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Fuzzy-logic-based Judgment Aggregation

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Sebastian Uhl, BSc
Matrikelnummer 11778248

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Betreuung: Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Christian Fermüller

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Sebastian Uhl   Christian Fermüller

Technische Universität Wien
A-1040 Wien ▪ Karlsplatz 13 ▪ Tel. +43-1-58801-0 ▪ www.tuwien.at

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Fuzzy-logic-based Judgment Aggregation

DIPLOMA THESIS

submitted in partial fulfillment of the requirements for the degree of

Diplom-Ingenieur

in

Logic and Computation

by

Sebastian Uhl, BSc
Registration Number 11778248

to the Faculty of Informatics

at the TU Wien

Advisor: Ao.Univ.Prof. Dipl.-Ing. Dr.techn. Christian Fermüller

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Sebastian Uhl

Christian Fermüller

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Sebastian Uhl, BSc

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Kurzfassung

In dieser Arbeit geben wir eine in sich geschlossene Einführung in die Urteilsaggregation. Darüber hinaus wollen wir die Literatur über mehrwertige Urteilsaggregation erweitern, indem wir ein mehrwertiges Framwework einführen und ein Möglichkeits- und ein Unmöglichkeitsresultat in der Kleene-Zadeh-Logik zeigen.

Zunächst wird das klassische Framework für Urteilsaggregation vorgestellt. Darüber hinaus werden Unmöglichkeitsresultate des klassischen Frameworks überprüft, indem das ursprüngliche Unmöglichkeitsresultat von [LP02] analysiert und die *Ultrafilter Beweistechnik* zur Charakterisierung der Unmöglichkeit eingeführt wird.

Zweitens werden die Relaxationen und Methoden zur Bewältigung von Unmöglichkeitsresultaten untersucht. Dabei analysieren wir Lockerungen der kollektiven Rationalität, der Unabhängigkeit und der universellen Domäne. Darüber hinaus besprechen wir das von [Die07] vorgeschlagene Framework, das die betrachtete Logik auf einige minimale Eigenschaften verallgemeinert, die es dennoch erlauben, Unmöglichkeitsresultate zu beweisen.

Drittens führen wir ein mehrwertiges Framework für die Urteilsaggregation ein, das die Einstellung verallgemeinert, die ein Individuum gegenüber einer Aussage haben kann. Konkret untersuchen wir die Verwendung der Kleene-Zadeh-Logik für die Urteilsaggregation. Dabei geben wir ein Möglichkeitsresultat in der Kleene-Zadeh-Logik an. Dieses Ergebnis zeigt, dass Aggregationsfunktionen, die eine begrenzte Systematizitätsbedingung erfüllen, konsistente kollektive Urteile für bestimmte Arten von Agenden und Urteilsprofilen liefern. Schließlich ergänzen wir unser positives Ergebnis durch ein Unmöglichkeitsresultat. Unser Ergebnis zeigt, dass unter den verallgemeinerten Bedingungen von [LP02], bestimmte Profile immer zu inkonsistenten kollektiven Urteilen führen.

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Abstract

In this thesis, we give a self-contained introduction to judgment aggregation. Furthermore, we aim to extend the literature on many-valued judgment aggregation by introducing a many-valued framework and showing a possibility and an impossibility result in Kleene-Zadeh logic.

First, the classical framework for judgment aggregation is introduced. Moreover, impossibility results of the classical framework are reviewed by analyzing the original impossibility result of [LP02] and introducing the *ultrafilter proof technique* for characterizing impossibility.

Second, relaxations and methods for coping with impossibility results are surveyed. Thereby, we analyze relaxations of collective rationality, independence, and universal domain. Furthermore, we review the *general logics* framework proposed by [Die07], which generalizes the logic under consideration to some minimal properties that still allow proving impossibility results.

Third, we introduce a many-valued framework for judgment aggregation that generalizes the attitude an individual can have toward a proposition. Specifically, we investigate the use of Kleene-Zadeh logic for judgment aggregation. Thereby, we state a possiblity result in Kleene-Zadeh logic. This result shows that aggregation functions satisfying a limited systematicity condition yield consistent collective judgments for certain types of agendas and judgment profiles. Finally, we complement our positive result with an impossibility result. Our result shows that under the generalized conditions of [LP02], certain profiles always lead to inconsistent collective judgments.

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CHAPTER 1

Introduction

Finding collective decisions occurs in nearly every area of society, from voting in politics, finding a verdict in jurisprudence, to a group that simply wants to decide in which restaurant it should eat. Additionally, most collective decisions are not independent but have to include additional constraints in the finding process. For example, judges that try to find a legitimate verdict need to ground it on current law. Such questions of general decision making and decision processes are studied in the field of *social choice theory* [Lis22b].

One of the first people systematically studying collective decision making were Nicolas de Condorcet and Jean-Charles de Borda [Lis22b]. However, social choice theory was not formalized until the 20th century which was also strongly influenced by the works about preference aggregation and its impossibility results studied by Kenneth Arrow [Pig21, GP14, Arr51]. This ultimately led to the development of the mathematical theory of *judgment aggregation*, which formalizes collective decision making by using formal logic [GP14]. Judgment aggregation usually focuses on aggregating individual judgments of propositions that have logical relations and are not independent of each other.

However, the aggregation of individual judgments about logically interrelated propositions turned out to be rather difficult, as witnessed by the so-called *doctrinal paradox* [KS93] and, more general, the *discursive dilemma* [Pet01].

Example 1 (Discursive dilemma)**.** *An example of this dilemma would be that three judges have to reach a verdict in a case where they have to decide if:*

- v*: the contract is valid*
- b*: the defendant breached the contract, and in conclusion if*
- g*: the defendant is guilty*

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Each of the judges adjudicates on the basis of applicable law, which states that the defendant is guilty if, and only if, the contract is valid and the defendant breached the contract, i.e., $g \leftrightarrow v \wedge b$*. In this case, the applicable law, i.e.,* $g \leftrightarrow v \wedge b$*, can be seen as a kind of integrity constraint where* v *and* b *are the premises and* v *the conclusion, and every judge holds this law to be true.*

	v	b	g	$g \leftrightarrow v \wedge b$
Judge 1	1	1	1	1
Judge 2	1	0	0	1
Judge 3	0	1	0	1
Majority	1	1	0	1

Table 1.1: Example of the discursive dilemma

If we consider the individual judgments as given in Table 1.1, then each of the judges has come to a consistent individual judgment set. If we apply majority voting on the judgments of the individual judges, then as a result, we get that the majority of the judges holds the contract to be valid and that the defendant breached the contract. However, at the same time, the majority does not hold the defendant liable, which thus results in an inconsistent collective judgment set. Instead of aggregating the whole individual judgment sets, the judges could use majority voting only on the premises v *and* b *(*premise-based approach*) or only on the conclusion* g *(*conclusion-based approach*) [GP14, DM10]. In the case of the premise-based approach, the majority holds the contract valid and believes that the defendant breached the contract. Thus the majority concludes that the defendant is liable. Whereas when using the conclusion-based approach, the majority does not hold the defendant liable and thus contradicts the premise-based approach. Hence, resulting in a paradoxical or dilemmatic situation where it is unclear how we should proceed to reach a legitimate verdict [KS93].*

As we observe in Example 1, even though all the individual judgments of the three judges are consistent, the resulting aggregated collective judgment can still be inconsistent [LP04].

1.1 The relation between judgment and preference aggregation

Judgment and preference aggregation are historically strongly connected and are both subfields of the more general field of (computational) social choice theory[1] [CELM07]. Both judgment and preference aggregation formalize collective decision making. However, preference aggregation focuses on aggregating individual preferences, like agent i prefers option a over option b, into a collective set of preferences. Whereas judgment

[1]Computational social choice theory is an interdisciplinary field that intersects social choice theory and computer science by utilizing results and techniques from computer science in social choice theory (and vice versa) [CELM07].

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aggregation aggregates individual judgments that represent the acceptance or rejection of certain propositions, like agent i accepts proposition p but rejects proposition q. In particular, preference aggregation can be embedded into judgment aggregation by using so-called *preference agendas*, which are built by using a simple predicate logic that allows representing preferences, i.e., a preference relation, by using first-order logic with binary predicate symbols [DL07a, LP09]. Another method that only uses classic propositional logic maps preference aggregation problems to judgment aggregation problems by defining a propositional formula for every possible preference [Sla16]. Moreover, introducing so-called *implicative agendas*, which represent preferences as implications, and a propositional many-valued semantics for judgment sets by using a ranking function led to a correspondence result between preference aggregation and judgment aggregation over such implicative agendas [Gro09]. Additionally, to the result that judgment aggregation subsumes preference aggregation, [Gro10] showed a correspondence between classical propositional judgment aggregation and Boolean preference aggregation, which also builds on the semantics by using ranking functions established in [Gro09].

1.2 Related topics

Related topics of judgment aggregation are, in general, the subsuming field of *(computational) social choice theory* that copes in general with the design and analysis of methods for collective decision making and its subfields as *voting theory* (see [BF02, Pac19]), *belief merging* (see [KPP11, Pig21, EP05]) and *preference aggregation* [CELM07, Lis22b]. Preference aggregation has a special relation to judgment aggregation as judgment aggregation subsumes preference aggregation (see Section 1.1) and due to its impact on the development of judgment aggregation [DL07a, GP14]. Another related topic that copes with collective decision making for responsibility ascription and, therefore, partly utilizes judgment aggregation is the field of *collective responsibility* [Pet07, Lis22a].

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CHAPTER 2

Classical judgment aggregation

Judgment aggregation is an abstract aggregation framework for modeling and coping with complex collective decision making [Sla16]. In the classic case, the aggregation framework only uses classical (bivalent) propositional logic for modeling. However, other approaches like many-valued [SJ11] or fuzzy-logic frameworks [SBK16] have also been considered in the literature.

In judgment aggregation, we typically restrict our attention to problems that contain logically interrelated judgments in contrast to *simple* aggregation problems like common majority voting [Sla16]. The case of *simple* judgments with propositions that have no logical relation among each other is a strongly restricted subset of general judgments.

The chapter consists of 5 main sections. In Section 2.1, we define logical preliminaries and introduce basic notions of judgment aggregation. Then Section 2.2 copes with restrictions of the agenda. In Sections 2.3 and 2.4, we introduce how judgments are aggregated and which conditions can be induced on aggregation functions and their results, respectively. The chapter concludes with Section 2.5 in which we show how constraints of a judgment aggregation problem can be modeled explicitly.

2.1 Preliminaries

For stating and describing the problems and propositions under consideration, we need a formal logical language $\mathcal{L}$ that defines the well-formed formulas. Classical judgment aggregation, which covers the majority of literature, is based on classical propositional logic [GP14].

Definition 1 (Language of classical propositional logic [GP14])**.** *Let At be a countable set of atomic propositions (atoms). The* language of classical propositional logic $\mathcal{L}_p$ *is defined as*

$$\mathcal{L}_p \ni \phi, \psi ::= p \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid \phi \leftrightarrow \psi$$

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for $p \in At$ the set of atoms.

If it is unambiguous which language we use, we simply denote it by $\mathcal{L}$.

In the following, we assume familiarity with the semantics of classical propositional logic. By '$\models$', we denote the logical consequence relation. Hence, e.g., $\Gamma \models \bot$ means that the set of formulas Γ is unsatisfiable (inconsistent).

Definition 2 (Judgment aggregation problem [LP09, GP14])**.** *Let $\mathcal{L}$ be a language over the atoms At. An ($\mathcal{L}$-)*judgment aggregation problem *is a tuple $\mathcal{J} = \langle N, \mathcal{A} \rangle$, where*

- *N is a finite non-empty set of agents (or individuals)*[1]
- *$\mathcal{A} = I \cup \{\neg\phi \mid \phi \in I\}$ where the finite issue set $I \subseteq \mathcal{L}$ only contains positive formulas. $\mathcal{A}$ is called the agenda.*

If Λ is a logic over the language $\mathcal{L}$, then we denote a $\mathcal{L}$-judgement aggregation problem over logic Λ as a Λ-judgment aggregation problem.

The agenda $\mathcal{A}$ is by Definition 2 closed under negation, i.e., if $\phi \in \mathcal{A}$, then $\neg\phi \in \mathcal{A}$[2]. The set $\mathcal{A}^+ = \{\phi \in \mathcal{A} \mid \phi \text{ is non-negated } \} = I$ is called the *issues* or *pre-agenda* under consideration [GP14]. Judging whether the issue $\phi \in \mathcal{A}^+$ is accepted or rejected is equal to deciding which of the formulas $\phi, \neg\phi \in \mathcal{A}$ is selected [Sla16]. Moreover, for an issue set $I \subseteq \mathcal{L}$ of positive (non-negated) formulas, we introduce the notation $\pm I$ to mean the resulting agenda, i.e., closure under negation of the issue set I. In particular, for an agenda $\mathcal{A}$ we have $\mathcal{A} = \pm\mathcal{A}^+$.

Example 2. *Consider the set of issues $\mathcal{A}^+ = \{p, q, r \wedge s\}$. The corresponding agenda is*

$$\mathcal{A} = \pm\{p, q, r \wedge s\} = \{p, \neg p, q, \neg q, r \wedge s, \neg(r \wedge s)\}$$

The set N of individuals defines the totality of agents that have to decide on an individual level which of the issues $\phi \in \mathcal{A}^+$ are accepted or rejected. The aim is then to aggregate these individual judgments to gain a collective judgment that satisfies certain properties.

2.1.1 Judgment sets and profiles

Definition 3 (Judgment sets [Lis11, GP14])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem. A* judgment set *for $\mathcal{J}$ is a set of formulae $J \subseteq \mathcal{A}$ such that:*

[1] The set N of agents is sometimes called judges or voters depending on the context under consideration, e.g., in jurisprudence or voting theory.

[2] Note, that we identify a doubly negated proposition $\neg\neg\phi$ with its non-negated proposition ϕ. In some underlying logical frameworks, this identification of doubly negated propositions with its non-negated ones could be retracted.

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- *J is consistent, i.e., $J \not\models \bot$*
- *J is complete, i.e., if $\phi \in \mathcal{A}^+$, then $\phi \in J$ or $\neg\phi \in J$*

We call a judgment set (fully) rational *if, and only if, it is consistent and complete [Die07].*

The set of all judgment sets over an agenda $\mathcal{A}$ is denoted by $\mathbf{J}(\mathcal{A})$. Note that $\mathbf{J}(\mathcal{A}) \subseteq 2^{\mathcal{A}}$, where each $J \in \mathbf{J}(\mathcal{A})$ is rational. The condition of consistency and completeness for judgment sets mirrors the condition of rationality for human decisions [GP14]. However, in general, human decisions are not always fully rational and sometimes even contradictory. Thus under certain modeling circumstances, consistency or completeness may be omitted. Completeness represents that individuals have an opinion on every formula under consideration, i.e., an individual explicitly judges the formula to be either accepted or rejected.

Example 3 (Judgment sets)**.** *Consider the agenda $\mathcal{A} = \{p, \neg p, q, \neg q, p \wedge q, \neg(p \wedge q)\}$.*

- *The set $J_1 = \{p, \neg q, \neg(p \wedge q)\}$ is a judgment set.*
- *The set $J_2 = \{p, q, \neg(p \wedge q)\}$ is not a judgment set.*
  - *We have $p, q \in J_2$, so $J_2 \models p \wedge q$. However, $J_2 \models \neg(p \wedge q)$ and thus J_2 is inconsistent.*
- *The set $J_3 = \{p, p \wedge q\}$ is not a judgment set.*
  - *J_3 is consistent but not complete since $q \in \mathcal{A}$ but $q \notin J_3$ and $\neg q \notin J_3$.*

As we observe in Example 3, an individual accepts or rejects the issue $\phi \in \mathcal{A}^+$ by selecting one of the pair $\{\phi, \neg\phi\} \subseteq \mathcal{A}$ for its judgment set.

Further properties of a judgment set $J \subseteq \mathcal{A}$ that can be used are *weak consistency* and *deductive closure* [Die07].

Definition 4 (Weak consistency[3] [DL07b])**.** *A (judgment) set J is* weakly consistent *if, and only if, for every formula pair $\{\phi, \neg\phi\} \subseteq \mathcal{A}$ at most one of ϕ and $\neg\phi$ is in J.*

Definition 5 (Deductive closure [DL07b])**.** *A judgment set J is* deductively closed *if, and only if, for every formula $\phi \in \mathcal{A}$, if $J \models \phi$, then $\phi \in J$.*

As we observe given Definition 4, the condition of consistency imposed on judgment sets (see Definition 3) implies weak consistency. Hence, every (consistent) judgment set J is also weakly consistent. However, depending on the modeling situation, the typical conditions for judgment sets, i.e., consistency and completeness, may be weakened to, e.g., only weak consistency.

[3] weak consistency

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Example 4 (Weak consistency and deductive closure)**.** *Consider the set of issues* $\mathcal{A}^+ = \{p, p \to q\}$ *and its corresponding agenda.*

- *The set* $J_1 = \{p\}$ *is weakly consistent, since of the pair* $\{p, \neg p\}$ *is at most one and of the pair* $\{p \to q, \neg(p \to q)\}$ *there is non in* J_1*. Notice with regard to Example 3 that in this example* J_1 *is not complete since non of* $\{p \to q, \neg(p \to q)\}$ *is in* J_1*.*
- *The set* $J_2 = \{p, p \to q\}$ *is a judgment set, but it is not deductively closed. Observe that* $J_2 \models q$ *but* $q \notin J_2$*. Accordingly, the set* $J_2' = J_2 \cup \{q\}$ *would be a deductively closed judgment set.*

Definition 6 (Profiles [Lis11])**.** *Let* $\mathcal{J} = \langle N, \mathcal{A} \rangle$ *be a judgment aggregation problem. A* (judgment) profile $P = \langle J_i \rangle_{i \in N} \in \mathbf{J}(\mathcal{A})^{|N|}$ *is an* $|N|$*-tuple of judgment sets.*

By $P_\phi = \{i \in N \mid \phi \in J_i, J_i \in P\}$ *we denote the set of individuals that accept* ϕ*. For every formula* $\phi \in \mathcal{A}$ *in the minimum case, every individual rejects* ϕ*, and in the maximum case, every individual accepts* ϕ*, i.e.,* $\emptyset \subseteq P_\phi \subseteq N$*. Thus* $0 \leq |P_\phi| \leq |N|$ *for each formula* $\phi \in \mathcal{A}$*. Moreover, we denote the set of all judgment profiles as* $\mathbf{P}(\mathcal{A})$*.*

An (individual) judgment set $J_i \in P$ states the opinion of the individual $i \in N$ over the given agenda $\mathcal{A}$. So the formulas $\phi \in J_i$ represent the propositions individual i accepts, whereas the formulas $\psi \notin J_i$ represent the propositions individual i rejects. The set $\overline{J}_i = \mathcal{A} \setminus J_i$ contains the propositions individual i rejects. A profile P represents the judgments or opinions of a certain set of individuals, e.g., the opinion of a group of individuals that judges about a topic. In particular, considering Example 1 again, the profile consists of the judgment sets of the three judges that have to find a verdict in the given case.

Remark 1. *Consider a judgment set* J_i*, which is by definition consistent and complete. Note that its complement, i.e., the set* $\overline{J_i} = \mathcal{A} \setminus J_i$ *of formulas individual* i *rejects, contains exactly the not accepted formulas of every formula pair* $\{\phi, \neg\phi\}$ *in the agenda* $\mathcal{A}$*.*

2.2 Agenda restrictions

The agenda $\mathcal{A} \subseteq \mathcal{L}$ specifies the formulas on which the individuals and the collective have to judge if they accept or reject them [DM10]. The usual restriction for the agenda is that it is closed under negation, i.e., if $\phi \in \mathcal{A}$, then $\neg\phi \in \mathcal{A}$ [Lis11].

2.2.1 Non-simplicity

In general we do not exclude agendas that are not logically interrelated. Such agendas are called simple [GP14].

Example 5 (Simple agenda)**.** *Consider the simple agenda* $\mathcal{A} = \{p, q, r, \neg p, \neg q, \neg r\}$*. The only minimally inconsistent subsets of* $\mathcal{A}$ *are* $\{p, \neg p\}$*,* $\{q, \neg q\}$ *and* $\{r, \neg r\}$*.*

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As we observe in Example 5, when considering a simple agenda $\mathcal{A}$, the only possibility to form an inconsistent set $J \subseteq \mathcal{A}$ is that for some formula $\phi \in \mathcal{A}^+$ we have that $\{\phi, \neg\phi\} \subseteq J$, i.e., $\{\phi, \neg\phi\}$ is a minimally inconsistent subset of J.

To ensure a certain degree of logical interrelatedness for the agendas under consideration, we introduce so-called *non-simple* agendas.

Definition 7 (Minimal inconsistency[LP09])**.** *A set $X \subseteq \mathcal{A}$ is* minimally inconsistent *if*

- *X is inconsistent, and*
- *for all Y such that $Y \subset X$ is consistent*

Definition 8 (Non-simple agendas [Lis11, GP14])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda over the language $\mathcal{L}$. $\mathcal{A}$ is* non-simple *if, and only if, $\exists X$ such that $X \subseteq \mathcal{A}$ and:*

- $3 \leq |X|$
- *X is minimally inconsistent.*

Obviously, an agenda is non-simple if, and only if, it is not simple.

Example 6 (Non-simple agenda)**.** *Consider the agenda $\mathcal{A} = \pm\{p, q, s, p \to s\}$. The subset $X = \{p, p \to s, \neg s\}$ of $\mathcal{A}$ is inconsistent since $X \models s$ but $\neg s \in X$. Moreover, X is minimally inconsistent since every subset is consistent. Hence, the agenda $\mathcal{A}$ is non-simple.*

2.2.2 Closure under atoms

Definition 9 (Closure under atoms [Mon08, Sla16])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda over the language $\mathcal{L}$ over the atoms At. Let $At(\mathcal{A}) \subseteq At$ be the set of atoms occurring in the agenda $\mathcal{A}$. The agenda $\mathcal{A}$ is* closed under atoms *if, and only if, $At(\mathcal{A}) \subseteq \mathcal{A}$.*

Example 7. *Consider the agenda $\mathcal{A}_1 = \pm\{p, r, r \wedge s\}$. The set of atoms occurring in $\mathcal{A}_1$ is $At(\mathcal{A}_1) = \{p, r, s\}$. Since $s \in At(\mathcal{A}_1)$ but $s \notin \mathcal{A}_1$, the agenda $\mathcal{A}_1$ is not closed under atoms.*

Consider the agenda $\mathcal{A}_2 = \pm\{p, r, s, r \wedge s\}$. We observe that $At(\mathcal{A}_2) = \{p, r, s\} \subset \mathcal{A}_2$ and thus $\mathcal{A}_2$ is closed under atoms.

The idea behind closure under atoms is that the individuals are forced to judge on all occurring atoms that are used to build the composite formulas in the agenda.

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2.2.3 k-median property

Definition 10 (k-median property [LP09, EGP12])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda over the language $\mathcal{L}$. The agenda $\mathcal{A}$ satisfies the* k-median property *for some $k \geq 2$ if, and only if, every inconsistent subset $X \subseteq \mathcal{A}$ has an inconsistent subset $Y \subseteq X$ of size $|Y| \leq k$.*

The 2-median property is often simply called *median property.* We observe that an agenda $\mathcal{A}$ that satisfies the median property can have only minimally inconsistent subsets of size at most 2. Thus an agenda $\mathcal{A}$ satisfies the median property if, and only if, $\mathcal{A}$ is simple [Sla16].

2.2.4 Evenly negatability

Definition 11 (Evenly negatable agendas [Lis11, DL13])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda over the language $\mathcal{L}$. We say $\mathcal{A}$ is* evenly negatable[4] *if, and only if, $\exists X \subseteq \mathcal{A}$ and $\exists Y \subseteq X$ of even size (size 2) such that X is minimally inconsistent and $(X \setminus Y) \cup \{\neg\phi \mid \phi \in Y\}$ is consistent.*

Example 8 (Evenly negatability)**.** *Consider the agenda $\mathcal{A} = \pm\{p, q, p \to q\}$. $X = \{p, p \to q, \neg q\} \subseteq \mathcal{A}$ is a minimally inconsistent set. Moreover, $Y = \{p, \neg q\} \subseteq X$ and we have that*

$$(X \setminus Y) \cup \{\neg p, \neg\neg q\} = \{\neg p, p \to q, \neg\neg q\}$$

is consistent. Hence, $\mathcal{A}$ is evenly negatable.

In general evenly negatability can also be defined for subsets of the minimally inconsistent set of size exactly 2 [DL13]. So the property of envenly negatability describes agendas that have at least one minimally inconsistent subset X which can be made consistent by negating two (or an even number) of the formulas contained in X (as seen in Example 8).

Example 9 (Non evenly negatable agenda)**.** *Consider the agenda $\mathcal{A} = \pm\{p, p \leftrightarrow q, q\}$. The minimally inconsistent subsets are:*

(1) $\{p, p \leftrightarrow q, \neg q\}$

(2) $\{\neg p, p \leftrightarrow q, q\}$

(3) $\{p, \neg(p \leftrightarrow q), q\}$

(4) $\{\neg p, \neg(p \leftrightarrow q), \neg q\}$

[4] Evenly negatability is sometimes also referred to as *even-number negatability* [Lis11], or *even-number-negation property* [LP09].

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As we observe, every one of the minimally inconsistent subsets of the agenda from (1) to (4) cannot be made consistent by negating two formulas of the set.

Note that the logically very similar agenda $\mathcal{A}' = \pm\{p, p \to q, q \to p, q\}$ *is evenly negatable since the minimally inconsistent subset* $\{p, p \to q, \neg q\}$ *can be made consistent by negating* p *and* $\neg q$*, resulting in the consistent set* $\{\neg p, p \to q, q\}$*.*

2.2.5 Path-connectedness

Definition 12 (Conditional entailment [Sla16, GP14, DL13])**.** *Let* $\phi, \psi \in \mathcal{A}$*. The formula* ϕ conditionally entails ψ*, i.e.,* $\phi \models_c \psi$ *if there is a consistent subset* $X \subseteq \mathcal{A}$ *such that* $X \cup \{\phi\}$ *and* $X \cup \{\neg\psi\}$ *are consistent and* $\{\phi\} \cup X \models \psi$*.*

The transitive closure of conditional entailment $\phi \models_c^* \psi$ *is satisfied, if there exists a sequence* $\langle \phi_1, ..., \phi_n \rangle$ *of elements of* $\mathcal{A}$ *such that* $\phi = \phi_1, \phi_n = \psi$ *and for all* $1 \leq i < n$*,* $\phi_i \models_c \phi_{i+1}$ *holds.*

Example 10 (Conditional entailment)**.** *Consider the agenda* $\mathcal{A} = \pm\{v, b, v \wedge b\}$*.*

- $\{v\} \models_c v \wedge b$
  - $\{b\} \cup \{v\}$ *is consistent*
  - $\{b\} \cup \{\neg(v \wedge b)\}$ *is also consistent*
  - $\{v, b\} \models v \wedge b$
- $\{\neg v\} \not\models_c v \wedge b$
  - *There is no* $X \subseteq \mathcal{A}$ *such that* $\{\neg v\} \cup X \models v \wedge b$ *since* $\{\neg v\} \cup \{v \wedge b\}$ *is inconsistent.*
- $\neg v \not\models_c^* b$ *as seen in Figure 2.1 from Example 12.*

Definition 13 (Path-connected agendas [DL13, GP14])**.** *Let* $\mathcal{A} \subseteq \mathcal{L}$ *be an agenda over the language* $\mathcal{L}$*.* $\mathcal{A}$ *is* path-connected[5] *if, and only if, for all* $\phi, \psi \in \mathcal{A}$*,* $\phi \models_c^* \psi$ *holds.*

Regarding non-simplicity (see Defintion 8), we observe that a path-connected agenda is also non-simple, as in simple agendas, there is no possibility that $\phi \models_c^* \neg\phi$ for any formula $\phi \in \mathcal{A}$, i.e., simplicity implies path-disconnectedness and thus by contrapositive it follows that path-connectedness implies non-simplicity.

Lemma 1 ([Lis11])**.** *Let* $\mathcal{A}$ *be a path-connected agenda. Then* $\mathcal{A}$ *is non-simple.*

[5] Sometimes path-connectedness is also referred to *total-blockedness*, e.g., in [LP09, KE09] as for finite agendas both are equivalent [DL08].

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Example 11 (Path-connected agenda)**.** *Consider the discursive dilemma again (see Example 1). The agenda of the example* $\mathcal{A} = \pm\{v, b, g, g \leftrightarrow v \wedge b\}$ *is path-connected.*

We find the following cyclic sequence of conditional entailments that starts and ends with $g \leftrightarrow v \wedge b \in \mathcal{A}$*:*

(1) $\{g \leftrightarrow v \wedge b\} \cup \{g\} \models v$

(2) $\{v\} \cup \{g, g \leftrightarrow v \wedge b\} \models b$

(3) $\{b\} \cup \{v, g \leftrightarrow v \wedge b\} \models g$

(4) $\{g\} \cup \{\neg v, \neg b\} \models \neg(g \leftrightarrow v \wedge b)$

(5) $\{\neg(g \leftrightarrow v \wedge b)\} \cup \{g, b\} \models \neg v$

(6) $\{\neg v\} \cup \{g, \neg(g \leftrightarrow v \wedge b)\} \models \neg b$

(7) $\{\neg b\} \cup \{g \leftrightarrow v \wedge b\} \models \neg g$

(8) $\{\neg g\} \cup \{\neg v, \neg b\} \models g \leftrightarrow v \wedge b$

This proves that for all $\phi, \psi \in \mathcal{A}$*,* $\phi \models_c^* \psi$ *holds and thus by Definition 13 of path-connectedness, the agenda is path-connected.*

Example 12 (Path-disconnected agenda)**.** *Consider the agenda* $\mathcal{A} = \pm\{v, b, v \wedge b\}$ *compared to the agenda of the discursive dilemma in Example 11 above. To show that the agenda is path-disconnected, we have to find a pair of formulas* $\phi, \psi \in \mathcal{A}$ *such that* $\phi \not\models_c^* \psi$*.*

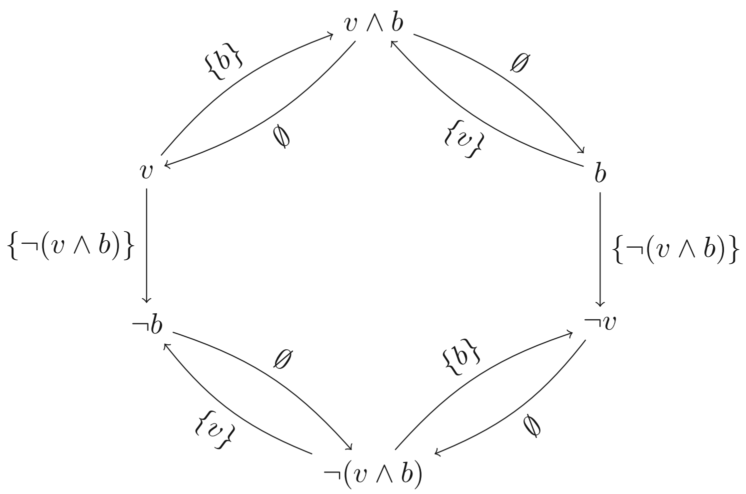

Figure 2.1: Graph of the conditional entailment relation over agenda $\mathcal{A} = \pm\{v, b, v \wedge b\}$, i.e., the graph $(\mathcal{A}, \models_c)$.

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As we observe in Figure 2.1, the lower part of the graph, that represents the conditional entailment relation over the agenda, is not connected to the upper part. We observe that $\neg b \not\models_c^* v$ *and thus the agenda* $\mathcal{A}$ *is not path-connected. Moreover, the agenda in Figure 2.1 would only be path-connected if the corresponding graph* $(\mathcal{A}, \models_c)$ *is strongly connected.*

Remark 2. *Regarding path-connected agendas (see Definition 13) and Example 11 and 12, we may observe that path-connected agendas are strongly related to (strongly) connected graphs, i.e., every two nodes (formulas) are connected by a path (the transitive closure of the conditional entailment relation* $\models_c^*$*).*

Definition 14 (Strongly connected graph)**.** *The directed graph* (V, E) *is* strongly connected *if, and only if, for all* $v, w \in V$*, there is a path from* v *to* w*.*

Theorem 1 ([NP10][6])**.** *Let* $\mathcal{A}$ *be an agenda.* $\mathcal{A}$ *is path-connected if, and only if, the corresponding graph* $(\mathcal{A}, \models_c)$ *is strongly connected.*

Proof. If $\mathcal{A}$ is path-connected, then by definition $\phi \models_c^* \psi$ for all $\phi, \psi \in \mathcal{A}$. By definition $\phi \models_c^* \psi$ if there exists a sequence $\langle \phi_1, \dots, \phi_n \rangle$ of elements of $\mathcal{A}$ such that $\phi = \phi_1, \phi_n = \psi$ and for all $1 \leq i < n$, $\phi \models_c \phi_{i+1}$ holds. This sequence $\langle \phi_1, \dots, \phi_n \rangle$ of elements of $\mathcal{A}$ forms by construction of $(\mathcal{A}, \models_c)$ a path in the graph. Hence, the graph $(\mathcal{A}, \models_c)$ is strongly connected.

The proof of the other direction is similar. □

2.3 Aggregation of judgments

The aggregation of individual judgment sets to a collective judgment set is the main aim of judgment aggregation. The solution of such an aggregation should represent the totality of the individual judgments and thus should be *most-representative* regarding the given profile [Sla16].

Definition 15 (Aggregation function [Lis11, GP14])**.** *Let* $\mathcal{J} = \langle N, \mathcal{A} \rangle$ *be a judgment aggregation problem. An* aggregation function[7] *for* $\mathcal{J}$ *is a function* $F : \mathbf{P}(\mathcal{A}) \to 2^{\mathcal{A}}$*. Let* $P = \langle J_i \rangle_{i \in N}$ *be a profile over* $\mathcal{A}$*, then the result of the aggregation function* $F(P)$ *is called* collective set*.*

Remark 3. *In general, the result* $F(P)$ *of an aggregation function* F *for a profile* P *is not necessarily a judgment set, i.e., it can be inconsistent or incomplete. If the aggregation result* $F(P)$ *is consistent and complete, we will denote it as a collective judgment set in contrast to a collective set.*

[6] [NP10] do not give a proof, but only mention it on the side.

