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Titel der Dissertation

The Field of a gravitational Shock Wave generated by a massless point-like Particle in a stationary Black Hole Background

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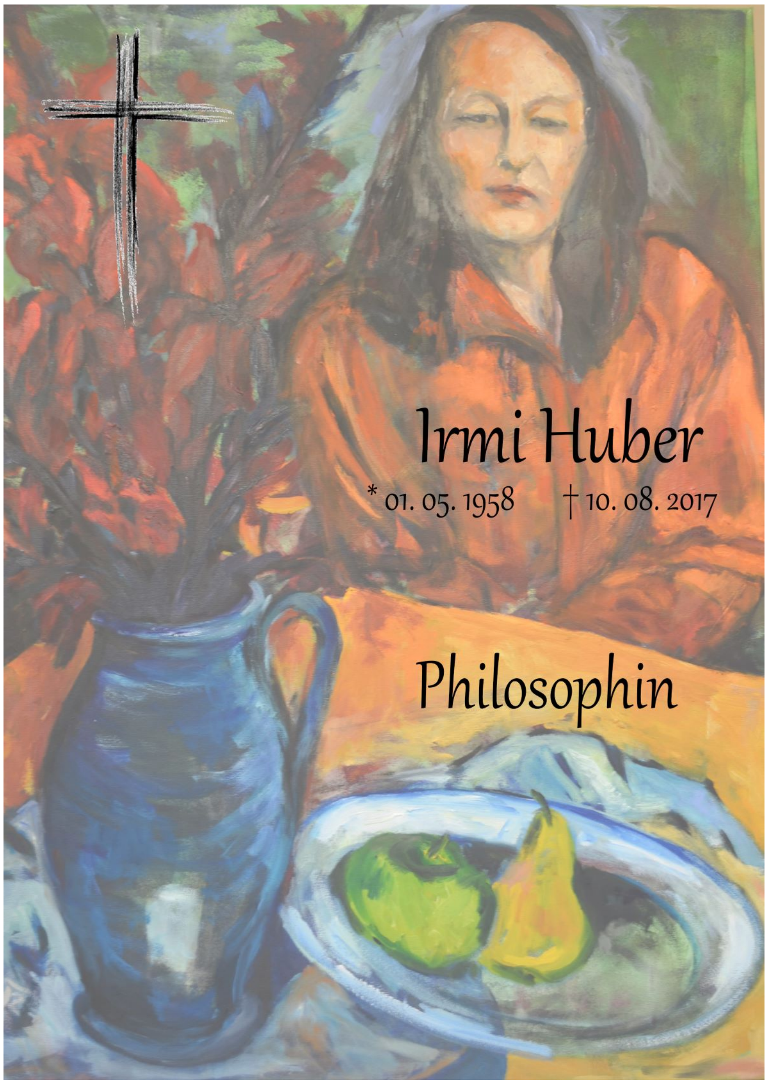
Meiner Familie - allen voran meiner Mutter Irmi - in Liebe gewidmet.



Contents

Abstract

In the present work, the field of a gravitational shockwave caused by a massless point-like particle is calculated at the event horizon of a Kerr-Newman black hole. Following the geometric setting given in [10], using the geometric framework of generalized Kerr-Schild deformations in combination with the spin-coefficient formalism of Newman and Penrose, it is shown that the field equations of the theory, at the event horizon of the black hole, can be reduced to a single linear ordinary differential equation for the so-called profile function of the geometry. This differential relation is solved exactly and, based on the results obtained, a physical interpretation is given for the found shockwave spacetime. In addition, it is clarified how these results lead back to those of previous works on the subject which deal with the simpler cases of gravitational shockwaves in static black hole backgrounds.

Zusammenfassung

Die vorliegende Arbeit widmet sich der Berechnung des Feldes einer gravitativen Schockwelle, die von einem masselosen punktartigen Teilchen am Ereignishorizont eines Kerr-Newman schwarzen Loches erzeugt wird. Unter Berücksichtigung des in [10] erarbeiteten geometrischen Settings, d.h. unter Verwendung verallgemeinerter Kerr-Schild Deformationen der zugehörigen Hintergrundmetrik und des Spin-Koeffizienten Formalismus von Newman und Penrose, wird hierbei zunächst gezeigt, dass sich die Einsteinschen Feldgleichungen - speziell am Ereignishorizont des schwarzen Lochs - auf eine einzige lineare gewöhnliche Differentialgleichung zweiter Ordnung für die Profilfunktion der Geometrie reduzieren lassen. Ferner wird eine exakte Lösung der besagten Differentialgleichung aufgezeigt und - basiered auf den erhaltenen Resultaten - eine physikalische Interpretation für die aufgefundene Schockwellengeometrie abgegeben. Zudem wird klargestellt, wie sich die gefundenen Ergebnisse mit verschiedenen einfacheren Spezialfällen von gravitativen Schockwellen in jenen sphärisch-symmetrischen Hintergründen vergleichen lassen, welche die Geometrien von statischen schwarzen Löchern beschreiben.

Conventions and Notation, Figures

Throughout the entire work, abstract index notation and other conventions based on the books of Penrose/Rindler [50, 51] and Wald [55] are frequently used. The only major difference is that in the given work the signature is fixed to $(-,+,+,+)$ and therefore will not be changed, as for example in [55], for spinor based calculations. Tensor indices are denoted by Latin letters $a, b, c...$, whereas spinor indices are denoted by capital Latin letters $A, B, C...$. General lightlike directions are denoted in standard Newman-Penrose fashion by ℓ^a and its associated lightlike co-directions by n^a. Special lightlike directions, as occurring in the Kerr-Schild context, are alternatively denoted by l^a and k^a. For the sake of simplicity, natural units are used, according to which $\hbar = G = c = e = k = 1$ applies by definition. The conformal diagrams presented in Section 2.3 have all been taken from the following sources: [33].

Introduction

As can be observed in the literature on the subject, there is, in many respects, a continuing interest in the construction of gravitational shockwave spacetimes in general relativistic models. This interest is based on a number of reasons; one of which is certainly the fact that the spacetimes mentioned could play a role in the description of high-energy particle collision events in which the gravitational interaction becomes dominant. This, in turn, is due to the fact that such spacetimes typically characterize the gravitational fields of extremely short radiation pulses propagating at the speed of light, thereby producing models that provide the perfect playground to describe collisions of very high-energetic particles in general relativity. Furthermore, there is another important reason: the fact that suitably constructed shockwave spacetimes could allow a physical treatment of geometric backreaction effects caused by black hole evaporation [36, 62]; effects that are expected to occur, as predicted by Hawking in his famous work on particle creation by black holes [31], because black holes constantly emit thermal radiation at a fixed temperature (which is exactly proportional to their surface gravity) until they are no longer stable and evaporate, and therefore may even disappear in potentially very violent explosions.

And although the determination of the geometric structure of shockwave spacetimes could prove interesting for many other reasons as well, their construction is often difficult because their associated geometries are usually generated by point-like gravitational sources, whose fields cannot easily be described within the theoretical framework of general relativity because of their highly nontrivial physical and mathematical properties.

This, in turn, is mainly due to the fact that the inclusion of the exact concept of a point particle in the framework of general relativity (although often avoided in practice by the consideration of idealized test particles) has proved to be tricky precisely because of the highly localized nature of such particles, therefore requiring the consideration of singular energy densities, which can only be treated by the use of distributional techniques. The 'standard' distributional techniques, however, are only defined in a linear context, and are therefore incompatible with the nonlinear character of Einstein's equations and thus with the theory of Einstein-Hilbert gravity as a whole. This is true even in the light of Colombeau's theory of generalized functions [9, 17, 18, 27], which, though capable of addressing a wide variety of problems associated with the treatment of fields of point-like gravitational sources, nevertheless, does not allow a rigorous treatment of the simultaneously singular and nonlinear field equations that are associated with the existence of point particles in general relativity.

This difficulty notwithstanding, there exist a number of approaches that, based on a rather more physically motivated than mathematically exact reasoning, allow a description of the motion of point-like 'corpuscles' in a general relativistic setting. Paying special attention to the motion of particles moving close to the speed of light, so-called ultrarelativistic gravitational sources, a fundamental work in that direction has been presented by Aichelburg and Sexl [1], who were the first to calculate the gravitational field of such a massless point-like particle. They did this by performing a Lorentz boost of the Schwarzschild geometry in isotropic coordinates and then executing the so-called ultrarelativistic limit, which is an operation based on first boosting the line element of a given geometry, taking into account terms proportional to the constant velocity v and the mass m in the resulting expression and taking then the combined limits $v \to 1$ and $m \to 0$. By means of this singular coordinate transformation, the authors obtained the metric of a deformed singular gravitational field which contains a term (generally known as the Brinkmann form) that is proportional to a delta distribution and thus has compact support in a single lightlike hyperplane in Minkowski space. The problem with this derivation, however, is that there is an ambiguity in the subtraction of occurring singular terms, which requires one to make an appropriate choice of a specific reference frame. As a result, however, due the fact that performing the said ultrarelativistic limit requires a particular choice of coordinates, the author's original derivation suffers from the problem of being observer-dependent, which, in turn, appears to be in contradiction to the coordinate independence generally expected from solutions of Einstein's field equations.

Fortunately, however, this observer dependency of Aichelburg's and Sexl's original derivation does not represent a major issue for various reasons; one particular reason being that the spacetime geometry discovered by the authors occurs as a special case of another class of solutions of Einstein's equations belonging to the so-called Robinson-Trautmann class of spacetimes, that is, in particular, to the class of so-called Kundt class of spacetimes (or even more precisely to the class of so-called impulsive pp-wave spacetimes), whose derivation is completely observer-independent. The said Kundt class is of great interest because it contains some more spacetimes whose geometries are deformed by point-like particles. Of particular note in this respect are the spacetimes of Dray and 't Hooft [20] and Sfetsos [58], which provide the precise form of a gravitational shock wave caused by a massless particle in static black hole and cosmological backgrounds.

These solutions, which generalize Aichelburg's and Sexl's work to curved Schwarzschild and Reissner-Nordström black hole backgrounds, characterize the geometric structure of general relativistic two-body systems consisting of a black hole and an additional point-like source located at the associated black hole event horizons. In both cases, the solutions mentioned were obtained as a byproduct of a specific distributional method known as Penrose's 'cut-and-paste' procedure [47], which is a procedure based on the idea of performing a specific coordinate shift in one of the components of the metric in double null coordinates.

While the application of this method has the interesting effect of giving rise to a confined particle-like source that generates a gravitational shock wave that skims along the black hole event horizon, there is the problem that the said method does not always produce mathematically well-defined quantities. This is due to the fact that the resulting geometric expression for the gravitational shock wave is proportional to a delta-shaped profile alias shift function (which is actually a distribution), whose precise structure must be obtained by explicitly solving the field equations of theory. As it turns out, this requires the treatment of geometric backreaction effects caused by the singular field of the point-like particle, which further requires performing nonlinear operations on distributions. Those, however, are generally ill-defined, which manifests itself in the fact that the nonlinear curvature fields associated with the deformed spacetime metric typically contain highly problematic, distributionally ill-defined terms ('squares' of the delta distribution), which cannot be properly treated and thus, as Dray and 't Hooft stated, must be 'blithely ignored' in practice.

As first stated by Alonso and Zamorano (but only in the context of rather special geometric circumstances) [2], this lack of mathematical rigour can elegantly be overcome by using a geometrically more appealing approach, known as the generalized Kerr-Schild framework. This framework, which makes it possible to rigorously deal with quantities of low regularity, is based on performing a specific null-geometric deformation of a given background spacetime, commonly referred to as a generalized Kerr-Schild deformation, which usually leads to a geometrically completely different, so-called generalized Kerr-Schild spacetime.

Due to its linearity, the said geometric framework is tailor-made for dealing with the low regularity of the components of the deformed field equations. More precisely, because of the fact that the mixed deformed Einstein tensor of the geometry (and hence also the mixed deformed Ricci) is always linear in the profile function, standard distribution theory can be used for solving the mixed Einstein equations with respect to a given background; a definitely indispensable advantage in the treatment of distributionally defined, singular gravitational fields in general relativity.

Using this particular geometric setting, to be discussed in chapter four of this work, a general approach to the situation was presented in [10]. Based on the fact that this approach does not require the consideration of a particular background geometry, it was shown that the corresponding geometric framework allows one to derive a generalized version of the Dray-'t Hooft relation for a gravitational shock wave concentrated on a null hypersurface, from which, given the particular choices of Schwarzschild and Reissner-Nordström black hole backgrounds, both the geometries of Dray and 't Hooft and Sfetsos can be derived as a special case. In addition, it was shown that the Aichelburg-Sexl geometry can be calculated as another special case of this setting.

As a bonus, the results obtained pointed to the possibility of extending the associated geometric setting to the much more complicated case of a gravitational shockwave generated by a massless particle-like source at the event horizon of a stationary, axisymmetric Kerr-Newman black hole. Motivated by this preliminary work, the present treatise is devoted to the construction of a corresponding solution of Einstein's field equations.

The basic strategy for putting the idea into action is to generalize the said approach via applying the Newman-Penrose spin-coefficient formalism to the generalized Kerr-Schild framework, or, more precisely, by formulating the gravitational field equations in terms of spin-coefficients. Based on this strategy, which is discussed in section four of this work, a system of five coupled second order partial differential relations for the profile function of the geometry is derived, which applies to any generalized Kerr-Schild geometry and is therefore completely general. Moreover, in order to then connect to the results of [10], it is shown that this system of equations is reducible (under rather basic assumptions) to an ordinary differential equation at the event horizon of the stationary black hole, which shall be referred to as generalized Dray-'t Hooft relation from now on. As will be shown, considering a specific null geodesic Kerr-Schild frame, this equation turns out to be exactly the same (modulo some differing formal conventions) as that obtained in [10].

Taking advantage of the freedom to perform null rescalings of the corresponding Kerr-Schild vector field that do not alter its geodeticity, but change the form of the profile function, the said generalized Dray-'t Hooft relation is tamed and brought into a much simpler form in section five of this work. It then turns out that the remaining relation belongs to the so-called Fuchsian class of second order differential equations with coefficients with five regular singular points, so that one immediately knows that it must have exactly two linearly independent solutions that can be superimposed to a single (distributionally defined) solution for the profile function of the geometry. In this context, however, the problem arises that within the theory of ordinary linear differential equations with variable coefficients, equations having five singular points are much less well studied and understood than the simpler cases of equations having three or four singular points. Thus, it is not surprising that the solution found in chapter five cannot be traced back to a previously known solution, but must rather be specified on the level of infinite power series, which, however, as it turns out, is completely sufficient in order to obtain an exact distributionally defined solution of Einstein's equations given by a generalized Kerr-Schild ansatz.

This is of importance not least because a prior work on the subject presented by BenTov and Swearngrin [16], which, like this work, is dedicated to the the construction of a Kerr-Schild shockwave in a Kerr-Newman background, failed to give an exact expression for the profile function of the geometry; despite its claim to have found an exact solution to Einstein's equations. However, as must be acknowledged, it appears that the geometric setting considered in this work is considerably different from that considered in [16]. As such, the two approaches can hardly be reasonably compared to each other.

Ultimately, a brief overview of the structure of the present work shall be given: In order to lay the formal foundations for both deriving and solving the generalized Dray-'t Hooft relation, the first chapter of this treatise deals with the causal structure of spacetimes in order to prepare the physical framework necessary for the definition of a black hole. This definition is then given in the second section, alongside a brief overview of the theory of gravitational collapse and black hole physics. Following this, the third section provides a detailed introduction to null geometry and the physics of null geodesic congruences, with particular emphasis on the spin-coefficient formalism of Newman and Penrose. It is only in section four that metric deformations are treated and the aforementioned system of coupled partial differential equations is derived from Einstein's equations, which specify the components of the deformed part of the Einstein tensor of the generalized Kerr-Schild class. From this system, the generalized Dray-'t Hooft equation is extracted, which is finally solved in the fifth and final section of this work. A discussion of the results obtained forms the conclusion of this thesis.

1 The Causal Structure of Spacetime

This incipient chapter intends to present a compendium of fundamental aspects and conceptions of the so-called theory of causality, which, generically associated to Lorentzian manifolds, will provide a formal basis for the definition of black hole spacetimes.

The addressed conceptions will, however, be covered merely in a recapitulatory fashion in order to provide a brief collection of non-redundant information needed subsequently. The comprised theoretical toolkit presented in this way will, however, heavily rely on more detailed treatments of the subject, as, for example, given in [33, 55].

1.1 Observers and Motion, Induced Submanifolds

In this section, a characterization of motions of observers associated with a given spacetime manifold (M, g) will be given. Intermediately, a number of selected properties of causal curves will be discussed, before finally the structure of submanifolds of spacetime, particularly that of hypersurfaces (and ordinary surfaces), will be reviewed.

Throughout this work, a spacetime (M, g) will be referred to as a four-dimensional, Hausdorff, differentiable, orientable and time orientable Lorentzian manifold M of signature $(-, +, +, +, ...)$, favorably of dimension 4, but in principle of any dimension n, which is assumed to be characterized by a fundamental field $g_{ab} = g_{ab}(x)$, called the metric. This metric or metrical field induces a non-degenerate pseudo-scalar product structure on M, by which the different forms of propagation or motion along trajectories between future and past events can be distinguished. For an observer moving along such a trajectory, usually referred to as a curve, being characterized by a tangent vector field $v^a = v^a(x)$, one can distinguish the following possible ways of motion by means of the associated non-degenerate bilinear form $g_{ab}v^a v^b$:

1.) *Timelike motion* : $g_{ab}v^a v^b < 0$,
2.) *Lightlike motion* : $g_{ab}v^a v^b = 0$,
3.) *Spacelike motion* : $g_{ab}v^a v^b > 0$.

This characterization is of great importance due to the fact that the local causal structure of a given spacetime (M, g) is determined by the behavior of events connected by smooth curves throughout (M, g), representing the analoga of so-called worldlines of observers in flat spacetime. Taking into account that any tangent vector field v^a associated with a curve determines the same completely, one distinguishes accordingly timelike, spacelike and lightlike curves $\gamma : I \to M$, where I is an interval of the real line. In this context, the timelike or lightlike structures associated with a respective curve $\gamma : I \to M$, induced by the tangent vector field v^a, completely determine the causal nature of events lying on such a trajectory. Therefore the local causal structure is subject to the expression $g_{ab}v^a v^b \leq 0$, specifying all non-spacelike motions. The according tangent field then can either be future directed or past directed, depending on the signature of the temporal component of v^a.

The given characterization is sometimes referred to as Einstein causality and represents, as already indicated, an inherent feature of Lorentzian geometry. It describes the causal relations of different events and observers in spacetime, or, more philosophically speaking, it determines in which way events are related with each other such that (under non-pathological circumstances) a given cause causes a certain effect, not vice versa.

The set of all curves (with associated tangent vectors) leading through a point p defines then the tangent space of the manifold M. The dimension of this tangent space T_pM is the same as that of the manifold M itself.

A timelike (resp. causal) curve $\gamma : I \to M$ is said to be past (or future) directed if each tangent vector $\dot{\gamma}$ is past (or future) directed. A point $p \in M$ is called the endpoint of a future directed causal curve $\gamma : I \to M$ if for every open neighborhood $\mathfrak{O}_p$ of p there exists a value t_0 such that $\gamma(t) \in \mathfrak{O}$ for all $t > t_0$. Accordingly a point $p \in M$ is the endpoint of a past directed causal curve $\gamma : I \to M$ if for every open neighborhood $\mathfrak{O}_p$ of p there exists a value t_0 such that $\gamma(t) \in \mathfrak{O}$ for all $t_0 < t$.

A curve $\gamma : I \to M$ is called future inextendible if it has no future endpoint. Accordingly, a curve $\gamma : I \to M$ is called past inextendible if it has no past endpoint.

A hypersurface (or surface for spacetimes of lower dimension) is an induced submanifold of spacetime with dimension $n - q$, where

$q = 1, 2, ..., n$ applies, although favorably one has $n = 4$ and $q = 1$ in most applications of interest. To any given hypersurface there exists then a particular, strictly associated scalar field, which remains constant along it.

Hypersurfaces possess a causal structure themselves, depending on the causal structure of a system of vector fields orthogonal to them. By default, a distinction is made here between spacelike hypersurfaces, whose orthogonal vectors are timelike, null hypersurfaces, whose orthogonal vectors are lightlike and timelike hypersurfaces, whose orthogonal vectors are spacelike. However, it has to be emphasized that there are also hypersurfaces whose causal structure varies locally in accordance with that of their associated orthogonal vectors.

A collection of hypersurfaces which covers the entire Lorentzian manifold is a so-called foliation. Depending on the causal structure of these leaves or folia, the foliation can either be spacelike, timelike or lightlike.

A given hypersurface Σ inherits generically geometrical structures from a given spacetime (M, g), which are usually subsumed as intrinsic geometric structures of Σ. These intrinsic geometric properties have to be strictly distinguished from extrinsic geometric properties, which describe geometric structures away from Σ.

1.2 Future and Past Developments, Domain of Dependence and Causality Conditions

At the beginning of this section, two different kinds of ordering relations connecting spacetime points shall be introduced and their main properties shall be analyzed thereupon. This will be necessary in order to be able to properly formulate the concept of chronological and causal pasts and futures of events, which, introduced in the further course of this section, will play a relevant role for the main definition of a black hole spacetime later on.

Considering two points p and q in M, the relation $p \prec\prec q$ denotes the fact that there exists a future directed, timelike curve $\gamma : [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. One says, then, that p chronologically precedes q.

Accordingly, considering again two points p and q in M, the relation $p \preceq q$ denotes the fact that there exists a future directed, causal curve $\gamma : [a, b] \to M$ such that $\gamma(a) = p$ and $\gamma(b) = q$. One says, then, that p causally precedes q.

Note that both of these ordering relations, the chronological as well as the causal relation, are determined by the expression $g_{ab}v^a v^b \leq 0$ evaluated in the points p and q if and only if v^a is the tangent vector to the given curve γ. Completely independently of the causal structure of spacetime, the said relations define a transitive, reflexive and antisymmetric preordering relation, which means that there holds $p \prec\prec q \prec\prec r \;\Rightarrow p \prec\prec r,\; \forall p, q, r \in M$ and $p \prec\prec q \prec\prec p \;\Rightarrow p = q,\; \forall p, q \in M$ as well as $p \preceq q \preceq r \;\Rightarrow p \preceq r,\; \forall p, q, r \in M$ and $p \preceq q \preceq p \;\Rightarrow p = q,\; \forall p, q \in M$. By excluding the possibility of closed timelike or causal curves, which is a natural restriction on the causal structure of spacetime, which will be discussed in more detail in the remainder of this section, the said relations become irreflexive (strict) partial ordering relations in respect to which $p \not\prec\not\prec p,\; \forall p \in M$ or $p \npreceq p \; \forall p \in M$ applies.

The given formulation of the chronological ordering relation $\prec\prec$ allows for the definition of what is called the chronological past $I^-(p)$ and the chronological future $I^+(p)$ of this very point. Strictly speaking, the chronological future $I^+(p)$ of p is defined as the set of all points q which chronologically precede p, i.e.

$$I^+(p) = \{q \in M | p \prec\prec q\},$$

while the chronological past $I^-(p)$ of p is defined as the set of all points $q \in M$ for which p chronologically precedes q, i.e.

$$I^-(p) = \{q \in M | q \prec\prec p\}.$$

Analogously, the formulation of the causal ordering relation $\preceq$ allows for the definition of the so-called causal past $J^-(p)$ and the causal future $J^+(p)$ of an event p with regard to the set of all points q, for which there holds

$$J^+(p) = \{q \in M | p \preceq q\}$$

and

$$J^-(p) = \{q \in M | q \preceq p\}.$$

The sets $I^{\pm}(p)$ and $J^{\pm}(p)$ then collectively define the local causal structure of an event $p \in M$. Elements of this class additionally determine the orientation and time orientation of the manifold M with regard to a point p lying on a curve γ.

For an open subset $\mathcal{O} \subset M$, representing a whole collection of different events, one defines accordingly

$$I^{\pm}(\mathcal{O}) = \bigcup_{p \in \mathcal{O}} I^{\pm}(p)$$

and

$$J^{\pm}(\mathcal{O}) = \bigcup_{p \in \mathcal{O}} J^{\pm}(p).$$

A subset $\mathcal{O} \subset M$ is called achronal if it is not intersected more than once by any future directed timelike curve, or, in other words, if there is no pair of chronologically related points lying in $\mathcal{O}$. Alternatively, it is often written here briefly and concisely $I^{+}(\mathcal{O}) \cap \mathcal{O} = \emptyset$.

Taking these definitions into account, the future and past domains of dependence of a region $\mathcal{O} \subset M$ can be defined. These definitions read

$$D^{+}(\mathcal{O}) = \{p \in \mathcal{O} | \textit{every past inextendible causal curve} \\ \gamma : I \rightarrow M \textit{ trough } p \textit{ intersects } \mathcal{O}\}$$

and

$$D^{-}(\mathcal{O}) = \{p \in \mathcal{O} | \textit{every future inextendible causal curve} \\ \gamma : I \rightarrow M \textit{ trough } p \textit{ intersects } \mathcal{O}\}.$$

The total domain of dependence of a region $\mathcal{O} \subset M$ is then defined by

$$D(\mathcal{O}) = D^{+}(\mathcal{O}) \cup D^{-}(\mathcal{O}).$$

Using these definitions, a Cauchy hypersurface Σ is defined as a closed achronal set for which $D(\Sigma) = M$. It is therefore a hypersurface whereupon through each of its points there runs one and only one unique, orthogonal geodesic γ.

Using these definitions, a number of reasonable constraints on the causal structure of a spacetime (M, g) can be made, in order to provide realistic physical circumstances. An employment of these constraints has become a standard procedure and is therefore approached customarily in the pertinent literature on the subject. It has, for example, also served as the keystone for the establishment of a so-called causal hierarchy of spacetimes, as presented in [42].

In order to pose a minimal condition on the causal structure of a spacetime (M, g), one may demand it to be a chronological spacetime, which is actually tantamount to requiring that it should not contain any closed timelike curves, i.e. $p \notin I^+(p)$ for all $p \in M$. This condition can easily be extended via requiring that (M, g) additionally should not contain closed causal curves, i.e. $p \notin J^+(p)$ for all $p \in M$. Spacetimes that obey such a requirement are called causal spacetimes. And although there are spacetimes that are neither chronological nor causal, these requirements are quite standard as they allow to avoid paradoxical conclusions being inconsistent with observation that can occur in the non-chronological, non-causal framework, such as those inconsistent conclusions that can be drawn from the infamous grandfather's paradox[1].

Furthermore, it seems to be convenient to require a given spacetime (M, g) to be future and past distinguishing, which may be stated in the form $I^{\pm}(p) = I^{\pm}(q) \Rightarrow p = q$ with respect to two fixed points $p, q \in M$. The main characteristic of future- and past-distinguishing spacetimes is that they allow a continuous choice of forward (or backward) light cone structures of observers over the entire spacetime. However, since the introduced concept of a spacetime (M, g) has been that of a time orientable manifold, which means that there should exist a consistent, continuous selection of future and past directed, causal tangent vector fields to a given curve γ throughout (M, g), this request should automatically be fulfilled anyway.

Additionally, there are two further restrictions on the notion of a causal spacetime, namely strong causality and stable causality. The main requirement for strong causality is the existence of a causally convex neighborhood V to a point $p \in M$, which guarantees that any future-directed (and hence also any past-directed) causal curve $\gamma : I \to M$ with endpoints at V is entirely contained in a larger convex normal neighborhood U with $V \subset U$, so that the two endpoints of the causal curve cannot get arbitrarily close together. Stable causality, on the other hand, is guaranteed to hold if a given spacetime (M, g) permits the existence of a continuous differentiable generalized time function $t \in \mathbb{R}$ on M, which gives rise to a continuous, non-vanishing, timelike vector field $\tau_a = \nabla_a t$ with which the metric can be perturbed, with the result that the causal structure of space-

[1]This goes hand in hand with the famous chronology protection conjecture of Hawking [32], which intends to generally rule out non-chronological spacetimes and the physical feasibility of time travel associated with such spacetimes.

time does not change. This requirement is so stringent that it also implies the weaker requirement for the validity of strong causality. Moreover, it is found in the given case that the manifold topology agrees with the so-called Alexandrov topology. For further details see [34, 39, 65].

Although the obtained class of spacetimes is already quite restrictive, one may furthermore demand that any causal diamond $J^+(p) \cap J^-(q)$, also called Alexandrov set, is compact for each $p, q \in M$, where (M, g) of course should be time-orientable by definition. This, however, is tantamount to the fact that (M, g) contains a closed achronal subset $\Sigma \subset M$ whose domain of dependence coincides with M itself, ergo a Cauchy hypersurface. As proven by Geroch [22, 25], spacetimes which do obey such restrictions are so-called globally hyperbolic spacetimes and do possess the topology $\mathbb{R} \times \Sigma$. Global hyperbolicity is considered as a necessary condition on the causal structure of spacetime, particularly in the initial value formulation for Einstein's equations, in that it guarantees, given any initial data set, that there exists a single maximum global hyperbolic solution of the field equations. This is because it represents a sufficient condition for the consideration of a differentiable function $t \in \mathbb{R}$ on M, which can be chosen in such a way that each $t = const.$ hypersurface is a Cauchy hypersurface, illustrating therefore that globally hyperbolic spacetimes can be foliated by Cauchy hypersurfaces and therefore exhibit a spacelike foliation. The existence of such spacetimes thus provides the formal basis for the formulation of a well-defined Cauchy problem in general relativity.

For the present work, however, neither the demand for global hyperbolicity nor the existence of a spacelike foliation of spacetime is necessary, which is of importance not least because none of the geometric models to be discussed below proves to be globally hyperbolic. Instead, as will be seen, the existence of a lightlike foliation of spacetime will play a role later on. To make this clear, however, the physics of black holes shall first be discussed in more detail, which will be done in the following chapter of this work.

2 Gravitational Collapse and Black Holes

The present chapter aims to provide a summary of the basic foundations of the theory of black holes and black hole event horizons. Relying on several ideas of the previous section as well as on more detailed and profound reviews on black hole physics, as, for example, given in [33, 53, 55], the focus of the proposed summary will lie on the development of an exact theoretical and phenomenological characterization of black holes and the introduction of black hole spacetimes.

While usually such a characterization is given in the literature along a number of theorems and relevant characteristics associated with black holes, such as, for example, the black hole uniqueness theorems, the area theorem or the laws of black hole mechanics, the focus of the present chapter will lie more on providing useful insights on the geometric characteristics of the gravitational fields of black hole spacetimes, i.e. of spacetimes that belong to the so-called Kerr-Newman family of spacetimes. Since black holes generally appear as a possible final configuration of the gravitational collapse of a compact massive object, a proper treatment of the topic also appears to require a brief overview of the phenomenology of these particular objects and the mechanisms leading to their collapse. Such an overview will therefore be given right at the beginning of this chapter.

2.1 Stellar Evolution, Gravitational Collapse and Black Holes

From a phenomenological point of view, the formation of a black hole is nothing more than an extreme consequence of the gravitational collapse of a star. More precisely, the formation of a black hole represents, or, rather is generally assumed to represent the final period in the evolution of a star exhibiting a sufficient amount of mass, during which an extreme, but stable stationary configuration of the geometric structure of spacetime is reached with the effect that the stellar matter is forced into an extreme physical state due to the gravitational collapse of the compact massive object. The reached geometric configuration is unique and independent of the retained material structures and only depends, quite similar to the case of elementary particles, on three independent parameters, namely the mass M, the angular momentum J and the charge e of the black hole. This is subject of the famous no-hair theorem of black holes [30, 37, 56], whose validity is generally accepted and in full accordance with the so-called black hole uniqueness theorems. These theorems state that in the absence of accreted matter any stationary black hole spacetime approaches to be a Kerr-Newman spacetime, which is entirely characterized by only these three independent parameters. Certainly the classical conclusion that no alternative 'degrees of freedom' exist in stationary black hole spacetimes is quite astonishing, taking into account the variety and complexity of the processes taking place during different epochs of stellar evolution, especially in the rotating case, for which no such exact, unique and particularly simple description exists to this day.

