The structural complexity of models of arithmetic

joint work with Antonio Montalbán

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It is easy to answer this questions for the standard model $\mathbb{N}$: *It is structurally easy.*

But what about non-standard models?

Let us give a framework to answer this questions.

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For every countable structure $\mathcal{A}$ there is a sentence in the infinitary logic $L_{\omega_1\omega}$ – its Scott sentence – characterizing $\mathcal{A}$ up to isomorphism among countable structures.

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The most useful definition is due to Ash and Knight:

Definition

1. $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if all atomic fromulas true of $\bar{b}$ are true of $\bar{a}$ and vice versa.
2. For non-zero $\gamma < \omega_1$, $(\mathcal{A}, \bar{a}) \leq_\gamma (\mathcal{B}, \bar{b})$ if for all $\beta < \gamma$ and $\bar{d} \in B^{<\omega}$ there is $\bar{c} \in A^{<\omega}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

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In an attempt to measure structural complexity, various notions of ranks have been used.

E.g. $r(\mathcal{A})$ is the least α such that for all $\bar{a}, \bar{b} \in A$ if $\bar{a} \leq_\alpha \bar{b}$, then $\bar{a} \leq_\beta \bar{b}$ for all $\beta > \alpha$.

1. A formula is $\Sigma_0^{\mathrm{in}} = \Pi_0^{\mathrm{in}}$ if it is a finite quantifier free formula.
2. A formula is $\Sigma_\alpha^{\mathrm{in}}$ for $\alpha > 0$, if it is of the form $\bigvee\!\!\!\bigvee_{i\in\omega} \exists \bar{x}_i \psi_i(\bar{x}_i)$ where all $\psi_i \in \Pi_{\beta_i}^{\mathrm{in}}$ for $\beta_i < \alpha$.
3. A formula is Π_α^{in} for $\alpha > 0$, if it is of the form $\bigwedge\!\!\!\bigwedge_{i\in\omega} \forall \bar{x}_i \psi_i(\bar{x}_i)$ where all $\psi_i \in \Sigma_{\beta_i}^{\mathrm{in}}$ for $\beta_i < \alpha$.
4. $L_{\omega_1\omega} = \bigcup_{\alpha<\omega_1} \Pi_\alpha^{\mathrm{in}}$

For example, let p_n denote the (formal term) for the nth prime in PA and let $X \subseteq \omega$. Then

$$\varphi = \exists x \left(\bigwedge\!\!\!\bigwedge_{n\in X} \exists y (y \cdot p_n = x) \wedge \bigwedge\!\!\!\bigwedge_{n\notin X} \forall y (y \cdot p_n \neq x) \right)$$

is a Σ_3^{in} formula and $\mathcal{A} \models \varphi$ iff X is in the Scott set of $\mathcal{A}$.

Theorem (Montalbán 2015)

The following are equivalent for countable $\mathcal{A}$ and $\alpha < \omega_1$.

1. *Every automorphism orbit of $\mathcal{A}$ is $\Sigma^{\mathrm{in}}_{\alpha}$-definable without parameters.*
2. *$\mathcal{A}$ has a $\Pi^{\mathrm{in}}_{\alpha+1}$ Scott sentence.*
3. *$\mathcal{A}$ is uniformly $\mathbf{\Delta}^0_{\alpha}$-categorical.* $(\exists \Phi \exists X \forall \mathcal{B} \cong \mathcal{C} \cong \mathcal{A} (\Phi^{X \oplus (\mathcal{C} \oplus \mathcal{B})^{(\alpha)}} : \mathcal{B} \cong \mathcal{C})$
4. *$Iso(\mathcal{A})$ is $\mathbf{\Pi}^0_{\alpha+1}$.*
5. *No tuple in $\mathcal{A}$ is α-free.*

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The standard model $\mathbb{N}$ of PA has Scott rank 1: Every element is the nth successor of $\dot{0}$ for some $n \in \omega$, so the automorphism orbits are definable by $s(s(\ldots(\dot{0})\ldots)) = x$.

Theorem (Karp)

For two countable structures $\mathcal{A}$ the following are equivalent.

1. $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$.
2. *All $\Sigma_\alpha^{\mathrm{in}}$ sentences true of $\bar{b}$ in $\mathcal{B}$ are true of $\bar{a}$ in $\mathcal{A}$.*
3. *All Π_α^{in} sentences true of $\bar{a}$ in $\mathcal{A}$ are true of $\bar{b}$ in $\mathcal{B}$.*

In other words, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ iff $\Pi_\alpha^{\mathrm{in}}\text{-}tp^{\mathcal{A}}(\bar{a}) \subseteq \Pi_\alpha^{\mathrm{in}}\text{-}tp^{\mathcal{B}}(\bar{b})$.

