DIPLOMARBEIT

Portfolio optimisation under integer constraints

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Stefan Gerhold

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Miloš Jovanović

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Abstract

The goal of this thesis is to study and analyze different models of portfolio optimization. We start with the basic models such as mean-variance model presented by Markowitz (1952), mean-absolute-deviation model proposed by Konno and Yamazaki (1991), minimax model developed by Young (1998). Furthermore, we extend models with various constraints, of which the most important is the integer constraint. The numerical implementation of these models is presented in addition and its results are discussed. This work is based largely on the two papers:
An Exact Solution Approach for Portfolio Optimization Problems Under Stochastic and Integer Constraints published by Pierre Bonami and Miguel Lejeune [1] and *Portfolio-optimization models for small investors* paper by Philipp Baumann and Norbert Trautmann [2]

Kurzfassunng

Ziel dieser Arbeit ist es, verschiedene Modelle der Portfoliooptimierung zu untersuchen und zu analysieren. Wir beginnen mit den Grundmodellen wie dem von Markowitz vorgestellten Mean-Variance Modell (1952), dem von Konno und Yamazaki vorgeschlagenen Mean-Absolute-Deviation Modell (1991) und dem von Young entwickelten Minimax Modell (1998). Darüber hinaus erweitern wir Modelle mit verschiedenen Einschränkungen, von denen die wichtigste die ganzzahlige Einschränkung ist. Zusätzlich wird die numerische Implementierung dieser Modelle vorgestellt und deren Ergebnisse diskutiert.

Diese Arbeit basiert sich hauptsächlich auf zwei Papers: *An Exact Solution Approach for Portfolio Optimization Problems Under Stochastic and Integer Constraints* von Pierre Bonami und Miguel Lejeune [1] und *Portfoliooptimization models for small investors* von Philipp Baumann und Norbert Trautmann [2]

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I must also express my very profound gratitude to my family, to my parents and to my brother for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them.

Thank you!

Statutory Declaration

I declare in lieu of an oath that I have written this master thesis myself and that I have not used any sources or resources other than stated for its preparation. This master thesis has not been submitted elsewhere for examination purposes.

Vienna, 01.02.2023

Milos Jovanovic

Contents

1 Introduction

Financial mathematics is a relatively new branch of applied mathematics, dealing with the application of mathematical methods to financial problems. In general, the application of financial mathematics can be divided into two branches of finance that require advanced quantitative techniques: derivative pricing and risk- and portfolio management.

Portfolio management involves building and overseeing a selection of investments that will meet the long-term financial goals and risk tolerance of an investor [25]. In order to achieve this, various portfolio optimization models are used. Portfolio optimization model should assist in the selection of the most efficient portfolio by analyzing various possible portfolios of the given securities. Assuming the financial market is arbitrage free, all the investments are exposed to some kind of downside risk. By Financial Times lexicon the volatility is defined as the extent to which the price of a security or commodity, or the level of the market, interest rate or currency, changes over time. High volatility implies rapid and large upward and downward movements over a relatively short period of time and low volatility implies much smaller and less frequent changes in value. In other words, volatility gives us the idea about our investment risk by showing the range to which the price may change while keeping the direction of the change unrevealed. [27] The portfolio optimization model can be based on the diverse risk measures. The modern portfolio theory was introduced by Harry Markowitz in 1952 with his means variance model. This model considered the trade off between expected return and variance. Some of the other risk measures that are later used are Roy's safety first risk criterion (Roy 1952), which belongs to a family of downside risk measures, value at risk (Morgan Guaranty 1994), conditional value at risk (Rockafellar and Uryasev 2000), stochastic dominance (Dentcheva and Ruszczynski 2003), semideviation (Ogryczak and Ruszczynski 1999), excess probabilities (Schultz and Tiedemann 2006), mean-absolute deviation (Konno and Yamamoto 1991), semiabsolute deviation (Feinstein and Tappa 1993) [1].

In Chapter 2 we describe in detail the standard portfolio optimization models and the risk measures used. In Chapter 3 our models and analysis are presented. The results of analysis and conclusion are also given. Chapter 4 gives insights into the Cumulative Prospect Theory and possible further developments in this direction.

2 Portfolio optimization

Portfolio optimization involves the optimal choice of investments for the set of available investments and the given amount of capital. At the time of decision it is not known what will happen to the cash flow that describes the investment. One can only estimate the expected cash flow, therefore the investment return rate is a random variable that has the expected value and variance. There are also investments where the cash flow is known in advance in the case the yield is a deterministic value and the standard deviation is zero. Such investments are called risk-free assets. Although academics agree that there are no risk-free assets in practice because even the safest financial instruments carry a small amount of risk, financial instruments with a fixed interest rate such as government bonds are often used in portfolio optimization to determine the risk-free interest rate.

2.1 Portfolio return rates

We suppose that our portfolio consists only of risky assets. Suppose we purchase an asset at time t_0 for some price P_0 an then we sell it later at time t_1 for a price P_1. Then the ratio

$$R = \frac{P_1}{P_0}$$

is called the return of the asset.
The rate of return r is given by:

$$r = \frac{P_1 - P_0}{P_0}.$$

We can also write $r = R - 1$.

Now we assume that our portfolio consists of n assets and let our initial budget be B. We want to assign the initial budget to the assets i,

$i = 1, \cdots, n$. The weight of stock i is represented by x_i. The budget constraint require that weights sum to 1

$$\sum_{i=1}^{n} x_i = 1. \tag{2.1}$$

In this work we will not consider the short selling. Therefore we need to assure that the weights must be non negative, i.e. $x_i \geq 1$, $i = 1, \cdots, n$. If the R_i, $i = 1, \cdots, n$ denotes the return of a risky asset i, then is total portfolio return:

$$R = \sum_{i=1}^{n} R_i x_i,$$

as well as return rate is given by:

$$r = R - 1 = \sum_{i=1}^{n} R_i x_i - \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (R_i - 1) x_i = \sum_{i=1}^{n} r_i x_i.$$

We denote expected return of an asset i by $\overline{R}_i$, variance of asset i by σ_i^2 or σ_{ii} and covariance between asset i and asset j as σ_{ij}.

2.2 Portfolio optimization models [3]

The ground model of the modern portfolio theory is the Mean-variance model. It was introduced by Harry Markowitz in an essay in 1952 [3], for which he was later rewarded the Nobel Prize in Economics. Therefore it is also called Markowitz model (or HM model). As its name says, it is based on trade-off between two measures: expected return (mean) and the variance of the diverse portfolios. Due to Markowitz, the process of selecting a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio.

Initially to develop the model, Markovitz started with the following assumptions[4]:

- Risk of a portfolio is based on the variability of returns from the said portfolio.

- The investor is risk averse.
- The investor prefers to increase consumption.
- The investor's utility function is concave and increasing.
- Analysis is based on a single period model of investment.

In order to choose the best portfolio, it is necessary to determine the set of efficient portfolios, **efficient frontier**. The efficient frontier represents those portfolios for which the expected return is the highest for any level of risk, and for which the risk is the lowest for any level of expected return [22].

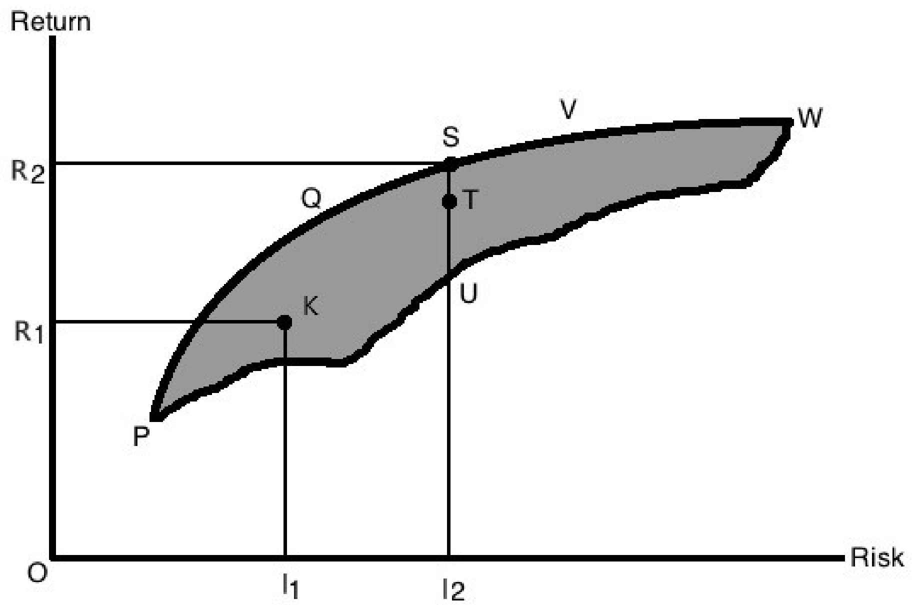


Figure 2.1: Risk-Return of Possible Portfolios ([23])