Interestingly, however, the evolution of a star does not always need to end with the formation of a black hole. There are in fact several different scenarios demonstrating what can happen in the final stage of the lifetime of a star, in which only three different possible final mass configurations are assumed to be left behind. The first of these scenarios naturally appears if the star has a total mass beyond the famous Chandrasekhar mass limit M_c ($\sim 1,4\, M_{\odot}$). In this case, it occurs that the electron degeneracy pressure, which contributes on a microscopic level to the macroscopic thermal and radiation pressures that are pre-dominant in the star, is no longer sufficient to support the star against gravity and no further nuclear reactions occur to guarantee the stability of the star. As a direct result, a part of the interior of the star will undergo gravitational collapse and a white dwarf will form as a direct consequence. Such a situation does not occur in the case of the second scenario, in which the remaining mass configuration finally forms a neutron star due to the collapse of the interior of the star. This addressed situation naturally appears if the mass of the collapsing part of the star is below the so-called cold matter upper mass limit ($\sim 2\, M_{\odot}$), so that the neutron degeneracy pressure is sufficient to halt the according collapse. Something different happens in the third possible scenario, according to which it occurs that the mass of the star is above the said cold matter upper mass limit, so that even the neutron degeneracy pressure is excelled by gravity and the star will eventually undergo complete gravitational collapse. This is assumed to happen in such a way that the involved matter configurations reach a state of maximal compactness, due to the fact that all persisting material forces are evanescent compared to the extraordinarily high opposing forces of mass attraction. In this final scenario, the spacetime curvature grows indefinitely near the remaining gravitational source, which now has shrunk toward its own critical size. The according region is what is called a black hole, or, more precisely, a black hole region.

From a theoretical point of view, the processes associated with the latter scenario lead to the formation of a so-called singularity of spacetime, which, roughly speaking, represents a part of the spacetime at which the curvature of the gravitational field grows unboundedly large. More precisely, it accords with a point or a whole region where the Riemann curvature tensor or related quantities like the Einstein tensor do neither appear to be suitably differentiable, nor regular, nor to be well-defined at all in a usual smooth spacetime continuum interpretation. To give an exact geometric characterization of a singularity associated with the incompleteness of causal geodesics can therefore prove to be a non-trivial problem in the generic case. Consequently, it is often assumed that a singular spacetime, i.e. a spacetime that contains a singularity or even a whole number of singularities, is the maximal manifold on which the metric is suitably smooth, subject to the condition that the occurrence of singularities goes hand in hand with the existence of incomplete geodesics that cannot be extended to infinite values of the affine parameter. In this vein, a spacetime is called singular if it is timelike or null geodesically incomplete, but cannot be embedded in a larger spacetime.

From a purely pragmatic point of view, this definition is quite meaningful - not least because a singularity is generally expected not to be observable by an external observer anyway because it is surrounded by a non-material, spherical boundary called the event horizon. This basically alludes to the famous (and unproven) cosmic censorship conjecture formulated first by Penrose [48, 49], stating that nature evades so-called naked singularities, which are singularities occurring to be visible for external observers. Although the Einstein equations permit solutions involving such singularities, the presence of naked singularities generally causes severe problems in the framework of the general theory of relativity, because by the existence of so-called Cauchy horizons they spoil the predictability of the theory. Hence, in order to cope with this significant obstacle, the cosmic censorship conjecture has been introduced in the way above, although there are in fact two different versions, the weak and the strong one, both of which demand some technical background and therefore will be approached merely in a superficial manner in the relevant part of the given chapter. However, by imposing the averaged null energy condition Cauchy horizons appear to be inherently unstable anyway, so that the standard approach, in which the mentioned boundary surface of a black hole, the event horizon, prevents the singularity from being observed from the outside, is justified by stability arguments.

From a phenomenological point of view, this given characterization of an event horizon, in the sense of being a hypersurface along which any light ray emitted from the black region can no longer be noticed by an outside observer, delivers striking consequences, as it basically suggests that nothing, not even a massless photon, could possibly resist the gravitational attraction exerted by a black hole after it has passed the said horizon - at least in the standard classical interpretation. In this classical picture (which has been revised several times due to incorporation of quantum effects), all particles of the emitted light ray must necessarily encounter the singularity instead of escaping to the external universe. Due to this property, the act of crossing the event horizon is sometimes referred to as reaching the so-called point of no return.

In conclusion, it is clear that black holes could never, under the assumption that the cosmic censorship conjecture holds, be directly observed in our universe. This, however, does not mean that black holes cannot be observed in nature in an indirect fashion instead, i.e. with regard to visible objects they attract. In fact, numerous different instances of such indirect observations of potential black hole candidates have accrued over the years, all of which are compatible with predictions from the theory. These observations suggest that there do in fact exist several different classes or types of black holes, which can be distinguished by the magnitude of their masses. One distinguishes stellar black holes ($\sim 2 - 3\, M_{\odot}$), large stellar black holes ($\sim 10 - 30\, M_{\odot}$), intermediate-mass black holes ($\sim 100 - 1000\, M_{\odot}$) and supermassive black holes ($\sim 10^6 - 10^9\, M_{\odot}$). The strongest and earliest observational evidence speaking for the existence of black holes stems from the discovery of candidates for so-called black hole binary systems, which are astrophysical configurations consisting of a visible star and a dark companion in close orbit around each other. In the case of stellar black holes, the most prominent of such binary systems appears to be Cygnus X-1, discovered in the eponymous Cygnus constellation. Here, an evaluation of the orbital parameters of the visible part of the binary allowed a recursive determination of a lower mass limit of the dark companion, settled in the range ($\sim 14 - 15\, M_{\odot}$). Due to the notion that the invisible attractive source of the binary appears to be too compact to be a normal star, the suspicion substantiated that it could, in accordance with theoretical predictions, be identified as a black hole. Ever since then, and especially since the commissioning of the famous Hubble space telescope, dozens of discoveries of comparable stellar black hole candidates have been established in binary systems. Furthermore, ultraluminous compact X-ray sources affecting mass concentrations in so-called star burst galaxies, i.e. in galaxies characterized by rapid star formation, have been observed. The observed objects are deemed to correspond to black holes of intermediate size, which in general tend to be strictly larger than stellar black holes, but fairly less massive than supermassive ones. Multiple times, similar dark objects have been observed in neighboring galaxies as well as at the centers of globular clusters, which are dense star systems that orbit a galactic core as a satellite. Gravitational lensing effect measurements and gamma-ray burst detections have additionally confirmed the view that the discovered objects can be identified as galactic black holes in agreement with theoretical predictions. The aforementioned gamma-ray bursts have been subject to intense study in the past decades, mainly due to the reason that the longest detected flashes have to be associated to extremely energetic, extraterrestrial explosions, representing in particular the most highly intense electromagnetic radiative pulse detections known to this day. The strongest of these gamma-ray bursts are thereby supposed to result from exceptionally energetic supernovae, so-called hypernovae, which are generally assumed to be linked to the formation of a black hole belonging to the largest part of the observed mass range, a supermassive black hole. Together with observations concerning the study of active galaxies and the investigation of the central region of our galaxy, ultra-long gamma-ray bursts provide the strongest evidence for the existence of dark objects lying in this mass range. The observations suggest that the most likely opportunity for the formation of supermassive black holes measuring up to such scales is provided at the center of galaxies. Therefore, it is not a coincidence that a strong, compact radio source named Sagittarius A^* has been observed at the center of our own galaxy, which may be identified as a supermassive galactic black hole configuration. Moreover, further observations seem to indicate that according black holes do in fact reside at the center of every galaxy in the universe. This, however, is far from being confirmed in an unassailable fashion.

From a theoretical point of view, black holes with masses much smaller than that of stellar black holes ($<< M_{\odot}$) may furthermore be predicted to exist in nature, although evidence for these objects is inconclusive so far. Such small black holes may have existed in form of so-called primordial black holes in the early epochs of the universe where ultra-high densities, required for the formation of such small black holes, dominated the landscape of the universe at the time. Additionally, so-called micro black holes, which are supposed to be transient objects which could be formed in high-energy collisions that achieve sufficient densities, may, in principle, exist in nature, as there is no mechanism per se preventing a given massive object from going below its own Schwarzschild radius. However, the ultra-high densities required for such an act of formation go far beyond those of known stellar objects and are not expected to be simulated in any terrestrial laboratory any time soon.

All in all, from a theoretical and phenomenological point of view, the deployment of black holes and black hole physics has not only been proven to be consistent with observations, it has also initiated some of the most remarkable developments in theoretical physics by revealing basically unforeseen connections between otherwise distinct areas of physics, such as general relativity, quantum physics and statistical mechanics. Black hole physics therefore seem to represent a building block needed for a proper understanding of the theory of gravity, both at the classical and the quantum level. This particular subject shall be depicted in more detail in upcoming sections, in which geometries that are strictly associated with black hole spacetimes shall be explored in full detail.

2.2 Local, global and asymptotic Structures of Spacetime

In order to be able to give a mathematically precise definition of a black hole and a black hole event horizon, a few definitions and pieces of rather technical background information play an essential role. The given section therefore aims to provide relevant technical information by analyzing certain local, global and asymptotical structures of a special class of spacetime manifolds suitable for the said purpose.

The according special class of spacetimes, usually referred as the class of asymptotically flat spacetimes, does in fact provide the addressed structures and thereby proves to be of great importance in order to pursue black hole physics. The main reason for this is that black holes are supposed to represent ideally isolated objects and therefore should intuitively be characterized by the property that the metric of a given black hole spacetime should become flat at large distances from the attracting source. For this reason, the definition of a black hole demands the introduction of spacetimes that comply with according ideally isolated systems, being intimately specified in general relativity by the addressed class of asymptotically flat spacetimes.

Here, however, there is a problem: In the literature, there is not just a single, but rather several different definitions of asymptotic flatness and asymptotically flat spacetimes. Introduced first by Penrose [45, 46] in the context of conformal compactifications of spacetime, approached briefly during the further course of this section, an alternative definition has been given separately by Geroch [23, 24], which was based more on methods developed in the celebrated work of Arnowitt, Deser and Misner [3]. These two notions were later combined into a single one by Ashtekar and Hansen [7]. It is largely this final combined notion which shall now be introduced and then be used throughout the course of the given work. The intuitive idea of this approach (and of the others as well) is to map infinite distances to finite ones. This can be achieved in practice by use of the mentioned technique of conformal compactification of spacetime, i.e. by trying to associate with a given spacetime (M, g) an 'unphysical' spacetime $(\tilde{M}, \tilde{g})$ such that $M \subseteq \tilde{M}$ via considering a conformal isometry $\psi : M \to \tilde{M}$ with $\psi^* \tilde{g}_{ab} = \Omega^2 g_{ab}$, where $\Omega \in C^\infty(\psi[M], \mathbb{R}_+)$ is some scalar function, in order to attach suitable boundary structures representing points at future and past infinity. Due to the fact that the transformed metric $\tilde{g}_{ab}$ is not supposed to possess a decent physical behavior, the metrics related by the conformal factor Ω^2 are called the physical and the unphysical metric. Although the unphysical metric $\tilde{g}_{ab}$ may not possess direct physical significance in general, the asymptotics of the physical metric g_{ab} become accessible if the alternative line element $d\tilde{s}^2 = \Omega^2 ds^2$ is analyzed on the compactified manifold. By this procedure the boundary ∂M of a given spacetime (M, g) can be divided (at least in the ideal case) in the independent parts

$i)$ $Future\ timelike\ Infinity\ i^+$
$ii)$ $Past\ timelike\ Infinity\ i^-$
$iii)$ $Spacelike\ Infinity\ i^0$
$iv)$ $Future\ null\ Infinity\ \mathcal{I}^+$
$v)$ $Past\ null\ Infinity\ \mathcal{I}^-$,

even though $i^{\pm}$ need not exist in general. These different parts of the boundary are specified by the property that all infinite spacelike, null or timelike curves which do not possess a certain endpoint located in (M, g) reach a point lying on the corresponding part of the boundary. In this respect, the image of the physical spacetime under the introduced conformal isometry between the physical and the unphysical spacetimes provides a precise notion of infinity. While the regions i^+, i^- and i^0 are 2-surfaces, the lightlike boundaries appear to be three-dimensional submanifolds.

The geometric procedure of conformal compactification now has the advantage that one is not forced to impose certain falloff conditions on the spacetime metric in a particular coordinate system or suchlike, due to the fact that this notion is manifestly coordinate independent. Within the addressed framework it is also possible to define conserved quantities associated with a spacetime itself, as for example the total energy of the gravitational field of a given spacetime, in a rigorous manner. Furthermore, the associated techniques enable one to represent an entire spacetime in a compact region in such a way that the causal structure is identically preserved, which is an inherent feature of any conformal isometry. Besides, the given method of conformal compactification by no means demands the validity of Einstein's field equations and therefore is also accessible to certain alternative theories of gravity in which the Einstein equations are not supposed to hold, although this work will focus on theories in which these fundamental equations of gravity are supposed to hold anyway.

Turning now again to the definition of asymptotic properties, a vacuum spacetime is called asymptotically flat at null or spatial infinity if there exists a related spacetime $(\tilde{M}, \tilde{g})$ in addition to a conformal isometry $\psi : M \to \tilde{M}$ with a conformal factor Ω such that $\psi^* \tilde{g}_{ab} = \Omega^2 g_{ab}$ with pull-back $\psi^* : T^*(\tilde{M}) \to T(M)$, where $\psi^* \tilde{g}_{ab} \in \Gamma(T(M))$, which fulfills the following conditions:

i) There exists a $2-sphere$ $i^0 \in \tilde{M}$ *such that* $\overline{J^+(i^0)} \cup \overline{J^-(i^0)} = \tilde{M} \backslash \psi[M]$.

ii) There exists an open neighborhood $\mathfrak{O}$ *of* ∂M *in* $\tilde{M}$ *such that* $(\mathfrak{O}, \tilde{g})$ *is strongly causal.*

iii) The function Ω *can be extended to a function on whole* $\tilde{M}$ *which is smooth and contiuously differentiable (except for* i^0 *where it has to be* C^2*).*

iv) The function Ω *must vanish on* $\mathcal{I}^{\pm}$ *and* $\tilde{\nabla}_a \Omega \neq 0$ *additionally therein. Furthermore, on* i^0*, it holds that* $\Omega(i^0) = 0$ *and*

$$\lim_{i^0} \tilde{\nabla}_a \Omega = 0, \ \lim_{i^0} \tilde{\nabla}_a \tilde{\nabla}_b \Omega = 2 g_{ab}(i^0).$$

v) The map of all null directions at i^0 *into the space of integral curves* $n^a := \tilde{g}^{ab} \tilde{\nabla}_b \Omega$ *on* $\mathcal{I}^{\pm}$ *is a diffeomorphism and, furthermore, for any choice of a function* $\omega \in C^{\infty}(\tilde{M} \backslash \{i^0\})$ *with* $\omega > 0$ *on* $\mathcal{I}^+ \cup \mathcal{I}^- = \{\psi[M] \cup J^{\pm}(i^0)\} \backslash \{i^0\}$ *and* $\tilde{\nabla}_a(\omega^4 n^a) = 0$ *on* $\mathcal{I}^{\pm} = \partial J^{\pm}(i^0) \backslash \{i^0\}$*, the vector field* $\omega^{-1} n^a$ *is assumed to be complete.*

While conditions *i)* and *ii)* disclose information about the causal structure of the boundary of the physical spacetime (at infinity), seen as an open subset of the unphysical one, condition *iii)* implies on the one hand that the conformal factor must be a well-behaved function on the entire unphysical spacetime and on the other hand that the Penrose compactification process is highly non-unique because of the arbitrariness in fixing the conformal factor of the unphysical metric. This can either be seen if one directly changes the occurring conformal factor Ω or if one again associates the unphysical spacetime $(\tilde{M}, \tilde{g})$ to another unphysical spacetime $(\tilde{\tilde{M}}, \tilde{\tilde{g}}) = (\tilde{\tilde{M}}, \tilde{\Omega}^2 \tilde{g})$, according to which now $\tilde{\Omega}$ should equally possess all desired properties of Ω. By setting $\tilde{\tilde{\Omega}} := \tilde{\Omega}\Omega$ one clearly re-obtains the same construction, but now with regard to a different conformal factor, which is sometimes referred to as gauge freedom of the construction in the literature. Condition *iv*), in the meantime, entails that the unphysical spacetime behaves in such a way that the boundary of its associated Lorentzian manifold looks like that of Minkowski space at infinity and, based on the listed completeness assumption, condition *v*) entails the fact that the topology of the boundary should be $\mathbb{R} \times \mathbb{S}_2$, since the flow generated by the corresponding null directions defines by assumption a one-parameter group of diffeomorphisms.

A spacetime is called strongly asymptotically predictable if in the unphysical spacetime there exists an open region $\tilde{\mathfrak{O}} \subset \tilde{M}$ with $\overline{M \cap J^-(\mathcal{I}^+)} \subset \tilde{\mathfrak{O}}$ such that $(\tilde{\mathfrak{O}}, \tilde{g})$ is globally hyperbolic. This is of interest because strongly asymptotically predictable spacetimes are those which are usually assumed to contain a black hole. More precisely, strongly asymptotically predictable spacetimes are spacetimes on which it is possible to distinguish so-called inner and outer trapped surfaces, whose presence usually indicates the existence of a singularity. What is meant by trapped surface is a closed spacelike 2-surface Δ, whose future-pointing outgoing null geodesics have negative expansion[2]. This actually means that there are two systems of null geodesics emerging orthogonally from Δ that in turn locally converge there as well. Note that this definition is a purely quasilocal one, since it involves only quantities defined on Δ. An outer trapped surface Δ on the other hand is an orientable trapped surface, i.e. a compact spacelike 2-surface Δ with a certain spatial orientation, contained in the future development of a partial Cauchy hypersurface Σ such that a system of outgoing null geodesics emerging orthogonally from Δ locally converge at this surface. The same holds true for an inner trapped surface Δ, but instead for null geodesics emerging orthogonally from Δ that are now ingoing. A partial Cauchy hypersurface in this context is a Cauchy hypersurface, which basically has to be asymptotically flat in addition, or, more precisely, which has to be simply connected and strong future asymptotically predictable as well. Note finally that a trapped

[2]In the language of proceeding sections this means actually that there is a pair of congruences of null geodesics emanating from Δ for which the expansions of the associated lightlike vector fields, say ℓ^a and n^a, of the congruences are manifestly non-positive, i.e. $\Theta = \Theta_{(\ell)} \leq 0$ and $\tilde{\Theta} = \Theta_{(n)} \leq 0$. This of course implies $\Theta\tilde{\Theta} \geq 0$. If now one of the included subcases $\Theta = \Theta_{(\ell)} = 0$ or $\tilde{\Theta} = \Theta_{(n)} = 0$ is fulfilled, Δ is referred to as a so-called marginally trapped surface.

region is simply a region containing trapped surfaces.

2.3 Black Holes and Black Hole Spacetimes

The given section will now focus on the elaboration of a mathematically precise and physically comprehensible definition of black holes and black hole spacetimes. For this purpose, an appropriate stationary and asymptotically flat setting will be re-engaged, which, in the spirit of the preceding section, complies with the requirement of taking into account the only recently featured local, global and asymptotical characteristics of ideally isolated systems. In this very context, a number of the compiled aspects of causality theory will prove to be a cornerstone in the definitions to be provided. The according theory of black holes established therefrom will then be delivered with regard to a selected number of black hole spacetimes, all of which lie in the celebrated Kerr-Newman family of black hole spacetimes.

To proceed as anticipated, a strongly asymptotically predictable, physical spacetime (M, g) shall be considered, which is conformally isometric to a related unphysical spacetime $(\tilde{M}, \tilde{g})$ containing a region $\tilde{\mathfrak{O}} \subset \tilde{M}$ with $\overline{M \cap J^{-}(\mathcal{I}^{+})} \subset \tilde{\mathfrak{O}}$. Based on this geometric setting, a black hole region $\mathcal{B}$ can be identified as the complement of $J^{-}(\mathcal{I}^{+})$, i.e.

$$\mathcal{B} := M \backslash J^{-}(\mathcal{I}^{+}).$$

This definition has been depicted from the idea that there is a general impossibility for any observer to escape from a given region $\mathcal{B}$ to future null infinity and that this appropriately describes the notion of a black hole in such spacetimes. Note that a spacetime containing a black hole region has been denoted as black hole spacetime as usual.

The event horizon $\mathcal{H}$ of a black hole is defined, in the given context, as the boundary of the mentioned region $J^{-}(\mathcal{I}^{+})$

$$\mathcal{H} := \partial J^{-}(\mathcal{I}^{+}).$$

By this definition, a future event horizon $\mathcal{H}^{+}$ is defined as the boundary of the causal past of future null infinity and a past event horizon $\mathcal{H}^{-}$ as the boundary of the causal past of future null infinity. The whole event horizon is, as already indicated, supposed to be subject to either the strong or the weak cosmic censorship conditions, stating that the parts $\mathcal{I}^{\pm}$ of the boundary are complete and that either the whole manifold M is globally hyperbolic or at least the domain of outer communication $J^{-}(\mathcal{I}^{+})$. It is therefore a smooth lightlike hypersurface, according to which a singularity of spacetime always appears to be 'invisible' to any observer in $\{M \cap \tilde{\mathfrak{O}}\} \supset \{M \cap \overline{J^{-}(\mathcal{I}^{+})}\}$ (subject to the condition that $M \cap \tilde{\mathfrak{O}}$ can be assumed to be foliated by a family of spacelike Cauchy hypersurfaces Σ_t). This allows one to cope with causal deficits caused by the presence of Cauchy horizons on such spacetimes; those which had already been ruled out for the purposes of this work by the causality requirement for spacetimes.

Black hole regions can be characterized by trapped surfaces, which are spacelike surfaces of co-dimension 2 in a Lorentzian spacetime with two independent forward-in-time pointing, lightlike, normal null directions, on which both expansions of the said null directions become negative. Following [41], these trapped surfaces can neither be closed trapped surfaces nor marginally trapped surfaces with regard to the proposed stationary setting in general. Nevertheless, in the case of spherical symmetry in vacuo, which implies staticity by Birkhoff's theorem, the event horizon $\mathcal{H}$ may be viewed as a collection of outermost marginally outer trapped surfaces (MOTS) of spherical topology. More generally, given the case that the event horizon coincides with a so-called isolated horizon (to be discussed in the next chapter of the present work), it may be thought of as a non-expanding null hypersurface foliated by so-called apparent horizons, which are compact, orientable 2-surfaces $\mathcal{A}$ (beyond which outgoing light rays cannot expand outward) lying inside a spacelike Cauchy hypersurface Σ such that $\mathcal{A}$ is a connected component of the outer boundary of the trapped region of Σ.

Bearing this in mind, it may now be the time to discuss, case by case, different black hole spacetimes, all of which lie in the famous Kerr-Newman spacetime family. The addressed solutions are exact and describe stationary axisymmetric or even static spherically symmetric vacuum spacetimes, representing the exterior field of a 'maximally densitized' mass configuration. As will be illustrated at a later point of this work, all of the discussed solutions belong to the so-called Kerr-Schild class of spacetimes and can easily be described within the geometric framework associated with this particular class of geometries, the so-called Kerr-Schild framework.

2.3.1 The Schwarzschild Black Hole

The Schwarzschild spacetime is a vacuum spacetime, describing the empty exterior field of an extended spherically symmetric massive body or the field of an eternal black hole for the whole spacetime. While in the former case the exterior Schwarzschild solution can be joined by an interior Schwarzschild solution describing the interior of an idealized spherically symmetric star, in the latter case the geometry of spacetime is completely determined by a single static solution of Einstein's equations. The focus of this subsection shall lie on the latter case.

In spherical coordinates, the line element of Schwarzschild spacetime takes the form

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2. \tag{1}$$

It can be seen that this spacetime is a static spacetime; neither does its metric depend explicitly on t nor does it contain any mixed terms that could violate the invariance of the line element under time reversal transformations. Therefore, it is invariant under temporal translations and under time reversals as well. By the validity of Birkhoff's famous theorem, the staticity of Schwarzschild spacetime is implied by spherical symmetry and the in vacuo condition. What is additionally proven by Birkhoff's theorem is uniqueness: the Schwarzschild spacetime is the only existing, spherically symmetric vacuum spacetime in general relativity[3].

As the corresponding metrical field is invariant under time translations, there exists a timelike Killing vector field $\xi^a = \partial_t^a$, a solution of Killing's equation $\nabla_{(a}\xi_{b)} = 0$, with $g_{ab}\xi^a\xi^b = g_{tt} = -(1 - \frac{2M}{r})$, that incorporates this symmetry. In addition, the given line element is invariant under the group of isometries $SO(3)$ operating on M, whose orbits are spacelike 2-spheres. The introduced radial coordinate r therefore is intrinsically related to the area of the according

[3]However, as shown in [12] using distributional methods, the mass parameter M can be interpreted as one associated with a singular matter source that generates the gravitational field, which, from a purely interpretive point of view, most likely has occurred in the past as a result of the gravitational collapse of some mass accumulation to a black hole. In this sense, it probably would be wrong to simply and naively interpret the Schwarzschild geometry as a vacuum geometry.

transitive 2-surfaces, which is encoded by the relation $r = \sqrt{\frac{A}{4\pi}}$, whereas A is the area of a given 2-sphere.

The Schwarzschild spacetime is asymptotically flat in complete agreement with all assumptions of the previous section. Its metric can be brought into the form $g_{ab} = \eta_{ab} + h_{ab}$ with $|h_{ab}| = O(r^{-1})$ and $|\partial_c h_{ab}| = O(r^{-2})$ for a, b fixed in the case of large r. In the given black hole case the Schwarzschild spacetime can be considered a vacuum geometry for all values of r. Looking at possible problematic regions, one immediately finds that the given line element becomes singular for $r = 2M$ and $r = 0$. While an investigation of all the scalar invariants of the Riemann tensor (such as in particular so-called Kretschmann scalar $R_{abcd}R^{abcd}$) shows that the $r = 2M$-region, located on the event horizon of the given black hole, appears to be a coordinate singularity, which can easily be removed by a transformation to less singular coordinates, the $r = 0$-region turns out to depict a real singularity of the given gravitational field.

In order to remove the occurring coordinate singularity one may introduce the so-called advanced and retarded null coordinates v and u given by $v = t + r^*$ and $u = t - r^*$, where $r^* = \int \frac{dr}{1-\frac{2M}{r}} = r + 2M \ln|\frac{r}{2M} - 1|$. According to those new coordinates the line element can be re-written in the form

$$ds^2 = -(1 - \frac{2M}{r})dvdu + r^2 d\Omega^2, \tag{2}$$

where now $r = r(u, v)$. Using further the conversions $v = u + 2r^*$ and $u = v - 2r^*$ one can distinguish so-called ingoing and outgoing Eddington-Finkelstein coordinates. According to those coordinates, the Schwarzschild line element takes the form

$$ds^2 = -(1 - \frac{2M}{r})du^2 - 2dudr + r^2 d\Omega^2 \tag{3}$$

in the ingoing case and

$$ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2 d\Omega^2$$

in the outgoing case. Although these ingoing and outgoing Eddington-Finkelstein coordinates remain regular on the horizon, they do not provide an analytic extension covering the whole manifold, but merely one half of it. Nevertheless they will be of rather indirect importance for later purposes of the given work, as they are intimately related to so-called Kerr-Schild coordinates. These coordinates are obtained by setting $v = \bar{t} + r$, which yields the line element

$$ds^2 = -d\bar{t}^2 + dr^2 + r^2 d\Omega^2 + \frac{2M}{r}(d\bar{t} + dr)^2 \tag{4}$$

for the ingoing case and

$$ds^2 = -d\bar{t}^2 + dr^2 + r^2 d\Omega^2 + \frac{2M}{r}(d\bar{t} - dr)^2 \tag{5}$$

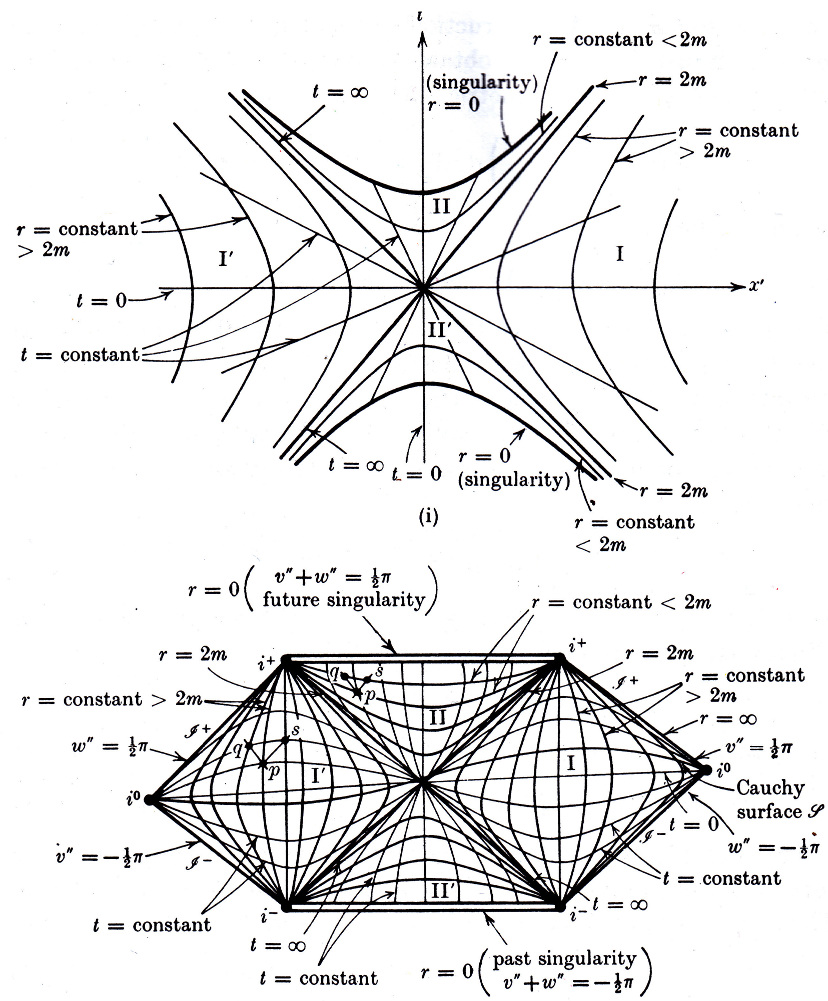
for the outgoing case, which occurs as a consequence of the fact that the Lorentzian manifold M of Schwarzschild spacetime has an atlas with two different charts, which taken together give the entire manifold.

In this comparatively simple case, however, it even turns out that a maximally analytic extension exists that covers the whole manifold, as follows directly from the consideration of so-called Kruskal-Szekeres coordinates. These are obtained from Eddington-Finkelstein coordinates by re-writing $1 - \frac{2M}{r} = \frac{2M \cdot e^{-\frac{r}{2M}}}{r} e^{\frac{v-u}{4M}}$ and by introducing $U = -4Me^{-\frac{u}{4M}}$ and $V = 4Me^{\frac{v}{4M}}$ such that the line element

$$ds^2 = -\frac{2M}{r} e^{-\frac{r}{2M}} dU dV + r^2 d\Omega^2 \tag{6}$$

results; knowing that $-\infty < U, V < \infty$. Evidently, as can be read off from the given form of the line element, Schwarzschild spacetime is no longer stationary in these coordinates.

Next, by transforming further to coordinates $\tilde{U} = \arctan(\frac{U}{4M})$ and $\tilde{V} = \arctan(\frac{V}{4M})$, where one has $-\frac{\pi}{2} < \tilde{U}, \tilde{V} < \frac{\pi}{2}$, a conformal compactification of Schwarzschild geometry can be reached unambiguously. Although an according conformal compactification can of course be constructed for all black hole spacetimes of the Kerr-Newman spacetime family, it is not possible to do so in general with respect to one single coordinate chart. In this respect, Schwarzschild definitely represents an exception from the general case. Surpressing the angular coordinates (θ, ϕ), one can draw the Kruskal and Penrose-Carter diagrams



, whereas only the latter diagram provides an illustration of cornformal infinity as well as the two singularities of spacetime occuring in this particular chart.

Considering the structure of the Weyl curvature tensor of the Schwarzschild geometry, it is found that Schwarzschild belongs to the class of Petrov-type D spacetimes. This is of great interest for the later purposes of the present work, since it means in accordance with the famous Goldberg-Sachs theorem, that all but one coefficient of the Weyl tensor can be set to zero by choosing a suitable null geodesic frame.

2.3.2 The Reissner-Nordström Black Hole

The Reissner-Nordström spacetime is a solution of Einstein's equations describing the electrovac exterior field of a collapsed, charged material object producing a spherically symmetric gravitational field. It is a unique solution of the Einstein-Maxwell equations for the ansatz of an energy-momentum tensor of a spherically symmetric electromagnetic field. In spherical coordinates, the line element of this spacetime is given by

$$ds^2 = -(1 - \frac{2M}{r} + \frac{e^2}{r^2})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{e^2}{r^2})} + r^2 d\Omega^2. \qquad (7)$$

In these coordinates, one immediately sees that the Reissner-Nordström geometry describes a charged black hole, since it looks like the Schwarzschild solution with an additional static electromagnetic potential term. Hence, by carefully taking the limit $e \to 0$, one recovers the geometry of a Schwarzschild black hole as a special case.