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Definition

A tuple $\bar{a}$ in $\mathcal{A}$ is *α-free* if

$$\forall(\beta < \alpha)\forall\bar{b}\exists\bar{a}'\bar{b}'(\bar{a}\bar{b} \leq_\beta \bar{a}'\bar{b}' \wedge \bar{a} \not\leq_\alpha \bar{a}').$$

Definition (Makkai 1981)

The *Scott spectrum* of a theory T is the set

$$SS(T) = \{\alpha \in \omega_1 : \text{there is a countable model of } T \text{ with Scott rank } \alpha\}.$$

Here T might be a sentence in $L_{\omega_1\omega}$.

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- Ash (1986) characterized back-and-forth relations of well-orderings. The following is a corollary: $SR(n) = 1$, $SR(\omega^\alpha) = 2\alpha$, $SR(\omega^\alpha + \omega^\alpha) = 2\alpha + 1$.
- $SS(LO) = \omega_1 - 0$
- $1 \in SS(PA)$

Throughout this talk $\mathcal{M}$ and $\mathcal{N}$ denote countable non-standard models of PA.

- Back-and-forth relations accept tuples of arbitrary length.
- Makes it impossible to formalize in first order logic.
- In PA we can talk about being n-bf equivalent up to some length a of tuples for $a \in M$.

Let $Tr_{\Delta_1^0}$ be a truth predicate for bounded formulas and define the bounded back-and-forth relations by induction on n:

$$\bar{u} \leq_0^a \bar{v} \Leftrightarrow \forall (x \leq a)(Tr_{\Delta_1^0}(x, \bar{u}) \rightarrow Tr_{\Delta_1^0}(x, \bar{v}))$$

$$\bar{u} \leq_{n+1}^a \bar{v} \Leftrightarrow \forall \bar{x} \exists \bar{y} \Big(|\bar{x}| \leq a \rightarrow (|\bar{y}| \leq a \wedge \bar{v}\bar{x} \leq_n^a \bar{u}\bar{y}) \Big)$$

Proposition

The bounded back-and-forth relations $\leq_n^x$ satisfy the following properties for all n:

1. $PA \vdash \forall \bar{u}, \bar{v}, a, b((a \leq b \wedge \bar{u} \leq_n^b \bar{v}) \rightarrow \bar{u} \leq_n^a \bar{v})$
2. $PA \vdash \forall \bar{u}, \bar{v}, a(\bar{u} \leq_{n+1}^a \bar{v} \rightarrow \bar{u} \leq_n^a \bar{v})$

Proposition

Let $\bar{a}, \bar{b} \in M$. Then $\bar{a} \leq_n \bar{b} \Leftrightarrow \forall (m \in \omega) \mathcal{M} \models \bar{a} \leq_n^{\dot{m}} \bar{b}$. Furthermore, if there is $c \in M - \mathbb{N}$ such that $\mathcal{M} \models \bar{a} \leq_n^c \bar{b}$, then $\bar{a} \leq_n \bar{b}$.

Lemma

For every $\bar{a}, \bar{b} \in M^{<\omega}$, $\bar{a} \leq_\omega \bar{b}$ *if and only if* $tp(\bar{a}) = tp(\bar{b})$.

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Recall that $\mathcal{M}$ is ***homogeneous*** if every partial elementary map $M \to M$ is extendible to an automorphism.

Lemma

If $\mathcal{M}$ is not homogeneous then $SR(\mathcal{M}) > \omega$.

Proposition

If $\mathcal{M}$ is homogeneous, then $SR(\mathcal{M}) \leq \omega + 1$.

Note that every completion T of PA has an atomic model. Take $\mathcal{M} \subseteq T$ and the subset of all Skolem terms without parameters. This is an elementary substructure and all types realized are isolated. By the least number principle this model is rigid and its automorphism orbits in $\mathcal{M}$ are singletons.

Theorem (Montalbán, R.)

If $\mathcal{M}$ is atomic, then $SR(\mathcal{M}) = \omega$.

Theorem (Montalbán, R.)

For any nonstandard model $\mathcal{M}$, $SR(\mathcal{M}) \geq \omega$. In particular $(1, \omega) \cap SS(PA) = \emptyset$. If $T \supseteq PA$ does not have a standard model, then $1 \notin SS(T)$.

