The structural complexity of models of arithmetic

joint work with Antonio Montalbán

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Theorem (Scott 1963)
For every countable structure $\mathcal{A}$ there is a sentence in the infinitary logic $L_{\omega_1\omega}$ – its Scott sentence – characterizing $\mathcal{A}$ up to isomorphism among countable structures.

The proof heavily relies on the analysis of the α-back-and-forth relations for countable ordinals α. The most useful definition is due to Ash and Knight:

Definition

1. $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if all atomic fromulas true of $\bar{b}$ are true of $\bar{a}$ and vice versa.
2. For non-zero $\gamma < \omega_1$, $(\mathcal{A}, \bar{a}) \leq_\gamma (\mathcal{B}, \bar{b})$ if for all $\beta < \gamma$ and $\bar{d} \in B^{<\omega}$ there is $\bar{c} \in A^{<\omega}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

In an attempt to measure structural complexity, various notions of ranks have been used.

Theorem (Montalbán 2015)
The following are equivalent for countable $\mathcal{A}$ and $\alpha < \omega_1$.

1. *Every automorphism orbit of $\mathcal{A}$ is $\Sigma^{\mathrm{in}}_{\alpha}$-definable without parameters.*
2. *$\mathcal{A}$ has a $\Pi^{\mathrm{in}}_{\alpha+1}$ Scott sentence.*
3. *$\mathcal{A}$ is uniformly $\mathbf{\Delta}^0_{\alpha}$-categorical.*
4. *$Iso(\mathcal{A})$ is $\mathbf{\Pi}^0_{\alpha+1}$.*
5. *No tuple in $\mathcal{A}$ is α-free.*

The least α satisfying the above is the (parameterless) Scott rank of $\mathcal{A}$.

A robust Scott rank

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Recently, an even more fine-grained notion has received some interest.

Definition
The Scott complexity of a structure $\mathcal{A}$ is the least complexity among $\Sigma^{\mathrm{in}}_{\alpha}$, $\Pi^{\mathrm{in}}_{\alpha}$, and d-$\Sigma^{\mathrm{in}}_{\alpha}$ of a Scott sentence for $\mathcal{A}$.

This notion is even more robust than the above as it corresponds to the Wadge degree of the isomorphism classes of $\mathcal{A}$ (A. Miller 1983, AGH-TT).

Connection to $\leq_\alpha$

Theorem (Ash, Knight)
For two countable structures $\mathcal{A}$ the following are equivalent.

1. $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$.
2. *All $\Sigma^{\mathrm{in}}_\alpha$ sentences true of $\bar{b}$ in $\mathcal{B}$ are true of $\bar{a}$ in $\mathcal{A}$.*
3. *All Π^{in}_α sentences true of $\bar{a}$ in $\mathcal{A}$ are true of $\bar{b}$ in $\mathcal{B}$.*

Definition
A tuple $\bar{a}$ in $\mathcal{A}$ is *α-free* if

$$\forall(\beta < \alpha)\forall \bar{b} \exists \bar{a}' \bar{b}' (\bar{a}\bar{b} \leq_\beta \bar{a}'\bar{b}' \wedge \bar{a} \not\leq_\alpha \bar{a}').$$

Scott ranks in classes of structures

- A main topic in computable structure theory is to investigate computability theoretic properties in classes of structures
- *computable categoricity* – "Measure the complexity of the isomorphisms between computable copies of a given structure $\mathcal{A}$"
- *index set complexity* – "How hard is it to identify the indices of computable structures isomorphic to a given structure $\mathcal{A}$"
- In many cases answers to this questions are obtained by (implicitly) calculating the Scott rank.
- Often these results are obtained by giving reductions from a well understood classes.

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- Often these results are obtained by giving reductions from a well understood classes.

Example: Ash (1986) characterized the back-and-forth relations of well-orderings. The following is a corollary of his analysis: $SR(n) = 1$, $SR(\omega^\alpha) = 2\alpha$, $SR(\omega^\alpha + \omega^\alpha) = 2\alpha + 1$.

Question: What about the class of countable models of Peano arithmetic?

First results

By Peano arithmetic (PA) we mean the usual semiring axioms together with the first-order induction scheme in the language $(\dot{0}, \dot{1}, +, \cdot, <)$.