As a formal basis - similar to Schwarzschild spacetime - the symmetry requirement is that the metric field of the Reissner-Nordström geometry should be invariant under time translations. As a result of this requirement, one knows that there must exist a timelike Killing vector field $\xi^a = \partial_t^a$, once more a solution of Killing's equation $\nabla_{(a}\xi_{b)} = 0$, with $g_{ab}\xi^a\xi^b = g_{tt} = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})$, incorporating this symmetry. Additionally, it is required from the very beginning - again similar to the Schwarzschild case - that the given line element is invariant under rotations. Therefore, the metric is again required to be spherically symmetric, so that the introduced radial coordinate r can again be intrinsically related to the area of transitive 2-surfaces, which is encoded by the relation $r = \sqrt{\frac{A}{4\pi}}$, where A is the area of a given 2-sphere. Of course, the metric of Reissner-Nordström can be brought into the form $g_{ab} = \eta_{ab} + h_{ab}$ with $|h_{ab}| = O(r^{-1})$ and $|\partial_c h_{ab}| = O(r^{-2})$ for a, b fixed in the case of large r. Looking again at possible problematic regions, one immediately finds that in the case $M^2 < e^2$ the metric remains non-singular everywhere except for the real, irremovable naked singularity at $r = 0$. However, this case is physically uninteresting. The interesting and physically feasible case is rather $M^2 \geq e^2$. Here, the given line element becomes singular for the values $r_\pm = M \pm \sqrt{M^2 - e^2}$. According to these values the above line element can be re-written in the form

$$ds^2 = -(1 - \frac{r_+}{r})(1 - \frac{r_-}{r})dt^2 + \frac{dr^2}{(1 - \frac{r_+}{r})(1 - \frac{r_-}{r})} + r^2 d\Omega^2. \quad (8)$$

Using this expression for the line element, it can easily be seen that the Reissner-Nordström geometry remains regular in the regions defined by $\infty > r > r_+$, $r_+ > r > r_-$ and $r_- > r > 0$, whereas in the case $M^2 = e^2$ the second region does not exist. In this case, the black hole is called an extremal Reissner-Nordström black hole. The event horizon is located at $r = r_+$.

As in the Schwarzschild context, the coordinate singularities at r_+ and r_- may be removed by introducing suitable coordinates and extending the manifold to obtain a maximally analytic extension. In this context, it is possible to proceed along the same lines as presented in the previous subsection. Therefore, once again advanced and retarded null coordinates v and u given by $v = t + r^*$ and $u = t - r^*$ shall be introduced; where now $r^* = \int \frac{dr}{1 - \frac{2M}{r} + \frac{e^2}{r^2}}$ giving the following values for $r > r_+$:

$$\begin{aligned} r^* &= r + \frac{r_+^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_-^2}{r_+ - r_-} \ln |r - r_-| \quad if\ e^2 < M^2, \\ r^* &= r + 2M \ln |r - M| - \frac{2}{r - M} \quad if\ e^2 = M^2, \\ r^* &= r + M \ln |r^2 - 2Mr + e^2| - \frac{2}{e^2 - M^2} \arctan |\frac{r - M}{e^2 - M^2}| \quad if\ e^2 > M^2. \end{aligned} \quad (9)$$

The third case can be ruled out, because it is unphysical to demand $e^2 > M^2$ due to the fact that in this particular regime the radicand in $r_\pm$ can become imaginary such that no event horizon is supposed to exist. In such a situation Reissner-Nordström would contain a naked singularity, which, however, is forbidden by the weak and strong cosmic censorship conjectures.

By transforming in the manner discussed above, the line element of Reissner-Nordström spacetime takes the dual null form

$$ds^2 = -(1 - \frac{2M}{r} + \frac{e^2}{r^2}) dv du + r^2 d\Omega^2. \quad (10)$$

Using further the conversions $v = u + 2r^*$ and $u = v - 2r^*$ one can distinguish again similar to the Schwarzschild case ingoing and outgoing null coordinates. This yields

$$ds^2 = -(1 - \frac{2M}{r} + \frac{e^2}{r^2})du^2 - 2dudr + r^2 d\Omega^2 \tag{11}$$

in the ingoing case and

$$ds^2 = -(1 - \frac{2M}{r} + \frac{e^2}{r^2})dv^2 + 2dvdr + r^2 d\Omega^2 \tag{12}$$

in the outgoing case. By setting $v = \bar{t} + r$, one obtains the line element in Kerr-Schild coordinates, which gives

$$ds^2 = -d\bar{t}^2 + dr^2 + r^2 d\Omega^2 + (\frac{2M}{r} - \frac{e^2}{r^2})(d\bar{t} + dr)^2 \tag{13}$$

for the ingoing case and

$$ds^2 = -d\bar{t}^2 + dr^2 + r^2 d\Omega^2 + (\frac{2M}{r} - \frac{e^2}{r^2})(d\bar{t} - dr)^2 \tag{14}$$

for the outgoing case.

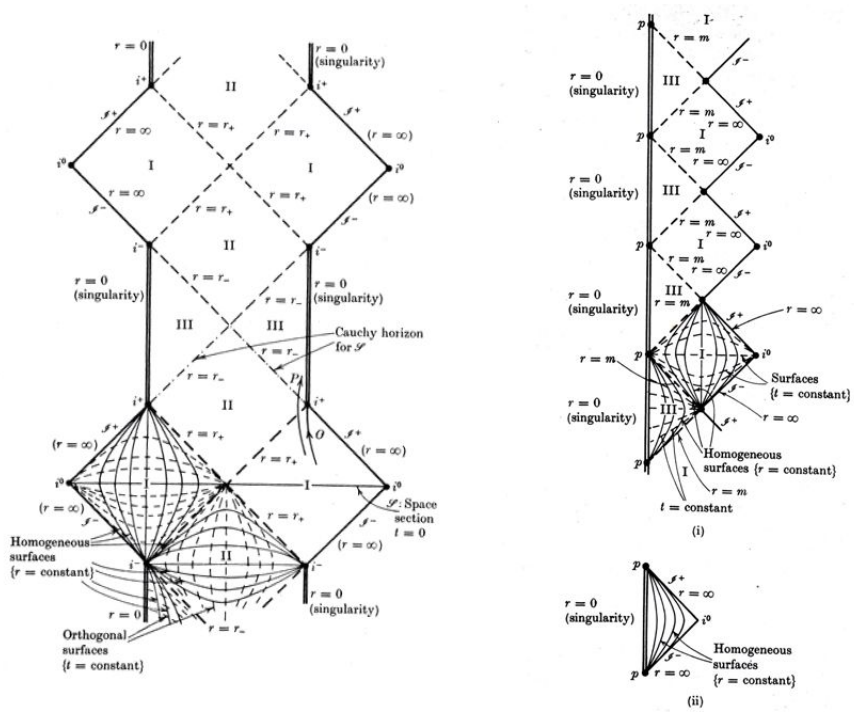
In order to try to obtain a maximal extension of Reissner-Nordström spacetime, one may perform the transformation $V' = \arctan|exp(\frac{r_+ - r_-}{4r_+^2}v)|$ and $U' = \arctan|-exp(-\frac{r_+ - r_-}{4r_+^2}u)|$ so that

$$ds^2 = -(1 - \frac{2M}{r} + \frac{e^2}{r^2}) \cdot 64 \frac{r_+^4}{(r_+ - r_-)^2} \csc|2V'| \csc|2U'| dV' dU' + r^2 d\Omega^2, \tag{15}$$

where r is now implicitly given by

$$\tan|V'| \tan|U'| = -exp((\frac{r_+ - r_-}{2r_+^2})r) \cdot (r - r_+)^{\frac{1}{2}} (r - r_-)^{\frac{\alpha}{2}} \tag{16}$$

with $\alpha := (\frac{r_-}{r_+})^2$. However, as can readily be seen, one has to use different 'Kruskal charts' to cover the whole manifold, which leads to an infinite conformal diagram. Including the pathologocal case of a naked singularity, while surpressing again the angular coordinates (θ, ϕ), there are three possibilities of drawing a Penrose-Carter diagram of maximally extended Reissner-Nordström spacetime



,

depending on whether $M^2 > e^2$, $M^2 = e^2$ or $M^2 < e^2$.

The resulting metric is analytic everywhere except for $r = r_-$, where it is degenerate. This can be remedied by transforming to new coordinates $V'' = \arctan|exp(\frac{r_+ - r_-}{2nr_-^2}v)|$ and $U'' = \arctan|-exp(-\frac{r_+ - r_-}{2nr_-^2}u)|$, where n is some integer fulfilling $n \geq 2(\frac{r_+}{r_-})^2$. In these coordinates, the metric is analytic everywhere except for $r = r_+$, where it is again degenerate. However, in the obtained atlas the metric is analytic everywhere, which is then the starting point for a conformal compactification and the introduction of boundary regions in the spirit of preceding sections. For further details, see [33].

Considering finally the structure of the Weyl curvature tensor of the Reissner-Nordström geometry, it is found that it belongs - like Schwarzschild - to the class of Petrov-type D spacetimes, which means that all but one component of the Weyl and Einstein-Maxwell tensors, respectively, can be set to zero by the use of a suitable null geodesic frame.

2.3.3 The Kerr and the Kerr-Newman Black Holes

The Kerr and the Kerr-Newman spacetimes are geometries describing the empty exterior gravitational field produced by an either rotating or rotating and charged black hole. The corresponding geometries are stationary, axisymmetric and - similar to the case of elementary particle physics - determined by three real positive definite scalar parameters, i.e. their mass, their charge and their angular momentum.

Both geometries, the geometry of Kerr and Kerr-Newman, are related in a way similar as the Schwarzschild and Reissner-Nordström spacetimes, which means that they should not be treated on the same footing. Nevertheless, this is exactly what will be done in the further course of this subsection.

This is of course problematic in that both of these geometries do describe physically distinct situations; one may view Kerr-Newman as an extended version of the Kerr geometry that is not only a solution to Einstein's equations but rather to the Einstein-Maxwell equations for the ansatz of an energy-momentum tensor of an axisymmetric electromagnetic field in addition. Kerr spacetime, as already indicated, describes the gravitational field of a rotating black hole; Kerr-Newman, on the other hand, that of a charged rotating black hole. The metric of both spacetime geometries can be read off from a line element of the form

$$ds^2 = -dt^2 + \Sigma(\frac{dr^2}{\Delta} + d\theta^2) + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{\chi}{\Sigma}(dt - a \cdot \sin^2\theta d\phi)^2 \tag{17}$$

in so-called Boyer-Lindquist coordinates. However, while for Kerr one has $\Delta = \Delta(r) = r^2 - 2Mr + a^2$ and $\chi = \chi(r) = 2Mr$, for Kerr-Newman instead there holds $\Delta = \Delta(r) = r^2 - 2Mr + a^2 + e^2$and $\chi = \chi(r) = 2Mr - e^2$. In both cases, one has $\Sigma = \Sigma(r, \theta) = r^2 + a^2\cos^2\theta$. Indeed, by carefully taking the limit $e \to 0$, one recovers the geometry of Kerr as a special case of the Kerr-Newman geometry.

In the weak field limit one certainly finds that $J = Ma$ can be interpreted as the associated angular momentum of the black hole. The quantity a therefore measures the angular momentum per mass. Accordingly, by carefully taking the limit $a \to 0$ in the Kerr-Newman case one re-obtains the line element of Reissner-Nordström spacetime and by taking the limits $a \to 0$ and $e \to 0$, one obtains that of Schwarzschild spacetime. Both, the Kerr and Kerr-Newman spacetime can be brought into the form $g_{ab} = \eta_{ab} + h_{ab}$ with $|h_{ab}| = O(r^{-1})$ and $|\partial_c h_{ab}| = O(r^{-2})$ for a, b fixed. Therefore, both of these geometries reduce to Minkowski space in polar coordinates in the limit $r \to \infty$. In the limit $M \to 0$ the given line element in fact rather reduces to Minkowski space in spheroidal coordinates; a case which needs a little attention. As M goes to zero one ends up with the line element

$$ds^2 = -dt^2 + \frac{\Sigma}{r^2 + a^2} dr^2 + (r^2 + a^2) \sin^2 \theta d\phi^2. \tag{18}$$

By performing the transformations $x = \sqrt{r^2 + a^2} \sin\theta \cos\phi$, $y = \sqrt{r^2 + a^2} \sin\theta \sin\phi$ and $z = r\cos\theta$, one then finds that the obtained line element is equal to that of Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{19}$$

as it ought to be. As a consequence of the executability of all these limits, one may realize that the given line element encodes the geometry of a whole family of spacetimes. This family is the previously addressed Kerr-Newman family of spacetimes.

Looking for potentially problematic regions, one immediately finds that the Kerr metric becomes singular for $\Delta = 0$ and $\Sigma = 0$. As the evaluation of $R_{abcd}R^{abcd}$ then shows, there is the real geometric singularity $\Sigma = 0$ and the coordinate singularity $\Delta = 0$ similar to the Schwarzschild and Reissner-Nordström cases. In closer range to the black hole singularity, i.e. in the limit $r \to 0$, one finds a disk-like singularity located at $r = 0,\ \theta = \frac{\pi}{2}$.

As opposed to Schwarzschild, which possesses a point singularity, it has the form of a ring and is therefore called a ring singularity. There are once more two values $r_\pm = M \pm \sqrt{M^2 - a^2}$ in the Kerr case and $r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}$ in the Kerr-Newman case, according to which the introduced coordinates become singular. These are solutions of the equation $\Delta = 0$, existing generically only in the restricted regime $a^2 \leq M^2$. In the cases $a^2 > M^2$ and $e^2 + a^2 > M^2$, respectively, the equation $\Delta = 0$ has no real solution so that there is no horizon hiding the singularity located at $r = 0,\ \theta = \frac{\pi}{2}$ and neither Kerr nor Kerr-Newman describe a black hole spacetime. Similarly to Reissner-Nordström, the according situations appear to be unphysical ones and are additionally ruled out once more by the cosmic censorship conjecture due to the fact that under the given assumptions both spacetimes would contain a naked singularity. Hence it suffices to look at the restricted values $a^2 \leq M^2$ and $e^2 + a^2 \leq M^2$. In this context one finds in both cases that $\Delta = \Delta(r) = (r - r_+)(r - r_-)$, whereas, once more analogous to the Reissner-Nordström situation, the event horizon is located at $r = r_+$. Any object crossing the surface located at $r = r_+$ appears to be infinitely redshifted to an observer whose worldline resides outside this surface and approaches the future i^+. In the cases $a^2 = M^2$ and $a^2 + e^2 = M^2$, respectively, both black hole solutions are referred to as extremal.

The given coordinate singularities can be removed by transforming to more suitable coordinates that remain regular on the horizon. This can be achieved by transiting to new coordinates $v = t + T$ and $\bar{\phi} = \phi + \Phi$, where $T = T(r) = r + M \ln |\Delta| + \frac{2M^2 - e^2}{r_+ - r_-} \ln |\frac{r - r_+}{r - r_-}|$ and $\Phi = \Phi(r) = \frac{a}{r_+ - r_-} \ln |\frac{r - r_+}{r - r_-}|$ are scalar functions such that $dv = dt + \partial_r T \cdot dr = dt + \frac{r^2 + a^2}{\Delta} dr$ and $d\bar{\phi} = d\phi + \partial_r \Phi \cdot dr = d\phi + \frac{a}{\Delta} dr$. In these coordinates, the line element takes the form

$$ds^2 = -(1 - \frac{2Mr - e^2}{\Sigma})dv^2 + 2dvdr + \Sigma d\theta^2 + \tag{20}$$
$$+ \frac{((r^2 + a^2)^2 - \Delta a \sin^2 \theta) \sin^2 \theta}{\Sigma} d\bar{\phi}^2 - 2a \sin^2 \theta dr d\bar{\phi} - \frac{4Mr}{\Sigma} a \sin^2 \theta dv d\bar{\phi}.$$

As it turns out, the resulting expression can be re-written in the following much more compact fashion

$$ds^2 = -dv^2 + 2dvdr + \Sigma d\theta^2 + (r^2 + a^2) \sin^2 \theta d\bar{\phi}^2 - 2a \sin^2 \theta dr d\bar{\phi} +$$
$$+ \frac{2Mr - e^2}{\Sigma}(dv - a \cdot \sin^2 \theta d\bar{\phi})^2. \tag{21}$$

By introducing then the coordinate $\bar{t} = v - r$ one can assess further

$$ds^2 = -d\bar{t}^2 + dr^2 + \Sigma d\theta^2 + (r^2 + a^2) \sin^2 \theta d\bar{\phi}^2 - 2a \sin^2 \theta dr d\bar{\phi} +$$
$$+ \frac{2Mr - e^2}{\Sigma}(d\bar{t} + dr - a \cdot \sin^2 \theta d\bar{\phi})^2. \tag{22}$$

The introduced coordinates are referred to as Kerr-coordinates in the literature.

Another set of coordinates that should be mentioned in this context are Kerr-Schild coordinates $(\bar{t}, x, y, z)$, which can be introduced with regard to Boyer-Lindquist coordinates by means of the coordinate transformations $x = \sqrt{r^2+a^2}\sin\theta\cos(\bar{\phi}+\varphi)$, $y = \sqrt{r^2+a^2}\sin\theta\sin(\bar{\phi}+\varphi)$, $z = r\cos\theta$ with $\varphi := \arctan\frac{a}{r}$. Here x,y,z can be thought of as Euclidean coordinates fulfilling

$$x^2+y^2 = (r^2+a^2)\sin^2\theta,\ z^2 = r^2\cos^2\theta \tag{23}$$

and hence

$$\frac{x^2+y^2}{(r^2+a^2)} + \frac{z^2}{r^2} = 1. \tag{24}$$

Furthermore, the surfaces of constant radius r are then specified by

$$\frac{x^2+y^2}{a^2\sin^2\theta} + \frac{z^2}{a^2\cos^2\theta} = 1. \tag{25}$$

The line element is now of the form

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + dx^2 + dy^2 + dz^2 + \\ &+ \frac{2Mr^3}{r^4+a^2z^2}(d\bar{t} + \frac{r(xdx+ydy) - a(xdy - ydx)}{r^2+a^2} + \frac{zdz}{r})^2. \end{aligned} \tag{26}$$

Defining now

$$f := \frac{2Mr^3}{r^4+a^2z^2}, \tag{27}$$

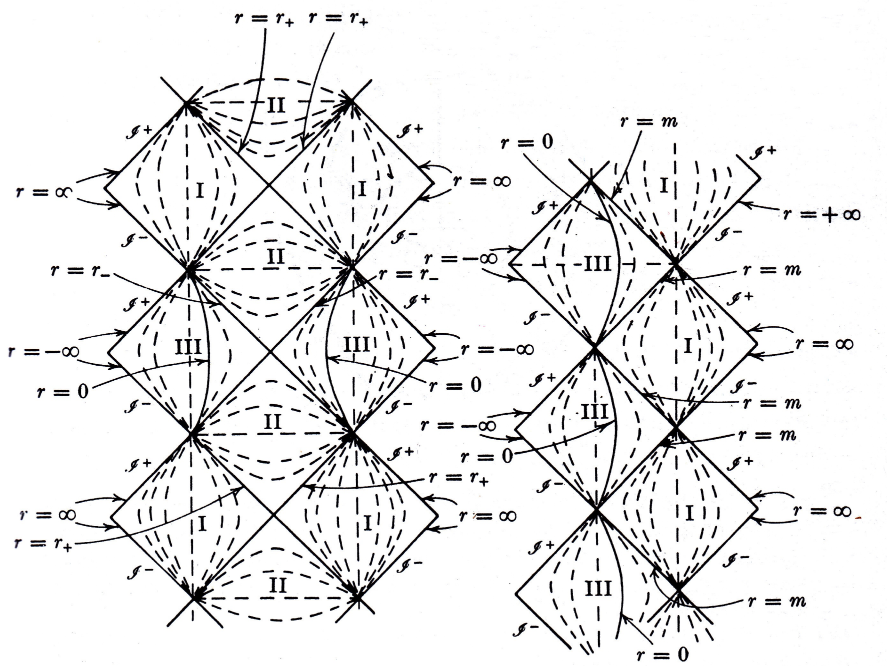
one sees using $k_a dx^a = d\bar{t} + \frac{r(xdx+ydy)-a(xdy-ydx)}{r^2+a^2} + \frac{zdz}{r}$ that the metrical field now possesses the structure

$$g_{ab} = \eta_{ab} + f k_a k_b, \tag{28}$$

defining what is called the Kerr-Schild form. In this geometry the $r = const.$ surfaces become confocal ellipsoids with focus on the ring $\rho^2 = x^2 + y^2 = a^2$, $z = 0$. For $r = 0$ the ellipsoid degenerates into a double cover of the disk $\rho \le a$, $z = 0$. In the latter region, the coordinates essentially reduce to polar coordinates (ρ, ϕ) with radius $\rho = a\sin\theta$.

Now that this has been made clear, it may be pointed out that neither the Kerr nor the Kerr-Newman solution is spherically symmetric, so that one cannot draw a Penrose diagram for these solutions. However, if one considers the submanifold of the spacetime corresponding to the axis of symmetry ($\theta = 0$ or $\theta = \pi$) then, since this submanifold is two-dimensional, one can actually draw a

Penrose diagram for it. However, similar to the case of Reissner-Nordström spacetime, the Lorentzian manifold of both these spacetimes cannot be covered by a single 'Kruskal-like' chart like Schwarzschild, which leads once again to an infinite conformal diagram of the form:



(i) (ii)

As already indicated, the given spacetimes are stationary and axisymmetric, which actually means that metric coefficients of this spacetime depend neither on the time variable t nor on the angular variable ϕ and, moreover, its associated line element is invariant, if they are combined, under time reflection transformations $t \mapsto -t$ and under transformations $\phi \mapsto -\phi$ that change the orientation of the axis of rotation. As a consequence, the given spacetimes possess a two-parameter group of isometries. There are thus two associated unique Killing vector fields ${}^{(t)}\xi^a = \partial_t^a$ and ${}^{(\phi)}\xi^a = \partial_\phi^a$, which can be combined to one vector field ξ^a via linear combination. This yields, on the horizon, the well-known Killing vector

$$\xi^a|_{\mathcal{H}_+} = \partial_t^a + \omega_+ \partial_\phi^a, \tag{29}$$

representing a solution of Killing's equation $\nabla_{(a}\xi_{b)} = 0$ with constant angular velocity factor ω_+ explicitly given by $\omega_+ = \frac{d\phi}{dt}|_{r=r_+} =$

$\frac{\dot{\phi}}{\dot{t}}|_{r=r_+} = \frac{a^2}{r_+^2+a^2}$. This constant factor ω_+ may be viewed as a sort of 'angular velocity' of the black hole as observed by a 'zamo', i.e. a special co-rotating zero angular momentum observer, at $r = r_+$.

Finally, considering the structure of the respective Weyl curvature tensors, it is found that both the Kerr and the Kerr-Newman geometries belong to the class of Petrov-type D spacetimes, which means once again that all but one component of the Weyl tensor or the Weyl and the Einstein-Maxwell tensors can be set to zero by the use of a suitable null geodesic frame. Since these spacetimes serve as background for geometric deformation in the further course of the work, this will be explained in more detail later.

3 Null Geometry

The main focus of this chapter is to give an overview of null geometric methods, in particular the spin-coefficient methods of Newman and Penrose as well as that of Geroch, Held and Penrose, and to give a brief introduction to the related theory of null geodesic congruences in general relativity. In addition, in order to lay the foundation for later arguments, the geometric framework of the lightlike foliations of spacetime and the associated 2+2-formulation of general relativity will be discussed. In between, the physics of null frames and embedded null hypersurfaces will be briefly outlined, which is deeply related with the theory of null geodesic congruences, which in turn will (retrospectively) permit a physical interpretation of the previously mentioned spin-coefficient formalisms. Finally, an overview of the hierarchy of horizons in the general theory of relativity is given, which on the one hand is important for later calculations and on the other hand provides a definition of black hole horizons, which does not presuppose knowledge of the entire past and future development of the corresponding black hole spacetime.

3.1 Null Frames, Null Foliations and embedded Null Hypersurfaces of Spacetime

This section deals with the null tetrad formulation and the closely related $2+2$-formulation of general relativity, which will serve as a geometric basis for subsequent sections of this chapter. These formulations can be used as a starting point for the construction of lightlike foliations of spacetime, which is an important point insofar as in a later phase of this work it will become necessary to identify the event horizon of a Kerr-Newman black hole as a non-expanding null hypersurface, which is embedded in a corresponding lightlike foliation of spacetime (as can be shown for any event horizon of a stationary black hole). For this reason, both the basic idea of a null foliation of spacetime and the definition of embedded null hypersurfaces are discussed below.

As a starting point, consider the following fact: Given a spacetime (M, g), its associated metric g_{ab} and the corresponding inverse g^{ab} can be decomposed with respect to a pair of so-called tetrad and co-tetrad fields E^a_μ and $e^\mu_{\ a}$, also called vierbein fields, according to the rule

$$g_{ab} = \eta_{\mu\nu} e^{\mu}_{\ a} e^{\nu}_{\ b}, \ g^{ab} = \eta^{\mu\nu} E^{a}_{\ \mu} E^{b}_{\ \nu}. \tag{30}$$

Of course, this also applies in the case that the null frame $(\ell^a, n^a, m^a, \bar{m}^a)$ and its associated co-frame $(-n_a, -\ell_a, \bar{m}_a, m_a)$ are identified as the respective null tetrad field E^a_μ and its co-tetrad $e^\mu_{\ a}$. This naturally leads to the $2+2$-decompositions

$$g_{ab} = -2\ell_{(a} n_{b)} + 2m_{(a}\bar{m}_{b)}, \ g^{ab} = -2\ell^{(a} n^{b)} + 2m^{(a}\bar{m}^{b)} \tag{31}$$

of the metric and its inverse, which, as will be seen later, play an important role not only for the given null geometric approach to general relativity, but also for the spin-coefficient framework, which will be introduced later in this chapter. It should be noted, however, that the assumption that the corresponding basis is orthonormal, which is the basis for the fact that

$$-\ell_a n^a = m_a \bar{m}^a = 1 \tag{32}$$

applies in the given context, is introduced at this point for simplicity's sake only, which is worth emphasizing, as such a choice usually proves to be unnecessary, but nevertheless practical.

In this regard, also turns out to be a practical to suppress an index of the associated vierbein fields and on this basis to consider said fields as vector-valued (or co-vector-valued) one-forms, which is the usual practice of the theory of differential forms on semi-Riemannian manifolds. In this way, it is very easy to calculate the concrete form of the exterior derivative of the said objects, which results in

$$de^{\mu} = -\omega^{\mu}_{\ \nu} \wedge e^{\nu} \tag{33}$$

and

$$dE_{\mu} = \omega^{\nu}_{\ \mu} \wedge E_{\nu}, \tag{34}$$

respectively, where, just for illustrative purposes, the former relation can be written down in the form $\nabla_{[a} e^{\nu}_{\ b]} = \partial_{[a} e^{\nu}_{\ b]} = e^{\nu}_{\ [a} \omega_{b]\ \nu}^{\ \mu}$ in index notation; provided that the occurring coefficients $\omega_{b\ \nu}^{\ \mu}$, called the Ricci rotation coefficients, are subject to the relation

$$\omega_{a\mu}^{\ \ \ \nu} = E^{b}_{\ \mu} \nabla_a e^{\nu}_{\ b} = -e^{\nu}_{\ b} \nabla_a E^{b}_{\ \mu} = -\omega_{a\ \mu}^{\ \nu}. \tag{35}$$

But what the elegant framework of differential forms is especially well suited for is the calculation of the curvature tensor, which can be achieved by considering the expression

$$d^2 E_{\ \mu} = R^{\nu}_{\ \mu} \wedge E_{\nu}, \tag{36}$$

which entails the definition

$$R^{\sigma}_{\ \rho} = d\omega^{\sigma}_{\ \rho} + \omega^{\sigma}_{\ \nu} \wedge \omega^{\nu}_{\ \rho} = 0. \tag{37}$$

As is commonly known, equations (34) and (37) are generally referred to as Cartan's equations of structure.

By using the flat Minkowski metric $\eta_{\rho\sigma} = g_{ab} E^a_{\ \rho} E^b_{\ \sigma}$, one may write down relation (37) in the form

$$\begin{aligned} R_{\rho\sigma\mu\nu} = & E^a_{\ \rho} \nabla_a \omega_{\sigma\mu\nu} - E^a_{\ \sigma} \nabla_a \omega_{\rho\mu\nu} - \\ & -\eta^{\alpha\beta} \{\omega_{\rho\beta\mu}\omega_{\sigma\alpha\nu} - \omega_{\sigma\beta\mu}\omega_{\rho\alpha\nu} + \omega_{\rho\beta\sigma}\omega_{\alpha\mu\nu} - \omega_{\sigma\beta\rho}\omega_{\alpha\mu\nu}\} = \\ & = E^a_{\ \rho} \partial_a \omega_{\sigma\mu\nu} - E^a_{\ \sigma} \partial_a \omega_{\rho\mu\nu} - \\ & -\eta^{\alpha\beta} \{\omega_{\rho\beta\mu}\omega_{\sigma\alpha\nu} - \omega_{\sigma\beta\mu}\omega_{\rho\alpha\nu} + \omega_{\rho\beta\sigma}\omega_{\alpha\mu\nu} - \omega_{\sigma\beta\rho}\omega_{\alpha\mu\nu}\} \end{aligned} \tag{38}$$

in index notation.

However, the two equations of structure (34) and (37) are not only useful for calculating curvature expressions. They can also be used to define the basic equations of the spin-coefficient formalism, whereas the main advantage of the latter approach is that it provides a collection of complex scalar equations which, in special cases, can often be studied separately. As will be shown later in the special case of the generalized Kerr-Schild class, the said approach can also be used to convert the tensorially defined Einstein equations into a system of coupled scalar equations. However, before this is explained in greater detail, a few restrictions on the null tetrad and co-tetrad fields E^a_μ and $e^\mu_{\ a}$ shall first be made.

For instance, a quite natural restriction that can be made with respect to any spacetime (M, g) is the restriction that the null tetrad E^a_μ and its co-tetrad $e^\mu_{\ a}$ should be null geodesic frames. This means that either ℓ^a or n^a should be null geodesic vector fields and should therefore satisfy either

$$(\ell\nabla)\ell^a = 0 \tag{39}$$

or

$$(n\nabla)n^a = 0. \tag{40}$$

For the sake of simplicity, only the former case shall be considered from now on whenever the term geodesic null vector field is used.

The existence of a null geodesic vector field and an associated null geodesic frame is always guaranteed if the spacetime (M, g) can be decomposed into lightlike hypersurfaces and thus a so-called null foliation of spacetime exists. Such a null foliation of spacetime is by definition a collection of embedded null hypersurfaces $\mathcal{H} = \mathcal{H}_\sigma$, also called level sets, which is defined as a collection of slices $\{\mathcal{H}_\sigma\}$ which vary smoothly in σ such that

$$M = \bigcup_\sigma \mathcal{H}_\sigma.$$

Any given null hypersurface $\mathcal{H}$ contained in (M, g) is therefore assumed to be labelled by a smooth parameter function σ, which has to be constant along $\mathcal{H}$. Therefore, it can be considered as a representative of the set $\{\mathcal{H}_\sigma\}$, which may be viewed as an ordering prescription for all lightlike hypersurfaces of (M, g), ergo as a null foliation of spacetime.

The null property of the hypersurface $\mathcal{H}$ encodes the fact that the line bundle associated with surface co-cormal $-d\sigma_a = -\nabla_a\sigma$ is lightlike, i.e.

$$g^{ab} d\sigma_a d\sigma_b = 0. \tag{41}$$

Due to the fact that this equation is identical to the Eikonal equation, one knows that a solution of this equation exists on any spacetime and that σ coincides with a phase function, which is referred to as a so-called optical function on occasion.

Setting $\ell^a := -\nabla^a\sigma$, one therefore determines a non-vanishing, smooth, lightlike vector field, which defines a closed form, meaning that there holds $\nabla_{[a}\nabla\sigma_{b]} = 0$ in full agreement with the fact that in Lorentzian geometry the torsion tensor is always identically zero. This allows one to conclude

$$\ell_a \ell^a = 0, \ \nabla_{[a}\ell_{b]} = 0, \tag{42}$$

which holds on any spacetime (M, g). A vector field $\ell^a \in T_p(\mathcal{H})$, which fulfills exactly these two properties, is called a lightlike generator of a given null hypersurface, in the given case of $\mathcal{H}$.

