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Convolution of valuations on manifolds

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Institut für Diskrete Mathematik und Geometrie
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unter der Anleitung von

Univ.Prof. Dr. Franz Schuster

durch

Simon Ellmeyer
Matrikelnummer: 01526374
Tandelmarktgasse 9
1020, Wien

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Introduction

Our main object of interest is the study of valuations. These are functions that take values in an Abelian semigroup and satisfy the inclusion-exclusion principle, i.e.,

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B),$$

for suitable A and B. In particular, we examine valuations on nonempty, convex and compact sets, $\mathcal{K}(V)$, in a Euclidean n-dimensional vector space V and valuations on smooth manifolds with corners. We will restrict our attention to continuous valuations in an appropriate sense. Moreover, in the convex case, it is fruitful to examine valuations that are invariant under linear transformations. The characterization of the space of continuous and rigid motion invariant valuations, by Hadwiger, as the linear span of the intrinsic volumes allowed for effortless proofs of kinematic formulas:

$$\int_{E_n} \mu_k(A \cap gK)\, dg = \sum_{i=0}^{n-k} \begin{bmatrix} i+k \\ k \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} \mu_{k+i}(A)\mu_{n-i}(K).$$

Since formulas of this kind play an important role in integral geometry, we want to generalize them.

In Section 1, we start by giving an overview of properties of $\mathcal{K}(V)$ and smooth manifolds that we are going to need in the following. Moreover, we introduce a representation theoretic approach to smooth vectors of a Frechet space.

Next, in Section 2, we introduce spaces of valuations. First, the Banach space of translation invariant continuous valuations, $Val(V)$, will be defined. Alesker introduced in [2] a dense subspace, $Val^{sm}(V)$, with nice algebraic and topological properties. Moreover, we will take a look at another important approach to valuations, which is also due to Alesker [3]. Finally, we will be able to define smooth valuations on smooth manifolds, $\mathcal{V}^\infty(M)$, as it is done in [4]. One of the most important properties, that we are going to use in this thesis is the identification of smooth valuations with smooth differential forms.

In Section 3 we define the product and the convolution on $Val^{sm}(V)$, we follow Bernig's and Alesker's approach from [15] and [2], respectively. To highlight the importance of these algebraic structures, we give a brief introduction to more general kinematic formulas. Alesker showed in [2], that the space G-invariant valuations with a closed subgroup $G \subseteq SO(n)$, that acts transitively on the sphere, Val^G, is finite dimensional. For a basis $\phi_1, \ldots, \phi_m$ we therefore obtain the additive and intersectional kinematic formulas:

$$\int_V \int_G \phi_k(K \cap (x + gL))\, dgdx = \sum_{i,j=1}^{m} a_{i,j}^k \phi_i(K)\phi_j(L),$$

$$\int_G \phi_k(K + gL)\, dg = \sum_{i,j=1}^{m} b_{i,j}^k \phi_i(K)\phi_j(L).$$

The associated operators $k_G : Val^G \to Val^G \otimes Val^G$ defined by

$$\phi_k \mapsto \sum_{i,j}^{m} a_{i,j}^k \phi_i \otimes \phi_j,$$

and $a_G : Val^G \to Val^G \otimes Val^G$ defined by

$$\phi_i \mapsto \sum_{i,j}^{m} a_{i,j}^k \phi_i \otimes \phi_j,$$

are called kinematic operators. In [13], Bernig showed the connection between the algebraic opertions and the kinematic operators in the Fundamental Theorem of Algebraic Integral Geometry (FTAIG). We give a short description of this theorem: Let $m_G : Val^G \otimes Val^G \to Val^G$ be the restriction of the product to Val^G and let $pd_G : Val^G \to (Val^G)^*$ be the induced embedding into the dual space. We obtain

$$(pd_G \otimes pd_G) \circ k_G = m_G^* \circ pd_G.$$

Similar, we proceed with the convolution. Hence, let $c_G : Val^G \otimes Val^G \to Val^G$ be the restriction of the convolution to Val^G, then

$$(pd_G \otimes pd_G) \circ a_G = c_G^* \circ pd_G.$$

Therefore if we want to state kinematic formulas in a more general way, we should try to lift the introduced operations to smooth valuations on manifolds. This is done in Section 4. Similar to the product on $Val^{sm}(V)$, we obtain a dense embedding into the dual space of compactly supported smooth valuations, $pd : \mathcal{V}^\infty(M) \to (\mathcal{V}_c^\infty(M))^* := \mathcal{V}^{-\infty}(M)$. This embedding is described in Section 5. Since we already mentioned that the space of smooth valuations can be seen as a space of differential forms, we can identify the corresponding dual spaces. Proposition 5.2.7 is due to Bernig and describes the precise connection of these spaces.

Finally, in Section 5 we want to extend the convolution to compactly supported valuations on smooth manifolds. It was introduced by Bernig and Alesker in [8]. The convolution is defined in terms of generalized valuations, i.e. elements of $\mathcal{V}^{-\infty}(M)$, by the push-forward of the exterior product under the addition map. In Proposition 6.2.10, we get a representation in the sense of generalized valuations and Proposition 6.2.12 yields the desired extension.

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Contents

1 General background

1.1 Convex Bodies

In this thesis we are often dealing with the set of convex bodies on a vector space V. We need some basic definitions and properties which we are going to recall in the following. For proofs and more details, we refer to [29].

1.1.1 Support functions

Let V be an n-dimensional vector space with a fixed scalar product $(\cdot,\cdot)$ and $\mathcal{K}(V)$ the set of compact, convex and nonempty sets in V with respect to this Euclidean topology. The map $d : \mathcal{K}(V) \times \mathcal{K}(V) \to \mathbb{R}$ defined by

$$d(A, B) = \inf\{\epsilon > 0 : A \subseteq B + B_\epsilon(0), B \subseteq A + B_\epsilon(0)\}.$$

defines a metric on $\mathcal{K}(V)$. It is called the Hausdorff metric. Equipped with this metric we have the following important properties:

Proposition 1.1.1. *The topological space $(\mathcal{K}(V), d)$ is a complete, locally compact metric space.*

Every $K \in \mathcal{K}(V)$ is determined by a function on V that gives a correspondence between convex bodies and continuous functions: The support function $h_K : V \to \mathbb{R}$ of K is defined by

$$h_K(y) = h(K, y) := \max_{x \in K}(x, y).$$

Since these functions are 1-homogeneous it is useful to work with their restriction to the sphere S^{n-1}. Hence we get the following result:

Theorem 1.1.2. *The map $\varphi : \mathcal{K}(V) \to C(S^{n-1}), K \mapsto h(K, \cdot)$ is an isometric imbedding of $\mathcal{K}(V)$ into the Banach space $C(S^{n-1})$, i.e., we have*

$$d(K, L) = \|h(K, \cdot) - h(L, \cdot)\|_\infty,$$

for $K, L \in \mathcal{K}(V)$.

Definition 1.1.3. We say $K \in \mathcal{K}(V)$ has a smooth boundary, if its boundary is a smooth manifold. Let us denote the set of convex bodies with smooth boundary by $\mathcal{K}^\infty(V)$ and the set of all bodies which contain 0 in their interior with $\mathcal{K}_0(V)$.

Lemma 1.1.4. *For $K \in \mathcal{K}^\infty(V)$, we have $h_K \in C^\infty(S^{n-1}, \mathbb{R})$ and $h_K(y) = (\nabla h_K(y), y)$.*

Another important property of the space of convex bodies is that the volume $vol_n : \mathcal{K}(V) \to \mathbb{R}$ is continuous.

Theorem 1.1.5. *The volume $vol_n : \mathcal{K}(V) \to \mathbb{R}$ is a continuous map with respect to the Hausdorff metric.*

The set of nonempty compact subsets of V is also a complete metric space, when endowed with the Hausdorff metric, but it is important to mention, that the volume is not continuous on this space. The volume induces an important class of real-valued functions:

Definition 1.1.6. The mixed volume of $K_1, \dots, K_n \in \mathcal{K}(V)$ is defined by

$$V(K_1, \dots, K_n) = \frac{1}{n!}\sum_{k=1}^{n}(-1)^{k+n} \sum_{1 \le i_1 < \dots < i_m \le n} vol_n(K_{i_1} + \dots + K_{i_m}).$$

We need the following theorem concerning mixed volumes to define a finite measure on S^{n-1} called the surface area measure of a convex body K:

Theorem 1.1.7. *Let $K_1, \ldots, K_{n-1} \in \mathcal{K}(V)$. There is a finite measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on S^{n-1}, called the mixed area measure of $K_1, \ldots, K_{n-1}$, such that*

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{S^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u)$$

for all $K \in \mathcal{K}(V)$.

Definition 1.1.8. The surface area measure $S_{n-1}(K, \cdot)$ of a convex body $K \in \mathcal{K}(V)$ is the measure on S^{n-1} defined by

$$S_{n-1}(K, \cdot) := S(K, \ldots, K, \cdot).$$

A body $K \in \mathcal{K}(V)$ has a curvature function $s_{n-1}(K, \cdot) \in C(S^{n-1})$, if

$$dS_{n-1}(K, \cdot) = s_{n-1}(K, \cdot) d\sigma,$$

where σ is the rotation-invariant probability measure on S^{n-1}. We say K has positive curvature, if $s_{n-1}(K, \cdot)$ is a positive function. The convex bodies with positive curvature form a dense subset of $\mathcal{K}(V)$:

Theorem 1.1.9. *Let $K \in \mathcal{K}(V)$ and $\epsilon > 0$. Then there is a smooth convex body $L \in \mathcal{K}(V)$ with positive curvature and $d(K, L) \leq \epsilon$.*

Theorem 1.1.10. *If $f \in C^2(S^{n-1})$, then there exists a convex body $K \in \mathcal{K}(V)$ and $s > 0$ such that*

$$f(x) = h_K(x) - s h_{B_1(0)}(x), x \in S^{n-1},$$

with $\|h_K\| \leq c\|f\|_{C^2}$. In particular, every $f \in C^2(S^{n-1})$ can be written as the difference of two support functions.

It is crucial that if $f \in C^k(S^{n-1})$, we can choose $K \in \mathcal{K}(V)$ with $h_K \in C^k(S^{n-1})$ in the theorem above, such that $\|h_K\|_{C^k(S^{n-1})} \leq \|f\|_{C^k(S^{n-1})}$. Therefore, every $f \in C^\infty(S^{n-1})$ can be written as a difference of smooth support functions.

1.2 Smooth Manifolds

We want to recall some basic definitions from the theory of smooth manifolds, which we will need throughout this thesis. For proofs and more details, we refer to [25, 11] and [24].

1.2.1 Fiber Bundles

Definition 1.2.1. Let M, F be smooth manifolds. A smooth fiber bundle over M with model fiber F is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ with the property that for each $x \in M$, there exists a neighbourhood U of x in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times F$, called local trivialization of E over U such that the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\
& U &
\end{tikzcd}$$

The most important example of a fiber bundle is given by a smooth vector bundle.

Definition 1.2.2. Let M be a smooth manifold. A real smooth vector bundle of rank k over M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ satisfying the following conditions:

1. For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.

2. For each $p \in M$, there exist a neighbourhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ satisfying

   (a) $\pi_U \circ \Phi = \pi$

   (b) for each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k$.

A rank-1 vector bundle is called a line bundle. The space E is called the total space of the bundle and M is called its base. Furthermore π is called the projection.

Example 1.2.3. Let TM be the tangent bundle of a smooth n-manifold with or without boundary. The topology and smooth structure on TM are the unique ones with respect to which $\pi : TM \to M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections. A proof of this can be found in [25, Proposition 10.4].
The bundle of covariant k-tensors has as well a natural structure as smooth vector bundles over M.

Definition 1.2.4. Let $\pi : E \to M$ be a smooth vector bundle. A section of E is a section of the map π, that is a continuous map $\sigma : M \to E$ satisfying $\pi \circ \sigma = Id_M$, i.e., $\sigma(p)$ is an element of the fiber E_p for each $p \in M$. A local section is a continuous map $\sigma : U \to E$ defined on some open subset $U \subseteq M$ and satisfying $\pi \circ \sigma = Id_U$.
The zero section of E is the global section $0 : M \to E$ defined by

$$0(p) = 0 \in E_p, \ \forall p \in M.$$

The support of a section σ is the closure of the set

$$\{p \in M : \sigma(p) \neq 0\}.$$

Example 1.2.5. We can identify the space of smooth real-valued functions on M, denoted by $C^\infty(M)$ with the space of smooth sections of the line bundle $M \times \mathbb{R} \to M$.

More generally we can define the space of smooth sections.

Definition 1.2.6. If $E \to M$ is a smooth vector bundle, the set of all smooth global sections of E is a vector space, which we denote by $C^\infty(M, E)$.

Definition 1.2.7. Given a smooth vector bundle $\pi_E : E \to M$, a smooth subbundle of E is a smooth vector bundle $\pi_D : D \to M$ in which D is an embedded submanifold of E and π_D is the restriction of π_E to D, such that for each $p \in M$, the subset $D_p = D \cap E_p$ is a linear subspace of E_p.

The following examples of subbundles will appear in the upcoming sections.

Example 1.2.8. Let M be a smooth manifold.

- The bundle of alternating tensors $\Lambda^k T^*M$ is a smooth subbundle of the bundle of covariant k-tensors, T^kT^*M, with rank $\binom{n}{k}$. Sections of $\Lambda^k T^*M$ are called k-forms and we denote the space of smooth k-forms by $\Omega^k(M)$.

- Let $S \subseteq M$ be an embedded submanifold. The conormal bundle of S is defined to be the subset
$$N^*_M S = \{(q,\eta) \in T^*M : q \in S, \eta|_{T_qS} \equiv 0\} \subseteq T^*M.$$
$N^*_M S$ is a smooth subbundle of $T^*M|_S$. Let us denote the smooth embedding by $\iota : S \to M$, then we can compute the conormal bundle by considering $\ker d\iota^*$. This follows since we identify T_qS with $d\iota_q(T_qS) \subseteq M$. Therefore we get for $(q,\eta) \in N^*_M S$
$$\eta(d\iota_q v) = 0 \ \forall v \in T_qS \Leftrightarrow d\iota_q^*(\eta)(v) = 0 \ \forall v \in T_qS.$$

- Let V be a vector space and let us denote by $\mathbb{P}_+(V)$ the manifold defined by
$$\mathbb{P}_+(V) := (V \setminus \{0\}) / \mathbb{R}_+.$$
Let E be a vector bundle over a smooth manifold M. We define the vector bundle over M whose fiber over a point $p \in M$ is given by $\mathbb{P}_+(E_p)$. We will denote it by $\mathbb{P}_E$. In the special case of the cotangent bundle T^*M we call it the cosphere bundle of M. It is a $(2n-1)$-dimensional manifold of all tuples $(p,[\xi])$, with $p \in M, \xi \in T^*_pM \setminus \{0\}$ and the equivalence class $[\xi]$ is with respect to the relation given above. It inherits a map $s := \mathbb{P}_M \to \mathbb{P}_M$ given by $s(x,[\xi]) = (x,[-\xi])$.

There are several operations defined on a manifold that induce operations on the vector bundle. The following definition is taken from [11].

Definition 1.2.9. Let $\pi_E : E \to M$ be a smooth fiber bundle with n-dimensional fibers, such that both E and M are oriented. If $\nu \in \Omega^k_c(E)$, its integral over the fibres is the differential form $\pi_*\nu \in \Omega^{k-n}(M)$ such that
$$\int_M \pi_*\nu \wedge \eta = \int_E \nu \wedge \pi_E^*\eta,$$
for all differential forms η on the base M, where $\pi_E^*\eta$ denotes the pull-back of the differential form.

Proposition 1.2.10. *Let $\pi_E : E \to M$ be a smooth fiber bundle. The integral over the fibers commutes with the exterior differential, in the sense that*
$$d\pi_*\alpha = \pi_*(d\alpha).$$
We also have
$$\pi_*(\alpha \wedge \pi^*\beta) = \pi_*\alpha \wedge \beta.$$

Definition 1.2.11 (Pull-back of manifold)**.** Let M_1, M_2 and N be smooth manifolds and let $f : M \to N$ be a smooth submersion. The pull-back of N is defined to be the embedded submanifold of $M_1 \times M_2$ given by
$$M_1 \times_N M_2 := M_1 \times_{f_1,N,f_2} M_2 := \{(p_1,p_2) \in M_1 \times M_2 : f_1(p_1) = f_2(p_2)\}$$

with the universal property: For each pair of smooth maps $g_i : O \to M_i$ with $f_1 \circ g_1 = f_2 \circ g_2$, there exists a uniquely determined smooth map $g : O \to M_1 \times_N M_2$ with $\pi_{M,N} \circ g = g_i$.

$$\begin{tikzcd}
O \arrow[dr, dashed, "g"] \arrow[drr, bend left, "g_2"] \arrow[ddr, bend right, "g_1"'] & & \\
& M_1 \times_N M_2 \arrow[r, "\pi_{M_1}"] \arrow[d, "\pi_{M_2}"'] & M_2 \arrow[d, "f_2"] \\
& M_1 \arrow[r, "f_1"] & N
\end{tikzcd} \tag{1.1}$$

The tangent space in a point $(p_1, p_2) \in M_1 \times_N M_2$ is given by

$$T_{(p_1,p_2)}(M_1 \times_N M_2) = \{(v_1, v_2) \in T_{p_1}M_1 \oplus T_{p_2}M_2 : df_1 v_1 = df_2 v_2\}. \tag{1.2}$$

The following lemma combines properties of the pull-back of a manifold and the integration along fibers. For details, we refer to [7, Lemma 2.20].

Lemma 1.2.12. *Let M_1, M_2 and N be smooth oriented manifold and let the diagram in* (1.1) *be a cartesian square. Then for $\nu \in \Omega^*(M_2)$, we have*

$$\pi_{M_1 *} \circ \pi^*_{M_2} \nu = f_1^* \circ f_{2*} \nu.$$

An important tool related to integration over a smooth manifold is the concept of densities:

Definition 1.2.13. Let V be a real n- dimensional vector space. A multilinear map

$$\mu : V \times \cdots \times V \to \mathbb{R}$$

is called a density if for any linear map $T : V \to V$, we have

$$\mu(Tv_1, \ldots, Tv_n) = |det\ T| \mu(v_1, \ldots, v_n).$$

The following proposition can be found in [25, Proposition 16.35].

Proposition 1.2.14. *If $\omega \in \Lambda^n(V^*)$, the map $|\omega| : V \times \cdots \times V \to \mathbb{R}$ defined by*

$$|\omega|(v_1, \ldots, v_n) = |\omega(v_1, \ldots, v_n)|$$

is a density. Moreover the space of densities over V, denoted by $\mathcal{D}V$ is a one-dimensional vector space, spanned by $|\omega|$, for any nonzero $\omega \in \Lambda^n(V^*)$.

Definition 1.2.15. Let M be a smooth manifold with or without boundary. The set

$$\mathcal{D}V = \coprod_{p \in M} \mathcal{D}(T_p M)$$

is called the density bundle of M.

The density bundle is a smooth line bundle over M. The construction of the local trivialisation is as follows. Let $(U, (x^i))$ be any smooth coordinate chart on M. Let $\nu = dx^1 \wedge \cdots \wedge dx^n$. Since $|\nu_p|$ is a basis for $\mathcal{D}T_p M$ at each point $p \in U$, the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}$ given by $\Phi(c|\nu_p|) = (p, c)$ is a bijection. For more details, we refer to [25].

1.2.2 Lie Groups

Definition 1.2.16. A Lie group is a smooth manifold G that is also a group with the property that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ are smooth.

Example 1.2.17. The group $GL(V)$ is a Lie group. A famous theorem of Cartan states, that the closed subgroups of a Lie group are Lie subgroups. Hence all closed subgroups of $GL(V)$ are Lie groups as well.

Definition 1.2.18. Let G be a Lie group and let M be smooth manifold. A left action of G on M is a map $a : G \times M \to M$ with

$$(g, p) \mapsto a(g, p) =: g \cdot p$$

that satisfies

$$\begin{aligned} a(g_1, a(g_2, p)) = a(m(g_1, g_2), p), \forall g_1, g_2 \in G \quad \text{and} \quad p \in M \\ a(e, p) = p \ \forall p \in M. \end{aligned}$$

Right actions are defined analogously. An action is said to be transitive if for every pair of points $p, q \in M$, there exists $g \in G$ such that $g \cdot p = q$. We call a subset of $S \subseteq M$ invariant if $G \cdot S \subseteq S$.

Example 1.2.19. The Lie Group $GL(V)$ acts on the vector space V by linear transformations, where the action is just the usual matrix-vector product.

Definition 1.2.20. Suppose G is a Lie group and M, N are smooth manifolds endowed with smooth left G-actions. A map $F : M \to N$ is said to be G-equivariant with respect to the given G actions if for each $g \in G$,

$$F(g \cdot p) = g \cdot F(p).$$

1.2.3 Contact manifolds

Definition 1.2.21. A smooth distribution on M of rank k is a rank-k smooth subbundle of TM. If D is a rank-k distribution on a smooth n-manifold M, any $n - k$ linearly independent 1-forms $\omega_1, \ldots, \omega_{n-k}$ defined on an open subset $U \subseteq M$ and satisfying

$$D_q = ker\ \omega_1|_q \cap \cdots \cap ker\ \omega_{n-k}|_q$$

for each $q \in U$ are called local defining forms for D. More generally, if $0 \leq p \leq n$, we say that a p-form $\omega \in \Omega^p(M)$ annihilates D if $\omega(X_1, \ldots, X_p) = 0$, whenever $X_1, \ldots, X_p$ are local secions of D.

Definition 1.2.22. A 2-covector on a finite-dimensional vector space V is said to be nondegenerate if the linear map $\hat{\omega} : V \to V^*$ defined by $\hat{\omega}(v) = v \lrcorner \omega$ is invertible. A nondegenerate 2-covector is called a symplectic tensor. A vector space endowed with a symplectic tensor is called a symplectic vector space.

Definition 1.2.23. Let M be an odd-dimensional smooth manifold. A contact form on M is a nonvanishing smooth 1-form α with the property that for each $p \in M$ the restriction of $d\alpha_p$ to the subspace $ker\ \alpha_p \subseteq T_pM$ is nondegenerate. A contact structure on M is a smooth distribution $H \subseteq TM$ of rank $2n$ with the property that any smooth local defining form α for H is a contact form. A contact manifold is a smooth manifold M together with a contact structure on M.

We have a criterion that ensures that a 1-form is a contact form. Sometimes this is used to define contact forms. For a proof of the following statements, we refer to [25, Proposition 22.24].

Proposition 1.2.24. *A smooth* 1*-form* α *on a* $(2n-1)$*-dimensional manifold is a contact form if and only if* $\alpha \wedge d\alpha^n$ *is nonzero everywhere on* M*.*

Example 1.2.25.

- A canonical contact structure on $V \times S^{n-1}$ is defined by $\alpha := \sum_{k=1}^{n} y_i dx_i$, where the x_i, y_i are coordinates in V with $\|y\|_2 = 1$.
- As a generalization of $\mathbb{P}_V = \mathbb{P}_+(T^*V) = V \times \mathbb{P}_+(V) = V \times S^{n-1}$, we can define a canonical contact structure on $\mathbb{P}_M$ given by restriction of the canonical 1-form to $\mathbb{P}_M$.

A form $\omega \in \Omega^*(M)$ is called vertical if $\alpha \wedge \omega = 0$, for a contact form α. Given a local contact form α, then ω is vertical if and only if $\omega = \alpha \wedge \phi$ for some $\phi \in \Omega^*(M)$. The following theorem concerning vertical forms will be important in later sections. A proof can be found in [28] and [18, Proposition 2.48].

Theorem 1.2.26 (Rumin operator)**.** *Let* M *be a contact manifold of dimension* $2n-1$*. Given* $\omega \in \Omega^{n-1}(M)$ *there exists a unique vertical form* $\omega' \in \Omega^{n-1}(M)$ *such that* $d(\omega + \omega')$ *is vertical. We define the projection operator* $Q : \Omega^{n-1}(M) \to \Omega^{n-1}(M)$

$$Q(\omega) = \omega + \omega'.$$

Furthermore we define $D := d \circ Q : \Omega^{n-1}(M) \to \Omega^n(M)$*.* D *is called the Rumin operator.*

1.3 Representation theory

We can examine smooth maps between more general spaces then manifolds. More details to this topic can be found in [30] and [22].

Definition 1.3.1. A Frechet space is a topological vector space F satisfying the following properties:

- F is Hausdorff.
- The topology is induced by a countable family of semi norms $\|\cdot\|_k, k \in \mathbb{N}$.
- F is complete with respect to the metric induced by the family of semi norms.

Definition 1.3.2. Let E, F be Frechet spaces and U open in E. Let $f : U \to F, x \in U$ and $h \in F$. We say that f is differentiable at x in direction of h if

$$D_x f(h) := \lim_{h \to 0} \frac{1}{t}(f(x+th) - f(x))$$

exists. If this limit exists for any h we say that f is differentiable at x. We say f is continuously differentiable if $D_x f(h)$ is continuous in both x and h.

For a Frechet space F and a smooth manifold M let us denote by $C^\infty(M, F)$ the Frechet space of smooth F-valued functions on M with the topology of uniform convergence of all derivatives on compact subsets of M.

Definition 1.3.3. Let G be a Lie group and let F be a Frechet space. A homomorphism $\rho : G \to Aut(F)$ of G into the automorphisms on F is called a representation. It is called continuous if the map $(g, x) \mapsto \rho(g)x =: g \cdot x$ is a continuous map from $G \times F$ to F.

By definition, for a continuous representation (ρ, G) and fixed $x \in F$, the map $g \mapsto \rho(g)(x)$ is a continuous map from G to F and therefore $\rho(\cdot)(x) \in C(G, F)$. This motivates the following definition:

Definition 1.3.4. Let (ρ, G) be a continuous representation of a Lie group G on a Frechet space F. We call a vector $v \in F$ smooth if $\rho(\cdot)(v) \in C^\infty(G, F)$. The subspace of all smooth vetcors in F is denoted by F^∞. A representation on a Frechet space F is called smooth if $F^\infty = F$.

Lemma 1.3.5. *Let (ρ, G) be a continuous representation of G on a Frechet space F. Then the $\rho(\cdot) : F \to \rho(\cdot)(F) \subseteq C(G, F)$ is a homeomorphic isomorphism and the image is closed in $C(G, F)$.*

We define the Garding topology on F^∞ as the relative topology on $\Phi_\pi(F^\infty)$. We obtain the following useful lemma:

Lemma 1.3.6. *Let ρ and π be continuous representations of a Lie Group G on Banach spaces X and Y and let $T : X \to Y$ be a continuous and G-equivariant linear map. Then $T(X^\infty) \subseteq Y^\infty$ and the induced map $T : X^\infty \to Y^\infty$ is continuous.*

Let E, F be Banach spaces. If we take a look at the diagonal representation of a Lie group defined by

$$g \cdot (e, f) = (g \cdot e, g \cdot f),$$

it is immediate, that Lemma 1.3.6 can be extended to multilinear maps, i.e., let $m : E \times F \to G$ be a G-equivariant continuous map. Then $m(E^\infty \times F^\infty) \subseteq G^\infty$ and the induced map $m : E^\infty \times F^\infty \to G^\infty$ is continuous.

The following lemma is taken from [3, Lemma 1.1.7].

