MASTER'S THESIS

Randomized Isoperimetric Inequalities for L_p-Centroid Bodies

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by

Christoph FRITZ

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Affidavit

I declare that I have authored this thesis independently, that I have not used other than the declared sources/resources, and that I have explicitly indicated all material which has been quoted either literally or by content from the sources used.

Vienna, January 2023

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Christoph FRITZ

Abstract

This thesis is concerned with the L_p-Busemann-Petty centroid inequality, an affine isoperimetric inequality, which compares the volume of a convex body in $\mathbb{R}^n$ with that of its L_p-centroid body as an extension of the classical Busemann-Petty centroid inequality. Isoperimetric-type inequalities not only occupy a central role in the field of geometric convexity but also have numerous applications to fields such as ordinary and partial differential equations, functional analysis, the geometry of numbers, discrete geometry and polytopal approximations, stereology and stochastic geometry, and Minkowskian geometry. On the one hand, we present a direct proof of the L_p-Busemann-Petty centroid inequality by Campi and Gronchi [10] which does not use the L_p-analog of the Petty projection inequality but instead uses shadow systems. On the other hand, we present a randomized version of the same inequality due to Paouris and Pivovarov [28] using an extension of Groemer's theorem [16] to the class $\mathcal{P}_{[n]}$ of all probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure and rearrangement inequalities. Additionally, we present a randomized version of the polar L_p-Busemann-Petty centroid inequality due to Cordero-Erausquin, Fradelizi, Paouris and Pivovarov [12] which combines methods and ideas from both topics above.

Contents

Notation

We mainly follow the notation from [32].

The setting in this thesis is the n-dimensional ($n \geq 2$) real Euclidean vector space $\mathbb{R}^n$ with origin o, standard inner-product $\langle \cdot, \cdot \rangle$, standard Euclidean norm $\|\cdot\|_2$ and standard unit vector basis $e_1, \ldots, e_n$. Furthermore, we denote the orthogonal complement $A^\perp$ of a subset $A \subseteq \mathbb{R}^n$ with $A^\perp := \{x \in \mathbb{R}^n : \langle x, y \rangle = 0 \text{ for all } y \in A\}$.

We denote by $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$ the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$,

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\},$$

and by $B^n \subseteq \mathbb{R}^n$ the closed unit ball in $\mathbb{R}^n$,

$$B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}.$$

We write $\mathrm{vol}_n(\cdot)$ for the n-dimensional Lebesgue measure λ_n and denote the volume of B^n by

$$\omega_n = \mathrm{vol}_n(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})},$$

where Γ denotes the Gamma function $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$.

Additionally, we reserve D_n for the Euclidean ball of volume one, i.e., $D_n = \omega_n^{-1/n} B^n$.

The vector $x \in \mathbb{R}^n$ is a linear combination of the vectors $x_1, \ldots, x_n \in \mathbb{R}^n$ if $x = \lambda_1 x_1 + \cdots + \lambda_n x_n$ with suitable $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. If $\lambda_1 + \cdots + \lambda_n = 1$, then x is an affine combination of $x_1, \ldots, x_n$. For $A \subseteq \mathbb{R}^n$, $\mathrm{lin} A$ ($\mathrm{aff} A$) denotes the linear hull (affine hull) of A which is the set of all linear (affine) combinations of elements of A and at the same time the smallest linear subspace (affine subspace) of $\mathbb{R}^n$ containing A.

Points $x_1, \ldots, x_n \in \mathbb{R}^n$ are affinely independent if none of them is an affine combination of the others, i.e., if

$$\sum_{i=1}^k \lambda_i x_i = o \quad \text{with } \lambda_i \in \mathbb{R} \text{ and } \sum_{i=1}^k \lambda_i = 0$$

implies that $\lambda_1 = \cdots = \lambda_n = 0$.

For $A, B \subseteq \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we define

$$A + B := \{a + b : a \in A, b \in B\}, \quad \lambda A := \{\lambda a : a \in A\}$$

and write $-A$ for $(-1)A$, $A - B$ for $A + (-B)$ and $A + x$ for $A + \{x\}$, where $x \in \mathbb{R}^n$.

A hyperplane of $\mathbb{R}^n$ can be written as

$$H_{u,\alpha} = \{x \in \mathbb{R}^n : \langle x, u \rangle = \alpha\}$$

with $u \in \mathbb{R}^n \backslash \{o\}$ and $\alpha \in \mathbb{R}$. The hyperplane $H_{u,\alpha}$ bounds the two closed halfspaces

$$H^-_{u,\alpha} := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq \alpha\} \quad H^-_{u,\alpha} := \{x \in \mathbb{R}^n : \langle x, u \rangle \geq \alpha\}$$

For a subset $A \subseteq \mathbb{R}^n$ we denote the indicator function of A by $\mathbb{1}_A$, that is $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \in \mathbb{R}^n \backslash A$.

Furthermore, we denote the power set of a set A, i.e., the set of all subsets of A including the empty set and A itself, by $P(A)$.

1 Introduction

Affine isoperimetric inequalities occupy a central role in many different fields of mathematics such as geometric convexity, geometry of numbers, functional analysis and Minkowskian geometry just to name a few. Depending on the geometric bodies of interest, there are many different isoperimetric-type inequalities but, at least in convex geometry, the Euclidean ball is the extremal case in most of them. Therefore, we will focus on one specific isoperimetric-type inequality, namely the L_p-Busemann-Petty centroid inequality which compares the volume of a convex body in $\mathbb{R}^n$ with that of its L_p-centroid body and is an extension of the classical Busemann-Petty centroid inequality. It turns out that the classical inequality as well as the L_p-equivalent inequality are in close relationships with other affine isoperimetric inequalities like the Petty projection inequality or the Busemann random simplex inequality. Although, those other isoperimetric-type inequalities are not of much interest for the topic of this thesis, we still present them and their relationships to give a rough overview and motivation.

Chapter 2 and Chapter 3 introduce and recap the most important definitions, results and notation from the field of convex geometry as well as measure and probability theory. Since these chapters only serve as an entrypoint to be able to follow the next chapters, proofs are mostly omitted.

Chapter 4 deals with the L_p-Busemann-Petty centroid inequality and, in contrast to Lutwak, Yang and Zhang [23], who proved the inequality by using the L_p-analog of the Petty projection inequality, presents the direct proof given by Campi and Gronchi [10] who used shadow systems and Steiner symmetrization.

Chapter 5 extends the L_p-Busemann-Petty centroid inequality presented in the previous chapter in a randomized way following Paouris and Pivovarov [28]. They extended a theorem by Groemer [16] on the expected volume of a random polytope in a convex body and obtained the L_p-Busemann-Petty centroid inequality as an application of it.

Chapter 6 covers the dual inequality of Chapter 5, the randomized polar L_p-Busemann-Petty centroid inequality (for general measures), due to Cordero-Erausquin, Fradelizi, Paouris and Pivovarov [12] which combines methods and ideas from both Chapter 4 and Chapter 5 for the main result.

Of course, this is just a small insight into affine isoperimetric inequalities and their generalizations. Further generalizations, e.g. to Orlicz-bodies or different types of convex bodies such as spherical centroid bodies are not part of this thesis but can be found in the available literature, e.g. [3], [15], [18], [24], [25] and [27].

2 Background: Geometry

This chapter serves as an introduction and a reminder of notation, definitions and results from the field of convex geometry required for the understanding of the following chapters. Therefore, proofs are mostly omitted but more details, further results as well as complete proofs can be found in [14] and [32].

2.1 Convex Bodies

Convex bodies form the central geometrical object in this thesis which is why we start with general definitions and results about convex bodies. The starting point for this whole area of mathematics is the convex set.

Definition 2.1 (Convex Set). *A set $A \in \mathbb{R}^n$ is convex if together with any two points x, y it contains the segment $[x, y]$, thus, if*

$$(1-\lambda)x + \lambda y \in A \quad \textit{for all } x, y \in A, 0 \leq \lambda \leq 1.$$

An immediate consequence of the definition is that intersections of convex sets are convex, affine images of convex sets are convex and, if A, B are convex sets then $A + B$ and λA with $\lambda \in \mathbb{R}$ are convex.

We now restrict affine and linear combinations to non-negative coefficients.

Definition 2.2 (Convex Combination). *The point $x \in \mathbb{R}^n$ is a convex combination of the points $x_1, \dots, x_k \in \mathbb{R}^n$ if there are numbers $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that*

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \textit{with } \lambda_i \geq 0 (i = 1, \dots, k), \sum_{i=1}^{k} \lambda_i = 1.$$

Definition 2.3 (Convex Hull). *For $A \subseteq \mathbb{R}^n$, the set of all convex combinations of any finitely many elements of A is called the convex hull of A and is denoted by* $\operatorname{conv}(A)$.

Results regarding the convex hull and convex sets can be found in [32]. However, we are mostly interested in supporting halfspaces which can be used to characterize convex sets.

Definition 2.4. *Let $A \subseteq \mathbb{R}^n$ be a subset and $H \subseteq \mathbb{R}^n$ be a hyperplane and let H^+, H^- denote the two closed halfspaces bounded by H. We say that H supports A at x if*

$x \in A \cap H$ and either $A \subseteq H^+$ or $A \subseteq H^-$. Further, H is a support plane of A or supports A if H supports A at some point x, which is necessarily a boundary point of A. If $H = H_{u,\alpha}$ supports A and $A \subseteq H^-_{u,\alpha} = \{y \in \mathbb{R}^n : \langle y, u\rangle \leq \alpha\}$, then $H^-_{u,\alpha}$ is called a supporting halfspace of A and u is called an outer (or exterior) normal vector of both $H_{u,\alpha}$ and $H^-_{u,\alpha}$. If, moreover, $H_{u,\alpha}$ supports A at x, then u is an outer normal vector of A at x.

Theorem 2.5. *Let $A \subseteq \mathbb{R}^n$ be convex and closed. Then through each boundary point of A there is a support plane of A. If $A \neq \emptyset$ is bounded, then for each vector $u \in \mathbb{R}\backslash\{o\}$ there is a support plane to A with outer normal vector u.*

Theorem 2.6. *Let $A \subseteq \mathbb{R}^n$ be a closed set such that $\mathrm{int}A \neq \emptyset$ and such that through each boundary point of A there is a support plane to A. Then A is convex.*

Therefore, the last two theorems yield the following characterization of convex sets.

Corollary 2.7. *Every nonempty closed convex set in $\mathbb{R}^n$ is the intersection of its supporting halfspaces.*

We close this section with the main definition and geometrical object of this thesis. Of course, all results presented above apply to convex bodies as well.

Definition 2.8 (Convex Body)**.** *A nonempty, compact, convex subset of $\mathbb{R}^n$ is called a convex body. The set of all convex bodies is denoted by $\mathcal{K}^n$.*

2.2 Convex Functions

The investigation of convex sets or convex bodies is closely linked to convex functions which play an important role throughout this thesis. We only present basic properties and results in this section and refer, again, to [32] for more details.

Definition 2.9 (Convex Function)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in \mathbb{R}^n$ and for $0 \leq \lambda \leq 1$.

A closely related concept to convex functions are the concave functions.

Definition 2.10 (Concave Function)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called concave if*

$$f((1-\lambda)x + \lambda y) \geq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in \mathbb{R}^n$ and for $0 \leq \lambda \leq 1$.

An important subset of convex functions are so-called sublinear functions which must satisfy two properties. It turns out that sublinear functions play a vital role in the description of convex bodies.

Definition 2.11 (Sublinear Functions)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called positively homogeneous if*

$$f(\lambda x) = \lambda f(x) \quad \text{for all } \lambda \geq 0 \text{ and all } x \in \mathbb{R}^n,$$

and f is called subadditive if

$$f(x+y) \leq f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}^n.$$

A function that is positively 1-homogeneous and subadditive is called sublinear.

In addition to recalling the definition of a convex function, we state two useful and well-known criteria for differentiable functions of one variable to be convex.

Theorem 2.12. *Let $D \subseteq \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ a differentiable function. Then f is convex if and only if f' is increasing.*

Corollary 2.13. *Let $D \subseteq \mathbb{R}$ be an open interval and $f : D \to \mathbb{R}$ a twice differentiable function. Then f is convex if and only if $f'' \geq 0$.*

2.3 Support Functions

There are many ways in which a convex body can be described by real functions but, especially in the Brunn-Minkowski theory and convex geometry, the concept of the support function is fundamental. We have seen in Corollary 2.7 that a closed convex set is the intersection of its supporting halfspaces. Therefore, it can be described by specifying the position of its support planes by their outer normal vectors. This is exactly the support function.

Definition 2.14 (Support Function)**.** *Let $K \subseteq \mathbb{R}^n$ be a closed convex set with $\emptyset \neq K \neq \mathbb{R}^n$. The support function $h(K, \cdot)$ of K is defined by*

$$h(K, u) := \sup\{\langle z, u \rangle : z \in K\}, \quad \text{for } u \in \mathbb{R}^n.$$

When working with nonempty, compact convex sets (convex bodies) we can replace the supremum with the maximum. We can also extend the notions of support plane and supporting halfspace of K in the following way. For $u \in \mathbb{R}^n \backslash \{o\}$,

$$H(K, u) := \{x \in \mathbb{R}^n : \langle x, u \rangle = h(K, u)\},$$
$$H^-(K, u) := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h(K, u)\},$$

each with outer normal vector u.

Therefore, we have

$$x \in K \Leftrightarrow \langle x, u \rangle \leq h(K, u)$$

which means that for a unit vector $u \in \mathbb{R}^n \backslash \{o\}$ the number $h(K, u)$ is the signed distance of the support plane to K with outer normal vector u from the origin. The distance is negative if and only if u points into the open halfspace containing the origin.

The support function of a convex body K is continuous but also convex and sublinear.

Proposition 2.15. *Let $K \in \mathcal{K}^n, \lambda \geq 0$ and $u, v \in \mathbb{R}^n$, then*

1. $h(K, \lambda u) = \lambda h(K, u)$,
2. $h(K, u + v) \leq h(K, u) + h(K, v)$.

This sublinearity is already enough to characterize a support function.

Theorem 2.16. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a sublinear function, then there is a unique convex body $K \in \mathcal{K}^n$ with support function f.*

Another important property of the support function is its additive behavior in the first argument.

Theorem 2.17. *For $K, L \in \mathcal{K}^n$ one has*

$$h(K + L, \cdot) = h(K, \cdot) + h(L, \cdot).$$

2.4 The space of convex bodies $\mathcal{K}^n$

We now shift our attention to the space of convex bodies $\mathcal{K}^n$ and its structures.

The additivity of the support function shows that the equality $K + M = L + M$ for convex bodies $K, L, M \in \mathcal{K}^n$ implies $K = L$ which makes $(\mathcal{K}^n, +)$ a commutative semigroup with cancellation law. Furthermore, multiplication with nonnegative real numbers satsifies the rules $\lambda(K + L) = \lambda K + \lambda L, (\lambda + \mu)K = \lambda K + \mu K, \lambda(\mu K) = (\lambda\mu)K, 1K = K$.

We are now able to equip $\mathcal{K}^n$ with a metric.

Definition 2.18 (Hausdorff Distance)**.** *The Hausdorff distance of the sets $K, L \in \mathcal{K}^n$ is defined by*

$$\delta(K, L) := \max \left\{ \max_{x \in K} \min_{y \in L} \|x - y\|, \max_{x \in L} \min_{y \in K} \|x - y\| \right\}$$

or, equivalently, by

$$\delta(K, L) = \min\{\lambda \geq 0 : K \subseteq L + \lambda B^n, L \subseteq K + \lambda B^n\}.$$

Due to Blaschke's selection theorem [32, Theorem 1.8.7], $(\mathcal{K}^n, \delta)$ is a locally compact complete metric space. Since support functions are 1-homogenous we can restrict them to the sphere $\mathbb{S}^{n-1}$. This brings us to following fundamental theorem which is used to verify convergence of convex bodies in the Hausdorff metric.

Theorem 2.19. *The map $\phi : \mathcal{K}^n \to \mathbf{C}(\mathbb{S}^{n-1}), K \mapsto h(K, \cdot)$ is an isomorphic embedding of $\mathcal{K}^n$ into the Banach space $\mathbf{C}(\mathbb{S}^{n-1})$ of continuous functions in $\mathbb{S}^{n-1}$ by*

$$\delta(K, L) = \|h(K, \cdot) - h(L, \cdot)\|_\infty$$

for $K, L \in \mathcal{K}^n$.

We close this section with the fact that pointwise convergence of support functions implies uniform convergence and that the volume functional $\mathrm{vol}_n(\cdot)$ is continuous on $\mathcal{K}^n$.

Theorem 2.20. *If a sequence of support functions converges pointwise (on $\mathbb{R}^n$ or, equivalently, on $\mathbb{S}^{n-1}$), then it converges uniformly on $\mathbb{S}^{n-1}$ to a support function.*

Theorem 2.21. *The volume functional $vol_n(\cdot)$ is continuous on $\mathcal{K}^n$.*

2.5 Centroid, Projection and Polar Bodies

With the definitions and results from the previous chapter we are now able to define some special convex bodies which are needed to be able to present some affine isoperimetric inequalities in the next section.

Definition 2.22 (Centroid Body). *The centroid body $\Gamma(K)$ of $K \in \mathcal{K}^n$ containing the origin o is the convex body whose support function is*

$$h(\Gamma(K), x) = \frac{1}{\mathrm{vol}_n(K)} \int_K |\langle x, y \rangle| dy \quad \text{for } x \in \mathbb{R}^n.$$

One interesting property of an origin symmetric body K is that the boundary of $\Gamma(K)$ is the locus of the centroids of all the halves of K obtained by cutting K with hyperplanes through the origin.

In order to be able to define projection bodies we first need to introduce the concept of mixed volumes.

Definition 2.23 (Mixed Volume). *The mixed volume of $K_1, \dots, K_n \in \mathcal{K}^n$ is defined by*

$$V(K_1, \dots, K_n) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{k+n} \sum_{1 \le i_1 < \dots < i_m \le n} V(K_{i_1}) + \dots + V(K_{i_m}).$$

The following theorem lets us define a finite measure on $\mathbb{S}^{n-1}$, namely the mixed area measure of $K_1, \ldots, K_{n-1}$, which we can use to define the surface area measure of a convex body K.

Theorem 2.24. *Let $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$. There exists a finite measure $S(K_1, \ldots, K_{n-1}, \cdot)$ on $\mathbb{S}^{n-1}$, called the mixed area measure of $K_1, \ldots, K_{n-1}$, such that*

$$V(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u) dS(K_1, \ldots, K_{n-1}, u)$$

for all $K \in \mathcal{K}^n$.