[7] Sometimes the terms *aggregation rule* [LP10], *social judgment function* [Mon08], or *aggregator* [Sla16] are also used to refer to aggregation functions.

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Remark 4. *In Definition 15, we assume that an aggregation function is defined over the whole set of all different judgment profiles* $\mathbf{P}(\mathcal{A})$*. This property is called* universal domain[8] *[GP14]. However, not all aggregation functions are defined over all judgment profiles, but only over a restricted subset. This restriction enables, in certain cases, the formulation of consistent aggregation functions that would be inconsistent over the whole set of profiles* $\mathbf{P}(\mathcal{A})$*.*

2.3.1 Threshold-based aggregation functions

One of the most studied aggregation functions are (propositionwise) *threshold-based rules* that include a proposition in the collective set if it is true in a certain number of individual judgment sets defined by the *threshold*.

Propositionwise quota rule

The propositionwise quota rule is the most general propositionwise threshold-base rule since it allows for individual thresholds for each formula of the agenda in contrast to a single threshold for multiple formulas. All propositionwise threshold-based rules that do not define an individual threshold for every formula of the agenda are special cases of the propositionwise quota rule [GP14].

Definition 16 (Propositionwise quota rule [DL07b, GP14])**.** *Let* $\mathcal{J} = \langle N, \mathcal{A} \rangle$ *be a judgment aggregation problem,* $P \in \mathbf{P}(\mathcal{A})$ *be a profile, and let* $t = \langle t_\phi \rangle_{\phi \in \mathcal{A}}$ *be a vector that contains a threshold value* t_ϕ *for every formula* $\phi \in \mathcal{A}$*. The* propositionwise quota rule *is defined as:*

$$F_t(P) = \{\phi \in \mathcal{A} \mid |P_\phi| \geq t_\phi\} \tag{2.1}$$

The quota rule accepts a formula ϕ in the collective set if the formula is contained in at least t_ϕ individual judgments.

Example 13 (Propositionwise quota rule)**.** *Let* $\mathcal{A} = \pm\{p, q, p \wedge q\}$*. Consider the profile* $P = \langle J_1, J_2, J_3 \rangle$ *over this agenda, where the individual judgments are:*

- $J_1 = \{p, q, p \wedge q\}$
- $J_2 = \{\neg p, q, \neg(p \wedge q)\}$
- $J_2 = \{\neg p, \neg q, \neg(p \wedge q)\}$

The threshold values are as follows:

- $t_p = t_q = 1$

[8] Restrictions or conditions on the domain (input) of an aggregation function, like universal domain, are sometimes also referred to as *input conditions* [Lis11].

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- $t_{\neg p} = t_{\neg q} = 3$
- $t_{p \wedge q} = t_{\neg(p \wedge q)} = 2$

This results in the collective set $F_t(P) = \{p, q, \neg(p \wedge q)\}$*, which is obviously inconsistent and thus not a judgment set.*

Moreover, we could even set $t_p = t_q = 3$*, which would result in the collective set* $F_t(P) = \{\neg(p \wedge q)\}$ *that contains only one formula of the agenda and hence is not complete.*

Propositionwise majority rule

The (propositionwise) majority rule is common in voting by a number N of individuals. For example, a party wins the election if the majority of individuals ($> \frac{|N|}{2}$) voted for this party.

Definition 17 (Propositionwise majority rule [DL07b, GP14])**.** *Let* $\mathcal{J} = \langle N, \mathcal{A} \rangle$ *be a judgment aggregation problem and* $P \in \mathbf{P}(\mathcal{A})$ *a profile. The* propositionwise majority rule *is defined as:*

$$F_{maj}(P) = \left\{ \phi \in \mathcal{A} \mid |P_\phi| \geq \left\lceil \frac{|N|+1}{2} \right\rceil \right\} \tag{2.2}$$

We observe that the propositionwise majority rule is a special case of the propositionwise quota rule (see Definition 16), where the threshold value of every formula $\phi \in \mathcal{A}$ of the threshold vector $t = \langle t_\phi \rangle_{\phi \in \mathcal{A}}$ is set to $t_\phi = \left\lceil \frac{|N|+1}{2} \right\rceil$. Rules that assign the same value m to every threshold value, i.e., for every formula $\phi \in \mathcal{A}$ we have $t_\phi = m$, are called *uniform* quota rules [GP14].

Example 14 (Propositionwise majority rule)**.** *Like in Example 13, let* $\mathcal{A} = \pm\{p, q, p \wedge q\}$ *and the corresponding the profile* $P = \langle J_1, J_2, J_3 \rangle$*, where:*

- $J_1 = \{p, q, p \wedge q\}$
- $J_2 = \{\neg p, q, \neg(p \wedge q)\}$
- $J_2 = \{\neg p, \neg q, \neg(p \wedge q)\}$

The individuals are $N = \{1, 2, 3\}$*. The resulting collective set of the propositionwise majority rule (see Definition 17) is* $F_{maj} = \{\neg p, q, \neg(p \wedge q)\}$*, which is consistent and thus a judgment set.*

Since the individual judgment sets are consistent and thus do not contain a formula and its negation, we observe that for odd N *the propositionwise majority rule results always in a complete collective set.*

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2.4 Aggregation conditions

Aggregation conditions are restrictions imposed on aggregation functions (see Section 2.3). Usually, these conditions are structured in *input*, *output* and *mapping* (or *responsiveness*) conditions [Lis11, GP14]. On the one hand, input conditions restrict the admissible inputs, i.e., they constrain the domain X of an aggregation function $F : X \to Y$. On the other hand, output conditions constrain the admissible outputs, i.e., they restrict the codomain Y of an aggregation function $F : X \to Y$. The mapping conditions neither restrict the input, nor the output. However, they restrict how the aggregation, or informally the aggregation process, of the given individual judgments, can occur.

2.4.1 Input conditions

Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and $F : X \to Y$ an aggregation function for $\mathcal{J}$. An input condition imposes a restriction on the (input) domain X.

Universal domain

Note that in Section 2.3, we defined an aggregation function as a mapping from profiles, i.e., tuples of judgment sets, to an arbitrary subset of the agenda, i.e., $F : \mathbf{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ (see Definition 15). This input condition that restricts the admissible inputs to tuples of fully rational subsets of the agenda, i.e., judgment sets (see Defintion 3), is called *universal domain* [Lis11].

Definition 18 (Universal domain [Lis11])**.** *Let $\langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for it. F satisfies* universal domain *if, and only if, the domain of F is the set of* all *profiles of fully rational individual judgment sets (see Definition 3), i.e., $F : \mathbf{P}(\mathcal{A}) \to 2^{\mathcal{A}}$.*

Universal domain is usually assumed since we want to focus on rational individual judgments, despite the fact that in real-world applications judgments could also be irrational.

2.4.2 Output conditions

Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem, F an aggregation function for $\mathcal{J}$, and $P \in \mathbf{P}(\mathcal{A})$ a profile. The output conditions of aggregation functions require that the collective set $F(P)$, i.e., the output of the aggregation, must satisfy certain conditions.

Consistency and completeness

Definition 19 (Consistent aggregation functions [GP14])**.** *F is* consistent *if, and only if, $\forall P \in \mathbf{P}(\mathcal{A})$ the collective set $F(P)$ is consistent, i.e., $F(P) \not\models \bot$ (see Definition 3).*

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Definition 20 (Complete aggregation functions [GP14])**.** *F is* complete *if, and only if,* $\forall \phi \in \mathcal{A} \forall P \in \mathbf{P}(\mathcal{A})$, $\phi \in F(P)$ *or* $\neg\phi \in F(P)$ *(see Definition 3).*

Remark 5. *Note that consistency (see Definition 19) and completeness (see Definition 20) for an aggregation function must be satisfied over every possible input profile in* $\mathbf{P}(\mathcal{A})$ *in comparison to consistency for judgment sets given in Definition 3.*

Example 15 (Consistent aggregation function)**.** *Consider as an aggregation function the unanimity rule*

$$F_u(P) = \{\phi \in \mathcal{A} \mid |P_\phi| = |N|\} \tag{2.3}$$

which accepts a formula ϕ *in the collective set if, and only if, all of the individuals in the given profile accept* ϕ*. The collective set* $F_u(P)$ *contains by definition only formulas that are contained in any of the individual judgment sets* $J_i \in P$*. Hence, for every individual judgment set* $J_i \in P$*, it holds that* $F_u(P) \subseteq J_i$*. By definition of judgment sets (see Definition 3), every individual judgment set* J_i *is consistent, and thus* $F_u(P)$ *must also be consistent.*

However, $F_u(P)$ *is, in general, not complete. Consider the profile* $P = \langle J_1, J_2 \rangle$ *over the agenda* $\mathcal{A} = \pm\{p, q, p \vee q\}$ *where*

- $J_1 = \{p, q, p \vee q\}$
- $J_2 = \{p, \neg q, p \vee q\}$

Then $F_u(P) = \{p, p \vee q\}$*, which is consistent but neither contains* q *nor* $\neg q$ *and is thus incomplete (see Definition 20).*

Remark 6. *Consider the unanimity rule given by Equation 2.3 (see Example 15). Since it contains exactly those formulas of the agenda that are accepted by every individual judgment set, it can also be defined by a (finite) intersection of the individual judgments in the given profile as*

$$F_u(P) = \bigcap_{J_i \in P} J_i \tag{2.4}$$

Example 16 (Complete aggregation function)**.** *Consider the following uniform threshold-based aggregation function*

$$F(P) = \left\{\phi \in \mathcal{A} \mid |P_\phi| \geq \frac{|N|}{2}\right\} \tag{2.5}$$

Note that $F(P)$ *coincides with the propositionwise majority rule (see Section 2.3.1) if* $|N|$ *is odd.*

By definition, every individual judgment set $J_i \in P$ *is complete and consistent (see Definition 3). So for every formula* $\phi \in \mathcal{A}^+$ *either* $\phi \in J_i$ *or* $\neg\phi \in J_i$*. Thus for* $\phi \in \mathcal{A}^+$

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we have $|P_\phi| \geq \frac{|N|}{2}$ *or* $|P_{\neg\phi}| \geq \frac{|N|}{2}$*. Hence, for every* $\phi \in \mathcal{A}^+$ *at least one of* ϕ *or* $\neg\phi$ *is in* $F(P)$*.*

In general, $F(P)$ *is not consistent. Consider the profile* $P = \langle\{p, q\}, \{p, \neg q\}\rangle$*. In this case,* $F(P) = \{p, q, \neg q\}$ *and is thus inconsistent but complete.*

Collective rationality

Definition 21 (Rational aggregation functions [LP09])**.** *F is* (collectively) rational[9] *if, and only if, F is consistent and complete. If F is rational, we call the collective set* $F(P)$ *a* collective judgment *set.*

Remark 7. *Note that rationality for aggregation functions and rationality for judgment sets are both defined in terms of consistency and completeness. However, consistency and completeness for aggregation functions (see Definitions 19 and 20) are not fully congruent with the definition of consistency and completeness for judgment sets (see Definition 3).*

If an aggregation function is rational, then its collective set is, by definition, a judgment set. Hence, a rational aggregation function is defined as $F : \mathbf{P}(\mathcal{A}) \rightarrow \mathbf{J}(\mathcal{A})$ [GP14].

Example 17 (Rational aggregation function)**.** *Consider the following aggregation function*

$$F_d^j(P) = \{\phi \in \mathcal{A} \mid \phi \in J_j, J_j \in P\} \tag{2.6}$$

for some fixed j *with* $0 \leq j \leq |N|$*. The collective set* $F_d^j(P)$ *accepts exactly the formulas that are accepted by the* j*-th individual judgment set of the given profile* P*. This means that* $F_d^j(P) = J_j$*, i.e., it is the projection function to the* j*-th element of the given profile* P*. Since every individual judgment set* $J_i \in P$ *is by definition rational, it follows that* $F_d^j(P)$ *is also rational.*

Remark 8. *The aggregation function given in Equation 2.6 in Example 17 belongs to a certain kind of aggregation functions called* dictatorial *[GP14]. As the name suggests, dictatorial aggregation functions result always in a collective set that corresponds to a certain individual judgment set and thus ensures a rational collective set.*

Deductive closure

Definition 22 (Deductive closed aggregation functions [GP14])**.** *F is* deductively closed *(with respect to the agenda* $\mathcal{A}$*) if, and only if,*

$$\forall\phi \in \mathcal{A}\, \forall P \in \mathbf{P}(\mathcal{A}),\ \textit{if } F(P) \models \phi,\ \textit{then } \phi \in F(P)$$

See also deductive closure for judgment sets in Definition 5.

[9] Rationality on the collective set is often called *collective rationality* to emphasize that it refers to the aggregated collective judgment [Lis11].

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Note that there is a connection between rationality and deductive closure of an aggregation function, i.e., rationality implies deductive closure.

Theorem 2. *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. If F is rational, then it is also deductively closed.*

Proof. Assume that F is rational. Let $\phi \in \mathcal{A}$ be an arbitrary formula of the agenda and let $P \in \mathbf{P}(\mathcal{A})$ be an arbitrary profile over $\mathcal{A}$.

If $F(P) \not\models \phi$, then there is nothing to show.

Assume that $F(P) \models \phi$. By definition of rationality, and since F is rational, we conclude that F is consistent and complete. Since F is complete either $\phi \in F(P)$ or $\neg\phi \in F(P)$. If $\neg\phi \in F(P)$, then since F is consistent and $F(P) \models \phi$ by assumption, it would lead to a contradiction. Hence $\phi \in F(P)$ must be the case. □

Remark 9. *Recall Theorem 2, note that deductive closure (see Definition 22) is a weaker condition than collective rationality (see Definition 21). For example, many (collective) judgment sets can be incomplete but still deductively closed [DL08].*

2.4.3 Mapping conditions

Additional to restricting the resulting collective sets of the aggregation functions through output conditions[10], we can also restrict the aggregation function by specifying how the mapping should occur, i.e., which properties the aggregation of the individual judgments must satisfy.

Unanimity

The unanimity of an aggregation function ensures that if all individuals accept a formula, then the collective will also accept this formula [LP09, GP14]. Hence, it is usually a desired property of aggregation functions since it properly represents the sum of individual judgments.

Definition 23 (Unanimous aggregation functions [DL07a, DL08])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* unanimous[11] *if, and only if, for ever $\phi \in \mathcal{A}$ and for every profile $P \in \mathbf{P}(\mathcal{A})$, if every $i \in N$ accepts ϕ, then $\phi \in F(P)$.*

Example 18 (Unanimous aggregation function)**.** *Consider the propositionwise majority rule (see Equation 2.2)*

$$F_{maj}(P) = \left\{ \phi \in \mathcal{A} \mid |P_\phi| \geq \left\lceil \frac{|N|+1}{2} \right\rceil \right\}$$

[10] Output conditions are sometimes also referred to as *responsiveness conditions* [Lis11].
[11] Unanimity is sometimes also called *unanimity preservation*, e.g., in [Lis11].

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F_{maj} is unanimous since it accepts a formula ϕ in the collective set if a majority accepts ϕ and all individuals are trivially a majority.

Moreover the unanimity rule (see Equation 2.3) from Example 15

$$F_u(P) = \{\phi \in \mathcal{A} \mid |P_\phi| = |N|\}$$

is also trivially a unanimous aggregation function as it directly implements the unanimity condition and only accepts the formulas that are unanimously accepted by all individuals.

Remark 10. *As we observe regarding unanimous aggregation functions, for every unanimous aggregation function F, if $P = \langle J, J, \ldots, J\rangle$ is a profile that consists only of identical individual judgment sets, then $F(P) = J$ [LP09].*

Dictatorship

Dictatorship defines an aggregation function where a single individual always decides the collective set [Lis11]. Hence, it is usually not a desirable mapping condition, as we want to find collective sets most representative regarding the individual judgments of the given profile.

Definition 24 (Dictatorial aggregation functions [Lis11, GP14])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* dictatorial *if, and only if, there exists an individual $i \in N$ such that for every profile $P \in \mathbf{P}(\mathcal{A})$, $F(P) = J_i$.*

If such an individual exists, it is called dictator*.*

If an aggregation function F does not satisfy the mapping condition of dictatorship, then it is called a *non-dictatorial* aggregation function [LP09]. An example of a dictatorial aggregation function is given in Example 17.

Example 19 (Dictatorial aggregation function)**.** *Consider the judgment aggregation problem $\langle\{1, 2, 3, 4\}, \mathcal{A}\rangle$. Consider now the dictatorial rule from Equation 2.6 from Example 17 for the dictator $j = 3$, i.e., $F_d^3(P) := J_3$. For any profile of the form $P = \langle J_1, J_2, J_3, J_4\rangle$, the collective set will be J_3, i.e., the output of the aggregation is always the third element of the given profile.*

However, there are not only classical dictatorships (see Definition 24) but also so-called inverse dictatorships, where an individual enforces the opposite of his own views.

Definition 25 (Inverse dictatorial aggregation function [DL07a])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* inversely dictatorial *if, and only if, there exists an individual $i \in N$ such that for every profile $P \in \mathbf{P}(\mathcal{A})$, $F(P) = \{\neg\phi \mid \phi \in J_i\}$.*

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Example 20 (Inverse dictatorial aggregation function)**.** *Consider the judgment aggregation problem* $\langle\{1,2,3,4\},\mathcal{A}\rangle$ *as given in Example 19. In this case, as opposed to Example 19, let the individual* $j = 4$ *be an inverse dictator for the dictatorial rule given in Equation 2.6, i.e., for any profile* $P \in \mathbf{P}(\mathcal{A})$ *let* $F_d^4(P) := \{\neg\phi \mid \phi \in J_4\}$*.*

Oligarchy

An oligarchic aggregation function is a generalization of the above defined dictatorship (see Definition 24), i.e., in the case of an oligarchy, there is not necessarily an individual that decides the collective set, but rather a group of individuals that decides it. In particular, a dictatorship is the specific case of an oligarchy where the oligarchic group consists of a single individual.

Definition 26 (Oligarchic aggregation functions [DL08])**.** *Let* $\mathcal{J} = \langle N, \mathcal{A}\rangle$ *be a judgment aggregation problem and* F *an aggregation function for* $\mathcal{J}$*.* F *is (strong*[12]*)* oligarchic *if, and only if, there is a non-empty set* $O \subseteq N$ *of individuals such that for every profile* $P \in \mathbf{P}(\mathcal{A})$*,* $F(P) = \bigcap_{i\in O} J_i$*. The non-empty set of individuals* O *is called* oligarchs*.*

Remark 11. *Note that the notion of* oligarchy *sometimes refers to the so-called* weak oligarchy*, where the "=" symbol of Definition 26 is exchanged with a "⊆" symbol, i.e., the collective set needs not to coincide with the intersection of the oligarchs' individual judgment sets, but rather it must be contained in it [DL08, Gär06].*

As the dictatorial aggregation function is a special case of an oligarchic aggregation function, we observe that Example 19 is also an example of an oligarchic aggregation function. Another oligarchic aggregation function is the unanimity rule F_u (see Equation 2.3) as seen in Examples 15 and 18.

Anonymity

Anonymity describes the condition that all individual judgments are weighted the same in the aggregation [GP14, LP09]. In particular, this means that the output of the aggregation function, i.e., the collective set, should not depend on the order of the individual judgment sets of the given input profile [LP02, LP04].

Definition 27 (Anonymous aggregation functions [LP02, Lis11])**.** *Let* $\mathcal{J} = \langle N, \mathcal{A}\rangle$ *be a judgment aggregation problem and* F *an aggregation function for* $\mathcal{J}$*.* F *is* anonymous *if, and only if, for all profiles* $P, P' \in \mathbf{P}(\mathcal{A})$ *such that* P' *is a permutation of* P *it holds that* $F(P) = F(P')$*.*

Example 21 (Non-anonymous aggregation function)**.** *An example of a non-anonymous aggregation function would be the dictatorial aggregation function given in Example 17 and specifically in Example 19 as the collective set depends on the order of the judgment set.*

[12] [Gär06] refers to this notion of oligarchy as *strict oligarchy*.

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In particular, consider again the judgment aggregation problem $\langle\{1,2,3,4\},\mathcal{A}\rangle$ *and the aggregation function* $F_d^3(P) := J_3$ *from Example 19. Moreover, consider the two profiles* $\langle A,B,C,D\rangle$ *and* $\langle D,B,A,C\rangle$[13]*, which are permutations of each other. Observe that*

$$F_d^3(\langle A,B,C,D\rangle) = C \neq A = F_d^3(\langle D,B,A,C\rangle)$$

and thus, the aggregation function F_d^3 *is not anonymous since the weight of the individual judgment for the collective set depends on its position in the given profile.*

Example 22 (Anonymous aggregation function)**.** *Reconsider the uniform threshold function given in Example 16, i.e.,* $F(P) = \left\{\phi \in \mathcal{A} \mid |P_\phi| \geq \frac{|N|}{2}\right\}$*. In contrast to the aggregation function in Example 21, it is anonymous since the order of the individual judgments in the profile is irrelevant in the aggregation. A formula simply occurs in the collective set if the overall number of individuals in the profile accepting it is big enough. Specifically, consider the judgment aggregation problem* $\langle\{1,2,3\},\mathcal{A}\rangle$ *with the simple agenda* $\mathcal{A} = \pm\{p,q\}$ *(see Definition 8). Let* $J_1 = \{p,q\}$*,* $J_2 = \{p,\neg q\}$*, and* $J_3 = \{\neg p,\neg q\}$*. We observe that for any permutation* P *of the three individual judgment sets* J_1*,* J_2*, and* J_3 *the collective set is the same, i.e.,* $F(P) = \{p,\neg q\}$*.*

Independence

Independence defines the condition that if all the individuals of two profiles agree on the acceptance of a formula, then the resulting collective of the two profiles applying the same aggregation function will also agree on the acceptance of that formula [GP14, LP09]. Thus this condition ensures that collective judgments on a formula ϕ depend only on the individual judgments of the same formula ϕ and not on the judgments of other formulas, whereby a propositionwise aggregation approach (see Section 2.3.1) is ensured [LP09, Lis11].

Definition 28 (Independent aggregation functions [PvH06, DL07a, GP14])**.** *Let* $\mathcal{J} = \langle N,\mathcal{A}\rangle$ *be a judgment aggregation problem and* F *an aggregation function for* $\mathcal{J}$*.* F *is* independent *(of irrelevant alternatives*[14]*) if, and only if, for every formula* $\phi \in \mathcal{A}$ *and all profiles* $P, P' \in \mathbf{P}(\mathcal{A})$*, if [for every individual* $i \in N$*,* $\phi \in J_i \iff \phi \in J'_i$*] , then [*$\phi \in F(P) \iff \phi \in F(P')$*].*

Since independence ensures a propositionwise aggregation approach, all propositionwise quota rules satisfy independence (see Section 2.3.1), e.g., the propositionwise quota rule (see Definition 16), the propositionwise majority rule (see Equation 2.2), and the unanimity rule (see Equation 2.3).

[13] In this example, we use capital letters without any index for the individual judgment sets to reduce ambiguities that could happen due to the indices.

[14] Independence is sometimes also called *independence of irrelevant alternatives* [Mon08].

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Example 23 (Independent aggregation function)**.** *Let $\langle N, \mathcal{A}\rangle$ be a judgment aggregation problem and consider the unanimity rule (see Equation 2.3) defined as:*

$$F_u(P) = \{\phi \in \mathcal{A} \mid |P_\phi| = |N|\}$$

Let P and P' be two profiles and assume that for every individual $i \in N$, $\phi \in J_i$ if, and only if $\phi \in J_i'$. If $\phi \in F_u(P)$, then by the definition of the unanimity rule every individual in the profile P accepts ϕ. Since $\phi \in J_i \iff \phi \in J_i'$ we can follow that every individual in the profile P' also accepts ϕ. So, by the definition of the unanimity rule $\phi \in F_u(P')$. Moreover, the same argument holds for the other direction. Hence, the unanimity rule satisfies independence.

Example 24 (Dependent aggregation function)**.** *Consider the aggregation problem $\langle\{1,2\}, \mathcal{A}\rangle$, with the agenda $\mathcal{A} = \pm\{p, q, p \to q\}$ (see Example 8). Consider the following profiles $P = \langle J_1, J_2\rangle$ with the individual judgments*

- $J_1 = \{p, q, p \to q\}$
- $J_2 = \{p, q, p \to q\}$

and $P' = \langle J_1', J_2'\rangle$ with the individual judgments

- $J_1' = \{\neg p, q, p \to q\}$
- $J_2' = \{\neg p, q, p \to q\}$

We define the dependent aggregation function F_{dep} over the given aggregation problem as follows:

$$F_{dep}(P) = \{\phi \in \mathcal{A} \mid |P_\phi| = |N| \text{ and } \forall p \in At(\phi) \cap \mathcal{A} : |P_p| = |N|\} \tag{2.7}$$

The aggregation function F_{dep} accepts atoms that are unanimously accepted and only composite formulas that are unanimously accepted, where all atoms of the formula that are in the agenda are also unanimously accepted.

We observe that for $p \to q \in \mathcal{A}$ and the profiles P and P' that for both (all) individuals $i \in \{1,2\}$, $p \to q \in J_i \iff p \to q \in J_i'$. However, $p \to q \in F_{dep}(P)$ but $p \to q \notin F_{dep}(P')$ since $p \in At(p \to q)$ but $|P_p'| = 0 < |\{1,2\}| = |N|$. Hence, the aggregation function F_{dep} is not independent of irrelevant alternatives, and the collective acceptance of a formula depends also on other formulas.

As we observe in Example 23 and Example 24, independence is a very strong condition since it restricts the judgment of a formula ϕ completely on the individual judgments of ϕ in the given profiles, although two profiles could coincide in this single formula ϕ but could completely differ for all other formulas in the agenda [Sla16]. Consider, for

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example, the two profiles P and P' of Example 24. In the profile P, every individual accepts the proposition p. However, in profile P', every individual rejects it. To loosen this strong condition of independence of irrelevant alternatives, [Mon08] introduced the weaker condition of *independence of irrelevant propositional alternativs*, where the independence condition must only hold for the literals in the agenda [Mon08].

Neutrality

Neutrality[15] ensures that if all individuals of a profile agree on the acceptance or rejection of any two formulas, then the collective set must also agree on the acceptance or rejection of those two formulas [GE13, GP14, Sla16]. In particular, this means that for any formula, the condition to be collectively accepted has to be the same [Lis11].

Definition 29 (Neutral aggregation functions [Lis11, GP14])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* neutral *if, and only if, for all formulas $\phi, \psi \in \mathcal{A}$ and for every profile $P \in \mathbf{P}(\mathcal{A})$, if [for every individual $i \in N$, $\phi \in J_i \iff \psi \in J_i$], then [$\phi \in F(P) \iff \psi \in F(P)$].*

Example 25 (Neutral aggregation function)**.** *Consider the propositionwise majority rule F_{maj} (see Definition 17) that accepts a formula in the collective set if, and only if, a majority accepts it. Let $\langle N, \mathcal{A} \rangle$ be a judgment aggregation problem. Let $P \in \mathbf{P}(\mathcal{A})$ be an arbitrary profile, $\phi, \psi \in \mathcal{A}$ two arbitrary formulas and assume that for every individual $i \in N$, $\phi \in J_i$ if, and only if, $\psi \in J_i$. If $\phi \in F_{maj}(P)$, then by definition of F_{maj} a majority of individuals accept ϕ. By assumption, the same individuals that accept ϕ also accept ψ. So ψ is also accepted by a majority of the individuals and thus $\psi \in F_{maj}(P)$. Moreover, the argument holds for the other direction. Hence, the propositionwise majority rule satisfies neutrality.*

Example 26 (Non-neutral aggregation function)**.** *Consider again Example 24 of the dependent aggregation function F_{dep}. We observe that this aggregation function is also neutral, since the condition for being collectively accepted is different for composite formulas built from atoms contained in the agenda, than for atoms and composite formulas that are built from atoms not occurring in the agenda. This becomes more obvious if we consider the definition of F_{dep} again, which is given in Equation 2.7 as:*

$$F_{dep}(P) = \{\phi \in \mathcal{A} \mid |P_\phi| = |N| \text{ and } \forall p \in At(\phi) \cap \mathcal{A} : |P_p| = |N|\}$$

Considering the definition of the resulting set, if ϕ is an atom then the second condition of the definition is already satisfied. Hence, we can rewrite the definition of F_{dep} the following way:

$$F_{dep}(P) = \{\phi \in \mathcal{A} \mid |P_\phi| = |N| \text{ and if } \phi \notin At(\mathcal{A}), \text{ then } \forall p \in At(\phi) \cap \mathcal{A} : |P_p| = |N|\}$$

As we observe, composite formulas have to satisfy a different condition in contrast to atomic propositions.

[15]The neutrality property is sometimes also called *issue-neutrality* in comparison to *domain-neutrality* where the individuals agree on the non-equivalence relation of two formulas [GE13, Sla16].

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Systematicity

Systematicity is a condition that combines the properties of independence (see Section 2.4.3) and neutrality (see Section 2.4.3) [LP09]. This means that collective judgments on a formula depend only on the individual judgments on this same formula and that the condition for being collectively accepted is the same for all formulas [Lis11, Sla16].

Definition 30 (Systematic aggregation functions [LP02, PvH06, DL07a])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* systematic *if, and only if, for all formulas $\phi, \psi \in \mathcal{A}$ and for all profiles $P, P' \in \mathbf{P}(\mathcal{A})$, if [for every individual $i \in N$, $\phi \in J_i \iff \psi \in J'_i$], then [$\phi \in F(P) \iff \psi \in F(P')$].*

Theorem 3 ([Sla16, LP09])**.** *A judgment aggregation function F is systematic if, and only if, it is independent (see Definition 28) and neutral (see Definition 29).*

Proof. Let $\langle N, \mathcal{A} \rangle$ be any judgment aggregation problem, and let F be an aggregation function for it.

$\Longrightarrow$ Assume that F is systematic. The condition of independency is trivially satisfied by systematicity if we set the formulas ϕ and ψ to the same formula. Moreover, neutrality is also trivially satisfied by systematicity if we set the two profiles P and P' to the same profile. Thus F is independent and neutral.

$\Longleftarrow$ Assume that F is independent and neutral. Let $\phi, \psi \in \mathcal{A}$ be two arbitrary formulas, and let $P, P' \in \mathbf{P}(\mathcal{A})$ be any two profiles. Assume that for every individual $i \in N$, $\phi \in J_i \iff \phi \in J'_i$ and that for every $i \in N$, $\phi \in J'_i \iff \psi \in J'_i$. Since F is independent $\phi \in F(P) \iff \phi \in F(P')$. Moreover, F is neutral and thus $\phi \in F(P') \iff \psi \in F(P')$. Thus, by independence together with neutrality, we get as a result that $\phi \in F(P) \iff \psi \in F(P')$. Combining the two assumptions leads to the overall assumption that for every individual $i \in N$, $\phi \in J_i \iff \psi \in J'_i$. Hence, F satisfies the condition of systematicity and is thus systematic. □

Example 27 (Systematic aggregation function)**.** *Consider again the propositionwise majority rule F_{maj} (see Definition 17) that collectively accepts formulas that are accepted by the majority of the individuals. We can use a similar argument as given in Example 25 for showing systematicity. Observe that if in two different profiles, P and P', ϕ is accepted in P by the same number of individuals as ψ is accepted in P', then ϕ is accepted by a majority in P if, and only if, ψ is accepted by a majority in P'. Hence, ϕ is collectively accepted in P if, and only if, ψ is collectively accepted in P' when using the propositionwise majority rule F_{maj}. Hence, the propositionwise majority rule is systematic.*

Example 28 (Systematic aggregation function)**.** *Another systematic rule is the unanimity rule F_u (see Equation 2.3). We already showed in Example 23 that F_u is independent. So, if we can additionally show that F_u is neutral, then by Theorem 3 it is also systematic. We can again use a similar argument as in Example 25. If a formula ϕ is accepted by the*

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same individuals as another formula ϕ in a profile P, then ϕ is unanimously accepted if, and only if, ψ is unanimously accepted. Hence, F_u is neutral. Since, the unanimity rule is independent and neutral, by Theorem 3, F_u is systematic.

Remark 12. *Consider Example 28. By Theorem 3 and since the aggregation function is independent and neutral it follows that the aggregation function is also systematic. Furthermore, we can use Theorem 3 to conclude that an aggregation function is independent and neutral if it is systematic. For example, by using the result from Example 27, i.e., that the propositionwise majority rule is systematic, it follows that it is also neutral.*

Responsiveness

The condition of responsiveness ensures that an aggregation function cannot be constant, i.e., that it does not return the same collective set for every profile. In particular, this means that every formula $\phi \in \mathcal{A}$ can be collectively accepted by some profile, and also collectively rejected by some profile.

Definition 31 (Responsive aggregation functions [PvH06, Die10])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* responsive *if, and only if, for all formulas $\phi \in \mathcal{A}$ (such that ϕ is contingent), there are profiles $P, P' \in \mathbf{P}(\mathcal{A})$ such that $\phi \in F(P)$ and $\phi \notin F(P')$.*

Remark 13. *[Die06, PvH06] also consider a weak responsiveness condition that is weaker than responsiveness from Definition 31 and requires that there is at least some formula $\phi \in \mathcal{A}$ and profiles $P, P' \in \mathbf{P}(\mathcal{A})$ such that $\phi \in F(P)$ and $\phi \notin F(P)$. Note that if the agenda $\mathcal{A} \neq \emptyset$, then trivially responsiveness implies weak responsiveness. Moreover, [vH07] considers a minimal responsiveness condition, that is, in general, weaker than responsiveness from Definition 31 and stronger than weak responsiveness. However, in classical bivalent logic weak and minimal responsiveness are equivalent.*

Example 29 (Responsive aggregation function)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. Consider the propositionwise majority rule (see Definition 17), i.e.*

$$F_{maj}(P) = \left\{ \phi \in \mathcal{A} \mid |P_\phi| \geq \left\lceil \frac{|N| + 1}{2} \right\rceil \right\}$$

and assume that F_{maj} satisfies universal domain (see Definition 18). We observe that for any formula $\phi \in \mathcal{A}$ there are at least the trivial profiles $P, P' \in \mathbf{P}(\mathcal{A})$ such that $P_\phi = N$ and $P'_\phi = \emptyset$. By the definition of the propositionwise majority rule, we have that since $|P_\phi| = |N|$, that $\phi \in F(P)$ and due to $|P'_\phi| = |\emptyset| = 0$ that $\phi \notin F(P')$. Hence, we observe that F_{maj} satisfies the mapping condition of responsiveness.