Lemma 1.3.7. *Let M_1, M_2 be two smooth manifolds such that M_2 is compact. Let E_1 and E_2 be smooth finite dimensional vector bundles over M_1 and M_2 respectively. Let $k \in \mathbb{N}$ be an integer. Let $G \subseteq M_1$ be a compact subset. Then there exists a compact subset $\tilde{G} \subseteq M_1$ containing G, an integer $l \in \mathbb{N}$ and a constant C such that for any $f \in C^\infty(M_1 \times M_2, E_1 \otimes E_2)$ there exists a representation*

$$f = \sum_{i=1}^{\infty} g_i \otimes h_i$$

such that $g_i \in C^\infty(M_1, E_1), h_i \in C^\infty(M_2, E_2)$ and

$$\sum_{i=1}^{\infty} \|g_i\|_{C^k(G)} \|h_i\|_{C^k(M_2)} \leq C \|f\|_{C^l(\tilde{G} \times M_2)}.$$

Remark 1.3.8. Lemma 1.3.7 gives us a description of the dense subset $C^\infty(M_1 \times M_2, E_1 \otimes E_2)$. In other words we have

$$\mathrm{cl}(\mathrm{span}\{g \otimes h : g \in C^\infty(M_1, E_1), h \in C^\infty(M_2, E_2)\}) = C^\infty(M_1 \times M_2, E_1 \otimes E_2).$$

This property is crucial when it comes to defining a product on the space of smooth valuations.

Theorem 1.3.9 (L. Schwartz Kernel Theorem)**.** *Let M_1 and M_2 be compact smooth manifolds. Let E_1 and E_2 be smooth finite dimensional vector bundles over M_1 and M_2. Let F be a Frechet space and $B : C^\infty(M_1, E_1) \times C^\infty(M_2, E_2) \to F$ be a continuous bilinear map. Then there exists a unique continuous linear operator*

$$b : C^\infty(M_1 \times M_2, E_1 \otimes E_2) \to F$$

such that $b(f_1 \otimes f_2) = B(f_1, f_2)$ for any $f_i \in C^\infty(M_i, E_i), i = 1, 2$.

The following construction is due to [20, Chapter VII], [19, Chapter VI] and also can be found in [6]. We are now going to introduce the wave fronts of elements of the dual space of $C_c^\infty(\mathbb{R}^n)$, denoted by $C^{-\infty}(\mathbb{R}^n)$. Therefore let $u \in C^{-\infty}(\mathbb{R}^n)$, then its Fourier transform is defined and is a continuous function on $\mathbb{R}^n$. We now introduce the cone $\Sigma(u)$ as the complement of the set of points x, where

$$|\hat{u}(y)| \leq c_n(1+|y|)^{-n}, n \in \mathbb{N} \tag{1.3}$$

holds for all y in an open $\mathbb{R}_{>0}$-invariant neighbourhood C of x. Therefore, we have

$$\Sigma(u) = \{x \in \mathbb{R}^n \setminus 0 : x \text{ has no conic neighbourhood } C, \text{ such that (1.3) holds } \forall y \in C\}.$$

As we can see, $\Sigma(u)$ is a closed $\mathbb{R}_{>0}$-invariant subset of $\mathbb{R}^n$. If we fix a point $x \in \mathbb{R}^n$, we can define

$$\Sigma_x(u) = \bigcap_{\substack{\phi \in C_c^\infty(U) \\ \phi(x) \neq 0}} \Sigma(\phi u).$$

Finally, the wave front of u is defined by

$$WF(u) := \{(x,y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) : y \in \Sigma_x(u)\}.$$

For a fixed closed $\mathbb{R}_{>0}$-invariant subset $\Lambda \subseteq T^*\mathbb{R}^n \setminus 0$ we define $C_\Lambda^{-\infty}(\mathbb{R}^n)$ to be the subspace of $C^{-\infty}(\mathbb{R}^n)$ consisting of elements u with $WF(u) \subseteq \Lambda$. Then, we can define a topology on the space of generalized sections u with $WF(u) \subseteq \Lambda$:
Let $\phi \in C_c^\infty(\mathbb{R}^n)$ and let $C \subseteq \mathbb{R}^n$ be any closed $\mathbb{R}_{>0}$-invariant subset such that

$$\Lambda \cap (\operatorname{supp}(\phi) \times C) = \emptyset.$$

We define the semi-norm on $C_\Lambda^{-\infty}(\mathbb{R}^n)$ by

$$\|u\|_{\phi,C,n} := \sup_{y \in C} |y|^n \left|\widehat{\phi u}(y)\right|.$$

We equip $C_\Lambda^{-\infty}(\mathbb{R}^n)$ with the weakest locally convex topology which is stronger than the weak topology on $C_\Lambda^{-\infty}(\mathbb{R}^n)$ and such that all semi-norms $\|\cdot\|_{\phi,C,n}$ are continuous.

2 Valuations

Our main interest is the class of valuations. These are real or complex valued functions satisfying a certain additivity property similar to measures but more general. In the following we want to give an overview of their important properties.

2.1 Valuations on convex bodies

First we consider the classical case of valuations on convex bodies. For more details, we refer to [21, 26] and [27].

Definition 2.1.1. A valuation is a real-valued map ϕ on $\mathcal{K}(V)$ such that

$$\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$$

whenever $A, B, A \cup B \in \mathcal{K}(V)$.

As groups act on vector spaces, we can try to transfer this behaviour to valuations. Therefore, let G be a group that acts on the vector space V by linear transformations. Then the action of G on a valuation ϕ is given by

$$[g \cdot \phi](K) = \phi(g^{-1}K), \quad K \in \mathcal{K}(V). \tag{2.1}$$

A valuation ϕ is called G-invariant if $[g \cdot \phi](K) = \phi(K)$ for all $K \in \mathcal{K}(V), g \in G$. A subset W of the set of valuations is called invariant if $g \cdot W = \{g \cdot \phi : \phi \in W\} \subseteq W$. With the following definitions we will try to classify certain valuations with some additional topological and algebraic properties.

Definition 2.1.2. A valuation ϕ on $\mathcal{K}(V)$ is called continuous if the map

$$K \mapsto \phi(K)$$

is continuous with respect to the Hausdorff metric. ϕ is called translation-invariant if $\phi(K+v) = \phi(K)$ for all $K \in \mathcal{K}(V)$ and $v \in V$. It is said to be of degree i if $\phi(\lambda K) = \lambda^i \phi(K)$ for all $K \in \mathcal{K}(V)$ and even if $\phi(-K) = \phi(K)$; odd if $\phi(K) = -\phi(-K)$. ϕ is simple, if $\phi(K) = 0$ for any $K \in \mathcal{K}(V)$ with empty interior and rigid motion invariant, if it is $SO(V)$-invariant and translation-invariant.

Example 2.1.3. There are several well known valuations on $\mathcal{K}(V)$:

1. The volume $vol_n : \mathcal{K}(V) \to \mathbb{R}$.
2. The Euler characteristic χ, defined by $\chi(K) = 1$ for $K \in \mathcal{K}(V)$.

With pointwise operations the set of continuous and translation invariant valuations forms a vector space that we will denote by $Val(V)$. We write $Val_i(V)$ for the subset consisting of valuations of degree i. Moreover the subspace of even and odd valuations is denoted by $Val^+(V)$ and $Val^-(V)$, respectively. Finally we set $Val_i^{\pm}(V) := Val_i(V) \cap Val^{\pm}(V)$. We define a norm on $Val(V)$ by

$$\|\phi\| := \sup\{|\phi(K)| : K \subseteq B_1(0), K \in \mathcal{K}(V)\}, \quad \phi \in Val(V) \tag{2.2}$$

With the above definitions we can state several interesting properties that combine the algebraic and topological structures.

Theorem 2.1.4 (McMullen). *Let $\phi \in Val(V)$. If $K_1, \ldots, K_k \in \mathcal{K}(V)$ and $\lambda_1, \ldots \lambda_k \geq 0$, then*

$$\phi(\lambda_1 K_1 + \cdots + \lambda_k K_k)$$

is a polynomial in $\lambda_1, \ldots \lambda_k$ of degree at most n. The coefficients of $\lambda_1^{d_1} \cdots \lambda_k^{d_k}$ are translation invariant and continuous valuations in K_j of degree d_j for all $j \in \{1, \ldots, k\}$.

Proposition 2.1.5. *$(Val(V), \| \cdot \|)$ is a Banach space.*

Proof. Let $\phi_j \in Val(V)$ be a Cauchy sequence. First, we are going to show, that the pointwise limit exists, i.e., $\phi(K) := \lim_{j \to \infty} \phi_j(K)$ is well-defined for all $K \in \mathcal{K}(V)$. Therefore, let us take $K \in \mathcal{K}(V)$. By compactness and the Heine-Borel theorem, K is bounded and therefore, there is $r \in \mathbb{R}_{>0}$, such that $K \subseteq B_r(0)$ and, hence, $\frac{1}{r}K \subseteq B_1(0)$. We obtain by Theorem 2.1.4,

$$\left| \sum_{i=1}^{n} \frac{1}{r^i} \phi_i^j (K) \right| = \left| \phi_j \left(\frac{1}{r} K \right) \right| \leq \|\phi_j\| \leq C,$$

since $\|\phi_j\|$ is a Cauchy sequence in $\mathbb{R}$. We can deduce that $|\phi_j(K)|$ is bounded and, thus, the pointwise limit exists. Obviously, the obtained function is a continuous and translation invariant valuation. Hence, $Val(V)$ is complete. ■

It is easily seen that with respect to the norm given in (2.2), $Val_i^+(V)$ and $Val_i^-(V)$ are closed subspaces of $Val(V)$.

Definition 2.1.6. For $i \in \mathbb{N}$ there is a continuous, rigid motion invariant valuation μ_i, which is homogeneous of degree i, called the i-th intrinsic volume. For $\dim V = n$, we have $\mu_n = vol_n$ and $\chi = \mu_0$.

$Val(V)$ is an infinite dimensional Banach space. Rigid motion invariant valuations form a small set in the sense that they span a finite dimensional subspace. The following theorem is an important part of the classification of rigid motion invariant and simple valuations.

Theorem 2.1.7. *Let μ be a continuous, rigid motion invariant, simple valuation. Then there is $c \in \mathbb{R}$ such that $\mu = c vol_n$.*

This is a theorem of Hadwiger (cf. [21]):

Theorem 2.1.8 (Hadwiger). *The intrinsic volumes $\mu_0, \ldots, \mu_n$ form a basis of the space of continuous, rigid motion invariant valuations.*

We also have a decomposition of $Val(V)$ which is a corollary of Theorem 2.1.4.

Corollary 2.1.9.

$$Val = \bigoplus_{i=0}^{n} Val_i^+ \oplus Val_i^-$$

Moreover, Val_0 and Val_n are one-dimensional, where Val_0 is spanned by χ and Val_n is spanned by the volume vol_n.

The group action described in (2.1) leads to the study of the following space.

Definition 2.1.10. A valuation $\phi \in Val(V)$ is called smooth if the map

$$\begin{aligned} GL(V) &\to Val(V), \\ g &\mapsto g \cdot \phi, \end{aligned}$$

is smooth as a map from the Lie Group $GL(V)$ to the Banach space $Val(V)$. The space of translation invariant smooth valuations is denoted by $Val^\infty(V)$.

We should always bare in mind that topological statements concerning $Val^\infty(V)$ are stated with respect to the Garding topology (see [16]).

The following theorem by Alesker (see [1]) concerning irreducible subspaces has far reaching consequences.

Theorem 2.1.11 (Alesker's Irreducibility theorem)**.** *The representation of $GL(V)$ on $Val^\pm(V)$ is irreducible, i.e., there are no nontrivial, $GL(V)$-invariant, and closed subspaces of $Val_i^\pm(V)$.*

Next we are going to introduce a class of valuations that are generated by the volume:

Definition 2.1.12. Let $A \in \mathcal{K}(V)$. We define the valuation μ_A by

$$\mu_A := vol_n(K + A).$$

We denote the subspace of valuations spanned by μ_A for $A \in \mathcal{K}(V)$ by

$$MVol := span\{\mu_A : A \in \mathcal{K}(V)\}.$$

The subspace of smooth valuations with respect to the $GL(V)$-action is denoted by $MVol^\infty(V)$.

Theorem 2.1.11 implies an important property of the space $MVol^\infty$.

Theorem 2.1.13. *The space $MVol_i^\infty(V)$ is dense in Val_i^∞, furthermore $MVol^\infty(V)$ is dense in $Val^\infty(V)$.*

2.1.1 Representation of translation-invariant smooth valuations

We want to express valuations as integrals over a certain set called the normal cycle. We follow [3] and [4].

Definition 2.1.14. Let $K \in \mathcal{K}(V)$ and $x \in K$. The tangent cone $T_x K$ at the point x is defined by

$$T_x K := \mathrm{cl}\left(\{y \in V : \exists \epsilon > 0, x + \epsilon y \in K\}\right).$$

We define the normal cone at x as the set

$$Nor(K, x) = \{w \in V : (y, w) \le 0, \forall y \in T_x K\}.$$

Moreover we set $Nor(K) := \bigcup_{x \in K}(\{x\} \times Nor(K, x))$.
Next the conic normal cycle of K is given by

$$\vec{N}(K) = \{(x; u) : x \in \partial K, u \in Nor(K, x) \cap S^{n-1}\} \subseteq V \times S^{n-1}.$$

Finally we define the normal cycle by

$$\vec{N}_B(K) = \{(x; u) : x \in K, u \in Nor(K, x) \cap B_1(0)\} \subseteq V \times V.$$

Since we are interested in an integral representation of translation invariant valuations, we also need certain properties of differential forms. Therefore we need the following definitions.

Definition 2.1.15. Let $\omega \in \Omega^*(V \times S^{n-1})$ or $\omega \in \Omega^*(V \times V)$.

- We say ω is translation-invariant, if it is translation-invariant in the first component, i.e., $\tau_x^* \omega = \omega$ for all $x \in V$, where $\tau_x(a, b) := (a + x, b)$.

- We say that ω is of bi-degree (k,l) if $\omega \in \Omega^k(V) \wedge \Omega^l(X)$.
- We say $\omega \in \Omega^*(V \times V)$ is homogeneous of degree k, if it is k-homogeneous in the second component, i.e., $h_\lambda^* \omega = \lambda^k \omega$, with $h_\lambda(x,y) = (x, \lambda y)$.

We denote by Ω_B^k the space of all smooth, translation-invariant, $(n-k)$-homogeneous, closed, n-differential forms ω of bi-degree $(k, n-k)$ on $V \times V$, such that $\omega|_{V \times S^{n-1}}$ is vertical. We set

$$\Omega_B := \bigoplus_{k=0}^{n-1} \Omega_B^k \oplus \operatorname{span}\{\pi_V^* dvol_n\},$$

where π_V denotes the canonical projection onto V and $dvol_n = dx_1 \wedge \cdots \wedge dx_n$. We denote by Ω_T^k the space of all smooth, translation invariant, $(n-1)$-differential forms ω of bi-degree $(k, n-k-1)$ on $V \times S^{n-1}$, such that $d\omega$ is vertical. We set

$$\Omega_T := \bigoplus_{k=0}^{n-1} \Omega_T^k.$$

With the definitions above, we get the following theorem.

Theorem 2.1.16. *Given $\omega \in \Omega_T^k$, we define*

$$\omega_B := \begin{cases} d(\tilde{r}^{n-k} p^* \omega) & \text{on } TV \setminus (V \times \{0\}), \\ 0 & \text{on } V \times \{0\}, \end{cases} \tag{2.3}$$

where $\tilde{r}(x,y) := |y|, x, y \in V$ and $p : V \times V \setminus (V \times \{0\}) \to V \times S^{n-1}$ by

$$(x,y) \mapsto \left(x, \frac{y}{\tilde{r}(x,y)}\right)$$

Then the linear map $\Phi : \Omega_T^k \to \Omega_B^k, \omega \mapsto \omega_B$ is surjective and $\omega_B|_{V \times S^{n-1}} = d\omega$.

With these preparations we can state our first classification of smooth translation-invariant valuations on a vector space V.

Theorem 2.1.17. *Let $\omega \in \Omega_T^k$. Then the map $\nu(\omega) : \mathcal{K}(V) \to \mathbb{R}$*

$$K \mapsto \int_{\vec{N}(K)} \omega,$$

is a smooth, translation-invariant valuation of degree k. The map $\nu : \Omega_T^k \to Val_k^\infty(V)$ is surjective and $\ker \nu = \{\omega \in \Omega_T^k : \omega \text{ is exact}\}$.

We also get the following corollary.

Corollary 2.1.18. *Let $\omega \in \Omega_T^k$. Then*

$$\nu(\omega)(K) = \int_{\vec{N}_B(K)} \omega_B.$$

In particular, the map $\nu_B : \Omega_B^k \to Val_k^\infty(V)$ given by

$$\nu_B(\theta)(K) = \int_{\vec{N}_B(K)} \theta, \quad \theta \in \Omega_B^k,$$

is surjective.

The observations of this section yield the following diargam:

$$
\begin{tikzcd}
 & Val_k(V) \\
\Omega_B^k \arrow[r, "2.1.18", "\nu_B"'] & Val_k^\infty(V) \arrow[u, "\cup\mid"] \\
 & \Omega_T^k \arrow[u, "\nu", "2.1.17"'] \arrow[ul, "\Phi"', "2.1.16"]
\end{tikzcd}
\tag{2.4}
$$

If we define $N(K) := \{(x, -v) \in \partial K \times S^{n-1} : K \subseteq H_{v,x}\}$, where $H_{v,x} := \{y \in V : (y, v) \leq (x, v)\}$, and set $N^*(K)$ as the image of $N(K)$ under the map $V \times S^{n-1} \to V \times \mathbb{P}_+(V^*)$ induced by the inner product, we can state the following theorem.

Theorem 2.1.19. *The map* $(\mathbb{R}\ dvol_n) \oplus \Omega_{tr}^{n-1}(V \times \mathbb{P}_+(V^*)) \to Val^\infty(V)$,

$$
(c\ dvol_n, \omega) \mapsto \left[K \mapsto c\ dvol_n(K) + \int\limits_{N^*(K)} \omega \right],
$$

is onto.

We obtain a map $\Psi : \Omega_{tr}^{n-1}(V \times \mathbb{P}_+(V^*)) \to Val^\infty(V)$ defined by

$$
\omega \mapsto \int\limits_{N^*(.)} \omega.
$$

The following theorem describes its kernel:

Theorem 2.1.20 (Kernel Theorem)**.** *Let V be an n-dimensional real vector space. Then $\omega \in ker(\Psi)$ if and only if $D\omega = 0$ and $\pi_{V*}\omega = 0$.*

2.1.2 Spaces of valuations

We have already introduced the spaces $Val^\infty(V)$ and $Val(V)$. For the study of valuations on more general sets than vector spaces, we need different approaches to smooth valuations. We follow [3] and [4].

Definition 2.1.21. Let V be an n-dimensional real vector space. Let us denote by $CV(V)$ the space of continuous valuations on $\mathcal{K}(V)$ with respect to the Hausdorff metric. Equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$, $CV(V)$ becomes a Frechet space. Let $QV(V)$ denote the space of continuous valuations on V which satisfy that the map given by

$$
\begin{cases}
\mathcal{K}(V) \to C^n([0,1] \times V), \\
K \mapsto \phi(tK + x),
\end{cases}
$$

is continuous. Let us call such valuations quasi-smooth. For a compact subset $G \subseteq V$, we define a seminorm on $QV(V)$ by

$$
\|\phi\|_G := \sup_{K \subseteq G} \|\phi(tK + x)\|_{C^n([0,1]\times G)}.
$$

The family of seminorms obtained by taking all compact subsets $G \subseteq V$ induces a Frechet topology on $QV(V)$.

Definition 2.1.22. The natural representation of the group of affine transformations of V in the space $QV(V)$ is continuous. We will denote by $SV(V)$ the elements of the affine-smooth vectors in $QV(V)$.

There are several descriptions of the space $SV(V)$, which will become important in the upcoming sections. The first one is an integral representation which is similar to the translation invariant case, where $N(K)$ is the normal cycle of K defined in Definition 2.2.1. For a proof, we refer to [3, Theorem 5.2.2].

Theorem 2.1.23. *The map* $\Psi : C^\infty(V, \mathcal{D}V) \oplus C^\infty(V \times S^{n-1}, \Lambda^{n-1}(V \times S^{n-1}) \otimes \pi_V^* o) \to SV(V)$

$$(\nu, \eta) \mapsto \left[K \mapsto \int_K \nu + \int_{N(K)} \eta \right]$$

is onto.

For the next representation, we need some preparations.

Theorem 2.1.24. *Let V be a vector space, let $\bar{K} = (K_1, \dots, K_s)$ with $K_i \in \mathcal{K}(V)$ and let $\mu \in C^\infty(V, \mathcal{D}V)$. We define $M_{\bar{K}} : \mathbb{R}_+^s \to \mathbb{R}$ by*

$$(\lambda_1, \dots, \lambda_s) \mapsto \mu \left(\sum_{i=1}^s \lambda_i K_i \right).$$

$M_{\bar{K}}$ has the following properties

1. *$M_{\bar{K}}\mu \in C^\infty(\mathbb{R}_+^s)$ and $M_{\bar{K}}$ is a continuous operator from $C^\infty(V, \mathcal{D}V)$ to $C^\infty(\mathbb{R}_+^s)$.*

2. *For sequences $\mu_j \to \mu$ in $C^\infty(V, \mathcal{D}V)$ and $K_i^j \to K_i$ in $\mathcal{K}(V)$, we have convergence $M_{\bar{K}^j}\mu_j \to M_{\bar{K}}\mu$ in $C^\infty(\mathbb{R}_+^s)$.*

According to Theorem 2.1.24 we have

$$(\Theta_s'(\mu, A_1, \dots, A_s))(K) := \frac{\partial^s}{\partial\lambda_1 \cdots \partial\lambda_s}\bigg|_0 \mu \left(K + \sum_{j=1}^s \lambda_j A_j \right) \in SV(V). \tag{2.5}$$

Due to Theorem 1.1.10 we can represent an element of $h \in C^\infty(S^{n-1}, \mathbb{R})$ as a difference of two support functionals of compact, convex bodies, i.e. $h = h_A - h_B$ with $A, B \in \mathcal{K}(V)$. Let us define

$$\Theta_s'(\mu, h, \dots, h_s) := \Theta_s'(\mu, h_A, \dots, h_s) - \Theta_s'(\mu, h_B, \dots, h_s).$$

It needs to be shown that this map is independend of the choice of the support functionals. Therefore let $h = h_{A_1} - h_{B_1} = h_{A_2} - h_{B_2}$. By Minkowski linearity of the support functionals we get $h_{A_1+B_2} = h_{A_2+B_1}$. Now Minkowski linearity of (2.5) yields

$$\begin{aligned}
&\Theta_s'(\mu, h_{A_1}, \dots, h_s) - \Theta_s'(\mu, h_{B_1}, \dots, h_s) = \Theta_s'(\mu, h_{A_2}, \dots, h_s) - \Theta_s'(\mu, h_{B_2}, \dots, h_s) \\
\Leftrightarrow &\Theta_s'(\mu, h_{A_1}, \dots, h_s) + \Theta_s'(\mu, h_{B_2}, \dots, h_s) = \Theta_s'(\mu, h_{A_2}, \dots, h_s) + \Theta_s'(\mu, h_{B_1}, \dots, h_s) \\
\Leftrightarrow &\Theta_s'(\mu, h_{A_1} + h_{B_2}, \dots, h_s) = \Theta_s'(\mu, h_{A_2} + h_{B_1}, \dots, h_s) \\
\Leftrightarrow &\Theta_s'(\mu, h_{A_1+B_2}, \dots, h_s) = \Theta_s'(\mu, h_{A_2+B_1}, \dots, h_s).
\end{aligned}$$

Hence we obtain the multilinear continuous map

$$\Theta_s' : C^\infty(V, \mathcal{D}V) \times (C^\infty(S^{n-1}, \mathbb{R}))^s \to SV(V),$$

that induces the map

$$\Theta'_s : C^\infty(V \times (S^{n-1})^s, \mathcal{D}V \otimes \mathbb{R}^{\otimes s}) \to SV(V)$$

by Theorem 1.3.9. Let us set $\Theta_k = \sum_{i=0}^k \Theta'_i$. Moreover let us define

$$\Theta_V := \bigoplus_{k=0}^n \Theta_k : \bigoplus_{k=0}^n C^\infty(V \times (S^{n-1})^k, \mathcal{D}V \otimes \mathbb{R}^{\otimes k}) \to SV(V).$$

To see that the map Θ_V is an epimorphism, we have to introduce a filtration of $SV(V)$: We define the filtration W_i by

$$W_i := \left\{ \phi \in SV(V) : \frac{d^k}{dt^k}\phi(tK+x)|_{t=0} = 0 \ \forall k < i, K \in \mathcal{K}(V), x \in V \right\}. \tag{2.6}$$

By [3, Proposition 3.1.1] we obtain

$$SV(V) = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_n \supseteq W_{n+1} = 0,$$

which yields a grading of $SV(V)$

$$SV(V) = \bigoplus_{k=0}^n W_k/W_{k+1} =: gr_W SV(V). \tag{2.7}$$

We therefore expect a connection between Θ_k and the filtration, which is given in the next proposition.

Proposition 2.1.25. *The image of the map* $\Theta_k : \bigoplus_{i=0}^k C^\infty(V \times (S^{n-1})^i, \mathcal{D}V \otimes \mathbb{R}^{\otimes i}) \to SV(V)$ *is equal to* W_{n-k}.

Finally our second description of $SV(V)$ is given by:

Theorem 2.1.26. *The continuous linear map Θ_V is onto. It is uniquely defined by the property*

$$\Theta_V(\mu \otimes h_{A_1} \otimes \cdots \otimes h_{A_k})(K) = \left.\frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\right|_0 \mu\left(K + \sum_{i=1}^k \lambda_i A_i\right), \quad \lambda_i \geq 0,$$

for $k = 0, \ldots, n, \mu \in C^\infty(V, \mathcal{D}V), A_1, \ldots, A_k \in \mathcal{K}^\infty(V), K \in \mathcal{K}(V)$.

For our next representation of elements of $SV(V)$, we need the vector bundle over V whose fiber over $x \in V$ is equal to the space of translation invariant $GL(T_xV)$-smooth valuations on the tangent space T_xV, i.e., $Val^\infty(T_xV)$. Let us denote this vector bundle by $Val(TV)$. For a proof of the following proposition we refer to [3, Proposition 3.1.5].

Proposition 2.1.27. *The map* $\Lambda_k : W_k \to C^\infty(V, Val_k^\infty(TV))$ *defined by*

$$\phi \mapsto \left[K \mapsto \frac{1}{k!}\frac{d^k}{dt^k}|_{t=0}\phi(tK+x)\right]$$

has the following properties

1. $\Lambda_k : W_k \to C^\infty(V, Val_k^\infty(TV))$ *is an epimorphism.*
2. $\ker \Lambda_k = W_{k+1}$. *Moreover* W_k/W_{k+1} *is isomorphic to* $C^\infty(V, Val_k^\infty(TV))$.

To keep track of the introduced spaces and properties we sum up in the following diagram. We will abbreviate $F_V := \bigoplus_{k=0}^{n} C^\infty(V \times (S^{n-1})^k, \mathcal{D}V \otimes \mathbb{R}^{\otimes k})$ and $\Lambda^{n-1} = \Lambda^{n-1}(V \times S^{n-1})$.

$$
\begin{tikzcd}
& CV(V) & \\
& QV(V) \arrow[u, "\cup\!|"'] & \\
C^\infty(V, \mathcal{D}V) \oplus C^\infty(V \times S^{n-1}, \Lambda^{n-1} \otimes \pi_V^* o) \arrow[r, "2.1.23"'] & SV(V) \arrow[u, "\cup\!|"'] \arrow[r, leftrightarrow, "\Theta_V", "2.1.26"'] \arrow[d, leftrightarrow] & F_V \\
\bigoplus_{k=0}^{n} C^\infty(V, Val_k^\infty(TV)) \arrow[r, leftrightarrow, "2.1.27", "\oplus\Lambda_k"'] & \bigoplus_{k=0}^{n} W_k/W_{k+1} &
\end{tikzcd}
\tag{2.8}
$$

Next we are going to introduce another class of sets that are the domains of functionals that satisfy the exclusion-inclusion principle, which can even be defined for smooth manifolds:

Definition 2.1.28. Let M be an n-dimensional smooth manifold, $P \subseteq M$. We call P a submanifold with corners of M if it is compact, connected and locally diffeomorphic to $\mathbb{R}^{n-k} \times \mathbb{R}^k_{\geq 0}$, where k is fixed for P. We denote the family of these sets by $\mathcal{P}(M)$.