Definition 2.25 (Surface Area Measure)**.** *The surface area measure $S_{n-1}(K, \cdot)$ of a convex body $K \in \mathcal{K}^n$ is the measure on $\mathbb{S}^{n-1}$ defined by*

$$S_{n-1}(K, \cdot) := S(K, \ldots, K, \cdot).$$

Definition 2.26 (Projection Body)**.** *The projection body ΠK of $K \in \mathcal{K}^n$ is the convex body whose support function is*

$$h(\Pi K, v) = V_{n-1}(K|v^{\perp}) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| dS_{n-1}(K, u) \quad \text{for } v \in \mathbb{S}^{n-1}$$

We finish this section with the definition of the polar body K° of a convex body K. In the dual concept of a convex set the origin o plays a role which is why we denote by $\mathcal{K}_o^n$ the set of all convex bodies in $\mathcal{K}^n$ containing o, and by $\mathcal{K}_{(o)}^n$ the subset of all convex bodies with o as an interior point.

Definition 2.27 (Polar Body)**.** *For a convex body $K \in \mathcal{K}^n$ with $o \in \operatorname{int} K$ the polar body K° is defined as*

$$K^\circ = \{u \in \mathbb{R}^n : \langle u, v \rangle \leq 1 \text{ for all } v \in K\}.$$

It turns out that for $K \in \mathcal{K}_{(o)}^n$ the set K° is in fact a true dual object.

Theorem 2.28. *Let $K \in \mathcal{K}_{(o)}^n$. Then $K^\circ \in \mathcal{K}_{(o)}^n$ and $K^{\circ\circ} = K$.*

2.6 Some Affine Isoperimetric Inequalities

We are now able to give a rough overview of some of the most important affine isoperimetric inequalities and the relations between them. We mainly follow the survey article by Lutwak [21].

We start with the Blaschke-Groemer inequality which can be stated as follows.

Theorem 2.29 (Blaschke-Groemer Inequality)**.** *If K is a convex body with nonempty interior in $\mathbb{R}^n$, then*

$$\frac{1}{\mathrm{vol}_n(K)^{n+1}} \int_K \cdots \int_K [x_0, x_1, \ldots, x_n] dx_0 dx_1 \cdots dx_n \geq \omega_n \mathrm{vol}_n(K),$$

where $[x_0, x_1, \ldots, x_n]$ denotes the volume of the simplex in $\mathbb{R}^n$ whose vertices are the points $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$. Equality holds if and only if K is an ellipsoid.

Closely related to this inequality is the Busemann random simplex inequality.

Theorem 2.30 (Busemann Random Simplex Inequality)**.** *If K is a convex body with nonempty interior in $\mathbb{R}^n$ and $x_0 \in K$, then*

$$\frac{1}{\mathrm{vol}_n(K)^n} \int_K \cdots \int_K [x_0, x_1, \ldots, x_n] dx_1 \cdots dx_n \geq \frac{2\omega_{n+1}^{n-1}}{(n+1)!\omega_n^{n+1}} \mathrm{vol}_n(K),$$

where equality holds if and only if K is an ellipsoid centered at x_0.

Petty was able to give an integral representation of the mixed volume of centroid bodies and, as a special case, the volume of the centroid body $\Gamma(K)$ of $K \in \mathcal{K}_o^n$ as

$$\mathrm{vol}_n(\Gamma(K)) = \frac{2^n}{\mathrm{vol}_n(K)^n} \int_K \cdots \int_K [0, x_1, \ldots, x_n] dx_1 \cdots dx_n$$

which ultimately allowed Petty to reformulate the Busemann random simplex inequality as the Busemann-Petty centroid inequality.

Theorem 2.31 (Busemann-Petty Centroid Inequality)**.** *If $K \in \mathcal{K}_o^n$, then*

$$\mathrm{vol}_n(\Gamma(K)) \geq \left(\frac{2\omega_{n-1}}{(n+1)\omega_n} \right)^n \mathrm{vol}_n(K),$$

where equality holds if and only if K is an origin symmetric ellipsoid.

Another fundamental affine isoperimetric inequality is the Petty Projection Inequality.

Theorem 2.32 (Petty Projection Inequality)**.** *If K is a convex body with nonempty interior in $\mathbb{R}^n$, then*

$$\mathrm{vol}_n(K)^{n-1} \mathrm{vol}_n(\Pi^\circ(K)) \leq \left(\frac{\omega_n}{\omega_{n-1}} \right)^n,$$

where equality holds if and only if K is an ellipsoid.

Here $\Pi^{\circ}(K)$ is the polar body of the projection body $\Pi(K)$ of $K \in \mathcal{K}^n$. Petty [30] proved that the Busemann-Petty centroid inequality implies the Petty projection inequality. Conversly, Lutwak [20] proved that the Petty projection inequality implies the Busemann-Petty centroid inequality.

In the following chapters we present generalizations of the Busemann-Petty centroid inequality in the form of the L_p-Busemann-Petty centroid inequality, the randomized version of the L_p-Busemann-Petty centroid inequality and the randomized polar L_p-Busemann-Petty centroid inequality for general measures.

2.7 Shadow Systems and Steiner Symmetrization

Another important concept that we need in later chapters is that of shadow systems (or linear parameter systems) introduced by Rogers and Shephard [31] [33].

Definition 2.33 (Shadow System). *A shadow system (or a linear parameter system) along the direction v is a family of convex bodies $K_t \subseteq \mathbb{R}^n$ that can be defined by*

$$K_t = \operatorname{conv}\{z + \alpha(z)tv : z \in A \subseteq \mathbb{R}^n\},$$

where A is an arbitrary bounded set of points, α is a real bounded function on A, and the parameter t runs in an interval of the real axis.

One particular type of shadow system are parallel chord movements.

Definition 2.34. *A parallel chord movement along the direction v is a family of convex bodies K_t in $\mathbb{R}^n$ defined by*

$$K_t = \{z + \beta(x)tv : z \in K, x = z - \langle z, v\rangle v\},$$

where K is a convex body in $\mathbb{R}^n$, β is a continuous real function on $v^{\perp}$ and the parameter t runs in an interval of the real axis, say $t \in [0,1]$.

Basically, to each chord of $K = K_0$ parallel to the direction v we assign a speed vector $\beta(x)v$, where x is the projection of the chord onto $v^{\perp}$. We then let the chords move for a time t and denote by K_t their union. The only restriction when defining the speed function β is that the union K_t has to be convex.

We close this chapter with one of the most important and heavily used symmetrization procedures for convex bodies - the Steiner Symmetrization.

Definition 2.35 (Steiner Symmetrization). *Let $v^{\perp} \in \mathbb{R}^n$ be a hyperplane and let $K \subseteq \mathcal{K}^n$. The Steiner symmetral of K with respect to $v^{\perp}$ is the set $S_{v^{\perp}}K$ with the property that, for each line v orthogonal to $v^{\perp}$ and meeting K, the set $v \cap S_{v^{\perp}}K$ is a closed segment with midpoint on $v^{\perp}$ and length equal to that of the set $v \cap K$. The mapping $S_{v^{\perp}} : K \to S_{v^{\perp}}K$ is the Steiner symmetrization with respect to $v^{\perp}$.*

It turns out that the movement related to the Steiner Symmetrization is a special instance of a shadow system. If we fix a direction v, then a convex body can be written as

$$K = \{x + yv \in \mathbb{R}^n : x \in K|v^{\perp}, y \in \mathbb{R}, f(x) \leq y \leq g(x)\}$$

where f and $-g$ are convex functions on $K|v^{\perp}$. The parallel chord movement with speed function $\beta(x) = -(f(x) + g(x))$ and $t \in [0,1]$ is such that $K_0 = K$ and $K_1 = K^v$, the reflection of K in the hyperplane $v^{\perp}$, and $K_{1/2}$ is the Steiner symmetral of K with respect to $v^{\perp}$.

3 Background: Measure and Probability Theory

Similarly to Chapter 2, this chapter serves as an overview of important definitions and results needed for the following chapters. We follow [13] and refer to it for more details and proofs.

3.1 Measure Theory

In order to be able to define measure spaces and probability spaces we need some standard definitions of additive functions.

Definition 3.1 (Finitely Additive Function)**.** *Let X be a set, $P(X)$ its power set and $\mathcal{C} \subseteq P(X)$ a collection of subsets. A function μ from $\mathcal{C}$ into $[-\infty, \infty]$ is said to be finitely additive if $\mu(\emptyset) = 0$ and whenever $A_i \in \mathcal{C}$ are disjoint for $i = 1, \ldots, n$, and*

$$A := \bigcup_{i=1}^{n} A_i \in \mathcal{C}, \quad \textit{we have} \quad \mu(A) = \sum_{i=1}^{n} \mu(A_i).$$

Definition 3.2 (Countably Additive Function)**.** *Let X be a set, $P(X)$ its power set and $\mathcal{C} \subseteq P(X)$ a collection of subsets. A function μ from $\mathcal{C}$ into $[-\infty, \infty]$ is said to be countably additive if $\mu(\emptyset) = 0$ and whenever $A_n \in \mathcal{C}$ are disjoint for $n = 1, \ldots,$ and*

$$A := \bigcup_{n \geq 1} A_n \in \mathcal{C}, \quad \textit{we have} \quad \mu(A) = \sum_{n \geq 1} \mu(A_n).$$

Furthermore, we need the concept of σ-algebras as the underlying structures.

Definition 3.3 (σ-Algebra)**.** *Let X be a set and $P(X)$ its power set. A collection $\mathcal{A} \subseteq P(X)$ is called a ring if $\emptyset \in \mathcal{A}$ and for all A and B in $\mathcal{A}$, we have $A \cup B \in \mathcal{A}$ and $B \setminus A \in \mathcal{A}$. A ring $\mathcal{A}$ is called an algebra if $X \in \mathcal{A}$. An algebra $\mathcal{A}$ is called a σ-algebra if for any sequence $(A_n)_{n \in \mathbb{N}}$ of sets in $\mathcal{A}$, $\bigcup_{n \geq 1} A_n \in \mathcal{A}$.*

One of the most important σ-algebras is the Borel σ-algebra containing Borel sets.

Definition 3.4 (Borel Sets). *The σ-algebra generated by the open sets of $\mathbb{R}^n$, that is the intersection of all the σ-algebras containing the family of open sets, is called the Borel σ-algebra. The sets in it are called Borel sets.*

Definition 3.5 (Measure Space). *A countably additive function μ from a σ-algebra $\mathcal{S}$ of subsets of the set X into $[0,\infty]$ is called a measure. Then $(X,\mathcal{S},\mu)$ is called a measure space.*

We also need the notion of measurable functions and product measures for the subsequent fundamental theorems from measure theory.

Definition 3.6 (Measurable Function). *If $(X,\mathcal{S})$ and $(Y,\mathcal{B})$ are measure spaces and f is a function from X into Y, then f is called measurable if $f^{-1}(B) \in \mathcal{S}$ for all $B \in \mathcal{B}$.*

Theorem 3.7 (Dominated Convergence). *Let f_n and g be in $L^1(X,\mathcal{S},\mu)$, $|f_n(x)| \leq g(x)$ and $f_n(x) \to f(x)$ for all x. Then $f \in L^1$ and $\int f_n d\mu \to \int f d\mu$.*

Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be any two measure spaces. In $X \times Y$ let $\mathcal{R}$ be the collection of all $B \times C$ with $B \in \mathcal{B}$ and $C \in \mathcal{C}$. For such sets let $\rho(B \times C) := \mu(B)\nu(C)$. It can be shown (see [13, Theorem 4.4.4]) that ρ extends uniquely to a measure on $\mathcal{B} \times \mathcal{C}$ and is called the product measure $\mu \times \nu$.

Theorem 3.8 (Product Measure Existance Theorem). *Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be two σ-finite measure spaces. Then ρ extends uniquely to a measure on $\mathcal{B} \otimes \mathcal{C}$ such that for all $E \in \mathcal{B} \otimes \mathcal{C}$,*

$$\rho(E) = \int \int \mathbb{1}_E(x,y)d\mu(x)d\nu(y) = \int \int \mathbb{1}_E(x,y)d\nu(y)d\mu(x).$$

This brings us to Fubinis theorem, the most important theorem on integrals for product measures.

Theorem 3.9 (Fubini). *Let $(X,\mathcal{B},\mu)$ and $(Y,\mathcal{C},\nu)$ be σ-finite and let f be a function from $X \times Y$ into $[0,\infty]$ measurable for $\mathcal{B} \otimes \mathcal{C}$, or $f \in L^1(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \times \nu)$. Then*

$$\int f d(\mu \times \nu) = \int \int f(x,y)d\mu(x)d\nu(y) = \int \int f(x,y)d\nu(y)d\mu(x).$$

Here $\int f(x,y)d\mu(x)$ is defined for ν-almost all y and $\int f(x,y)d\nu(y)$ for μ-almost all x.

Another standard result is the layer cake formula or layer cake representation which can be stated as follows (see [19, Theorem 1.13]).

Theorem 3.10 (Layer Cake Formula). *Let ν be a measure on the Borel sets of the positive real line $[0,\infty)$ such that*

$$\phi(t) := \nu([0,t))$$

is finite for every $t > 0$. Now let $(\Omega, \mathcal{S}, \mu)$ be a measure space and f any nonnegative measurable function on Ω. Then

$$\int_\Omega \phi(f(x))d\mu(x) = \int_0^\infty \mu(\{x : f(x) > t\})d\nu(t).$$

In particular, by choosing $d\nu(t) = pt^{p-1}dt$ for $p > 0$, we have

$$\int_\Omega f(x)^p d\mu(x) = p\int_0^\infty t^{p-1}\mu(\{x : f(x) > t\})dt.$$

3.2 L^p Spaces

In this section we quickly recall the concept of L^p spaces and, especially, the Minkowski inequality which finds heavy use in Chapter 4.

Definition 3.11 (L^p Space)**.** *For any measure space $(X, \mathcal{S}, \mu)$ and $0 < p < \infty$, $L^p(X, \mathcal{S}, \mu) := L^p(X, \mathcal{S}, \mu, \mathbb{R})$ denotes the set of all measurable functions f on X such that $\int |f|^p d\mu < \infty$ and the values of f are real numbers except possibly on a set of measure 0, where f may be undefined or infinite. For $1 \leq p < \infty$, $\|f\|_p := (\int |f|^p d\mu)^{1/p}$, s called the L^p norm or L_p-norm of f.*

We close this very short section with two useful inequalities for L_p-norms.

Theorem 3.12. *For any integrable function f with values in $\mathbb{R}$, $|\int f d\mu| \leq \int |f| d\mu$.*

Theorem 3.13 (Minkowski Inequality)**.** *For $1 \leq p \leq \infty$, if f and g are in $L^p(X, \mathcal{S}, \mu)$, then $f + g \in L^p(X, \mathcal{S}, \mu)$ and $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

3.3 Probability Theory

Probability spaces are measure spaces with a special, normalized measure called probability measure.

Definition 3.14 (Probability Space)**.** *A measure space $(\Omega, \mathcal{S})$ is a set Ω with a σ-algebra $\mathcal{S}$ of subsets of Ω. A probability measure $\mathbb{P}$ is a measure on $\mathcal{S}$ with $\mathbb{P}(\Omega) = 1$. $(\Omega, \mathcal{S}, \mathbb{P})$ is called a probability space.*

Members of $\mathcal{S}$ are called measurable sets, e.g., Borel sets, in measure theory and events in a probability space. Furthermore, if $(\Omega, \mathcal{S}, \mathbb{P})$ is a probability space and $(A, \mathcal{B})$ any measure space, a measurable function $X : \Omega \to A$ is called a random variable. The image measure $\mathbb{P} \circ X^{-1}$ defined on $\mathcal{B}$ is also a probability measure and called law of X, or $\mathcal{L}(X)$. Random variables X_j are identically distributed if $\mathcal{L}(X_n) = \mathcal{L}(X_1)$ for all n. Lastly, two events A and B are called independet for a probability measure $\mathbb{P}$ if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

Definition 3.15 (Expectation). *The expectation or mean $\mathbb{E}(X)$ of a real-valued random variable X is defined as $\int X d\mathbb{P}$ if and only if the integral exists.*

Like any integral, the expectation is linear.

Theorem 3.16. *For any two random variables X and Y such that the expectations $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ are both defined and finite, and any constant c, $\mathbb{E}(cX+Y) = c\mathbb{E}(X) + \mathbb{E}(Y)$.*

Let $(X, \mathcal{S}, \mathbb{P})$ be a probability space and $X_1, X_2, \dots$ be real random variables on X. Let $S_n := X_1 + \cdots + X_n$. Any event with probability 1 is said to happen almost surely (a.s.). Therefore, a sequence Y_n of random variables is said to converge almost surely to a random variable Y if $\mathbb{P}(Y_n \to Y) = 1$.

This brings us to the strong law of large numbers which plays a vital role in the last two chapters.

Theorem 3.17 (Strong Law of Large Numbers). *For independent, identically distributed real X_j, if $\mathbb{E}(X_1) < \infty$, then the strong law of large numbers holds, that is, $S_n/n \to \mathbb{E}(X_1)$ almost surely. If $\mathbb{E}(X_1) = +\infty$, then almost surely S_n/n does not converge to any finite limit.*

The strong law of large numbers can also be stated for random compact sets [2].

Definition 3.18. *A selection from the random set X is a random vector x such that $x \in X$ almost surely. Let X be a random set such that each selection has finite expectation $\mathbb{E}(x)$. The expectation of X, written $\mathbb{E}(X)$, is the set $\{\mathbb{E}(x) : x \text{ is a selection of } X\}$.*

Theorem 3.19. *Let $X_i, i = 1, 2, \dots$ be independent, identically distributed random sets such that $\mathbb{E}(X_i) < \infty$. Then*

$$S_N = \frac{X_1 + \cdots X_N}{N} \to \mathbb{E}(\mathrm{conv}(X)) \quad \textit{(a.s.)}.$$

3.4 Rearrangements of Functions

Typically, Steiner symmetrization is used on $\mathcal{K}^n$ but, especially in the last two chapters, we make use of rearrangement inequalities. Therefore, we present some definitions and results concerning the rearrangement of functions in this section but refer to [8] and [19] for further material on this subject.

Definition 3.20. *Let A be a Borel subset of $\mathbb{R}^n$ with finite Lebesgue measure. The symmetric rearrangement A^* of A is the open ball with center at the origin, whose volume is equal to the measure of A. Since we choose A^* to be open, $\mathbb{1}_{A^*}$ is lower semicontinuous. The symmetric decreasing rearrangement of $\mathbb{1}_A$ is defined by*

$$\mathbb{1}_A^* = \mathbb{1}_{A^*}.$$

We only consider Borel measurable functions $f : \mathbb{R}^n \to \mathbb{R}^+$ which satisfy the following condition: for every $t > 0$, the set $\{x \in \mathbb{R}^n : f(x) > t\}$ has finite Lebesgue measure. For such functions, we say that f vanishes at infinity.