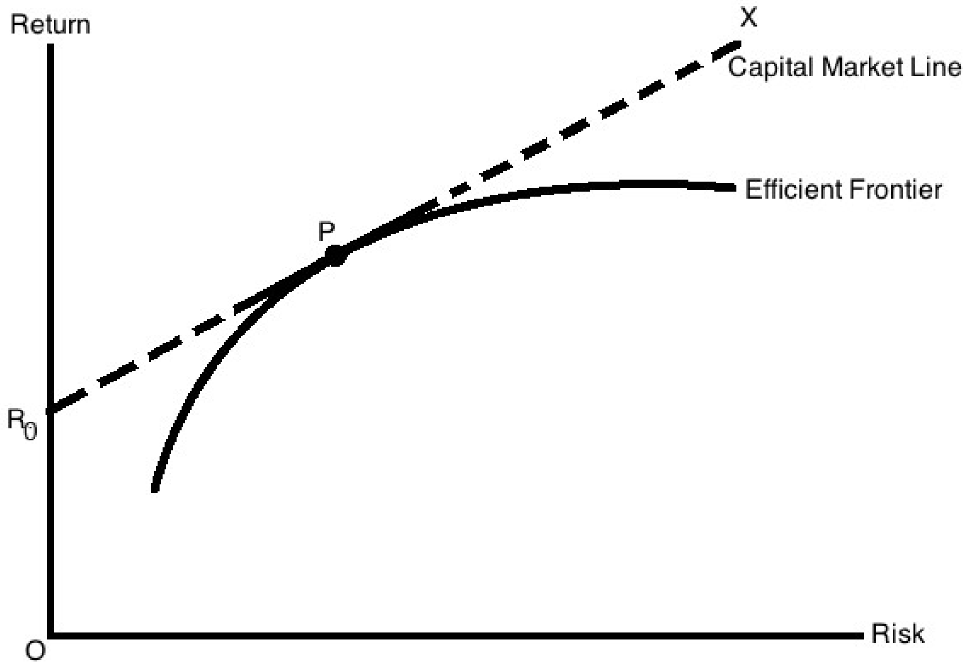


Figure 2.2: The Combination of Risk-Free Securities with the efficient frontier and CML ([23])

In the first Figure above, R_1 and R_2 represents the returns of the portfolios 1 and 2 respectively and l_1 and l_2 represents their risk levels. In general various risk measures could be considered (ie. variance of the portfolio). Shaded area represents the set of parameter pairs of feasible portfolios. The boundary PQVW represents the efficient frontier. All portfolios that lie below the Efficient Frontier are not good enough because the return would be lower for the given risk. Portfolios that lie to the right of the Efficient Frontier would not be good enough, as there is higher risk for a given rate of return. All portfolios lying on the boundary of PQVW are called Efficient Portfolios [24]. For example, at risk level l_2, there are three portfolios S, T, U. The portfolio S is called the efficient portfolio as it has the highest return, R_2, compared to T and U. The Efficient Frontier is the same for all investors, as all investors want maximum return with the lowest possible risk and they are risk averse [24].
The second Figure also contains the efficient frontier, R_0 is return of the risk-free asset. R_1PX is drawn so that it is tangent to the efficient fron-

tier. Capital Market Line represents the risk-return trade off in the capital market [24]. The CML is an upward sloping line, which means that the investor will take higher risk if the return of the portfolio is also higher [24]. The portfolio P represents the most efficient portfolio, as it lies on both the CML and Efficient Frontier, and every investor would prefer to attain this portfolio, P .[24]

On these assumptions, the original Markowitz model was created. The original Markowitz model assumes that expected return $\overline{R}_i$ of a risky assets i, $i = 1, \cdots, n$ as well as variance-covariance matrix ($\sigma_{ij}, \quad i = 1, \cdots, n, \quad j = 1, \cdots, n$) of the returns are known. There are several formulation of the mean-variance portfolio selection problems. One of them has a goal to achieve minimal risk provided that a prescribed return level R_{min} is attained [1]. This model is formulated by:

$$
\begin{aligned}
\min \ & \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \\
\text{subject to } & R_0 x_0 + \sum_{i=1}^{n} R_i x_i \geq R_{min}, \\
& \sum_{j=0}^{n} x_j = 1, \qquad (2.2) \\
& R_i \in \mathbb{R}\ , i = 0, \cdots, n.
\end{aligned}
$$

In this problem, the variables x_j represent the proportion of capital invested in the risky assets j, where we have n risky assets, ie. $j = 1, \cdots, n$. On the other hand x_0 is the fraction of capital invested in the money market or so called riskless asset with known return R_0. The objective function aims at minimizing the variance of the portfolio $\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j$.[1]
The constraint

$$x_0 + \sum_{j=1}^{n} x_j = 1,$$

so called budget constraint, shows that the sum of investment is equal to one, so the whole capital B is used. It it clear, that the investor can allocate part of the available capital to the money market. [1]

Another formulation of a Markowitz model has a goal to achieve the maximal expected return so that risk level does not exceed the risk level prescribed by the investor. This variant of a model is formulated by [2]:

$$\max \sum_{i=1}^{n} \overline{R}_i x_i$$

$$\text{s.t.} \ \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \leq \rho,$$

$$\sum_{i=1}^{n} x_i = 1,$$

$$x_i \geq 0 \ \ i = 1, \cdots, n.$$

Here ρ represents the maximal risk level. We will use this formulation of a model as one of a basic model for the numerical analysis.

Since this Markowitz work there have been a lot of attempts of extending it and making the modern portfolio theory more practical and applicable. Main lack of this theory is that it required the perfect knowledge of the expected returns of the assets and the variance-covariance matrix. These returns are unknown and unpredictable. It can be tried to obtain the accurate estimation of these returns, but even that is really complicated and imprecise. Many possible choices of error, like instability of data, impossibility to obtain a sufficient number of data samples, unpredicted investors behaviour etc., affect estimation of returns and lead to the so-called estimation risk in the portfolio selection.[1][5] On the other hand, investors would often rather trade off some return for a more secure portfolio that performs well under a wide set of realisations of the random variables, so the need for constructing portfolios that are much less impacted by inaccuracies in the estimation of the expected return and the variance of the return is therefore clear.[1]
The main disadvantage of the model is the uncertainty associated with the estimation of the expected returns. It was said in the work of Bonami and Lejeune [1] that a widespread belief among portfolio managers is that the portfolio estimation risk is due mainly to errors in the estimation of the expected return and not so much to errors in the estimation of the variance-covariance matrix. Those assumptions were also shown by Broadie[6] as

well as by Chopra and Ziemba[7].
In order to improve this disadvantages some researchers have developed other approaches and applied different risk measures. One of them is Roy's safety-first criterion [8]. It is based on the Roy's criterion that the probability of the portfolio's return falling below a minimum desired threshold is minimized. Assuming that the portfolios have normally distributed returns, Roy's first-safety criterion can be reduced to the maximization of the safety first, defined by:

$$\text{SFRatio}_i = \frac{E(R_i) - R_{min}}{\sqrt{Var(R_i)}}$$

where $E(R_i)$ is the expected return of the portfolio, $\sqrt{Var(R_i)}$ is the standard deviation of the portfolio's return and R_{min} is the minimum acceptable return[9].This measure is similar to the Sharpe ratio[10], which maximizes the ratio of the excess return of risk. Under normality, it is equal to:

$$\text{Sharpe ratio}_i = \frac{E(R_i) - R_0}{\sqrt{Var(R_i)}}$$

where R_0 is the risk free return.
There are many other risk measures that can be used. I will present some of them hereafter. One of the basic and significant measures is Value at risk (VaR). It is the quantile of the loss function. The Value at risk for a given confidence level indicates the amount of loss that will not be exceeded within a given period with this probability. It is most used by regulators and firms in the financial industry. Value at risk was first developed and introduced in 1994 by J.P. Morgan[11]. Mathematically VaR with confidence level $1-\alpha$ is defined as $(1-\alpha)$- quantile from loss function.

$$VaR_{1-\alpha}(X) = F_X^{-1}(1-\alpha) = \inf\left\{x \in \mathbb{R} : F_X(x) \geq 1-\alpha\right\},$$

where X describes the random variable of the loss function of the portfolio over the period under consideration. F_X denotes the associated distribution function. The disadvantage of Var is that it is not a coherent risk measure, because it violates the sub-additivity property. Therefore Conditional value at risk ($CVaR$) was developed. It is defined as[12]:

$$CVaR_{1-\alpha}(X) = E[X|X > Var_{1-\alpha}] = E[X|X > F_X^{-1}(1-\alpha)].$$

While the VaR represents the maximum loss that will not be exceeded wit a confidence level $1 - \alpha$, the $CVaR$ implies the average loss outside of the confidence level, so in other α worst cases. It is increasingly used in risk management, portfolio management as well as in portfolio optimization. The difference between the two measures can be clearly seen in the following graphic.

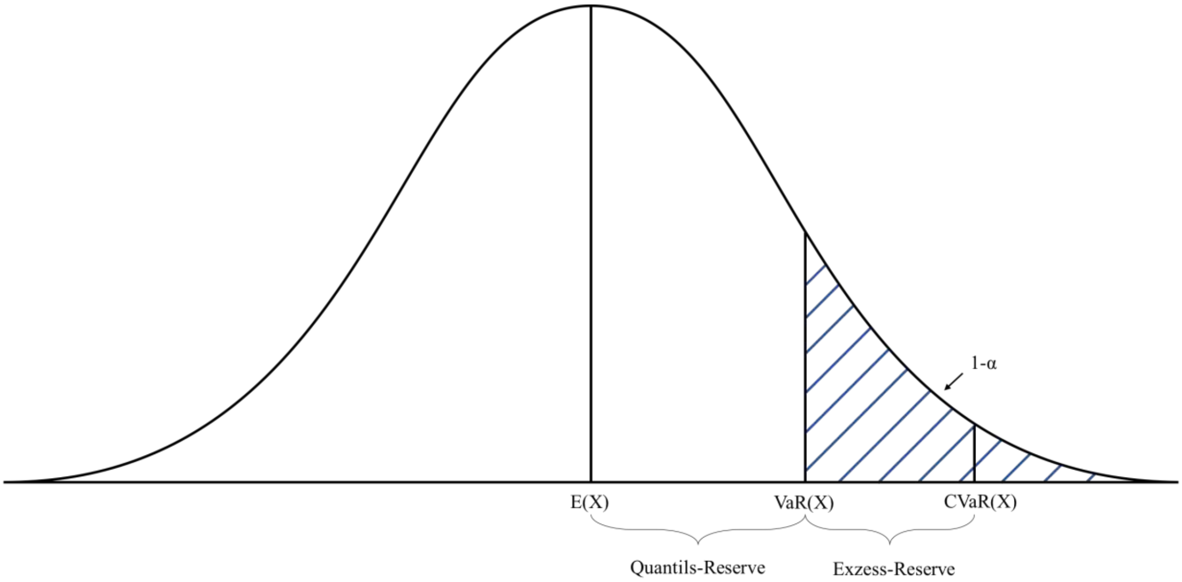


Figure 2.3: Differentiation of the CVaR from the VaR ([13])