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Definition (Harrison-Trainor, Miller, Montalbán 2018)

A structure $\mathcal{A} = (A, P_0^{\mathcal{A}}, \dots)$ is *infinitary interpretable* in $\mathcal{B}$ if there exists a $L_{\omega_1\omega}$ definable in $\mathcal{B}$ sequence of relations $(Dom_{\mathcal{A}}^{\mathcal{B}}, \sim, R_0, \dots)$ such that

1. $Dom_{\mathcal{A}}^{\mathcal{B}} \subseteq B^{<\omega}$,
2. $\sim$ is an equivalence relation on $Dom_{\mathcal{A}}^{\mathcal{B}}$,
3. $R_i \subseteq (B^{<\omega})^{a_{P_i}}$ is closed under $\sim$ on $Dom_{\mathcal{A}}^{\mathcal{B}}$,

and there exists a function $f_{\mathcal{B}}^{\mathcal{A}} : (Dom_{\mathcal{A}}^{\mathcal{B}}, R_0, \dots)/\sim \, \cong (A, P_0^{\mathcal{A}}, \dots)$, the *interpretation of $\mathcal{A}$ in $\mathcal{B}$*. If the formulas in the interpretation are $\Delta_\alpha^{\text{in}}$ then $\mathcal{A}$ is $\Delta_\alpha^{\text{in}}$ interpretable in $\mathcal{B}$.

Definition (Harrison-Trainor, Miller, Montalbán 2018)

Two structures $\mathcal{A}$ and $\mathcal{B}$ are ***bi-interpretable*** if there are infinitary interpretations of one in the other such that the compositions

$$f^{\mathcal{A}}_{\mathcal{B}} \circ \hat{f}^{\mathcal{B}}_{\mathcal{A}} : Dom_{\mathcal{B}}^{Dom^{\mathcal{B}}_{\mathcal{A}}} \to \mathcal{B} \quad \text{and} \quad f^{\mathcal{B}}_{\mathcal{A}} \circ \hat{f}^{\mathcal{A}}_{\mathcal{B}} : Dom_{\mathcal{A}}^{Dom^{\mathcal{A}}_{\mathcal{B}}} \to \mathcal{A}$$

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Theorem (Harrison-Trainor, Miller, Montalbán 2018)

$\mathcal{A}$ and $\mathcal{B}$ are infinitary bi-interpretable iff their automorphism groups are Borel-measurably isomorphic.

Theorem (Gaifman 1976)

Let T be a completion of PA and $\mathcal{L}$ a linear order. Then there is a model $\mathcal{N}_{\mathcal{L}}$ of T such that $Aut(\mathcal{N}_{\mathcal{L}}) \cong Aut(\mathcal{L})$.

- Indiscernible construction with a ***minimal type*** $p(x)$.
- For every $l_1 < \cdots < l_n \in L$ obtain a model $\mathcal{N}(l_1) \dots (l_n)$ where each l_i realizes $p(x)$.
- Take the direct limit of all these models to be $\mathcal{N}_{\mathcal{L}}$

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Theorem (cf. Gaifman)

$(\{x \in N_L : x \vDash p(x)\}, \leq^{\mathcal{N}_{\mathcal{L}}}) \cong \mathcal{L}$

Thus, $\mathcal{L}$ is $\Delta_{\omega+1}^{\text{in}}$ interpretable in $\mathcal{N}_{\mathcal{L}}$. (As the universe is Π_{ω}^{in}.)

- $\mathcal{L}$ and $\mathcal{N}_{\mathcal{L}}$ are $\Delta^{\text{in}}_{\omega+1}$ bi-interpretable
- The complexities are asymmetric
  - The elementary diagram of $\mathcal{N}_{\mathcal{L}}$ is Δ^{in}_{1} interpretable in $\mathcal{L}$
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Definition

Given a τ-structure $\mathcal{A}$ and a countable ordinal $\alpha > 0$ fix an injective enumeration $(\bar{a}_i)_{i\in\omega}$ of the α-back-and-forth equivalence classes. The *canonical structural α-jump* $\mathcal{A}_{(\alpha)}$ of $\mathcal{A}$ is the structure in the vocabulary $\tau_{(\alpha)}$ obtained by adding to τ relation symbols R_i interpreted as

$$\bar{b} \in R_i^{\mathcal{A}_{(\alpha)}} \Leftrightarrow \bar{a}_i \leq_\alpha \bar{b}.$$

We will use the convention that $\mathcal{A}_{(0)} = \mathcal{A}$.

Proposition

Let $\mathcal{A}$ be a structura and $\alpha, \beta < \omega_1$ with $\beta > 0$. Then
$(\mathcal{A}_{(\alpha)}, \bar{a}) \leq_\beta (\mathcal{A}_{(\alpha)}, \bar{b}) \Leftrightarrow (\mathcal{A}, \bar{a}) \leq_{\alpha+\beta} (\mathcal{A}, \bar{b})$.

Proposition

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Corollary

For any structure $\mathcal{A}$ and non-zero $\alpha, \beta < \omega_1$, $SR(\mathcal{A}) = \alpha + \beta$ if and only if $SR(\mathcal{A}_{(\alpha)}) = \beta$.

Recall that two Δ_1^{in} bi-interpretable structures have the same Scott rank. So if $\mathcal{B}$ is Δ_1^{in} bi-interpretable with $\mathcal{A}_{(\alpha)}$, then $SR(\mathcal{A}) = \alpha + SR(\mathcal{B})$.