The two listed relations entail furthermore the instance that ℓ^a is a hypersurface orthogonal null vector field, which then further implies that it is automatically geodesic as well, which can easily be concluded from the fact that

$$0 = \ell^b \nabla_{[a} \ell_{b]} = \ell^b \nabla_a \ell_b - (\ell \nabla) \ell_a = \frac{1}{2} \nabla_a (\underbrace{\ell_b \ell^b}_{=0}) - (\ell \nabla) \ell_a.$$

The hypersurface orthogonality of ℓ_a imposes the requirement that the integral curves generated by it have to be orthogonal to $\mathcal{H}$, which, in the given context, is tantamount to the fact that ℓ_a has to fulfill the Frobenius theorem[4]

$$\ell_{[a} \nabla_b \ell_{c]} = 0. \tag{43}$$

Since $\mathcal{H}$ is lightlike, this requires ℓ^a to be an affine geodesic vector field.

However, that does not mean that ℓ_a is uniquely defined: One may realize that the listed properties of a generator remain unchanged after a multiplication with an arbitrary function $\chi = \chi(\sigma)$, by which, as a consequence, one may obtain another generator of $\mathcal{H}$ by setting $\ell_a^{'} := -d\chi_a = -\chi(\sigma) d\sigma_a$. Thus, there exists in fact a whole equivalence class of generators of $\mathcal{H}$, which shall be denoted by $[\ell]$.

While in the given case of null foliations of spacetime each generator can be assumed to be normalized with respect to an associated non-tangential null normal $n^a \in T_p(M)$ so that (32) applies, this can no longer be assumed in the case of double null foliations, also called dual null foliations. These foliations are defined by considering collections of slices $\{\mathcal{H}_\sigma\}$ and $\{\bar{\mathcal{H}}_{\bar{\sigma}}\}$, which vary smoothly in the optical parameters σ and $\bar{\sigma}$ such that

$$M = \bigcup_\sigma \mathcal{H}_\sigma$$

and

$$M = \bigcup_{\bar{\sigma}} \bar{\mathcal{H}}_{\bar{\sigma}}.$$

Therefore, the existence of such a foliation requires the existence of a smooth optical function $\bar{\sigma}$ and a closed 1-from $-d\bar{\sigma}_a$ satisfying

$$g^{ab} d\bar{\sigma}_a d\bar{\sigma}_b = 0 \tag{44}$$

[4]This is in fact clear, as $\ell_{[a} \nabla_b \ell_{c]} = 0 \iff {}^*\{\nabla_{[a} \ell_{b]}\} \ell^a = 0$.

globally on (M, g). Setting now $n_a := -d\bar{\sigma}_a$ in this context, this means that there should exist next to the generator ℓ_a another non-vanishing, smooth lightlike and hypersurface orthogonal co-vector field, which is affinely parametrized and fulfills

$$n_a n^a = 0, \ \nabla_{[a} n_{b]} = 0, \tag{45}$$

along the three-dimensional lightlike co-hypersurface $\bar{\mathcal{H}}$, which implies that also n_a fulfills Frobenius' theorem and thus

$$n_{[a} \nabla_b n_{c]} = 0. \tag{46}$$

Apart from one special case - that being the simplest case of the flat Minkowski space - the existence of a further generator strongly indicates that the corresponding pair of null normals can no longer be normalized, which in turn implies that for double null foliations, equation (32) is generally violated. Rather, there is an additional (unknown) function e^m with $m = m(x)$, which, as it will soon turn out, must be considered as an analogue to the lapse function of the $3+1$-approach to general relativity. This function is defined via the relations

$$g^{ab} d\sigma_a d\bar{\sigma}_b = -e^m, \tag{47}$$

and

$$g_{ab} \partial^a_\sigma \partial^b_{\bar{\sigma}} = -e^{-m}, \tag{48}$$

from which it can be concluded that the corresponding generating vector fields and associated co-vectors should be given by expressions of the form

$$\ell^a = e^m (\partial^a_{\bar{\sigma}} - L^a), \ \ell_a = -d\sigma_a \tag{49}$$

and

$$n^a = e^m (\partial^a_\sigma - N^a), \ n_a = -d\bar{\sigma}_a \tag{50}$$

respectively. From this, in turn, it follows that

$$\ell_a n^a = g^{ab} \ell_a n_b = g_{ab} \ell^a n^b = -e^m \tag{51}$$

applies by definition, which together with the above relations depicts that in the given dual null context can legitimately be viewed as shift vector fields, sometimes referred to as equivariant vector fields.

As a direct result, the following $2+2$-decomposition of the line element of the associated spacetime (M, g)

$$ds^2 = -2e^{-m} d\sigma d\bar{\sigma} + q_{ab}(dx^a + L^a d\sigma + N^a d\bar{\sigma})(dx^b + L^b d\sigma + N^b d\bar{\sigma}) \tag{52}$$

is obtained in the given dual null approach.

Since a simple rescaling of the form $\ell_a \to \chi \ell_a$, $n_a \to \chi^{-1} n_a$, where $\chi = \chi(\sigma)$ shall be assumed to be valid, does not change anything about the above form of the line element, but instead converts a given double null foliation into a null foliation of spacetime, it becomes clear that null foliations yield, in general, the same type of splitting of the line element and the corresponding metric. Moreover, it becomes clear that by considering different types of null geodesic frames, different types of foliations are obtained. Note here in particular that the construction of null Gaussian coordinates [10, 21, 43] results in a particularly interesting type of null foliation that is associated with a null Gaussian frame, which enables a much simpler splitting of the line element and the metric of spacetime, which is given by

$$ds^2 = -\phi d\sigma^2 + 2d\sigma d\rho + 2\beta_b dx^b d\sigma + q_{bc} dx^b dx^c, \tag{53}$$

where ϕ, β_b and q_{bc} all are functions of (σ, ρ, x^2, x^3). However, these coordinates are generally defined only locally and their existence therefore does not guarantee that spacetime is globally foliated by lightlike hypersurfaces. Nonetheless, such coordinates allow a simplified calculation of the curvature of spacetime, which undoubtedly legitimates their consideration in many cases of interest.

At this point, however, it may be emphasized that it is generally not necessary to construct a foliation of spacetime just to obtain a null geodesic vector field. Of much greater interest for the present work is, at any rate, the case of a null geodesic vector field, which is orthogonal only with respect to a single lightlike null hypersurface and thus fulfills

$$\ell_a \ell^a = 0,\ (\ell\nabla)\ell^a = 0,\ \nabla_{[a}\ell_{b]}|_{\mathcal{N}} = 0 \tag{54}$$

in respect to some lightlike hypersurface $\mathcal{N}$.

However, to prove the existence of a vector field exhibiting such properties, one cannot argue purely intrinsically. Rather, it proves necessary to identify $\mathcal{N}$ as a particular folium of a corresponding null foliation of spacetime, to ensure that certain scalar quantities, i.e. certain spin-coefficients, are well-defined quantities that can be obtained by applying the directional derivative $n^a\nabla_a$ of the non-tangential vector field n^a to the different components of $(\ell^a, n^a, m^a, \bar{m}^a)$. In particular, since a black hole event horizon is always a lightlike hypersurface (in the case that the black hole is stationary), its identification as a particular folium $\mathcal{H} = \mathcal{H}_0$ of a family of null hypersurfaces $\{\mathcal{H}_\sigma\}$ will allow one to convert non-intrinsically defined quantities into intrinsically defined ones, which will be necessary later to ensure the existence of a geodesic null vector field of the above form. The fact that such an identification is actually possible is clarified in a related work on the subject [11], which deals specifically with the problem of constructing lightlike foliations of black hole spacetimes. This will be very useful for the 'Kerr-Schild program' introduced later in this work, which will serve as the starting point for the calculation of the gravitational field of an ultrarelativistic particle at the event horizon of a black hole.

3.2 Spin-Coefficients

In this section, different mathematical instrumentaria for the development of the theory of null geodesic congruences are introduced. These instrumentaria are on the one hand the well-known spin-coefficient formalism of Newman and Penrose (NP formalism) and on the other hand its extension, the formalism of Geroch, Held and Penrose (GHP formalism), both of which expand the scope of the standard null tetrad formalism discussed in the previous section. After the introduction of these formalisms, fundamental aspects of the theory of null congruences are discussed retrospectively, which are then used for the physical interpretation of the above abstract mathematical methods. As an adjunct, further aspects of the theory of null congruences are presented, which will play a role in the further course of this work.

3.2.1 The Newman-Penrose Spin-Coefficient Formalism

As already anticipated, the tetrad formalism introduced in the previous section is closely related to the spin-coefficient formalism to be presented in this section. In fact, the spin-coefficient approach can even be considered a natural foundation of the tetrad formalism in that it provides exactly the same results, but deals with complex-valued so-called spinor fields (of which vector fields and tensor fields are a special case), which are often easy to manipulate and handle in practice. In this regard, the major plus of the spin-coefficient method is that it not only has a naturally inherent geometric interpretation, but, because it leads to the same scalar geometric invariants as the null tetrad formalism, it also allows the extraction of systems of complex-valued scalar equations from often very complicated relations between vector or tensor fields. In the end, this is why it has been widely used in finding elegant solutions to difficult technical and geometric problems in the dual null approach as well as in other areas of the broad field of general relativity.

In any case, due to the variety and richness of the said formalism, it shall be made clear from the outset that the present section deals only with those aspects of spin-coefficient formalism that are useful for the given work, despite being based on more detailed introductions to both formalisms, as they are given for example in [50, 51] and additionally in [25, 44]. In this respect, this section focuses primarily on the presentation of relevant characteristics of the above-mentioned formalism and only briefly addresses some aspects of this broad area of research in a nutshell.

Correspondingly, in order to introduce the relevant concepts and ideas of the spin-coefficient formalism, now again a null tetrad field $(\ell^a, n^a, m^a \bar{m}^a)$ shall be considered, so that

$$g_{ab} = -\ell_a n_b - n_a \ell_b + m_a \bar{m}_b + \bar{m}_a m_b. \tag{55}$$

Then one of the main observations, which allows what may be called a 'spinorial re-formulation' of the given approach, is the observation that any of the given vector fields can be re-written as a product of certain associable spinor fields $(o^A, \iota^A; o^{A'}, \iota^{A'})$, which form a basis of an associated spinor bundle $\mathfrak{S}$. For the geometrical interpretation of the corresponding spin-frame one may imagine, following [50, 51], each spinor to be visualized by a flag which possesses a flagpole and lies in a certain flag plane.

Given this geometric input, the subject shall now be approached by considering the so-called Infeld-van der Waerden symbols $g_a^{AA'}$ and ${g_{AA'}}^a$, which allow one re-express the null tetrad frame $(\ell^a, n^a, m^a \bar{m}^a)$

given above in terms of the spin-vectors $(o^A, \iota^A; o^{A'}, \iota^{A'})$ in the form

$$\ell^a = g^a_{AA'} o^A o^{A'},\ m^a = g^a_{AA'} o^A \iota^{A'},\ \bar{m}^a = g^a_{AA'} \iota^A o^{A'},\ n^a = g^a_{AA'} \iota^A \iota^{A'} \tag{56}$$

where $o^A \iota_A = o^{A'} \iota_{A'} = 1$ applies by definition. A consistent choice in this regard, which seems natural in view of the convention $(-, +, +, +)$, is $g_{AA'}{}^a := i\sigma_{AA'}{}^a$ and $g^{AA'}_a := i\sigma^{AA'}{}_a$, where the specified objects $\sigma_{AA'}{}^a$ and $\sigma^{AA'}{}_a$ can be defined in matrix language using the two-dimensional unit matrix as zeroth component and the famous Pauli matrices $\sigma_i := (\sigma_{AA'})^i = {}_i(\sigma^{AA'})$ as remaining components, where $i = 1, 2, 3$.

By means of these Infeld-van der Waerden symbols $g^{AA'}_a$ and $g_{AA'}{}^a$, the spinor based derivative operators $\nabla_{AA'} = g_{AA'}{}^a \nabla_a$ and $\nabla^{AA'} = g^{AA'}_a \nabla^a$ can now be formed. With respect to these particular quantities, the four different derivative operators

$$\begin{aligned} D &:= o^A o^{A'} \nabla_{AA'} = \ell^a \nabla_a \\ \delta &:= o^A \iota^{A'} \nabla_{AA'} = m^a \nabla_a \\ \delta' &:= \iota^A o^{A'} \nabla_{AA'} = \bar{m}^a \nabla_a \\ D' &:= \iota^A \iota^{A'} \nabla_{AA'} = n^a \nabla_a \end{aligned} \tag{57}$$

can be defined. In addition, the object ε_{AB}, which represents the two-dimensional, spinorial counterpart of the metrical fundamental form g_{ab} of the four-dimensional spacetime setting, can then be decomposed according to that choice as $\varepsilon_{AB} = o_A \iota_B - \iota_A o_B$.

Given this setting, it should however be emphasized that this particular choice for the Infeld-van der Waerden symbols is often viewed as a rather unusual route toward the spinor formalism, in that it requires one not to consider the set of objects $Herm(\mathfrak{S} \otimes T(M))$, where $\mathfrak{S}$ is the spinor bundle and $T(M)$ the tangent bundle associated with M, which is usually done for spacetimes with the signature convention $(+, -, -, -)$, but rather the set $Antiherm(\mathfrak{S} \otimes T(M))$, which is reflected by the fact that the corresponding soldering form defines an anti-Hermitian rather than a Hermitian algebraic structure, so that $\bar{g}_{AA'}{}^a = -g_{AA'}{}^a$ applies in the given context instead of $\bar{g}_{AA'}{}^a = g_{AA'}{}^a$ as usual.

And although such a choice is of course always consistent, one may take another route toward the subject by realizing that - based on an identification of $T(M)$ with $\mathfrak{S} \otimes \bar{\mathfrak{S}}$ - tensor fields (and thus of course vector fields as well) can always be regarded as specific spinor fields, which can be constructed by considering products of basis spinors. To be more specific, it is always possible re-express the said null tetrad $(\ell^a, n^a, m^a \bar{m}^a)$ by using abstract index notation [50], according to which each individual tensor index is re-interpreted as a corresponding pair of (capital) spinor indices, one unprimed and the other primed (e.g. $a = AA'$), and therefore to give alternative definitions to (56) via considering the expressions

$$\ell^a = o^A o^{A'},\ m^a = o^A \iota^{A'},\ \bar{m}^a = \iota^A o^{A'},\ n^a = \iota^A \iota^{A'}, \tag{58}$$

whereas it may be assumed that $(o^A)' = i\iota^A$, $(\iota^A)' = io^A$ and thus $o_{A'}\iota^{A'} = (o_A \iota^A)' = -\iota_A o^A = o_A \iota^A = 1$ applies in the given context.

Using the fact that there holds $\varepsilon_0{}^A = o^A$ and $\varepsilon_1{}^A = \iota^A$, it is possible to give an exact definition of spin-coefficients by considering the expressions

$$\gamma_{AA'\underline{C}}{}^{\underline{B}} := \varepsilon_D{}^{\underline{B}} \nabla_{AA'} \varepsilon_{\underline{C}}{}^D = -\varepsilon_{\underline{C}}{}^D \nabla_{AA'} \varepsilon_D{}^{\underline{B}}, \tag{59}$$

where those indices that are fixed are identified by the use of the underbar symbol. The corresponding spin-coefficients have the explicit form

$$
\begin{aligned}
&\epsilon := \gamma_{00'0}{}^0 = \iota^A D o_A = \tfrac{1}{2}(-n^a D\ell_a + m^a D\bar{m}_a)\\
&\kappa := -\gamma_{00'0}{}^1 = o^A D o_A = -m^a D\ell_a\\
&\tau' := -\gamma_{00'1}{}^0 = -\iota^A D\iota_A = -\bar{m}^a D n_a\\
&\gamma' := \gamma_{00'1}{}^1 = -o^A D\iota_A = \tfrac{1}{2}(-\ell^a D n_a + m^a D\bar{m}_a)\\
&\beta := \gamma_{01'0}{}^0 = \iota^A \delta o_A = \tfrac{1}{2}(-n^a \delta\ell_a + m^a \delta\bar{m}_a)\\
&\sigma := -\gamma_{01'0}{}^1 = o^A \delta o_A = -m^a \delta\ell_a\\
&\rho' := -\gamma_{01'1}{}^0 = -\iota^A \delta\iota_A = -\bar{m}^a \delta n_a\\
&\alpha' := \gamma_{01'1}{}^1 = -o^A \delta\iota_A = \tfrac{1}{2}(-\ell^a \delta n_a + m^a \delta\bar{m}_a)\\
&\alpha := \gamma_{10'0}{}^0 = \iota^A \delta' o_A = \tfrac{1}{2}(-n^a \delta'\ell_a + m^a \delta'\bar{m}_a)\\
&\rho := -\gamma_{10'0}{}^1 = o^A \delta' o_A = -m^a \delta'\ell_a\\
&\sigma' := -\gamma_{10'1}{}^0 = -\iota^A \delta'\iota_A = -\bar{m}^a \delta' n_a\\
&\beta' := \gamma_{10'0}{}^1 = -o^A \delta'\iota_A = \tfrac{1}{2}(-\ell^a \delta' n_a + m^a \delta'\bar{m}_a)\\
&\gamma := \gamma_{11'0}{}^0 = \iota^A D' o_A = \tfrac{1}{2}(-n^a D'\ell_a + m^a D'\bar{m}_a)\\
&\tau := -\gamma_{11'01}{}^0 = o^A D' o_A = -m^a D'\ell_a\\
&\kappa' := -\gamma_{11'1}{}^0 = -\iota^A D'\iota_A = -\bar{m}^a D' n_a\\
&\epsilon' := \gamma_{11'1}{}^1 = -o^A D'\iota_A = \tfrac{1}{2}(-\ell^a D' n_a + m^a D'\bar{m}_a).
\end{aligned}
$$

However, there is a simpler way to introduce the given set of spin-coefficients, namely by simply considering the basic spinorial relations

$$
\begin{aligned}
Do_A &= \epsilon o_A - \kappa \iota_A, \\
\delta o_A &= \beta o_A - \sigma \iota_A, \\
\delta' o_A &= \alpha o_A - \rho \iota_A, \\
D' o_A &= \gamma o_A - \tau \iota_A, \\
D \iota_A &= \gamma' \iota_A - \tau' o_A, \\
\delta \iota_A &= \alpha' \iota_A - \rho' o_A, \\
\delta' \iota_A &= \beta' \iota_A - \sigma' o_A, \\
D' \iota_A &= \epsilon' \iota_A - \kappa' o_A,
\end{aligned} \tag{60}
$$

where the associated dual relations can be easily calculated by raising the occurring index by means of the skew-symmetric quantity $\varepsilon^{AB} = o^A \iota^B - \iota^A o^B$. Next, by looking at the decomposition relation $\nabla_{AA'} = -\iota_A \iota_{A'} D' - o_A o_{A'} D + \iota_A o_{A'} \delta + o_A \iota_{A'} \delta'$, it is straightforward to write down the components of $\gamma_{AA'\underline{C}}{}^{\underline{B}}$ in full agreement with relation (59).

The physical relevance of the resulting scalar quantities shall be illustrated below by considering the null tetrad approach and the physics of null geodesic congruences to be presented in subsequent sections of this work. However, their geometric and mathematical relevance can easily be illustrated by emphasizing that the listed spin-coefficients are nothing but linear combinations of certain components of the Ricci rotation coefficients introduced at the beginning of the previous section and therefore are null curvature expressions.

To see this explicitly, define $e^0_a = -n_a$, $e^1_a = -\ell_a$, $e^2_a = \bar{m}_a$, $e^3_a = m_a$ and $E^a_0 = \ell^a$, $E^a_1 = n^a$, $E^a_2 = m^a$, $E^a_3 = \bar{m}^a$. According to these definitions, one finds

$$
\begin{aligned}
&\kappa := \omega_{020} \\
&\epsilon := \tfrac{1}{2}(\omega_{010} - \omega_{023}) \\
&\rho := \omega_{231} \\
&\alpha := \tfrac{1}{2}(\omega_{301} - \omega_{323}) \\
&\sigma := \omega_{202} \\
&\beta := \tfrac{1}{2}(\omega_{201} - \omega_{232}) \\
&\tau := \omega_{102} \\
&\gamma := \tfrac{1}{2}(\omega_{101} - \omega_{123})
\end{aligned}
$$

$\kappa' := \omega_{131}$
$\epsilon' := \frac{1}{2}(\omega_{110} - \omega_{132})$
$\rho' := \omega_{302}$
$\alpha' := \frac{1}{2}(\omega_{210} - \omega_{223})$
$\sigma' := \omega_{313}$
$\beta' := \frac{1}{2}(\omega_{310} - \omega_{323})$
$\tau' := \omega_{013}$
$\gamma' := \frac{1}{2}(\omega_{010} - \omega_{023})$.

The main advantage of the introduction of the complex valued spin-coefficients over null Ricci rotation coefficients and Christoffel symbols is that one only has to introduce twelve complex quantities rather than twenty-four real quantities needed to define the tetrad structure or the forty independent coefficients needed to define the Christoffel symbols.

The operators D, δ, δ', D' possess the following spin-coefficient expansion when applied to the four null vector fields ℓ^a, n^a, m^a, $\bar{m}^a$:

$$
\begin{aligned}
&D\ell^a = (\epsilon+\bar{\epsilon})\ell^a - \bar{\kappa}m^a - \kappa\bar{m}^a, \quad Dn^a = (\gamma'+\bar{\gamma}')n^a - \tau' m^a - \bar{\tau}'\bar{m}^a \\
&Dm^a = (\epsilon+\bar{\gamma}')m^a - \bar{\tau}'\ell^a - \kappa n^a, \quad D\bar{m}^a = (\gamma'+\bar{\epsilon})\bar{m}^a - \tau'\ell^a - \bar{\kappa}n^a \\
&D'\ell^a = (\gamma+\bar{\gamma})\ell^a - \bar{\tau}m^a - \tau\bar{m}^a, \quad D'n^a = (\epsilon'+\bar{\epsilon}')n^a - \kappa' m^a - \bar{\kappa}'\bar{m}^a \\
&D'm^a = (\gamma+\bar{\epsilon}')m^a - \bar{\kappa}'\ell^a - \tau n^a, \quad D'\bar{m}^a = (\epsilon'+\bar{\gamma})\bar{m}^a - \kappa'\ell^a - \bar{\tau}n^a \\
&\delta\ell^a = (\beta+\bar{\alpha})\ell^a - \bar{\rho}m^a - \sigma\bar{m}^a, \quad \delta n^a = (\alpha'+\bar{\beta}')n^a - \rho' m^a - \bar{\sigma}'\bar{m}^a \\
&\delta m^a = (\beta+\bar{\beta}')m^a - \bar{\sigma}'\ell^a - \sigma n^a, \quad \delta\bar{m}^a = (\alpha'+\bar{\alpha})\bar{m}^a - \rho'\ell^a - \bar{\rho}n^a \\
&\delta'\ell^a = (\alpha+\bar{\beta})\ell^a - \bar{\sigma}m^a - \rho\bar{m}^a, \quad \delta' n^a = (\beta'+\bar{\alpha}')n^a - \sigma' m^a - \bar{\rho}'\bar{m}^a \\
&\delta' m^a = (\alpha+\bar{\alpha}')m^a - \bar{\rho}'\ell^a - \rho n^a, \quad \delta'\bar{m}^a = (\beta'+\bar{\beta})\bar{m}^a - \sigma'\ell^a - \bar{\sigma}n^a.
\end{aligned}
$$

In addition, commutator relations between arbitrary combinations of the four differential operators D,D', δ, δ' can be expressed in terms of spin-coefficients, i.e.

$$
\begin{aligned}
&DD' - D'D = (\gamma+\bar{\gamma})D - (\gamma'+\bar{\gamma}')D' + (\tau' - \bar{\tau})\delta - (\tau - \bar{\tau}')\delta' \\
&\delta D - D\delta = (\beta + \bar{\alpha} + \bar{\tau}')D + \kappa D' - \sigma\delta - (\epsilon + \bar{\gamma}' + \bar{\rho})\delta \\
&\delta D' - D'\delta = \bar{\kappa}' D + (\tau + \alpha' + \bar{\beta}')D' - \bar{\sigma}'\delta - (\gamma + \bar{\epsilon}' + \rho')\delta' \\
&\delta D' - D'\delta = (\rho' - \bar{\rho}')D - (\rho - \bar{\rho})D' + (\alpha' + \bar{\alpha})\delta - (\alpha + \bar{\alpha}')\delta'.
\end{aligned}
$$

Note that the requirement that $o^A\iota_A = o^{A'}\iota_{A'} = 1$ reduces the number of independent spin-coefficients, since this implies that $\alpha = -\beta'$,

$\epsilon = -\gamma'$ and accordingly that $\beta = -\alpha'$, $\gamma = -\epsilon'$. It is customary to employ the symbols π, λ, μ, ν for the spin-coefficients $-\tau', -\sigma', -\rho', -\kappa'$ in this case.

Due to the two-dimensionality of the spin space, a general spinor $X^{CD...}_{AB...}$ can always be decomposed in a purely symmetric part $X^{(CD...)}_{(AB...)}$ and a purely antisymmetric part $X^{[CD...]}_{[AB...]}$ in such a way that the corresponding antisymmetric part is fully specified by contractions of the skew-symmetric objects ε_{AB}, ε^{A}_{B} and ε^{AB}. Accordingly, calculating the expressions

$$\Delta_{AB}\alpha^{C} = X_{ABD}{}^{C}\alpha^{D}$$

and

$$\Delta_{AB}\alpha^{C'} = \Phi_{ABD'}{}^{C'}\alpha^{D'}$$

with respect to some spinor α^{C}, where $\Delta_{AB} := \nabla_{A'(A}\nabla^{A'}_{B)} = \varepsilon^{A'B'}\nabla_{[a}\nabla_{b]}$ and

$$X_{ABD}{}^{C} = \Psi_{ABD}{}^{C} + \Pi\sigma_{ABD}{}^{C}$$

are given by definition with respect to the symmetrizer $\sigma_{CD}{}^{AB} = \varepsilon^{A}_{C}\varepsilon^{B}_{D} + \varepsilon^{B}_{C}\varepsilon^{A}_{D}$ and the scalar field $\Pi = \frac{R}{24}$, yields two relevant spinor fields $\Phi_{ABD}{}^{C}$ and $\Psi_{ABD}{}^{C}$, whose individual components coincide with those of the Weyl and the stress-energy tensors. More precisely, using the definitions $\Psi_0 = \Psi_{ABCD}o^{A}o^{B}o^{C}o^{D}$, ..., and by comparing these components with those of the said tensors, it is found that

$$\begin{aligned}
\Psi_0 &= C_{abcd}\ell^a m^b \ell^c m^d, \ \ \Psi_1 = C_{abcd}\ell^a m^b \ell^c n^d,\\
\Psi_2 &= C_{abcd}\ell^a m^b \bar{m}^c n^d, \ \ \Psi_3 = C_{abcd}\ell^a n^b \bar{m}^c n^d,\\
\Psi_4 &= C_{abcd}\bar{m}^a n^b \bar{m}^c n^d, \ \ \Phi_{00} = \frac{1}{2}R_{ab}\ell^a\ell^b\\
\Phi_{01} &= \frac{1}{2}R_{ab}\ell^a m^b, \ \ \Phi_{02} = \frac{1}{2}R_{ab}m^a m^b,\\
\Phi_{10} &= \frac{1}{2}R_{ab}\ell^a \bar{m}^b, \ \ \Phi_{11} = \frac{1}{2}R_{ab}\ell^a n^b + 3\Pi,\\
\Phi_{12} &= \frac{1}{2}R_{ab}m^a n^b, \ \ \Phi_{20} = \frac{1}{2}R_{ab}\bar{m}^a \bar{m}^b,\\
\Phi_{21} &= \frac{1}{2}R_{ab}n^a \bar{m}^b, \ \ \Phi_{22} = \frac{1}{2}R_{ab}n^a n^b.
\end{aligned}$$

As is well known, these components are subject to a set of coupled first order differential relations built from different combinations of spin-coefficients and associated directional derivatives of these spin-coefficients, which are generally referred to as the Newman-Penrose spin-coefficient equations [50].

The next section will show that there are natural transformations of spin-coefficients and that the act of choosing a certain basis naturally leads to an extended version of the investigated formalism of spin-coefficients, the formalism of Geroch, Held and Penrose, also called GHP formalism for short.

3.2.2 The Geroch-Held-Penrose Spin-Coefficient Formalism

After fixing a spin basis, the components of spinors have a simple scaling behavior. As it then turns out, spin quantities (strictly speaking: the spinors o^A and ι^A) can be transformed either by so-called boost-weighting or by so-called spin-weighting transformations. This can be seen in the spin-coefficient formalism of Geroch, Held and Penrose, which takes this exact fact into account.

Regarding once again the setting introduced in the previous section, the most general change in the dyad fields preserving the orthonormality condition $o^A \iota_A = o^{A'} \iota_{A'} = 1$ is

$$o^A \mapsto \lambda o^A, \quad \iota^A \mapsto \lambda^{-1} \iota^A. \tag{61}$$

By carrying out this transformation, the four-dimensional lightlike vector fields entering the null tetrad E^a_α rescale in the following way

$$\ell^a \mapsto \lambda\bar{\lambda}\ell^a, \; m^a \mapsto \lambda\bar{\lambda}^{-1} m^a, \; \bar{m}^a \mapsto \bar{\lambda}\lambda^{-1}\bar{m}^a, \; n^a \mapsto (\lambda\bar{\lambda})^{-1} n^a. \tag{62}$$

For some real R and some complex phase factor $e^{i\theta}$, one can then define w.l.o.g. $\lambda := \sqrt{Re^{i\theta}}$, which allows one to re-express (62) in the form

$$\ell^a \mapsto R\ell^a, \; n^a \mapsto R^{-1} n^a, \quad m^a \mapsto e^{i\theta} m^a, \; \bar{m}^a \mapsto e^{-i\theta}\bar{m}^a. \tag{63}$$

Any given weighted scalar spinorial quantity $\eta = \eta(o, \iota)$, obtained by a certain combination of contractions of o^A, $o^{A'}$ and ι^B, $\iota^{B'}$ with some multi-indexed object $\eta_{A \dots A' \dots B \dots B' \dots}$, can then be used to transform the said multilinear form $\eta = \eta(o, \iota)$ to an object $\tilde{\eta} = \tilde{\eta}(\tilde{o}, \tilde{\iota}) = \eta(\tilde{o}, \tilde{\iota})$ with $\tilde{o}^A := \lambda o^A$, $\tilde{o}^{A'} := \bar{\lambda} o^{A'}$ and $\tilde{\iota}^B := \lambda^{-1}\iota^B$, $\tilde{\iota}^{B'} := \bar{\lambda}^{-1}\iota^{B'}$ according to the rule

$$\eta \mapsto \lambda^p \bar{\lambda}^q \eta =: \tilde{\eta}. \tag{64}$$

In this context, η is usually referred to as a spinor of type $\{p, q\}$, which possesses the spin-weight $s := \frac{1}{2}(p-q)$ and the boost-weight $b := \frac{1}{2}(p+q)$. In this vein, o^A, $o^{A'}$ and ι^A, $\iota^{A'}$ may themselves be viewed as spinors of type $\{1,0\}$, $\{0,1\}$, $\{-1,0\}$ and $\{0,-1\}$ and l^a, n^a, m^a, $\bar{m}^a$ as vectors of type $\{1,1\}$, $\{-1,-1\}$, $\{1,-1\}$ and $\{-1,1\}$ respectively. Of course, not all spin-coefficients do retain their structure under the given transformations. Rather, transforming o^A, $o^{A'}$ and ι^A, $\iota^{A'}$ divides the whole set of spin-coefficients into two different classes: into one class of coefficients, consisting of $\{\epsilon, \alpha, \beta, \gamma; \epsilon', \alpha', \beta', \gamma'\}$, which are not only weighted, but also are shifted by derivatives of λ and λ^{-1} respectively, and into another class of coefficients, consisting of $\{\kappa, \rho, \sigma, \tau; \kappa', \rho', \sigma', \tau'\}$, which are just weighted, but are not shifted by according terms involving derivatives of λ and λ^{-1}. These coefficients have the weights $\kappa : \{3,1\}$, $\sigma : \{3,-1\}$, $\rho : \{1,1\}$, $\tau : \{1,-1\}$, $\kappa' : \{-3,-1\}$, $\sigma' : \{-3,1\}$, $\rho' : \{-1,-1\}$, $\tau' : \{-1,1\}$.