Another theoretical approach to the portfolio selection problem is that of stochastic dominance [14]. This approach is related to the investor's risk aversion. The stochastic dominance gives a partial order between random variables. The concept arises in decision theory and decision analysis in situations where one gamble can be ranked as superior to another gamble for a broad class of decision-makers[15]. First we come with the definition of the **first-order stochastic dominance**. Random variable X dominates the random variable Y (first-order stochastic dominance) if for any outcome k, X gives at least as high probability of receiving at least k as does Y and for some $\hat{k}$, X gives a higher probability of receiving at least $\hat{k}$. In notation form it can be written as $P[X \geq k] \geq P[Y \geq k]$, for all outcomes k and there is at least one outcome $\hat{k}$ for which $P[X \geq \hat{k}] > P[Y \geq \hat{k}]$.
From the portfolio point of view the **second order stochastic dominance** is more important. The definition of the second-order dominance is equivalent to: X dominates Y in the second order if and only if

$E[u(X)] \geq E[u(Y)]$ for all non decreasing concave functions u for which expected values are finite[14]. Second-order dominance describes the shared preferences of a smaller class of decision-makers than does first-order dominance. Risk-averse decision makers will prefer portfolio with return X than portfolio with return Y.

In the work of Dentcheva and Ruszczyński [14], the new, dominance-constrained portfolio problem was introduced. In order to introduce a new model, they started from a basic model of stochastic application, which can be formulated as:

$$\max_{z \in Z} E[\varphi(z, \omega)].$$

In this formulation, ω denotes an elementary event in a probability space $(\Omega, \mathcal{F}, P)$. z is a decision vector in an appropriate space $\mathcal{Z}$, and $\varphi : \mathcal{Z} \times \Omega \longrightarrow \mathbb{R}$. The set $Z \subset \mathcal{Z}$ is defined either explicitly, or via some constraints that may involve the elementary event ω and must hold with some prescribed probability. [14]

Initially, a solution was sought for the following problems. They formulated the mean-risk portfolio optimization problem as follows:

$$\max_{X \in C} E[X] - \lambda \rho(X).$$

In this approach C represents the set of all random variables X such that, for some $z \in Z$, one has $X(\omega) \leq \varphi(z, \omega)$ a.s. $\rho(X)$ is a risk measure that represents the variability of the X. In contrast to the Markowitz model, it does not necessarily have to be the variance of the portfolio. Many other measures are possible here. λ is a nonnegative parameter that represents desirable exchange rate of mean for risk, so the parameter for risk aversion. If λ is equal to 0, then the risk has no value and we choose to maximize the mean. We can also select a utility function u, which should be concave and non decreasing in order to represent risk adverse investors. Then the the optimization problem is formulated as:

$$\max_{X \in C} E[u(X)].$$

Dentcheva and Ruszczyński introduced the alternative approach. They made a comparison to a reference outcome, based on the stochastic dominance relation. Assume that a reference random outcome $Y \subset \mathcal{L}^1(\Omega, \mathcal{F}, P)$ has an available finite expected value. Their intention was to have the new outcome X preferably over Y. They had the following optimization problem:

$$\max f(X) \tag{2.3}$$
$$\text{subject to } X \succeq_{(2)} Y, \tag{2.4}$$
$$X \in C \tag{2.5}$$

where $f : C \longrightarrow \mathbb{R}$ is a real concave continuous function, $C \subset \mathcal{L}^1(\Omega, \mathcal{F}, P)$ is convex and closed and $Y \subset \mathcal{L}^1(\Omega, \mathcal{F}, P)$[14]. It is shown that with a selection of suitable utility functions, that build convex cone, we can define a Lagranian of (2.3) – (2.5) as follows:

$$L(X, u) = f(X) + E[u(X)] - E[u(Y)].$$

In the [14] it was shown that if there exist an optimizer $\hat{X}$ for the first optimization problem ((2.3) – (2.5)), then there exists a utility function $\hat{u}$ such that

$$L(\hat{X}\hat{u}) = \max_{X \in C} L(X, \hat{u}).$$

It was shown that, in general, the Markowitz classical mean-variance model is not consistent with stochastic dominance rules. The are some issues about using standard deviation alone. It does not tell much about skewness. That is, does the distribution of returns "lean" in one direction or another? Are you more likely to get a gain or a loss? Stocks are more likely to have a gain than a loss, so they have "negative skew" [17].

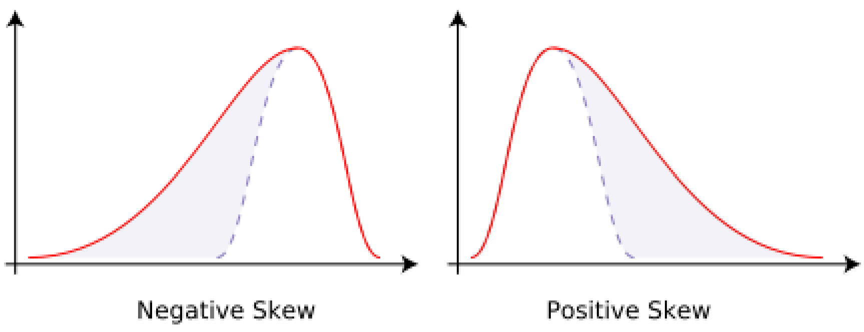


Figure 2.4: Diagrams illustrating negative and positive skew.
Author: Rodolfo Hermans (Godot)

It also does not tell anything about kurtosis or spread. That is, how tall is the peak of the curve and how fat are the tails? [17]

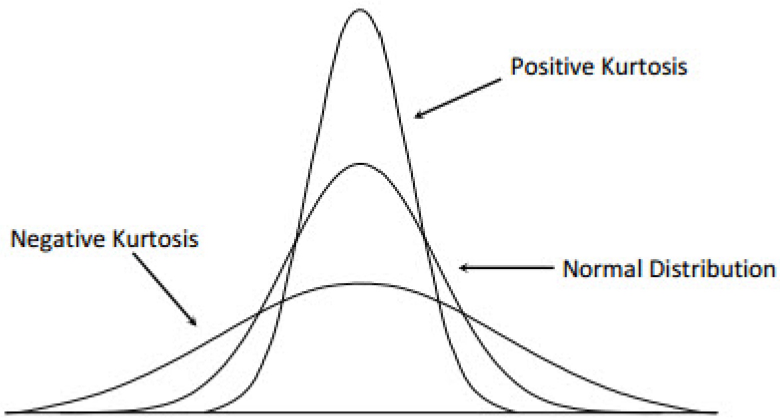


Figure 2.5: Kurtosis ([17])

But an even bigger problem is that when we use standard deviation as a measure of risk, then we are saying that any variance is bad. Even when our returns are more than the average, that is still considered bad. In order to make it consistent we need to create a risk measure, that takes in account all possible measures below the mean. In the work by Ogryczak and Ruszczynski [16] two measures were considered: **absolute semideviation** and **standard semideviation**.
Absolute semideviation is defined as [16]:

$$\bar{\delta}_X = \int_{-\infty}^{\mu_X} (\mu_X - \xi) P_X(d\xi) = \frac{1}{2} \int_{-\infty}^{\infty} |\mu_X - \xi| P_X(d\xi),$$

where X represents the random variable, P_X its probability measure and $\mu_X = E[X]$.
The standard semideviation is defined as [16]:

$$\bar{\sigma}_X = \left(\int_{-\infty}^{\mu_X} (\mu_X - \xi)^2 P_X(d\xi) \right)^{\frac{1}{2}}.$$

The standard semi deviation can be written in discrete cases as:

$$\overline{\sigma}_X = \sqrt{\frac{1}{n}\sum_{\xi_t<\mu}^{n}(\mu - \xi_t)^2}.$$

The weak relation of mean–risk dominance is defined as follows: [40]

$$X \succeq_{\mu/\sigma} Y \Leftrightarrow \mu_X \geq \mu_Y \text{ and } \sigma_X \leq \sigma_Y.$$

A mean–risk model is said to be consistent with the stochastic dominance relation of degree i if: [40]

$$X \succeq_{(i)} Y \Rightarrow X \succeq_{\mu/\sigma} Y.$$

Ogryczak and Ruszczynski [16] showed that mean-risk models using standard or absolute semideviations as risk measures are consistent with the stochastic dominance orders, if a bounded set of mean-risk trade-offs is considered.