Definition 3.21 (Symmetric Decreasing Rearrangement)**.** *In the case that f vanishes at infinity, the symmetric decreasing rearrangement f^* is defined by*

$$f^*(x) = \int_0^\infty \mathbb{1}^*_{\{f>t\}}(x)dt = \int_0^\infty \mathbb{1}_{\{f>t\}^*}dt.$$

The Steiner symmetrization for convex sets was already introduced in Section 2.7 but we also define it here for measurable functions vanishing at infinity.

Definition 3.22 (Steiner Symmetrization for Measurable Functions)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^+$ be a measurable function vanishing at infinity. For $\theta \in \mathbb{S}^{n-1}$, we fix a coordinate system such that $e_1 := \theta$. The Steiner symmetrization $f^*(\cdot|\theta)$ of f with respect to $\theta^\perp$ is defined as follows: for $x_2, \dots, x_n \in \mathbb{R}$, we set $h(t) = f(t, x_2, \dots, x_n)$ and define*

$$f^*(t, x_2, \dots, x_n|\theta) := h^*(t).$$

In other words, $f^*(\cdot|\theta)$ is obtained by rearranging f along every line parallel to θ.

We are now able to give an overview of results related to the Rogers/Brascamp-Lieb-Luttinger rearrangement inequality proved in [7].

Theorem 3.23 (Rogers/Brascamp-Lieb-Luttinger Rearrangement Inequality)**.** *Let $f_1, \dots, f_M : \mathbb{R} \to \mathbb{R}^+$ be non-negative measurable functions. Let $u_1, \dots, u_M \in \mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, u_i \rangle)dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^m f_i^*(\langle x, u_i \rangle)dx.$$

For symmetric convex sets $K = -K \subseteq \mathbb{R}^n$ we obtain the following

Corollary 3.24. *Let K be a symmetric convex set in $\mathbb{R}^n$. Suppose that $f_1, \dots, f_n$ are non-negative measurable functions defined on $\mathbb{R}$. Then*

$$\int_K \prod_{i=1}^n f_i(x_i)dx \leq \int_K \prod_{i=1}^m f_i^*(x_i)dx.$$

With the definition of quasi-concave and quasi-convex functions, we can show an immediate but important consequence of the above corollary. The proof is taken from [28].

Definition 3.25 (Quasi-Concave Function)**.** *We say that $F : \mathbb{R}^N \to \mathbb{R}^+$ is quasi-concave if for all s, the set $\{x : F(x) > s\}$ is convex.*

Definition 3.26 (Quasi-Convex Function). *We say that $F : \mathbb{R}^N \to \mathbb{R}^+$ is quasi-convex if for all s, the set $\{x : F(x) < s\}$ is convex.*

Corollary 3.27. *Let $F : \mathbb{R}^N \to \mathbb{R}^+$ be an even quasi-concave function and g_i be real non-negative integrable functions. Then*

$$\int_{\mathbb{R}^N} F(t) g_1(t_1) \cdots g_N(t_N) dt \leq \int_{\mathbb{R}^N} F(t) g_1^*(t_1) \cdots g_N^*(t_N).$$

If $F : \mathbb{R}^N \to \mathbb{R}^+$ is even and quasi-convex then

$$\int_{\mathbb{R}^N} F(t) g_1(t_1) \cdots g_N(t_N) dt \geq \int_{\mathbb{R}^N} F(t) g_1^*(t_1) \cdots g_N^*(t_N).$$

Proof. For $s \geq 0$ let $K(s) := \{x : F(x) > s\}$. Then $K(s)$ is symmetric and convex. Using the Layer Cake Formula (see Theorem 3.10), Fubini's Theorem (see 3.9) and Corollary 3.24, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} F(t) g_1(t_1) \cdots g_N(t_N) dt &= \int_0^\infty \int_{K(s)} g_1(t_1) \cdots g_N(t_N) dt ds \\
&\leq \int_0^\infty \int_{K(s)} g_1^*(t_1) \cdots g_N^*(t_N) dt ds \\
&= \int_{\mathbb{R}^N} F(t) g_1^*(t_1) \cdots g_N^*(t_N) dt.
\end{aligned}$$

The second assertion follows from the fact that $\mathbb{1}_{\{F \leq s\}} + \mathbb{1}_{\{F > s\}} = 1$. □

4 The L_p-Busemann-Petty Centroid Inequality

In this chapter we start with our discussion of the L_p-Busemann-Petty centroid inequality which states that the ratio between the volume of the L_p-centroid body of a convex body K in $\mathbb{R}^n$ and the volume of K attains its minimum if and only if K is an origin symmetric ellipsoid.

Theorem 4.1 (L_p-Busemann-Petty Centroid Inequality). *If K is a convex body with nonempty interior in $\mathbb{R}^n$, then for $1 \leq p < \infty$*

$$\mathrm{vol}_n(\Gamma_p(K)) \geq \mathrm{vol}_n(K),$$

where equality holds if and only if K is an origin symmetric ellipsoid.

This inequality extends the classical Busemann-Petty centroid inequality (see Theorem 2.31) and, along with the L_p-analog of the Petty projection inequality (see Theorem 2.32), was first proved by Lutwak, Yang and Zhang [23]. However, we will present a direct proof by Campi and Gronchi [10] which does not rely on the L_p-Petty projection inequality.

4.1 L_p-Centroid Bodies and Inequalities

The definition of L_p-centroid bodies of compact sets for each real number $p \geq 1$ is due to Lutwak and Zhang [26] and can be stated as follows.

Definition 4.2 (L_p-Centroid Body). *Let K be a compact subset of $\mathbb{R}^n$ with nonempty interior. The L_p-centroid body of K, denoted $\Gamma_p(K)$, is the convex body whose support function is*

$$h\left(\Gamma_p(K)), u\right) = \left\{ \frac{1}{c_{n,p}\mathrm{vol}_n(K)} \int_K |\langle u, z\rangle|^p dz \right\}^{\frac{1}{p}}, \quad u \in \mathbb{R}^n,$$

where the integration is with respect to Lebesgue measure and

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

Some authors prefer to define the L_p-centroid body without the normalization constant which we will also do in Chapter 5 and Chapter 6. However, in this chapter we keep it since it is chosen in a way that $\Gamma_p(B^n) = B^n$ which simplifies some inequalities. Note that for $p = 1$ we do not get the centroid body defined in Chapter 2 but rather $\Gamma_1(K) = c_{n,1}^{-1}\Gamma(K)$.[1] Similarly to Section 2.6 we state some generalized affine inequalities before presenting the direct proof of the L_p-Busemann-Petty centroid inequality.

For $p = 2$, the L_p-centroid body is also well known as the ellipsoid of inertia of K. This is the ellipsoid that has the same moments of inertia as K about every axis. One special case of the L_p-Busemann-Petty centroid inequality for $p = 2$ goes back to Blaschke [4] who proved for $n = 3$ that

Theorem 4.3. *If K is a convex body in $\mathbb{R}^n$, then*

$$\mathrm{vol}_n(\Gamma_2(K)) \geq \mathrm{vol}_n(K),$$

where equality holds if and only if K is an origin symmetric ellipsoid.

For general n this inequality was proved by John [17] and later by Lutwak, Yang and Zhang [22]. Futhermore, Lutwak and Zhang [26] conjectured that the classical Busemann-Petty centroid inequality and Theorem 4.3 are just special instances of the more general L_p-Busemann-Petty centroid inequality which they first proved together with Yang in [23]. They also showed that the L_p-analog of the Petty projection inequality[2].

Theorem 4.4 (L_p-Petty Projection Inequality)**.** *If K is a convex body with nonempty interior in $\mathbb{R}^n$, then for $1 \leq p < \infty$*

$$\mathrm{vol}_n(K)^{\frac{n-p}{p}}\mathrm{vol}_n(\Pi_p^\circ(K)) \leq \omega_n^{n/p},$$

where equality holds if and only if K is an origin symmetric ellipsoid.

Lutwak, Yang and Zhang showed that the L_p-Petty projection inequality implies the L_p-Busemann-Petty centroid inequality. Conversely, it can be shown that the L_p-Busemann-Petty centroid inequality implies the L_p-Petty projection inequality so that we can deduce one inequality from the other. Furthermore, the L_p-Busemann-Petty centroid inequality strengthens the following inequality proved by Lutwak and Zhang [26].

[1] This is the reason why some of the inequalities presented in this chapter differ from the inequalities presented in Section 2.6 exactly by the normalization constant above.

[2] We refer to [23] for the definition of the L_p-projection body $\Pi_p(K)$, its polar body $\Pi_p^\circ(K)$ and the proof.

Theorem 4.5. *If K is a convex body in $\mathbb{R}^n$, then for $1 \leq p \leq \infty$*

$$\mathrm{vol}_n(K)\mathrm{vol}_n(\Gamma_p^\circ(K)) \leq \omega_n^2,$$

where equality holds if and only if K is an origin symmetric ellipsoid.

The body $\Gamma_p^\circ(K)$ is the polar body of $\Gamma_p(K)$ of K and is given by $\Gamma_p^\circ = \{z \in \mathbb{R}^n : h(\Gamma_p(K), z) \leq 1\}$. We can interpret Γ_∞ as the limit of Γ_p as $p \to \infty$ and get $\Gamma_\infty(K) = \mathrm{conv}(K \cup (-K))$. Therefore, if K is a centered convex body the body Γ_∞° is just the polar body K° of K and we get the Blaschke-Santalò inequality.

Theorem 4.6 (Blaschke-Santalò Inequality)**.** *If K is a convex body in $\mathbb{R}^n$, then*

$$\mathrm{vol}_n(K)\mathrm{vol}_n(K^\circ) \leq \omega_n^2,$$

with equality if and only if K is an ellipsoid.

Note that if we apply Theorem 4.6 to $\Gamma_p(K)$ and use the L_p-Busemann-Petty centroid inequality we get Theorem 4.5.

$$\mathrm{vol}_n(K)\mathrm{vol}_n(\Gamma_p^\circ(K)) \leq \mathrm{vol}_n(\Gamma_p(K))\mathrm{vol}_n(\Gamma_p^\circ(K)) \leq \omega_n^2$$

4.2 Direct Proof of the L_p-Busemann-Petty Centroid Inequality

In contrast to Lutwak, Yang and Zhang [23], Campi and Gronchi [10] gave a direct proof of the L_p-Busemann-Petty centroid inequality (without the use the L_p-Petty projection inequality) by looking at the behaviour of $\Gamma_p(K)$ under special transformations, namely parallel chord movments, acting on K. This section presents their direct proof with some smaller results, which were obtained as parts of longer proofs in [10], pulled out and stated as lemmas similar to the structure in [18].

The notion of shadow systems (or linear parameter systems), parallel chord movements and Steiner symmetrization needed for this section were already introduced in Chapter 2 so that we can focus on the results.

Campi and Gronchi asked an interesting question: Suppose that a general parallel chord movement is applied to a convex body K. What happens to the corresponding L_p-centroid bodies? They answered this question with the following theorem.

Theorem 4.7. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then $\Gamma_p(K_t)$ is a shadow system along the same direction v.*

Before we can proof this theorem we still need a few more results. At first, we show the following.

Lemma 4.8. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then the orthogonal projection of $\Gamma_p(K_t)$ onto $v^\perp$ is independent of t.*

Proof. By Definition 4.2 and Definition 2.34, it holds that

$$\begin{aligned}
h\left(\Gamma_p(K_t), u\right) &= \left\{\frac{1}{c_{n,p}\mathrm{vol}_n(K_t)} \int_{K_t} |\langle u, z\rangle|^p dz\right\}^{\frac{1}{p}} \\
&= \left\{\frac{1}{c_{n,p}\mathrm{vol}_n(K_0)} \int_{K_0} |\langle u, z + \beta(z|v^\perp)tv\rangle|^p dz\right\}^{\frac{1}{p}} \\
&= \left\{\frac{1}{c_{n,p}\mathrm{vol}_n(K_0)} \int_{K_0} |\langle u, z\rangle + \beta(z|v^\perp)t\langle u, v\rangle|^p dz\right\}^{\frac{1}{p}} \\
&= \left\{\frac{1}{c_{n,p}\mathrm{vol}_n(K_0)} \int_{K_0} |\langle u, z\rangle|^p\right\}^{\frac{1}{p}} = h\left(\Gamma_p(K_0), u\right)
\end{aligned}$$

since $\langle u, v\rangle = 0$ for $u \in v^\perp$. Therefore, for every $u \in v^\perp, h\left(\Gamma_p(K_t), u\right) = h\left(\Gamma_p(K_0), u\right)$. □

Next, we prove some smaller but helpful results concerning the support function of $\Gamma_p(K_t)$.

Lemma 4.9. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then the support function of $\Gamma_p(K_t)$ given by*

$$\begin{aligned}
h\left(\Gamma_p(K_t), u\right) &= \left\{\frac{1}{c_{n,p}\mathrm{vol}_n(K_0)} \int_{K_0} |\langle u, z\rangle + \beta(z|v^\perp)t\langle u, v\rangle|^p dz\right\}^{\frac{1}{p}} \\
&= \|\langle u, \cdot\rangle + \beta(\cdot|v^\perp)t\langle u, v\rangle\|_p, \quad u \in \mathbb{R}^n
\end{aligned}$$

where $\|q(\cdot)\|_p = \{\frac{1}{c_{n,p}\mathrm{vol}_n(K_0)} \int_{K_0} |q(z)|^p dz\}^{\frac{1}{p}}$ is a convex function of t for every $u \in \mathbb{R}^n$.

Proof. In order to see that $h\left(\Gamma_p(K_t), \cdot\right)$ is a convex function of t for every $u \in \mathbb{R}^n$ we show that

$$2h\left(\Gamma_p(K_{\frac{t_1+t_2}{2}}), \cdot\right) \leq h\left(\Gamma_p(K_{t_1}), \cdot\right) + h\left(\Gamma_p(K_{t_2}), \cdot\right)$$

with the help of the Minkowski inequality for L_p-norms (see Theorem 3.13):

$$\begin{aligned} 2h\left(\Gamma_p\left(K_{\frac{t_1+t_2}{2}}\right),\cdot\right) &= 2\left\|\langle u,\cdot\rangle + \beta(\cdot,v^{\perp})\frac{t_1+t_2}{2}\langle u,v\rangle\right\|_p \\ &= \|\langle 2u,\cdot\rangle + \beta(\cdot,v^{\perp})(t_1+t_2)\langle u,v\rangle\|_p \\ &= \|\langle u+u,\cdot\rangle + \beta(\cdot,v^{\perp})t_1\langle u,v\rangle + \beta(\cdot,v^{\perp})t_2\langle u,v\rangle\|_p \\ &= \|\langle u,\cdot\rangle + \beta(\cdot,v^{\perp})t_1\langle u,v\rangle + \langle u,\cdot\rangle + \beta(\cdot,v^{\perp})t_2\langle u,v\rangle\|_p \\ &\leq \|\langle u,\cdot\rangle + \beta(\cdot,v^{\perp})t_1\langle u,v\rangle\|_p + \|\langle u,\cdot\rangle + \beta(\cdot,v^{\perp})t_2\langle u,v\rangle\|_p \\ &= h\left(\Gamma_p(K_{t_1}),\cdot\right) + h\left(\Gamma_p(K_{t_2}),\cdot\right) \end{aligned}$$

□

Lemma 4.10. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then*

$$|h\left(\Gamma_p(K_{t_1}),u\right) - h\left(\Gamma_p(K_{t_2}),u\right)| \leq \|\beta(\cdot|v^{\perp})\langle u,v\rangle\|_p|t_1-t_2|.$$

Proof. This result also follows from the Minkowski inequality for L_p-norms:

$$\begin{aligned} &|h\left(\Gamma_p(K_{t_1}),u\right) - h\left(\Gamma_p(K_{t_2}),u\right)| \\ &= \|\langle u,\cdot\rangle + \beta(\cdot|v^{\perp})t_1\langle u,v\rangle\|_p - \|\langle u,\cdot\rangle + \beta(\cdot|v^{\perp})t_2\langle u,v\rangle\|_p \\ &\leq \|\langle u,\cdot\rangle\|_p + \|\beta(\cdot|v^{\perp})t_1\langle u,v\rangle\|_p - \|\langle u,\cdot\rangle\|_p - \|\beta(\cdot|v^{\perp})t_2\langle u,v\rangle\|_p \\ &= |t_1|\|\beta(\cdot|v^{\perp})\langle u,v\rangle\|_p - |t_2|\|\beta(\cdot|v^{\perp})\langle u,v\rangle\|_p \\ &\leq \|\beta(\cdot|v^{\perp})\langle u,v\rangle\|_p|t_1-t_2| \end{aligned}$$

□

The rest of this section still follows the ideas of [10] but is supplemented with work and arguments from [18]. Since $\Gamma_p(K_t)$ is an origin symmetric convex body for every $t \in [0,1]$, we can write it as

$$\Gamma_p(K_t) = \{x + yv : x \in (\Gamma_p(K_0))|v^{\perp}, f_t(x) \leq y \leq g_t(x)\},$$

where $f(t)$ and $-g_t(x)$ are suitable convex functions defined on $(\Gamma_p(K_t))|v^{\perp}$.

Lemma 4.11. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the unit direction v, then for every $x \in (\Gamma_p K_0)|v^{\perp}$,*

$$g_t(x) = \inf_{u \in v^{\perp}}\{h\left(\Gamma_p(K_t),u+v\right) - \langle x,u\rangle\}$$

and

$$f_t(x) = \sup_{u \in v^{\perp}}\{\langle x,u\rangle - h\left(\Gamma_p(K_t),u+v\right)\}$$

Proof. Let $u \in v^{\perp}$. For $x \in \Gamma_p(K_t)|v^{\perp}$, we have

$$x + g_t(x)v \in \Gamma_p(K_t), \quad x + f_t(x)v \in \Gamma_p(K_t).$$

Since $x \in \Gamma_p(K_t)$ if and only if $\langle x, u\rangle \leq h\left(\Gamma_p(K_t)\right)$ for every $u \in \mathbb{R}^n$, we have

$$\langle x + g_t(x)v, u + v\rangle \leq h\left(\Gamma_p(K_t), u + v\right)$$

and

$$\langle x + f_t(x)v, u - v\rangle \leq h\left(\Gamma_p(K_t), u - v\right).$$

Therefore,

$$\langle x, u\rangle + g_t(x) \leq h\left(\Gamma_p(K_t), u + v\right), \quad \langle x, u\rangle - f_t(x) \leq h\left(\Gamma_p(K_t), u - v\right)$$

for all $u \in v^{\perp}$.

