\|X_n(t,x)\|_2^p] \leq C_{p,d}(1 + \|x\|_2^p), && t\in[0,1],\ x\in\mathbb{R}^d,\ n\in\mathbb{N} \\ (ii) \quad & \mathbb{E}[\|X_n(t,x) - X_n(t,y)\|_2^p] \leq C_{p,d}\|x-y\|_2^p, && t\in[0,1],\ x,y\in\mathbb{R}^d,\ n\in\mathbb{N} \end{aligned}$$

<u>Proof:</u> For the proof of (i) we will only write $X_n(t)$ whenever it is clear that we consider a solution X_n starting at $X_n(0) = x$. Then

$$\begin{aligned} \mathbb{E}[\|X_n(t,x)\|_2^p] &\overset{(2.13)}{=} \mathbb{E}\Big[\Big(\|x + \int_0^t b(\tilde{t}_n, \tilde{s}_n, X_n(s_n))ds + \int_0^t \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n))dW_s\|_2\Big)^p\Big] \\ &\leq \mathbb{E}\Big[\Big(\|x\|_2 + \|\int_0^t b(\tilde{t}_n, \tilde{s}_n, X_n(s_n))ds\|_2 + \|\int_0^t \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n))dW_s\|_2\Big)^p\Big] \\ &\overset{(A.2)}{\leq} C_p\Big(\mathbb{E}[\|x\|_2^p] + \mathbb{E}[\|\int_0^t b(\tilde{t}_n, \tilde{s}_n, X_n(s_n))ds\|_2^p] \\ &\quad + \mathbb{E}[\|\int_0^t \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n))dW_s\|_2^p]\Big) \\ &=: C_p(\|x\|_2^p + I_1 + I_2) \end{aligned}$$

$$
\begin{aligned}
I_1 &\leq \mathbb{E}[(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, X_n(s_n))\|_2 \; ds)^p] \\
&= \mathbb{E}[(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, X_n(s_n)) - b(\tilde{t}_n, \tilde{s}_n, 0) + b(\tilde{t}_n, \tilde{s}_n, 0)\|_2 \; ds)^p] \\
&\leq \mathbb{E}[(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, 0)\|_2 \; ds + \int_0^t \|b(\tilde{t}_n, \tilde{s}_n, X_n(s_n)) - b(\tilde{t}_n, \tilde{s}_n, 0)\|_2 \; ds)^p] \\
&\overset{(A.2)}{\leq} C_p\Big(\mathbb{E}[(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, 0)\|_2 \; ds)^p] + \mathbb{E}[(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, X_n(s_n)) - b(\tilde{t}_n, \tilde{s}_n, 0)\|_2 \; ds)^p]\Big) \\
&\overset{(2.15),(A.5)}{\leq} C_p\Big(\mathbb{E}[(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, 0)\|_2^\beta \; ds)^{p/\beta} 1^{p/\beta^*}] + \mathbb{E}[(\int_0^t K_1(\tilde{t}_n, \tilde{s}_n)\|X_n(s_n)\|_2 \; ds)^p]\Big) \\
&\overset{(2.15),(A.5)}{\leq} C_p\Big(1 + \mathbb{E}[(\int_0^t (K_1(\tilde{t}_n, \tilde{s}_n))^\alpha ds)^{p/\alpha}(\int_0^t \|X_n(s_n)\|_2^{\alpha^*} ds)^{p/\alpha^*}]\Big) \\
&\overset{(2.15)}{\leq} C_p\Big(1 + \int_0^t \mathbb{E}[\|X_n(s_n)\|_2^p] ds\Big) \\
&\leq C_p + C_p \int_0^t \sup_{u \leq s} \mathbb{E}[\|X_n(u)\|_2^p] ds
\end{aligned}
$$

where $\alpha* = \frac{\alpha}{\alpha-1}$ is the Hoelder conjugate of α. For the last inequality we also used Jensen on the $(...)^{p/\alpha^*}$-term (applicable because $p \geq \frac{2\alpha}{\alpha-1}$, hence $p/\alpha* \geq 1$) and afterwards Fubini to commute expectation value und integral. Jensen is applicable since

$$
\int_0^t (...) d\lambda(s) = \int_0^1 (...) \mathbb{1}_{\{s \leq t\}} d\lambda(s)
$$

and λ is a probability measure on $[0, 1]$.

$$
\begin{aligned}
I_2 &\leq \mathbb{E}\Big[\Big(\sum_{i=1}^d \Big|\sum_{j=1}^m \int_0^t \sigma_{ij}(\tilde{t}_n, \tilde{s}_n, X_n(s_n) dW_s^{(j)}\Big|\Big)^p\Big] \\
&\overset{(A.2)}{\leq} C_p \sum_{i=1}^d \mathbb{E}\Big[\Big(\sup_{u \leq t}\Big|\sum_{j=1}^m \int_0^u \sigma_{ij}(\tilde{u}_n, \tilde{s}_n, X_n(s_n) dW_s^{(j)}\Big|\Big)^p\Big] \\
&\overset{(A.3)}{\leq} C_p \sum_{i=1}^d \mathbb{E}\Big[\Big[\sum_{j=1}^m \int_0^\cdot \sigma_{ij}(\tilde{\cdot}_n, \tilde{s}_n, X_n(s_n) dW_s^{(j)}\Big]_t^{p/2}\Big] \\
&= C_p \sum_{i=1}^d \mathbb{E}[(\int_0^t \sum_{j=1}^m (\sigma_{ij}(\tilde{t}_n, \tilde{s}_n, X_n(s_n)))^2 ds)^{p/2}] \\
&\leq C_p d \; \mathbb{E}[(\int_0^t \|\sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n))\|_F^2 ds)^{p/2}] \\
&\leq C_{p,d} + C_{p,d} \int_0^t \sup_{u \leq s} \mathbb{E}[\|X_n(u)\|_2^p] ds
\end{aligned}
$$

The last inequality follows analogously to the chain of inequalities of I_1.
Let $u(t) := \sup_{u \leq t} \mathbb{E}[\|X_n(u)\|_2^p]$. Since $u > 0$, we conclude

$$
u(t) \leq C_{p,d}(1 + \|x\|_2^p) + C_{p,d} \int_0^t u(s) ds
$$

The claimed statement now follows from $\mathbb{E}[\|X_n(t)\|_2^p] \leq \sup_{u\leq t} \mathbb{E}[\|X_n(u)\|_2^p]$ and Gronwall's inequality. The proof of (ii) follows analogously. $\square$

Lemma 3.2.
Under assumption 1 & 2 let $X_n(\cdot, x)$ be a solution of (2.13). Then there exists a $\theta > 0$ such that for all $p \geq \max\{\frac{2\alpha}{\alpha-1}, \frac{2\beta}{\beta-1}\}$ there exists a constant $C_{p,d} > 0$ such that

$$\mathbb{E}[\|X_n(t', x) - X_n(t, x)\|_2^p] \leq C_{p,d}(1 + \|x\|_2^p)|t' - t|^{\theta p}, \quad t, t' \in [0, 1],\ x \in \mathbb{R}^d,\ n \in \mathbb{N}$$

Proof: Again we will write $X_n(t)$ whenever it is clear that we consider a solution X_n starting at $X_n(0) = x$. W.l.o.g. let $t \leq t'$. Analogous steps to the proof of Lemma 3.1 will be denoted by (a). Then

$$\begin{aligned}
\mathbb{E}[\|X_n(t', x) - X_n(t, x)\|_2^p] \quad &\leq \quad \mathbb{E}[\| \int_t^{t'} b(\tilde{t}'_n, \tilde{s}_n, X_n(s_n))ds + \int_t^{t'} \sigma(\tilde{t}'_n, \tilde{s}_n, X_n(s_n))dW_s \\
&\quad + \int_0^t (b(\tilde{t}'_n, \tilde{s}_n, X_n(s_n)) - b(\tilde{t}_n, \tilde{s}_n, X_n(s_n)))ds \\
&\quad + \int_0^t (\sigma(\tilde{t}'_n, \tilde{s}_n, X_n(s_n)) - \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n)))dW_s\|_2^p] \\
&\overset{(A.2),(a)}{\leq} C_p\Big(\mathbb{E}[(\int_t^{t'} \|b(\tilde{t}'_n, \tilde{s}_n, X_n(s_n))\|_2 ds)^p] \\
&\quad + \mathbb{E}[(\int_t^{t'} \|\sigma(\tilde{t}'_n, \tilde{s}_n, X_n(s_n))\|_F^2 ds)^p] \\
&\quad + \mathbb{E}[(\int_0^t \|b(\tilde{t}'_n, \tilde{s}_n, X_n(s_n)) - b(\tilde{t}_n, \tilde{s}_n, X_n(s_n))\|_2 ds)^p] \\
&\quad + \mathbb{E}[(\int_0^t \|\sigma(\tilde{t}'_n, \tilde{s}_n, X_n(s_n)) - \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n))\|_F^2 ds)^{p/2}]\Big) \\
&=: C_p(I_1 + I_2 + I_3 + I_4)
\end{aligned}$$