Another risk measure that can be used in portfolio optimization theory is the mean-absolute deviation (short MAD). As the name indicates the mean-absolute deviation of a set
$R = \{R_1, R_2, \cdots, R_n\}$ is equal to $\frac{1}{n}\sum_{i=1}^{n}|R_i - m(R)|$. $m(R)$ represents the measure of central tendency. Usually it is mean, median or mode. This measure was considered in the work of Konno and Yamazaki [18]. They proposed L1 risk model that, after suitable reduction, leads to the linear program. The model was presented as following [19]:

$$\min \ \frac{1}{T}\sum_{t=1}^{T} y_t$$

$$\text{s.t. } y_t + \sum_{j=1}^{n} a_{jt}x_j \geq 0, \ t = 1, \cdots, T, \tag{2.6}$$

$$y_t - \sum_{j=1}^{n} a_{jt}x_j \geq 0, \ t = 1, \cdots, T, \tag{2.7}$$

$$\sum_{j=1}^{n} \overline{R_j}x_j \geq R_{min}.$$

$$\sum_{j=1}^{n} x_j = 1,$$

$$x_j \geq 0, \ j = 1, \cdots, n,$$

where $y_i = \left| \sum_{j=1}^{n} (R_{jt} - \overline{R_j}) x_j \right|$, x_j is the proportion of funds to be invested in asset j, R_{jt} is a realization of returns for the asset j in time period t, and $a_{jt} = R_{jt} - \overline{R_j}$, where $\overline{R_j} = \frac{1}{T} \sum_{t=1}^{T} R_{jt}$. Scalar R_{min} represents the minimum return required by an investor. Model from Konno and Yamazaki minimizes the L1 risk function. This function is given by:

$$\frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j=1}^{n} a_{jt} x_j \right| .$$

This shows that only one of the constraint sets (2.6) or (2.7) is required to find optimal solutions to the problem, so that either one of (2.6) or (2.7) is redundant and can be removed. With T fewer effective rows, one consequence of this manoeuvre is that solutions to the model have at most $T + 2$ nonzero assets in the solution [19]. Same as in the mean variance model from Markowitz, we can formulate the MAD model such a way that it aims to achieve the maximum expected return so that risk level does not exceed prescribed risk level ρ. This variant of the MAD model looks like:

$$\max \ \sum_{j=1}^{n} \overline{R_j} x_j$$

$$\text{s.t. } y_t + \sum_{j=1}^{n} a_{jt} x_j \geq 0, \ t = 1, \cdots, T,$$

$$y_t - \sum_{j=1}^{n} a_{jt} x_j \geq 0, \ t = 1, \cdots, T,$$

$$\frac{1}{T}\sum_{t=1}^{T} y_t \leq \rho,$$

$$\sum_{j=1}^{n} x_j = 1,$$

$$x_j \geq 0, \ j = 1, \cdots, n.$$

The Konno and Yamazaki model was modified by Feinstein and Thapa [20]. In their model the nonnegative variables ϕ_t and ψ_t are applied to constraints (2.6) and (2.7) as follows [19]:

$$y_t + \sum_{j=1}^{n} a_{jt} x_j = 2\phi_t, \ t = 1, \cdots, T,$$

$$y_t - \sum_{j=1}^{n} a_{jt} x_j = 2\psi_t, \ t = 1, \cdots, T.$$

ϕ_t and ψ_t represent the positive and negative deviation of portfolio return, respectively.

Then y_t can be eliminated and we can transform the objective and the constraints (2.6) and (2.7) so that model will look like: [20]

$$\min \sum_{t=1}^{T} (\phi_t + \psi_t)$$

$$\text{s.t. } \phi_t - \psi_t = \sum_{j=1}^{n} (R_{jt} - \overline{R_j}) x_j, \ t = 1, \cdots, T,$$

$$\sum_{j=1}^{n} \overline{R_j} x_j \geq R_{min},$$

$$\sum_{j=1}^{n} x_j = 1,$$

$$x_j \geq 0, \ j = 1, \cdots, n,$$

$$\phi_t, \psi_t \geq 0,\ t = 1, \cdots, T.$$

The other formulation of a model, which maximizes the expected return will look like:

$$\max \sum_{j=1}^{T} \overline{R_j} x_j$$

$$\text{s.t. } \phi_t - \psi_t = \sum_{j=1}^{n} (R_{jt} - \overline{R_j}) x_j,\ t = 1, \cdots, T,$$

$$\sum_{t=1}^{T} (\phi_t + \psi_t) \leq \rho,$$

$$\sum_{j=1}^{n} x_j = 1,$$

$$x_j \geq 0,\ j = 1, \cdots, n,$$

$$\phi_t, \psi_t \geq 0,\ t = 1, \cdots, T.$$

The formulation above will be used as basic model in numerical analysis.

Another model that can be considered is the Minimax model developed by Young [21]. In this model the optimal portfolio is defined as that one which would minimize the maximum loss over all past historical periods, subject to a restriction on the minimum acceptable average return across all observed time periods. According to Young this model can accommodate fixed transaction costs and have logical advantages when the returns are non normally distributed, and when investor has a strong form of risk aversion. The model contains the following variables[21]:

$R_{jt} \rightarrow$ return of the security j in time period t

$\overline{R_j} = \frac{1}{T} \sum_{t=1}^{T} R_{jt} \rightarrow$ average return on security j

$x_j \rightarrow$ portfolio allocation to security j

$$R_{pt} = \sum_{j=1}^{n} x_j R_{jt} \rightarrow \text{return on portfolio in period } t$$

$$E_p = \sum_{j=1}^{n} x_j \overline{R_j} \rightarrow \text{average return on portfolio}$$

$$M_p = \min_t R_{pt} \rightarrow \text{minimum return on portfolio}$$

The Minimax model maximizes the M_p, subject to the constraint that E_p exceeds some minimum prescribed level and to the budget constraint. This means that minimax portfolio minimizes the maximum loss, where loss is defined as negative gain or, alternatively, maximizes the minimum gain [21]. The model has been originally written as [21]:

$$\begin{aligned} &\max_{M_p, x} M_p \\ \text{s.t.} \quad &\sum_{j=1}^{n} x_j R_{jt} - M_p \geq 0, \quad t = 1, \cdots, T, \\ &\sum_{j=1}^{n} x_j \overline{R}_j \geq \hat{R}_{min}, \\ &\sum_{j=1}^{n} x_j = 1, \\ &x_j \geq 0, \quad j = 1, \cdots, n. \end{aligned} \tag{2.8}$$

The minimum level we need to exceed is $\hat{R}_{min}$. From the equation (2.8) we see that it guarantees that M_p will be bounded from above by the minimum portfolio return. We can also make an equivalent formulation of the model, that seeks to maximize expected return, subject to a constraint that the portfolio return exceeds some level R_{min} in each observation period [21]. It is described as following:

$$\max \sum_{j=1}^{n} x_j \overline{R}_j$$

$$\text{s.t} \sum_{j=1}^{n} x_j R_{jt} \geq R_{min}, \quad t = 1, \cdots, T,$$

$$\sum_{j=1}^{n} x_j = 1,$$

$$x_j \geq 0, \quad j = 1, \cdots, n.$$

We will use this formulation as a basic model for the numerical analysis. This portfolio is being optimized with respect to the data set $\{R_{jt}\}$. We can take this data set from historical observations or from some probabilistic model for future returns. If one lacks both historical data on past returns and a predictive probability model for future return, the minimax model will not be applicable.

3 Numerical analysis

3.1 Basic models

Here we will present the basic models that are used for analysis. All basic models are taken from Baumann and Trautmann[2] and then expanded through various constraints. The objective of each model is to maximize the expected return of the portfolio so that the portfolio risk, which is measured by various risk measures, does not exceed the risk level prescribed by the investor, so called risk constraint. The other general constraint is the budget constraint, i.e. the entire budget should be invested. In our project we will use the variant of mean-variance model, which maximize the expected return so that the prescribed risk level is not exceeded. This model was also used in the paper from Baumann and Trautmann [2]. We have already presented this model in section 2.2. For a better overview, we also provide a summary of the model here:

$$(MV) \begin{cases} \max \sum_{i=1}^{n} \overline{R}_i x_i \\ \text{s.t. } \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \leq \rho \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \; (i = 1, \cdots, n) \end{cases}$$

,in which

n number of stocks

σ_{ij} covariance beetween return of stock i and return of stock j

$\overline{R}_i$ expected return of stock i

ρ maximal risk level

x_i weight of stock i in portfolio

From the model we can see that first constraint represents the risk constraint with the maximum risk level ρ and second constraint represents the budget constraint.

The second basic model that we will consider, is mean-absolute deviation model from Konno and Yamazaki. As already seen in the last chapter, the measure of this model is the mean absolute deviation of the portfolio return from the expected portfolio return in all periods. In this case we will use the variant of this model, described by Baumann and Trautmann [2]:

$$(MAD)\begin{cases} \max \sum_{i=1}^{n} \overline{R}_i x_i \\ \text{s.t. } \phi_t - \psi_t = \frac{1}{T}\sum_{i=1}^{n}(R_{it} - \overline{R}_i)x_i \ \ (t = 1, \cdots, T) \\ \sum_{t=1}^{T}(\phi_t + \psi_t) \leq \rho \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \ (i = 1, \cdots, n) \\ \phi_t, \psi_t \geq 0 \ \ (t = 1, \cdots, T), \end{cases}$$

where is:

T number of periods

R_{it} return of stock i in period t

ϕ_t positive deviation of portfolio return

ψ_t negative deviation of portfolio return

As in previous model, we can see that the first constraint represents the risk constraint,and the second represents the budget constraint. It is also to notice that the deviation in period t is computed as sum of two negative variables ϕ_t and ψ_t, which correspond to the positive and negative deviations. It has been done in this way in order to obtain a linear model.

The third basic model that we consider, is the Minimax (MM) model. As already mentioned in the previous chapter, in this model the portfolio risk is measured by the minimum portfolio return over all periods. The model reads as follows [2][21] :

$$(MM) \begin{cases} \max \sum_{i=1}^{n} \overline{R}_i x_i \\ \text{s.t.} \ \sum_{i=1}^{n} x_i R_{it} \geq R_{min} \quad (t = 1, \cdots, T) \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \ (i = 1, \cdots, n) \end{cases}$$

As in the previous model, the first and the second constraints are risk and budget constraints, respectively. The risk constraint in minimax model ensures that the portfolio return does not fall below the minimum return R_{min} in any period.

3.2 Analysis design and results

In this chapter we want to evaluate the models from second chapter. As the basic models, we took the models from Baumann and Trautmann [2], that have already been described in previous chapter. These models are mean-variance model (MV), mean-absolute-deviation model (MAD) and minimax model (MM). In addition to these models we have built extensions with integer constraints. The extended models were built similarly as in the reserach from Baumann and Trautmann [2]. The first additional constraint is the integer constraint, which means that the number of units of any stock has to be integral, i.e

$$z_i \in \mathbb{Z}_{\geq 0} \ (i = 1, \cdots, n)$$

Assuming P_i denotes the price of the stock i at time of the purchase and B denotes the budget, we define:

$$\lambda_i := \frac{P_i}{B}.$$

Then the weight of the stock i is defined as

$$x_i := \lambda_i z_i \ (i = 1, \cdots, n). \tag{3.1}$$

Because of previous constraint it would not be possible to satisfy the budget constraint in lot of cases. Therefore budget constraint has to be modified and replaced by:

$$1 - \delta \leq \sum_{i=1}^{n} x_i \leq 1 + \delta, \tag{3.2}$$

with a small positive constant $\delta > 0$.

