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DISSERTATION

Fractional perimeters and symmetrization

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Unterschrift

Declaration

I hereby declare that I have written the present thesis on my own and that I have used only specified literature. The thesis has not been submitted elsewhere for the application of a scientific degree. Parts of this thesis are based on work that have been submitted by the author to scientific journals.

Vienna, March 2021

Andreas Kreuml

Kurzfassung

In der vorliegenden Dissertation werden Konvergenz und Symmetrisierung von fraktionellen Perimetern in verschiedenen Räumen untersucht.

Zu allererst wird eine Klassifizierung aller Gleichheitsfälle in der anisotropen fraktionellen isoperimetrischen Ungleichung angegeben unter der Annahme, dass die zugrundeliegende Einheitskugel symmetrisch zu jeder Koordinatenhyperebene und strikt konvex ist. Mit deren Hilfe wird gezeigt, dass die anisotrope Symmetrisierung bezüglich dieser Gleichheitsfälle wohldefiniert ist und eine anisotrope fraktionelle Pólya-Szegő-Ungleichung wird für diese Symmetrisierung hergeleitet.

Als Nächstes werden fraktionelle Seminormen und Perimeter auf Riemannschen Mannigfaltigkeiten eingeführt und deren Konvergenz zur Sobolev-Seminorm beziehungsweise zum Perimeter für $s \nearrow 1$ wird gezeigt. Für fraktionelle Perimeter auf der Sphäre wird ein alternativer Beweis für dieses Resultat mittels sphärischer Integralgeometrie angegeben. In diesem Speziallfall wird die Konvergenz von geeignet renormalisierten fraktionellen Perimetern gegen ein Volumsfunktional für $s \searrow -\infty$ gezeigt. Schlussendlich werden isoperimetrische Ungleichungen für sphärische fraktionelle Perimeter mit einer vollständigen Beschreibung aller Gleichheitsfälle hergeleitet.

Einige Resultate in dieser Dissertation sind in Zusammenarbeit mit Olaf Mordhorst entstanden.

Abstract

In this thesis, convergence and symmetrization of fractional perimeters in different settings are studied.

First, a classification of minimizers of the anisotropic fractional isoperimetric inequality is given whenever the unit ball of the space is unconditional and strictly convex. With its help it is shown that anisotropic symmetrization with respect to these minimizers is well-defined and an anisotropic fractional Pólya-Szegő principle for this symmetrization is established.

Next, fractional seminorms and perimeters are introduced on Riemannian manifolds and their convergence to the Sobolev seminorm and the perimeter, respectively, is shown as $s \nearrow 1$. For fractional perimeters on the sphere an alternative proof for this result using spherical integral geometry is presented. In this special case, the convergence of suitably normalized fractional perimeters towards a volume functional as $s \searrow -\infty$ is shown. Finally, isoperimetric-type inequalities for spherical fractional perimeters with a complete classification of equality cases are derived.

Some results of this thesis are joint work together with Olaf Mordhorst.

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Contents

Chapter 1

Introduction

For $0 < s < 1$ the *fractional s-perimeter* of a Lebesgue measurable set $E \subseteq \mathbb{R}^n$ is defined as

$$P_s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{n+s}} \, \mathrm{d}y \, \mathrm{d}x$$

where $E^c := \mathbb{R}^n \backslash E$ is the complement of E in $\mathbb{R}^n$ and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. It is invariant under translations and rotations, as well as positively homogeneous of degree $n-s$, i.e. for $\lambda > 0$ and every measurable set $E \subseteq \mathbb{R}^n$ it holds that $P_s(\lambda E) = \lambda^{n-s} P_s(E)$, where $\lambda E := \{\lambda x : x \in E\}$ is the dilate of E by the factor λ. Thus, fractional perimeters can be seen as $(n-s)$-dimensional perimeter functionals. Perhaps the most striking difference compared to other well-known perimeter functionals such as surface area or Hausdorff measures is that fractional perimeters are *non-local* in the sense that $P_s(E)$ is not determined by the behaviour of E in a neighborhood of the boundary ∂E.

Closely related to fractional s-perimeters are *fractional Sobolev s-seminorms*. For $1 \leq p < \infty$ and $0 < s < 1$ the fractional s-seminorm of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$|f|_{s,p} := \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

If $E \subseteq \mathbb{R}^n$ is measurable, then $P_s(E) = \frac{1}{2} |\chi_E|_{s,1}$, where $\mathbb{1}_E$ is the characteristic function of E, i.e. $\mathbb{1}_E(x) = 1$, whenever $x \in E$, and $\mathbb{1}_E(x) = 0$ otherwise. Thus, results obtained for fractional seminorms, where $p = 1$, can be easily translated into results for fractional perimeters. Fractional seminorms and corresponding Sobolev spaces were introduced in the 1950's independently by Aronszajn, Gagliardo and Slobodeckij, and have found a multitude of applications thereafter (see e.g. [DNPV12] and the references therein). However, the systematic study of fractional perimeters did not start until the early 2010's, when they arose as energy functionals in nonlocal phase transition problems (see e.g. [CRS10, ADPM11, SV12]).

One natural question is to ask how fractional perimeters are related to the standard surface area of a set. An answer given by Dávila [Dáv02], based on the work of Bourgain, Brezis & Mironescu [BBM01, BBM02], is to consider the behaviour of suitably rescaled s-perimeters as $s \nearrow 1$. They showed that

$$\lim_{s \nearrow 1} (1-s) P_s(E) = \alpha_n P(E) \tag{1.1}$$

where $\alpha_n > 0$ is a constant only depending on n, and $P(E)$ is the perimeter of E which coincides with the $(n-1)$-dimensional Hausdorff measure of ∂E for smooth sets. On the other hand, for $s \searrow 0$, Maz'ya and Shaposhnikova [MS02] showed that

$$\lim_{s \searrow 0} s P_s(E) = \beta_n |E| \tag{1.2}$$

where $\beta_n > 0$ is a constant only depending on n, and $|E|$ is the volume, i.e. the n-dimensional Lebesgue measure, of E. In this sense, fractional perimeters interpolate between the surface area and the volume.

Another big field of study is to consider geometric variational problems and inequalities for fractional perimeters. Perhaps the most famous inequality comparing perimeter and volume is the isoperimetric inequality which states that, given a fixed volume, only balls minimize the surface area, up to sets of measure zero. The following fractional isoperimetric inequality was proven by Frank & Seiringer [FS08]: If $E \subseteq \mathbb{R}^n$ is a Borel set then

$$P_s(E) \geq \gamma_{n,s} |E|^{\frac{n-s}{n}}$$

for a optimal constant $\gamma_{n,s} > 0$ only depending on n and s, with equality precisely for balls up to sets of measure zero.

The goal of this thesis to consider convergence and isoperimetric problems for fractional perimeters defined on certain spaces where the metric is different from the Euclidean case. In particular, anisotropic fractional perimeters and fractional perimeters defined on Riemannian manifolds are studied. The thesis is structured as follows:

Chapter 2 provides background material on fractional perimeters and Sobolev spaces, as well as from Riemannian geometry, and presents symmetrization techniques used in the following chapters.

In Chapter 3 we study anisotropic fractional perimeters and the corresponding isoperimetric problem. Anisotropic fractional perimeters and seminorms were first introduced by Ludwig in [Lud14a, Lud14b]. Here, the closed unit ball $K \subseteq \mathbb{R}^n$ induced by the norm $\|\cdot\|_K$ may differ from the Euclidean unit ball, and the anisotropic fractional s-perimeter of a measurable set $E \subseteq \mathbb{R}^n$ with respect to K is defined by

$$P_s(E, K) := \int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \, \mathrm{d}y \, \mathrm{d}x.$$

The question of convergence was fully answered by Ludwig and led to the following result which uncovered a surprising connection to convex geometry: If $E \subseteq \mathbb{R}^n$ is measurable, then

$$\lim_{s \nearrow 1} (1-s) P_s(E, K) = P(E, Z_1 K),$$

where $P(E, Z_1 K)$ is the anisotropic perimeter of E with respect to the *moment body* $Z_1 K$ of K. Furthermore, she argued that an optimal anisotropic fractional isoperimetric inequality,

$$P_s(E, K) \geq \gamma_{n,s}(K) |E|^{\frac{n-s}{n}}, \tag{1.3}$$

must hold true. In contrast to the (classical) anisotropic isoperimetric inequality, Ludwig observed that the minimizers of (1.3) in general cannot be homothetic to the unit ball K for values of s close to 1, and as of yet, the complete classification of minimizers is open (see also [XY17]). In this thesis, we give a partial answer for the special case that the unit ball K is an unconditional strictly convex body. In this case each minimizer must be, up to translation and Lebesgue nullsets, an unconditional star body. This enables us to define an anisotropic symmetrization with respect to these minimizers which we use to show a Pólya-Szegő principle for anisotropic fractional seminorms.

Next, in Chapter 4 we consider fractional perimeters defined on Riemannian manifolds. The question of convergence of fractional perimeters as $s \nearrow 1$ in this setting was raised in [FMP+18]. We fully answer this question and show that formula (1.1) also holds true in the setting of compact Riemannian manifolds. In the more general context of fractional seminorms, we show that the corresponding limit can be used to characterize Sobolev functions and functions of bounded variation on Riemannian manifolds which is well-known in the Euclidean case (see [BBM01]). A key technique for our proofs is the use of a covering of the manifold by charts in which the manifold locally looks like a Euclidean space.

Finally, Chapter 5 deals with the special case of the sphere $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. Based on techniques used in [Lud14a, Lud14b], we give an alternative proof for the convergence of fractional perimeters as $s \nearrow 1$ using integral geometric formulas for the sphere. Even though the analogue of fomula (1.2) for the limit $s \searrow 0$ does not lead to an interesting result, we show that a similar limit involving the volume appears for rescaled fractional perimeters as $s \searrow -\infty$. Lastly, we present isoperimetric-type inequalities for fractional perimeters on the sphere which partly are a direct consequence of results by Beckner [Bec92] for $-n < s < 1$, and extend their proof to the range $-\infty < s \leq -n$.

The results for anisotropic fractional perimeters in Chapter 3 are published in [Kre21]. Chapters 4 and 5 are based on joint work with Olaf Mordhorst. The work on fractional perimeters on Riemannian manifolds in Chapter 4 can be found in [KM19], the results on the sphere from Chapter 5 in [KM20].

Chapter 2

Background and Notation

The purpose of this chapter is to give an overview over symmetrization methods and results for fractional perimeters and seminorms in the Euclidean case, as well as to shortly present all definitions and results needed for our work in Riemannian geometry.

2.1 Basic notation and norms on $\mathbb{R}^n$

We always assume $n \in \mathbb{N}$ and $n \geq 1$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ we denote by $|x| := (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ the *Euclidean norm* of x and by $x \cdot y := \sum_{i=1}^n x_i y_i$ the *inner product* of x and y. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by

$$B_r^n(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$$

the *open Euclidean ball* around x with radius r and write $B_r^n := B_r^n(o)$ for balls centered at the origin o. Furthermore, we write

$$\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$$

for the closed Euclidean unit ball and

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$$

for the Euclidean unit sphere.

The *characteristic function* $\mathbb{1}_E$ of a set E is the function which satisfies $\mathbb{1}_E(x) = 1$, whenever $x \in E$, and $\mathbb{1}_E(x) = 0$ otherwise. If $E \subseteq \mathbb{R}^n$, then we denote by $E^c := \mathbb{R}^n \backslash E$ the complement of E in $\mathbb{R}^n$. If X is a topological space and $E \subseteq X$, then we denote the interior, closure and boundary of E as $\operatorname{int} E, \overline{E}$ and ∂E, respectively. The *support* of a function $f : X \to V$, where V is a vector space, is defined as

$$\operatorname{spt} f := \overline{\{x \in X : f(x) \neq o\}}.$$

The (n-dimensional) Lebesgue measure of a set $E \subseteq \mathbb{R}^n$ is denoted by $|E|$ and for $k \in \mathbb{N}$ the k-dimensional Hausdorff measure of E is denoted by $\mathcal{H}^k(E)$. We say that two sets are equivalent if they differ only by a set of Lebesgue measure zero.

We call a set $K \subseteq \mathbb{R}^n$ a *convex body* if it is compact, convex and has non-empty interior. Furthermore K is *strictly convex* if

$$(1 - \lambda)x + \lambda y \in \operatorname{int} K \quad \text{for all } x, y \in K, x \neq y, \text{ and } 0 < \lambda < 1.$$

If K is an origin-symmetric convex body, then the map $\|\cdot\|_K$ defined by

$$\|x\|_K := \min\{\lambda \geq 0 : x \in \lambda K\}, \quad x \in \mathbb{R}^n,$$

is a norm on $\mathbb{R}^n$ with closed unit ball K ($\|\cdot\|_K$ is oftentimes also called *gauge function* or *Minkowski functional* of K in the literature). On the other hand, each norm on $\mathbb{R}^n$ can be written in the form $\|\cdot\|_K$ where K is its closed unit ball. With this notation we have $|\cdot| = \|\cdot\|_{\mathbb{B}^n}$ for the Euclidean norm. Note that every norm $\|\cdot\|_K$ in $\mathbb{R}^n$ is equivalent to the Euclidean norm, i.e. there exist constants $0 < \alpha \leq \beta$ such that

$$\alpha|x| \leq \|x\|_K \leq \beta|x|, \quad \text{for all } x \in \mathbb{R}^n. \tag{2.1}$$

If $K \subseteq \mathbb{R}^n$ is an origin-symmetric convex body, then its *polar body*

$$K^\circ := \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in K\}$$

is again an origin-symmetric convex body. Its norm $\|\cdot\|_{K^\circ}$ coincides with the dual norm of $\|\cdot\|_K$ given by $\|y\|_K^* := \max_{\|x\|_K \leq 1} x \cdot y$ for $y \in \mathbb{R}^n$. Also note that $(K^\circ)^\circ = K$ such that any origin-symmetric convex body is uniquely determined by its polar body.

2.2 Symmetrization

In this section we present common methods of symmetrizing sets and function, the most general of which is known under the name of convex or anisotropic symmetrization. Originally introduced in [AFTL97], this symmetrization is taken with respect to origin-symmetric convex bodies, and the well-known Schwarz symmetrization, or symmetric decreasing rearrangement, is the special case where all symmetrized sets are dilates of the Euclidean unit ball. Van Schaftingen [VS06] further extended this notion to asymmetric unit balls. To derive Pólya-Szegő inequalities where the symmetrization is taken with respect to minimizers of the anisotropic fractional isoperimetric inequality, we introduce an extension to the case where we symmetrize with respect to star-shaped bodies. For a general reference on star-shaped sets and bodies we refer to the books of Gardner [Gar06] and Schneider [Sch14].

A set $L \subseteq \mathbb{R}^n$ is called *star-shaped* (with respect to the origin o) if for every $x \in L$ the line segment $[o, x] := \{\lambda x : 0 \leq \lambda \leq 1\}$ connecting the origin o with x lies entirely in L. If L is bounded and star-shaped then its *radial function* $\rho_L : \mathbb{R}^n \backslash \{o\} \to [0, \infty)$ is defined by

$$\rho_L(x) := \sup\{\lambda \geq 0 : \lambda x \in L\}.$$

Since radial functions are positively homogeneous of degree -1, i.e. for every $x \in \mathbb{R}^n \backslash \{o\}$ and $\lambda > 0$

$$\rho_L(\lambda x) = \lambda^{-1} \rho_L(x),$$

they are completely determined by their values on the Euclidean unit sphere $\mathbb{S}^{n-1}$. We call a bounded star-shaped set $L \subseteq \mathbb{R}^n$ a *star body* if it contains the origin in its interior and its radial function is continuous. Note that the unit ball K of an arbitrary norm $\|\cdot\|_K$ on $\mathbb{R}^n$ is a star body with radial function given by $\rho_K(x) = \frac{1}{\|x\|_K}$ for $x \in \mathbb{R}^n \backslash \{o\}$.

Definition 2.1. Let $L \subseteq \mathbb{R}^n$ be a star body. Then the *(anisotropic) symmetrization* E^L of the set $E \subseteq \mathbb{R}^n$ with respect to L is defined as follows: If $|E| = \infty$, then $E^L := \mathbb{R}^n$. If $|E| < \infty$, then

$$E^L := rL$$

where $rL = \{r\ell : \ell \in L\}$ and $r \geq 0$ is chosen such that $|E^L| = |E|$.

Note that in case $|E| < \infty$ the factor $r \geq 0$ is uniquely determined by the relation $|E^L| = r^n|L| = |E|$. Since L has a continuous radial function bounded away from 0 on $\mathbb{S}^{n-1}$, every point $x \in \mathbb{R}^n$ lies on the boundary of precisely one of the dilates rL with $r \geq 0$. Furthermore, this notion of symmetrization does not depend on the scaling of L, i.e. if $\tilde{L} = \lambda L$ for $\lambda > 0$, then $E^{\tilde{L}} = E^L$.

Example 2.2.

1. If L is an origin-symmetric convex body, then the symmetrization with respect to L was introduced by Alvino et al. [AFTL97] under the name of convex symmetrization and extended to non-symmetric convex bodies by Van Schaftingen [VS06].

2. Symmetrization with respect to $L = \mathbb{B}^n$, the Euclidean unit ball, is called *Schwarz symmetrization* and denoted by $\cdot^*$, i.e. $E^* = E^{\mathbb{B}^n}$. For the decomposition $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ we write $x \in \mathbb{R}^n$ as $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. If $A \subseteq \mathbb{R}^n$ and $x' \in \mathbb{R}^{n-1}$, the section $A_{x'}$ is defined as

$$A_{x'} := \{y \in \mathbb{R} : (x', y) \in A\}.$$

The *Steiner symmetrization* $A^\#$ of A with respect to the hyperplane $\{x_n = 0\}$ (or simply with respect to x_n) is then defined by

$$[A^\#]_{x'} = [A_{x'}]^*,$$

for every $x' \in \mathbb{R}^{n-1}$, where $[A_{x'}]^*$ is the Schwarz symmetrization of the set $A_{x'}$ in $\mathbb{R}$. ▲

In the following, if $f : A \to \mathbb{R}$ is a function on $A \subseteq \mathbb{R}^n$ and $\tau \in \mathbb{R}$, we write

$$\{f > \tau\} = \{x \in A : f(x) > \tau\}$$

for the level sets of f.

Definition 2.3. Let $L \subseteq \mathbb{R}^n$ be a star body and $f : \mathbb{R}^n \to \mathbb{R}$ a measurable function such that all level sets $\{|f| > \tau\}$ for $\tau > 0$ have finite measure. Then the *(anisotropic) symmetrization* $f^L : \mathbb{R}^n \to [0, \infty)$ of f with respect to L is defined as

$$f^L(x) := \sup\left\{\tau > 0 : x \in \{|f| > \tau\}^L\right\},$$

where $\{|f| > \tau\}^L$ is the symmetrization of the set $\{|f| > \tau\}$ with respect to L.

Again, symmetrization of functions with respect to L does not depend on the scaling of L.

Example 2.4. In the case of Schwarz symmetrization, f^* is also commonly known as the *symmetric decreasing rearrangement* of f (cf. [LL01]). For $x' \in \mathbb{R}^{n-1}$ we define the section $f_{x'} : \mathbb{R} \to \mathbb{R}$ of f as

$$f_{x'}(y) := f(x', y).$$

Then the Steiner symmetrization $f^{\#}$ of a function f with respect to the hyperplane $\{x_n = 0\}$ is defined by

$$f^{\#}(x', x_n) := \sup\left\{\tau > 0 : x_n \in \{y \in \mathbb{R} : f(x', y) > \tau\}^*\right\},$$

for $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, i.e. $[f^{\#}]_{x'} = [f_{x'}]^*$. ▲

The next result shows that the level sets of a symmetrized function f^L are obtained by symmetrizing the corresponding level sets of f. It is well-known for symmetric decreasing rearrangement (see e.g. [LL01, Chapter 3.3]) and the proof for symmetrization with respect to star-shaped bodies follows along the same lines, since it only relies on measure-theoretic properties of the symmetrization.

Proposition 2.5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable with $|\{|f| > \tau\}|$ finite for all $\tau > 0$. Then*

$$\left\{f^L > \tau\right\} = \{|f| > \tau\}^L$$

for all $\tau > 0$.

Proof. First note that from $f^L(x) = \int_0^\infty \mathbb{1}_{\{|f|>s\}^L}(x)\,\mathrm{d}s > \tau$ and $\{|f| > s_1\}^L \supseteq \{|f| > s_2\}^L$ for $s_1 \le s_2$ it follows that $x \in \{|f| > \tau\}^L$.

For the other direction we note that the distribution function $s \mapsto |\{|f| > s\}|$ is continuous from the right, so $x \in \{|f| > \tau\}^L$ implies that $x \in \{|f| > \tau + \delta\}^L$ for some $\delta > 0$ and eventually

$$f^L(x) = \int_0^\infty \mathbb{1}_{\{|f|>s\}^L}(x)\,\mathrm{d}s \ge \tau + \delta,$$

so $x \in \left\{f^L > \tau\right\}$. ■

We will use the following strict version of Riesz's rearrangement inequality (cf. [Lie77]):

Theorem 2.6 (Riesz's rearrangement inequality). *Let f, g and k be non-negative measurable functions on $\mathbb{R}^n$ such that all their level sets have finite measure. Then,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)k^*(x-y)g^*(y)\,\mathrm{d}y\,\mathrm{d}x, \tag{2.2}$$

where $\cdot^*$ denotes symmetric decreasing rearrangement (as introduced in Example 2.4).

Furthermore, if k is strictly symmetric decreasing (i.e. $k(x) = k(y)$ whenever $|x| = |y|$, and $k(x) > k(y)$ whenever $|x| < |y|$), then equality holds in (2.2) if and only if there exists $c \in \mathbb{R}^n$ such that $f(x) = f^*(x - c)$ and $g(x) = g^*(x - c)$ almost everywhere.

We conclude this section with a result by Van Schaftingen [VS06] that Riesz's rearrangement inequality is in general not true, if Schwarz symmetrization is replaced by symmetrization with respect to a unit ball different from $\mathbb{B}^n$.

Theorem 2.7 ([VS06]). *Let K be a convex body with $o \in \operatorname{int} K$. If*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)k(x-y)g(y)\,\mathrm{d}y\,\mathrm{d}x \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^K(x)k^K(x-y)g^K(y)\,\mathrm{d}y\,\mathrm{d}x$$

for all non-negative continuous functions f, g and k with compact support, then K is an ellipsoid.

2.3 Sobolev spaces and BV functions on $\mathbb{R}^n$

Throughout this section we assume that $\Omega \subseteq \mathbb{R}^n$ is open and that $K \subseteq \mathbb{R}^n$ is an origin-symmetric convex body. As a general reference for Sobolev spaces we refer to the book by Adams & Fournier [AF03] and for functions of bounded variation we recommend the monograph by Ambrosio, Fusco & Pallara [AFP00].

For $k \in \mathbb{N} \cup \{\infty\}$ we denote by $C_c^k(\Omega)$ and $C_c^k(\Omega;\mathbb{R}^n)$ the space of all k-times continuously differentiable functions with compact support in Ω and values in $\mathbb{R}$ and $\mathbb{R}^n$ respectively. For $1 \le p < \infty$ we denote by $L^p_{loc}(\Omega)$ and $L^p_{loc}(\Omega;\mathbb{R}^n)$ the space of all locally p-integrable functions on Ω with values in $\mathbb{R}$ and $\mathbb{R}^n$ respectively. If $f \in L^1_{loc}(\Omega)$, we say that the vector field $\nabla f \in L^1_{loc}(\Omega;\mathbb{R}^n)$ is the *weak gradient* of f if

$$\int_\Omega \phi \nabla f \,\mathrm{d}x = -\int_\Omega f \nabla \phi \,\mathrm{d}x$$

for all $\phi \in C_c^\infty(\Omega)$.

For $1 \le p < \infty$ the *Sobolev space* $W^{1,p}(\Omega)$ is defined as

$$W^{1,p}(\Omega) := \{f \in L^p(\Omega) : \nabla f \in L^p(\Omega;\mathbb{R}^n)\}\,.$$

If $f \in W^{1,p}(\Omega)$, then the *anisotropic (Sobolev) seminorm* of f with respect to K is defined as

$$|f|_{1,p,K} := \left(\int_\Omega \|\nabla f(x)\|^p_{K^\circ} \,\mathrm{d}x\right)^{\frac{1}{p}},$$

see e.g. [Gro]. In the Euclidean case $K = \mathbb{B}^n$ we simply write $|f|_{1,p} := |f|_{1,p,\mathbb{B}^n}$ and since $(\mathbb{B}^n)^\circ = \mathbb{B}^n$ we have the standard seminorm

$$|f|_{1,p} = \left(\int_\Omega |\nabla f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Furthermore we adopt the convention that $|f|_{1,p} = \infty$ if $f \in L^p(\Omega)$, but $f \notin W^{1,p}(\Omega)$.