$$\begin{aligned}
I_1 &\overset{(A.2),(A.5),(a)}{\leq} C_p\Big(\mathbb{E}[(\int_t^{t'} \|b(\tilde{t}'_n, \tilde{s}_n, 0)\|^\beta ds)^{p/\beta}|t' - t|^{p/\beta^*}] \\
&\quad + \mathbb{E}[(\int_t^{t'} K_1(\tilde{t}'_n, \tilde{s}_n)\|X_n(s_n)\|_2 ds)^p]\Big) \\
&\overset{(2.15),(a)}{\leq} C_p|t' - t|^{p/\beta^*} + C_p \int_t^{t'} \mathbb{E}[\|X_n(s_n, x)\|_2^p]ds \\
&\overset{3.1}{\leq} C_{p,d}|t' - t|^{p/\beta^*} + C_{p,d}(1 + \|x\|_2^p)|t' - t| \\
&\leq C_{p,d}(1 + \|x\|_2^p)(|t' - t|^{p/\beta^*} + |t' - t|)
\end{aligned}$$

Analogously to I_1 it follows for I_2

$$I_2 \leq C_{p,d}(1 + \|x\|_2^p)(|t' - t|^{p/\beta^*} + |t' - t|)$$

$$
\begin{aligned}
I_3 &\overset{(2.16)}{\leq} \mathbb{E}[(\int_0^t F_1(\tilde{t}'_n, \tilde{t}_n, \tilde{s}_n)(1 + \|X_n(s_n)\|_2)ds)^p] \\
&= (\mathbb{E}[(\int_0^t F_1(\tilde{t}'_n, \tilde{t}_n, \tilde{s}_n)(1 + \|X_n(s_n)\|_2)ds)^p]^{1/p})^p \\
&\overset{(A.6)}{\leq} \Big(\int_0^t (F_1(\tilde{t}'_n, \tilde{t}_n, \tilde{s}_n)^p \mathbb{E}[(1 + \|X_n(s_n)\|_2)^p]\big)^{1/p} ds\Big)^p \\
&\overset{(2.16),3.1}{\leq} C_{p,d}(1 + \|x\|_2^p)|t' - t|^{\gamma p}
\end{aligned}
$$

Analogously to I_3 it follows for I_4

$$
I_4 \leq C_{p,d}(1 + \|x\|_2^p)|t' - t|^{\gamma p/2}
$$

In conclusion there exists a $\theta > 0$ such that

$$
\begin{aligned}
\mathbb{E}[\|X_n(t', x) - X_n(t, x)\|_2^p] &\leq C_{p,d}(1 + \|x\|_2^p)(|t' - t|^{p/\beta^*} + |t' - t| + |t' - t|^{\gamma p/} + |t' - t|^{\gamma p/2}) \\
&\leq C_{p,d}(1 + \|x\|_2^p)|t' - t|^{\theta p}
\end{aligned}
$$

Thus, the statement of this lemma is shown. □

Remark 3.3.
If we would not have chosen 1 as our time horizon for simplicity reasons the estimation constant of lemma 3.2 would also depend on the arbitrary but fixed time horizen T. To be more precise we should denote that constant by $C_{p,d,1}$.

Theorem 3.4.
Under assumptions 1 - 4 let $X_n(\cdot, x)$ be a solution of (2.13) and $X(\cdot, x)$ a solution (2.11). Then there exists an $\eta > 0$ such that for all $p \geq \max\{\frac{2\alpha}{\alpha-1}, \frac{2\beta}{\beta-1}\}$ there exists a constant $C_{p,d} > 0$ such that

$$
\sup_{t\in[0,1]} \mathbb{E}[\|X_n(t, x) - X(t, x)\|_2^p] \leq C_{p,d}(1 + \|x\|_2^p)2^{-n\eta p}, \quad n \in \mathbb{N}
$$

Proof: Again we will write $X_n(t)$ and $X(t)$ respectively whenever it is clear that we consider solutions starting in x. Analogous steps to the proof of Lemma 3.1 and 3.2 will be denoted by (a). Then

$$
\begin{aligned}
\mathbb{E}[\|X_n(t, x) - X(t, x)\|_2^p] &= \mathbb{E}[\| \int_0^t (\ b(\tilde{t}_n, \tilde{s}_n, X_n(s_n)) - b(t, sX(s))\ )ds \\
&\quad + \int_0^t (\ \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n)) - \sigma(t, sX(s))\ )dW_s\|_2^p] \\
&\overset{(A.2)}{\leq} C_p\Big(\mathbb{E}[\| \int_0^t (\ b(t, s, X(s)) - b(t, s, X_n(s))\ )ds\|_2^p] \\
&\quad + \mathbb{E}[\| \int_0^t (\ b(t, s, X_n(s)) - b(t, s, X_n(s_n))\ )ds\|_2^p] \\
&\quad + \mathbb{E}[\| \int_0^t (\ b(t, s, X_n(s_n)) - b(\tilde{t}_n, s, X_n(s_n))\ )ds\|_2^p]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{E}[\| \int_0^t (\ b(\tilde{t}_n, s, X_n(s_n)) - b(\tilde{t}_n, \tilde{s}_n, X_n(s_n)) \ )ds\|_2^p] \\
&+\mathbb{E}[\| \int_0^t (\ \sigma(t, s, X(s)) - \sigma(t, s, X_n(s)) \ )dW_s\|_2^p] \\
&+\mathbb{E}[\| \int_0^t (\ \sigma(t, s, X_n(s)) - \sigma(t, s, X_n(s_n)) \ )dW_s\|_2^p] \\
&+\mathbb{E}[\| \int_0^t (\ \sigma(t, s, X_n(s_n)) - \sigma(\tilde{t}_n, s, X_n(s_n)) \ )dW_s\|_2^p] \\
&+\mathbb{E}[\| \int_0^t (\ \sigma(\tilde{t}_n, s, X_n(s_n)) - \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n)) \ )dW_s\|_2^p]\Big) \\
=: \ & C_p(I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8)
\end{aligned}
$$

$$
\begin{aligned}
I_1 &\overset{(a)}{\leq} C_p \int_0^t \mathbb{E}[\|X(s,x) - X_n(s,x)\|_2^p ds \\
I_5 &\overset{(a)}{\leq} C_p \int_0^t \mathbb{E}[\|X(s,x) - X_n(s,x)\|_2^p ds \\
I_2 &\overset{(a)}{\leq} C_p \int_0^t \mathbb{E}[\|X_n(s,x) - X_n(s_n,x)\|_2^p ds \\
&\overset{3.2}{\leq} C_{p,d} \int_0^t (1 + \|x\|_2^p)|s - s_n|^{\theta p} \\
&\overset{(2.12)}{\leq} C_{p,d}(1 + \|x\|_2^p)2^{-n\theta p} ds \\
I_6 &\overset{(a)}{\leq} C_{p,d}(1 + \|x\|_2^p)2^{-n\theta p} \\
I_3 &\overset{(a)}{\leq} C_p \mathbb{E}[(\int_0^t F_1(\tilde{t}_n, t, s)(1 + \|X_n(s_n)\|_2)ds)^p] \\
&\overset{(a)}{\leq} C_{p,d}(1 + \|x\|_2^p)|\tilde{t}_n - t|^{\theta p} \\
&\overset{(2.12)}{\leq} C_{p,d}(1 + \|x\|_2^p)2^{-n\theta p} \\
I_7 &\overset{(a)}{\leq} C_{p,d}(1 + \|x\|_2^p)2^{-n\theta p} \\
I_2 &\overset{(a)}{\leq} C_p \mathbb{E}[(\int_0^t F_3(\tilde{t}_n, t, \tilde{s}_n)(1 + \|X_n(s_n)\|_2)ds)^p] \\
&\overset{(2.17),(a)}{\leq} C_{p,d}(1 + \|x\|_2^p)2^{-n\delta p} \\
I_8 &\overset{(a)}{\leq} C_{p,d}(1 + \|x\|_2^p)2^{-n\delta p}
\end{aligned}
$$

