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Numerical validation of the analytical estimate for future discretionary benefits

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Kurzfassung

Seit 01.01.2016 sind Versicherungsunternehmen mit Sitz in der europäischen Union aufgrund der Richtlinie Solvency II unter anderem dazu verpflichtet, die zukünftige Überschussbeteiligung (future discretionary benefits, FDB) zu melden. Normalerweise wird diese mit aufwändigen Monte Carlo Algorithmen berechnet, allerdings wurde im Paper *Estimation of future discretionary benefits in traditional life insurance* [GH22] von Gach und Hochgerner eine neue Methode entwickelt um die FDB im Kontext von Lebensversicherungen mit Gewinnbeteiligung abzuschätzen.

Der Fokus dieser Arbeit liegt auf der Überprüfung der Annahmen, die Gach und Hochgerner in ihrem Paper getroffen haben. Dies wird mit einem Asset-Liability-Modell in R durchgeführt. Es zeigt sich, dass der Großteil der Annahmen verifiziert werden kann und alle Annahmen zumindest so weit bestätigt werden können, dass deren Anwendung im Paper gerechtfertigt ist. Nach der Validierung der Annahmen wird die FDB in verschieden simulierten Zinsszenarien numerisch und analytisch berechnet, der Schätzer für die FDB bewertet und die verschiedenen Ergebnisse verglichen. Zusammenfassend kann die Abschätzung der FDB in allen simulierten Szenarien als erfolgreich bezeichnet werden, zusätzlich ist der Schätzfehler immer sehr klein.

Abstract

Since 01.01.2016, insurance companies domiciled in the European Union are obligated to report the future discretionary benefits (FDB) due to the regulation Solvency II. The FDB are normally calculated with computationally expensive Monte Carlo algorithms, but a new method to estimate the FDB in the context of life insurance with profit participation was developed by Gach and Hochgerner in their paper *Estimation of future discretionary benefits in traditional life insurance* [GH22].

The focus of this thesis is to validate the assumptions made by Gach and Hochgerner in their paper. This is done with an Asset Liability Management model in R. It turns out that the majority of the assumptions can be verified and that all assumptions can be confirmed at least to the extent that their application in the paper is justified. After validating the assumptions, the FDB are calculated numerically and analytically in differently simulated interest rate scenarios, the estimation of the FDB is evaluated and the different results are compared. Overall, the estimation of the FDB can be considered successful in all simulated scenarios, in addition the estimation error is always very small.

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Eidesstattliche Erklärung

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Wien, am 24.02.2023

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Contents

1 Introduction

Solvency II, a directive of the European Union which harmonises the EU insurance regulation, came into effect on 01.01.2016. Among other demands, it requires insurance companies to report the best estimate (BE) on a quarterly basis. The Directive of the European Parliament of 25.11.2009 defines the best estimate as follows:

> *The best estimate shall correspond to the probability weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure.* [Eurb]

This thesis focuses on life insurance companies which are subject to the Solvency II regulatory regime and sell traditional life insurance products with profit participation, the following was written according to [GH22]. In the context of life insurance with profit participation the policyholder's expected payoff consists of a guaranteed part and a bonus benefit. Essentially, the guaranteed part is defined by a guaranteed technical interest rate and the bonus benefit depends on the performance of the insurer's asset portfolio. In general, the company's gross surplus is shared between policyholder, shareholder and tax office.

The representation

$$BE = GB + FDB,$$

where GB are the benefits which are guaranteed at valuation time (guaranteed benefits) and FDB are the future discretionary benefits which depend on the company's gross surplus, is one possibility to decompose the best estimate when considering life insurance with profit participation. Both the GB and the FDB must be reported on a quarterly basis. While the guaranteed benefits GB can be calculated with methods which are close to classical actuarial computations, the calculation of the FDB, which we are interested in, is more complicated. This is caused by their direct dependence on the company's gross surplus and therefore on management actions and financial revenue.

The FDB are often calculated with computationally expensive Monte Carlo algorithms, but in the paper [GH22] the authors Gach and Hochgerner have found a new method to evaluate them by calculating an upper and a lower bound for the FDB and using the average of these two bounds as an estimator. This procedure works easier and faster than the calculation with Monte Carlo algorithms and the aim of this thesis is to validate the assumptions Gach and Hochgerner needed to be able to make their calculations. Furthermore, the FDB are calculated analytically and numerically in differently simulated interest rate scenarios, the estimation of the FDB is evaluated and the different results are compared. All this is done with an Asset Liability Management model (ALM model) which is implemented in R.

In Chapter 2, we start with an introduction of the later used notation and derive a closed formula for the FDB. Chapter 3 briefly describes our ALM model which is implemented in R. The validation of the assumptions which were made by Gach and Hochgerner in [GH22] is performed in Chapter 4, which is the main part of this thesis. Finally, upper and lower bounds for the FDB are derived, the analytical estimator $\widehat{FDB}$ is calculated and the numerical FDB and the analytical $\widehat{FDB}$ are compared in differently simulated interest rate scenarios in Chapter 5. Chapter 6 provides a quick review of existing literature.

actuarial provisions	2020	2019
insurance with profit participation	**253 428 793**	**245 126 705**
best estimate	268 604 237	242 665 093
of which guaranteed benefits	223 865 226	195 223 681
of which future profit participation	44 739 011	47 441 412
of which options and guarantees	3 397 937	3 339 925
risk margin	2 485 520	2 461 613

Table 1.1: Extract from the solvency and financial condition report of the Allianz Lebensversicherungs-AG Germany in 2020 and 2019, values are in thousand euros. [Allb], [Alla]

	2020	2019
eligible own funds	40 318 958	28 919 681
solvency capital requirement	11 345 148	8 977 868
solvency ratio	355%	322%
minimum capital requirement	5 105 317	4 040 040

Table 1.2: Extract from the solvency and financial condition report of the Allianz Lebensversicherungs-AG Germany in 2020, values are in thousand euros. [Allb]

Tables 1.1 and 1.2 show a few key figures from the solvency and financial condition report of the Allianz Lebensversicherungs-AG Germany in 2020 so that the reader has an idea of the magnitude of the BE and the relationship between BE, FDB and own funds.

Disclaimer. The opinions expressed in this thesis are those of the author and do not necessarily reflect the official position of the Austrian Financial Market Authority.

2 Preliminaries

2.1 Notation

To make the subsequent parts of this thesis more readable, many of the later used abbreviations from [GH22] are defined and explained in the following section. However, all connections and definitions are also listed in the Appendix: List of symbols.

Firstly, a yearly time grid $t = 0, 1, \dots, T$ is fixed, where $t = 0$ corresponds to valuation time and the time horizon T may be large. Furthermore it is assumed that there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ such that all stochastic processes are adapted to $(\mathcal{F}_t)$ and all expected values are computed with respect to the risk neutral spot measure $\mathbb{Q}$ [Fil09, Sec. 11.4]. The increments of a time dependent quantity f_t are denoted by $\Delta f_t = f_t - f_{t-1}$ and c^+, c^- describe the positive and negative parts of a number c.

Let C_t denote the amount of cash held by the company at time t, MV_t the market value and BV_t the book value of the portfolio at time t. MV_t and BV_t are defined as

$$MV_t = \sum_{a \in \mathcal{A}_t} MV_t^a + C_t, \qquad BV_t = \sum_{a \in \mathcal{A}_t} BV_t^a + C_t$$

where $\mathcal{A}_t$ denotes the set of assets (excluding cash) in the portfolio at time t and BV_t^a, MV_t^a are the book value and the market value of an asset $a \in \mathcal{A}_t$ at time t. The unrealized gains at time t, UG_t, are defined as the difference $UG_t = MV_t - BV_t$. The simple one year forward rate between t and $t+1$ is defined as F_t, the bank account at time t is specified as

$$B_t = \prod_{j=0}^{t-1} (1 + F_j).$$

Let

$$D(t, s) = \prod_{j=t}^{s-1} (1 + F_j)^{-1}$$

denote the discount factor from s to $t < s$, and thus the value of a zero coupon bond at time t which pays one unit of currency at time s can be specified as

$$P(t, s) = \mathbb{E}[D(t, s)].$$

The book value return of the portfolio at time t is denoted by

$$ROA_t = \sum_{a \in \mathcal{A}_{t-1}} ROA_t^a + F_{t-1}C_{t-1}$$

where

$$ROA_t^a = cf_t^a + \Delta BV_t^a$$

is the book value return of an asset $a \in \mathcal{A}_t$ at time t and cf_t^a is the cash flow of an asset $a \in \mathcal{A}_t$ at time t which includes coupon, nominal, dividend or rental income.

Let L_t denote the book value of liabilities at time t, we assume that $L_t = BV_t$ at all time steps t and that it can be expressed as $L_t = LP_t + SF_t$. Thereby LP_t denotes the life assurance provision at time t which can be itemized as follows:

$$LP_t = V_t + DB_t + DB_t^{\leq 0}.$$

In this equation V_t describes the mathematical reserve at time t which only depends on the survival rates of the policyholders but not on future surplus declarations. The declared bonuses after valuation time t are denoted by DB_t and $DB_t^{\leq 0}$ defines the declared bonuses up to and including valuation time at time t, $DB_t^{\leq 0}$ also only depends on the survival rates of the policyholders and not on future surplus declarations. As we also model the projection of model points which have started before valuation time $t = 0$, this distinction between past and future is necessary. The total declared bonuses at time t are then defined as $TDB_t = DB_t^{\leq 0} + DB_t$.

The surplus fund at time t, SF_t, belongs to the collective of policyholders and consists of those profits that have not yet been declared to the policyholders. The fraction of declarations of SF_{t-1} to DB_t at time t is η_t, management may choose the value of η_t at each time step t.

The gross surplus at time t is denoted by gs_t, it can be expressed as

$$gs_t = ROA_t - \rho_t V_{t-1} + \gamma_t LP_{t-1}, \tag{2.1}$$

where ρ_t denotes the technical interest rate at time t and γ_t is the fraction of technical gains at time t. It is used to compute the cost of guarantees

$$COG = \mathbb{E}[B_t^{-1} gs_t^-].$$

If the gross surplus gs_t is positive at time t, it has to be shared between policyholder, shareholder and tax office, if it is negative, it is covered by the shareholder. Thereby the policyholder accounting flow, the shareholder cash flow and the tax cash flow at time t are denoted by ph_t^*, sh_t and tax_t and

$$gs_t = sh_t + ph_t^* + tax_t$$

applies. The policyholder share in gross surplus is denoted by gph and therefore we have

$$ph_t^* = gph \cdot gs_t^+.$$

Note that a fundamental principle in traditional life insurance is that ph_t^* is not necessarily declared to specific policyholder accounts to its full extent because profit sharing is not equal to profit declaration. The declaration fraction of ph_t^*, that the management may choose at each accounting step t, is ν_t and therefore $\nu_t \cdot ph_t^*$ is declared to the policyholder accounts at time t. The shareholder share in gross surplus and the tax paid on gross surplus are defined as gsh and $gtax$, we have

$$gsh + gph + gtax = 1$$

and

$$sh_t = gsh \cdot gs_t^+ - gs_t^-, \qquad tax_t = gtax \cdot gs_t^+.$$

The time value of the accounting flows ph_t^* is denoted by

$$PH^* = \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} ph_t^*],$$

the time value of tax_t by

$$TAX = \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} tax_t]$$

and the value of in-force business by

$$VIF = \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} sh_t].$$

The declaration to the policyholder accounts at time t is defined as

$$bd_t = \nu_t \cdot ph_t^* + \eta_t \cdot SF_{t-1} \tag{2.2}$$

and the amount of discretionary benefits paid out at time t is denoted by ph_t, this amount is provided by DB_t. As the declarations at valuation time $t = 0$ belong to $DB_t^{\leq 0}$ and the resulting cash flows are already guaranteed at $t = 0$, we have $ph_1 = 0$. Moreover, gbf_t are the guaranteed benefits at time t and sg_t^* is the surrender fee at time t. Finally, the future discretionary benefits are defined as

$$FDB = \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} ph_t].$$

2.2 A representation of the future discretionary benefits

In combination with a representation of PH^*, Gach and Hochgerner derived a closed formula for the FDB in [GH22]. This is outlined below.

To obtain the desired representation of PH^*, the no-leakage principle [GH19] is needed. This convention only uses no-arbitrage theory and generally accepted accounting principles which define the cash flows that characterize the quantities BE, VIF and TAX. It states that

$$MV_0 = BE + VIF + TAX + \mathbb{E}[B_T^{-1} MV_T]$$

and gives rise to the following representation of PH^*, for which some definitions and relations defined in the previous section 2.1 have been used.

$$\begin{aligned}
PH^* &:= \mathbb{E}[\sum B_t^{-1} ph_t^*] \\
&= gph \cdot \mathbb{E}[\sum B_t^{-1} gs_t^+] \\
&= gph \cdot \mathbb{E}[\sum B_t^{-1} gs_t] + gph \cdot \mathbb{E}[\sum B_t^{-1} gs_t^-] \\
&= gph \cdot \mathbb{E}\big[\sum B_t^{-1}(sh_t + ph_t^* + tax_t)\big] + gph \cdot COG \\
&= gph \cdot (VIF + PH^* + TAX + COG) \\
&= gph \cdot \big(MV_0 - \mathbb{E}[B_T^{-1} MV_T] - TAX - BE + PH^* + TAX + COG\big) \\
&= gph \cdot \big(MV_0 - \mathbb{E}[B_T^{-1} MV_T] - GB - FDB + COG\big) + gph \cdot PH^* \\
\Leftrightarrow & \\
PH^* &= \frac{gph}{1-gph}\big(MV_0 - \mathbb{E}[B_T^{-1} MV_T] - GB - FDB + COG\big) \qquad (2.3)
\end{aligned}$$

Using the integration by parts formula

$$\Delta(f_t g_t) = (\Delta f_t) g_{t-1} + f_t \Delta g_t,$$

the relations

$$DB_t = DB_{t-1} + \eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* - ph_t - sg_t^*$$

and

$$SF_t = SF_{t-1} + (1-\nu_t) \cdot ph_t^* - \eta_t \cdot SF_{t-1}$$

and the facts that

$$\begin{aligned}
\Delta(DB_t + SF_t) &= DB_t + SF_t - DB_{t-1} - SF_{t-1} \\
&= \eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* - ph_t - sg_t^* + (1-\nu_t) \cdot ph_t^* - \eta_t \cdot SF_{t-1} \\
&= ph_t^* - ph_t - sg_t^*
\end{aligned}$$

and

$$\begin{aligned}
\Delta B_t^{-1} &= B_t^{-1} - B_{t-1}^{-1} \\
&= \prod_{j=0}^{t-1}(1+F_j)^{-1} - \prod_{j=0}^{t-2}(1+F_j)^{-1} \\
&= \prod_{j=0}^{t-2}(1+F_j)^{-1} \cdot \big((1+F_{t-1})^{-1} - 1\big) \\
&= B_{t-1}^{-1} \cdot \big(\frac{1}{1+F_{t-1}} - 1\big) \\
&= -F_{t-1} \cdot B_t^{-1}
\end{aligned}$$

leads to another helpful relation:

$$\begin{aligned}
\sum_{t=1}^{T} B_t^{-1}(ph_t^* - ph_t - sg_t^*) &= \sum_{t=1}^{T} B_t^{-1}\Delta(DB_t + SF_t) \\
&= \sum_{t=1}^{T} \Delta\big(B_t^{-1}(DB_t + SF_t)\big) - \sum_{t=1}^{T} \Delta B_t^{-1}(DB_{t-1} + SF_{t-1}) \\
&= B_T^{-1}(DB_T + SF_T) - SF_0 + \sum_{t=1}^{T} F_{t-1}B_t^{-1}(DB_{t-1} + SF_{t-1}).
\end{aligned} \tag{2.4}$$