We have also limited the maximum weight of one stock in portfolio by 30%. Extended models were used for all 3 basic models.

In order to build our models we have built datasets from stocks, which were included in ATX (Austrian Traded Index). For this purpose we used weekly stock returns from 2009 to 2019. For this period we created 10 dataset. First year of each dataset was used to construct the portfolio with 3 basic and 3 extended models. Based on the performances of the first year of each dataset, we have computed the maximal risk level ρ, variance for the MV model, the mean absolute deviation for the MAD and the minimum return for the Minimax model. The second year of the dataset was used to evaluate the performance of the portfolios.

Computations were performed on the Microsoft Office Excel Solver. This tool uses several algorithms to find optimal solutions. We used The GRG Nonlinear Solving Method for nonlinear optimization, which uses the Generalized Reduced Gradient (GRG2) code, which was developed by Leon Lasdon, University of Texas at Austin, and Alan Waren, Cleveland State University, and enhanced by Frontline Systems, Inc. [26]. The standard Microsoft Excel Solver uses a implementation of the Branch and Bound method to solve Mixed-Integer Programming (MIP) problems. Its speed limitations make it suitable only for problems with a small number (perhaps 50 to 100) integer variables[26]. Having in mind that we considered only stocks from ATX, this number of variables was suitable for this implementation.

As described in a Paper from Barreto, Haffner, Pereira and Bauer [29] the Branch and Bound algorithm is an enumerative technique, in which a solution is found based on the construction of a tree in which nodes represent the problems candidates and branches represent the new restrictions to be considered. Through this tree, all integer solutions of the problem feasible region are listed explicitly or implicitly ensuring that all the optimal solutions will be found.

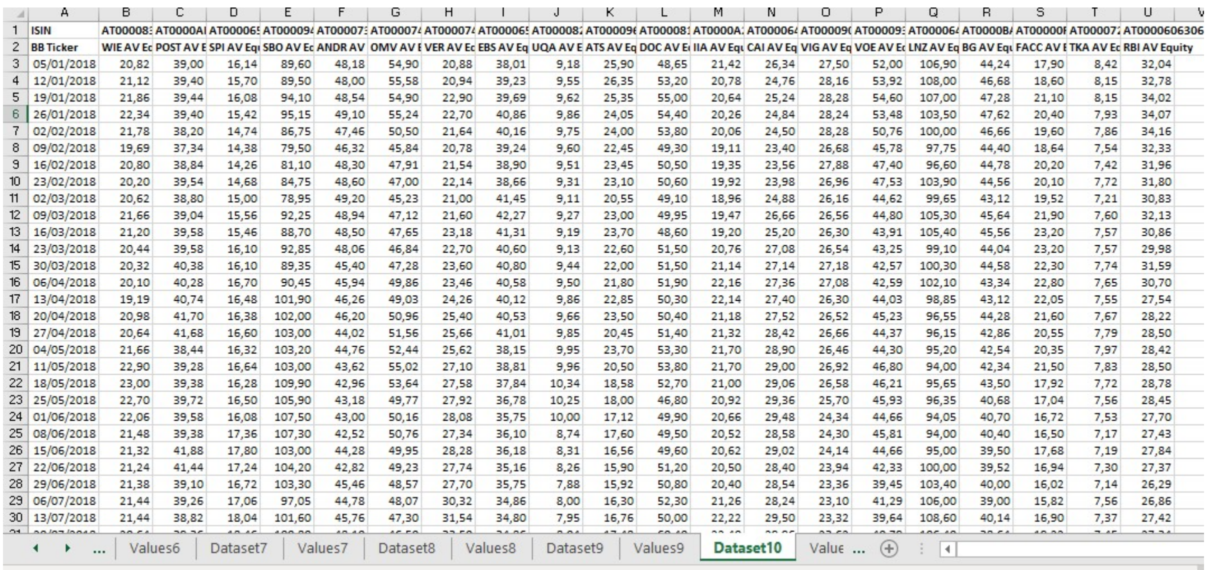
The overall structure has three key elements, separation, relaxation and pruning. Separation uses the tactic of "divide and conquer" in order to solve the problem (P). In order to find the solution of P, it is decomposed into two or more descendant subproblems, generating a list of candidate problems (CP). At a subsequent step in the algorithm, a candidate is selected from the list of candidate problems and the algorithm tries to solve that problem. If a solution can not be found to that problem, that problem is again decomposed and its descendants are added to the list of candidate problems. If the selected problem can be solved, then a new solution is obtained. The objective function value of this new solution is then compared with the value of the incumbent solution, which is the best feasible solution known so far. If the new solution is better than the incumbent solution, it becomes the new incumbent. Then, the algorithm returns to the list and selects the next candidate. This procedure is repeated until the list is empty, and the solution of the problem is taken as the final incumbent solution [29].

The usual way of separation of an integer programming problem is through contradictory constraints in a single integer variable (separation variable or branching variable). Thus, from the original problem (called node zero), two new descendant subproblems are created, which are easier to solve than the original one, since a constraint was added to the separation variable. Each generated node has an associated candidate subproblem and each branch indicates the addition of a constraint related to the variable used in the separation. Therefore, as the algorithm moves down in the tree, the viable region of the generated descendants becomes more restricted [29].

Among the possible ways of relaxation, the most common is the elimination of the integral constraints (integer variables), which converts the integer or mixed integer problem to a general LP. At first, through the relaxation, it assumes that the original integer variables may take fractional values and then the resulting LP problem is solved. The obtained optimal solution usually has several variables with non-integer values. Among these variables, one must be selected for separation. Once the separation and inclusion of new descendants in candidates list is done, the algorithm must select from among the stored candidates the next to be evaluated and, if necessary, separated successively until the linear problem solution becomes integer, or unfeasible or worse than the incumbent, meaning that the candidate sub-problem can be removed from the list (pruned), not producing any more descendants. This procedure repeats until the list of candidates' subproblems becomes empty[29].

Due to program limitation, there is so called MIP gap (or Integer Optimality option) in Excel Solver, which allows the Solver to stop if it finds an integer solution that is within the specified percentage of the best known bound on the optimal solution [26]. The default and the value used for this implementation is 1%. This means that there is a possible 1% deviation in the selection of the best solution for extended models.

In the following we show an example of how the data sets and the models look like:



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
1	ISIN	AT00008	AT0000AI	AT00006	AT00009	AT00007	AT00007	AT00007	AT00006	AT00008	AT00009	AT00008	AT0000A	AT00006	AT00009	AT00009	AT00006	AT0000B	AT00000	AT00007	AT0000606306
2	BB Ticker	WIE AV Ec	POST AV E	SPI AV Equ	SBO AV Ec	ANDR AV	OMV AV E	VER AV Ec	EBS AV Eq	UQA AV E	ATS AV Eq	DOC AV E	IIA AV Equ	CAI AV Eq	VIG AV Eq	VOE AV Ec	LNZ AV Eq	BG AV Equ	FACC AV E	TKA AV Ec	RBI AV Equity
3	05/01/2018	20,82	39,00	16,14	89,60	48,18	54,90	20,88	38,01	9,18	25,90	48,65	21,42	26,34	27,50	52,00	106,90	44,24	17,90	8,42	32,04
4	12/01/2018	21,12	39,40	15,70	89,50	48,00	55,58	20,94	39,23	9,55	26,35	53,20	20,78	24,76	28,16	53,92	108,00	46,68	18,60	8,15	32,78
5	19/01/2018	21,86	39,44	16,08	94,10	48,54	54,90	22,90	39,69	9,62	25,35	55,00	20,64	25,24	28,28	54,60	107,00	47,28	21,10	8,15	34,02
6	26/01/2018	22,34	39,40	15,42	95,15	49,10	55,24	22,70	40,86	9,86	24,05	54,40	20,26	24,84	28,24	53,48	103,50	47,62	20,40	7,93	34,07
7	02/02/2018	21,78	38,20	14,74	86,75	47,46	50,50	21,64	40,16	9,75	24,00	53,80	20,06	24,50	28,28	50,76	100,00	46,66	19,60	7,86	34,16
8	09/02/2018	19,69	37,34	14,38	79,50	46,32	45,84	20,78	39,24	9,60	22,45	49,30	19,11	23,40	26,68	45,78	97,75	44,40	18,64	7,54	32,33
9	16/02/2018	20,80	38,84	14,26	81,10	48,30	47,91	21,54	38,90	9,51	23,45	50,50	19,35	23,56	27,88	47,40	96,60	44,78	20,20	7,42	31,96
10	23/02/2018	20,20	39,54	14,68	84,75	48,60	47,00	22,14	38,66	9,31	23,10	50,60	19,92	23,98	26,96	47,53	103,90	44,56	20,10	7,72	31,80
11	02/03/2018	20,62	38,80	15,00	78,95	49,20	45,23	21,00	41,45	9,11	20,55	49,10	18,96	24,88	26,16	44,62	99,65	43,12	19,52	7,21	30,83
12	09/03/2018	21,66	39,04	15,56	92,25	48,94	47,12	21,60	42,27	9,27	23,00	49,95	19,47	26,66	26,56	44,80	105,30	45,64	21,90	7,60	32,13
13	16/03/2018	21,20	39,58	15,46	88,70	48,50	47,65	23,18	41,31	9,19	23,70	48,60	19,20	25,20	26,30	43,91	105,40	45,56	23,20	7,57	30,86
14	23/03/2018	20,44	39,58	16,10	92,85	48,06	46,84	22,70	40,60	9,13	22,60	51,50	20,76	27,08	26,54	43,25	99,10	44,04	23,20	7,57	29,98
15	30/03/2018	20,32	40,38	16,10	89,35	45,40	47,28	23,60	40,80	9,44	22,00	51,50	21,14	27,14	27,18	42,57	100,30	44,58	22,30	7,74	31,59
16	06/04/2018	20,10	40,28	16,70	90,45	45,94	49,86	23,46	40,58	9,50	21,80	51,90	22,16	27,36	27,08	42,59	102,10	43,34	22,80	7,65	30,70
17	13/04/2018	19,19	40,74	16,48	101,90	46,26	49,03	24,26	40,12	9,86	22,85	50,30	22,14	27,40	26,30	44,03	98,85	43,12	22,05	7,55	27,54
18	20/04/2018	20,98	41,70	16,38	102,00	46,20	50,96	25,40	40,53	9,66	23,50	50,40	21,18	27,52	26,52	45,23	96,55	44,28	21,60	7,67	28,22
19	27/04/2018	20,64	41,68	16,60	103,00	44,02	51,56	25,66	41,01	9,85	20,45	51,40	21,32	28,42	26,66	44,37	96,15	42,86	20,55	7,79	28,50
20	04/05/2018	21,66	38,44	16,32	103,20	44,76	52,44	25,62	38,15	9,95	23,70	53,30	21,70	28,90	26,46	44,30	95,20	42,54	20,35	7,97	28,42
21	11/05/2018	22,90	39,28	16,64	103,00	43,62	55,02	27,10	38,81	9,96	20,50	53,80	21,70	29,00	26,92	46,80	94,00	42,34	21,50	7,83	28,50
22	18/05/2018	23,00	39,38	16,28	109,90	42,96	53,64	27,58	37,84	10,34	18,58	52,70	21,00	29,06	26,58	46,21	95,65	43,50	17,92	7,72	28,78
23	25/05/2018	22,70	39,72	16,50	105,90	43,18	49,77	27,92	36,78	10,25	18,00	46,80	20,92	29,36	25,70	45,93	96,35	40,68	17,04	7,56	28,45
24	01/06/2018	22,06	39,58	16,08	107,50	43,00	50,16	28,08	35,75	10,00	17,12	49,90	20,66	29,48	24,34	44,66	94,05	40,70	16,72	7,53	27,70
25	08/06/2018	21,48	39,38	17,36	107,30	42,52	50,76	27,34	36,10	8,74	17,60	49,50	20,52	28,58	24,30	45,81	94,00	40,40	16,50	7,17	27,43
26	15/06/2018	21,32	41,88	17,80	103,00	44,28	49,95	28,28	36,18	8,31	16,56	49,60	20,62	29,02	24,14	44,66	95,00	39,50	17,68	7,19	27,84
27	22/06/2018	21,24	41,44	17,24	104,20	42,82	49,23	27,74	35,16	8,26	15,90	51,20	20,50	28,40	23,94	42,33	100,00	39,52	16,94	7,30	27,37
28	29/06/2018	21,38	39,10	16,72	103,30	45,46	48,57	27,70	35,75	7,88	15,92	50,80	20,40	28,54	23,36	39,45	103,40	40,00	16,02	7,14	26,29
29	06/07/2018	21,44	39,26	17,06	97,05	44,78	48,07	30,32	34,86	8,00	16,30	52,30	21,26	28,24	23,10	41,29	106,00	39,00	15,82	7,56	26,86
30	13/07/2018	21,44	38,82	18,04	101,60	45,76	47,30	31,54	34,80	7,95	16,76	50,00	22,22	29,50	23,32	39,64	108,60	40,14	16,90	7,37	27,42

Figure 3.1: Dataset 10 example

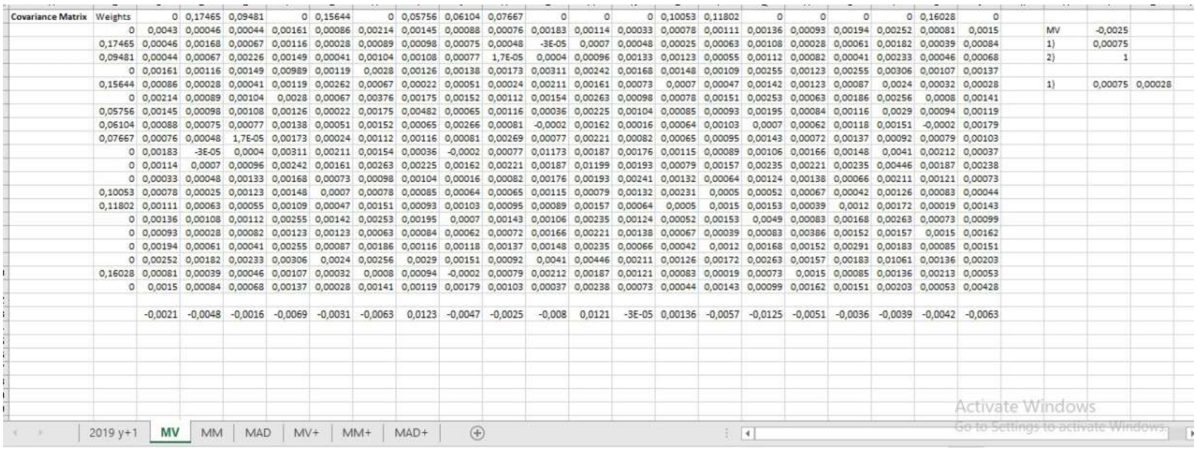


Figure 3.2: MV model, portfolio construction

By the same principle the construction of the remaining five models was made. As already said, the second year of the dataset was used for evaluation of the portfolio. In the following table we present the results of the different models and the comparison to the real ATX in those years.

	MV	MV+	MAD	MAD+	MM	MM+	ATX real
2019	0.16	0.16	0.21	0.21	0.18	0.16	0.16
2018	-0.15	-0.17	-0.10	-0.12	-0.25	-0.26	-0.22
2017	0.33	0.32	0.36	0.35	0.16	0.18	0.28
2016	0.04	0.05	0.03	0.03	-0.24	0.05	0.12
2015	0.08	0.10	0.16	0.18	0.48	0.41	0.11
2014	0.06	0.04	0.11	0.16	-0.04	-0.12	-0.15
2013	0.05	0.02	0.09	0.01	0.20	-0.03	0.02
2012	0.12	0.12	0.19	0.27	0.25	0.27	0.11
2011	-0.27	-0.27	-0.10	-0.11	-0.13	-0.15	-0.34
2010	1.04	1.02	0.80	1.04	0.28	0.15	0.13
Ø	0.15	0.14	0.18	0.20	0.08	0.07	0.02
Var	0.11	0.11	0.06	0.10	0.05	0.04	0.03

Table 3.1: Return rate of the portfolios after one year

We will show the return rate of the portfolio as a chart:

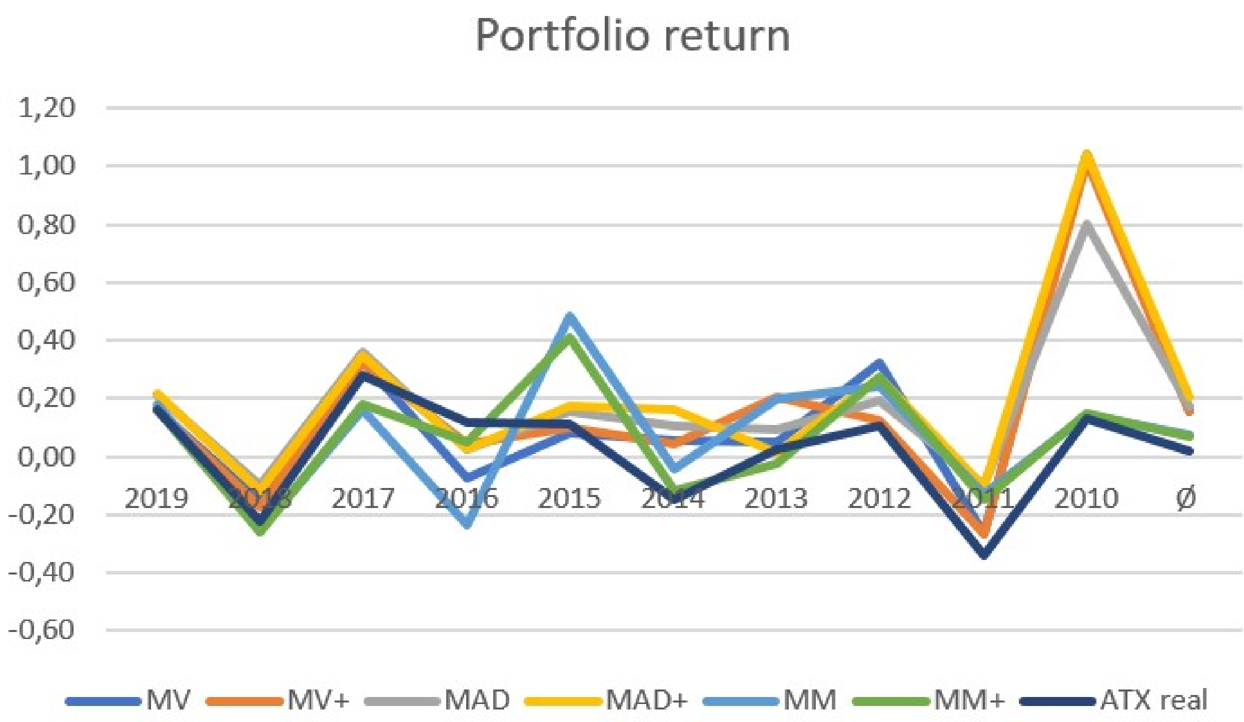


Figure 3.3: Portfolio return

As we can see, the return rate of the portfolio in the first two datasets, which were evaluate from the 2010 and 2011 data, significantly differ from other observations. The reason for it is the high volatility in the stock values in those years. If we consider the portfolio returns only from 2011/2012 - 2018/2019 datasets, we obtain following results:

	MV	MV+	MAD	MAD+	MM	MM+	ATX real
2019	0.16	0.16	0.21	0.21	0.08	0.16	0.16
2018	-0.15	-0.17	-0.10	-0.12	-0.25	-0.26	-0.22
2017	0.33	0.32	0.36	0.35	0.05	0.18	0.28
2016	0.04	0.05	0.03	0.03	-0.33	0.05	0.12
2015	0.08	0.10	0.16	0.18	0.48	0.41	0.11
2014	0.06	0.04	0.11	0.16	-0.04	-0.12	-0.15
2013	0.05	0.02	0.09	0.01	0.31	-0.03	0.02
2012	0.12	0.12	0.19	0.27	0.15	0.27	0.11
Ø	0.09	0.08	0.13	0.14	0.06	0.08	0.05
Var	0.02	0.02	0.02	0.02	0.06	0.04	0.02

Table 3.2: Return rate of the portfolios, stable years

We can see from both tables that the portfolios yield on average is higher that the return from ATX, independent of which model is used. However we can also see, particularly from the Table 3.1, that the variation of the portfolio return is higher than the variation of the ATX return. These results are in the line with the from Baumann and Trautmann presented numerical results [2]. We have also analysed the performance of return per unit of standard deviation. In this term all the models outperform ATX . From all expending models, the MAD+ model achieves the best result, followed by MV+ and MM+ model. These results differ from the Baumann and Trautmann results, where MV+ was slightly better then MAD+. In contrast to the result of Trautmann and Baumann, MM and MM+ have the lowest variance and standard deviation compared to the other models, but they are still outperformed by the other models. We can see in Table 3.3 that extended models MV+ and MM+ have a slightly worse performance than standard models and MM+ has a same performance as MM. Although the difference is smaller than it was shown in the results from Trautmann and Baumann, this is also in line with their results as well as with the value of flexibility concept in Lüthi and Doege, which is defined by the (marginal) risk absorption capacity of a technology or certain types of resources [28]. The overview of the performances of the models is given in following tables:

	MV	MV+	MAD	MAD+	MM	MM+	ATX real
Ø	0.15	0.14	0.18	0.20	0.08	0.07	0.02
Var	0.11	0.11	0.06	0.10	0.05	0.04	0.03
Std	0.34	0.33	0.25	0.32	0.22	0.20	0.18
Ø/Std	0.44	0.42	0.71	0.64	0.34	0.34	0.12

Table 3.3: Overview of the performance of the models (entire period)

	MV	MV+	MAD	MAD+	MM	MM+	ATX real
Ø	0.09	0.08	0.13	0.14	0.09	0.08	0.05
Var	0.02	0.02	0.02	0.02	0.06	0.04	0.02
Std	0.13	0.13	0.13	0.14	0.24	0.20	0.15
Ø/Std	0.68	0.62	1.04	0.95	0.39	0.42	0.34

Table 3.4: Overview of the performance of the models (without periods with extreme performances)

A possible reason for the slightly different results than in the Trautmann and Baumman model is the selection of the stocks we analyse. We decided to analyse stocks from the ATX index, which has a smaller number of stocks compared to the SPI (Swiss Performance Index) adopted by Trautmann and Baumann. There are also differences in the performance of the two indexes over the observation period. The SPI shows a smaller decline and a better recovery after the financial crisis in 2008. The comparison of both indexes can be seen in the chart below.

Figure 3.4: Comparison SPI and ATX. Source: Google Finance

4 Cumulative Prospect Theory

4.1 Prospect Theory

Prospect theory is a theory of behavioral economics and behavioral finance that was developed by Daniel Kahneman and Amos Tversky [30]. The theory explored numerous behavioral biases leading to sub-optimal decisions making. Kahneman and Tversky found that people are biased in their real estimation of probability of events happening. They tend to overweight both low and high probabilities and underweight medium probabilities [30].In their work they questioned the use of Expected Utility Theory and proposed the alternative account of choice under risk.
Decision making under risk can be viewed as a choice between prospects or gambles. A prospect $(R_1, p_1; \cdots; R_n, p_n)$ is a contract that yields outcome R_i with probability p_i, where $p_l + p_2 + ... + p_n = 1$. The application of expected utility theory to the selection between potential prospects is based on the following three principles [30].

1. **Expectation**: $u(R_1, p_1; \cdots; R_n, p_n) = p_1 u(R_1) + \cdots p_n u(R_n)$.
2. **Asset Integration**: $(R_1, p_1; \cdots; R_n, p_n)$ is acceptable at asset position w iff $u(w + R_1, p_1; \cdots; w + R_n, p_n) > u(w)$.
3. **Risk Aversion**: u is concave.

In order to doubt the Expected Utility Theory, the Prospect Theory starts with the concept of loss aversion, an asymmetric form of risk aversion, from the observation that people react differently between potential losses and potential gains. Thus, people make decisions based on the potential gain or losses relative to their specific situation (the reference point) rather than in absolute terms; this is referred to as reference dependence.

- Faced with a risky choice leading to gains, individuals are risk-averse, preferring solutions that lead to a lower expected utility but with a higher certainty (concave value function)[30] [31].

- Faced with a risky choice leading to losses, individuals are risk-seeking, preferring solutions that lead to a lower expected utility as long as it has the potential to avoid losses (convex value function).

These two examples are thus in contradiction with the expected utility theory, which only considers choices with the maximum utility. Also, the concavity for gains and convexity for losses implies diminishing marginal utility with increasing gains/losses. In other words, someone who has more money has a lower desire for a fixed amount of gain (and lower aversion to a fixed amount of loss) than someone who has less money.[30] [31]
The theory continues with a second concept, based on the observation that people attribute excessive weight to events with low probabilities and insufficient weight to events with high probability. For example, individuals may unconsciously treat an outcome with a probability of 99% as if its probability were 95%, and an outcome with probability of 1% as if it had a probability of 5%. Under- and overweighting of probabilities is importantly distinct from under- and overestimating probabilities, a different type of cognitive bias observed for example in the overconfidence effect.[30] [31]

Positive prospects				Negative prospects			
Problem 3:	(4,000, .80)	<	(3,000).	Problem 3′:	(−4,000, .80)	>	(−3,000).
$N = 95$	[20]		[80]*	$N = 95$	[92]*		[8]
Problem 4:	(4,000, .20)	>	(3,000, .25).	Problem 4′:	(−4,000, .20)	<	(−3,000, .25).
$N = 95$	[65]*		[35]	$N = 95$	[42]		[58]
Problem 7:	(3,000, .90)	>	(6,000, .45).	Problem 7′:	(−3,000, .90)	<	(−6,000, .45).
$N = 66$	[86]*		[14]	$N = 66$	[8]		[92]*
Problem 8:	(3,000, .002)	<	(6,000, .001).	Problem 8′:	(−3,000, .002)	>	(−6,000, .001).
$N = 66$	[27]		[73]*	$N = 66$	[70]*		[30]

Figure 4.1: Preferences beetween positive and negative prospects [30]

The following 5 phenomena, which violate the the standard model have been confirmed and described in various works of Tversky and Kahnemann: [30][34][35]

- **Framing effect**[34] : The rational theory of choice assuumed decription invariance: equivalent formulations of a choice problems should give rise to the same preference order. Contrary to this assumption there is evidence that variation in the framing of options, particularly in terms of gains or losses, yield systematically different preferences.

- **Nonlinear preferences**[34] : The utility of a risky prospect is linear in outcome probabilities, according to the expectation principle. In the [30] it was shown that the small and high probabilities are over and under weighted.

- **Source dependence**[34]: Willingness to bet on an uncertain event depends not only on the degree of uncertainty but also on its source. There are a lot of indication that people often prefer a bet on an event in their area of competence over a bet on a matched chance event.

- **Risk seeking**[34]: Risk aversion is generally assumed (i.e. Expected Utility Theory). However, the evidence show that risk seeking is prevalent when people must choose between a sure loss and a substantial probability of a large loss.

- **Loss aversion**

Prospect Theory distinguishes two phases in the choice process [30]:

1. early phase of editing

2. subsequent phase of evaluation.

The editing phase consists of a preliminary analysis of the offered prospects, which often yields a simpler representation of these prospects. This phase can be viewed as composed of coding, combination, segregation, cancellation, simplification and detection of dominance. In the subsequent evaluation phase, people behave as if they would compute a value (utility), based on the potential outcomes and their respective probabilities, and then choose the alternative having a higher utility.[31] The overall value of an edited prospect, denoted V, is expressed as:

$$V = \sum_{i=1}^{n} \pi(p_i) v(R_i).$$

The scale π, associates each probability p_i the decision weight $\pi(p_i)$. That reflects the impact of p on the over-all value of the prospect. However

π is not the probabilistic measure since $\pi(p) + \pi(1-p)$ is typically smaller than 1.
The second scale v assigns to each outcome R a number $v(R)$, which reflects the subjective value of that outcome. Recall that outcomes are defined relative to a reference point, which serves as the zero point of the value scale. Hence, v measures the value of deviations from that reference point, i.e., gains and losses [30].

The value function that passes through reference point is s-shaped and asymmetrical. Therefore the new utility function, S-shaped utility was introduced. It has following properties:[33]

1. increasing

2. convex on the left

3. concave on the right

4. non-differentiable at the origin

5. asymmetrical: negative events are considered worse than positive events are considered good.

It can also be written as:

$$u(R) = R^{\gamma}\mathbb{1}_{\{R\geq 0\}} - \lambda(-R)^{\gamma}\mathbb{1}_{\{R\geq 0\}},$$

for a zero benchmark level, with $\lambda > 0$ and $\gamma \in (0, 1]$.