The body $\Gamma_p(K_t)$ has two supporting hyperplanes at $x + g_t(x)v, x + f_t(x)v \in \partial(\Gamma_p(K_t))$ and for $x \in \operatorname{relint}\left((\Gamma_p(K_t))|v^{\perp}\right)$ there exist two vectors $u' + v$ and $u'' - v$ with $u', u'' \in v^{\perp}$ such that

$$\langle x + g_t(x)v, u' + v\rangle = h\left(\Gamma_p(K_t), u' + v\right)$$

and

$$\langle x + f_t(x)v, u'' - v\rangle = h\left(\Gamma_p(K_t), u'' - v\right).$$

Thus, if $x \notin \operatorname{relint}\left((\Gamma_p(K_t))|v^{\perp}\right)$, it is possible that $g_t(x) = f_t(x) = 0$ and that we cannot find $u', u'' \in v^{\perp}$ such that

$$\langle x + g_t(x)v, u' + v\rangle = \langle x, u'\rangle = h\left(\Gamma_p(K_t), u' + v\right)$$

and

$$\langle x + f_t(x)v, u'' - v\rangle = \langle x, u''\rangle = h\left(\Gamma_p(K_t), u'' - v\right).$$

Since the support functions are continuous, we can take the infimum and supremum for all $u \in v^{\perp}$ and get

$$g_t(x) = \inf_{u \in v^{\perp}} \{h\left(\Gamma_p(K_t), u + v\right) - \langle x, u\rangle\},$$

and

$$f_t(x) = \sup_{u \in v^{\perp}} \{\langle x, u\rangle - h\left(\Gamma_p(K_t), u - v\right)\}$$

which is exactly what we wanted to show. $\square$

We have seen that $h\left(\Gamma_p(K_t), x\right)$ is a Lipschitz function of t which shows that also $g_t(x)$ and $f_t(x)$ are continuous with respect to t. We can also show that $g_t(x)$ and $-f_t(x)$ are convex.

Lemma 4.12. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the unit direction v, then for every $x \in (\Gamma_p(K_0))|v^\perp$ the functions $g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in $[0,1]$.*

Proof. We first show that if $u_1, u_2 \in v^\perp$, then

$$h\left(\Gamma_p\left(K_{\frac{t_1+t_2}{2}}\right), u_1+u_2+2v\right) \le h\left(\Gamma_p(K_{t_1}), u_1+v\right) + h\left(\Gamma_p(K_{t_2}), u_2+v\right).$$

Expanding the definition and applying the Minkowski inequality yields

$$\begin{aligned}
&h\left(\Gamma_p\left(K_{\frac{t_1+t_2}{2}}\right), u_1+u_2+2v\right)\\
&\quad= \left\| \langle u_1+u_2+2v,\cdot\rangle + \beta(\cdot|v^\perp)\frac{t_1+t_2}{2}\langle u_1+u_2+2v, v\rangle \right\|_p\\
&\quad= \|\langle u_1+v,\cdot\rangle + \langle u_2+v,\cdot\rangle + \beta(\cdot|v^\perp)\langle 2v,v\rangle\frac{t_1+t_2}{2}\|_p\\
&\quad= \|\langle u_1+v,\cdot\rangle + \beta(\cdot|v^\perp)t_1 + \langle u_2+v,\cdot\rangle + \beta(\cdot|v^\perp)t_2\|_p\\
&\quad= \|\langle u_1+v,\cdot\rangle + \beta(\cdot|v^\perp)t_1\langle u_1+v,v\rangle + \langle u_2+v,\cdot\rangle + \beta(\cdot|v^\perp)t_2\langle u_2+v,v\rangle\|_p\\
&\quad\le \|\langle u_1+v,\cdot\rangle + \beta(\cdot|v^\perp)t_1\langle u_1+v,v\rangle\|_p\\
&\qquad\quad + \|\langle u_2+v,\cdot\rangle + \beta(\cdot|v^\perp)t_2\langle u_2+v,v\rangle\|_p\\
&\quad= h\left(\Gamma_p(K_{t_1}), u_1+v\right) + h\left(\Gamma_p(K_{t_2}), u_2+v\right).
\end{aligned}$$

By Lemma 4.11 and the above inequality we see that

$$\begin{aligned}
2g_{\frac{t_1+t_2}{2}} &= \inf_{u\in v^\perp}\left\{h\left(\Gamma_p\left(K_{\frac{t_1+t_2}{2}}\right), 2(u+v)\right) - \langle x, 2u\rangle\right\}\\
&= \inf_{u_1,u_2\in v^\perp}\left\{h\left(\Gamma_p\left(K_{\frac{t_1+t_2}{2}}\right), u_1+u_2+2v\right) - \langle x, u_1+u_2\rangle\right\}\\
&\le \inf_{u_1,u_2\in v^\perp}\left\{h\left(\Gamma_p(K_{t_1}), u_1+v\right) + h\left(\Gamma_p(K_{t_2}), u_2+v\right) - \langle x, u_1+u_2\rangle\right\}\\
&= \inf_{u_1,u_2\in v^\perp}\left\{h\left(\Gamma_p(K_{t_1}), u_1+v\right) + h\left(\Gamma_p(K_{t_2}), u_2+v\right) - \langle x, u_1\rangle - \langle x, u_2\rangle\right\}\\
&= \inf_{u_1\in v^\perp}\left\{h\left(\Gamma_p(K_{t_1}), u_1+v\right) - \langle x, u_1\rangle\right\}\\
&\qquad + \inf_{u_2\in v^\perp}\left\{h\left(\Gamma_p(K_{t_2}), u_2+v\right) - \langle x, u_2\rangle\right\}\\
&= g_{t_1}(x) + g_{t_2}(x).
\end{aligned}$$

The convexity of $-f_t(x)$ can be in proved the same way. $\square$

Lemma 4.13. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement along the direction v, then for every $x \in (\Gamma_p(K_0))|v^\perp$ it holds that*

$$f_{\lambda t_1+(1-\lambda)t_2}(x) \le \lambda g_{t_1}(x) + (1-\lambda)f_{t_2}(x) \le g_{\lambda t_1+(1-\lambda)t_2}(x)$$

for every $t_1, t_2, \lambda \in [0,1]$.

Proof. We first show that if $u_1, u_2 \in v^\perp$, then

$$h\left(\Gamma_p(K_{t_2}), u_2 - \lambda u_1 - (1-\lambda)v\right) \\ \leq h\left(\Gamma_p(K_{t_1}), -\lambda u_1 + \lambda v\right) + h\left(\Gamma_p(K_{\lambda t_1 + (1-\lambda)t_2}), u_2 - v\right).$$

Again, using the Minkowski inequality, we obtain

$$\begin{aligned} &h\left(\Gamma_p(K_{t_2}), u_2 - \lambda u_1 - (1-\lambda)v\right) \\ &= \|\langle u_2 - \lambda u_1 - (1-\lambda)v, \cdot\rangle + \beta(\cdot|v^\perp)t_2\langle u_2 - \lambda u_1 - (1-\lambda)v, v\rangle\|_p \\ &= \|\langle u_2 - v, \cdot\rangle + \langle -\lambda u_1 + \lambda v, \cdot\rangle - \beta(\cdot|v^\perp)t_2\langle(1-\lambda)v, v\rangle\|_p \\ &= \|\langle u_2 - v, \cdot\rangle + \langle -\lambda u_1 + \lambda v, \cdot\rangle - \beta(\cdot|v^\perp)t_2(1-\lambda) + \lambda t_1 - \lambda t_1\|_p \\ &\leq \|\langle u_2 - v, \cdot\rangle - \beta(\cdot|v^\perp)\left(\lambda t_1 + (1-\lambda)t_2\right)\|_p \\ &\qquad + \|\langle -\lambda u_1 + \lambda v, \cdot\rangle - \beta(\cdot|v^\perp)\lambda t_1\|_p \\ &= \|\langle u_2 - v, \cdot\rangle + \beta(\cdot|v^\perp)\left(\lambda t_1 + (1-\lambda)t_2\right)\langle u_2 - v, v\rangle\|_p \\ &\qquad + \|\langle -\lambda u_1 + \lambda v, \cdot\rangle + \beta(\cdot|v^\perp)t_1\langle -\lambda u_1 + \lambda v, v\rangle\|_p \\ &= h\left(\Gamma_p(K_{t_1}), -\lambda u_1 - \lambda v\right) + h\left(\Gamma_p(K_{\lambda t_1 + (1-\lambda)t_2}), u_2 - v\right). \end{aligned}$$

It is enough to prove the first inequality as the other one follows by interchanging t_1 with t_2 and x with $-x$. By Lemma 4.11 and the above inequality, we see that

$$\begin{aligned} &(1-\lambda)f_{t_2}(x) \\ &\quad= \sup_{u \in v^\perp}\{\langle x, (1-\lambda)u\rangle - h\left(\Gamma_p(K_{t_2}), (1-\lambda)(u - v)\right)\} \\ &\quad= \sup_{-u_1, u_2 \in v^\perp}\{\langle x, u_2 - \lambda u_1\rangle - h\left(\Gamma_p(K_{t_2}), u_2 - \lambda u_1 - (1-\lambda)v\right)\} \\ &\quad\geq \sup_{-u_1, u_2 \in v^\perp}\{\langle x, u_2 - \lambda u_1\rangle - h\left(\Gamma_p(K_{t_1}), -\lambda u_1 + \lambda v\right) \\ &\qquad\qquad - h\left(\Gamma_p(K_{\lambda t_1 + (1-\lambda)t_2}), u_2 - v\right)\} \\ &\quad= \sup_{-u_1 \in v^\perp}\{\langle x, -\lambda u_1\rangle - h\left(\Gamma_p(K_{t_1}), -\lambda u_1 + \lambda v\right) \\ &\qquad\qquad + \sup_{u_2 \in v^\perp}\{\langle x, u_2\rangle - h\left(\Gamma_p(K_{\lambda t_1 + (1-\lambda)t_2}), u_2 - v\right)\}\} \\ &\quad= -\lambda g_{t_1}(x) + f_{\lambda t_1 + (1-\lambda)t_2}(x). \end{aligned}$$

□

We require one last lemma before we can proof Theorem 4.7 which was also proved by Campi and Gronchi [10]. It provides necessary and sufficient conditions for a family of convex bodies with constant orthogonal projections onto a fixed hyperplane to be a shadow system. For the relatively long proof we refer to [10].

Lemma 4.14. *Let $\{H_t : t \in [0,1]\}$ be a one-parameter family of convex bodies such that $H_t|v^\perp$ is independent of t. Assume the bodies H_t are defined by*

$$H_t = \{x + yv : x \in H_t|v^\perp, y \in \mathbb{R}, f_t(x) \leq y \leq g_t(x)\}, \quad \forall t \in [0,1]$$

for suitable functions g_t, f_t. Then $\{H_t : t \in [0,1]\}$ is a shadow system along the direction v if and only if for every $x \in H_0|v^\perp$,

1. *$g_t(x)$ and $-f_t(x)$ are convex functions of the parameter t in $[0,1]$,*
2. *$f_{\lambda t_1+(1-\lambda)t_2}(x) \leq \lambda g_{t_1}(x) + (1-\lambda) f_{t_2}(x) \leq g_{\lambda t_1+(1-\lambda)t_2}(x)$, for every $t_1, t_2, \lambda \in [0,1]$.*

Proof of Theorem 4.7. Let $\{K_t : t \in [0,1]\}$ be a parallel chord movement along the direction v. We have seen in Lemma 4.8 that the orthogonal projection of $\Gamma_p(K_t)$ onto the hyperplane $v^\perp$ is independent of t. Therefore, the family $\Gamma_p(K_t)$ meets the assumptions of Lemma 4.14 and it is sufficient to show that both conditions are satisfied. Actually, Lemma 4.12 shows the first condition and 4.13 shows the second condition. Therefore, we see that $\Gamma_p(K_t)$ is a shadow system along the direction v. □

Shadow systems have many interesting properties but the one we are interested in was also proved by Shephard [33] and states that $\mathrm{vol}_n(\Gamma_p(K_t))$ is a convex function of the parameter t.

Theorem 4.15. *Let $\{K_t : t \in [0,1]\}$ be a shadow system along the direction v, then the volume $\mathrm{vol}_n(K_t)$ is a convex function of the parameter t.*

Together, Theorem 4.7 and Theorem 4.15 imply that the volume $\mathrm{vol}_n(\Gamma_p(K_t))$ is a convex function of t and that, if $\{K_t : t \in [0,1]\}$ is the parallel chord movement related to Steiner Symmetrization along v, then

$$\begin{aligned}
\mathrm{vol}_n(\Gamma_p(K_{1/2})) &= \mathrm{vol}_n\left(\frac{1}{2}\Gamma_p(K_0) + \frac{1}{2}\Gamma_p(K_1)\right) \\
&= \mathrm{vol}_n\left(\frac{1}{2}\Gamma_p(K_0) + \left(1 - \frac{1}{2}\right)\Gamma_p(K_1)\right) \\
&\leq \frac{1}{2}\mathrm{vol}_n(\Gamma_p(K_0)) + \left(1 - \frac{1}{2}\right)\mathrm{vol}_n(\Gamma_p(K_1)) = \mathrm{vol}_n(\Gamma_p(K))
\end{aligned}$$

since $\mathrm{vol}_n(\Gamma_p(K_0)) = \mathrm{vol}_n(\Gamma_p(K_1)) = \mathrm{vol}_n(\Gamma_p(K))$. This shows that the volume of the L_p-centroid body is not increased after Steiner symmetrization. Following standard theorem states that every convex body can be transformed into a ball by a sequence of suitable Steiner symmetrizations (see [32, Theorem 10.3.2]).

Theorem 4.16. *If K is a convex body and $\mathcal{S}(K)$ is the set of convex bodies that arise from K by applying iterated Steiner symmetrizations, then $\mathcal{S}(K)$ contains a sequence that converges to a ball.*

Therefore, the ratio $\mathrm{vol}_n(\Gamma_p(K))/\mathrm{vol}_n(K)$, which is continuous in the Hausdorff metric, attains its minimum value when K is a ball. To characterize the minimizers Campi and Gronchi [10] proved that

Theorem 4.17. *If $\{K_t : t \in [0,1]\}$ is a parallel chord movement with speed function β, then the volume $\Gamma_p(K_t)$ is a strictly convex function of t unless β is linear, that is $\beta(x) = \langle x, u\rangle$ for some vector u.*

Proof. By Fubini's theorem we have

$$\mathrm{vol}_n(\Gamma_p(K_t)) = \int_{(\Gamma_p(K_0))|v^\perp} [g_t(x) - f_t(x)]dx,$$

where the integral is with respect to Lebesgue measure. Since $-f(t)$ and $g_t(x)$ are convex functions with respect to t, so is the volume. Assume that

$$\mathrm{vol}_n(\Gamma_p(K_{(t_1+t_2)/2})) = \frac{1}{2}\mathrm{vol}_n(\Gamma_p(K_{t_1})) + \frac{1}{2}\mathrm{vol}_n(\Gamma_p(K_{t_2})),$$

for some $t_1, t_2 \in [0,1]$, then we obtain from the continuity of g_t, f_t with respect to x that

$$g_{\frac{t_1+t_2}{2}}(x) - f_{\frac{t_1+t_2}{2}}(x) = \frac{1}{2}(g_{t_1}(x) + g_{t_2}(x)) - \frac{1}{2}(f_{t_1}(x) + f_{t_2}(x)),$$

for almost every $x \in (\Gamma_p(K_0))|v^\perp$. Furthermore, by the continuity of f_t, g_t, this equality holds everywhere. If x is a point from the interior of $(\Gamma_p(K_0))|v^\perp$, then there exist $u_1, u_2, u_3, u_4 \in v^\perp$ such that

$$\begin{aligned}
&\frac{1}{2}(g_{t_1}(x) + g_{t_2}(x)) - \frac{1}{2}(f_{t_1}(x) + f_{t_2}(x)) \\
&\quad = \frac{1}{2}\big(h\left(\Gamma_p(K_{t_1}), u_1 + v\right) + h\left(\Gamma_p(K_{t_2}), u_2 + v\right) + h\left(\Gamma_p(K_{t_1}), u_3 - v\right) \\
&\qquad\quad + h\left(\Gamma_p(K_{t_2}), u_4 - v\right) - \langle x, u_1\rangle - \langle x, u_2\rangle - \langle x, u_3\rangle - \langle x, u_4\rangle\big).
\end{aligned}$$

We use the inequality from the proof of Lemma 4.12 again to obtain

$$\begin{aligned}
\frac{1}{2}(g_{t_1}(x) + g_{t_2}(x)) - \frac{1}{2}(f_{t_1}(x) + f_{t_2}(x)) \qquad&\\
\geq h\left(\Gamma_p\left(K_{\frac{t_1+t_2}{2}}\right), \frac{u_1+u_2}{2} + v\right) - \left\langle x, \frac{u_1+u_2}{2}\right\rangle&\\
+ h\left(\Gamma_p\left(K_{\frac{t_1+t_2}{2}}\right), \frac{u_1+u_2}{2} - v\right) - \left\langle x, \frac{u_3+u_4}{2}\right\rangle&\\
\geq g_{\frac{t_1+t_2}{2}}(x) - f_{\frac{t_1+t_2}{2}}(x)&
\end{aligned}$$

Therefore, by our assumption, the equality condition for the inequality has to hold which means that there exists a constant c such that

$$\langle u_1 + v, z\rangle + \beta(z|v^\perp)t_1 = c\langle u_2 + v, z\rangle + c\beta(z|v^\perp)t_2,$$

for every $z \in K_0$ due to β being continuous. If we set $z = z' + \lambda v$ and differentiate with respect to the parameter λ, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\Big(\langle u_1 + v, z' + \lambda v\rangle + \beta(z' + \lambda v|v^\perp)t_1\Big)\\
&\quad= \frac{\partial}{\partial\lambda}\Big(\langle u_1, z'\rangle + \langle u_1, \lambda v\rangle + \langle v, z'\rangle + \langle v, \lambda v\rangle + \beta(z' + \lambda v|v^\perp)t_1\Big)\\
&\quad= \frac{\partial}{\partial\lambda}\Big(0 + 0 + 0 + \lambda + \beta(z' + \lambda v|v^\perp)t_1\Big)\\
&\quad= \frac{\partial}{\partial\lambda}\lambda + \frac{\partial}{\partial\lambda}\beta(z' + \lambda v|v^\perp)t_1 = 1 + 0 = 1
\end{aligned}$$

and, furthermore,

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\Big(c\langle u_2 + v, z' + \lambda v\rangle + c\beta(z' + \lambda v|v^\perp)t_2\Big)\\
&\quad= \frac{\partial}{\partial\lambda}\Big(c\langle u_1, z'\rangle + c\langle u_1, \lambda v\rangle + c\langle v, z'\rangle + c\langle v, \lambda v\rangle + c\beta(z' + \lambda v|v^\perp)t_2\Big)\\
&\quad= \frac{\partial}{\partial\lambda}\Big(0 + 0 + 0 + c\lambda + c\beta(z' + \lambda v|v^\perp)t_2\Big)\\
&\quad= \frac{\partial}{\partial\lambda}c\lambda + \frac{\partial}{\partial\lambda}c\beta(z' + \lambda v|v^\perp)t_2 = c + 0 = c
\end{aligned}$$

which shows $c = 1$ and that β is a linear function. □

If the speed function β of the parallel chord movement is linear, then K_t is a linear image of K for every t in the range of the movement. Furthermore, if K is not an origin symmetric ellipsoid, then there exists a direction v such that the Steiner symmetral of K along the direction v is not an image of K under a linear transformation. Therefore, $\mathrm{vol}_n(\Gamma_p(K))/\mathrm{vol}_n(K)$ attains its minimum value if and only if K is an origin symmetric ellipsoid which finishes the proof of the L_p-Busemann-Petty centroid inequality (see Theorem 4.1).