Taking the expectation of equation (2.4) and using the representation of PH^* (2.3) then leads to the following result.

$$PH^* - FDB - \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} sg_t^*]$$

$$= \mathbb{E}\big[B_T^{-1}(DB_T + SF_T)\big] - SF_0 + \mathbb{E}\big[\sum_{t=1}^{T} F_{t-1}B_t^{-1}(DB_{t-1} + SF_{t-1})\big]$$

$$\Leftrightarrow$$

$$PH^* - FDB$$

$$= \mathbb{E}\big[B_T^{-1}(DB_T + SF_T)\big] - SF_0 + \mathbb{E}\big[\sum_{t=1}^{T} F_{t-1}B_t^{-1}(DB_{t-1} + SF_{t-1})\big] + \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} sg_t^*]$$

$$\Leftrightarrow$$

$$\frac{gph}{1-gph}\big(MV_0 - \mathbb{E}[B_T^{-1}MV_T] - GB - FDB + COG\big) - \frac{1-gph}{1-gph} \cdot FDB = RHS$$

$$\Leftrightarrow$$

$$\frac{gph}{1-gph}\big(MV_0 - \mathbb{E}[B_T^{-1}MV_T] - GB + COG\big) - \frac{1}{1-gph} \cdot FDB = RHS \tag{2.5}$$

Rearranging equation (2.5) finally induces a closed formula for the FDB:

$$\begin{aligned}
FDB &:= \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} ph_t] \\
&= (1-gph) \cdot \Big(\frac{gph}{1-gph} \cdot \big(MV_0 - \mathbb{E}[B_T^{-1}MV_T] - GB + COG\big) + SF_0 \\
&\quad - \mathbb{E}\big[B_T^{-1}(DB_T + SF_T)\big] - \mathbb{E}\big[\sum_{t=1}^{T} F_{t-1}B_t^{-1}(DB_{t-1} + SF_{t-1})\big] - \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} sg_t^*]\Big) \\
&= gph \cdot (LP_0 + SF_0 + UG_0 - \mathbb{E}[B_T^{-1}MV_T] - GB + COG) + (1-gph) \cdot SF_0 \\
&\quad - (1-gph) \cdot \mathbb{E}\big[B_T^{-1}(DB_T + SF_T)\big] - (1-gph) \cdot \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} sg_t^*] \\
&\quad - (1-gph) \cdot \mathbb{E}\big[\sum_{t=1}^{T} F_{t-1}B_t^{-1}(DB_{t-1} + SF_{t-1})\big] \\
&= gph \cdot (LP_0 + UG_0 - GB) - gph \cdot \mathbb{E}[B_T^{-1}MV_T] + gph \cdot COG + SF_0 \\
&\quad - (1-gph) \cdot \mathbb{E}\big[B_T^{-1}(DB_T + SF_T)\big] - III - II
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
II &:= (1-gph) \cdot \mathbb{E}[\sum_{t=2}^{T} B_t^{-1} sg_t^*] \\
III &:= (1-gph) \cdot \mathbb{E}\big[\sum_{t=1}^{T} F_{t-1}B_t^{-1}(DB_{t-1} + SF_{t-1})\big].
\end{aligned}$$

The component

$$gph \cdot \mathbb{E}[B_T^{-1}MV_T] + (1-gph) \cdot \mathbb{E}[B_T^{-1}(DB_T + SF_T)]$$

of (2.6) is considered separately:

$$
\begin{aligned}
&gph \cdot \mathbb{E}[B_T^{-1} MV_T] + (1-gph) \cdot \mathbb{E}\big[B_T^{-1}(DB_T + SF_T)\big] \\
&= gph \cdot \mathbb{E}\big[B_T^{-1}(UG_T + LP_T + SF_T)\big] + (1-gph) \cdot \mathbb{E}\big[B_T^{-1}(DB_T + SF_T)\big] \\
&= \mathbb{E}[B_T^{-1} SF_T] + gph \cdot \mathbb{E}[B_T^{-1}(UG_T + V_T + DB_T^{\leq 0} + DB_T)] + (1-gph) \cdot \mathbb{E}[B_T^{-1} DB_T] \\
&= \mathbb{E}\Big[B_T^{-1}\big(DB_T + SF_T + gph \cdot (UG_T + V_T + DB_T^{\leq 0})\big)\Big] \\
&=: I.
\end{aligned}
$$

Altogether, the following representation of the FDB is obtained [GH22]:

$$
FDB = SF_0 + gph \cdot (LP_0 + UG_0 - GB) + gph \cdot COG - I - II - III \tag{2.7}
$$

with

$$
I := \mathbb{E}\Big[B_T^{-1}\big(DB_T + SF_T + gph \cdot (UG_T + V_T + DB_T^{\leq 0})\big)\Big] \tag{2.8}
$$

$$
II := (1-gph) \cdot \mathbb{E}[\sum_{t=2}^{T} B_t^{-1} sg_t^*] \tag{2.9}
$$

$$
III := (1-gph) \cdot \mathbb{E}\big[\sum_{t=1}^{T} F_{t-1} B_t^{-1}(DB_{t-1} + SF_{t-1})\big]. \tag{2.10}
$$

In this representation, the first part $SF_0 + gph \cdot (LP_0 + UG_0 - GB)$ is determined by balance sheet items and classical actuarial computations (GB), the second part $gph \cdot COG - I - II - III$ has to be estimated.

3 The cash-flow model

The aim of this chapter is to briefly introduce the Asset Liability Management model we implemented in R. The model is capable of simulating statutory balance sheets and it was implemented in order to calculate the GB and the FDB numerically. While the GB can be calculated independently from the asset module and the management rules, the calculation of the FDB is more difficult since it is in interaction with the asset module (Section 3.3), the liability module (Section 3.4) and the management rules (Section 3.5). Life insurance with profit participation in general is described in Section 3.1 and Section 3.2 introduces the Economic Scenario Generator (ESG).

The content of this chapter has in slightly adapted form been taken from the paper *Numerical validation of analytic FDB estimation* [GHKS]. This paper, which is currently still in progress, is written by Gach, Hochgerner, Schachinger and myself.

3.1 Life insurance with profit participation

Our ALM model is concerned with classical life insurance contracts. Each contract (or, more generally, model point) x has a specified maturity payment M^x which is guaranteed and may participate in the company's earnings. More precisely, each contract x

- has a specific maturity time T^x;
- has associated best estimate mortality and surrender tables;
- has a constant technical interest rate ρ^x;
- pays a constant premium pr^x up to $T^x - 1$;
- has an associated mathematical reserve V_t^x calculated according to classical actuarial assumptions involving e.g. ρ^x;
- has an associated bonus account of total declared benefits TDB_t^x depending on the company's profits;
- receives either a maturity benefit $M^x + TDB^x_{T^x-1}$ at T_x; or a death benefit $M^x + TDB^x_{t-1}$ at $t < T_x$; or, in case of surrender at $t < T_x$, a surrender benefit $\kappa_t(V^x_{t-1} + TDB^x_{t-1})$ where κ_t is a penalty term which is linear in t such that $\kappa_0 = 0.9$ and $\kappa_{T^x} = 1$.

3.2 Economic Scenario Generator (ESG)

Our ALM model is able to generate market and book values for four asset classes: cash, bonds (without default), equity and property.

Cash is modelled as a bond with maturity of one year. This corresponds to the roll-over definition of the implied money market account.

Market values of property and equity are assumed to follow a geometric Brownian motion where the drift depends on the prevailing one-year forward interest rate F_t. Further, the drift may depend on a fixed rate d, modelling rental income or dividend yield. Finally, each property or equity asset may have its own fixed volatility σ. Hence the market value MV_t^a of a given property or equity asset is assumed to be projected according to

$$MV_t^a = MV_0^a \cdot \exp\big((F_t - d - \frac{\sigma^2}{2})t + \sigma W_t\big) \tag{3.1}$$

where MV_0^a is the initial market value and W_t^a is the assets Brownian motion whose correlation with other assets' stochastic drivers remains to be specified. We emphasize that $d \geq 0$ and $\sigma > 0$ are assumed to be fixed numbers while F_t is white in time, i.e. follows its own stochastic process.

Specifying this stochasticity is the main task of choosing an appropriate ESG. The mean-field Libor market model (MF-LMM) of [DHOT22] is an interest rate model that has been developed precisely for the purpose of valuating long term guarantees as they are present in traditional life insurance contracts. We have thus chosen this model to generate the stochastic scenarios for our ALM model. In the following we provide a short introduction to this MF-LMM.

Let us fix a tenor structure $0 = t_0 < t_1 = \delta < \ldots < t_N = N\delta$. In the ALM model we consider yearly time steps such that $\delta = 1$ and N corresponds to the projection horizon. For $i = 0, \ldots, N-1$ the i-th forward (Libor) rate valid on $[t_i, t_{i+1}]$ at time $t \leq t_i$ is

$$L_t^i := \frac{1}{\delta} \frac{P(t, t_i) - P(t, t_{i+1})}{P(t, t_{i+1})}$$

where $P(t, t_i)$ is the value of a zero coupon bond at time t paying 1€ at time t_i.

The classical Libor market model is now specified as follows. For each index $i = 0, \ldots, N-1$, there should be an $\mathbb{R}^d$-valued diffusion coefficient $\sigma_i = \sigma_i(t)$, a forward measure $\mathbb{Q}^i$ and a corresponding d-dimensional Brownian motion W^i ($d \geq 1$), such that the dynamics of L^i under $\mathbb{Q}^i$ are given by

$$dL_t^i = L_t^i \sigma_i^\top \, dW_t^i, \qquad t \leq t_i.$$

The mean-field extension consists now of a generalized dependency of σ_i. Namely, we assume that $\sigma_i : [0, t_i] \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$:

$$\sigma_i = \sigma_i(t, \mu_t^i) = \lambda^i(t, \mathbb{V}_{\mathbb{Q}^i}[L_t^i])$$

where

- $\mathcal{P}_2(\mathbb{R}) = \{\text{probability measures } \mu : \int_{\mathbb{R}} x^2 \, d\mu(x) < \infty\}$
- $\mathbb{Q}$ is the spot measure
- $\mu_t^i = \mathrm{Law}(L_t^i) = (L_t^i)_\sharp \mathbb{Q}^i$ is the law of L_t^i under $\mathbb{Q}^i$
- $\mathbb{V}_{\mathbb{Q}^i}[L_t^i] = \mathbb{V}_{\mu^i}$ denotes the variance of L_t^i under $\mathcal{Q}_i$ which can be expressed as

$$\mathbb{V}_{\mathbb{Q}^i}[L_t^i] = \mathbb{E}_{\mathbb{Q}}\Big[\big(L_t^i - \mathbb{E}_{\mathbb{Q}}[L_t^i Y_t^i]\big)^2 Y_t^i\Big] =: \Psi_t^i$$

  where

$$Y_t^i := \frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{Q}}\Big|_{\mathcal{F}_t} = \mathbb{E}_{\mathbb{Q}}\big[\frac{\mathrm{d}\mathbb{Q}^i}{\mathrm{d}\mathbb{Q}}\big|\mathcal{F}_t\big]$$

- $\lambda^i : [0, t_i] \times \mathbb{R}^+ \to \mathbb{R}^d$ is a deterministic function.

Then the dynamics of the mean-field Libor Market Model under the spot measure $\mathbb{Q}$ are defined as follows:

$$\mathrm{d}L_t^i = L_t^i\Big(\sum_{k=\eta(t)}^{i} \frac{\delta_k L_t^k}{\delta_k L_t^k + 1} \lambda^k(t, \Psi_t^k)^\top \lambda^i(t, \Psi_t^i)\mathrm{d}t + \lambda^i(t, \Psi_t^i)^\top \mathrm{d}W_t^*\Big), \tag{3.2}$$

$$\mathrm{d}Y_t^i = -Y_t^i \sum_{k=\eta(t)}^{i} \frac{\delta_k L_t^k}{\delta_k L_t^k + 1} \lambda^k(t, \Psi_t^k)^\top \mathrm{d}W_t^*, \tag{3.3}$$

with $Y_0^i = 1$. Equations (3.2) and (3.3) are a mean field system of SDEs since the Ψ_t^i depend on the joint law $\widetilde{\mu_t^i}$ of L_t^i and Y_t^i under $\mathbb{Q}$. Those Ψ_t^i can be denoted by

$$\Psi_t^i(\widetilde{\mu_t^i}) = \int_{\mathbb{R}^2} \Big(x - \int_{\mathbb{R}^2} x'y' \, \mathrm{d}\widetilde{\mu_t^i}(x', y')\Big)^2 y \, \mathrm{d}\widetilde{\mu_t^i}(x, y) = \mathbb{V}_{\mathbb{Q}^i}[L_t^i],$$

where Ψ_t^i is a map $\Psi_t^i : \mathcal{P}_2(\mathbb{R}^2) \to \mathbb{R}$.

Projection along tenor dates

If $t = t_j$ is a tenor date, then the conditional expectation can be expressed as

$$Y^i_{t_j} = B^*(t_j)^{-1} \frac{D(j,i)}{D(0,i)},$$

see [Fil09, Sec. 7.1], where

$$B^*(t_j) = (1 + \delta_{j-1} L^{j-1}_{t_{j-1}}) B^*(t_{j-1}), \quad B^*(t_0) = 1$$

is the implied money market account (i.e., the numeraire) and

$$D(j,i) = \prod_{l=j}^{i-1} (1 + \delta_l L^l_{t_j})^{-1}$$

is the discount factor from i to $j < i$. With

$$\Psi^i_j := \mathbb{E}_{\mathbb{Q}}\Big[\Big(L^i_{t_j} - \mathbb{E}_{\mathbb{Q}}\big[L^i_{t_j} B^*(t_j)^{-1} \frac{D(j,i)}{D(0,i)}\big]\Big)^2 B^*(t_j)^{-1} \frac{D(j,i)}{D(0,i)}\Big]$$

it follows that the evolution along the tenor dates of the mean-field Libor market model with respect to the spot measure is given by

$$dL^i_{t_j} = L^i_{t_j} \Big(\sum_{k=j+1}^{i} \frac{\delta_k L^k_{t_j}}{\delta_k L^k_{t_j} + 1} \lambda^k(t_j, \Psi^k_j)^\top \lambda^i(t_j, \Psi^i_j) \mathrm{d}t + \lambda^i(t_j, \Psi^i_j)^\top \mathrm{d}W^*_t \Big)$$

since $\eta(t_j) = j + 1$.