Example 30 (Unresponsive aggregation function)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. Consider the constant aggregation*

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function F_{const}, which is defined for all profiles $P \in \mathbf{P}(\mathcal{A})$ as:

$$F_{const}(P) = J \qquad \text{for some } J \in \mathbf{J}(\mathcal{A}) \tag{2.8}$$

For example, under the agenda $\mathcal{A} = \pm\{p, q, p \rightarrow q\}$ (see Examples 8 and 24) let $F_{const}(P) = \{p, q, p \rightarrow q\}$ for all profiles. We note that there is no profile $P \in \mathbf{P}(\mathcal{A})$ such that $p \notin F_{const}(P)$. Hence, the constant aggregation function F_{const} is not responsive, i.e., unresponsive (see Definition 31).

Monotonicity

The condition of monotonicity describes that if a collective accepts a formula $\phi \in \mathcal{A}$ and the support for ϕ increases, then the formula ϕ should still be collectively accepted. In particular, more support for a formula cannot lead to a sudden collective rejection.

Definition 32 (Monotonic aggregation functions [KE09][16])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* monotonic *if, and only if, for all formulas $\phi \in \mathcal{A}$ and all profiles $P, P' \in \mathbf{P}(\mathcal{A})$, if $\phi \in F(P)$ and $P_\phi \subset P'_\phi$, then $\phi \in F(P')$.*

Example 31 (Monotonic aggregation function)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem. Consider the propositionwise majority rule F_{maj} (see Definition 17), that accepts a formula $\phi \in \mathcal{A}$ if it is accepted by a majority of the input profile $P \in \mathbf{P}(\mathcal{A})$, i.e., $|P_\phi| > \frac{|N|}{2}$. Let $\phi \in \mathcal{A}$ be any formula, and $P \in \mathbf{P}(\mathcal{A})$ be any profile such that $\phi \in F(P)$ is collectively accepted. Now consider any other profile P' such that $P_\phi \subset P'_\phi$. By Definition 17 of the propositionwise majority rule, we have that $\frac{|N|}{2} < |P_\phi| < |P'_\phi|$ and thus $\phi \in F_{maj}(P')$ must also be the case.*

Example 32 (Non-monotonic aggregation function)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem. Consider the propositionwise minority rule F_{min}, that accepts a formula $\phi \in \mathcal{A}$ if it is accepted by a minority of the input profile $P \in \mathbf{P}(\mathcal{A})$, i.e., if $|P_\phi| \leq \frac{|N|}{2}$, then $\phi \in F_{min}(P)$. In particular, it is the converse of the propositionwise majority rule, i.e., for any profile $P \in \mathbf{P}(\mathcal{A})$ and any formula $\phi \in \mathcal{A}$*

$$\phi \in F_{min}(P) \iff \phi \notin F_{maj}$$

Let $\phi \in \mathcal{A}$ and $P \in \mathbf{P}(\mathcal{A})$ such that $|P_\phi| = \left\lfloor \frac{|N|}{2} \right\rfloor$. Moreover, let $P' \in \mathbf{P}(\mathcal{A})$ such that $P_\phi \subset P'_\phi$. Thus we have that $|P_\phi| \leq \frac{|N|}{2} < |P'_\phi|$. Observe that $\phi \in F_{min}(P)$, but $\phi \notin F_{min}(P')$. Hence, the monotonic condition is not satisfied, and so F_{min} is non-monotonic.

Example 32 shows that monotonic aggregation functions (see Definition 32) prohibit inverse acceptance, which would lead to rather counterintuitive collective sets [KE09].

[16] There are also other equivalent definitions of monotonicity, e.g., in [DM10, Lis11, GP14]. However, [KE09] provides a rather straightforward definition.

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2.5 Modelling constraints explicitly

Until now, we considered judgment aggregation problems without imposing any restricting constraints on the individual judgments. However, reconsider the discursive dilemma from Example 1: there are three judges that have to decide if the contract is valid (v), the defendant breached the contract (b), and if the defendant is guilty (g). The applicable law in the example is represented by the formula $v \wedge b \leftrightarrow g$, and we modeled the individual judgments to hold this formula true for every judge. By Definition 2 of a judgment aggregation problem, there is no constraint that ensures that the formula $v \wedge b \leftrightarrow g$ must be considered as true by any judge. So, for modeling such constraints that must be true in every judgment, we can extend the general definition of a judgment aggregation problem $\langle N, \mathcal{A} \rangle$ to a judgment aggregation problem with constraints $\mathcal{J} = \langle N, \mathcal{A}, \Gamma \rangle$ with $\Gamma \subseteq \mathcal{L}$ the set of constraints that must be satisfied [Sla16, EGdHL16].

Definition 33 (Judgment aggregation problem with constraints [Sla16])**.** *Let $\mathcal{L}$ be a language over the atoms At. A* ($\mathcal{L}$-)judgment aggregation problem with constraints *is a tuple $\mathcal{J} = \langle N, \mathcal{A}, \Gamma \rangle$, where*

- *N is a finite non-empty set of individuals.*
- *$\mathcal{A} \subseteq \mathcal{L}$ such that $\mathcal{A}$ is closed under negation. $\mathcal{A}$ is called the agenda.*
- *$\Gamma \subseteq \mathcal{L}$ is a set of constraints.*

The set of individuals N and the agenda $\mathcal{A}$ are as in Definition 2, i.e., we can also use the agenda definition based on a positive set of issues.

Definition 34 (Judgment set with constraints [Sla16])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}, \Gamma \rangle$ be a $\mathcal{L}$-judgment aggregation problem with constraints Γ. A* judgment set *for $\mathcal{J}$ is a set of formulae $J \subseteq \mathcal{A}$ such that*

- *J is consistent*
- *J is consistent with the constraints Γ, i.e., $J \cup \Gamma \not\models \bot$*
- *J is complete*

The only difference between judgment sets of judgment problems with and without constraints is the consistency with and without regarding the set of constraints Γ. However, explicitly modeling the constraints as, e.g., in the discursive dilemma (Example 1) allows us to reduce the set of (admissible) judgment sets. In particular, $\mathbf{J}(\mathcal{A}, \Gamma) \subseteq \mathbf{J}(\mathcal{A})$, where $\mathbf{J}(\mathcal{A}, \Gamma)$ denotes the set of all judgment sets over agenda $\mathcal{A}$ with constraints Γ, i.e., admissible judgment sets of $\mathbf{J}(\mathcal{A})$ that are additionally consistent with the constraints Γ.

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Example 33. *Consider the discursive dilemma (see Example 1) with the agenda* $\mathcal{A} = \pm\{v, b, g, v \wedge b \leftrightarrow g\}$*. By using the modeling with constraints, we can define that every individual judge has to adhere to current law. So if a judge considers the contract valid (*v*) and also that the defendant breached the contract (*b*), then at the same time, he must accept the defendant is guilty (*g*). In particular, the current law formulated as* $v \wedge b \leftrightarrow g$ *is a constraint.*

Hence, we have the following judgment aggregation problem (with constraints) $\mathcal{J} = \langle N, \mathcal{A}, \Gamma \rangle$ *with*

- $N = \{1, 2, 3\}$
- $\mathcal{A} = \pm\{v, b, g, v \wedge b \leftrightarrow g\}$
- $\Gamma = \{v \wedge b \leftrightarrow g\}$

Consider the following examples:

- $J_1 = \{v, b, \neg g, \neg(v \wedge b \leftrightarrow g)\}$

  Regarding the previous Definition 3 of judgment sets without constraints, J_1 *would be a judgment set since it is a complete and consistent subset of the agenda* $\mathcal{A}$*. However, regarding Definition 34 that includes the constraint set* Γ*, we observe that* $J_1 \cup \Gamma$ *is inconsistent since* $\{\neg(v \wedge b \leftrightarrow g), v \wedge b \leftrightarrow g\} \subseteq J_1 \cup \Gamma$*.*

- $J_2 = \{v, b, g, v \wedge b \leftrightarrow g\}$

  The set J_2 *is a judgment set satisfying the constraints and thus is also a judgment set regarding Definition 34.*

Theorem 4. *Let* $\mathcal{J} = \langle N, \mathcal{A}, \Gamma \rangle$ *be a judgment problem with constraints (see Definition 34). If* $J \in \mathbf{J}(\mathcal{A}, \Gamma)$ *is a judgment set of* $\mathcal{J}$*, then* $\Gamma \cap \mathcal{A} \subseteq J$*.*

Proof. Let J be a judgment set of the judgment problem with constraints $\langle N, \mathcal{A}, \Gamma \rangle$. Let ϕ be any formula in $\Gamma \cap \mathcal{A}$. It follows that $\phi \in \mathcal{A}$ and by the Definition 34 J is consistent and complete with regard to the agenda $\mathcal{A}$. Hence, if $\phi \in \mathcal{A}$, then either $\phi \in J$ or $\neg\phi \in J$[17]. Assume that $\neg\phi \in J$. Since $\phi \in \Gamma$ it follows that J is not consistent with the constraints Γ, i.e., $J \cup \Gamma \not\models \bot$. However, this contradicts the assumption that J is a judgment set. Thus $\phi \in J$. □

Remark 14. *Regarding Theorem 4, observe that we cannot generalize the theorem such that* $\Gamma \subseteq J$ *if* J *is a judgment set. By the definition of the constraints* Γ *(see Definition 2), they are a subset of the language, i.e.,* $\Gamma \subseteq \mathcal{L}$*. Hence, in general, a judgment set does not judge about the formulas in* $\Gamma \setminus \mathcal{A}$*. If* $\Gamma \subseteq \mathcal{A}$*, then we trivially have the case that if* J *is a judgment set, then also* $\Gamma \subseteq J$*.*

[17] Note that we identify a doubly negated formula $\neg\neg\phi$ with its corresponding non-negated formula ϕ.

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When considering judgment aggregation problems without constraints (see Definition 2) in comparison with judgment aggregation problems with constraints (see Definition 33), we observe that judgment aggregation problems without constraints are simply a special case of judgment aggregation problems with constraints where the set of constraints is empty. However, in the classical propositional case, both definitions have the same expressibility and are even equally succinct, i.e., converting a judgment aggregation problem with constraints into an equivalent judgment aggregation problem without constraints (and vice versa) is bounded polynomially by the size of the problem [EGdHL16].

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CHAPTER 3

Impossibility results

The aim of judgment aggregation and especially of using aggregation functions is to find a collective (judgment) set that is most representative regarding the aggregated individual judgments [Sla16]. However, as many so-called *impossibility results* [LP02, PvH06, Die06, DL08, DM10] have shown, it is not always the case that even “adequately representative” aggregated collective judgment sets can be found. In particular, this means that for agendas with certain restrictions (see Section 2.2), aggregation functions satisfying certain aggregation conditions (see Section 2.4) either do not exist or fail to be even slightly democratic, e.g., being dictatorial (see Section 2.4.3) [Lis11].

The chapter consists of 4 main sections. In Section 3.1 at first, we informally consider why certain aggregation conditions (see Section 2.4) should be satisfied and then show that an aggregation function (see Section 2.3) satisfying all these conditions does not exist, which was originally shown in [LP02]. Section 3.2 copes with the general technique for proving impossibility theorems by using the so-called *ultrafilter proof technique* [KE09, NP10, DM10, GP14]. In Section 3.3, we will conclude by reviewing some impossibility results and their implications for the aggregation of judgments.

3.1 A simple impossibility theorem

In the following Section 3.1.1, we will reconsider the discursive dilemma of Example 1 and argue why certain properties are desirable for aggregation functions. However, as we will see in Section 3.1.2 under certain agenda restrictions (see Section 2.2), such an aggregation function cannot exist, as originally shown in [LP02].

3.1.1 Desirable properties for aggregation functions

Consider again Example 1 introducing the discursive dilemma, where three judges want to find a verdict in a case. If we consider this specific example in the field of jurisprudence,

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then we would want that the resulting collective verdict is also rational and that the views of all three judges are considered equally in the ultimate verdict [LP02]. For example, regarding the discursive dilemma example where the judges have to find a collective verdict, we can imagine that the defendant would simply not accept the verdict if it is not consistent and the premises in the collective set contradict the conclusion of the collective set (see Example 1).

Moreover, the collective set should also be complete. Imagine, e.g., that the collective set of the discursive dilemma is incomplete, i.e., $\{v, g \leftrightarrow v \wedge b, g\}$. So the collegial court would accept the current law ($g \leftrightarrow v \wedge b$), that the contract is valid (c), and at the same time collectively conclude that the defendant is guilty (g). Considering that v and b are the premises and g the conclusion, it makes no sense to conclude g without both v and b. Hence, overall we observe that the collective set should satisfy the same rationality demands as we have for individual judgments, i.e., collective rationality should be satisfied (see Definition 21).

As we further observe, the collegial court consisting of the three judges should find the verdict in an anonymous (see Definition 27) way. Otherwise, it would contradict its own purpose of reducing the power of a single judge and splitting it equally to all members of the collegial court to ensure a fairer verdict.

Furthermore, it is a desired property that a court should only reach a collective verdict by considering only the judges' individual judgments on the case and nothing more, which corresponds to the property of independence (see Definition 28). Additionally, every proposition of the collective judgment of the collegial court should be accepted under the same condition, which corresponds to neutrality (see Definition 29). As Theorem 3 shows, an aggregation function is independent and neutral if, and only if, it is systematic (see Definition 30). Hence, the actual property the collegial court should satisfy is systematicity.

Last but not least, we usually demand that individual members of a collegial court can have every rational individual judgment, which corresponds to the property of universal domain (see Definition 18).

Hence, overall we want that a collective that has to find a collective judgment, e.g., like a collegial court, should satisfy the output condition *fully rationality* (see Section 2.4.2), the input condition *universal domain* (see Section 2.4.1), and the mapping conditions *anonymity* and *systematicity* (see Section 2.4.3). However, as [LP02] showed, there is no aggregation function satisfying all these conditions for non-simple agendas (see Definition 8).

Remark 15. *Note that Example 1 of the discursive dilemma is already a counterexample which shows that the propositionwise majority rule (see Definition 17) cannot be an aggregation function that satisfies all the above-mentioned properties for non-simple agendas since the collective set of the example does not satisfy rationality. In particular, the collective set of Example 1 is inconsistent.*

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3.1.2 The impossibility of an aggregation function

The properties for aggregation functions the above Section 3.1.1 argues for seem desirable for collective judgments. However, as we will see in this section, such an aggregation function does not exist [LP02].

Lemma 2 (Anonymous and systematic aggregation functions [LP02][1])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an anonymous and systematic (see Section 2.4.3) aggregation function for $\mathcal{J}$. For every $\phi, \psi \in \mathcal{A}$, if $|P_\phi| = |P_\psi|$, then $\phi \in F(P) \iff \psi \in F(P)$.*

Proof. Let $\langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an anonymous and systematic aggregation function for this problem. Moreover, let $\phi, \psi \in \mathcal{A}$ be arbitrary formulas such that $|P_\phi| = |P_\psi|$.

By assumption $|P_\phi| = |P_\psi|$, so we know that the same number of individuals (but not necessarily the same individuals) accept the formulas ϕ and ψ. Let P' be the permutation of P such that $\phi \in J_i \iff \psi \in J'_i$, which must exist since the number of individuals that accept ϕ and ψ are equal. Thus we have that $\phi \in J_i \iff \psi \in J'_i$, and by systematicity, we get that $\phi \in F(P) \iff \psi \in F(P')$. Moreover, P' is a permutation of P, and by anonymity, it must hold that $F(P) = F(P')$. Hence, we conclude that $\phi \in F(P) \iff \psi \in F(P')$. □

Theorem 5 ([LP02])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem with $|N| \geq 2$. If for some formulas $\phi, \psi \in \mathcal{A}$ we have $\{\phi, \psi, \phi \circ \psi, \neg(\phi \circ \psi)\} \subseteq \mathcal{A}$ for $\circ \in \{\wedge, \vee, \rightarrow\}$, then there is no collectively rational (see Section 2.4.2) judgment aggregation function F that satisfies universal domain, anonymity and systematicity (see Section 2.4.3).*

Proof. We proceed by proving the theorem for the conjunctive case, i.e., $\circ = \wedge$. The cases for $\vee$ and $\rightarrow$ are similar. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem with $\{\phi, \psi, \phi \wedge \psi, \neg(\phi \wedge \psi)\} \subseteq \mathcal{A}$ and let F be an aggregation function for $\mathcal{J}$ that satisfies collective rationality, universal domain, anonymity, and systematicity. We have to distinguish two cases of $|N|$ many individuals. The case where $|N|$ is even and the case where $|N|$ is odd.

$|N|$ **is even** : Since $|N|$ is even, by universal domain (see Definition 18) there must be a profile $P \in \mathbf{P}(\mathcal{A})$ such that $|P_{(\phi \wedge \psi)}| = |P_{\neg(\phi \wedge \psi)}|$, e.g., the one half of individuals accepts $\phi \wedge \psi$ and the other half of individuals rejects $\phi \wedge \psi$. Since F is collectively rational (see Definition 21) it is also complete. Hence, at least one of have $(\phi \wedge \psi)$ and $\neg(\phi \wedge \psi)$ must be in $F(P)$. Moreover, by assumption F is anonymous and systematic, thus by Lemma 2, we get that $(\phi \wedge \psi) \in F(P) \iff \neg(\phi \wedge \psi) \in F(P)$ and together with completeness it follows that $(\phi \wedge \psi) \in F(P)$ and $\neg(\phi \wedge \psi) \in F(P)$. Hence, $F(P)$ is inconsistent, which contradicts that F is collectively rational.

[1] In [LP02] the proof of Lemma 2 is only a step of the proof of Theorem 5 but done with different definitions.

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$|N|$ **is odd** : If $|N|$ is odd, then by universal domain, there must be a profile P such that $|P_\phi| = |P_\psi| = |P_{\neg(\phi\wedge\psi)}|$. For example, in the case of $\phi \wedge \psi$ the first individual accepts ϕ and $\neg\psi$, the second accepts $\neg\phi$ and ψ and the third accepts ϕ and ψ. For every additional two individuals, one accepts $\neg\phi$ and $\neg\psi$, and the other one accepts ϕ and ψ, as also seen in Table 3.1. So since $|P_\phi| = |P_\psi| = |P_{\neg(\phi\wedge\psi)}|$, either $\phi, \psi, \neg(\phi \wedge \psi) \in F(P)$ or none of the formulas are in $F(P)$.

If $\phi, \psi \in F(P)$ and since F is collectively rational, then by Theorem 2 it follows that $(\phi \wedge \psi) \in F(P)$. So $\neg(\phi \wedge \psi) \in F(P)$ and $(\phi \wedge \psi) \in F(P)$ and thus $F(P)$ is inconsistent, which again contradicts that F is collectively rational.

If $\phi, \psi \notin F(P)$, then by the case assumption we get that $\neg(\phi\wedge\psi) \notin F(P)$. However, F is complete so we can conclude that $(\phi \wedge \psi) \in F(P)$ and $\neg\phi, \neg\psi \in F(P)$. If $(\phi \wedge \psi) \in F(P)$ and since F is collectively rational and thus by Theorem 2 also deductively closed, we get that $\phi, \psi \in F(P)$. So $\phi \in F(P)$ and $\neg\phi \in F(P)$, which contradicts that F is collectively rational.

So overall, we showed that if $|N|$ is even and also if $|N|$ is odd, it leads to a contradiction. Thus we can conclude for general $|N| \geq 2$ that such an aggregation function does not exist. □

Individual	ϕ	ψ	$\neg(\phi \wedge \psi)$
1	1	0	1
2	0	1	1
3	1	1	0
$2 \cdot i$	0	0	1
$2 \cdot (i+1)$	1	1	0

Table 3.1: Example of a profile P for the case if $|N|$ odd, for $i \geq 2$

Remark 16 (Agenda restrictions)**.** *Considering Theorem 5. Note that the agenda $\mathcal{A}$ is defined has having the certain subset $\{\phi, \psi, \phi \wedge \psi, \neg(\phi \wedge \psi)\} \subseteq \mathcal{A}$, e.g., for $\circ = \wedge$. Hence, $\mathcal{A}$ satisfies non-simplicity (see Definition 8) since the set $X = \{\phi, \psi, \neg(\phi \wedge \psi\} \subseteq \mathcal{A}$ is minimally inconsistent (see Definition 7) and $3 \leq |X|$. Moreover, $\mathcal{A}$ is also evenly negatable (see Definition 11) as for the same minimally inconsistent subset $X = \{\phi, \psi, \neg(\phi \wedge \psi\} \subseteq \mathcal{A}$, there is a subset $Y = \{\phi, \psi\} \subseteq X$ of even size ($|Y| = 2$) such that the set*

$$(X \setminus Y) \cup \{\neg\phi \mid \phi \in Y\} = \{\neg\phi, \neg\psi, \neg(\phi \wedge \psi)\}$$

is consistent. In particular, the agenda of Theorem 5 is minimally connected*, i.e., non-simple and evenly negetable [DL07a].*

Theorem 5 shows that imposing desirable conditions on the aggregation of individual judgments leads actually to the case that there is no aggregation function satisfying all these conditions. In particular, this is the general strategy of impossibility results in

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judgment aggregation, i.e., we impose desirable conditions and show that there is either no aggregation function satisfying these conditions or that the resulting aggregation function is degenerate, e.g., dictatorial [GP14, Lis11].

3.2 The ultrafilter proof technique

In the above Section 3.1, we showed in Theorem 5 that imposing certain conditions like collective rationality, universal domain, anonymity, and systematicity on the aggregation of individual judgments leads to the impossibility of finding an aggregation function for such problems. However, Theorem 5 only states a sufficient condition for leading to that impossibility, but not a necessary condition for it. Hence, to characterize sufficient and necessary conditions of impossibility, we will introduce the ultrafilter proof technique and the notion of winning coalitions in the following.

3.2.1 Winning coalitions for formulas and profiles

Winning coalitions for formulas and profiles describe sets of individuals that can decide the acceptance of a formula in all profiles or the acceptance of every formula in one profile, respectively.

Winning coalitions for formulas

Definition 35 (Winning coalitions for formulas [KE09, GP14])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. We call a set $W \subseteq N$ of individuals* winning for a formula $\phi \in \mathcal{A}$ *in a profile $P \in \mathbf{P}(\mathcal{A})$ if, and only if, $W = P_\phi$ and $\phi \in F(P)$.*

Moreover, we call a set $W \subseteq N$ a winning coalition for a formula $\phi \in \mathcal{A}$ *if, and only if*

$$\forall P \in \mathbf{P}(\mathcal{A}) : \text{ if } W = P_\phi, \text{ then } \phi \in F(P)$$

We denote the set of all winning coalitions[2] *for a formula $\phi \in \mathcal{A}$ over the judgment aggregation problem $\mathcal{J}$ and the aggregation function F as $\mathcal{W}_\phi(\mathcal{J}, F)$.*

If a set W of individuals is a winning coalition for some formula $\phi \in \mathcal{A}$, then we also say that W *is winning* for ϕ. Informally regarding Definition 35, the winning coalitions define those individuals that can together decide if a formula of the agenda is collectively accepted or not [GP14].

Example 34 (Winning coalition for the unanimity rule)**.** *Consider the unanimity rule F_u (see Section 2.4.3) that collectively accepts a formula $\phi \in \mathcal{A}$ exactly if it is accepted by every individual of the given profile $P \in \mathbf{P}(\mathcal{A})$. Note that for every formula $\phi \in \mathcal{A}$,*

[2] Sometimes winning coalitions are also called *decisive coalitions*, e.g., in [KE09]. Moreover, [DM10] describes the set $\mathcal{W}_\phi(\mathcal{J}, F)$ as *generating* F on ϕ.

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the only winning coalition for F_u is $W = N$ since, by definition, a formula can only be collectively accepted if all individuals accept it.

Example 35 (Winning coalition for the propositionwise majority rule)**.** *Consider the propositionwise majority rule F_{maj} (see Definition 17) and the judgment aggregation problem $\mathcal{J} = \langle N, \mathcal{A} \rangle$. By the definition of the rule (see Equation 2.2), a formula $\phi \in \mathcal{A}$ is collectively accepted if, and only if, ϕ is accepted by at least a majority of the individuals. Hence, we can define the set of all winning coalitions for any formula $\phi \in \mathcal{A}$ for the propositionwise majority rule as*

$$\mathcal{W}_\phi(\mathcal{J}, F_{maj}) = \left\{ W \subseteq N \mid |W| \geq \left\lceil \frac{|N|+1}{2} \right\rceil \right\}$$

Consider now any formula $\phi \in \mathcal{A}$, any profile $P \in \mathbf{P}(\mathcal{A})$ and assume that $W \in \mathcal{W}_\phi(\mathcal{J}, F_{maj})$. Then if $W = P_\phi$ and since by definition $|W| \geq \lceil \frac{|N|+1}{2} \rceil$, we can conclude that ϕ is accepted by a majority and thus $\phi \in F_{maj}(P)$. Hence, the coalitions $W \in \mathcal{W}_\phi(\mathcal{J}, F_{maj})$ are truly winning for ϕ.

Remark 17. *Consider winning coalitions for a formula $\phi \in \mathcal{A}$ (see Definition 35) and Examples 34 and 35. Note that we can also define aggregation functions by directly specifying the winning coalitions that should lead to the collective acceptance of a formula ϕ, i.e., $F(P) = \{\phi \in \mathcal{A} \mid P_\phi \in \mathcal{W}_\phi(\mathcal{J})\}\}$, where the winning coalitions $\mathcal{W}_\phi(\mathcal{J})$ no longer depend on the aggregation function but are used for the definition of the aggregation function in contrast to Definition 35 [Lis11]. Then, trivially $\mathcal{W}_\phi(\mathcal{J}, F) = \mathcal{W}_\phi(\mathcal{J})$ by definition.*

Lemma 3 (Characterization of independence by winning coalitions for a formula [DM10][3])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is independent (see Definition 28) if, and only if*

$$\forall \phi \in \mathcal{A}\, \forall P \in \mathbf{P}(\mathcal{A}) : \phi \in F(P) \iff P_\phi \in \mathcal{W}_\phi(\mathcal{J}, F)$$

i.e., $W = P_\phi$ is winning for ϕ in one profile $P \in \mathbf{P}(\mathcal{A})$ if, and only if, W is winning for ϕ in all profiles $P' \in \mathbf{P}(\mathcal{A})$.

Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function of $\mathcal{J}$.

$\Longrightarrow$: For the *if*-direction, we proceed by proof of contradiction. Let F be an independent aggregation function, and $\phi \in \mathcal{A}$ be any formula. Assume that $P \in \mathbf{P}(\mathcal{A})$ such that $W = P_\phi$ and $\phi \in F(P)$, i.e., W is a winning for ϕ in the profile P, but that W is not winning for ϕ in all profiles, i.e., $W = P_\phi \notin \mathcal{W}_\phi(\mathcal{J}, F)$. Hence, there

[3] In comparison to the characterization of Lemma 3 see also the definition of *voting by issue* given in [NP08, LP09] and a similar characterization of independence in [KE09].

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must be a profile $P' \in \mathbf{P}(\mathcal{A})$ such that $W = P'_\phi$ but $\phi \notin F(P)$. By definition of independency, for all profiles $P' \in \mathbf{P}(\mathcal{A})$ if $P'_\phi = P_\phi$ and $\phi \in F(P)$, then $\phi \in F(P')$. Since $P_\phi = W = P'_\phi$ it must hold that $\phi \in F(P')$, which is a contradiction. Hence, W must be winning for ϕ in all profiles, i.e., $W = P_\phi \in \mathcal{W}_\phi(\mathcal{J}, F)$.

For the *only-if*-direction, assuming that $P_\phi \in \mathcal{W}_\phi(\mathcal{J}, F)$ by Definition 35 implies that $\phi \in F(P)$.

$\Longleftarrow$: We proceed by proof of contradiction. Assume that for any formula $\phi \in \mathcal{A}$ and any profile $P \in \mathbf{P}(\mathcal{A})$ that $\phi \in F(P) \iff P_\phi \in \mathcal{W}_\phi(\mathcal{J}, F)$. But assume that F is not independent. Since F is not independent, there is a formula $\phi \in \mathcal{A}$ and profiles $P, P' \in \mathbf{P}(\mathcal{A})$ such that $P_\phi = P'_\phi$ but w.l.o.g. $\phi \in F(P)$ and $\phi \notin F(P')$. By assumption and since $\phi \in F(P)$, we can follow that $W = P_\phi \in \mathcal{W}_\phi(\mathcal{J}, F)$. Moreover, $W = P_\phi = P'_\phi$ ís a winning coalition for ϕ. Thus P'_ϕ is also winning for ϕ in P' and thus $\phi \in F(P')$. However, this contradicts our assumption that F is not independent. Hence, F must be independent. □

Remark 18. *In particular, the characterization of independence by winning coalitions for a formula, as given in Lemma 3, states that the sets of individuals able to collectively decide the acceptance or rejections of a formula are invariant of the profile if and only if, the aggregation function is independent (see also Definition 28).*

Note that both aggregation functions F_u and F_{maj} as seen in Example 34 and Example 35 respectively, are both independent, and thus Lemma 3 holds.

Recall the mapping condition monotonicity (see Definition 32) and the notion of winning coalitions for formulas (see Definition 35). Note that imposing monotonicity on an aggregation function F naturally gives rise to superset closed sets of winning coalitions for formulas $\mathcal{W}_\phi(\mathcal{J}, F)$, as the following Lemma 4 shows.

Lemma 4 (Monotonic winning coalitions for formulas)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. If F is monotonic (see Definition 32), then*

$$\forall \phi \in \mathcal{A}\ \forall W, W' \subseteq N : \textit{if } W \in \mathcal{W}_\phi(\mathcal{J}, F) \textit{ and } W \subset W',\ \textit{then } W' \in \mathcal{W}_\phi$$

i.e., W_ϕ is superset closed for any formula $\phi \in \mathcal{A}$.

Proof. We proceed with a proof by contradiction. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. Let F be monotonic, but assume that $\mathcal{W}_\phi(\mathcal{J}, F)$ is not superset closed for every formula $\phi \in \mathcal{A}$. Thus there is some formula $\phi \in \mathcal{A}$ and sets $W, W' \subseteq N$ such that $W \in \mathcal{W}_\phi(\mathcal{J}, F)$ and $W \subset W'$, but $W' \notin \mathcal{W}_\phi(\mathcal{J}, F)$. So by Definition 35 of winning coalitions for formulas, there is a profile $P' \in \mathbf{P}(\mathcal{A})$ such that $W' = P'_\phi$ but $\phi \notin F(P')$. Moreover, let $P \in \mathbf{P}(\mathcal{A})$ be any profile such that $W = P_\phi$. Since $W \in \mathcal{W}_\phi(\mathcal{J}, F)$, it follows that $\phi \in F(P)$. We have that $P_\phi = W \subset W' = P'_\phi$. Thus by Definition 32 of monotonicity, it follows that $\phi \in F(P')$.

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However, this contradicts the assumption that $W' \notin \mathcal{W}_\phi(\mathcal{J}, F)$. Hence, $W' \in \mathcal{W}_\phi(\mathcal{J}, F)$ must be the case. □

Winning coalitions for profiles

Definition 36 (Winning coalitions for profiles)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. We call $W \subseteq N$ a* winning coalition for a profile *$P \in \mathbf{P}(\mathcal{A})$ if, and only if*

$$\forall \phi \in \mathcal{A}: \text{ if } W = P_\phi, \text{ then } \phi \in F(P)$$

We denote the set of all winning coalitions for a profile *$P \in \mathbf{P}(\mathcal{A})$ over the judgment aggregation problem $\mathcal{J}$ and the aggregation function F as $\mathcal{W}_P(\mathcal{J}, F)$.*

By comparing Definition 35 and Definition 36, note that the only difference lies in regard to the object of quantification. In particular, winning coalitions for formulas decide the acceptance of a certain formula over all profiles, whereas winning coalitions for profiles decide the acceptance of all formulas in a certain profile.

Example 36 (Winning coalition for the dictatorial rule)**.** *Consider the aggregation following aggregation function (see Equation 2.6 and Example 17)*

$$F_d^j(P) = \{\phi \in \mathcal{A} \mid \phi \in J_j, J_j \in P\}$$

for some fixed $j \in N$. In this specific case, trivially $\mathcal{W}_P(\mathcal{J}, F_d^j) = \{j\}$ for any P as j decides the acceptance of every formula $\phi \in \mathcal{A}$.

Lemma 5 (Characterization of neutrality by winning coalitions for a profile)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is neutral (see Definition 29) if, and only if*

$$\forall \phi \in \mathcal{A}\, \forall P \in \mathbf{P}(\mathcal{A}) : \phi \in F(P) \iff P_\phi \in \mathcal{W}_P(\mathcal{J}, F)$$

i.e., $W = P_\phi$ is winning for one formula $\phi \in \mathcal{A}$ in P if, and only if, W is winning for all formulas $\psi \in \mathcal{A}$ in P.

Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$.

$\implies$: For the *if*-direction, we proceed with a direct proof. Let F be a neutral aggregation function, and $P \in P(\mathcal{A})$ be any profile. Assume that there is a formula $\phi \in \mathcal{A}$ such that $W = P_\phi$ and $\phi \in F(P)$, i.e., W is a winning for ϕ in P. By definition of neutrality it holds that $\forall \psi \in \mathcal{A}$ if $P_\phi = P_\psi$, then $\psi \in F(P)$. Hence, $W = P_\phi \in \mathcal{W}_P(\mathcal{J}, F)$, i.e., P_ϕ is winning for all formulas in P.