5 Randomized L_p-Busemann-Petty Centroid Inequality

In this chapter we continue our investigation of the L_p-Busemann-Petty centroid inequality and present it in a randomized version. We follow Paouris and Pivovarov [28] who extended a theorem of Groemer (see [16]) on the expected volume of a random polytope in a convex body and proved a randomized version of the L_p-Busemann-Petty centroid inequality as an application of their generalization.

As we have already seen in Chapter 2 and Chapter 4, various functionals $\phi : \mathcal{K}^n \to \mathbb{R}^+$ are minimized (or maximized) on the Euclidean ball. This is also true for the functional

$$\phi(K) := \frac{1}{\mathrm{vol}_n(K)^N} \int_K \cdots \int_K \mathrm{vol}_n(\mathrm{conv}\{x_1, \dots, x_N\}) dx_1, \dots, dx_N \quad (K \in \mathcal{K}^n)$$

which defines the expected volume of the convex hull of independent random points sampled in K. In this notation Groemer's theorem can be written as follows.

Theorem 5.1 (Groemer's Theorem)**.** *Let K be a convex body in $\mathbb{R}^n$. Then, if B^n denotes the n-dimensional Euclidean ball with radius one,*

$$\phi(K) \geq \phi(B^n).$$

Equality holds if and only if K is an ellipsoid.

Paouris and Pivovarov [28] extended Groemer's theorem by working in the class $\mathcal{P}_{[n]}$ of all probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. Thus, instead of the Steiner symmetrization we use rearrangement inequalities. Furthermore, if $N \geq n$ and $x_1, \dots, x_N$ are independent random points with x_i distributed according to $\mu_i \in \mathcal{P}_{[n]}$, we treat the $n \times N$ random matrix $[x_1 \dots x_N]$ as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$. Applying this operator to a convex body K yields a random convex set in $\mathbb{R}^n$,

$$[x_1 \dots x_N]K = \left\{ \sum_{i=1}^{N} k_i x_i : (k_i) \in K \right\}.$$

This brings us to one of the main results in Paouris and Pivovarov [28]

Theorem 5.2 (Extension of Groemer's Theorem). *Let $N \geq n$ and $\mu_1, \dots, \mu_N \in \mathcal{P}_{[n]}$; denote the density of μ_i by f_i. Let K be a convex body in $\mathbb{R}^N$ and set*

$$\mathcal{F}_K(f_1, \dots, f_N) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathrm{vol}_n([x_1, \dots, x_N]K) \prod_{i=1}^{N} f_i(x_i) dx_N \dots dx_1.$$

If $\|f_i\|_\infty \leq 1$ for $i = 1, \dots, N$, then

$$\mathcal{F}_K(f_1, \dots, f_N) \geq \mathcal{F}_K(\mathbb{1}_{D_n}, \dots, \mathbb{1}_{D_n}),$$

where $D_n \subseteq \mathbb{R}^n$ is the Euclidean ball of volume one.

In the next sections we present the necessary definitions and results to be able to prove this theorem and apply it to obtain a randomized version of the L_p-Busemann-Petty centroid inequality due to Paouris and Pivovarov.

5.1 An Extension of Groemer's Theorem

For a function $F : \bigotimes_{i=1}^{N} \mathbb{R}^n \to \mathbb{R}$, let

$$\mathcal{F}_F(f_1, \dots, f_N) := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} F(x_1, \dots, x_N) f_1(x_1), \dots, f_N(x_N) dx_1 \dots dx_N.$$

It turns out that it is possible to isolate a condition on F from which one can conclude a minimization result such as the extension of Groemer's Theorem (see Theorem 5.2). This condition is called Groemer's convexity condition (GCC).

Definition 5.3 (Groemer's Convexity Condition). *We say that $F : \bigotimes_{i=1}^{N} \mathbb{R}^n \to \mathbb{R}^+$ satisfies Groemer's convexity condition, or simply (GCC), if for every $z \in \mathbb{R}^n \backslash \{0\}$ and for every $Y = \{y_1, \dots, y_N\} \subseteq z^\perp$, the function $F_Y : \mathbb{R}^N \to \mathbb{R}^+$ defined by*

$$F_Y(t) = F(y_1 + t_1 z, \dots, y_N + t_N z)$$

is even and convex.

Our goal is to indicate how rearrangement inequalities are useful in the presence of Groemer's convexity condition. The first of two results is given in the next propostion. The proof given by Paouris and Pivovarov [28] is analogous to one given in Christ [11, Theorem 4.2] and uses the following lemma from [7].

Lemma 5.4. *If $g : \mathbb{R}^n \to \mathbb{R}^+$ is a measurable function with compact support, then there exists a sequence of functions g_k, where $g_0 = g$ and $g_{k+1} = g_k^*(\cdot|\theta_k)$ for some $\theta_k \in \mathbb{S}^{n-1}$ such that*

$$\lim_{k \to \infty} \|g_k - g^*\|_{L_1} = 0.$$

Proposition 5.5. *Let $f_1, \dots, f_N$ be non-negative integrable functions on $\mathbb{R}^n$. Suppose that $F : \bigotimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ satisfies the following condition: for each $z \in \mathbb{S}^{n-1}$ and for each $Y = \{y_1, \dots, y_N\} \subseteq z^\perp$, the function*

$$F_Y(t) = F(y_1 + t_1 z, \dots, y_N + t_N z)$$

is even and quasi-convex. Then

$$\mathcal{F}_F(f_1, \dots, f_N) \geq \mathcal{F}_F(f_1^*, \dots, f_N^*).$$

Proof. Let $\theta \in \mathbb{S}^{n-1}$. We show first that

$$\mathcal{F}_F(f_1, \dots, f_N) \geq \mathcal{F}_F(f_1^*(\cdot|\theta), \dots, f_N^*(\cdot|\theta)),$$

where $f^*(\cdot|\theta)$ is the Steiner symmetrization of f with respect to $\theta^\perp$ (see Definition 3.22). For fixed $y_1, \dots, y_N \in \theta^\perp$, we set $h_i(t_i) = f_i(y_i + t_i\theta)$. Using Fubini's Theorem (see Theorem 3.9) and Corollary 3.27 we get

$$\begin{aligned}
\mathcal{F}_F(f_1, \dots, f_N) &= \int_{(\theta^\perp)^N} \int_{\mathbb{R}^N} F(y_1 + t_1\theta, \dots, y_N + t_N\theta) \prod_{i=1}^N f_i(y_i + t_i\theta) dt d\bar{y} \\
&= \int_{(\theta^\perp)^N} \int_{\mathbb{R}^N} F(t_1, \dots, t_N) h_1(t_1) \cdots h_N(t_N) dt d\bar{y} \\
&\geq \int_{(\theta^\perp)^N} \int_{\mathbb{R}^N} F(t_1, \dots, t_N) h_1^*(t_1) \cdots h_N^*(t_N) dt d\bar{y} \\
&= \mathcal{F}_F(f_1^*(\cdot|\theta), \dots, f_N^*(\cdot|\theta))
\end{aligned}$$

since $f_i^*(\cdot|\theta)$ is the function obtained by rearranging f_i along every line parallel to θ and where $dt = dt_1 \dots dt_N$ and $d\bar{y} = dy_1 \dots dy_N$.

We still need to get to the symmetric decreasing rearrangement f_i^* for each $f_i, i \leq N$ which can be achieved by suitable successive symmetrizations with respect to $n-1$ dimensional subspaces by Lemma 5.4. □

The reverse inequality is due to Christ [11] and is stated for completeness only.

Proposition 5.6. *Let $f_1, \dots, f_N$ be non-negative integrable functions on $\mathbb{R}^n$. Suppose that $F : \bigotimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ satisfies the following condition: for each $z \in \mathbb{S}^{n-1}$ and for each $Y = \{y_1, \dots, y_N\} \subseteq z^\perp$, the function*

$$F_Y(t) = F(y_1 + t_1 z, \dots, y_N + t_N z)$$

is even and quasi-concave. Then

$$\mathcal{F}_F(f_1, \dots, f_N) \leq \mathcal{F}_F(f_1^*, \dots, f_N^*).$$

For further studies we require the definition of rotationally invariant densities.

Definition 5.7. *Let f be an integrable function with $\int_{\mathbb{R}^n} f(x)dx = 1$. We say that f is rotationally invariant if $f(x) = f(y)$ whenever $\|x\|_2 = \|y\|_2$. We denote by $\mathcal{RP}_{[n]} \subseteq \mathcal{P}_{[n]}$ the subclass of measures with rotationally invariant densities.*

Proposition 5.5 allows us to pass to densities that are rotationally invariant because if F satisfies Groemer's convexity condition, then

$$\inf_{\mathcal{P}_{[n]}} \mathcal{F}_F(f_1,\dots,f_N) = \inf_{\mathcal{RP}_{[n]}} \mathcal{F}_F(f_1,\dots,f_N),$$

where the f_i's are the densities of measures in $\mathcal{P}_{[n]}$ and $\mathcal{RP}_{[n]}$, respectively.

If we look at $\inf_{\mathcal{RP}_{[n]}} \mathcal{F}_F(f_1,\dots,f_N)$ under the additional assumption that $\|f_i\|_\infty \le 1$, for $i \le i \le N$, then we can prove the following useful lemmas [28].

Lemma 5.8. *Let $f : \mathbb{R}^+ \to [0,1]$ be a measurable function and assume that*

$$A := \int_0^\infty f(t)t^{n-1}dt < \infty.$$

Let $g = \mathbb{1}_{[0,(nA)^{1/n}]}$. Then for any increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\int_0^\infty \phi(t)f(t)t^{n-1}dt \ge \int_0^\infty \phi(t)g(t)t^{n-1}dt.$$

Proof. Set $B = (nA)^{1/n}$ and note that

$$\int_0^\infty f(t)t^{n-1}dt \ge \int_0^B t^{n-1}dt.$$

Then

$$\begin{aligned}
\int_0^\infty \phi(t)f(t)t^{n-1}dt &= \int_0^B \phi(t)f(t)t^{n-1}dt + \int_B^\infty \phi(t)f(t)t^{n-1}dt \\
&\ge \int_0^B \phi(t)f(t)t^{n-1}dt + \phi(B)\int_B^\infty f(t)t^{n-1}dt \\
&= \int_0^B \phi(t)f(t)t^{n-1}dt + \phi(B)\int_0^B (1-f(t))t^{n-1}dt \\
&\ge \int_0^B \phi(t)f(t)t^{n-1}dt + \int_0^B \phi(t)(1-f(t))t^{n-1}dt \\
&= \int_0^B \phi(t)t^{n-1}dt.
\end{aligned}$$

□

Lemma 5.9. *Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $\mathbb{E}$ denote expectation with respect to $\mathbb{P}$. Let $X : \Omega \to \mathbb{R}^n$ be a symmetric random vector. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a function such that*

$$\mathbb{R} \ni s \mapsto \rho(sx)$$

is convex for each $x \in \mathbb{R}^n$. Then

$$\mathbb{R}^+ \ni s \mapsto \mathbb{E}\rho(sx)$$

is an increasing function.

Proof. It is sufficient to show that

$$\mathbb{E}\rho(aX) \leq \mathbb{E}\rho(X)$$

for any $0 \leq a \leq 1$. We can write $a = b(1) + (1-b)(-1)$ with $0 \leq b \leq 1$ and use the convexity assumption

$$\rho(aX) \leq b\rho(X) + (1-b)\rho(-X),$$

which yields

$$\begin{aligned}
\mathbb{E}\rho(aX) \leq \mathbb{E}(b\rho(X) + (1-b)\rho(-X)) &= \mathbb{E}(b\rho(X)) + \mathbb{E}((1-b)\rho(-X)) \\
&= \mathbb{E}(b\rho(X)) + \mathbb{E}\rho(-X) - \mathbb{E}(b\rho(X)) \\
&= \mathbb{E}\rho(-X) = \mathbb{E}\rho(X),
\end{aligned}$$

since X is a symmetric random vector. □

Lemma 5.10. *If $F : \bigotimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ satisfies Groemer's convexity condition, then for any $x_1, \dots, x_N \in \mathbb{R}^n$ and any $1 \leq i \leq N$, the function*

$$\mathbb{R} \ni s \mapsto F(x_1, \dots, sx_j, \dots, x_N)$$

is convex.

Proof. In this proof we use the fact that the restriction of a convex function to a line is itself convex. Let us fix j as in the assumption and for each $i \neq j$, we write $x_i = x_i' + s_i x_j$ with $s_i \in \mathbb{R}$ and $x_i' \perp x_j$. Since F satisfies Groemer's convexity condition, we can take $z = x_j, y_j = 0$ and $y_i = x_i'$ for all $i \neq j$ and $Y = \{y_1, \dots, y_N\}$. Then the function $G_Y : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$\begin{aligned}
G_Y(t) &:= F(y_1 + t_1 s_1 z, \dots, t_j z, \dots, y_N + t_N s_N z) \\
&= F_Y(t_1 s_1, \dots, t_j, \dots, t_N s_N)
\end{aligned}$$

is convex by Groemer's convexity condition. On the other hand, we have

$$G_Y(t) = F(x_1 + (t_1 - 1)s_1 x_j, \dots, x_j + (t_j - 1)x_j, \dots, x_N + (t_N - 1)s_N x_j),$$

since

$$y_i + t_i s_i z = x_i' + t_i s_i z = x_i - s_i z + t_i s_i z = x_i + (t_i - 1) s_i z$$

and, hence, the restriction of G_Y to the line $\{t \in \mathbb{R}^N : t_j = s \in \mathbb{R}, t_i = 1 \text{ for each } i \neq j\}$ is just the function $F(x_1, \dots, sx_j, \dots, x_N)$ above. □

The second result of this section is given by the following proposition [28].

Proposition 5.11. *Let $f_i : \mathbb{R}^n \to [0,1]$ be rotationally invariant probability densities. Suppose that $F : \bigotimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ satisfies Groemer's convexity condition. Then*

$$\mathcal{F}_F(f_1, \dots, f_N) \geq \mathcal{F}_F(\mathbb{1}_{D_n}, \dots, \mathbb{1}_{D_n}).$$

Proof. We use spherical coordinates for each $x_i \in \mathbb{R}^n$ and write

$$x_i := r_i \theta_i, \quad \text{with } 0 \leq r_i < \infty, \text{ and } \theta_i \in \mathbb{S}^{n-1} \text{ for } i = 1, \dots, N.$$

Then

$$\mathcal{F}_F(f_1, \dots, f_N) = (n\omega_n)^N \int_{(\mathbb{R}^+)^N} \int_{(\mathbb{S}^{n-1})^N} F(r_1\theta_1, \dots, r_N\theta_N) \prod_{i=1}^N f_i(r_i\theta_i) r_i^{n-1} d\bar{\theta} dr,$$

where $d\bar{\theta} = d\sigma(\theta_1) \dots d\sigma(\theta_N)$ and $dr = dr_1 \dots dr_N$. Fix $1 \leq j \leq N$ and suppose that $r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_N$ are fixed non-negative scalars. Suppose that $\theta_1, \dots, \theta_N \in \mathbb{S}^{n-1}$ are fixed vectors, then, by Lemma 5.10, the function

$$\mathbb{R}^+ \ni r_j \mapsto F(r_1\theta_1, \dots, r_j\theta_j, \dots, r_N\theta_N)$$

is convex. If we now regard θ_j as a random vector uniformly distributed on $\mathbb{S}^{n-1}$ and averaging, then Lemma 5.9 implies that

$$\mathbb{R}^+ \ni r_j \mapsto \int_{\mathbb{S}^{n-1}} F(r_1\theta_1, \dots, r_j\theta_j, \dots, r_N\theta_N) d\sigma(\theta_j)$$

is increasing. Due to f_j being a rotationally invariant probability density, we have

$$1 = \int_{\mathbb{R}^n} f_j(x) dx = n\omega_n \int_0^\infty \int_{\mathbb{S}^{n-1}} f_j(r_j\theta_j) r_j^{n-1} d\sigma(\theta_j) dr_j.$$

Since f_j depends only on the value of r_j, we have that for any $\theta_j \in \mathbb{S}^{n-1}$,

$$\int_0^\infty f_j(r_j\theta_j) r_j^{n-1} dr_j = (n\omega_n)^{-1}.$$

Thus we can apply Lemma 5.8 with $A = (n\omega_n)^{-1}$ to see that

$$\int_0^\infty \int_{\mathbb{S}^{n-1}} F(r_1\theta_1, \ldots, r_j\theta_j, \ldots, r_N\theta_N) f_j(r_j\theta_j) r_j^{n-1} d\sigma(\theta_j) dr_j$$
$$\geq \int_0^{\omega_n^{-1/n}} \int_{\mathbb{S}^{n-1}} F(r_1\theta_1, \ldots, r_j\theta_j, \ldots, r_N\theta_N) r_j^{n-1} d\sigma(\theta_j) dr_j$$

Applying Fubini's theorem (see Theorem 3.9) iteratively, we then obtain

$$\mathcal{F}_F(f_1, \ldots, f_N) \geq (n\omega_n)^N \int_{[0,\omega_n^{-1/n}]^N} \int_{(S^{n-1})^N} F(r_1\theta_1, \ldots, r_N\theta_N) \prod_{i=1}^N r_i^{n-1} d\bar{\theta} dr$$
$$= \mathcal{F}_F(\mathbb{1}_{D_n}, \ldots, \mathbb{1}_{D_n}).$$

□

We summarize the results of Proposition 5.5 and 5.11 in the following theorem.