Corollary

For all countable ordinals α and β, the following are equivalent.

1. $\mathcal{A}_{(\gamma)}$ *is* Δ_1^{in} *bi-interpretable with* $\mathcal{B}_{(\alpha)}$.
2. $\mathcal{A}$ *is infinitary bi-interpretable with* $\mathcal{B}$ *such that*
   2.1 *the interpretation of* $\mathcal{A}$ *in* $\mathcal{B}$ *and* $f_{\mathcal{B}}^{\mathcal{A}} \circ \tilde{f}_{\mathcal{A}}^{\mathcal{B}}$ *are* $\Delta_{\alpha+1}^{\text{in}}$ *in* $\mathcal{B}$,
   2.2 *the interpretation of* $\mathcal{B}$ *in* $\mathcal{A}$ *and* $f_{\mathcal{A}}^{\mathcal{B}} \circ \tilde{f}_{\mathcal{B}}^{\mathcal{A}}$ *are* $\Delta_{\gamma+1}^{\text{in}}$ *in* $\mathcal{A}$,
   2.3 *for every* $\bar{a} \in Dom_{\mathcal{A}}^{\mathcal{B}}$, $\{\bar{c} : (\mathcal{A}^{\mathcal{B}}, \bar{c}) \models \Pi_{\gamma}^{\text{in}}\text{-}tp^{\mathcal{A}^{\mathcal{B}}}(\bar{a})\}$ *is* $\Delta_{\alpha+1}^{\text{in}}$ *definable in* $\mathcal{B}$,
   2.4 *for every* $\bar{b} \in Dom_{\mathcal{B}}^{\mathcal{A}}$, $\{\bar{c} : (\mathcal{B}^{\mathcal{A}}, \bar{c}) \models \Pi_{\alpha}^{\text{in}}\text{-}tp^{\mathcal{B}^{\mathcal{A}}}(\bar{b})\}$ *is* $\Delta_{\gamma+1}^{\text{in}}$ *definable in* $\mathcal{A}$.

Recall that $\mathcal{N}_{\mathcal{L}}$ is Δ_1^{in} interpretable in $\mathcal{L}$ and $\mathcal{L}$ is $\Delta_{\omega+1}^{\text{in}}$ interpretable in $\mathcal{N}_{\mathcal{L}}$. Hence, taking $\mathcal{A} = \mathcal{L}$ and $\mathcal{B} = \mathcal{N}_{\mathcal{L}}$, 2.1, 2.2 are satisfied for $\alpha = \omega$, $\gamma = 0$. It remains to show that the elements satisfying a fixed Π_{ω}^{in}-type in $\mathcal{N}_{\mathcal{L}}$ are both Δ_1^{in} definable in $\mathcal{L}$.

Recall that the elementary diagram of $\mathcal{N}_{\mathcal{L}}$ is interpetable in $\mathcal{L}$ and that $\{\bar{b} \vDash \Pi^{\text{in}}_{\omega}\text{-}tp(\bar{a})\} = \{\bar{b} \vDash tp(\bar{a})\}$ for any $\bar{a}$. The sets $\{\bar{b} \vDash tp(\bar{a})\}$ are Π^{in}_{1} definable in $\mathcal{L}$.

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To show that it is also Π^{in}_{1} definable notice that the following claim holds.

Lemma

Let s be a Skolem term and $a = s(l_1, \dots, l_n)$ where $l_1 < \dots < l_n \in L$. If $b = s(k_1, \dots, k_n)$ for some $k_1 < \dots < k_n \in L$ then $b \vDash tp(a)$.

Thus every set $\{\bar{b} \vDash tp(\bar{a})\}$ can be written as a union of Skolem terms with parameters ordered $\mathcal{L}$-tuples. Thus, the set is Π^{in}_{1} definable.

Theorem (Montalbán, R.)

Given a completion T of PA, there is a reduction via Δ^{in}_{1} bi-interpretability between $\mathcal{L}$ and the structural ω-jumps of its models.

Theorem (Montalbán, R.)

1. $SS(PA) = 1 \cup \{\alpha : \omega \leq \alpha \leq \omega_1\}$
2. *If* $\mathcal{M}$ *is non-homogeneous, then* $SR(\mathcal{M}) \geq \omega + 1$.
3. *If* $\mathcal{M}$ *is non-standard atomic , then* $SR(\mathcal{M}) = \omega$.
4. *If* $\mathcal{M}$ *is non-standard homogeneous, then* $SR(\mathcal{M}) \in [\omega, \omega + 1]$.
5. *For any completion* T *of* PA*, there is a* T*-computable model* $\mathcal{M}$ *with* $SR(\mathcal{M}) = \omega_1^T + 1$.

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Thank you!