For the *only-if*-direction, assuming that $P_\phi \in \mathcal{W}_P(\mathcal{J}, F)$ by Definition 36 implies that $\phi \in F(P)$.

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$\Longleftarrow$: We proceed by proof of contradiction. Assume that for every formula $\phi \in \mathcal{A}$ and any profile $P \in \mathbf{P}(\mathcal{A})$ that $\phi \in F(P) \iff P_\phi \in \mathcal{W}_P(\mathcal{J}, F)$. But assume that F is not neutral. Since F is not neutral, there are formulas $\phi, \psi \in \mathcal{A}$ and a profile $P \in \mathbf{P}(\mathcal{A})$ such that $P_\phi = P_\psi$ but w.l.o.g. $\phi \in F(P)$ and $\psi \notin F(P)$. By assumption and since $P_\phi = P_\psi$ it follows that $\psi \in F(P)$, which leads to a contradiction. Hence, F must be neutral. □

Regarding Lemma 5 and since we know that the propositionwise majority rule F_{maj} (see Section 2.3.1) satisfies neutrality (see Example 25), we can conclude that a formula ϕ is collectively accepted by F_{maj} in a profile P exactly if it is accepted by a winning coalition for the profile P.

3.2.2 Winning coalitions

Definition 37 (Winning coalitions [DL07a, Die07, KE09])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. We call a set $W \subseteq N$ a* winning coalition *(for any formula in any profile) if, and only if*

$$\forall \phi \in \mathcal{A}\ \forall P \in \mathbf{P}(\mathcal{A}):\ \textit{if } W = P_\phi, \textit{ then } \phi \in F(P)$$

We denote the set of all winning coalitions over the judgment aggregation problem $\mathcal{J}$ and the aggregation function F as $\mathcal{W}(\mathcal{J}, F)$.

The set of all winning coalitions $\mathcal{W}(\mathcal{J}, F)$ with respect to Definition 37 defines in this sense exactly the sets of individuals that can decide the acceptance of every formula in every profile. In particular, this means that the coalitions in $\mathcal{W}(\mathcal{J}, F)$ in some sense form a dictatorial or oligaric set of individuals (see Section 2.4.3), which is an important part of the proof technique using winning coalitions as ultrafilters.

Remark 19. *Regarding Definition 37, we observe that the set of winning coalitions is simply the intersection of all winning coalitions for a formula $\phi \in \mathcal{A}$ (see Definition 35), i.e.*

$$\mathcal{W}(\mathcal{J}, F) = \bigcap_{\phi \in \mathcal{A}} \mathcal{W}_\phi(\mathcal{J}, F)$$

Moreover, we can also view the set of winning coalitions as the intersection of all winning coalitions for a profile $P \in \mathbf{P}(\mathcal{A})$ (see Definition 36), i.e.

$$\mathcal{W}(\mathcal{J}, F) = \bigcap_{P \in \mathbf{P}(\mathcal{A})} \mathcal{W}_P(\mathcal{J}, F)$$

Lemma 6. *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. For any formula $\phi \in \mathcal{A}$ and any profile $P \in \mathbf{P}(\mathcal{A})$*

$$\mathcal{W}(\mathcal{J}, F) \subseteq \mathcal{W}_\phi(\mathcal{J}, F) \cap \mathcal{W}_P(\mathcal{J}, F)$$

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Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. We proceed by proof by contradiction. Assume that $W \in \mathcal{W}(\mathcal{J}, F)$, but that $W \notin \mathcal{W}_\phi(\mathcal{J}, F) \cap \mathcal{W}_P(\mathcal{J}, F)$ for some formula $\phi \in \mathcal{A}$ and profile $P \in \mathbf{P}(\mathcal{A})$ Hence, by set intersection $W \notin \mathcal{W}_\phi(\mathcal{J}, F)$ or $W \notin \mathcal{W}_P(\mathcal{J}, F)$.

Case $W \notin \mathcal{W}_\phi(\mathcal{J}, F)$: By Definition 35 of winning coalitions for formulas, there exists a profile $P' \in \mathbf{P}(\mathcal{A})$ such that $W = P'_\phi$ but $\phi \notin F(P')$. However, by Definition 37 of winning coalitions, this profile P' together with ϕ then contradicts the assumption that $W \in \mathcal{W}(\mathcal{J}, F)$. Hence, $W \in \mathcal{W}_\phi(\mathcal{J}, F)$ must be the case.

Case $W \notin \mathcal{W}_P(\mathcal{J}, F)$: By Definition 36 of winning coalitions for profiles, there exists a formula $\psi \in \mathcal{A}$ such that $W = P_\psi$ but $\psi \notin F(P)$. However, by Definition 37 of winning coalitions, this formula ψ together with P contradicts the assumption that $W \in \mathcal{W}(\mathcal{J}, F)$. Thus, $W \in \mathcal{W}_P(\mathcal{J}, F)$ must be the case.

So $W \in \mathcal{W}_\phi(\mathcal{J}, F)$ and $W \in \mathcal{W}_P(\mathcal{J}, F)$ which contradicts the assumption. Thus we conclude that $\mathcal{W}(\mathcal{J}, F) \subseteq \mathcal{W}_\phi(\mathcal{J}, F) \cap \mathcal{W}_P(\mathcal{J}, F)$. □

Lemma 7 (Characterization of systematicity by winning coalitions [DM10, GP14][4])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is systematic (see Definition 30) if, and only if*

$$\forall \phi \in \mathcal{A}\, \forall P \in \mathbf{P}(\mathcal{A}) : \phi \in F(P) \iff P_\phi \in \mathcal{W}(\mathcal{J}, F)$$

Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$.

$\Longrightarrow$: For the *if*-direction, we proceed with a direct proof. Let F be a systematic aggregation function, and $P \in \mathbf{P}(\mathcal{A})$ be any profile. Let $\phi \in \mathcal{A}$ be an arbitrary formula and assume that $\phi \in F(P)$. By systematicity for every $P' \in \mathbf{P}(\mathcal{A})$ such that $P_\phi = P'_\phi$, then $\phi \in F(P')$. Hence, $P_\phi \in \mathcal{W}(\mathcal{J}, F)$ is a winning coalition.

For the *only if*-direction, assuming that $P_\phi \in \mathcal{W}(\mathcal{J}, F)$ by Definition 37 implies that $\phi \in F(P)$ for any formula $\phi \in \mathcal{A}$ and profile $P \in \mathbf{P}(\mathcal{A})$.

$\Longleftarrow$: We proceed by proof of contradiction. Assume that for every formula $\phi \in \mathcal{A}$ and every profile $P \in \mathbf{P}(\mathcal{A})$ that $\phi \in F(P) \iff P_\phi \in \mathcal{W}(\mathcal{J}, F)$. But assume that F is not systematic. Since F is not systematic, there are formulas $\phi, \psi \in \mathcal{A}$ and profiles $P, P' \in \mathbf{P}(\mathcal{A})$ such that $P_\phi = P'_\psi$ and w.l.o.g. $\phi \in F(P)$ but $\psi \notin F(P')$. By assumption and since $\phi \in F(P)$ it follows that $P_\phi \in \mathcal{W}(\mathcal{J}, F)$. And since $P_\phi = P'_\psi$ we can conclude that $\psi \in F(P')$. However, this contradicts our assumption. Thus F must be systematic. □

[4] See also [GP14] for a proof similar to the proofs given for Lemma 3 and Lemma 5

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Example 37 (Winning coalitions for systematic rules)**.** *Reconsider Example 35. Since the propositionwise majority rule F_{maj} is systematic (see Example 27, and by Lemma 7 we know that if a formula $\phi \in \mathcal{A}$ is collectively accepted, then the set of individuals P_ϕ accepting it is a winning coalition. Hence, we get the following set of winning coalitions, which is equal to the winning coalitions of any formula*

$$\mathcal{W}(\mathcal{J}, F_{maj}) = \mathcal{W}_\phi(\mathcal{J}, F_{maj}) = \left\{W \subseteq N \mid |W| \geq \left\lceil \frac{|N|+1}{2} \right\rceil \right\} \quad (\textit{for any } \phi)$$

Moreover, consider again Example 34. The unanimity rule F_u is also systematic (see Example 28). Hence, in this case the are also exactly the coalitions of individuals P_ϕ for which the formula is accepted in the collective set. So we get the following set of winning coalitions, which is, once again, equal to the winning coalitions of any formula

$$\mathcal{W}(\mathcal{J}, F_u) = \mathcal{W}_\phi(\mathcal{J}, F_u) = \{N\}$$

In particular, by definition of the unanimity rule (see Equation 2.3), there is only one winning coalition, viz the set of all individuals N.

Note that in Example 37 the set of winning coalitions $\mathcal{W}(\mathcal{J}, F)$ corresponds with the set of winning coalitions for a formula $\mathcal{W}_\phi(\mathcal{J}, F)$ for any $\phi \in \mathcal{A}$ for both aggregation functions $F = F_u$ and $F = F_{maj}$. This is no coincidence, it is rather a result of F being systematic, as we see in the following theorem.

Theorem 6. *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. If F is systematic (see Definition 30), then*

$$\forall \phi \in \mathcal{A} : \mathcal{W}(\mathcal{J}, F) = \mathcal{W}_\phi(\mathcal{J}, F)$$

Moreover, if F is systematic, then also $\forall P \in \mathbf{P}(\mathcal{A})$, $\mathcal{W}(\mathcal{J}, F) = \mathcal{W}_P(\mathcal{J}, F)$.

Proof. We give the proof for the case of winning coalitions for formulas. The proof for the case of winning coalitions for profiles is similar.

Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. Assume that F is systematic.

$\subseteq$: By Lemma 6 for any $\phi \in \mathcal{A}$ and every profile $P \in \mathbf{P}(\mathcal{A})$ we have that

$$\mathcal{W}(\mathcal{J}, F) \subseteq \mathcal{W}_\phi(\mathcal{J}, F) \cap \mathcal{W}_P(\mathcal{J}, F)$$

Moreover, by set intersection, it is the case that

$$\mathcal{W}_\phi(\mathcal{J}, F) \cap \mathcal{W}_P(\mathcal{J}, F) \subseteq \mathcal{W}_\phi(\mathcal{J}, F)$$

Thus by the transitivity of the $\subseteq$-relation it follows that $\mathcal{W}(\mathcal{J}, F) \subseteq \mathcal{W}_\phi(\mathcal{J}, F)$ for any $\phi \in \mathcal{A}$.

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$\supseteq$: We proceed by a proof by contradiction. Assume that there is a formula $\phi \in \mathcal{A}$ such that $\mathcal{W}(\mathcal{J}, F) \not\supseteq \mathcal{W}_\phi(\mathcal{J}, F)$. So there is a $W \subseteq N$ such that $W \in \mathcal{W}_\phi(\mathcal{J}, F)$ but $W \notin \mathcal{W}(\mathcal{J}, F)$. It follows that there is a formula $\psi \in \mathcal{A}$ and a profile $P \in \mathbf{P}(\mathcal{A})$ such that $W = P_\psi$ but $\psi \notin F(P)$. By Definition 35 of winning coalitions for formulas, it must be the case that if $W = P_\phi$, then $\phi \in F(P)$. So if $P_\phi = W = P_\psi$, then by systematicity, it must be the case that $\psi \in F(P)$. However, this contradicts our assumption. Thus $\mathcal{W}(\mathcal{J}, F) \supseteq \mathcal{W}_\phi(\mathcal{J}, F)$

Hence, we can conclude that for every $\phi \in \mathcal{A}$ it is the case that $\mathcal{W}(\mathcal{J}, F) = \mathcal{W}_\phi(\mathcal{J}, F)$. □

Lemma 8 (Characterization of unanimity by winning coalitions [DM10][5])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is unanimous (see Definition 23) if, and only if, $N \in \mathcal{W}(\mathcal{J}, F)$.*

Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. We proceed by a proof by contradiction for the *if* and also the *only if*-direction.

$\Longrightarrow$: By assumption, F is unanimous (see Definition 23). Hence, for every $\phi \in \mathcal{A}$ and every profile $P \in \mathbf{P}(\mathcal{A})$ if $P_\phi = N$, then $\phi \in F(P)$. Thus N is, by definition, a winning coalition, and so $N \in \mathcal{W}(\mathcal{J}, F)$.

$\Longleftarrow$: Let $N \in \mathcal{W}(\mathcal{J}, F)$ but assume that F is not unanimous. Thus there is a formula $\phi \in \mathcal{A}$ and a profile $P \in \mathbf{P}(\mathcal{A})$ such that $P_\phi = N$ but $\phi \notin F(P)$. However, by definition of winning coalitions (see Definition 37), this contradicts that $N \in \mathcal{W}(\mathcal{J}, F)$. Hence, F must be unanimous. □

3.2.3 Ultrafilters

In this section, we will introduce the notion of so-called ultrafilters and then use winning coalitions as ultrafilters to deduce the impossibility results.

Definition 38 (Ultrafilters [KE09, GP14])**.** *Let N be a nonempty set of elements (individuals). A collection $\mathcal{W} \subset 2^N$ of subsets (coalitions) of N is an* ultrafilter *on N if, and only if*

(1) $N \in \mathcal{W}$ (and $\emptyset \notin \mathcal{W}$), i.e., the ultrafilter contains the underlying set itself (but not the emptyset[6])

(2) $W \in \mathcal{W} \iff \overline{W} \notin \mathcal{W}$ where $\overline{W} = N \setminus W$ is the complement of W, i.e., for every subset of N either itself or its negation is an element of the ultrafilter

[5] [DM10] only states the *if*-direction, which is usually enough for the following aimed proofs.

[6] An ultrafilter by definition contains either a set or its complement but not both. Moreover, the underlying set N is an element of the ultrafilter. Thus the empty set can never be an element of the ultrafilter already by definition. However, it is a necessary condition for a filter, which by definition satisfies only conditions (1), (3), and (4) [EH18].

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(3) If $W \in \mathcal{W}$ and $W \subseteq W'$, then $W' \in \mathcal{W}$, i.e., the ultrafilter is superset closed

(4) If $W, W' \in \mathcal{W}$, then $W \cap W' \in \mathcal{W}$, i.e., the ultrafilter is closed under (finite) intersection

The following Lemma 12 shows that, if the agenda and aggregation function under consideration satisfy certain conditions, then the winning coalition is actually an ultrafilter. Therefore, at first, we have to prove some lemmas that show that the set of winning coalitions $\mathcal{W}(\mathcal{J}, F)$ really satisfies all the conditions necessary to classify as an ultrafilter, as given in Definition 38.

Lemma 9 ([DM10, DL07a][7])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F be an aggregation function for $\mathcal{J}$. If F satisfies collective rationality (see Definition 21) and systematicity (see Definition 30), then for each $W \subseteq N$, $W \in \mathcal{W}(\mathcal{J}, F) \iff \overline{W} \notin \mathcal{W}(\mathcal{J}, F)$, where $\overline{W} = N \setminus W$ is the complement of W.*

Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F be an aggregation function for $\mathcal{J}$. Let F be collectively rational and systematic.

$\Longrightarrow$: For the *if*-direction, we proceed by a proof by contradiction. Assume that there is a set $W \subseteq N$ such that $W \in \mathcal{W}(\mathcal{J}, F)$ and also $\overline{W} \in \mathcal{W}(\mathcal{J}, F)$. Let $P \in \mathbf{P}(\mathcal{A})$ be any profile and $\phi \in \mathcal{A}$ be any formula such that $P_\phi = W$. By definition of the complement, it follows that $\overline{W} = P_{\neg\phi}$. Since by assumption $W \in \mathcal{W}(\mathcal{J}, F)$ and $\overline{W} \in \mathcal{W}(\mathcal{J}, F)$ it follows that $\phi \in F(P)$ and $\neg\phi \in F(P)$. So $F(P)$ is inconsistent. However, this contradicts that F is collectively rational (see Definition 21). Hence, $\overline{W} \notin \mathcal{W}(\mathcal{J}, F)$ must be the case.

$\Longleftarrow$: For the *only if*-direction, we proceed again with a proof by contradiction. Assume that there is a set $W \subseteq N$ such that $\overline{W} \notin \mathcal{W}(\mathcal{J}, F)$ but $W \notin \mathcal{W}(\mathcal{J}, F)$. By the characterization of systematicity by winning coalitions (see Lemma 7) and since F is systematic, it follows that for every formula $\phi \in \mathcal{A}$ and every profile $P \in \mathbf{P}(\mathcal{A})$ if $P_\phi = W \notin \mathcal{W}(\mathcal{J}, F)$, then $\phi \notin F(P)$. Moreover, F satisfies collective rationality. So $F(P)$ must, by definition, also be complete (see Definition 21). Thus for all $\phi \in \mathcal{A}$ and $P \in \mathbf{P}(\mathcal{A})$ if $P_\phi = W$, then $\neg\phi \in F(P)$ by collective rationality (completeness) of F since $\phi \notin F(P)$. In particular, in these cases if $P_\phi = W$, then $P_{\neg\phi} = \overline{W}$ by definition of the complement. So we overall get that for all $\phi \in \mathcal{A}$ and $P \in \mathbf{P}(\mathcal{A})$ if $P_{\neg\phi} = \overline{W}$, then $\neg\phi \in F(P)$. Thus by definition of winning coalitions $\overline{W} \in \mathcal{W}(\mathcal{J}, F)$ which contradicts our assumption. Hence, $W \in \mathcal{W}(\mathcal{J}, F)$ must be the case.

So overall $W \in \mathcal{W}(\mathcal{J}, F) \iff \overline{W} \notin \mathcal{W}(\mathcal{J}, F)$. □

[7] [DM10] considers a slightly different lemma regarding only independence and winning coalitions for formulas.

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Note that Lemma 9 already establishes that if we consider the winning coalitions of a judgment aggregation problem with an aggregation function that satisfies collective rationality and systematicity, then condition (2) necessary for classifying would be satisfied.

Lemma 10 ([DM10, DL07a, GP14])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment problem and $\mathcal{A}$ evenly negatable (see Definition 11) and F an aggregation function for $\mathcal{J}$. If F is collectively rational, systematic and satisfies universal domain, then*

$$\forall W, W' \subseteq N: \text{ if } W \in \mathcal{W}(\mathcal{J}, F) \text{ and } W \subseteq W', \text{ then } W' \in \mathcal{W}(\mathcal{J}, F)$$

i.e, $\mathcal{W}(\mathcal{J}, F)$ is superset closed.

Proof. We proceed by a proof by contradiction. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment problem and $\mathcal{A}$ evenly negatable and F an aggregation function for $\mathcal{J}$ such that F is collectively rational and systematic. Assume that $W \in \mathcal{W}(\mathcal{J}, F)$ and $W \subseteq W'$ but that $W' \notin \mathcal{W}(\mathcal{J}, F)$. By construction, the agenda $\mathcal{A}$ is evenly negatable (see Definition 11). Thus there is a minimally inconsistent set $X \subseteq \mathcal{A}$ and a subset $\{\phi, \psi\} \subseteq X$ of even 2 such that $(X \setminus \{\phi, \psi\}) \cup \{\neg\phi, \neg\psi\}$ is consistent. Moreover, since X is minimally inconsistent and $\phi, \psi \in X$, it follows that the sets $(X \setminus \{\phi\}) \cup \{\neg\phi\}$ and $(X \setminus \{\psi\}) \cup \{\neg\psi\}$ are consistent.

In the following, we consider a profile that, by the condition of universal domain (see Definition 18), must exist but contradicts that F satisfies collective rationality. Therefore we define the following partition of the set of individuals N

- W
- $W' \setminus W$
- $N \setminus W'$

which are by construction disjoint. In particular, $N = W \cup (W' \setminus W) \cup (N \setminus W')$ and $W \cap (W' \setminus W) \cap (N \setminus W') = \emptyset$. Moreover, let $P = \langle J_i \rangle_{i \in N}$ be the profile such that

- $J_i = (X \setminus \{\phi\}) \cup \{\neg\phi\}$ for $i \in W$
- $J_i = (X \setminus \{\phi, \psi\}) \cup \{\neg\phi, \neg\psi\}$ for $i \in (W' \setminus W)$
- $J_i = (X \setminus \{\psi\}) \cup \{\neg\psi\}$ for $i \in (N \setminus W')$

where for all $i \in N$ the individual judgment sets J_i of the profile P are completed such that they stay consistent. By universal domain, such a profile P must exist as a possible input to F, and $F(P)$ must be consistent by collective rationality.

$(X \setminus \{\phi, \psi\}) \subseteq F(P)$: We observe that every individual $i \in N$ accepts the set $(X \setminus \{\phi, \psi\})$ in profile P. Thus by unanimity, it follows that $(X \setminus \{\phi, \psi\}) \subseteq F(P)$.

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$\psi \in F(P)$: By assumption that $W \in \mathcal{W}(\mathcal{J}, F)$, $\psi \in (X \setminus \{\phi\}) \cup \{\neg\phi\}$ and since $\psi \notin (X \setminus \{\phi, \psi\}) \cup \{\neg\phi, \neg\psi\}$ and $\psi \notin (X \setminus \{\psi\}) \cup \{\neg\psi\}$ we have that $W = P_\psi$. Hence, by definition of winning coalitions (see Definition 37) $\psi \in F(P)$.

$\phi \in F(P)$: By assumption we have that $W' \notin \mathcal{W}(\mathcal{J}, F)$. Since F is collectively rational and systematic, by Lemma 9, which we already proved, it follows that $N \setminus W' = \overline{W'} \in \mathcal{W}(\mathcal{J}, F)$. Since $\phi \in (X \setminus \{\psi\}) \cup \{\neg\psi\}$ and ϕ is not accepted by other individuals in P, it follows that $\phi \in F(P)$.

Overall, we thus have that $(X \setminus \{\phi, \psi\}) \subseteq F(P)$, $\psi \in F(P)$ and $\phi \in F(P)$. Hence, $X \subseteq F(P)$. However, X is inconsistent, and thus $F(P)$ must be inconsistent, which contradicts the assumption that F satisfies collective rationality. So $W' \in \mathcal{W}(\mathcal{J}, F)$ must be the case. □

Lemma 11 ([DM10, DL07a])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment problem, $\mathcal{A}$ non-simple (see Definition 8) and evenly negatable (see Definition 11), and F an aggregation function for $\mathcal{J}$. If F is collectively rational, unanimous, systematic and satisfies universal domain, then*

$$\forall W, W' \subseteq N: \text{ if } W, W' \in \mathcal{W}(\mathcal{J}, F), \text{ then } W \cap W' \in \mathcal{W}(\mathcal{J}, F)$$

i.e, $\mathcal{W}(\mathcal{J}, F)$ closed under (finite) intersection.

Proof. We, once again, proceed with a proof by contradiction similar to the proof of condition (3). Assume that $W, W' \in \mathcal{W}(\mathcal{J}, F)$ but that $W \cap W' \notin \mathcal{W}(\mathcal{J}, F)$. By construction, the agenda $\mathcal{A}$ is non-simple (see Definition 8). Thus by definition, there is a set $X \subseteq \mathcal{A}$ such that $3 \leq |X|$ and X is minimally inconsistent. In particular, there are at least three formulas $\phi, \psi, \chi \in \mathcal{A}$ such that $\{\phi, \psi, \chi\} \subseteq X$. Moreover, since X is minimally inconsistent it follows that for every $\xi \in \{\phi, \psi, \chi\}$ we have that the set $(X \setminus \{\xi\}) \cup \{\neg\xi\}$ is consistent.

In the following, we consider a profile that, by the assumption of universal domain (see Definition 18), must exist but contradicts that F satisfies collective rationality. Therefore we define the following sets of individuals N

- $W \cap W'$
- $W' \setminus W$
- $N \setminus W'$

which are, in this case, not disjoint and thus also not a real partition of N. Moreover, let $P = \langle J_i \rangle_{i \in N}$ be the profile such that

- $J_i = (X \setminus \{\phi\}) \cup \{\neg\phi\}$ for $i \in (W \cap W')$
- $J_i = (X \setminus \{\psi\}) \cup \{\neg\psi\}$ for $i \in (W' \setminus W)$

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- $J_i = (X \setminus \{\chi\}) \cup \{\neg\chi\}$ for $i \in (N \setminus W')$.

where for all $i \in N$ the individual judgment sets J_i of the profile P are completed such that they stay consistent. By universal domain, such a profile P must exist as a possible input to F, and $F(P)$ must be consistent by collective rationality.

$(X \setminus \{\phi, \psi, \chi\}) \subseteq F(P)$: We observe that every individual $i \in N$ accepts by construction of P all formulas in $(X \setminus \{\phi, \psi, \chi\})$. Since F is, by assumption, unanimous (see Definition 23), it follows that $(X \setminus \{\phi, \psi, \chi\}) \subseteq F(P)$.

$\phi \in F(P)$: By assumption $(W \cap W') \notin \mathcal{W}(\mathcal{J}, F)$. We observe that

$$(N \setminus W') \cup (W' \setminus W) = N \setminus (W' \cap W) = \overline{(W' \cap W)}$$

Thus by condition (2) of an ultrafilter, which we already proved, it must be the case that $\overline{(W' \cap W)} = (N \setminus W') \cup (W' \setminus W) \in \mathcal{W}(\mathcal{J}, F)$. Moreover, we have by construction of P that $P_\phi = (N \setminus W') \cup (W' \setminus W) \in \mathcal{W}(\mathcal{J}, F)$. Hence, by definition of a winning coalition $\phi \in F(P)$.

$\psi \in F(P)$: By assumption, we have that $W \in \mathcal{W}(\mathcal{J}, F)$. By construction, it is the case that the set $(N \setminus W') \cup (W \cap W')$ of individuals contains exactly those individuals that accept ψ in P, i.e., $P_\psi = (N \setminus W') \cup (W \cap W')$. Moreover, we observe that

$$(N \setminus W') \cup (W \cap W') \supseteq W$$

since all elements that are in W and W' and removed from N by W' are added again by $(W \cap W')$. Since $\mathcal{A}$ is evenly negatable and F is collectively rational and systematic, by Lemma 10, which we already proved, we have that if $W \in \mathcal{W}(\mathcal{J}, F)$ then it is also the case that every superset is in $\mathcal{W}(\mathcal{J}, F)$. Hence, it follows that $(N \setminus W') \cup (W \cap W') \in \mathcal{W}(\mathcal{J}, F)$. Thus $P_\psi \in \mathcal{W}(\mathcal{J}, F)$ and it follows by definition of winning coalitions that $\psi \in F(P)$.

$\chi \in F(P)$: By assumption $W' \in \mathcal{W}(\mathcal{J}, F)$. Moreover, by construction the set $(W' \setminus W) \cup (W \cap W')$ contains exactly those individuals that accept χ, i.e., $P_\chi = (W' \setminus W) \cup (W \cap W')$. We observe that

$$(W' \setminus W) \cup (W \cap W') = W'$$

as the individuals that are in W and removed from W are exactly those that are again added through $W \cap W'$. So $P_\chi = W' \in \mathcal{W}(\mathcal{J}, F)$ and thus by definition of winning coalitions, it follows that $\chi \in F(P)$.

So overall, we have $(X \setminus \{\phi, \psi, \chi\}) \subseteq F(P)$, $\phi \in F(P)$, $\psi \in F(P)$ and $\chi \in F(P)$. Hence, the minimally inconsistent set $X \subseteq F(P)$ and so $F(P)$ must be inconsistent, which contradicts that F satisfies collective rationality. Thus $W \cap W' \in \mathcal{W}(\mathcal{J}, F)$ must be the case. □

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Lemma 12 (Ultrafilter lemma [GP14]). *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. If the agenda $\mathcal{A}$ satisfies non-simplicity and evenly negatability (see Section 2.2), and F is collectively rational (see Section 2.4.2), satisfies universal domain*[8] *(see Section 2.4.1), unanimity and systematicity (see Section 2.4.3), then the set of winning coalitions $\mathcal{W}(\mathcal{J}, F)$ is an ultrafilter.*

Proof. We proceed by showing that under the assumed conditions for the agenda and the aggregation function, the set of winning coalitions satisfies all properties (1)-(4) necessary for classifying as an ultrafilter (see Definition 38).

Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. Let the agenda $\mathcal{A}$ satisfy non-simplicity and evenly negatability, and F be collectively rational and satisfy unanimity and systematicity.

(1) Since F is unanimous, by Lemma 8 for the characterization of unanimity by winning coalitions, it follows that $N \in \mathcal{W}(\mathcal{J}, F)$.

(2) As F is collectively rational and systematic, by Lemma 9, it follows directly $W \in \mathcal{W}(\mathcal{J}, F) \iff \overline{W} \notin \mathcal{W}(\mathcal{J}, F)$.

(3) We have that $\mathcal{A}$ is evenly negatable and F is collectively rational and systematic. By Lemma 10, it follows directly that the set of winning coalitions $\mathcal{W}(\mathcal{J}, F)$ is superset closed.

(4) Since $\mathcal{A}$ is non-simple and evenly negatable, and F is collectively rational, unanimous, systematic, and satisfies universal domain, by Lemma 11, it follows that the set of winning coalitions $\mathcal{W}(\mathcal{J}, F)$ is closed under (finite) intersection.

Thus overall, under the given conditions, $\mathcal{W}(\mathcal{J}, F)$ is an ultrafilter. □

So Lemma 12, if the agenda $\mathcal{A}$ is non-simple and evenly negatable, and F is collectively rational, unanimous, systematic, and satisfies universal domain, then the set of all winning coalitions is an ultrafilter. Recalling the properties of ultrafilters (see Definitin 38), we observe that the closure under intersection and since the empty set is not a winning coalition entails that there must be a winning coalition of minimal size. In particular, the following lemma shows that every ultrafilter on a finite set of elements contains an element of size 1.

Lemma 13 (Ultrafilters on finite sets contain dictator [GP14]). *Let N be a finite and nonempty set, and let $\mathcal{W}$ be an ultrafilter on N. There is an element $i \in N$ such that $\{i\} \in \mathcal{W}$ and for every $W \in \mathcal{W}$, $i \in W$.*

[8] In [GP14] the condition of F satisfying universal domain is not explicitly stated. However, by the definition of aggregation functions given in [GP14], it is implicitly satisfied.

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Proof. We proceed with a proof by contradiction. Let $\mathcal{W}$ be an ultrafilter on the finite, nonempty set N. But assume that for every element $i \in N$, $\{i\} \notin \mathcal{W}$. By condition (1) of Definition 38 of ultrafilters, we have that $N \in \mathcal{W}$. Since for every $i \in N$, $\{i\} \notin \mathcal{W}$, it follows by condition (2) of ultrafilters that for every $i \in N$, $\overline{\{i\}} = (N \setminus \{i\}) \in \mathcal{W}$. Hence, for every $i \in N$ there is a set in $\mathcal{W}$, viz $(N \setminus \{i\}) \in \mathcal{W}$, such that i is not in it. Moreover, $|\mathcal{W}| \leq 2^{|N|}$ and thus finite, since N is finite. Thus by condition (4) of ultrafilters, the intersection over all elements in $\mathcal{W}$ is also an element of $\mathcal{W}$, i.e., $\bigcap \mathcal{W} \in \mathcal{W}$. However, since for every $i \in N$ there is a set $(N \setminus \{i\}) \in \mathcal{W}$ such that i is not in it, it follows that $\bigcap \mathcal{W} = \emptyset \in \mathcal{W}$. But if $\emptyset \in \mathcal{W}$, then by condition (2) $N = \overline{\emptyset} \notin \mathcal{W}$, which contradicts that $\mathcal{W}$ is an ultrafilter. Hence, there must be an element $i \in N$ such that $\{i\} \in \mathcal{W}$. □

Observe now that by Lemma 12 it holds that we consider a judgment aggregation problem $\mathcal{J} = \langle N, \mathcal{A} \rangle$ where the agenda $\mathcal{A}$ is non-simple and evenly negatable, and the aggregation function F is collectively rational, unanimous, systematic and satisfies universal domain, then the set of all winning coalitions $\mathcal{W}(\mathcal{J}, F)$ is an ultrafilter. Moreover, Lemma 13 additionally shows that then there must be a single individual $i \in N$ that is a winning coalition, and thus by definition, a dictator. In particular, if the set of winning coalitions $\mathcal{W}(\mathcal{J}, F)$ is an ultrafilter, then F degenerates inevitably to a dictatorship (see Definition 24).

Theorem 7 ([DL07a])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ where $\mathcal{A}$ is non-simple and evenly negatable, and let F be an aggregation function for $\mathcal{J}$. F is collectively rational, unanimous, systematic, and satisfies universal domain if, and only if F is dictatorial.*

Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ and let F be an aggregation function for $\mathcal{J}$.

$\Longrightarrow$: Let $\mathcal{A}$ be non-simple and evenly negatable. And assume that F is collectively rational, unanimous, systematic and satisfies universal domain. Since F is systematic and thus by Lemma 7 of the characterization of systematicity, we have that

$$\forall \phi \in \mathcal{A}\, \forall P \in \mathbf{P}(\mathcal{A}) : \phi \in F(P) \iff P_\phi \in \mathcal{W}(\mathcal{J}, F)$$

Moreover, by the given conditions of the agenda $\mathcal{A}$ and the aggregation function F toghether with Lemma 12, it follows that $\mathcal{W}(\mathcal{J}, F)$ is an ultrafilter. Thus since $\mathcal{W}(\mathcal{J}, F)$ is an ultrafilter (and N contains only finitely many individuals) and by Lemma 13, we can conclude that there some individual $i \in N$ such that $\{i\} \in \mathcal{W}(\mathcal{J}, F)$ and for every winning coalition $W \in \mathcal{W}(\mathcal{J}, F)$, $i \in W$. Thus we get that

$$\phi \in F(P) \iff P_\phi \in \mathcal{W}(\mathcal{J}, F) \iff i \in P_\phi \iff \phi \in J_i$$

So overall, we have that $\phi \in F(P) \iff \phi \in J_i$, which ultimately means that $F(P) = J_i$ and thus by Definition 24 of a dictatorial aggregation function F is dictatorial.

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$\Longleftarrow$: Assume that F is dictatorial, i.e., $F(P) = J_i$ for some $i \in N$. We observe that trivially F satisfies universal domain.