Thus it remains to specify the functional form of $\lambda^i : [0, t_i] \times \mathbb{R}^+ \to \mathbb{R}^d$. In this regard we choose four different forms:

1. Libor market model: $\lambda^i = \lambda^i(t)$. This corresponds to the classical Libor market model without mean-field interaction. For technical reasons (stemming from notation in the R code) this choice will be referred to as *VolSwi2*.

2. Mean-field taming: we choose a variance threshold $\tilde{\sigma}$ and define

$$\lambda^i(t, \Psi^i_t) = \sigma^{(1)}_i(t) \exp\big(\frac{-\max\{\Psi^i_t - \tilde{\sigma}, 0\}}{\tilde{\sigma}}\big),$$

where $\sigma^{(1)}_i$ is the same time dependent structure as in the classical Libor market model. This choice will be referred to as *VolSwi25*.

3. Decorrelation beyond threshold: it is assumed that the mean-field interaction does not affect the scalar instantaneous volatility structure but rather leads to an exponential decay in the instantaneous correlation between different forward rates. This choice will be referred to as *VolSwi6*.

4. Anti-correlation beyond threshold: it is assumed that the mean-field interaction does not affect the scalar instantaneous volatility structure but rather leads to a negative instantaneous correlation between different forward rates. The effect of this modelling ansatz is that high rates tend to decrease while low rates tend to increase. This choice will be referred to as *VolSwi4*.

Details and a numerical study concerning these choices are provided in [DHOT22], where it is also shown numerically that, in the long run, the classical Libor market model can lead to exploding rates while the mean-field controlled models all significantly reduce blow-up probability.

3.3 Asset module

The purpose of the asset module is to model book and market values of assets under management, and to provide re- and deinvestment strategies. The latter are part of the management rules and described in Section 3.5.

The ALM model provides four different asset classes: cash, bonds, equity, and property. Bonds are assumed to be default-free, thus there is no distinction between corporate and government bonds.

Cash is equivalent to a bond with a maturity of 1, and correspondingly the interest earned by cash is the prevailing one-year forward rate. Further, book and market values, BV^c and MV^c, for cash coincide. That is, $BV_t^c = MV_t^c = (1 + F_{t-1})MV_{t-1}^c$.

A bond b consists of a maturity T^b, a nominal payment N^b, a coupon factor K^b such that the coupon payment is $K^b N^b$, a market value MV^b and a book value BV^b. At each time step $t < T^b$ the market value is determined by the prevailing yield curve, i.e.

$$MV_t^b = \sum_{s=t+1}^{T^b} P(t,s)K^b N^b + P(t,T_b)N^b$$

where $P(t,s)$ is the value of the zero-coupon bond from the mean-field Libor market ESG. The book value is determined by the strict lower-of-cost-or-market (LCM) principle, that is

$$BV_t^b = \min\left(BV_{t-1}^b, MV_t^b\right).$$

When a bond b with nominal N^b and maturity T^b is bought at time t the initial book value is $BV_t^b = N^b$, and the coupon factor K^b follows from the requirement $MV_t^b = N^b$ and the prevailing yield curve at t up to T^b. I.e., bonds are bought *at par*.

An equity position e consists of a market value, a book value, a (constant) volatility factor and a (constant) dividend factor. The latter is relevant for the company's surplus which is calculated according to local generally accepted accounting principles (GAAP), since the dividend affects the book value return. The market value development is given by the geometric Brownian motion (3.1). The book value is given by the strict LCM principle, that is $BV_t^e = \min(BV_{t-1}^e, MV_t^e)$.

Properties are modelled similar to equities with two distinctions: The dividend factor is interpreted not as a dividend but as rental income. Second, properties p have a depreciation time T^p such that $BV_{T^p}^p = 0$. This depreciation time (which is usually not more than 30 years) has to be provided as part of the initial data. The depreciation is linear, i.e. according to $\big(1 - (s - t + 1)(T^p - t + 1)\big)BV_{t-1}^p$ for $t - 1 \leq s \leq T^p$ and the strict LCM applies. Hence the book value development is given by

$$BV_t^p = \min\big((1 - \frac{1}{T^p - t + 1})BV_{t-1}^p, MV_t^p\big)$$

where MV_t^p follows (3.1). Consequently properties often carry comparatively large amounts of unrealized gains $UG_t^p = MV_t^p - BV_t^p$.

3.4 Liability module

All definitions and relations which are part of the liability module are already listed in Section 2.1 as they are also needed in the remaining part of this thesis. The ALM model assumes that the following quantities are given as deterministic functions of time:

- premium payments: pr_t
- guaranteed benefits, including surrender payments, due to V_t and $DB_t^{\leq 0}$: gbf_t
- cost payments: co_t
- mathematical reserve: V_t
- previously allocated bonuses: $DB_t^{\leq 0}$
- technical interest rate: ρ_t

In fact, all these quantities are given on the level of model points such that the quoted values are the aggregate sums. Moreover, these quantities have been constructed according to classical actuarial assumptions such that benefits, premiums and technical reserves are consistent.

The liability module gives rise to two management rules concerning the quantities ν_t and η_t in (2.2). These rules are specified in Section 3.5.

3.5 Management rules and profit declaration

Strategic asset allocation

Rule 1. *New bonds are bought at par with a time to maturity of* 10 *years.*

Rule 2. *The strategic asset allocation is kept approximately constant: the market value ratios cash over total market value, bonds over total market value, equities over total market value and properties over total market value must remain constant up to a deviation of at most* $\pm 10\%$*. When an asset class breaches this bound the portfolio is rebalanced such that the original market value ratios at* $t = 0$ *are restored. The rebalancing is such that placement of assets with minimal unrealized gains is prioritized (to avoid unintended book value return).*

Negative surplus

Rule 3. *When the gross surplus is negative, unrealized gains are realized until the gross surplus equals* 0 *or no more positive unrealized gains exist. The selling order is bonds before equity before property, and within those classes positions with large amounts of positive unrealized gains are prioritized.*

This rule is in place in order to avoid shareholder capital injections as much as possible.

Rule 4. *After three consecutive years of negative gross surplus and when, at the same time, no more positive unrealized gains are available, the surplus fund is depleted in order to (try to) achieve a non-negative gross surplus.*

This rule constitutes the only instance where the surplus fund may increase the gross surplus. Concretely, if gs_t is smaller than 0 and the requirements of Rule 4 hold, then the gross surplus may be augmented as $gs_t^{(1)} = gs_t + \min(-gs_t, SF_{t-1})$; at the same time the surplus fund is reduced according to $SF_t = SF_{t-1} - \min(-gs_t, SF_{t-1})$.

Positive surplus

Fix $\vartheta = SF_0/LP_0$. Let τ_t denote the declared total participation rate at t. Notice that a participation rate of τ_t means that the amount of bonus declarations is given by

$$bd_t = \sum_{x \in \mathcal{X}_t} \left(\tau_t - \rho^x\right)_+ V_{t-1}^x$$

where $\mathcal{X}_t$ is the set of model points x which are active at t, and ρ^x and V_{t-1}^x are the technical interest rate and previous mathematical reserve. The following rule applies if $gs_t > 0$.

Rule 5. *Let* $\upsilon = 5/1000$, $\tau_t^* = (\tau_{t-1} + L_t^{10})/2$ *and* $ta_t = \sum_{x \in \mathcal{X}_t} (\tau_t^* - \rho^x)_+ V_{t-1}^x$. *We distinguish two cases:*

1. *If* $ph_t^* \geq ta_t$ *we define*

$$\nu_t = \min\left(1, \sum_{x \in \mathcal{X}_t} \frac{(\tau_t^* + \upsilon - \rho^x)_+ V_{t-1}^x}{ph_t^*}\right)$$

$$\eta_t = \max\left(0, \frac{SF_{t-1} - \vartheta(V_t + DB_t^{\leq 0} + \nu_t\, ph_t^*)}{(1+\vartheta) SF_{t-1}}\right)$$

unless $SF_{t-1} = 0$ *in which case we set* $\eta_t = 0$.

2. *If* $ph_t^* < ta_t$ *we define* $\nu_t = 1$ *and*

$$\eta_t^{(1)} = \frac{\left(\sum_{x \in \mathcal{X}_t} (\tau_t^* - \upsilon - \rho^x)_+ V_{t-1}^x - ph_t^*\right)_+}{SF_{t-1}}$$

$$\eta_t^{(2)} = \min\left(\frac{1}{2}, \eta_t^{(1)}\right)$$

$$\eta_t^{(3)} = \frac{SF_{t-1} - \vartheta(V_t + DB_t^{\leq 0} + ph_t^*)}{(1+\vartheta) SF_{t-1}}$$

$$\eta_t = \max\left(\eta_t^{(2)}, \eta_t^{(3)}\right)$$

unless $SF_{t-1} = 0$ *in which case we set* $\eta_t = 0$.

This rule aims at avoiding large jumps in the profit participation rate. The choice for υ means that the participation rate deviates by at most 5 basis points from the specified target, unless the ϑ-term is positive. Notice that the target ta_t is defined in terms of the previous participation rate τ_{t-1} and the prevailing 10Y-forward rate L_t^{10}. Thus the target is a combination of previous profit participation and general market expectation. However, to reach this target not more than half of the existing surplus fund is provided.

Moreover, this rule ensures that, in cases of positive gross surplus, $SF_t \leq \vartheta LP_t$ so that the surplus fund does not become unrealistically high.

Order of management rules

Rule 6. *The rules are applied in the order in which they are stated.*

As a consequence of this rule it may happen, within one accounting step, that Rule 3 is carried out after the rebalancing step in Rule 2. Thus capital gains may be realized by, e.g., selling an equity position so that the distribution of assets may no longer be in line with the strategic asset allocation. In such a case a misalignment of asset positions is carried forward along one accounting year and then rebalanced at the end of this year.

This rule is chosen nevertheless in this form since, firstly, short term misalignment is acceptable and, secondly, it is very difficult to implement the alternative: if rebalancing were to occur at the end of each accounting step, the gross surplus would change and the profit participation would have to be calculated once over, thus potentially leading to a multiple loop.

4 Numerical validation of assumptions

Calculating the model dependent quantities I, II, III and COG, which were derived in Section 2.2, is just as difficult as calculating the FDB itself. For the purpose of estimating the terms I, II and III and to derive a lower and an upper bound for the FDB, a few assumptions are needed. Gach and Hochgerner defined ten assumptions in their paper [GH22] and the goal of this chapter is to verify those.

Since all of the quantities which are part of the assumptions are also part of our ALM model in R, it is possible to validate the assumptions with it. In this chapter, the assumptions are tested with simulated data and the projection horizon is set to $T = 60$ years. In general, 1000 interest rate scenarios simulated with *VolSwi4* are considered. Furthermore, the effect of a 2% increase in the interest curve on the assumptions is tested with 1000 interest rate scenarios simulated with *VolSwi4*.

4.1 Assumption 1

> *The projection horizon T corresponds to the run-off time of the liability portfolio such that $SF_T = LP_T = UG_T = 0$.* [GH22]

Solvency II requires a run-off approach for best estimate calculation. This means that the valuation of liabilities is restricted to the existing business at valuation time $t = 0$ and it is assumed that the company does not write new business in the future. All future cash flows associated with existing contracts are considered for the calculation of the best estimate, a contract is no longer recognised as an existing contract if its obligation is discharged, cancelled or expired. Therefore the existing portfolio is steadily on the decrease and expiring at the projection horizon T. This can also be recognised in figures 4.1 and 4.2. The values of SF_T, LP_T and UG_T might probably not exactly be 0 since a few contracts might not have expired at T, but their values are very small compared to the initial market value MV_0 (less than 0.5% of the initial market value MV_0), and therefore the assumption holds.

The corresponding Solvency II article in the commission delegated regulation of the European Union [Eura] states:

> *The cash flow projection used in the calculation of the best estimate shall include all of the following cash flows, to the extent that these cash flows relate to existing insurance and reinsurance contracts:*

(a) *benefit payments to policy holders and beneficiaries;*

(b) *payments that the insurance or reinsurance undertaking will incur in providing contractual benefits that are paid in kind;*

(c) *payments of expenses as referred to in point (1) of Article 78 of Directive 2009/138/EC;*

(d) *premium payments and any additional cash flows that result from those premiums;*

(e) *payments between the insurance or reinsurance undertaking and intermediaries related to insurance or reinsurance obligations;*

(f) *payments between the insurance or reinsurance undertaking and investment firms in relation to contracts with index-linked and unit-linked benefits;*

(g) *payments for salvage and subrogation to the extent that they do not qualify as separate assets or liabilities in accordance with international accounting standards, as endorsed by the Commission in accordance with Regulation (EC) No 1606/2002;*

(h) *taxation payments which are, or are expected to be, charged to policy holders or are required to settle the insurance or reinsurance obligations.*

4.2 Assumption 2

The expected life assurance provisions $\mathbb{E}[LP_t]$ *decrease geometrically: there is a fixed* $1 \leq h < T$ *such that* $\mathbb{E}[LP_t] = l_t^h \cdot LP_0$ *where* $l_t^h := 2^{-t/h}$ *for* $t < T$ *and* $l_T^h := 0$. [GH22]

Since the portfolio is in run-off, there is a time h where $\mathbb{E}[LP_h] = LP_0/2$. If the business model of the considered company is stable over time, we have $\mathbb{E}[LP_{h+h}] = LP_0/4$. Continuing like this leads to the assumption that the run-off of the liability book is geometric. This approximation works really well if the company under consideration has a longer history and has not taken up business very recently.

The geometrical decrease of LP_t and $V_t + DB_t^{\leq 0}$ can be recognised in the corresponding plots 4.1 and 4.2, which were generated with the ALM model in R. Figure 4.1 shows the geometrical decrease with respect to the risk-free curve as of 31.01.2022 of annual zero-coupon spot rates from EIOPA, figure 4.2 shows the analogous but with a 2% upward shift in the spot rates. The blue curve shows in each plot the development of the sum of the mathematical reserve V_t and the declared bonuses up to and including valuation time $DB_t^{\leq 0}$, these values are given and deterministic. The red curve respectively shows the decline of the total reserve LP_t, which is part of the model in R and stochastic.

Since the life assurance provision LP_t at time t is defined as $LP_t = V_t + DB_t^{\leq 0} + DB_t$ and $DB_0 = 0$, the blue and the red curves have the same starting points in both figures. In the following time steps the red curves are above the blue ones because the difference DB_t is greater than or equal to zero at all time steps t. All reserves are paid out until the projection horizon T is reached and therefore the blue and the red curves meet in the end.