Definition 2.1.29 (Push-forward)**.** Let M be a smooth manifold, $f : M \to \mathbb{R}^n$ be a smooth, proper submersion and let ϕ be a function with the property $\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$, for A, B and $A \cup B, A \cap B \in \mathcal{P}(X)$. We define the push-forward by

$$f_*\phi(K) = \phi(f^{-1}(K)).$$

Obviously $\mathcal{K}(V) \cap \mathcal{P}(V)$ is a dense subset of $\mathcal{K}(V)$, hence for every continuous valuation given on this dense subset, we get a uniquely defined extension to a valuation on $\mathcal{K}(V)$.

Definition 2.1.30. Let M be a smooth manifold. A functional $\phi : \mathcal{P}(M) \to \mathbb{R}$ that satisfies $\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B)$, wherever it is well-defined, is called a smooth valuation if for any point $p \in M$ there is a neighbourhood $U_p \subseteq M$ and a chart $\varphi : U_p \to \mathbb{R}^n$, such that the restriction of the push-forward $\varphi_*\phi : \mathcal{K}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ extends uniquely by continuity from the dense subset $\mathcal{K}(V)$ to a valuation of $SV(\mathbb{R}^n)$. The space of valuations with this property will be denoted by $\mathcal{V}^\infty(M)$.

Due to its construction, there is a strong connection between the spaces $SV(V)$ and $\mathcal{V}^\infty(V)$, see [4, Proposition 2.4.12]:

Proposition 2.1.31. *Let V be an n-dimensional real vector space. There is an isomorphism of Frechet spaces between $\mathcal{V}^\infty(V)$ and $SV(V)$.*

Finally we can sum up the connection between $\mathcal{V}^\infty(V)$ and $SV(V)$.

$$
\begin{tikzcd}
SV(V) \arrow[d, leftrightarrow, "2.1.31"'] & C^\infty(V, \mathcal{D}V) \oplus C^\infty(V \times S^{n-1}, \Lambda^{n-1} \otimes \pi_V^* o) \arrow[l, "2.1.23"'] \arrow[dl, "2.2.2"] \\
\mathcal{V}^\infty(V) &
\end{tikzcd}
\tag{2.9}
$$

Due to the definition of $\mathcal{V}^\infty(M)$ we can define a filtration by considering open sets that are diffeomorphic to $\mathbb{R}^n$:

Definition 2.1.32. Let M be a smooth manifold and let W_i be the filtration given in (2.6). We define $W_i(M)$ by

$$W_i(M) := \{\phi \in \mathcal{V}^\infty(M) : \exists (U,\varphi) \text{ with } \varphi : U \to \mathbb{R}^n \text{ diffeomorphism }, \varphi_*\phi \in W_i\}.$$

It is not surprising that for vector spaces the filtration coincides, see [4, Proposition 3.1.3].

Proposition 2.1.33. *Let V be a vector space. Then $W_i(V) = W_i$.*

2.2 Valuations on smooth manifolds

For sake of simplicity, we will always assume that the manifold is oriented. Similar as in Theorem 2.1.23, we want a representation of smooth valuations as an integral over a certain set. This motivates the following definition:

Definition 2.2.1. Let M be a smooth manifold and let $P \in \mathcal{P}(M)$. The normal cycle of P is a subset $N(P) \subseteq \mathbb{P}_M$ defined by

$$N(P) := \bigcup_{p \in P} ((T_pP)^\circ \setminus \{0\})/\mathbb{R}_{>0},$$

where

$$T_pP := \{\xi \in T_pM : \exists \gamma : [0,1] \to M, \gamma \in C^1([0,1], M), \gamma'(0) = \xi\}$$

and $(T_pP)^\circ$ denotes the dual cone of T_pP.

The proof of the following statement can be found in [4, Lemma 2.4.8].

Lemma 2.2.2. *Let M be a smooth manifold.*

- *Let $\nu \in \Omega^n(M)$ and $\eta \in \Omega^{n-1}(\mathbb{P}_M)$. Then $P \mapsto \int_P \nu + \int_{N(P)} \eta$ defines a smooth valuation on M.*
- *Let $\phi \in \mathcal{V}^\infty(M)$ and $p \in M$. Then there exists an open neighbourhood U of p and a pair of differential forms $(\nu, \eta) \in \Omega^n(U) \times \Omega^{n-1}(\mathbb{P}_U)$ such that for any $P \in \mathcal{P}(U)$ one has*

$$\phi(P) = \int_P \nu + \int_{N(P)} \eta \tag{2.10}$$

Remark 2.2.3. A smooth valuation is sometimes defined as a functional which can be represented in the form (2.10). By Lemma 2.2.2 these definitions are equivalent.

The support of a smooth valuation $\phi \in \mathcal{V}^\infty(M)$ is defined to be the set

$$\operatorname{supp} \phi := M \setminus \{p \in M : \exists U \subseteq M \text{ open}, p \in U \text{ with } \phi|_U = 0\}.$$

The space of all $\phi \in \mathcal{V}^\infty(M)$ with compact support is a subspace of $\mathcal{V}^\infty(M)$. We will denote it by $\mathcal{V}_c^\infty(M)$. There is a useful property of compactly supported valuations. A proof of this can be found in [12, Lemma 2.3].

Lemma 2.2.4. *Let M be a smooth manifold and let $\phi \in \mathcal{V}_c^\infty(M)$. Then ϕ can be represented by a pair $(\nu, \eta) \in \Omega^n(M) \times \Omega^{n-1}(\mathbb{P}_M)$ of compactly supported differential forms.*

From Lemma 2.2.2 we obtain a surjective map $\tilde{\Psi} : \Omega^n(M) \oplus \Omega^{n-1}(\mathbb{P}_M) \to \mathcal{V}^\infty(M)$ given by

$$(\nu, \eta) \mapsto \left[P \mapsto \int_P \nu + \int_{N(P)} \eta \right] \tag{2.11}$$

It will give rise to a topology on the space of smooth valuations. Since this map is not injective, we have to determine its kernels to gain a representation in the sense of topological vector spaces. Since the representation as an integral over the normal cycle carries over to the case of a smooth manifold, we expect that we can state a theorem which is similar to Theorem 2.1.20. For a proof of Theorem 2.2.5 we refer to [14, Theorem 1].

Theorem 2.2.5. *Let M be a smooth oriented manifold of dimension n and let $\tilde{\Psi}$ be the map from* (2.11)*. Then $(\nu, \eta) \in \ker \tilde{\Psi}$ if and only if*

- $D\eta + \pi_M^* \nu = 0$,
- $\pi_{M*}\eta = 0$,

where D is the Rumin operator that was introduced in Theorem 1.2.26 and π_{M*} is the integral along fibers from Proposition 1.2.10. Moreover, if $D\eta + \pi_M^* \nu = 0$, then $\tilde{\Psi}(\nu, \eta) = r\chi$, $r \in \mathbb{R}$.

Remark 2.2.6. By continuity of the given operators, we see that the kernel of $\tilde{\Psi}$ is closed in $\Omega^n(M) \oplus \Omega^{n-1}(\mathbb{P}_M)$. We equip the space of smooth valuations on a smooth manifold M with the topology induced by $\tilde{\Psi}$. Therefore

$$\Psi : \Omega^n(M) \oplus \Omega^{n-1}(\mathbb{P}_M)/\ker \tilde{\Psi} \to \mathcal{V}^\infty(M)$$

is a linear homeomorphism of Frechet spaces.

Since most of the constructions concerning manifolds are local, it is useful to have a partition of unity which is also possible for valuations. At this point we call χ, the Euler characteristic, a unit and the explanation will follow in Section 4. This proposition can be found in [5, Proposition 6.2.1].

Proposition 2.2.7. *Let $(U_\alpha)_{\alpha \in A}$ be a locally finite open covering of a smooth manifold M. Then there exists a family $(\phi_\alpha)_{\alpha \in A} \subseteq \mathcal{V}^\infty(M)$ such that*

$$\operatorname{supp}(\phi_\alpha) \subseteq U_\alpha \quad \textit{and} \quad \sum_{\alpha \in A} \phi_\alpha \equiv \chi,$$

where the sum is locally finite.

3 Product and Convolution of translation invariant valuations on convex bodies

There are at least two important algebraic operations on the space of translation invariant valuations on convex bodies. They give rise to the definition of similar operations on valuations on smooth manifolds which we will examine in sections 4 and 6.

3.1 Definition of the Product of smooth translation invariant valuations

Let us recall Theorem 2.1.13. The space $MVol^\infty$ is dense in $Val^\infty(V)$. We will make use of this fact in the following to define a product on $Val^\infty(V)$. We follow [2].

Definition 3.1.1. Let $A, B \in \mathcal{K}(V)$ and μ_A, μ_B be the respective valuations from Definition 2.1.12. Then we define the exterior product of μ_A and μ_B by

$$(\mu_A \boxtimes \mu_B)(K) = vol_{2n}(K + (A \times B)), \tag{3.1}$$

with $K \in \mathcal{K}(V \times V)$.

We obtain the following formula:

Lemma 3.1.2. *For $A, B \in \mathcal{K}(V)$, we have*

$$(\mu_A \boxtimes \mu_B)(K) = \int_V \mu_B((K + (A \times \{0\}) \cap (\{x\} \times V)dx.$$

By this integral representation, we get a continuous extension to $MVol(V)$:

Proposition 3.1.3. *The bilinear map from* (3.1) *uniquely extends to a bilinear, associative and commutative product on $MVol$ that is continuous in each argument.*

Hence we can define:

Definition 3.1.4. Let $\phi, \psi \in MVol(V)$. We define $\cdot : MVol(V) \times MVol(V) \to MVol(V)$ by

$$(\phi, \psi) \mapsto \phi \cdot \psi := \Delta^*(\phi \boxtimes \psi),$$

with the diagonal embedding $\Delta : V \to V \times V$.

Due to the integral representation it is immediate that the Euler characteristic χ acts as a unit. We make sure that the product is $GL(V)$-equivariant:

$$\begin{aligned} g \cdot (\mu_A \cdot \mu_B)(K) &= vol_{2n}(\Delta(g^{-1}K) + A \times B) \\ &= det(g^{-1})^2 vol_{2n}(\Delta(K) + g \cdot A \times g \cdot B) \\ &= ((g \cdot \mu_A) \cdot (g \cdot \mu_B))(K). \end{aligned}$$

Thus this product induces a $GL(V)$ equivariant product on $MVol^\infty(V)$. To obtain a continuous product on $Val^\infty(V)$ we proceed similar to Theorem 2.1.26 to gain the following representation.

Theorem 3.1.5. *For every $0 \le k \le n$, there is an open and continuous epimorphism*

$$\Theta_k : Val_n \otimes C^\infty((S^{n-1})^k) \to Val^\infty_{n-k}(V)$$

which is $GL(V)$-equivariant with respect to the action given by

$$(g \cdot f)(x_1, \dots, x_k) = \|gx_1\| \cdots \|gx_k\| f\left(\frac{g^{-1}x_1}{\|g^{-1}x_1\|}, \cdots, \frac{g^{-1}x_k}{\|g^{-1}x_k\|}\right).$$

With the help of this representation we consider $\mu_A, \mu_B \in MVol^\infty(V)$ as image of $\Theta = \sum_{k=0}^n \Theta_k$ of some $F \in C^\infty((S^{n-1})^k)$ and $G \in C^\infty((S^{n-1})^l)$. By the Stone-Weierstrass theorem, we obtain that

$$\operatorname{span}\{f_1(x_1)\cdots f_k(x_k) : f_i \in C^\infty(S^{n-1})\}$$

is dense in $C^\infty((S^{n-1})^k)$. Moreover, we already know that every $C^\infty(S^{n-1})$-function can be represented as the difference of two support functions of convex bodies. Therefore we reduce the problem to the level of support functions. By construction of Θ_k, we have

$$\Theta_k((cvol_n) \otimes (h_{A_1}, \dots, h_{A_k}) = V(K[n-k], A_1, \dots, A_{n-k}).$$

From this we can show that the exterior product is a continuous map $\boxtimes : MVol^\infty(V) \times MVol^\infty(V) \to Val(V)$. Finally since $MVol^\infty(V)$ is dense in $Val^\infty(V)$ this extends by continuity. With the above observations, we can finally state the following theorem.

Theorem 3.1.6. *The exterior product $\boxtimes : Val^\infty(V) \times Val^\infty(V) \to Val(V \times V)$ is continuous.*

The theorem above implies that the product $\cdot : Val^\infty(V) \times Val^\infty(V) \to Val$ is continuous. We make use of Lemma 1.3.6 to infer that the product maps to $Val^\infty(V)$. We thus obtain the following theorem:

Theorem 3.1.7. *We can extend the product on $MVol^\infty(V)$ to a product*

$$\cdot : Val^\infty(V) \times Val^\infty(V) \to Val^\infty(V)$$

with the following properties:

- *$\cdot : Val^\infty(V) \times Val^\infty(V) \to Val^\infty(V)$ is continuous.*
- *The Euler characteristic χ acts as a unit.*
- *$Val_k^\infty(V) \cdot Val_i^\infty(V) \subseteq Val_{k+i}^\infty(V)$, i.e., if $\phi \in Val_k^\infty(V)$ and $\psi \in Val_i^\infty(V)$, then $\phi \cdot \psi \in Val_{k+i}^\infty$ where $Val_m^\infty(V) = \{0\}$ if $m > n$.*
- *The product of valuations with given parity is only odd if their parity is different.*
- *Let $W \subseteq V$ be a linear subspace. The restriction map $r_W : Val^\infty(V) \to Val^\infty(W)$ commutes with the product, i.e., $r_W(\phi \cdot \psi) = r_W(\phi) \cdot r_W(\psi)$.*
- *The product is $GL(V)$-equivariant, that is,*

$$g \cdot (\phi \cdot \psi) = (g \cdot \phi) \cdot (g \cdot \psi), g \in GL(V).$$

3.1.1 Embedding of continuous translation invariant valuations I

Furthermore, we can extend this bilinear map to a larger set. It will give us an embedding of continuous translation invariant valuations into the dual space of $Val^\infty(V)$. We will state a similar result in Section 5. For details, we refer to [20].

Theorem 3.1.8. *There is a continuous extension*

$$\hat{\cdot} : Val(V) \times Val^\infty(V) \to Val(V)$$

of the product given in Theorem 3.1.7.

To obtain the desired embedding, we need Theorem 2.1.4, from that we get the projection

$$\kappa_n : Val(V) \to Val_n(V)$$

which is a projection to a one dimensional space, that is spanned by vol_n. With a fixed scalar product we can identify Val_n with $\mathbb{R}$, by sending vol_n to 1. Hence we obtain a continuous functional on $Val(V)$, since it is a composition of continuous maps. This motivates the following definition:

Definition 3.1.9. Let $\psi \in Val(V)$ and $\hat{\cdot}$ the product given in Theorem 3.1.8. Then we can define $pd(\psi)(\phi) : Val(V) \to (Val^\infty(V))^*$ by

$$\psi \mapsto [\phi \mapsto \kappa_n(\psi \hat{\cdot} \phi)] \, .$$

Hence, we obtain the following:

Theorem 3.1.10. *The map pd is an embedding with dense image of the space of continuous translation invariant valuations into the dual space of smooth translation invariant valuations.*

The dual space of $Val^\infty(V)$ is often called the space of generalized valuations.

3.2 Definition of the Convolution of smooth translation invariant valuations

There is another algebraic structure on $Val^\infty(V)$ called the convolution. We follow [15].

Definition 3.2.1. Let I be a monotone multi index and set $I^c := \{1, \ldots, n\} \setminus I$. The Hodge $*$-operator $* : \Lambda^k V \to \Lambda^{n-k} V$ is determined by

$$*dx_I = \sigma_{I,I^c} dx_{I^c}$$

where σ_{I,I^c} is the sign of the permutation mapping (I, I^c) to $\{1, \ldots, n\}$.

For futher reading and properties of the Hodge $*$-operator, we refer to [25].

Definition 3.2.2. Let X be either S^{n-1} or V. The operator $*_1 : \Lambda^k V \wedge \Lambda X \to \Lambda^{n-k} V \wedge \Lambda X$ is defined by

$$*_1(\omega_1 \wedge \omega_2) = (-1)^{\binom{n-k}{2}} (*\omega_1) \wedge \omega_2$$

for $\omega_1 \in \Lambda^k V, \omega_2 \in \Lambda X$.

Definition 3.2.3. Let $\alpha_1 \in \Omega^k_T, \alpha_2 \in \Omega^l_T, \beta_1 \in \Omega^k_B, \beta_2 \in \Omega^l_B$ and let us define

$$\begin{aligned}
\alpha_1 * \alpha_2 &:= (2n-k-l)^{-1} *_1^{-1} \left[(n-k) *_1 \alpha_1 \wedge *_1 d\alpha_2 + (n-l) *_1 \alpha_2 \wedge *_1 d\alpha_1\right] \\
\beta_1 *_B \beta_2 &:= *_1^{-1}(*_1\beta_1 \wedge *_1\beta_2), \\
(\pi_V^* vol_n) *_B \beta_1 &:= \beta_1.
\end{aligned}$$

Let $A \in \mathcal{K}^\infty(V)$ and denote its support function by $h_A : S^{n-1} \to \mathbb{R}$. We define

$$\eta_A(y) := \begin{cases} r\nabla h_A(y), & y \neq 0, \\ 0, & y = 0, \end{cases}$$

and the Lipschitz map $G_A : V \times V \to V$

$$(x, y) \mapsto x + \eta_A(y). \tag{3.2}$$

It is easy to see that $G_A(\vec{N}_B(K)) = K + A$. We set $\theta_A := G_A^*(dvol_n)$. Now, let us recall Corollary 2.1.18. The following identification of $\mu_A \in MVol^\infty(V)$ with the differential forms given above is crucial. For a proof we refer to [15, Lemma 2.8].

Theorem 3.2.4. *For $A \in \mathcal{K}^\infty(V)$, we have $\theta_A \in \Omega_B$ and $\nu_B(\theta_A) = \mu_A$.*

Now we are ready to define the convolution on smooth, translation-invariant valuations.

Theorem 3.2.5. *Let $\phi, \psi \in Val^\infty(V), \beta \in \Omega_B^k$ and $\gamma \in \Omega_T^l$ with*

$$\nu_B(\beta) = \phi, \nu_B(\gamma) = \psi.$$

Then the bilinear map, given by

$$* : \begin{cases} Val^\infty(V) \times Val^\infty(V) \to Val^\infty(V) \\ \phi * \psi = \nu_B(\beta *_B \gamma) \end{cases}$$

is called the convolution and defines an associative, commutative and continuous product of degree $-n$ on $Val^\infty(V)$. Moreover, it satisfies

$$\mu_A * \mu_B = \mu_{A+B}, \tag{3.3}$$

for all $A, B \in \mathcal{K}(V)$.

Equation (3.3) will help us to show that the convolution in Section 6 actually is an extension of the operation above.

3.2.1 Embedding of continuous translation invariant valuations II

Similar to the product we want to extend the convolution to a map $* : Val(V) \times Val^\infty(V)$. We therefore need the definition of a push-forward of a continuous valuation.

Lemma 3.2.6. *Let V, W be vector spaces and $f : V \to W$ be a surjective linear map. Let k be the dimension of $\ker f$ and fix Lebesgue measures $vol_{\ker f}$ and vol_W on $\ker f$ and $vol_V = vol_{\ker f} \otimes vol_W$. Let $A \in \mathcal{K}(V), B \in \mathcal{K}(W)$, then*

$$\frac{1}{k!} \frac{\partial^k}{\partial \epsilon^k}\bigg|_{\epsilon=0} vol_V(A + \epsilon B) = vol_{\ker f}(B) vol_W(f(A)).$$

The above lemma motivates that for $\phi \in Val(V)$ and a surjective linear map $f : V \to W$ we can define a valuation $\tau \in Val(\ker f)$ for fixed $K \in \mathcal{K}(V)$ by

$$\frac{1}{k!} \frac{\partial^k}{\partial \epsilon^k}\bigg|_{\epsilon=0} \phi(K + \epsilon S) =: \tau(S).$$

By McMullen's decomposition, $\phi(K + \epsilon S)$ is a polynomial in ϵ with degree less or equal k and the corresponding coefficients are k-homogeneous, translation invariant and continuous valuations. Hence there is $C(K, \phi) \in \mathbb{R}$ such that

$$\tau = C(K, \phi) vol_{\ker f}.$$

Next, we define the pull-back of f by

$$f_* \phi(K) := C(\tilde{K}, \phi), \ K \in \mathcal{K}(W) \text{ with } f(\tilde{K}) = K.$$

This definition is independent of the choice of $\tilde{K}$. Together with these preparations we obtain the following theorem.

Theorem 3.2.7. *Let $a : V \times V \to V$ be the addition map and let $\mu_A, \mu_B \in Val(V)$. Then $a_*(\mu_A \boxtimes \mu_B) = \mu_{A+B}$.*

Due to (3.3) we obtain the desired extension:

Theorem 3.2.8. *There is a continuous extension of $*$ to $Val(V) \times Val^\infty(V)$ given by $a_* \circ \boxtimes$. We denote this extension by $\hat{*}$.*

Similar to Definition 3.1.9 this yields an embedding, where we identify the space $Val_0(V)$ with $\mathbb{R}$ by sending χ to 1.

Definition 3.2.9. Let $\psi \in Val(V)$ and $\hat{*}$ be the convolution given in Theorem 3.2.8. Then we can define the map $pd'(\psi)(\phi) : Val(V) \to (Val^\infty(V))^*$ by

$$\psi \mapsto [\phi \mapsto \kappa_0(\phi \hat{*} \psi)]$$

We obtain the following diagram:

$$\begin{tikzcd}
Val(V) \times Val^\infty(V) \arrow[d, "\hat{\cdot}"] & Val(V) \arrow[dd, "pd"] \\
Val(V) \arrow[r, Rightarrow] \arrow[d, "\kappa_n"] & {} \\
\mathbb{R} & (Val^\infty(V))^*
\end{tikzcd}
\qquad
\begin{tikzcd}
Val(V) \times Val^\infty(V) \arrow[d, "\hat{*}"] & Val(V) \arrow[dd, "pd'"] \\
Val(V) \arrow[r, Rightarrow] \arrow[d, "\kappa_0"] & {} \\
\mathbb{R} & (Val^\infty(V))^*
\end{tikzcd}$$

4 Product of valuations on smooth manifolds

In this section we want to introduce a multiplicative structure on the space of smooth valuations on manifolds. At first we have to define a product on valuations on vector spaces, then we will make use of the strong connection between the spaces $SV(\mathbb{R}^n)$ and $\mathcal{V}^\infty(\mathbb{R}^n)$ to obtain a similar result on smooth manifolds. We follow [5].

4.1 Product of valuations in $SV(V)$

Let V be a vector space of dimensions n. We give a short introduction to a product defined on $SV(V) \times SV(V)$. Therefore we need the description of $SV(V)$ in terms of smooth sections given in Theorem 2.1.26. Let us define a map $M : C^\infty(V, \mathcal{D}V) \times C^\infty(S^{n-1}, \mathbb{R})^l \times C^\infty(V, \mathcal{D}V) \times C^\infty(S^{n-1}, \mathbb{R})^m \to QV(V \times V)$ by

$$\begin{aligned} &M(\mu, \xi_1, \ldots, \xi_k, \nu, \eta_1, \ldots, \eta_l)(K) = \\ &\frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k} \frac{\partial^l}{\partial\beta_1 \cdots \partial\beta_l}\bigg|_0 (\mu \times \nu) \left(K + \left(\sum_{i=1}^k \lambda_i A_i \times \sum_{j=1}^l \beta_j B_j \right) \right), \end{aligned}$$

where the ξ_i and η_j are the support functions of the $A_i \in \mathcal{K}(V)$ and $B_j \in \mathcal{K}(V)$. By Theorem 1.3.9 we obtain a map $M : \bigoplus_l C^\infty(V \times (S^{n-1})^l, \mathcal{D}V \otimes \mathbb{R}^{\otimes l}) \times \bigoplus_m C^\infty(V \times (S^{n-1})^m, \mathcal{D}V \otimes \mathbb{R}^{\otimes m}) \to Q(V \times V)$.
The construction of the product on $SV(V)$ is given in the following diagram. We abbreviate the notation by $F_V := \bigoplus_l C^\infty(V \times (S^{n-1})^l, \mathcal{D} \otimes \mathbb{R}^{\otimes l})$:

$$\begin{tikzcd} & F_V \times F_V \arrow[r, "M"] \arrow[dl, "\Theta_V \times \Theta_V"', "2.1.26"] \arrow[d] & QV(V \times V) \arrow[d, "\Delta^*"] \\ SV(V) \times SV(V) \arrow[r, leftrightarrow, "\cong"] & (F_V \times F_V)/\ker(\Theta_V \times \Theta_V) \arrow[ur, "4.1.1"] \arrow[r, dashed, "M'=\boxtimes"] & SV(V) \end{tikzcd} \tag{4.1}$$

This is similar to the idea of constructiong the product on translation invariant valuations. It thus needs to be shown that $\ker(\Theta_V \times \Theta_V) \subseteq \ker M$.
To prove this claim we need a decomposition of $\phi \in SV(V)$. First Lemma 1.3.7 implies that every element $\psi \in C^\infty(V \times (S^{n-1})^k, \mathcal{D}V \otimes \mathbb{R}^{\otimes k})$ can be decomposed into

$$\psi = \sum_{i=1}^\infty \mu_i \otimes h_{A_1^i} \otimes \cdots \otimes h_{A_k^i},$$

where $\mu_i \in \mathcal{D}V$ and the $h_{A_j^i}$ are support functions of some smooth convex bodies A_j^i. Therefore we can decompose $\phi = \sum_{j=0}^n \phi_j$. Moreover, by Proposition 2.1.25, we have $\Theta_j(\psi_j) = \phi_j \in W_{n-j}$ for some $\psi_j \in C^\infty(V \times (S^{n-1})^j, \mathcal{D}V \otimes \mathbb{R}^{\otimes j})$. Hence, we obtain

$$\phi = \sum_{j=0}^n \phi_j = \sum_{j=0}^n \sum_{i=1}^\infty \mu_i^j \otimes h_{A_1^i} \otimes \cdots \otimes h_{A_k^i}.$$

Notice that this decomposition is not unique since the kernel of Θ_k is not trivial.

Lemma 4.1.1. *The bilinear map M introduced above admits a unique factorization to a continuous bilinear map*

$$M' : SV(V) \times SV(V) \to QV(V \times V)$$

such that $M = M' \circ (\Theta_V \times \Theta_V)$. We will denote $M'(\phi, \psi)$ by $\phi \boxtimes \psi$.

Proof. By (4.1) it suffices to show that $M(f,g) = 0$ for fixed $f \in F_V$ and arbirtrary $g \in F_W$. Since the considered expressions are continuous, it is enough to show that $M(f,g) = 0$ for $g = \mu \otimes h_1 \otimes \cdots \otimes h_k$. We are going to prove that

$$M(f,g)(K) = \frac{\partial^k}{\partial\theta_1 \cdots \partial\theta_k}\bigg|_0 \int_{w\in W} \Theta_V(f)\left(\left(K + \sum_{j=1}^k \theta_j(0 \times B_j)\right) \cap (V \times \{w\})\right) d\nu(w). \quad (4.2)$$

By our assumption, it then follows that $M(f,g)(K) = 0$ and thus the claim. In order to prove the above identity, we will restrict us to the case $f = \mu \otimes g_1 \otimes \cdots \otimes g_l$. Due to the obeservations above, the claim will follow by continuity of M. We get

$$\begin{aligned} M(f,g) &= \frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\frac{\partial^l}{\partial\theta_1 \cdots \partial\theta_l}\bigg|_0 (\mu \times \nu)\left(K + \sum_{i=1}^k \lambda_i(A_i \times 0) + \sum_{j=1}^l \theta_i(0 \times B_j)\right) \\ &= \frac{\partial^l}{\partial\theta_1 \cdots \partial\theta_l}\frac{\partial^k}{\partial\lambda_1 \cdots \partial\lambda_k}\bigg|_0 \int_{w\in W} \mu\left(\left(K + \sum_{j=1}^k \theta_j(0 \times B_j)\right) \cap (V \times \{w\})\right) d\nu(w) \\ &= \frac{\partial^l}{\partial\theta_1 \cdots \partial\theta_l}\bigg|_0 \int_{w\in W} \Theta_V(f)\left(\left(K + \sum_{j=1}^k \theta_j(0 \times B_j)\right) \cap (V \times \{w\})\right) d\nu(w). \end{aligned}$$

By continuity, this equation holds for all $f \in F_V$. ■

Bare in mind that the integral representation given in (4.2) holds for all $f \in F_V$. We will make use of this fact in the upcoming section.