Collective rationality : Since the individual judgment set J_i of the dictator is by definition complete and consistent and thus also rational. By definition of a dictatorial aggregation function, it follows that $F(P) = J_i$ is also collectively rational.

Unanimity : If all individuals $j \in N$ accept a formula $\phi \in \mathcal{A}$, then i will also accept it, i.e., $\phi \in J_i$. Thus it follows that if all individuals accept ϕ, then $\phi \in J_i = F(P)$.

Systematicity : If all individuals accept a formula $\phi \in \mathcal{A}$ in profile $P \in \mathbf{P}(\mathcal{A})$ if, and only if, they accept another formula $\psi \in \mathcal{A}$ in profile $P' \in \mathbf{P}(\mathcal{A})$, then trivially i accepts ϕ if, and only if, it accepts ψ. Hence, $\phi \in F(P) = J_i$ if, and only if, $\psi \in F(P) = J_i$, which thus guarantees systematicity. □

Comparing Theorem 5 with Theorem 7, we observe that both theorems restrict their judgment aggregation problems to non-simple and evenly negatable agendas (see Remark 16). Despite that the agenda in Theorem 5 is a rather specific non-simple and evenly negatable agenda, the only real difference between the two theorems lies in their imposed mapping conditions (see Section 2.4.3). In particular, Theorem 5 imposes anonymity and systematicity, whereas Theorem 7 imposes unanimity and systematicity. Hence, we observe that exchanging anonymity (see Definition 27) with unanimity (see Definition 23) results no longer in the impossibility of an aggregation function itself, but only in the impossibility of a non-dictatorial one.

3.2.4 Impossibility characterized by ultrafilters

Reconsidering Theorem 5, we note that it only states a sufficient condition that results in no existing aggregation function for the given conditions [LP02]. Thus it yields no indication which conditions need to be removed or weakened to get either an aggregation function at all or even a non-degenerate aggregation result. In particular, [PvH06] state a similar theorem that shows that there is no aggregation rule satisfying both mapping conditions (see Section 2.4.3) anonymity and systematicity, i.e., we have to remove or weaken at least one of them, as it is done in Theorem 7. On the other hand, Theorem 7 states a sufficient and also necessary condition for dictatorial aggregation functions under the given agenda restrictions. In particular, such impossibility theorems with sufficient and necessary conditions come with the advantage that they also entail which properties we have to weaken or remove to achieve non-dictatorial or non-degenerate aggregation results. Hence, if we want to find a non-dictatorial aggregation function, then Theorem 7 suggests that we weaken or remove at least one of collective rationality, unanimity, systematicity, or universal domain. For example, the property of systematicity seems to be rather demanding and could be weakened to independence, i.e., removing the neutrality mapping condition (see Section 2.4.3).

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3.3 Impossibility results

In the following, we will review some impossibility results on judgment aggregation. In particular, we will review the different impossibility results that emerge by imposing different agenda restrictions (see Section 2.2) and mapping conditions (see Section 2.4.3). As input condition (see Section 2.4.1) and output condition (see Section 2.4.2), we fix universal domain (see Definition 18) and collective rationality (see Definition 21) respectively.

3.3.1 Impossibility with systematicity

Systematicity (see Definition 30) is a very demanding mapping condition, i.e., weakening or removing conditions other than systematicity still leads to impossibility results as shown in [DL07a, NP10]. Considering Theorem 7, we observe that the proof was only possible because of the ultrafilter lemma (see Lemma 12) which in turn needed Lemma 11 which depends upon unanimity (see Definition 23). In particular, unanimity was only needed for a small part of the impossibility proof, in contrast to systematicity. Hence, removing unanimity from Theorem 7 could possibly still lead to an impossibility result, despite weaker restrictions. This is actually the case with just a small 'tradeoff', as the following Theorem 8 by [DL07a] shows.

Theorem 8 ([DL07a])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem such that the agenda $\mathcal{A}$ is non-simple and evenly negatable, and let F be an aggregation function for $\mathcal{J}$. F satisfies universal domain, collective rationality, and systematicity if, and only if, F is dictatorial or inversely dictatorial.*

Comparing Theorem 7 and Theorem 8, we observe that Theorem 8 does not need the mapping condition of unanimity (see Definition 23). However, thus the resulting aggregation function is not necessarily a classical dictatorship, but it can also be an inverse dictatorship (see Definition 25). As opposed to a dictatorship, an inverse dictatorship does not directly enforce its own judgment but the complement of its judgment. Nevertheless, it remains that a single individual is able to determine the collective judgment for any profile for the imposed conditions by Theorem 8.

On the other hand, we can weaken the agenda restrictions, i.e., remove evenly negatability (see Definition 11). Regarding the proof of Theorem 7 and the ultrafilter lemma (Lemma 12), we observe that evenly negatability was a main ingredient for proving Lemma 10. However, as [NP10] showed, if we additionally impose the mapping condition of monotonicity (see Definition 32), then resulting aggregation rules can still only be dictatorial.

Theorem 9 ([NP10])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem such that the agenda $\mathcal{A}$ is non-simple, and let F be an aggregation function for $\mathcal{J}$. F satisfies universal domain, collective rationality, systematicity, and monotonicity if, and only if, F is dictatorial.*

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The mapping condition of monotonicity is a desirable property, as it prohibits aggregation functions that behave rather counterintuitive due to inverse acceptance (see Section 2.4.3). Moreover, we usually want to focus on interrelated propositions, i.e., on at least non-simple agendas, as simple agendas are trivial. Hence, Theorem 9 shows that, in general, systematicity is in combination with collective rationality and universal domain a too demanding property. Thus weakening systematicity to independence without the neutrality condition (see Theorem 3) would be a plausible step.

3.3.2 Impossibility with independence

Imposing only independence (see Definition 28) without the neutrality conditions (see Definition 29) still ensures propositionwise aggregation functions. However, this propositionwise aggregation method is not necessarily the same over all propositions (see Section 2.4.3.

However, as [DH10] have shown, agendas restricted to path-connectedness (see Definition 13) and evenly negatability are enough to aggregation functions satisfying collective rationality, universal domain, independence, and unanimity are also dictatorial.

Theorem 10 ([DH10])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem such that the agenda $\mathcal{A}$ is path-connected and evenly negatable, and let F be an aggregation function for $\mathcal{J}$. F satisfies universal domain, collective rationality, independence, and unanimity if, and only if, F is dictatorial.*

Note that every path-connected agenda is also non-simple (see Lemma 1), i.e., if we restrict the agenda to satisfy path-connectedness, then we do not need to additional impose the restriction of non-simplicity.

Once again, we can remove evenly negatability from the agenda restrictions. However, by additionally imposing the mapping conditions of monotonicity, the result stays the same as shown by [NP10].

Theorem 11 ([NP10])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem such that the agenda $\mathcal{A}$ is path-connected, and let F be an aggregation function for $\mathcal{J}$. F satisfies universal domain, collective rationality, independence, unanimity, and monotonicity if, and only if, F is dictatorial.*

Note that Theorem 11 imposes only mapping conditions that are desirable (see also Section 3.1). In particular, unanimity and monotonicity are intuitive properties of judgments, whereas propositionwise aggregation is usually preferred but could also be abandoned under certain circumstances.

Hence, we observe further possibilities to cope with impossibilities and, in particular, to avoid dictatorships, is by weakening or removing independence or the general assumptions of universal domain and collective rationality. This will be examined in more detail in the next Chapter 4.

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CHAPTER 4

Coping with impossibility

The previous Chapter 3 showed that assuming the desirable conditions of collective rationality and universal domain (see Section 2.4), even with the agenda restriction non-simplicity leads to an impossibility (see Theorem 9) [NP10]. Relaxing non-simplicity (see Definition 8) would lead to rather trivial aggregation problems. Hence, relaxing the usually desired properties like independence [Lis04, DL07a, DM10, Sla16, SJ11, BB12], universal domain [Lis03, DL10] and collectively rationality [DL08] is the next step. For a general overview of coping with impossibility and relaxing the imposed conditions, see [LP04, LP09, Lis11, GP14].

The chapter consists of 3 main sections. In Section 4.1, we review the results on the aggregation of individual judgments with relaxed output conditions (see Section 2.4.2). In Section 4.2, we consider aggregation functions that satisfy weaker conditions than independence (see Section 2.4.3). Finally, Section 4.3 copes with the usually imposed input condition (see Section 2.4.1) of universal domain and reviews certain domain restrictions.

4.1 Relaxing output conditions

Usually, we assume that not only all individual judgments are rational but also that the collective (judgment) set is rational (see Section 2.4.2) as we view the individuals as rational beings and also desire collectively rational views that enable better comprehensibility of judgments (see Section 3.1.1). However, as [Gär06] notes, completeness (see Definition 20) is a rather unnatural condition since, e.g., if we have the case that all individuals hold no judgment about a certain issue, then it would make no sense that the collective should decide upon this issue. In this case, we also have to remove completeness as a condition for individual judgments (see Definition 3), i.e., weaken rationality to consistency. In particular, [Gär06] allows any individual to either accept, reject or abstain from a judgment about an issue.

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However, as [Gär06] showed, even discarding completeness and thus rationality (see Section 2.4.2) only allows (weak) oligarchic aggregation functions (see Definition 26). Thus still resulting in only degenerate aggregation of individual judgments. Moreover, [DL08] built on the results of [Gär06] and showed even stronger results, with consistency and deductive closure (see Definition 5) as the only output condition that still leads to degenerate aggregation results as the following theorems show. Note that, as Theorem 2 shows, deductive closure is weaker than collective rationality (see Section 2.4.2) [DL08].

Theorem 12 ([DL08])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem such that the agenda $\mathcal{A}$ is non-simple and evenly negatable, and let F be an aggregation function for $\mathcal{J}$. F satisfies universal domain, consistency, deductive closure, systematicity, and unanimity if, and only if, F is oligarchic.*

Theorem 12, that weakening collective rationality (see Definition 21) still leads to impossibility results in the sense of oligarchies. Furthermore, [DL08] additionally shows that removing the agenda restriction of even negatability but adding the mapping condition (see Section 2.4.3) of monotonicity leads to the same result. However, systematicity (see Definition 30) is a rather demanding mapping condition. Nevertheless, adding the agenda restriction of path-connectedness (see Definition 13) allows deducing the same results with independence instead of systematicity.

Theorem 13 ([DL08])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem such that the agenda $\mathcal{A}$ is evenly negatable and path-connected, and let F be an aggregation function for $\mathcal{J}$. F satisfies universal domain, consistency, deductive closure, independence, and unanimity if, and only if, F is oligarchic.*

Once again, adding monotonicity to the output condition in Theorem 13 allows removing evenly negatabilty from the agenda restrictions. As a result, it follows that the unanimity rule F_u (see Equation 2.3) is the only (oligarchic) aggregation function that satisfies the conditions given in Theorem 12 respectively Theorem 13 and the additional mapping condition anonymity (see Definition 27) [DL08]. Concluding, the results of [DL08] and [Gär06] anticipate that the mapping condition of independence is still a too demanding condition for escaping degenerate aggregation of individual judgments.

4.2 Relaxing independence

Independence (see Definition 28) ensures that the collective judgment is reached in a propositionwise approach. However, the main interest in judgment aggregation lies in coping with agendas that contain logically interconnected formulas. In particular, at least the agenda restriction of non-simplicity (see Section 2.2) should be satisfied since otherwise, aggregation would be trivial. Thus a fully propositionwise approach would, in some sense, contradict the aim to aggregate logically interconnected formulas, as it would be reasonable to not only consider propositions per se but also the premises it is built on for deciding about its acceptance or rejection [Mon08].

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4.2.1 The premise- and conclusion-based approach

Premise- and conclusion-based aggregation functions are one of the most studied aggregators in judgment aggregation that can be used as an alternative to propositionwise and majority aggregation functions [LP09]. The idea behind such aggregation functions is based on a special agenda type that allows viewing the agenda $\mathcal{A} = \mathcal{A}_p \cup \mathcal{A}_c$ as two distinct, disjoint subagendas $\mathcal{A}_p$ and $\mathcal{A}_c$ [Sla16]. One subagenda contains the premises $\mathcal{A}_p$, which are mutually independent propositions, and one subagenda contains the conclusions $\mathcal{A}_c$, which are logically entailed by the premises $\mathcal{A}_p$ [LP09, Sla16]. Such agendas that can be viewed as two distinct and disjoint subagendas of premises and conclusions are called *truth-functional* as the conclusions are truth-functionally determined by the premises [NP08]. The main idea is then to aggregate only the premises (or conclusions) and, in the case of the premises, then include the conclusions that are entailed by the collectively accepted premises.

Definition 39 (Truth-functional agendas [NP08, Sla16])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda. $\mathcal{A}$ is called* truth-functional *if and only if there is a subagenda $\mathcal{A}_p \subseteq \mathcal{A}$ of premises, such that for every formula $\phi \in \mathcal{A}_c = \mathcal{A} \setminus \mathcal{A}_p$ in the conclusions, there is a consistent subset $X \subseteq \mathcal{A}_p$ such that $X \models \phi$. I.e., the conclusions $\mathcal{A}_c$ are truth-functionally determined by the premises $\mathcal{A}_p$.*

Example 38 (Truth-functional agenda)**.** *Consider the judgment aggregation problem $\langle\{1,2,3\},\mathcal{A}\rangle$ with the agenda $\mathcal{A} = \pm\{v, b, g, g \leftrightarrow v \wedge b\}$ discursive dilemma Example 1. This agenda is truth-functional. Let $\mathcal{A}_p = \pm\{v, b, g\}$ be the premises, and thus $\mathcal{A}_c = \pm\{g \leftrightarrow v \wedge b\}$ the conclusions. $(g \leftrightarrow v \wedge b)$ is entailed by the set $\{v, b, g\} \subseteq \mathcal{A}_p$, and $\neg(g \leftrightarrow v \wedge b)$ is entailed by the set $\{v, b, \neg g\}$.*

Definition 40 (Partial profiles [Sla16])**.** *Let $\langle N, \mathcal{A}\rangle$ be a judgment aggregation problem and $P \in \mathbf{P}(\mathcal{A})$ be a profile. For an agenda subset $\mathcal{A}_p \subseteq \mathcal{A}$, we call*

$$P^{\downarrow \mathcal{A}_p} = \langle J_1 \cap \mathcal{A}_p, \ldots, J_n \cap \mathcal{A}_p\rangle$$

the partial profile *w.r.t. the subagenda $\mathcal{A}_p$.*

In particular, for an truth-functional agenda, $\mathcal{A} = \mathcal{A}_p \cup \mathcal{A}_c$ with premises and conclusions, the profile $P^{\downarrow \mathcal{A}_p}$ would be the profile w.r.t. the premises and $P^{\downarrow \mathcal{A}_c}$ would be the profile w.r.t. the conclusions.

Example 39 (Partial profile)**.** *Consider the judgment aggregation problem $\langle\{1,2,3\},\mathcal{A}\rangle$ with the agenda $\mathcal{A} = \pm\{v, b, v \wedge b\}$ which is similar to the judgment aggregation problem in the discursive dilemma (see Example 1 and Example 38). We have the premises $\mathcal{A}_p = \pm\{v, b\}$ and the conclusion $\mathcal{A}_c = \pm\{v \wedge b\}$. Let $P = \langle\{v, b, v \wedge b\}, \{v, \neg b, \neg(v \wedge b)\}, \{\neg v, b, \neg(v \wedge b)\}\rangle$ be the profile representing the discursive dilemma. The partial profile w.r.t. the premises is*

$$P^{\downarrow \mathcal{A}_p} = \langle\{v, b\}, \{v, \neg b\}\{\neg v, b\}\rangle$$

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and the partial profile w.r.t. the conclusion is

$$P^{\downarrow \mathcal{A}_c} = \langle \{v \wedge b\}, \{\neg(v \wedge b)\}\{\neg(v \wedge b)\}\rangle$$

As we observe in Example 39, for a truth-functional agenda $\mathcal{A} = \mathcal{A}_p \cup \mathcal{A}_c$, we can form the partial profiles only considering the premises or the conclusions and then use, e.g., the propositionwise majority rule (see Section 2.3.1) to get a consistent, non-contradicting collective judgment set.

The premise-based approach

Definition 41 (Premise-based majority rule [DM10, GP14])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ be a judgment aggregation problem such that the agenda $\mathcal{A} = \mathcal{A}_p \cup \mathcal{A}_c$ is truth-functional (respectively let $\mathcal{A}_p \subseteq \mathcal{A}$ be any set of premises and the conclusions $\mathcal{A}_c = \mathcal{A} \setminus \mathcal{A}_p$ [DM10]). The* premise-based majority aggregation rule *F_{pbm} is for any profile $P \in \mathbf{P}(\mathcal{A})$ defined as*

$$F_{pbm}(P) = F_{maj}(P^{\downarrow \mathcal{A}_p}) \cup \{\phi \in \mathcal{A}_c \mid F_{maj}(P^{\downarrow \mathcal{A}_p}) \models \phi\}$$

Remark 20. *The premise-based majority rule, as given in Definition 41, was first introduced by using truth-functional agendas [NP08]. However, it was later generalized by [DM10] for arbitrary 'premise' subagendas $\mathcal{A}_p \subseteq \mathcal{A}$ such that the conclusions are simply the complement of the premises, i.e., $\mathcal{A}_c = \mathcal{A} \setminus \mathcal{A}_p$, without any restriction to truth-functionality or logical entailment.*

Example 40 (Premise-based approach)**.** *Consider the judgment aggregation problem $\langle\{1, 2, 3\}, \mathcal{A}\rangle$ with the truth-functional agenda $\mathcal{A} = \pm\{v, b, v \wedge b\}$ with the premises $\mathcal{A}_p = \pm\{v, b\}$ and the conclusion $\mathcal{A}_c = \pm\{v \wedge b\}$ as in Example 39.*

Let $P = \langle\{v, b, v \wedge b\}, \{v, \neg b, \neg(v \wedge b)\}, \{\neg v, b, \neg(v \wedge b)\}\rangle$ be the profile representing the discursive dilemma (see Example 1). The partial profile w.r.t. the premises is

$$P^{\downarrow \mathcal{A}_p} = \langle \{v, b\}, \{v, \neg b\}\{\neg v, b\}\rangle$$

Hence, applying the premise-based majority rule from Definition 41 results in the collective judgment $F_{pbm} = \{v, b, v \wedge b\}$ since $F_{maj}(P^{\downarrow \mathcal{A}_p}) = \{v, b\}$ and $\{v, b\} \models v \wedge b$ with $v \wedge b \in \mathcal{A}_c$.

As Example 40 shows, using the premise-based majority rule no longer leads to an inconsistent collective set for the profile representing the discursive dilemma. However, the conclusion that $v \wedge b \in F_{pbm}(P^{\downarrow \mathcal{A}_p})$ in the collective judgment is still actually not accepted by a majority of the individuals in P. Moreover, as the following Example 41 shows, the premise-based majority rule can violate unanimity in contrast to the propositionwise majority rule [DM10].

Example 41 (Violation of unanimity)**.** *Consider the judgment aggregation problem $\langle\{1, 2, 3\}, \mathcal{A}\rangle$ with the truth-functional agenda $\mathcal{A} = \pm\{v, b, g, g \leftrightarrow v \wedge b\}$ with the premises $\mathcal{A}_p = \pm\{v, b, g\}$ and the conclusion $\mathcal{A}_c = \pm\{g \leftrightarrow v \wedge b\}$ as in the discursive dilemma (see Example 1). Furthermore, also consider the profile given in Example 1 of the discursive dilemma, i.e., $P = \langle J_1, J_2, J_3\rangle$ with*

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- $J_1 = \{v, b, g, (g \leftrightarrow v \wedge b)\}$
- $J_2 = \{v, \neg b, \neg g, (g \leftrightarrow v \wedge b)\}$
- $J_3 = \{\neg v, b, \neg g, (g \leftrightarrow v \wedge b)\}$

The partial profile w.r.t. the premises is

$$P^{\downarrow \mathcal{A}_p} = \langle \{v, b, g\}, \{v, \neg b, \neg g\} \{\neg v, b, \neg g\} \rangle$$

Hence, applying the premise-based majority rule from Definition 41 results in the collective judgment $F_{pbm} = \{v, b, \neg g, \neg(g \leftrightarrow v \wedge b)\}$ *since* $F_{maj}(P^{\downarrow \mathcal{A}_p}) = \{v, b, \neg g\}$ *and* $\{v, b, \neg g\} \models \neg(g \leftrightarrow v \wedge b)$ *with* $\neg(g \leftrightarrow v \wedge b) \in \mathcal{A}_c$.

Example 41 shows that despite each of the three individuals accepts $(g \leftrightarrow v \wedge b)$, the collective judgment rejects it on the basis of the premises.

The conclusion-based approach

Definition 42 (Conclusion-based majority rule [DM10, GP14])**.** *Let* $\mathcal{J} = \langle N, \mathcal{A} \rangle$ *be a judgment aggregation problem such that the agenda* $\mathcal{A} = \mathcal{A}_p \cup \mathcal{A}_c$ *is truth-functional (respectively let* $\mathcal{A}_p \subseteq \mathcal{A}$ *be any set of premises and the conclusions* $\mathcal{A}_c = \mathcal{A} \setminus \mathcal{A}_p$ *[DM10]). The* conclusion-based majority aggregation rule F_{cbm} *is for any profile* $P \in \mathbf{P}(\mathcal{A})$ *defined as*

$$F_{cbm}(P) = F_{maj}(P^{\downarrow \mathcal{A}_c})$$

Note that the conclusion-based majority rule of Definition 42 only aggregates the conclusions $\mathcal{A}_c$. In particular, for truth-functional agendas, this means that the collective judgment $F_{cbm}(P)$ will be incomplete (see Definition 20) for any $P \in \mathbf{P}(\mathcal{A})$ (at least if the conclusions $\mathcal{A}_c$ are contingent formulas). For arbitrary sets of premises $\mathcal{A}_p \subseteq \mathcal{A}$ of the agenda $F_{cbm}(P)$ is incomplete if $\mathcal{A}_p$ is non-empty and contains at least a formula and its negation.

Example 42 (Conclusion-based approach)**.** *Consider the judgment aggregation problem* $\langle \{1, 2, 3\}, \mathcal{A} \rangle$ *with the truth-functional agenda* $\mathcal{A} = \pm\{v, b, v \wedge b\}$ *with the premises* $\mathcal{A}_p = \pm\{v, b\}$ *and the conclusion* $\mathcal{A}_c = \pm\{v \wedge b\}$ *as in Examples 40 and 39.*

Let $P = \langle \{v, b, v \wedge b\}, \{v, \neg b, \neg(v \wedge b)\}, \{\neg v, b, \neg(v \wedge b)\} \rangle$ *be the profile representing the discursive dilemma (see Example 1). The partial profile w.r.t. the conclusions is*

$$P^{\downarrow \mathcal{A}_c} = \langle \{v \wedge b\}, \{\neg(v \wedge b)\} \{\neg(v \wedge b)\} \rangle$$

Hence, applying the conclusion-based majority rule from Definition 42 results in the collective judgment $F_{cbm} = \{\neg(v \wedge b)\}$ *since this conclusion is accepted by a majority.*

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Considering Example 40 of the premise-based approach and Example 42 of the conclusion-based approach, we get (as anticipated by the discursive dilemma of Example 1) that the collective judgment of the premise-based majority rule $F_{pbm}(P)$ contradicts the collective judgment of the conclusion-based majority rule $F_{cbm}(P)$. In particular, $F_{pbm}(P)$ accepts $(v \wedge b)$ whereas $F_{cbm}(P)$ rejects it.

Remark 21. *Regarding Definition 41 of the premise-based majority rule and Definition 42 of the aggregation of the premises $\mathcal{A}_p$ and conclusions $\mathcal{A}_c$ is done by using the propositionwise majority rule F_{maj} (see Definition 17). This is a typical approach as the propositionwise majority rule satisfies many desirable properties. However, the premise-based and conclusion-based approaches do not depend on a specific aggregation function, i.e., the aggregation rule for the approach can be arbitrary.*

Impossibility results

As [DM10] showed, the premise-based approach still leads to impossibility results, i.e., to degenerate cases of the aggregation of individual judgments.

Theorem 14 (Premise-based impossibility [DM10])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem such that $|N| \geq 3$ and the agenda $\mathcal{A}$ is non-simple, evenly negatable, and path-connected w.r.t. the set of premises $\mathcal{A}_p \subseteq \mathcal{A}$[1]. Furthermore, let F be an aggregation function for $\mathcal{J}$. F satisfies universal domain, collective rationality (respectively consistency and deductive closure), independence on $\mathcal{A}_p$, and unanimity if, and only if, F is dictatorial (respectively oligarchic).*

As Theorem 14 shows, for sufficiently rich agendas imposing independence only on the premises and unanimity are still conditions strong enough to allow only dictatorial or oligarchic aggregation functions. Moreover, if we only focus on premise-based aggregation functions that do not satisfy unanimity, then we could be left with counterintuitive outcomes where a formula is accepted by all individuals but not by the collective itself, as seen in Example 41. On the other hand, as the premise-based and the conclusion-based approach can contradict each other, the approach overall is manipulable, as one could choose exactly the approach that yields the better outcome regarding the own judgment and views [Pig06]. In particular, as the aim of premise-based approaches is to weaken the independence condition (see Definition 28), it would follow to impose conditions on the aggregation functions that weaken the independence condition even more. This led to the development of the so-called *sequential priority approach*, which is a generalization of the premise-based approach as we will see in the next Section 4.2.2 [LP09].

4.2.2 The sequential priority approach

The *sequential priority approach* introduces a sequential order of propositions of the agenda in which the aggregation process takes place [LP09]. In particular, prior propositions

[1] For the explicit definitions of non-simplicity, evenly negatability and path-connectedness w.r.t. a set of premises, we refer to [DM10].

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in the sequence can logically constrain or even decide later propositions in contrast to the usual concurrent aggregation rules like the propositionwise majority rule (see Definition 17), the unanimity rule (see Equation 2.3) or the premise-based majority rule (see Definition 41) [DL07b]. The sequence in which the propositions of the agenda are considered is defined by a *decision-path* function, as follows.

Definition 43 (Decision-path [Lis04, DL07b])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be a set of propositions with $|\mathcal{A}| = k$. A* decision-path *(function) is a one-to-one function $\Omega : \{1, \dots, k\} \rightarrow \mathcal{A}$ such that Ω generates a priority sequence $\langle \Omega(1), \dots, \Omega(k) \rangle$ of all propositions $\Omega(i) \in \mathcal{A}$ for $1 \leq i \leq k$ in which sequence the propositions are considered in the aggregation process.*

A decision-path (see Definition 43) can be interpreted in multiple ways. For example, it can model a temporal sequence in which the propositions are aggregated over time, but it can also be interpreted as a priority over the importance of certain propositions or a sequence of dependence from later propositions to prior ones [Lis04]. To apply the decision-path of propositions in the aggregation process, we define the following sequential aggregation rule.

Definition 44 (Sequential quota rule [Lis04, DL07b])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem with $|N| = n$ and $|\mathcal{A}| = k$. And let $t = \langle t_\phi \rangle_{\phi \in \mathcal{A}}$ a threshold vector (see Definition 16), and let Ω be any decision-path (see Definition 43) for $\mathcal{A}$.*

The sequential quota rule *$F_{\Omega,t}$ is for every profile $P = \langle J_i, \dots, J_n \rangle \in \mathbf{P}(\mathcal{A})$ defined as*

$$F_{\Omega,t}(P) := \Phi_k$$

where the set of formulas Φ_k is defined recursively (for $l = 0, \dots, k$) as

$$\Phi_l = \begin{cases} \emptyset, & \text{if } l = 0 \\ \Phi_{l-1} \cup \{\Omega(l)\}, & \text{else if } \Phi_{l-1} \models \Omega(l) \\ \Phi_{l-1} \cup \{\Omega(l)\}, & \text{else if } \Phi_{l-1} \cup \Omega(l) \text{ is consisten and } |P_{\Omega(l)}| \geq t_{\Omega(l)} \\ \Phi_{l-1}, & \text{otherwise} \end{cases}$$

Remark 22. *Regarding Definition 44 of the sequential quota rule, observe that the collective judgment set $F_{\Omega,t}(P)$ is by construction always consistent as the empty set is consistent, the deductive closure of a consistent set is consistent, and a consistent set is by definition consistent. Moreover, the propositionwise quota rule (see Definition 16) is very similar to the sequential quota rule. However, note that for small enough thresholds, the propositionwise quota rule can be inconsistent, whereas the sequential quota rule is, by definition, always consistent.*

Definition 45 (Sequential majority rule)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem, and let $F_{\Omega,t}$ be a sequential quota rule (see Definition 44). The* sequential majority rule *is defined as*

$$F_{\Omega,maj} := F_{\Omega,t} \qquad \text{with } t_\phi = \left\lceil \frac{|N| + 1}{2} \right\rceil \in t \text{ for all } \phi \in \mathcal{A}$$

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The sequential majority rule of Definition 45 is the corresponding sequential aggregation rule of the propositionwise majority rule (see Definition 17). However, observe that since the threshold is set to a majority, in this case, not only the sequential majority rule is always consistent, but also the propositionwise majority rule.

Example 43 (Sequential majority aggregation)**.** *Consider the judgment aggregation problem* $\langle \{1,2,3\}, \mathcal{A} \rangle$ *with the agenda* $\mathcal{A} = \pm\{v, b, v \wedge b\}$ *(see Examples 40, and 42). Furthermore, let*

$$P = \langle \{v, b, v \wedge b\}, \{v, \neg b, \neg(v \wedge b)\}, \{\neg v, b, \neg(v \wedge b)\} \rangle$$

be the profile representing the discursive dilemma (see Example 1). Consider the following decision path

$$\Omega = \{1 \mapsto v, 2 \mapsto \neg v, 3 \mapsto b, 4 \mapsto \neg b, 5 \mapsto (v \wedge b), 6 \mapsto \neg(v \wedge b)\}$$

By aggregating the individual judgments with the sequential majority rule $F_{\Omega,maj}$ *(see Definition 45), we get the following collective judgment*

$$F_{\Omega,maj}(P) = \{v, b, (v \wedge b)\}$$

In particular, we decide in the sequence $\langle v, \neg v, b, \neg b, (v \wedge b), \neg(v \wedge b) \rangle$ *as follows*

(1) $\Phi_1 = \emptyset \cup \{v\}$*, since* $|P_v|$ *is accepted by a majority*

(2) $\Phi_2 = \Phi_1 = \{v\}$*, since* $\{v\} \cup \{\neg v\}$ *is inconsistent and does not satisfy any other condition*

(3) $\Phi_3 = \Phi_2 \cup \{b\}$*, since* $|P_b|$ *is accepted by a majority*

(4) $\Phi_4 = \Phi_3 = \{v, b\}$ *since* $\{v, b\} \cup \{\neg b\}$ *is inconsistent and does not satisfy any other condition*

(5) $\Phi_5 = \Phi_4 \cup \{(v \wedge b)\}$ *since* $\Phi_4 \models (v \wedge b)$

(6) $\Phi_6 = \Phi_5$ *since* $\{v, b, (v \wedge b)\} \cup \{\neg(v \wedge b)\}$ *is inconsistent and does not satisfy any other condition*

Remark 23. *Regarding Example 43, observe that applying the sequential majority rule (see Definition 45) results in exactly the same collective judgment as the application of the premise-based majority rule (see Definition 41) in Example 40. This is the case because of the specific decision-path* Ω *used in Example 43, as it projects the premise-based approach by prioritizing the premises over the conclusion. In particular, in both Example 40 and 43, the premises* v *and* b *are selected by propositionwise majority aggregation into the collective judgment, and the conclusion* $(v \wedge b)$ *is selected by entailment of the premises. Thus this shows how the premise-based approach can be modeled with the sequential priority approach by using certain decision-paths that prioritize the premises over the conclusions.*

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Nevertheless, as we will observe in the following, different decision-paths can also lead to completely different collective judgment sets.

Example 44 (Decision-path dependence)**.** *Consider again the judgment aggregation problem $\langle\{1,2,3\},\mathcal{A}\rangle$ with the agenda $\mathcal{A} = \pm\{v, b, v \wedge b\}$ (see Example 43). Furthermore, let*

$$P = \langle\{v, b, v \wedge b\}, \{v, \neg b, \neg(v \wedge b)\}, \{\neg v, b, \neg(v \wedge b)\}\rangle$$

be the profile representing the discursive dilemma (see Example 1). Consider the decision path from Example 43

$$\Omega_1 = \{1 \mapsto v, 2 \mapsto \neg v, 3 \mapsto b, 4 \mapsto \neg b, 5 \mapsto (v \wedge b), 6 \mapsto \neg(v \wedge b)\}$$

and the following one

$$\Omega_2 = \{1 \mapsto (v \wedge b), 2 \mapsto \neg(v \wedge b), 3 \mapsto v, 4 \mapsto \neg v, 5 \mapsto b, 6 \mapsto \neg b\}$$

We already know from Example 43 that the collective set by using the sequential majority rule for Ω_1 is $F_{\Omega_1,maj} = \{v, b, (v \wedge b)\}$. However, observe that $F_{\Omega_2,maj} = \{\neg(v \wedge b), v, \neg b\}$ since at first we decide about $(v \wedge b)$, which is rejected because it is neither entailed nor accepted by a majority. Then $\neg(v \wedge b)$ is added because it is accepted by a majority, and since v still does not yield inconsistency, it is also accepted because of majority. Moreover, b is rejected since collectively accepting b would lead to inconsistency. Last but not least, $\{\neg(b, \wedge b), v\}$ entails $\neg b$ and thus $\neg b$ is also collectively accepted.

As Example 44 shows, the collective judgment set of sequential rules may depend on the chosen decision-path. This allows us to introduce the following notion of decision-path dependence.

Definition 46 (Decision-path dependence [Lis04, DL07b])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ be a judgment aggregation problem and F_Ω a sequential aggregation function. F_Ω is* decision-path dependent *if, and only, if there are decision-paths Ω_1, Ω_2, a profile $P \in \mathbf{P}(\mathcal{A})$ and a formula $\phi \in \mathcal{A}$ such that $\phi \in F_{\Omega_1}(P)$ and $\phi \notin F_{\Omega_2}(P)$.*

Furthermore, F_Ω is strongly decision-path dependent if, and only if, there are decision-paths Ω_1, Ω_2, a profile $P \in \mathbf{P}(\mathcal{A})$ and a formula $\phi \in \mathcal{A}$ such that $\phi \in F_{\Omega_1}(P)$ and $\neg\phi \in F_{\Omega_2}(P)$.