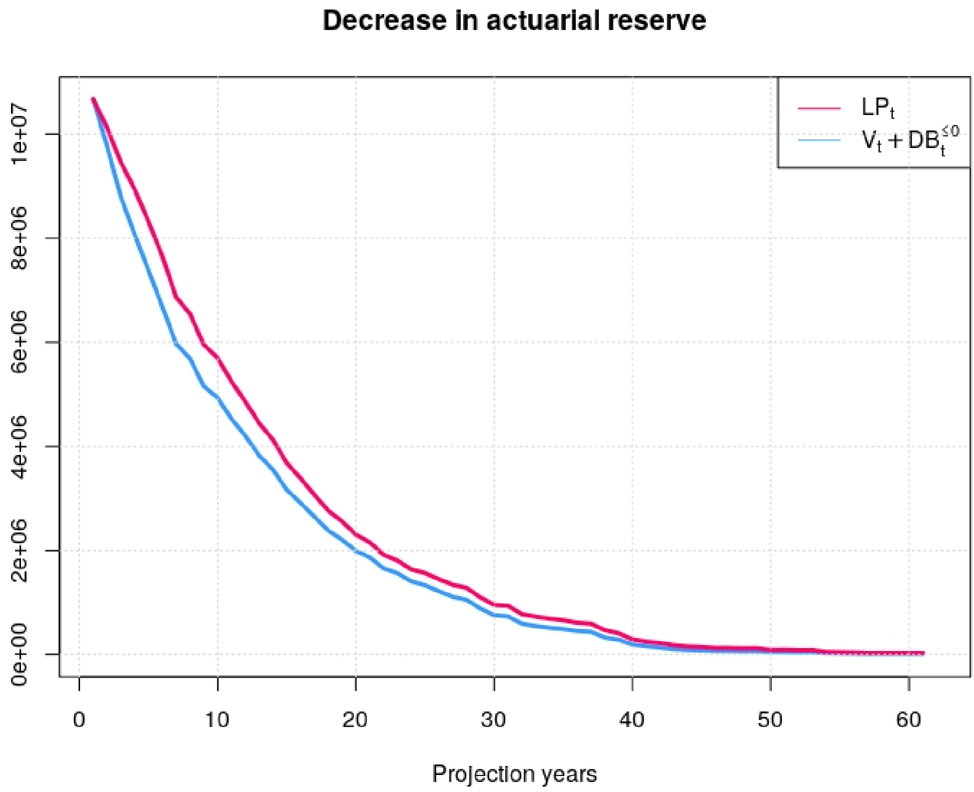

Figure 4.1: Geometrical decrease of LP_t and $V_t + DB_t^{\leq 0}$ with respect to the risk-free curve as of 31.01.2022 of annual zero-coupon spot rates from EIOPA.

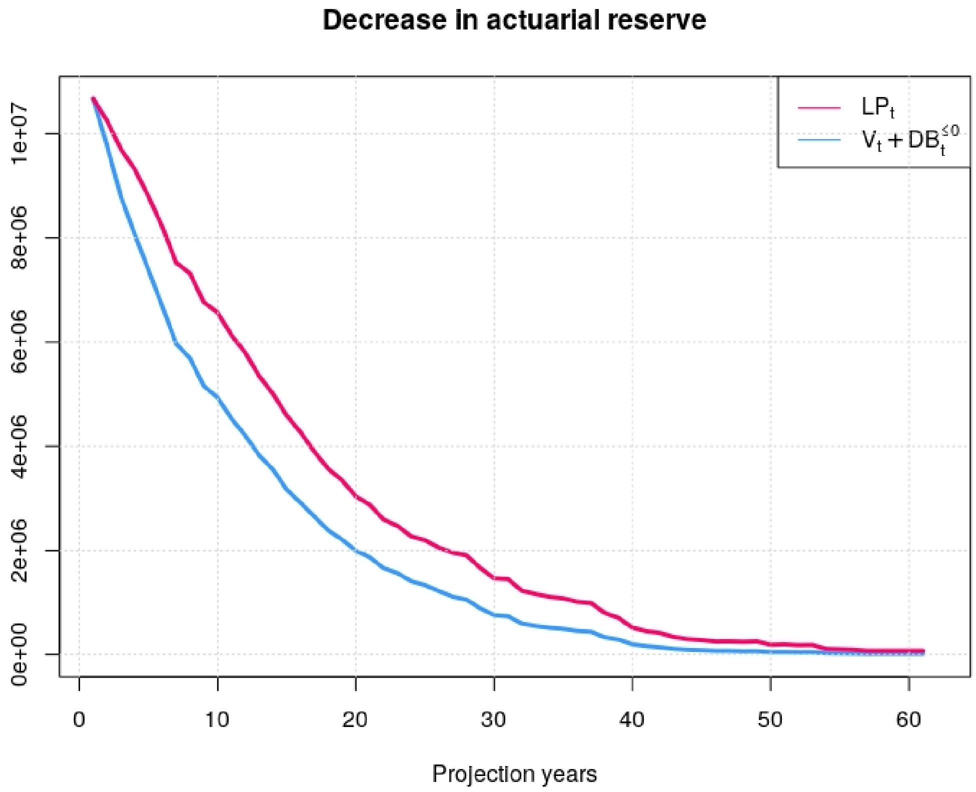

Figure 4.2: Geometrical decrease of LP_t and $V_t + DB_t^{\leq 0}$ with respect to the risk-free curve as of 31.01.2022 of EIOPA's annual zero-coupon spot rates increased by 2%.

As an explicit value of h is needed to check some of the following assumptions, our ALM model in R is used to compute a value h_t for every time step t and to decide whether a fixed h exists or not. The values of LP_t are known in the model and therefore it is possible to calculate $\mathbb{E}[LP_t]$ and to rearrange the equation in Assumption 2 to compute h_t for every time step t:

$$h_t = \frac{-t}{log_2(\frac{\mathbb{E}[LP_t]}{LP_0})}.$$

Those calculated h_t can be considered in the left table of table 4.1, they start at $t = 1$ because h_0 is arbitrary since $2^{-0/h} = 1$ for all h. Some statistical key figures of h_t can be recognized in the left table of table 4.2. They show that the assumption of the existence of a fixed h is appropriate, especially because the variance of the computed h_t is very small.

In the following a constant h is fixed as the mean

$$h = \text{mean}(h_t) = 8.66,$$

this will be used later for further computations.

t	h_t	t	h_t
1	13.26	31	8.40
2	11.39	32	8.46
3	11.67	33	8.56
4	11.07	34	8.69
5	10.35	35	8.73
6	9.38	36	8.89
7	9.91	37	8.48
8	9.47	38	8.35
9	9.91	39	7.82
10	9.75	40	7.74
11	9.73	41	7.70
12	9.53	42	7.47
13	9.55	43	7.41
14	9.19	44	7.45
15	9.16	45	7.42
16	9.00	46	7.58
17	8.83	47	7.66
18	8.84	48	7.88
19	8.70	49	7.54
20	8.81	50	7.78
21	8.61	51	7.78
22	8.73	52	7.97
23	8.63	53	7.34
24	8.79	54	7.36
25	8.80	55	7.35
26	8.82	56	7.16
27	8.97	57	7.31
28	8.69	58	7.44
29	8.49	59	7.56
30	8.75	60	7.72

t	$\widetilde{h_t}$	t	$\widetilde{h_t}$
1	17.18	31	10.13
2	14.48	32	10.20
3	15.21	33	10.35
4	14.16	34	10.51
5	13.09	35	10.55
6	11.81	36	10.76
7	12.80	37	10.20
8	12.09	38	10.01
9	12.79	39	9.29
10	12.54	40	9.29
11	12.49	41	9.26
12	12.05	42	8.92
13	12.04	43	8.83
14	11.44	44	8.89
15	11.32	45	8.84
16	11.01	46	9.05
17	10.77	47	9.19
18	10.77	48	9.48
19	10.50	49	9.02
20	10.64	50	9.45
21	10.34	51	9.42
22	10.49	52	9.65
23	10.40	53	8.76
24	10.62	54	8.78
25	10.66	55	8.78
26	10.67	56	8.53
27	10.89	57	8.74
28	10.48	58	8.90
29	10.20	59	9.07
30	10.65	60	9.35

Table 4.1: Values of h_t and $\widetilde{h_t}$, h_t calculated with respect to the base case, $\widetilde{h_t}$ computed with respect to the interest curve increased by 2%, both for each time step t.

$\text{mean}(h_t) = 8.66$	$\text{mean}(\widetilde{h}_t) = 10.61$
$\min(h_t) = 7.16$	$\min(\widetilde{h}_t) = 8.53$
$\max(h_t) = 13.26$	$\max(\widetilde{h}_t) = 17.18$
$\text{var}(h_t) = 1.41$	$\text{var}(\widetilde{h}_t) = 3.05$
$\text{CV}(h_t) = 0.1370$	$\text{CV}(\widetilde{h}_t) = 0.16$

Table 4.2: Statistical key figures of the h_t and $\widetilde{h}_t$ noted in table 4.1.

Subsequently, the same calculations are performed with respect to the by 2% increased interest curve, the results for these $\widetilde{h}_t$ can be considered in the right table of table 4.1. The statistical key figures of $\widetilde{h}_t$ in the right table of table 4.2 show that the assumption is again fulfilled, especially because the variance of $\widetilde{h}_t$ is small.

A constant $\widetilde{h}$ is then fixed as the mean

$$\widetilde{h} = \text{mean}(\widetilde{h}_t) = 10.61.$$

4.3 Assumption 3

In expectation the total declared bonuses are a fixed fraction of the life assurance provisions:

$$\mathbb{E}[DB_t^{\leq 0} + DB_t] = \sigma \cdot \mathbb{E}[LP_t] \quad \forall\ 0 \leq t \leq T \ \textit{ and } \ 0 \leq \sigma \leq 1 \ \textit{ fixed.}$$

Moreover, $\mathbb{E}[DB_t]$ *does not vanish too quickly:*

$$\mathbb{E}[DB_t] \leq \sigma_t \cdot \mathbb{E}[LP_t] \quad \textit{where } \sigma_t := \begin{cases} t\sigma/h, & \text{for } t \leq h \\ \sigma & \text{for } t > h \end{cases} \quad \textit{with } h \textit{ as in Assumption 2.}$$

[GH22]

The existence of a $\sigma \in [0,1]$ for each time step t is obvious since $LP_t = V_t + DB_t^{\leq 0} + DB_t$ and these quantities are all greater than or equal to zero at all time steps t. Whether σ remains constant throughout the time period and if the second estimation holds for $t \leq h$ is tested with our ALM model in R.

As the values of $DB_t^{\leq 0}$, DB_t and LP_t are known in the model, the expected values and therefore σ can be computed for each time step t by rearranging the equation in Assumption 3:

$$s_t := \frac{\mathbb{E}[DB_t^{\leq 0} + DB_t]}{\mathbb{E}[LP_t]},$$

the calculated values can be considered in table 4.3. It is apparent that these values are not constant throughout the time period but increasing. Therefore we cannot assume that there exists a constant σ and the first part of the assumption does not hold. Mean, minimum, maximum and the coefficient of variation of the computed s_t can be considered in table 4.4.

t	s_t	t	s_t
0	0.1667	31	0.3102
1	0.1866	32	0.3202
2	0.2073	33	0.3281
3	0.2227	34	0.3315
4	0.2293	35	0.3371
5	0.2346	36	0.3430
6	0.2327	37	0.3808
7	0.2288	38	0.3879
8	0.2263	39	0.4309
9	0.2216	40	0.4651
10	0.2209	41	0.4958
11	0.2176	42	0.5124
12	0.2183	43	0.5361
13	0.2177	44	0.5516
14	0.2174	45	0.5604
15	0.2164	46	0.5669
16	0.2124	47	0.5886
17	0.2129	48	0.5898
18	0.2105	49	0.6061
19	0.2065	50	0.6157
20	0.2091	51	0.6211
21	0.2094	52	0.6192
22	0.2084	53	0.6263
23	0.2127	54	0.6456
24	0.2182	55	0.6728
25	0.2269	56	0.6954
26	0.2382	57	0.6943
27	0.2454	58	0.6877
28	0.2618	59	0.6812
29	0.2762	60	0.6793
30	0.2899		

t	s_t^w	t	s_t^w
0	0.0125	31	0.0018
1	0.0133	32	0.0018
2	0.0138	33	0.0017
3	0.0140	34	0.0017
4	0.0134	35	0.0016
5	0.0126	36	0.0016
6	0.0112	37	0.0014
7	0.0105	38	0.0012
8	0.0095	39	0.0010
9	0.0089	40	0.0010
10	0.0082	41	0.0009
11	0.0075	42	0.0008
12	0.0069	43	0.0007
13	0.0064	44	0.0007
14	0.0057	45	0.0006
15	0.0052	46	0.0006
16	0.0047	47	0.0006
17	0.0042	48	0.0006
18	0.0039	49	0.0005
19	0.0034	50	0.0005
20	0.0033	51	0.0005
21	0.0029	52	0.0005
22	0.0027	53	0.0003
23	0.0025	54	0.0003
24	0.0025	55	0.0003
25	0.0024	56	0.0002
26	0.0023	57	0.0002
27	0.0023	58	0.0002
28	0.0021	59	0.0002
29	0.0019	60	0.0002
30	0.0020		

t	$f(t)$	t	$f(t)$
0	0.0465	31	-0.0049
1	0.0296	32	-0.0052
2	0.0136	33	-0.0054
3	0.0029	34	-0.0053
4	-0.0012	35	-0.0052
5	-0.0041	36	-0.0054
6	-0.0027	37	-0.0057
7	-0.0007	38	-0.0053
8	0.0004	39	-0.0049
9	0.0023	40	-0.0051
10	0.0024	41	-0.0052
11	0.0034	42	-0.0045
12	0.0029	43	-0.0043
13	0.0029	44	-0.0042
14	0.0026	45	-0.0038
15	0.0027	46	-0.0039
16	0.0033	47	-0.0040
17	0.0029	48	-0.0041
18	0.0031	49	-0.0032
19	0.0035	50	-0.0035
20	0.0029	51	-0.0032
21	0.0025	52	-0.0033
22	0.0025	53	-0.0021
23	0.0018	54	-0.0020
24	0.0010	55	-0.0019
25	0.0000	56	-0.0016
26	-0.0011	57	-0.0016
27	-0.0017	58	-0.0016
28	-0.0028	59	-0.0016
29	-0.0035	60	-0.0016
30	-0.0045		

Table 4.3: Values of s_t, s_t^w and $f(t) = (\sigma - s_t) \cdot \mathbb{E}[LP_t]/MV_0$, calculated in R with respect to the base case.

$$\begin{aligned} \text{mean}(s_t) &= 0.37 \\ \min(s_t) &= 0.17 \\ \max(s_t) &= 0.70 \\ \text{CV}(s_t) &= 0.48 \end{aligned}$$

Table 4.4: Statistical key figures of the s_t noted in table 4.3.