With these preparations we can define the expression $\phi \cdot \psi := \Delta^*(\phi \boxtimes \psi)$, with Δ being the diagonal imbedding. We have the following properties:

Theorem 4.1.2. *Let V be a vector space and $\phi, \psi \in SV(V)$. We define a bilinear map with the following properties:*

- *The product $\phi \cdot \psi \in SV(V)$.*
- *The product is a continuous map $SV(V) \times SV(V) \to SV(V)$.*
- *$SV(V)$ becomes an associative commutative algebra with the Euler characteristic as a unit.*
- *The filtration introduced in Equation (2.6) is compatible with this multiplication, i.e.*

$$W_i \cdot W_j \subseteq W_{i+j}.$$

Proof. The map $\Delta : V \to V \times V$ induces a continuous map from $\mathcal{K}(V)$ to $\mathcal{K}(V \times V)$. Thus let $\phi \in QV(V \times V)$. The map $K \mapsto \phi(t\Delta(K) + x)$ is continuous with respect to the Hausdorff metric as a composition of the continuous maps $K \mapsto \phi(tK + x)$ with $K \in \mathcal{K}(V \times V)$ and $K \mapsto \Delta(K)$ with $K \in \mathcal{K}(V)$. We obtain $\Delta^*\phi \in QV(V)$. Hence, by definition, it follows $SV(V) \cdot SV(V) \subseteq QV(V)$. Moreover the map Δ^* is continuous with respect to the Frechet topology given on $QV(V)$, since it is a contraction. The product commutes with the action of affine transformations and, hence, the product of affine-smooth vectors is affine-smooth. Therefore $SV(V) \cdot SV(V) \subseteq SV(V)$ and continuity follows in the same way. For the other properties we refer to [3, Theorem 4.1.2]. ■

Hence we obtain the following definition:

Definition 4.1.3. Let $\phi, \psi \in SV(V)$. The bilinear map $\cdot : SV(V) \times SV(V) \to SV(V)$

$$(\phi, \psi) \mapsto \phi \cdot \psi := \Delta^*(\phi \boxtimes \psi).$$

defines a product on $SV(V)$.

Theorem 4.1.4. *The graded algebra $gr_W SV(V)$ is canonically isomorphic to the graded algebra $C^\infty(V, Val^\infty(V))$ with the pointwise multiplication on V and the k-th graded term of it is equal to $C^\infty(V, Val_k^\infty(V))$.*

4.2 Product of valuations in $\mathcal{V}^\infty(M)$

As already mentioned, the definition of the product on $\mathcal{V}^\infty(M)$ is strongly connected to the product on $SV(\mathbb{R}^n)$. Our first attempt to define the product will a priori not be independent of the chosen chart:

Let M be a smooth n-dimensional manifold and $\phi, \psi \in \mathcal{V}^\infty(M)$. Let (U, φ) be a chart of M with U diffeomorphic to $\mathbb{R}^n$. Then, by Definition 2.1.30, $\varphi_*\phi \in SV(\mathbb{R}^n)$ and hence by (2.7) can be decomposed as

$$\varphi_*\phi = \phi_0 + \cdots + \phi_n,$$

with $\phi_j \in W_{n-j}(\mathbb{R}^n)$. Now recall that Proposition 2.1.27 yields a decomposition of ϕ_j into

$$\phi_j(S) = \sum_{N=1}^{\infty} \frac{\partial^j}{\partial\lambda_1 \ldots \partial\lambda_j}\bigg|_0 \mu_N^j\left(S + \lambda_i A_N^{ij}\right),$$

with $\mu_N^j \in C^\infty(V, \mathcal{D}V)$. Obvoiously we obtain the same decomposition for $\varphi^*\psi$ with $\nu_N^j \in C^\infty(V, \mathcal{D}V)$. Due to Section 4.1, we have

$$\phi|_U \cdot \psi|_U := (\varphi_*\phi \cdot \varphi_*\psi)(K) = \sum_{j,j'=0}^{n} \sum_{N,N'} \frac{\partial^{j+j'}}{\partial\lambda_1 \ldots \partial\lambda_j \partial\mu_1 \ldots \partial\mu'_j}\bigg|_{\bar{\mu}=\bar{\lambda}=0} \tag{4.3}$$

$$(\mu_N^j \times \nu_{N'}^{j'}) \left(\Delta(K) + \left(\sum_{i=1}^{j} \lambda_i A_N^{i,j}\right) \times \left(\sum_{i'=1}^{j'} \mu_{i'} B_{N'}^{i',j'}\right)\right). \tag{4.4}$$

Therefore if we want to define a product by

$$(\phi \cdot \psi)|_U := \phi|_U \cdot \psi|_U,$$

we have to take into account that the expressions (4.3) and (4.4) heavily depend on the chosen chart (U, φ). To prove independence, we have to consider the expressions above on the overlap of two arbitrary different charts (U, φ) and $(\tilde{U}, \tilde{\varphi})$. Let us denote the valuation constructed above by $\phi|_U \circ_\varphi \psi|_U$ and $\phi|_{\tilde{U}} \circ_{\tilde{\varphi}} \psi|_{\tilde{U}}$, hence we aim for

$$(\phi|_U \circ_\varphi \psi|_U)|_{U\cap\tilde{U}} = (\phi|_{\tilde{U}} \circ_{\tilde{\varphi}} \psi|_{\tilde{U}})|_{U\cap\tilde{U}}. \tag{4.5}$$

In this section we will show that (4.3) and indeed defines a product of valuations on smooth manifolds that extends the linear case from Section 4.1. At first we have to make some technical preparations. A proof can be found in [9, Lemma 3.1.14].

Lemma 4.2.1. *Let M be a smooth n-manifold. Let $\varphi_1, \varphi_2 : M \to \mathbb{R}^n$ be two smooth maps which map M diffeomorphically onto open subsets $\varphi_1(M), \varphi_2(M)$. Let $\phi \in \mathcal{V}^\infty(M)$. Assume that $\phi(K) = 0$ for any compact domain $K \subseteq M$ with smooth boundary such that both $\varphi_1(K)$ and $\varphi_2(K)$ are convex. Then $\phi \equiv 0$.*

By means of Lemma 4.2.1, it suffices to show that

$$(\varphi_*\omega \cdot \varphi_*\psi)(\varphi(K)) = (\tilde{\varphi}_*\omega \cdot \tilde{\varphi}_*\psi)(\tilde{\varphi}(K))$$

for any compact domain $K \subseteq U \cap \tilde{U}$ with smooth boundary, such that $\varphi(K)$ and $\tilde{\varphi}(K)$ are convex. The independence of charts is now reduced to showing that

$$\sum_{j,j'}^{n} \sum_{N,N'}^{\infty} \frac{\partial^{j+j'}}{\partial\lambda_1 \ldots \partial\lambda_j \partial\mu_1 \ldots \partial\mu_{j'}}\bigg|_0 (\mu_N^{j'} \times \nu_{N'}^{j'}) \left((\varphi \times \varphi)(K) + \left(\sum_{i=1}^{j} \lambda_i A_N^{ij} \times \sum_{i'=1}^{j'} \mu_{i'} B_{N'}^{i'j'}\right)\right) =$$
$$\sum_{j,j'}^{n} \sum_{N,N'}^{\infty} \frac{\partial^{j+j'}}{\partial\lambda_1 \ldots \partial\lambda_j \partial\mu_1 \ldots \partial\mu_{j'}}\bigg|_0 (\tilde{\mu}_N^{j'} \times \tilde{\nu}_{N'}^{j'}) \left((\tilde{\varphi} \times \tilde{\varphi})(K) + \left(\sum_{i=1}^{j} \lambda_i \tilde{A}_N^{ij} \times \sum_{i'=1}^{j'} \mu_{i'} \tilde{B}_{N'}^{i'j'}\right)\right)$$

As this problem is symmetric in $\nu_N^{j'}$, the crucial equality is

$$\sum_{j,j'}^{n} \sum_{N,N'}^{\infty} \frac{\partial^{j+j'}}{\partial\lambda_1 \ldots \partial\lambda_j \partial\mu_1 \ldots \partial\mu_{j'}}\bigg|_0 (\mu_N^{j'} \times \tilde{\nu}_{N'}^{j'}) \left((\varphi \times \tilde{\varphi})(K) + \left(\sum_{i=1}^{j} \lambda_i A_N^{ij} \times \sum_{i'=1}^{j'} \mu_{i'} \tilde{B}_{N'}^{i'j'}\right)\right) = \tag{4.6}$$

$$\sum_{j,j'}^{n} \sum_{N,N'}^{\infty} \frac{\partial^{j+j'}}{\partial\lambda_1 \ldots \partial\lambda_j \partial\mu_1 \ldots \partial\mu_{j'}}\bigg|_0 (\mu_N^{j'} \times \nu_{N'}^{j'}) \left((\varphi \times \varphi)(K) + \left(\sum_{i=1}^{j} \lambda_i A_N^{ij} \times \sum_{i'=1}^{j'} \mu_{i'} B_{N'}^{i'j'}\right)\right). \tag{4.7}$$

Remark 4.2.2. Due to these preparations, we can restrict our observations to a fixed compact smooth n-dimensional submanifold $M' \subseteq V \times V$ which projects diffeomorphically onto its images in V under both projections $p_1, p_2 : V \times V \to V$. Moreover let us fix a compact submanifold with boundary $M \subseteq M'$ such that $M \cap \partial M = \emptyset$ and let $\tilde{p}_1, \tilde{p}_2 : M' \to V$ be the restrictions of the projections to M'. The following lemmas give an integral representation of the expressions above. We will have to consider the space $\mathcal{H}$ of smooth functions on $V^* \setminus \{0\}$ that are homogeneous of degree one. Moreove we will need the maps $\xi : \mathcal{H} \times V \times \mathbb{P}_+(V^*) \times [0,1] \to V$ defined by $\xi(h,p,n,t) = p + t\nabla h(n)$, the smooth map $\Xi : \mathcal{H} \times C^\infty(V, \mathcal{D}V) \to C^\infty(V \times \mathbb{P}_+(V^*) \times [0,1], \Lambda^n \otimes \pi_V^* o)$

$$(h, \mu) \mapsto (\xi(h, \cdot))^* \mu$$

and finally let us define the map $\Theta : C^\infty(V \times \mathbb{P}_+(V^*) \times [0,1], \Lambda^n \otimes \pi_V^* o) \to \mathcal{V}^\infty(V)$ by

$$\omega \mapsto \left[P \mapsto (\Theta(\omega))(P) = \int_{N^*(P) \times [0,1]} \omega \right].$$

By [4], every smooth valuation can be represented as an image of Θ. We follow [9, Section 3 and 4].

Lemma 4.2.3. *Let V be an n-dimensional vector space.*

1. *The map $\gamma : V^\infty(V) \times C^\infty(V, \mathcal{D}V) \times \mathcal{K}(V)^{k+1} \times \mathbb{R}^k_{\geq 0} \to \mathbb{R}$*

$$(\phi, \mu, K, A^1, \ldots, A^k, \lambda_1, \ldots, \lambda_k) \mapsto \int_{x \in V} \phi\left(K \cap (x - \sum_{i=1}^{k} \lambda_i A^i)\right) d\mu(x)$$

is a continuous function which is smooth on $V^\infty(V) \times C^\infty(V, \mathcal{D}V) \times \mathbb{R}^k_{\geq 0}$ for fixed $(K, A^1, \ldots, A^k) \in \mathcal{K}(V)^{k+1}$.

2. *Let $R > 0$ and $k \in \mathbb{N}$. Then there exist a constant $C > 0$, a positive integer $L \in \mathbb{N}$, and continuous seminorms $\|\cdot\|$ and $\|\cdot\|'$ on $\mathcal{V}^\infty(V)$ and $C^\infty(V, \mathcal{D}V)$, respectively, depending on n, k and R, such that for any strictly convex compact sets $A_1, \dots, A_k \in \mathcal{K}^\infty(V)$ and any $K \in \mathcal{K}(V)$ such that K is contained in the centered Euclidean ball of radius R, one has an estimate*

$$\left| \int_{x \in V} \phi(K \cap (x - \sum_{i=1}^k \lambda_i A_i)) d\mu(x) \right| \leq \|\phi\| \|\mu\|' \prod_{i=1}^k \|h_{A_i}\|_{C^L(S^{n-1})}.$$

Lemma 4.2.4. *Let $\psi \in V^\infty(V)$ be a smooth valuation of the form*

$$\psi = (\Theta \circ \Xi_l) \left(\sum_{N=1}^\infty \nu_N \otimes h_{B_N^1} \cdots \otimes h_{B_N^l} \right)$$

with $\nu_N \subseteq C^\infty(V, \mathcal{D}V)$ being smooth densities, and $B_N^i \in \mathcal{K}^\infty(V)$, containing the origin in the interior, and such that for any compact subset $T \subseteq V$ and any $L \in \mathbb{N}$,

$$\sum_{N=1}^\infty \|\nu_N\|_{C^L(T)} \prod_{i=1}^l \|h_{B_N^i}\|_{C^L(S^{n-1})} < \infty.$$

Let $A \in \mathcal{K}(V)$. Let $\mu \in C^\infty(V, \mathcal{D}V)$. Let $M \subseteq M' \subseteq V \times V$ be as in Remark 4.2.2. Assume that $p_1(M)$ is convex. Then the series

$$\sum_{N=1}^\infty \frac{\partial^l}{\partial \mu_1 \dots \partial \mu_l}\bigg|_0 (\mu_N \boxtimes \nu_N) \left(M + A \times \left(\sum_{i=1}^l \mu_i B_N^i \right) \right)$$

converges absolutely and its sum is equal to

$$\int_{x \in V} ((\tilde{p}_1 \circ \tilde{p}_2^{-1})_* \psi)(p_1(M) \cap (x - A)) d\mu(x).$$

With the preparations above we are able to justify that the product is well-defined.

Lemma 4.2.5. *The equality given in* (4.6) *holds.*

Proof. Our main goal is to reduce the calculation to

$$\sum_{j'}^n \sum_{N'}^\infty \frac{\partial^{j'}}{\partial \mu_1 \dots \partial \mu_{j'}} (\mu_N^j \times \nu_{N'}^{j'}) \left((\varphi \times \varphi)(K) + \left(\sum_{i=1}^j \lambda_i A_N^{ij} \times \sum_{i'=1}^{j'} \mu_{i'} B_{N'}^{i'j'} \right) \right) = \tag{4.8}$$

$$\sum_{j'}^n \sum_{N'}^\infty \frac{\partial^{j'}}{\partial \mu_1 \dots \partial \mu_{j'}}\bigg|_0 (\mu_N^j \times \tilde{\nu}_{N'}^{j'}) \left((\varphi \times \tilde{\varphi})(K) + \left(\sum_{i=1}^j \lambda_i A_N^{ij} \times \sum_{i'=1}^{j'} \mu_{i'} B_{N'}^{i'j'} \right) \right). \tag{4.9}$$

The claim then follows from Lemma 4.2.4, if we insert $M = (\varphi \times \varphi)(K)$, $\psi = \varphi_* \psi$ and $\tilde{p}_1 = \tilde{p}_2 = \varphi$ for (4.8) and $\psi = \tilde{\varphi}_* \psi$, $M = (\varphi \times \tilde{\varphi})(K)$ and $\tilde{p}_1 = \varphi, \tilde{p}_2 = \tilde{\varphi}$ for (4.9). We then obtain

$$\int_{x \in V} ((\varphi \circ \varphi^{-1})_* (\varphi_* \psi))(\varphi(K) \cap (x - A)) d\mu(x) =$$
$$\int_{x \in V} ((\varphi \circ \tilde{\varphi}^{-1})_* (\tilde{\varphi}_* \psi))(\varphi(K) \cap (x - A)) d\mu(x).$$

This is obviously true. Therefore let us show that we just have to consider expressions like (4.8) and (4.9). At first by [9], we have absolute convergence and smoothness of

$$\sum_{j,j'}^{n}\sum_{N,N'}^{\infty}\frac{\partial^{j+j'}}{\partial\lambda_1\dots\partial\lambda_j\partial\mu_1\dots\partial\mu_{j'}}\bigg|_0(\mu_N^j\times\nu_{N'}^{j'})\left((\varphi\times\varphi)(K)+\left(\sum_{i=1}^{j}\lambda_iA_N^{ij}\times\sum_{i'=1}^{j'}\mu_{i'}\tilde{B}_{N'}^{i'j'}\right)\right).$$

Therefore we can interchange sum and differentiation and obtain

$$\sum_{j}^{n}\sum_{N}^{\infty}\frac{\partial^{j}}{\partial\lambda_1\dots\partial\lambda_j}\sum_{j'}^{n}\sum_{N'}^{\infty}\frac{\partial^{j'}}{\partial\mu_1\dots\partial\mu_{j'}}\bigg|_0(\mu_N^j\times\nu_{N'}^{j'})\left((\varphi\times\varphi)(K)+\left(\sum_{i=1}^{j}\lambda_iA_N^{ij}\times\sum_{i'=1}^{j'}\mu_{i'}\tilde{B}_{N'}^{i'j'}\right)\right).$$

By Lemma 4.2.3, we can use the first part of the proof and conclude that (4.6) holds. ■

The previous statements and the preparations from the beginning of this section now justify the following definition:

Definition 4.2.6. We define the product of $\phi,\psi\in\mathcal{V}^\infty(M)$ locally for any open chart U by

$$(\phi\cdot\psi)|_U:=\phi|_U\cdot\psi|_U.$$

By compactness of $P\in\mathcal{P}(M)$, we can cover it with a finite number of maps and thus, by finite additivity, $(\phi\cdot\psi)(P)$ is well-defined.

To sum up, we get the following theorem:

Theorem 4.2.7. *The product from Definition 4.2.6* $\cdot:\mathcal{V}^\infty(M)\times\mathcal{V}^\infty(M)\to\mathcal{V}^\infty(M)$*,*

$$(\phi,\psi)\mapsto\phi\cdot\psi$$

is a continuous, commutative, and associative bilinear map with the Euler characteristic as unit and therefore $\mathcal{V}^\infty(M)$ *becomes an algebra.*

Lemma 4.2.8. *For any* $\phi,\psi\in\mathcal{V}^\infty(M)$*,*

$$\operatorname{supp}(\phi\cdot\psi)\subseteq\operatorname{supp}\phi\cap\operatorname{supp}\psi.$$

Proof. By Definition 5.2.3 we have $(\phi\cdot\psi)|_U=\phi|_U\cdot\psi|_U$. Hence for $p\in(\operatorname{supp}\phi)^c\cup(\operatorname{supp}\psi)^c$ there is U such that either $\phi|_U=0$ or $\psi|_U=0$. Therefore we get

$$(\phi\cdot\psi)|_U=\phi|_U\cdot\psi|_U=0.$$

It follows that $p\in\operatorname{supp}(\phi\cdot\psi)^c$ and this implies

$$\operatorname{supp}(\phi\cdot\psi)\subseteq(\operatorname{supp}\phi)\cap(\operatorname{supp}\psi).$$

■

Theorem 4.2.9. *We have*

$$W_i(M)\cdot W_j(M)\subseteq W_{i+j}(M)$$

Proof. This follows from the construction of the product and Theorem 4.1.2. ■

Definition 4.2.10. Let $S\subseteq M$. We denote the space of smooth valuations with $\operatorname{supp}\phi\subseteq S$ by $\mathcal{V}_S^\infty(M)$. We denote the space of valuations with compact support by $\mathcal{V}_c^\infty(M)$ and endow it with the topology of inductive limit of the spaces $\mathcal{V}_S^\infty(M)$ (with the relative topology), where S runs through all compact subsets of M.

Similar to the linear case, Proposition 2.1.27, we have the following statement. For a proof, we refer to [5, Theorem 3.1.2].

Lemma 4.2.11. *Let* M *be a smooth manifold. We have*

$$W_{i,c}(M)/W_{i+1,c}(M)\cong C_c^\infty(M,Val_i^\infty(TM)).$$

5 The dual space of $\mathcal{V}_c^\infty(M)$

In the following we are going to introduce the dual space of $\mathcal{V}_c^\infty(M)$. We will make use of the identifications obtained in Section 2.

5.1 Currents

By Remark 2.2.6, we have a correspondence between smooth valuations and smooth differential forms in terms of topological vector spaces. If we want to consider continuous linear functionals on $\mathcal{V}_c^\infty(M)$ and recall Lemma 2.2.4, the following definitions are of special interest. We follow [20, 5] and [8].

Definition 5.1.1. We set $\mathcal{D}_k(M) := \left(\Omega_c^k(M)\right)^*$. The elements of this space are called k-currents. The boundary of a k-current T is the $(k-1)$ current ∂T such that $\langle \partial T, \omega\rangle = \langle T, d\omega\rangle, \omega \in \Omega_c^{k-1}(X)$. If $\partial T = 0$, then T is called a cycle. The support of a current T is defined as

$$\operatorname{supp} T := M \setminus \{p \in M : \exists U \subseteq M \text{ open}, p \in U, T(\phi) = 0 \text{ with } \operatorname{supp}\phi \subseteq U\}.$$

Example 5.1.2. A very important example of a k-current is

$$\langle [[S]], \eta\rangle := \int_S \eta,$$

where S is a k-dimensional closed oriented submanifold of M. By Stoke's theorem, we have

$$\langle \partial[[S]], \eta\rangle = \langle [[S]], d\eta\rangle = \int_S d\eta = \int_{\partial S} \eta = \langle [[\partial S]], \eta\rangle$$

and hence $\partial[[S]] = [[\partial S]]$.
Moreover we can identify a smooth compactly supported differential form ν with a current by

$$\langle [[M]] \llcorner \nu, \eta\rangle := \langle [[M]], \nu \wedge \eta\rangle.$$

These kind of currents form a large subset of the space of currents.

We can define several useful operations on currents which are based on pull-backs and push-forwards:

Definition 5.1.3. Let M and N be smooth manifolds, let $T \in \mathcal{D}_k(M)$ and let $f : M \to N$ be a smooth map with $f|_{\operatorname{supp} T}$ proper. Then we can define the push-forward $f_*T \in \mathcal{D}_k(N)$ by $\langle f_*T, \eta\rangle := \langle T, \zeta f^*\eta\rangle$, where $\zeta \in C_c^\infty(M)$ is equal to 1 in a neighbourhood of $(\operatorname{supp} T) \cap (\operatorname{supp} f^*\eta)$ and f^* denotes the pullback of a k-form.

Definition 5.1.4. For $T \in \mathcal{D}_k(M)$ and $S \in \mathcal{D}_l(N)$ there is a unique current $T \boxtimes S \in \mathcal{D}_{k+l}(M \times N)$ such that

$$\langle T \boxtimes S, \pi_M^*\nu \wedge \pi_N^*\eta\rangle = \langle T, \nu\rangle\langle S, \eta\rangle, \tag{5.1}$$

where $\nu \in \Omega_c^k(M), \eta \in \Omega_c^l(N)$ and π_M, π_N are the projections from $M \times N$ to M and N, respectively.

Example 5.1.5. We want to compute the exterior product of smooth differential forms given by the identification of Example 5.1.2. Therefore let $\nu, \eta \in \Omega_c^n(M)$ and $\pi_1, \pi_2 : M \times M \to M$ the canonical projections. Since the product is uniquely defined by (5.1) we compute

$$\langle [[M]] \llcorner \nu \boxtimes [[M]] \llcorner \eta, \pi_1^* \omega \wedge \pi_2^* \phi \rangle = \langle [[M]] \llcorner \nu, \omega \rangle \langle [[M]] \llcorner \eta, \phi \rangle = \left(\int\limits_M \nu \wedge \omega \right) \left(\int\limits_M \eta \wedge \phi \right).$$

On the other hand we have

$$\begin{aligned} \langle [[M \times M]] \llcorner (\pi_1^* \nu \wedge \pi_2^* \eta), \pi_1^* \omega \wedge \pi_2^* \phi \rangle &= \int\limits_{M \times M} (\pi_1^* \nu \wedge \pi_2^* \eta) \wedge (\pi_1^* \omega \wedge \pi_2^* \phi) \\ &= \int\limits_{M \times M} \pi_1^* (\nu \wedge \omega) \wedge \pi_2^* (\eta \wedge \phi) \\ &= \left(\int\limits_M \nu \wedge \omega \right) \left(\int\limits_M \eta \wedge \phi \right). \end{aligned}$$

Therefore, we conclude that $[[M]] \llcorner \nu \boxtimes [[M]] \llcorner \eta = [[M \times M]] \llcorner (\pi_1^* \nu \wedge \pi_2^* \eta)$.

Remark 5.1.6. Let $\Lambda \subseteq T^*M \setminus 0$ be a closed conical set, in the sense that it is conical restricted to every tangent space T_pM for all $p \in M$. We set $\overline{\Lambda} := \Lambda \cup 0 \subseteq T^*M$.

The definition in Section 1 concerning the wave fronts carries over to the case of the dual space of k-differential forms.

Proposition 5.1.7. *Let T be a current. Then the following holds:*

1. *The wave front $WF(T)$ is a closed $\mathbb{R}_{>0}$-invariant subset of $T^*M \setminus 0$.*
2. *$WF(T) = \emptyset$ if and only if T is smooth.*

Let M be a smooth manifold and let $\Gamma \subseteq T^*M \setminus 0$ be a closed conical set. Then we set

$$\mathcal{D}_{k,\Gamma} := \{T \in \mathcal{D}_k(M) : WF(T) \subseteq \Gamma\}.$$

Let us equip this space with the topology described in Section 1.3. We obtain the following proposition, see [20, Theorem 8.2.3].

Proposition 5.1.8. *Given $T \in \mathcal{D}_{k,\Gamma}(M)$, there exists a sequence of compactly supported k-currents $(\omega_j)_{j \in \mathbb{N}} \in \Omega_c^{n-k}(M)$ such that*

$$([[M]] \llcorner \omega_j)_{j \in \mathbb{N}} \to T \quad in \quad \mathcal{D}_{k,\Gamma}(M)$$

and hence we can identify compactly supported smooth forms with a dense subspace of $\mathcal{D}_{k,\Gamma}(M)$.

The operations that were introduced at the beginning of this section are continuous maps under some assumptions on the wave fronts.