If F_Ω is not decision-path dependent, then we call it decision-path independent. If F_Ω is not strongly decision-path dependent, then we call it weakly decision-path independent.

Note that trivially every strongly decision-path dependent sequential aggregation function F_Ω is also decision-path dependent since the collective set of a sequential aggregation function is by definition consistent and thus $\neg\phi \in F_{\Omega_2}(P)$ implies that $\phi \notin F_{\Omega_2}(P)$.

The problem of sequential decision-path dependent aggregation functions (see Definition 46) is that their collective judgment can be manipulated by changing the decision-path, as Example 44 shows [DL07b].

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4.2.3 The distance-based approach

The distance-based approach[2] was originally introduced by [Pig06] into judgment aggregation by building on the results of belief merging. The main idea of this approach is to circumvent the problems of the premise-based and conclusion-based approaches (see Section 4.2.1), i.e., that the premise-based approach can contradict the conclusion-based approach (see Examples 40 and 42). Hence, in contrast to a sequential priority approach (see Section 4.2.2), like the premise-based and conclusion-based approach, the propositions are not sequentially prioritized, but rather all propositions have the same priority, and the aim is to minimize the differences over all individual judgments in the given profile [Lis11].

Definition 47 (Distance metric [Pig06, EP05, MO09])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem. A* distance metric *or* distance *is a function $d : \mathbf{J}(\mathcal{A}) \times \mathbf{J}(\mathcal{A}) \rightarrow \mathbb{R}^+$ such that*

- $d(J, J') = 0 \iff J = J'$
- $d(J, J') = d(J', J)$

for any judgment sets $J, J' \in \mathbf{J}(\mathcal{A})$.

A distance metric, in this sense, defines a value that represents the difference between two judgment sets. In particular, if the judgment sets are equal, then by definition, the distance is 0. However, if the difference between judgments is big, then the distance should also be big. Thus we can use the notion of distance between judgment sets to find a collective judgment set with the minimal distance w.r.t. the given profile.

Definition 48 (Minimal sum distance rule [Pig06, Lis11, GP14])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and $d : \mathbf{J}(\mathcal{A}) \times \mathbf{J}(\mathcal{A}) \rightarrow \mathbb{R}^+$ a distance function. The* minimal sum distance rule *$F_{\Sigma,d}$ is defined as follows*

$$F_{\Sigma,d}(P) = \text{argmin}_{J \in \mathbf{J}(\mathcal{A})} \sum_{J_i \in P} d(J_i, J)$$

Regarding Definition 48 of the minimal sum distance rule, we observe that this aggregation function tries to find the best compromise between all individual judgments in the profile by minimizing the sum over all distances.

Definition 49 (Hamming distance [Lis11])**.** *The* Hamming distance *$d_H : \mathbf{J}(\mathcal{A}) \times \mathbf{J}(\mathcal{A}) \rightarrow \mathbb{R}^+$ is defined for any judgment sets $J, J' \in \mathbf{J}(\mathcal{A})$ as*

$$d_H(J, J') = |\{\phi \in \mathcal{A} \mid \phi \in J \not\Leftrightarrow \phi \in J'\}|$$

In particular, the Hamming distance is the number of formulas of the agenda on which the two given judgment sets differ.

[2] [Pig06] called this approach originally the *argument-based procedure*, but the term *distance-based approach* is more widely used in the literature.

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Example 45 (Minimal sum distance rule)**.** *Consider again the judgment aggregation problem* $\langle\{1,2,3,4\},\mathcal{A}\rangle$ *with the agenda* $\mathcal{A} = \pm\{v, b, v \wedge b\}$*. Furthermore, let*

$$P = \langle\{v, b, v \wedge b\}, \{v, \neg b, \neg(v \wedge b)\}, \{\neg v, b, \neg(v \wedge b)\}, \{v, \neg b, \neg(v \wedge b)\}\rangle$$

be a profile, and consider the Hamming distance (see Definition 49) as the used distance function.

Then the minimal sum distance rule F_{Σ,d_H} *(see Definition 48) yields the following collective set*

$$F_{\Sigma,d_H}(P) = \{v, \neg b, \neg(v \wedge b)\}$$

with an overall sum of Hamming distances of $\sum_{J_i \in P} d(J_i, F_{\Sigma,d_H}(P)) = 4$*.*

Since the judgment set $\{v, \neg b, \neg(v \wedge b)\}$ *occurs twice in* P*, the collective set gets "moved" in this direction as it lowers the overall sum of Hamming distances.*

Note that despite distance-based aggregation rules do not satisfy independence (see Definition 28), using distance functions that treat all propositions equally, like the Hamming distance (see Definition 49), can at least lead to a "kind of neutrality" (see Definition 29) [Lis11]. Moreover, as Example 45 shows, desirable mapping conditions (see Section 2.4.3) that are satisfied for the minimal sum distance rule F_{Σ,d_H} are anonymity, as the order of individual judgment sets is completely irrelevant for the outcome, monotonicity and unanimity, as both conditions lead to a smaller overall sum of distances.

4.3 Relaxing universal domain

Universal domain (see Definition 18) is a relatively little criticized input condition (see Section 2.4.1) since we usually want to allow all possible rational individual judgments. For example, regarding democracy, we can view universal domain as the condition that every opinion is allowed if it is rational. Nevertheless, restricting the domain can lead to possibility results as [Lis03] showed.

4.3.1 Unidimensional alignment

Unidimensional alignment is a property of profiles that ensures collective rational aggregation by using the propositionwise majority rule (see Definition 17) [GP14]. In particular, a unidimensional aligned profile can be ordered such that for every proposition $\phi \in \mathcal{A}$, the individuals that accept ϕ are all to the left or to the right of the individuals that reject ϕ.

Definition 50 (Unidimensional alignment [Lis03])**.** *Let* $\langle N, \mathcal{A}\rangle$ *be a judgment aggregation problem, and let* $P \in \mathbf{P}(\mathcal{A})$ *be any profile. Let for any formula* $\phi \in \mathcal{A}$ *be*

$$P_{\in,\phi} = \{i \in N \mid \phi \in J_i, J_i \in P\} \text{ and } P_{\notin,\phi} = \{i \in N \mid \phi \notin J_i, J_i \in P\}$$

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the set of individuals in P that accept ϕ and the set of individuals in P that reject ϕ respectively.

Furthermore for any linear ordering $< \subseteq N \times N$ on N and any two sets of individuals $P_{\in,\phi}, P_{\notin,\phi} \subseteq N$ we define

$$P_{\in,\phi} < P_{\notin,\phi} \iff \forall i \in P_{\in,\phi}\, \forall j \in P_{\notin,\phi} : i < j.$$

A profile $P \in \mathbf{P}(\mathcal{A})$ satisfies unidimensional alignment *if and only if there exists a linear ordering $<$ on N such that*

$$\forall \phi \in \mathcal{A} : \text{ either } P_{\in,\phi} < P_{\notin,\phi} \text{ or } P_{\notin,\phi} < P_{\in,\phi}.$$

An ordering $< \subseteq N \times N$ that satisfies the condition of unidimensional alignment is called structuring ordering *on N.*

If $|N|$ is odd, then $m \in N$ is the median individual *w.r.t. the ordering $<$ if and only if $|\{i \in N \mid i < m\}| = |\{i \in N \mid m < i\}|$.*

If $|N|$ is even, then $m_1, m_2 \in N$ are the median pair *w.r.t. $<$ if and only if*

- $m_1 < m_2$,
- *for all $i \in N \setminus \{m_1, m_2\}$: $i < m_1$ or $m_2 < i$, and*
- $|\{i \in N \mid i < m_1\}| = |\{i \in N \mid m_2 < i\}|$.

Note that a formula $\phi \in \mathcal{A}$ can be accepted (rejected) by every individual in the profile, i.e., $P_{\notin,\phi} = \emptyset$ ($P_{\in,\phi} = \emptyset$). Moreover, by Definition 3, a judgment set $J \in \mathbf{J}(\mathcal{A})$ is complete and consistent, i.e., fully rational. Thus $\phi \in J_i \iff \neg\phi \notin J_i$, which implies that $P_{\in,\phi} < P_{\notin,\phi} \iff P_{\notin,\neg\phi} < P_{\in,\neg\phi}$. Hence, we can restrict the linear ordering of unidimensional alignment (see Definition 50) to the pre-agenda $\mathcal{A}^+$, since it implies an inverse ordering on $\overline{A^+} = \mathcal{A} \setminus \mathcal{A}^+$. In particular, if a profile P is unidimensionally aligned over the pre-agenda $\mathcal{A}^+$, then it is unidimensionally aligned over the whole agenda.

Example 46 (Unidimensional alignment for odd N)**.** *Let $\langle N, \mathcal{A}\rangle$ be a judgment aggregation problem with $\mathcal{A} = \pm\{v, b, g, (g \leftrightarrow v \wedge b)\}$ the agenda of the discursive dilemma (see Example 1). Moreover, consider the profile given in the following Table 4.1.*

The profile P of Table 4.1 satisfies unidimensional alignment (see Definition 50) since there is an ordering of the individuals (which is given in the table) such that $P_{\in,\phi} < P_{\notin,\phi}$ or $P_{\notin,\phi} < P_{\in,\phi}$ for every $\phi \in \mathcal{A}$. Furthermore, the median individual is J_2 and as Table 4.1 shows $F_{maj}(P)(P) = J_2$.

In contrast to Example 46, the profile of the discursive dilemma as given in Example 1 does not satisfy unidimensional alignment.

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$P =$	$\langle$ J_1,	J_2,	J_3 $\rangle$	$F_{maj}(P)$
v	1	1	1	1
b	1	0	0	0
g	0	0	0	0
$g \leftrightarrow v \wedge b$	0	1	1	1

Table 4.1: Example of unidimensional alignment for an odd number of individuals

$P =$	$\langle$ J_1,	J_2,	J_3,	J_4 $\rangle$	$F_{maj}(P)$
v	1	1	1	1	1
b	1	1	0	0	-
g	0	0	0	1	0
$g \leftrightarrow v \wedge b$	0	0	1	1	-

Table 4.2: Example of unidimensional alignment for an even number of individuals

Example 47 (Unidimensional alignment for even N)**.** *Let $\langle N, \mathcal{A}\rangle$ be a judgment aggregation problem with $\mathcal{A} = \pm\{v, b, g, (g \leftrightarrow v \wedge b)\}$ (see Example 46). Moreover, consider the profile given in the following Table 4.2.*

The profile P of Table 4.2 satisfies unidimensional alignment (see Definition 50) since there is an ordering of the individuals (which is given in the table) such that $P_{\in,\phi} < P_{\notin,\phi}$ or $P_{\notin,\phi} < P_{\in,\phi}$ for every $\phi \in \mathcal{A}$. The number of individuals $|N|$ is even. Thus there is no single median individual but a median pair of individuals (see Definition 50), viz. $\langle J_2, J_3\rangle$. In this case $F_{maj}(P) = J_2 \cap J_3$, but since J_2 and J_3 disagree on b and $(g \leftrightarrow v \wedge b)$, the collective set $F_{maj}(P)$ is incomplete.

Note that in Example 46 and Example 47, it is not a coincidence that the collective set equals the median individual, respectively, the intersection of the median pair.

Theorem 15 (Collective set for unidimensional alignment [Lis03])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ be a judgment aggregation problem and F_{maj} the propositionwise majority rule (see Definition 17). For any profile $P \in \mathbf{P}(\mathcal{A})$ that satisfies unidimensional alignment*

- *if $|N|$ is odd, then for any structuring ordering $<$ of P where m is the median individual, it holds that $F_{maj}(P) = J_m$.*
- *if $|N|$ is even, then for any structuring ordering $<$ of P where $\langle m_1, m_2\rangle$ are the median pair, it holds that $F_{maj}(P) = J_{m_1} \cap J_{m_2}$.*

Informally, Theorem 15 holds since if for odd $|N|$ the median individual accepts a formula $\phi \in \mathcal{A}$, then due to the structuring ordering a majority accepts ϕ (see Example 46). For even $|N|$, if both individuals of the median pair accept a formula $\phi \in \mathcal{A}$, then due to the structuring ordering, a majority accepts ϕ. If only one individual of the median

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pair accepts a formula ϕ, but the other individual does not, then exactly half of the individuals accept ϕ. Thus the formula is not collectively accepted by the propositionwise majority rule. Furthermore, if the median pair disagrees on a formula, then the collective set of the propositionwise majority rule is incomplete (see Example 47).

Theorem 16 ([Lis03])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$.*

- *If $|N|$ is odd, then [F is collectively rational, anonymous and systematic (see Section 2.4) if and only if $F = F_{maj}$].*
- *If $|N|$ is even, then [F is consistent, deductively closed, anonymous, and systematic if and only if $F = F_{maj}$].*

Note that w.r.t. Theorem 16 if $|N|$ is odd, then the collective set $F_{maj}(P)$ equals an individual judgment set (see Theorem 15). Thus by Definition 3 of judgment sets, it holds that the collective set $F_{maj}(P)$ is rational, i.e., complete and consistent. If $|N|$ is even, then due to the possible discrepancy between the median pair, the collective set can be incomplete (see Theorem 15).

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CHAPTER 5

General frameworks

Up until now, we considered judgment aggregation in a specific logical framework, i.e., in classical propositional logic (see Chapter 2). However, there is also the so-called *abstract* or *binary aggregation framework* [DH10, NP10]. This framework does not depend on any specific logical language. Rather $\{0,1\}^m$ vectors represent the acceptance/rejection for every of the n individuals over all m issues. The aim is then to study aggregation functions as functions from profiles, which are $\{0,1\}^{n\times m}$ matrices, to individual acceptance/rejections vectors. Thereby, the abstract aggregation framework abstracts away from any logical view. Note that every classical judgment aggregation problem can be translated into an abstract aggregation problem. However, this translation leads to a loss of information since the logical interpretation is lost [Lis11].

The *general logics framework*, on the other hand, generalizes the logic under consideration. In particular, in this framework, the logic is not fixed to a specific one. Rather classes of logics that satisfy certain basic properties are considered [GP14]. In the following chapter, we focus on the general logics framework based on the results of [Die07] that uses at least some logical language for modeling judgment aggregation problems.

In Section 5.1, we introduce the general logics framework and some basic conditions that allow deriving impossibility results. Section 5.2 then shows the impossibility result of [Die07] that hold for a variety of logics.

5.1 The general logics framework

In the classical propositional logic framework of judgment aggregation (see Chapter 2), complex propositions are only built by the connectives $\wedge$, $\vee$, and $\neg$[1]. However, some

[1] Note that every other truth-functional connective can also be used. However, the set of connectives $\{\wedge, \vee, \neg\}$ is functional complete. Thus every other truth-functional operator is equivalent to a combination of those three connectives.

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aggregation problems may need the expressibility of first-order logic, like quantification and relations. In particular, to embed preference aggregation into judgment aggregation, a simple first-order logic with binary relation symbols is needed [DL07a]. Moreover, expressing propositions of necessity and obligation like "the defendant is obliged to say the truth" or propositions of permit like "the defendant is permitted to refrain from incriminating himself" need a modal logic to be properly expressed [BVW07]. Therefore [Die07] introduces the general logics framework that does not fix the concrete underlying logic but describes a class of logics that commonly satisfy certain conditions, like *self-entailment*, *monotonicity* and *completeability*.

Definition 51 (Logics with negation [Die07]). *A* logic (with negation) *is a pair* $(\mathcal{L}, \models)$[2] *where*

- $\mathcal{L} \neq \emptyset$ *is a* formal language *(see Definition 1) closed under negation* $\neg$ *of* propositions *or* formulas*, i.e., if* $\phi \in \mathcal{L}$*, then* $\neg\phi \in \mathcal{L}$*.*
- $\models \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ *is an* entailment relation *between sets* $\Gamma \subset \mathcal{L}$ *and single propositions* $\phi \in \mathcal{L}$*. In particular,* $\Gamma \models \phi$ *means that the proposition* ϕ *follows from the set of propositions* Γ*.*

Note that the language of classical propositional logic $\mathcal{L}_p$ (see Definition 1) and the entailment relation of the classical propositional logic $\models$ build the classical propositional logic $(\mathcal{L}_p, \models)$ as described by Definition 51.

Definition 52 ((In-)Consistency in logics with negation [Die07]). *Let* $(\mathcal{L}, \models)$ *be a logic (see Definition 51).*

- *A set* $\Gamma \subseteq \mathcal{L}$ *is called*
  - inconsistent *(written* $\Gamma \models \bot$*) if there is a formula* $\phi \in \mathcal{L}$ *such that* $\Gamma \models \phi$ *and* $\Gamma \models \neg\phi$*.*
  - consistent *if it is not inconsistent.*
- *A formula* $\phi \in \mathcal{L}$ *is called*
  - *a* contradiction *if* $\{\phi\}$ *is inconsistent.*
  - *a* tautology *(written* $\models \phi$*) if* $\{\neg\phi\}$ *is inconsistent.*
  - contingent *if* $\{\phi\}$ *and* $\{\neg\phi\}$ *are consistent.*

[2] Note that usually a logic is defined as a triple $(\mathcal{L}, \vdash, \models)$, i.e., a formal language $\mathcal{L}$ with a deductive system $\vdash$ and a semantics (entailment relation) $\models$ [SKK22]. In the case of judgment aggregation, we usually only need one of the two relations.

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Observe that this notion of consistency and inconsistency (see Definition 52) for a logic with negation $(\mathcal{L}, \models)$ (see Definition 51) is enough to define the notions of judgment aggregation problems (see Definition 2), judgment sets (see Definition 3), agenda restrictions (see Section 2.2) and aggregation conditions (see Section 2.4).

To derive impossibility results for such a general logic with negation, the logic must only satisfy some mild conditions, which we describe in the following.

Definition 53 (Logic conditions [Die07])**.** *Let $(\mathcal{L}, \models)$ be a logic with negation (see Definition 51). We say that $(\mathcal{L}, \models)$ satisfies*

(L1) self-entailment *if and only if for any formula $\phi \in \mathcal{L}$, $\{\phi\} \models \phi$.*

(L2) monotonicity *if and only if for any formula $\phi \in \mathcal{L}$ and sets $\Gamma, \Delta \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Delta$, if $\Gamma \models \phi$, then $\Delta \models \phi$.*

(L3) completability *if and only if the empty set $\emptyset$ is consistent, and for every consistent set $\Gamma \subseteq \mathcal{L}$ there is a consistent set $\Delta \subseteq \mathcal{L}$ such that $\Gamma \subseteq \Delta$ and for every formula $\phi \in \mathcal{L}$, either $\phi \in \Delta$ or $\neg\phi \in \Delta$.*

(L4) compactness *if and only if for any formula $\phi \in \mathcal{L}$ and set $\Gamma \subseteq \mathcal{L}$, if $\Gamma \models \phi$, then there is a finite set $\Delta \subseteq \mathcal{L}$ such that $\Delta \models \phi$.*

The monotonicity condition (L2) ensures that adding new knowledge does not alter previous knowledge. Note that monotonicity excludes all non-monotonic logics, like commonsense reasoning, from the considered spectrum. Completability (L3) ensures that every consistent set can be extended to a complete set, which is necessary for establishing full rationality judgment sets (see Definition 3) and collective rationality (see Definition 21).

The notions of a $\mathcal{L}$-judgment aggregation problem (see Definition 2), a judgment set (see Definition 3), agenda restrictions (see Section 2.2), and aggregation conditions (see Section 2.4) are defined as for classical judgment aggregation in Chapter 2. Furthermore, we introduce the following additional agenda restriction.

Definition 54 (Asymmetric agendas [Die07])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda (see Definition 2). $\mathcal{A}$ is* asymmetric *if and only if there is a subset $X \subseteq \mathcal{A}$ such that either X or $X^\neg$ is consistent but not both, where $X^\neg = \{\neg\phi \mid \phi \in X\}$.*

Example 48 (Asymmetric agenda)**.** *Let $\mathcal{A} = \pm\{p, q, p \leftrightarrow q\}$. The subset $X = \{p, q, p \leftrightarrow q\} \subseteq \mathcal{A}$ is consistent, whereas $X^\neg = \{\neg p, \neg q, \neg(p \leftrightarrow q)\}$ is inconsistent. Thus by Definition 54, $\mathcal{A}$ is an asymmetric agenda.*

Note that in the general logics framework, we do not consider specific logics, but classes of logics that satisfy the imposed conditions given in Definition 53.

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5.2 General impossibility results

The impossibility theorems we reviewed in Chapter 3 only hold in a specific logical framework, i.e., in the classical propositional logic framework (see Chapter 2). Thus we cannot apply these impossibility results directly to other logical languages but have to prove them anew. However, by using the general logics framework (see Section 5.1), we describe whole classes of logics at once. Thus we can apply the impossibility results to every specific logic that satisfies the imposed conditions.

Theorem 17 (A general impossibility [Die07][3]). *Let $(\mathcal{L}, \models)$ any logic with negation (see Definition 51) that satisfies self-entailment, monotonicity and completability (see Definition 53 (L1-L3)). Furthermore, let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a $\mathcal{L}$-judgment aggregation problem where $\mathcal{A}$ is non-simple (see Definition 8), evenly negatable (see Definition 11) and asymmetric (see Definition 54). Let F be an aggregation function for $\mathcal{J}$.*

If F satisfies collective rationality (see Definition 21), universal domain (see Definition 18), and systematicity (see Definition 30), then F is dictatorial (see Definition 24).

Theorem 17 is a generalization of Theorem 5 of [LP02], where anonymity is dropped. Thus resulting in aggregation functions that are existent but dictatorial. Note that Theorem 17 can be applied in every logic with negation satisfying the conditions self-entailment, monotonicity, and completability. Thus this theorem introduces an impossibility result to a wide variety of logics.

Moreover, if the logic under consideration additionally satisfies compactness, then for $|N| \geq 3$, the converse of Theorem 17 does also hold.

Theorem 18 (A general impossibility characterization [Die07]). *Let $(\mathcal{L}, \models)$ any logic with negation that satisfies self-entailment, monotonicity, completability, and compactness (see Definition 53 (L1-L4)). Furthermore, let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a $\mathcal{L}$-judgment aggregation problem with $|N| \geq 3$ where $\mathcal{A}$ is non-simple, evenly negatable and asymmetric. Let F be an aggregation function for $\mathcal{J}$. F satisfies collective rationality, universal domain, and systematicity if and only if F is dictatorial.*

For example, first-order logic $(\mathcal{L}_{FO}, \models)$ satisfies self-entailment, monotonicity, and completability and is also compact. Hence, we can apply Theorem 18. Thus for any $(\mathcal{L}_{FO})$-judgment aggregation problem $\mathcal{J}$, every aggregation function F of $\mathcal{J}$ satisfies collective rationality, universal domain, and systematicity if and only if it is dictatorial.

Concluding, note that the general logics framework abstracts away from any concrete logical language and thus generalizes impossibility (or possibility) results for classes of logics. Nevertheless, some results depend on the specific logical structure of a certain language and thus must be stated in this specific logic to hold.

[3]The proof in [Die07] is done by using the *ultrafilter proof technique* as described in Section 3.2.

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CHAPTER 6

Many-valued judgment aggregation

Classical judgment aggregation, as thoroughly introduced in Chapter 2, only utilizes classical propositional logic. This also led to the notion of a judgment set (see Definition 3) in the classical sense as a subset of formulas of the agenda. In particular, in classical bivalent logic, a formula can either be accepted or rejected but not more. Thus leading to the definition of issues as pairs of formulas, i.e., a formula and its negation. However, as [PvH06, vH07, DP13, Her13, Fer23] show, the classical judgment aggregation framework can be generalized to a many-valued framework by simply exchanging judgment sets with (judgment) valuation functions, which can model any value of acceptance. Thus generalizing the attitude an individual can have toward a proposition [Die07].

The chapter consists of 5 main sections. Section 6.1 describes the general many-valued framework with its adapted language and notions. In Section 6.2, we consider the aggregation of judgments in the many-valued case. In particular, we introduce the average aggregation rule for general many-valued judgments. Section 6.3 introduces the adapted aggregation conditions for the many-valued framework similar to Section 2.4 in Chapter 2. In Section 6.4, we review impossibility results for many-valued judgment aggregation shown by [PvH06]. Finally, Section 6.5 concludes by reviewing further and related work to many-valued judgment aggregation.

6.1 The many-valued framework

In contrast to the classical definition of judgment sets as subsets of the agenda (see Definition 3), we can also view judgments as valuation functions. In this sense, a judgment J is (in the classical propositional case) equivalent to the characteristic function of its corresponding judgment set of Definition 3. However, classical judgment sets do not

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directly map to the many-valued case. In the classical logic framework (see Chapter 2), a judging individual i can accept a formula ϕ ($\phi \in J_i$), or it can reject the formula ϕ ($\neg\phi \in J_i$)[1]. States as partially accepting a formula ϕ can not be expressed by this definition. However, the modeling approach via valuation functions allows arbitrary states of acceptance and thus enables many-valued judgments. Thereby the many-valued framework generalizes the classical propositional framework (see Chapter 2) by allowing arbitrary attitudes toward formulas.

6.1.1 General propositional logic

For coping with many-valued judgments, we first introduce a general propositional language (logic) that can have arbitrary operators and truth values or values of acceptance. Then we only need to fix the operators and the set of truth values to get a specific many-valued propositional logic.

Definition 55 (General propositional languages [Fer23]). *Let At be a countable set of atomic propositions (atoms), and let $Op_{\mathcal{L}}$ be a set of connectives for the language $\mathcal{L}$. The* language of (general) propositional logic *$\mathcal{L}$ is defined inductively as*

- *$At \subseteq \mathcal{L}$, and*
- *if $\phi_1, \ldots, \phi_n \in \mathcal{L}$, then $\circ(\phi_1, \ldots, \phi_n) \in \mathcal{L}$, for every n-ary connective $\circ \in Op_{\mathcal{L}}$.*

For example, the language of classical propositional logic $\mathcal{L}_p$ (see Definition 1) is built by the set of operators $Op_{\mathcal{L}_p} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$ as described in Definition 55.

Definition 56 (Many-valued logics [PvH06, Fer23]). *A propositional* many-valued logic *Λ over the set of truth values T is general propositional language $\mathcal{L}$ (see Definition 55) together with a semantics that is specified by an* interpretation function *$\widetilde{v}$ that maps every n-ary connective $\circ \in Op_{\mathcal{L}}$ to a corresponding* truth function *$\widetilde{v}(\circ) : T^n \rightarrow T$.*

A Λ-valuation *$v : X \rightarrow T$ of a set of formulas $X \subseteq \mathcal{L}$ is an assignment of truth values in T to the formulas in X such that for any n-ary connective $\circ \in Op_{\mathcal{L}}$ and formulas $\phi_1, \ldots, \phi_n \in X$*

$$v(\circ(\phi_1, \ldots, \phi_n)) = \widetilde{v}(\circ)(v(\phi_1), \ldots, v(\phi_n)),$$

i.e., it respects the truth functions. An arbitrary assignment $w : X \rightarrow T$ is called Λ-consistent *if and only if it is a Λ-valuation. Otherwise w is called* Λ-inconsistent.

Example 49 (Classical propositional logic). *Consider Definition 56 of propositional many-valued logics. If we restrict the set of truth values to $T = \{0, 1\}$ and define the set of operators $Op_{\mathcal{L}} = \{\neg, \wedge, \vee\}$ with the following truth functions for formulas $\phi, \psi \in \mathcal{L}$ as*

$$v(\neg\phi) = 1 - v(\phi)$$
$$v(\phi \wedge \psi) = \min(v(\phi), v(\psi))$$
$$v(\phi \vee \psi) = \max(v(\phi), v(\psi))$$

[1] Sometimes it is also allowed that the judging individual can be indifferent about the formula, i.e., neither $\phi \in J_i$ nor $\neg\phi \in J_i$ [Gär06].

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Then it results in classical propositional logic.

Moreover, the so-called Kleene-Zadeh logic KZ generalizes classical propositional logic (see Example 49) from the set of truth values $\{0, 1\}$ to the real interval of truth values $[0, 1]$. Thereby, the logic KZ allows a non-finite number of truth values.

Definition 57 (Kleene-Zadeh logic KZ [Fer23])**.** *The set of all KZ-formulas $\mathcal{L}$ is built up as usual from atomic formulas At using the connectives $\wedge$, $\vee$, and $\neg$ (see Definition 55). The semantics of KZ is given by the following truth functions that extend any given valuation v, i.e. any, assignment $At \to [0, 1]$, from atomic formulas to arbitrary formulas.*

$$v(\neg\phi) = 1 - v(\phi)$$
$$v(\phi \wedge \psi) = \min(v(\phi), v(\psi))$$
$$v(\phi \vee \psi) = \max(v(\phi), v(\psi))$$

Note that Kleene-Zadeh logic turns into classical logic if we restrict $[0, 1]$ to the bivalent set $\{0, 1\}$.

Example 50 (Kleene-Zadeh logic)**.** *Consider the set of atoms $At = \{p, q, r\}$ and the valuation*

$$v = \{p \mapsto 0.5, q \mapsto 0.3, r \mapsto 1\}$$

We get the following formula valuation values:

- $v(\neg q) = 0.7$
- $v(p \wedge r) = 0.5$
- $v(p \vee q) = 0.5$

Remark 24. *Note that this many-valued framework differs from the general logics framework of Chapter 5. The general logics framework generalizes the expressibility of propositions, whereas the many-valued framework generalizes the attitude an individual can have towards a proposition [Die07]. In particular, the general propositional language for this many-valued framework (see Definition 55) is still a propositional language. The generalization lies in the arbitrary truth values T that can be assigned to the formulas in the considered many-valued logic (see Definition 56).*

6.1.2 Judgments and profiles

Similar to the classical notions of judgment sets (see Definition 3) and profiles (see Definition 6), we introduce the corresponding many-valued equivalents. Note that judgment aggregation problems (see Definition 2) are defined as in Chapter 2[2].

[2]Recall that a judgment aggregation problem does only depend on the underlying formal language, but not on the semantics of this language.

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Definition 58 (Judgment aggregation problem [Her13, Fer23]). *An* ($\mathcal{L}-$)judgment aggregation problem *is a tuple* $\mathcal{J} = \langle N, \mathcal{A} \rangle$*, where*

- N *is a finite non-empty set of agents (individuals)*
- $\mathcal{A} \subseteq \mathcal{L}$ *is a finite and non-empty set of formulas.* $\mathcal{A}$ *is called the* agenda*.*

If Λ *is a logic over the language* $\mathcal{L}$*, then we denote a* $\mathcal{L}$*-judgment aggregation problem over the logic* Λ *as a* Λ*-judgment aggregation problem.*

Note that the agenda of a judgment aggregation problem in the many-valued framework (see Definition 58) does not need to be closed under negation in contrast to the classical framework (see Definition 2).

Definition 59 (Judgment valuations [Her13, Fer23]). *Let* $\mathcal{J} = \langle N, \mathcal{A} \rangle$ *be a* Λ*-judgment aggregation problem (see Definition 58) over the logic* Λ *with the set of truth values* T*. A* Λ-judgment (valuation) *for* $\mathcal{J}$ *is a* Λ*-valuation* $J : \mathcal{A} \to T$ *(see also Definition 3).*

The set of all Λ*-judgment valuations over agenda* $\mathcal{A}$ *is denoted by* $\mathbf{J}_\Lambda(\mathcal{A})$*.*

Example 51 (Many-valued judgment valuation). *Let* $\langle N, \mathcal{A} \rangle$ *be a* KZ*-judgment aggregation problem with the agenda* $\mathcal{A} = \pm\{v, b, v \wedge b\}$*.*

- $J_1 = \{v \mapsto 0.5, b \mapsto 0.4, (v \wedge b) \mapsto 0.4\}$ *is a* KZ*-judgment since it is a* KZ*-valuation, i.e.,* $J_1(v \wedge b) = \min(J_1(v), J_1(b))$*.*
- $J_2 = \{v \mapsto 0.6, b \mapsto 0.4, (v \wedge b) \mapsto 0.5\}$ *is not a* KZ*-judgment. In particular,* J_2 *is* KZ*-inconsistent since* $J_2(v \wedge b) \neq \min(J_2(v), J_2(b))$*. Thus* J_2 *is not a* KZ*-valuation.*

Definition 60 (Profiles [Her13, Fer23]). *Let* $\mathcal{J} = \langle N, \mathcal{A} \rangle$ *be a* Λ*-judgment aggregation problem (see Definition 2). A* (Λ-judgment) profile $P = \langle J_i \rangle_{i \in N} \in \mathbf{J}_\Lambda(\mathcal{A})^{|N|}$ *is an* $|N|$*-tuple of* Λ*-judgments.*

We denote the set of all (Λ*-judgment) profiles as* $\mathbf{P}_\Lambda(\mathcal{A})$*.*

If the underlying logic Λ is clear from the context, then we omit it.

Note that in both the many-valued (see Definition 60) and the classical framework (see Definition 6), profiles are tuples of individual judgments, i.e., the difference lies in the individual judgments.

6.1.3 Equivalence between classical judgment sets and classical judgment valuations

We show that in classical propositional logic, both judgment sets (see Definition 3) and judgment valuations (see Definition 59) are equivalent. Thus judgment valuations are a true generalization of the classical notion of judgment sets.