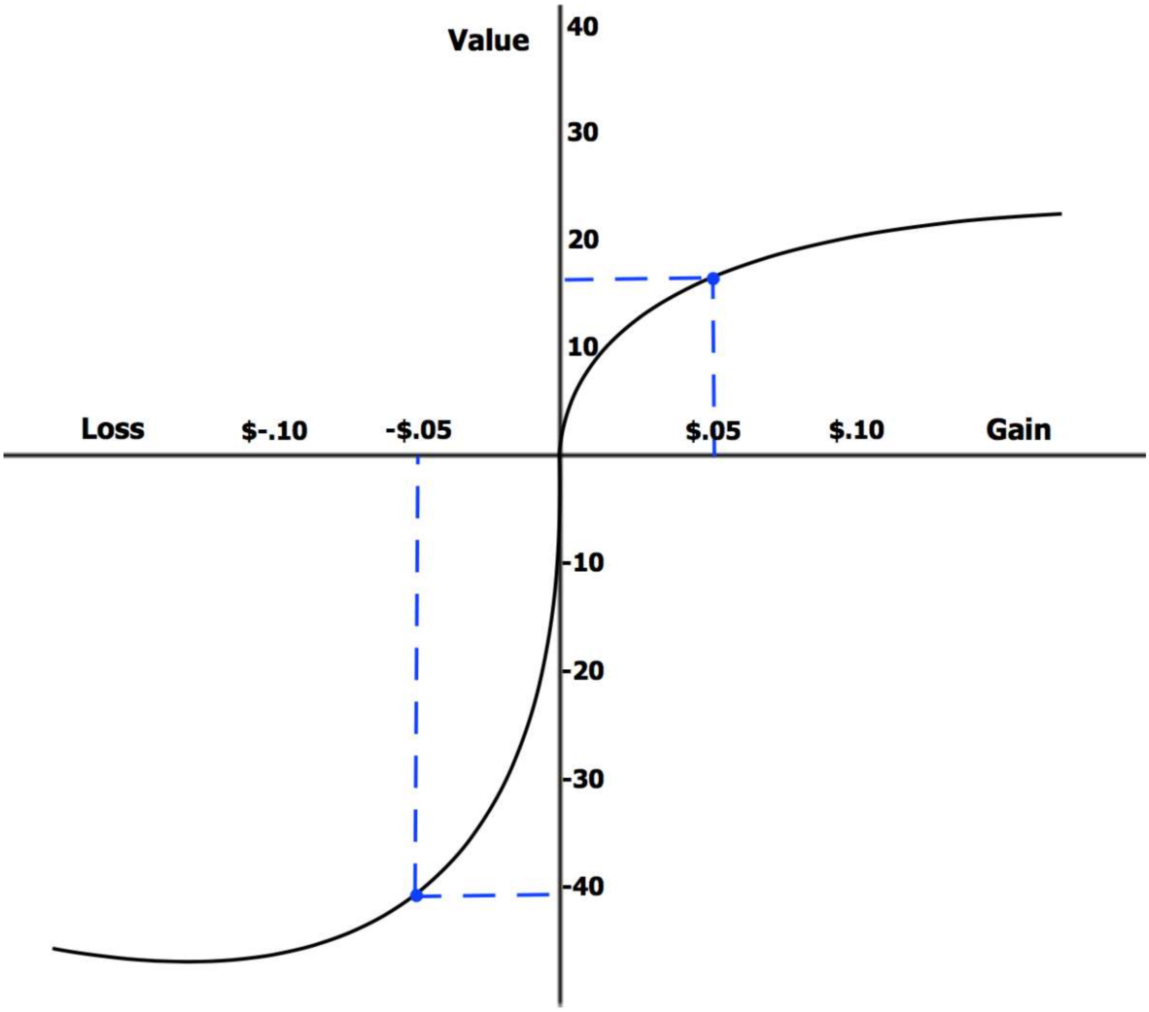


Figure 4.2: S-shaped utility [32]

4.2 Cumulative Prospect Theory

The Cumulative Prospect Theory is a further development and variant of Prospect Theory. This theory is also developed by Daniel Kahneman and Amos Tversky [34]. The theory uses cumulative rather than separable decision weights. In the original prospect theory, the utility of a prospect is the sum of utilities of the outcomes, each weighted by its probabilities [34]. The new theory suggested the two major modification[34]:

1. the carriers of value are gains and losses, not final assets.
2. the value of each outcome is multiplied by a decision weight, not by an additive probability.

Since cumulative probabilities are transformed, it leads to overweighting of extreme events which occur with small probability, rather than to an overweighting of all small probability events. The modification helps to avoid a violation of first order stochastic dominance and makes the generalization to arbitrary outcome distributions easier [37]. In the CPT the weighting function is given separately for gains and losses as w^+ and w^- respectively. The CPT value of the prospect $(R_1, p_1; \cdots; R_n, p_n)$ is given by the formula: [38][34]

$$\sum_{i=1}^{k} \pi_i^- v(R_i) + \sum_{i=k+1}^{n} \pi_i^+ v(R_i),$$

where the decision weights are defined by:

$$\pi_1^- = w^-(p_1),$$

$$\pi_i^- = w^-(p_1 + \cdots + p_i) - w^-(p_1 + \cdots + p_{i-1}),\ 2 \leq i \leq k.$$

$$\pi_n^+ = w^+(p_n),$$

$$\pi_i^+ = w^+(p_i + \cdots + p_n) - w^+(p_{i+1} + \cdots + p_n),\ k+1 \leq i \leq n-1.$$

A comparison between PT and CPT weighting functions is given below.

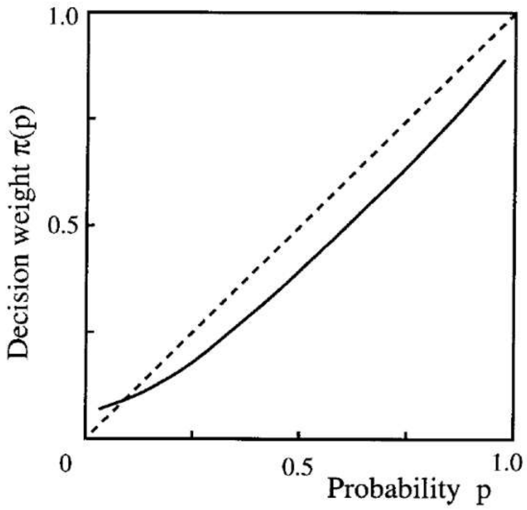


Figure 4.3: A typical weighting function for PT [38]

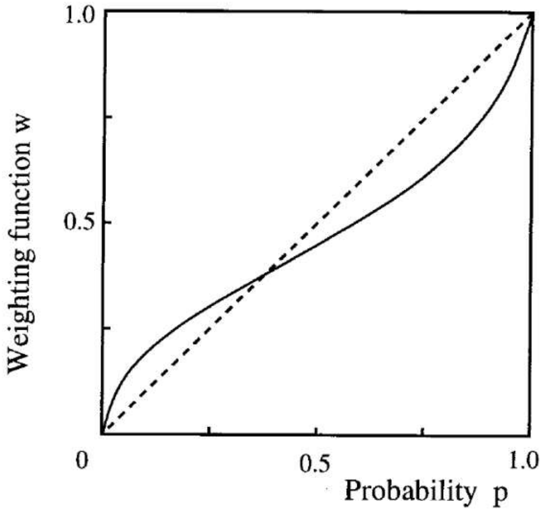


Figure 4.4: A typical CPT weighting function [38]

In the work by Deng and Pirvu [39] some of the possible solutions for the application of the portfolio optimization models under CPT were presented. They considered the multi period investment strategies with one risk free and one risky asset. The results revealed the effect of CPT investors' psychology on the optimal portfolio choice. They found out that the ratio of wealth invested in the risky asset was decreasing when the coefficient of relative risk aversion increased. Moreover the investment in the risky asset was decreasing in the risky asset volatility and increasing in the risky asset return. The effect of the model parameters on optimal strategies was slightly diminishing as time went by.[39]
At the time of writing this work, the author was not aware of any applications with integer constraints. Nevertheless we think that there is a lot of potential, particularly in the portfolio optimization strategies under CPT for small, individual investors. Therefore our recommendation is to consider CPT and make the application for the optimal portfolio under integer constraints and under cumulative prospect theory in some of the further works.

5 Conclusion

In this work we used the models, described in the paper from Baumann and Trautmann [2], as basic models for the numerical analysis. The models were extended with similar constraints as in the paper [2], particularly with integer constraint. We have used the stocks, included in ATX, in order to built relevant data sets for the analysis. All models yield on average a higher return and achieve a better risk-return ratio than the ATX. The best risk-return ratio has the MAD model from Konno and Yamazaki, what differs from the results by Baumman and Trautmann, where MV model of Markowitz outperformed MAD. Extended models MV+ and MAD+ have a slightly worse performance than the corresponding standard models, while the MM+ has the same performance as MM. Although the difference is smaller than expected, this is also in line with the results from papers and the value of flexibility concept, described by Lüthi and Doege [28]. The numerical application of the Cumulative Prospect Theory under integer constraints is left for some further research. We think that CPT has a big potential and eventually could be successfully implemented as the model for individual and small investors.

6 Notations and Conventions

- P_0, P_1 - price of the risky asset at time t_0 and time t_1 respectively....10
- R - return....10
- R_i, $i = 1, \cdots, n$ - return of a risky asset i....11
- $\overline{R}_i$ - average expected return of asset i....11
- R_0 - return of the risk free asset....13
- r - rate of return....10
- n - number of assets in portfolio....10
- B - initial budget 10
- x_i - weight of stock i....11
- σ_i^2, σ_{ii} - variance of asset i....11
- σ_{ij} - covariance between asset i and asset j....11
- l_i - risk level of the Portfolio i 13
- R_{min} - minimum acceptable return....14
- Sharpe ratio....16
- X - random variable 16
- VaR - Value at risk....16
- $CVaR$ - Conditional value at risk....16
- u - non decreasing concave function....17

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