Proposition 5.1.9. *Let M, N be smooth manifolds, $\Gamma_1 \subseteq T^*M \setminus 0, \Gamma_2 \subseteq T^*N \setminus 0$ closed conical subsets. Set $\bar{\Gamma} := \bar{\Gamma}_1 \times \bar{\Gamma}_2 \subseteq T^*(M \times N)$. Then the exterior product is a jointly sequentially continuous bilinear map*

$$\boxtimes : \mathcal{D}_{k,\Gamma_1}(M) \times \mathcal{D}_{l,\Gamma_2}(N) \to \mathcal{D}_{k+l,\Gamma}(M \times N).$$

Proposition 5.1.10. *Let $f : M \to N$ be a smooth and proper map between smooth manifolds and $\Gamma \subseteq T^*X \setminus 0$ be a closed conical set. Let*

$$f_*\Gamma := \{(q,\eta) \in T^*N \setminus 0 : \exists p \in f^{-1}(q), (p, df_p^*(\eta)) \in \Gamma\}.$$

Then the push-forward map f_* of currents is a sequentially continuous map

$$f_* : \mathcal{D}_{k,\Gamma}(M) \to \mathcal{D}_{k,f_*\Gamma}(N).$$

Proposition 5.1.11. *Let $f : M \to N$ be a smooth map between smooth manifolds, where M may have a boundary B. Let $\Gamma \subseteq T^*N \setminus 0$ be a closed conical set satisfying the following transversality conditions:*

1. *If $p \in M, (f(p), \eta) \in \Gamma$, then $df_p^*(\eta) \neq 0$.*
2. *If $p \in B, (f(p), \eta) \in \Gamma$, then $(df|_B)_p^*(\eta) \neq 0$.*

Define

$$f^*\Gamma := \{(p, df_p^*(\eta)) \in T^*M \setminus 0 : p \in M, (f(p), \eta) \in \Gamma\}.$$

Then there exists a unique sequentially continuous map

$$f^* : \mathcal{D}_\Gamma(N) \to \mathcal{D}_{(f^*\bar{\Gamma} + N_M^* B)\setminus\{0\}}(M)$$

extending the pull-back of smooth forms, i.e., $\langle f^*[[N]]\llcorner\nu, \omega\rangle = \langle [[M]]\llcorner f^*\nu, \omega\rangle$.

In other words Condition 2 of Proposition 5.1.11 reads

$$f^*\Gamma \cap N_M^* B = \emptyset.$$

According to the definition of the pull-back and push-forward, we have

$$WF(f^*T) \subseteq f^*\Gamma \quad \text{and} \quad WF(f_*T) \subseteq f_*\Gamma, \tag{5.2}$$

for $T \in \mathcal{D}_{k,\Gamma}$ and a smooth map satisfying 1 and 2.

Example 5.1.12. Let $F : M \to N$ be a diffeomorphism and let $\Phi : M \to O$ be a smooth submersion. We want to give a description of $F_*\Phi^*\Gamma$, for a closed conical set $\Gamma \subseteq T^*N \setminus 0$. Elements of $\Phi^*\Gamma$ are of the form $(p, d\Phi_p^*(\eta))$ and elements of $F_*\Gamma'$ are of the form (y, η) with $dF^*_{F^{-1}(y)}(\eta) \in \Gamma'$, we obtain that elements of $F_*\Phi^*\Gamma$ satisfy

$$dF^*_{F^{-1}(y)}(\xi) = d\Phi_y^*(\eta) \Rightarrow \xi = (dF^{-1})^*_{F^{-1}(y)} d\Phi_y^*(\eta) = d(\Phi \circ F^{-1})^*_{F^{-1}(y)}(\eta),$$

with $(p, \eta) \in \Gamma$.

To define the next operation, let $\Delta : M \to M \times M$ be the diagonal embedding and let T_1, T_2 be currents and. Then, the intersection of currents is the bilinear map defined by

$$T_1 \cap T_2 := \Delta^*(T_1 \boxtimes T_2),$$

whenever these expressions exist.

Proposition 5.1.13. *Let M be a smooth manifold of dimension n and $\Gamma_1, \Gamma_2 \subseteq T^*M \setminus 0$ be closed conical sets such that the following transversality condition is satisfied:*

$$\Gamma_1 \cap s\Gamma_2 = \emptyset.$$

Set

$$\overline{\Gamma} := \overline{\Gamma}_1 + \overline{\Gamma}_2 = \{(p, \xi_1 + \xi_2) : (p, \xi_1) \in \overline{\Gamma}_1, (p, \xi_2) \in \overline{\Gamma}_2\}.$$

Then the intersection is a jointly sequentially continuous map

$$\cap : \mathcal{D}_{k,\Gamma_1}(M) \times \mathcal{D}_{l,\Gamma_2}(M) \to \mathcal{D}_{k+l-n,\Gamma}(M).$$

Proof. Since the intersection is defined as the pull-back of the diagonal-imbedding Δ we have to consider $f = \Delta$ in Proposition 5.1.11. To verify the transversality condition, let $p \in M$ and $(\Delta(p), \eta) \in \Gamma_1$. By identifying $T_{(p,p)}(M \times M)$ with $T_pM \oplus T_pM$ via the map $\alpha(v) = (d(\pi_1)_{(p_1,p_2)}(v), d(\pi_2)_{(p_1,p_2)}(v))$, we get

$$\alpha(d\Delta_p(v)) = (d(\pi_1)_{(p,p)}(d\Delta_p(v)), d(\pi_1)_{(p,p)}(d\Delta_p(v))) = (v, v)$$

and, furthermore, by the identification $T^*_{(p,p)}(M \times M) = T^*_pM \oplus T^*_pM$, we have

$$d\Delta^*_p(\eta, \xi)v = (\eta, \xi)(v, v) = \eta v + \xi v. \tag{5.3}$$

Now, by our assumption, $\eta \neq -\xi$, for $\eta \in \Gamma_1$ and $\xi \in \Gamma_2$. Hence, $\Delta^*(T_1 \boxtimes T_2)$ is well defined. The corresponding wave fronts satisfy

$$\overline{WF}(\Delta^*(T_1 \boxtimes T_2)) \subseteq \Delta^*\overline{WF}(T_1 \boxtimes T_2) \subseteq \Delta^*(\overline{WF}(T_1) \times \overline{WF}(T_2))$$

The computations in (5.3) show that this yields $\overline{WF}(\Delta^*(T_1 \boxtimes T_2) \subseteq \overline{\Gamma}$.Thus the claim follows. ∎

The support of the current $T_1 \cap T_2$ has the following property

$$\operatorname{supp}(T_1 \cap T_2) = \operatorname{supp}(\Delta^*(T_1 \boxtimes T_2)) \subseteq \Delta^{-1}(\operatorname{supp} T_1 \times \operatorname{supp} T_2)) \subseteq \operatorname{supp} T_1 \cap \operatorname{supp} T_2.$$

Hence, we have $\operatorname{supp}(T_1 \cap T_2) \subseteq \operatorname{supp} T_1 \cap \operatorname{supp} T_2$.

Proposition 5.1.14. *Let $S \subseteq M$ be a compact oriented k-dimensional submanifold and $[[S]]$ the k-dimensional current of integration against S. Then*

$$\overline{WF}([[S]]) = N^*_M S,$$

where $N^*_M S$ is the conormal bundle from Example 1.2.8.

For a proof of the following theorem, we refer to [20, Example 8.2.5].

Proposition 5.1.15. *Let T be a k-dimensional current on M and ω an l-form on M. Then*

$$WF(T \llcorner \phi) \subseteq WF(T).$$

5.2 The space of generalized valuations

5.2.1 Embedding of smooth valuations

Definition 5.2.1. Define the space of generalized valuations by

$$\mathcal{V}^{-\infty}(M) := (\mathcal{V}^\infty_c(M))^*$$

equipped with the weak topology on the dual space.

First we can observe that $\mathcal{V}^\infty(M)$ is densely embedded into $\mathcal{V}^{-\infty}(M)$. This is a consequence of Lemma 4.2.8.

Definition 5.2.2. Let M be a smooth manifold and fix a compact subset S. Let us choose a compact subset S' with smooth boundary and such that S is contained in the interior of S'. Then $S' \in \mathcal{P}(M)$. For any $\phi \in \mathcal{V}^\infty_S(M)$ we define

$${}^S\!\!\int \phi := \phi(S').$$

The map ${}^{S}\!\int : \mathcal{V}^{\infty}_{S}(M) \to \mathbb{R}$ is a continuous linear functional. Moreover for fixed S the definition is independent of S' containing S. To see this let us take another compact subset with smooth boundary containing S in the interior, denoted by S''. This leaves us to show that $\phi(S) = \phi(S'')$. Using the additivity property we get

$$\phi(S'') = \phi(S') + \phi(\overline{S'' \setminus S'}) - \phi(\partial S).$$

Now since $S \subseteq \operatorname{int} S'$ and by definition $\operatorname{supp} \phi \subseteq S$, we have

$$\phi(S') + \phi(\overline{S'' \setminus S'}) - \phi(\partial S) = \phi(S')$$

and thus the claim holds. This motivates the following definition.

Definition 5.2.3. Let M be a smooth manifold. Then we define

$$\int : \begin{cases} \mathcal{V}^{\infty}_{c}(M) \to \mathbb{R} \\ \phi \mapsto \int \phi \end{cases}$$

the integration functional on $\mathcal{V}^{\infty}_{c}(M)$, which acts as ${}^{S}\!\int$ for $\phi \in \mathcal{V}^{\infty}_{S}(M)$ for any compact subset S of M.

As a consequence of Lemma 4.2.8, for $\phi \in \mathcal{V}^{\infty}_{c}(M)$ and $\psi \in \mathcal{V}^{\infty}(M)$, we have that $\operatorname{supp}(\phi \cdot \psi)$ is compact and, hence, $pd : \mathcal{V}^{\infty}(M) \times \mathcal{V}^{\infty}_{c}(M) \to \mathbb{R}$

$$(\phi, \psi) \mapsto \int \phi \cdot \psi. \tag{5.4}$$

is well-defined.

Theorem 5.2.4.

- *For any $\phi \in W_i(M) \setminus W_{i+1}(M)$, there exists $\psi \in W_{n-i,c}(M)$ such that $\int \phi \cdot \psi \neq 0$.*
- *For any $\phi \in W_{i,c}(M) \setminus W_{i+1,c}(M)$ there exists $\psi \in W_{n-i}(M)$ such that $\int \phi \cdot \psi \neq 0$.*

Proof. The proof follows by considering Theorem 4.2.9 and the representation in Lemma 4.2.11. ■

Theorem 5.2.5. *Consider the bilinear form from (5.4). It is a perfect pairing. More precisely, the induced map $\iota : \mathcal{V}^{\infty}(M) \to \mathcal{V}^{-\infty}(M)$ given by*

$$\phi \mapsto \iota(\phi) = [\psi \mapsto pd(\phi)(\psi)] \tag{5.5}$$

is injective and has dense image in $\mathcal{V}^{-\infty}(M)$.

Proof. Let $S = \operatorname{cl}(\iota(\mathcal{V}^{\infty}(M))) \subseteq \mathcal{V}^{-\infty}(M)$. By a corollary of the Hahn-Banach theorem we get a linear homeomorphism

$$(\mathcal{V}^{-\infty}(M)/S)^{*} \to S^{\perp}.$$

This leads us to determine $S^{\perp}$. We have

$$\begin{aligned} S^{\perp} &= \{\omega \in \mathcal{V}^{-\infty}(M) : \langle \phi, \omega \rangle = 0, \phi \in \mathcal{V}^{\infty}(M)\} \\ &= \left\{\omega \in \mathcal{V}^{-\infty}(M) : \int \omega \cdot \phi = 0, \phi \in \mathcal{V}^{\infty}(M)\right\} = \{0\}, \end{aligned}$$

since the map ι is injective by Theorem 5.2.5. Therefore, $S = \operatorname{cl}(\iota(\mathcal{V}^{\infty}(M))) = \mathcal{V}^{-\infty}(M)$. ■

For $\psi \in \mathcal{V}^{-\infty}(M)$, we define the support of ψ by

$$\operatorname{supp}\psi := M \setminus \{p \in M : \exists U \subseteq M \text{ open}, p \in U, \psi(\phi) = 0,\ \forall \phi \text{ with } \operatorname{supp}\phi \subseteq U\}.$$

Moreover, let $S \subseteq M$. We denote the space of generalized valuations with support contained in S by $\mathcal{V}_S^{-\infty}(M)$ and by $\mathcal{V}_c^{-\infty}(M)$ the space of generalized valuations with compact support. We endow $\mathcal{V}_c^{-\infty}(M)$ with the topology of inductive limit when each space $\mathcal{V}_S^{-\infty}(M)$, with S compact, is endowed with the relative topology on $\mathcal{V}^{-\infty}(M)$. The following statement can be found in [5].

Proposition 5.2.6. *The integration functional*

$$\int : \mathcal{V}_c^{\infty}(M) \to \mathbb{R}$$

extends uniquely by continuity to a functional

$$\int : \mathcal{V}_c^{-\infty}(M) \to \mathbb{R}$$

Proof. For any $\alpha \in \mathcal{V}_c^{\infty}(M)$ we obtain a functional $\alpha : \mathcal{V}^{\infty}(M) \to \mathbb{R}$ defined by

$$\phi \mapsto \int \alpha \cdot \phi$$

This is possible since the support of $\alpha \cdot \phi$ is compact for any $\phi \in \mathcal{V}^{\infty}(M)$. Furthermore we can extend this functional to a map $\hat{\alpha} : \mathcal{V}^{-\infty}(M) \to \mathbb{R}$ by continuity, by

$$\hat{\alpha}(\phi) := \lim_{j\to\infty} \alpha(\phi_j)$$

for any sequence with $\phi_j \in \mathcal{V}^{\infty}(M)$ converging to ϕ in $\mathcal{V}^{-\infty}(M)$. To obtain a functional on $\mathcal{V}_c^{-\infty}(M)$ let us fix an arbitrary compact subset $S \subseteq M$ and a smooth compactly supported valuation $\alpha \in \mathcal{V}_c^{\infty}(M)$ such that α equals the Euler chracteristic in a neighbourhood of S. We claim that the restriction of $\hat{\alpha}$ to $\mathcal{V}_S^{-\infty}(M)$ is the desired extension of the integration functional to $\mathcal{V}_S^{-\infty}(M)$. To check this, let us fix a compact neighbourhood S' of S such that the restriction of α to S' is still equal to the Euler characteristic. By [5, Proposition 7.3.4], every valuation from $\mathcal{V}_S^{-\infty}(M)$ can be approximated by a net from $\mathcal{V}_{S'}^{\infty}(M)$. Hence, it is enough to check that for any $\phi \in \mathcal{V}_{S'}^{\infty}(M)$ one has

$$\int \alpha \cdot \phi = \int \phi.$$

This follows since $\phi \cdot (\alpha - \chi) = 0$. ■

5.2.2 Generalized valuations as pairs of currents

Due to the representation of a smooth valuation as a pair of differential forms, we obtain an identification of $\mathcal{V}^{-\infty}(M)$ with a space of currents. Together with Remark 2.2.6 we obtain the following result. A proof of this theorem can be found in [7, Section 8].

Proposition 5.2.7. *The space $\mathcal{V}^{-\infty}(M)$ of generalized valuations is in one to one correspondence with the space of pairs of currents $(C, T) \in \mathcal{D}_n(M) \times \mathcal{D}_{n-1}(\mathbb{P}_M)$ such that T is a Legendrian cylce and $\pi_* T = \partial C$.*

With the duality given in Proposition 5.2.7, we obtain for $\psi \in \mathcal{V}^{-\infty}(M)$ represented by the pair of currents $(C, T) \in \mathcal{D}_n(M) \times \mathcal{D}_{n-1}(\mathbb{P}_M)$ and $\phi \in \mathcal{V}_c^{\infty}(M)$ represented by the pair of compactly supported differential forms $(\nu, \eta) \in \Omega_c^n(M) \times \Omega_c^{n-1}(\mathbb{P}_M)$,

$$\langle \psi, \phi \rangle := \langle C, \nu \rangle + \langle T, \eta \rangle$$

and hence

$$\begin{gathered} \Omega_c^n(M) \oplus \Omega_c^{n-1}(\mathbb{P}_M)/\ker \tilde{\Psi} \xleftrightarrow{\quad \Psi \quad} \mathcal{V}_c^{\infty}(M), \\ (\Omega_c^n(M) \oplus \Omega_c^{n-1}(\mathbb{P}_M)/\ker \tilde{\Psi})^* \xleftrightarrow{\Psi^*} \mathcal{V}^{-\infty}(M) = (\ker \tilde{\Psi})^{\perp}. \end{gathered} \tag{5.6}$$

Here $(\ker \tilde{\Psi})^{\perp}$ denotes the set $\{(C, T) \in \mathcal{D}_n(M) \times \mathcal{D}_{n-1}(\mathbb{P}_M) : \langle \nu, C \rangle + \langle \eta, T \rangle = 0, (\nu, \eta) \in \ker \tilde{\Psi}\}$. We can identify submanifolds with corners as generalized valuations by an injection which has dense image due to the weak topology:

$$\begin{aligned} \mathcal{P}(M) &\to \mathcal{V}^{-\infty}(M), \\ P &\mapsto [\mu \mapsto \mu(P)]. \end{aligned}$$

Therefore it is useful to describe $P \in \mathcal{P}(M)$ and $\phi \in \mathcal{V}^{\infty}(M)$ in terms of currents, where we again use the identification of ϕ with (ν, η):

$$P \leftrightarrow ([[P]], [[N(P)]])\,, \tag{5.7}$$

$$(\nu, \eta) \leftrightarrow ([[M]] \llcorner \pi_{M*}\eta, [[\mathbb{P}_M]] \llcorner s^*(D\eta + \pi_M^* \nu))\,, \tag{5.8}$$

where π_M is the projection $\pi_M : \mathbb{P}_M \to M$. For a proof we refer to [7, Theorem 2].

6 Convolution of valuations on manifolds

To introduce a convolution that acts similar to the one we introduced in Section 3 we need some operations on generalized valuations. Among these operations we will get an extension of the product introduced in Section 4. Recall that we established generalized valuations as the dual space of compactly supported smooth valuations and hence we identified them with pairs of currents. This strategy will become important in the upcoming section. We follow [6, 7] and [8].

6.1 Operations on generalized valuations

To define an extension of the convolution we need the exterior product of generalized valuations. Therefore, let M_1, M_2 be smooth manifolds without boundary of dimensions n_1, n_2 and set $M := M_1 \times M_2$. Moreover we define

$$\mathcal{M}_1 := \mathbb{P}_{M_1} \times M_2 = \{(p_1, p_2, [\xi_1 : 0]), p_1 \in M_1, p_2 \in M_2, \xi_1 \in T^*_{p_1} M_1 \setminus \{0\}\} \subseteq \mathbb{P}_M,$$
$$\mathcal{M}_2 := M_1 \times \mathbb{P}_{M_2} = \{(p_1, p_2, [0 : \xi_2]), p_1 \in M_1, x_2 \in M_2, \xi_2 \in T^*_{p_2} M_2 \setminus \{0\}\} \subseteq \mathbb{P}_M.$$

Let us consider the fiber bundle A over M, given by

$$A := \coprod_{(p_1,p_2)\in M} \{(p_1, p_2, [\xi_1 : \xi_2], [\xi_1'], [\xi_2']) : [\xi_1 : \xi_2] \in \mathbb{P}_+ T^*_{(p_1,p_2)} M, [\xi_1'] \in \mathbb{P}_+ T^*_{p_1} M_1, [\xi_2'] \in \mathbb{P}_+ T^*_{p_2} M_2\}.$$

We set

$$\hat{\mathbb{P}}_M := \mathrm{cl}\left(\coprod_{(p_1,p_2)\in M} \left\{(p_1, p_2, [\xi_1 : \xi_2], [\xi_1'], [\xi_2']) \in A_{(p_1,p_2)} : \xi_1, \xi_2 \neq 0, [\xi_1] = [\xi_1'], [\xi_2] = [\xi_2']\right\}\right) \subseteq A.$$

$\hat{\mathbb{P}}_M$ is a manifold of dimension $2(n_1 + n_2) - 1$ with boundary $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, where

$$\mathcal{N}_1 = \coprod_{(p_1,p_2)\in M} \{(p_1, p_2, [\xi_1 : 0], [\xi_1], [\eta_2]) : \xi_1, \eta_2 \neq 0\} \subseteq \hat{\mathbb{P}}_M,$$
$$\mathcal{N}_2 = \coprod_{(p_1,p_2)\in M} \{(p_1, p_2, [0 : \xi_2], [\eta_1], [\xi_2]) : \eta_1, \xi_2 \neq 0\} \subseteq \hat{\mathbb{P}}_M.$$

Then we define $F : \hat{\mathbb{P}}_M \to \mathbb{P}_M$ by

$$(p_1, p_2, [\xi_1 : \xi_2], [\xi_1'], [\xi_2']) \mapsto (p_1, p_2, [\xi_1 : \xi_2]).$$

We see that $F(\mathcal{N}) = \mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathbb{P}_M$. Next we define $\Phi : \hat{\mathbb{P}}_M \to \mathbb{P}_{M_1} \times \mathbb{P}_{M_2}$ by

$$(p_1, p_2, [\xi_1 : \xi_2], [\xi_1'], [\xi_2']) \mapsto ((p_1, [\xi_1]), (p_2, [\xi_2])).$$

We also need the projections

$$\begin{aligned} p_1 &: \mathbb{P}_{M_1} \times M_2 \to \mathbb{P}_{M_1} \\ p_2 &: M_1 \times \mathbb{P}_{M_2} \to \mathbb{P}_{M_2} \\ \tilde{p}_i &: M_1 \times M_2 \to M_i, i = 1, 2 \\ q_i &: \mathbb{P}_{M_1} \times \mathbb{P}_{M_2} \to \mathbb{P}_{M_i}, i = 1, 2. \end{aligned}$$

Finally let $i_1 : \mathbb{P}_{M_1} \times M_2 \to \mathbb{P}_M$ and $i_2 : M_1 \times \mathbb{P}_{M_2} \to \mathbb{P}_M$ be defined by $i_1(p_1, [\xi_1], p_2) = (p_1, p_2, [\xi : 0])$ and $i_2(p_1, p_2, [\xi_2]) = (p_1, p_2, [0 : \xi_2])$. Then we obtain $\mathcal{M} = \operatorname{im} i_1 \cup \operatorname{im} i_2$.
We will make use of the following diagram:

$$
\begin{tikzcd}
\mathbb{P}_{M_1} & \mathbb{P}_{M_1} \times \mathbb{P}_{M_2} \arrow[l, "q_1"'] \arrow[r, "q_2"] & \mathbb{P}_{M_2} \\
 & \hat{\mathbb{P}}_M \arrow[u, "\Phi"] \arrow[d, "F"] & \\
\mathbb{P}_{M_1} \times M_2 \arrow[r, "i_1"] \arrow[d, "p_1"] & \mathbb{P}_M \arrow[dd, "\pi_M"] & M_1 \times \mathbb{P}_{M_2} \arrow[l, "i_2"'] \arrow[d, "p_2"] \\
\mathbb{P}_{M_1} \arrow[d, "\pi_{M_1}"] & & \mathbb{P}_{M_2} \arrow[d, "\pi_{M_2}"] \\
M_1 & M_1 \times M_2 \arrow[l, "q_1"'] \arrow[r, "q_2"] & M_2
\end{tikzcd}
\tag{6.1}
$$

Definition 6.1.1 (Exterior product of generalized valuations)**.** Let $\psi_i \in \mathcal{V}^{-\infty}(M)$ be given by pairs of currents (C_i, T_i). Then the generalized valuation $\psi_1 \boxtimes \psi_2 \in \mathcal{V}^{-\infty}(M)$ is defined by the pair of currents

$$
\begin{aligned}
C &:= \; C_1 \boxtimes C_2, &(6.2)\\
T &:= \; F_* \Phi^*(T_1 \boxtimes T_2) + (\tilde{p}_1 \circ \pi_M)^* C_1 \cap (i_{2*} p_2^* T_2) + (i_{1*} p_1^* T_1) \cap (\tilde{p}_2 \circ \pi_M)^* C_2. &(6.3)
\end{aligned}
$$

For a proof of the following continuity statement, we refer to [6, Claim 2.1.9].

Proposition 6.1.2. *The exterior product defined in* (6.2) *and* (6.3) *defines a bilinear jointly sequentially continuous map*

$$
\boxtimes : \begin{cases} \mathcal{V}^{-\infty}(M_1) \times \mathcal{V}^{-\infty}(M_2) \to \mathcal{V}^{-\infty}(M) \\ (\psi_1, \psi_2) \mapsto \psi_1 \boxtimes \psi_2. \end{cases}
$$

From the above definition we obtain

$$
\operatorname{supp}(\psi_1 \boxtimes \psi_2) \subseteq \operatorname{supp}(\psi_1) \times \operatorname{supp}(\psi_2). \tag{6.4}
$$

To see this, we examine each current separately. We are going to show that $(\operatorname{supp} C_1 \times \operatorname{supp} C_2)^c \subseteq (\operatorname{supp} C)^c$. The argument for T follows in a similar way. Let us take $(p_1, p_2) \in (\operatorname{supp} C_1 \times \operatorname{supp} C_2)^c$. Then, without loss of generality, there is an open neighbourhood U_1 of p_1 with $C_1(\nu) = 0$ for all $\nu \in \Omega_c^n(M)$ with $\operatorname{supp} \nu \subseteq U_1$. The subset of $(n_1 + n_2)$-forms of $\Omega_c^{n_1+n_2}(M_1 \times M_2)$ spanned by $\pi_{M_1}^* \nu_1 \wedge \pi_{M_2}^* \nu_2$ with $\nu_i \in \Omega_c^{n_i}(M_i)$ is dense. By Definition 5.1.4, we get

$$
\langle C_1 \boxtimes C_2, \pi_{M_1}^* \nu_1 \wedge \pi_{M_2}^* \nu_2 \rangle = \langle C_1, \nu_1 \rangle \langle C_2, \nu_2 \rangle = 0
$$

with $\operatorname{supp}(\nu_1) \subseteq U_1$ and hence by linearity and continuity, we get $(p_1, p_2) \in (\operatorname{supp} C)^c$.