Our next approach is to calculate weighted s_t^w for each time step t as

$$s_t^w := w_t \cdot s_t = \frac{\mathbb{E}[LP_t]}{\sum_k \mathbb{E}[LP_k]} \cdot \frac{\mathbb{E}[DB_t^{\leq 0} + DB_t]}{\mathbb{E}[LP_t]} \quad \text{with} \quad w_t := \frac{\mathbb{E}[LP_t]}{\sum_k \mathbb{E}[LP_k]}.$$

These values, which can be recognized in table 4.3, are also not constant but decreasing. Nevertheless a fixed σ is computed as the weighted average

$$\sigma = \sum_t s_t^w = 0.23.$$

As the fixed σ is not very far away from the computed s_t in the first 25 time steps, this choice is not too bad because the existing portfolio is steadily on the decrease.

The main part of the paper [GH22] where the considered relation $\mathbb{E}[DB_t^{\leq 0} + DB_t] = \sigma \cdot \mathbb{E}[LP_t]$ is needed is the computation of $\widehat{gs}_t$, which will be considered in Section 4.7. There the relation is used for the equation

$$\mathbb{E}[\rho_t \cdot V_{t-1}] = \rho_t \cdot (1-\sigma) \cdot \mathbb{E}[LP_{t-1}],$$

which can be rewritten as

$$\begin{aligned} \mathbb{E}[\rho_t \cdot V_{t-1}] &= \mathbb{E}\big[\rho_t \cdot (LP_{t-1} - DB_{t-1} - DB_{t-1}^{\leq 0})\big] \\ &= \rho_t \cdot \mathbb{E}[LP_{t-1}] \cdot (1 - s_{t-1}) \\ &= \rho_t \cdot (1-\sigma) \cdot \mathbb{E}[LP_{t-1}] + \rho_t \cdot (\sigma - s_{t-1}) \cdot \mathbb{E}[LP_{t-1}]. \end{aligned} \tag{4.1}$$

Equation (4.1) shows the desired result if the second summand is vanishing small. Hence

$$\frac{(\sigma - s_t) \cdot \mathbb{E}[LP_t]}{MV_0}$$

is computed for every time step t. The ideal outcome would be that all of the computed values are smaller than 0.5%, and almost all of the results shown in table 4.3 satisfy this. However, since these calculated values are multiplied with the percentage ρ_t in (4.1), the results will indeed be vanishing small even if some values of $\big((\sigma - s_t) \cdot \mathbb{E}[LP_t]\big)/MV_0$ are not smaller than 0.5%. Furthermore, the sum of all error terms

$$\sum_t \frac{(\sigma - s_t) \cdot \mathbb{E}[LP_t]}{MV_0} = 6.51 \cdot 10^{-18}$$

is vanishing small. Therefore the choice of the fixed σ is appropriate for the application of Assumption 3 in the paper of Gach and Hochgerner [GH22].

t	$\widetilde{s}_t$	t	$\widetilde{s}_t$
0	0.1667	31	0.5550
1	0.1963	32	0.5656
2	0.2276	33	0.5769
3	0.2543	34	0.5819
4	0.2704	35	0.5903
5	0.2865	36	0.5964
6	0.2996	37	0.6283
7	0.3095	38	0.6364
8	0.3185	39	0.6704
9	0.2716	40	0.7053
10	0.3347	41	0.7288
11	0.3421	42	0.7408
12	0.3490	43	0.7566
13	0.3562	44	0.7690
14	0.3639	45	0.7765
15	0.3694	46	0.7813
16	0.3717	47	0.7972
17	0.3812	48	0.7991
18	0.3869	49	0.8119
19	0.3881	50	0.8255
20	0.3973	51	0.8279
21	0.4042	52	0.8271
22	0.4102	53	0.8338
23	0.4243	54	0.8451
24	0.4352	55	0.8596
25	0.4515	56	0.8727
26	0.4663	57	0.8736
27	0.4772	58	0.8716
28	0.4968	59	0.8699
29	0.5134	60	0.8747
30	0.5348		

t	$\widetilde{s_t^w}$	t	$\widetilde{s_t^w}$
0	0.0106	31	0.0042
1	0.0120	32	0.0041
2	0.0132	33	0.0040
3	0.0141	34	0.0039
4	0.0141	35	0.0038
5	0.0140	36	0.0037
6	0.0134	37	0.0032
7	0.0135	38	0.0029
8	0.0128	39	0.0023
9	0.0127	40	0.0023
10	0.0122	41	0.0022
11	0.0118	42	0.0018
12	0.0111	43	0.0016
13	0.0107	44	0.0016
14	0.0099	45	0.0014
15	0.0094	46	0.0015
16	0.0086	47	0.0015
17	0.0081	48	0.0015
18	0.0077	49	0.0012
19	0.0070	50	0.0013
20	0.0069	51	0.0012
21	0.0063	52	0.0013
22	0.0061	53	0.0008
23	0.0058	54	0.0008
24	0.0058	55	0.0007
25	0.0056	56	0.0006
26	0.0055	57	0.0006
27	0.0054	58	0.0006
28	0.0050	59	0.0006
29	0.0046	60	0.0006
30	0.0048		

t	$\widetilde{f(t)}$	t	$\widetilde{f(t)}$
0	0.1383	31	-0.0192
1	0.1110	32	-0.0191
2	0.0831	33	-0.0194
3	0.0619	34	-0.0192
4	0.0482	35	-0.0188
5	0.0355	36	-0.0188
6	0.0254	37	-0.0175
7	0.0195	38	-0.0160
8	0.0137	39	-0.0135
9	0.0105	40	-0.0139
10	0.0053	41	-0.0136
11	0.0019	42	-0.0116
12	-0.0009	43	-0.0108
13	-0.0035	44	-0.0105
14	-0.0057	45	-0.0097
15	-0.0070	46	-0.0099
16	-0.0070	47	-0.0100
17	-0.0089	48	-0.0104
18	-0.0097	49	-0.0083
19	-0.0091	50	-0.0094
20	-0.0106	51	-0.0087
21	-0.0108	52	-0.0088
22	-0.0114	53	-0.0056
23	-0.0129	54	-0.0054
24	-0.0142	55	-0.0051
25	-0.0158	56	-0.0043
26	-0.0170	57	-0.0044
27	-0.0180	58	-0.0044
28	-0.0181	59	-0.0044
29	-0.0179	60	-0.0047
30	-0.0205		

Table 4.5: Values of $\widetilde{s}_t$, $\widetilde{s_t^w}$ and $\widetilde{f(t)} = (\widetilde{\sigma} - \widetilde{s}_t) \cdot \mathbb{E}[LP_t]/MV_0$, calculated in R with respect to the interest curve increased by 2%.

The same observations are made when performing these calculations regarding the by 2% increased interest curve. The results can be considered in table 4.5. Thereby, the values of the error terms are not as small as in the base case, but still small enough because the computed term will be multiplied with the percentage ρ_t as described in the base case. Besides that, the sum of the error terms is

$$\sum_t \frac{(\widetilde{\sigma} - \widetilde{s}_t) \cdot \mathbb{E}[LP_t]}{MV_0} = 5.08 \cdot 10^{-17},$$

which is vanishing small. Altogether, the assumption works with analogous argumentation as in the base case and the constant $\widetilde{\sigma}$ is fixed as

$$\widetilde{\sigma} = \sum_t \widetilde{s_t^w} = 0.35.$$

The second part of Assumption 3, the inequation

$$\mathbb{E}[DB_t] \leq \sigma_t \cdot \mathbb{E}[LP_t] \quad \textit{where } \sigma_t := \begin{cases} t\sigma/h, & \text{for } t \leq h \\ \sigma & \text{for } t > h \end{cases} \quad \textit{with } h \textit{ as in Assumption 2}$$

holds per assumption for all t greater than h because we have fixed a constant σ in the first part of Assumption 3 that satisfies $\mathbb{E}[DB_t^{\leq 0} + DB_t] = \sigma \cdot \mathbb{E}[LP_t]$, and $DB_t^{\leq 0}$ is greater than or equal to 0 at all time steps t. An explicit calculation of $\mathbb{E}[DB_t]/\mathbb{E}[LP_t]$ and $\sigma_t = t\sigma/h$ in the ALM model in R for t smaller than or equal to h shows that the assumption does not hold in all cases, the results can be considered in table 4.6. Nevertheless, the error is not too bad.

t	1	2	3	4	5	6	7	8
$\frac{\mathbb{E}[DB_t]}{\mathbb{E}[LP_t]}$	0.03	0.07	0.10	0.11	0.12	0.13	0.13	0.13
σ_t	0.03	0.05	0.08	0.10	0.13	0.16	0.18	0.21

Table 4.6: $\mathbb{E}[DB_t]/\mathbb{E}[LP_t] \leq \sigma_t$ tested by explicit computation in R for $t \leq h$ with respect to the base case, values start at $t = 1$ because $DB_0 = 0$.

Considering the second part of Assumption 3 regarding the by 2% increased interest curve shows that the assumption is also not fulfilled in some cases of t smaller than or equal to $\widetilde{h}$. However, the results, which can be considered in table 4.7, show that the error is again not too bad. The inequation $\mathbb{E}[DB_t]/\mathbb{E}[LP_t] \leq \widetilde{\sigma}_t$ always holds per assumption for t greater than $\widetilde{h}$ since we have fixed a constant $\widetilde{\sigma}$ in the first part of Assumption 3.

t	1	2	3	4	5	6	7	8	9	10
$\frac{\mathbb{E}[DB_t]}{\mathbb{E}[LP_t]}$	0.05	0.09	0.13	0.16	0.18	0.20	0.22	0.24	0.25	0.26
σ_t	0.03	0.07	0.10	0.13	0.16	0.20	0.23	0.26	0.29	0.33

Table 4.7: $\mathbb{E}[DB_t]/\mathbb{E}[LP_t] \leq \widetilde{\sigma}_t$ tested by explicit computation in R for $t \leq \widetilde{h}$ with respect to the by 2% increased interest curve, values start at $t = 1$ because $DB_0 = 0$.

4.4 Assumption 4

The relation $SF_0/LP_0 =: \vartheta$ remains constant in expectation:
$\mathbb{E}[SF_t] = \vartheta \cdot \mathbb{E}[LP_t] \quad \forall\ 0 \leq t \leq T$. [GH22]

This assumption is tested by computing $\mathbb{E}[SF_t]$ and $\mathbb{E}[LP_t]$ and in the following

$$\vartheta_t = \frac{\mathbb{E}[SF_t]}{\mathbb{E}[LP_t]}$$

with the ALM model in R for all time steps t.

The resulting values, which can be recognized in the left table of table 4.9, are quite stable and have very small variance, therefore ϑ can be fixed as the mean of ϑ_t and the assumption holds. The same conclusion can be made for the values $\widetilde{\vartheta}_t$, which were calculated analogously to ϑ_t but with respect to the by 2% increased interest curve. These values can be considered in the right table of table 4.9. Some statistical key figures of ϑ_t and $\widetilde{\vartheta}_t$ are listed in table 4.8.

$$\begin{aligned} \vartheta &:= \text{mean}(\vartheta_t) = 0.05 \\ \text{median}(\vartheta_t) &= 0.05 \\ \min(\vartheta_t) &= 0.04 \\ \max(\vartheta_t) &= 0.08 \\ \text{var}(\vartheta_t) &= 3.48 \cdot 10^{-5} \\ \text{CV}(\vartheta_t) &= 0.12 \end{aligned}$$

$$\begin{aligned} \widetilde{\vartheta} &:= \text{mean}(\widetilde{\vartheta}_t) = 0.05 \\ \text{median}(\widetilde{\vartheta}_t) &= 0.05 \\ \min(\widetilde{\vartheta}_t) &= 0.04 \\ \max(\widetilde{\vartheta}_t) &= 0.08 \\ \text{var}(\widetilde{\vartheta}_t) &= 4.20 \cdot 10^{-5} \\ \text{CV}(\widetilde{\vartheta}_t) &= 0.12 \end{aligned}$$

Table 4.8: Statistical key figures of the ϑ_t and $\widetilde{\vartheta}_t$ noted in table 4.9.

t	ϑ_t	t	ϑ_t
0	0.04	31	0.05
1	0.04	32	0.05
2	0.04	33	0.05
3	0.04	34	0.05
4	0.04	35	0.05
5	0.05	36	0.05
6	0.05	37	0.05
7	0.05	38	0.05
8	0.05	39	0.06
9	0.05	40	0.05
10	0.05	41	0.05
11	0.05	42	0.06
12	0.05	43	0.05
13	0.05	44	0.05
14	0.05	45	0.05
15	0.05	46	0.05
16	0.04	47	0.05
17	0.04	48	0.05
18	0.04	49	0.06
19	0.04	50	0.05
20	0.04	51	0.05
21	0.04	52	0.05
22	0.04	53	0.08
23	0.05	54	0.06
24	0.05	55	0.06
25	0.05	56	0.06
26	0.05	57	0.05
27	0.05	58	0.05
28	0.05	59	0.05
29	0.05	60	0.05
30	0.05		

t	$\widetilde{\vartheta}_t$	t	$\widetilde{\vartheta}_t$
0	0.04	31	0.06
1	0.04	32	0.05
2	0.04	33	0.05
3	0.04	34	0.05
4	0.05	35	0.05
5	0.05	36	0.05
6	0.05	37	0.06
7	0.05	38	0.06
8	0.05	39	0.06
9	0.05	40	0.06
10	0.05	41	0.06
11	0.05	42	0.06
12	0.05	43	0.06
13	0.05	44	0.05
14	0.05	45	0.06
15	0.05	46	0.05
16	0.05	47	0.05
17	0.05	48	0.05
18	0.05	49	0.07
19	0.05	50	0.05
20	0.05	51	0.06
21	0.05	52	0.05
22	0.05	53	0.08
23	0.05	54	0.06
24	0.05	55	0.06
25	0.05	56	0.07
26	0.05	57	0.05
27	0.05	58	0.05
28	0.05	59	0.05
29	0.05	60	0.05
30	0.05		

Table 4.9: Values of ϑ_t, calculated with respect to the base case and values of $\widetilde{\vartheta}_t$, computed with respect to the interest curve increased by 2%, both for each time step t.