Closely related to Sobolev spaces are functions of bounded variation where the weak derivative is a vector-valued Radon measure. For $f \in L^1(\Omega)$ and $U \subseteq \Omega$ open we define the *variation* of f in U as

$$V(f,U) := \sup\left\{\int_U f \operatorname{div} T \,\mathrm{d}x : T \in C_c^1(U;\mathbb{R}^n), |T| \le 1\right\}. \tag{2.3}$$

It can be shown that $V(f,\cdot)$ can be extended to a Borel measure on Ω. We say that f is a function of *bounded variation* if $V(f,\Omega) < \infty$ or equivalently if there exists a finite $\mathbb{R}^n$-valued Radon measure Df on Ω such that

$$\int_\Omega f \operatorname{div} T \,\mathrm{d}x = -\int_\Omega T \cdot \mathrm{d}Df$$

for all vector fields $T \in C_c^1(\Omega;\mathbb{R}^n)$. In this case, $|Df| = V(f,\cdot)$ as (positive) Radon measures on Ω (see e.g. [AFP00]). The space of all functions of bounded variation on Ω is denoted by $BV(\Omega)$ and $|f|_{BV} := |Df|(\Omega)$ is a seminorm on $BV(\Omega)$. For functions $f \in L^1(\Omega)$ which are not of bounded variation we put $|f|_{BV} = \infty$. Replacing the Euclidean norm in (2.3) by the norm $\|\cdot\|_K$ leads to an anisotropic seminorm $|f|_{BV,K}$ on $BV(\Omega)$,

$$|f|_{BV,K} := \sup\left\{\int_\Omega f \operatorname{div} T \,\mathrm{d}x : T \in C_c^1(\Omega;\mathbb{R}^n), \|T\|_K \le 1\right\}.$$

It holds that $W^{1,1}(\Omega) \subseteq BV(\Omega)$ and that for $f \in W^{1,1}(\Omega)$

$$|f|_{BV,K} = \int_\Omega \|\nabla f(x)\|_{K^\circ} \,\mathrm{d}x = |f|_{1,1,K},$$

see [AB94].

Next, we consider fractional seminorms and the corresponding fractional Sobolev spaces. For $0 < s < 1$ and $1 \le p < \infty$ the *anisotropic fractional seminorm* of a measurable function $f : \Omega \to \mathbb{R}$ was defined in [Lud14b] by

$$|f|_{s,p,K} := \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{\|x-y\|_K^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

In the special case that $K = \mathbb{B}^n$ is the Euclidean unit ball we simply write $|f|_{s,p} := |f|_{s,p,\mathbb{B}^n}$ for the fractional seminorm, i.e.

$$|f|_{s,p} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

The *fractional Sobolev space* $W^{s,p}(\Omega)$ is defined by

$$W^{s,p}(\Omega) := \{ f \in L^p(\Omega) : |f|_{s,p} < \infty \}.$$

It is a Banach space equipped with the norm $\|f\|_{W^{s,p}} := (\|f\|_{L^p}^p + |f|_{s,p}^p)^{\frac{1}{p}}$, see e.g. [DNPV12].

We point out that all anisotropic seminorms introduced in this section are by (2.1) equivalent to their corresponding Euclidean version. In particular, the definition of the spaces of BV and (fractional) Sobolev functions does not depend on the choice of the underlying norm on $\mathbb{R}^n$.

Next, we present standard density results for spaces of Sobolev and BV functions which also provide a classification of those spaces (see e.g. [AFP00, AF03]):

Theorem 2.8. *Let $f \in L^p(\Omega)$ with $1 \le p < \infty$.*

1. *If $p > 1$, then $f \in W^{1,p}(\Omega)$ if and only if there exists a sequence $(f_j) \subseteq C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $f_j \to f$ in L^p as $j \to \infty$, and*

$$L := \lim_{j\to\infty} \int_\Omega |\nabla f_j(x)|^p \,\mathrm{d}x < \infty.$$

   In this case, $L = \int_\Omega |\nabla f(x)|^p \,\mathrm{d}x$.

2. *If $p = 1$, then $f \in BV(\Omega)$ if and only if there exists a sequence $(f_j) \subseteq C^\infty(\Omega)$ such that $f_j \to f$ in L^1 as $j \to \infty$, and*

$$L := \lim_{j\to\infty} \int_\Omega |\nabla f_j(x)| \,\mathrm{d}x < \infty.$$

   In this case, $L = |Df|(\Omega)$.

For all $1 \le p < \infty$ and $0 < s \le t < 1$ we have $W^{t,p}(\Omega) \subseteq W^{s,p}(\Omega)$, see [DNPV12, Prop. 2.1]. This suggests that the Sobolev space $W^{1,p}(\Omega)$ is contained in all fractional Sobolev spaces which, however, is only true under certain additional assumptions on Ω. We say that the open set $\Omega \subseteq \mathbb{R}^n$ is an extension domain if for all $0 < s < 1$ and $1 \le p < \infty$ the following holds: There exists a constant $C = C(s,p,\Omega) > 0$ such that for every $f \in W^{s,p}(\Omega)$ there exists a function $\bar{f} \in W^{s,p}(\mathbb{R}^n)$ with $\bar{f}|_\Omega = f$ and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\bar{f}(x)-\bar{f}(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \le C \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x.$$

If $\Omega \subseteq \mathbb{R}^n$ is an extension domain, then the following chains of inclusions hold where each inclusion is strict:

$$W^{1,1}(\Omega) \subseteq BV(\Omega) \subseteq \bigcap_{s\in(0,1)} W^{s,1}(\Omega),$$
$$W^{1,p}(\Omega) \subseteq \bigcap_{s\in(0,1)} W^{s,p}(\Omega), \quad p > 1,$$

see e.g. [Lom15].

We conclude this section with convergence results for fractional seminorms as the parameter s tends to 0 or 1. Here, we first state the results for Euclidean fractional seminorms as we refer to this special case in Chapters 4 and 5, and ultimately present all statements for the more general anisotropic fractional seminorms.

For the discussion of the behaviour of fractional seminorms as $s \nearrow 1$, Bourgain, Brezis & Mironescu [BBM01] considered a more general family of kernel functions, namely radial mollifiers. A family of functions $\rho_\sigma : (0,\infty) \to [0,\infty), 0 < \sigma < 1$, is called a family of *radial mollifiers* if they satisfy the following properties:

$$\int_0^\infty \rho_\sigma(r) r^{n-1}\,\mathrm{d}r = \frac{1}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}, \qquad \forall\, 0 < \sigma < 1,$$
$$\lim_{\sigma\searrow 0} \int_\delta^\infty \rho_\sigma(r) r^{n-1}\,\mathrm{d}r = 0, \qquad \forall\, \delta > 0.$$

The following theorem is mostly due to Bourgain, Brezis & Mironescu, with the exception of the computation of the precise limit (2.6) for $p = 1$, which is due to Dávila [Dáv02].

Theorem 2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain, and $f \in L^p(\Omega)$ with $1 \le p < \infty$. Furthermore, let (ρ_σ) be a family of radial mollifiers.*

1. *If $p > 1$, then*

$$\lim_{\sigma\searrow 0} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\sigma(|x-y|)\,\mathrm{d}y\,\mathrm{d}x = K_{p,n}|f|^p_{1,p}, \tag{2.4}$$

where the constant $K_{p,n}$ is defined as

$$K_{p,n} := \frac{1}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} |e\cdot u|^p \,\mathrm{d}\mathcal{H}^{n-1}(u), \tag{2.5}$$

and $e \in \mathbb{S}^{n-1}$ is any unit vector. In particular, $f \in W^{1,p}(\Omega)$ if and only if

$$\liminf_{\sigma\searrow 0} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\sigma(|x-y|)\,\mathrm{d}y\,\mathrm{d}x < \infty.$$

2. *If $p = 1$, then*

$$\lim_{\sigma \searrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|} \rho_\sigma(|x-y|)\, \mathrm{d}y\, \mathrm{d}x = K_{1,n} |f|_{BV}, \tag{2.6}$$

with the constant $K_{1,n}$ defined in (2.5)*. In particular, $f \in BV(\Omega)$ if and only if*

$$\liminf_{\sigma \searrow 0} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|} \rho_\sigma(|x-y|)\, \mathrm{d}y\, \mathrm{d}x < \infty.$$

Van Schaftingen & Willem [VSW04] proved the previous characterization and convergence result for $\Omega = \mathbb{R}^n$, and Brezis & Nguyen [BN16] established stronger pointwise convergence results. By a counterexample of Brezis [Bre02], the results (2.4) and (2.6) fail to hold in general on non-smooth open sets Ω. Still, Leoni & Spector [LS11] recovered a variant of Theorem 2.9 for arbitrary open sets.

Choosing a suitable family of radial mollifiers leads to the convergence of fractional seminorms (see also [BBM01]):

Theorem 2.10. *Let $\Omega \subseteq \mathbb{R}^n$ a smooth bounded domain, and $f \in L^p(\Omega)$ with $1 \leq p < \infty$.*

1. *If $p > 1$, then*

$$\lim_{s \nearrow 1} (1-s) \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, \mathrm{d}y\, \mathrm{d}x = \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) K_{p,n}}{p} |f|_{1,p}^p.$$

2. *If $p = 1$, then*

$$\lim_{s \nearrow 1} (1-s) \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{n+s}}\, \mathrm{d}y\, \mathrm{d}x = 2\mathcal{H}^{n-1}(\mathbb{B}^{n-1}) |f|_{BV}.$$

Regarding the other endpoint in the family of seminorms, Maz'ya & Shaposhnikova [MS02] proved the convergence of suitably rescaled seminorms to the L^p-norm as $s \searrow 0$:

Theorem 2.11. *Let $1 \leq p < \infty$ and $f \in W^{s_0,p}(\mathbb{R}^n)$ for an $s_0 \in (0,1)$. Then*

$$\lim_{s \searrow 0} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\, \mathrm{d}y\, \mathrm{d}x = \frac{2n}{p} |\mathbb{B}^n| \int_{\mathbb{R}^n} |f(x)|^p\, \mathrm{d}x.$$

In the more general setting of anisotropic fractional seminorms, Ludwig [Lud14b] showed that in the limit as $s \nearrow 1$ the unit ball in the seminorm changes. If $K \subseteq \mathbb{R}^n$ is an origin-symmetric convex body, and $1 \leq p < \infty$, then the *L_p-moment body* $Z_p K$ of K is defined by the gauge function of its polar body,

$$\|u\|_{Z_p^\circ K} := \left(\frac{n+p}{2} \int_K |u \cdot x|^p\, \mathrm{d}x \right)^{\frac{1}{p}},$$

such that

$$Z_p K = \left\{ x \in \mathbb{R}^n : x \cdot u \leq 1 \text{ for all } u \in \mathbb{R}^n \text{ with } \|u\|_{Z_p^\circ K} \leq 1 \right\}. \tag{2.7}$$

Theorem 2.12 ([Lud14a, Lud14b]). *Let* $\Omega = \mathbb{R}^n$.

1. *If* $1 \le p < \infty$ *and* $f \in W^{1,p}(\mathbb{R}^n)$ *is a function with compact support, then*

$$\lim_{s \nearrow 1}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x = |f|^p_{1,p,Z_pK}.$$

2. *If* $f \in BV(\mathbb{R}^n)$, *then*

$$\lim_{s \nearrow 1}(1-s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}y\,\mathrm{d}x = 2|f|_{BV,Z_1K}.$$

If, in particular, $K = \mathbb{B}^n$, then $Z_p^\circ \mathbb{B}^n$ is a multiple of $\mathbb{B}^n$ such that this result recovers the limits in Theorem 2.10 with the right constants.

For the other endpoint $s = 0$, Ludwig showed:

Theorem 2.13 ([Lud14b]). *Let* $1 \le p < \infty$ *and* $f \in W^{s_0,p}(\mathbb{R}^n)$ *for an* $s_0 \in (0,1)$ *be a function with compact support. Then,*

$$\lim_{s \searrow 0} s\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x = \frac{2n}{p}|K|\int_{\mathbb{R}^n}|f(x)|^p\,\mathrm{d}x.$$

2.3.1 Perimeter functionals

In this section we put $\Omega = \mathbb{R}^n$ and restrict the study of seminorms to characteristic functions of sets which leads to set functionals sharing similar properties with classical surface area measures. We first recall the definition of the (anisotropic) perimeter. The *anisotropic perimeter* of a Borel set $E \subseteq \mathbb{R}^n$ with respect to K (cf. [AB94]) is defined by

$$P(E,K) := |\mathbb{1}_E|_{BV,K} = \sup\left\{\int_E \operatorname{div} T\,\mathrm{d}x : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \|T\|_K \le 1\right\}.$$

In the special case of the Euclidean unit ball $K = \mathbb{B}^n$, we recover the definition of the *perimeter* introduced by de Giorgi [DG53], i.e.

$$P(E) := \sup\left\{\int_E \operatorname{div} T\,\mathrm{d}x : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), |T| \le 1\right\}.$$

If $E \subseteq \mathbb{R}^n$ is a set with smooth boundary and $\nu_E : \partial E \to \mathbb{S}^{n-1}$ is the vector field of outer unit normals, then

$$P(E,K) = \int_{\partial E}\|\nu_E\|_{K^\circ}\,\mathrm{d}\mathcal{H}^{n-1},$$

see [AB94]. In particular, $P(E) = \mathcal{H}^{n-1}(\partial E)$ for sets with smooth boundary, so the perimeter extends the notion of surface area to a broader class of sets.

The geometric counterparts of fractional seminorms are fractional perimeters: For $s \in (0,1)$ the anisotropic fractional perimeter of a Borel set $E \subseteq \mathbb{R}^n$ with respect to K is defined by

$$P_s(E,K) := \int_E\int_{E^c}\frac{1}{\|x-y\|_K^{n+s}}\,\mathrm{d}y\,\mathrm{d}x, \tag{2.8}$$

where, again, for $K = \mathbb{B}^n$ we simply write $P_s(E) := P_s(E, \mathbb{B}^n)$ for the fractional perimeter. Here the relation to fractional seminorms

$$P_s(E, K) = \frac{1}{2}|\mathbb{1}_E|_{s,1,K}$$

follows directly from Fubini's theorem.

In the following, we list some properties of geometric interest which all perimeter functionals we have presented so far have in common. For their proofs we refer to [Mag12], and [CN18] for the fractional versions. To provide a simple unified notation for these functionals, let $\mathcal{P}_s$ denote the anisotropic fractional perimeter with respect to K if $s \in (0,1)$ and the anisotropic perimeter[1] with respect to K if $s = 1$.

Let $E \subseteq \mathbb{R}^n$ be a Borel set. Then,

- $\mathcal{P}_s(E) = \mathcal{P}_s(E^c)$,
- $\mathcal{P}_s$ is invariant under translations, i.e. if $y \in \mathbb{R}^n$, then $\mathcal{P}_s(E + y) = \mathcal{P}_s(E)$, where $E + y := \{x + y : x \in E\}$.

  If $K = \mathbb{B}^n$, then $\mathcal{P}_s$ is also invariant under rotations, i.e. if $\theta \in SO(n)$ is a rotation, then $\mathcal{P}_s(\theta E) = \mathcal{P}_s(E)$, where $\theta E := \{\theta x : x \in E\}$,
- $\mathcal{P}_s$ is $(n-s)$-homogeneous, i.e. if $\lambda > 0$, then $\mathcal{P}_s(\lambda E) = \lambda^{n-s}\mathcal{P}_s(E)$, where $\lambda E := \{\lambda x : x \in E\}$,
- $\mathcal{P}_s$ is lower semicontinuous with respect to $L^1(\mathbb{R}^n)$-convergence, i.e. if $\int_{\mathbb{R}^n} |\mathbb{1}_{E_i} - \mathbb{1}_E| \, \mathrm{d}x \to 0$ for Borel sets $E_i, E \subseteq \mathbb{R}^n$ as $i \to \infty$, then

$$\mathcal{P}_s(E) \leq \liminf_{i\to\infty} \mathcal{P}_s(E_i).$$

The convergence of anisotropic fractional perimeters as s tends to 0 or 1 follows immediately from the corresponding results for seminorms, Theorem 2.12 and 2.13 (for $s \searrow 0$ in the Euclidean case see also [DFPV13]):

Theorem 2.14 ([Lud14a]). *Let $E \subseteq \mathbb{R}^n$ be a bounded Borel set of finite perimeter. Then*

$$\lim_{s\nearrow 1}(1-s)P_s(E, K) = P(E, Z_1 K),$$

where $Z_1 K$ is the moment body of K defined in (2.7), and

$$\lim_{s\searrow 0} sP_s(E, K) = n|K||E|. \tag{2.9}$$

Visintin [Vis90] showed the following coarea formula by which the $W^{s,1}(\mathbb{R}^n)$-seminorm of a function can be computed by the fractional perimeters of its level sets (see also [ADPM11, Lemma 10] and [Lud14a, (23)]):

[1] The case $s = 1$ in the definition of anisotropic fractional perimeters (2.8) leads to a functional different from the anisotropic perimeter, since the integrals in (2.8) do not converge, unless E or E^c is a set of measure 0 (cf. [Bre02]). This justifies the need for the new notation $\mathcal{P}_s$. However, Theorem 2.14 shows that the anisotropic perimeter is the endpoint in the scale of anisotropic fractional perimeters in a certain sense.

Theorem 2.15 (generalized coarea formula). *For $f \in L^1(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{\|x - y\|_K^{n+s}} \, \mathrm{d}y \, \mathrm{d}x = 2 \int_0^\infty P_s(\{|f| > \tau\}, K) \, \mathrm{d}\tau.$$

To conclude this section, we present two types of functionals which are closely related to fractional perimeters. The first type stems from the anisotropic Riesz potential which was defined by Hou, Xiao & Ye [HXY18] for $\alpha \in [0, n)$ and bounded measurable sets $\Omega \subseteq \mathbb{R}^n$ by

$$I_\alpha(\Omega, K; y) := \int_\Omega \frac{1}{\|x - y\|_K^\alpha} \, \mathrm{d}x, \quad y \in \mathbb{R}^n.$$

In the same paper the authors introduced a mixed volume based on this potential, which satisfies a reverse Minkowski-type inequality. In the isotropic case, $K = \mathbb{B}^n$, O'Hara & Solanes [OS18] investigated the analytic continuation of Riesz energies

$$E_\Omega(z) := \int_\Omega \int_\Omega \frac{1}{|x - y|^z} \, \mathrm{d}y \, \mathrm{d}x,$$

where $z \in \mathbb{C}$ with Re $z > -n$ and $\Omega \subseteq \mathbb{R}^n$ is a compact regular domain with smooth boundary. In the context of this thesis, it is worth noting that they studied Riesz energies of closed submanifolds $M \subseteq \mathbb{R}^n$, where the distance between two points is still measured by the Euclidean norm (compared to the fractional perimeters on manifolds introduced in Chapter 4).

A second type of functionals, which generalize anisotropic fractional perimeters, are *nonlocal* perimeters, as introduced in [CN18]. Let $k : \mathbb{R}^n \to [0, \infty)$ be a measurable function such that $\min(|\cdot|, 1)k \in L^1(\mathbb{R}^n)$. Then the nonlocal perimeter Per_k is defined for Borel sets $E \subseteq \mathbb{R}^n$ as

$$\mathrm{Per}_k(E) := \int_E \int_{E^c} k(x - y) \, \mathrm{d}y \, \mathrm{d}x.$$

The family of anisotropic fractional perimeters is a special case of this definition with the kernel function $k(\xi) = \|\xi\|_K^{-(n+s)}$.

2.3.2 Geometric and functional inequalities

In this section we recall isoperimetric inequalities for the perimeter functionals introduced in the previous section and their close connection to Pólya-Szegő inequalities for the corresponding seminorms.

The anisotropic isoperimetric inequality is due to Minkowski for convex bodies (cf. [Sch14]), and in full generality proven in [Tay78] for Borel sets:

Theorem 2.16 (anisotropic isoperimetric inequality). *Let $E \subseteq \mathbb{R}^n$ be a Borel set with $|E| < \infty$. Then,*

$$P(E, K) \geq n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}, \tag{2.10}$$

with equality if and only if E is homothetic to K, i.e. $E = \lambda K + x$ for some $\lambda > 0$ and $x \in \mathbb{R}^n$, up to sets of measure 0.

In the Euclidean case, $K = \mathbb{B}^n$, inequality (2.10) together with its equality cases is a direct consequence of the following Steiner inequality (see e.g. [Mag12]):

Theorem 2.17 (Steiner inequality). *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter with $|E| < \infty$. Then $E^\#$ is a set of finite perimeter, and*

$$P(E) \geq P(E^\#). \tag{2.11}$$

Moreover, if equality holds in (2.11), then, for a.e. $x' \in \mathbb{R}^{n-1}$, the section $E_{x'}$ is equivalent to an interval.

Note that the condition for the equality cases in Theorem 2.17 is necessary for equality to hold, but not sufficient, as counterexamples can be easily constructed.

Closely related to the anisotropic isoperimetric inequality is the following anisotropic Pólya-Szegő inequality which was proved in [AFTL97]. The characterization of equality cases under suitable assumptions was established in [ET04].

Theorem 2.18 (anisotropic Pólya-Szegő inequality). *Let $f \in W^{1,p}(\mathbb{R}^n)$ be a function with compact support. Then,*

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^\circ}^p \, \mathrm{d}x \geq \int_{\mathbb{R}^n} \|\nabla f^K(x)\|_{K^\circ}^p \, \mathrm{d}x, \tag{2.12}$$

where f^K is the anisotropic symmetrization of f with respect to K. Moreover, if f is non-negative and such that

$$|\left\{\nabla f^K = o\right\} \cap \left\{0 < f^K < \operatorname{ess\,sup} f\right\}| = 0,$$

then there is equality in (2.12) if and only if $f = f^K$ up to some translation.

We remark that the symmetrization in this Pólya-Szegő inequality is taken with respect to the minimizers of the anisotropic isoperimetric inequality (2.10).

Next, we present the anisotropic fractional isoperimetric inequality and give a detailed discussion on the existence of minimizers following Ludwig [Lud14a]. One crucial tool is the following isoperimetric inequality for Euclidean fractional perimeters which was proved by Frank & Seiringer in [FS08] using symmetrization results by Almgren & Lieb [AL89] (see Theorem 2.20):

Theorem 2.19 (fractional isoperimetric inequality). *There exists a sharp constant $\gamma_{n,s} > 0$ such that for all bounded Borel sets $E \subseteq \mathbb{R}^n$*

$$P_s(E) \geq \gamma_{n,s} |E|^{\frac{n-s}{n}}, \tag{2.13}$$

with equality if and only if E is homothetic to $\mathbb{B}^n$ up to sets of measure 0.

The proof of Theorem 2.19 relies on Riesz's rearrangement inequality, Theorem 2.6, which is used in [FS08] to show the following fractional Pólya-Szegő inequality with the full description of equality cases. Inequality (2.14) without the classification of equality cases had already been proved before by Almgren & Lieb in [AL89].

Theorem 2.20 (fractional Pólya-Szegő inequality). *If $f \in W^{s,p}(\mathbb{R}^n)$ and $p \geq 1$, then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x \geq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f^*(x) - f^*(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}y \, \mathrm{d}x. \tag{2.14}$$

Moreover, equality holds in (2.14) if and only if

1. *f is proportional to a translate of a symmetric decreasing function almost everywhere if $p > 1$, or*
2. *f is proportional to a non-negative function g such that the level sets $\{g > \tau\}$ are balls for a.e. $\tau > 0$ if $p = 1$.*

The anisotropic fractional isoperimetric inequality established in [Lud14a] reads as

$$P_s(E, K) \geq \gamma_{n,s}(K)|E|^{\frac{n-s}{n}} \tag{2.15}$$

for every bounded Borel set $E \subseteq \mathbb{R}^n$, where $\gamma_{n,s}(K) > 0$ is the optimal constant given by

$$\gamma_{n,s}(K) := \inf\left\{P_s(E, K)|E|^{-\frac{n-s}{n}} : E \subseteq \mathbb{R}^n \text{ bounded }, |E| > 0\right\}.$$

We first remark that a proof analogous to the proof of (2.13) by Frank & Seiringer [FS08] is not possible since by Theorem 2.7 a Riesz-type rearrangement inequality does not hold true for anisotropic symmetrization. To prove the existence of minimizers we apply the Frechet-Kolmogorov compactness criterion to a minimizing sequence for $\gamma_{n,s}(K)$ to show that it has a converging subsequence. We recall the compactness criterion for L^1-functions and refer to [Bre02] for its proof:

Theorem 2.21 (Frechet-Kolmogorov compactness criterion). *Let $\mathcal{F}$ be a bounded set in $L^1(\mathbb{R}^n)$. If*

$$\lim_{h\to 0} \sup_{f\in\mathcal{F}} \int_{\mathbb{R}^n} |f(x+h) - f(x)|\, \mathrm{d}x = 0,$$

then the closure of $\mathcal{F}|_\Omega$ in $L^1(\Omega)$ is compact for any measurable set $\Omega \subseteq \mathbb{R}^n$ with finite measure. Here, $\mathcal{F}|_\Omega$ denotes the restrictions of the functions in $\mathcal{F}$ to Ω.

First, the equivalence of norms (2.1) implies that

$$\beta^{-(n+s)} P_s(E) \leq P_s(E, K) \leq \alpha^{-(n+s)} P_s(E)$$

for all Borel sets $E \subseteq \mathbb{R}^n$ and thus by the Euclidean fractional isoperimetric inequality, Theorem 2.19, $0 < \gamma_{n,s}(K) < \infty$. Furthermore, if $E \subseteq \mathbb{R}^n$ is a bounded Borel set and $\tilde{E} = \lambda E$ for a $\lambda > 0$, then by homogeneity

$$P_s(\tilde{E}, K)|\tilde{E}|^{-\frac{n-s}{n}} = P_s(E, K)|E|^{-\frac{n-s}{n}}.$$

Thus

$$\gamma_{n,s}(K) = \inf\left\{P_s(E, K)|E|^{-\frac{n-s}{n}} : E \subseteq B^n_R, |E| > 0\right\}, \tag{2.16}$$

where $R > 0$ is any fixed real number. Without loss of generality, we assume that any set E appearing in (2.16) has volume one. Now let (E_j) be a sequence of Borel sets contained in B^n_R with $|E_j| = 1$ such that $\gamma_{n,s}(K) = \lim_{j\to\infty} P_s(E_j, K)$. By [ADPM11, (4)], for all $h \in \mathbb{R}^n$ with $|h| < R$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathbb{1}_{E_j}(x+h) - \mathbb{1}_{E_j}(x)|\, \mathrm{d}x &= \int_{B^n_{2R}} |\mathbb{1}_{E_j}(x+h) - \mathbb{1}_{E_j}(x)|\, \mathrm{d}x \\ &\leq C(n,s)|h|^s P_s(E_j, K)\end{aligned}$$

where $C(n,s)$ is a constant depending on n and s. Since $\sup_{j\in\mathbb{N}} P_s(E_j,K) < \infty$ by the Frechet-Kolmogorov compactness criterion, Theorem 2.21, there exists a subsequence of (E_j) which converges in $L^1(B^n_{2R})$ to a Borel set E with $E \subseteq B_R$ and $|E| = 1$. By lower semicontinuity of the perimeter

$$P_s(E,K) \le \liminf_{j\to\infty} P_s(E_j,K) = \gamma_{n,s}(K) \le P_s(E,K)$$

which shows that E is a minimizer of the anisotropic fractional isoperimetric inequality (2.15).