In conclusion there exists an $\eta > 0$ such that

$$\mathbb{E}[\|X_n(t,x) - X(t,x)\|_2^p] \leq C_{p,d}(1 + \|x\|_2^p)2^{-n\eta p}$$

and since the right hand side is independent of t it holds that

$$\sup_{t\in[0,1]} \mathbb{E}[\|X_n(t,x) - X(t,x)\|_2^p] \leq C_{p,d}(1 + \|x\|_2^p)2^{-n\eta p}$$

Thus, the statement of this theorem is shown. □

Theorem 3.5.
Under assumptions 1 - 4 let $X_n(\cdot,x)$ and $X(\cdot,x)$ be a solution of (2.13) and (2.11) respectively. Then for all $p \geq \max\{\frac{2\alpha}{\alpha-1}, \frac{2\beta}{\beta-1}\}$ and all $R > 0$ it holds that

$$(i) \quad \exists C_{p,d,R} > 0,\ \eta > 0 : \mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2^p] \leq C_{p,d,R} 2^{-n\eta p}$$

$$(ii) \quad \mathbb{P}(\lim_{n\to\infty} \sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2 = 0) = 1$$

<u>Proof:</u> To prove (i) we will show that the statements of lemma 3.1 and theorem 3.4 also hold for

$$\mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2^p]$$

Therefore we show their validity for the first part of the proof of lemma 3.1, the rest follows analogously. Again, analogous steps of the proof will be highlighted by (a). For $T \in [0,1]$ it holds that

$$\begin{aligned}
\mathbb{E}[\sup_{t\in[0,T],\ \|x\|_2\leq R} \|X_n(t,x)\|_2^p] &\overset{(a)}{\leq} C_p\Big(\mathbb{E}[\sup_{t,x} \|x\|_2^p] \\
&\quad +\mathbb{E}[\sup_{t,x} \|\int_0^t b(\tilde{t}_n, \tilde{s}_n, X_n(s_n,x))ds\|_2^p] \\
&\quad +\mathbb{E}[\sup_{t,x} \|\int_0^t \sigma(\tilde{t}_n, \tilde{s}_n, X_n(s_n,x))dW_s\|_2^p]\Big) \\
&\leq C_p(R^p + I_1 + I_2)
\end{aligned}$$

$$\begin{aligned}
I_1 &\overset{(a)}{\leq} C_p\Big(\mathbb{E}[\sup_{t,x}(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, 0)\|_2 ds)^p] \\
&\quad +\mathbb{E}[\sup_{t,x}(\int_0^t \|b(\tilde{t}_n, \tilde{s}_n, X_n(s_n,x)) - b(\tilde{t}_n, \tilde{s}_n, 0)\|_2 ds)^p]\Big) \\
&\overset{(a)}{\leq} C_p + C_p\mathbb{E}[\sup_{t,x}(\int_0^t (K_1(\tilde{t}_n, \tilde{s}_n))^\alpha ds)^{p/\alpha}(\int_0^t \|X_n(s_n,x)\|_2^{\alpha^*} ds)^{p/\alpha^*}] \\
&\overset{(a)}{\leq} C_p + C_p\mathbb{E}[\sup_{t\in[0,T],x} \int_0^t \|X_n(s_n,x)\|_2^p ds] \\
&\leq C_p + C_p\mathbb{E}[\int_0^T \sup_{\|x\|_2\leq R} \|X_n(s_n,x)\|_2^p ds] \\
&\leq C_p + C_p \int_0^T \mathbb{E}[\sup_{u\in[0,s],\ \|x\|_2\leq R} \|X_n(u,x)\|_2^p] ds
\end{aligned}$$

Consequently, we get the analogous results

$$(1) \quad \mathbb{E}[\sup_{t\in[0,T],\ \|x\|_2\leq R} \|X_n(t,x)\|_2^p] \leq C_{p,d}(1+R^p), \qquad n \in \mathbb{N}$$

$$(2) \quad \mathbb{E}[\sup_{t\in[0,T],\ \|x\|_2\leq R} \|X_n(t,x) - X_n(t,y)\|_2^p] \leq C_{p,d} 2^p R^p, \qquad n \in \mathbb{N}$$

$$(3) \quad \mathbb{E}[\sup_{t\in[0,T],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2^p] \leq C_{p,d}(1+R^p)2^{-n\eta p}, \qquad n \in \mathbb{N}$$

With (3) and the choice $T = 1$ statement (i) is shown. From (i) and the Markov inequality (M) it follows that for all $\varepsilon > 0$

$$
\begin{aligned}
\mathbb{P}(\sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2 > \varepsilon) &\overset{(M)}{\leq} \frac{1}{\varepsilon}\,\mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2] \\
&\overset{(A.5)}{\leq} \frac{1}{\varepsilon}\,(\mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2^p])^{1/p} \\
&\overset{(i)}{\leq} \frac{C_{p,d,R}}{\varepsilon} 2^{-n\eta}
\end{aligned}
$$

Then

$$
\sum_{n\in\mathbb{N}} \mathbb{P}(\sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2 > \varepsilon) < \infty
$$

and from Borel-Cantelli we get

$$
\mathbb{P}(\limsup_{n\to\infty} \sup_{t\in[0,1],\ \|x\|_2\leq R} \|X_n(t,x) - X(t,x)\|_2 > \varepsilon) = 0
$$

The validity of the last equality for all $\varepsilon > 0$ is equivalent to the $\mathbb{P}$-as convergence (see also [Kus14, Lemma 7.78]), thus statement (ii) is shown. □

Theorem 3.6.
Under assumptions 1 - 4 let $D_R := \{x \in \mathbb{R}^d : \|x\|_2 \leq R\}$, $x \in D_R$, $X_n(\cdot, x)$ a solution of (2.13) and $X(\cdot, x)$ a solution of (2.11). If we look at X_n and X as mappings from $[0,1] \times \mathbb{R}^d$ to $\mathbb{R}^d$, it holds that they are continuous.

<u>Proof:</u> Let $p > \max\{\frac{2\alpha}{\alpha-1}, \frac{2\beta}{\beta-1}, d, \frac{d}{\theta}\}$. From lemma 3.1 and lemma 3.2 if follows that

$$
\begin{aligned}
\mathbb{E}[\|X_n(t',x) - X_n(t,y)\|_2^p] &\leq C_p\mathbb{E}[\|X_n(t',x) - X_n(t,x)\|_2^p + \|X_n(t,x) - X_n(t,y)\|_2^p] \\
&\leq C_{p,d}(|t'-t|^{\theta p} + \|x-y\|_2^p) \\
&\leq C_{p,d}\|\binom{t'}{x} + \binom{t}{y}\|_2^{\min\{\theta,1\}p}
\end{aligned}
$$

From Kolmogorov's continuity theorem A.11 it follows that $X_n \in C([0,1]\times\mathbb{R}^d;\mathbb{R}^d)$. In addition, by theorem 3.5 we have

$$
\begin{aligned}
\mathbb{E}[\|X(t',x) - X(t,y)\|_2^p] &\leq C_p\mathbb{E}[\|X(t',x) - X_n(t',x)\|_2^p + \|X_n(t',x) - X_n(t,y)\|_2^p \\
&\quad + \|X_n(t,y) - X(t,y)\|_2^p] \\
&\leq C_{p,d}(2^{-n\delta p} + |t'-t|^{\theta p} + \|x-y\|_2^p)
\end{aligned}
$$

Sending n to infinity and again using Kolmogorov's continuity theorem A.11 we get the statement of this theorem. □

Chapter 4

Equivalence of the Large Deviation Principle and the Laplace Principle

As indicated in the title we will state the equvalence of the LDP and the Laplace Principle in this section. Afterwards we will present a representation formula of the good rate function for certain families of random variables (see also [Aze80]).

4.1 Equivalence of the Principles

Definition 4.1. (good rate function)
Let E be a Polish space. A mapping $I : E \to [0, \infty]$ is called a good rate function if and only if the set $\{I \leq a\}$ is compact for all $a < \infty$.