4.5 Assumption 5

The surrender gains, $sg_t^* = \chi_t \cdot DB_{t-1}$, can be estimated on average with the same factor γ_t as the technical gains: $\mathbb{E}[sg_t^*] \leq \mathbb{E}[\gamma_t \cdot DB_{t-1}]$. [GH22]

The technical gains depend on costs, life tables and surrender tables and these are modeled independently of the scenarios in R. Therefore the inequation

$$\mathbb{E}[sg_t^*] \leq \mathbb{E}[\gamma_t \cdot DB_{t-1}]$$

can be rearranged to

$$\frac{\mathbb{E}[sg_t^*]}{\mathbb{E}[DB_{t-1}]} \leq \mathbb{E}[\gamma_t].$$

t	$\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$	t	$\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$	t	$\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$
0	0.00000	21	0.00016	41	0.00018
1	NaN	22	0.00016	42	0.00018
2	0.00031	23	0.00015	43	0.00022
3	0.00029	24	0.00016	44	0.00023
4	0.00027	25	0.00016	45	0.00023
5	0.00025	26	0.00016	46	0.00025
6	0.00024	27	0.00016	47	0.00023
7	0.00024	28	0.00015	48	0.00023
8	0.00022	29	0.00017	49	0.00021
9	0.00023	30	0.00019	50	0.00029
10	0.00021	31	0.00017	51	0.00027
11	0.00021	32	0.00020	52	0.00027
12	0.00020	33	0.00020	53	0.00022
13	0.00019	34	0.00019	54	0.00038
14	0.00018	35	0.00018	55	0.00037
15	0.00018	36	0.00017	56	0.00035
16	0.00017	37	0.00015	57	0.00045
17	0.00018	38	0.00016	58	0.00041
18	0.00018	39	0.00016	59	0.00036
19	0.00017	40	0.00018	60	0.00030
20	0.00017				

Table 4.10: Values of $\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$, computed in R with respect to the base case.

According to [GH22, Table 6], it is reasonable to assume that the fraction of technical gains γ_t is greater than or equal to 0.5%. Since the values of $\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$, which were computed with our ALM model in R and can be considered in table 4.10, are smaller than 0.5% and Assumption 5 uses an upward estimate, the assumption holds. This works similarly if the considered interest curve is increased by 2%, the values of $\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$ are then also smaller than 0.5% and therefore the assumption is correct as well. Those values can be considered in table 4.11.

t	$\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$
0	0.00000
1	NaN
2	0.00031
3	0.00029
4	0.00027
5	0.00025
6	0.00024
7	0.00025
8	0.00023
9	0.00023
10	0.00022
11	0.00022
12	0.00021
13	0.00020
14	0.00019
15	0.00019
16	0.00018
17	0.00018
18	0.00018
19	0.00018
20	0.00017

t	$\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$
21	0.00016
22	0.00016
23	0.00016
24	0.00016
25	0.00016
26	0.00016
27	0.00016
28	0.00016
29	0.00017
30	0.00019
31	0.00017
32	0.00020
33	0.00020
34	0.00019
35	0.00018
36	0.00017
37	0.00015
38	0.00016
39	0.00015
40	0.00018

t	$\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$
41	0.00018
42	0.00018
43	0.00022
44	0.00023
45	0.00024
46	0.00025
47	0.00024
48	0.00023
49	0.00021
50	0.00030
51	0.00027
52	0.00027
53	0.00023
54	0.00038
55	0.00037
56	0.00036
57	0.00046
58	0.00042
59	0.00037
60	0.00031

Table 4.11: Values of $\mathbb{E}[sg_t^*]/\mathbb{E}[DB_{t-1}]$, computed in R with respect to the interest curve increased by 2%.

4.6 Assumption 6

There is a fixed $0 < \nu < 1$ *such that the declarations satisfy*
$\eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* \geq \nu \cdot ph_t^* \quad \forall \quad 1 \leq t \leq T.$ [GH22]

The reference value for ν in our ALM model in R is $\nu = 0.7$ and the purpose of the following analysis is to show that this assumption is appropriate. This will be done by computing ν for every time step t and each scenario j, thereby the calculated ν for every time step t and each scenario j are denoted by ξ_t^j to avoid notational confusion because there already are ν_t on the left hand side of the inequation in Assumption 6 and $\nu \neq \nu_t$. The base case and the interest curve increased by 2% will be considered.

Quantiles	ξ_t^j
0%	0.00
0.910%	0.69
0.915%	0.71
5%	1.00
10%	1.00
15%	1.00
20%	1.00
25%	1.00
30%	1.01
35%	1.04
40%	1.06
45%	1.07
50%	1.10
55%	1.13
60%	1.16
65%	1.20
70%	1.25
75%	1.32
80%	1.41
85%	1.54
90%	1.77
95%	2.29
100%	1530.87

Quantiles	$\widetilde{\xi_t^j}$
0%	0.49
0.025%	0.69
0.030%	0.71
5%	0.99
10%	1.00
15%	1.02
20%	1.02
25%	1.04
30%	1.05
35%	1.05
40%	1.06
45%	1.07
50%	1.08
55%	1.09
60%	1.11
65%	1.12
70%	1.13
75%	1.15
80%	1.18
85%	1.22
90%	1.29
95%	1.44
100%	177.29

Table 4.12: Quantiles of ξ_t^j, computed with respect to the base case and quantiles of $\widetilde{\xi_t^j}$, computed with respect to the interest curve increased by 2%.

For $ph_t^* > 0$, the inequation can be rearranged to

$$\frac{\eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^*}{ph_t^*} \geq \nu,$$

in that case

$$\xi_t^j = \frac{\eta_t^j \cdot SF_{t-1}^j + \nu_t^j \cdot ph_t^{*j}}{ph_t^{*j}}$$

is calculated for every time step t and each scenario j with the model in R. Then ξ_t^j fulfill the inequation in Assumption 6 with equality for all t, j ($\nu \mathrel{\widehat{=}} \xi_t^j$). If $ph_t^* = 0$, the inequation always holds because η_t and SF_t are both greater than or equal to zero at every time step t and each scenario j and therefore ξ_t^j is arbitrary, anyway we fix $\xi_t^j = 1$ in this case.

The quantiles of ξ_t^j in the left table of table 4.12 show that $\nu = 0.7$ is a good approach when considering the base case because 99.085% of the computed ξ_t^j are greater than 0.7. This applies because Assumption 6 uses an estimation downwards. According to the quantiles of $\widetilde{\xi_t^j}$ in the right table of table 4.12, which were computed with respect to the by 2% increased interest curve, the assumption of $\nu = 0.7$ also holds in this case. This is similarly due to the fact that 99.97% of the computed $\widetilde{\xi_t^j}$ are greater than 0.7. The quantiles for some time steps t of both cases can be considered in table 4.13.

t	0%	25%	50%	75%	100%
5	1.00	1.17	1.21	1.27	1.73
10	0.78	1.20	1.5	4 2.24	473.22
15	0.12	1.00	1.00	1.20	29.18
20	0.00	1.00	1.00	1.08	46.13
25	0.00	1.00	1.10	1.29	34.87
30	0.00	1.00	1.08	1.21	315.48
35	0.00	1.02	1.10	1.24	176.98
40	0.98	1.08	1.25	1.54	103.82
45	0.59	1.05	1.11	1.21	22.66
50	0.97	1.11	1.28	1.55	37.90
55	0.96	1.08	1.16	1.28	135.49
60	0.32	0.98	1.00	1.03	11.42

t	0%	25%	50%	75%	100%
5	1.05	1.07	1.07	1.08	1.30
10	1.01	1.07	1.10	1.13	1.51
15	0.84	1.14	1.22	1.34	9.23
20	0.49	1.08	1.14	1.25	6.71
25	0.63	1.03	1.06	1.09	13.85
30	0.96	1.02	1.05	1.09	3.90
35	0.80	1.02	1.04	1.07	6.92
40	0.98	1.09	1.18	1.29	3.03
45	0.97	1.03	1.05	1.08	2.05
50	0.97	1.07	1.15	1.26	5.71
55	0.98	1.06	1.09	1.13	12.89
60	0.93	0.97	0.99	1.00	1.70

Table 4.13: Quantiles of ξ_t^j computed with respect to the base case for some time steps t in the left table and quantiles of $\widetilde{\xi_t^j}$ computed with respect to the interest curve increased by 2% for some time steps t in the right table.

4.7 Assumption 7

Assume that μ_k^{s+1} is determined by the geometric run-off assumption
Assumption 2: $\mu_k^{s+1} = \frac{l_s^h - l_{s+1}^h}{l_k^h}$. [GH22]

In the paper [GH22], Gach and Hochgerner define μ_k^t as the fraction of the bonus declarations $\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*$ at k that is either paid out as a surrender fee (in the case of premature contract termination) or paid out as a future discretionary benefit (in the case of contract maturity or mortality). The defining relation is

$$ph_t + sg_t^* = \sum_{k=1}^{t-1} \mu_k^t \cdot (\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*), \tag{4.2}$$

this relation is now used to show that Assumption 7 holds by simply inserting μ_k^t as defined in Assumption 7 into (4.2) and checking whether the equality holds or not. This is possible with our ALM model in R since all of the other needed variables are known in the model.

t	$\mathbb{E}$	SD
0	0.0000	0.0000
1	0.0000	0.0000
2	-0.0018	0.0000
3	-0.0022	0.0001
4	-0.0030	0.0001
5	-0.0035	0.0001
6	-0.0045	0.0002
7	0.0005	0.0000
8	-0.0029	0.0002
9	0.0007	0.0000
10	-0.0007	0.0002
11	-0.0003	0.0002
12	-0.0002	0.0004
13	0.0003	0.0005
14	-0.0015	0.0010
15	-0.0001	0.0012
16	-0.0014	0.0012
17	-0.0004	0.0010
18	0.0002	0.0011
19	-0.0006	0.0010
20	0.0011	0.0008
21	-0.0003	0.0006
22	0.0008	0.0004
23	0.0000	0.0004
24	0.0014	0.0003
25	0.0006	0.0003
26	0.0006	0.0003
27	0.0010	0.0002
28	-0.0012	0.0010
29	-0.0011	0.0009
30	0.0011	0.0004
31	-0.0013	0.0014
32	0.0005	0.0002
33	0.0008	0.0002
34	0.0008	0.0003
35	0.0002	0.0002
36	0.0010	0.0003
37	-0.0014	0.0015
38	-0.0008	0.0007
39	-0.0021	0.0018
40	0.0002	0.0004
41	0.0006	0.0004
42	-0.0005	0.0007
43	0.0005	0.0003
44	0.0009	0.0005
45	0.0005	0.0003
46	0.0013	0.0008
47	0.0011	0.0006
48	0.0013	0.0008
49	-0.0008	0.0009
50	0.0012	0.0007
51	0.0005	0.0003
52	0.0010	0.0006
53	-0.0015	0.0014
54	0.0005	0.0003
55	0.0006	0.0003
56	0.0001	0.0003
57	0.0008	0.0005
58	0.0007	0.0005
59	0.0007	0.0004
60	0.0006	0.0004

Table 4.14: Expected values and standard deviation of $\left(\sum_{k=1}^{t-1} \mu_k^t \cdot (\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*) - ph_t - sg_t^*\right)/MV_0$, computed with respect to the base case.

After computing μ_k^t for each time step t as

$$\mu_k^t = \frac{2^{-(t-1)/h} - 2^{-t/h}}{2^{-k/h}}$$

with h as in Assumption 2, equation (4.2) can be rearranged to

$$\frac{\sum_{k=1}^{t-1} \mu_k^t \cdot (\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*) - ph_t - sg_t^*}{MV_0} = 0 \tag{4.3}$$

and the left hand side is computed for each time step t and every scenario j. Equation (4.2) was divided by MV_0 since the error should be small in comparison to MV_0, in this case the outcome of the left hand side of (4.3) should be smaller than 0.5%.

t	$\mathbb{E}$	SD
0	0.0000	0.0000
1	0.0000	0.0000
2	-0.0029	0.0000
3	-0.0040	0.0001
4	-0.0056	0.0001
5	-0.0067	0.0001
6	-0.0085	0.0002
7	-0.0012	0.0001
8	-0.0071	0.0002
9	-0.0006	0.0001
10	-0.0042	0.0004
11	-0.0031	0.0004
12	-0.0038	0.0006
13	-0.0017	0.0004
14	-0.0056	0.0010
15	-0.0016	0.0007
16	-0.0036	0.0009
17	-0.0022	0.0010
18	-0.0003	0.0006
19	-0.0024	0.0009
20	0.0018	0.0003

t	$\mathbb{E}$	SD
21	-0.0021	0.0012
22	0.0016	0.0002
23	-0.0008	0.0008
24	0.0029	0.0002
25	0.0007	0.0006
26	0.0004	0.0007
27	0.0022	0.0002
28	-0.0041	0.0019
29	-0.0034	0.0016
30	0.0033	0.0006
31	-0.0046	0.0025
32	0.0012	0.0004
33	0.0020	0.0003
34	0.0024	0.0005
35	0.0009	0.0005
36	0.0027	0.0005
37	-0.0047	0.0031
38	-0.0017	0.0014
39	-0.0059	0.0037
40	0.0008	0.0008

t	$\mathbb{E}$	SD
41	0.0018	0.0008
42	-0.0011	0.0016
43	0.0014	0.0007
44	0.0030	0.0009
45	0.0018	0.0006
46	0.0041	0.0014
47	0.0035	0.0012
48	0.0041	0.0016
49	-0.0017	0.0020
50	0.0037	0.0014
51	0.0017	0.0006
52	0.0032	0.0013
53	-0.0041	0.0037
54	0.0019	0.0006
55	0.0020	0.0007
56	0.0008	0.0009
57	0.0027	0.0012
58	0.0025	0.0011
59	0.0023	0.0010
60	0.0021	0.0009

Table 4.15: Expected values and standard deviation of $\left(\sum_{k=1}^{t-1} \widetilde{\mu_k^t} \cdot (\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*) - ph_t - sg_t^*\right)/MV_0$, computed with respect to the interest curve increased by 2%.

Taking a look at the expected values and the standard deviation of the results of (4.3) for each time step t shows that the assumption holds because those values, which can be observed in table 4.14, are vanishing small. This result is also obtained if the expected values and standard deviation of (4.3) are considered with respect to the interest curve increased by 2%, the corresponding values can be considered in table 4.15. The quantiles of the results of (4.3) for both cases can be considered in tables 4.16 and 4.17 for some time steps t.

t	0%	25%	50%	75%	100%
0	0.0000	0.0000	0.0000	0.0000	0.0000
5	-0.0038	-0.0036	-0.0035	-0.0035	-0.0028
10	-0.0020	-0.0008	-0.0007	-0.0006	-0.0004
15	-0.0094	0.0002	0.0003	0.0003	0.0004
20	-0.0061	0.0012	0.0013	0.0013	0.0017
25	-0.0032	0.0006	0.0007	0.0007	0.0009
30	0.0004	0.0009	0.0010	0.0012	0.0042
40	-0.0043	0.0001	0.0003	0.0004	0.0013
50	0.0003	0.0007	0.0010	0.0014	0.0086
60	0.0001	0.0004	0.0005	0.0007	0.0046

Table 4.16: Quantiles of $\left(\sum_{k=1}^{t-1} \mu_k^t \cdot (\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*) - ph_t - sg_t^*\right)/MV_0$, computed with respect to the base case for some time steps t.

t	0%	25%	50%	75%	100%
0	0.0000	0.0000	0.0000	0.0000	0.0000
5	-0.0071	-0.0068	-0.0067	-0.0067	-0.0058
10	-0.0058	-0.0045	-0.0042	-0.0040	-0.0035
15	-0.0052	-0.0018	-0.0014	-0.0011	-0.0005
20	-0.0007	0.0017	0.0018	0.0019	0.0023
25	-0.0048	0.0005	0.0008	0.0010	0.0016
30	0.0022	0.0029	0.0032	0.0035	0.0070
40	-0.0093	0.0006	0.0010	0.0012	0.0024
50	0.0017	0.0028	0.0034	0.0042	0.0172
60	0.0010	0.0016	0.0019	0.0024	0.0105

Table 4.17: Quantiles of $\left(\sum_{k=1}^{t-1} \widetilde{\mu_k^t} \cdot (\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*) - ph_t - sg_t^*\right)/MV_0$, computed with respect to the interest curve increased by 2% for some time steps t.