The full classification of minimizers of (2.15) remains an open problem. In the following we give a short overview over some partial results obtained in the last few years. The following theorem by Ludwig [Lud14a, Theorem 7] deals with the convergence of minimizers for a family of isoperimetric inequalities where $s \nearrow 1$:

Theorem 2.22. *Let $E_{s_i} \subseteq \mathbb{R}^n$ be bounded Borel sets such that*

$$P_{s_i}(E_{s_i},K) = \gamma_{n,s_i}(K)|E_{s_i}|^{\frac{n-s_i}{n}},$$

and let $E_1 \subseteq \mathbb{R}^n$ be a bounded Borel set. If $s_i \nearrow 1$ and $E_{s_i} \to E_1$ in $L^1(\mathbb{R}^n)$, i.e. $\int_{\mathbb{R}^n} |\mathbb{1}_{E_{s_i}} - \mathbb{1}_{E_1}|\,\mathrm{d}x \to 0$, as $i \to \infty$, then there exists $c \ge 0$ such that $E_1 = cZ_1K$ up to a set of measure zero. Here, Z_1K is the L_1-moment body of K defined in (2.7).

In particular, this theorem implies that minimizers of the anisotropic fractional isoperimetric inequality for s close to 1 are in general *not* homothetic to the unit ball K, which is a striking difference compared to its non-fractional version, Theorem 2.16.

A lower bound for $\gamma_{n,s}(K)$ was found by Xiao & Ye [XY17], who showed that

$$\gamma_{n,s}(K) > \frac{n}{s}|K|^{\frac{n+s}{n}}.$$

The anisotropic fractional isoperimetric inequality implies the anisotropic fractional Sobolev inequality, as shown in [Lud14a, Theorem 9], and their equivalence as well as equivalence to the anisotropic fractional isocapacitary inequality was proved in [XY17, Theorem 4.5].

In contrast to the anisotropic fractional isoperimetric problem, it is not known in general if there exist minimizers for the nonlocal isoperimetric problem,

$$\inf\left\{\mathrm{Per}_k(E) : E \subset \mathbb{R}^n \text{ bounded}, |E| = m\right\}, \tag{2.17}$$

where $k : \mathbb{R}^n \to [0,\infty)$ is measurable such that $\min(|\cdot|,1)k \in L^1(\mathbb{R}^n)$, and $m > 0$ is fixed. For partial results on the existence of minimizers we refer to [CN18].

2.4 Riemannian geometry

In this section all basic notions necessary for the discussion of fractional perimeters on Riemannian manifolds are briefly presented. All results in this chapter are taken from [Lee13] and [Lee97]. We also refer the interested reader to these books for a thorough introduction on Riemannian geometry.

Let M be an n-dimensional Riemannian manifold of class C^∞ equipped with the Riemannian metric g. If $\phi : U \to \mathbb{R}^n$ is a chart on M and we denote the coordinate frame with respect to this chart by $\left(\frac{\partial}{\partial x^\alpha}\right)_{\alpha=1}^n$, then we denote the components of the metric g with respect to this chart by $g_{\alpha\beta} := g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right)$. Furthermore, if $f : M \to \mathbb{R}$ is a function, then we put $\hat{f} : \phi(U) \to \mathbb{R}$ for its coordinate representation, i.e. $\hat{f} = f \circ \phi^{-1}$. We denote the tangent bundle of M by TM and the function $|V|_g := \sqrt{g(V,V)}$ defines a norm on each tangent space. The Riemannian volume form $\mathrm{d}V_g$ gives rise to a measure Vol_g on M defined by $\mathrm{Vol}_g(A) := \int_A \mathrm{d}V_g$, where $A \subseteq M$ is a Borel set. If $\phi : U \to \mathbb{R}^n$ is a chart on M, then the integration of a function $f : U \to \mathbb{R}$ can be expressed in coordinates by the formula

$$\int_U f \,\mathrm{d}V_g = \int_{\phi(U)} \hat{f}(\xi)\sqrt{\det(\hat{g}_{\alpha\beta}(\xi))}\,\mathrm{d}\xi.$$

If $\gamma : [a,b] \to M$ is a piecewise smooth curve segment, i.e. γ is continuous and there exists a finite subdivision $a = a_0 < a_1 < \cdots < a_k = b$ such that $\gamma|_{[a_{i-1},a_i]}$ is smooth for all $i = 1, \ldots k$, then the length of γ is defined by

$$\ell(\gamma) := \int_a^b |\gamma'(t)|_g \,\mathrm{d}t.$$

If M is a connected manifold, then the *geodesic distance* $d(x,y)$ between two points $x, y \in M$ is given by

$$d(x,y) := \inf \ell(\gamma),$$

where the infimum ranges over all piecewise smooth curve segments connecting x and y. With the geodesic distance function, M is a metric space and for $x \in M$ and $r > 0$ we denote the open ball around x with radius r in this space by $B_r^M(x) := \{y \in M : d(x,y) < r\}$. We remark that if M is compact, then for every pair of points $x, y \in M$ there exists a smooth curve γ connecting x and y such that $d(x,y) = \ell(\gamma)$, and this curve is a geodesic.

Around each point $x \in M$ there exists a coordinate chart which is compatible with the geodesic distance in the following sense:

Theorem 2.23 ([Lee97, Prop. 5.11]). *For each point $x \in M$ there exist coordinates $\phi : U \to \mathbb{R}^n$, called* normal coordinates, *with $\phi(x) = o$ and*

1. *For any $V = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i} \in T_xM$ the geodesic γ_V with $\gamma_V(0) = x$ and $\gamma_V'(0) = V$ is given by $\gamma_V(t) = \phi^{-1}(tV^1, \ldots, tV^n)$ as long as γ_V stays within U.*

2. *Any geodesic ball $B_r^M(x)$ around x contained in U is mapped to the Euclidean ball $B_r^n = \{\xi \in \mathbb{R}^n : |\xi| < r\}$.*

3. *The components of the metric at x are $g_{\alpha\beta} = \delta_{\alpha\beta}$ (Kronecker-delta).*

4. *The first partial derivatives of $g_{\alpha\beta}$ vanish at x.*

If $f : M \to \mathbb{R}$ is a smooth function, we define the gradient of f as the smooth vector field $\operatorname{grad} f$ satisfying $g(\operatorname{grad} f, X) = df(X)$ for all smooth vector fields X on M, where df is the differential of f. The divergence $\operatorname{div} X$ of a smooth vector field X on M is defined as the Lie derivative of $\mathrm{d}V_g$ with respect to X, i.e. $\operatorname{div} X = L_X(\mathrm{d}V_g)$. In local coordinates with respect to a chart ϕ, the vector field $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$ gives rise to a vector field $T = (X^1 \circ \phi^{-1}, \ldots, X^n \circ \phi^{-1})^T$ in $\mathbb{R}^n$ and the divergence of X can be expressed as

$$(\text{div } X)(x) = \frac{1}{\sqrt{\det(g_{\alpha\beta}(x))}} \text{div}_{\mathbb{R}^n} \left(\sqrt{\det(\hat{g}_{\alpha\beta}(\cdot))}\, T \right) (\phi(x)), \tag{2.18}$$

where $\text{div}_{\mathbb{R}^n}$ denotes the divergence operator in $\mathbb{R}^n$.

Chapter 3

Anisotropic fractional perimeters

In this chapter we investigate the anisotropic fractional isoperimetric problem and present a result concerning the geometry of minimizers in a special case. The proof of this result heavily relies on rearrangement inequalities for Steiner symmetrization which appear in many different forms in the literature. In Section 3.1 we present a version adapted to the application to anisotropic fractional perimeters and seminorms. We introduce the notion of box bodies which is closely related to anti-blocking bodies in discrete geometry. We prove that if the unit ball K is an unconditional strictly convex body, then all minimizers in the isoperimetric problem are box bodies.

The results of this chapter are published in [Kre21].

3.1 Rearrangement inequalities for Steiner symmetrization

The main result of this section is a Pólya-Szegő inequality for anisotropic fractional seminorms, where the unit ball K of the norm $\|\cdot\|_K$ is symmetric with respect to the hyperplane $\{x_n = 0\}$ and the symmetrization is Steiner symmetrization with respect to the same hyperplane.

Theorem 3.1. *Let $s \in (0,1)$ and $1 \leq p < \infty$, and let the unit ball K of the norm $\|\cdot\|_K$ be symmetric with respect to the hyperplane $\{x_n = 0\}$. If $f \in W^{s,p}(\mathbb{R}^n)$ and $f^\#$ is the Steiner symmetrization of f with respect to x_n, then $f^\# \in W^{s,p}(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^\#(x)-f^\#(y)|^p}{\|x-y\|_K^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x. \tag{3.1}$$

Furthermore, assume that K is strictly convex.

(a) If $p > 1$, equality holds in (3.1) if and only if there exists $c \in \mathbb{R}$ such that for a.e. $x' \in \mathbb{R}^{n-1}$

$$f(x', x_n) = f^\#(x', x_n - c) \text{ for a.e. } x_n \in \mathbb{R}.$$

(b) If $p = 1$, equality holds in (3.1) if and only if for almost every $\tau > 0$ there exists $c_\tau \in \mathbb{R}$ such that for a.e. $x' \in \mathbb{R}^{n-1}$ the level sets $\{x_n : f(x', x_n) > \tau\}$ are equivalent to intervals centered around c_τ.

We postpone the proof to the end of this section. Although inequality (3.1) can be deduced from a result by Beckner [Bec92, Theorem 3], as well as the equality cases for $p > 1$, it does not provide the equality cases for $p = 1$. For this reason we give an alternative proof of Theorem 3.1 which includes a characterization of equality cases for $p = 1$.

In the following, Theorem 3.1 is used to deduce a Steiner inequality for anisotropic fractional perimeters. The equality cases of this fractional Steiner inequality are different from those of the classical Steiner inequality, Theorem 2.17, even in the isotropic case where K is the Euclidean unit ball. In the following fractional version equality holds precisely for sets for which almost all slices are equivalent to intervals centered around the same point.

Corollary 3.2 (Steiner inequality for anisotropic fractional perimeters). *Let $E \subseteq \mathbb{R}^n$ be a Borel set of finite measure and K an origin-symmetric convex body which is symmetric with respect to the hyperplane $\{x_n = 0\}$. If $E^\#$ is the Steiner symmetrization of E with respect to x_n, then*

$$P_s(E, K) \geq P_s(E^\#, K). \tag{3.2}$$

Furthermore, assume that K is strictly convex. Then equality holds if and only if E is equivalent to a translate of $E^\#$.

Proof. The corollary follows easily from

$$P_s(E, K) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\mathbb{1}_E(x) - \mathbb{1}_E(y)|}{\|x - y\|_K^{n+s}} \,\mathrm{d}y \,\mathrm{d}x$$

and the case $p = 1$ for $f = \mathbb{1}_E$ in Theorem 3.1. ■

The key result used in the proof of Theorem 3.1 is the following general rearrangement inequality for functionals of the form

$$\mathcal{E}[f, g] = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J(f(x) - g(y)) k(x - y) \,\mathrm{d}y \,\mathrm{d}x,$$

where J is a non-negative convex function on $\mathbb{R}$ and $k \in L^1(\mathbb{R}^m)$ is symmetric decreasing. For the case $f = g$ it was proved by Frank & Seiringer [FS08] and we will follow their methods closely in our proof. We point out that we need the statement in its full generality for two functions for the following reason: By Fubini we split the integrals in the definition of the seminorm,

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{\|x - y\|_K^{n+sp}} \,\mathrm{d}y \,\mathrm{d}x \\ &\qquad = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u_{x'}(x_n) - u_{y'}(y_n)|^p}{\|(x' - y', x_n - y_n)\|_K^{n+sp}} \,\mathrm{d}y_n \,\mathrm{d}x_n \right) \mathrm{d}y' \,\mathrm{d}x'. \end{aligned}$$

The expression in parentheses depends on the sections $u_{x'}$ and $u_{y'}$ which are in general two different functions on $\mathbb{R}$.

We emphasize that for the equality cases the two functions f and g or their level sets, respectively, share the *same* center which plays a crucial role in the discussion of minimizers for the anisotropic fractional isoperimetric inequality.

Proposition 3.3. *Let J be a non-negative, convex function on $\mathbb{R}$ with $J(0) = 0$ and let $k \in L^1(\mathbb{R}^m)$ be a symmetric decreasing function. For non-negative measurable functions f and g on $\mathbb{R}^m$ define*

$$\mathcal{E}[f,g] := \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} J(f(x)-g(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x \tag{3.3}$$

and suppose that $|\{f>\tau\}|$ and $|\{g>\tau\}|$ are finite for all $\tau > 0$.

1. *The functional $\mathcal{E}$ does not increase under symmetric decreasing rearrangement, i.e.*
$$\mathcal{E}[f,g] \geq \mathcal{E}[f^*,g^*]. \tag{3.4}$$
2. *Furthermore, suppose that $\mathcal{E}[f,g] < \infty$ and that k is strictly symmetric decreasing.*
   (a) *If J is strictly convex then equality in (3.4) holds if and only if there exists a point $c \in \mathbb{R}^m$ such that for a.e. $x \in \mathbb{R}^m$*
   $$f(x) = f^*(x-c) \quad \text{and} \quad g(x) = g^*(x-c),$$
   i.e. f and g are symmetric decreasing around the same center c almost everywhere.
   (b) *If $J(t) = |t|$ then equality in (3.4) holds if and only if the level sets $\{f>\tau\}$ and $\{g>\tau\}$ are equivalent to balls around the same center $c_\tau \in \mathbb{R}^m$ for a.e. $\tau > 0$.*

Proof. Throughout the proof we assume that $\mathcal{E}[f,g] < \infty$ since otherwise the inequality (3.4) holds trivially.

First, we decompose J into

$$J = J_+ + J_-$$

where $J_+(t) = J(t)$ for $t \geq 0$ and $J_+(t) = 0$ for $t \leq 0$. Correspondingly, $\mathcal{E}$ can be decomposed into $\mathcal{E} = \mathcal{E}_+ + \mathcal{E}_-$. Since $\mathcal{E}_-[f,g] = \tilde{\mathcal{E}}_+[g,f]$ where the corresponding function $\tilde{J}_+(t) := J_-(-t)$ vanishes for $t \leq 0$ we only need to show the assumptions for the functional $\mathcal{E}_+$. The proof consists of two steps: In the first step we prove all assertions for bounded f and g and in the second step we remove the restriction that the functions are bounded. In both cases, the essential tool will be Riesz's rearrangement inequality, Theorem 2.6.

Step 1:

We assume first that f and g are bounded. Since J_+ is convex, the right derivative J_+' exists everywhere and is non-decreasing. So we can express $J_+(f(x)-g(y))$ as an integral via

$$J_+(f(x)-g(y)) = \int_{-\infty}^{f(x)-g(y)} J_+'(s)\,\mathrm{d}s = \int_0^\infty J_+'(f(x)-\tau)\mathbb{1}_{\{g\leq\tau\}}(y)\,\mathrm{d}\tau.$$

By Fubini's theorem

$$\mathcal{E}_+[f,g] = \int_0^\infty e_\tau^+[f,g]\,\mathrm{d}\tau$$

where

$$e_\tau^+[f,g] := \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} J_+'(f(x)-\tau)k(x-y)\mathbb{1}_{\{g\le\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Note that we cannot apply the Riesz rearrangement inequality yet since the level sets $\{\mathbb{1}_{\{g\le\tau\}} > t\}$ have infinite measure for $t<1$. Instead by the boundedness of f and

$$\int_{\mathbb{R}^m} J_+'(f(x)-\tau)\,\mathrm{d}x = \int_{\{f>\tau\}} J_+'(f(x)-\tau)\,\mathrm{d}x \le |\{f>\tau\}|\,J_+'(\sup f) < \infty$$

we can split the integral in $e_\tau^+[f,g]$ using $\mathbb{1}_{\{g\le\tau\}}(y) = 1-\mathbb{1}_{\{g>\tau\}}(y)$, so

$$e_\tau^+[f,g] = \|k\|_{L^1}\int_{\mathbb{R}^m} J_+'(f(x)-\tau)\,\mathrm{d}x - \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} J_+'(f(x)-\tau)k(x-y)\mathbb{1}_{\{g>\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x.$$

Since J_+' is non-decreasing, the first integral does not change by replacing f with f^*, and for the second integral Riesz's rearrangement inequality gives

$$\begin{aligned}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} & J_+'(f(x)-\tau)k(x-y)\mathbb{1}_{\{g>\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} J_+'(f^*(x)-\tau)k(x-y)\mathbb{1}_{\{g^*>\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Together with the same argument for $\tilde{\mathcal{E}}_+[g,f]$ this proves inequality (3.4) for bounded functions.

Next we settle the conditions for equality in this case: For a.e. $\tau>0$ we have

$$\begin{aligned}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} & J_+'(f(x)-\tau)k(x-y)\mathbb{1}_{\{g>\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} J_+'(f^*(x)-\tau)k(x-y)\mathbb{1}_{\{g^*>\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x, \qquad (3.5)\\ \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} & \tilde{J}_+'(g(x)-\tau)k(x-y)\mathbb{1}_{\{f>\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \tilde{J}_+'(g^*(x)-\tau)k(x-y)\mathbb{1}_{\{f^*>\tau\}}(y)\,\mathrm{d}y\,\mathrm{d}x. \qquad (3.6)\end{aligned}$$

If we assume k to be strictly decreasing, then by the equality cases in Riesz's rearrangement inequality, Theorem 2.6, there must exist $c_\tau, d_\tau \in \mathbb{R}^m$ such that up to sets of measure zero

$$\text{by (3.5)} \qquad J_+'(f(x)-\tau) = J_+'(f^*(x-c_\tau)-\tau) \quad \text{and} \qquad (3.7a)$$
$$\{g>\tau\} = \{x : g^*(x-c_\tau) > \tau\}, \qquad (3.7b)$$

$$\text{by (3.6)} \qquad \tilde{J}_+'(g(x)-\tau) = \tilde{J}_+'(g^*(x-d_\tau)-\tau) \quad \text{and} \qquad (3.8a)$$
$$\{f>\tau\} = \{x : f^*(x-d_\tau) > \tau\}, \qquad (3.8b)$$

so the level sets of f and g are equivalent to balls for a.e. $\tau>0$. If furthermore J is strictly convex and thus J_+' and $\tilde{J}_+'$ are strictly increasing on $[0,\infty)$, from (3.7a) and (3.8a) we deduce that $f(x) = f^*(x-c_\tau)$ and $g(x) = g^*(x-d_\tau)$ almost everywhere. Since these equalities hold true for almost every $\tau>0$, the centers c_τ and d_τ do not depend on τ and we simply write c and d for them. On one hand, by $f(x) = f^*(x-c)$ the level sets of f are equivalent to balls centered around c, but on the other hand, by (3.8b) almost all level sets are centered around d which is only possible if $c = d$.

If $J(t) = |t|$, then (3.7a) and $J'_+(t) = \mathbb{1}_{[0,\infty)}(t)$ imply that for a.e. $\tau > 0$ it holds that $\{f > \tau\} = \{x : f^*(x - c_\tau) > \tau\}$, so the level sets are equivalent to balls centered around c_τ. But (3.8b) implies that these level sets are also centered around d_τ which can only happen if $c_\tau = d_\tau$.

Step 2:

We now remove the assumption that f and g are bounded. We put $f_N := \min(f, N)$ for $N > 0$ and notice that $(f_N)^* = (f^*)_N =: f_N^*$ as well as $f_N \nearrow f$ pointwise as $N \to \infty$. Since for every $x, y \in \mathbb{R}^m$ the expression $J_+(f_N(x) - g_N(y))$ is non-decreasing in N, by step 1 and the monotone convergence theorem we get the inequality

$$\mathcal{E}_+[f, g] \geq \mathcal{E}_+[f^*, g^*].$$

Finally we turn our attention to the cases of equality whenever k is strictly decreasing. We decompose $f = f_N + f_u$ and $g = g_N + g_u$ with f_N and g_N defined as before and f_u and g_u possibly unbounded. In particular,

$$\begin{aligned}
\mathcal{E}_+[f_N, g_N] = &\int_{\{f\leq N\}} \int_{\{g\leq N\}} J_+(f(x) - g(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x \, + \\
&\int_{\{f>N\}} \int_{\{g\leq N\}} J_+(N - g(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x, \\
\mathcal{E}_+[f_u, g_u] = &\int_{\{f>N\}} \int_{\{g\leq N\}} J_+(f_u(x))k(x-y)\,\mathrm{d}y\,\mathrm{d}x \, + \\
&\int_{\{f>N\}} \int_{\{g>N\}} J_+(f(x) - g(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_+(f_u(x) + N - g_N(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\{f\leq N\}} \int_{\{g\leq N\}} J_+(N - g(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x \, + \\
&\quad \int_{\{f>N\}} \int_{\{g\leq N\}} J_+(f(x) - g(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x \, + \\
&\quad \int_{\{f>N\}} \int_{\{g>N\}} J_+(f_u(x))k(x-y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

such that

$$\mathcal{E}_+[f, g] = \mathcal{E}_+[f_N, g_N] + \mathcal{E}_+[f_u, g_u] + \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} I_N(f_u(x), g_N(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x \quad (3.9)$$

where

$$I_N(f, g) := J_+(f + N - g) - J_+(f) - J_+(N - g).$$

If we assume that $0 < N - g \leq f$ then by convexity of J_+ we have

$$\frac{J_+(N-g) - J_+(0)}{N-g} \leq \frac{J_+(f + N - g) - J_+(f)}{N-g}$$

and an analogous inequality holds for exchanged roles of $N - g$ and f. Using $J_+(0) = 0$ we get that $I_N(f,g) \geq 0$ for $0 \leq g \leq N$ and $f \geq 0$. In particular, all integrals in (3.9) are non-negative and finite. Since $\{f_u > \tau\} = \{f > \tau + N\}$ it holds that $(f_u)^* = (f^*)_u$, so that by rearranging f and g all of the functions appearing on the right hand side of (3.9) are replaced by their rearrangements. We claim that the last integral in (3.9) does not increase when replacing f_u and g_N by their rearrangements $(f_u)^*$ and g_N^*. If $\mathcal{E}[f,g] = \mathcal{E}[f^*,g^*]$ then this would imply that $\mathcal{E}_+[f_N, g_N] = \mathcal{E}_+[f_N^*, g_N^*]$ for all $N > 0$ which would eventually lead to the equality cases established in step 1.

Finally, we prove the claim that the double integral in (3.9) does not increase under rearrangement: Since J_+ is convex its right derivative J_+' is the distribution function of a non-negative measure μ. In particular, $J_+'(s) = \int_0^s \mathrm{d}\mu(\tau)$ and

$$J_+(t) = \int_0^\infty (t-\tau)_+ \,\mathrm{d}\mu(\tau).$$

This implies that

$$I_N(f,g) = \int_0^\infty \iota_{N,\tau}(f,g)\,\mathrm{d}\mu(\tau)$$

where

$$\iota_{N,\tau}(f,g) := (f + N - g - \tau)_+ - (f-\tau)_+ - (N - g - \tau)_+$$

so it suffices to prove that for all τ the double integral

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \iota_{N,\tau}(f_u(x), g_N(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x$$

does not increase under rearrangement. In order to apply the Riesz rearrangement inequality we write

$$\iota_{N,\tau}(f,g) = \iota_{N,\tau}^{(1)}(f) - \iota_{N,\tau}^{(2)}(f,g)$$

where

$$\begin{aligned}\iota_{N,\tau}^{(1)}(f) &:= f - (f-\tau)_+,\\ \iota_{N,\tau}^{(2)}(f,g) &:= f - (f + N - g - \tau)_+ + (N - g - \tau)_+ = \min(f, (g - N + \tau)_+).\end{aligned}$$

Since $\iota_{N,\tau}^{(1)}$ is bounded from above by τ and non-decreasing in v, and since by $|\{f > N\}| < \infty$ the support of f_u has finite measure, the integral

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \iota_{N,\tau}^{(1)}(f_u(x))k(x-y)\,\mathrm{d}y\,\mathrm{d}x = \|k\|_{L^1}\int_{\mathbb{R}^m} \iota_{N,\tau}^{(1)}(f_u(x))\,\mathrm{d}x$$

is finite and does not change under rearrangement. For the $\iota_{N,\tau}^{(2)}$-integral we use the representation of $\iota_{N,\tau}^{(2)}$ as a minimum and the layer-cake formula,

$$\begin{aligned}&\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \iota_{N,\tau}^{(2)}(f_u(x), g_N(y))k(x-y)\,\mathrm{d}y\,\mathrm{d}x = \\ &\qquad \int_0^\infty \left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \mathbb{1}_{\{f_u > t\}}(x)k(x-y)\mathbb{1}_{\{(g_N - N + \tau)_+ > t\}}(y)\,\mathrm{d}y\,\mathrm{d}x\right)\mathrm{d}t.\end{aligned}$$

By the Riesz rearrangement inequality the double integral in parentheses does not decrease under rearrangement which shows the claim. ∎

Next, we generalize the previous result to the case where symmetry of k is only assumed for one of the factors in the decomposition $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ and we use Steiner symmetrization instead of full-dimensional Schwarz symmetrization. Although we only consider the case $f = g$ in the proof of Theorem 3.1, we state the next result for two possibly different functions f and g as this might be of independent interest.