Proposition 6.1.3. *Let M_1, M_2 be smooth manifolds and let $\Lambda_i \subseteq T^*M_i \setminus 0, \Gamma_i \subseteq T^*\mathbb{P}_{M_i} \setminus 0$ be closed conical sets. Set*

$$
\begin{aligned}
\bar{\Lambda} &:= \; \bar{\Lambda}_1 \times \bar{\Lambda}_2 \subseteq T^*(M),\\
\bar{\Gamma} &:= \; F_*(\Phi^*(\bar{\Gamma}_1 \times \bar{\Gamma}_2) + N^*_{\hat{\mathbb{P}}_M} \mathcal{N}) \cup [(\tilde{p}_1 \circ \pi_M)^* \bar{\Lambda}_1 + i_{2*} p_2^* \bar{\Gamma}_2] \cup [i_{1*} p_1^* \bar{\Gamma}_1 + (\tilde{p}_2 \circ \pi_M)^* \bar{\Lambda}_2] \subseteq T^*\mathbb{P}_M
\end{aligned}
$$

Then the exterior product given by (6.2) *and* (6.3) *is a jointly sequentially continuous map*

$$
\mathcal{V}^{-\infty}_{\Lambda_1, \Gamma_1}(M_1) \times \mathcal{V}^{-\infty}_{\Lambda_2, \Gamma_2}(M_2) \to \mathcal{V}^{-\infty}_{\Lambda, \Gamma}(M).
$$

Proof. We have to verify the conditions from Section 5.1. Since $\mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M_1 \times M_2)$ is a vector space, we can consider each term of the sum in (6.2) and (6.3) separately:

- $WF(C_1 \boxtimes C_2) \subseteq \Gamma_1 \times \Gamma_2$ due to Proposition 5.1.9.
- By 5.2 and Proposition 5.1.10, we get that
$$WF(F_*(\Phi^*(T_1 \boxtimes T_2)) \subseteq F_*(WF(\Phi^*(T_1 \boxtimes T_2))).$$
Since $\hat{\mathbb{P}}_M$ has boundary $\mathcal{N}$, we obtain by Proposition 5.1.11,
$$WF(\Phi^*(T_1 \boxtimes T_2)) \subseteq \Phi^* WF(T_1 \boxtimes T_2) + N^*_{\hat{\mathbb{P}}_M}\mathcal{N} \subseteq \Phi^*(\Gamma_1 \times \Gamma_2) + N^*_{\hat{\mathbb{P}}_M}\mathcal{N}$$
and, to sum up, we get,
$$WF(F_*(\Phi^*(T_1 \boxtimes T_2)) \subseteq F_*(\Phi^* WF(T_1 \boxtimes T_2) + N^*_{\hat{\mathbb{P}}_M}\mathcal{N}) \subseteq F_*(\Phi^*(\Gamma_1 \times \Gamma_2) + N^*_{\hat{\mathbb{P}}_M}\mathcal{N}).$$
- Considering $(\tilde{p}_1 \circ \pi_M)^* C_1 \cap (i_{2*} p_2^* T_2)$, Proposition 5.1.13 yields
$$WF((\tilde{p}_1 \circ \pi_M)^* C_1 \cap (i_{2*} p_2^* T_2)) \subseteq WF((\tilde{p}_1 \circ \pi_M)^* C_1) + WF((i_{2*} p_2^* T_2)).$$
and, furthermore, we have
$$WF((\tilde{p}_1 \circ \pi_M)^* C_1) \subseteq (\tilde{p}_1 \circ \pi_M)^* WF(C_1) \subseteq (\tilde{p}_1 \circ \pi_M)^* \Lambda_1$$
and
$$WF((i_{2*} p_2^* T_2)) \subseteq i_{2*}(WF(p_2^* T_2)) \subseteq i_{2*}(p_2^* WF(T_2)) \subseteq i_{2*}(p_2^* \Gamma_2)$$
and, thus,
$$WF((\tilde{p}_1 \circ \pi_M)^* C_1 \cap (i_{2*} p_2^* T_2)) \subseteq (\tilde{p}_1 \circ \pi_M)^* \bar{\Lambda}_1 + i_{2*}(p_2^* \bar{\Gamma}_2).$$
- At last, we examine $(i_{1*} p_1^* T_1)) \cap (\tilde{p}_2 \circ \pi_M)^* C_2$. Similar to the above arguments we get
$$WF((i_{1*} p_1^* T_1)) \cap (\tilde{p}_2 \circ \pi_M)^* C_2) \subseteq [(\tilde{p}_1 \circ \pi_M)^* \bar{\Lambda}_1 + i_{2*} p_2^* \bar{\Gamma}_2] \cup [i_{1*} p_1^* \bar{\Gamma}_1 + (\tilde{p}_2 \circ \pi_M)^* \bar{\Lambda}_2].$$

Now the claim follows. ■

To define another operation on generalized valuations, we need the following preperations. The upcoming proposition then yields a dense embedding of smooth valuations into generalized valuations with wave fronts as specified below. Therefore, we consider the fiber bundle over a smooth manifold M consisting of tuples

$$B := \coprod_{p \in M} \left\{(p, [\xi : \eta], [\xi'], [\eta'], [\zeta]) : [\xi : \eta] \in \mathbb{P}_+(T^*_{(p,p)}(M \times M)), [\xi'], [\eta'], [\zeta] \in \mathbb{P}_+(T^*_p M)\right\},$$

and define

$$\bar{\mathbb{P}} := \mathrm{cl}\left(\coprod_{p \in M} \{(p, [\xi : \eta], [\xi'], [\eta'], [\zeta]) \in B : p \in M, \xi, \eta, \xi + \eta \neq 0, [\xi'] = [\xi], [\eta'] = [\eta], [\xi + \eta] = [\zeta]\}\right).$$

$\bar{\mathbb{P}}$ is a $(3n-1)$-dimensional manifold whose boundary $\bar{\mathcal{N}}$ consists of the three manifolds

$$\begin{aligned}
\bar{\mathcal{N}}_0 &:= \{(p, [\xi : -\xi'], [\xi], [-\xi], [\zeta]) : p \in M, \xi, \zeta \in T^*_p M \setminus \{0\}\}, \\
\bar{\mathcal{N}}_1 &:= \{(p, [\xi : 0], [\xi], [\eta'], [\xi]) : p \in M, \xi, \eta' \in T^*_p M \setminus \{0\}\}, \\
\bar{\mathcal{N}}_2 &:= \{(p, [0 : \eta], [\xi'], [\eta], [\eta]) : p \in M, \eta, \xi' \in T^*_p M \setminus \{0\}\}.
\end{aligned}$$

Let $\bar{\Phi} : \bar{\mathbb{P}} \to \mathbb{P}_M \times_M \mathbb{P}_M$ be the map, defined by

$$(p, [\xi : \eta], [\xi'], [\eta'], [\zeta]) \mapsto ((p, [\xi']), (p, [\eta']))$$

and let $\bar{p} : \bar{\mathbb{P}} \to \mathbb{P}_M$ be defined by

$$(p, [\xi : \eta], [\xi'], [\eta'], [\zeta]) \mapsto (p, [\zeta]).$$

Hence, we get the following diagram

$$\begin{tikzcd}
 & \mathbb{P}_M \arrow[dr, "\pi_M"] & \\
 & \bar{\mathbb{P}} \arrow[u, "\bar{p}"'] \arrow[r, "\bar{\pi}"] \arrow[d, "\bar{\Phi}"] & M \\
\mathbb{P}_M & \mathbb{P}_M \times_M \mathbb{P}_M \arrow[l, "q_1"'] \arrow[r, "q_2"] & \mathbb{P}_M \arrow[u, "\pi_M"']
\end{tikzcd}, \tag{6.5}$$

with q_1, q_2, π_M and $\bar{\pi}$ being the corresponding projections. This construction is taken from [8]. Together with the tools developed at the beginning of this section, we can give a representation of the product defined in Section 4 by identifying smooth valuations with differential forms. For a proof we refer to [7, Theorem 2].

Proposition 6.1.4. *Let M be an n-dimensional smooth oriented manifold and let $\phi_1, \phi_2 \in \mathcal{V}^\infty(M)$ be represented by $(\nu_1, \eta_1), (\nu_2, \eta_2) \in \Omega^n(M) \times \Omega^{n-1}(\mathbb{P}_M)$. Then the product $\phi_1 \cdot \phi_2$ is represented by*

$$\begin{aligned}
\nu &= \pi_{M*}(\eta_1 \wedge s^*(D\eta_2 + \pi_M^*\nu_2)) + \nu_1 \wedge \pi_{M*}\eta_2 \in \Omega(M), \\
\eta &= \bar{p}_*\bar{\Phi}^*(q_1^*\eta_1 \wedge q_2^* D\eta_2) + \eta_1 \wedge \pi_M^*\pi_{M*}\eta_2 \in \Omega(\mathbb{P}_M).
\end{aligned}$$

Moreover, we have

$$\pi_{M*}\eta = \pi_{M*}\eta_1 \wedge \pi_{M*}\eta_2, \tag{6.6}$$

$$D\eta + \pi^*\nu = \bar{p}_*\bar{\Phi}^*(q_1^*(D\eta_1 + \pi_M^*\nu_1) \wedge q_2^*(D\eta_2 + \pi_M^*\nu_2)) + \tag{6.7}$$

$$\pi_M^*\pi_{M*}\eta_1 \wedge (D\eta_2 + \pi_M^*\nu_2) + \pi_M^*\pi_{M*}\eta \wedge (D\eta_1 + \pi_M^*\nu_1). \tag{6.8}$$

Proposition 6.1.4 motivates the following definition:

Definition 6.1.5. Let $\psi_1 \in \mathcal{V}^{-\infty}_{\Lambda_1,\Gamma_1}$ and $\psi_2 \in \mathcal{V}^{-\infty}_{\Lambda_2,\Gamma_2}$ and $(C_1, T_1), (C_2, T_2)$ be pairs of the corresponding currents. Then we define a bilinear map

$$\cdot : \mathcal{V}^{-\infty}_{\Lambda_1,\Gamma_1}(M) \times \mathcal{V}^{-\infty}_{\Lambda_2,\Gamma_2}(M) \to \mathcal{V}^{-\infty}(M),$$

by defining currents in the following way

$$C := C_1 \cap C_2 \in \mathcal{D}_n(M), \tag{6.9}$$

$$T := (-1)^n \bar{p}_*\bar{\Phi}^*(q_1^*T_1 \cap q_2^*T_2) + \pi_M^*C_1 \cap T_2 + T_1 \cap \pi_M^*C_2 \in \mathcal{D}_{n-1}(\mathbb{P}_M). \tag{6.10}$$

To see that the map given in Definition 6.1.5 equals the product introduced in Section 4 of smooth valuations, we make use of Proposition 6.1.4. By (5.8) the product of $\phi_1 \cdot \phi_2$ corresponds to the currents

$$(\pi_{M*}\eta, s^*(D\eta + \pi_M^*\nu)).$$

Therefore, we need to show that

$$\begin{aligned} \pi_{M*}\eta &= (C_1 \cap C_2), \\ s^*(D\eta + \pi_M^*\nu) &= (-1)^n \bar{p}_* \bar{\Phi}^*(q_1^* T_1 \cap q_2^* T_2) + \pi_M^* C_1 \cap T_2 + T_1 \cap \pi_M^* C_2, \end{aligned}$$

with $C_i = \pi_{M*}\eta_i$ and $T_i = s^*(D\eta_i + \pi_M^*\nu_i)$. We are just going to prove the claim for C. For T we refer to [7, Theorem 8.3]. By definition of $C_1 \cap C_2$ and by Example 5.1.5, we have

$$\begin{aligned} \langle C_1 \cap C_2, \omega \rangle &= \langle \Delta^*([[M]] \llcorner \eta_1 \boxtimes [[M]] \llcorner \eta_2), \omega \rangle \\ &= \langle \Delta^*[[M \times M]] \llcorner (\pi_1^*\pi_{M*}\eta_1 \wedge \pi_2^*\pi_{M*}\eta_2), \omega \rangle \\ &= \langle [[M]] \llcorner \Delta^*(\pi_1^*\pi_{M*}\eta_1 \wedge \pi_2^*\pi_{M*}\eta_2), \omega \rangle \\ &= \langle [[M]] \llcorner (\pi_{M*}\eta_1 \wedge \pi_{M*}\eta_2), \omega \rangle. \end{aligned}$$

Hence $C_1 \cap C_2 = [[M]] \llcorner (\pi_{M*}\eta_1 \wedge \pi_{M*}\eta_2)$.

For a proof of the continuity statement below, we refer to [7, Theorem 8.3].

Theorem 6.1.6. *Let M be a smooth manifold and let $\Lambda_i \subseteq T^*M \setminus 0, \Gamma_i \subseteq T^*\mathbb{P}_M \setminus 0$ be closed conical sets. Suppose that the following conditions are satisfied:*

1. *$\Lambda_1 \cap s(\Lambda_2) = \emptyset$.*
2. *$\Gamma_1 \cap s(\pi_M^*\Lambda_2) = \emptyset$.*
3. *$\Gamma_2 \cap s(\pi_M^*\Lambda_1) = \emptyset$.*
4. *If for some $(p, [\xi]) \in \mathbb{P}_M$ we have $\eta_1 \in \Gamma_1|_{(p,[\xi])}$ and $\eta_2 \in \Gamma_2|_{(p,[\xi])}$ vanish on the fibers of π_M, then $\eta_1 \neq -\eta_2$.*
5. *Let $(p, [\xi], \eta_1) \in \Gamma_1$ with $(p, [\xi]) \in \mathbb{P}_M$ and $\eta_1 \in T^*_{(p,[\xi])}\mathbb{P}_M \setminus \{0\}$ and $(p, [-\xi], \eta_2) \in \Gamma_2$ with $(p, [-\xi]) \in \mathbb{P}_M$ and $\eta_2 \in T^*_{(p,[-\xi])}\mathbb{P}_M \setminus \{0\}$ and let $\tau : \mathbb{P}_M \times_M \mathbb{P}_M \to \mathbb{P}_M \times \mathbb{P}_M$ be the natural embedding. Set $\zeta := d\tau^*_{(p,[\xi],[-\xi])}(\eta_1, \eta_2) \in T^*_{(p,[\xi],[-\xi])}\mathbb{P}_M \times_M \mathbb{P}_M$. Then*

   $$d\theta^*(\zeta) \notin N^*_{\mathbb{P}_M \times_M \mathbb{P}_M}\Delta|_{(p,[\xi],[\xi])},$$

   where

   $$\theta(p, [\xi_1], [\xi_2]) := (p, [\xi_1], [-\xi_2]),$$

   and Δ is the diagonal in $\mathbb{P}_M \times_M \mathbb{P}_M$.

Then the product of smooth valuations extends to a unique jointly sequentially continuous bilinear map

$$\mathcal{V}^{-\infty}_{\Lambda_1,\Gamma_1}(M) \times \mathcal{V}^{-\infty}_{\Lambda_2,\Gamma_2}(M) \to \mathcal{V}^{-\infty}(M).$$

In the following proposition we will refine the statement about the product given in Theorem 6.1.6.

Proposition 6.1.7. *Let $\Lambda \subseteq T^*M \setminus 0, \Gamma \subseteq T^*\mathbb{P}_M \setminus 0$ be closed conical sets such that*

$$\pi_M^*(\Lambda) \subseteq \Gamma \quad \textit{and} \quad \pi_M^*\pi_{M*}\Gamma \subseteq \Gamma \tag{6.11}$$

Then the multiplication map

$$\mathcal{V}^{\infty}(M) \times \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M) \to \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M).$$

is well-defined and jointly sequentially continuous.

Proof. Let $\phi_1 \in \mathcal{V}^\infty(M), \phi_2 \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$ and (C_i, T_i) be the corresponding currents. The currents associated to the product $\phi_1 \cdot \phi_2$ are given in Definition 6.1.5. As in the proof of Proposition 6.1.3, we can consider each summand of each current separately:

1. C: Since C_1 is smooth, we have by Proposition 5.1.13,

$$\overline{WF}(C) = \overline{WF}(C_1 \cap C_2) \subseteq \overline{WF}(C_1) + \overline{WF}(C_2) = \bar{\Lambda}.$$

   Therefore the claim holds for C.

2. T: We have to examine all summands of T separately:

   (a) $(-1)^n \bar{p}_* \bar{\Phi}^*(q_1^* T_1 \cap q_2^* T_2)$:
   First, let us mention that

$$WF(q_1^* T_1 \cap q_2^* T_2) \subseteq WF(q_1^* T_1) + WF(q_2^* T_2).$$

   By smoothness of T_1, we get

$$WF(q_1^* T_1 \cap q_2^* T_2) \subseteq q_2^* WF(T_2).$$

   Hence, by Proposition 5.1.11, to prove that $\bar{\Phi}^*(q_1^* T_1 \cap q_2^* T_2)$ is well defined, we have to consider the boundary $\bar{\mathcal{N}}$ of $\bar{\mathbb{P}}$ and therefore

$$WF(\bar{\Phi}^*(q_1^* T_1 \cap q_2^* T_2)) \subseteq \bar{\Phi}^* WF(q_2^* T_2) \subseteq \bar{\Phi}^*(q_2^* \overline{\Gamma}) + N^*_{\bar{\mathbb{P}}} \bar{\mathcal{N}}$$

   This leads us to show that

$$(q_2 \circ \bar{\Phi})^* \Gamma \cap N^*_{\bar{\mathbb{P}}} \bar{\mathcal{N}} = \emptyset, \tag{6.12}$$

   so that $\bar{\Phi}^*(q_1^* T_1 \cap q_2^* T_2)$ is well defined and

$$\bar{p}_*((q_2 \circ \bar{\Phi})^* \Gamma + N^*_{\bar{\mathbb{P}}} \bar{\mathcal{N}}) \subseteq \Gamma, \tag{6.13}$$

   to make sure that $WF((-1)^n \bar{p}_* \bar{\Phi}^*(q_1^* T_1 \cap q_2^* T_2)) \subseteq \Gamma$. Now let us fix a point $x \in \bar{\mathbb{P}}$ and let us set $y_1 := \bar{p}(x) \in \mathbb{P}_M, y_2 := (q_2 \circ \bar{\Phi})(x) \in \mathbb{P}_M$. Moreover, we fix

$$\alpha \in (\bar{\Phi}^*(q_2^* \Gamma) + N^*_{\bar{\mathbb{P}}} \bar{\mathcal{N}})\Big|_x, \tag{6.14}$$

$$\beta \in T^*_{y_1} \mathbb{P}_M, \text{ with } d\bar{p}_x^*(\beta) = \alpha. \tag{6.15}$$

   We are aiming for $\alpha \neq 0$ and $\beta \in \Gamma|_{y_1}$. The desired equalities in (6.12) and (6.13) will then follow, since

$$\begin{aligned}(x, d\bar{p}_x^*(\beta)) = (x, \alpha) \in \left(\bar{\Phi}^*(q_2^* \Gamma) + N^*_{\bar{\mathbb{P}}} \bar{\mathcal{N}}\right)\Big|_x &\Rightarrow (y_1, \beta) \in \bar{p}_* \left(\bar{\Phi}^*(q_2^* \Gamma) + N^*_{\bar{\mathbb{P}}} \bar{\mathcal{N}}\right)\Big|_x \\ &\Rightarrow (y_1, \beta) \in \Gamma.\end{aligned}$$

   In order to prove this, we are going to distinguish between the case when x is in the interior and when x is in the boundary of $\bar{\mathbb{P}}$. We are just going to describe the cases $x \in \bar{\mathbb{P}} \setminus \bar{\mathcal{N}}$ and $x \in \bar{\mathcal{N}}$. The remaining cases are similar. We will denote the projection of $\bar{\mathbb{P}}$ to M by $\bar{\pi}$:

- Let $x \in \bar{\mathbb{P}} \setminus \bar{\mathcal{N}}$: $\alpha \neq 0$ is satisfied, since

$$d(q_2 \circ \bar{\Phi}) : T_x\bar{\mathbb{P}} \to T_{y_2}\mathbb{P}_M$$

is onto and, therefore, the dual map is injective. According to (6.5), we have

$$\begin{aligned}\bar{\pi}(p, [\xi : \eta], [\xi'], [\eta'], [\zeta]) = p &= (\pi_M \circ q_2 \circ \bar{\Phi})(p, [\xi : \eta], [\xi'], [\eta'], [\zeta]) \\ &= (\pi_M \circ \bar{p})(p, [\xi : \eta], [\xi'], [\eta'], [\zeta]).\end{aligned}$$

This shows that $d(\pi_M \circ \bar{p}) = d(\pi_M \circ q_2 \circ \bar{\Phi}) = d\bar{\pi}$ and, hence, that the kernels of these maps are equal. Moreover, we can conclude that

$$\ker(d(q_2 \circ \bar{\Phi})) + \ker(d\bar{p}) \subseteq \ker(d\bar{\pi}).$$

We even get equality since the kernel of $d\pi_M$ is disjoint to the image of $d(q_2 \circ \bar{\Phi})$ and $d\bar{p}$, respectively. By the choice of α, there is $\gamma \in \Gamma|_{y_2}$ with

$$\alpha = d(q_2 \circ \bar{\Phi})^* \gamma.$$

Hence, we can infer that α is in the image of $d\bar{\pi}^*$. This implies that

$$\gamma \in \Gamma|_{y_2} \cap d\pi_M^*(T^*M)|_{y_2},$$

i.e., $\gamma = d\pi_{M,y_2}^*\delta \in \Gamma|_{y_2}$ with $\delta \in T_{y_2}^*M$. Now, we are making use of the surjectivity of $d\bar{p}_x$:

$$\begin{aligned}d\bar{p}_x^*(\beta) = \alpha = d(q_2 \circ \bar{\Phi})^*\gamma \Rightarrow \beta &= (d\bar{p}_x^*)^{-1} \circ d(q_2 \circ \bar{\Phi})^* d\pi_{M,y_2}^*\delta \\ &= (d\bar{p}_x^*)^{-1} \circ d(\pi_{M,y_2} \circ q_2 \circ \bar{\Phi})^*\delta \\ &= (d\bar{p}_x^*)^{-1} \circ d\bar{p}_x^* \circ d\pi_{M,y_2}^*\delta = d\pi_{M,y_2}^*\delta.\end{aligned}$$

Hence, $\beta \in \pi_M^*\pi_{M*}\Gamma$ and, by assumption, $\pi_M^*\pi_{M*}\Gamma \subseteq \Gamma|_{y_1}$.
- Let $x \in \bar{\mathcal{N}}$: We can write

$$\alpha = d(q_2 \circ \bar{\Phi})^*(\gamma) + n = d\bar{p}^*(\beta),$$

where $\gamma \in \Gamma|_{y_1}, n \in T^*_{\bar{\mathcal{N}}}\bar{\mathbb{P}}|_x, \alpha \in T^*_{y_1}\mathbb{P}_M$. Since the restriction

$$(q_2 \circ \bar{\Phi})|_{\bar{\mathcal{N}}_0} : \bar{\mathcal{N}}_0 \to \mathbb{P}_M$$

is a submersion, we can deduce the properties in the same way as in the case above.

Hence the first summand satisfies the conditions.

(b) $\pi_M^*C_1 \cap T_2$:
The smoothness of C_1 and T_1 imply

$$\overline{WF}(\pi_M^*C_1 \cap T_2) \subseteq \pi_M^*\overline{WF}(C_1) + \overline{WF}(T_2) = \{0\} + \overline{WF}(T_2) \subseteq \bar{\Gamma}.$$

From this, it follows that the second summand also satisfies $WF(\pi_M^*C_1 \cap T_2) \subseteq \Gamma$.

(c) $T_1 \cap \pi_M^*C_2$:
At last, the smoothness of T_1 implies

$$\overline{WF}(T_1 \cap \pi_M^*C_2) \subseteq \{0\} + \pi_M^*\overline{WF}(C_2) \subseteq \pi_M^*\Lambda \subseteq \Gamma.$$

By these computation, the claim follows. ∎

As a crucial consequence of the above proposition, we get Proposition 6.1.9. It is a powerful tool since it is often easier to prove statements for smooth valuations. At first we need the case of $\mathbb{R}^n$. For a proof, we refer to [7, Lemma 8.2].

Lemma 6.1.8. *Let $\phi \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(\mathbb{R}^n)$ be a generalized valuation on $\mathbb{R}^n$. Then there exists a sequence of smooth valuations $(\phi_j)_{j\in\mathbb{N}}$ such that $\phi_j \to \phi$ in $\mathcal{V}^{-\infty}_{\Lambda,\Gamma}(\mathbb{R}^n)$.*

Proposition 6.1.9. *Let $\Lambda \subseteq T^*M \setminus 0, \Gamma \subseteq T^*\mathbb{P}_M \setminus 0$ be closed conical sets satisfying (6.11). Then $\mathcal{V}^\infty(M)$ is sequentially dense in $\mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$.*

Proof. Let us take a locally finite open covering (U_α) of the manifold M and open sets (O_α) diffeomorphic to $\mathbb{R}^n$ such that the closure of U_α in M is compact and contained in O_α. Now by Proposition 2.2.7 there exists a partition of unity in valuations subordinate to the covering (U_α). Let $\zeta \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$. Then

$$\zeta = \sum_\alpha \psi_\alpha \cdot \zeta.$$

As we have shown above in Proposition 6.1.7, $\psi_\alpha \cdot \zeta \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$. This reduces the proof to the case $\psi_\alpha \cdot \zeta$ for some fixed α, hence we can assume that $\zeta \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$ is a compactly supported valuation with support contained in an open set $O_\alpha \subseteq M$. Then, by Lemma 6.1.8, there exists a sequence $\zeta_j \in \mathcal{V}^\infty(O_\alpha)$ which converges to $\zeta|_{O_\alpha}$ in $\mathcal{V}^{-\infty}_{\Lambda,\Gamma}(O_\alpha)$.
Let us choose a smooth compactly supported valuation τ on O_α which is equal to the Euler characteristic χ in a neighbourhood of $\operatorname{supp}(\zeta)$. Then it follows

$$\tau \cdot \zeta_j \to \tau \cdot \zeta|_{O_\alpha} = \zeta|_{O_\alpha} \quad \text{in} \quad \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(O_\alpha),$$

and $\operatorname{supp}(\tau \cdot \zeta_j) \subseteq \operatorname{supp}(\tau)$. If we extend ζ_j by zero to M the result follows. ∎

We need the following diagram to keep track of the operations considered in the upcoming propositions.

$$\begin{tikzcd}
\mathbb{P}_M \arrow[dr, "\pi_M"'] & M \times_{f,\pi_M} \mathbb{P}_N \arrow[l, "df^*"'] \arrow[r, "p"] \arrow[d, "\pi_1"] & \mathbb{P}_N \arrow[d, "\pi_N"] \\
& M \arrow[r, "f"] & N
\end{tikzcd}, \tag{6.16}$$

where $p : M \times_{f,\pi_M} \mathbb{P}_N \to \mathbb{P}_N$ is the projection map with $p(x,[\eta]) = (f(x),[\eta])$. Since f is a submersion, the dual map of the differential is injective and hence it induces an injective map $df^* : M \times_{f,\pi_N} \mathbb{P}_N \to \mathbb{P}_M$ wtih

$$df^*(p,[\xi]) = (p,[df_p^*\xi]).$$

Therefore, we can consider $M \times_{f,\pi_N} \mathbb{P}_N$ as a subset of $\mathbb{P}_M$. We will make use of this, to define a pull-back operator as a dual to the push forward on smooth valuations. In order to do that, we need a description of the push forward in terms of differential forms. A proof of this can be found in [6, Proposition 3.2.3].

Proposition 6.1.10. *Let $\phi \in \mathcal{V}^\infty_c(M)$ be represented by the pair $(\nu,\eta) \in \Omega^n_c(M) \times \Omega^{n-1}_c(\mathbb{P}_M)$ and let $f : M \to N$ be a smooth, proper submersion. Then the valuation $f_*\phi$ is represented by the pair $(f_*\nu, p_*(df^*)^*\eta)$.*

Definition 6.1.11. Let M, N be smooth manifolds and let $f : M \to N$ be a smooth submersion. We define the pull-back operator $f^* : \mathcal{V}^{-\infty}(N) \to \mathcal{V}^{-\infty}(M)$ as the adjoint operator to $f_* : \mathcal{V}^\infty_c(M) \to \mathcal{V}^\infty_c(N)$, i.e., let $\psi \in \mathcal{V}^{-\infty}(N)$, then,

$$\langle f^*\psi, \phi\rangle := \langle \psi, f_*\phi\rangle, \ \forall \phi \in \mathcal{V}^\infty(M).$$

With the help of Proposition 6.1.10 we get the following representation of the pull-back operator in terms of currents:

Proposition 6.1.12. *Let M, N be smooth manifolds and let $f : M \to N$ be a smooth submersion. If $\psi \in \mathcal{V}^{-\infty}(N)$ corresponds to a pair of currents (C, T), then $f^*\psi \in \mathcal{V}^{-\infty}(M)$ corresponds to the pair of currents*

$$C' = f^*C, \tag{6.17}$$
$$T' = (df^*)_* p^*T. \tag{6.18}$$

Proof. By Lemma 2.2.4, $\phi \in \mathcal{V}_c^\infty(M)$ can be represented by $(\nu, \eta) \in \Omega_c^n(M) \times \Omega_c^{n-1}(\mathbb{P}_M)$. By Proposition 6.1.10, we have

$$\langle f^*\psi, \phi\rangle = \langle \psi, f_*\phi\rangle = \langle C, f_*\nu\rangle + \langle T, p_*(df^*)^*\eta\rangle = \langle f^*C, \nu\rangle + \langle (df^*)_* p^*T, \eta\rangle.$$

■

By Proposition 6.1.12, we can compute the wave fronts of $f^*\psi$. Since C is smooth, we have

$$\overline{WF}(f^*C) \subseteq f^*\overline{WF}(C) = \{0\}.$$

For T, we have to consider once more Proposition 5.1.11,

$$\overline{WF}((df^*)_* p^*T) \subseteq (df^*)_*\overline{WF}(p^*T) \subseteq (df^*)_* p^*\overline{WF}(T) + N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N)$$

Since T is smooth, we obtain $\overline{WF}((df^*)_* p^*T) \subseteq N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N)$. Thus, this gives rise to an operator $f^* : \mathcal{V}_c^\infty(N) \to \mathcal{V}^{-\infty}_{\emptyset, N^*_{\mathbb{P}_M}(M\times_{f,\pi_N}\mathbb{P}_N), c}(M)$.
Due to Proposition 5.2.6 and (6.4), we get a continuous functional $\int : \mathcal{V}^{-\infty}(M) \to \mathbb{R}$ by considering

$$\psi \mapsto \int \psi \cdot \phi \tag{6.19}$$

with $\phi \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma,c}$. This is well-defined since $\psi \cdot \phi$ has compact support.