Theorem 5.12. *Let $\mu_1, \ldots, \mu_N \in \mathcal{P}_{[n]}$ and denote the density of μ_i by f_i. Suppose that $F : \bigotimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ satisfies Groemer's convexity condition and set*

$$\mathcal{F}_F(f_1, \ldots, f_N) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} F(x_1, \ldots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \ldots dx_N.$$

Then

$$\mathcal{F}(f_1, \ldots, f_N) \geq \mathcal{F}_F(f_1^*, \ldots, f_N^*).$$

Moreover, if $f_i = f_i^*$ and $\|f_i\| \leq 1$ for $i = 1, \ldots, N$, we also have

$$\mathcal{F}_F(f_1, \ldots, f_N) \geq \mathcal{F}_F(\mathbb{1}_{D_n}, \ldots, \mathbb{1}_{D_n}).$$

In the proof of Groemer's Theorem [16] (see Theorem 5.1) a main technical step is to show that the integrand $F(x_1, , \ldots, x_N) = \mathrm{vol}_n(\mathrm{conv}\{x_1, \ldots, x_N\})$ satisfies (GCC). We now show that this is also the case in our setting.

Let K be a symmetric convex body in $\mathbb{R}^n$. For $x_1, \ldots, x_N \in \mathbb{R}^n$, let $T(x_1, \ldots, x_N) = [x_1 \cdots x_N]$ be the $n \times N$ matrix with x_i as the columns. Furthermore, let $F : \bigotimes_{i=1}^N \mathbb{R}^n \to \mathbb{R}^+$ be the function

$$F(x_1, \ldots, x_N) := \mathrm{vol}_n(T(x_1, \ldots, x_N)K).$$

We can see that for any $S \in SL(n)$ we have

$$F(Sx_1, \ldots, Sx_N) = F(x_1, \ldots, x_N)$$

since for any $n \times n$ matrix $M \in SL(n)$ we have

$$\begin{aligned} F(Mx_1, \dots, Mx_N) &= \mathrm{vol}_n([Mx_1 \dots Mx_N]K) \\ &= \mathrm{vol}_n(M[x_1 \dots x_N]K) \\ &= |\det(M)| F(x_1, \dots, x_N) \\ &= F(x_1, \dots, x_N) \end{aligned}$$

Proposition 5.13. *Let F be defined by*

$$F(x_1, \dots, x_N) := \mathrm{vol}_n(T(x_1, \dots, x_N)K)$$

and let $\theta \in \mathbb{S}^{n-1}$ and $y_1, \dots, y_N \in \theta^\perp$. Set $Y := [y_1 \dots y_N]$. Furthermore, let $T_Y(t) := [y_i + t_i\theta]$ and define $F_Y : \mathbb{R}^N \to \mathbb{R}^+$ by

$$F_Y(t) = \mathrm{vol}_n(T_Y(t)K).$$

The function F_Y is even and convex. In particular, F satisfies Groemer's convexity condition.

Proof. The proof presented here (taken from [28]) is analogous of the proof given by Groemer in [16]. We start by proving that F_Y is even. Note that

$$[y_1 + t_1\theta \dots y_N + t_N\theta]K = \left\{ \sum_{i=1}^N k_i(y_i + t_i\theta) : (k_i) \in K \right\},$$

while

$$[y_1 - t_1\theta \dots y_N - t_N\theta]K = \left\{ \sum_{i=1}^N k_i(y_i - t_i\theta) : (k_i) \in K \right\}.$$

These sets are reflections of each other onto $\theta^\perp$, hence $F_Y(t) = F_Y(-t)$. For the second assertion, let $P_{\theta^\perp}$ be the orthogonal projection onto $\theta^\perp$. For any compact, convex set $A \subseteq \mathbb{R}^n$, define functions $f_A, g_A : P_{\theta^\perp} A \to \mathbb{R}$ by

$$f_A(y) := \sup\{\lambda : y + \lambda\theta \in A\}$$

and

$$g_A(y) := \inf\{\lambda : y + \lambda\theta \in A\}.$$

Then f_A is concave and g_A is convex. Furthermore, let $s, t \in \mathbb{R}^N$ and consider the functions

$$f_{T_Y(s)K}, g_{T_Y(s)K} : P_{\theta^\perp} T_Y(s)K \to \mathbb{R}$$

and

$$f_{T_Y(t)K}, g_{T_Y(t)K} : P_{\theta^\perp} T_Y(t)K \to \mathbb{R}$$

defined as f_A and g_A above. Since $P_{\theta^\perp}$ is the orthogonal projection on $\theta^\perp$, we have

$$P_{\theta^\perp} T_Y(s)K = P_{\theta^\perp}[y_i + s_i\theta]K = [y_i]K = P_{\theta^\perp}[y_i + s_i\theta]K = P_{\theta^\perp} T_Y(t)K.$$

Thus setting $D = P_{\theta^\perp} T_Y(s)K = P_{\theta^\perp} T_Y(t)K$, we can define $f, g : D \to \mathbb{R}$ by

$$f = \frac{1}{2} f_{T_Y(s)} + \frac{1}{2} f_{T_Y(t)}, \quad g = \frac{1}{2} g_{T_Y(s)} + \frac{1}{2} g_{T_Y(t)},$$

Set $\hat{K}$ to be the associated compact convex set

$$\hat{K} := \{y + \lambda\theta : y \in D, g(y) \leq \lambda \leq f(y)\}.$$

If we can show that $T_Y(s/2 + t/2)K \subseteq \hat{K}$, then we obtain

$$\begin{aligned}
&\mathrm{vol}_d(T_Y(s/2 + t/2)K) \\
&\qquad \leq \mathrm{vol}_d(\hat{K}) = \int_D f(y) - g(y) dy \\
&\qquad = \int_D \frac{1}{2} f_{T_Y(s)}(y) + \frac{1}{2} f_{T_Y(t)}(y) - \frac{1}{2} g_{T_Y(s)}(y) + \frac{1}{2} g_{T_Y(t)}(y) \\
&\qquad = \frac{1}{2} \int_D f_{T_Y(s)}(y) + g_{T_Y(s)}(y) + \frac{1}{2} \int_D f_{T_Y(s)}(y) - g_{T_Y(t)}(y) \\
&\qquad = \frac{1}{2} \mathrm{vol}_d(T_Y(s)K) + \frac{1}{2} \mathrm{vol}_d(T_Y(t)K).
\end{aligned}$$

which shows that F_Y is convex.

Indeed, let $x \in T_Y(s/2 + t/2)K$ so that for some $k = (k_1, \ldots, k_N) \in K$, we have

$$\begin{aligned}
x = \sum_{i=1}^N k_i(y_i + (s_i/2 + t_i/2)\theta) &= \sum_{i=1}^N k_i y_i + \sum_{i=1}^N k_i(s_i/2 + t_i/2)\theta \\
&= y + \sum_{i=1}^N k_i(s_i/2 + t_i/2)\theta.
\end{aligned}$$

Note that

$$y + \left(\sum_{i=1}^N k_i s_i\right)\theta = \sum_{i=1}^N k_i(y_i + s_i\theta) \in T_Y(s)K$$

which yields

$$g_{T_Y(s)}(y) \leq \sum_{i=1}^N k_i s_i \leq f_{T_Y(s)}(y), \quad g_{T_Y(t)}(y) \leq \sum_{i=1}^N k_i t_i \leq f_{T_Y(t)}(y).$$

Therefore, we finally have

$$\begin{aligned} g(y) &= \frac{1}{2} g_{T_Y(s)}(y) + \frac{1}{2} g_{T_Y(t)}(y) \\ &\leq \frac{1}{2} \sum_{i=1}^{N} k_i s_i + \frac{1}{2} \sum_{i=1}^{N} k_i t_i \\ &\leq \frac{1}{2} f_{T_Y(s)}(y) + \frac{1}{2} f_{T_Y(t)}(y) \\ &= f(y) \end{aligned}$$

which shows that $x = y + \sum_{i=1}^{N} k_i(s_i/2 + t_i/2)\theta \in \hat{K}$ and concludes this proof. $\square$

Finally, we have everthing we need to proof Theorem 5.2.

Proof of Theorem 5.2. From Proposition 5.13 we see that the integrand of

$$\mathcal{F}_C(f_1, \dots, f_N) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathrm{vol}_n([x_1, \dots, x_N]K) \prod_{i=1}^{N} f_i(x_i) dx_N \dots dx_1$$

given by $F = \mathrm{vol}_n([x_1 \dots x_N]K) = \mathrm{vol}_n(T(x_1, \dots, x_N)K)$ satsifies Groemer's Convexity Condition. Therefore, we can apply Theorem 5.12 which yields the desired inequality

$$\mathcal{F}_C(f_1, \dots, f_N) \geq \mathcal{F}_C(\mathbb{1}_{D_n}, \dots, \mathbb{1}_{D_n}),$$

if $\|f_i\|_\infty \leq 1$ for $i = 1, \dots, N$. $\square$

5.2 Randomized version of the L_p-Busemann-Petty Centroid Inequality

We follow [28] and describe the probabilistic setting. First, we consider a sequence of convex bodies $(K_N)_{N=n}^{\infty}$ with $K_N \subseteq \mathbb{R}^n$ and assume that $\mu_1, \mu_2, \dots$ are probability measures in $\mathcal{P}_{[n]}$ and f_i denotes the density of μ_i for $i = 1, 2, \dots$. All random vectors are defined on a common underlying probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. Furthermore, suppose that we have the following sequence of independent random vectors

- $X_1, X_2, \dots$ with X_i distributed according to f_i;
- $X_1^*, X_2^*, \dots$ with X_i^* distributed according to f_i^*;
- $Y_1, Y_2, \dots$ with Y_i distributed according to $\mathbb{1}_{D_n}$;

For $k = 1,2,3$ and for each $N \geq n$ let us denote the corresponding linear operators $T_N^{(k)} : \mathbb{R}^N \to \mathbb{R}^n$ represented by $n \times N$ matrices as

$$T_N^{(1)} = [X_1 \cdots X_N], \quad T_N^{(2)} = [X_1^* \cdots X_N^*], \quad T_N^{(3)} = [Y_1 \cdots Y_N]. \tag{5.1}$$

Then $(T_N^{(k)} K_N)_{N=n}^{\infty}$ is a sequence of random convex bodies in $\mathbb{R}^n$ for each $k = 1,2,3$. We now proof a corollary of Theorem 5.2 which enables us to obtain isoperimetric inequalities for non-random sets by using the classical strong law of large numbers (see Theorem 3.17) and a suitable choice of (K_N).

Corollary 5.14. *Suppose that $(K_N)_{N=n}^{\infty}$ is a sequence of convex bodies with $K_N \subseteq \mathbb{R}^N$. For each $k = 1,2,3$ and $N \geq n$, let $T_N^{(k)} : \mathbb{R}^N \to \mathbb{R}^n$ be the random linear operators defined by (5.1). Suppose $K^{(k)}$ are (random) convex bodies in $\mathbb{R}^n$ defined by the following*

$$K^{(k)} := \lim_{N\to\infty} T_N^{(k)} K_N \quad (a.s.) \tag{5.2}$$

for $k = 1,2,3$, where the convergence is in the Hausdorff metric. Let $M \in L_1(\Omega, \mathcal{S}, \mathbb{P})$ and suppose further that for each $k = 1,2,3$,

$$\mathrm{vol}_n(T_N^{(k)} K_N) \leq M \quad (a.s.). \tag{5.3}$$

Then

$$\mathbb{E}\mathrm{vol}_n(K^{(1)}) \geq \mathbb{E}\mathrm{vol}_n(K^{(2)})$$

and, if $\|f_i\|_\infty \leq 1$ for each $i = 1,2,\ldots$, then

$$\mathbb{E}\mathrm{vol}_n(K^{(2)}) \geq \mathbb{E}\mathrm{vol}_n(K^{(3)}).$$

Proof. In the notation of this section we see that Theorem 5.2 and its proof imply that for each $N \geq n$, we have

$$\mathbb{E}\mathrm{vol}_n(T_N^{(1)} K_N) \geq \mathbb{E}\mathrm{vol}_n(T_N^{(2)} K_N)$$

and if $\|f_i\|_\infty \leq 1$ for each $i = 1,2,\ldots$ then

$$\mathbb{E}\mathrm{vol}_n(T_N^{(2)} K_N) \geq \mathbb{E}\mathrm{vol}_n(T_N^{(3)} K_N).$$

Since $K^{(k)} = \lim_{N\to\infty} T_N^{(k)} K_N$ converges in the Hausdorff metric almost surely to some random convex body $K^{(k)}$ by assumption, we obtain

$$\mathrm{vol}_n(T_N^{(k)} K_N) \to \mathrm{vol}_n(K^{(k)}) \quad \text{as } N \to \infty$$

almost surely as $\mathrm{vol}_n(\cdot)$ is continuous with respect to the Hausdorff metric in $\mathcal{K}^n$. Furthermore, since $\mathrm{vol}_n(T_N^{(k)} K_N) \leq M$ almost surely by assumption, we have dominated convergence of $\mathrm{vol}_n(T_N^{(k)} K_N)$ (see Theorem 3.7). Thus, the above limit implies

$$\lim_{N\to\infty} \mathbb{E}\mathrm{vol}_n(T_N^{(k)} K_N) = \mathbb{E}\mathrm{vol}_n(K^{(k)})$$

which concludes the proof. □

Before we can finally proof a randomized version of the L_p-Busemann-Petty centroid inequality, we require a bit more notation. If $T : \mathbb{R}^N \to \mathbb{R}^n$ is a linear operator denote its adjoint by $T^t : \mathbb{R}^n \to \mathbb{R}^N$. If $K \subseteq \mathbb{R}^N$ is an arbitrary convex body, the support function of TK is given by

$$h(TK, y) = \sup\{\langle Tx, y\rangle : x \in K\} = \sup\{\langle x, T^t y\rangle : x \in K\} = h(K, T^t y) \tag{5.4}$$

for any $y \in \mathbb{S}^{n-1}$. Furthermore, if $T_N = [x_1, \dots, x_N]$, then $T_N^t : \mathbb{R}^n \to \mathbb{R}^N$ is given by

$$T_N^t y = (\langle x_1, y\rangle, \dots, \langle x_N, y\rangle) \quad (y \in \mathbb{R}^n)$$

In contrast to the definition of L_p-centroid bodies given in Chapter 4 , Paouris and Pivovarov [28] define the L_p-centroid body without the normalization constant $c_{n,p}$ as follows

Definition 5.15. *Let $K \subseteq \mathbb{R}^n$ be a bounded Borel measurable set with $\mathrm{vol}_n(K) = 1$. For $p \geq 1$, let $\Gamma_p(K)$ denote the L_p-centroid body of K, i.e., the body with support function*

$$h(\Gamma_p(K), y) = \left(\int_K |\langle x, y\rangle|^p dx\right)^p$$

Corollary 5.14 now gives a short proof of a randomized version of the L_p-Busemann-Petty centroid inequality.

Corollary 5.16 (Randomized L_p-Busemann-Petty Centroid Inequality)**.** *Let $K \subseteq \mathbb{R}^n$ be a bounded Borel measurable set with $\mathrm{vol}_n(K) = 1$. Then*

$$\mathrm{vol}_n(\Gamma_p(K)) \geq \mathrm{vol}_n(\Gamma_p(D_n)),$$

where D_n is the Euclidean ball of volume one.

Proof. We want to use Corollary 5.14 and, therefore, need to check if $\Gamma_p(K)$ and $\Gamma_p(D_n)$ satisfy (5.2) and (5.3).

Take $f_i = \mathbb{1}_K$ for $i = 1, 2, \dots$. Note that $f_i^* = \mathbb{1}_{D_n}$ and hence the random operators $T_N^{(2)}$ and $T_N^{(3)}$ have the same distribution. Let B_q^N denote the closed unit ball in l_q^N, where $1/p + 1/q = 1$. If we set $K = N^{-1/p} B_q^N$, the support function of $T_N^{(1)} K_N$ is

$$h(N^{-1/p} T_N^{(1)} B_q^N, y)^p = h(N^{-1/p} B_q^N, (T_N^{(1)})^t y)^p = \frac{1}{N} \sum_{i=1}^N |\langle X_i, y\rangle|^p$$

for each $y \in \mathbb{S}^{n-1}$, see (5.4). By the strong law of large numbers (see Theorem 3.17) the empirical mean converges to the actual mean almost surely

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} |\langle X_i, y\rangle|^p = \int_K |\langle x, y\rangle|^p dx \quad \text{(a.s.)}.$$

Thus, for any $y \in \mathbb{S}^{n-1}$, we have

$$\lim_{N\to\infty} h(N^{-1/p} T_N^{(1)} B_q^N, y) = \left(\int_K |\langle x, y\rangle|^p dx\right)^p \quad \text{(a.s.)}.$$

Theorem 2.20 states that pointwise convergence of support functions implies uniform convergence. Therefore, we see that (5.2) follows from

$$\Gamma_p(K) = \lim_{N\to\infty} N^{-1/p} T_N^{(1)} B^N \quad \text{(a.s.)},$$

in the Hausdorff metric. Finally, we denote by $R(K)$ the circumradius of K,

$$R(K) = \inf\{R > 0 : K \subseteq RB^n\}.$$

Since $|\langle X_i, y\rangle| \leq R(K)$, it holds that $N^{-1/p} T_N^{(1)} B_q^N \subseteq R(K)B^n$, hence (5.3) is satisfied as well. The same reasoning applies when $T_N^{(1)}$ is replaced by $T_N^{(2)}$ and K by D_n. Therefore, Corollary 5.14 yields that

$$\text{vol}(\Gamma_p(K)) = \mathbb{E}\text{vol}_n(\Gamma_p(K)) \geq \mathbb{E}\text{vol}_n(\Gamma_p(D_n)) = \text{vol}_n(\Gamma_p(D_n)).$$

$\square$

6 Randomized Polar L_p-Busemann-Petty Centroid Inequality for General Measures

In the last chapter of this thesis we present a dual, or polar, version of the randomized L_p-Busemann-Petty centroid inequality and follow [12]. One celebrated and fundamental result in the classical setting is the Blaschke-Santalò inequality (see Theorem 4.6). It states that among symmetric convex bodies K of fixed volume, the volume of the polar body K° is maximized by the Euclidean ball and, due to $SL(n)$-invariance, also by ellipsoids.

The following theorem extends this inequality to random sets.

Theorem 6.1. *Let $N, n \geq 1$. In the class of N-tuples $(X_1, \ldots, X_N)$ of independent random vectors in $\mathbb{R}^n$ whose laws have a density bounded by one, the expectation of the volume of the set*

$$(\mathrm{conv}\{\pm X_1, \ldots, \pm X_N\})^\circ$$

is maximized by N independent random vectors uniformly distributed in the Euclidean ball $D_n \subseteq \mathbb{R}^N$ of volume one.

Note that in this chapter the density of a measure on $\mathbb{R}^n$ always refers to the density with respect to the Lebesgue measure on $\mathbb{R}^n$.