Corollary 3.4. *Let J be a non-negative, convex function on $\mathbb{R}$ with $J(0) = 0$, and let $k \in L^1(\mathbb{R}^n)$ be such that $k_{x'}$ is a symmetric decreasing function on $\mathbb{R}$ for every $x' \in \mathbb{R}^{n-1}$. For non-negative measurable functions f and g on $\mathbb{R}^n$ we define $\mathcal{E}[f,g]$ as in (3.3) with integration over $\mathbb{R}^n$. Suppose that for a.e. $x' \in \mathbb{R}^{n-1}$ the values $\mathcal{H}^1(\{f_{x'} > \tau\})$ and $\mathcal{H}^1(\{g_{x'} > \tau\})$ are finite for all $\tau > 0$.*

1. *The functional $\mathcal{E}$ does not increase under Steiner symmetrization, i.e.*

$$\mathcal{E}[f,g] \geq \mathcal{E}[f^{\#}, g^{\#}], \tag{3.10}$$

where $f^{\#}$ denotes the Steiner symmetrization of f with respect to x_n.

2. *Furthermore, suppose that $\mathcal{E}[f,g] < \infty$ and that $k_{x'}$ is strictly symmetric decreasing for every $x' \in \mathbb{R}^{n-1}$.*

   (a) *If J is strictly convex then equality in (3.10) holds if and only if there exists a point $c \in \mathbb{R}$ such that for almost every $x' \in \mathbb{R}^{n-1}$*

$$f(x', x_n) = f^{\#}(x', x_n - c) \quad \textit{and} \quad g(x', x_n) = g^{\#}(x', x_n - c)$$

   for a.e. $x_n \in \mathbb{R}$.

   (b) *If $J(t) = |t|$ then equality in (3.10) holds if and only if for a.e. $\tau > 0$ there exists $c_\tau \in \mathbb{R}$ such that for a.e. $x' \in \mathbb{R}^{n-1}$ the level sets*

$$\{x_n : f(x', x_n) > \tau\} \quad \textit{and} \quad \{x_n : g(x', x_n) > \tau\}$$

   are equivalent to intervals around the same center c_τ.

Proof. By Fubini we decompose the integration,

$$\mathcal{E}[f,g] = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} J(f_{x'}(x_n) - g_{y'}(y_n)) k_{x'-y'}(x_n - y_n) \, \mathrm{d}y_n \, \mathrm{d}x_n \right) \mathrm{d}y' \, \mathrm{d}x',$$

where for the double integration in parentheses we can apply Proposition 3.3 to immediately see inequality (3.10). For the discussion of equality cases we remark that equality in (3.10) implies that for a.e. $x', y' \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} J(f_{x'}(x_n) - g_{y'}(y_n)) k_{x'-y'}(x_n - y_n) \, \mathrm{d}y_n \, \mathrm{d}x_n \\ &\qquad = \int_{\mathbb{R}} \int_{\mathbb{R}} J(f^*_{x'}(x_n) - g^*_{y'}(y_n)) k_{x'-y'}(x_n - y_n) \, \mathrm{d}y_n \, \mathrm{d}x_n. \end{aligned}$$

Now observe that the centers $c_{x',y'}$ (resp. $c_{\tau,x',y'}$) obtained by the equality cases of Proposition 3.3 cannot depend on x' and y' since for fixed x' we have $f_{x'}(x_n) = f^*_{x'}(x_n - c_{x',y'})$ (resp. $\{f_{x'} > \tau\}$ is centered around $c_{\tau,x',y'}$) for all y' and vice versa for fixed y' and $g_{y'}$. ■

Proof of Theorem 3.1. Since $|f(x) - f(y)| \geq ||f(x)| - |f(y)||$ with equality if and only if f is proportional to a non-negative function we assume that f is non-negative throughout the proof. Note that the kernel function $\|x-y\|_K^{-(n+sp)}$ is not integrable so first we rewrite the seminorm following an idea of Almgren & Lieb [AL89, p. 770]:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\|x-y\|_K^{n+sp}}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{\Gamma(\frac{n+sp}{2})}\int_0^\infty I_\alpha[f]\alpha^{\frac{n+sp}{2}-1}\,\mathrm{d}\alpha,$$

where Γ is the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$ and

$$I_\alpha[f] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)-f(y)|^p e^{-\alpha\|x-y\|_K^2}\,\mathrm{d}y\,\mathrm{d}x.$$

This can be seen by Fubini and computing

$$\int_0^\infty \alpha^{\frac{n+sp}{2}-1}e^{-\alpha\|x-y\|_K^2}\,\mathrm{d}\alpha = \int_0^\infty \frac{t^{\frac{n+sp}{2}-1}}{\|x-y\|_K^{n+sp-2}}e^{-t}\cdot\frac{1}{\|x-y\|_K^2}\,\mathrm{d}t = \frac{\Gamma(\frac{n+sp}{2})}{\|x-y\|_K^{n+sp}},$$

where we substituted $t = \alpha\|x-y\|_K^2$ in the first equality. Now we apply Corollary 3.4 with $J(t) = |t|^p$ and $k(\xi) = e^{-\alpha\|\xi\|_K^2}$ to $I_\alpha[f] = \mathcal{E}[f,f]$. ■

We conclude this section with the remark that a Steiner inequality analogous to (3.2) also holds for general nonlocal perimeters Per_k whenever the kernel function k is symmetric decreasing in the x_n-coordinate, with the same equality cases if k is strictly decreasing in the x_n-coordinate.

3.2 A Pólya-Szegő inequality for anisotropic symmetrization

We recall that a set $E \subseteq \mathbb{R}^n$ is called *unconditional* if it is symmetric with respect to every coordinate hyperplane. Furthermore, it is convenient to introduce the following notion: Let $L \subseteq \mathbb{R}^n$ be a bounded set with $|L| > 0$. We call L a *box body* if for every $x = (x_1, \dots, x_n) \in L$ the box $[-|x_1|, |x_1|] \times \dots \times [-|x_n|, |x_n|]$ is fully contained in L. Under certain assumptions on the unit ball, every minimizer of the anisotropic fractional isoperimetric inequality is a box body:

Theorem 3.5. *Let $K \subseteq \mathbb{R}^n$ be an unconditional strictly convex body. Then every minimizer $M \subseteq \mathbb{R}^n$ of the anisotropic fractional isoperimetric inequality,*

$$P_s(E,K) \geq \gamma_{n,s}(K)|E|^{\frac{n-s}{n}}, \quad E \subseteq \mathbb{R}^n \text{ bounded Borel,} \tag{3.11}$$

is up to translation equivalent to a box body.

Before we prove the theorem, let us first investigate some geometric properties of box bodies. Clearly, every box body is unconditional and thus completely determined by its intersection with the orthant $\mathbb{R}^n_+$ of points in $\mathbb{R}^n$ with non-negative coordinates. Conversely, if a set $L_+ \subseteq \mathbb{R}^n_+$ with $|L_+| > 0$ is down-monotone (i.e. $x = (x_1, \dots, x_n) \in L_+$ implies that $y = (y_1, \dots, y_n) \in L_+$ whenever $0 \leq y_i \leq x_i$ for all $i = 1, \dots, n$), then $L_+ = L \cap \mathbb{R}^n_+$ for a box body $L \subseteq \mathbb{R}^n$.

To the author's knowledge subsets L_+ of $\mathbb{R}^n_+$ of this form were mostly studied for the case that L_+ is a convex body or polyhedron. In this case they are called anti-blocking bodies or polyhedra, respectively (cf. [Ful71], see also [Sch03]). Our definition of box bodies does not demand that the sets are convex. However, box bodies are still star-shaped as the following lemma shows.

Lemma 3.6. *Every box body is an unconditional star body.*

Proof. Let $L \subseteq \mathbb{R}^n$ be a box body. Then L is star-shaped since for every $x \in L$ the line segment $[o, x]$ is a half-diagonal of the box $[-|x_1|, |x_1|] \times \cdots \times [-|x_n|, |x_n|]$ which is fully contained in L. Furthermore, since $|L| > 0$ there exists a point $x \in L$ such that $x_i \neq 0$ for all $i = 1, \ldots, n$, so the box spanned by x and consequently L contains the origin in the interior.

Next, we show that the radial function ρ_L is continuous. For $u \in \mathbb{S}^{n-1}$ and all $0 < \alpha < \rho_L(u)$ the point $x := (\rho_L(u) - \alpha)u$ is contained in L. Denote by ρ_α the radial function of the box $[-|x_1|, |x_1|] \times \cdots \times [-|x_n|, |x_n|]$ spanned by x. Then, by our assumption, the radial function ρ_L of L is bounded from below by ρ_α, i.e.

$$\rho_L(v) \geq \rho_\alpha(v) \quad \text{for all } v \in \mathbb{S}^{n-1}. \tag{3.12}$$

Suppose that ρ_L is not continuous at u. Then there exists $\varepsilon > 0$ such that in every neighbourhood of u there is a point v with

$$|\rho_L(u) - \rho_L(v)| \geq \varepsilon. \tag{3.13}$$

On the other hand, ρ_α is continuous so that $\rho_\alpha(w) > \rho_\alpha(u) - \frac{\varepsilon}{2}$ for every w in a certain neighbourhood of u. We only consider the case that $\rho_L(u) \geq \rho_L(v) + \varepsilon$ in (3.13) for a point v in this neighbourhood, since for the case that $\rho_L(u) \leq \rho_L(v) - \varepsilon$ one can use similar arguments. Since $\rho_\alpha(u) = \rho_L(u) - \alpha$ putting the inequalities together yields

$$\rho_\alpha(v) > \rho_\alpha(u) - \frac{\varepsilon}{2} = (\rho_L(u) - \alpha) - \frac{\varepsilon}{2} \geq \rho_L(v) + \frac{\varepsilon}{2} - \alpha > \rho_L(v)$$

for all $\alpha < \frac{\varepsilon}{2}$ which is a contradiction to (3.12). ■

Note that the converse statement, that every unconditional star body is a box body, is not true. For example, take $L := R_1 \cup R_2 \subseteq \mathbb{R}^2$ with rectangles $R_1 := [-2, 2] \times [-1, 1]$ and $R_2 := [-1, 1] \times [-2, 2]$ and rotate L around the origin by $\frac{\pi}{4}$. The resulting set is an unconditional star body but not a box body.

We recall that for a Borel set $E \subseteq \mathbb{R}^n$ the set $E^{(1)}$ of points of density one, or Lebesgue points, is defined by

$$E^{(1)} := \left\{ x \in \mathbb{R}^n : \lim_{r \searrow 0} \frac{|E \cap B_r^n(x)|}{|B_r^n(x)|} = 1 \right\}.$$

Since $E^{(1)}$ differs from E only on a set of measure zero we can restrict the study of the anisotropic fractional isoperimetric problem to sets consisting only of Lebesgue points. To establish symmetry of minimizers we need the following lemma which is stated in [Fus04].

Lemma 3.7. *Let $E \subseteq \mathbb{R}^n$ be a Borel set such that for a.e. $x' \in \mathbb{R}^{n-1}$ the section $E_{x'}$ is equivalent to an interval. Then the set of points of density one, $E^{(1)}$, of E has the property that for every $x' \in \mathbb{R}^{n-1}$ the section $(E^{(1)})_{x'}$ is an interval.*

Proof of Theorem 3.5. Since K is symmetric with respect to every coordinate hyperplane $\{x_i = 0\}$, $i = 1, \ldots, n$, by the classification of equality cases in the Steiner inequality, Corollary 3.2, almost all sections of a minimizer M in x_i-direction are equivalent to intervals centered around the same center c_i. By translation invariance we may assume that $c_i = 0$ for all $i = 1, \dots, n$ and by passing to $M^{(1)}$ by Lemma 3.7 we also may assume that *all* sections in every coordinate direction are centered around 0. This implies that for every $x \in M$ the box $[-|x_1|, |x_1|] \times \cdots \times [-|x_n|, |x_n|]$ is fully contained in M which finishes the proof. ■

Remark 3.8. Assume that the kernel k of the nonlocal perimeter Per_k is strictly symmetric decreasing in every coordinate direction. If the nonlocal isoperimetric problem (2.17) has a minimizer, then we can repeat all arguments in the proof of Theorem 3.5 to see that each minimizer is up to translation equivalent to a box body. ▲

The next result is a Pólya-Szegő principle for anisotropic fractional perimeters where the symmetrization is carried out with respect to minimizers of the anisotropic fractional isoperimetric inequality.

Theorem 3.9. *Let $K \subseteq \mathbb{R}^n$ be an unconditional strictly convex body and M a minimizer of the anisotropic fractional isoperimetric inequality* (3.11)*. Then the anisotropic rearrangement f^M with respect to M is well-defined, and*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}y\,\mathrm{d}x \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^M(x)-f^M(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}y\,\mathrm{d}x \tag{3.14}$$

for all $f \in L^1(\mathbb{R}^n)$.

Proof. The rearrangement with respect to M yields a well-defined function, since by Theorem 3.5 the minimizer M is a box body and thus, in particular, a star body. To show (3.14) we apply the coarea formula for anisotropic fractional perimeters (see Theorem 2.15) and get

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}y\,\mathrm{d}x &= 2\int_0^\infty P_s(\{|f|>\tau\}, K)\,\mathrm{d}\tau \\ &\geq 2\int_0^\infty P_s(\{|f|>\tau\}^M, K)\,\mathrm{d}\tau \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f^M(x)-f^M(y)|}{\|x-y\|_K^{n+s}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

■

Chapter 4

Fractional perimeters on Riemannian manifolds

In this chapter we prove a convergence and characterization result for Sobolev norms on manifolds which is the Riemannian analogue of results by Bourgain, Brezis & Mironescu and Dávila (see Theorem 2.10). After defining all function spaces and norms in the Riemannian setting, we compare differential operators on manifolds with the corresponding operators on $\mathbb{R}^n$ represented in suitable coordinates. This leads us to the notion of weighted BV functions introduced by Baldi [Bal01]. Together with a suitable covering of the manifold we use techniques from [BBM01] and [Dáv02] as well as from [LS11] to prove the convergence of fractional perimeters to the standard perimeter. Similarly to the Euclidean case, this convergence result extends to functionals which are represented by radial mollifiers.

This chapter is based on joint work together with Olaf Mordhorst published in [KM19].

Throughout this chapter we denote by M a compact connected n-dimensional Riemannian manifold of class C^∞ equipped with the metric g.

4.1 Sobolev spaces and BV functions on manifolds

We define the *weak gradient* of a function $f \in L^1(M)$ as the unique vector field Y on M, such that $\int_M |Y|_g \, \mathrm{d}V_g < \infty$ and for all smooth vector fields X on M

$$\int_M g(Y, X) \, \mathrm{d}V_g = - \int_M f \text{ div } X \, \mathrm{d}V_g$$

holds. Here, uniqueness is understood up to sets of measure zero. We denote it by $\operatorname{grad} f$ and justify this notation by remarking that for smooth functions the (standard) gradient and the weak gradient coincide.

For $1 \le p < \infty$ we define the Sobolev space $W^{1,p}(M)$ by

$$W^{1,p}(M) := \{ f \in L^p(M) : \text{the weak gradient } \operatorname{grad} f \text{ exists and } |\operatorname{grad} f|_g \in L^p(M) \} .$$

Equipped with the norm

$$\|f\|_{W^{1,p}} := \left(\|f\|_{L^p}^p + \int_M |\operatorname{grad} f|_g^p \, \mathrm{d}V_g \right)^{\frac{1}{p}} ,$$

$W^{1,p}(M)$ is a Banach space (cf. [Heb99, p. 21]).

We need an alternative characterization of Sobolev spaces, which works only if $p > 1$, since only in this case the spaces $L^p(M)$ and $W^{1,p}(M)$ are reflexive (cf. [Heb99, Prop. 2.3]; compare also with Theorem 2.8):

Proposition 4.1. *Let $f \in L^p(M)$, $p > 1$. Then $f \in W^{1,p}(M)$ if and only if there exists a sequence $(f_j)_j \subset C_c^\infty(M)$ such that the following two statements hold:*

1. $f_j \overset{j\to\infty}{\to} f$ *in* $L^p(M)$, *and*
2. $L := \lim_{j\to\infty} \int_M |\operatorname{grad} f_j|_g^p \, \mathrm{d}V_g < \infty.$

In this case $L = \int_M |\operatorname{grad} f|_g^p \, \mathrm{d}V_g$.

In analogy to the Euclidean case, the *variation* of a function $f \in L^1(M)$ is introduced in [MPPP07] as a measure given on open sets $U \subseteq M$ by

$$|Df|(U) := \sup \left\{ \int_M f \operatorname{div} X \, \mathrm{d}V_g : X \in \Gamma_c(TM), \operatorname{spt} X \subseteq U, |X|_g \leq 1 \right\}, \tag{4.1}$$

where $\Gamma_c(TM)$ denotes the space of all compactly supported vector fields of class C^∞. The definition works also for not necessarily compact manifolds but since we only work on compact M, the condition that the vector fields involved are compactly supported can of course be dropped. We say that f is of *bounded variation* and write $f \in BV(M)$, if $|Df|(M) < \infty$.

If $f \in C^\infty(M)$, then

$$|Df|(U) = \int_U |\operatorname{grad} f|_g \, \mathrm{d}V_g$$

for all open $U \subseteq M$. This can be seen as follows: Since M is a manifold without boundary, the divergence theorem implies

$$0 = \int_M \operatorname{div}(fX) \, \mathrm{d}V_g = \int_M f \operatorname{div} X \, \mathrm{d}V_g + \int_M g(\operatorname{grad} f, X) \, \mathrm{d}V_g,$$

for every smooth vector field $X \in \Gamma_c(TM)$. Thus, we can approximate the supremum in (4.1) by a sequence of smooth vector fields converging to $-\mathbb{1}_{\{\operatorname{grad} f \neq 0\}} \frac{\operatorname{grad} f}{|\operatorname{grad} f|_g}$.

A related concept is the notion of *weighted BV functions*, as introduced in [Bal01] for the Euclidean case. Let $\Omega \subseteq \mathbb{R}^n$ open and Ω_0 an open neighbourhood of $\overline{\Omega}$. We call a lower semicontinuous function $w \in L^1_{loc}(\Omega_0)$, $w > 0$, satisfying

$$\frac{1}{|B_r^n(x)|} \int_{B_r^n(x)} w(y) \, \mathrm{d}y \leq C w(x)$$

for all balls $B_r^n(x) \subseteq \Omega_0$ with a constant $C > 0$, a weight. The variation of a function $f \in L^1(\Omega; w \, \mathrm{d}x)$ with respect to the weight w is defined as

$$|Df|_w(\Omega) := \sup \left\{ \int_\Omega f \operatorname{div} T \, \mathrm{d}x : T \in C_c^1(\Omega; \mathbb{R}^n), |T(x)| \leq w(x) \text{ for all } x \in \Omega \right\},$$

and the space $BV(\Omega; w)$ consists of those functions f such that $|Df|_w(\Omega) < \infty$. In accordance to the case of unweighted BV functions, the map $f \mapsto |Df|_w(\Omega)$,

$f \in BV(\Omega; w)$, is lower semicontinuous with respect to $L^1(\Omega; w\,\mathrm{d}x)$-convergence, see [Bal01, Theorem 3.2].

The following lemma establishes a link between the notions of variation on a manifold and weighted variation in $\mathbb{R}^n$, as well as an analogous result for weak gradients. A short proof of the second statement was given in [MPPP07]. Some arguments of the proof are not accessible to the author, so we include an alternative proof.

Lemma 4.2. *Let $\phi : U \to \mathbb{R}^n$ be a chart on M such that the operator norm of $d\phi|_x : (T_xM, |\cdot|_g) \to (\mathbb{R}^n, |\cdot|)$ satisfies $\|d\phi|_x\| \le 1+\varepsilon$ for all $x \in U$.*

1. *If $f \in W^{1,p}(U)$, then for all $\xi \in \phi(U)$*

$$|\mathrm{grad}\, f(\phi^{-1}(\xi))|_g \le (1+\varepsilon)|\nabla(f\circ\phi^{-1})(\xi)|\ . \tag{4.2}$$

2. *If $f \in BV(U)$, then*

$$|Df|(U) \le (1+\varepsilon)|D(f\circ\phi^{-1})|_w(\phi(U)) \tag{4.3}$$

with weight $w = \sqrt{\det(\hat{g}_{\alpha\beta})}$.

Proof. Let $f \in W^{1,p}(U)$. Since $\nabla(f\circ\phi^{-1})$ is the weak gradient of $f\circ\phi^{-1}$, we have for all smooth compactly supported vector fields $T \in C_c^\infty(\phi(U);\mathbb{R}^n)$:

$$\begin{aligned}
-\int_{\phi(U)} \nabla(f\circ\phi^{-1})(\xi)\cdot T(\xi)\,\mathrm{d}\xi &= \int_{\phi(U)} (f\circ\phi^{-1})(\xi)\,\mathrm{div}_{\mathbb{R}^n} T(\xi)\,\mathrm{d}\xi \\
&= \int_{\phi(U)} f(\phi^{-1}(\xi))\,\mathrm{div}_{\mathbb{R}^n}\left(\sqrt{\det(\hat{g}_{\alpha\beta})}\frac{T}{\sqrt{\det(\hat{g}_{\alpha\beta})}}\right)(\xi)\,\mathrm{d}\xi.
\end{aligned}$$

If $\xi = \phi(x)$ and $T(\xi) = (T^1(\xi),\dots,T^n(\xi))^T \in \mathbb{R}^n \cong T_\xi\phi(U)$, then $d(\phi^{-1})|_\xi(T(\xi)) = \sum_{i=1}^n T^i(\phi(x))\frac{\partial}{\partial x^i}|_x$ with respect to the chart ϕ , so $d(\phi^{-1})$ is a bijection between vector fields on $\phi(U)$ and vector fields on U. We put $X(x) := d(\phi^{-1})|_\xi(T(\xi))$ and use the representation (2.18) of the divergence in coordinates to obtain

$$\begin{aligned}
&\int_{\phi(U)} f(\phi^{-1}(\xi))\,\mathrm{div}_{\mathbb{R}^n}\left(\sqrt{\det(\hat{g}_{\alpha\beta})}\frac{T}{\sqrt{\det(\hat{g}_{\alpha\beta})}}\right)(\xi)\,\mathrm{d}\xi \\
&= \int_U f(x)\sqrt{\det(g_{\alpha\beta}(x))}\ \mathrm{div}\ \left(\frac{X}{\sqrt{\det(g_{\alpha\beta})}}\right)(x)\frac{1}{\sqrt{\det(g_{\alpha\beta}(x))}}\,\mathrm{d}V_g(x) \\
&= -\int_U g(\mathrm{grad}\, f(x), X(x))\frac{1}{\sqrt{\det(g_{\alpha\beta}(x))}}\,\mathrm{d}V_g(x).
\end{aligned}$$

In analogy to the differential of a smooth function, we denote by $df|_x$ the covector field $X(x) \mapsto g(\mathrm{grad}\, f(x), X(x)), X(x) \in T_xM$, and further rewrite the last integral as

$$-\int_U df|_x(X(x))\frac{1}{\sqrt{\det(g_{\alpha\beta}(x))}}\,\mathrm{d}V_g(x) = -\int_{\phi(U)} df|_{\phi^{-1}(\xi)}(d(\phi^{-1})|_\xi(T(\xi)))\,\mathrm{d}\xi,$$

which proves the chain rule $\nabla(f\circ\phi^{-1})(\xi) = (df|_{\phi^{-1}(\xi)}\circ d(\phi^{-1})|_\xi)^T$ for weak gradients. It is equivalent to $d(f\circ\phi^{-1})|_\xi \circ d\phi|_{\phi^{-1}(\xi)} = df|_{\phi^{-1}(\xi)}$, so by duality we obtain the estimate

$$\begin{aligned}|\mathrm{grad}\, f(\phi^{-1}(\xi))|_g &= \|d(f\circ\phi^{-1})|_\xi \circ d\phi|_{\phi^{-1}(\xi)}\| \\ &\le \|d(f\circ\phi^{-1})|_\xi\| \cdot \|d\phi|_{\phi^{-1}(\xi)}\| \le (1+\varepsilon)|\nabla(f\circ\phi^{-1})(\xi)|,\end{aligned}$$

which shows the first statement.