Estimating the gross surplus

In order to estimate terms III and COG, Gach and Hochgerner [GH22] use a simplified model of gs_t where all quantities except F_{t-1} are replaced by their expected values. The simplified model is denoted by $\widehat{gs}_t$, its derivation according to [GH22] is illustrated below.

To find a useful representation of gs_t and therefore also for $\widehat{gs}_t$, the following representation of ROA_t is needed:

$$\begin{aligned} ROA_t &= \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] + ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] \\ &= \sum_{a\in\mathcal{A}_{t-1}} \mathbb{E}[cf_t^a \mid \mathcal{F}_{t-1}] + \mathbb{E}[\Delta BV_t^a \mid \mathcal{F}_{t-1}] + F_{t-1}C_{t-1} + ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] \\ &= \sum_{a\in\mathcal{A}_{t-1}} (1+F_{t-1})\cdot MV_{t-1}^a - \mathbb{E}[MV_t^a \mid \mathcal{F}_{t-1}] + \mathbb{E}[\Delta BV_t^a \mid \mathcal{F}_{t-1}] + F_{t-1}C_{t-1} \\ &\qquad + ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] \\ &= F_{t-1}\cdot(BV_{t-1}+UG_{t-1}) - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}] + ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}]. \end{aligned}$$

The gross surplus gs_t can then be denoted by

$$\begin{aligned} gs_t &= ROA_t - \rho_t V_{t-1} + \gamma_t LP_{t-1} \\ &= F_{t-1}BV_{t-1} + F_{t-1}UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}] + ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] \\ &\quad - \rho_t V_{t-1} + \gamma_t LP_{t-1} \\ &= F_{t-1}BV_{t-1} + P(0,t)^{-1}(l_{t-1}^d - l_t^d)\cdot UG_0 - \rho_t V_{t-1} + \gamma_t LP_{t-1}, \end{aligned}$$

where Assumption 9 was used for the last equation. Compared to the representation of gs_t in [GH22], this one is correct because there is a typing error in the paper.

By replacing all quantities in gs_t except F_{t-1} with their expected values and using Assumption 2, Assumption 3, Assumption 4 and Assumption 8, the simplified representation $\widehat{gs}_t$ can be derived:

$$\begin{aligned} \widehat{gs}_t &= F_{t-1}\cdot\mathbb{E}[BV_{t-1}] + P(0,t)^{-1}(l_{t-1}^d - l_t^d)\cdot UG_0 + \mathbb{E}[\gamma_t\cdot LP_{t-1} - \rho_t\cdot V_{t-1}] \\ &= F_{t-1}\cdot(l_{t-1}^h\cdot LP_0 + \vartheta\cdot l_{t-1}^h\cdot LP_0) + P(0,t)^{-1}(l_{t-1}^d - l_t^d)\cdot UG_0 \\ &\quad + \gamma_t\cdot l_{t-1}^h\cdot LP_0 - \rho_t\cdot l_{t-1}^h\cdot LP_0\cdot(1-\sigma) \\ &= l_{t-1}^h\cdot LP_0\cdot(1+\vartheta)\cdot\big(F_{t-1} + P(0,t)^{-1}\frac{l_{t-1}^d - l_t^d}{l_{t-1}^h}\frac{UG_0}{LP_0\cdot(1+\vartheta)} + \frac{\gamma_t - \rho_t\cdot(1-\sigma)}{1+\vartheta}\big). \end{aligned}$$

4.8 Assumption 8

In $\widehat{gs}_t$ the technical interest rate ρ_t and the fraction of technical gains γ_t are deterministic functions of t. [GH22]

This assumption is fulfilled in our ALM model in R because ρ_t and γ_t were modelled deterministically. Hence the assumption cannot be checked here.

4.9 Assumption 9

In $\widehat{gs}_t$ the return ROA_t is predictable, i.e. $\mathcal{F}_{t-1}$-measurable, and realizations of unrealized gains are determined by a fixed number $1 < d < T$: [GH22]

1. $ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] = 0$
2. $F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}] = P(0,t)^{-1}(l_{t-1}^d - l_t^d) \cdot UG_0$ *where* $l_t^d := 2^{-t/d}$ *for* $t < T$ *and* $l_T^d := 0$.

t	0%	25%	50%	75%	100%
5	-0.02575	-0.00230	-0.00078	0.00103	0.02781
10	-0.01773	-0.00110	-0.00000	0.00071	0.01682
15	-0.05026	-0.00294	-0.00000	0.00289	0.02701
20	-0.06999	-0.00421	0.00023	0.00604	0.05939
25	-0.11959	-0.00501	-0.00000	0.00516	0.13070
30	-0.21022	-0.00878	-0.00042	0.00898	0.70315
35	-0.28231	-0.00613	0.00000	0.00696	0.12570
40	-0.29872	-0.01177	0.00016	0.01444	0.36713
45	-0.62105	-0.00964	0.00037	0.00863	0.49382
50	-0.33968	-0.01143	0.00006	0.01117	0.59588
55	-0.31601	-0.01308	-0.00018	0.00983	0.42577
60	-0.35104	-0.01335	0.00024	0.01330	0.49764

Table 4.18: Quantiles of $\big(ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}]\big)/BV_{t-1}$, computed with respect to the base case.

The first part of this assumption is tested by computing

$$\frac{ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}]}{BV_{t-1}} \tag{4.4}$$

for each time step t and every scenario j, the term is divided by BV_{t-1} because the error should be small in comparison to BV_{t-1}. The quantiles of these values can be considered in table 4.18 for some time steps t. Then the expected values and the standard deviation of (4.4) are computed for every time step t. The results, which can be considered in table 4.19, are very small and therefore the assumption holds.

The corresponding expected values and standard deviation of (4.4) which were computed with respect to the by 2% increased interest curve can be considered in table 4.20, the assumption applies with analogous argumentation as in the base case. The corresponding quantiles of the results of (4.4) can be considered in table 4.21 for some time steps t.

t	$\mathbb{E}$	SD	t	$\mathbb{E}$	SD	t	$\mathbb{E}$	SD
1	-0.00002	0.00017	21	-0.00077	0.01158	41	0.00015	0.03107
2	-0.00025	0.00098	22	0.00024	0.01315	42	0.00020	0.04178
3	0.00045	0.00111	23	0.00012	0.01332	43	-0.00082	0.04377
4	-0.00061	0.00340	24	-0.00023	0.01749	44	-0.00040	0.03398
5	-0.00039	0.00375	25	0.00041	0.01719	45	-0.00052	0.04266
6	0.00057	0.00192	26	0.00074	0.01979	46	0.00279	0.04935
7	-0.00004	0.00090	27	-0.00094	0.02101	47	-0.00065	0.03284
8	0.00003	0.00208	28	0.00043	0.02212	48	0.00112	0.05021
9	-0.00010	0.00176	29	-0.00005	0.02333	49	0.00103	0.03926
10	-0.00015	0.00338	30	0.00142	0.04279	50	0.00250	0.06343
11	-0.00007	0.00389	31	-0.00071	0.02386	51	-0.00073	0.04381
12	0.00009	0.00596	32	0.00012	0.02478	52	-0.00172	0.05430
13	0.00055	0.00632	33	0.00015	0.02713	53	0.00102	0.04638
14	0.00002	0.00665	34	0.00055	0.02688	54	-0.00194	0.03922
15	-0.00032	0.00684	35	0.00023	0.02474	55	-0.00035	0.04135
16	0.00040	0.00872	36	0.00035	0.03389	56	0.00075	0.04801
17	-0.00013	0.00923	37	-0.00067	0.02739	57	-0.00166	0.04396
18	-0.00008	0.01011	38	0.00024	0.03535	58	-0.00106	0.04738
19	-0.00027	0.01056	39	-0.00037	0.03593	59	-0.00217	0.04666
20	0.00075	0.01183	40	0.00315	0.04341	60	0.00176	0.05639

Table 4.19: Expected values and standard deviation of $\big(ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}]\big)/BV_{t-1}$, computed with respect to the base case.

The simplest way to incorporate interest rate dependencies (discount factors and market value projections) in our Asset Liability Management model is to consider an evolution along the initial term structure. This corresponds to a deterministic modelling of asset prices and discount factors. The relevant initial term structure in this context is the risk-free curve as of 31.01.2022 of annual zero-coupon spot rates from EIOPA with which forward rates are computed. When considering this approach (EIOPA scenario), the first part of Assumption 9 is always fulfilled because in this case there is no volatility in property, stocks and the interest rate scenario. Therefore the market movements are always as expected. In other words, the information available at $t = 0$ is complete and

$$\begin{aligned} ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] &= \mathbb{E}[ROA_t \mid \mathcal{F}_0] - \mathbb{E}[\mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}] \mid \mathcal{F}_0] \\ &= \mathbb{E}[ROA_t \mid \mathcal{F}_0] - \mathbb{E}[ROA_t \mid \mathcal{F}_0] \\ &= 0 \end{aligned}$$

applies. The calculated values can be considered in table 4.22.

t	$\mathbb{E}$	SD
1	-0.00001	0.00004
2	-0.00027	0.00085
3	0.00037	0.00098
4	-0.00101	0.00326
5	0.00004	0.00438
6	0.00075	0.00226
7	-0.00008	0.00079
8	-0.00064	0.00416
9	-0.00027	0.00215
10	0.00002	0.00663
11	0.00042	0.00365
12	-0.00034	0.00494
13	0.00134	0.00747
14	0.00023	0.00606
15	0.00044	0.00434
16	0.00128	0.00596
17	-0.00104	0.00584
18	-0.00069	0.00605
19	0.00003	0.00480
20	0.00032	0.00571

t	$\mathbb{E}$	SD
21	-0.00079	0.00691
22	0.00048	0.00871
23	0.00119	0.00985
24	-0.00069	0.01183
25	0.00094	0.01305
26	-0.00039	0.01184
27	-0.00048	0.01028
28	0.00067	0.01287
29	-0.00018	0.00848
30	-0.00079	0.01898
31	-0.00080	0.01420
32	-0.00011	0.01679
33	0.00340	0.02056
34	0.00004	0.01503
35	0.00032	0.01471
36	-0.00014	0.01654
37	-0.00039	0.01430
38	0.00018	0.01193
39	0.00041	0.01047
40	-0.00063	0.03614

t	$\mathbb{E}$	SD
41	-0.00181	0.01882
42	0.00084	0.02386
43	0.00261	0.03485
44	0.00099	0.02092
45	0.00121	0.01904
46	-0.00159	0.01909
47	0.00041	0.01531
48	-0.00055	0.01464
49	0.00050	0.01559
50	-0.00124	0.04854
51	-0.00067	0.01895
52	-0.00003	0.02780
53	0.00549	0.03710
54	-0.00009	0.02044
55	0.00039	0.01947
56	0.00063	0.02233
57	0.00124	0.01790
58	-0.00018	0.01403
59	0.00060	0.01384
60	0.00096	0.06258

Table 4.20: Expected values and standard deviation of $\big(ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}]\big)/BV_{t-1}$, computed with respect to the interest curve increased by 2%.

t	0%	25%	50%	75%	100%
10	-0.02900	-0.00135	-0.00022	0.00096	0.02630
15	-0.04233	-0.00134	0.00028	0.00248	0.02253
20	-0.04774	-0.00218	0.00005	0.00223	0.03604
25	-0.07056	-0.00359	0.00019	0.00448	0.09206
30	-0.16073	-0.00863	-0.00108	0.00736	0.12158
35	-0.07168	-0.00361	0.00001	0.00293	0.12785
40	-0.23737	-0.01219	-0.00080	0.01024	0.26846
45	-0.09249	-0.00313	0.00044	0.00431	0.17629
50	-0.20602	-0.01515	-0.00023	0.01039	0.54216
55	-0.11135	-0.00392	0.00043	0.00517	0.14458
60	-0.38042	-0.01289	-0.00015	0.01402	0.77437

Table 4.21: Quantiles of $\big(ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}]\big)/BV_{t-1}$, computed with respect to the interest curve increased by 2%.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$f(t)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
t	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
$f(t)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
t	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	
$f(t)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
t	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	
$f(t)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

Table 4.22: Values of $f(t) = ROA_t - \mathbb{E}[ROA_t \mid \mathcal{F}_{t-1}]$ in the EIOPA Scenario.

Validating the second part of Assumption 9 is more complicated because a fixed number d has to be determined and it is not possible to solve the equation for d at every time step t. We start by rearranging

$$F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}] = P(0,t)^{-1}(l_{t-1}^d - l_t^d) \cdot UG_0$$

to

$$\frac{P(0,t)\big(F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}]\big)}{UG_0} = l_{t-1}^d - l_t^d \tag{4.5}$$

and computing the standard deviation of the left hand side of (4.5) for every time step t. The results with respect to the base case can be considered in table 4.24, the results with respect to the interest curve which has been increased by 2% are denoted in table 4.26. The standard deviation is slightly higher when regarding the increased interest curve, but since it is still very small in each case it is possible to perform a non linear regression analysis with the expected values of the left hand side of (4.5) in both cases for every time step t. The goal is to derive fixed values for d (base case) and $\widetilde{d}$ (shifted interest curve). The corresponding expected values can be considered in tables 4.24 and 4.26, some quantiles of the results of the left hand side of (4.5) can be observed in tables 4.25 and 4.27.