To see that the above theorem actually generalizes the Blaschke-Santalò inequality, let K be a symmetric convex body and assume that $\mathrm{vol}_n(K) = 1$. Let $X_1, \ldots, X_N, \ldots$ be a sequence of independent random vectors uniformly distributed in K which means that the law of X_i is λ_K (the Lebesgue measure restricted to K) which density $\mathbb{1}_K$. One can show that $\mathrm{conv}\{\pm X_1, \ldots, \pm X_N\}$ converges almost surely to K in the Hausdorff metric as $N \to \infty$. Of course, this also holds if $K = D_n$ and therefore, we derive from the theorem, in the limit, that $\mathrm{vol}_n(K^\circ) \leq \mathrm{vol}_n(D_n^\circ)$ under the assumption that $\mathrm{vol}_n(K) = \mathrm{vol}_n(D_n) = 1$. This is exactly the Blaschke-Santalò inequality.

However, Cordero-Erausquin, Fradelizi, Paouris and Pivovarov [12] proved a more general inequality which extends the above theorem in three ways

1. The result holds in distribution and not only in expectation;

2. The Lebesgue measure can be replaced by any rotationally invariant, radially decreasing measure;
3. We can perform more general (convex) operations than the convex hull.

Before we can state the theorem we need two more definitions.

Definition 6.2. *A convex body $K \in \mathbb{R}^N$ is unconditional if it is invariant under coordinate reflections, i.e., if $(k_1, \dots, _N) \in K$, then $(\pm k_1, \dots, \pm k_N) \in K$.*

Similarly to Chapter 5, $\mathcal{P}_{[n]}$ denotes the class of all Borel probability measures on $\mathbb{R}^n$ that have an L^1-density with respect to Lebesgue measure bounded by 1. This set includes Lebesgue measure restricted to sets of volume one, and after proper rescaling, any Borel probability measure that is absolutely continuous with respect to Lebesgue measure and that has a bounded density.

Theorem 6.3. *Let K be an unconditional convex body in $\mathbb{R}^N$ and ν be a radical measure on $\mathbb{R}^n$ of the form $d\nu(x) = \rho(|x|)dx$ with $\rho : [0, +\infty) \to [0, +\infty)$ decreasing. If $X_1, \dots, X_N$ are N independent random vectors in $\mathbb{R}^n$ whose laws are in $\mathcal{P}_{[n]}$, then*

$$\mathbb{E}[\nu\left(([X_1 \cdots X_N]K)^\circ\right)] \leq \mathbb{E}[\nu\left(([Z_1 \cdots Z_N]K)^\circ\right)],$$

where $Z_1, \dots, Z_N$ are independent random vectors uniformly distributed in the Euclidean ball $D_n \subseteq \mathbb{R}^n$ of volume one.

Moreover, if $\rho^{-1/(n+1)} : [0, +\infty) \to [0, +\infty]$ is convex, then, with the same notation, we also have that

$$\forall t > 0, \quad \mathbb{P}[\nu\left(([X_1 \cdots X_N]K)^\circ\right) \geq t] \leq \mathbb{P}[\nu\left(([Z_1 \cdots Z_N]K)^\circ\right) \geq t].$$

The proof of this theorem is the content of the next section.

6.1 A General Inequality for Random Polar Convex Bodies

We start this section with two definitions due to Borell [5], [6] which are required for some preliminary lemmas.

Definition 6.4. *Let $s \in [-\infty, 1]$. A Borel measure μ on $\mathbb{R}^n$ is called s-concave if*

$$\mu((1-\lambda)A + \lambda B) \geq ((1-\lambda)\mu(A)^s + \lambda\mu(B)^s)^{1/s}$$

for all compact sets $A, B \subseteq \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$. For $s = 0$, one says that μ is log-concave and the inequality is read as $\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^\lambda$. For $s = -\infty$, the measure is said to be convex and the inequality is replaced by

$$\mu((1-\lambda)A + \lambda B) \geq \min\{\mu(A), \mu(B)\}.$$

Definition 6.5. *A non-negative, non-identically zero function ψ is γ-concave if*

1. *for $\gamma > 0$, ψ^γ is concave on $\{\psi > 0\}$,*
2. *for $\gamma = 0$, $\log \psi$ is concave on $\{\psi > 0\}$,*
3. *for $\gamma < 0$, ψ^γ is convex on $\{\psi > 0\}$.*

The next theorem and proposition are stated for completeness only as we are mostly interested in the subsequent corollary which plays a key role in the proof of Theorem 6.3. For complete proofs of these results we refer to [12]. The first one is an extension of a result by Campi and Gronchi [9] which uses shadow systems and was originally proved for Lebesgue measure.

Theorem 6.6. *Let ν be a measure on $\mathbb{R}^n$ with a density ψ which is even and γ-concave on $\mathbb{R}^n$ for some $\gamma \geq -1/(n+1)$. Let (K_t) be a shadow system of centrally symmetric convex sets. Then the function $t \mapsto \nu(K_t^\circ)^{-1}$ is convex.*

Theorem 6.7. *Let n, N be positive integers and K be a centrally symmetric closed convex set in $\mathbb{R}^n \times \mathbb{R}^N$. Let $\theta \in \mathbb{S}^{n-1}$. For $t \in \mathbb{R}^N$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^N$, we define $P_t(x, y) = x + \langle y, t\rangle\theta$ and $K_t = P_t(K)$. Let ν be a measure on $\mathbb{R}^n$ with a density ψ with respect to Lebesgue measure that is even and $-1/(n+1)$-concave on $\mathbb{R}^n$. Then*

1. *$t \mapsto \nu(K_t^\circ)^{-1}$ is convex on $\mathbb{R}^N$,*
2. *if K and ψ are symmetric with respect to $\theta^\perp$, then $t \mapsto \nu(K_t^\circ)^{-1}$ is even and convex on $\mathbb{R}^N$.*

Corollary 6.8. *Let $r \geq 0$, K be an origin-symmetric convex set in $\mathbb{R}^N$, let $\theta \in \mathbb{S}^{n-1}$ and $y_1, \ldots, y_N \in \theta^\perp$. Let ν be a measure on $\mathbb{R}^n$ with a density ψ which is $-1/(n+1)$-concave on $\mathbb{R}^n$, even and symmetric with respect to $\theta^\perp$. Then, the map*

$$(t_1, \ldots, t_N) \mapsto \nu(([y_1 + t_1\theta \cdots y_N + t_N\theta]K + rB^n)^\circ)^{-1}$$

is even and convex on $\mathbb{R}^N$.

The following lemma from [29] will be used in the last stage of the proof of Theorem 6.10.

Lemma 6.9. *Let $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ be a function that is coordinate-wise decreasing in the sense that*

$$\begin{aligned} &\forall x_1, \ldots, x_N \in \mathbb{R}^n, \quad (0 \leq s_i \leq t_i, \forall i \leq N) \\ &\quad \Rightarrow F(s_1x_1, \ldots, s_Nx_N) \geq F(t_1x_1, \ldots, t_Nx_N). \end{aligned}$$

If $g_1, \ldots, g_N : \mathbb{R}^+ \to [0,1]$ are nonnegative, bounded by 1, integrable functions with $\int_{\mathbb{R}^n} g_i(|x|)dx = 1$ for all $i = 1, \ldots, N$, then

$$\int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N) \prod_{i=1}^{N} g_i(|x_i|) dx_1 \ldots dx_N \\ \leq \int_{(\mathbb{R}^n)^N} F(x_1, \ldots, x_N) \prod_{i=1}^{N} \mathbb{1}_{[0,r_n]}(|x_i|) dx_1 \ldots dx_N,$$

where r_n is the radius of D_n.

Proof. By Fubini's theorem (see Theorem 3.9), it is sufficient to treat each coordinate successively. Therefore, the statement boils down to

$$\int_{\mathbb{R}^n} F(x) g(|x|) dx \leq \int_{\mathbb{R}^n} F(x) \mathbb{1}_{[0,r_n]}(|x|) dx,$$

for $N = 1$ where g is a nonnegative function bounded by 1 with $\int_{\mathbb{R}^n} g(|x|)dx = 1$, and F is an even function on $\mathbb{R}^n$ satisfying $F(sx) \geq F(x)$ for all $x \in \mathbb{R}^n$ and $s \in [0,1]$. This implies that the function $r \mapsto F(rx_0)$ is decreasing on $\mathbb{R}^+$ for any fixed $x_0 \in \mathbb{R}^n$. We change to polar coordinates and integrate, where it suffices to prove that

$$\int_0^{+\infty} f(r) g(r) r^{n-1} dr \leq \int_0^{r_n} f(r) r^{n-1} dr,$$

with f a decreasing function and g with values in $[0,1]$ with $\int_0^{+\infty} g(r) r^{n-1} dr = \int_0^{r_n} r^{n-1} dr$. Let $\alpha(r) := (\mathbb{1}_{[0,r_n]}(r) - g(r)) r^{n-1}$ and note that

$$\int_0^{+\infty} f(r)\alpha(r) dr = \int_0^{+\infty} (f(r) - f(r_n))\alpha(r) dr \geq 0$$

since the integrand in the second integral is pointwise nonnegative. □

Let us now recall the spherically-invariant measures ν on $\mathbb{R}^n$ with

$$d\nu(x) = \rho(|x|)dx \quad \text{with } \rho : [0, +\infty) \to [0, +\infty) \text{ decreasing,} \tag{6.1}$$

together with the subclass of those measures of the form

$$d\nu(x) = k^{-(n+1)}(|x|)dx \quad \text{with } k : [0, +\infty) \to [0, +\infty] \text{ convex increasing.} \tag{6.2}$$

The following statement is even more general than Theorem 6.3 which it is a special case of the second point below, with $r = 0$.

Theorem 6.10. *Let $X_1, \ldots, X_N$ be N independent random vectors in $\mathbb{R}^n$ whose laws are in $\mathcal{P}_{[n]}$ and let $r \geq 0$.*

1. *If K is an origin-symmetric convex body in $\mathbb{R}^N$ and ν a measure on $\mathbb{R}^n$ of the form (6.1), then*

$$\mathbb{E}[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right)] \leq \mathbb{E}[\nu\left(([X_1^* \cdots X_N^*]K + rB^n)^\circ\right)], \tag{6.3}$$

where $X_1^*, \ldots, X_N^*$ are independent random vectors in $\mathbb{R}^n$ whose densities are the symmetric decreasing rearrangement of the densities of $X_1, \ldots, X_N$. Moreover, if ν is of the form (6.2), we also have for every $t \geq 0$,

$$\mathbb{P}[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right) \geq t] \leq \mathbb{P}[\nu\left(([X_1^* \cdots X_N^*]K + rB^n)^\circ\right) \geq t]. \tag{6.4}$$

2. *If K is an unconditional convex body in $\mathbb{R}^N$ and ν a measure on $\mathbb{R}^n$ of the form (6.1), then*

$$\mathbb{E}[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right)] \leq \mathbb{E}[\nu\left(([Z_1 \cdots Z_N]K + rB^n)^\circ\right)], \tag{6.5}$$

where $Z_1, \ldots, Z_N$ are independent random vectors distributed according to λ_{D_n}. Moreover, if ν is of the form (6.2), we also have for every $t \geq 0$,

$$\mathbb{P}[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right) \geq t] \leq \mathbb{P}[\nu\left(([Z_1 \cdots Z_N]K + rB^n)^\circ\right) \geq t]. \tag{6.6}$$

Proof. The proof will also be split into two parts.

1. Let G and F be defined on $(\mathbb{R}^n)^N$ by

$$G(x_1, \ldots, x_N) = \nu(([x_1 \cdots x_N]K + rB^n)^\circ) \quad \text{and} \quad F = \mathbb{1}_{\{G > \alpha\}}.$$

Furthermore, let $\theta \in \mathbb{S}^{n-1}$ and $Y = (y_1, \ldots, y_N) \subseteq (\theta^\perp)^N$ and let $F_{\theta,Y}$ and $G_{\theta,Y}$ be the restrictions of F and G given by

$$G_{\theta,Y} = G(y_1 + t_1\theta, \ldots, y_N + t_N\theta) \quad \text{and} \quad F_{\theta,Y} = \mathbb{1}_{\{G_{\theta,Y} > \alpha\}}.$$

We first assume the stronger assumption from equation (6.2). Due to the rotational invariance and the convexity assumption on the density, we see that the assumptions of Corollary 6.8 are satisfied. Therefore,

$$(t_1, \ldots, t_N) \mapsto \nu(([y_1 + t_1\theta \cdots y_N + t_N\theta]K + rB^n)^\circ)^{-1} = G_{\theta,Y}^{-1}$$

is even and convex on $\mathbb{R}^N$. Hence, $G_{\theta,Y}$ and therefore $F_{\theta,Y}$ are quasi-concave and even. We can now apply Proposition 5.6 to the function F to see that

$$F = \mathbb{1}_{\{G > \alpha\}} = \mathbb{1}_{\{\nu(([x_1 \cdots x_N]K + rB^n)^\circ) \geq \alpha\}} \leq \mathbb{1}_{\{\nu\left(([x_1^* \cdots x_N^*]K + rB^n)^\circ\right) \geq \alpha\}}$$

which yields

$$\mathbb{P}[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right) \geq t] \leq \mathbb{P}[\nu\left(([X_1^* \cdots X_N^*]K)^\circ\right) \geq t].$$

If ν satisfies the weaker condition from equation (6.1), we apply the previous result in the case of Lebesgue measure restricted to the Euclidean ball with radius $R > 0$. The density $\mathbb{1}_{RB^n}$ satisfies the assumptions since it is $+\infty$-concave and, therefore, $-1/(n+1)$-concave. We have

$$\begin{aligned}\mathbb{P}\left[|([X_1 \cdots X_N]K + rB^n)^\circ \cap RB^n| \geq t\right] \\ \leq \mathbb{P}\left[|([X_1^* \cdots X_N^*]K + rB^n)^\circ \cap RB^n| \geq t\right]\end{aligned}$$

for all $t > 0$. Note that for $t \in (0, \rho(0))$, the set $\{\rho \geq t\}$ is a Euclidean ball which can be open or closed but the difference is of Lebesgue measure zero which is why we can choose closed balls with corresponding radius $R(t)$. For any Borel set $A \subseteq \mathbb{R}^n$, we can write by Fubini (see Theorem 3.9)

$$\nu(A) = \int_0^{+\infty} |A \cap \{\rho \geq t\}| dt = \int_0^{\rho 0} |A \cap R(t)B^n| dt,$$

which gives

$$\begin{aligned}&\mathbb{E}\left[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right)\right] \\ &= \mathbb{E}\left[\int_0^{+\infty} |([X_1 \cdots X_N]K + rB^n)^\circ \cap R(t)B^n| dt\right] \\ &= \int_0^{+\infty}\int_0^{+\infty} \mathbb{P}\left[|([X_1 \cdots X_N]K + rB^n)^\circ \cap R(t)B^n| > \alpha\right] d\alpha dt \\ &\leq \int_0^{+\infty}\int_0^{+\infty} \mathbb{P}\left[|([X_1^* \cdots X_N^*]K + rB^n)^\circ \cap R(t)B^n| > \alpha\right] d\alpha dt \\ &= \mathbb{E}\left[\int_0^{+\infty} |([X_1^* \cdots X_N^*]K + rB^n)^\circ \cap R(t)B^n| dt\right] \\ &= \mathbb{E}\left[\nu\left(([X_1^* \cdots X_N^*]K + rB^n)^\circ\right)\right].\end{aligned}$$

This finishes the first part.

2. In the last step we arrived at radially decreasing probability distributions. It remains to go to the uniform distributions on D_n with C being an unconditional convex body. In this case, the functions F and G defined above are coordinate-wise decreasing in the sense that

$$\begin{aligned}\forall x_1, \ldots, x_N \in \mathbb{R}^n, \quad (0 \leq s_i \leq t_i, \forall i \leq N) \\ \Rightarrow F(s_1 x_1, \ldots, s_N x_N) \geq F(t_1 x_1, \ldots, t_N x_N).\end{aligned}$$

This follows because for such s_i's and t_i's the unconditionality of K implies that, for every $x_1, \ldots, x_N \in \mathbb{R}^n$,

$$[s_1 x_1 \cdots s_N x_N]K \subseteq [t_1 x_1 \cdots t_N x_N]K.$$

Therefore, the functions F and G satisfy the assumptions of Lemma 6.9, where g_i is the law of X_i^* and of the function F. Note that the the law of X_i^* satisfies the assumptions since the density of X_i is bounded by 1 which implies that its radial rearrangement is also bounded by 1. This yields

$$\mathbb{P}[\nu\left(([X_1^* \cdots X_N^*]K + rB^n)^\circ\right) \geq t] \leq \mathbb{P}[\nu\left(([Z_1 \cdots Z_N]K + rB^n)^\circ\right) \geq t]$$

which shows that

$$\mathbb{P}[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right) \geq t] \leq \mathbb{P}[\nu\left(([Z_1 \cdots Z_N]K + rB^n)^\circ\right) \geq t].$$

If we apply the same argument to the function G instead of F we get

$$\mathbb{E}[\nu\left(([X_1^* \cdots X_N^*]K + rB^n)^\circ\right)] \leq \mathbb{E}[\nu\left(([Z_1 \cdots Z_N]K + rB^n)^\circ\right)]$$

which shows that

$$\mathbb{E}[\nu\left(([X_1 \cdots X_N]K + rB^n)^\circ\right)] \leq \mathbb{E}[\nu\left(([Z_1 \cdots Z_N]K + rB^n)^\circ\right)].$$

Alternatively, we could have used the same argument as in the first part of this proof.

□

6.2 Randomized Polar L_p-Busemann-Petty Centroid Inequality

Cordero-Erausquin, Fradelizi, Paouris and Pivovarov also showed in [12] that we can pass to the limit in Theorem 6.10 when there is almost sure convergence in the Hausdorff metric. Before we can state and prove this theorem we need two small lemmas where the first one is a standard result.

Lemma 6.11. *Let $K, L, K_1, K_2, \cdots \in \mathcal{K}_o^n$ be such that $K_N \to K$ as $N \to \infty$ in the Hausdorff metric. Then*

1. *$K_N^\circ \to K^\circ$ as $N \to \infty$,*
2. *$K_N \cap L \to K \cap L$ as $N \to \infty$,*
3. *$K_N + L \to K + L$ as $N \to \infty$,*

with convergence in the Hausdorff metric.

Lemma 6.12. *Let ν be a measure on $\mathbb{R}^n$ with a spherically-symmetric, decreasing density. Then ν is continuous on $\mathcal{K}_o^n$ with respect to the Hausdorff metric.*

Proof. It is sufficient to show continuity for sets included in some compact set and we may assume that ν has a density of the form $f_\nu = \frac{d\nu}{dx}$ by uniform approximation as

$$f_\nu(x) = \sum_{j=1}^{M} a_j \mathbb{1}_{r_j B^n}(x) \quad (x \in \mathbb{R}^n),$$

where $a_j > 0, j = 1, \ldots, M$ and $r_1 > r_2 > \cdots > r_M > 0$. Suppose that $K, K_1, K_2, \cdots \in \mathcal{K}_o^n$ and that $K_N \to K$ as $N \to \infty$ in the Hausdorff metric. Applying Lemma 6.11 as $N \to \infty$ yields

$$\nu(K_N) = \sum_{j=1}^{M} a_j |K_N \cap (r_j B^n)| \to \sum_{j=1}^{M} a_j |K \cap (r_j B^n)| = \nu(K).$$

□

This brings us to the main result of this section which we then use to recover Blaschke-Santalò type inequalities, especially the Polar L_p-Busemann-Petty centroid inequality.