If $f\in BV(U)$ and if X is a compactly supported vector field in U with $|X(x)|_g \le 1$ for all $x\in U$, then the vector field T on $\phi(U)$ defined by $T(\xi) := d\phi|_{\phi^{-1}(\xi)}(X(\phi^{-1}(\xi)))$ is smooth, compactly supported and satisfies the inequality $|T(\xi)| \le 1+\varepsilon$, since

$$\begin{aligned}|T(\xi)| &= |(d\phi|_{\phi^{-1}(\xi)} \circ d(\phi^{-1})|_\xi)(T(\xi))| \\ &\le (1+\varepsilon)|X(\phi^{-1}(\xi))|_g \le 1+\varepsilon.\end{aligned}$$

We apply formula (2.18) for the divergence in coordinates and compute

$$\int_U f \,\operatorname{div}\, X \,\mathrm{d}V_g = \int_{\phi(U)} (f\circ\phi^{-1})\,\operatorname{div}_{\mathbb{R}^n}\left(\sqrt{\det(\hat{g}_{\alpha\beta})}T\right)\,\mathrm{d}\xi.$$

Thus,

$$\begin{aligned}|Df|(U) &= \sup\left\{\int_U f\,\operatorname{div}\, X\,\mathrm{d}V_g : X\in\Gamma_c(TM), \mathrm{spt}\ X\subset U, |X(x)|_g\le 1\ \forall\, x\in M\right\} \\ &\le \sup\left\{\int_{\phi(U)} (f\circ\phi^{-1})\,\operatorname{div}_{\mathbb{R}^n}\left(\sqrt{\det(\hat{g}_{\alpha\beta})}T\right)\,\mathrm{d}\xi : T\in C_c^\infty(\phi(U);\mathbb{R}^n),\right. \\ &\qquad\qquad \left. |T(\xi)|\le 1+\varepsilon\ \forall\,\xi\in\phi(U)\right\} \\ &\le (1+\varepsilon)\sup\left\{\int_{\phi(U)} (f\circ\phi^{-1})\,\operatorname{div}_{\mathbb{R}^n}\tilde{T}\,\mathrm{d}\xi : \tilde{T}\in C_c^\infty(\phi(U);\mathbb{R}^n),\right. \\ &\qquad\qquad \left. |\tilde{T}(\xi)| \le \sqrt{\det(\hat{g}_{\alpha\beta}(\xi))}\ \forall\,\xi\in\phi(U)\right\} \\ &= (1+\varepsilon)|D(f\circ\phi^{-1})|_w(\phi(U))\end{aligned}$$

with weight $w = \sqrt{\det(\hat{g}_{\alpha\beta})}$, which concludes the proof of the second statement. ∎

The authors in [MPPP07] used formula (4.3) to show the following (compare with Theorem 2.8):

Proposition 4.3 ([MPPP07], Prop. 1.4). *Let $f\in L^1(M)$. Then $f\in BV(M)$ if and only if there exists a sequence $(f_j)_j\subset C_c^\infty(M)$ such that the following two statements hold:*

1. $f_j \overset{j\to\infty}{\to} f$ *in* $L^1(M)$, *and*
2. $L := \lim_{j\to\infty}\int_M |\mathrm{grad}\, f_j|_g\,\mathrm{d}V_g < \infty.$

In this case $L = |Df|(M)$ and $\lim_{j\to\infty}\int_U |\mathrm{grad}\, f_j|_g\,\mathrm{d}V_g = |Df|(U)$ for every open $U\subseteq M$.

The previous proposition provides a different approach to the space of BV function via approximation by smooth functions. The authors of [AGM15] give even further definitions of BV functions, which all agree on Riemannian manifolds.

For special weights w, Baldi gave a description of the space $BV(\Omega; w)$:

Proposition 4.4 ([Bal01], Prop. 3.5). *Let w be a Lipschitz continuous weight function on Ω. Then a function f belongs to $BV(\Omega; w)$ if and only if $f \in BV(\Omega)$ and $w \in L^1(d|Df|)$. In this case*

$$|Df|_w(\Omega) = \int_\Omega w \, \mathrm{d}|Df|. \tag{4.4}$$

In analogy to the Euclidean case, the *perimeter* of a measurable set $E \subseteq M$ is defined by

$$P(E) := |D\mathbb{1}_E|(M) = \sup \left\{ \int_E \operatorname{div} X \, \mathrm{d}V_g : X \text{ smooth vector field, } |X|_g \leq 1 \right\}.$$

If the boundary of E is a closed hypersurface of class C^∞ equipped with the metric $\tilde{g}$ inherited by M, then

$$P(E) = \mathrm{Vol}_{\tilde{g}}(\partial E) = \mathcal{H}^{n-1}(\partial E),$$

which follows by isometric embedding of M into a Euclidean ambient space of suitable dimension and the result therein (cf. [Mag12, Example 12.5]).

For $0 < s < 1$ and $1 \leq p < \infty$ the *fractional seminorm* of a measurable function $f : M \to \mathbb{R}$ is defined by

$$|f|_{s,p} := \left(\int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^{n+sp}} \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) \right)^{\frac{1}{p}},$$

where $d(x, y)$ is the geodesic distance between x and y. The *fractional perimeter* of a measurable set $E \subseteq M$ is defined by

$$P_s(E) := \int_E \int_{M \setminus E} \frac{1}{d(x,y)^{n+s}} \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x).$$

Computing the fractional seminorm with $p = 1$ of the indicator function $\mathbb{1}_E$ of E yields $|\mathbb{1}_E|_{s,1} = 2P_s(E)$.

4.2 Convergence of fractional seminorms as $s \nearrow 1$

The main result of this section is the following theorem, which is the Riemannian variant of Theorem 2.10:

Theorem 4.5. *Let M be a compact connected n-dimensional Riemannian manifold and $f \in L^p(M), p \geq 1$.*

1. *If $p > 1$, then*

$$\lim_{s \nearrow 1} (1 - s) \int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^{n+sp}} \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) = \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1}) K_{p,n}}{p} \int_M |\mathrm{grad}\, f(x)|_g^p \, \mathrm{d}V_g(x),$$

where the constant $K_{p,n}$ is given by

$$K_{p,n} := \frac{1}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} |e \cdot u|^p \, \mathrm{d}\mathcal{H}^{n-1}(u), \tag{4.5}$$

and $e \in \mathbb{S}^{n-1}$ is any unit vector.

2. *If $p = 1$, then*

$$\lim_{s \nearrow 1} (1-s) \int_M \int_M \frac{|f(x) - f(y)|}{d(x,y)^{n+s}} \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) = 2\mathcal{H}^{n-1}(\mathbb{B}^{n-1}) |Df|(M).$$

In particular, we get the convergence of fractional perimeters to the standard perimeter:

Corollary 4.6. *Let $E \subseteq M$ be a measurable set. Then*

$$\lim_{s \nearrow 1} (1-s) P_s(E) = \mathcal{H}^{n-1}(\mathbb{B}^{n-1}) P(E).$$

Proof. This follows immediately from Theorem 4.5, 2., with $f = \mathbb{1}_E$. ■

The convergence of fractional seminorms in Theorem 4.5 follows from a more general result, where the functionals in question are represented by radial mollifiers. For this reason we postpone the proof of Theorem 4.5 to the end of this section.

We adapt the definition of radial mollifiers, given in [BBM01] (see Section 2.3), to the manifold setting: We call a family of functions $\rho_\sigma : (0, \infty) \to [0, \infty), 0 < \sigma < 1$, a family of *radial mollifiers* if they satisfy the following properties:

$$\int_0^\infty \rho_\sigma(r) r^{n-1} \, \mathrm{d}r = \frac{1}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}, \quad \forall\, 0 < \sigma < 1, \tag{4.6}$$

$$\lim_{\sigma \searrow 0} \int_\delta^\infty \rho_\sigma(r) r^{n-1} \, \mathrm{d}r = 0, \quad \forall\, \delta > 0, \tag{4.7}$$

$$\rho_\sigma \text{ is monotonically decreasing on } (0, \infty). \tag{4.8}$$

Note that this definition differs from the definition in Section 2.3, since we additionally demand each mollifier to be monotonically decreasing, (4.8). As a consequence of the monotonicity and (4.7) we have the following uniform convergence on compact sets:

$$\lim_{\sigma \searrow 0} \sup_{r \in K} \rho_\sigma(r) = 0, \quad \forall\, K \subset (0, \infty) \text{ compact.} \tag{4.9}$$

Now the corresponding convergence and characterization result for functionals involving radial mollifiers reads as follows (compare with the Euclidean version, Theorem 2.9):

Theorem 4.7. *Let M be a compact connected n-dimensional Riemannian manifold and $f \in L^p(M)$ with $p \geq 1$. Furthermore, let $(\rho_\sigma)_\sigma$ be a family of radial mollifiers satisfying* (4.6)-(4.8)*, and $K_{p,n}$ be the constant defined in* (4.5).

1. *If* $p > 1$, *then*

$$\lim_{\sigma \searrow 0} \int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_\sigma(d(x,y)) \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) = K_{p,n} \int_M |\mathrm{grad}\, f(x)|_g^p \, \mathrm{d}V_g(x).$$

   In particular, $f \in W^{1,p}(M)$ *if and only if*

$$\liminf_{\sigma \searrow 0} \int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_\sigma(d(x,y)) \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) < \infty.$$

2. *If* $p = 1$, *then*

$$\lim_{\sigma \searrow 0} \int_M \int_M \frac{|f(x) - f(y)|}{d(x,y)} \rho_\sigma(d(x,y)) \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) = K_{1,n} |Df|(M).$$

   In particular, $f \in BV(M)$ *if and only if*

$$\liminf_{\sigma \searrow 0} \int_M \int_M \frac{|f(x) - f(y)|}{d(x,y)} \rho_\sigma(d(x,y)) \, \mathrm{d}V_g(y) \, \mathrm{d}V_g(x) < \infty.$$

We define the distance of a point $x \in M$ to a set $E \subseteq M$ by

$$d(x,E) := \inf \{ d(x,y) : y \in E \},$$

and for $\tau > 0$ we define the τ-neighbourhood of a set $E \subseteq M$ by

$$E^\tau := \{ x \in M : d(x,E) < \tau \}.$$

For our calculations we want to work with families of finitely many open sets in M, such that on each set the geodesic distance $d(x,y)$ can be controlled by the Euclidean distance on a corresponding chart (cf. [MPPP07, proof of Prop. 1.4]):

Lemma 4.8. *If $E \subseteq M$ is a compact set, then for each $\varepsilon \in (0,1)$ there exists a finite family $(U_k)_{k=1}^N$, $N = N(\varepsilon)$, of open sets of M such that*

1. $U_k \cap U_l = \emptyset, \quad \forall\, k \neq l$,

2. *there exists $\tau_0 > 0$ such that for all $0 < \tau < \tau_0$ the family $(U_k^\tau)_{k=1}^N$ is an open covering of E and $U_k^{\tau_0}$ lies in the domain of a coordinate chart (V_k, ϕ_k), where*

$$(1 - \varepsilon)|\phi_k(x) - \phi_k(y)| \leq d(x,y) \leq (1 + \varepsilon)|\phi_k(x) - \phi_k(y)|, \tag{4.10}$$

$$1 - \varepsilon \leq \sqrt{\det(g_{\alpha\beta}(x))} \leq 1 + \varepsilon \tag{4.11}$$

   for every $x, y \in V_k$.

3. *The operator norm $\|d\phi_k|_x\|$ of $d\phi_k|_x : (T_xM, |\cdot|_g) \to (\mathbb{R}^n, |\cdot|)$ is bounded by*

$$1 - \varepsilon \leq \|d\phi_k|_x\| \leq 1 + \varepsilon \tag{4.12}$$

   for every $x \in V_k$.

4. $\int_{\partial U_k} \mathrm{d}V_g = 0$ *for all* $k = 1, \dots, N$.

Furthermore, given a function $f \in BV(M)$*, the sets can be chosen in such a way that*

4'. $|Df|(\partial U_k) = 0$ *for all* $k = 1, \dots, N$.

Proof. For each point $p \in E$ there exists a normal coordinate chart (V_p, ϕ_p) around p such that the inequalities (4.10) and (4.11) are satisfied, see e.g. Theorem 2.23 as well as [HZ16, p.8]. Since the operator norm of $d\phi_p$ at p is equal to 1 we may choose V_p so small around p such that inequality (4.12) holds. The compactness of E ensures the existence of $\tau_0 > 0$ such that for every p there exists an open subset $W_p \subset V_p$ around p such that $W_p^{\tau_0} \subseteq V_p$. Since the geodesic spheres $\{y \in E : d(p,y) = r\} \cap W_p$ form a disjoint uncountable covering of W_p and both $\mathrm{d}V_g$ and $|Df|$ are finite Radon measures, there exists an open geodesic ball $B_p \subseteq W_p$ such that both $\int_{\partial B_p} \mathrm{d}V_g = 0$ and $|Df|(\partial B_p) = 0$ hold.

By compactness of E there exists an open subcovering $(B_{p_k})_{k=1}^N$ of $(B_p)_{p\in E}$, which can be made disjoint by setting

$$U_1 := B_{p_1},$$
$$U_k := B_{p_k} \setminus \bigcup_{i=1}^{k-1} \overline{U}_i, \; k = 2, \dots, N.$$

Note that the new family does not cover E anymore, but still satisfies conditions 4 and 4', because $\partial U_k \subseteq \bigcup_{i=1}^N \partial B_{p_i}$. ■

For the case $p = 1$ in the main result we establish that the total variation $|Df|$ of a BV function f on M is a limit of certain integrals. So it is convenient to introduce the following notion, which is also appropriate to use if $p > 1$. For each $\sigma > 0$ and $p \geq 1$ we define the Radon measure $\mu_{\sigma,p}$ on M by

$$\mu_{\sigma,p}(E) := \int_E \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_\sigma(d(x,y)) \,\mathrm{d}V_g(y) \,\mathrm{d}V_g(x), \; E \subseteq M \text{ Borel.} \tag{4.13}$$

The outline of the proof of our main results follows [Dáv02] and [LS11], adapted to the manifold setting.

Proposition 4.9. *Let* $E \subseteq M$ *be a compact set.*

If $p > 1$ *and* $f \in W^{1,p}(M)$*, then for every* $\varepsilon \in (0,1)$ *there exist* $R_0 > 0$ *and a function* G_ε *independent of* σ *such that for every* $0 < R < R_0$

$$\mu_{\sigma,p}(E) \leq \frac{(1+\varepsilon)^{p+2}}{(1-\varepsilon)^{p+n}} K_{p,n} \int_{E^{2R}} |\mathrm{grad}\, f|_g^p \,\mathrm{d}V_g + \frac{\alpha_\sigma}{R^p} \|f\|_{L^p(M)}^p + G_\varepsilon(R), \tag{4.14}$$

where $\lim_{\sigma \searrow 0} \alpha_\sigma = 0$ *and* $\lim_{R \searrow 0} G_\varepsilon(R) = 0$.

If $p = 1$ *and* $f \in BV(M)$*, then* (4.14) *holds with* $\int_{E^{2R}} |\mathrm{grad}\, f|_g^p \,\mathrm{d}V_g$ *replaced by* $|Df|(E^{2R})$.

Proof. We may assume without loss of generality that $\varepsilon < \frac{1}{3}$. We divide the proof into two steps:

Step 1: An upper estimate for $\displaystyle\int_E \int_{d(x,y)<R} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x).$

First, assume $f \in C^1(M)$. Fix $\varepsilon \in (0,1/3)$ and let $(U_k)_{k=1}^N, N = N(\varepsilon)$, be a family of open sets as in Lemma 4.8 and (V_k,ϕ_k) the corresponding charts such that $U_k \subseteq V_k$ and (4.10) and (4.11) hold. The following computations are carried out for fixed $k \in \{1,\dots,N\}$, which we will omit for better readability, and with respect to the aforementioned chart. Using (4.10) and (4.11) as well as the monotonicity of ρ_σ we have $(\xi := \phi(x), \eta := \phi(y))$

$$
\begin{aligned}
&\int_{U\cap E}\int_{B_R^M(x)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x)\\
&\le \int_{\phi(U\cap E)}\int_{B^n_{\frac{R}{1-\varepsilon}}(\xi)} \frac{|\hat f(\xi)-\hat f(\eta)|^p}{(1-\varepsilon)^p|\xi-\eta|^p}\rho_\sigma((1-\varepsilon)|\xi-\eta|)\sqrt{\det \hat g_{\alpha\beta}(\xi)}\sqrt{\det \hat g_{\alpha\beta}(\eta)}\,\mathrm{d}\eta\,\mathrm{d}\xi\\
&\le \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p}\int_{\phi(U\cap E)}\int_{B^n_{\frac{R}{1-\varepsilon}}(o)} \frac{|\hat f(\xi)-\hat f(\xi+h)|^p}{|h|^p}\rho_\sigma((1-\varepsilon)|h|)\,\mathrm{d}h\,\mathrm{d}\xi\\
&\le \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p}\int_{\phi(U\cap E)}\int_{B^n_{\frac{R}{1-\varepsilon}}(o)}\int_0^1 \frac{|\nabla\hat f(\xi+th)\cdot h|^p}{|h|^p}\rho_\sigma((1-\varepsilon)|h|)\,\mathrm{d}t\,\mathrm{d}h\,\mathrm{d}\xi\\
&\overset{\tilde\xi=\xi+th}{\le} \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p}\int_{B^n_{\frac{R}{1-\varepsilon}}(o)}\int_0^1\int_{\phi(U\cap E)^{\frac{R}{1-\varepsilon}}} \frac{|\nabla\hat f(\tilde\xi)\cdot h|^p}{|h|^p}\rho_\sigma((1-\varepsilon)|h|)\,\mathrm{d}\tilde\xi\,\mathrm{d}t\,\mathrm{d}h,
\end{aligned}
$$

where we applied Fubini's theorem in the last step. Choosing a unit vector $e \in \mathbb{S}^{n-1}$, which can be thought of as $\frac{\nabla\hat f(\tilde\xi)}{|\nabla\hat f(\tilde\xi)|}$ for all $\tilde\xi$, for which $\nabla\hat f(\tilde\xi) \neq o$, we factorize the last expression in our chain of inequalities as

$$
\frac{(1+\varepsilon)^2}{(1-\varepsilon)^p}\left(\int_{\phi(U\cap E)^{\frac{R}{1-\varepsilon}}} |\nabla\hat f(\tilde\xi)|^p\,\mathrm{d}\tilde\xi\right)\int_{B^n_{\frac{R}{1-\varepsilon}}(o)}\left|e\cdot\frac{h}{|h|}\right|^p\rho_\sigma((1-\varepsilon)|h|)\,\mathrm{d}h. \tag{4.15}
$$

We introduce spherical coordinates for h and further rewrite the second integral as

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-1}} |e\cdot u|^p\,\mathrm{d}\mathcal{H}^{n-1}(u)\cdot\int_0^{\frac{R}{1-\varepsilon}}\rho_\sigma((1-\varepsilon)r)r^{n-1}\,\mathrm{d}r\\
&= \mathcal{H}^{n-1}(\mathbb{S}^{n-1})K_{p,n}(1-\varepsilon)^{-n}\int_0^R \rho_\sigma(r)r^{n-1}\,\mathrm{d}r \le (1-\varepsilon)^{-n}K_{p,n},
\end{aligned}
$$

since $\int_0^\infty \rho_\sigma(r)r^{n-1}\,\mathrm{d}r = \frac{1}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}$.

Finally, we transform the integration in $\tilde\xi$ in (4.15) back to an integral over a subset of M: The equivalence of Euclidean and geodesic distance (4.10) on one hand implies

$$
\phi^{-1}(\phi(U\cap E)^{\frac{R}{1-\varepsilon}}) \subseteq (U\cap E)^{\frac{1+\varepsilon}{1-\varepsilon}R} \subseteq (U\cap E)^{2R},
$$

and we choose $R > 0$ in such way, that $4R < \tau_0$ in condition 2 of Lemma 4.8 (the factor 2 ensures the validity of equation (4.16), where U_k is replaced by U_k^{2R}, which we need later). On the other hand condition 3 of the same Lemma assures that $|\nabla \hat{f}(\tilde{\xi})| = |d\phi|_x(\operatorname{grad} f(x))| \leq (1+\varepsilon)|\operatorname{grad} f(x)|_g$, where $\phi(x) = \tilde{\xi}$, so using (4.11)

$$\int_{\phi(U\cap E)^{\frac{R}{1-\varepsilon}}} |\nabla \hat{f}(\tilde{\xi})|^p \,\mathrm{d}\tilde{\xi} \leq \frac{(1+\varepsilon)^p}{1-\varepsilon} \int_{(U\cap E)^{2R}} |\operatorname{grad} f(x)|_g^p \,\mathrm{d}V_g(x).$$

After reintroducing the index k the inequality we have proved so far reads as

$$\begin{aligned}\int_{U_k\cap E}\int_{B_R^M(x)} &\frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x)\\ &\leq \frac{(1+\varepsilon)^{p+2}}{(1-\varepsilon)^{p+n+1}} K_{p,n} \int_{(U_k\cap E)^{2R}} |\operatorname{grad} f(x)|_g^p \,\mathrm{d}V_g(x).\end{aligned} \tag{4.16}$$

By Propositions 4.1 and 4.3, as well as Fatou's lemma, this inequality holds true for all $f \in W^{1,p}(M)$ if $p > 1$ or $f \in BV(M)$ if $p = 1$, respectively, where in the latter case $\int_{(U_k\cap E)^{2R}} |\operatorname{grad} f(x)|_g^p \,\mathrm{d}V_g(x)$ needs to be replaced by $|Df|(U_k \cap E^{2R})$. First, assume that $p > 1$ and $f \in W^{1,p}(M)$:

The domain of integration $(U_k \cap E)^{2R}$ in (4.16) is contained in the intersection

$$U_k^{2R} \cap E^{2R} = (U_k \cap E^{2R}) \cup ((U_k^{2R}\backslash U_k) \cap E^{2R}),$$

so if we sum up over all k and note that the U_k cover E up to a set of measure zero by Lemma 4.8, 4., we have

$$\begin{aligned}&\int_E\int_{B_R^M(x)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x)\\ \leq& \frac{(1+\varepsilon)^{p+2}}{(1-\varepsilon)^{p+n+1}} K_{p,n} \left(\sum_{k=1}^N \int_{U_k\cap E^{2R}} |\operatorname{grad} f|_g^p \,\mathrm{d}V_g + \sum_{k=1}^N \int_{U_k^{2R}\backslash U_k} |\operatorname{grad} f|_g^p \,\mathrm{d}V_g\right)\\ \leq& \frac{(1+\varepsilon)^{p+2}}{(1-\varepsilon)^{p+n+1}} K_{p,n} \left(\int_{E^{2R}} |\operatorname{grad} f|_g^p \,\mathrm{d}V_g + \sum_{k=1}^N \int_{U_k^{2R}\backslash U_k} |\operatorname{grad} f|_g^p \,\mathrm{d}V_g\right).\end{aligned}$$

The sets $U_k^{2R}\backslash U_k$ converge to ∂U_k as $R \searrow 0$, which by Lemma 4.8, 4., satisfy $\int_{\partial U_k} \mathrm{d}V_g = 0$. Thus, put

$$G_\varepsilon(R) = \frac{(1+\varepsilon)^{p+2}}{(1-\varepsilon)^{p+n+1}} K_{p,n} \left(\sum_{k=1}^N \int_{U_k^{2R}\backslash U_k} |\operatorname{grad} f|_g^p \,\mathrm{d}V_g\right) \quad . \tag{4.17}$$

In the case of $p = 1$ and $f \in BV(M)$ all computations up to (4.17) carry over verbatim, where all integrals of the form $\int_A |\operatorname{grad} f|_g^p \,\mathrm{d}V_g$ need to be replaced by $|Df|(A)$ and we need to apply property 4' of Lemma 4.8 to show that $\lim_{R\searrow 0} G_\varepsilon(R) = 0$.

Step 2: An upper estimate for $\displaystyle\int_E\int_{d(x,y)\geq R} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x)$.

For the remaining region consisting of all pairs (x, y) such that $x \in E$ and $d(x, y) \geq R$ we estimate

$$\int_E\int_{M\backslash B_R^M(x)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \leq \frac{2^{p-1}}{R^p}(I_1 + I_2),$$

where

$$I_1 := \int_E |f(x)|^p \int_{M\setminus B_R^M(x)} \rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x), \text{ and}$$
$$I_2 := \int_E \int_{M\setminus B_R^M(x)} |f(y)|^p \rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x).$$

By monotonicity of ρ_σ, we estimate

$$I_1 \le \mathrm{Vol}_g(M)\rho_\sigma(R)\|f\|^p_{L^p(M)},$$

where $\rho_\sigma(R)$ tends to zero as $\sigma \searrow 0$.

For I_2, we observe that the set $K := \{d(x,y) : x \in E, y \in M\setminus B_R^M(x)\}$ is closed and therefore compact, such that

$$I_2 \le C_\sigma \mathrm{Vol}_g(M)\|f\|^p_{L^p(M)},$$

where the sequence $C_\sigma := \sup_{r\in K} \rho_\sigma(r)$ converges to zero by locally uniform convergence.

Therefore, putting $\alpha_\sigma := 2^{p-1}\mathrm{Vol}_g(M)(\rho_\sigma(R) + C_\sigma)$, we have

$$\int_E \int_{M\setminus B_R^M(x)} \frac{|f(x)-f(y)|^p}{d(x,y)^p}\rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \le \frac{\alpha_\sigma}{R^p}\|f\|^p_{L^p(M)}.$$

■

Analogously to [Dáv02] we have the following result of weak-* convergence of Radon measures:

Theorem 4.10. *If $p > 1$ and $f \in W^{1,p}(M)$, the Radon measures $\mu_{\sigma,p}$ defined in (4.13) weakly-* converge to $K_{p,n}|\mathrm{grad}\, f|_g^p\,\mathrm{d}V_g$ as $\sigma \searrow 0$.*

If $p = 1$ and $f \in BV(M)$, the measures $\mu_{\sigma,1}$ weakly- converge to $K_{1,n}|Df|$ as $\sigma \searrow 0$.*

Proof. Proposition 4.9 shows that for $p \ge 1$ and every compact set $E \subseteq M$

$$\sup_{0<\sigma<1} \mu_{\sigma,p}(E) < \infty,$$

so by weak-* compactness there exists a subsequence $\mu_{\sigma_i,p} =: \mu_{i,p}$ and a limit measure μ_p such that $\mu_{i,p} \overset{i\to\infty}{\to} \mu_p$ with respect to the weak-* topology. We need to show, that for every such subsequence $\mu_p = K_{p,n}\nu_p$, where the measure ν_p is defined as

$$\nu_p(A) := \begin{cases} \int_A |\mathrm{grad}\, f|_g^p\,\mathrm{d}V_g, & \text{if } p > 1, \\ |Df|(A), & \text{if } p = 1, \end{cases} \tag{4.18}$$

for every Borel set $A \subseteq M$.