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We can compute an isomorphism between any copies $\mathcal{A}, \mathcal{B}$ by mapping $a = (\dot{0} + \dot{n})^{\mathcal{A}}$ to the unique element satisfying the equality in $\mathcal{B}$. Thus, $\mathbb{N}$ is uniformly Δ^0_1 categorical.

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Definition
The *Scott spectrum* of a theory T is the set

$$SS(T) = \{\alpha \in \omega_1 : \text{there is a countable model of } T \text{ with Scott rank } \alpha\}.$$

Goal:

- What is $SS(PA)$?
- What are the Scott ranks of well-understood models?

Formalizing back-and-forth relations

Throughout this talk $\mathcal{M}$ and $\mathcal{N}$ denote countable non-standard models of PA.

Recall that $\mathcal{M}$-finite sets can be coded by single elements, i.e., given $S \subseteq_{fin} M$ code it using $\sum_{s \in S} 2^s$. Thus finite strings $\bar{u} \in M^{<\omega}$ can be considered as the $\mathcal{M}$-finite set $\{\langle i, \bar{u}(i)\rangle : i < |\bar{u}|\}$.

Let $Tr_{\Delta_1^0}$ be a truth predicate for bounded formulas and define the formal back-and-forth relations by induction on n:

$$\bar{u} \leq_0^a \bar{v} \Leftrightarrow \forall(x \leq a)(Tr_{\Delta_1^0}(x, \bar{u}) \rightarrow Tr_{\Delta_1^0}(x, \bar{v}))$$
$$\bar{u} \leq_{n+1}^a \bar{v} \Leftrightarrow \forall \bar{x} \exists \bar{y} \Big(|\bar{x}| \leq a \rightarrow (|\bar{y}| \leq a \wedge \bar{u}\bar{x} \leq_n^a \bar{v}\bar{y})\Big)$$

Proposition
The formal back-and-forth relations $\leq_n^x$ satisfy the following properties for all n:

1. $PA \vdash \forall \bar{u}, \bar{v}, a, b((a \leq b \wedge \bar{u} \leq_n^b \bar{v}) \to \bar{u} \leq_n^a \bar{v})$
2. $PA \vdash \forall \bar{u}, \bar{v}, a(\bar{u} \leq_{n+1}^a \bar{v} \to \bar{u} \leq_n^a \bar{v})$

Proposition
Let $\bar{a}, \bar{b} \in M$. Then $\bar{a} \leq_n \bar{b} \Leftrightarrow \forall (m \in \omega) \mathcal{M} \vDash \bar{a} \leq_n^{\dot{m}} \bar{b}$. Furthermore, if there is $c \in M - \mathbb{N}$ such that $\mathcal{M} \vDash \bar{a} \leq_n^c \bar{b}$, then $\bar{a} \leq_n \bar{b}$.

Lemma

For every $\bar{a}, \bar{b} \in M^{<\omega}$, $\bar{a} \leq_\omega \bar{b}$ if and only if $tp(\bar{a}) = tp(\bar{b})$.

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Recall that $\mathcal{M}$ is ***homogeneous*** if every partial elementary map $M \to M$ is extendible to an automorphism.

Lemma
If $\mathcal{M}$ is not homogeneous then $SR(\mathcal{M}) > \omega$.

Proposition
If $\mathcal{M}$ is homogeneous, then $SR(\mathcal{M}) \leq \omega + 1$.

Note that every completion T of PA has an atomic model. Take $\mathcal{M} \subseteq T$ and the subset of all Skolem terms without parameters. This is an elementary substructure and all types are isolated. By the least number principle this model is rigid and its automorphism orbits in $\mathcal{M}$ are singletons.

Theorem (Montalbán, R.)
If $\mathcal{M}$ is atomic, then $SR(\mathcal{M}) = \omega$.

Theorem (Montalbán, R.)
For any nonstandard model $\mathcal{M}$, $SR(\mathcal{M}) \geq \omega$. In particular $(1, \omega) \notin SS(PA)$. If $T \supseteq PA$ does not have a standard model, then $1 \notin SS(T)$.