As can easily be verified, the derivative operators D, D' and δ,δ' do not map weighted quantities to weighted quantities. A standard approach within the given formalism is therefore the introduction of the derivative operators ð (eth) and þ (thorn), whereas ð, ð′ and þ, þ′ can be viewed as extensions of the four derivative operators D, D' and δ,δ'. With regard to some spinorial quantity η of type $\{p, q\}$ there is the connection

$$\begin{aligned} þ\eta &= (D - p\epsilon - q\bar{\epsilon})\eta, \\ ð\eta &= (\delta - p\beta - q\bar{\alpha})\eta, \\ ð'\eta &= (\delta' - p\alpha - q\bar{\beta})\eta, \\ þ'\eta &= (D' - p\gamma - q'\bar{\gamma})\eta. \end{aligned}$$

between the differential operators ð, ð′ and þ, þ′ and D, D' and δ,δ'. The weighting transformations then yield the transitions

$$\begin{aligned} þ\eta &\mapsto \lambda^{p+1}\bar{\lambda}^{q+1}þ\eta, \\ ð\eta &\mapsto \lambda^{p+1}\bar{\lambda}^{q-1}ð\eta, \\ ð'\eta &\mapsto \lambda^{p-1}\bar{\lambda}^{q+1}ð'\eta, \\ þ'\eta &\mapsto \lambda^{p-1}\bar{\lambda}^{q-1}þ'\eta. \end{aligned}$$

Thus the occurring operators have the types þ : $\{1,1\}$,ð : $\{1,-1\}$, ð′ : $\{-1,1\}$, þ′ : $\{-1,-1\}$.

The extra terms entering the definition of $\eth$, $\eth'$ and þ, $\text{þ}'$ are chosen in such a way that they cancel the occurring derivatives of λ and μ in the definitions of the weighted spin-coefficients $\{\epsilon, \alpha, \beta, \gamma; \epsilon', \alpha', \beta', \gamma'\}$. For the spinor fields o^A, $o^{A'}$ and ι^A, $\iota^{A'}$ one has the differential relations

$$
\begin{aligned}
&\text{þ} o^A = -\kappa\iota^A, \ \ \text{þ}\iota^A = -\tau' o^A, \ \ \text{þ} o^{A'} = -\bar{\kappa}\iota^{A'}, \ \ ,\text{þ}\iota^{A'} = -\bar{\tau}' o^{A'},\\
&\eth o^A = -\sigma\iota^A, \ \ \eth\iota^A = -\rho' o^A, \ \ \eth o^{A'} = -\bar{\rho}\iota^{A'}, \ \ ,\eth\iota^{A'} = -\bar{\sigma}' o^{A'},\\
&\eth' o^A = -\rho\iota^A, \ \ \eth\iota^A = -\sigma' o^A, \ \ \eth o^{A'} = -\bar{\sigma}\iota^{A'}, \ \ ,\eth\iota^{A'} = -\bar{\rho}' o^{A'},\\
&\text{þ}' o^A = -\tau\iota^A, \ \ \text{þ}'\iota^A = -\kappa' o^A, \ \ \text{þ}' o^{A'} = -\bar{\tau}\iota^{A'}, \ \ ,\text{þ}'\iota^{A'} = -\bar{\kappa}' o^{A'}.
\end{aligned}
$$

In addition, one may form the following commutators of the regarded differential operators, which, when applied to a $\{p;q\}$-scalar η, read

$$
\begin{aligned}
&\text{þ}\text{þ}' - \text{þ}'\text{þ} = (\bar{\tau} - \tau')\eth + (\tau - \bar{\tau}')\eth' - p(\kappa\kappa' - \tau\tau' + \Psi_2 + \Phi_{11} - \Pi) -\\
&- q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\Psi}_2 + \Phi_{11} - \Pi),\\
&\text{þ}\eth - \eth\text{þ} = \bar{\rho}\eth + \sigma\eth' - \kappa\text{þ} - \kappa\text{þ}' - p(\rho'\kappa - \tau'\sigma + \Psi_1) -\\
&- q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}' + \Phi_{01}),\\
&\eth\eth' - \eth'\eth = (\bar{\rho}' - \rho')\text{þ} + (\rho - \bar{\rho})\text{þ}' + p(\rho\rho' - \sigma\sigma' + \Psi_2 + \Phi_{11} - \Pi) -\\
&- q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \bar{\Psi}_2 + \Phi_{11} - \Pi).
\end{aligned}
$$

It is also possible to convert the covariant d'Alembertian $\Box_g := \nabla^2 = \nabla_a\nabla^a$ by means of the defined differential operators $\eth$, $\eth'$ and þ, $\text{þ}'$. It takes the form

$$
\Box_g = 2(\text{þ}\text{þ}' - \eth'\eth - \bar{\rho}'\text{þ} - \rho\text{þ}' - \bar{\tau}\eth - \tau\eth') \tag{65}
$$

when applied to $\{0,0\}$-scalars. This does not actually yield a simpler expression, as a mixture of first and second derivatives of $\eth$, $\eth'$ and þ, $\text{þ}'$ are involved in the obtained differential operator.

3.3 Null Geodesic Congruences

Having discussed some important aspects of the null tetrad and the standard and compacted spin-coefficient approaches, the mathematical techniques required for the description of null geodesic congruences shall be presented next. For the sake of simplicity, the geometric setting used as a starting point for this description is chosen in this regard to be essentially the same as that given to obtain a null foliation of spacetime. In view of this, the best way to start the discussion is probably to first define a null geodesic congruence in relation to the said geometric setting, and then to draw similarities to the spin-coefficient formalisms discussed in the previous sections.

In this respect, considering a spacetime (M, g) and an open subset $\mathcal{O} \subset M$, a congruence is a family of non-intersecting spacetime filling curves having a fixed causal structure (so that their associated tangent fields are either spacelike, timelike or null), defined in such a way that through $\mathcal{O}$ there passes precisely one curve of the given family. Generated by a collection of non-intersecting, parametrized integral curves associated with a non-vanishing continuous vector field tangent to $\mathcal{O}$, a line congruence (that is, a congruence of curves) does possess a certain causal structure inherited by its tangent fibers. Therefore, a congruence is called either timelike, spacelike or null if and only if there is an appropriate family of continuous timelike, spacelike or null curves generating it. Accordingly, a congruence is called geodesic if its generating vector field is geodesic.

Focusing exclusively on null geodesic congruences that generate freely propagating light rays, the simplest case of an an affinely parametrized, lightlike, normalized hypersurface orthogonal vector field ℓ^a fulfilling

$$\ell_a \ell^a = 0, \ \nabla_{[a} \ell_{b]} = 0, \tag{66}$$

shall thus be considered. The said vector field shall be completed to a normalized null geodesic tetrad $(\ell^a, n^a, m^a, \bar{m}^a)$ for convenience. This is the same as to demand $\ell_a = -df_a = -f(\sigma) d\sigma_a$ and $-\ell_a n^a = m_a \bar{m}^a = 1$ to hold, which, regarding the results from preceding sections concerning null foliations, amounts to choosing the vector field ℓ^a to be the generator of a foliation of null hypersurfaces $\{\mathcal{H}_\sigma\}$. In this context, then, assuming that $f = f(\sigma)$ is chosen so that ℓ^a always points into the future and never vanishes, the tangent-space elements of the considered congruence of null curves correspond exactly to the null foliation of spacetime.

Again, a multiplication with a function $\chi = \chi(f)$ does not influence the lightlikeness and hypersurface orthogonality of the generator, opening up the opportunity to pass over to the generating fields $\ell'_a = -\chi(f) df_a$ without changing the essential structure, so that there holds

$$\ell'_a \ell'^a = 0,\ \nabla_{[a}\ell'_{b]} = 0. \tag{67}$$

The equivalence class $[\ell]$ is then obtained directly as a result of the given considerations.

A further possible assumption could now be that $(\ell^a, n^a, m^a, \bar{m}^a)$ is chosen in such a way that the tetrad is kept constant along the null congruence generated by ℓ^a, meaning that it is actually parallelly propagated along the flow of the vector field ℓ^a, which in fact would simplify some of the following calculations. However, this choice, while justified in advance, is by no means necessary and, as stated below, is only made in the present context in order to explain relevant concepts of the approach.

Bearing this in mind, it may be noted that such a congruence gives rise to a transversal (and possibly even orthogonal) deviation vector field η^a, defined and decomposable with regard to the given normalized null frame $(\ell^a, n^a, m^a, \bar{m}^a)$, which measures the displacement to a nearby geodesic, while itself generating by definition a purely orthogonal flow. Accordingly, such a vector field necessarily fulfills

$$L_\ell \eta^a = 0 \tag{68}$$

or equivalently

$$D\eta^a = \ell^c \nabla_c \eta^a = \eta^c \nabla_c \ell^a, \tag{69}$$

which shows that $\nabla_c \ell^a$ measures the failure of being parallelly transported. In this context, therefore, the following necessarily applies: $\ell_a L_\ell \eta^a = L_\ell(\eta_a \ell^a) = D(\eta_a \ell^a) = 0$, so that it can be concluded that the inner product remains constant under the flow generated by ℓ^a, i.e. $\eta_a \ell^a = const.$

The introduced deviation vector field η^a, which is carried along with the flow of ℓ^a, represents a solution to the geodesic deviation equation, as can easily be verified by checking that

$$\begin{aligned} D^2\eta^a &= D((\eta\nabla)\ell^a) = D\eta^c \nabla_c \ell^a + \eta^c D \nabla_c \ell^a = \\ &= (\eta\nabla)\ell^c \nabla_c \ell^a + \eta^c \ell^d \nabla_d \nabla_c \ell^a = \\ &= (\eta\nabla)(\underbrace{(\ell\nabla)\ell^a}_{=0}) - \ell^c (\eta\nabla)\nabla_c \ell^a + \eta^c \ell^d \nabla_d \nabla_c \ell^a = \\ &= 2\eta^d \ell^c \nabla_{[c}\nabla_{d]}\ell^a = R^a_{\ bcd}\ell^b \ell^c \eta^d \end{aligned}$$

holds, which leads to the geodesic deviation equation

$$D^2\eta^a + R^a_{\ bdc}\ell^b\ell^c\eta^d = 0. \tag{70}$$

This shows that the non-zero vector field η^a, which measures the 'distance' to a nearby reference geodesic with respect to the conjugate points p and q [5], is a solution of the geodesic deviation equation if and only if $L_\ell\eta^a = 0$ is fulfilled. Meanwhile, η^a can either be an abreast vector field, meaning that it moves completely in the orthogonal direction to ℓ^a fulfilling $\eta_a\ell^a = 0$, or a non-abreast vector field, which is not solely moving in the orthogonal direction to ℓ^a, fulfilling rather $\eta_a\ell^a \neq 0$, which, however, is related to the fact that $D(\eta_a\ell^a) = 0$, from which it can be concluded that the inner product $\eta_a\ell^a$ remains constant with respect to the flow of ℓ^a. It is therefore always possible to assume either the validity of the abreast case by assuming that $\eta_a\ell^a = 0$ or of the non-abreast case by assuming that $\eta_a\ell^a \neq 0$, where the former case of course occurs as a special case of the latter.

Considering first the abreast case for reasons of simplicity, the decomposition

$$\eta^a = \lambda\ell^a + \bar{\zeta}m^a + \zeta\bar{m}^a \tag{71}$$

can be made, which, after assuming that the frame $(\ell^a, n^a, m^a, \bar{m}^a)$ is parallelly transported along ℓ^a such that $D\ell^a = Dm^a = 0$ and therefore $\pi = 0$, straightforwardly yields the relations

$$\begin{aligned} D\zeta &= -\rho\zeta - \sigma\bar{\zeta}, \\ D\lambda &= \bar{\tau}\zeta + \tau\bar{\zeta}, \end{aligned} \tag{72}$$

where $\tau = \bar{\alpha} + \beta$, $\bar{\tau} = \alpha + \bar{\beta}$ has been used, which is possible, since ℓ_a is a gradient field by assumption. Accordingly, given these results one further finds - using the geodesic deviation equation (70) and the decomposition relation $R^{ab}_{cd} = C^{ab}_{cd} + 2\delta^{[a}_{[c}R^{b]}_{d]} - \frac{1}{6}\delta^{ab}_{cd}R$, where C^{ab}_{cd} is the Weyl tensor, i.e. the trace-free part of the Riemann curvature tensor - the scalar relations

$$\begin{aligned} D^2\zeta &= -\Phi_{00}\zeta - \Psi_0\bar{\zeta}, \\ D^2\lambda &= (\bar{\Psi}_1 + \Phi_{10})\zeta + (\Psi_1 + \Phi_{01})\bar{\zeta} \end{aligned} \tag{73}$$

[5] Conjugate points are initial and final intersection points at which the deviation vector field has to vanish, i.e. $\eta^a|_p = \eta^a|_q = 0$. Vice versa, two points p and q are conjugate if there is a Jacobi field connecting them.

are found. Focusing first on the ζ part, the important set of relations

$$D\rho - \rho^2 - \sigma\bar{\sigma} - \Phi_{00} = 0, \qquad (74)$$
$$D\sigma - 2\rho\sigma - \Psi_0 = 0,$$

known as the famous Sachs equations, describing the dynamics of the so-called optical scalars ρ and σ, are found. Additionally, the useful relation

$$D\tau + \rho\tau + \sigma\bar{\tau} + \Psi_1 + \Phi_{01} = 0 \qquad (75)$$

is found in addition, from which it can be concluded that the scalar equations (74) and (75) naturally result from calculating the Lie bracket (68) and the geodesic deviation equation (70) with respect to a given Jacobi vector field of the form (71).

In the non-abreast case, assuming that m^a and $\bar{m}^a$ are again chosen in a 'smart' way such that $D\ell^a = Dm^a = 0$ and therefore $\pi = \bar{\pi} = 0$, which is in complete agreement with the standard gauge freedom associated with the spin-coefficient formalism, the decomposition

$$\eta^a = \lambda\ell^a + \omega n^a + \bar{\zeta}m^a + \zeta\bar{m}^a, \qquad (76)$$

is now found in turn, yielding the extended relations

$$D\zeta = -\tau\omega - \rho\zeta - \sigma\bar{\zeta}, \qquad (77)$$
$$D\lambda = (\gamma + \bar{\gamma})\omega + \bar{\tau}\zeta + \tau\bar{\zeta},$$
$$D\omega = 0,$$

where $\epsilon' = -\gamma$ and $\bar{\epsilon}' = -\bar{\gamma}$ has been used in the present context. Note that $\tau = \alpha + \bar{\beta}$ has been used here similarly to the abreast case.

As a bonus, one finds the further relations

$$D(\gamma + \bar{\gamma}) - 2\tau\bar{\tau} - \Psi_2 - \bar{\Psi}_2 - 2\Phi_{11} + 2\Pi = 0, \qquad (78)$$

in addition to the remaining equations of the abreast case, from which it can be concluded that solving the scalar equations (74), (75) and (78) with respect to a Jacobi vector field of the form (76) is equivalent to solving the tensorially defined geodesic deviation equation (70) with respect to the same vector field. Consequently, a first point of contact of the physics of null geodesic congruences and the spin-coefficient formalism becomes clear at this juncture.

From the spinor perspective, the results obtained are based on specific restrictions on the spinor fields o^A and ι^A. To be exact, the main requirements here are that the spin-frame is chosen such that $Do^A \propto o^A$, which implies that the vector field ℓ^a is affinely parametrized, and so that the flag planes of the spin-frame are parallelly propagated ($\epsilon \overset{!}{=} \bar{\epsilon}$), which can always be achieved by means of an appropriate rotation (that is a rescaling of $o^A \mapsto e^{i\theta} o^A$ with $\theta \in \mathbb{R}$ within the spin-coefficient framework). In fact, the condition $Do^A = 0$ is a necessary and sufficient condition for relations $\epsilon = \kappa = 0$ to hold, which both guarantee that ℓ^a is a null geodesic vector field, i.e. $\ell_a \ell^a = 0$ and $D\ell^a = 0$. The given so-called extremal case, according to which there holds $\epsilon + \bar{\epsilon} = 0$, turns out to be especially interesting for the Kerr-Schild framework, since this case will allow one to choose ℓ^a as a possible Kerr-Schild vector field at a later point of this work. The additional condition $D\iota^A = 0$ meanwhile ensures that $\pi = 0$ and thus $Dm^a = 0$ applies. However, to ensure that the much stronger condition $\nabla_{[a}\ell_{b]} = 0$ is also satisfied in the given context, $\epsilon = \kappa = \rho - \bar{\rho} = \bar{\alpha} + \beta - \tau = 0$ must necessarily hold, which then implies that the null congruence is both geodetic and twist-free, so that ℓ_a in fact is a gradient co-vector field. This in turn means that the so-called volume expansion function $\Theta = \rho + \bar{\rho}$ (called expansion for short), which in the present context coincides with the gradient of the null vector field ℓ^a (so that $\Theta = \nabla_a \ell^a$) and which can be derived from an associated tensor field Θ_{ab} (to be defined below), has no imaginary part and the scalar twist function $\omega = \sqrt{\omega_{ab}\omega^{ab}} = -i(\rho - \bar{\rho})$ (called twist for short) vanishes identically.

In view of these facts, it can thus be concluded that geometric constraints for spinor basis fields lead to geometric constraints for null frames and their associated null geodesic congruences. An important restriction in this context, which is often required in practice, is that the null geodesic congruence is also shear free, which means that the trace-free part σ_{ab} of Θ_{ab}, the so-called shear tensor, should be zero. However, this is exactly the case when $\sigma = 0$, which then implies that $\delta o_A \propto o^A$ needs to be satisfied.

The other spin-coefficients have interesting physical interpretations as well. For instance, from the decomposition relations presented in section 3.2 it becomes clear that $2Re(\epsilon) = \epsilon + \bar{\epsilon}$ can be interpreted as an acceleration parameter associated with ℓ^a, which can be eliminated by a rescaling of the form $\ell^a \to \lambda \ell^a$. Accordingly, it can be concluded that the coefficient ϵ measures deflections of a given curve parameter with respect to an affine parameter. The coefficients $\alpha - \bar{\beta}$ and $\bar{\alpha} - \beta$, which can be used for the definition of ð and ð′, in the meantime, determine the curvature of a submanifold generated by m^a and $\bar{m}^a$, while the spin coefficient τ measures the change of ℓ^a in the direction of n^a. These further points of contact between null geodesic congruences and the spin-coefficient method shall thus be highlighted in more detail below.

To accomplish this, however, some further relations, which have so far been only partially discussed, are now to be defined and physically interpreted by taking into account intrinsic and extrinsic geometric quantities associated with $\mathcal{H}$, which are obtained by defining projections onto $\mathcal{H}$ or onto a two-surface Δ lying in $\mathcal{H}$. In fact, this can be achieved by considering the projectors

$$^{(\Delta)}\Pi_a^{\ c} = q_a^{\ c} = \delta_a^{\ c} + \ell_a n^c + n_a \ell^c = m_a \bar{m}^c + \bar{m}_a m^c \tag{79}$$

and

$$^{(\mathcal{H})}\Pi^a_{\ b} = \iota^a_{\ b} = \delta^a_{\ b} + n^a \ell_b = q^a_{\ b} - \ell^a n_b, \tag{80}$$

which map from $T_p(M)$ either to $T_p(\Delta)$ or directly to $T_p(\mathcal{H})$, while annihilating transverse vector fields. This last statement is implied by the relations $q^a_b n^b = q^a_b n_a = 0$, $q^a_b \ell^b = q^a_b \ell_a = 0$ and $q^a_b m^b = m^a$, $q^a_b m_a = m_b$, as well as by $\iota^a_b \ell^b = \ell^a$, $\iota^a_b \ell_a = 0$, $\iota^a_b n^b = 0$, $\iota^a_b n_a = n_b$ and $\iota^a_b m^b = m^a$, $\iota^a_b m_a = m_b$, respectively, which require the existence of intersecting null hypersurfaces $\mathcal{H}$ and $\bar{\mathcal{H}}$,so that Δ is the associated apex and q_{ac} is the metric on Δ and ι_{ac} on the degenerate metric on $\mathcal{H}$; however, as can easily be verified, only the previously mentioned fundamental tensor is manifestly non-degenerate, which follows from the fact that a null hypersurface does not generally have a unique Levi-Civita connection that is compatible with its metric. Therefore, additional structures are needed to choose a preferred connection on $\mathcal{H}$, which can be determined by imposing additional conditions on the geometric nature of the considered hypersurface $\mathcal{H}$. A way to establish such extra structures shall be discussed later in the context of non-expanding null hypersurfaces, which are either weakly isolated or isolated horizons.

In preparation for this, the derivative operator

$$\underleftarrow{\nabla_a} := \iota_a^{\ b} \nabla_b, \tag{81}$$

which selects derivatives along $\mathcal{H}$, shall be introduced. This allows the expression

$$\underset{\leftarrow}{\nabla}_a \ell^b = \iota_a^{\ c} \iota_d^{\ b} \nabla_c \ell^d = (q_a^{\ c} - n_a \ell^c)(q_d^{\ b} - n_d \ell^b) \nabla_c \ell^d = \qquad (82)$$
$$= (q_d^{\ b} - n_d \ell^b) q_a^{\ c} \nabla_c \ell^d = q_a^{\ c} q_d^{\ b} \nabla_c \ell^d - q_a^{\ c} D^{'} \ell_c \ell^b =$$
$$= \Theta_a^{\ b} - q_a^{\ c} D^{'} \ell_c \ell^b = \Theta_a^{\ b} - \iota_a^{\ c} D^{'} \ell_c \ell^b$$

to be calculated, where the definition

$$\Theta_{ab} := \frac{1}{2} q_a^{\ c} q_b^{\ d} L_\ell q_{cd} = \qquad (83)$$
$$= \frac{1}{2} q_a^{\ c} q_b^{\ d} L_\ell g_{cd} = q_a^c q_b^d \nabla_d \ell_c =$$
$$= (\delta_a^{\ c} + \ell_a n^c + n_a \ell^c)(\delta_b^{\ d} + \ell_b n^d + n_b \ell^d) \nabla_c \ell_d =$$
$$= (\delta_a^{\ c} + \ell_a n^c + n_a \ell^c)(\nabla_c \ell_b + D^{'} \ell_c \cdot \ell_b) =$$
$$= \nabla_a \ell_b + 2 D^{'} \ell_{(a} \cdot \ell_{b)} + D^{'} \ell_c n^c \cdot \ell_a \ell_b$$

has been used. Clearly, the presented relations hold globally on (M, g), as $\nabla_c \ell^d = \nabla^d \ell_c$ applies by assumption.

As can be seen, the introduced field Θ_{ab} describes the symmetric part of the projection of the gradient $\nabla_a \ell_b$ onto a cross-section of $\mathcal{H}$. It can be split in its trace and trace-free parts such that

$$\Theta_{ab} = \frac{1}{2} \Theta q_{ab} + \sigma_{ab}, \qquad (84)$$

where $\Theta = \nabla_a \ell^a$ denotes the so-called scalar expansion, while the trace-free part $\sigma_{ab} = \Theta_{ab} - \frac{1}{2} \Theta q_{ab}$ defines the so-called shear tensor of the congruence. While the expansion Θ can be interpreted as a measure for the average expansion of infinitesimally nearby geodesics, the shear tensor σ_{ab} measures any tendency of the initial shape of an extended object to become distorted, or, somewhat more model-related, it measures changes in the cross-sectional area orthogonal to the flow lines of the null congruence (which encloses a fixed number of geodesics) as one moves along these lines. In general, also the antisymmetric part ω_{ab}, called the twist or vorticity tensor, which usually measures any tendency of an object with initially fixed shape to change its structure due to rotations, would play a role in these considerations. However, it is zero as a consequence of the assumption that $\nabla_{[a} \ell_{b]} = 0$.

Defining now the field $\omega_a = \ell^c \underleftarrow{\nabla_a} n_c = -n_c \underleftarrow{\nabla_a} \ell^c$, one may thus equally write

$$\underleftarrow{\nabla_a} \ell^b = \frac{1}{2}\Theta q_a^{\ b} + \sigma_a^{\ b} + \omega_a \cdot \ell^b, \tag{85}$$

such that for the special case of vanishing scalar expansion and the shear tensor there holds

$$\underleftarrow{\nabla_a} \ell^b = \omega_a \cdot \ell^b. \tag{86}$$

This special case is of interest, as it allows the determination of a suitable Levi-Civita connection on $\mathcal{H}$.

The given definition opens up the possibility to re-express the gradient of the null generator locally via using $-2D^{'}\ell_{(a} \cdot \ell_{b)} = \tilde{\kappa}\ell_a\ell_b + 2\omega_{(a}\ell_{b)}$ and $D^{'}\ell_c n^c = \tilde{\kappa} = \epsilon' + \bar{\epsilon}' = -\gamma - \bar{\gamma}$ such that

$$\nabla_a \ell_b = \frac{1}{2}\Theta q_{ab} + \sigma_{ab} + 2\omega_{(a} \cdot \ell_{b)} + \tilde{\kappa} \cdot \ell_a \ell_b, \tag{87}$$

on the horizon. Note that the vector field ω_b represents the adapted so-called Hájiček 1-form [26, 28, 29], which in terms of spin-coefficients is given by $\omega_b = q_b^e \omega_e = m_b \bar{m}^e \omega_e + \bar{m}_b m^e \omega_e = \bar{\tau} m_b + \tau \bar{m}_b$. By considering this particular 1-form, it can be seen that the corresponding spin-coefficient τ actually measures the change of ℓ^a in the direction of n^a.

Note, however, that in contrast to the standard literature on the topic, the rotation 1-form ω_a in the present context is completely contained in the tangent space of the section Δ of $\mathcal{H}$, meaning that there holds $\omega_a \ell^a = 0$ and $\omega_a n^a = 0$. Nevertheless, both ω_a expressions, i.e. the expression presented in this work and the one addressed in the literature, coincide exactly under the given geometric circumstances. From this perspective, it seems justified to treat them on an equal footing.

By defining the vector field $\phi_a := \omega_a + \frac{1}{2}\tilde{\kappa} \cdot \ell_a$ the obtained result can be re-written in the form

$$\nabla_a \ell_b = \frac{1}{2}\Theta q_{ab} + \sigma_{ab} + 2\phi_{(a} \cdot \ell_{b)}, \tag{88}$$

which on a non-expanding horizon $\mathcal{H}$ reduces to

$$\nabla_a \ell_b = 2\phi_{(a} \cdot \ell_{b)}. \tag{89}$$

In fact, the same results can easily be obtained using the spin-coefficient approach, which can easily be verified by calculating

$$\begin{aligned}\Theta_{ab} &= 2q_a^c q_b^d \nabla_{(d}\ell_{c)} = \\ &= (m_a\bar{m}^c + \bar{m}_a m^c)(m_b\bar{m}^d + \bar{m}_b m^d)\nabla_d \ell_c = \\ &= -\bar{\sigma} m_a m_b - \sigma \bar{m}_a \bar{m}_b - 2\rho m_{(a}\bar{m}_{b)}\end{aligned} \tag{90}$$

where $\rho = \bar{\rho}$ has been used, so that it follows that

$$\sigma_{ab} = -\bar{\sigma} m_a m_b - \sigma \bar{m}_a \bar{m}_b \tag{91}$$

and

$$\Theta = -2\rho, \tag{92}$$

for the given case of a vanishing twist. In addition, one finds

$$\begin{aligned}\nabla_a \ell_b = -(\gamma + \bar{\gamma})\ell_a \ell_b + 2\bar{\tau}\ell_{(a}m_{b)} + 2\tau \ell_{(a}\bar{m}_{b)} - \\ - 2\rho m_{(a}\bar{m}_{b)} - \bar{\sigma} m_a m_b - \sigma \bar{m}_a \bar{m}_b.\end{aligned} \tag{93}$$

for the gradient of the generating null normal, which reduces to

$$\nabla_a \ell_b = -(\gamma + \bar{\gamma})\ell_a \ell_b + 2\bar{\tau}\ell_{(a}m_{b)} + 2\tau \ell_{(a}\bar{m}_{b)} \tag{94}$$

on a non-expanding null hypersurface $\mathcal{H}$, i.e. in particular on the horizon of a black hole. Accordingly, it is found that the introduced vector field ϕ^a reads

$$\phi^a = -(\gamma + \bar{\gamma})\ell^a + \bar{\tau} m^a + \tau \bar{m}^a \tag{95}$$

in its spin-coefficient representation.

However, there are more similarities between the spin-coefficient method and the physics of null geodesic congruences, as one easily learns by looking at the well-known identity

$$2\nabla_{[c}\nabla_{b]}\ell_a = R_{cbad}\ell^d. \tag{96}$$

By contracting this relation with ℓ^c, one obtains

$$\begin{aligned}D\nabla_b \ell_a &= \ell^c(\nabla_b \nabla_c \ell_a + R_{adcb}\ell^d) = \\ &= \nabla_b(\underbrace{D\ell_a}_{=0}) - \nabla_b \ell^c \cdot \nabla_c \ell_a + R_{adcb}\ell^d \ell^c.\end{aligned}$$

Bringing now the terms from the right-hand side to the left one, while using the decomposition $\nabla_a \ell_b = \Theta_{ab} + 2\phi_{(a} \cdot \ell_{b)}$, one finds out that

$$D\Theta_{ab} + 2D\phi_{(a} \cdot \ell_{b)} + \Theta_a^{\ c}\Theta_{cb} + 2\ell_{(a}\Theta_{b)c}\phi^c + \\ + \phi_c\phi^c \cdot \ell_a\ell_b + R_{acdb}\ell^c\ell^d = 0. \tag{97}$$

By recognizing that $L_\ell\phi_a = D\phi_a + \Theta_{ac}\phi^c + \phi_c\phi^c \cdot \ell_a$ applies in this context, the obtained result can alternatively be re-written in the form

$$D\Theta_{ab} + 2L_\ell\phi_{(a} \cdot \ell_{b)} + \Theta_a^{\ c}\Theta_{cb} + R_{acdb}\ell^c\ell^d = 0. \tag{98}$$

All dynamical quantities defined on $\mathcal{H}$ are now subject to contractions and projections of the given relation. To see this, one may at first contract it with the inverse metric g^{ab}, which immediately leads to the result

$$D\Theta + \Theta_{ab}\Theta^{ab} + R_{ab}\ell^a\ell^b = 0. \tag{99}$$

Using then once more the decomposition $\Theta_{ab} = \frac{1}{2}\Theta q_{ab} + \sigma_{ab}$, this scalar relation can be re-written in the form

$$D\Theta + \frac{1}{2}\Theta^2 + \sigma_{ab}\sigma^{ab} + R_{ab}\ell^a\ell^b = 0, \tag{100}$$

which is just the nullgeometric version of the famous Raychaudhuri equation, usually simply referred to as null Raychaudhuri equation.

Furthermore, a projection of this equation onto $\mathcal{H}$ by means of objects ι_b^a leads to

$$\iota^e_a\iota^f_b D\Theta_{cd} + \Theta_a^{\ c}\Theta_{cb} + \iota^e_a\iota^f_b R_{ecfd}\ell^c\ell^d = \\ = q^e_a q^f_b D\Theta_{cd} + \Theta_a^{\ c}\Theta_{cb} + q^e_a q^f_b R_{ecfd}\ell^c\ell^d = 0.$$

By combining this with $D\Theta_{ab} = £_\ell\Theta_{ab} - \Theta_a^{\ c}\Theta_{cb} - \Theta_{ac}\Theta^c_b$, it follows

$$£_\ell\Theta_{ab} - 2\Theta^{\ c}_{(a}\Theta_{b)c} + 2q^e_{(a}q^f_{b)}R_{ecfd}\ell^c\ell^d = 0, \tag{101}$$

where $£_\ell$ denotes the Lie-derivative along ℓ^a projected onto a section Δ of $\mathcal{H}$. Setting $\Theta_{ab} = \frac{1}{2}\Theta q_{ab} + \sigma_{ab}$ once again, one finds initially

$$\frac{1}{2}(£_\ell\Theta + \frac{1}{2}\Theta^2)q_{ab} + £_\ell\sigma_{ab} + \sigma_a^{\ c}\sigma_{cb} + q^e_a q^f_b R_{ecfd}\ell^c\ell^d = 0. \tag{102}$$

Utilizing then the splitting $R^{ab}_{cd} = C^{ab}_{cd} + 2\delta^{[a}_{[c}R^{b]}_{d]} - \frac{1}{6}\delta^{ab}_{cd}R$ next to the identity $\sigma_a^{\ c}\sigma_{cb} = \frac{1}{2}\sigma_{cd}\sigma^{cd} \cdot q_{ab}$, one obtains

$$\frac{1}{2}(\pounds_\ell\Theta + \frac{1}{2}\Theta^2 + \sigma_{cd}\sigma^{cd} + R_{cd}\ell^c\ell^d)\cdot q_{ab} + \pounds_\ell\sigma_{ab} - \sigma_{cd}\sigma^{cd}q_{ab} + q^e_{\ a}q^f_{\ b}C_{ecfd}\ell^c\ell^d = 0.$$

By realizing now that $\pounds_\ell\Theta = D\Theta$, it becomes obvious that the first part of the result is identical to the null Raychaudhuri equation and therefore zero. Hence one is left with

$$\pounds_\ell\sigma_{ab} - \sigma_{cd}\sigma^{cd}q_{ab} + q^e_{\ a}q^f_{\ b}C_{ecfd}\ell^c\ell^d = 0, \tag{103}$$

which alternatively can be re-written in the form

$$D\sigma_{ab} + 2\sigma_a^{\ c}\Theta_{cb} - \sigma_{cd}\sigma^{cd}q_{ab} + q^e_{\ a}q^f_{\ b}C_{ecfd}\ell^c\ell^d = 0.$$

Setting once more $\Theta_{ab} = \frac{1}{2}\Theta q_{ab} + \sigma_{ab}$, it can be concluded that

$$D\sigma_{ab} + \Theta\sigma_{ab} + q^e_{\ a}q^f_{\ b}C_{ecfd}\ell^c\ell^d = 0. \tag{104}$$

Now, by recalling that $\Theta = 2\rho$, $\sigma_{ab} = \bar{\sigma}m_a m_b + \sigma\bar{m}_a\bar{m}_b$, $\Phi_{00} = \frac{1}{2}R_{ab}\ell^a\ell^b$, $\Psi_0 = C_{ecfd}m^e\ell^c m^f\ell^d$ and $\bar{\Psi}_0 = C_{ecfd}\bar{m}^e\ell^c\bar{m}^f\ell^d$, it becomes obvious that the attained evolution equations directly reduce to the Sachs equations deduced at the beginning of this section. As a consequence, a decent geometrical interpretation for these equations is found, as it is now clear that they describe the evolution of a geodesic congruence null curves and therefore measure the rate of change of contraction and distortion of these curves along the flow of the generator ℓ^a of the hypersurface $\mathcal{H}$.