Definition 4.2. (Large Deviation Principle)
Let E be a Polish space, I a good rate function and $\{X^\varepsilon, \varepsilon > 0\}$ a family of E-valued random variables. Then this family is said to satisfy the Large Deviation Principle with rate function I if and only if

$$\begin{aligned} (i) \quad & \limsup_{\varepsilon \to 0} \varepsilon \mathbb{P}(X^\varepsilon \in F) \leq -J(F), \quad F \subset E \text{ closed} \\ (ii) \quad & \liminf_{\varepsilon \to 0} \varepsilon \mathbb{P}(X^\varepsilon \in G) \leq -J(G), \quad G \subset E \text{ open} \end{aligned}$$

where $J(A) := \inf_{x \in A} I(x)$.

Definition 4.3. (Laplace Principle)
Let E be a Polish space, $I : E \to [0, \infty]$ a good rate function and $\{Y^\varepsilon, \varepsilon > 0\}$ a family of E-valued random variables. Then this family satisfies the Laplace Principle with good rate function I if and only if for all continuous, bounded functions $h : E \to \mathbb{R}$ it holds that

$$\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}[\exp(-h(Y^\varepsilon)/\varepsilon)] = - \inf_{x \in E} \{h(x) + I(x)\} \tag{4.1}$$

Theorem 4.4. Equivalence of the Principles
Let E be a Polish Space. The Laplace Principle is equivalent to the Large Deviations Principle (with respect to the same good rate function).

Proof: According to the remark at the beginning of [DE11, section 10.2] the proof follows analogously to the discrete case proven in [DE11, Th. 1.2.1 and Th. 1.2.3].

Lemma 4.5. Uniqueness of the good rate function
Let E be a Polish space. A family of probability measures defined on E can satisfy the LDP with at most one good rate function.

Proof: see also [DZ10, Lemma 4.1.4]. $\square$

4.2 Deriving the good rate function

Definition 4.6. (Cramer transform and functional)
Let $(E, \|\cdot\|)$ be a separable Banach space, μ a probability measure on $(E, \mathcal{T}_{\|\cdot\|})$. Let E' denote the dual space of E and $\int_E |\langle t, x\rangle| d\mu(x) < \infty$ for all $t \in E'$. We define

$$\hat{\mu} : E' \to [0, \infty] : t \mapsto \int_E \exp(\langle t, x\rangle) d\mu(x)$$
$$\lambda : E \to [0, \infty] : x \mapsto \sup_{t \in E'} \{\langle t, x\rangle - \ln \hat{\mu}(t)\} \quad \text{(Cramer transform)}$$
$$\Lambda : \mathcal{P}(E) \to [0, \infty] : A \mapsto \inf_{x \in A} \lambda(x) \quad \text{(Cramer functional)}$$

Remark 4.7.
Obviously, those mappings are also well defined in a more general setting (see also [Aze80, Ch. I. 2.1, Def. 4.6 and 5.2]) but for our application we will need a separable Banach space.

Theorem 4.8.
Let E be separable Banach space, $\mu \sim N(0, \sigma)$ a probability measure on E and λ, Λ as given in definition 4.6. Let $X : \Omega \to E$ be a random variable with $X \sim \mu$. Then λ is a good rate function and $\{\sqrt{\varepsilon} X, \varepsilon > 0\}$ satisfies the LDP with λ. For all $A \subset E$ it holds that

$$-\Lambda(A^\circ) \leq \liminf_{\varepsilon \to 0} \varepsilon^2 \ln \mathbb{P}(\varepsilon X \in A) \leq \liminf_{\varepsilon \to 0} \varepsilon^2 \ln \mathbb{P}(\varepsilon X \in A) \leq -\Lambda(\bar{A}) \tag{4.2}$$

where A° denotes the interior and $\bar{A}$ the closure of the set A.

Proof: see also [Aze80, Ch. II. Th. 1.6 and Ch. I. Lemma 6.1]. $\square$

From definition 2.1 und 2.2 and remark 2.3 it follows that $\{\sqrt{\varepsilon} W\}$ satisfies the LDP. The next two lemmata will give us a representation of the Cramer transform.

Lemma 4.9.
Let $\beta : \Omega \to \mathbb{R}^n$ be a centered, quadratic integrable process on the interval $[0, 1]$ with independent increments, $\rho(s, t) := \mathbb{E}[\langle \beta_s, \beta_t\rangle_2]$. Then for all $f, g \in L^2([0, 1]; \mathbb{R}^n)$ it holds that

$$\mathbb{E}[\langle f, \beta\rangle_{L^2} \langle g, \beta\rangle_{L^2}] = \langle f, Rg\rangle_{L^2}$$

where $R : L^2 \to L^2 : g \mapsto \int_0^1 \rho(\cdot, t) g(t) dt$.

Proof: Let $s, t \in [0,1]$. Then

$$\begin{aligned}
\langle f(s), \beta_s \rangle_2 \langle g(t), \beta_t \rangle_2 &= \sum_{i=1}^n f_i(s)\beta_s^{(i)} \sum_{j=1}^n g_j(t)\beta_t^{(j)} \\
&= \sum_{i,j=1}^n f_i(s) g_j(t) \beta_s^{(i)} \beta_t^{(j)}
\end{aligned}$$

From the centeredness (Z) and independence of the increments (I) it follows that $\rho(s,t)$ is a diagonal matrix $\rho(s,t)_{ii} = \mathbb{E}[\beta_s^{(i)}\beta_t^{(i)}]$. From Fubini (F) it follows that

$$\begin{aligned}
\mathbb{E}[\langle f, \beta \rangle_{L^2} \langle g, \beta \rangle_{L^2}] &\overset{(F)}{=} \int_0^1 \int_0^1 \mathbb{E}[\langle f(s), \beta_s \rangle_2 \langle g(t), \beta_t \rangle_2] \; dt \; ds \\
&= \int_0^1 \int_0^1 \sum_{i,j=1}^n f_i(s) g_j(t) \mathbb{E}[\beta_s^{(i)} \beta_t^{(j)}] \; dt \; ds \\
&\overset{(Z),(UI)}{=} \int_0^1 \int_0^1 \sum_{i=1}^n f_i(s) \mathbb{E}[\beta_s^{(i)} \beta_t^{(i)}] g_i(t) \; dt \; ds \\
&= \langle f, Rg \rangle_{L^2}
\end{aligned}$$

Thus, the statement of this lemma is shown. □

Lemma 4.10.
Let $W : \Omega \to E$ be a m-dimensional Brownian motion, E the space of all continuous, $\mathbb{R}^m$-valued functions on $[0,1]$ starting in zero - hence E is a separable Banach space - and μ the law of W on E. The the Cramer transform λ is given by

$$\lambda(f) = \begin{cases} \frac{1}{2}\int_0^1 \|f'(t)\|_2^2 dt, & f \text{ absolutely continuous} \\ \infty & \text{otherwise} \end{cases} \tag{4.3}$$

Proof: see also [Aze80, Ch. II. Prop. 3.6]. □

Remark 4.11.
In [Aze80, Ch. II. Prop. 3.6] the statement is only proven for the real case, but with lemma 4.9 it follows analogously that the statement holds also for the multidimensional case.

Considering a SDE of the form

$$\begin{aligned}
dY_t^\varepsilon &= b(Y_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(Y_t^\varepsilon)dW_t \\
Y_0^\varepsilon &= x
\end{aligned}$$

it is shown in [Aze80, Ch. III. Th. 2.13] that the family $\{Y^\varepsilon\}$ of the solutions satisfies the LDP with the good rate function

$$\tilde{\lambda}(g) = \inf_{f = B^{-1}g} \lambda(f) \tag{4.4}$$

where B maps an absolutely continuous $f \in E$ to the solution of $g'(t) = b(g(t)) + \sigma(g(t))f'(t)$ with $g(0) = x$. In theorem 5.9 we will generalise this result to the stochastic Volterra integral equation case with varying starting value.

Chapter 5

Large Deviation Principle

In this section we will prove that the family $\{X^\varepsilon, \varepsilon > 0\}$ satisfies the Large Deviation Principle where each X^ε solves the integral equation (2.14) with $H \equiv 0$. This will be done in theorem 5.9 by showing that this family satisfies the Laplace Principle, the equivalence of which has already been stated in theorem 4.4.