Bi-interpretability

In order to obtain a characterization of the set of possible Scott ranks, a first try is to see if there is a reduction from linear orders to models of PA.

Ash's results show that $SS(LO) = \omega_1 - 0$.

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Definition (Harrison-Trainor, R. Miller, Montalbán 2018)
A structure $\mathcal{A} = (A, P_0^{\mathcal{A}}, \dots)$ is *infinitary interpretable* in $\mathcal{B}$ if there exists a $L_{\omega_1\omega}$ definable in $\mathcal{B}$ sequence of relations $(Dom_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, \dots)$ such that

1. $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$,
3. $R_i \subseteq (B^{<\omega})^{a_{P_i}}$ is closed under $\sim$ on $Dom_{\mathcal{A}}^{\mathcal{B}}$,

and there exists a function $f_{\mathcal{B}}^{\mathcal{A}} : (Dom_{\mathcal{A}}^{\mathcal{B}}, R_0, \dots)/\sim \,\cong (A, P_0^{\mathcal{A}}, \dots)$, the *interpretation of $\mathcal{A}$ in $\mathcal{B}$*. If the formulas in the interpretations are $\Delta_\alpha^{\text{in}}$ then $\mathcal{A}$ is $\Delta_\alpha^{\text{in}}$ interpretable in $\mathcal{B}$.

Definition (Harrison-Trainor, R. Miller, Montalbán 2018)
Two structures $\mathcal{A}$ and $\mathcal{B}$ are *bi-interpretable* if there are infinitary interpretations of one in the other such that the compositions

$$f_{\mathcal{B}}^{\mathcal{A}} \circ \hat{f}_{\mathcal{A}}^{\mathcal{B}} : Dom_{\mathcal{B}}^{Dom_{\mathcal{A}}^{\mathcal{B}}} \to \mathcal{B} \quad \text{and} \quad f_{\mathcal{A}}^{\mathcal{B}} \circ \hat{f}_{\mathcal{B}}^{\mathcal{A}} : Dom_{\mathcal{A}}^{Dom_{\mathcal{B}}^{\mathcal{A}}} \to \mathcal{A}$$

are inf. definable in $\mathcal{B}$ and $\mathcal{A}$ respectively.

Theorem (Harrison-Trainor, R. Miller, Montalbán 2018)
If $\mathcal{A}$ and $\mathcal{B}$ are infinitary bi-interpretable, then $Aut(\mathcal{A}) \cong Aut(\mathcal{B})$.

Theorem (Harrison-Trainor, R. Miller, Montalbán 2018)
A structure $\mathcal{A}$ is Δ_α^0 interpretable in $\mathcal{B}$ iff there is a functor $F : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ where the operators $\Phi : Iso(\mathcal{B}) \to Iso(\mathcal{A})$ and $\Phi_* : Hom(\mathcal{B}) \to Hom(\mathcal{A})$ are Δ_α^0.

If $\mathcal{A}$ and $\mathcal{B}$ are bi-interpretable by Δ_1^0 formulas, then $SR(\mathcal{A}) = SR(\mathcal{B})$. If that is not the case, the story is not that clear.

Theorem (Gaifman 1976)
Let T be a completion of PA and $\mathcal{L}$ a linear order. Then there is a model $\mathcal{N}_{\mathcal{L}}$ of T such that $Aut(\mathcal{N}_{\mathcal{L}}) \cong Aut(\mathcal{L})$.

- A *cut* of a model $\mathcal{M}$ is a non-empty initial segment of $\mathcal{M}$ closed under successor.
- $\mathcal{N}$ is an *end-extension* of $\mathcal{M}$ if $\mathcal{M} \preccurlyeq \mathcal{N}$ and $\mathcal{M}$ is a cut of $\mathcal{N}$.
- $\mathcal{N}$ is a minimal extension of $\mathcal{M}$ if there is no $\mathcal{K}$ with $\mathcal{M} \prec \mathcal{K} \prec \mathcal{N}$.

Theorem (Gaifman 1976)
Let $\mathcal{M}$ be any model of PA, then $\mathcal{M}$ has a minimal end extension.