The remaining dynamical equations then are easily obtained by either projecting the 'main initial relation' with $\iota^a_{\ b}$ and n^a or by contracting it with $n^a n^b$. They take the form

$$\pounds_\ell\omega_a + \iota^e_{\ a}R_{ecbd}\ell^c n^b\ell^d = 0 \tag{105}$$

and

$$\pounds_\ell\tilde{\kappa} + \omega_a\omega^a + R_{acbd}n^a\ell^c n^b\ell^d = 0. \tag{106}$$

As can be seen, if the decompositions $\omega_a = \bar{\tau}m_a + \tau\bar{m}_a$ and $\tilde{\kappa} = -\gamma - \bar{\gamma}$ are taken into account in the present context, the given relations (105) and (106) reduce exactly to equations (75) and (78), while in the case that once more the decomposition $\sigma_{ab} = \bar{\sigma}m_a m_b + \sigma\bar{m}_a\bar{m}_b$ is here taken into account in addition, equation (104) reduces exactly to the second Sachs relation in (74). Thus, in effect, it turns out that the evolution equations (100), (103), (105) and (106), derived from relation (96), all reduce to the scalar relations (74), (75) and (78) if the spin-coefficient framework is taken into account, which is not surprising inasmuch as one of the main advantages of the spin-coefficient framework is that it allows to find scalar relations from associated tensor relations; even in much more general cases, including the case of less well-adapted null geodesic frames and associated null geodesic congruences.

Since the spin-coefficient method works so well in deducing scalar relations from vector or tensor relations, the question arises as to whether or not similar scalar relations could be derived directly for the more complicated case of Einstein's equations. Unsurprisingly, it turns out that, considering the manageable case of the so-called generalized Kerr-Schild class, which consists of metrics that have been deformed by a so-called generalized Kerr-Schild ansatz (for which, incidentally, the consideration of a null geodesic congruence is necessary), this question can be answered in the affirmative, which turns out to be a major point in the construction of gravitational fields of ultrarelativistic sources in the external field of a stationary black hole. This will be shown in the next two chapters, whereas the singular geometry of a corresponding ultrarelativistic source at the event horizon of a charged rotating Kerr-Newman black hole constructed only in the very last chapter of this work.

4 Metric Deformations and the generalized Kerr-Schild Framework

After making it clear in the previous chapter that the spin-coefficient formalism can be used to obtain scalar equations from vectorial or tensorial equations in the theory of null congruences, it shall now be shown that the same also works for the field equations of general relativity in the case of the so-called generalized Kerr-Schild framework. More precisely, it is shown that the deformed Einstein tensor of the generalized Kerr-Schild class, which is usually obtained by performing a Kerr-Schild transformation of a given background metric, can be decomposed with respect to a basis that contains a null geodesic Kerr-Schild vector field. Because of this fact, the corresponding formalism will serve as the basis for calculating the geometric structure of the gravitational field of a massless particle from Einstein's field equations in the final chapter of this work.

4.1 Metric Deformations

This section provides an introduction to the theory of metric deformations, with particular reference to so-called generalized Kerr-Schild deformations, which will later be used to calculate the distributionally valued metric of the gravitational field of an ultrarelativistic particle at the event horizon of a stationary Kerr-Newman black hole.

By a deformation, in this context, any backreaction is meant that modifies the geometric properties of a given spacetime metric with respect to a comparable background metric and an associated class or group of deformation fields that propagate on that background in a geometrically consistent manner.

To make this final statement precise, consider a metric field g_{ab} associated with the background spacetime (M, g). In addition, consider the inverse metric g^{ab}, defined by the relation $g_{ac}g^{cb} = \delta_a^{\ b}$. Given this geometric setting, the main idea of the theory of metric deformations is to modify the structure of the metric g_{ab} and its inverse g^{ab} by considering deformation relations of the form

$$\tilde{g}_{ab} = g_{ab} + e_{ab} \tag{107}$$

and

$$\tilde{g}^{ab} = g^{ab} + f^{ab}, \tag{108}$$

which are introduced in order to obtain a new spacetime $(\tilde{M}, \tilde{g})$ with possibly completely different geometric characteristics. These relations are well-defined if and only if the consistency relations

$$e_a^{\ b} + f_a^{\ b} + e_a^{\ c} f_c^{\ b} = 0 \tag{109}$$

are satisfied, which follow directly from the identity $\tilde{g}_{ac}\tilde{g}^{cb} = \delta_a^{\ b}$. If this is the case, the given deformation leads to the existence of a difference connection tensor of the form

$$C^a_{bc} = \frac{1}{2}(g^{ad} + f^{ad})(\nabla_b e_{dc} + \nabla_c e_{bd} - \nabla_d e_{bc}), \tag{110}$$

which leads to modifications of the curvature tensor. Here, one finds

$$\tilde{R}^a_{bcd} = R^a_{bcd} + E^a_{bcd}, \tag{111}$$

where $E^a_{bcd} = 2\nabla_{[c}C^a_{d]b} + 2C^a_{d[e}C^e_{c]b}$ applies by definition. By contracting indices, one finds that the associated Ricci tensors are of the form

$$\tilde{R}_{bd} = R_{bd} + E_{bd}, \tag{112}$$

where, of course, $E_{bd} = 2\nabla_{[a}C^a_{d]b} + 2C^a_{d[e}C^e_{a]b}$. By repeating that procedure, the associated Ricci scalars

$$\tilde{R} = R + g^{bd}E_{bd} + f^{bd}R_{bd} + f^{bd}E_{bd} \tag{113}$$

can be obtained. However, as a direct consequence, Einstein's equations

$$\tilde{G}_{ab} = 8\pi\tilde{T}_{ab} \tag{114}$$

read

$$G_{ab} + \rho_{ab} = 8\pi\tilde{T}_{ab}, \tag{115}$$

after being decomposed with respect to the local metric of (M, g); at least by virtue of the fact that $\rho_{ab} = \psi_{ab} - \frac{1}{2}g_{ab}(f^{cd}R_{cd} + f^{cd}E_{cd}) - \frac{1}{2}e_{ab}(R + f^{cd}R_{cd} + g^{cd}E_{cd} + f^{cd}E_{cd})$ with $\psi_{ab} = E_{ab} - \frac{1}{2}g_{ab}(g^{cd}E_{cd})$ holds in the given context. If the given equations reduce to the restricted local Einstein equations $G_{ab} = 8\pi T_{ab}$ on M, it becomes clear that the remaining equations are

$$\rho_{ab} = 8\pi\tau_{ab}, \tag{116}$$

where $\tau_{ab} = \tilde{T}_{ab} - T_{ab}$. These equations considerably simplify if an important subclass of the given class of metric deformations can be selected by requiring either

$$f^{ab} = -e^{ab} \tag{117}$$

or

$$e_{ab} = f^{ab} = e^{ab} = 0. \tag{118}$$

While metric deformations fulfilling the former conditions, containing the theory of metric perturbations as a special case, shall from now on be referred to as linear, deformations which globally fulfill the latter conditions, on the other hand, from now on shall on be referred to as trivial deformations.

In case of trivial deformations, obviously no change of the geometric structure results. In case of linear deformations, on the other hand, the introduced system of equations considerably simplifies, as, for example, the metrical consistency condition reduces to

$$e_a^{\ c} e_c^{\ b} = 0. \tag{119}$$

Additionally, the form of any of the introduced fields considerably simplifies due to the fact that now they can be built from a single field on (M, g).

An important point here is that perturbatively defined metric deformations must be distinguished from well-defined ones. The most common perturbatively defined deformations are linear perturbations, such as for example those linear perturbations of Minkowski space that Einstein once used to derive from Newtonian gravity as a special case of general relativity and to show the existence of gravitational waves. And even today, linear (and often nonlinear) perturbations in general relativity are continuously considered, for example to show the stability of solutions of the field equations or to approximate the behavior of complex gravitational fields. In contrast, exact metric deformations are considered much less often; even though many well-known solutions of Einstein's field equations can be written (in appropriate coordinates) as deformations of flat Minkowski spacetime. These are all exact solutions of the fully nonlinear field equations, which can be written in the form (107) and whose associated inverse can be written in the form (108), but with the general metric replaced by that of Minkowski space. The undoubtedly most important class of such deformations is the so-called Kerr-Schild class, which includes, for instance, all solutions of the Kerr-Newman family of spacetimes, as can easily be concluded by recapitulating the results of section 2.3 of this work. This already very important class of spacetimes can be extended to the so-called generalized Kerr-Schild class, which is defined in terms of an a priori completely unspecified background geometry. As discussed in the next section, this class deals with linear deformations that exactly satisfy equations (114),(116) and (117), which is ultimately made possible by considering a given null geodesic vector field, a so-called Kerr-Schild vector field, and an associated null geodesic congruence of curves. It is this very class of metrics of Einstein's theory of general relativity which the spin-coefficient method presented in the previous chapter shall be applied to next: first with respect to any background geometry; then, in the last chapter of this work, specifically regarding the metric of a stationary axisymmetric Kerr-Newman black hole. As it turns out, taking into account a few mild geometric constraints, this then establishes the direct formal basis for the calculation of the gravitational field of a massless particle at the event horizon of a rotating charged black hole, which is done - as already anticipated several times - in the fifth and final chapter of this work.

4.2 The generalized Kerr-Schild Framework

The consideration of the generalized Kerr-Schild framework usually involves considering a background spacetime (M, g) with metric g_{ab}, a scalar function f that represents the profile function of the geometry and a co-vector field l_a, whose index can be raised and lowered with the background metric, so that in particular $l_a = g_{ab}l^b$ applies. In this context, the corresponding vector field is assumed to be null geodetic, which means that it must satisfy the relations $g_{ab}l^a l^b = 0$ and $Dl^a = 0$, where $D := l^b\nabla_b$ applies by definition. The background metric g_{ab} is often called the seed metric and l^a the Kerr-Schild vector field of the geometry.

By combining all these objects in a rather obvious way, the background metric g_{ab} can be deformed into a new metric $\tilde{g}_{ab}$, which is given by the expression

$$\tilde{g}_{ab} = g_{ab} + f l_a l_b. \tag{120}$$

Due to the fact that l^a is lightlike, it is easily found that the inverse metric is given by

$$\tilde{g}^{ab} = g^{ab} - f l^a l^b. \tag{121}$$

Using both of these relations, the affine connection

$$C^a_{\ bc} = \frac{1}{2}\nabla_b(fl^a l_c) + \frac{1}{2}\nabla_c(fl^a l_b) - \frac{1}{2}\nabla^a(fl_b l_c) + \frac{1}{2}fl^a D(fl_b l_c) \tag{122}$$

can be defined, which allows us to calculate the Riemann tensor

$$\tilde{R}^a_{\ bcd} = R^a_{\ bcd} + E^a_{\ bcd} \tag{123}$$

where $E^a_{\ bcd} = 2\nabla_{[c}C^a_{\ d]b} + 2C^a_{\ d[e}C^e_{\ c]b}$ holds in the present context. Moreover, using the definition $E^a_{\ b} = 2(g^{ad} - fl^a l^d)(\nabla_{[m}C^m_{\ d]b} + 2C^m_{\ d[n}C^n_{\ m]b})$, the corresponding Ricci tensor is found to be of the form

$$\tilde{R}^a_{\ b} = R^a_{\ b} + E^a_{\ b} - \frac{1}{2}fR^a_{\ c}l^c l_b - \frac{1}{2}fR^c_{\ b}l_c l^a \tag{124}$$

and the corresponding Ricci scalar reads

$$\tilde{R} = R + E - fR^a_{\ c}l^c l_a, \tag{125}$$

where $E = \delta^b_{\ a}E^a_{\ b}$ holds again by definition. Accordingly, the Einstein tensor of the deformed spacetime $(\tilde{M}, \tilde{g})$

$$\tilde{G}^a_{\ b} = \tilde{R}^a_{\ b} - \frac{1}{2}\delta^a_{\ b}\tilde{R} \tag{126}$$

has the form

$$\tilde{G}^a_{\ b} = G^a_{\ b} + \rho^a_{\ b}, \tag{127}$$

if it is decomposed with respect to the background metric g_{ab}. This conclusion is valid only if the corresponding expression is

$$\begin{aligned}\rho^a_{\ b} = E^a_{\ b} - \frac{1}{2}fR^a_{\ c}l^c l_b - \frac{1}{2}fR^c_{\ b}l_c l^a \\ - \frac{1}{2}\delta^a_{\ b}(E - fR^d_{\ c}l^c l_d).\end{aligned} \tag{128}$$

This can alternatively be brought into the form

$$\begin{aligned}\rho^a_{\ b} = -\frac{1}{2}fR^a_{\ c}l^c l_b - \frac{1}{2}fR^c_{\ b}l_c l^a + \frac{1}{2}\delta^a_{\ b}(fR^d_{\ c}l_d l^c - \nabla_d\nabla^c(fl^d l_c)) + \\ + \frac{1}{2}(\nabla_c\nabla^a(fl^c l_b) + \nabla^c\nabla_b(fl_c l^a) - \nabla_c\nabla^c(fl^a l_b)).\end{aligned} \tag{129}$$

What is remarkable about this result is that the given mixed deformed Einstein tensor is obviously linear in the profile function f; an instance that holds neither with regard to the deformed Einstein tensor $\tilde{G}_{ab}$ with lowered indices nor with respect to its counterpart $\tilde{G}^{ab}$ with raised indices.

Based on the fact that the considered Kerr-Schild vector field l^a can always be completed to a normalized null geodesic frame $(l^a, k^a, m^a, \bar{m}^a)$ whose components satisfy $-k_a l^a = m_a \bar{m}^a = 1$, the spin-coefficient method of Newman and Penrose [50, 51] can be applied to the given problem. This allows one to introduce the decomposition $Dl^a = (\epsilon + \bar{\epsilon}) l^a - \bar{\kappa} m^a - \kappa \bar{m}^a$, from which it can be inferred that the condition $Dl^a = 0$ is tantamount to requiring $\epsilon + \bar{\epsilon} = \kappa = 0$. In turn, this can be used as a basis for setting up the decomposition relations

$$
\begin{aligned}
\nabla_a l_b &= -(\gamma + \bar{\gamma}) l_a l_b + \bar{\tau} l_a m_b + \tau l_a \bar{m}_b + (\alpha + \bar{\beta}) m_a l_b + \\
&+ (\bar{\alpha} + \beta) \bar{m}_a l_b - \bar{\sigma} m_a m_b - \sigma \bar{m}_a \bar{m}_b - \bar{\rho} \bar{m}_a m_b - \rho m_a \bar{m}_b, \qquad (130)\\
\nabla_a k_b &= (\gamma + \bar{\gamma}) l_a k_b - \pi k_a m_b - \bar{\pi} k_a \bar{m}_b - \nu l_a m_b - \bar{\nu} l_a \bar{m}_b - (\alpha + \bar{\beta}) m_a k_b - \\
&- (\bar{\alpha} + \beta) \bar{m}_a k_b + \lambda m_a m_b + \bar{\lambda} \bar{m}_a \bar{m}_b + \mu \bar{m}_a m_b + \bar{\mu} m_a \bar{m}_b, \qquad (131)\\
\nabla_a m_b &= (\bar{\gamma} - \gamma) l_a m_b - \bar{\nu} l_a l_b + \tau l_a k_b + (\bar{\epsilon} - \epsilon) k_a m_b - \bar{\pi} k_a l_b + \qquad (132)\\
&+ (\alpha - \bar{\beta}) m_a m_b + (\beta - \bar{\alpha}) \bar{m}_a m_b + \bar{\mu} m_a l_b - \rho m_a k_b + \bar{\lambda} \bar{m}_a l_b - \sigma \bar{m}_a k_b,
\end{aligned}
$$

by means of which one finds, after a lengthy computation, that the deformed part $\rho^a_{\ b}$ of the mixed Einstein tensor $\tilde{G}^a_{\ b}$ can be decomposed in the form

$$
\begin{aligned}
\rho^a_{\ b} &= \mathfrak{G}_1 l^a l_b + \mathfrak{G}_2 (l^a k_b + k^a l_b) + \bar{\mathfrak{G}}_3 (l^a m_b + m^a l_b) + \qquad (133)\\
&+ \mathfrak{G}_3 (l^a \bar{m}_b + \bar{m}^a l_b) + \mathfrak{G}_4 (m^a \bar{m}_b + \bar{m}^a m_b) + \bar{\mathfrak{G}}_5 m^a m_b + \mathfrak{G}_5 \bar{m}^a \bar{m}_b,
\end{aligned}
$$

where the exact expressions read

$$\mathfrak{G}_1 = -\frac{1}{2}\mathcal{D}^2 f + (\bar\tau - 2(\alpha + \bar\beta))\delta f + (\tau - 2(\bar\alpha + \beta))\delta' f + \frac{1}{2}(\rho + \bar\rho)D' f -$$
$$- (\gamma + \bar\gamma - \frac{1}{2}(\mu + \bar\mu))Df + \delta(\bar\tau - \alpha - \bar\beta)f + \delta'(\tau - \bar\alpha - \beta)f +$$
$$+ (\bar\tau - \alpha - \bar\beta)(\tau + \beta - \bar\alpha)f + (\tau - \bar\alpha - \beta)(\bar\tau + \bar\beta - \alpha)f - 4|\alpha + \bar\beta|f +$$
$$+ \bar\tau(\bar\alpha + \beta)f + \tau(\alpha + \bar\beta)f + (\gamma + \bar\gamma)(\rho + \bar\rho)f + (\mu - \bar\mu)(\rho - \bar\rho)f -$$
$$- (\Psi_2 + \bar\Psi_2 - 4\Pi)f, \tag{134}$$

$$\mathfrak{G}_2 = -\frac{1}{2}[(\rho + \bar\rho)Df + D(\rho + \bar\rho)f - 2(\rho^2 + \bar\rho^2)f], \tag{135}$$

$$\mathfrak{G}_3 = \frac{1}{2}[D\delta f + (2(\bar\alpha + \beta) - \tau - \bar\pi)Df + (\bar\epsilon - \epsilon + \rho - 2\bar\rho)\delta' f +$$
$$+ \sigma\delta f + 2D(\bar\alpha + \beta)f + +\delta(\rho - \bar\rho)f + 2(\bar\alpha + \beta)(\bar\epsilon - \epsilon)f +$$
$$+ (\bar\rho - \rho)(\bar\alpha + \beta - \tau)f - 2\tau\rho f - 2\bar\rho(\bar\alpha + \beta + \bar\pi)f +$$
$$+ 2\sigma(\alpha + \bar\beta - \bar\tau - \pi)f - 2\Phi_{01}f] \tag{136}$$

$$\mathfrak{G}_4 = \frac{1}{2}[-D^2 f + (\rho + \bar\rho)Df + (\rho - \bar\rho)^2 f + 2\Phi_{00}f], \tag{137}$$

$$\mathfrak{G}_5 = -\sigma Df - D\sigma f + 2\sigma(\bar\rho + \epsilon - \bar\epsilon)f. \tag{138}$$

Note that according to the usual conventions, the definitions $\Pi = \frac{R}{24}$ and $\mathcal{D}^2 = \mathcal{D}^a\mathcal{D}_a = \delta\delta' + \delta'\delta + (\beta - \bar\alpha)\delta' + (\bar\beta - \alpha)\delta$ have been used in this context.

The derived equations, which are valid with respect to any generalized Kerr-Schild metric, encode the geometric structure of the deformed Einstein tensor of the generalized Kerr-Schild class in spin-coefficient form. They are therefore too general and must be restricted in the next chapter of this work in order to provide the exact specifications that lead to the structure of the geometric field of an ultrarelativistic point-like particle in a stationary black hole background.

5 Einstein's Field Equations and the generalized Dray-'t Hooft Relation

Finally, in this last chapter of the present work, the gravitational field of a massless particle is calculated at the event horizon of a stationary axisymmetric Kerr-Newman black hole. To this end, the general setup given in [10] is followed, which means that the geometric framework of generalized Kerr-Schild deformations, in combination with the spin-coefficient formalism of Newman and Penrose, is used to show that the deformed field equations derived in the previous section can be reduced - at the event horizon of the black hole - to a single linear ordinary differential equation for the so-called profile function of the geometry. As a basis for this approach, one exploits the fact that the event horizon of the black hole is an isolated horizon (which is actually a Killing horizon) [4, 5, 6, 8] and therefore a non-expanding null hypersurface on which both convergence and scalar shear of the associated generating vector field vanish, which in turn implies the vanishing of certain null components of the Ricci and Weyl tensors of the corresponding background geometry. With some additional, but legitimate constraints on the geometric structure of the null geodesic frame of the generator, it then follows that the deformed Einstein tensor presented in the previous section can be fully described by a single differential relation at the event horizon of the black hole. It is the main task of this chapter to derive this equation and then to solve it step by step using methods of the theory of differential equations with variable coefficients, briefly discussed in the appendix of this work.

5.1 Einstein's Field Equations I: Deriving the generalized Dray-'t Hooft Relation

Taking into account the general considerations of the previous section, the next step shall be the application of the corresponding mathematical approach to the special case of a Kerr-Newman black hole background. This background can be classified as a Petrov-type D spacetime, which brings about a decisive simplification in that, according to the Goldberg-Sachs theorem, it means that the coefficients Ψ_0, Ψ_1, Ψ_3, Ψ_4 and ϕ_0, ϕ_2 of the Weyl and Einstein-Maxwell tensors can all be set to zero by the use of a suitable null geodesic frame.

Consequently, in order to be able to deal with a maximally simplified geometric setting, it seems reasonable to consider the Kerr-Newman metric g_{ab} in Kerr coordinates. As is well-known, the individual components of the Kerr-Newman metric in these coordinates can be read off from the line element

$$\begin{aligned} ds^2 = &-(1-\frac{2Mr-e^2}{\Sigma})dv^2 + 2(dv - a\sin^2\theta d\phi)dr + \Sigma d\theta^2 + \\ &+ \frac{\Pi\sin^2\theta}{\Sigma}d\phi^2 - \frac{2(2Mr-e^2)}{\Sigma}a\sin^2\theta dv d\phi, \end{aligned} \tag{139}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Pi = (r^2+a^2)^2 - \Delta a^2\sin^2\theta$ and $\Delta = r^2 + a^2 - 2Mr + e^2$.

There are two main reasons for the given choice of Kerr coordinates: On the one hand, these coordinates are regular at the internal and external event horizons located at $r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}$. On the other hand, they allow one to directly read off the only two principal null directions of the geometry, which lead to associated null congruences that are globally shear free.

In view of this comparatively simple geometric setting, it makes sense to consider now once more the previously introduced decomposition relation (120), which results in a new deformed metric belonging to the generalized Kerr-Schild class of the given Kerr-Newman black hole background. In this context, g_{ab} is usually called the seed or background metric and l^a the Kerr-Schild vector field, which must satisfy $l_a = g_{ab}l^b$ and $\tilde{D}l^a = Dl^a = 0$.

An appropriate candidate for such a vector field is found by performing a 2+2-decomposition of the given seed metric and its inverse, which leads to the expressions $g_{ab} = -2l_{(a}k_{b)} + 2m_{(a}\bar{m}_{b)}$ and $g^{ab} = -2l^{(a}k^{b)} + 2m^{(a}\bar{m}^{b)}$. This gives a null tetrad $(l^a, k^a, m^a, \bar{m}^a)$

that is of the form

$$
\begin{aligned}
l^a &= \frac{r^2+a^2}{\Sigma}\partial_v^a + \frac{a}{\Sigma}\partial_\phi^a + \frac{\Delta}{2\Sigma}\partial_r^a \\
k^a &= -\partial_r^a \\
m^a &= \frac{1}{\sqrt{2}\Gamma}(\partial_\theta^a - \frac{i}{\sin\theta}(\partial_\phi^a + a\sin^2\theta\partial_v^a)) \\
\bar{m}^a &= \frac{1}{\sqrt{2}\bar{\Gamma}}(\partial_\theta^a + \frac{i}{\sin\theta}(\partial_\phi^a + a\sin^2\theta\partial_v^a)),
\end{aligned} \tag{140}
$$

where $\Gamma = r + ia\cos\theta$ and $\bar{\Gamma} = r - ia\cos\theta$.

In respect to this particular choice of a null frame, all the nonzero spin-coefficients can be calculated. These coefficients are

$$
\begin{aligned}
\epsilon = \frac{r-M}{2\Sigma} - \frac{\Delta}{2\Sigma\Gamma},\ \alpha = -\frac{\cot\theta}{2\sqrt{2}\bar{\Gamma}},\ \tau = \frac{ia\sin\theta}{\sqrt{2}\Gamma^2} \\
\beta = \frac{ia\sin\theta}{\sqrt{2}\Gamma^2} + \frac{\cot\theta}{2\sqrt{2}\Gamma},\ \pi = -\frac{ia\sin\theta}{\sqrt{2}\Sigma},\ \rho = -\frac{\Delta}{2\Sigma\Gamma},\ \mu = -\frac{1}{\Gamma}, \\
\phi_1 = \frac{e}{\sqrt{2}\Gamma^2},\ \Psi_2 = -\frac{M}{\Gamma^3} + \frac{e^2}{\Gamma^2\Sigma}.
\end{aligned} \tag{141}
$$

As can be seen, $\epsilon + \bar{\epsilon} \neq 0$ applies according to the given choice of the null frame. Hence, to actually convert the given expression into a null geodesic frame, a null rescaling of the form $l^a \to Be^{-\kappa v}l^a$ and $k^a \to B^{-1}e^{\kappa v}k^a$ needs to be applied, where $B = B(r,\theta) = B_0\Sigma\exp(-\frac{r}{M} - \frac{2M^2-e^2}{2M(r_+^2+a^2)}[r+2M\ln|r-r_-|])$ with $B_0 = \frac{\exp(\frac{r_+}{M} + \frac{2M^2-e^2}{2M(r_+^2+a^2)}[r_+ + 2M\ln|r_+ - r_-|])}{r_+^2+a^2\cos^2\theta}$ and $\kappa = \frac{r_+-M}{r_+^2+a^2}$. Here, the factors $Be^{-\kappa v}$ are chosen in such a way that $D(Be^{-\kappa v}) = 0$ applies and, as can readily be seen, the resulting expression remains regular at the outer event horizon, that is, for a radial parameter value of $r = r_+$.

By performing this rescaling, the condition $\epsilon + \bar{\epsilon} = 0$ is now exactly satisfied. As a consequence, the structure of the associated spin-coefficients changes according to the rule $\epsilon \to \epsilon + \frac{1}{2}D\ln|Be^{-\kappa v}|$, $\gamma \to \gamma + \frac{1}{2}D'\ln|Be^{-\kappa v}|$, $\alpha \to \alpha + \frac{1}{2}\delta'\ln|Be^{-\kappa v}|$, $\beta \to \beta + \frac{1}{2}\delta\ln|Be^{-\kappa v}|$, $\rho \to Be^{-\kappa v}\rho$, $\mu \to B^{-1}e^{\kappa v}\mu$; all other coefficients are zero or remain unaltered.

Before explicitly calculating the deformed part of the total Einstein tensor $\tilde{G}^a_b$, it shall first be noted that the geometric field of a null particle field at the horizon must satisfy $\tilde{G}^a_{\ b}|_{\mathcal{H}^+} \propto l^a l_b$. However, since this obviously implies that $\tilde{R}|_{\mathcal{H}^+} = 0$ as a consequence of $R = 24\Pi = 0$, there must further hold $\nabla_d \nabla^c (f l^d l_c) = 0$ locally at the event horizon of the black hole background. This is certainly satisfied if the local condition

$$\nabla^c (f l^d l_c)|_{\mathcal{H}^+} = 0 \tag{142}$$

is met by the profile 'function' f of the geometry.

To describe the distributional profile of a point-like null particle, it is appropriate to consider an ansatz that contains a delta distribution. More precisely, a generic choice for $f = f(v, r, \theta, \phi)$ that is consistent with the above requirements should locally be of the form $f|_{\mathcal{H}^+} = \tilde{f}(v, \theta, \phi)\delta(r - r_+) = U(\theta)F(\theta, \phi - \omega_+ v)e^{\kappa v}\delta(r - r_+)$, where $\omega_+ = \frac{a}{r_+^2 + a^2}$ and $U = U(\theta)$ and $F(\theta, \phi - \omega_+ v)$ are free functions that are to be specified in the further course of this work.

Given the fact that the profile function has compact support on the exterior event horizon $\mathcal{H}^+$, it suffices to calculate the local object

$$\rho^a_{\ b} = [G^a_{\ b}] = \tilde{G}^a_{\ b}|_{\mathcal{H}^+} - G^a_{\ b}|_{\mathcal{H}^+} \tag{143}$$

in order to determine the geometric structure of the entire deformed spacetime $(\tilde{M}, \tilde{g})$. However, this drastically simplifies the deformed field equations, since it follows that the system (134) – (138) of equations presented in the previous section can be reduced locally to a simplified version of equation (134). This means de facto that they are found to reduce to a single differential relation of the type

$$\rho^a_{\ b} = -\boxtimes f l^a l_b, \tag{144}$$

which is given with respect to the definition

$$\begin{aligned}\boxtimes := &\frac{1}{2}\mathcal{D}^2 + (2(\alpha + \bar{\beta}) - \bar{\tau})\delta + (2(\bar{\alpha} + \beta) - \tau)\delta' + \frac{1}{2}D'(\rho + \bar{\rho})f \\ &+ \delta(\alpha + \bar{\beta} - \bar{\tau}) + \delta'(\bar{\alpha} + \beta - \tau) + (\alpha + \bar{\beta} - \bar{\tau})(\tau + \beta - \bar{\alpha}) + \\ &+ (\bar{\alpha} + \beta - \tau)(\bar{\tau} + \bar{\beta} - \alpha) + 4|\alpha + \bar{\beta}| - \bar{\tau}(\bar{\alpha} + \beta) - \\ &- \tau(\alpha + \bar{\beta}) + (\Psi_2 + \bar{\Psi}_2)f.\end{aligned} \tag{145}$$

Note that the relation $\frac{1}{2}[(\rho + \bar{\rho})D'f, \cdot] = -\frac{1}{2}[D'(\rho + \bar{\rho})f, \cdot]$ was used to determine the concrete form of this differential operator.

Unfortunately, by looking at the result obtained, it immediately becomes clear that the derived differential relation is too complicated to be solved directly in its present form. Consequently, in order to further simplify the geometric structure of the Kerr-Schild deformed Einstein tensor, it proves beneficial to try to satisfy the local conditions

$$\nabla_{[a}l_{b]}|_{\mathcal{H}^+} = 0, \tag{146}$$

which can be satisfied by performing a null rotation $l^a \to l^a$, $k^a \to k^a + \bar{\zeta}m^a + \zeta\bar{m}^a + |\zeta|l^a$, $m^a \to m^a + \zeta l^a$, $\to \bar{m}^a \to \bar{m}^a + \bar{\zeta}l^a$, where $\zeta = \frac{i\omega_+ e^{\kappa v}\bar{\Gamma}\sin\theta}{\sqrt{2}B}$.

Given this specific transformation, the structure of the complex valued null normals m^a and $\bar{m}^a$ changes in such a way that $\bar{\alpha} + \beta = \tau$ is satisfied on $\mathcal{H}^+$ and thus also $\nabla_{[a}l_{b]}|_{\mathcal{H}^+} = 0$ as a further consequence.