5.1 Variational Representation Formula

Theorem 5.1.
Let $g : C([0,1];\mathbb{R}^d) \to \mathbb{R}$ be a bounded Borel measurable function, $\mathcal{A}_b$ as defined by (2.6). Then

$$-\ln(\mathbb{E}[\exp(-g(W))]) = \inf_{H\in\mathcal{A}_b} \mathbb{E}[g(W+H) + \frac{1}{2}\|H\|^2_{\mathbb{H}}]$$

Proof: In the statement of [BD98, Theorem 3.1] the infimum is taken over the set $\mathcal{A}$ of all $\mathbb{H}$-valued, $\mathbb{F}$-progressive processes H but if you look at the proof in detail the statement is even shown for $\mathcal{A}_b$. □

Theorem 5.2.
Let $f : C([0,1];\mathbb{R}^d) \to \mathbb{R}$ be a bounded Borel measurable function, X, X^H the solution of (2.11) and (2.14) respectively. Then

$$-\ln(\mathbb{E}[\exp(-f(X))]) = \inf_{H\in\mathcal{A}_b} \mathbb{E}[f(X^H) + \frac{1}{2}\|H\|^2_{\mathbb{H}}]$$

Proof: Per definition of W (see (2.2)) it holds that

$$(X \circ W)(\omega) = X(W(\omega)) = X(\omega)$$

With this representation we have for $X \circ (W+H)$ where $H \in \mathcal{A}_b$

$$\begin{aligned} X(W+H)(t) &= X(0) + \int_0^t b(t,s,X(W+H)(s))ds \\ &\quad + \int_0^t \sigma(t,s,X(W+H)(s))d(W(s)+H(s)) \\ &= X(0) + \int_0^t b(t,s,X(W+H)(s))ds \\ &\quad + \int_0^t \sigma(t,s,X(W+H)(s))H'(s)ds + \int_0^t \sigma(t,s,X(W+H)(s))dW(s) \end{aligned}$$

Hence, $X \circ (W+H)$ solves the same SDE as X^H. Since solutions of (2.14) are unique they have to coincide. Therefore we get

$$\begin{aligned} -\ln(\mathbb{E}[\exp(-f(X))]) &= -\ln(\mathbb{E}[\exp(-f(X \circ W))]) \\ &\overset{5.1}{=} \inf_{H \in \mathcal{A}_b} \mathbb{E}[f(X \circ (W+H)) + \frac{1}{2}\|H\|_{\mathbb{H}}^2] \\ &= \inf_{H \in \mathcal{A}_b} \mathbb{E}[f(X^H) + \frac{1}{2}\|H\|_{\mathbb{H}}^2] \end{aligned}$$

Thus, the statement of this theorem is shown. □

5.2 The good rate function

In this section we will show that the mapping I as defined in lemma 5.7 is indeed a good rate function.

Lemma 5.3.
Let $B_N \subset \mathbb{H}$ be the ball with radius N.

$$B_N := \{h \in \mathbb{H} \mid \|h\|_{\mathbb{H}} \leq N\} \tag{5.1}$$

Then, the space $(B_N, \mathcal{T}_{\|\cdot\|_{\mathbb{H}}}|_{B_N})$ is a compact Polish space where $\mathcal{T}_{\|\cdot\|_{\mathbb{H}}}$ denotes the induced topology from $\|\cdot\|_{\mathbb{H}}$ and $\mathcal{T}_{\|\cdot\|_{\mathbb{H}}}|_{B_N}$ the corresponding trace topology.

Proof: We have to prove the following properties: (i) metrisable, (ii) separable, (iii) complete, (iv) compact. (i) The metrisability is clear since $\|\cdot\|_{\mathbb{H}}$ induces the metric $d_{\mathbb{H}}$

$$d_{\mathbb{H}}(g,h) := \|g-h\|_{\mathbb{H}}, \quad g,h \in \mathbb{H} \tag{5.2}$$

(ii) Since $(\mathbb{H}, \mathcal{T}_{\|\cdot\|_{\mathbb{H}}})$ is separable this also holds for $(B_N, \mathcal{T}_{\|\cdot\|_{\mathbb{H}}}|_{B_N})$ as its subspace. (iii) Let $(h_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in B_N (and in particular in $\mathbb{H}$ too). Because $\mathbb{H}$ is complete, there exists a $h \in \mathbb{H}$ such that $h_n \to h$ in $\mathbb{H}$. Since

$$\|h\|_{\mathbb{H}}^2 = \int_0^t \|h'(s)\|_2^2 ds = \lim_{n\to\infty} \int_0^t \|h_n'(s)\|_2^2 ds \leq N^2$$

it follows that $h \in B_N$. Thus, B_N is complete. (iv) Since the topology is induced by a metric the compactness of B_N follows from its completeness and boundedness. □

Definition 5.4.
Let $h \in \mathbb{H}$, $x \in \mathbb{R}^d$. We consider the ODE

$$S(h)(t) = S(h)(0) + \int_0^t b(t,s,S(h)(s))ds + \int_0^t \sigma(t,s,S(h)(s))h'(s)ds \tag{5.3}$$
$$S(h)(0) = x$$

To highlight that a solution $S(h)$ of (5.3) is starting at $S(h)(0) = x$ we will write $S(h)(\cdot, x)$.

Lemma 5.5.
For all $N > 0$ the set $\{S(h) : h \in B_N\}$ is relatively compact in $C([0,1] \times D_R; \mathbb{R}^d)$.

Proof: By Arzela-Ascoli we only need to prove that the set is uniformly equicontinuous and that $\{S(h)(t,x) : h \in B_N\}$ is uniformly bounded for all $(t,x) \in [0,1] \times D_R$. Let $p > 0$ sufficiently large. Then

$$\begin{aligned}
\|S(h)(t,x)\|_2^p &\overset{(A.2)}{\leq} C_p(\|x\|_2^p + (\int_0^t \|b(t,s,S(h)(s,x))\|_2 ds)^p \\
&\quad + (\int_0^t \|\sigma(t,s,S(h)(s,x))h'(s)\|_2 ds)^p) \\
&\overset{(a)}{\leq} C_p(R^p + (\int_0^t \|b(t,s,S(h)(s,x))\|_2 ds)^p \\
&\quad + (\int_0^t \|\sigma(t,s,S(h)(s,x))\|_F^2 ds)^{p/2} \|h\|_{\mathbb{H}}^p) \\
&\overset{(a)}{\leq} C_{p,R} + C_{p,R} \int_0^t \|S(h)(s,x)\|_2^p ds
\end{aligned}$$

The uniformly boundedness now follows from Gronwall and $\|\cdot\|_2 = (\|\cdot\|_2^p)^{1/p}$. Furthermore it holds that

$$\begin{aligned}
\|S(h)(t',x) - S(h)(t,y)\|_2^p &\overset{(A.2)}{\leq} C_p(\|S(h)(t',x) - S(h)(t,x)\|_2^p + \|S(h)(t,x) - S(h)(t,y)\|_2^p) \\
&\overset{(a)}{\leq} C_p(|t'-t|^{\theta p} + \|x-y\|_2^p)
\end{aligned}$$

Hence, for all $\varepsilon > 0$ and $(t,x) \in [0,1] \times D_R$ there is $\delta > 0$ such that for all $(t',y) \in [0,1] \times D_R$ satisfying $\|(t-t', x-y)\|_2 \leq \delta$ the p-th root of the last inequality is not bigger than ε. Thus, we have shown that $\{S(h) : h \in B_N\}$ is uniformly equicontinuous. $\square$

Lemma 5.6.
The mapping $\Phi : \mathbb{H} \to C([0,1] \times D_R; \mathbb{R}^d) : h \mapsto S(h)$ is continuous.