Step 1: $\mu_p(A) \le K_{p,n}\nu_p(A)$ for every Borel set $A \subseteq M$.

By inner regularity of Radon measures, it suffices to prove the inequality for compact sets $E \subseteq M$. We apply Proposition 4.9 with E replaced by $\overline{E^{2R}}$ for $\varepsilon > 0$

and $R < R_0$. Note that the weak-* convergence of the sequence $(\mu_{i,p})$ implies that $\mu_p(E^{2R}) \leq \liminf_{i\to\infty} \mu_{i,p}(E^{2R})$, so we get

$$\mu_p(E) \leq \mu_p(E^{2R}) \leq \liminf_{i\to\infty} \mu_{i,p}(E^{2R}) \leq (1+o_\varepsilon)K_{p,n}\nu_p(E^{4R}) + G_\varepsilon(2R).$$

Letting $R \searrow 0$ and then $\varepsilon \searrow 0$ we obtain the desired inequality, since by compactness $E^{4R} \to E$ as $R \searrow 0$.

Step 2: $\mu_p(M) \geq K_{p,n}\nu_p(M)$.

This step uses a regularization argument similar to the proofs in [LS11]; consider a regularization kernel $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi \,\mathrm{d}x = 1$ and spt $\psi \subseteq B_1^n(o)$, and for $\delta > 0$ set

$$\psi_\delta(x) := \frac{1}{\delta^n}\psi\left(\frac{x}{\delta}\right), \; x \in \mathbb{R}^n.$$

For $U \subseteq \mathbb{R}^n$ open we define the mollification of a function $g \in L^1_{loc}(U)$ for every $x \in U$ with $d(x, \partial U) > \delta$ by

$$g_\delta(x) := \int_{\mathbb{R}^n} f(x-\zeta)\psi_\delta(\zeta)\,\mathrm{d}\zeta.$$

Note that g_δ is a C^∞ function. Furthermore, fix $\varepsilon \in (0,1)$ and consider a finite family of open sets $(U_k)_{k=1}^N$ and corresponding charts (V_k, ϕ_k) as in Lemma 4.8 with $E = M$. Then define the functions $f_{k,\delta} : U_k \to \mathbb{R}$ for $k = 1,\dots,N$ and $\delta > 0$ sufficiently small by $f_{k,\delta}(x) := (f \circ \phi_k^{-1})_\delta(\phi_k(x))$, i.e.

$$f_{k,\delta}(x) = \int_{B_\delta^n} (f \circ \phi_k^{-1})(\phi_k(x) - \zeta)\psi_\delta(\zeta)\,\mathrm{d}\zeta. \tag{4.19}$$

Note that $f_{k,\delta}$ is defined for every $x \in U_k$ since by property 2. of Lemma 4.8 the function ϕ_k is defined on an U_k^τ for some $\tau > 0$. Again, $f_{k,\delta}$ is a C^∞ function on U_k. The following calculations take place in only one U_k for k fixed, so we oppress the index k for the sake of readability. We denote the radial mollifiers corresponding to the subsequence $\mu_{i,p}$ by ρ_i. Putting $\xi := \phi(x)$ and $\eta := \phi(y)$ we estimate

$$\int_U \int_U \frac{|f_\delta(x) - f_\delta(y)|^p}{d(x,y)^p}\rho_i(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \tag{4.20}$$

$$\leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p}\int_{\phi(U)}\int_{\phi(U)} \frac{|\int_{B_\delta^n}((f\circ\phi^{-1})(\xi-\zeta) - (f\circ\phi^{-1})(\eta-\zeta))\psi_\delta(\zeta)\,\mathrm{d}\zeta|^p}{|\xi-\eta|^p}\times$$
$$\times\rho_i((1-\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\xi$$

$$\leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p}\int_{\phi(U)}\int_{\phi(U)} \frac{\int_{B_\delta^n}|(f\circ\phi^{-1})(\xi-\zeta) - (f\circ\phi^{-1})(\eta-\zeta)|^p\psi_\delta(\zeta)\,\mathrm{d}\zeta}{|\xi-\eta|^p}\times$$
$$\times\rho_i((1-\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\xi$$

$$\leq \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p}\int_{\phi(U)^\delta}\int_{\phi(U)^\delta}\int_{B_\delta^n} \frac{|(f\circ\phi^{-1})(\xi) - (f\circ\phi^{-1})(\eta)|^p}{|\xi-\eta|^p}\times$$
$$\times\psi_\delta(\zeta)\rho_i((1-\varepsilon)|\xi-\eta|)\,\mathrm{d}\zeta\,\mathrm{d}\eta\,\mathrm{d}\xi \tag{4.21}$$

$$
\begin{aligned}
&= \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p} \int_{\phi(U)^\delta} \int_{\phi(U)^\delta} \frac{|(f\circ\phi^{-1})(\xi) - (f\circ\phi^{-1})(\eta)|^p}{|\xi-\eta|^p} \times \\
&\qquad\qquad \times \rho_i((1-\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\xi \int_{B_\delta^n} \psi_\delta(\zeta)\,\mathrm{d}\zeta \\
&= \frac{(1+\varepsilon)^2}{(1-\varepsilon)^p} \int_{\phi(U)^\delta} \int_{\phi(U)^\delta} \frac{|(f\circ\phi^{-1})(\xi) - (f\circ\phi^{-1})(\eta)|^p}{|\xi-\eta|^p} \rho_i((1-\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&\le \frac{(1+\varepsilon)^{p+2}}{(1-\varepsilon)^{p+2}} \int_{U^{(1+\varepsilon)\delta}} \int_{U^{(1+\varepsilon)\delta}} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i\left(\frac{1-\varepsilon}{1+\varepsilon} d(x,y)\right) \mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \\
&\le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{p+2} \int_{U^{(1+\varepsilon)\delta}} \int_M \frac{|f(x)-f(y)|^p}{d(x,y)^p} \rho_i\left(\frac{1-\varepsilon}{1+\varepsilon} d(x,y)\right) \mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \qquad (4.22) \\
&= \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-n+p+2} \int_{U^{(1+\varepsilon)\delta}} \int_M \frac{|f(x)-f(y)|^p}{d(x,y)^p} \tilde{\rho}_i(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \\
&= \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-n+p+2} \left(\tilde{\mu}_{i,p}(U) + \tilde{\mu}_{i,p}\left(U^{(1+\varepsilon)\delta} \backslash U\right)\right), \qquad (4.23)
\end{aligned}
$$

where $\tilde{\rho}_i(r) := \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n \rho_i\left(\frac{1-\varepsilon}{1+\varepsilon} r\right)$ and $\tilde{\mu}_{i,p}$ is the measure defined by replacing ρ_i with $\tilde{\rho}_i$ in (4.13).

On the other hand (4.20) can be estimated from below via

$$
\begin{aligned}
&\int_U \int_U \frac{|f_\delta(x) - f_\delta(y)|^p}{d(x,y)^p} \rho_i(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \\
&\ge \frac{1-\varepsilon}{(1+\varepsilon)^p} \int_{\phi(U)} \sqrt{\det(\hat{g}_{\alpha\beta}(\xi))} \int_{\phi(U)} \frac{|(f\circ\phi^{-1})_\delta(\xi) - (f\circ\phi^{-1})_\delta(\eta)|^p}{|\xi-\eta|^p} \times \\
&\qquad\qquad \times \rho_i((1+\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\xi, \qquad (4.24)
\end{aligned}
$$

where the inner integral converges to

$$
(1+\varepsilon)^{-n} K_{p,n} |\nabla (f\circ\phi^{-1})_\delta(\xi)|^p
$$

as $i \to \infty$, see [BBM01, (6)]. Since the integrand of the outer integral can be estimated by Lipschitz continuity of $(f\circ\phi^{-1})_\delta$ with Lipschitz constant $L_\delta > 0$ via

$$
\begin{aligned}
&\sqrt{\det(\hat{g}_{\alpha\beta}(\xi))} \int_{\phi(U)} \frac{|(f\circ\phi^{-1})_\delta(\xi) - (f\circ\phi^{-1})_\delta(\eta)|^p}{|\xi-\eta|^p} \rho_i((1+\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta \\
&\le \sqrt{\det(\hat{g}_{\alpha\beta}(\xi))} \int_{\phi(U)} \frac{L_\delta^p |\xi-\eta|^p}{|\xi-\eta|^p} \rho_i((1+\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta \le L_\delta^p \sqrt{\det(\hat{g}_{\alpha\beta}(\xi))},
\end{aligned}
$$

we can apply the dominated convergence theorem for the ξ-integration in (4.24).

Now we put the estimates (4.23) and (4.24) together:

$$
\begin{aligned}
&\frac{1-\varepsilon}{(1+\varepsilon)^p} \int_{\phi(U)} \sqrt{\det(\hat{g}_{\alpha\beta}(\xi))} \int_{\phi(U)} \frac{|(f\circ\phi^{-1})_\delta(\xi) - (f\circ\phi^{-1})_\delta(\eta)|^p}{|\xi-\eta|^p} \times \\
&\qquad\qquad \times \rho_i((1+\varepsilon)|\xi-\eta|)\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&\le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{-n+p+2} (\tilde{\mu}_{i,p}(U) + \tilde{\mu}_{i,p}(U^{(1+\varepsilon)\delta} \backslash U)). \qquad (4.25)
\end{aligned}
$$

We claim that

$$\limsup_{i\to\infty} \tilde{\mu}_{i,p}(\overline{U}) = \limsup_{i\to\infty} \mu_{i,p}(\overline{U}) + o_\varepsilon, \tag{4.26}$$

where $o_\varepsilon \to 0$ as $\varepsilon \searrow 0$. First observe that the sequence $(\tilde{\rho}_i)_{i\in\mathbb{N}}$ is a sequence of radial mollifiers (for $i \to \infty$) itself, such that for $f \in C^1(M)$ we can repeat the calculations in the proof of Proposition 4.9, but rather than using one mollifier, we plug in the difference $\rho_i\left(\frac{1-\varepsilon}{1+\varepsilon} d(x,y)\right) - \rho_i(d(x,y))$, which is non-negative by monotonicity of ρ_i, instead:

$$\begin{aligned}
&\int_{\overline{U}} \int_{B_R^M(x)} \frac{|f(x)-f(y)|^p}{d(x,y)^p} \left(\rho_i\left(\frac{1-\varepsilon}{1+\varepsilon} d(x,y)\right) - \rho_i(d(x,y))\right) \mathrm{d}V_g(y)\,\mathrm{d}V_g(x) \\
&\le (1+\varepsilon)^2 \int_{\phi(\overline{U})} \int_{B^n_{\frac{R}{1-\varepsilon}}} \frac{|\hat{f}(\xi)-\hat{f}(\eta)|^p}{(1-\varepsilon)^p |\xi-\eta|^p} \times \\
&\qquad \times \left(\rho_i\left(\frac{(1-\varepsilon)^2}{1+\varepsilon}|\xi-\eta|\right) - \rho_i((1+\varepsilon)|\xi-\eta|)\right) \mathrm{d}\eta\,\mathrm{d}\xi,
\end{aligned}$$

where on the right-hand side we used the equivalence of distances (4.10) accordingly. Following the proof of Proposition 4.9 up to (4.15) with the obvious modifications, we see that the last expression does not exceed

$$(1+o_\varepsilon) K_{p,n} \nu_p(U^{2R}) \int_0^{\frac{R}{1-\varepsilon}} \left(\rho_i\left(\frac{(1-\varepsilon)^2}{1+\varepsilon} r\right) - \rho_i((1+\varepsilon)r)\right) r^{n-1}\,\mathrm{d}r, \tag{4.27}$$

and this estimate from above still holds true for $f \in W^{1,p}(M)$ or $f \in BV(M)$, if $p = 1$, respectively, as can be seen by approximation. As $i \to \infty$ (4.27) converges to a remainder o_ε, which is 0 as $\varepsilon \searrow 0$. The integral over the remaining domain, consisting of all pairs $x \in \overline{U}, y \notin B_R^M(x)$, is zero in the limit, which we already have seen in Step 2 in the proof of Proposition 4.9, thus verifying (4.26).

Applying the limit in (4.25), and noting that by weak-* convergence $\limsup_{i\to\infty} \mu_{i,p}(\overline{U}) \le \mu_p(\overline{U})$, we obtain

$$K_{p,n} \int_{\phi(U)} \sqrt{\det(\hat{g}_{\alpha\beta}(\xi))} |\nabla (f\circ\phi^{-1})_\delta(\xi)|^p \,\mathrm{d}\xi \le (1+o_\varepsilon)(\mu_p(\overline{U}) + \mu_p(\overline{U^{(1+\varepsilon)\delta}\setminus U})). \tag{4.28}$$

Now weed need to distinguish, whether $f \in W^{1,p}(M)$ or $f \in BV(M)$:

First, let $p > 1$ and $f \in W^{1,p}(M)$. Since $|\nabla(f\circ\phi^{-1})_\delta|$ tends to $|\nabla(f\circ\phi^{-1})|$ in $L^p(\phi(U))$ as $\delta \searrow 0$, and $|\nabla(f\circ\phi^{-1})(\xi)|^p \ge \frac{1}{(1+\varepsilon)^p} |\operatorname{grad} f(\phi^{-1}(\xi))|_g^p$ by Lemma 4.2, we have that

$$K_{p,n} \int_U |\operatorname{grad} f|_g^p \,\mathrm{d}V_g \le (1+o_\varepsilon)(\mu_p(\overline{U}) + \mu_p(\partial U)).$$

If $p = 1$ and $f \in BV(M)$, by (4.4) the integral on the left-hand side of (4.28) is equal to the weighted variation $|D(f\circ\phi^{-1})_\delta|_w(\phi(U))$ with weight $w(\xi) = \sqrt{\det(\hat{g}_{\alpha\beta}(\xi))}$. The convolutions $(f\circ\phi^{-1})_\delta$ converge in $L^1(\phi(U))$ to the function $f\circ\phi^{-1}$, and furthermore

$$\begin{aligned}
&\int_{\phi(U)} |(f\circ\phi^{-1}) - (f\circ\phi^{-1})_\delta| \sqrt{\det(\hat{g}_{\alpha\beta})}\,\mathrm{d}x \\
&\qquad \le (1+\varepsilon) \int_{\phi(U)} |(f\circ\phi^{-1}) - (f\circ\phi^{-1})_\delta|\,\mathrm{d}x \overset{\delta\searrow 0}{\to} 0.
\end{aligned}$$

Since the map $u \mapsto |Du|_w(\phi(U))$ is lower semicontinuous with respect to convergence in $L^1(\phi(U), w\,\mathrm{d}x)$, letting $\delta \searrow 0$ we obtain

$$K_{1,n}|D(f \circ \phi^{-1})|_w(\phi(U)) \leq (1 + o_\varepsilon)(\mu_1(\overline{U}) + \mu_1(\partial U)).$$

By Lemma 4.2 the left-hand side can further be estimated by $\frac{1}{(1+\varepsilon)} K_{1,n}|Df|(U)$ from below.

For both cases $p > 1$ and $p = 1$ the resulting inequality can be written as

$$K_{p,n}\nu_p(U) \leq (1 + o_\varepsilon)(\mu_p(\overline{U}) + \mu_p(\partial U)).$$

By our assumptions 4 and 4' of Lemma 4.8 on the mass on the boundary of U, Step 1 guarantees $\mu_p(\partial U) \leq K_{p,n}\nu_p(\partial U) = 0$, and in consequence

$$K_{p,n}\nu_p(U) \leq (1 + o_\varepsilon)\mu_p(U).$$

Summing up over all k and letting $\varepsilon \searrow 0$ yields the desired inequality.

Step 3: $\mu_p(A) \geq K_{p,n}\nu_p(A)$ for every Borel set $A \subseteq M$.

Since μ_p is a finite measure, for each Borel set $A \subseteq M$ it holds that

$$\mu_p(A) = \mu_p(M) - \mu_p(M \backslash A) \geq K_{p,n}\nu_p(M) - K_{p,n}\nu_p(M \backslash A) = K_{p,n}\nu_p(A)$$

by the preceding steps 1 and 2. ■

With the weak-* convergence at hand, the proof of Theorem 4.7 is not difficult anymore:

Proof of Theorem 4.7. First, suppose that $f \in W^{1,p}(M)$, if $p > 1$, and $f \in BV(M)$, if $p = 1$. Since M is both open and compact, the weak-* convergence of $\mu_{\sigma,p}$ to $K_{p,n}\nu_p$ (with ν_p defined in (4.18)), which is established in Theorem 4.10, implies

$$K_{p,n}\nu_p(M) \leq \liminf_{\sigma \searrow 0} \mu_{\sigma,p}(M) \leq \limsup_{\sigma \searrow 0} \mu_{\sigma,p}(M) \leq K_{p,n}\nu_p(M),$$

which is the desired result.

On the other hand, suppose that

$$\liminf_{\sigma \searrow 0} \int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_\sigma(d(x,y))\,\mathrm{d}V_g(y)\,\mathrm{d}V_g(x) < \infty. \tag{4.29}$$

We show that $f \in W^{1,p}(M)$ and, if $p = 1$, that $f \in BV(M)$. By Propositions 4.1 and 4.3 it is enough to construct a family $(f_\delta)_{\delta > 0}$ of C^∞ functions on M, such that $f_\delta \to f$ in $L^p(M)$ as $\delta \searrow 0$ and

$$\liminf_{\delta \searrow 0} \int_M |\mathrm{grad}\,(f_\delta)|_g^p\,\mathrm{d}V_g < \infty \quad . \tag{4.30}$$

For $\varepsilon \in (0,1)$ introduce the modified metric $\tilde{g} := \frac{1+\varepsilon}{1-\varepsilon} g$. Note that the corresponding distance function satisfies $\tilde{d}(x,y) = \frac{1+\varepsilon}{1-\varepsilon} d(x,y)$ for all $x, y \in M$, and the volume form transforms as $\mathrm{d}V_{\tilde{g}} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{n}{2}} \mathrm{d}V_g$. Furthermore, a function on M is of bounded variation with respect to $\tilde{g}$ if and only if it is with respect to g, and the variations coincide up to a factor dependent on ε.

Let $(U_k)_{k=1}^N$ be chosen according to Lemma 4.8 and put $W_k = U_k^\tau$ for some $\tau < \tau_0$. Then W_k is a covering of M by open sets, i.e. $M = \bigcup_{k=1}^N W_k$. Let $(\chi_k)_{k=1}^n$ be an underlying smooth partition of unity, i.e. smooth functions $\chi_k : M \to [0,1]$ compactly supported in W_k with $\sum_{k=1}^N \chi_k = 1$. If $\delta > 0$ is sufficiently small we are able to define regularization functions $f_{k,\delta}$ on W_k according to (4.19). Putting $f_\delta := \sum_{k=1}^N \chi_k f_{k,\delta}$ yields a family of C^∞ functions such that $f_\delta \to f$ in $L^p(M)$ as $\delta \searrow 0$.

We estimate

$$\begin{aligned}
&\int_M \int_M \frac{|f_\delta(x) - f_\delta(y)|^p}{\tilde{d}(x,y)^p} \rho_\sigma(\tilde{d}(x,y)) \,\mathrm{d}V_{\tilde{g}}(y) \,\mathrm{d}V_{\tilde{g}}(x) \\
&\leq N^{p-1} \sum_{k=1}^N \Bigg(\int_{W_k} \int_{W_k} \frac{|\chi_k(x) f_{k,\delta}(x) - \chi_k(y) f_{k,\delta}(y)|^p}{\tilde{d}(x,y)^p} \rho_\sigma(\tilde{d}(x,y)) \,\mathrm{d}V_{\tilde{g}}(y) \,\mathrm{d}V_{\tilde{g}}(x) \\
&\qquad + \int_{M \setminus W_k} \int_{W_k} \frac{|\chi_k(x) f_{k,\delta}(x) - \chi_k(y) f_{k,\delta}(y)|^p}{\tilde{d}(x,y)^p} \rho_\sigma(\tilde{d}(x,y)) \,\mathrm{d}V_{\tilde{g}}(y) \,\mathrm{d}V_{\tilde{g}}(x) \Bigg),
\end{aligned} \tag{4.31}$$

where the integrals over $M \setminus W_k$ tend to 0 as $\sigma \searrow 0$, since the support of χ_k is compact in W_k and therefore $\tilde{d}(x,y) \geq R > 0$. The remaining summands can be estimated by

$$\begin{aligned}
&\int_{W_k} \int_{W_k} \frac{|\chi_k(x) f_{k,\delta}(x) - \chi_k(y) f_{k,\delta}(y)|^p}{\tilde{d}(x,y)^p} \rho_\sigma(\tilde{d}(x,y)) \,\mathrm{d}V_{\tilde{g}}(y) \,\mathrm{d}V_{\tilde{g}}(x) \\
&\qquad \leq 2^{p-1} \Bigg(\int_{W_k} |f_{k,\delta}(x)|^p \int_{W_k} \frac{|\chi_k(x) - \chi_k(y)|^p}{\tilde{d}(x,y)^p} \rho_\sigma(\tilde{d}(x,y)) \,\mathrm{d}V_{\tilde{g}}(y) \,\mathrm{d}V_{\tilde{g}}(x) \\
&\qquad\qquad + \int_{W_k} \int_{W_k} \frac{|f_{k,\delta}(x) - f_{k,\delta}(y)|^p}{\tilde{d}(x,y)^p} \rho_\sigma(\tilde{d}(x,y)) \,\mathrm{d}V_{\tilde{g}}(y) \,\mathrm{d}V_{\tilde{g}}(x) \Bigg) \\
&\qquad =: 2^{p-1}(I_{k,1} + I_{k,2}).
\end{aligned}$$

Since χ_k is smooth, the inner integral in $I_{k,1}$ converges to $K_{p,n} |\mathrm{grad}\, \chi_k(x)|_{\tilde{g}}^p$ as $\sigma \searrow 0$, and by dominated convergence we have

$$\lim_{\sigma \searrow 0} I_{k,1} = K_{p,n} \int_{W_k} |f_{k,\delta}(x)|^p |\mathrm{grad}\, \chi_k(x)|_{\tilde{g}}^p \,\mathrm{d}V_{\tilde{g}}(x) \leq C \int_{W_k} |f_{k,\delta}(x)|^p \,\mathrm{d}V_{\tilde{g}}(x),$$

where $C := \max_{x \in \mathrm{spt}\, \chi_k} |\mathrm{grad}\, \chi_k(x)|_{\tilde{g}}$. By the L^p-convergence of $f_{k,\delta}$ as $\delta \searrow 0$, we furthermore get that $\lim_{\sigma \searrow 0} I_{k,1}$ is uniformly bounded in δ.

For the second integrals $I_{k,2}$ we can repeat the calculations of (4.20) up to (4.22), leading to

$$\begin{aligned}
I_{k,2} &\leq (1 + o_\varepsilon) \int_M \int_M \frac{|f(x) - f(y)|^p}{\tilde{d}(x,y)^p} \rho_\sigma \left(\frac{1-\varepsilon}{1+\varepsilon} \tilde{d}(x,y) \right) \,\mathrm{d}V_{\tilde{g}}(y) \,\mathrm{d}V_{\tilde{g}}(x) \\
&= (1 + o_\varepsilon) \int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_\sigma(d(x,y)) \,\mathrm{d}V_g(y) \,\mathrm{d}V_g(x),
\end{aligned}$$

where in the last line we switched back to the metric g and absorbed the occurring factors into o_ε, which converges to 0 as $\varepsilon \searrow 0$. Thus, the limit superior of $I_{k,2}$ as $\sigma \searrow 0$ is finite by our assumption (4.29), and even uniformly bounded in $\delta > 0$.

We conclude by observing that taking the limit $\sigma \searrow 0$ in the left-hand side of (4.31) yields $K_{p,n} \int_M |\mathrm{grad}\, f_\delta(x)|_{\tilde{g}}^p \,\mathrm{d}V_{\tilde{g}}$, thus passing to the original metric, we have showed (4.30). ■

Using suitable radial mollifiers leads to the s-seminorm and thus to Theorem 4.5:

Proof of Theorem 4.5. Define radial mollifiers $\rho_\sigma, \sigma > 0$, by

$$\rho_\sigma(r) := \begin{cases} \frac{\sigma p}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \frac{1}{r^{n-\sigma p}}, & 0 < r < 1, \\ 0, & r \geq 1. \end{cases}$$

and set $s := 1 - \sigma$. We claim that

$$\begin{aligned} &\lim_{\sigma \searrow 0} \int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^p} \rho_\sigma(d(x,y)) \,\mathrm{d}V_g(y) \,\mathrm{d}V_g(x) \\ &\qquad = \lim_{s \nearrow 1} \frac{(1-s)p}{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})} \int_M \int_M \frac{|f(x) - f(y)|^p}{d(x,y)^{n+sp}} \,\mathrm{d}V_g(y) \,\mathrm{d}V_g(x), \end{aligned}$$

where by Theorem 4.7 the left-hand side is equal to either $K_{p,n} \int_M |\mathrm{grad}\, f|_g^p \,\mathrm{d}V_g$, if $p > 1$, or $K_{1,n}|Df|(M)$, if $p = 1$. To see this, we only need to show that

$$\lim_{s \nearrow 1} (1-s) \int_M \int_{\{y : d(x,y) \geq 1\}} \frac{|f(x) - f(y)|^p}{d(x,y)^{n+sp}} \,\mathrm{d}V_g(y) \,\mathrm{d}V_g(x) = 0.$$

But this is a simple consequence of

$$\begin{aligned} \int_M \int_{\{y : d(x,y) \geq 1\}} \frac{|f(x) - f(y)|^p}{d(x,y)^{n+sp}} \,\mathrm{d}V_g(y) \,\mathrm{d}V_g(x) &\leq \int_M \int_M |f(x) - f(y)|^p \,\mathrm{d}V_g(y) \,\mathrm{d}V_g(x) \\ &\leq 2^p \mathrm{Vol}_g(M) \|f\|_{L^p}^p. \end{aligned}$$

■

Chapter 5

Spherical fractional perimeters

In the last chapter we turn our attention to the special case of fractional perimeters and seminorms defined on the n-dimensional Euclidean unit sphere $\mathbb{S}^n$. Here, we exploit the Grassmannian-like structure of great circles to give an alternative proof for the convergence to the perimeter as $s \nearrow 1$. For this we use spherical Blaschke-Petkantschin and Crofton formulas, following ideas established by Ludwig in [Lud14a]. Since a formula analogous to (2.9) for the convergence to the volume as $s \searrow 0$ does not hold, we extend the range of possible exponents for the perimeter functional to the negative half-axis $s \in (-\infty, 0]$. We show that the limit of suitably normalized fractional perimeters as $s \searrow -\infty$ measures the volume of all points such that their reflection lie in the complement. Finally, we derive isoperimetric-type inequalities for fractional perimeters on the sphere using rearrangement techniques presented by Beckner in [Bec92].