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Definition 61 (Classical valuations)**.** *Let $\mathcal{L}_p$ be the set of well-formed propositional formulas, and At be a set of atoms (see Definition 1). A* classical valuation *(see Definition 56 and Example 49) is a function $v : \mathcal{L}_p \to \{0, 1\}$ satisfying:*

- $v(\neg\phi) = 1 - v(\phi)$
- $v(\phi \wedge \psi) = \min(v(\phi), v(\psi))$
- $v(\phi \vee \psi) = \max(v(\phi), v(\psi))$

Definition 62 (Classical judgment valuations)**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ be a $\mathcal{L}_p$-judgment aggregation problem (see Definitions 1 and 2). A* classical judgment (valuation) *for $\mathcal{J}$ is a classical valuation $J : \mathcal{A} \to \{0, 1\}$ (see Definition 61).*

Example 52 (Classical judgment valuation)**.** *Consider the agenda $\mathcal{A} = \pm\{p, q, p \wedge q\}$.*

- *The function $J_1 = \{p \mapsto 1, \neg p \mapsto 0, q \mapsto 0, \neg q \mapsto 1, p \wedge q \mapsto 0, \neg(p \wedge q) \mapsto 1\}$ is a judgment valuation and corresponds with the judgment set $J_1' = \{p, \neg q, \neg(p \wedge q)\}$.*
- *The function $J_2 = \{p \mapsto 1, \neg p \mapsto 0, q \mapsto 1, \neg q \mapsto 0, p \wedge q \mapsto 0, \neg(p \wedge q) \mapsto 1\}$ is not a judgment valuation.*
  - *We have $J_2(p \wedge q) = 0 \neq 1 = \min(J_2(p), J_2(q))$ and thus J_2 is not a classical valuation at all.*
  - *However, J_2 is the characteristic function of the set $J_2' = \{p, q, \neg(p \wedge q)\}$ which is not a judgment set since it is inconsistent (see Definition 3).*

Recall Definition 62 of classical judgment valuations and Definition 3 of judgment sets. We will usually use the more general expression 'judgment' to refer to both judgment sets and judgment valuations. In particular, as the following Lemma 14 shows, in the framework of classical propositional logic, judgment valuations and judgment sets are equivalent and model the same notion of judgment.

Lemma 14. *Let $\mathcal{A} \subseteq \mathcal{L}_p$ be a classical propositional agenda and let $J : \mathcal{A} \to \{0, 1\}$ be the characteristic function of a set $J' \subseteq \mathcal{A}$. The function J is a judgment valuation if, and only if, the set J' is a judgment set.*

Proof. Let $\mathcal{A}$ be a classical propositional agenda and let $J : \mathcal{A} \to \{0, 1\}$ be the characteristic function of the set $J' \subseteq \mathcal{A}$.

$\Longleftarrow$: Assume that J' is a judgment set. We proceed as follows:

$J(\neg\phi) = 1 - J(\phi)$: J' is a judgment set, so either $\phi \in J'$ or $\neg\phi \in J'$ but not both. If $\phi \in J'$, then $\neg\phi \notin J'$. Hence, $J(\phi) = 1 = 1 - J(\neg\phi)$. If $\neg\phi \in J'$, then $\phi \notin J'$. Hence, $J(\phi) = 1 = 1 - J(\neg\phi)$.

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$J(\phi \wedge \psi) = \min(J(\phi), J(\psi))$:

- Assume that $\phi \wedge \psi \in J'$. Since J' is consistent $J' \models \phi$ and $J' \models \psi$. We get that $J(\phi \wedge \psi) = 1 = \min(J(\phi), J(\psi))$.
- Assume that $\phi \wedge \psi \notin J'$. Since J' is consistent $\phi \notin J'$ or $\psi \notin J'$. We get that $J(\phi \wedge \psi) = 0 = \min(J(\phi), J(\psi))$.

$J(\phi \vee \psi) = \max(J(\phi), J(\psi))$:

- Assume that $\phi \vee \psi \in J'$. Since J' is consistent, $\phi \in J'$ or $\psi \in J'$. We get that $J(\phi \vee \psi) = 1 = \max(J(\phi), J(\psi))$.
- Assume that $\phi \vee \psi \notin J'$. Since J' is consistent, both $\phi \notin J'$ and $\psi \notin J'$. We get that $J(\phi \vee \psi) = 0 = \max(J(\phi), J(\psi))$.

$\Longrightarrow$: Assume that J is a judgment valuation. We proceed as follows:

J' **is complete** : Since $J(\neg\phi) = 1 - J(\phi)$ for every $\phi \in \mathcal{A}$ and J is the characteristic function of J', we can follow that either $\phi \in J'$ or $\neg\phi \in J'$ but not both for every $\phi \in \mathcal{A}$.

J' **is consistent** : Assume that J' is inconsistent. Hence, there is a formula $\phi \in \mathcal{A}$ such that $J' \models \phi$ and $J' \models \neg\phi$. We already know that J' is complete. So $\phi \in J'$ and $\neg\phi \in J'$ must be the case. Since, by definition, J is the characteristic function of J', it must hold that $J(\phi) = 1$ and also $J(\neg\phi) = 1$. However, then $J(\phi) = 1 \neq 0 = 1 - J(\neg\phi)$, which contradicts the initial assumption that J is a judgment valuation. Hence, J' must be consistent. □

Note that judgment valuations (see Definition 59) allow generalizing the classical judgment aggregation framework to any many-valued framework by generalizing classical judgment valuations from the values $\{0, 1\}$ to arbitrary sets of values, e.g., to $[0, 1]$ for fuzzy-logic [CFN15].

6.2 Aggregation of judgments

The aggregation of judgments in a many-valued framework works similarly to the classical logic framework (see Section 2.3). However, note that the notions of aggregation functions (see Definition 15) and aggregation conditions (see Section 2.4) have to be slightly adapted.

Definition 63 (Aggregation functions [Her13, Fer23])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a Λ-judgment aggregation problem. An* aggregation function *for $\mathcal{J}$ is a function $F : \mathbf{P}_\Lambda(\mathcal{A}) \to T^{\mathcal{A}}$. Let $P \in \mathbf{P}_\Lambda(\mathcal{A})$ be a profile, then $F(P)$ is called* collective judgment*. $F(P)$ is called* Λ-consistent *(or a collective judgment valuation) if and only if $F(P)$ is a Λ-valuation.*

Example 53 (Propositionwise majority rule for classical judgment valuations)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a classical aggregation problem. The propositionwise majority rule (see*

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Definition 17) for profiles of classical judgment valuations is defined as

$$F_{maj}(P) = \left\{ \phi \in \mathcal{A} \mid \sum_{i \in N} J_i(\phi) \geq \left\lceil \frac{|N|+1}{2} \right\rceil \right\} \tag{6.1}$$

Note that the propositionwise majority rule as defined in Example 53 is not directly applicable for logics with more than two truth values since the partial acceptance of a formula should not count as (full) acceptance. A more natural propositionwise aggregation function for arbitrary sets of truth values is the following propositionwise average rule F_{av}.

Definition 64 ((Propositionwise) average rule [BB12, Fer23])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a Λ-judgment aggregation problem. For every profile formula $\phi \in \mathcal{A}$ and each profile $P \in \mathbf{P}_\Lambda(\mathcal{A})$ the* (propositionwise) average rule *is defined as*

$$F_{av}(P)(\phi) = \frac{\sum_{i \in N} J_i(\phi)}{|N|} \tag{6.2}$$

The average rule F_{av} is systematic (see Section 2.4.3) since the collective judgment on a formula $\phi \in \mathcal{A}$ depends only on the individual judgments on ϕ and the method for aggregating the individual judgments is the same for any formula in the agenda $\mathcal{A}$. Furthermore, the average rule is anonymous (see Section 2.4.3) since the order of the individual judgments in the given profile is irrelevant to the collective judgment.

Example 54 (Average rule)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be KZ-judgment aggregation problem with the agenda $\mathcal{A} = \pm\{v, b, v \wedge b\}$. Moreover, let $P = \langle J_1, J_2, J_2 \rangle$ be a profile with the following individual judgments*

- $J_1 = \{v \mapsto 0.5, b \mapsto 0.3, (v \wedge b) \mapsto 0.3\}$
- $J_2 = \{v \mapsto 0.7, b \mapsto 1, (v \wedge b) \mapsto 0.7\}$
- $J_3 = \{v \mapsto 1, b \mapsto 1, (v \wedge b) \mapsto 1\}$

This leads to the following collective judgment

$$F(P) = \left\{ v \mapsto \frac{2.2}{3}, b \mapsto \frac{2.3}{3}, (v \wedge b) \mapsto \frac{2}{3} \right\}$$

$F(P)$ is not KZ-inconsistent since $F(P)(v \wedge b) < \min(F(P)(v), F(P)(v))$.

Note that the average rule (see Definition 64) can also be used to aggregate classical judgment profiles. However, the aggregation will result in a many-valued collective judgment.

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P	v	b	$v \wedge b$
Judge 1	1	1	1
Judge 2	0	0	0
Judge 3	0	1	0
$F_{av}(P)$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

Table 6.1: Example of the average rule applied to classical profiles

Example 55 (Average rule with classical profiles)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ with the agenda $\mathcal{A} = \pm\{v, b, v \wedge b\}$ and $|N| = 3$. Moreover, let P be the classical profile given in Table 6.1.*

The collective judgment $F_{av}(P)$ for a formula $\phi \in \mathcal{A}$ given in Table 6.1 equals the fraction of individuals that accept ϕ. Furthermore, note that using the logic KZ *for the collective judgment, in this case, leads to a* KZ*-consistent collective judgment $F_{av}(P)$ since $F_{av}(P)(v \wedge b) = \min(F_{av}(P)(v), F_{av}(P)(b))$.*

Moreover, the average rule F_{av} enables a redefinition of the propositionwise majority rule to an accumulated majority rule, which generalizes the classical propositionwise majority rule.

Definition 65 (Accumulated majority rule [Fer23])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a Λ-judgment aggregation problem. For every profile formula $\phi \in \mathcal{A}$ and each profile $P \in \mathbf{P}_\Lambda(\mathcal{A})$ the* accumulated majority rule *is defined as*

$$F_{amaj}(P)(\phi) = \begin{cases} 1 & \text{if } F_{av}(P)(\phi) > 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{6.3}$$

Note that the accumulated majority rule of Definition 65 does not accept a formula if it is (fully) accepted by a majority of the individuals. Rather it accepts a formula if the average acceptance (or judgment) value is bigger than 0.5.

6.3 Aggregation conditions

In the many-valued framework (see Section 6.1), aggregation conditions have the same meaning as their classical counterparts (see Section 2.4). However, we no longer have judgment sets (see Definition 3). Thus we have to adapt the formal definitions of some aggregation conditions for judgment valuations.

Definition 66 (Collective rationality [Her13])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a Λ-judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* collectively rational *if, and only if, for any admissible profile $P \in \mathbf{P}_\Lambda(\mathcal{A})$, $F(P)$ is Λ-consistent, i.e., it is a Λ-valuation (see Definition 56).*

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Note that collective rationality in the many-valued framework under a logic Λ is equivalent to Λ-consistency (see Definition 66) in contrast to collective rationality in the classical framework (see Chapter 2) which is equivalent to (classical) consistency and completeness (see Definition 21). In particular, in a logic Λ with negation $\neg$, a set of formulas Γ can only be Λ-consistent if it implicitly satisfies completeness since it satisfies the truth function of $\neg$ (see Lemma 14 and Example 52).

Definition 67 (Anonymity with permutations [LP02, PvH06])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ with $N = \{1, \ldots, n\}$ be a judgment aggregation problem and F an aggregation function for $\mathcal{J}$. F is* anonymous *if, and only if, for any permutation $\sigma : N \to N$, every profile $P = \langle J_i, \ldots, J_n\rangle \in \mathbf{P}(\mathcal{A})$, and every formula $\phi \in \mathcal{A}$*

$$F(P)(\phi) = F(J_{\sigma(1)}(\phi), \ldots, J_{\sigma(n)}(\phi)).$$

Note that the Definition 67 is equivalent to anonymity in the classical framework (see Definition 27). Both the propositionwise majority rule (see Example 53) and the average rule (see Definition 64) satisfy anonymity. For example, recall Examples 54 and 55. In both cases, the order of the individual judgments of the profile is irrelevant since the judgment values are anonymously added up to a collective value. For a non-anonymous aggregation function, see Example 21.

Definition 68 (Independence as a function [PvH06])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ with $N = \{1, \ldots, n\}$ be a judgment aggregation problem and F and aggregation function for $\mathcal{J}$. F is* independent *if, and only if, for every formula $\phi \in \mathcal{A}$ there is some function f_ϕ such that*

$$\forall P = \langle J_1, \ldots, J_n\rangle \in \mathbf{P}(\mathcal{A})\ \forall \phi \in \mathcal{A} : F(P)(\phi) = f_\phi(J_1(\phi), \ldots, J_n(\phi)).$$

Note that usually, an independence condition similar to the classical independence (see Definition 28), where judgment valuations exchange judgment sets, is used. However, [PvH06] showed that Definition 68 follows from the 'classical' notion.

Definition 69 (Systematicity as a function [LP02, PvH06, Her13])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ with $N = \{1, \ldots, n\}$ be a judgment aggregation problem and F and aggregation function for $\mathcal{J}$. F is* systematic *if, and only if, there is some function f such that*

$$\forall P = \langle J_1, \ldots, J_n\rangle \in \mathbf{P}(\mathcal{A})\ \forall \phi \in \mathcal{A} : F(P)(\phi) = f(J_1(\phi), \ldots, J_n(\phi)).$$

Recall the classical definition of systematicity (see Definition 30) and Theorem 3. Systematicity means that the collective judgment of a formula depends only on the individual judgments of this formula (independence). Hence, the collective judgment must be a function depending only on the individual judgments of the profile. Moreover, the condition or method for accepting a collective judgment is the same for all formulas (neutrality). Thus the function depending only on the individual judgments must be the same for all formulas.

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Note that the average rule (see Definition 64) is systematic since the aggregation function depends only on the individual judgments of the input profile and is the same for all formulas.

6.4 Impossibility results

Impossibility results have been mainly studied in the classical framework of judgment aggregation (see Chapter 2). However, as [PvH06, vH07] have shown, allowing arbitrary degrees of acceptance does not directly reverse impossibility results. Thus in this section, we review some impossibility results in finite many-valued logic.

The many-valued logic used by [PvH06] allows finitely many values of acceptance, i.e., the set of truth-values is $T = \{0, \ldots, t-1\}$ with $|T| \geq 2$. For the remainder of this section, we will refer to the logic introduced in [PvH06] as PvH-logic to prevent ambiguities.

Definition 70 (PvH-logic [PvH06])**.** *The set of all PvH-formulas $\mathcal{L}$ is built up as usual from atomic formulas At using the connectives $\wedge$, and $\neg$ (see Definition 55). The semantics of PvH over the truth-values T with $t = |T| \geq 2$ is given by the following truth functions that extend any given valuation v, i.e. any, assignment $At \rightarrow T$, from atomic formulas to arbitrary formulas.*

$$v(\neg\phi) = v(\phi) + 1(\text{mod } t)$$
$$v(\phi \wedge \psi) = \min(v(\phi), v(\psi))$$

Note PvH-logic over the truth-values $T = \{0, 1\}$ corresponds to classical propositional logic (see Example 49), i.e., a PvH-valuation over the truth-values $T = \{0, 1\}$ is equivalent to a classical valuation (see Definition 61). However, for $|T| > 2$, the semantics of negation is rather unintuitive, as the following example shows.

Example 56 (PvH-logic)**.** *Consider PvH-logic over the set $T = 0, 1, 2$ of truth values. Moreover, let $At = \{p, q\}$.*

- *If $v(p) = 0$, then $v(\neg p) = 1$.*
- *If $v(p) = 1$, then $v(\neg p) = 2$.*
- *If $v(p) = 2$, then $v(\neg p) = 0$.*
- *If $v(p) = 0$ and $v(q) = 1$, then $v(p \wedge q) = 0$.*

Example 56 shows that the negation of a formula maps its truth-value to the 'next bigger' truth-value w.r.t. mod t. The semantics of conjunction is as usual, e.g., like for KZ-logic (see Definition 57).

Definition 71 (Atomically closed agendas [PvH06])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda. $\mathcal{A}$ is* atomically closed *if and only if it is closed under atoms (see Definition 9) and if two literals $l, l' \in \mathcal{A}$, then $l \wedge l' \in \mathcal{A}$.*

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Note that atomically closed agendas are thus an extension of agendas closed under atoms by additionally closing them under the conjunction of literals.

Theorem 19 ([PvH06])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a PvH-judgment aggregation problem where $\mathcal{A}$ is atomically closed, and there are at least two atoms in the agenda, i.e., $|At(\mathcal{A})| \geq 2$ (see Definition 9). Moreover, let F be an aggregation function for $\mathcal{J}$. F satisfies collective rationality, universal domain, responsiveness, and independence if and only if it is dictatorial.*

Theorem 19 shows that even in a many-valued framework, judgment aggregation with aggregation functions that satisfy desirable properties (see Section 3.1.1) lead to only degenerate cases. Moreover, the aggregation conditions in Theorem 19, i.e., responsiveness and independence, are much weaker than in Theorem 5 of [LP02]. However, note that the agenda condition of atomic closure (see Definition 71) is more restrictive than the condition in Theorem 5. Furthermore, [PvH06] have shown that in the case of $|T| = 2$, the result holds if we weaken responsiveness to weak responsiveness (see Definition 31).

6.5 Further and related work

There has also been some work on using the distance-based approach (see Section 4.2.3) for aggregating many-valued judgments since distance-based methods can be easily implemented for many-valued or fuzzy judgments. For example, [SJ11] introduced a three-valued approach for modeling the acceptance, rejection, and abstention of a proposition. For aggregating such three-valued judgments, they proposed a distance-based approach with weights that is based on distance minimization similar to Definition 48 of [Pig06]. Further related work has been conducted by [DP18] that considers fuzzy collective preference relations. Moreover, [IS18, IS19] use the methods of (classical) judgment aggregation for pooling probabilistic opinions over interrelated logical propositions.

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CHAPTER 7

Characterizations for many-valued judgment aggregation

The search for impossibility theorems dominates the majority of literature about judgment aggregation (see Chapter 3). However, there are also possibility results (see Chapter 4), e.g., by using the sequential priority approach [Lis04, DL07b] (see Section 4.2.2), like the premise- and conclusion-based approach [DM10] (see Section 4.2.1), or by restricting the domain of the aggregation functions, e.g., unidimensional alignment [Lis03] (see Section 4.3).

In this chapter, we introduce characterizations for many-valued judgment aggregation by showing an impossibility result and by showing some consistency criteria building on results of [Fer23]. All results in this chapter apply to Kleene-Zadeh logic KZ (see Definition 57).

Section 7.1 reviews the possibility result of [Fer23]. In Section 7.2, we generalize the consistency criteria introduced by [Fer23]. Section 7.3 introduces and examines a new possibility characterization for average aggregation. Last but not least, in Section 7.4, we prove an impossibility theorem for fuzzy-logic-based judgment aggregation in the logic KZ.

7.1 Some consistency criteria for many-valued judgment aggregation

The following section reviews the results of [Fer23][Fer23] shows that, if one restricts the universal domain (see Definition 18) to profiles that satisfy the condition of *order compatibility*, then the average rule (see Definition 64) yields KZ-consistent collective judgments. This condition applies to both, profiles of classical judgments and profiles of KZ-judgments.

Definition 72 (Closure under subformulas [Fer23])**.** *Let $\mathcal{A}$ be an agenda. $\mathcal{A}$ is* closed under subformulas *if and only if $\phi \in \mathcal{A}$ implies that every subformula of ϕ is also in $\mathcal{A}$.*

Example 57 (Agenda closed under subformulas)**.** *Consider the agenda*

$$\mathcal{A} = \pm\{v, b, g, g \leftrightarrow v \wedge b\}$$

of Example 1. The closure of $\mathcal{A}$ under subformulas is

$$\mathcal{A}^* = \pm\{v, b, g, g \leftrightarrow v \wedge b, v \wedge b\}.$$

Definition 73 (Order compatible profiles [Fer23])**.** *Let $\mathcal{J} = \langle N, \mathcal{A}\rangle$ be a KZ-judgment aggregation problem where the agenda $\mathcal{A}$ is closed under subformulas (see Definition 9). Moreover, let $P = \langle J_i\rangle_{i\in N} \in \mathbf{P}(\mathcal{A})$ a profile for $\mathcal{J}$. P is* order compatible *if and only if there exists a enumeration $\langle p_1, \ldots, p_n\rangle$ of all atoms occurring in the agenda $\mathcal{A}$ such that*

$$\forall i \in N : J_i(p_1) \leq \cdots \leq J_i(p_n)$$

Example 58 (Order compatible profile)**.** *Let $\mathcal{J} = \langle\{1, 2, 3\}, \mathcal{A}\rangle$ be a KZ-judgment aggregation problem with the agenda $\mathcal{A} = \pm\{v, b, v \wedge b\}$ which is closed under subformulas (see Definition 72). Consider the profile $P = \langle J_1, J_2, J_3\rangle$ with the following individual judgments (see Example 54)*

- $J_1 = \{v \mapsto 0.5, b \mapsto 0.3, (v \wedge b) \mapsto 0.3\}$
- $J_2 = \{v \mapsto 0.7, b \mapsto 1, (v \wedge b) \mapsto 0.7\}$
- $J_3 = \{v \mapsto 1, b \mapsto 1, (v \wedge b) \mapsto 1\}$

Then P is not order compatible since $J_1(v) > J_2(b)$ and $J_2(v) < J_2(b)$.

Furthermore, let $P' = \langle J'_1, J'_2, J'_3\rangle$ be the classical profile with the following individual judgments (see Example 55)

- $J'_1 = \{v \mapsto 1, b \mapsto 1, (v \wedge b) \mapsto 1\}$
- $J'_2 = \{v \mapsto 0, b \mapsto 0, (v \wedge b) \mapsto 0\}$
- $J'_3 = \{v \mapsto 0, b \mapsto 1, (v \wedge b) \mapsto 0\}$

Then P' satisfies order compatibility.

Additionally, note that for the profile P (see Example 54), the average rule yields an KZ-inconsistent collective judgment, whereas for the profile P' (see Example 55) the average rule yields a KZ-consistent collective judgment.

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Recall Definition 73 and Example 58. A profile P is order compatible if all individual judgments in the profile have the same relative acceptance degrees for all atomic propositions [Fer23].

Definition 74 (Positive agendas)**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda. $\mathcal{A}$ is* positive *if and only if every formula $\phi \in \mathcal{A}$ is negation-free.*

Note that a positive agenda $\mathcal{A}$ is not closed under negation, and thus $\mathcal{A}$ is not an agenda in the classical sense of Definition 2. However, it is an agenda in the many-valued framework since it is a non-empty and finite subset of the considered language (see Definition 58).

Lemma 15 (Extended order compatibility [Fer23])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem where $\mathcal{A}$ is positive (see Definition 74). Moreover, let $P = \langle J_i \rangle_{i \in N}$ be an order compatible KZ-judgment profile of $\mathcal{J}$. Then by order compatibility (see Definition 73), the existing enumeration of atoms can be extended to all formulas $\phi \in \mathcal{A}$. In particular, there exists an enumeration $\langle \phi_1, \dots, \phi_n \rangle$ of all formulas of $\mathcal{A}$ such that*

$$\forall i \in N : J_i(\phi_1) \leq \cdots \leq J_i(\phi_n).$$

The intuitive argument for Lemma 15 is that due to the definition of the logic KZ (see Definition 57), the acceptance value of the conjunction $J(\phi \wedge \psi)$ of two formulas $\phi, \psi \in \mathcal{A}$ is the minimum of the acceptance values of both formulas. Hence, it follows that $J(\phi \wedge \psi) = J(\phi)$ or $J(\phi \wedge \psi) = J(\psi)$. The argument for disjunction is accordingly.

Definition 75 (Internally positive agendas [Fer23])**.** *Let $\mathcal{A} \subseteq \mathcal{L}$ be an agenda. $\mathcal{A}$ is* internally positive *if and only if every formula $\phi \in \mathcal{A}$ either is negation-free or else is of the form $\neg\psi$ where ψ is negation-free.*

Note that if $\mathcal{A}$ is internally positive, then there is a positive agenda $\mathcal{A}'$ (see Definition 74) such that $\mathcal{A} = \{\neg\phi \mid \phi \in \mathcal{A}'\} \cup \mathcal{A}'$, i.e., $\mathcal{A}$ is the closure under negation of some positive agenda.

Theorem 20 (Possibility of average aggregation [Fer23])**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem, where the agenda $\mathcal{A}$ is internally positive and closed under subformulas. If P is an order compatible KZ-judgment profile for $\mathcal{J}$, then the average rule yields a KZ-consistent collective judgment $F_{av}(P)$.*

Theorem 20 states that if we restrict the domain of admissible profiles to order compatible profiles, then the (propositionwise) average rule (see Definition 64) yields KZ-consistent collective aggregation. Moreover, since the average rule is anonymous, it follows that the aggregation is also non-dictatorial.

Remark 25. *Note that the restriction that agendas are closed under subformulas can be omitted from Theorem 20. In particular, by the definition of KZ-judgment profiles (see Definition 60), the individual judgments are KZ-valuations. Thus the judgments for subformulas are given implicitly by the individual KZ-judgment valuations [Fer23].*

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7.2 Some consistency criteria for many-valued judgment aggregation - A generalization

In the following, we built on the result of [Fer23], which has been reviewed in Section 7.1. In particular, we generalize the possibility Theorem 20 to aggregation functions satisfying a special type of systematicity, which we call *linear systematicity*.

Theorem 21. *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem, where the agenda $\mathcal{A}$ is closed and positive. And let F be a systematic (see Definition 69) aggregation function of $\mathcal{J}$. Then F yields a* KZ*-consistent collective judgment $F(P)$ for every order compatible (see Definition 73)* KZ*-judgment profile $P \in \mathbf{P}(\mathcal{A})$.*

Proof. If all formulas in $\mathcal{A}$ are atomic, then the claim holds trivially since every assignment of values in $[0, 1]$ to propositional variables constitutes a valuation over any logic. We proceed inductively and consider for this theorem only the case where $\mathcal{A}$ contains only negation-free formulas.

$\phi \wedge \psi$: Let P be an order compatible KZ-judgment profile over the agenda $\mathcal{A}$. By simply ignoring the values for $\phi \wedge \psi$, P induces an order compatible judgment profile $P' = \langle J'_i \rangle_{i \in N}$ over $\mathcal{A}' = \mathcal{A} - \{\phi \wedge \psi\}$. The induction hypothesis states that $F(P')$ amounts to a KZ-consistent valuation of the propositions in $\mathcal{A}'$. In particular, by systematicity, we have

$$F(P')(\phi) = f(J'_1(\phi), \ldots, J'_n(\phi)) \quad \text{and}$$
$$F(P')(\psi) = f(J'_1(\psi), \ldots, J'_n(\psi)).$$

By order compatibility we know that either (1) $J_i(\phi) \leq J_i(\psi)$ for every $i \in N$ or (2) $J_i(\psi) \leq J_i(\phi)$ for every $i \in N$. Hence, we have the following two cases:

(1) $J_i(\phi \wedge \psi) = J_i(\phi)$ for all $i \in N$, or

(2) $J_i(\phi \wedge \psi) = J_i(\psi)$ for all $i \in N$

P and P' are identical for all formulas in $\mathcal{A}'$, hence in particular $J_i(\phi) = J'_i(\phi)$ and $J_i(\psi) = J'_i(\psi)$. Thus it follows that if (1), then

$$\begin{aligned} F(P)(\phi \wedge \psi) &= f(J_1(\phi \wedge \psi), \ldots, J_n(\phi \wedge \psi)) \\ &= f(J_1(\phi), \ldots, J_n(\phi)) \\ &= F(P)(\phi) \end{aligned}$$

Moreover, if (2), then

$$\begin{aligned} F(P)(\phi \wedge \psi) &= f(J_1(\phi \wedge \psi), \ldots, J_n(\phi \wedge \psi)) \\ &= f(J_1(\psi), \ldots, J_n(\psi)) \\ &= F(P)(\psi) \end{aligned}$$

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Hence, overall we get that

$$\begin{aligned} F(P)(\phi \wedge \psi) &= f(J_1(\phi \wedge \psi), \dots, J_n(\phi \wedge \psi)) \\ &= \min(f(J_1(\phi), \dots, J_n(\phi)), f(J_1(\psi), \dots, J_n(\psi))) \\ &= \min(F(P)(\phi), F(P)(\psi)) \end{aligned}$$

$\phi \vee \psi$**:** The argument is analogous to that for $\phi \wedge \psi$. □

Note that Theorem 21, does not generalize the result of [Fer23] completely, since systematicity is not enough to consistently negate internally positive formulas on a collective level. However, consistently negating internally positive formulas is enabled by a subset of systematic aggregation functions, which we introduce in the following.

Definition 76 (Linear systematic aggregation functions)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ with $N = \{1, \dots, n\}$ be a judgment aggregation problem and F and aggregation function for $\mathcal{J}$. F is* linear systematic *if, and only if, there is some function f with*

$$f(1 - x_1, \dots, 1 - x_n) = 1 - f(x_1, \dots, x_n) \quad \textit{for any } \langle x_1, \dots, x_n \rangle$$

such that

$$\forall P = \langle J_1, \dots, J_n \rangle \in \mathbf{P}(\mathcal{A}) \; \forall \phi \in \mathcal{A} : F(P)(\phi) = f(J_1(\phi), \dots, J_n(\phi)).$$

Remark 26. *Regarding Definition 76 of linear systematicity, we observe that every linear systematic aggregation function is also systematic, as the decision function is only restricted to a subset of systematic functions.*

Corollary 1. *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem, where the agenda $\mathcal{A}$ is closed under subformulas (see Definition 72) and internally positive (see Definition 75). And let F be a linear systematic aggregation function of $\mathcal{J}$. Then F yields a* KZ*-consistent collective judgment $F(P)$ for every order compatible* KZ*-judgment profile $P \in \mathbf{P}(\mathcal{A})$.*

Proof. The inductive part for closed and positive agendas $\mathcal{A}$ has already been shown in Theorem 21.

Hence, we only have to show that negation works for internally positive formulas in the agenda. In particular, for negated formulas in $\phi \in \mathcal{A}$ by Definition 76 of linear systematicity, we have that there is a decision function f such that

$$\begin{aligned} F(P)(\neg\phi) &= f(1 - J_1(\phi), \dots, 1 - J_n(\phi)) \\ &= 1 - f(J_1(\phi), \dots, J_n(\phi)) = 1 - F(P)(\phi) \end{aligned}$$

This means that the KZ-valuation provided by F remains KZ-consistent if extended from negation-free formulas to negations of such formulas if F is linear systematic. □

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Note that at first appearance, it is not obvious that Corollary 1 allows more aggregation function than the average rule F_{av} (see Definition 64) which satisfies the condition of linear systematicity. However, observe that the constant aggregation function $F_{0.5}(P)(\phi) = 0.5$ for each $P \in \mathbf{P}(\mathcal{A})$ and every formula $\phi \in \mathcal{A}$ satisfies both anonymity and linear systematicity. Another example satisfying both anonymity and linear systematicity is the accumulated majority rule (see Definition 65).

Example 59 (Linear systematicity of the accumulated majority rule)**.** *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ be a judgment aggregation problem, where the agenda $\mathcal{A}$ is closed under subformulas, and $|N|$ is odd. Let $P \in \mathbf{P}(\mathcal{A})$ be any KZ-judgment profile and $\phi \in \mathcal{A}$ any formula. Assume that $F_{amaj}(P)(\phi) = 1$. By Definition 65 of the accumulated majority rule, it holds that*

$$F_{av}(P)(\phi) = \frac{\sum_{i \in N} J_i(\phi)}{|N|} > 0.5.$$

Moreover, the average rule is linearly systematic. Thus, it follows that

$$F_{av}(P)(\neg\phi) = 1 - F_{av}(P)(\phi) < 0.5.$$

Since $F_{av}(P)(\neg\phi) < 0.5$ it follows that $F_{amaj}(P)(\neg\phi) = 0$. Hence, it holds that

$$F_{amaj}(P)(\neg\phi) = 0 = 1 - F_{amaj}(P)(\phi).$$

Thus (at least for odd $|N|$), the accumulated majority rule is linearly systematic.

7.3 Ordered profiles

Consider the classical profile given in Table 7.1. By using the average rule F_{av} for aggregating the individual judgments, we get, as a result, a KZ-consistent collective judgment $F(P)$, i.e., $F(P)(p \wedge q) = \min\{F(P)(p), F(P)(q)\}$.

P	p	q	$p \wedge q$
J_1	1	1	1
J_2	0	1	0
J_3	0	0	0
$F_{av}(P)$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$

Table 7.1: Example of (totally) ordered profile

Furthermore, note that J_1 is the only individual judgment such that $J_1(p \wedge q) = 1$, and by the consistency of individual judgments (see Definition 59), this can only be the case if both conjuncts have also the maximal acceptance value. Hence, if we consider an arbitrary conjunction $c = a_1 \wedge ... \wedge a_m$ of conjuncts $a_1, \ldots, a_m$, then we observe that c is already rejected if at least one of its conjuncts $a_1, \ldots, a_m$ is rejected. Moreover, c is accepted only if all of its conjuncts $a_1, \ldots, a_m$ are also accepted. Thus in some sense, an individual judgment J_i accepts a conjunct c, i.e., $J_i(c) = 1$, only if the individual

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judgment J_i restricted to $c, a_1, \ldots, a_m$ is maximal w.r.t. all possible judgments about $c, a_1, \ldots, a_m$, i.e., every formula of $c, a_1, \ldots, a_m$ is judged as being accepted. In particular, as Table 7.1 shows, the number of judgments that accept a conjunction would then be the number of maximal judgments w.r.t. to all possible judgments restricted to this conjunction and its conjuncts. Moreover, as Table 7.1 shows, then the average rule will yield a KZ-consistent collective judgment. Furthermore, note that the inverse holds for disjunction, i.e., a disjunction is only rejected if the judgment of the disjunction and all its disjuncts is a minimal element w.r.t. to all possible judgments about it.

7.3.1 Partial order relation on judgments

As the above insight shows, relating judgments among each other and classifying profiles according to this relation allows enabling KZ-consistent collective aggregation.