Definition 6.1.13. Let M, N be smooth manifolds and let $f : M \to N$ be a smooth, proper submersion. By (6.19) we can define $f_*\psi \in \mathcal{V}^{-\infty}(N)$ by

$$\langle f_*\psi, \phi\rangle := \int_M \psi \cdot f^*\phi, \quad \phi \in \mathcal{V}_c^\infty(N).$$

We obtain the following diagram

$$\begin{tikzcd}
 & \overset{\psi}{\mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)} \arrow[r, "f_*"] \arrow[dr] & \overset{f_*\psi}{\mathcal{V}^{-\infty}(N)} \\
 & & \overset{\psi\cdot f^*\phi}{\mathcal{V}_c^{-\infty}(M)} \arrow[u, dashed] \arrow[d, "(6.19)"', "\int \psi\cdot f^*\phi"] \\
\overset{\phi}{\mathcal{V}_c^\infty(N)} \arrow[r, "f^*"] & \overset{f^*\phi}{\mathcal{V}^{-\infty}_{\emptyset,\Gamma',c}(M)} \arrow[ur] & \mathbb{R}
\end{tikzcd} \tag{6.20}$$

.

with $\Gamma' = N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N)$.

Proposition 6.1.14. *Let $\psi \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$ and $f : M \to N$ be a smooth and proper submersion. Then the pushforward $f_*\psi \in \mathcal{V}^{-\infty}(N)$ is well-defined provided that*

$$\Gamma \cap N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N) = \emptyset. \tag{6.21}$$

The map

$$f_* : \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M) \to \mathcal{V}^{-\infty}(N) \tag{6.22}$$

is sequentially continuous.

Proof. We have to show that the product $\psi \cdot f^*\phi$ is defined for every $\phi \in \mathcal{V}^{\infty}(N)$ and therefore that conditions 1-5 in Theorem 6.1.6 hold for

$$(\Lambda_1, \Gamma_1) = (\Lambda, \Gamma) \quad \text{and} \quad (\Lambda_2, \Gamma_2) = (\emptyset, N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N)).$$

Obviously, the first and second conditions are satisfied since $\Lambda_2 = \emptyset$.
Next, $\pi_M \circ df^* : M \times_{f,\pi_N} \mathbb{P}_N \to M$ is a submersion, since

$$(df^*) \circ d\pi_M = \pi_1 \Rightarrow d((df^*) \circ d\pi_M) = d\pi_1$$

and π_1 is a submersion. Therefore, for $\eta \in N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N)$, we have

$$\eta \in N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N) = \ker(d(df^*) \circ d\pi_M)^* \Rightarrow (d(df^*) \circ d\pi_M)^*(\eta) = d\pi_1^*(\eta) = 0$$

and, hence, $\eta \equiv 0$. We can conclude that conditions 3 and 4 are satisfied. Finally, let $(x, [\xi], \eta_1) \in \Gamma_1, (x, [-\xi], \eta_2) \in \Gamma_2$. Then,

$$\begin{aligned} d(id \times s)^* \circ d\tilde{\tau}^*(\eta_1, \eta_2)(v, v) &= d\tilde{\tau}^*(\eta_1, \eta_2)(d(id \times s)(v, v)) \\ &= \eta_1(v) + \eta_2(ds(v)) = \eta_1(v) + ds^*(\eta_2(v)) \\ &= d\tilde{\tau}(\eta_1, ds^*(\eta_2))(v, v). \end{aligned}$$

Since Γ_2 is closed under ds^*, we have $ds^*(\eta_2) \in \Gamma_2$. Now $d\tilde{\tau}^*(\eta_1, ds^*(\eta_2)) \in N^*_{\mathbb{P}_M \times_M \mathbb{P}_M} \Delta|_{(x,[\xi],[\xi])}$ if and only if $\eta_1 = -ds^*(\eta_2)$ and, hence,

$$\Gamma \cap N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N) = \emptyset$$

implies 5. ∎

In the following we will need to consider the support of a generalized valuation obtained by a push-forward. We claim

$$\operatorname{supp}(f_*\psi) \subseteq f(\operatorname{supp}\psi). \tag{6.23}$$

To show that, we take $U \subseteq f(\operatorname{supp}\psi)^c$ open and $\phi \in \mathcal{V}^{\infty}_c(N)$ with $\operatorname{supp}\phi \subseteq U$. Since the product satisfies $\operatorname{supp}(\psi \cdot f^*\phi) \subseteq \operatorname{supp}(\psi) \cap \operatorname{supp}(f^*\phi)$ and $\operatorname{supp}(f^*\phi) \subseteq f^{-1}(\operatorname{supp}\phi)$, it follows that

$$(\operatorname{supp}\psi \cap f^{-1}(U))^c = \operatorname{supp}(\psi)^c \cup f^{-1}(U)^c \subseteq \operatorname{supp}(\psi)^c \cup f^{-1}(\operatorname{supp}\phi)^c \subseteq \operatorname{supp}(\psi \cdot f^*\phi)^c.$$

Now, a point in the intersection $x \in \operatorname{supp}\psi \cap f^{-1}(U)$ satisfies $f(x) \in f(\operatorname{supp}\psi)$. And the claim follows.

Proposition 6.1.15. *Let $f : M \to N$ be a smooth and proper submersion. Suppose Λ and Γ satisfy* (6.21)*. Let $\psi \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$ be represented by the pair of currents (C, T). Let the push-forward $f_*\psi$ be represented by the pair of currents (C', T'). Then*

$$T' = p_*(df^*)^* T. \tag{6.24}$$

Proof. First, we are going to prove the statement for smooth $\psi \in \mathcal{V}^\infty(M)$ and corresponding smooth forms (ω, ϕ). According to (5.8), we have $C = (\pi_M)_*\omega$ and $T = D\omega + \pi_M^*\phi$. We have already shown that the push-forward $f_*\psi$ has a representation $(f_*\phi, p_*(df^*)^*\omega)$. Translating into the language of currents, we get $C' = (\pi_N)_* p_*(df^*)^*\omega$ and $T' = D(p_*(df^*)^*\omega) + \pi_N^* f_*\phi$. By Lemma 1.2.12, (6.16) leads to

$$p_*(df^*)^*\pi_M^*\phi = p_*\pi_1^*\phi = \pi_N^* f_*\phi. \tag{6.25}$$

To verify the proposition for smooth ψ, we have to show that

$$D(p_*(df^*)^*\omega) = p_*(df^*)^* D\omega. \tag{6.26}$$

From this the claim follows since,

$$\begin{aligned} p_*(df^*)^*T &= p_*(df^*)^*(D\omega + \pi_M^*\phi) \\ &\overset{6.25}{=} D(p_*(df^*)^*\omega) + p_*(df^*)^*\pi_M^*\phi \\ &\overset{6.26}{=} D(p_*(df^*)^*\omega) + \pi_N^* f_*\phi = T'. \end{aligned}$$

Now, let ω be a smooth differential form. By definition of the Rumin operator $D = d \circ Q$ (see Theorem 1.2.26), Q maps ω to the unique form $\omega + \omega'$ such that ω' is vertical and $d(\omega + \omega')$ is vertical. Now, since the exterior derivative commutes with pullbacks and integration along fibers,

$$\begin{aligned} p_*(df^*)^*D\omega &= d(p_*(df^*)^*(Q(\omega))) = d(p_*(df^*)^*(\omega)) + d(p_*(df^*)^*(\omega')) \\ &= p_*(df^*)^*(d(\omega)) + d(p_*(df^*)^*(\omega')). \end{aligned}$$

Now, let us assume that $p_*(df^*)^*$ maps vertical forms to vertical forms. Then $p_*(df^*)^*D\omega = d(p_*(df^*)^*(\omega)) + d(p_*(df^*)^*(\omega'))$ is vertical. This means that the unique element $\tilde{\omega}$ such that $d(p_*(df^*)^*(\omega) + \tilde{\omega})$ is vertical, is given by $\tilde{\omega} = p_*(df^*)^*(\omega')$. On the other hand, we have

$$D(p_*(df^*)^*(\omega)) = d(p_*(df^*)^*(\omega)) + d(p_*(df^*)^*(\omega')).$$

Hence, it suffices to show that $p_*(df^*)^*$ maps vertical forms to vertical forms. This claim holds by [8, Proposition 4.6].
For the general case, we fix Λ, Γ with the given properties and take $\psi \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$. Since we can approximate ψ by a sequence of smooth valuations $\psi_n \in \mathcal{V}^\infty(M)$, we can use sequential continuity of $f_* : \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M) \to \mathcal{V}^{-\infty}(N)$ to get convergence of T_n' to T' in $\mathcal{D}(N)$. As we have seen above, we can represent T_n' as $p_*(df^*)^*T_n$. Again by sequential continuity, we obtain that $p_*(df^*)^*T_n \to p_*(df^*)^*T$ in $\mathcal{D}(N)$ and the claim follows. ∎

Corollary 6.1.16. *Let $f : M \to N$ be a smooth and proper submersion. Suppose Λ and Γ satisfy* (6.21)*. Then the push-forward map is a continuous map*

$$f_* : \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M) \to \mathcal{V}^{-\infty}_{\Lambda',\Gamma'}(N)$$

with $\Lambda' := T^*N \setminus 0$ and $\Gamma' := p_*(df^*)^*\Gamma$.

Proof. By Proposition 6.1.15, we have to check if propositions 5.1.11 and 5.1.10 hold for (6.24). We have,

$$
\begin{aligned}
&WF(C') \subseteq f_* WF(C) \subseteq T^*N, \\
&WF(T') = WF(p_*(df^*)^*T) \subseteq p_* WF((df^*)^*T) \subset p_*(df^*)^* WF(T) \subseteq p_*(df^*)^*\Gamma = \Gamma',
\end{aligned}
$$

since $WF(T) \subseteq \Gamma$. ∎

Proposition 6.1.17. *Let $f : M \to N, g : N \to O$ be smooth proper submersions between smooth manifolds. Let $\Lambda \subseteq T^*M \setminus 0, \Gamma \subseteq T^*\mathbb{P}_M \setminus 0$ be closed conical sets satisfying condition (6.21). Let $\Lambda' := T^*N \setminus 0$ and $\Gamma' := p_*(df^*)^*\Gamma$ and suppose Γ' satisfies (6.21). Then*

$$g_* \circ f_* = (g \circ f)_* \quad \text{on} \quad \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M).$$

In particular, both sides of this equation are well-defined maps.

Proof. In order to check if $(g \circ f)_*$ is well defined on $\mathcal{V}^{-\infty}_{\Lambda,\Gamma}$, let us take a look at the following diagram,

$$
\begin{tikzcd}
 & & \mathbb{P}_O \\
 & M \times_O \mathbb{P}_O \arrow[ur, "p_{MO}"] \arrow[r, "f \times id"] \arrow[d, "q"] \arrow[dl, "d(g\circ f)^*"'] & N \times_O \mathbb{P}_O \arrow[u, "p_{NO}"'] \arrow[d, "dg^*"] \\
\mathbb{P}_M & M \times_N \mathbb{P}_N \arrow[l, "df^*"] \arrow[r, "p_{MN}"'] & \mathbb{P}_N
\end{tikzcd}
$$

By Proposition 6.1.14, we have to show that $\Gamma \cap N^*_{\mathbb{P}_M}(M \times_O \mathbb{P}_O) = \emptyset$. Since $N^*_{\mathbb{P}_M}(M \times_O \mathbb{P}_O) = \ker(d(d(g \circ f)^*)^*) = \ker(d(df^* \circ dq)^*) = \ker(dq^* \circ d(df^*)^*)$, we have to show that $dq^*\xi \neq 0$ for all $\xi \in (df^*)^*\Gamma$.
Aiming for a contradiction, we assume that there exists $(x, [\eta]) \in M \times_O \mathbb{P}_O$ and $\xi \in T^*_{q(x,[\eta])}(M \times_N \mathbb{P}_N)$ with $\xi \neq 0, (q(x, [\eta]), \xi) \in (df^*)^*\Gamma$ and $dq^*_{(x,[\eta])}\xi = 0$. Due to (1.2), the kernel of the map

$$dp_{MN}|_{q(x,[\eta])} : T_{q(x,[\eta])}(M \times_N \mathbb{P}_N) \to T_{p_{MN}(q(x,[\eta]))}(\mathbb{P}_N)$$

consists of vectors of the form $(v, 0)$ with $df_x(v) = 0$. Since these vectors are therfore in the kernel of ξ and $\operatorname{im} dp^*_{MN} = (\ker dp_{MN})^\perp$, it follows that $\xi = dp^*_{MN_{q(x,[\eta])}}\xi'$ with $\xi' \in T^*_{p_{MN}(q(x,[\eta]))}N$. This implies that $(f(x), [dg^*_{f(x)}\eta], \xi') \in p_{MN_*}(df^*)^*\Gamma = \Gamma'$. The assumption leads to

$$d(dg^*)^*_{(f(x),[\eta])}\xi' \neq 0.$$

Since f is a submersion, the map $d(f \times id)^*_{(f(x),[\eta])}$ is injective and hence we obtain

$$d(f \times id)^*_{(x,[\eta])} \circ d(dg^*)^*_{(f(x),[\eta])}\xi' \neq 0.$$

This is a contradiction, since $dg^* \circ (f \times id) = p_{MN} \circ q$ and

$$dq^*_{(x,[\eta])} \circ dp^*_{MN_{q(x,[\eta])}}\xi' = dq^*_{(x,[\eta])}\xi = 0.$$

Hence, it follows that $(g \circ f)_*\psi$ is well defined for $\psi \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$.
For $\psi \in \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(M)$ we can approximate ψ, by Proposition 6.1.9, with a sequence of smooth valuations $(\psi_n)_{n\in\mathbb{N}} \in \mathcal{V}^\infty(M)$ if the condition (6.22) concerning the wave fronts is satisfied. We will show that $(\Lambda, \tilde{\Gamma})$ with $\tilde{\Gamma} := \pi^*_M(\Lambda) \cup \Gamma$ satisfy the desired conditions, that is

1. $\pi_M^*\Lambda \subseteq \tilde{\Gamma}$,
2. $\pi_M^*\pi_{M*}\tilde{\Gamma} \subseteq \tilde{\Gamma}$,
3. $\tilde{\Gamma} \cap N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N)$.

1 and 2 hold obviously. Finally, 3 is satisfied since an element of the intersection satisfies

$$\eta \in \tilde{\Gamma} \cap N^*_{\mathbb{P}_M}(M \times_{f,\pi_N} \mathbb{P}_N) \Rightarrow 0 = d(df^*)^*\eta = d(df^*)^* d\pi_M^*\alpha = d(\pi \circ df^*)^*\alpha = d\pi_1^*\alpha.$$

Since π_1 is a submersion, it follows that $\eta = 0$ and the claim follows. Analogously, we define $\tilde{\Gamma}' := \pi^*(\Lambda')$. Now by sequentially continuity of the maps

$$\begin{aligned} f_* : \mathcal{V}^{-\infty}_{\Lambda,\tilde{\Gamma}}(M) &\to \mathcal{V}^{-\infty}_{\Lambda',\tilde{\Gamma}'}(M), \\ g_* : \mathcal{V}^{-\infty}_{\Lambda',\tilde{\Gamma}'}(M) &\to \mathcal{V}^{-\infty}(O), \\ (g \circ f)_* : \mathcal{V}^{-\infty}_{\Lambda,\tilde{\Gamma}}(M) &\to \mathcal{V}^{-\infty}(O), \end{aligned}$$

we have

$$\begin{aligned} g_*(f_*(\psi_n)) &\to g_*(f_*(\psi)), \\ (g \circ f)_*\psi_n &\to (g \circ f)_*\psi \end{aligned}$$

in $\mathcal{V}^{-\infty}(O)$. Now, by definition of the pull-back on smooth valuations, it follows that $g_*(f_*(\psi_n)) = (g \circ f)_*\psi_n$ and we obtain $g_*(f_*(\psi)) = (g \circ f)_*\psi$. ∎

6.2 Convolution of generalized valuations

The definition of the convolution is similar to the construction in Theorem 3.2.8.

Definition 6.2.1. Let G be a Lie group acting on a smooth manifold M by a smooth map $a : G \times M \to M$. Let μ be a generalized valuation on G and ψ a generalized valuation on M. The convolution $\mu * \psi$ is defined as the generalized valuation $a_*(\mu \boxtimes \psi)$, provided that the push-forward exists.

We want to make use of (6.1) by setting $M_1 = G$ and $M_2 = M$ for a Lie Group G that acts transitively on a smooth manifold M and extend it by the following diagram

$$\begin{tikzcd} \mathbb{P}_G \times \mathbb{P}_M & & \\ \hat{\mathbb{P}}_{G\times M} \arrow[u, "\Phi"] \arrow[d, "F"] & (G \times M) \times_{a,\pi_M} \mathbb{P}_M \arrow[ul, "r"'] \arrow[dl, "da^*"] \arrow[r, "p"] & \mathbb{P}_M \\ \mathbb{P}_{G\times M} & & \end{tikzcd} \qquad (6.27)$$

Lemma 6.2.2. *Let G act transitively on M. Let $da^* : (G \times M) \times_{a,\pi_M} \mathbb{P}_M \to \mathbb{P}_{G\times M}$. Then*

$$da^*((G \times M) \times_{a,\pi_M} \mathbb{P}_M) \cap \mathcal{M} = \emptyset.$$

Proof. For $(g, p, [\tau]) \in (G \times M) \times_{a,\pi_M} \mathbb{P}_M$, we have $da^*(g, p, [\tau]) = (g, p, [da|^*_{(g,p))}(\tau)])$. Aiming for a contradiction suppose that $(g, p, [da^*_{(g,p)}(\tau)]) \in \mathcal{M}_1$, i.e., $[da^*_{(g,p)}(\tau)] = [\xi : 0]$ for some $\xi \in T_g^*G$. We obtain

$$0 = \langle da^*_{(g,p)}\tau, (0, v)\rangle = \langle \tau, da_{(g,p)}(0, v)\rangle$$

for $v \in T_pM$. But since

$$a(g^{-1}, a(g,p)) = g^{-1}gp = p = a(g, a(g^{-1},p))$$

holds for all $(g,p) \in G \times M$, the map $T_gG \oplus T_pM \to T_{gp}M, (0,v) \mapsto da_{(g,p)}(0,v)$ is an isomorphism, and thus we have $\tau = 0$, a contradiction.
On the other hand, if we have $(g,p,[da^*_{(g,p)}(\tau)]) \in \mathcal{M}_2$ and $u \in T_gG$, then

$$0 = \langle da^*_{(g,p)}(\tau), (u,0)\rangle = \langle \tau, da_{(g,p)}(u,0)\rangle.$$

The transitivity of the action of G on M implies that for $g \in G$ and $p \in M$, there is $q \in M$ with $a(g,q) = g \cdot q = p$. Hence, the map $a_g : G \to M$

$$p \mapsto a(g,p)$$

is surjective and $T_gG \oplus T_pM \to T_{gp}M, (u,0) \mapsto da_{(g,p)}(u,0)$ is onto. Therefore we can conclude $\tau = 0$, a contradiction. ■

Lemma 6.2.2 is crucial, since the map $F|_{\hat{\mathbb{P}}_{G\times M}\setminus\mathcal{N}}$ is a diffeomorphism and, hence, we can invert F on $\mathbb{P}_M \setminus \mathcal{M}$. We get the following result.

Lemma 6.2.3. *Let*

$$r := \Phi \circ F^{-1} \circ da^* : (G \times M) \times_{a,\pi_M} \mathbb{P}_M \to \mathbb{P}_G \times \mathbb{P}_M.$$

Then the map

$$((\pi_G \circ q_1) \times q_2) \circ r : (G \times M) \times_{a,\pi_M} \mathbb{P}_M \to G \times \mathbb{P}_M$$

is a diffeomorphism.

Proof. For $g \in G$ let $\iota_g : M \to G \times M$

$$p \mapsto (g,p).$$

Again let a_g be the multiplication by $g \in G$. We have $a \circ \iota_g = a_g$ and $d\iota_g$ is injective with inverse $d\pi_M$. Moreover, $(da_{(g,p)} \circ d\iota_g)_p = (da_g)_p$ is an isomorhpism and since this property transfers to the dual map, we get that $(d\iota_g)^*_p \circ da^*_{(g,p)} = (da_g)^*_p$ is an isomorphism. Now let $(g,p,[\xi]) \in (G \times M) \times_{a,\pi_M} \mathbb{P}_M$, then

$$\begin{aligned}((\pi_G \circ q_1) \times q_2) \circ r(g,p,[\xi]) &= ((\pi_G \circ q_1) \times q_2) \circ \Phi \circ F^{-1} \circ da^*(g,p,[\xi]) \\ &= ((\pi_G \circ q_1) \times q_2) \circ \Phi \circ F^{-1}(g,p,[da^*_{(g,p)}(\xi)]) \\ &= (g,p,[(d\iota_g)^*_p \circ da^*_{(g,p)}(\xi)]) \\ &= (g,p,[(da_g)^*_p\xi]).\end{aligned}$$

■

Definition 6.2.4. Let M be a smooth manifold. Let $\mathcal{V}_t^{-\infty}(M)$ be defined by

$$\mathcal{V}_t^{-\infty}(M) := \mathcal{V}^{-\infty}_{T^*M\setminus 0, d\pi^*_M T^*M\setminus 0}(M).$$

These generalized valuations are called tame.

Example 6.2.5. Let $K \subseteq \mathbb{R}^n$ be a smooth compact convex body. Then the generalized valuation τ_K given by

$$\langle \tau_K, \mu \rangle := \left.\frac{d}{dt}\right|_{t=0} \mu(tK), \quad \mu \in \mathcal{V}_c^\infty(\mathbb{R}^n)$$

is tame. For a proof of this, we refer to [8, Proposition 6.5].

Proposition 6.2.6. *Let G act transitively on M.*

1. *The convolution product $* : \mathcal{V}_{c,t}^{-\infty}(G) \times \mathcal{V}^{-\infty}(M) \to \mathcal{V}^{-\infty}(M)$ is well-defined and jointly sequentially continuous.*

2. *Let $\mu \in \mathcal{V}_{c,t}^{-\infty}(G)$ and $\psi \in \mathcal{V}^{-\infty}(M)$. Let $(C_1, T_1), (C_2, T_2)$ be the currents corresponding to μ and ψ respectively. Let (C, T) be the currents corresponding to $\mu * \psi$. Then*

$$T = p_* r^*(T_1 \boxtimes T_2).$$

3. $\operatorname{supp}(\mu * \psi) \subseteq a(\operatorname{supp} \mu \times \operatorname{supp} \psi)$.

Proof.

1. Let $\mu \in \mathcal{V}_{c,t}^{-\infty}(G)$ and $\psi \in \mathcal{V}^{-\infty}(M)$ and let (C_1, T_1) and (C_2, T_2) be the corresponding currents. By Definition 6.1.5, the exterior product $\mu \boxtimes \psi$ corresponds to currents (C', T') given by

$$C' = C_1 \boxtimes C_2,$$
$$T' = F_* \Phi^*(T_1 \boxtimes T_2) + (\tilde{p}_1 \circ \pi_{G\times M})^* C_1 \cap (i_{2*} p_2^* T_2) + i_{1*} p_1^* T_1 \cap (\tilde{p}_2 \circ \pi_{G \times M})^* C_2.$$

Let us denote the currents corresponding to $a_*(\mu \boxtimes \psi)$ by (C, T). Then Proposition 6.1.15 yields a representation of the current T as,

$$T = p_*(da^*)^* T'.$$

To justify that this expression is well defined, we have to consider (6.21). Recall that for a tame valuation μ we have, $WF(T_1) \subseteq d\pi_G^* T^*G \setminus 0$, therefore, we obtain

$$\overline{WF}(T_1 \boxtimes T_2) \subseteq \overline{WF}(T_1) \times \overline{WF}(T_2) \subseteq d\pi_G^*(T^*G) \times \overline{WF}(T_2).$$

Condition 2 is satisfied for the submersion Φ since the differential $d\Phi^*$ at any point is thus injective. Hence, $\Phi^*(T_1 \boxtimes T_2)$ is well-defined. For a point $q \in \hat{\mathbb{P}} \setminus \mathcal{N}$, we get

$$\overline{WF}(\Phi^*(T_1 \boxtimes T_2))|_q \subseteq \Phi^* \overline{WF}(T_1 \boxtimes T_2)|_{\Phi(q)} \subseteq \Phi^*(d\pi_G^*(T^*G) \times \overline{WF}(T_2)).$$

With Example 5.1.12 we infer for $u \in \mathbb{P}_{G\times M} \setminus \mathcal{M}$,

$$\overline{WF}(F_*\Phi^*(T_1 \boxtimes T_2))|_u \subseteq F_*\Phi^* \overline{WF}(T_1 \boxtimes T_2) \subseteq F_*\Phi^*(d\pi_G^*(T^*G) \times \overline{WF}(T_2)) \subseteq$$
$$\{d(\Phi \circ F^{-1})_u^*(y_1, y_2) : y_1 = d\pi_G^* x_1, x_1 \in T^*_{\pi_G(q_1(F^{-1}(u)))} G, y_2 \in \overline{WF}(T_2)|_{q_2(\Phi(F^{-1}(u)))}\}$$

Let us set

$$A := \{d(\Phi \circ F^{-1})_u^*(y_1, y_2) : u \in \mathbb{P}_{G\times M} \setminus \mathcal{M}, y_1 = d\pi_G^* x_1, x_1 \in T^*G\} \cup N^*_{\mathbb{P}_{G\times M}} \mathcal{M}.$$

Proposition 6.1.3 yields that the exterior product is a jointly sequentially continuous map

$$\boxtimes : \mathcal{V}_{c,t}^{-\infty}(G) \times \mathcal{V}^{-\infty}(M) \to \mathcal{V}^{-\infty}_{T^*(G\times M)\setminus 0, A\setminus 0}(G \times M).$$

Hence the map $(T_1, T_2) \mapsto F_*\Phi^*(T_1 \boxtimes T_2)$ is jointly sequentially continuous as a map $\mathcal{D}_{d\pi_G^*(T^*G)}(\mathbb{P}_G) \times \mathcal{D}(\mathbb{P}_M) \to \mathcal{D}_A(\mathbb{P}_{G\times M})$. Since

$$\operatorname{supp}(i_{2*}p_2^*T_2) \subseteq i_2(p_2^{-1}(\operatorname{supp} T_2)) \subseteq \mathcal{M}_2,$$

we obtain $\operatorname{supp}((\tilde{p}_1 \circ \pi_M)^*C_1 \cap i_{2*}p_2^*T_2) \subseteq \mathcal{M}_2$ and, therefore,

$$\overline{WF}((\tilde{p}_1 \circ \pi_M)^*C_1 \cap (i_{2*}p_2^*T_2)) \subseteq N^*_{\mathbb{P}_{G\times M}}\mathcal{M}_2 \subseteq A.$$

The map $(C_1, T_2) \mapsto (\tilde{p}_1 \circ \pi_M)^*C_1 \cap (i_{2*}p_2^*T_2)$ is a jointly sequentially continuous map $\mathcal{D}(G) \times \mathcal{D}(\mathbb{P}_M) \to \mathcal{D}_A(\mathbb{P}_{G\times M})$. Similarly, since $i_{1*}p_1^*T_1$ is supported on $\mathcal{M}_1$, we get

$$\overline{WF}((i_{1*}p_1^*T_1) \cap (\tilde{p}_2 \circ \pi_M)^*C_2) \subseteq N^*_{\mathbb{P}_{G\times M}}\mathcal{M}_1 \subseteq A.$$

The map $(T_1, C_2) \mapsto (i_{1*}p_1^*T_1) \cap (\tilde{p}_2 \circ \pi_M)^*C_2$ is a jointly sequentially continuous map $\mathcal{D}_{d\pi_G^*(T^*G)}(\mathbb{P}_G) \times \mathcal{D}(M) \to \mathcal{D}_A(\mathbb{P}_{G\times M})$. Thus, we have $\overline{WF}(T') \subseteq A$ and, therefore,

$$\boxtimes : \mathcal{V}^{-\infty}_{c,t}(G) \times \mathcal{V}^{-\infty}(M) \to \mathcal{V}^{-\infty}_{T^*(G\times M)\setminus 0, A\setminus 0}(G \times M).$$

By Lemma 6.2.2, the image of da^* is disjoint from $\mathcal{M}$. Take an element $c \in A \setminus N^*_{\mathbb{P}_{G\times M}}\mathcal{M}$. Then

$$c = d(\Phi \circ F^{-1})^*_u(d\pi_G^* x_1, y_2) \tag{6.28}$$

with $u \in \mathbb{P}_{G\times M} \setminus \mathcal{M}$. We then obtain

$$\begin{aligned} d(da^*)^*c = d(\Phi \circ F^{-1} \circ da^*)^*(d\pi_G^* x_1, y_2) &= dr^* \circ d((\pi_G \circ q_1) \times q_2)^*(x_1, y_2) && (6.29)\\ &= d(((\pi_G \circ q_1) \times q_2) \circ r)^*(x_1, y_2). && (6.30) \end{aligned}$$

If $c \neq 0$, by (6.28), we have $(x_1, y_2) \neq (0, 0)$ and, since the map $((\pi_G \circ q_1) \times q_2) \circ r$ is a diffeomorphism, we get that $d(da^*)^*c \neq 0$. Hence, by Corollary 6.1.16, the map

$$a_* : \mathcal{V}^{-\infty}_{T^*(G\times M)\setminus 0, A\setminus 0}(G \times M) \to \mathcal{V}^{-\infty}(M)$$

is defined and sequentially continuous. Putting things together, the convolution is well-defined and jointly sequentially continuous.