Proof: Let $(h_n)_{n\in\mathbb{N}}, h \in \mathbb{H}$, $h_n \to h$, and $N := \|h\|_{\mathbb{H}}$. For all $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\|h_n - h\|_{\mathbb{H}} \leq \varepsilon \quad \implies \quad \|h_n\|_{\mathbb{H}} \leq \varepsilon + \|h\|_{\mathbb{H}} < \infty$$

Then

$$\begin{aligned}
\|S(h_n)(t,x) - S(h)(t,x)\|_2^p &\overset{(A.2)}{\leq} C_p((\int_0^t \|b(t,s,S(h_n)(s,x)) - b(t,s,S(h)(s,x))\|_2 ds)^p \\
&\quad +(\int_0^t \|(\sigma(t,s,S(h_n)(s,x)) - \sigma(t,s,S(h)(s,x)))h_n'(s)\|_2 ds)^p \\
&\quad +(\int_0^t \|\sigma(t,s,S(h)(s,x))(h_n'(s) - h'(s))\|_2 ds)^p) \\
&\overset{(a)}{\leq} C_p \int_0^t \|S(h_n)(s,x) - S(h)(s,x)\|_2^p ds \\
&\quad +C_p \int_0^t \|S(h_n)(s,x) - S(h)(s,x)\|_2^p ds \ (N+\varepsilon)^p \\
&\quad +C_p(\int_0^t \|\sigma(t,s,S(h)(s,x))\|_F^2 ds)^{p/2} \|h_n - h\|_{\mathbb{H}}^p \\
&\overset{(a)}{\leq} C_p \int_0^t \|S(h_n)(s,x) - S(h)(s,x)\|_2^p ds \\
&\quad +C_p \int_0^t \|S(h_n)(s,x) - S(h)(s,x)\|_2^p ds \ (N+\varepsilon)^p \\
&\quad +C_p(1 + \int_0^t \|S(h)(s,x))\|_2^p ds) \|h_n - h\|_{\mathbb{H}}^p \\
&\overset{(a')}{\leq} C_p \int_0^t \|S(h_n)(s,x) - S(h)(s,x)\|_2^p ds \\
&\quad +C_p \int_0^t \|S(h_n)(s,x) - S(h)(s,x)\|_2^p ds \ (N+\varepsilon)^p \\
&\quad +C_p(1 + \|x\|_2^p) \|h_n - h\|_{\mathbb{H}}^p
\end{aligned}$$

The statement of lemma 3.1 can be shown analogously for $S(h)$. This is used at the (a') inequality. It now follows from Gronwall and $\|x\|_2 \leq R$ that

$$\|S(h_n)(t,x) - S(h)(t,x)\|_2^p \leq C_{p,R,h} \|h_n - h\|_{\mathbb{H}}^p$$

Hence

$$\begin{aligned}
&\|S(h_n)(t,x) - S(h)(t,x)\|_2 = (\|S(h_n)(t,x) - S(h)(t,x)\|_2^p)^{1/p} \\
&\leq C_{p,R,h}(\|h_n - h\|_{\mathbb{H}}^p)^{1/p} = C_{p,R,h} \|h_n - h\|_{\mathbb{H}}
\end{aligned}$$

This upper bound is independent of (t,x) and so the inequality still holds true if we build the supremum over $[0,1] \times D_R$. We then get

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,1]\times D_R} \|S(h_n)(t,x) - S(h)(t,x)\|_2 \leq \lim_{n\to\infty} \sup_{(t,x)\in[0,1]\times D_R} C_{p,R,h} \|h_n - h\|_{\mathbb{H}} = 0$$

Thus, the mapping Φ is continuous. □

Lemma 5.7.
For each $R > 0$ the mapping

$$I : C([0,1] \times D_R; \mathbb{R}^d) \to [0,\infty] : f \mapsto \frac{1}{2} \inf_{h\in\mathbb{H}, S(h)=f} \|h\|_{\mathbb{H}}^2 \tag{5.4}$$

is a good rate function (see definition 4.1).

Proof: From Lemma 4.10 we already now that I is a good rate function if we restrict it to $C([0,1] \times \{x\}; \mathbb{R}^d)$ for any $x \in D_R$. We will now prove that this can be "generalised" as stated in this lemma. Let $f \in C([0,1] \times D_R; \mathbb{R}^d)$. Per definition of I there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{H}$ such that $\|h_n\|_{\mathbb{H}}^2 \to 2I(f)$. Let $N := \sup_{n\in\mathbb{N}} \|h_n\|_{\mathbb{H}}$ and w.l.o.g. $N < \infty$. Then $h_n \in B_N$ for all $n \in \mathbb{N}$. By lemma5.3 there exists a $h \in B_N$ and a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that $h_{n_k} \to h$. This h then satisfies

$$\|h\|_{\mathbb{H}}^2 = \lim_{k\to\infty} \|h_{n_k}\|_{\mathbb{H}}^2 = 2I(f)$$

As shown in lemma 5.6 the mapping $h \mapsto S(h)$ is continuous, hence it holds that

$$S(h) = \lim_{k\to\infty} S(h_{n_k}) = f$$

From there $2I(f) = \|h\|_{\mathbb{H}}^2$ follows. Let $a < \infty$, $A := \{f \in C([0,1] \times D_R; \mathbb{R}^d) \mid I(f) \leq a^2\}$. Since $A \subset \{S(h) : h \in B_{2a}\}$, which is relatively compact by lemma 5.5, we only need to prove that A is closed. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in A converging to an $f \in \mathbb{C}([0,1] \times \mathbb{R}^d; \mathbb{R}^d)$. For each f_n there exists a $h_n \in B_{2a}$ satisfying $2I(f_n) = \|h_n\|_{\mathbb{H}}^2$ and $S(h_n) = f_n$. Again there exists a $h \in B_N$ and a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that $h_{n_k} \to h$. Since $f_{n_k} = S(h_{n_k}) \to S(h)$ it follows that $f = S(h)$ and thus $I(f) \leq 1/2\|h\|_{\mathbb{H}}^2 = a^2$, which shows that $f \in A$. As a subset of a relatively compact set the compactness of A now follows from its completeness. □

5.3 X^ε satisfies the LDP

Lemma 5.8.
For $\varepsilon > 0$, $N > 0$ let $H^\varepsilon \in \mathcal{A}_N$ such that the mapping $\varepsilon \mapsto H^\varepsilon$ is continuous in the sense that $\varepsilon \mapsto H^\varepsilon(\omega)$ is countinuous with respect to $\|\cdot\|_{\mathbb{H}}$ for all $\omega \in \Omega$. Let $\varepsilon_0 > 0$. Then for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0, \varepsilon_0]$ there exists a subsequence (ε_{n_k}) and $\varepsilon^* \in [0, \varepsilon_0]$, $H \in \mathcal{A}_N$ such that

$$\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2 \leq R} \|X^{\varepsilon_n, H^{\varepsilon_n}}(t,x) - X^{\varepsilon^*, H}(t,x)\|_2^p] = 0$$

In particular, $\{H^\varepsilon, \varepsilon \in (0, \varepsilon_0]\}$ and $\{X^{\varepsilon, H^\varepsilon}, \varepsilon \in (0, \varepsilon_0]\}$ are tight families in B_N and $C([0,1] \times D_R; \mathbb{R}^d)$ respectively.

Proof: Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0, \varepsilon_0]$ and w.l.o.g. there exists a $\varepsilon^* \in [0, \varepsilon_0]$ and a $H \in \mathcal{A}_N$ such that $\varepsilon_n \to \varepsilon^*$ and $H^{\varepsilon_n} \to H$ (otherwise there would be a subsequence satisfying this statement). Then

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[\|H_t^{\varepsilon_n} - H_t\|_2] &\leq \lim_{n\to\infty} \mathbb{E}[\|H^{\varepsilon_n} - H\|_{\mathbb{H}}] \\
&= \mathbb{E}[\lim_{n\to\infty} \|H^{\varepsilon_n} - H\|_{\mathbb{H}}] \\
&= 0
\end{aligned}$$

From there it follows that the laws of H^{ε_n} are converging in distribution to the law of H. Since B_N is a Polish space we get that $\{H^\varepsilon\}$ is a tight family by lemma A.7. Next, we want to show the same result for $\{X^{\varepsilon, H^\varepsilon}\}$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ as before. Then for all p sufficiently large and using the notation $X^n := X^{\varepsilon_n, H^{\varepsilon_n}}$ and $X^* := X^{\varepsilon^*, H}$ it follows that

$$
\begin{aligned}
\mathbb{E}[\|X^n(t,x) - X^*(t,x)\|_2^p] \quad &\leq \quad C_p\Big(\mathbb{E}[\| \int_0^t (\ b(t,s,X^n(s,x)) - b(t,s,X^*(s,x)\ )ds\|_2^p] \\
&\quad + \mathbb{E}[\| \int_0^t (\ \sqrt{\varepsilon_n}\sigma(t,s,X^n(s,x)) - \sqrt{\varepsilon^*}\sigma(t,s,X^*(s,x))\ )dW_s\|_2^p] \\
&\quad + \mathbb{E}[\| \int_0^t (\ \sigma(t,s,X^n(s,x))(H^{\varepsilon_n})'_s - \sigma(t,s,X^*(s,x))H'_s\ )ds\|_2^p]\Big) \\
&=: \quad C_p(I_1 + I_2 + I_3)
\end{aligned}
$$