This chapter is based on joint work with Olaf Mordhorst. A paper containing its contents is submitted for publication (see [KM20] for the preprint).

5.1 Results from spherical convex and integral geometry

The aim of this chapter is to provide all definitions and results from spherical convex and integral geometry which are needed later on. For a general reference on this topic we recommend [Gla95] and [SW08, Chapter 6.5]. To increase the readability of constants we put $\omega_k := \mathcal{H}^{k-1}(\mathbb{S}^{k-1})$ for $k \geq 1$.

The Grassmannian of 2-dimensional subspaces of $\mathbb{R}^{n+1}$ is denoted by $G(n+1,2)$ and equipped with the Haar measure $\mathrm{d}L$ such that $\int_{G(n+1,2)} \mathrm{d}L = 1$. For any two-dimensional plane $L \in G(n+1,2)$ the intersection $L \cap \mathbb{S}^n$ is called a *great circle*.

The geodesic distance between two points $x, y \in \mathbb{S}^n$ on the sphere, already defined in the context of general Riemannian manifolds in Section 2.4, allows for a simpler geometric description: $d(x,y)$ is equal to the length of any of the shortest great circle arcs connecting x and y. For $x \in \mathbb{S}^n$ and $r \geq 0$, the geodesic ball $B_r^{\mathbb{S}^n}(x)$ is a (possibly empty) spherical cap, which we aptly denote in this setting by $C_r(x)$,

$$C_r(x) := \{y \in \mathbb{S}^n : d(x,y) < r\}.$$

Note that $C_0(x) = \emptyset, C_\pi(x) = \mathbb{S}^n \setminus \{-x\}$, and $C_r(x) = \mathbb{S}^n$ if $r > \pi$. As introduced in Section 4.2, we denote by

$$d(E,x) := \inf \{d(x,y) : y \in E\},$$

the distance between $x \in \mathbb{S}^n$ and $E \subseteq \mathbb{S}^n$.

A non-empty subset $K \subseteq \mathbb{S}^n$ is called a (spherically) convex body if the cone

$$\mathrm{pos}(K) := \{\lambda x : \lambda \geq 0, x \in K\}$$

generated by K is a closed convex subset of $\mathbb{R}^{n+1}$. Note that for any pair of points $x, y \in K, y \neq -x$, the shorter geodesic line segment connecting x and y lies entirely in K. We further remark that for each $x \in \mathbb{S}^n$ such that $0 \leq d(K,x) < \pi/2$ there is a unique point $p(K,x)$ in K that is nearest to x. We say that the set $E \subseteq \mathbb{S}^n$ is *polyconvex* if it can be written as a finite union of convex bodies.

A convex body $P \subseteq \mathbb{S}^n$ is called a (spherical) polytope, if its cone $\mathrm{pos}(P)$ is the intersection of finitely many halfspaces. We call F a k-face of P, $k \in \{0, \dots, n\}$, if $F = \tilde{F} \cap \mathbb{S}^n$, where $\tilde{F}$ is a $(k+1)$-face of the polyhedral cone $\mathrm{pos}(P)$.

If $K \subseteq \mathbb{S}^n$ is a convex body, its polar body K° is defined by

$$K^\circ := \{x \in \mathbb{S}^n : x \cdot y \leq 0 \text{ for all } y \in K\}.$$

The normal cone $N(K,x)$ of K at $x \in \partial K$ is then defined by

$$N(K,x) := \{y \in K^\circ : x \cdot y = 0\}.$$

For our integral-geometric treatment of the fractional perimeter we will use spherical curvature measures, which satisfy a local spherical Steiner formula. For a convex body $K \subseteq \mathbb{S}^n$, a Borel set $A \subseteq \mathbb{S}^n$ and $0 < \varepsilon < \pi/2$ we put

$$M_\varepsilon(K,A) := \{x \in \mathbb{S}^n : d(K,x) \leq \varepsilon,\ p(K,x) \in A\}.$$

Then the *curvature measures* $\phi_0(K,\cdot), \dots, \phi_{n-1}(K,\cdot)$ are the uniquely determined Borel measures on $\mathbb{S}^n$ such that for all Borel sets $A \subseteq \mathbb{S}^n$ and $0 < \varepsilon < \pi/2$

$$\mathcal{H}^n(M_\varepsilon(K,A)) = \sum_{j=0}^{n-1} g_{n,j}(\varepsilon)\phi_j(K,A),$$

where

$$g_{n,j}(\varepsilon) = \omega_{j+1}\omega_{n-j}\int_0^\varepsilon \cos^j t\, \sin^{n-j-1} t \,\mathrm{d}t,$$

see e.g. [SW08, Theorem 6.5.1] for generalized curvature measures.

The top- and bottom-degree curvature measures ϕ_{n-1} and ϕ_0 occur in the proof of the following spherical Crofton formula which identifies the average number of intersections of a polyconvex subset $E \subseteq \mathbb{S}^n$ with great circles as its perimeter:

Theorem 5.1 (spherical Crofton formula). *Let $E \subseteq \mathbb{S}^n$ be an n-dimensional polyconvex subset of $\mathbb{S}^n$. Then*

$$\int_{G(n+1,2)} \mathcal{H}^0(\partial E \cap L)\,\mathrm{d}L = \frac{2}{\omega_n}\mathcal{H}^{n-1}(\partial E).$$

Proof. We rewrite the spherical Crofton formula for curvature measures,

$$\int_{G(n+1,2)} \varphi_0(E \cap L, \mathbb{S}^n \cap L)\, \mathrm{d}L = \varphi_{n-1}(E, \mathbb{S}^n), \tag{5.1}$$

see [SW08, p. 261], in terms of Hausdorff measures as follows:

For spherical convex polytopes $P \subseteq \mathbb{S}^n$ and $m \in \{0, \dots, n-1\}$, we have the formula

$$\varphi_m(P, A) = \frac{1}{\omega_{m+1}\omega_{n-m}} \sum_{F \in \mathcal{F}_m(P)} \int_F \int_{N(P,F)} \mathbb{1}_{A \times \mathbb{S}^n}(x, u)\, \mathrm{d}\mathcal{H}^{n-m-1}(u)\, \mathrm{d}\mathcal{H}^m(x),$$

where $\mathcal{F}_m(P)$ denotes the set of all m-dimensional faces of P and $N(P, F)$ is the normal cone to P for F (see [SW08, Theorem 6.5.1]).

Since $E \cap L$ is a finite union of disjoint arcs and $\varphi_0(\cdot, A)$ is a valuation, it suffices to calculate $\varphi_0(P, S)$, where P is an arc on the great sphere $S = \mathbb{S}^n \cap L$. The set of 0-dimensional faces $\mathcal{F}_0(P) = \{p_1, p_2\}$ consists of the endpoints of the arc, and for every normal vector $u \in N(P, p_i)$, $i = 1, 2$ we have $\mathbb{1}_{S \times \mathbb{S}^n}(p_i, u) = 1$, so it remains to compute $\mathcal{H}^{n-1}(N(P, p_i))$.

We can think of P as an intersection of the great sphere S with two hemispheres, with normal vectors lying in the same plane as S each. Therefore, the polar body P° of P is again an intersection of two hemispheres, and since for $i = 1, 2$ the normal cone $N(P, p_i)$ consists of one of the bounding $(n-1)$-dimensional hemispheres, we have $\mathcal{H}^{n-1}(N(P, p_i)) = \frac{\omega_n}{2}$.

In conclusion, for any polyconvex set $E \subseteq \mathbb{S}^n$ and any plane $L \in G(n+1, 2)$, counting the components of $E \cap L$ yields

$$\varphi_0(E \cap L, \mathbb{S}^n \cap L) = \frac{\mathcal{H}^0(\partial E \cap L)}{2\omega_1}.$$

Regarding the right-hand side of (5.1), [Gla95, Satz 4.4.3] identifies the curvature measure $\varphi_{n-1}(E, \mathbb{S}^n)$ as $\frac{1}{2\omega_n}\mathcal{H}^{n-1}(\partial E)$. ■

In [AZ91, Theorem 1] a higher order kinematic formula on the sphere is proven from which a spherical Blaschke-Petkantschin formula follows. The direct statement of the spherical Blaschke-Petkantschin formula with a shorter and better accessible proof can be found in [HT19, Lemma 5.3]. In the case of double integrals the formula reads as follows.

Theorem 5.2 (spherical Blaschke-Petkantschin formula). *Let $f : \mathbb{S}^n \times \mathbb{S}^n \to [0, \infty)$ be a measurable function. Then*

$$\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} f(x, y)\, \mathrm{d}\mathcal{H}^n(y)\, \mathrm{d}\mathcal{H}^n(x) = c_n \int_{G(n+1,2)} \int_{\mathbb{S}^n \cap L} \int_{\mathbb{S}^n \cap L} f(x, y) \nabla_2^{n-1}(x, y)\, \mathrm{d}\mathcal{H}^1(y)\, \mathrm{d}\mathcal{H}^1(x)\, \mathrm{d}L, \tag{5.2}$$

where $c_n := \frac{\omega_{n+1}\omega_n}{\omega_1\omega_2}$ and $\nabla_2(x, y)$ denotes the area of the parallelogram spanned by x and y.

Note that for points $x, y \in \mathbb{S}^n$ on the sphere

$$\nabla_2(x, y) = \sqrt{1 - (x \cdot y)^2} = \sin \sphericalangle(x, y),$$

where $\sphericalangle(x, y)$ is the unorientated angle between x and y.

5.2 Convergence of fractional perimeters as $s \nearrow 1$

We start with a result for subsets of intervals and show that only the behaviour in a neighbourhood of their boundary points contributes to the limit:

Lemma 5.3. *Let $0 < s < 1$ and $I \subseteq \mathbb{R}$ be a (possibly unbounded) closed interval. Suppose that $E = \bigcup_{i=1}^{M} [a_i, b_i] \subseteq I$, where $M \in \mathbb{N}$ and $a_1 < b_1 < a_2 < \cdots < a_M < b_M$, and that $I \backslash E = \bigcup_{k=1}^{N} J_k$ is the corresponding decomposition into pairwise disjoint intervals $J_k \subseteq \mathbb{R}$. Furthermore, let l, the minimal length of any of the intervals $[a_i, b_i]$ and J_k, be greater than 0. Then for any $0 < \varepsilon < \frac{l}{2}$*

$$\lim_{s \nearrow 1} (1-s) \int_E \int_{I\backslash E} \frac{1}{|x-y|^{1+s}} \, \mathrm{d}y \, \mathrm{d}x = \lim_{s \nearrow 1} (1-s) \iint\limits_{F_\varepsilon} \frac{1}{|x-y|^{1+s}} \, \mathrm{d}y \, \mathrm{d}x = \mathcal{H}^0(\partial E) \tag{5.3}$$

where $F_\varepsilon := \{(x,y) \in E \times (I \backslash E) : |x-y| < \varepsilon\}$ and ∂E is the boundary of E with respect to the relative topology on I.

Proof. The leftmost limit in (5.3) was evaluated in [Lud14a, Lemma 1] for the case $I = \mathbb{R}$.

For any interval $J_k \subseteq I \backslash E$ with endpoints $-\infty < \alpha < \beta < \infty$ (the case $\alpha = -\infty$ or $\beta = +\infty$ works analogously) we have

$$\int_\alpha^\beta \int_{a_i}^{b_i} \frac{1}{|x-y|^{1+s}} \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{s(1-s)} [(a_i - \alpha)^{1-s} - (a_i - \beta)^{1-s} + (b_i - \beta)^{1-s} - (b_i - \alpha)^{1-s}].$$

In the cases that $\beta = a_i$ or $\alpha = b_i$ we thus get

$$\lim_{s \nearrow 1} (1-s) \int_\alpha^\beta \int_{a_i}^{b_i} \frac{1}{|x-y|^{1+s}} \, \mathrm{d}y \, \mathrm{d}x = 1,$$

otherwise this limit is equal to 0. If $a_1 = \min I$, then a_1 does not lie in the boundary of E relative to I and is not an endpoint of any interval J_k, thus

$$\lim_{s \nearrow 1} (1-s) \sum_{k=1}^{N} \int_{a_1}^{b_1} \int_{J_k} \frac{1}{|x-y|^{1+s}} \, \mathrm{d}y \, \mathrm{d}x = 1,$$

since only the boundary point b_1 contributes to the limit. Otherwise, if $a_1 > \min I$, then the above limit is equal to 2. A similar distinction is necessary for b_M and $\max I$. Summing up over all $i = 1, \ldots, M$ leads to the first identity in (5.3).

To see the second identity in (5.3), we only need to evaluate integrals of the form

$$\int_a^{a+\varepsilon} \int_{x-\varepsilon}^{a} \frac{1}{|x-y|^{1+s}} \, \mathrm{d}y \, \mathrm{d}x = \int_{b-\varepsilon}^{b} \int_b^{x+\varepsilon} \frac{1}{|x-y|^{1+s}} \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{s} \left(\frac{\varepsilon^{1-s}}{1-s} - \varepsilon^{1-s} \right). \tag{5.4}$$

It is easy to see that after multiplying all sides with the factor $(1-s)$, the right-hand side tends to 1 as $s \nearrow 1$. Now a similar argument as before, taking into account the position of a_1 and b_M relative to I, yields the second equality in formula (5.3). ∎

As a simple application we get a convergence result for subsets on a curve:

Corollary 5.4. *Let $I \subseteq \mathbb{R}$ be a (possibly unbounded) closed interval and $\gamma : I \to \mathbb{R}^n$ be a simple C^1 curve. If $E \subseteq I$ is a finite union of closed and pairwise disjoint intervals, then*

$$\lim_{s \nearrow 1}(1-s)\int_{\gamma(E)}\int_{\gamma(I)\setminus\gamma(E)} \frac{1}{d_\gamma(x,y)^{1+s}}\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\mathcal{H}^1(x) = \mathcal{H}^0(\partial\gamma(E)), \tag{5.5}$$

where $\partial\gamma(E)$ is the boundary of $\gamma(E)$ relative to $\gamma(I)$ and $d_\gamma(\cdot,\cdot)$ denotes the distance on the curve, i.e. if $x = \gamma(t_1)$ and $y = \gamma(t_2)$, then

$$d_\gamma(x,y) = \left|\int_{t_1}^{t_2} |\gamma'(t)|\,\mathrm{d}t\right|.$$

Proof. Since line integrals do not depend on the parametrization of the curve, we can assume that γ is an arc-length parametrization, i.e. $|\gamma'(t)| = 1$ for all $t \in I$. The line integrals in (5.5) can thus be rewritten as

$$\int_{\gamma(E)}\int_{\gamma(I)\setminus\gamma(E)} \frac{1}{d_\gamma(x,y)^{1+s}}\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\mathcal{H}^1(x) = \int_E\int_{I\setminus E}\frac{1}{|t-u|^{1+s}}\,\mathrm{d}u\,\mathrm{d}t.$$

The result then follows from Lemma 5.3. ■

Now we give an alternative proof of the convergence result Corollary 4.6 adapted to the sphere:

Theorem 5.5. *Let E be a polyconvex subset of $\mathbb{S}^n$. Then*

$$\lim_{s \nearrow 1}(1-s)\int_E\int_{\mathbb{S}^n\setminus E}\frac{1}{d(x,y)^{n+s}}\,\mathrm{d}\mathcal{H}^n(y)\,\mathrm{d}\mathcal{H}^n(x) = \frac{\omega_{n+1}}{\omega_2}\mathcal{H}^{n-1}(\partial E). \tag{5.6}$$

Proof. If E is an $\mathcal{H}^n$-nullset, then both sides of (5.6) are equal to 0, so suppose $\mathcal{H}^n(E) > 0$.

We apply the spherical Blaschke-Petkantschin formula (5.2) to the left-hand side of (5.6) which results in

$$\begin{aligned}
&(1-s)\int_E\int_{\mathbb{S}^n\setminus E}\frac{1}{d(x,y)^{n+s}}\,\mathrm{d}\mathcal{H}^n(y)\,\mathrm{d}\mathcal{H}^n(x)\\
&\quad= c_n(1-s)\int_{G(n+1,2)}\int_{E\cap L}\int_{(\mathbb{S}^n\setminus E)\cap L}\frac{\nabla_2^{n-1}(x,y)}{d(x,y)^{n+s}}\,\mathrm{d}\mathcal{H}^1(y)\,\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}L\\
&\quad= c_n(1-s)\int_{G(n+1,2)}\int_{A_L}\int_{[0,2\pi]\setminus A_L}\frac{\sin^{n-1}(\delta(\phi,\psi))}{\delta(\phi,\psi)^{n+s}}\,\mathrm{d}\psi\,\mathrm{d}\phi\,\mathrm{d}L,
\end{aligned} \tag{5.7}$$

where in the last step we introduced an arc-length parametrization $\gamma_L : [0,2\pi] \to \mathbb{S}^n \cap L$ of the great circle $\mathbb{S}^n \cap L$ such that $d(\gamma_L(\phi),\gamma_L(\psi)) = \delta(\phi,\psi)$, where

$$\delta(\phi,\psi) = \begin{cases} |\phi-\psi|, & \text{if } |\phi-\psi| \le \pi,\\ 2\pi - |\phi-\psi|, & \text{else,}\end{cases}$$

and put $A_L := \gamma_L^{-1}(E\cap L)$. We only consider the case $E \cap L \neq \mathbb{S}^n \cap L$ (otherwise the inner integrals equal 0), such that $(\mathbb{S}^n\setminus E)\cap L$ is the finite union of nonempty open circular arcs and choose the parametrization γ_L such that $\gamma_L(0) = \gamma_L(2\pi)$ lies in one of the open arcs.

Taylor expansion of the nominator in the integrand yields

$$\frac{\sin^{n-1}(\delta(\phi,\psi))}{\delta(\phi,\psi)^{n+s}} = \frac{\delta(\phi,\psi)^{n-1}(1+O(\delta(\phi,\psi)^2))^{n-1}}{\delta(\phi,\psi)^{n+s}} = \frac{1+r(\delta(\phi,\psi))}{\delta(\phi,\psi)^{1+s}},$$

where for the remainder $r(t)$ there exists $\varepsilon > 0$ and a constant $C > 0$ such that $|r(t)| \leq Ct^2$ as long as $t < \varepsilon$. Now divide the domain of integration into

$$M_{<\varepsilon} := \{(\phi,\psi) \in A_L \times ([0,2\pi]\backslash A_L) : \delta(\phi,\psi) < \varepsilon\}, \text{ and}$$
$$M_{\geq\varepsilon} := \{(\phi,\psi) \in A_L \times ([0,2\pi]\backslash A_L) : \delta(\phi,\psi) \geq \varepsilon\}.$$

By our choice of parametrization, if ε is small enough, then all pairs of the form $(\phi, 0)$ or $(\phi, 2\pi)$, $\phi \in A_L$, do not lie in $M_{<\varepsilon}$ such that $\delta(\phi,\psi) = |\phi-\psi|$ in $M_{<\varepsilon}$. Since

$$\left| \iint\limits_{M_{<\varepsilon}} \frac{r(|\phi-\psi|)}{|\phi-\psi|^{1+s}} \,\mathrm{d}\psi\,\mathrm{d}\phi \right| \leq C \iint\limits_{M_{<\varepsilon}} |\phi-\psi|^{1-s} \,\mathrm{d}\psi\,\mathrm{d}\phi \leq C\mathcal{H}^2(M_{<\varepsilon})\varepsilon^{1-s} \leq (2\pi)^2 C\varepsilon^{1-s}$$

by Lemma 5.3 we get

$$\lim_{s\nearrow 1} (1-s) \iint\limits_{M_{<\varepsilon}} \frac{\sin^{n-1}|\phi-\psi|}{|\phi-\psi|^{n+s}} \,\mathrm{d}\psi\,\mathrm{d}\phi = \mathcal{H}^0(\partial A_L).$$

Since the integrand has no singularities in $M_{\geq\varepsilon}$ by dominated convergence we readily have

$$\lim_{s\nearrow 1} (1-s) \iint\limits_{M_{\geq\varepsilon}} \frac{\sin^{n-1}(\delta(\phi,\psi))}{\delta(\phi,\psi)^{n+s}} \,\mathrm{d}\psi\,\mathrm{d}\phi = 0.$$

By the finiteness of the measure on $G(n+1,2)$, we can exchange limit and integration over $G(n+1,2)$ in (5.7) to eventually obtain

$$\begin{aligned}\lim_{s\nearrow 1}(1-s) \int_E \int_{\mathbb{S}^n\backslash E} \frac{1}{d(x,y)^{n+s}} \,\mathrm{d}\mathcal{H}^n(y)\,\mathrm{d}\mathcal{H}^n(x) &= c_n \int_{G(n+1,2)} \mathcal{H}^0(\partial E \cap L)\,\mathrm{d}L \\ &= \frac{\omega_{n+1}}{\omega_2}\mathcal{H}^{n-1}(\partial E),\end{aligned}$$

where for the last equality we applied the spherical Crofton formula, Theorem 5.1. ■

5.3 Convergence of fractional perimeters as $s \searrow -\infty$

In view of the result due to Maz'ya & Shaposhnikova ([MS02], see Theorem 2.11) it would be natural to consider

$$\lim_{s\searrow 0} s \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{|f(x)-f(y)|}{d(x,y)^{n+s}} \,\mathrm{d}\mathcal{H}^n(y)\,\mathrm{d}\mathcal{H}^n(x). \tag{5.8}$$

However, the following considerations show that we do not get anything similar to the integral of f. In the Euclidean setting we already observe for the fractional 0-seminorm of a smooth function f with compact support in an open Euclidean ball B that

$$\int_B \int_B \frac{|f(x)-f(y)|}{|x-y|^n} \,\mathrm{d}y\,\mathrm{d}x \le \max_{\xi \in B} |\nabla f(\xi)| \int_B \int_B \frac{1}{|x-y|^{n-1}} \,\mathrm{d}y\,\mathrm{d}x < \infty.$$

For the sphere, the finiteness of fractional seminorms for smooth functions then follows from introducing suitable coordinates (cf. Lemma 4.8) and using the argument above. Hence, (5.8) is always 0 for smooth functions.

Since there are no singularities in the integrand for $s \le -n$ and the integrals converge whenever $-n < s \le 0$, we consider fractional seminorms and perimeters for $s \in (-\infty, 1)$. Denote by $\tilde{d}(x,y) = \frac{d(x,y)}{\pi} \in [0,1]$ the normalized geodesic distance between two points $x, y \in \mathbb{S}^n$. Furthermore we put $t := -s$.

Lemma 5.6. *Let $1 \le p < \infty$. For every $x \in \mathbb{S}^n$ and $\delta > 0$*

$$\lim_{t \nearrow \infty} t^n \int_{C(-x,\delta)} \tilde{d}(x,y)^{-n+tp} \,\mathrm{d}\mathcal{H}^n(y) = \frac{\omega_n \pi^n (n-1)!}{p^n},$$

where $C(-x,\delta) = \{y \in \mathbb{S}^n : d(-x,y) < \delta\}$ is the open spherical cap around $-x$ with radius δ (if $\delta > \pi$, then $C(-x,\delta) = \mathbb{S}^n$).