$\mathcal{L}$-canonical extension

The minimal end extension is obtained by taking $\mathcal{M}(a)$, the Skolem hull of $\mathcal{M}$ with a new element a having type $p(x)$ where

- $p(x)$ is *indiscernible*: for $I \subseteq M$ with every $i \in I$ having type $p(x)$ and ordered sequences $\bar{a}, \bar{b} \in I^{<\omega}$, $tp(\bar{a}) = tp(\bar{b})$,
- $p(x)$ is *unbounded*: there is no Skolem constant c such that $x \leq c \in p(x)$.

The version of Gaifman's theorem above is obtained by taking an $\mathcal{L}$-canonical extension for given $\mathcal{L}$ over the prime model $\mathcal{N}$, i.e., take an indiscernible, unbounded type $p(x)$, and construct the model

$$\mathcal{N}_{\mathcal{L}} = \bigcup_{l_1 \leq \cdots \leq l_{|l|} \in L^{<\omega}} \mathcal{N}(l_1)(l_2)\dots(l_{|l|})$$

This construction gives a functor $F : LO \to Mod(T)$. The functor is computable relative to T. This is equivalent to having that for any $\mathcal{L}$, $\mathcal{N}_{\mathcal{L}}$ is Δ^0_1 interpretable in $\mathcal{L}$.

We still need to recover $\mathcal{L}$ from $\mathcal{N}_{\mathcal{L}}$ to obtain a bi-interpretation

Definition
Fix $\mathcal{M} \models PA$ and let $\mathcal{F}$ be the set of definable functions $f : M \to M$ for which $x \leq f(x) \leq f(y)$ whenever $x \leq y$. For any $a \in M$ let $gap(a)$ be the smallest set S with $a \in S$ and and if $b \in S$, $f \in \mathcal{F}$, and $b \leq x \leq f(b)$ or $x \leq b \leq f(x)$, then $x \in S$.

Define $a =_g b$ as $a =_g b \Leftrightarrow a \in gap(b)$. The gap relation partitions $\mathcal{M}$ into intervals.

Theorem (Gaifman 1976)

- *If $a \in gap(b)$ and a, b both realize the same minimal type $p(x)$, then $a = b$.*
- *If $\mathcal{N}_{\mathcal{L}}/{=_g}$ is order isomorphic to $1 + \mathcal{L}$.*

So we can interpret $\mathcal{L}$ in $\mathcal{N}_{\mathcal{L}}$ using the interpretation given by

$a \in Dom^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}} \Leftrightarrow tp(a) = p(x)$	$a \sim b \Leftrightarrow a = b$	$a \leq b \Leftrightarrow a \leq^{\mathcal{N}_{\mathcal{L}}} b$
Π^{in}_{ω}	Δ^0_1	Δ^0_1

Properties of the interpretation

- Every element in $\mathcal{N}_{\mathcal{L}}$ is a Skolem term with parameters $f^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}}(\bar{a})$ for some $\bar{a} \in L$.
- So in particular, if we can define the automorphism orbits of the elements in $Dom^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}}$ we can get definitions for all tuples.

Lemma
For every $\alpha < \omega_1$ $\bar{a} \leq_\alpha \bar{b} \Leftrightarrow f^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}}(\bar{a}) \leq_{\omega+\alpha} f^{\mathcal{L}}_{\mathcal{N}_{\mathcal{L}}}(\bar{b})$.

Theorem (Montalbán, R.)
Let $\mathcal{L}$ *be a linear order with* $SR(\mathcal{L}) = \alpha$, *then* $SR(\mathcal{N}_{\mathcal{L}}) = \omega + \alpha$.

Theorem (Montalbán, R.)

1. $SS(PA) = 1 \cup \{\alpha : \omega \leq \alpha \leq \omega_1\}$
2. *If* $\mathcal{M}$ *is non-homogeneous, then* $SR(\mathcal{M}) \geq \omega + 1$.
3. *If* $\mathcal{M}$ *is non-standard atomic , then* $SR(\mathcal{M}) = \omega$.
4. *If* $\mathcal{M}$ *is non-standard homogeneous, then* $SR(\mathcal{M}) \in [\omega, \omega + 1]$.
5. *For any completion* T *of* PA*, there is a* T*-computable model* $\mathcal{M}$ *with* $SR(\mathcal{M}) = \omega_1^T + 1$.

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Thank you!