Definition 77 (Partial order relation on judgments)**.** *Let $\langle N, \mathcal{A}\rangle$ be a judgment aggregation problem where $\mathcal{A}$ is internally positive (see Definition 75), i.e., $\mathcal{A}^+$ contains only negation-free formulas. We define the* partial order relation *over the (admissible) judgments $\leq \subseteq \mathbf{J}(\mathcal{A}) \times \mathbf{J}(\mathcal{A})$ as*

$$\leq := \{(J_i, J_j) \mid \forall \phi \in \mathcal{A}^+ : J_i(\phi) \leq J_j(\phi),\ \text{for } J_i, J_j \in \mathbf{J}(\mathcal{A})\} \tag{7.1}$$

We denote the resulting partial ordered set by $\langle \mathbf{J}(\mathcal{A}), \leq\rangle$.

We will usually use infix notation, i.e., $J_i \leq J_j$ instead of $(J_i, J_j) \in \leq$. In particular, for two judgments $J_i, J_j \in \mathbf{J}(\mathcal{A})$ the following holds:

$$J_i \leq J_j \iff \forall \phi \in \mathcal{A}^+ : J_i(\phi) \leq J_j(\phi) \tag{7.2}$$

Remark 27. *Considering the partial order relation (see Definition 77) if we cope with classical judgment sets, then*

$$J_i \leq J_j \iff \forall \phi \in \mathcal{A}^+ : \phi \in J_i \implies \phi \in J_j$$

Example 60 (Partial order relation on judgments)**.** *Consider the following issue set $\mathcal{A}^+ = \{p, p \to q\}$ and the following judgments:*

- $J_1 = \{p \mapsto 0, (p \to q) \mapsto 0\}$
- $J_2 = \{p \mapsto 0, (p \to q) \mapsto 1\}$
- $J_3 = \{p \mapsto 1, (p \to q) \mapsto 0\}$
- $J_4 = \{p \mapsto 1, (p \to q) \mapsto 1\}$

Observe, that $J_1 \leq J_2$ and $J_2 \leq J_4$. Hence, we also have that $J_1 \leq J_4$ by transitivity. On the other hand, neither $J_2 \leq J_3$ nor $J_3 \leq J_2$ since the relation does not hold in either direction, i.e., J_2 and J_3 are incomparable. For a visual representation consider also the corresponding Hasse diagram in Figure 7.1.

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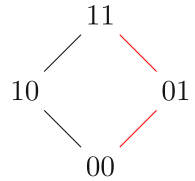

Figure 7.1: Hasse diagram of partial order set $\langle \mathbf{J}(\mathcal{A}^+), \leq \rangle$ with 2 issues in $\mathcal{A}^+$.

7.3.2 Totally ordered profiles

Definition 78 (Totally ordered profiles)**.** *Let $\langle N, \mathcal{A} \rangle$ be a judgment aggregation problem where $\mathcal{A}$ is internally positive. A profile $P = \langle J_i \rangle_{i \in N}$ is called* totally ordered *if for all $i, j \in N$*

$$J_i \leq J_j \quad \textit{or} \quad J_j \leq J_i$$

with respect to the partial order on judgments $(\mathbf{J}(\mathcal{A}), \leq)$ (see Definition 77).

Remark 28. *Note that $\langle P, \leq \rangle$ is a totally ordered set for any totally ordered profile P (see Definition 78). In particular, the definition of a totally ordered profile ensures that only judgments are selected that build a total order, which corresponds to a single path from the maximum element to the minimum element in the corresponding Hasse diagram (see, e.g., the red path in Figure 7.1).*

Theorem 22. *Let $\langle N, \mathcal{A} \rangle$ be a judgment aggregation problem, where $\mathcal{A}$ is closed with respect to subformulas and internally positive. Then the average rule F_{av} yields a KZ-consistent collective judgment $F_{av}(P)$ for every totally ordered classical judgment profile P.*

Proof. If all formulas in A are atomic, then the claim holds trivially since every assignment of values in $[0, 1]$ to propositional variables constitutes a valuation over any logic. We proceed inductively for composite formulas.

$\phi \wedge \psi$: Let P be a totally ordered classical judgment profile over the agenda $\mathcal{A}$. In particular, we have:

$$F_{av}(P)(\phi) = \frac{|P_\phi|}{|N|} \quad \text{and} \quad F_{av}(P)(\psi) = \frac{|P_\psi|}{|N|}$$

By assumption for all $i, j \in N$, we have that $J_i \leq J_j$ or $J_j \leq J_i$. Let $k \in N$ be the individual such that J_k is a minimal judgment with respect to the total ordered profile such that $J_k(\phi) = 1$. And let $l \in N$ be the individual such that J_l is also a minimal judgment with respect to the total ordered profile P such that $J_l(\psi) = 1$. Observe that $P_\phi = \{i \in N \mid J_k \leq J_i\}$ and $P_\psi = \{i \in N \mid J_l \leq J_i\}$. In particular, we have two cases (since the agenda is internally positive):

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$J_k \leq J_l$: Then $P_\psi \subseteq P_\phi$, since all of judgments bigger than J_l are also bigger than J_k, and if $J_k = J_l$, then $P_\psi = P_\phi$.

$J_l < J_k$: Then $P_\phi \subset P_\psi$, since all judgments that are bigger than J_k are also bigger than J_l.

Hence, $P_{\phi\wedge\psi} = P_\phi \cap P_\psi$ consists of exactly those individuals where both ϕ and ψ are accepted and $|P_{\phi\wedge\psi}| = \min(|P_\phi|, |P_\psi|)$, which is only the case since the agenda $\mathcal{A}$ is internally positive.

Thus we get that:

$$F_{av}(P)(\phi \wedge \psi) = \frac{|P_{\phi\wedge\psi}|}{|N|} = \frac{\min(|P_\phi|, |P_\psi|)}{|N|} = \min\Big(F_{av}(P)(\phi), F_{av}(P)(\psi)\Big)$$

$\phi \vee \psi$: The argument is analogous to the argument of $\phi \wedge \psi$.

$\neg\phi$: We have that:

$$F_{av}(P)(\phi) = \frac{|P_\phi|}{|N|}$$

Let $k \in N$ be the individual such that J_k is the smallest judgment with respect to the total ordered profile such that $J_k(\phi) = 1$. By the definition of the partial order relation, $P_\phi = \{i \in N \mid J_k \leq J_i\}$ is the set of individuals accepting ϕ. Moreover, all judgments rejecting ϕ are exactly the judgments J_i such that $J_i < J_k$. Hence, all individuals in $P_{\neg\phi} = \{i \in N \mid J_i < J_k\}$ reject ϕ. By total orderedness of the profile, we get that $P_{\neg\phi} = N \setminus P_\phi$. Thus, overall we get as result:

$$F_{av}(P)(\neg\phi) = \frac{|P_{\neg\phi}|}{|N|} = \frac{|N| - |P_\phi|}{|N|} = 1 - \frac{|P_\phi|}{|N|} = 1 - F_{av}(P)(\phi)$$

□

7.3.3 Totally ordered vs. order compatible profiles

Totally ordered profiles and order compatible profiles have some relation in the case of classical judgment profiles, i.e., they are equivalent restrictions on the input profiles. However, for general judgment profiles, they are incomparable.

Lemma 16. *Let $\mathcal{A}$ be an internally positive agenda. Then a classical judgment profile $P = \langle J_i \rangle_{i \in N}$ is called* order compatible *(see Definition 73) if there exists an ordering $\langle \phi_1, \ldots, \phi_m \rangle$ of all formulas in $\mathcal{A}^+$ such that for all $i \in N$:*

$$J_i(p_1) \leq \cdots \leq J_i(p_m)$$

In particular, for classical judgment profiles, this is equivalent to for all $i \in N$:

$$\phi_1 \in J_i \implies \cdots \implies \phi_m \in J_i$$

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Proof. The proof directly follows from the Definition 75 of internally positive agendas, Lemma 15 and since the judgment profile contains by assumption only classical individual judgments. □

Lemma 17. *Let $P = \langle J_1 \rangle_{i \in N}$ be a classical judgment profile. If P is totally ordered, then P is order compatible.*

Proof. Let $\langle N, \mathcal{A} \rangle$ be a judgment aggregation problem. Assume that P is a classical judgment profile, which is totally ordered, but not order compatible. Hence, there exist two individuals $i, j \in N$ and formulas $\phi, \psi \in \mathcal{A}^+$ such that w.l.o.g.

$$\phi \notin J_i, \text{ but } \psi \in J_i \quad \text{and} \quad \psi \notin J_j, \text{ but } \phi \in J_j$$

Since P is totally ordered by assumption, i.e., for all individuals $i, j \in N$, $J_i \leq J_j$ or $J_j \leq J_i$. Hence, we have to consider the following two cases:

$J_i \leq J_j$: By definition of the partial order relation $\leq$, for all formulas $\phi \in \mathcal{A}^+$

$$\phi \in J_i \implies \phi \in J_j$$

By assumption $\psi \in J_i$, hence since P is totally ordered, we can conclude that $\psi \in J_j$. However, this contradicts the assumption that $\psi \notin J_j$

$J_j \leq J_i$: The argument is analogous to $J_i \leq J_j$. By assumption $\phi \in J_j$ and since $J_j \leq J_i$ it follows that $\phi \in J_i$. Again, this contradicts the assumption that $\phi \notin J_i$.

Thus P must be order compatible. □

Lemma 18. *Let $P = \langle J_1 \rangle_{i \in N}$ be a classical judgment profile. If P is order compatible, then P is totally ordered.*

Proof. The proof is similar to the proof of Lemma 17.

Let $\langle N, \mathcal{A} \rangle$ be a judgment aggregation problem. Assume that P is a classical judgment profile, which is order compatible, but not totally ordered. Hence, there exist two individuals $i, j \in N$ such that $J_i \not\leq J_j$ and $J_j \not\leq J_i$. In particular, there is a formula $\phi \in \mathcal{A}^+$ such that

$$\phi \in J_i, \text{ but } \phi \notin J_j$$

and a formula $\psi \in \mathcal{A}^+$ such that

$$\psi \in J_j, \text{ but } \psi \notin J_i$$

Since P is order compatible either $J_i(\phi) \leq J_i(\psi)$ for all $i \in N$, or $J_i(\psi) \leq J_i(\phi)$ for all $i \in N$. Thus we have two cases to distinguish:

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$J_i(\phi) \leq J_i(\psi)$ **for all** $i \in N$: Since $\phi \in J_i$ and by case, we can follow that $\psi \in J_i$. However, this contradicts the assumption.

$J_i(\psi) \leq J_i(\phi)$ **for all** $i \in N$: Since $\psi \in J_j$ and by case, we can follow that $\phi \in J_j$. However, this again contradicts the assumption.

Since both cases lead to a contradiction, we can conclude that P must be totally ordered. □

Theorem 23. *Let $P = \langle J_i \rangle_{i \in J}$ be a classical judgment profile. P is totally ordered if, and only if, it is order compatible.*

Proof. Follows directly from Lemma 17 and Lemma 18. □

Remark 29. *Theorem 23 only holds for classical judgment profiles. Consider the following counterexample for* KZ*-judgments $J_i : \mathcal{A}^+ \to [0,1]$ for $\mathcal{A}^+ = \{p_1, p_2\}$:*

- $J_1 = \{p_1 \mapsto 0.5, p_2 \mapsto 0.6\}$
- $J_2 = \{p_1 \mapsto 0.4, p_2 \mapsto 0.8\}$
- $J_3 = \{p_1 \mapsto 0, p_2 \mapsto 0.5\}$
- $J_4 = \{p_1 \mapsto 1, p_2 \mapsto 0.6\}$

The profile $P_1 = \langle J_1, J_2 \rangle$ is order compatible, but not partial ordered since $J_1(p_1) > J_2(p_1)$ and $J_1(p_2) < J_2(p_2)$.

Moreover, the profile $P_3 = \langle J_3, J_4 \rangle$ is totally ordered, but not order compatible since $J_3(p_1) < J_3(p_2)$ but $J_4(p_1) > J_4(p_2)$.

Hence, for KZ*-judgment profiles, order compatibility and total orderedness are incomparable.*

7.4 A simple impossibility result for fuzzy-logic-based judgment aggregation

The original impossibility result of [LP02] (see Theorem 5) showed that if the agenda satisfies some restrictions, then there is no aggregation function that satisfies collective rationality (see Definition 21), universal domain (see Definition 18), anonymity (see Definition 67) and systematicity (see Definition 69). In this section, we show a generalization for this theorem in KZ-logic, i.e., in a non-finite many-valued logic. In particular, we state for which class of profiles an inconsistency or impossibility is inevitable.

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7.4.1 Inconsistency in the two-agent case for average aggregation

In the following section, we show that for the average rule (see Definition 64) already, a two-agent situation can lead to a KZ-inconsistent collective outcome. In particular, note that if the two individuals do not agree on the relative acceptance of the conjuncts of a conjunction, then the collective judgment of the average rule will be KZ-inconsistent.

Theorem 24. *Let $\langle N, \mathcal{A}\rangle$ with $N = \{1, 2\}$ be a KZ-judgment aggregation problem such that for every profile $P \in \mathbf{P}(\mathcal{A})$ there are formulas $\phi_k, \phi_l, \phi_k \circ \phi_l \in \mathcal{A}^+$ with $\circ = \wedge$ or $\circ = \vee$ such that*

$$J_1(\phi_k) < J_1(\phi_l) \quad \textit{but} \quad J_2(\phi_k) > J_2(\phi_l).$$

Then the average rule yields an KZ-inconsistent collective judgment $F_{av}(P)$.

Proof. Let $\langle N, \mathcal{A}\rangle$ with $N = \{1, 2\}$ be a KZ-judgment aggregation problem. Let $P \in \mathbf{P}(\mathcal{A})$ be any judgment profile such that there are formulas $\phi_k, \phi_l, \phi_k \wedge \phi_l \in \mathcal{A}^+$ such that

$$J_1(\phi_k) < J_1(\phi_l) \quad \text{but} \quad J_2(\phi_k) > J_2(\phi_l)$$

We proceed by considering the formula $\phi_k \wedge \phi_l$ and showing that the average rule yields an KZ-inconsistent collective judgment, i.e., that F_{av} cannot be collectively rational (see Definition 21). For ϕ_k and ϕ_l we have

$$F_{av}(P)(\phi_k) = \frac{J_1(\phi_k) + J_2(\phi_k)}{2} \quad \text{and} \quad F_{av}(P)(\phi_l) = \frac{J_1(\phi_l) + J_2(\phi_l)}{2}$$

Since $J_1(\phi_k) < J_1(\phi_l)$, we have

$$J_1(\phi_k \wedge \phi_l) = \min(J_1(\phi_k), J_1(\phi_l)) = J_1(\phi_k)$$

Since $J_2(\phi_k) > J_2(\phi_l)$, we have

$$J_2(\phi_k \wedge \phi_l) = \min(J_2(\phi_k), J_2(\phi_l)) = J_2(\phi_l)$$

Thus, the collective judgment of $\phi_k \wedge \phi_l$ is

$$F_{av}(P)(\phi_k \wedge \phi_l) = \frac{J_1(\phi_k \wedge \phi_l) + J_2(\phi_k \wedge \phi_l)}{2} = \frac{J_1(\phi_k) + J_2(\phi_l)}{2}$$

We have that $J_1(\phi_k) < J_1(\phi_l)$ and thus

$$F_{av}(P)(\phi_k \wedge \phi_l) = \frac{J_1(\phi_k) + J_2(\phi_l)}{2} < \frac{J_1(\phi_l) + J_2(\phi_l)}{2} = F_{av}(P)(\phi_l)$$

Moreover $J_2(\phi_l) < J_2(\phi_k)$ and so

$$F_{av}(P)(\phi_k \wedge \phi_l) = \frac{J_1(\phi_k) + J_2(\phi_l)}{2} < \frac{J_1(\phi_k) + J_2(\phi_k)}{2} = F_{av}(P)(\phi_k)$$

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It follows that

$$F_{av}(P)(\phi_k \wedge \phi_l) = \frac{J_1(\phi_k) + J_2(\phi_l)}{2} < \min(F_{av}(P)(\phi_k), F_{av}(P)(\phi_l))$$

and thus, the collective judgment is inconsistent.

The proof for $\phi_k \vee \phi_l$ is analogous to $\phi_k \wedge \phi_l$. □

7.4.2 An inconsistency criterion for average aggregation

Generalizing Theorem 24 to larger sets of individuals poses the following problem. Further individuals that have smaller acceptance values for the formula ϕ_k could compensate for the bigger acceptance values of the other formula ϕ_l and vice versa. In the following proof of Theorem 25 we show that this is not the case.

Theorem 25. *Let $\langle N, \mathcal{A} \rangle$ with $|N| \geq 2$ be a KZ-judgment aggregation problem such that for every profile $P \in \mathbf{P}(\mathcal{A})$ there are formulas $\phi_k, \phi_l, \phi_k \circ \phi_l \in \mathcal{A}^+$ with $\circ = \wedge$ or $\circ = \vee$ and two individuals $i, j \in N$ such that*

$$J_i(\phi_k) < J_i(\phi_l) \quad \textit{but} \quad J_j(\phi_k) > J_j(\phi_l)$$

Then the average rule yields a KZ-inconsistent collective judgment $F_{av}(P)$.

Proof. Let $\langle N, \mathcal{A} \rangle$ with $|N| \geq 2$ be a KZ-judgment aggregation problem. Let $P \in \mathbf{P}(\mathcal{A})$ be any judgment profile such that there are formulas $\phi_k, \phi_l, \phi_k \wedge \phi_l \in \mathcal{A}^+$ and individuals $i, j \in N$ such that

$$J_i(\phi_k) < J_i(\phi_l) \quad \text{but} \quad J_j(\phi_k) > J_j(\phi_l)$$

We proceed by considering the formula $\phi_k \wedge \phi_l$ and show that the average rule yields a KZ-inconsistent collective judgment. The case for $\phi_k \vee \phi_l$ is analogous. For ϕ_k and ϕ_l we have

$$F_{av}(P)(\phi_k) = \frac{\sum_{i \in N} J_i(\phi_k)}{|N|} \quad \text{and} \quad F_{av}(P)(\phi_l) = \frac{\sum_{i \in N} J_i(\phi_l)}{|N|}$$

For $\phi_k \wedge \phi_l$, we have the following collective judgment.

$$F_{av}(P)(\phi_k \wedge \phi_l) = \frac{\sum_{i \in N} J_i(\phi_k \wedge \phi_l)}{|N|}$$

By our assumption that there exist two individuals $i, j \in N$ and two formulas ϕ_k and ϕ_l such that

$$J_i(\phi_k) < J_i(\phi_l) \quad \text{and} \quad J_j(\phi_k) > J_j(\phi_l)$$

we can define three disjoint subsets of individuals:

- $N_< = \{i \in N \mid J_i(\phi_k) < J_i(\phi_l)\}$

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- $N_= = \{i \in N \mid J_i(\phi_k) = J_i(\phi_l)\}$
- $N_> = \{i \in N \mid J_i(\phi_k) > J_i(\phi_l)\}$

Note that by our assumption $i \in N_<$ and $j \in N_>$ and thus these two sets of individuals are non-empty. Observe that $N = N_< \cup N_= \cup N_>$. The set $N_= \supseteq \emptyset$ can be empty. If $N_= = \emptyset$, then $N_< = N \setminus N_>$.

We get the following results for the three disjoint subsets of individuals:

$\forall m \in N_<$: Since $J_m(\phi_k) < J_m(\phi_l)$, we have

$$J_m(\phi_k \wedge \phi_l) = \min(J_m(\phi_k), J_m(\phi_l)) = J_m(\phi_k)$$

$\forall m \in N_=$: Since $J_m(\phi_k) = J_m(\phi_l)$, we have

$$J_m(\phi_k \wedge \phi_l) = \min(J_m(\phi_k), J_m(\phi_l)) = J_m(\phi_k)$$

$\forall m \in N_>$: Since $J_m(\phi_k) > J_m(\phi_l)$, we have

$$J_m(\phi_k \wedge \phi_l) = \min(J_m(\phi_k), J_m(\phi_l)) = J_m(\phi_l)$$

Moreover, we use the individual sets to rearrange the collective judgment for $\phi_k \wedge \phi_l$.

$$F_{av}(P)(\phi_k \wedge \phi_l) = \frac{\sum_{i \in N} J_i(\phi_k \wedge \phi_l)}{|N|} = \frac{\sum_{i \in N_<} J_i(\phi_k) + \sum_{i \in N_=} J_i(\phi_k) + \sum_{i \in N_>} J_i(\phi_l)}{|N|}$$

We have that $J_m(\phi_k) < J_m(\phi_l)$ for all $m \in N_<$ with $N_< \neq \emptyset$ and $J_m(\phi_k) = J_m(\phi_l)$ for all $m \in N_=$. Thus $\sum_{i \in N_<} J_i(\phi_k) < \sum_{i \in N_<} J_i(\phi_l)$ and $\sum_{i \in N_=} J_i(\phi_k) = \sum_{i \in N_=} J_i(\phi_l)$. So we can conclude that

$$\begin{aligned} F_{av}(P)(\phi_k \wedge \phi_l) &= \frac{\sum_{i \in N_<} J_i(\phi_k) + \sum_{i \in N_=} J_i(\phi_k) + \sum_{j \in N_>} J_j(\phi_l)}{|N|} \\ &< \frac{\sum_{i \in N_<} J_i(\phi_l) + \sum_{i \in N_=} J_i(\phi_l) + \sum_{j \in N_>} J_j(\phi_l)}{|N|} \\ &= \frac{\sum_{i \in N} J_i(\phi_l)}{|N|} = F_{av}(P)(\phi_l) \end{aligned}$$

Moreover we have $J_m(\phi_l) < J_m(\phi_k)$ for all $m \in N_>$ with $N_> \neq \emptyset$ and thus $\sum_{i \in N_>} J_i(\phi_l) < \sum_{i \in N_>} J_i(\phi_k)$. As a consequence, we get

$$\begin{aligned} F_{av}(P)(\phi_k \wedge \phi_l) &= \frac{\sum_{i \in N_<} J_i(\phi_k) + \sum_{i \in N_=} J_i(\phi_k) + \sum_{j \in N_>} J_j(\phi_l)}{|N|} \\ &< \frac{\sum_{i \in N_<} J_i(\phi_k) + \sum_{i \in N_=} J_i(\phi_k) + \sum_{j \in N_>} J_j(\phi_k)}{|N|} \\ &= \frac{\sum_{i \in N} J_i(\phi_k)}{|N|} = F_{av}(P)(\phi_k). \end{aligned}$$

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It follows that
$$F_{av}(P)(\phi_k \wedge \phi_l) < \min(F_{av}(P)(\phi_k), F_{av}(P)(\phi_l))$$
and thus, the collective judgment is inconsistent.

The proof for $\phi_k \vee \phi_l$ is analogous to $\phi_k \wedge \phi_l$. □

Recall that the average rule F_{av} satisfies anonymity and systematicity (see Section 2.4.3). In the following section, we show that anonymity and systematicity are actually the general mapping conditions (in addition to the agenda and input restrictions) that suffice to ensure that the aggregation function cannot be collectively rational.

7.4.3 A simple fuzzy-logic-based impossibility result

In this section, we generalize Theorem 25, and thereby also Theorem 5 of [LP02]. Furthermore, note that collective rationality in the many-valued framework over the logic KZ is equivalent to KZ-consistency (see Definition 66).

Theorem 26. *Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ with $|N| \geq 2$ be a KZ-judgment aggregation problem such that for every profile $P \in \mathbf{P}(\mathcal{A})$ there are formulas $\phi_k, \phi_l, \phi_k \circ \phi_l \in \mathcal{A}^+$ with $\circ \in \{\wedge, \vee\}$ and two individuals $i, j \in N$ such that*

$$J_i(\phi_k) < J_i(\phi_l) \quad \textit{but} \quad J_j(\phi_k) > J_j(\phi_l)$$

Then there is no aggregation function F of $\mathcal{J}$ that is collective rational (see Definition 66), systematic, and anonymous.

Proof. Let $\mathcal{J} = \langle N, \mathcal{A} \rangle$ with $|N| \geq 2$ be a KZ-judgment aggregation problem and F an aggregation function for $\mathcal{J}$ that satisfies collective rationality, systematicity, and anonymity. Furthermore, let $P \in \mathbf{P}(\mathcal{A})$ be any judgment profile such that there are formulas $\phi_k, \phi_l, \phi_k \wedge \phi_l \in \mathcal{A}^+$ and individuals $i, j \in N$ such that

$$J_i(\phi_k) < J_i(\phi_l) \quad \text{but} \quad J_j(\phi_k) > J_j(\phi_l)$$

We proceed by considering the formula $\phi_k \wedge \phi_l$ and show that F yields an inconsistent collective judgment, the case for $\phi_k \vee \phi_l$ is analogous.

For ϕ_k and ϕ_l by anonymity and systematicity we have that for any permutations $\sigma, \pi : N \to N$ that

$$F(P)(\phi_k) = f(J_{\sigma(1)}(\phi_k), \ldots, J_{\sigma(n)}(\phi_k)) \quad \text{and} \quad F(P)(\phi_l) = f(J_{\pi(1)}(\phi_l), \ldots, J_{\pi(n)}(\phi_l))$$

For $\phi_k \wedge \phi_l$, we have for any permutation $\tau : N \to N$ the following collective judgment for $\phi_k \wedge \phi_l$
$$F(P)(\phi_k \wedge \phi_l) = f(J_{\tau(1)}(\phi_k \wedge \phi_l), \ldots, J_{\tau(n)}(\phi_k \wedge \phi_l))$$

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By our assumption that there exist two individuals $i, j \in N$ and two formulas ϕ_k and ϕ_l such that

$$J_i(\phi_k) < J_i(\phi_l) \quad \text{and} \quad J_j(\phi_k) > J_j(\phi_l)$$

we can define three disjoint subsets of individuals:

- $N_< = \{i \in N \mid J_i(\phi_k) < J_i(\phi_l)\}$
- $N_= = \{i \in N \mid J_i(\phi_k) = J_i(\phi_l)\}$
- $N_> = \{i \in N \mid J_i(\phi_k) > J_i(\phi_l)\}$

Note that by our assumption and the definition of the sets $i \in N_<$ and $j \in N_>$ and thus these two individual sets are non-empty. Observe that $N = N_< \cup N_= \cup N_>$. The set $N_= \supseteq \emptyset$ can be empty. If $N_= = \emptyset$, then $N_< = N \setminus N_>$.

We get the following results for the three disjoint subsets of individuals:

$\forall m \in N_<$: Since $J_m(\phi_k) < J_m(\phi_l)$, we have

$$J_m(\phi_k \wedge \phi_l) = \min(J_m(\phi_k), J_m(\phi_l)) = J_m(\phi_k)$$

$\forall m \in N_=$: Since $J_m(\phi_k) = J_m(\phi_l)$, we have

$$J_m(\phi_k \wedge \phi_l) = \min(J_m(\phi_k), J_m(\phi_l)) = J_m(\phi_k)$$

$\forall m \in N_>$: Since $J_m(\phi_k) > J_m(\phi_l)$, we have

$$J_m(\phi_k \wedge \phi_l) = \min(J_m(\phi_k), J_m(\phi_l)) = J_m(\phi_l)$$

The idea to show inconsistency is that by Definition 67 of anonymity, we can order the individual judgments of the profile such that at first come all the individual judgments of set $(N_< \cup N_=)$ and then all the individual judgments of set $N >$. Since F is anonymous, let $\tau : N \to N$ be such that $\tau(i_x) = x$ for $i_x \in (N_< \cup N_=) = \{i_1, \ldots, i_m\}$, i.e., $m = |(N_< \cup N_=)|$, and $\tau(i_x) = m + x$ for $i_x \in N_> = \{i_1, \ldots, i_{(n-m)}\}$.

We have that $J_i(\phi_k) < J_i(\phi_l)$ for all $i \in N_<$ with $N_< \neq \emptyset$ and $J_i(\phi_k) = J_i(\phi_l)$ for all $i \in N_=$. Hence, we get the following

$$\begin{aligned} F(P)(\phi_k \wedge \phi_l) &= f(J_{\tau(1)}(\phi_k \wedge \phi_l), \ldots J_{\tau(n)}(\phi_k \wedge \phi_l)) \\ &= f(J_1(\phi_k), \ldots, J_m(\phi_k), J_{m+1}(\phi_l) \ldots, J_n(\phi_l)) \\ &< f(J_1(\phi_l), \ldots, J_m(\phi_l), J_{m+1}(\phi_l), \ldots, J_n(\phi_l)) \\ &= f(J_{\tau(1)}(\phi_l), \ldots, J_{\tau(n)}(\phi_l)) = F(P)(\phi_l) \end{aligned}$$

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Moreover we have $J_j(\phi_l) < J_j(\phi_k)$ for all $j \in N_>$ with $N_> \neq \emptyset$ and thus $\sum_{j \in N_>} J_j(\phi_l) < \sum_{j \in N_>} J_j(\phi_k)$. As a consequence, we get

$$\begin{aligned}
F(P)(\phi_k \wedge \phi_l) &= f(J_{\tau(1)}(\phi_k \wedge \phi_l), \dots J_{\tau(n)}(\phi_k \wedge \phi_l)) \\
&= f(J_1(\phi_k), \dots, J_m(\phi_k), J_{m+1}(\phi_l) \dots, J_n(\phi_l)) \\
&< f(J_1(\phi_k), \dots, J_m(\phi_k), J_{m+1}(\phi_k), \dots, J_n(\phi_k)) \\
&= f(J_{\tau(1)}(\phi_k), \dots, J_{\tau(n)}(\phi_k)) = F(P)(\phi_k)
\end{aligned}$$

It follows that

$$F(P)(\phi_k \wedge \phi_l) < \min(F(P)(\phi_k), F(P)(\phi_l))$$

and thus, the collective judgment is inconsistent. However, this contradicts that F is collectively rational. Hence, such an aggregation function F cannot exist.

The proof for $\phi_k \vee \phi_l$ is analogous to $\phi_k \wedge \phi_l$. □

As we observe, Theorem 26 gives a sufficient condition for a variety of aggregation functions with desirable properties (see Section 3.1) to be unable to properly aggregate individual judgments.

Note that the condition imposed on the profile in Theorem 26, i.e., that there are two individuals $i, j \in N$ such that

$$J_i(\phi_k) < J_i(\phi_l) \quad \text{but} \quad J_j(\phi_k) > J_j(\phi_l)$$

thereby classifying the profiles that will lead to inconsistency. Furthermore, this condition trivially implies that the considered profile does not satisfy order compatibility (see Definition 73) if the considered formulas of the agenda restriction are negation-free.

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CHAPTER 8

Conclusion

In this thesis we gave a self-contained introduction to the field of judgment aggregation, considered a general propositional many-valued framework, and proved further possibility and impossibility results in a specific fuzzy-logic-based judgment aggregation setting.

In Chapter 1 we motivated the study of judgment aggregation by introducing the so-called *discursive dilemma* [Pet01], through Example 1, which shows that aggregating judgments by propositionwise majority aggregation (see Definition 17) can lead to inconsistent collective judgments.

Furthermore, Chapter 2 introduced the *classical framework* for judgment aggregation, which uses classical propositional logic to model judgments about a given agenda (see Section 2.1). Additionally, we introduced an equivalent classical framework that enables the modeling of explicit constraints (see Section 2.5).

In Chapter 3 we reviewed *impossibility results* for the classical framework of judgment aggregation. In particular, Section 3.1 surveyed the first impossibility result given by [LP02], which shows that imposing desirable properties on an aggregation function (see Section 3.1.1) leads to the non-existence of such an aggregation function, even under rather weak agenda conditions. Furthermore, we introduced the *ultrafilter proof technique* (see Section 3.2), which allows one to derive impossibility characterizations. Section 3.3 reviews a range of different impossibility results.

Chapter 4 considered different strategies for coping with impossibility results by relaxing the range of allowed inputs, outputs, and mapping conditions, respectively (see Section 2.4). These considerations showed that relaxing collective rationality (see Definition 21) still leads to impossibility results in many cases [Gär06, DL08]. Moreover, we reviewed the *sequential priority approach* [Lis04, DL07b] (see Section 4.2.2) with the special cases of the premise- and conclusion-based approach [DM10] (see Section 4.2.1) which relaxes the independence condition (see Definition 28). However, this approach had the disadvantage of being manipulable by changing the decision-path (see Definition 43). Furthermore, we considered the *distance-based approach* [Pig06, EP05] (see Section 4.2.3), which tries to minimize the differences over all individual judgments in the profile [Lis11]. Last but not least, we considered *unidimensional alignment* [Lis03] (see Section 4.3), which restricts the universal domain (see Definition 18) and thereby allows one to aggregate individual judgments in a propositionwise fashion.

In Chapter 5 we reviewed the *general logics framework* introduced by [Die07]. This framework abstracts away from any specific logic and considers classes of logics by only imposing rather weak general conditions that have to be satisfied by the considered logics. Hence, showing impossibility results in the general logics framework, i.e., for a whole class of logics, implies that the impossibility holds in every more specific logic in this class.

Chapter 6 introduced a general many-valued framework for judgment aggregation, which allows one to model arbitrary acceptance values for propositions.

Finally, in Chapter 7, we first reviewed possibility results for KZ-logic shown by [Fer23] and then built on these results. In Section 7.2, we proved a generalization of Theorem 20 [Fer23] by introducing the notion of *linear systematicity* (see Definition 76). Furthermore, in Section 7.3, we introduced a partial order relation on judgments and *totally ordered profiles* (see Definition 78), which characterize a class of profiles that enable a possibility result. However, by proving Theorem 23, we showed that in the case of classical judgment profiles, the class of totally ordered profiles is equivalent to order compatible profiles. Last but not least, Section 7.4 states an impossibility result for KZ-logic and thereby classifies profiles that lead even under weak agenda conditions to impossibility results.

Overall, the field of judgment aggregation already contains a wide variety of impossibility and possibility results for the classical framework. However, the literature in the many-valued framework is only slowly emerging and leaves many open questions about the characterizations of possibility and impossibility of various forms of judgment aggregation.

The general question asking which conditions an aggregation function for judgments about interrelated propositions should satisfy and how this aggregation should be applied remains open. However, by systematically studying various judgment aggregation scenarios and, in particular, by characterizing possibility and impossibility results, we obtain deeper insights into the problem of aggregating judgments rationally. One may hope that these theoretical efforts assist in finding, formulating and understanding better ways of collective and democratic decision making in diverse fields.

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