Proof. For $\varepsilon > 0$ there exist $\delta_0 \in (0,\delta)$ and a normal coordinate chart $\phi : C(-x,\delta_0) \to B^n_{\delta_0}$ such that

$$\tilde{d}(x,y) = 1 - \frac{|\phi(y)|}{\pi}, \qquad \text{and}$$
$$1 - \varepsilon \le \sqrt{\det(g_{\alpha\beta}(y))} \le 1 + \varepsilon$$

for all $y \in C(-x,\delta_0)$ (see Theorem 2.23). Observe that

$$t^n \int_{C(-x,\delta)\setminus C(-x,\delta_0)} \tilde{d}(x,y)^{-n+tp} \,\mathrm{d}\mathcal{H}^n(y) \le \mathcal{H}^n(\mathbb{S}^n) t^n \left(1 - \frac{\delta_0}{\pi}\right)^{-n+tp} \to 0$$

as $t \nearrow \infty$, thus

$$\lim_{t \nearrow \infty} t^n \int_{C(-x,\delta)} \tilde{d}(x,y)^{-n+tp} \,\mathrm{d}\mathcal{H}^n(y) = \lim_{t \nearrow \infty} t^n \int_{C(-x,\delta_0)} \tilde{d}(x,y)^{-n+tp} \,\mathrm{d}\mathcal{H}^n(y).$$

By our choice of coordinates we have

$$\begin{aligned} t^n \int_{C(-x,\delta_0)} \tilde{d}(x,y)^{-n+tp} \,\mathrm{d}\mathcal{H}^n(y) &\le (1+\varepsilon) t^n \int_{B^n_{\delta_0}} \left(1 - \frac{|\phi(y)|}{\pi}\right)^{-n+tp} \mathrm{d}y \\ &= (1+\varepsilon)\omega_n t^n \int_0^{\delta_0} \left(1 - \frac{r}{\pi}\right)^{-n+tp} r^{n-1} \,\mathrm{d}r \\ &= (1+\varepsilon)\omega_n \pi^n t^n \int_0^{\delta_0/\pi} (1-u)^{-n+tp} u^{n-1} \,\mathrm{d}u \end{aligned}$$

where we introduced the substitution $u = \frac{r}{\pi}$ in the last step. The last integral is equal to $B_{\frac{\delta_0}{\pi}}(n, -n+tp+1)$, where $B_T(a,b) = \int_0^T u^{a-1}(1-u)^{b-1}\,\mathrm{d}u$ is the incomplete Beta function. Note that

$$\lim_{t\nearrow\infty} t^n B_{\frac{\delta_0}{\pi}}(n, -n+tp+1) = \lim_{t\nearrow\infty} t^n B(n, -n+tp+1)$$

where $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\,\mathrm{d}u$ is the (complete) Beta function since

$$t^n \int_{\frac{\delta_0}{\pi}}^1 u^{n-1}(1-u)^{-n+tp}\,\mathrm{d}u \le t^n \left(1 - \frac{\delta_0}{\pi}\right)^{-n+tp+1} \to 0,$$

as $t \nearrow \infty$. Thus it suffices to determine the value of $\lim\limits_{t\nearrow\infty} t^n B(n, -n+tp+1)$. The identity $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ together with Stirling's formula

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \left(\frac{x}{e}\right)^x e^{\mu(x)}, \quad \text{where } 0 < \mu(x) < \frac{1}{12x}$$

can be used to deduce

$$\lim_{t\nearrow\infty} t^n B(n, -n+tp+1) = \Gamma(n) \lim_{t\nearrow\infty} \sqrt{\frac{tp+1}{tp+1-n}} e^n \left(1 - \frac{n}{tp+1}\right)^{tp+1} \times$$
$$\times \frac{t^n}{(tp+1-n)^n} \frac{e^{\mu(-n+tp+1)}}{e^{\mu(tp+1)}} = \frac{(n-1)!}{p^n}.$$

Similarly, one can prove that the reverse inequality

$$\lim_{t\nearrow\infty} t^n \int_{C(-x,\delta_0)} \tilde{d}(x,y)^{-n+tp}\,\mathrm{d}\mathcal{H}^n(y) \ge (1-\varepsilon)\frac{\omega_n \pi^n (n-1)!}{p^n}$$

holds. Thus the result follows from letting $\varepsilon \searrow 0$. ■

The next theorem shows the convergence of fractional seminorms as $s \searrow -\infty$. Note that the limit measures the reflection symmetry of a function in the L_p-sense, i.e. the integral in the right-hand side of (5.9) is 0 precisely for functions which are even almost everywhere and it is equal to $(2\|f\|_p)^p$ precisely for functions which are odd almost everywhere.

Theorem 5.7. *Let $1 \le p < \infty$ and $f \in L_p(\mathbb{S}^n)$. Then*

$$\lim_{t\nearrow\infty} t^n \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \frac{|f(x)-f(y)|^p}{\tilde{d}(x,y)^{n-tp}}\,\mathrm{d}\mathcal{H}^n(y)\,\mathrm{d}\mathcal{H}^n(x) = c_{n,p}\int_{\mathbb{S}^n} |f(x)-f(-x)|^p\,\mathrm{d}\mathcal{H}^n(x) \tag{5.9}$$

where $c_{n,p} = \frac{\omega_n \pi^n (n-1)!}{p^n}$.

Proof. We split the proof into two steps. In the first step we show (5.9) for continuous functions and use a density argument in the second step to extend the formula to general L_p-functions.

Step 1: Proof for continuous functions

Let g be a continuous function on $\mathbb{S}^n$. First, note that by the dominated convergence theorem and Lemma 5.6

$$\lim_{t\nearrow\infty} t^n \int_{\mathbb{S}^n} \left(\int_{\mathbb{S}^n} \frac{|g(x)-g(-x)|^p}{\tilde{d}(x,y)^{n-tp}} \, \mathrm{d}\mathcal{H}^n(y) \right) \mathrm{d}\mathcal{H}^n(x) = c_{n,p} \int_{\mathbb{S}^n} |g(x)-g(-x)|^p \, \mathrm{d}\mathcal{H}^n(x) \tag{5.10}$$

since the integrand for x-integration is dominated by $(2\|g\|_\infty)^p C$ with a constant $C > 0$ for sufficiently large t. Moreover, we have for every $0 < \delta < \pi$ that

$$\begin{aligned} &\lim_{t\nearrow\infty} t^n \int_{\mathbb{S}^n} \int_{\mathbb{S}^n \setminus C(-x,\delta)} \frac{|g(x)-g(-x)|^p}{\tilde{d}(x,y)^{n-tp}} \, \mathrm{d}\mathcal{H}^n(y) \, \mathrm{d}\mathcal{H}^n(x) \\ &\qquad = \lim_{t\nearrow\infty} t^n \int_{\mathbb{S}^n} \int_{\mathbb{S}^n \setminus C(-x,\delta)} \frac{|g(x)-g(y)|^p}{\tilde{d}(x,y)^{n-tp}} \, \mathrm{d}\mathcal{H}^n(y) \, \mathrm{d}\mathcal{H}^n(x) = 0, \end{aligned} \tag{5.11}$$

i.e. the inner integrals concentrate on the point $-x$ in the limit.

Now let $\varepsilon > 0$ and choose $\delta > 0$ be such that $|g(y)-g(-x)| < \varepsilon$ whenever $x \in \mathbb{S}^n$ and $y \in C(-x,\delta)$. Using Taylor's formula for the case $p > 1$ and the reverse triangle inequality for the case $p = 1$ we rewrite

$$|g(x)-g(y)|^p = |g(x)-g(-x)|^p + r(x,y)$$

where the remainder term satisfies $|r(x,y)| \le c \cdot \varepsilon$ for all $y \in C(-x,\delta)$ with a constant c independent of x and y. From this and (5.11) it follows that

$$\begin{aligned} \limsup_{t\nearrow\infty} \Bigg| t^n &\int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{|g(x)-g(y)|^p}{\tilde{d}(x,y)^{n-tp}} \, \mathrm{d}\mathcal{H}^n(y) \, \mathrm{d}\mathcal{H}^n(x) \\ &\qquad - t^n \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{|g(x)-g(-x)|^p}{\tilde{d}(x,y)^{n-tp}} \, \mathrm{d}\mathcal{H}^n(y) \, \mathrm{d}\mathcal{H}^n(x) \Bigg| \\ &\le c \cdot \varepsilon \limsup_{t\nearrow\infty} t^n \int_{\mathbb{S}^n} \int_{C(-x,\delta)} \tilde{d}(x,y)^{-n+tp} \, \mathrm{d}\mathcal{H}^n(y) \, \mathrm{d}\mathcal{H}^n(x). \end{aligned}$$

Now formula (5.9) for continuous functions follows from the arbitrariness of $\varepsilon > 0$ and (5.10).

Step 2: Proof for general $f \in L_p$

Let $f \in L_p(\mathbb{S}^n)$ and define $\mathcal{S}f : \mathbb{S}^n \to \mathbb{R}$ by $\mathcal{S}f(x) := c_{n,p}^{\frac{1}{p}} |f(x) - f(-x)|$, and for $t \in \mathbb{R}$ the function $F_t : \mathbb{S}^n \to \mathbb{R}$ by

$$F_t(x) := \left(t^n \int_{\mathbb{S}^n} \frac{|f(x)-f(y)|^p}{\tilde{d}(x,y)^{n-tp}} \, \mathrm{d}\mathcal{H}^n(y) \right)^{\frac{1}{p}}.$$

We show that $F_t \overset{t\nearrow\infty}{\to} \mathcal{S}f$ in $L_p(\mathbb{S}^n)$ which implies that $\lim_{t\nearrow\infty} \|F_t\|_p^p = \|\mathcal{S}f\|_p^p$ and thus formula (5.9). By density, for each $\varepsilon > 0$ there exists a continuous function g on $\mathbb{S}^n$ such that $\|f - g\|_p < \varepsilon$. With $\mathcal{S}g$ and G_t defined as above, we have

$$\|F_t - \mathcal{S}f\|_p \le \|F_t - G_t\|_p + \|G_t - \mathcal{S}g\|_p + \|\mathcal{S}g - \mathcal{S}f\|_p.$$

By step 1, the summand $\|G_t - \mathcal{S}g\|_p$ tends to 0 as t goes to infinity. Moreover, by rotation invariance of the Hausdorff measure,

$$\begin{aligned}\|\mathcal{S}g - \mathcal{S}f\|_p &= c_{n,p}^{\frac{1}{p}} \left(\int_{\mathbb{S}^n} \Big| |g(x) - g(-x)| - |f(x) - f(-x)| \Big|^p \,\mathrm{d}\mathcal{H}^n(x) \right)^{\frac{1}{p}} \\ &\leq 2c_{n,p}^{\frac{1}{p}} \|f - g\|_p < 2c_{n,p}^{\frac{1}{p}} \cdot \varepsilon.\end{aligned}$$

For the remaining summand, we first observe that

$$F_t(x) = t^{\frac{n}{p}} \left\| \frac{f(x) - f(\cdot)}{\tilde{d}(x,\cdot)^{\frac{n}{p}-t}} \right\|_p$$

and by the triangle inequality for L_p-norms

$$\begin{aligned}\|F_t - G_t\|_p &= \left(\int_{\mathbb{S}^n} |F_t(x) - G_t(x)|^p \,\mathrm{d}\mathcal{H}^n(x) \right)^{\frac{1}{p}} \\ &\leq t^{\frac{n}{p}} \left(\int_{\mathbb{S}^n} \left| \left\| \frac{f(x) - g(x)}{\tilde{d}(x,\cdot)^{\frac{n}{p}-t}} \right\|_p + \left\| \frac{f(\cdot) - g(\cdot)}{\tilde{d}(x,\cdot)^{\frac{n}{p}-t}} \right\|_p \right|^p \mathrm{d}\mathcal{H}^n(x) \right)^{\frac{1}{p}} \\ &\leq 2 \left(t^n \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} \frac{|f(x) - g(x)|^p}{\tilde{d}(x,y)^{n-tp}} \,\mathrm{d}\mathcal{H}^n(y)\,\mathrm{d}\mathcal{H}^n(x) \right)^{\frac{1}{p}} \\ &= 2 \left(\int_{\mathbb{S}^n} \frac{t^n}{\tilde{d}(e_1,y)^{n-tp}} \,\mathrm{d}\mathcal{H}^n(y) \right)^{\frac{1}{p}} \|f - g\|_p < \mathrm{const} \cdot \varepsilon,\end{aligned}$$

where $e_1 = (1, 0, \ldots, 0)$ and the constant does not depend on t. ■

Corollary 5.8. *Let $E \subseteq \mathbb{S}^n$ be a Borel set. Then,*

$$\lim_{t \nearrow \infty} t^n \int_E \int_{\mathbb{S}^n \setminus E} \frac{1}{\tilde{d}(x,y)^{n-t}} \,\mathrm{d}\mathcal{H}^n(y)\,\mathrm{d}\mathcal{H}^n(x) = c_{n,1} \mathcal{H}^n((-E) \cap (\mathbb{S}^n \setminus E)),$$

where $c_{n,1} = \omega_n \pi^n (n-1)!$.

Proof. The statement follows from Theorem 5.7 with $f = \mathbb{1}_E$ and $p = 1$. ■

5.4 The spherical isoperimetric inequality for s-perimeters

We show a spherical isoperimetric inequality for s-perimeters if $s > -n$ and a reverse isoperimetric-type inequality for $s < -n$.

Theorem 5.9. *Let $E \subseteq \mathbb{S}^n$ be a Borel set and C a spherical cap with $\mathcal{H}^n(E) = \mathcal{H}^n(C)$. Then*

$$P_s(E) \geq P_s(C) \tag{5.12}$$

for $-n < s < 1$ and

$$P_s(E) \leq P_s(C) \tag{5.13}$$

for $-\infty < s < -n$. Equality is attained if and only if E is itself a spherical cap up to a $\mathcal{H}^n$-nullset.

Note that $P_{-n}(E)$ does not change for all E of same measure. We present this fact in the proof of the theorem.

It is easy to see that the theorem can be reformulated as follows: Let $0 < \alpha \leq \omega_{n+1}$. Then there is a constant $\gamma_{n,s,\alpha}$ such that for every Borel set $E \subseteq \mathbb{S}^n$ with $\alpha = \mathcal{H}^n(E)$ we have

$$P_s(E) \geq \gamma_{n,s,\alpha}\mathcal{H}^n(E) = \gamma_{n,s,\alpha}\alpha$$

if $-n < s < 1$ and

$$P_s(E) \leq \gamma_{n,s,\alpha}\mathcal{H}^n(E) = \gamma_{n,s,\alpha}\alpha$$

if $-\infty < s < -n$. The constant $\gamma_{n,s,\alpha}$ is given by $\gamma_{n,s,\alpha} = \frac{P_s(C)}{\alpha}$, where $C \subseteq \mathbb{S}^n$ is a spherical cap with $\mathcal{H}^n(C) = \alpha$. Equality is attained if and only if E is a spherical cap up to a $\mathcal{H}^n$-nullset. It is easy to show that

$$\lim_{\alpha \nearrow \omega_{n+1}} \gamma_{n,s,\alpha} = 0$$

for every $-n < s < 1$. Hence, one cannot expect to have a uniform constant $\gamma_{n,s}$ in this case.

In order to prove the theorems we use rearrangement inequalities with respect to a fixed center of symmetry $e \in \mathbb{S}^n$ on the sphere. We use the same notations as for rearrangements in the Euclidean setting as there will be no confusions. The function $a : [0, \pi] \to [0, \omega_{n+1}]$, $a(r) = \mathcal{H}^n(C(v, r))$ does not depend on the choice of $v \in \mathbb{S}^n$ and is strictly increasing and bijective. For a Borel set $E \subseteq \mathbb{S}^n$ the spherical volume radius r_σ is defined by

$$r_\sigma(E) = a^{-1}(\mathcal{H}^n(E)).$$

and the spherical rearrangement of E by $E^* = C(e, r_\sigma(E))$. Let $f : \mathbb{S}^n \to \mathbb{R}$ be a measurable function. The spherical rearrangement of f is denoted by f^* and is defined by the layer cake representation $f^* : \mathbb{S}^n \to \mathbb{R}_{\geq 0}$, $f^*(v) = \int_0^\infty \mathbb{1}_{\{|f|>t\}^*}(v)\mathrm{d}t$. The following rearrangement inequality on the sphere can be found in [Bec92, Theorem 3].

Theorem 5.10. *Let ϕ, k and ρ be non-negative functions defined on $[0, \infty)$ such that*

1. *$\phi(0) = 0$, ϕ is convex and monotonically increasing, $\phi'' \geq 0$ and $t \mapsto t\phi'(t)$ is convex,*
2. *k is monotonically decreasing, and*
3. *ρ is monotonically increasing.*

Then for measurable functions f and g on $\mathbb{S}^n$

$$\int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \phi\left(\frac{|f(x) - g(y)|}{\rho(d(x, y))}\right) k(d(x, y))\, \mathrm{d}\mathcal{H}^n(y)\, \mathrm{d}\mathcal{H}^n(x)$$
$$\geq \int_{\mathbb{S}^n}\int_{\mathbb{S}^n} \phi\left(\frac{|f^*(x) - g^*(y)|}{\rho(d(x, y))}\right) k(d(x, y))\, \mathrm{d}\mathcal{H}^n(y)\, \mathrm{d}\mathcal{H}^n(x).$$

If k is strictly decreasing and ϕ is strictly convex, then equality holds if and only if $f(x) = \lambda f^*(\theta x)$ and $g(x) = \lambda g^*(\theta x)$ for a.e. $x \in \mathbb{S}^n$, where $\lambda \in \{+1, -1\}$ and $\theta \in SO(n+1)$.

Proof of Theorem 5.9. For a Borel set $E \subseteq \mathbb{S}^n$ note that

$$2 \cdot P_s(E) = \int_{\mathbb{S}^n} \int_{\mathbb{S}^n} |\mathbb{1}_E(x) - \mathbb{1}_E(y)|^p d(x,y)^{-(n+s)} \, \mathrm{d}\mathcal{H}^n(y) \, \mathrm{d}\mathcal{H}^n(x)$$

where $p > 1$ is arbitrary.

For $-n < s < 1$ apply Theorem 5.10 with $\phi(t) = t^p$, $k(t) = \frac{1}{(1+t)^{n+s}}$ and $\rho(t) = (\frac{t}{1+t})^{\frac{n+s}{p}}$. Note further that

$$P_{-n}(E) = \int_E \int_{\mathbb{S}^n \setminus E} 1 \, \mathrm{d}\mathcal{H}^n(y) \, \mathrm{d}\mathcal{H}^n(x) = \mathcal{H}^n(E)(\mathcal{H}^n(\mathbb{S}^n) - \mathcal{H}^n(E))$$

and this quantity is always the same for E's of same Hausdorff measure, especially for E^*. For $-\infty < s < -n$ apply Theorem 5.10 with $\phi(t) = t^p$, $k(t) = \pi^{-n-s} - t^{-n-s}$ and $\rho(t) = 1$. Then substract $2 \cdot \pi^{-n-s} P_{-n}(E)$ on both sides of the inequality. ∎

Bibliography

[AF03] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[AL89] F. J. Almgren, Jr. and E. H. Lieb. Symmetric decreasing rearrangement is sometimes continuous. *J. Amer. Math. Soc.*, 2(4):683–773, 1989.

[AFTL97] A. Alvino, V. Ferone, G. Trombetti, and P.-L. Lions. Convex symmetrization and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(2):275–293, 1997.

[AB94] M. Amar and G. Bellettini. A notion of total variation depending on a metric with discontinuous coefficients. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11(1):91–133, 1994.

[ADPM11] L. Ambrosio, G. De Philippis, and L. Martinazzi. Gamma-convergence of nonlocal perimeter functionals. *Manuscripta Math.*, 134(3-4):377–403, 2011.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[AGM15] L. Ambrosio, R. Ghezzi, and V. Magnani. BV functions and sets of finite perimeter in sub-Riemannian manifolds. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(3):489–517, 2015.

[AZ91] E. Arbeiter and M. Zähle. Kinematic relations for Hausdorff moment measures in spherical spaces. *Math. Nachr.*, 153:333–348, 1991.

[Bal01] A. Baldi. Weighted BV functions. *Houston J. Math.*, 27(3):683–705, 2001.

[Bec92] W. Beckner. Sobolev inequalities, the Poisson semigroup, and analysis on the sphere S^n. *Proc. Nat. Acad. Sci. U.S.A.*, 89(11):4816–4819, 1992.

[BBM01] J. Bourgain, H. Brezis, and P. Mironescu. Another look at Sobolev spaces. In *Optimal Control and Partial Differential Equations*, pages 439–455. IOS, Amsterdam, 2001.

[BBM02] J. Bourgain, H. Brezis, and P. Mironescu. Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. *J. Anal. Math.*, 87:77–101, 2002.

[Bre02] H. Brezis. How to recognize constant functions. A connection with Sobolev spaces. *Uspekhi Mat. Nauk*, 57(4(346)):59–74, 2002.

[BN16] H. Brezis and H.-M. Nguyen. The BBM formula revisited. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 27(4):515–533, 2016.

[CRS10] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010.

[CN18] A. Cesaroni and M. Novaga. The isoperimetric problem for nonlocal perimeters. *Discrete Contin. Dyn. Syst. Ser. S*, 11(3):425–440, 2018.

[Dáv02] J. Dávila. On an open question about functions of bounded variation. *Calc. Var. Partial Differential Equations*, 15(4):519–527, 2002.

[DG53] E. De Giorgi. Definizione ed espressione analitica del perimetro di un insieme. *Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8)*, 14:390–393, 1953.

[DNPV12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[DFPV13] S. Dipierro, A. Figalli, G. Palatucci, and E. Valdinoci. Asymptotics of the s-perimeter as $s \searrow 0$. *Discrete Contin. Dyn. Syst.*, 33(7):2777–2790, 2013.

[ET04] L. Esposito and C. Trombetti. Convex symmetrization and Pólya-Szegö inequality. *Nonlinear Anal.*, 56(1):43–62, 2004.

[FMP$^+$18] F. Ferrari, M. Miranda, Jr., D. Pallara, A. Pinamonti, and Y. Sire. Fractional Laplacians, perimeters and heat semigroups in Carnot groups. *Discrete Contin. Dyn. Syst. Ser. S*, 11(3):477–491, 2018.

[FS08] R. L. Frank and R. Seiringer. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.*, 255(12):3407–3430, 2008.

[Ful71] D. R. Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Math. Programming*, 1:168–194, 1971.

[Fus04] N. Fusco. The classical isoperimetric theorem. *Rend. Accad. Sci. Fis. Mat. Napoli (4)*, 71:63–107, 2004.

[Gar06] R. J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, second edition, 2006.

[Gla95] S. Glasauer. Integralgeometrie konvexer Körper im sphärischen Raum. *Dissertation, Universiät Freiburg*, 1995.

[Gro] M. Gromov. Isoperimetric inequalities in Riemannian manifolds. Appendix of (V. D. Milman and G. Schechtman), *Asymptotic Theory of Finite-Dimensional Normed Spaces*, Springer, Berlin, 1986.

[HZ16] Y. Han and M. Zhu. Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications. *J. Differential Equations*, 260(1):1–25, 2016.

[Heb99] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[HXY18] S. Hou, J. Xiao, and D. Ye. A mixed volume from the anisotropic Riesz-potential. *Trans. London Math. Soc.*, 5(1):71–96, 2018.

[HT19] D. Hug and C. Thäle. Splitting tessellations in spherical spaces. *Electron. J. Probab.*, 24:60 pp., 2019.

[Kre21] A. Kreuml. The anisotropic fractional isoperimetric problem with respect to unconditional unit balls. *Commun. Pure Appl. Anal.*, 20(2):783–799, 2021.

[KM19] A. Kreuml and O. Mordhorst. Fractional Sobolev norms and BV functions on manifolds. *Nonlinear Anal.*, 187:450–466, 2019.

[KM20] A. Kreuml and O. Mordhorst. Fractional perimeters on the sphere. *Preprint, arXiv:2011.11562*, 2020.

[Lee97] J. M. Lee. *Riemannian manifolds: An Introduction to Curvature*, volume 176 of *Graduate Texts in Mathematics*. Springer, New York, 1997.

[Lee13] J. M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[LS11] G. Leoni and D. Spector. Characterization of Sobolev and *BV* spaces. *J. Funct. Anal.*, 261(10):2926–2958, 2011.

[Lie77] E. H. Lieb. Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Appl. Math.*, 57(2):93–105, 1976/77.

[LL01] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[Lom15] L. Lombardini. Fractional perimeter and nonlocal minimal surfaces. *Master's thesis, Università degli Studi di Milano*, 2015.

[Lud14a] M. Ludwig. Anisotropic fractional perimeters. *J. Differential Geom.*, 96(1):77–93, 2014.

[Lud14b] M. Ludwig. Anisotropic fractional Sobolev norms. *Adv. Math.*, 252:150–157, 2014.

[Mag12] F. Maggi. *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012.

[MS02] V. Maz′ya and T. Shaposhnikova. On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.*, 195(2):230–238, 2002.

[MPPP07] M. Miranda, Jr., D. Pallara, F. Paronetto, and M. Preunkert. Heat semigroup and functions of bounded variation on Riemannian manifolds. *J. Reine Angew. Math.*, 613:99–119, 2007.

[OS18] J. O'Hara and G. Solanes. Regularized Riesz energies of submanifolds. *Math. Nachr.*, 291(8-9):1356–1373, 2018.

[SV12] O. Savin and E. Valdinoci. Γ-convergence for nonlocal phase transitions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(4):479–500, 2012.

[Sch14] R. Schneider. *Convex Bodies: the Brunn-Minkowski Theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[SW08] R. Schneider and W. Weil. *Stochastic and Integral Geometry*. Probability and its Applications (New York). Springer, Berlin, 2008.

[Sch03] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, volume 24 of *Algorithms and Combinatorics*. Springer, Berlin, 2003.

[Tay78] J. E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc.*, 84(4):568–588, 1978.

[VS06] J. Van Schaftingen. Anisotropic symmetrization. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(4):539–565, 2006.

[VSW04] J. Van Schaftingen and M. Willem. Set transformations, symmetrizations and isoperimetric inequalities. In *Nonlinear analysis and applications to physical sciences*, pages 135–152. Springer Italia, Milan, 2004.

[Vis90] A. Visintin. Nonconvex functionals related to multiphase systems. *SIAM J. Math. Anal.*, 21(5):1281–1304, 1990.

[XY17] J. Xiao and D. Ye. Anisotropic Sobolev capacity with fractional order. *Canad. J. Math.*, 69(4):873–889, 2017.

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A. Kreuml and O. Mordhorst. Fractional Sobolev norms and BV functions on manifolds. *Nonlinear Anal.*, 187: 450–466, 2019