As a direct result, it is found that the deformed Einstein tensor can be written exactly the same way as in equation (144), with the only difference being that the scalar part of the said relation now reads

$$\boxtimes f = \frac{1}{2}\mathcal{D}^2 f + \tau\delta' f + \bar{\tau}\delta f + \frac{1}{2}D'(\rho+\bar{\rho})f + 2|\tau|^2 f + \Psi_2 f + \bar{\Psi}_2 f. \tag{147}$$

This leads directly to the generalized Dray-'t Hooft equation

$$\boxtimes f = 2\pi b_0 \delta_N, \tag{148}$$

which will be the main subject of the remainder of this thesis.

Remarkably, relation (147), which was previously found with respect to other definitions and conventions in [10], is invariant under null rotations that leave the form of l^a invariant. Accordingly, the result obtained remains completely unchanged if a corresponding null rotation is introduced in the present context, which, however, ensures that complex null normals m^a and $\bar{m}^a$ are hypersurface forming and therefore satisfy $L_{\bar{m}}m^a \propto m^a, \bar{m}^a$ locally on $\mathcal{H}^+$. Consequently, it seems justified to continue using the null reference frame considered so far.

To simplify the derived relation, it is convenient to make the ansatz $U = U(\theta) = \Sigma_+^{-1}$ with $\Sigma_+ = \Sigma|_{r=r_+}$, where U is a solution to the equation $\delta \ln|U| \hat{=} \bar{\pi} - \tau$. From this it follows that the derived relation takes the form

$$\boxtimes f = e^{\kappa v}\Sigma_+\delta_+(\frac{1}{2}\mathcal{D}^2F + \frac{1}{2}(\pi+\bar{\tau})\delta F + \frac{1}{2}(\bar{\pi}+\tau)\delta' F + \frac{1}{2}D'(\rho+\bar{\rho})F -$$
$$+ \frac{1}{2}\delta(\pi-\bar{\tau})F + \frac{1}{2}\delta'(\bar{\pi}-\tau)F + |\pi|^2F + |\tau|^2F + \frac{1}{2}(\bar{\pi}-\tau)(\bar{\beta}-\alpha)F +$$
$$+ \frac{1}{2}(\pi-\bar{\tau})(\beta-\bar{\alpha})F + \Psi_2 F + \bar{\Psi}_2 F) \quad (149)$$

where $\delta_+ := \delta(r - r_+)$ applies by definition. Taking then into account that $\omega^a\mathcal{D}_a = (\alpha+\bar{\beta})\delta + (\bar{\alpha}+\beta)\delta' = \bar{\tau}\delta + \tau\delta' = -\pi\delta - \bar{\pi}\delta'$ applies locally at the event horizon of the black hole, where $\omega^a = (\alpha + \bar{\beta})m^a + (\bar{\alpha}+\beta)\bar{m}^a$ is the one-form potential of the theory of isolated horizons [4, 6], and that the spin-coefficient relations

$$D'\rho - \delta'\tau + |\tau|^2 - \tau(\bar{\beta}-\alpha) + \Psi_2 = 0 \quad (150)$$

and

$$\delta\pi - D\mu + |\pi|^2 + \pi(\bar{\beta}-\alpha) + \Psi_2 = 0 \quad (151)$$

and their respective complex conjugates can be used to convert the derived differential equation, whereas it has to be noted that the said equations hold in this form only locally at black hole event horizon $\mathcal{H}_+$ (and the inner Killing horizon $\mathcal{H}_+$), it is found that the geometric structure of the deformed Einstein tensor now takes the significantly simpler form

$$\rho^a_{\ b} = -\frac{e^{\kappa v}\Sigma_+\delta_+}{2}(\mathcal{D}^2F + D(\mu+\bar{\mu})F)l^al_b. \quad (152)$$

Using then the definition $V \equiv D(\mu+\bar{\mu})$, the resulting expression can be re-written in the form

$$\rho^a_{\ b} = -\frac{e^{\kappa v}\Sigma_+\delta_+}{2}(\mathcal{D}^2F + V\cdot F)l^al_b. \quad (153)$$

Given this final result, it is found that Einstein's equations reduce in the given case to the single relation

$$\Sigma_+(\mathcal{D}^2F + V\cdot F) = 2\pi b_0\delta_N, \quad (154)$$

which shall be referred to as reduced generalized Dray-'t Hooft equation from now on.

By considering the limit $a \to 0$, it is found that this relation reduces to

$$(\Delta_{\mathbb{S}_2} - c)F = 2\pi b_0 \delta_N, \tag{155}$$

where $c = 2\kappa r_+ = \frac{2(r_+ - M)}{r_+}$, which was first obtained by Sfetsos. In addition, by considering the combined limits $a, e \to 0$, it is found that the derived differential equation takes the even simpler form

$$(\Delta_{\mathbb{S}_2} - 1)F = 2\pi b_0 \delta_N, \tag{156}$$

which was first found by Dray and 't Hooft. Therefore, since obviously both of these relations result as a special case from the given model, it can be concluded that the generalized Dray-'t Hooft equation provides a viable extension of those fundamental equations, which have been used in the past to determine the profile functions corresponding to a gravitational shock wave in either Reissner-Nordström or Schwarzschild black hole backgrounds.

5.2 Einstein's Field Equations II: Solving the generalized Dray-'t Hooft Relation

Next, to actually solve the deduced differential relation, the line element associated with the induced two-metric of the Kerr-Newman geometry

$$d\sigma^2 = \Sigma_+ d\theta^2 + \frac{(r_+^2 + a^2)^2 \sin^2\theta}{\Sigma_+} d\phi^2 \tag{157}$$

shall be considered. By introducing the coordinate transformation $\xi = \cos\theta$, this line element can be re-written in the form

$$d\sigma^2 = \frac{\Sigma_+}{1 - \xi^2} d\xi^2 + \frac{(r_+^2 + a^2)^2 (1 - \xi^2)}{\Sigma_+} d\phi^2, \tag{158}$$

which provides a spacelike dyad $(m^a, \bar{m}^a)$ that is locally of the form

$$m^a = \frac{1}{\sqrt{2}\Gamma_+}\left(\sqrt{1 - \xi^2}\,\partial_\xi^a - \frac{i\Sigma_+}{(r_+^2 + a^2)\sqrt{1 - \xi^2}}\partial_\phi^a\right), \tag{159}$$

$$\bar{m}^a = \frac{1}{\sqrt{2}\bar{\Gamma}_+}\left(\sqrt{1 - \xi^2}\,\partial_\xi^a + \frac{i\Sigma_+}{(r_+^2 + a^2)\sqrt{1 - \xi^2}}\partial_\phi^a\right), \tag{160}$$

according to which, of course, $\Gamma_+ = r_+ + ia\xi$ and thus $\bar{\Gamma}_+ = r_+ - ia\xi$.

Given this setting, the left hand side of the generalized Dray-'t Hooft equation can be written down in the form

$$\boxtimes f = \frac{1}{2\Sigma_+}(\Delta_{\mathbb{S}_2} + \mathcal{V})f \tag{161}$$

where $\mathcal{V} \equiv \Sigma_+(D'(\rho + \bar{\rho}) + 4|\tau|^2 + 2\Psi_2 + 2\bar{\Psi}_2)$, which implies that the resulting equation could (at least in principle) be solved by expanding the profile function f in spherical harmonics. However, this is not the approach of the present work.

As it turns out, the reduced generalized Dray-'t Hooft equation can be re-written in these coordinates in the form

$$\Sigma_+(\mathcal{D}^2 F + V \cdot F) = \Sigma_+ \partial_\xi (\frac{1-\xi^2}{\Sigma_+}\partial_\xi)F + \tag{162}$$

$$+ \frac{\Sigma_+^2}{(r_+^2 + a^2)^2(1-\xi^2)}\partial_\phi^2 F - \frac{2r_+(r_+ - M)}{\Sigma_+}F.$$

By considering in the following the special case of a profile function $f = f(\xi)$ as well as the reduced profile function $F = F(\xi)$ which both do not depend on the angular variable ϕ and assuming that the symmetry axis of the system points through the 'north pole', i.e. through the point $\xi_0 = 1$ at which the particle shall be assumed to be located, it is possible to re-write the generalized Dray-'t Hooft relation in the form

$$\frac{d}{d\xi}((1-\xi^2)\frac{dF}{d\xi}) - \frac{2a^2\xi(1-\xi^2)}{\Sigma_+}\frac{dF}{d\xi} - \frac{2r_+(r_+ - M)}{\Sigma_+}F = 2\pi b_0 \delta(\xi - 1). \tag{163}$$

Note that the same step was also taken in the previous works of Dray and 't Hooft and Sfetsos, although, as one must admit, in the spherical case (quite contrary to the given axisymmetric case) such an approach does not result in any loss of generality.

Considering the homogeneous relation

$$\frac{d^2F}{d\xi^2} + b\frac{dF}{d\xi} + cF = 0, \tag{164}$$

in respect of which the coefficients b and c are given by $b := -\frac{2\xi}{1-\xi^2} - \frac{2a^2\xi}{\Sigma_+}$ and $c := -\frac{2r_+(r_+-M)}{\Sigma_+(1-\xi^2)}$, it is found that the coefficients of this linear differential relation give rise to the five singular points ± 1, $\pm\frac{ir_+}{a}$ and ∞. However, since one knows that $\lim_{\xi\to\xi_i}(\xi-\xi_i)b$ and $\lim_{\xi\to\xi_i}(\xi-\xi_i)^2 c$ remain finite, these points are in fact regular singular points, so that the derived differential equation belongs to the Fuchsian class of linear differential equations of second order with regular singular coefficients [19, 60], which is discussed in more detail in the appendix of this work.

As it turns out, the found equation belongs to the class of generalized Lamé differential equations, which will also be presented in the appendix. Solutions of this class of equations are the generalized Lamé functions, which represent the formal basis of the construction of generalized ellipsoidal harmonics; much like Legendre functions represent the basis of the construction of spherical harmonics. In contrast to the much simpler spherical case, however, one finds here that within the class of generalized Lamé equations polynomial solutions could already exist, but also that the existence of such solutions is subject to fixed restrictions, since they result from a three-term recursion relation instead of a trivial one-term recursion relation. This point is also dealt with in more detail in the later stages of this work.

Anyway, from the theory of differential equations of the Fuchsian class it can, in fact, be concluded that there exist two different solutions to equation (164), which can be written as generalized infinite power series. To determine the exact form of these solutions, the given relation shall be re-written in the form

$$\Sigma_+(1-\xi^2)\frac{d^2F}{d\xi^2}-[2\xi\Sigma_++2a^2\xi(1-\xi^2)]\frac{dF}{d\xi}-2r_+(r_+-M)F=0. \quad (165)$$

By doing so, it is found that the two solutions are of the form $F_1=\sum_{k=0}^{\infty}w_k\xi^k$ and $F_2\equiv F_1\Delta+G$, where $G=\sum_{k=0}^{\infty}u_k\xi^k$ and $\Delta=C_1(r_+^2+a^2)[\frac{1}{2}\ln|\frac{\xi+1}{\xi-1}|-a\omega_+\xi]+C_2$, where, in this context, Δ is then defined in precisely such a way that it forms a solution of $\Sigma_+(1-\xi^2)\frac{d^2\Delta}{d\xi^2}-[2\xi\Sigma_++2a^2\xi(1-\xi^2)]\frac{d\Delta}{d\xi}=0$.

Unfortunately, however, the derived differential relation (in its present form) does not allow for any polynomial solutions so that both $F_1=F_1(\xi)$ and $F_2=F_2(\xi)$ are in fact infinite power series. While this does not imply that there could not in principle exist another variable more appropriate than ξ, which allows for the definition of polynomial solutions and could therefore be used as a basis for the construction of generalized ellipsoidal harmonics, the finding that there are no such solutions in the given variable ξ nevertheless shows that another pair of solutions is needed at this stage in order to find a combined solution not only to the homogeneous, but also the inhomogeneous equation.

The independent solutions $F_1^{\pm} = F_1^{\pm}(\xi)$ and $F_2^{\pm} = F_2^{\pm}(\xi)$, which can be obtained straightforwardly by introducing a new set of variables $\xi \to \mp\xi - 1$, are of the form $F_1^{\pm} = \sum_{k=0}^{\infty}(-1)^k w_k^{\pm}(1 \pm \xi)^k$ and $F_2^{\pm} \equiv F_1^{\pm}\Delta + G^{\pm}$, where $G^{\pm} = \sum_{k=0}^{\infty}(-1)^k u_k^{\pm}(1 \pm \xi)^k$. As it turns out, the corresponding coefficients must coincide exactly in that $w_k^+ = w_k^-$ and $u_k^+ = u_k^-$ as long as the conditions $w_0^+ = w_0^-$ and $u_0^+ = u_0^-$ are satisfied, which is simply due to the invariance of the equation under 'parity transformations' in ξ. The coefficients $w_k^{\pm}$ have a rather complicated structure, as they have to be determined from the three-term recurrence relation

$$w_{k+1}^{\pm} = m_k w_k^{\pm} + n_k w_{k-1}^{\pm} + o_k w_{k-2}^{\pm}, \tag{166}$$

according to which $m_k = m(k) = -\frac{r_+^2}{2(r_+^2+a^2)}\frac{k(k+1)}{(k+1)^2} - \frac{a^2}{2(r_+^2+a^2)}\frac{(5k-3)k}{(k+1)^2} - \frac{\kappa r_+}{(k+1)^2}$, $n_k = n(k) = -\frac{2a^2}{r_+^2+a^2}\frac{(k-2)(k-1)}{(k+1)^2}$ and $o_k = o(k) = -\frac{a^2}{r_+^2+a^2}\frac{(k-2)(k-3)}{(k+1)^2}$. Using the fact that one may always choose $w_2^{\pm}$ and $w_1^{\pm}$ freely and thus in such a way that $w_2^{\pm} = m_1 w_1^{\pm} + n_1 w_0^{\pm}$ and $w_1^{\pm} = m_0 w_0^{\pm}$, the coefficients $w_{k+1}^{\pm}$ can be written down explicitly by using the notation $w_{k+1}^{\pm} = \ll m, n, o \gg_k w_0^{\pm}$, where the occurring symbol $\ll m, n, o \gg_k$ is a multi-linear form of the type

$$\ll m, n, o \gg_k = W_{a_k a_{k-1} \ldots a_0} X_k^{a_k} X_{k-1}^{a_{k-1}} \ldots X_0^{a_0}, \tag{167}$$

where each a_j runs from zero to three. The corresponding objects $X_j^{a_j} = X^{a_j}(j)$ have the components $X_j^0 = \theta(j)$, $X_j^1 = m(j)$, $X_j^2 = n(j)$ and $X_j^3 = o(j)$, where $\theta(j) := \begin{cases} 0 & if\ j<0 \\ 1 & if\ j\geq 0 \end{cases}$ is the Heaviside step function. In the meantime, the object $W_{a_k a_{k-1} \ldots a_0}$ is defined in such a way that all its components are either zero or one. All its non-zero components are exactly those for which on the one hand $\sum_{j=0}^{k} a_j = k+1$

applies and, moreover, all indices that take the value zero occur only as successors of those that take a value of two, and all pairs of indices that take the value zero combined occur only as successors of indices with a value of three. This implies in particular that all $W_{0a_{k-1}...a_0}$ and $W_{00a_{k-2}...a_0}$ are zero by definition.

To prove that $F_1^{\pm}$ is actually a solution of (165), it shall first be noted that the validity of relations (166) and (167) implies that

$$W_{a_k a_{k-1}...a_0} X_k^{a_k} X_{k-1}^{a_{k-1}} ... X_0^{a_0} = m_k W_{a_{k-1}...a_0} X_{k-1}^{a_{k-1}} ... X_0^{a_0} + \\ + n_k W_{a_{k-2}...a_0} X_{k-2}^{a_{k-2}} ... X_0^{a_0} + o_k W_{a_{k-3}...a_0} X_{k-3}^{a_{k-3}} ... X_0^{a_0}. \qquad (168)$$

The first case to be considered is therefore $k = 2$. In this case, using the definitions $W_{a_1 a_0} X_1^{a_1} X_0^{a_0} := W_{11} X_1^1 X_0^1 + W_{20} X_1^2 X_0^0 = m_1 m_0 + n_1$ and $W_{a_0} X_0^{a_0} := W_1 X_0^1 = m_0$, which are consistent with the choice $w_2 = m_1 w_1 + n_1 w_0$ and $w_1 = m_0 w_0$, the above relation reads

$$W_{a_2 a_1 a_0} X_2^{a_2} X_1^{a_1} X_0^{a_0} \overset{!}{=} m_2 W_{a_1 a_0} X_1^{a_1} X_0^{a_0} + \\ + n_2 W_{a_0} X_0^{a_0}, \qquad (169)$$

which is fully consistent with (166) due to the fact that $o_2 = 0$. Considering the fact that the only non-zero components of $W_{a_2 a_1 a_0}$ are W_{111}, W_{120} W_{201}, W_{300}, the only non-zero components of $W_{a_1 a_0}$ are W_{11}, W_{20} and the only non-zero component of W_{a_0} is W_1, one obtains the result $W_{a_2 a_1 a_0} X_2^{a_2} X_1^{a_1} X_0^{a_0} = m_2 m_1 m_0 + n_2 m_0 + m_2 n_1 + o_2 = m_2 W_{a_1 a_0} X_1^{a_1} X_0^{a_0} + n_2 W_{a_0} X_0^{a_0}$.

The next case to be considered is $k = 3$. In this case, one finds

$$W_{a_3 a_2 a_1 a_0} X_3^{a_3} X_2^{a_2} X_1^{a_1} X_0^{a_0} \overset{!}{=} m_3 W_{a_2 a_1 a_0} X_2^{a_2} X_1^{a_1} X_0^{a_0} + \\ + n_3 W_{a_1 a_0} X_1^{a_1} X_0^{a_0} + o_3 W_{a_0} X_0^{a_0}, \qquad (170)$$

so that, in consideration of the fact that the only non-zero components of $W_{a_3 a_2 a_1 a_0}$ are W_{1111}, W_{1120} W_{1201}, W_{2011}, W_{2020}, W_{1300}, one obtains the result $W_{a_3 a_2 a_1 a_0} X_3^{a_3} X_2^{a_2} X_1^{a_1} X_0^{a_0} = m_3 m_2 m_1 m_0 + m_3 n_2 m_0 + m_3 m_2 n_1 + n_3 m_1 m_0 + n_3 n_1 + m_3 o_2 + o_3 m_0 = m_3 W_{a_2 a_1 a_0} X_2^{a_2} X_1^{a_1} X_0^{a_0} + n_3 W_{a_1 a_0} X_1^{a_1} X_0^{a_0} + o_3 W_{a_0} X_0^{a_0}$.

Finally, the consistency of the induction step $k \to k+1$ must be demonstrated, which can be achieved by considering the fact that

$$W_{a_{k+1}a_k...a_0}X_{k+1}^{a_{k+1}}X_k^{a_k}...X_0^{a_0} = W_{1a_k...a_0}X_{k+1}^1X_k^{a_k}...X_0^{a_0}+$$
$$+W_{20a_{k-1}...a_0}X_{k+1}^2X_k^0X_{k-1}^{a_{k-1}}...X_0^{a_0}+W_{300a_{k-2}...a_0}X_{k+1}^3X_k^0X_{k-1}^0X_{k-2}^{a_{k-2}}...X_0^{a_0} =$$
$$\overset{!}{=} W_{a_k...a_0}X_{k+1}^1X_k^{a_k}...X_0^{a_0}+W_{a_{k-1}...a_0}X_{k+1}^2X_{k-1}^{a_{k-1}}...X_0^{a_0}+$$
$$W_{a_{k-2}...a_0}X_{k+1}^3X_{k-2}^{a_{k-2}}...X_0^{a_0} \tag{171}$$

applies whenever $W_{1a_k...a_0} - W_{a_k...a_0} = W_{20a_{k-1}...a_0}X_k^0 - W_{a_{k-1}...a_0} = W_{300a_{k-2}...a_0}X_k^0X_{k-1}^0 - W_{a_{k-2}...a_0} = 0$ is fulfilled. But since all non-zero components of both $W_{a_{k+1}a_k...a_0}$ and $W_{a_k...a_0}$ have the same value equal to one, assertion (171) defines a distributional relation which is actually fulfilled for all possible combinations of indices $a_k...a_0$, $a_{k-1}...a_0$ and $a_{k-2}...a_0$ due to the fact that $X_j^0 = X_{j-1}^0 = 1$ applies for all fixed non-negative values j and $j-1$. Therefore, it can be concluded that relation (168) is valid and that $F_1^\pm$ really is a solution of the differential equation (165).

Subsequently, taking into account the mathematical framework of the Fuchsian class [60], one immediately comes to the conclusion that $\lim\limits_{k\to\infty}|\frac{w_{k+1}}{w_k}| = \lim\limits_{k\to\infty}|\frac{\ll m,n,o\gg_k}{\ll m,n,o\gg_{k-1}}| \leq 1$ must apply in the present context.

The additional solution $F_2^\pm$ has a related, but even more complicated structure. The said solution results from the inhomogeneous differential equation

$$\Sigma_+(1-\xi^2)\frac{d^2G^\pm}{d\xi^2} - [2\xi\Sigma_+ + 2a^2\xi(1-\xi^2)]\frac{dG^\pm}{d\xi} - \tag{172}$$
$$-2r_+(r_+-M)G^\pm = -2C_1\Sigma_+^2\frac{dF_1^\pm}{d\xi}$$

for $G^\pm$, which results directly from inserting the second solution $F_2 = F_2(\xi)$ into the homogeneous part of the reduced generalized Dray-'t Hooft relation (165).

Using here $G^\pm = \sum\limits_{k=0}^{\infty}(-1)^k u_k^\pm(1\pm\xi)^k$ and $2C_1\Sigma_+^2\frac{dF_1^\pm}{d\xi} = \sum\limits_{k=0}^{\infty}(-1)^k\phi_k^\pm(1\pm\xi)^k$, where $\phi_0^\pm := 2C_1(r_+^2+a^2)^2w_1^\pm$, $\phi_1^\pm := 4C_1(r_+^2+a^2)w_2^\pm+8C_1a^2w_1^\pm$, $\phi_2^\pm := 6C_1(r_+^2+a^2)^2w_3^\pm + 16C_1a^2w_2^\pm + 4C_1a^2(r_+^2+3a^2)w_1^\pm$, $\phi_3^\pm := 8C_1(r_+^2+a^2)^2w_4^\pm + 24C_1a^2w_3^\pm + 8C_1a^2(r_+^2+3a^2)w_2^\pm + 8C_1a^4w_1^\pm$ and $\phi_k^\pm := 2C_1(r_+^2+a^2)^2(k+1)w_{k+1} + 8C_1(r_+^2+a^2)kw_k^\pm + 4C_1a^2(r_+^2+3a^2)(k-1)w_{k-1}^\pm + 8C_1a^4(k-2)w_{k-2}^\pm + 2C_1a^4(k-3)w_{k-3}^\pm$ shall apply by definition for $k \geq 4$, one finds the recurrence relation

$$u^{\pm}_{k+1} = m_k u^{\pm}_k + n_k u^{\pm}_{k-1} + o_k u^{\pm}_{k-2} + \Phi^{\pm}_k, \tag{173}$$

which contains the expression $\Phi^{\pm}_k := \frac{\phi^{\pm}_k}{2(r^2_+ + a^2)(k+1)^2}$, which can alternatively be written down in the form

$$\Phi^{\pm}_k = C_1 \sum_{i=0}^{4} \alpha_i \frac{k-i+1}{(k+1)^2} \ll m, n, o \gg_{k-i} w^{\pm}_0, \tag{174}$$

provided that $\alpha_0 := r^2_+ + a^2$, $\alpha_1 := 4a^2$, $\alpha_2 := 2a^2(1 + 2a\omega_+)$, $\alpha_3 := 4a^3\omega_+$, $\alpha_4 := a^3\omega_+$ applies in the given context.

Assuming now that all $w^{\pm}_k$ are zero for $k < 0$ and therefore all $X^{a_k}_k$ are zero as well, and furthermore that $\sum\limits_{j=0}^{m-1} a_j = m$ and $\sum\limits_{j=m}^{k} a_j = k - m + 1$ applies in the present context and that $u^{\pm}_2$ and $u^{\pm}_1$ are suitably chosen, the corresponding coefficients can be written down in the form

$$u^{\pm}_{k+1} = \lll \| m, n, o \| \ggg^{\pm}_k \tag{175}$$

provided that

$$\lll \| m, n, o \| \ggg^{\pm}_k = \ll m, n, o \gg_k u^{\pm}_0 + \tag{176}$$
$$+ \sum_{j=0}^{k} W_{a_k a_{k-1} \dots a_j} X^{a_k}_k X^{a_{k-1}}_{k-1} \dots \Phi^{\pm}_{k-j},$$

which can be re-written in the form

$$\lll \| m, n, o \| \ggg^{\pm}_k = \ll m, n, o \gg_k u^{\pm}_0 + \tag{177}$$
$$+ C_1 \sum_{i=0}^{4} \sum_{j=0}^{k} \alpha_i \frac{k-i-j+1}{(k-j+1)^2} W_{a_k a_{k-1} \dots a_j} W_{a_{k-i-j} \dots a_0} X^{a_k}_k X^{a_{k-1}}_{k-1} \dots X^{a_j}_j X^{a_{k-i-j}}_{k-i-j} \dots X^{a_0}_0 w^{\pm}_0.$$

In order to see now that this expression really solves the inhomogeneous differential relation (172), it is advisable to write (176) in the form

$$\lll \| m, n, o \| \ggg^{\pm}_k = \ll m, n, o \gg_k u^{\pm}_0 + \Phi^{\pm}_k + \tag{178}$$
$$+ m_k \Phi^{\pm}_{k-1} + (m_k m_{k-1} + n_k) \Phi^{\pm}_{k-2} + (m_k m_{k-1} m_{k-2} + n_k m_{k-2} + o_k) \Phi^{\pm}_{k-3} + \dots,$$

so that it can be concluded that (173) is solved if and only if

$$\begin{aligned}
&\ll m,n,o \gg_k u_0^\pm + \Phi_k^\pm + m_k\Phi_{k-1}^\pm + (m_k m_{k-1} + n_k)\Phi_{k-2}^\pm + \qquad (179)\\
&+ (m_k m_{k-1} m_{k-2} + n_k m_{k-2} + o_k)\Phi_{k-3}^\pm + ... \stackrel{!}{=} m_k \ll m,n,o \gg_{k-1} u_0^\pm +\\
&+ m_k\Phi_{k-1}^\pm + m_k m_{k-1}\Phi_{k-2}^\pm + ... + n_k \ll m,n,o \gg_{k-2} u_0^\pm + n_k\Phi_{k-2}^\pm +\\
&+ n_k m_{k-1}\Phi_{k-3}^\pm + ... + o_k \ll m,n,o \gg_{k-3} u_0^\pm + o_k\Phi_{k-3}^\pm + ... =\\
&= m_k \ll m,n,o \gg_{k-1} u_0^\pm + n_k \ll m,n,o \gg_{k-2} u_0^\pm + o_k \ll m,n,o \gg_{k-3} u_0^\pm +\\
&+\sum_{j=0}^{k} W_{a_k a_{k-1}...a_j} X_k^{a_k} X_{k-1}^{a_{k-1}} ...\Phi_{k-j}^\pm
\end{aligned}$$

is fulfilled. However, due to the validity of (168), it is not difficult to see that this is indeed the case, from which it can be concluded that (172) is actually solved by an ansatz of the form (176), which proves to be completely equivalent to the expression given in (177).

Next, due to the fact that $\lll\| m,n,o \|\ggg_k^\pm \simeq \ll m,n,o \gg_k u_0^\pm$ holds for large k, it is in fact not difficult to see that $\lim\limits_{k\to\infty}|\frac{u_{k+1}}{uk}| = \lim\limits_{k\to\infty}|\frac{\lll\|m,n,o\|\ggg_k^\pm}{\lll\|m,n,o\|\ggg_{k-1}^\pm}| \leq 1$ applies in the present context.

Hence, given the precise form of the coefficients $w_k^\pm$ and $u_k^\pm$, one can now make the ansatz

$$F = \Theta_+ F^+ + \Theta_- F^- \tag{180}$$

for the reduced profile function F of the geometry, where

$$F^\pm = c_1^\pm F_1^\pm + c_2^\pm F_2^\pm. \tag{181}$$

Since it is required that the solution is regular at $\xi_0 = -1$, it is known that the coefficient c_2^- must be identically zero. Accordingly, using the fact that one can always set w.l.o.g. $\Theta_+ + \Theta_- = 1$ and define $\Theta := \Theta_+$, the solution can be expressed in the form

$$F = \Theta(c_1^+ F_1^+ + c_2^+ F_2^+) + (1-\Theta)c_1^- F_1^-. \tag{182}$$

Note that the individual parts of this solution can be related w.l.o.g. by means of a linear transformation of the type

$$\begin{pmatrix} F_1^- \\ F_2^- \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} F_1^+ \\ F_2^+ \end{pmatrix}, \tag{183}$$

which yields $F_1^- = a_{11}F_1^+ + a_{12}F_2^+$.

Inserting the ansatz (182) in the differential relation (163), the conditions

$$c_1^+ F_1^+(0) + c_2^+ F_2^+(0) \stackrel{!}{=} c_1^- F_1^-(0), \tag{184}$$

$$c_1^+ F_1^{+\prime}(0) + c_2^+ F_2^{+\prime}(0) \stackrel{!}{=} c_1^- F_1^{-\prime}(0) \tag{185}$$

can be deduced. Using now the fact that $F_1^- = a_{11}F_1^+ + a_{12}F_2^+$ applies in the present context, these conditions reduce to the much simpler form

$$F_2^+(0) \stackrel{!}{=} -\frac{c_2^+ - a_{12}c_1^-}{c_1^+ - a_{11}c_1^-}F_1^+(0),\ F_2^{+\prime}(0) \stackrel{!}{=} -\frac{c_2^+ - a_{12}c_1^-}{c_1^+ - a_{11}c_1^-}F_1^{+\prime}(0), \tag{186}$$

so that it can be concluded that $\frac{F_2^{+\prime}(0)}{F_1^{+\prime}(0)} \stackrel{!}{=} \frac{F_2^+(0)}{F_1^+(0)}$.

Thus, assuming now that the condition $c_2^+ - a_{12}c_1^- \stackrel{!}{=} 0$ is met in the present context, it follows that differential equation (148) can exactly be solved in a distributional sense, which means that

$$[\boxtimes f, \cdot] = [2\pi b_0 \delta_N, \cdot] \tag{187}$$

is satisfied in relation to any test function with compact support; at least provided that $b_0 = C_1 c_2^+ w_0^+ (r_+^2 + a^2)^2$. As a basis for the validity of this equation, however, the consistency conditions

$$F_2^+(0) \stackrel{!}{=} 0,\ F_2^{+\prime}(0) \stackrel{!}{=} 0, \tag{188}$$

which follow directly from (186), both must be met simultaneously as well. But this is not much of a problem, because the coefficients C_1 and C_2 occurring in the definition of $\Delta = \Delta(\xi)$ can be freely chosen, so that one comes to the conclusion that f, which can be written in the form $f = f(v, r, \xi) := \tilde{f}(v, \xi)\delta(r - r_+) = \Sigma_+ F(\xi)e^{\kappa v}\delta(r - r_+)$ on the black hole horizon, actually represents a solution to differential relation (187). The fact that this conclusion is indeed justified can be explained as follows: Using (161) in combination with the standard identity $[\boxtimes f, \varphi] = \lim\limits_{\epsilon \to 0} \int\limits_{\mathbb{R}^2 \setminus B_\epsilon} \boxtimes f \varphi \omega_q$, according to which B_ϵ represents a two-dimensional ball of radius ϵ centered around the singularity and ω_q is the standard volume form of $\mathbb{R}^2$, consideration of Green's identities leads after the substitution $\xi = z + 1$ and a careful treatment of all relevant terms, i. e. those that depend on

ϵ, to the result

$$[\boxtimes f, \varphi] = 2\pi C_1 c_2^+ w_0^+ (r_+^2 + a^2)^2 \varphi(0), \tag{189}$$

from which it can be concluded that (187) is actually valid. Consequently, it becomes immediately clear that the resulting expression for the profile function solves both the differential equation (148) and its reduced form (154) exactly and that the resulting Kerr-Schild deformation of the Kerr-Newman metric therefore represents an exact, distributionally well-defined solution of Einstein's field equations.