Analogously to the steps in the proof of theorem 3.1 and lemma 5.6 we get the estimate for I_1

$$
I_1 \leq C_p \int_0^t \mathbb{E}[\|X^n(s,x) - X^*(s,x)\|_2^p]ds
$$

Denoting analogous steps again by (a) we get for I_2 and I_3

$$
\begin{aligned}
I_3 \quad &\overset{(a)}{\leq} \quad C_{p,d,H}\Big(\int_0^t \mathbb{E}[\|X^n(s,x) - X^*(s,x)\|_2^p]ds \\
&\quad + \mathbb{E}[(\int_0^t \|\sigma(t,s,X^*(s,x))\|_F ds \|H^{\varepsilon_n} - H\|_{\mathbb{H}})^p]\Big) \\
&\leq \quad C_{p,d,H}\Big(\mathbb{E}[...] + \mathbb{E}[((\int_0^t \|\sigma(t,s,X^*(s,x))\|_F^2 ds)^{1/2} \|H^{\varepsilon_n} - H\|_{\mathbb{H}})^p]\Big) \\
&\overset{(A.5)}{\leq} \quad C_{p,d,H}\Big(\mathbb{E}[...] + \mathbb{E}[(\int_0^t \|\sigma(t,s,X^*(s,x))\|_F^2 ds)^p]^{1/2} \mathbb{E}[\|H^{\varepsilon_n} - H\|_{\mathbb{H}}^{2p}]^{1/2}\Big) \\
&\overset{(a)}{\leq} \quad C_{p,d,H}\Big(\int_0^t \mathbb{E}[\|X^n(s,x) - X^*(s,x)\|_2^p]ds + (1 + \|x\|_2^{2p})^{1/2}(\mathbb{E}[\|H^{\varepsilon_n} - H\|_{\mathbb{H}}^{2p}])^{1/2}\Big) \\
I_2 \quad &\overset{(a)}{\leq} \quad C_{p,d}\mathbb{E}[(\int_0^t \|\sqrt{\varepsilon_n}\sigma(t,s,X^n(s,x)) - \sqrt{\varepsilon^*}\sigma(t,s,X^*(s,x))\|_F^2 ds)^{p/2}] \\
&\leq \quad C_{p,d}\Big(\varepsilon_n^p \mathbb{E}[(\int_0^t \|\sigma(t,s,X^n(s,x))\|_F^2 ds)^{p/2}] + (\varepsilon^*)^p \mathbb{E}[(\int_0^t \|\sigma(t,s,X^*(s,x))\|_F^2 ds)^{p/2}]\Big) \\
&\leq \quad C_{p,d}|\varepsilon_n - \varepsilon^*|^p \max\Big\{\mathbb{E}[(\int_0^t \|\sigma(t,s,X^n(s,x))\|_F^2 ds)^{p/2}], \mathbb{E}[(\int_0^t \|\sigma(t,s,X^*(s,x))\|_F^2 ds)^{p/2}]\Big\} \\
&\overset{(a)}{\leq} \quad C_{p,d}|\varepsilon_n - \varepsilon^*|^p\Big(1 + \max\Big\{\int_0^t \mathbb{E}[\|X^n(s,x)\|_2^p]ds, \int_0^t \mathbb{E}[\|X^*(s,x)\|_2^p]ds\Big\}\Big)
\end{aligned}
$$

Analogously to lemma 3.1 and lemma 5.5 it can be shown that $\mathbb{E}[\|X^{\varepsilon,\tilde{H}}(t,x)\|_2^p] \leq C_{p,d,\tilde{H}}(1 + \|x\|_2^p)$ for all $\varepsilon > 0$ and $\tilde{H} \in \mathcal{A}_b$. Putting all the parts together and using Gronwall

$$
\mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2 \leq R} \|X^n(t,x) - X^*(t,x)\|_2^p] \leq C_{p,d,H,R}(|\varepsilon_n - \varepsilon^*|^p + (\mathbb{E}[\|H^{\varepsilon_n} - H\|_{\mathbb{H}}^{2p}])^{1/2})
$$

and in conclusion

$$
\lim_{n\to\infty} \mathbb{E}[\sup_{t\in[0,1],\ \|x\|_2 \leq R} \|X^{\varepsilon_n, H^{\varepsilon_n}}(t,x) - X^{\varepsilon^*, H}(t,x)\|_2^p] = 0
$$

In particular, $\{X^{\varepsilon, H^\varepsilon}\}$ is a tight family too. $\square$

Theorem 5.9.
The family $\{X^\varepsilon, \varepsilon > 0\}$, $X^\varepsilon \in C([0,1] \times D_R; \mathbb{R}^d)$ satisfies the Laplace Principle with good rate function I as given in lemma 5.7.

Proof: Let g be a bounded, continuous function on $C([0,1] \times D_R; \mathbb{R}^d)$, $\|g\|_\infty =: M$. We now want to show that

$$\lim_{\varepsilon \to 0} -\varepsilon \ln \mathbb{E}[\exp(-g(X^\varepsilon)/\varepsilon)] = \inf_{x \in E}\{g(x) + I(x)\}$$

"$\geq$": From theorem 5.2 it follows that

$$\begin{aligned} -\varepsilon \ln \mathbb{E}[\exp(-g(X^\varepsilon)/\varepsilon)] &= \varepsilon \inf_{H \in \mathcal{A}_b} \mathbb{E}[g(X^{\varepsilon,\sqrt{\varepsilon}H})/\varepsilon + 1/2\ \|H\|_{\mathbb{H}}^2] \\ &= \inf_{H \in \mathcal{A}_b} \mathbb{E}[g(X^{\varepsilon,\sqrt{\varepsilon}H}) + 1/2\ \|\sqrt{\varepsilon}H\|_{\mathbb{H}}^2] \\ &= \inf_{H \in \mathcal{A}_b} \mathbb{E}[g(X^{\varepsilon,H}) + 1/2\ \|H\|_{\mathbb{H}}^2] \end{aligned}$$

where it can be shown analogously to theorem 5.2 that $X^\varepsilon \circ (W + H)$ solves the same SDE as $X^{\varepsilon,\sqrt{\varepsilon}H}$, hence it follows from the uniqueness that both processes have to be identical. Since $H^* \equiv 0$ is an element of $\mathcal{A}_b$ if follows that for all H satisfying $\|H\|_{\mathbb{H}}^2 \geq 4M$ it holds that

$$\mathbb{E}[g(X^{\varepsilon,H}) + 1/2\ \|H\|_{\mathbb{H}}^2] \geq \mathbb{E}[g(X^{\varepsilon,H})] + 2M \geq \mathbb{E}[g(X^\varepsilon)]$$

since $|g(X^{\varepsilon,H}) - g(X^\varepsilon)| \leq 2M$. Thus we can restrict the infimum to the set $\{\|H\|_{\mathbb{H}}^2 \leq 4M\} =: \mathcal{A}_{4M}$ without changing its value. Let $\delta > 0$. For all $H \in \mathcal{A}_b$ the mapping $\varepsilon \mapsto H$ is continuous, hence as shown in the proof of lemma 5.8 it holds that $\lim_{n\to\infty} \mathbb{E}[\|X^{\varepsilon_n,H} - X^H\|_2^p] \to 0$ for all $\varepsilon_n \to 0$. Hence, there exists a $n_0(H)$ such that for all $n \geq n_0(H)$

$$\begin{aligned} &|\mathbb{E}[g(X^{\varepsilon_n,H}) - g(X^H)]| \leq \delta \\ \Longrightarrow\quad &\mathbb{E}[g(X^{\varepsilon_n,H})\ +1/2\ \|H\|_{\mathbb{H}}^2] \geq \mathbb{E}[g(X^H)\ +1/2\ \|H\|_{\mathbb{H}}^2] - \delta \end{aligned}$$