Theorem 6.13. *Let (X_i) and (Z_i) be sequences of independent random vectors in $\mathbb{R}^n$ with each X_i distributed according to the same fixed $\mu \in \mathcal{P}_{[n]}$ and each Z_i according to λ_{D_n}. Assume that $K_N, K_{N+1}, \ldots$ are unconditional convex bodies with $K_N \subseteq \mathbb{R}^N$, $N = n, n+1, \ldots,$ such that*

$$[X_1 \cdots X_N]K_N \text{ converges to } \bigotimes_{i=1}^{\infty} \mu\text{-a.s. in the Hausdorff metric} \tag{6.7}$$

and

$$[Z_1 \cdots Z_N]K_N \text{ converges to } \bigotimes_{i=1}^{\infty} \lambda_{D_n}\text{-a.s. in the Hausdorff metric.} \tag{6.8}$$

Then, if ν is a measure on $\mathbb{R}^n$ with a spherically-symmetric, decreasing density, we have

$$\mathbb{E}\left[\nu\left(\left(\lim_{N\to\infty}[X_1 \cdots X_N]K_N\right)^\circ\right)\right] \le \mathbb{E}\left[\nu\left(\left(\lim_{N\to\infty}[Z_1 \cdots Z_N]K_N\right)^\circ\right)\right] \tag{6.9}$$

Proof. Let $\epsilon > 0$ and note that

$$\epsilon B^n \subseteq [X_1 \cdots X_N]K_N + \epsilon B^n$$

hence

$$\nu\left(([X_1 \cdots X_N]K_N + \epsilon B^n)^\circ\right) \le \nu(\epsilon^{-1} B^n)$$

for each $N \geq n$. Analogously, this holds for $Z_1, \ldots, Z_N$. We use dominated convergence (see Theorem 3.7), Lemmas 6.11 and 6.12 together with Theorem 6.10 to see that

$$\begin{aligned}
&\mathbb{E}\nu\left(\left(\lim_{N\to\infty}[X_1\cdots X_N]K_N+\epsilon B^n\right)^\circ\right)\\
&\quad=\mathbb{E}\lim_{N\to\infty}\nu\left(([X_1\cdots X_N]K_N+\epsilon B^n)^\circ\right)\\
&\quad=\lim_{N\to\infty}\mathbb{E}\nu\left(([X_1\cdots X_N]K_N+\epsilon B^n)^\circ\right)\\
&\quad\leq\lim_{N\to\infty}\mathbb{E}\nu\left(([Z_1\cdots Z_N]K_N+\epsilon B^n)^\circ\right)\\
&\quad=\mathbb{E}\lim_{N\to\infty}\nu\left(([X_1\cdots X_N]K_N+\epsilon B^n)^\circ\right)\\
&\quad=\mathbb{E}\nu\left(\left(\lim_{N\to\infty}[X_1\cdots X_N]K_N+\epsilon B^n\right)^\circ\right).
\end{aligned}$$

In the case that $\mathbb{E}\nu\left((\lim_{N\to\infty}[Z_1\cdots Z_N]K_N)^\circ\right) = \infty$, the result holds trivially. Otherwise, since

$$\lim_{N\to\infty}[Z_1\cdots Z_N]K_N \subseteq \lim_{N\to\infty}[Z_1\cdots Z_N]K_N+\epsilon B^n,$$

we have

$$\nu\left(\left(\lim_{N\to\infty}[Z_1\cdots Z_N]K_N+\epsilon B^n\right)^\circ\right)\leq\nu\left(\left(\lim_{N\to\infty}[Z_1\cdots Z_N]K_N\right)^\circ\right)$$

for each $\epsilon > 0$. If we use dominated convergence again and let $\epsilon \to 0$ we finally obtain

$$\mathbb{E}\left[\nu\left(\left(\lim_{N\to\infty}[X_1\cdots X_N]K_N\right)^\circ\right)\right]\leq\mathbb{E}\left[\nu\left(\left(\lim_{N\to\infty}[Z_1\cdots Z_N]K_N\right)^\circ\right)\right].$$

□

We already presented two definitions of the L_p-centroid body in this thesis, but we define it here one last time for measures $\mu \in \mathcal{P}_{[n]}$.

Definition 6.14. *The L_p-centroid body $\Gamma_p(\mu)$ of a measure $\mu \in \mathcal{P}_{[n]}$ is the convex body with support function*

$$h(\Gamma_p(\mu), y) = \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^p d\mu(x)\right)^{1/p} \qquad (y \in \mathbb{R}^n).$$

With that we can finally present the randomized polar L_p-Busemann-Petty centroid inequality which was stated as a corollary to Theorem 6.13 in [12].

Corollary 6.15. *Let ν be a measure on $\mathbb{R}^n$ with a spherically-symmetric, decreasing density. Let $\mu \in \mathcal{P}_{[n]}$, $p \geq 1$, and $\Gamma_p(\mu)$ be the L_p-centroid body of μ. Then,*

$$\nu(\Gamma_p^\circ(\mu)) \leq \nu(\Gamma_p^\circ(\lambda_{D_n})).$$

Proof. Recall from Chapter 5 that if the X_i's are sampled according to μ, then

$$\Gamma_p(\mu) = \lim_{N\to\infty} N^{-1/p}[X_1 \cdots X_N]B_q^N,$$

where $1/p + 1/q = 1$ and convergence occurs almost surely in the Hausdorff metric. This follows from the strong law of large numbers (see Theorem 3.17) as demonstrated in the proof of Theorem 5.16. Therefore, we can use Theorem 6.13 as the B_q^N are unconditional convex bodies and the measure ν is spherically-symmetric with decreasing density by assumption. Thus,

$$\begin{aligned}
\nu(\Gamma_p^\circ(\mu)) &= \mathbb{E}\nu\left(\Gamma_p^\circ(\mu)\right) \\
&= \mathbb{E}\nu\left(\left(\lim_{N\to\infty} N^{-1/p}[X_1 \cdots X_N]B_q^N\right)^\circ\right) \\
&\leq \mathbb{E}\nu\left(\left(\lim_{N\to\infty} N^{-1/p}[Z_1 \cdots Z_N]B_q^N\right)^\circ\right) \\
&= \mathbb{E}\nu\left(\Gamma_p^\circ(\lambda_{D_n})\right) = \nu(\Gamma_p^\circ(\lambda_{D_n})).
\end{aligned}$$

□

7 Outlook: Randomized Isoperimetric Inequalities for $p \in (-1, 1)$

To conclude this thesis we give a short overview of very recent discoveries in the field of (randomized) isoperimetric inequalities for sets that interpolate between intersection bodies and dual L_p-centroid bodies due to Adamczak, Paouris, Pivovarov and Simanjuntak [1]. This gap is exactly at $p \in (-1, 1)$ and, especially the setting of $p < 1$, is of significant interest in the L_p-Brunn-Minkowski theory and extends the results presented in previous chapters. Since we only state the most important definitions and result we refer to [1] for more details and complete proofs and to [32] for general definitions and results in the dual Brunn-Minkowski theory. Additionally, we use the same notation as in Chapter 6.

Similarly to the Brunn-Minkowski theory, which deals with the behavior of the volume of Minkowski sums of convex bodies, the dual Brunn-Minkowski theory deals with star-shaped bodies and radial addition.

Definition 7.1. *We call a set K in $\mathbb{R}^n$ star-shaped if $0 \in K$ and $\alpha x \in K$ whenever $x \in K$ and $\alpha \in [0, 1]$. The radial function of a star-shaped set K is defined as $\rho(K, u) = \sup\{\alpha \geq 0 : \alpha u \in K\}$ for $u \in \mathbb{S}^{n-1}$. Furthermore, K is a star-body if it is a compact, star-shaped set with the origin in its interior and its radial function is continuous.*

Like centroid bodies in the Brunn-Minkowski theory, intersection bodies play a crucial role in the dual Brunn-Minkowski theory.

Definition 7.2. *Let K be a star body. The intersection body of K, denoted $I(K)$, is the star body whose support function is*

$$\rho(I(K), u) = \mathrm{vol}_{n-1}(K \cap u^{\perp}).$$

One of the most important inequalities for intersection bodies is the Busemann intersection inequality.

Theorem 7.3 (Busemann Intersection Inequality). *Let K be a compact, nonempty subset of $\mathbb{R}^n$. Then*

$$\int_{\mathbb{S}^{n-1}} \mathrm{vol}_{n-1}(K \cap u^{\perp})^n du \leq \frac{\omega_{n-1}^n}{\omega_n^{n-1}} \mathrm{vol}_n(K)^{n-1},$$

where du denotes integration with respect to the normalized Haar probability measure on $\mathbb{S}^{n-1}$.

Furthermore, let us recall the Lutwak-Zhang inequality presented in Chapter 4 which states that for $1 \leq p \leq \infty$,

$$\mathrm{vol}_n(\Gamma_p^\circ(K)) \leq \mathrm{vol}_n(\Gamma_p^\circ(K^*));$$

where K^* is the dilate of the unit ball centered at the origin of the same volume as K.

One of the main results in [1] establishes a sharp isoperimetric inequality that extends the Lutwak-Zhang inequality to the case $p \in (0,1)$ and, therefore, provides a method to partially bridge the gap between the Busemann intersection inequality and the Lutwak-Zhang inequality.

To state this result we define the dual L_p-centroid body and its empirical version similarly to the previous chapter.

Definition 7.4. *For $f \in \mathcal{P}_{[n]}$ and $p \in (-1,1)$, define the dual L_p-centroid body $\Gamma_p^\diamondsuit(f)$ via its radial function with $vol_n(K) = 1$:*

$$\rho^{-p}(\Gamma_p^\diamondsuit(f), u) = \int_{\mathbb{R}^n} |\langle x, u\rangle|^p dx.$$

The bodies $\Gamma_p^\diamondsuit(K)$ interpolate between intersection bodies and polar L_p-centroid bodies using

$$\rho(I(K), u) = \lim_{p \to +1} \frac{p+1}{1} \int_K |\langle x, u\rangle|^p dx.$$

Note that for $p < 1$, the dual L_p-centroid body need not to be convex (it is if K is an origin-symmetric convex body).

Definition 7.5. *Let $N > n$ and consider independent random vectors $X_1, \ldots, X_N$ according to f above. We define the empirical version $\Gamma_{p,N}^\diamondsuit(f)$ as*

$$\rho^{-p}(\Gamma_{p,N}^\diamondsuit(f), u) = \frac{1}{N}\sum_{i=1}^N |\langle X_i, u\rangle|^p.$$

Theorem 7.6 ([1], Theorem 2.1)**.** *Let $f \in \mathcal{P}_{[n]}$ and let $0 < p < 1$. Then*

$$\mathrm{vol}_n(\Gamma_p^\diamondsuit(f)) \leq \mathrm{vol}_n(\Gamma_p^\diamondsuit(f^*)).$$

Moreover,

$$\mathbb{E}\left(\mathrm{vol}_n(\Gamma_{p,N}^\diamondsuit(f))\right) \leq \mathbb{E}\left(\mathrm{vol}_n(\Gamma_{p,N}^\diamondsuit(f^*))\right).$$

In contrast to the empirical approach in Chapter 6 where the non-random inequalities inspired the development of their empirical versions, Theorem 7.6 relies on first establishing the empirical version. In addition to the case $0 < p < 1$, Adamczak, Paouris, Pivovarov and Simanjuntak not only proved a generalization of Theorem 7.6 for $p \geq 0$ but also proved a first result for $p \in [-1, 0)$ where $n/|p| \in \mathbb{N}$. However, we refer to [1] for these results due to the limitation of this thesis.

Bibliography

[1] R. Adamczak, G. Paouris, P. Pivovarov, and P. Simanjuntak. *From intersection bodies to dual centroid bodies: a stochastic approach to isoperimetry*. 2022. DOI: 10.48550/ARXIV.2211.16263.

[2] Z. Artstein and R. A. Vitale. "A Strong Law of Large Numbers for Random Compact Sets". In: *The Annals of Probability* 3.5 (1975), pp. 879–882. DOI: 10.1214/aop/1176996275.

[3] F. Besau, T. Hack, P. Pivovarov, and F. E. Schuster. *Spherical centroid bodies*. 2019. DOI: 10.48550/ARXIV.1902.10614.

[4] W. Blaschke. "Affine Geometrie XIV: Eine Minimumaufgabe für Legendres Trägheitsellipsoid". In: *Ber. Verh. Sächs. Akad. Leipzig, Math.-Phys. Kl.* 70 (1918), pp. 72–75.

[5] C. Borell. "Convex measures on locally convex spaces". In: *Arkiv för Matematik* 12.1 (1974), pp. 239–252. DOI: 10.1007/BF02384761.

[6] C. Borell. "Convex set functions ind-space". In: *Periodica Mathematica Hungarica* 6.2 (1975), pp. 111–136. DOI: 10.1007/BF02018814.

[7] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger. "A general rearrangement inequality for multiple integrals". In: *Journal of Functional Analysis* 17.2 (1974), pp. 227–237. DOI: 10.1016/0022-1236(74)90013-5.

[8] A. Burchard. *A short course on rearrangement inequalities*. http://www.math.utoronto.ca/almut/rearrange.pdf. [Online; accessed 11-November-2022]. 2009.

[9] S. Campi and P. Gronchi. "On volume product inequalities for convex sets". In: *Proceedings of the American Mathematical Society* 134 (2006), pp. 2393–2402. DOI: 10.2307/4098278.

[10] S. Campi and P. Gronchi. "The Lp-Busemann–Petty Centroid Inequality". In: *Advances in Mathematics* 167.1 (2002), pp. 128–141. DOI: 10.1006/aima.2001.2036.

[11] M. Christ. "Estimates for the k-plane Transform". In: *Indiana University Mathematics Journal* 33.6 (1984), pp. 891–910.

[12] D. Cordero-Erausquin, M. Fradelizi, G. Paouris, and P. Pivovarov. "Volume of the polar of random sets and shadow systems". In: *Mathematische Annalen* 362.3 (2015), pp. 1305–1325. DOI: 10.1007/s00208-014-1156-x.

[13] R. M. Dudley. *Real Analysis and Probability*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002. DOI: 10.1017/CBO9780511755347.

[14] R. J. Gardner. *Geometric Tomography*. 2nd ed. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2006. DOI: 10.1017/CBO9781107341029.

[15] R. J. Gardner, D. Hug, W. Weil, and D. Ye. "The dual Orlicz–Brunn–Minkowski theory". In: *Journal of Mathematical Analysis and Applications* 430.2 (2015), pp. 810–829. DOI: 10.1016/j.jmaa.2015.05.016.

[16] H. Groemer. "On the mean value of the volume of a random polytope in a convex set". In: *Archiv der Mathematik* 25.1 (1974), pp. 86–90. DOI: 10.1007/BF01238645.

[17] F. John. "Polar correspondence with respect to a convex region". In: *Duke Mathematical Journal* 3.2 (1937), pp. 355–369. DOI: 10.1215/S0012-7094-37-00327-2.

[18] Ai-Jun Li and G. Leng. "A new proof of the Orlicz Busemann-Petty centroid inequality". In: *Proceedings of the American Mathematical Society* 139.4 (2011), pp. 1473–1481. DOI: 10.1090/S0002-9939-2010-10651-2.

[19] E. H. Lieb and M. Loss. *Analysis*. 2nd ed. Vol. 14. Graduate Studies in Mathematics. American Mathematical Soc., 2001.

[20] E. Lutwak. "On some affine isoperimetric inequalities". In: *Journal of Differential Geometry* 23.1 (1986), pp. 1–13. DOI: 10.4310/jdg/1214439900.

[21] E. Lutwak. "Selected Affine Isoperimetric Inequalities". In: *Handbook of Convex Geometry*. Ed. by P.M. Gruber and J.M. Wills. North-Holland, 1993, pp. 151–176. DOI: 10.1016/B978-0-444-89596-7.50010-9.

[22] E. Lutwak, D. Yang, and G. Zhang. "A new ellipsoid associated with convex bodies". In: *Duke Mathematical Journal* 104.3 (2000), pp. 375–390. DOI: 10.1215/S0012-7094-00-10432-2.

[23] E. Lutwak, D. Yang, and G. Zhang. "Lp Affine Isoperimetric Inequalities". In: *Journal of Differential Geometry* 56.1 (2000), pp. 111–132. DOI: 10.4310/jdg/1090347527.

[24] E. Lutwak, D. Yang, and G. Zhang. "Orlicz centroid bodies". In: *Journal of Differential Geometry* 84.2 (2010), pp. 365–387. DOI: 10.4310/jdg/1274707317.

[25] E. Lutwak, D. Yang, and G. Zhang. "Orlicz projection bodies". In: *Advances in Mathematics* 223.1 (2010), pp. 220–242. DOI: 10.1016/j.aim.2009.08.002.

[26] E. Lutwak and G. Zhang. "Blaschke-Santaló inequalities". In: *Journal of Differential Geometry* 47.1 (1997), pp. 1–16. DOI: 10.4310/jdg/1214460036.

[27] V. H. Nguyen. "Orlicz–Lorentz centroid bodies". In: *Advances in Applied Mathematics* 92 (2018), pp. 99–121. DOI: 10.1016/j.aam.2017.07.003.

[28] G. Paouris and P. Pivovarov. "A probabilistic take on isoperimetric-type inequalities". In: *Advances in Mathematics* 230.3 (2012), pp. 1402–1422. DOI: 10.1016/j.aim.2012.03.019.

[29] G. Paouris and P. Pivovarov. "Small-Ball Probabilities for the Volume of Random Convex Sets". In: *Discrete & Computational Geometry* 49.3 (2013), pp. 601–646. DOI: 10.1007/s00454-013-9492-2.

[30] C. M. Petty. "Isoperimetric Problems". In: *Proc. Conf. Convexity and Combinatorial Geometry* (1971), pp. 26–41.

[31] C. A. Rogers and G. C. Shephard. "Some extremal problems for convex bodies". In: *Mathematika* 5.2 (1958), pp. 93–102. DOI: 10.1112/S0025579300001418.

[32] R. Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. 2nd ed. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2013. DOI: 10.1017/CBO9781139003858.

[33] G. C. Shephard. "Shadow systems of convex sets". In: *Israel Journal of Mathematics* 2 (1964), pp. 229–236. DOI: 10.1007/BF02759738.