5.3 Geometric Limits, Uniqueness

As already mentioned, the generalized Dray-'t Hooft relation reduces to Sfetsos' relation in the limit $a \to 0$ and to the original Dray-'t Hooft relation in the limit $a, e \to 0$. In the more general case of the limit $a \to 0$, one therefore obtains

$$\frac{d}{d\xi}((1-\xi^2)\frac{dF}{d\xi}) - cF = 2\pi b_0 \delta(\xi - 1). \tag{190}$$

This equation can be solved by considering first the homogeneous equation

$$\frac{d}{d\xi}((1-\xi^2)\frac{dF}{d\xi}) - cF = 0, \tag{191}$$

which obviously matches Legendre's differential equation in case that the constant c can be written in the form $c = -l(l+1)$. In this comparatively simple case, the said equation admits two independent solutions known as Legendre functions of first and second kind, which result as a special case of Gauß's differential equation [60]. Accordingly, these functions may be expressed in the form $F_1(\xi) = F(l+1, -l, 1; \frac{1-\xi}{2}) =: P_l(\xi)$ and $F_2(\xi) = \frac{1}{2} P_l(\xi) \ln |\frac{1+\xi}{1-\xi}| - \sum\limits_{k=1}^{L} \frac{2l-4k+3}{(2l-1)(l-k+1)} P_{l-2k+1}(\xi)$, where $L = \frac{1}{2} l$ applies in the case that l is even and $L = \frac{1}{2}(l+1)$ applies in the case that l is odd. Of course, by making a transformation of the type $\xi \to -\xi$, different pairs of solutions $F_1^\pm = F_1^\pm(\xi)$ and $F_2^\pm = F_2^\pm(\xi)$ are obtained. These individual pairs of solutions can be glued together the same way as previously shown in the more general axisymmetric case, which yields the reduced profile function

$$F = \Theta_+ F^+ + \Theta_- F^-, \tag{192}$$

where

$$F^{\pm} = c_1^{\pm} F_1^{\pm} + c_2^{\pm} F_2^{\pm}. \tag{193}$$

Thus, using the same definitions as in the axisymmetric case, this solution can be re-written in the form

$$F = \Theta(c_1^+ F_1^+ + c_2^+ F_2^+) + (1-\Theta) c_1^- F_1^-. \tag{194}$$

By inserting then in (191), this yields the conditions

$$c_1^+ F_1^+(0) + c_2^+ F_2^+(0) \stackrel{!}{=} c_1^- F_1^-(0), \tag{195}$$

$$c_1^+ F_1^{+\prime}(0) + c_2^+ F_2^{+\prime}(0) \stackrel{!}{=} c_1^- F_1^{-\prime}(0) \tag{196}$$

which, of course, can also be met in the given spherically symmetric case. The reduced profile function F therefore fulfills Sfetsos' relation given above in the case that $c = -l(l+1)$.

However, since it may not always be possible to assume the validity of $c = -l(l+1)$, it may be necessary to proceed differently. As was first demonstrated by Dray and 't Hooft, a particular way to do so is to solve the inhomogeneous equation directly, which can be achieved by expanding the reduced profile function on the left hand side and the delta function on the right hand side simultaneously in Legendre polynomials. Since it is known that $\delta(x) = \sum\limits_{l=0}^{\infty}(l+\frac{1}{2})P_l(x)$, one obtains the solution

$$F(\xi) = -b_0 \sum_{l=0}^{\infty} \frac{l+\frac{1}{2}}{l(l+1)+c} P_l(\xi) \tag{197}$$

by solving the corresponding eigenvalue problem. An integral expression for this solution can be found by considering the generating function of the Legendre polynomials

$$\sum_{l=0}^{\infty} \frac{l+\frac{1}{2}}{l(l+1)+c} P_l(\xi) e^{-sl} = \frac{1}{\sqrt{1-2\xi e^{-s}+e^{-2s}}}$$

in addition to the fact that

$$\frac{l+\frac{1}{2}}{l(l+1)+\alpha^2+\frac{1}{4}} = \int_0^{\infty} e^{-s(l+\frac{1}{2})} \cos(\alpha s) ds.$$

This yields

$$F(\xi) = -\frac{b_0}{\sqrt{2}}\int\limits_0^\infty \frac{\cos(\sqrt{c-\frac{1}{4}}s)}{\sqrt{\cosh s - \xi}}ds. \tag{198}$$

While handling the whole subject this way obviously works well enough, there is actually another way to proceed in this regard. One may simply solve the corresponding differential equation directly without relying on the existence of polynomial solutions. This can be seen as follows: Starting with an ansatz of the form $F_1 = \sum\limits_{k=0}^{\infty} w_k(\xi-1)^k$ and $F_2 \equiv F_1\Delta + G$, according to which $G = \sum\limits_{k=0}^{\infty} u_k(\xi - 1)^k$ applies, one obtains two different solutions of Sfetsos' differential relation under the condition that the corresponding coefficients fulfill $w_{k+1} = \prod\limits_{j=0}^{k} m_j \cdot w_0$ with $m_j = m(j) = -\frac{j(j+1)+c}{2(j+1)^2}$ and $u_{k+1} = \prod\limits_{j=0}^{k} m_j \cdot (u_0 + \sum\limits_{j=0}^{k} \frac{2D_1 w_0}{k-j+1})$ and the logarithmic part of the Green function is given by $\Delta = \frac{D_1}{2}\ln|\frac{1+\xi}{1-\xi}| + D_2$, where D_1, D_2 are arbitrary constants. By performing then once again a transformation of the type $\xi \to -\xi$, different pairs of solutions $F_1^{\pm} = F_1^{\pm}(\xi)$ and $F_2^{\pm} = F_2^{\pm}(\xi)$ are obtained. These individual pairs of solutions can be glued together the same way as previously shown. This yields once again a reduced profile function of the form

$$F = \Theta(c_1^+ F_1^+ + c_2^+ F_2^+) + (1-\Theta)c_1^- F_1^-, \tag{199}$$

which, apart from a slightly different form of the integration constants, is exactly what is obtained by considering the limit $a \to 0$ of the previously obtained solution of the generalized Dray-'t Hooft equation. Therefore, it can be concluded that the solution of the generalized Dray-'t Hooft relation reduces exactly to Sfetsos' solution in the limit $a \to 0$ and to the original solution of Dray and 't Hooft in the limit $a, e \to 0$.

Accordingly, since this expression and the one previously obtained both solve the corresponding differential equations (155) and (156), it becomes clear that they must be identical in the limit $e \to 0$. Therefore, it can be concluded that they are different expressions of one and the same reduced profile function of the geometry.

In a specific sense, the obtained class of solutions is not unique;

at least not from a purely physical point of view. This is because by performing a null rescaling of the form $l^a \to \chi l^a$, $k^a \to \chi^{-1} k^a$, as long as the condition $D\chi = 0$ is fulfilled in the present context, one obtains another completely different Kerr-Schild geometry, whose associated deformed metric has the form

$$\tilde{\tilde{g}}_{ab} = g_{ab} + \tilde{f} l_a l_b = g_{ab} + \chi^2 f l_a l_b. \tag{200}$$

More precisely, using this metric, another local energy-momentum distribution of a null particle located on the event horizon of a black hole can be calculated, which does not match the original one deduced in this work. This provides an infinite set of solutions of Einstein's equations, which all yield the energy-momentum tensor of a point-like null particle located at the event horizon of the Kerr-Newman black hole background spacetime. Since the Einstein tensor of this geometry is linear in the profile function, a finite series of these solutions for the reduced profile function can in principle be superimposed to a many-body solution.

Discussion

In the present work, the geometric structure of the field of a gravitational shockwave generated by a null particle at the outer event horizon of a stationary Kerr-Newman black hole was determined. For this purpose, the exact distributional form of the profile function of the shockwave was calculated using the Kerr-Schild framework and the Newman-Penrose spin-coefficient formalism. Based on this result, it was shown that the resulting geometric model contains the solutions of Dray and 't Hooft, Sfetsos, and of course Aichelburg and Sexl as a special case, thus providing a whole family of generalized Kerr-Schild spacetimes in four dimensions.

Ultimately, the said class of solutions must be physically interpreted appropriately. As mentioned at the end of the previous section, the representatives of this class characterize the geometric field of a relativistic two-body system consisting of a black hole singularity and a singular null particle field on the black hole event horizon, which may in principle be extended to a many-body system by superimposing different expressions for the profile function of the geometry. This has the effect that the black hole event horizon is no longer a Killing horizon, but rather an extremal weakly isolated horizon in the sense of Ashtekar et al. Moreover, it has the effect that the geometry is no longer stationary and axisymmetric.

However, since the backreaction effects calculated in this work are local in the sense that they do not change the overall geometrical structure of the black hole spacetime but only the structure of the horizon of the black hole background, the resulting geometry can legitimately be interpreted as that of a dynamical classical black hole, which may eventually allow one to find out a few things about quasi-stationary black holes that emit Hawking radiation.

More specifically, given the literature on the simpler case of the Dray-'t Hooft model [38], one must seriously consider the possibility that the gravitational backreaction effects caused by a massless particle could be exactly the same as those arising from the Hawking effect and the phenomenon of black hole evaporation. This is not least because Hawking's original approach gives no indication of how geometric backreactions to the spacetime of an evaporating black hole should actually be given; so that one could legitimately conclude that the continuous emission of thermal radiation by a black hole is accompanied by the emission of gravitational shockwaves that propagate along the black hole event horizon. In this case, the deformed gravitational field of emitted shockwave pulses would then be curved by high-energetic null particles which escape along that horizon to future null infinity. Thus, if this consideration proves to be meaningful, the results of the present work could greatly contribute to a better understanding of the geometric structure of the gravitational field of an evaporating black hole, which then could turn out to be crucial for a deeper understanding of the quantum properties of these types of physical objects.

However, since the geometric backreaction effects caused by gravitational shockwaves in stationary black hole backgrounds are not yet fully understood and there is no generally accepted approach that relates the theory of gravitational shockwaves with that of scalar fields on such geometric backgrounds, it is worthwhile to be careful with such interpretation attempts.

Appendix

A The Fuchsian Class of second Order linear differential Equations with regular singular Coefficients

In the theory of ordinary homogeneous second-order linear differential equations with variable coefficients, which are known to be of interest for theoretical physics and in particular for the present work, it may happen that the corresponding coefficients are not globally well-behaved analytic functions but instead have singularities. In this rather common case, the behavior of solutions is usually studied in the immediate vicinity of those (typically isolated) singular points, where at least one of the coefficients of the equation diverges and it is therefore to be expected (based on the fact that the isolated singular points of said solutions are known to lie amongst those of their associated singular coefficients) that at least one of all the corresponding linearly independent solutions strives toward infinity as well. The way this occurs in detail depends very much on the nature of the singular point examined, that is, in particular, on whether the point in question is a so-called regular or irregular singular point. To explain the differences between these different types of points, note that any ordinary homogeneous second-order linear differential equation can be be written in the form

$$f'' + pf' + qf = 0, \tag{201}$$

where the prime denotes differentiation with respect to the (typically) complex variable ξ. Assuming the validity of the initial conditions $f|_{\xi=\xi_0} = w_0$, $f'|_{\xi=\xi_0} = w_1$, one may then concentrate on the case in which the coefficients $p = p(\xi)$ and $q = q(\xi)$ are complex-valued functions with altogether k different isolated singularities ξ_k. Given this setting, the necessary conditions for a point ξ_0 to be a regular singular point are that $p(\xi)$ has a pole of at most first order and $q(\xi)$ one of at most second order, so that the differential equation (201) may be re-written in the form

$$f'' + \frac{p_0}{\xi - \xi_0} f' + \frac{q_0}{(\xi - \xi_0)^2} f = 0, \tag{202}$$

where $p_0(\xi)$ and $q_0(\xi)$ are power series and thus regular in ξ_0. A prerequisite for this is that the coefficients $p(\xi)$ and $q(\xi)$ are holomorphic in an annulus K of radius r, i.e. in the local domain $0 < |\xi - \xi_0| < r$.

A differential equation with exactly this kind of singular behavior belongs to the so-called Fuchsian class of homogeneous second-order ordinary linear differential equations with regular singular coefficients, which is the class of equations that can be written in the form

$$f'' + \sum_{j=1}^{k} \frac{\gamma_j}{\xi - \xi_j} f' + \frac{V}{\prod_{j=0}^{k} (\xi - \xi_j)} f = 0, \tag{203}$$

where the γ_j are constant coefficients and $V = V(\xi)$ is a power series that reduces to the so-called Van Vleck polynomial, i.e. a polynomial of degree at most $k-2$, in the case that (203) has a polynomial solution.

One of the main differences to differential equations with irregular singular points, which show a stronger singular behavior, is that the coefficients of a differential equation with regular singular points can be expanded in the vicinity of a singular point ξ_0 in a Laurent series with a finite instead of an infinite number of negative exponents.

In purely formal terms, this means that one of the two linearly independent solutions must be of the form

$$\begin{aligned} f_1(\xi) = & \sum_{k=-m}^{\infty} u_k (\xi - \xi_0)^k = (\xi - \xi_0)^{-m} (u_{-m} + u_{-m+1}(\xi - \xi_0) + ...) =: \\ =: & \sum_{k=0}^{\infty} w_k (\xi - \xi_0)^{k-m} \end{aligned} \tag{204}$$

in full accordance with Fuchs' theorem [19, 59, 60]. Hence, it becomes clear that one of the solutions of (202) can be determined by making an ansatz in the form of a generalized Frobenius series

$$f_1(\xi) = \sum_{k=0}^{\infty} w_k (\xi - \xi_0)^{k+\rho_1}. \tag{205}$$

In the case of the second solution, on the other hand, due to the fact that (202) is a linear differential relation, the standard method of variation of parameters can be used to obtain another solution of the form $f_2 = f_1 \int\limits_{\xi_0}^{\xi} \frac{C}{f_1^2} e^{-\int\limits_{\xi_0'}^{\xi'} p(\xi'')d\xi''} d\xi'$. This again gives an expression of the form (205), but for a different critical exponent ρ_2. However, based on the fact that the integrant of this second solution can also be expanded in a generalized Frobenius series, one may realize - in the event that $\rho_2 = \rho_1 + m$, where m is some positive integer - that said solution is of the form

$$f_2(\xi) = A \sum_{k=0}^{\infty} w_k (\xi - \xi_0)^{k+\rho_1} \ln(\xi - \xi_0) + \sum_{k=0}^{\infty} v_k (\xi - \xi_0)^{k+\rho_1}, \quad (206)$$

where A is an arbitrary constant and ρ_1 and ρ_2 are the so-called characteristic exponents of the given pair of solutions. Of course, the corresponding power series have positive radii of convergence. It may be noted that a rescaling of the solutions $f_1(\xi)$ and $f_2(\xi)$ of (202) by a factor of $(\xi - \xi_0)^{\alpha}$, where α is some positive integer, yields again an equation of type (202).

Anyhow, since the concrete form of the linearly independent solutions in the vicinity of an isolated singular point is now known, the next question is how the form of the solution looks at other singular points of the equation. Here, one can take advantage of the following fact: By analytic continuation along any path not passing through the poles of $p(\xi)$ and $q(\xi)$, any set of linearly independent solutions, which is valid around a singular point of the said differential equation (202), gives a new set of solutions. However, it typically occurs in this context that the extended functions obtained from the analytic continuation of a given set of solutions in the vicinity of a given isolated singular point $\xi = \xi_0$ are multivalued complex functions, whose value at another point $\xi = \xi_j$ depends on the chosen curve from ξ_0 to ξ_j. In particular, by choosing a path around one of the singular points, it often happens that this point becomes a branch point, which has the effect that a given pair $f_1(\xi)$ and $f_2(\xi)$ of solutions transitions into a new pair $\tilde{f}_1(\xi)$ and $\tilde{f}_2(\xi)$. However, by taking advantage of the fact that pairs of solutions form a vector space, it becomes clear that there should be a linear transformation

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad (207)$$

commonly referred to as monodromy transformation, which, after a certain singularity has been circulated clockwise, turns one pair of solutions into another. In this context, the components a_{ik} of the corresponding monodromy matrix are constants, which are required to meet the condition $a_{11}a_{22} - a_{12}a_{21} \neq 0$ in order to ensure that the solutions $\tilde{f}_1(\xi)$ and $\tilde{f}_2(\xi)$ are linearly independent. The monodromy matrices are the generators of the monodromy group, which is prescribed by means of a finite-dimensional complex linear representation of the so-called fundamental group, which is the first and simplest homotopy group [52, 54].

The monodromy concept is important in this context not least because its definition reveals an important property of analytical continuations along curves between regular singular points. This can be seen if one moves alongside special paths around an isolated singularity, which all have the same start and endpoints and can be continuously deformed into one another, because in this case the analytic continuations along different curves will yield the same results at their common endpoint, which is subject to the famous monodromy theorem.

The eigenvalues λ_1 and λ_2 of the matrix given above, which can be determined as usual by solving the relation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0, \tag{208}$$

play an important role in the following. This is because these eigenvalues coincide, independent of the concrete choice of a basis, with those numbers by which a solution of the differential equation has to be multiplied in order to remain a solution. In order to get to this insight, it is generally exploited that the eigenvectors resulting from solving

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{209}$$

can be assembled into matrices, which can be used to diagonalize the matrix in (207), so that one obtains

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{210}$$

In the event that both eigenvalues coincide, so that $\lambda_1 = \lambda_2$, however, one usually considers the alternative system

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ a_{21} & \lambda_1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{211}$$

in relation to which, of course, it is still possible that $a_{21} = 0$.

Given these different settings, it is then found that for $\lambda_1 \neq \lambda_2$ both solutions of (202) must be of the form (205) (but for different characteristic exponents ρ_1 and ρ_2), whereas for $\lambda_1 = \lambda_2$ one solution must be of the form (205) and the other of the form (206). This is because, as can be deduced from the fact that the quotients $\frac{f_j}{(\xi-\xi_0)^{\rho_j}}$ are unique and regular near a given regular singular point $\xi = \xi_0$, there is a connection between the said critical exponents and the above-mentioned eigenvalues, i.e. $\lambda_j = e^{2i\pi\rho_j}$ for $j = 1, 2$, which is caused by the fact that the logarithm in $(\xi - \xi_0)^{\rho_j} = e^{\rho_j \ln(\xi-\xi_0)}$ grows by a multiple of $2i\pi$ by circumventing a singular point along a closed path in positive direction. The equality of the eigenvalues then requires $\rho_2 = \rho_1 + m$, where m is some positive integer, so that one knows why in this particular case the solution must be of the form (206).

Thus, in the case that an analytical continuation of a given pair of solutions in the vicinity of a fixed regular singular point can be defined, it is ensured that those pairs of solutions, which are valid in the vicinity of all other regular singular points, can be converted into the given one and into each other by simple linear transformations. The continuous analytical continuation of solutions along curves, which 'connect' in this way pairs of regular singular points with each other, defines a so-called Riemann surface, i. e. a one-dimensional complex manifold, which is a connected Hausdorff space that is endowed with an atlas of charts to the open unit disk of the complex plane (whereas the transition maps between two overlapping charts are required to be holomorphic).

And even though this approach can be used in principle to solve differential equation (203) around each singular point $\xi = \xi_j$ for fixed j, it turns out that it is much more advisable in practice to bring the equation into the form

$$\prod_{j=0}^{k}(\xi - \xi_j)f'' + \prod_{j=0}^{k}(\xi - \xi_j)(\sum_{j=1}^{k}\frac{\gamma_j}{\xi - \xi_j})f' + Vf = 0 \tag{212}$$

and to seek polynomial solutions of the resulting expression. These solutions, if they exist, are then given by what are called Heine–Stieltjes polynomials (sometimes also called Stieltjes polynomials) [35, 40, 61, 63], which form the basis for the construction of ellipsoidal harmonics and their generalizations [64]. In the case that no such polynomial solutions of (212) exist, the solution will be of the form (205). In order to obtain the second possible solution with the aid of the already determined one, one may then either use ansatz (206) or possibly first seek another solution by applying the method of variation of parameters in a slightly different way, namely by trying to find a solution $\Delta = \Delta(\xi)$ of (212) for $V = 0$ and then make an ansatz of the form

$$f_2 = f_1 \Delta + G, \tag{213}$$

where $G = G(\xi)$ can be assumed to be a generalized power series of the form $G(\xi) = \sum\limits_{k=0}^{\infty} u_k (\xi - \xi_j)^{k+\rho_2}$. Provided that the definition $\mho \equiv \Delta'$ is used in the present context, $\Delta(\xi)$ is then obtained by solving the first order relation

$$\mho' + \sum_{j=1}^{k} \frac{\gamma_j}{\xi - \xi_j} \mho = 0, \tag{214}$$

which follows directly from (212) for the special case $V(\xi) = 0$. Both methods are equivalent in that they lead to exactly the same results, from which it can be concluded that (206) and (213) represent one and the same solution of (203). The concrete choice of one of these methods for solving Fuchsian differential equations is therefore purely a matter of taste. Obviously, however, it is the latter method that is pursued in the present work.

As an application of the present investigation of Fuchs' mathematical framework for solving second-order linear differential equations with regular singular coefficients, special types of differential equations belonging to this class shall be considered next, which play an important role in mathematics and theoretical physics, since they are of particular importance for the solution theory of linear partial differential equations. Since, however, the mathematical literature pays much attention to the discussion of these special differential equations with a small number of regular singular points anyway, the remaining part of this appendix will be content with giving some relevant examples without discussing them in full detail on a case-by-case basis or listing the exact structure of the associated solutions and all their relevant properties. For a more detailed treatment of the subject, one should therefore rather consult the relevant mathematical literature, such as for instance the books by Smirnov or Slavyanov and Lay.

To begin, first the case of Fuchsian differential equations with three regular singular points shall be discussed. The most prominent representative of this class is indisputably the hypergeometric differential equation of Gauß, which has the form

$$f'' + [\frac{c}{\xi} + \frac{a+b-c+1}{\xi-1}]f' + \frac{ab}{\xi(\xi-1)}f = 0. \tag{215}$$

This equation, whose associated three singular points are 0, 1 and ∞, obviously belongs to the Fuchsian class of differential equations with three regular singular points. It is solved by the so-called hypergeometric function

$$F(a,b,c;\xi) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{\xi^n}{n!}, \tag{216}$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} x(x+1)....(x+n-1) & n\geq 1 \\ 1 & n=0 \end{cases}$ is the so-called Pochhammer symbol. As outlined above, this form of the solution results from making a Frobenius ansatz of the form (205), which yields the indicial equation $\rho(\rho+c-1) = 0$, from which it can be concluded that there exist two independent solutions of (215) in terms of convergent power series in the vicinity of $\xi_0 = 0$ if c is not an integer. These solutions $f_1(\xi) = F(a,b,c;\xi)$ and $f_2(\xi) = \xi^{1-c}F(a-c+1, b-c+1, 2-c;\xi)$ can be superimposed to a single solution due to the fact that (215) is linear. This also works in the case that c is a positive integer; a case, in which the second solution instead takes the form $f_2(\xi) = Af_1 \ln\xi + \xi^{1-c}G$, where $G = G(\xi)$ is again a convergent power series. It also works in the event that c is a negative integer or zero, in which case the solutions are $f_1(\xi) = \xi^{1-c}F(a-c+1, b-c+1, 2-c;\xi)$ and $f_2(\xi) = Bf_1 \ln\xi + H$, where $H = H(\xi)$ is once more a convergent power series. The solutions found can then easily be converted by linear transformation into those in the vicinity of the singular points 1 and ∞, so that the equation (215) has a solution pair of the form (205) and (206) in the vicinity of each regular singular point.

The hypergeometric differential equation leads to some important special cases, including the Legendre equation

$$(1-\xi^2)f'' - 2\xi f' + l(l+1)f = 0, \tag{217}$$

the Jacobi equation

$$(1-\xi^2)f'' + ([\beta - \alpha - (\alpha+\beta+2)\xi]f' + n(n+\alpha+\beta+1)f = 0, \tag{218}$$

the Chebyshev equation

$$(1-\xi^2)f'' - \xi f' + \alpha^2 f = 0 \tag{219}$$

and the Gegenbauer equation

$$(1-\xi^2)f'' - 2(\mu+1)\xi f' + (\nu-\mu)(\nu+\mu+1)f = 0, \tag{220}$$

all of which possess polynomial solutions that belong to the superordinate class of Heine-Stieltjes polynomials. In this context, it shall be mentioned only briefly that by performing once again a linear transformation, which allows one to introduce a limiting procedure by means of which it can be achieved that a singularity lying at a finite position is shifted into infinity and thus coincides with the singularity already existing there, the said equation can be further transformed into the confluent hypergeometric differential equation, which leads to other important special cases such as the Bessel, Hermite and Laguerre equations. It thus becomes apparent that a large number of the special functions relevant for mathematical physics are solutions of the hypergeometric equation in one form or another.

The next step now shall be the discussion of Fuchsian differential equations with four regular singular points. The most prominent representative of this class is most certainly Heun's differential equation

$$f'' + (\frac{\gamma}{\xi} + \frac{\delta}{\xi-1} + \frac{\epsilon}{\xi-a})f' + \frac{\alpha\beta\xi - q}{\xi(\xi-1)(\xi-a)}f = 0, \tag{221}$$

whose associated coefficients must satisfy $\alpha+\beta-\gamma-\delta-\epsilon+1=0$. Its regular singular points lie at 0, 1, a and ∞ and it admits no less than 192 solutions, usually called Heun functions. Assuming that there is an infinite set of discrete values q_m for the accessory parameter q, these solutions are often denoted by $Hf_m(a, q_m; \alpha, \beta, \gamma, \delta; \xi)$ and rarely written down explicitly, which is mainly due to the fact that the coefficients have to be determined by solving a two-term recursion relation. However, they are often written down as power series expressions in Riemann's P-symbols and thus in series of hypergeometric functions, which are solutions of Riemann's P-differential equation. The most prominent among these solutions are certainly the polynomial ones, generally known as Heun polynomials, which are again special types of Heine-Stieltjes polynomials. The Heun equation and its generalized version

$$f''+(\frac{1-\mu_0}{\xi}+\frac{1-\mu_1}{\xi-1}+\frac{1-\mu_2}{\xi-a}+\alpha)f'+\frac{\beta_0+\beta_1\xi+\beta_2\xi^2}{\xi(\xi-1)(\xi-a)}f=0, \quad (222)$$

which is usually referred to as generalized Heun equation in the literature [15, 57], are of interest in mathematics because they both contain the Mathieu, Lamé, Whittaker-Hill and Ince equations as a special case and are of relevance for solving the ellipsoidal wave equation. However, they are also of great relevance for theoretical physics, that is to say for both black holes perturbation theory and quantum field theory on curved backgrounds, where these equations have been shown to occur naturally in the process of solving the Dirac equation for massive Fermions on a Kerr-Newman geometric background and giving a description of static perturbations of non-extremal Reissner-Nordström black holes [13, 15]. Moreover, it seems to play a role in finding solutions of the Teukolsky equation of Petrov type D vacuum background spacetimes. It should also be mentioned that the confluent special cases of the Heun equation have proved significant for the characterization and solution of the Regge-Wheeler equations, the Zerelli equations and the Teukolsky equations.

The next case to be discussed is the case of Fuchsian differential equations with five regular points. The most prominent representative of this class is the generalized Lamé equation (keep in mind that several differential equations were named after Lamé)

$$\begin{aligned} f''+\frac{1}{2}(\frac{1}{\xi+b}+\frac{1}{\xi-b}+\frac{1}{\xi+c}+\frac{1}{\xi-c})f'+ \\ +\frac{p(b^2+c^2)-m(m+1)\xi}{(\xi^2-b^2)(\xi^2-c^2)}f=0 \end{aligned} \quad (223)$$

with the singular points $\pm b$, $\pm c$ and ∞. It is solved by so-called generalized Lamé functions, which form the basis for the construction of generalized ellipsoidal harmonics. As in the case of Heun's differential equation, these solutions are rarely written down explicitly, which is because they are subject to a three-term recursion relation. As stated, the process of solving the homogeneous generalized Dray-'t Hooft equation also leads to an ordinary differential equation that belongs to this largely unexplored class.

In the case of equations with more than six regular singular points, only a few examples are known. One is the so-called hyper-generalized Heun equation, which occurs in the context of quantum field theory on Kerr-Newman-de Sitter backgrounds, which has six regular singular points [14]. There may be other examples, but not too many that are well known.

What all the differential equations with more than three regular singular points have in common is that their solutions, which have to be of the form (205), are difficult to write down explicitly. However, a great advantage of the methods developed in the present work is that they allow generically, i. e. for any finite number of terms of the associated recursion relation, to write down the form of the corresponding generalized power series expression. For an n-term recursion relation of the form

$$w_{k+1} = m_{(1)k} w_k + m_{(2)k} w_{k-1} + m_{(3)k} w_{k-2} + ... + m_{(n)k} w_{k-n}, \quad (224)$$

this can be achieved by introducing the symbol $\ll m_{(1)}, m_{(2)}, ...m_{(n)} \gg_k$, which represents a multi-linear form of the type

$$\ll m_{(1)}, m_{(2)}, ...m_{(n)} \gg_k = W_{a_k a_{k-1} ... a_0} X_k^{a_k} X_{k-1}^{a_{k-1}} ... X_0^{a_0}, \quad (225)$$

where each a_j runs from zero to $n-1$; at least provided that one uses one's freedom to choose $w_1 := m_{(1)0} w_0$, $w_2 := m_{(1)1} w_1 + m_{(2)1} w_0 = (m_{(1)1} m_{(1)0} + m_{(2)1}) w_0$, ..., $w_n := m_{(1)n} w_{n-1} + m_{(2)n} w_{n-2} + m_{(3)n} w_{n-3} + ... + m_{(n)n} w_0$. The solution then is of the form

$$w_{k+1} = \ll m_{(1)}, m_{(2)}, ...m_{(n)} \gg_k w_0 \quad (226)$$

The corresponding objects $X_j^{a_j} = X^{a_j}(j)$ have the components $X_j^0 = \theta(j)$, $X_j^1 = m_{(1)}(j)$, $X_j^2 = m_{(2)}(j)$, ... $X_j^n = m_{(n)}(j)$, where $\theta(j) := \begin{cases} 0 & if\ j<0 \\ 1 & if\ j\geq 0 \end{cases}$ is the Heaviside step function. In the meantime, the object $W_{a_k a_{k-1} ... a_0}$ is defined in such a way that all its components are either zero or one. The non-zero components are exactly those for which on the one hand $\sum_{j=0}^{k} a_j = k + 1$ applies and, moreover, all indices that take the value zero occur only as successors of those that take a value of two, all pairs of indices that take the value zero combined occur only as successors of indices with a value of three, all triples of indices that take the value zero occur only as successors of indices with a value of four and so on and so forth. This implies in particular that all $W_{0a_{k-1}\dots a_0}$, $W_{00a_{k-2}\dots a_0}$, ... $W_{00\dots 0a_{k-n}\dots a_0}$ are zero. It is also assumed that all coefficients with negative values are zero by definition.

For the sake of illustration, consider the simplest non-trivial example of a two-term recursion relation of the form

$$w_{k+1} = m_{(1)k} w_k + m_{(2)k} w_{k-1}, \tag{227}$$

whose solution is of the form (225), at least provided that the corresponding objects $X_j^{a_j} = X^{a_j}(j)$ have the components $X_j^0 = \theta(j)$, $X_j^1 = m_{(1)}(j)$ and $X_j^2 = m_{(2)}(j)$. For $k = 0$, the said relation (225) reads $\ll m_{(1)}, m_{(2)} \gg_0 = W_{a_0} X_0^{a_0} = W_1 X_0^1 = m_{(1)0}$, which coincides with what is obtained from (227). For $k = 1$, relation (225) reads $\ll m_{(1)}, m_{(2)} \gg_1 = W_{a_1 a_0} X_1^{a_1} X_0^{a_0} = W_{11} X_1^1 X_0^1 + W_{20} X_1^2 X_0^0 = m_{(1)1} m_{(1)0} + m_{(2)0}$, which coincides once more with what is obtained from (227). For $k = 2$, relation (225) reads $\ll m_{(1)}, m_{(2)} \gg_2 = W_{a_2 a_1 a_0} X_2^{a_2} X_1^{a_1} X_0^{a_0} = W_{111} X_2^1 X_1^1 X_0^1 + W_{120} X_2^1 X_1^2 X_0^0 + W_{201} X_2^2 X_1^0 X_0^1 = m_{(1)2} m_{(1)1} m_{10} + m_{(1)2} m_{(2)0} + m_{(2)2} m_{(1)0}$, which also coincides with what is obtained from (227). By further iteration one finds then that solutions of (227) can actually be written down on the basis of the symbol represented in (225), so that it can be concluded that all solutions of, for example, the Heun equation with characteristic exponent $\rho = 0$ are therefore necessarily of exactly this form.

It must therefore be concluded that all non-trivial solutions of homogeneous equations of the Fuchsian class with vanishing critical exponent can be represented in exactly this way.

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