The value n_0 is (a priori) depending on H since constant C in the estimate of lemma 5.8 contains the value $\|H\|_{\mathbb{H}}$. Because we restrict ourselves to $\mathcal{A}_{4M}$ this dependence can be omited and we can choose a n_0 independent of H. Thus we get

$$\begin{aligned} \liminf_{\varepsilon\to 0} -\varepsilon \ln \mathbb{E}[\exp(-g(X^\varepsilon)/\varepsilon)] &= \liminf_{\varepsilon\to 0} \inf_{H \in \mathcal{A}_{4M}} \mathbb{E}[g(X^{\varepsilon,H}) + 1/2\ \|H\|_{\mathbb{H}}^2] \\ &\geq \inf_{\varepsilon_n \to 0} \liminf_{n\to\infty} \inf_{H \in \mathcal{A}_{4M}} \mathbb{E}[g(X^{\varepsilon_n,H}) + 1/2\ \|H\|_{\mathbb{H}}^2] \\ &\geq \inf_{H \in \mathcal{A}_{4M}} \mathbb{E}[g(X^H) + 1/2\ \|H\|_{\mathbb{H}}^2] - \delta \end{aligned}$$

By lemma 5.6 the mapping $h \mapsto g(X^h) + 1/2\ \|h\|_{\mathbb{H}}^2$ is continuous (where X^h and $S(h)$ denote the same process), hence there exists a constant process $H_0 \equiv h_0 \in B_{4M}$ such that

$$\inf_{H \in \mathcal{A}_{4M}} \mathbb{E}[g(X^H) + 1/2\ \|H\|_{\mathbb{H}}^2] \geq g(S(h_0)) + 1/2\ \|h_0\|_{\mathbb{H}}^2 - \delta$$

In conclusion we get that

$$\begin{aligned} \liminf_{\varepsilon\to 0} -\varepsilon \ln \mathbb{E}[\exp(-g(X^\varepsilon)/\varepsilon)] &\geq g(S(h_0)) + 1/2\ \|h_0\|_{\mathbb{H}}^2 - 2\delta \\ &\geq \inf_{f,h:S(h)=f}\{g(f) + 1/2\ \|h\|_{\mathbb{H}}^2\} - 2\delta \\ &\geq \inf_f\{g(f) + I(f)\} - 2\delta \end{aligned}$$

The statement now follows from sending $\delta \to 0$.

"$\leq$": Since I is taking finite values and g is bounded it follows that $\inf\{g(f)+I(f)\}$ is finite too. Let $\delta > 0$. Then there exists a f_0 such that

$$g(f_0) + I(f_0) \leq \inf_f \{g(f) + I(f)\} + \delta$$

Hence, $I(f_0) =: N$ has to finite. As shown in the proof of lemma 5.7 there exists a $h_0 \in B_{3N}$ such that

$$1/2\ \|h_0\|_{\mathbb{H}}^2 = I(f_0) \quad \wedge \quad f_0 = S(h_0)$$

Again, it follows from theorem 5.2 (where we identify h_0 with the constant process H_0)

$$\begin{aligned}
\limsup_{\varepsilon\to 0} -\varepsilon \ln \mathbb{E}[\exp(-g(X^\varepsilon)/\varepsilon)] &= \limsup_{\varepsilon\to 0} -\varepsilon \inf_{H\in\mathcal{A}_b} \mathbb{E}[g(X^{\varepsilon,H}) + 1/2\ \|H\|_{\mathbb{H}}^2] \\
&\leq \limsup_{\varepsilon\to 0} \mathbb{E}[g(X^{\varepsilon,h_0}) + 1/2\ \|h_0\|_{\mathbb{H}}^2] \\
&= g(S(h_0)) + 1/2\ \|h_0\|_{\mathbb{H}}^2 \\
&\leq g(f_0) + I(f_0) \\
&\leq \inf_f \{g(f) + I(f)\} + \delta
\end{aligned}$$

The statement now follows from sending $\delta \to 0$. $\square$

Chapter A

Appendix

Lemma A.1.
Let $a, b, r \geq 0$. Then

$$(a+b)^r \leq 2^r(a^r + b^r) \tag{A.1}$$

Proof: It holds that

$$(a+b)^r \leq \max\{(2a)^r, (2b)^r\} \leq 2^r(a^r + b^r)$$

which proves the statement. □

Corollary A.2.
Let $d \in \mathbb{N}$, $a_i > 0$ for all $i \in [1, d]_{\mathbb{N}}$ and $r > 0$. Then there exists a $C > 0$ such that

$$\left(\sum_{i=1}^{d} a_i\right)^r \leq C \sum_{i=1}^{d} a_i^r \tag{A.2}$$

A possible choice for the constant is $C = 2^{\lceil \log_2 d \rceil r}$.

Proof: The proof follows by iteratively splitting the sum into two sums with equal amount of summands and applaying lemma A.1. □

Lemma A.3. Burkholder-Davis-Gundy inequality (BDG-inequality)
Let $X_t^* := \sup_{s \leq t} \|X_s\|_2$. Then

$$\begin{aligned}&\forall p \geq 1\ \exists c_p, C_p\ \forall X \text{ real local martingale}, X_0 = 0\ \forall \tau \text{ stopping time}: \\ &\quad c_p \mathbb{E}[[X]_\tau^{p/2}] \leq \mathbb{E}[(X_t^*)^p] \leq C_p \mathbb{E}[[X]_\tau^{p/2}]\end{aligned} \tag{A.3}$$

If we additionally require X to be continuous the statement even holds for all $p > 0$.

Definition A.4. (Hoelder continuity)
Let (X, d_X), (Y, d_Y) be metric spaces. Then a function $f : X \to Y$ is called α-Hoelder continuous for a $\alpha > 0$

$$:\Leftrightarrow \quad \exists C > 0\ \forall x_1, x_2 \in X : d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)^\alpha \tag{A.4}$$

Lemma A.5. Hölder inequality
Let $f \in L_p(\mu)$, $g \in L_q(\mu)$ real valued where $1 \leq p, q \leq \infty$, $1/p + 1/q = 1$. Then

$$\int |fg| d\mu \leq \Big(\int |f|^p d\mu \Big)^{1/p} \Big(\int |g|^q d\mu \Big)^{1/q} \tag{A.5}$$

Definition A.6. (weakly relatively compact)
A family M of probability measures is called weakly relatively compact if and only if for all sequences in M there exists a weakly convergent subsequence. Weakly convergent means that for $\mu_n \to \mu$ the respective cumulative distribution functions F_n converge to F in every point of continuity of F.

Lemma A.7. Prokhorov Kriterium
Let (E, d) be a metric space, $\mathcal{B}$ the respective Borel σ-algebra. A family of probability measures on $(E, \mathcal{B})$ is weakly relatively compact, if it is tight. If in addition $(E, \mathcal{B})$ Polish space, the conversive statement holds true as well.

Lemma A.8. Portmanteau
Let $\mathbb{P}_n, \mathbb{P}$ be probability measure on the same measureable space which as given in lemma A.7 is of the form $(E, \mathcal{B})$. Then the following statements are equivalent.

$$(i) \quad \mathbb{P}_n \overset{\text{schw.}}{\to} \mathbb{P}$$

$$(ii) \quad \lim_{n\to\infty} \int f d\mathbb{P}_n = \int f d\mathbb{P}, \quad f \text{ stetig, beschränkt}$$

Remark A.9.
In the original statement there are some more equivalent properties but this one is the only of interest here for us.

Lemma A.10. extended Minkoski inequality
Let (S_1, μ_1), (S_2, μ_2) be two measure spaces, $F : S_1 \times S_2 \to \mathbb{R}$ measureable and $p > 1$. Then

$$\Big[\int_{S_2} \Big(\int_{S_1} |F(x,y)| d\mu_1(x) \Big)^p d\mu_2(y) \Big]^{1/p} \leq \int_{S_1} \Big(\int_{S_2} |F(x,y)|^p d\mu_2(y) \Big)^{1/p} d\mu_1(x) \tag{A.6}$$

Proof: see also [HLP88, Th. 202]. There it "only" says "dx" as the integrator but as is noted at the start of the chapter 'Integrals' this is meant to be read as "$d\mu(x)$".

Theorem A.11. Kolmogorov's continuity criterion
Let $(X_t)_{t\in\mathbb{R}^n}$ be a stochastic process taking values in a complete, seperable, metric space (S, d). Let $\alpha, C, \varepsilon > 0$ such that for all $s, t \in \mathbb{R}^n$

$$\mathbb{E}[d(X_s, X_t)^\alpha] \leq C \|s - t\|_2^{n+\varepsilon}$$

Then there exists a contiuous version of X which is Hoelder contiuous of order θ for all $\theta \in (0, \varepsilon/\alpha]$.

Proof: see also [RW00, Chapter I, Theorem 25.2]. □

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