2. We observe that the images of i_1 and i_2 are contained in $\mathcal{M}$, while the image of da^* is disjoint from $\mathcal{M}$, therefore, the second and the third term vanish. Hence, by (6.27), we get

$$r = \Phi \circ F^{-1} \circ da^* \Rightarrow r^* = (da^*)^* F_* \Phi^*,$$

where this equation holds at first for smooth differential forms but since they are dense, it holds for all $T \in \mathcal{D}_A(\mathbb{P}_{G\times M})$ and we obtain the desired formula.

3. By (6.4), we have

$$\operatorname{supp}(\mu \boxtimes \psi) \subseteq \operatorname{supp}(\mu) \times \operatorname{supp}(\psi)$$

and, by (6.23),

$$\operatorname{supp}(a_*(\mu \boxtimes \psi)) \subseteq a(\operatorname{supp}(\mu \boxtimes \psi)) \subseteq a(\operatorname{supp} \mu \times \operatorname{supp} \psi).$$

Thus the claim follows.

■

Proposition 6.2.7. *Let $\mu \in \mathcal{V}_{c,t}^{-\infty}(G)$. If $\psi \in \mathcal{V}^{-\infty}(M)$ is smooth or belongs to $\mathcal{V}_t^{-\infty}(M)$, then the same holds true for $\mu * \psi$. The maps*

1. $* : \mathcal{V}_{c,t}^{-\infty}(G) \times \mathcal{V}^{\infty}(M) \to \mathcal{V}^{\infty}(M)$,
2. $* : \mathcal{V}_{c,t}^{-\infty}(G) \times \mathcal{V}_t^{-\infty}(M) \to \mathcal{V}_t^{-\infty}(M)$,

are jointly sequentially continuous. In particular, if G acts on itself by multiplication, then $\mathcal{V}_{c,t}^{-\infty}(G)$ is closed under convolution.

Proof. The currents corresponding to $\mu * \psi$ are (C, T) with $T = p_*(da^*)^*T'$. To compute the wave front, let $s \in (G \times M) \times_{a,\pi_M} \mathbb{P}_M$. According to (6.29) we have

$$d(da^*)^*c = d(\Phi \circ F^{-1} \circ da^*)^*(d\pi_G^* x_1, y_2)$$

and, therefore,

$$\overline{WF}((da^*)^*T') \subseteq \{dr_s^*(d\pi_G^* x_1, y_2) : x_1 \in T^*_{\pi_G(q_1(r(s)))}G, y_2 \in \overline{WF}(T_2)|_{q_2 \circ r(s)}\}.$$

We conclude that the push-forward $T = p_*(da^*)^*T'$ satisfies

$$\begin{aligned} \overline{WF}(T)|_t =& \overline{WF}(p_*(da^*)^*T')|_t \subseteq p_*(da^*)^*\overline{WF}(T') && (6.31)\\ \subseteq& \{\eta \in T_t^*\mathbb{P}_M : \exists s \in p^{-1}(t), \exists \alpha_1 \in T^*_{\pi_G(q_1(r(s)))}G, && (6.32)\\ & \exists y_2 \in \overline{WF}(T_2)|_{q_2(r(s))} \text{ such that } dp_s^*(\eta) = dr_s^*(d\pi_G^* x_1, y_2)\} =: B && (6.33) \end{aligned}$$

with $t \in \mathbb{P}_M$.

1. For $\psi \in \mathcal{V}^{\infty}(M)$, we have T_2 is smooth and, therefore, $\overline{WF}(T_2) = 0$. Let $\eta \in B$, then there are $s = (g, p, [\xi]) \in p^{-1}(t)$ and $x_1 \in T_g^*G$ with
$$dp_s^*(\eta) = dr_s^*(d\pi_G^* x_1, 0).$$
Now, for $v \in T_t\mathbb{P}_M$, we can choose an element $\tilde{v} \in T_s((G \times M) \times_{a,\pi_M} \mathbb{P}_M)$ with $\tilde{v} = (v, 0)$. Since $\pi_G \circ q_1 \circ r : (G \times M) \times_{a,\pi_M} \mathbb{P}_M \to G$ is the projection on the first factor and p is the projection on the last factor, we have $dp_s\tilde{v} = v$ and $d(\pi_G \circ q_1 \circ r)_s\tilde{v} = 0$. Thus,
$$\begin{aligned} \langle \eta, v \rangle &= \langle \eta, dp_s\tilde{v} \rangle = \langle dp_s^*\eta, \tilde{v} \rangle = \langle dr_s^*(d\pi_G^* x_1, 0), \tilde{v} \rangle \\ &= \langle d(\pi_G \circ q_1 \circ r)_s^* x_1, \tilde{v} \rangle = \langle x_1, d(\pi_G \circ q_1 \circ r)_s \tilde{v} \rangle = 0. \end{aligned}$$
Hence, $\eta = 0$, which means that T is smooth.
2. For $\psi \in \mathcal{V}_t^{-\infty}(M)$, we have, by definition, $\overline{WF}(T_2) \subseteq d\pi_M^*(T^*M)$. Let $\eta \in B$. Then we have,
$$dp_s^*\eta = dr_s^*(d\pi_G^* x_1, d\pi_M^* x_2) = d(\Phi \circ F^{-1} \circ da^*)_s^*(d\pi_G^* x_1, d\pi_M^* x_2).$$
For $((g, p), [\xi_1 : \xi_2]) \in \mathbb{P}_{G \times M} \setminus \mathcal{M}$, we have
$$\begin{aligned} (\pi_G \times \pi_M) \circ \Phi \circ F^{-1}((g, p), [\xi_1 : \xi_2]) &= (\pi_G \times \pi_M)((g, [\xi_1]), (p, [\xi_2])) \\ &= (g, p) = \pi_{G \times M}((g, p), [\xi_1 : \xi_2]). \end{aligned}$$
We therefore, get
$$dp_s^*\eta = d(da^*)^* d\pi_{G \times M}^*(x_1, x_2).$$

Now, we take a vector $v \in T_t\mathbb{P}_M$ with $d\pi_M(v) = 0$. As in the first part of this proof, we can choose an element $\tilde{v} \in T_s((G\times M)\times_{a,\pi_M}\mathbb{P}_M)$ such that $dp_s(\tilde{v}) = v$ and $d(\pi_{G\times M}\circ da^*)_s\tilde{v} = 0$. We obtain

$$\langle \eta, v\rangle = \langle \eta, dp_s\tilde{v}\rangle = \langle dp_s^*\eta, \tilde{v}\rangle = \langle (x_1,x_2), d(\pi_{G\times M}\circ da^*)_s\tilde{v}\rangle = 0.$$

Once again, by use of $(\ker d\pi_M)^\perp = \operatorname{im} d\pi_M^*$, it follows that $\eta \in d\pi_M^*(T^*M)$ and $\phi * \psi \in \mathcal{V}_t^{-\infty}(M)$.

■

Finally, the convolution makes $\mathcal{V}_{c,t}^{-\infty}(G)$ to an algebra. To verify that, we need the associativity of the convolution:

Proposition 6.2.8. *Let G act transitively on M. Let $\mu_1,\mu_2 \in \mathcal{V}_{c,t}^{-\infty}(G)$ and $\phi \in \mathcal{V}^{-\infty}(M)$. Then*

$$\mu_1 * (\mu_2 * \phi) = (\mu_1 * \mu_2) * \phi.$$

Proof. Let us denote the multiplication on G by $m : G\times G \to G$ and the action of G on M by $a : G\times M \to M$. We also have to consider the map $\tilde{a} : G\times G\times M \to M$ defined by

$$(g_1, g_2, p) \mapsto g_1g_2p.$$

Moreover, we define maps $r_1 : (G\times G\times M)\times_{\tilde{a},\pi_M}\mathbb{P}_M \to \mathbb{P}_G \times ((G\times M)\times_{a,\pi_M}\mathbb{P}_M)$

$$(g_1, g_2, p, [\tau]) \mapsto ((g_1, [\xi_1]), (g_2, p, [\xi_2]))$$

for $da^*_{(g_1,g_2p)}(\tau) = (\xi_1,\xi_2) \in T^*_{g_1}G \times T^*_{g_2p}M$ and $r_2 : (G\times G\times M)\times_{\tilde{a},\pi_M}\mathbb{P}_M \to ((G\times G)\times_{m,\pi_G}\mathbb{P}_G)\times\mathbb{P}_M$ by

$$(g_1, g_2, p, [\tau]) \mapsto ((g_1, g_2, [\xi_1]), (p, [\xi_2]))$$

for $da^*_{(g_1g_2,p)}(\tau) = (\xi_1,\xi_2) \in T^*_{g_1g_2}G\times T^*_pM$. Finally, we need the maps $p_1, p_2 : (G\times G\times M)\times_{\tilde{a},\pi_M}\mathbb{P}_M \to (G\times M)\times_{a,\pi_M}\mathbb{P}_M$ with $p_1(g_1,g_2,p,[\tau]) = (g_1, g_2p, [\tau])$ and $p_2(g_1,g_2,p,[\tau]) = (g_1g_2, p, [\tau])$. Let r_M, r_G, p_M and p_G be defined as in (6.27). We obtain the following diagram

$$\begin{tikzcd}
(G\times G\times M)\times_{\tilde{a},\pi_M}\mathbb{P}_M \arrow[r, "r_1"] \arrow[d, "p_1"] & \mathbb{P}_G\times((G\times M)\times_{a,\pi_M}\mathbb{P}_M) \arrow[r, "id\times r_M"] \arrow[d, "id\times p_M"] & \mathbb{P}_G\times\mathbb{P}_G\times\mathbb{P}_M \\
(G\times M)\times_{a,\pi_M}\mathbb{P}_M \arrow[r, "r_M"'] \arrow[d, "p_M"] & \mathbb{P}_G\times\mathbb{P}_M & \\
\mathbb{P}_M & &
\end{tikzcd} \tag{6.34}$$

and the corresponding diagram

$$\begin{tikzcd}
(G\times G\times M)\times_{\tilde{a},\pi_M}\mathbb{P}_M \arrow[r, "r_2"] \arrow[d, "p_1"] & ((G\times G)\times M)\times_{m,\pi_G}\mathbb{P}_G)\times\mathbb{P}_M \arrow[r, "r_G\times id"] \arrow[d, "p_G\times id"] & \mathbb{P}_G\times\mathbb{P}_G\times\mathbb{P}_M \\
(G\times M)\times_{a,\pi_M}\mathbb{P}_M \arrow[r, "r_M"'] \arrow[d, "p_M"] & \mathbb{P}_G\times\mathbb{P}_M & \\
\mathbb{P}_M & &
\end{tikzcd} \tag{6.35}$$

We set

$$
\begin{aligned}
\tilde{r}_1 &:= (id \times r_M) \circ r_1, \\
\tilde{r}_2 &:= (r_g \times id) \circ r_2, \\
\tilde{p}_1 &:= p_M \circ p_1, \\
\tilde{p}_2 &:= p_M \circ p_2.
\end{aligned}
$$

Let $(C_1, T_1), (C_2, T_2), (C_3, T_3)$ and (C, T) be the corresponding currents representing μ_1, μ_2, ϕ and $\mu_1 * (\mu_2 * \phi)$. Then

$$
\mu_1 * (\mu_2 * \phi) : \quad
\begin{tikzcd}
T_1 \arrow[drr] & & \\
T_2 \arrow[dr] & & p_{M*}r_M^* \left(T_1 \boxtimes p_{M*}r_M^*(T_2 \boxtimes T_3)\right) \\
& p_{M*}r_M^*(T_2 \boxtimes T_3) \arrow[ur] & \\
T_3 \arrow[u, ur] & &
\end{tikzcd}
\tag{6.36}
$$

It follows with the reprsentation of the current T,

$$
\begin{aligned}
T &= p_{M*}r_M^* \left(T_1 \boxtimes p_{M*}r_M^*(T_2 \boxtimes T_3)\right) \\
&= (p_M)_* r_M^* (id \times p_M)_* (id \times r_M)^* (T_1 \boxtimes T_2 \boxtimes T_3) \\
&= (p_M)_* (p_1)_* r_1^* (id \times r_M)^* (T_1 \boxtimes T_2 \boxtimes T_3) \\
&= (\tilde{p}_1)_* (\tilde{r}_1)^* (T_1 \boxtimes T_2 \boxtimes T_3).
\end{aligned}
$$

On the other hand, let (C', T') be the pair of currents representing $(\mu_1 * \mu_2) * \phi$. Then

$$
(\mu_1 * \mu_2) * \phi : \quad
\begin{tikzcd}
T_1 \arrow[dr] & & \\
& p_{G*}r_G^*(T_1 \boxtimes T_2) \arrow[dr] & \\
T_2 \arrow[ur] & & p_{M*}r_M^* \left(T_3 \boxtimes p_{G*}r_G^*(T_1 \boxtimes T_2)\right) \\
T_3 \arrow[urr] & &
\end{tikzcd}
\tag{6.37}
$$

and again we obtain

$$
\begin{aligned}
T' &= p_{M*}r_M^* \left(T_3 \boxtimes p_{G*}r_G^*(T_1 \boxtimes T_2)\right) \\
&= (p_M)_* r_M^* (p_G \times id)_* (r_G \times id)^* (T_1 \boxtimes T_2 \boxtimes T_3) \\
&= (p_M)_* (p_2)_* r_2^* (r_G \times id)^* (T_1 \boxtimes T_2 \boxtimes T_3) \\
&= (\tilde{p}_2)_* (\tilde{r}_2)^* (T_1 \boxtimes T_2 \boxtimes T_3)
\end{aligned}
$$

Hence we can deduce that $T = T'$. Therefore, $\mu_1 * (\mu_2 * \phi) - (\mu_1 * \mu_2) * \phi$ is represented by $(\tilde{C}, 0)$ for some $\tilde{C}$. Since we can approximate with a sequence of smooth valuations, we are

going to examine the case $\mu_1, \mu_2 \in \mathcal{V}^\infty(G)$ and $\phi \in \mathcal{V}^\infty(M)$ at first. Hence, $\tilde{C}$ is smooth and, by Theorem 2.2.5, $\mu_1 * (\mu_2 * \phi) - (\mu_1 * \mu_2) * \phi$ is a multiple of the Euler characteristic on M. Aiming for $\mu_1 * (\mu_2 * \phi) = (\mu_1 * \mu_2) * \phi$, we can restrict ourselves to the case that the supports of the valuations μ_1, μ_2 and ϕ are contained in open sets $U_1, U_2 \subseteq G$ and $U_3 \subseteq M$ such that $\tilde{a}(U_1 \times U_2 \times U_3) \neq M$, this is possible since we can choose a partition of unity. Then $\mu_1 * (\mu_2 * \phi) - (\mu_1 * \mu_2) * \phi = r\chi$ is a multiple of the Euler characteristic and supported in a proper subset of M, therefore $r = 0$.
Next let $\mu_1, \mu_2 \in \mathcal{V}_{c,t}^{-\infty}(G)$ and $\phi \in \mathcal{V}^{-\infty}(M)$. By Proposition 6.1.9, it follows that there exist sequences $\mu_1^j, \mu_2^j \in \mathcal{V}_{c,t}^{-\infty}(G)$ and $\phi_j \in \mathcal{V}^\infty(M)$ converging to μ_1, μ_2 and ϕ respectively. Finally, by jointly sequential continuity, we have

$$\mu_1^j * (\mu_2^j * \phi^j) \to \mu_1 * (\mu_2 * \phi)$$

and

$$(\mu_1^j * \mu_2^j) * \phi^j \to (\mu_1 * \mu_2) * \phi.$$

By the case above, we have

$$\mu_1^j * (\mu_2^j * \phi^j) = (\mu_1^j * \mu_2^j) * \phi^j$$

and, hence,

$$\mu_1 * (\mu_2 * \phi) = (\mu_1 * \mu_2) * \phi.$$

■

Due to 3 in Proposition 6.2.6, the support of the convolution of two compactly supported generalized valuations is compact, since it is a closed subset of the set $a(\operatorname{supp}\mu \times \operatorname{supp}\psi)$, which is the image of a compact set under a continuous function. This and Proposition 6.2.8 now imply:

Corollary 6.2.9. *The space $(\mathcal{V}_{c,t}^{-\infty}(G), *)$ is an algebra.*

6.2.1 Extension of the convolution of valuations on vector spaces

As we will see in this section the convolution defined in Definition 6.2.1 is an extension of the convolution of Section 3.

For $A \in \mathcal{K}(V)$, we can define a generalized valuation

$$\langle \Gamma(A), \mu \rangle := \mu(A), \quad \mu \in \mathcal{V}_c^\infty(V).$$

If we take a compactly supported signed measure μ on V, we obtain the generalized valuation

$$\psi_{\mu,A} := \int_V \Gamma(y - A)\, d\mu(y) \in \mathcal{V}^{-\infty}(V).$$

With fixed μ, the map $\mathcal{K}(V) \to \mathcal{V}^{-\infty}(V)$

$$A \mapsto \psi_{\mu,A} \tag{6.38}$$

is continuous. This follows by dominated convergence and the fact that the normal cycle is a continuous map from $\mathcal{K}(V) \to \mathbb{R}$.

Proposition 6.2.10. *Let μ be a smooth compactly supported signed measure and let A be a smooth convex body. Then the image of the smooth valuation τ with $\tau(K) = \mu(K + A)$ under the injection $\mathcal{V}^\infty(V) \to \mathcal{V}^{-\infty}(V)$ equals $\psi_{\mu,A}$.*

Proof. This follows by the definition of the product and the perfect pairing. If $\nu \in \mathcal{V}_c^\infty(V)$, then

$$\langle \tau, \nu \rangle = \int \tau \cdot \nu = \int_V \nu(y - A) d\mu(y) = \int_V \langle \Gamma(y - A), \nu \rangle d\mu(y) = \langle \psi_{\mu,A}, \nu \rangle.$$

■

Proposition 6.2.11. *Let V_1, V_2 be affine spaces. Let μ_1, μ_2 be compactly supported signed measures and $A_i \in \mathcal{K}(V_i), i = 1, 2$. Then*

$$\psi_{\mu_1,A_1} \boxtimes \psi_{\mu_2,A_2} = \psi_{\mu_1 \boxtimes \mu_2, A_1 \times A_2}.$$

Proof. By the following computations, we have

$$\begin{aligned}
\psi_{\mu_1 \boxtimes \mu_2, A_1 \times A_2}(\nu_1, \nu_2) &= \int_{V_1 \times V_2} \langle \Gamma((y_1, y_2) - A_1 \times A_2), (\nu_1, \nu_2) \rangle d(\mu_1 \boxtimes \mu_2)(y_1, y_2) = \\
&= \int_{V_1} \int_{V_2} \langle \Gamma((y_1 - A_1) \times (y_2 - A_2)), (\nu_1, \nu_2) \rangle d\mu_1(y_1) d\mu_2(y_2) \\
&= \int_{V_1} \nu_1(y_1 - A_1) d\mu_1(y_1) \boxtimes \int_{V_2} \nu_2(y_2 - A_2) d\mu_2(y_2) \\
&= \int_{V_1} \langle \Gamma(y_1 - A_1), \nu_1 \rangle d\mu_1(y_1) \boxtimes \int_{V_2} \langle \Gamma(y_2 - A_2), \nu_2 \rangle d\mu_2(y_2) \\
&= (\psi_{\mu_1,A_1} \boxtimes \psi_{\mu_2,A_2})(\nu_1, \nu_2).
\end{aligned}$$

Thus the claim follows. ■

Finally we can show that the convolution on generalized valuations actually extends the convolution constructed in Section 3.

Proposition 6.2.12. *Let $a : V \times V \to V$ be the addition map. Let μ be a compactly supported smooth signed measure, $A_1, A_2 \in \mathcal{K}^\infty(V)$ with positive curvature. Then*

$$a_* \psi_{\mu, A_1 \times A_2} = \psi_{a_*\mu, A_1 + A_2}.$$

Proof. We define a sequence of convex bodies with smooth boundary and positive curvature, B_i, by their support functions. After fixing a Euclidean scalar product, we can take a sequence of smooth probability measures ρ_i on $SO(2n)$, with $\mathrm{supp}(\rho_i) \to Id$ and set

$$h(B_i, \xi) := \int_{SO(2n)} h(g(A_1 \times A_2), \xi) d\rho_i(g).$$

B_i has the desired properties, since we can interchange differentiation and integration and it has positive curvature, since $A_1 \times A_2$ has positive curvature. Now, let us take an arbitrary $\xi \in V^*$. The map $g \mapsto g(A_1 \times A_2)$ is continuous with respect to the topology on $SO(2n)$ and the Hausdorff metric. Due to the continuity of h, we get a neighbourhood W of $A_1 \times A_2$, such that $|h(A, \xi) - h(A_1 \times A_2, \xi)| < \epsilon$ for all $A \in W$. Furthermore let us take a neighbourhood U of the identity element in $SO(2n)$ and let i be sufficiently large such that $\mathrm{supp}(\rho_i) \subseteq U$ and $U(A_1 \times A_2) \subseteq W$. We obtain with the triangle inequality,

$$\left| \int_{SO(2n)} h(g(A_1 \times A_2), \xi) - h(A_1 \times A_2, \xi) d\rho_i(g) \right| < \epsilon.$$

Hence, we have convergence of $B_i \to A_1 \times A_2$. We aim for convergence of ψ_{μ,B_i} in $\mathcal{V}^{-\infty}_{\Lambda,\Gamma}(V \times V)$, therefore we describe the corresponding currents and seek for an integral representation of the form

$$\mu(K + B_i) = \int_K \nu + \int_{N(K)} \eta$$

for some $\nu \in \Omega^{2n}(V \times V)$ and $\eta \in \Omega^{2n-1}(\mathbb{P}_{V\times V})$.

- ν: By definition of the convolution of a measure with a measureable and bounded function, the current C_i is given by the smooth function $\mathbb{1}_{B_i} * \mu \in C_c^\infty(V)$. Therefore $\nu = \mathbb{1}_{B_i} * \mu$.
- η: Since B_i is smooth, the normal cycle of $K + B_i$ is given by $(\tau_i)_* N(K)$ (this is similar to the map G_A in Equation (3.2)), where $\tau_i : \mathbb{P}_{V\times V} \to \mathbb{P}_{V\times V}$ is given by $\tau_i(x_1, x_2, [\xi_1 : \xi_2]) = ((x_1, x_2) + d_{(\xi_1,\xi_2)} h_{B_i}, [\xi_1 : \xi_2])$. By the Poincare lemma, we can choose $\kappa \in \Omega^{2n-1}(V \times V)$ with $d\kappa = \mu$. With Stoke's theorem, we get

$$\mu(K + B_i) = \int_{K+B_i} \mu = \int_{K+B_i} d\kappa = \int_{\partial(K+B_i)} \kappa = \int_{N(K+B_i)} \pi^*_{V\times V}\kappa = \int_{N(K)} \tau_i^* \pi^*_{V\times V}\kappa.$$

  Hence $\eta = \tau_i^* \pi^*_{V\times V}\kappa$.

By 5.7 and 5.8 we infer that $C_i = \mathbb{1}_{B_i} * \mu$ and $T_i = D(\tau_i^* \pi^*_{V\times V}\kappa) = \tau_i^* \pi^*_{V\times V}\mu$. Next, let us consider the set $\mathcal{M}$ from the beginning of this section:

$$\mathcal{M} = V \times V \times S \subseteq \mathbb{P}_{V\times V} = \mathbb{P}_+(T^*(V \times V)) = (V \times V) \times \mathbb{P}_+((V \times V)^*),$$

where

$$S := \{[\xi_1 : 0] : \xi_1 \in V^* \setminus \{0\}\} \cup \{[0 : \xi_2] : \xi_2 \in V^* \setminus \{0\}\} \subseteq \mathbb{P}_+((V \times V)^*).$$

Let $\Lambda \subseteq T^*(V \times V) \setminus 0$ be the set generated by the positive multiples on the second factor in $T^*(V \times V) \setminus 0$ of $\mathcal{M}$ and set $\Gamma := \pi^*_{V\times V}\Lambda \subseteq T^*\mathbb{P}_{V\times V} \setminus 0$. Continuity of the convolution with respect to the Hausdorff metric yields convergence of $\mathbb{1}_{B_i} * \mu$ in $\mathcal{D}_{n,\emptyset}(V \times V)$ to $\mathbb{1}_{A_1\times A_2} * \mu$. The support function of $A_1 \times A_2$ is smooth outside S and $h_{B_i}|_{(V\times V)^*\setminus S} \to h_{A_1\times A_2}|_{(V\times V)^*\setminus S}$, in $C^\infty((V \times V)^* \setminus S)$. Therefore, τ_i converges in $C^\infty(\mathbb{P}_{V\times V} \setminus \mathcal{M}, \mathbb{P}_{V\times V} \setminus \mathcal{M})$. Hence the currents T_i converge smoothly to T outside Γ. We can conclude that $\psi_{\mu,B_i} \to \psi_{\mu,A_1\times A_2}$ in $\mathcal{V}^{-\infty}_{\Lambda,\Gamma}(V \times V)$. By Lemma 6.2.2, we have

$$\Gamma \cap N^*_{\mathbb{P}_{V\times V}}((V \times V) \times_{a,\pi_{V\times V}} \mathbb{P}_V) = \emptyset.$$

Therefore, Proposition 6.2.6 implies that the push-forward map

$$a_* : \mathcal{V}^{-\infty}_{\Lambda,\Gamma}(V \times V) \to \mathcal{V}^{-\infty}(V)$$

is a sequentially continuous map. By [6, Proposition 3.6.5] we have

$$a_*\psi_{\mu,B_i} = \psi_{a_*\mu,a(B_i)}$$

and with (6.38) we obtain the convergence of $a_*\psi_{\mu,B_i} = \psi_{a_*\mu,a(B_i)}$ to $\psi_{a_*\mu,A_1+A_2}$ in $\mathcal{V}^{-\infty}(V)$, since addition is continuous in the Hausdorff metric. Hence we get $a_*\psi_{\mu,B_i} \to a_*\psi_{\mu,A_1\times A_2}$ in the weak topology, and, moreover, on $a_*\psi_{\mu,A_1\times A_2} = \psi_{a_*\mu,A_1+A_2}$, as claimed. ∎

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