Each regression analysis is performed with the formula

$$\mathbb{E}[\frac{P(0,t)\big(F_{t-1}\cdot UG_{t-1}-\mathbb{E}[\Delta UG_t\mid \mathcal{F}_{t-1}]\big)}{UG_0}]\sim l^d_{t-1}-l^d_t.$$

The starting values, which were calculated with WolframAlpha for $t=1$, are respectively $d=3.09436$ and $\widetilde{d}=3.79862$. Transferring these parameters to R and applying a non linear least squares fit leads to the results in table 4.23, the dedicated plots 4.3 and 4.4 also show that the regression analysis works really well. It can be concluded that the assumption holds in both cases. This outcome is particularly remarkable because the assumption relies on a deterministic approximation of a stochastic quantity which was derived by rather crude heuristic arguments.

	d
Coef.	2.625
Std. error	0.089
t-stat.	29.515
p	0.000

	$\widetilde{d}$
Coef.	3.233
Std. error	0.132
t-stat.	24.572
p	0.000

Table 4.23: Results of the non-linear regression analysis with respect to the base case (left table) and with respect to the interest curve increased by 2% (right table).

t	$\mathbb{E}$	SD	t	$\mathbb{E}$	SD	t	$\mathbb{E}$	SD
1	0.2007	0.0000	21	-0.0017	0.0160	41	0.0010	0.0064
2	0.2105	0.0088	22	-0.0018	0.0226	42	0.0012	0.0056
3	0.1716	0.0168	23	0.0005	0.0190	43	0.0016	0.0052
4	0.1017	0.0157	24	0.0005	0.0165	44	0.0009	0.0047
5	0.0942	0.0101	25	0.0017	0.0199	45	0.0007	0.0039
6	0.0453	0.0138	26	0.0011	0.0190	46	0.0007	0.0033
7	0.0294	0.0145	27	0.0027	0.0194	47	-0.0000	0.0035
8	0.0333	0.0194	28	-0.0003	0.0191	48	0.0003	0.0028
9	0.0220	0.0183	29	0.0056	0.0187	49	0.0002	0.0030
10	0.0256	0.0184	30	0.0060	0.0206	50	0.0012	0.0029
11	0.0130	0.0197	31	-0.0003	0.0152	51	-0.0001	0.0029
12	0.0101	0.0189	32	0.0033	0.0152	52	0.0002	0.0030
13	0.0094	0.0201	33	0.0013	0.0134	53	0.0000	0.0023
14	0.0011	0.0161	34	-0.0004	0.0150	54	0.0010	0.0025
15	0.0093	0.0182	35	0.0000	0.0126	55	0.0003	0.0015
16	0.0066	0.0129	36	0.0003	0.0123	56	0.0002	0.0014
17	0.0080	0.0146	37	0.0000	0.0133	57	0.0003	0.0014
18	0.0030	0.0129	38	0.0028	0.0113	58	0.0001	0.0010
19	0.0007	0.0147	39	0.0017	0.0102	59	0.0000	0.0011
20	0.0005	0.0172	40	0.0047	0.0100	60	0.0002	0.0011

Table 4.24: Expected values and standard deviation of $P(0,t)\big(F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}]\big)/UG_0$, computed with respect to the base case.

t	0%	25%	50%	75%	100%
5	0.0130	0.0892	0.0939	0.0992	0.1293
10	-0.0790	0.0166	0.0287	0.0379	0.0938
15	-0.0695	0.0003	0.0051	0.0141	0.2182
20	-0.1392	-0.0013	-0.0001	0.0042	0.1172
25	-0.1754	-0.0034	0.0019	0.0090	0.0981
30	-0.3035	0.0001	0.0048	0.0116	0.1792
40	-0.0453	0.0008	0.0027	0.0060	0.0856
50	-0.0196	0.0002	0.0008	0.0017	0.0220
60	-0.0084	-0.0001	0.0001	0.0003	0.0110

Table 4.25: Quantiles of $P(0,t)\big(F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}]\big)/UG_0$, computed with respect to the base case for some time steps t.

t	$\mathbb{E}$	SD	t	$\mathbb{E}$	SD	t	$\mathbb{E}$	SD
1	0.1668	0.0000	21	-0.0055	0.0187	41	-0.0007	0.0088
2	0.1769	0.0077	22	-0.0013	0.0282	42	0.0033	0.0087
3	0.1582	0.0156	23	0.0061	0.0255	43	0.0039	0.0086
4	0.0808	0.0140	24	-0.0065	0.0218	44	0.0012	0.0060
5	0.0942	0.0105	25	0.0027	0.0259	45	0.0012	0.0057
6	0.0581	0.0193	26	0.0037	0.0255	46	0.0014	0.0055
7	0.0412	0.0189	27	0.0034	0.0269	47	-0.0001	0.0059
8	0.0344	0.0189	28	-0.0021	0.0264	48	0.0004	0.0046
9	0.0260	0.0194	29	0.0075	0.0260	49	0.0003	0.0048
10	0.0312	0.0256	30	0.0198	0.0266	50	0.0053	0.0070
11	0.0119	0.0302	31	-0.0039	0.0201	51	-0.0005	0.0059
12	0.0311	0.0317	32	0.0073	0.0216	52	0.0007	0.0053
13	0.0065	0.0269	33	0.0039	0.0187	53	0.0003	0.0048
14	0.0008	0.0243	34	-0.0036	0.0215	54	0.0026	0.0053
15	0.0257	0.0232	35	-0.0005	0.0191	55	0.0008	0.0029
16	0.0002	0.0150	36	0.0007	0.0164	56	0.0005	0.0028
17	0.0081	0.0212	37	-0.0001	0.0187	57	0.0010	0.0030
18	0.0164	0.0183	38	0.0059	0.0152	58	0.0002	0.0022
19	-0.0021	0.0196	39	0.0034	0.0134	59	0.0002	0.0027
20	0.0032	0.0205	40	0.0150	0.0170	60	0.0012	0.0031

Table 4.26: Expected values and standard deviation of $P(0,t)\big(F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}]\big)/UG_0$, computed with respect to the interest curve increased by 2%.

t	0%	25%	50%	75%	100%
5	0.0374	0.0885	0.0948	0.1015	0.1178
10	-0.0975	0.0194	0.0348	0.0471	0.1075
15	-0.0927	0.0122	0.0233	0.0378	0.1925
20	-0.1849	-0.0022	0.0031	0.0097	0.1518
25	-0.2573	-0.0034	0.0056	0.0140	0.0868
30	-0.3100	0.0085	0.0161	0.0288	0.2178
40	-0.0544	0.0056	0.0097	0.0186	0.1387
50	-0.0173	0.0017	0.0032	0.0062	0.0739
60	-0.0096	0.0002	0.0005	0.0014	0.0352

Table 4.27: Quantiles of $P(0,t)\big(F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}]\big)/UG_0$, computed with respect to the interest curve increased by 2% for some time steps t.

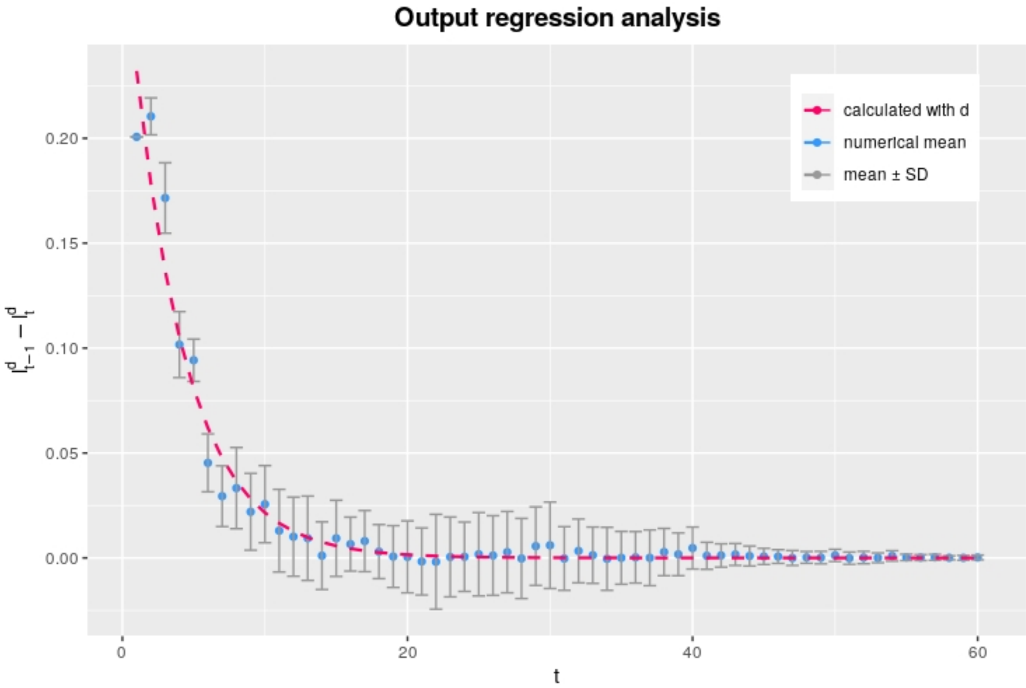

Figure 4.3: Plot of the output of the regression analysis for the base case: the red line shows the with the approximated value d calculated values of $l_{t-1}^d - l_t^d$, the blue dots show the expected values $\mathbb{E}\Big[P(0,t)\big(F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}]\big)/UG_0\Big]$ for every time step t and the grey lines denote mean $\pm$ standard deviation.

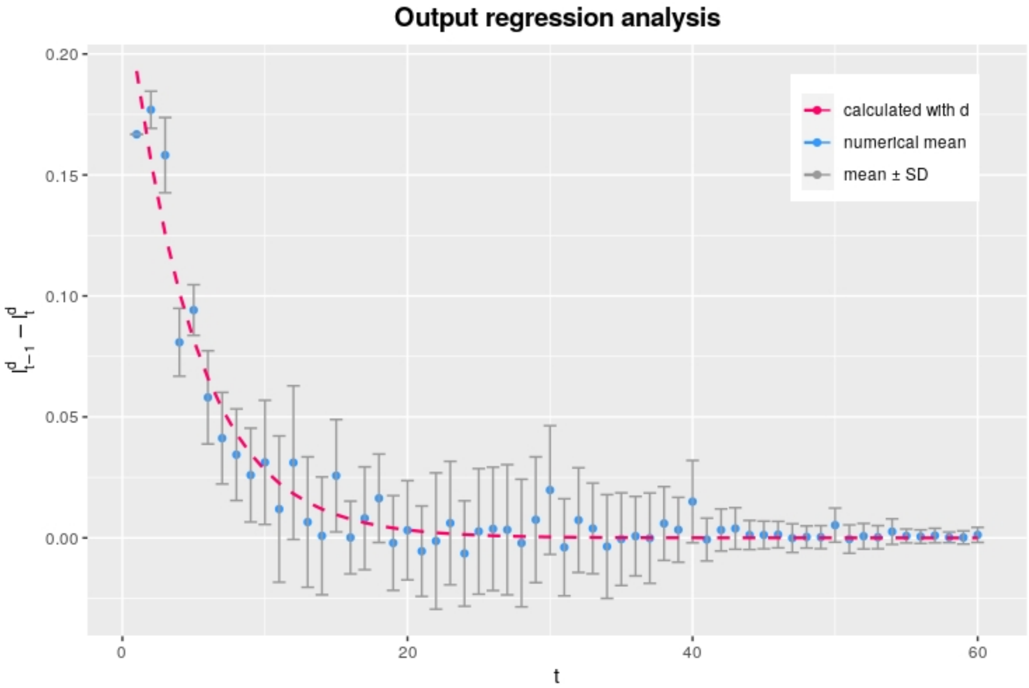

Figure 4.4: Plot of the output of the regression analysis for the interest curve increased by 2%: the red line shows the with the approximated value $\widetilde{d}$ calculated values of $l_{t-1}^{\widetilde{d}} - l_t^{\widetilde{d}}$, the blue dots show the expected values $\mathbb{E}\Big[P(0,t)\big(F_{t-1} \cdot UG_{t-1} - \mathbb{E}[\Delta UG_t \mid \mathcal{F}_{t-1}]\big)/UG_0\Big]$ for every time step t and the grey lines denote mean $\pm$ standard deviation.

4.10 Assumption 10

The coefficient of variation of book valued items is negligible in comparison to that of market movements. Concretely, the coefficients of variation of DB_t, LP_t and SF_t are assumed to be negligible in comparison to those of F_t and B_t^{-1}. [GH22]

The coefficients of variation of DB_t, LP_t, SF_t, F_t and B_t^{-1} are computed with the ALM model in R for every time step t. A part of the results with respect to the base case can be considered in table 4.29, some of the computed coefficients of variation regarding the interest curve which has been increased by 2% can be considered in table 4.30. Observing these results leads to the conclusion that the coefficients of variation of DB_t, LP_t and SF_t are indeed small in comparison to those of F_t in both cases. This especially holds because there are management rules to ensure that book valued items do not vary too much. The coefficient of variation of B_t^{-1} is in each case as small as those of DB_t, LP_t and SF_t and therefore the assumption does not hold for B_t^{-1}.

Altogether, the values of the coefficients of variation are not as small as we expected, but the application of Assumption 10 in [GH22] still works.

- In [GH22], Assumption 10 is needed in equation [5.29], where the estimation even gets sharper with the observed results because the correlations of the computed coefficients of variations, which can be considered in table 4.28, are positive.
- The assumption is also needed in equation [5.25] in [GH22], where the resulting values are very small and therefore the fact that the coefficients of variation are not as small as expected does not have a great impact.
- Lastly, the assumption is needed in equation [4.23] in [GH22], the main argument there is that the main contribution of the variance is derived from F_t, which is still true.

$\mathrm{Corr}\big(CV(DB_t), CV(F_t)\big)$	0.64
$\mathrm{Corr}\big(CV(DB_t), CV(B_t^{-1})\big)$	0.97
$\mathrm{Corr}\big(CV(LP_t), CV(F_t)\big)$	0.64
$\mathrm{Corr}\big(CV(LP_t), CV(B_t^{-1})\big)$	0.97
$\mathrm{Corr}\big(CV(SF_t), CV(F_t)\big)$	0.68
$\mathrm{Corr}\big(CV(SF_t), CV(B_t^{-1})\big)$	0.96

$\mathrm{Corr}\big(CV(DB_t), CV(F_t)\big)$	0.67
$\mathrm{Corr}\big(CV(DB_t), CV(B_t^{-1})\big)$	0.99
$\mathrm{Corr}\big(CV(LP_t), CV(F_t)\big)$	0.64
$\mathrm{Corr}\big(CV(LP_t), CV(B_t^{-1})\big)$	0.96
$\mathrm{Corr}\big(CV(SF_t), CV(F_t)\big)$	0.64
$\mathrm{Corr}\big(CV(SF_t), CV(B_t^{-1})\big)$	0.97

Table 4.28: Pearson correlations of the coefficients of variation computed with respect to the base case in the left table, computed with respect to the interest curve increased by 2% in the right table.

t	$CV(DB_t)$	$CV(LP_t)$	$CV(SF_t)$	$CV(F_t)$	$CV(B_t^{-1})$
0	NaN	NaN	NaN	-0.00	0.00
5	0.04	0.00	0.01	1.46	0.01
10	0.11	0.02	0.02	1.68	0.05
15	0.25	0.04	0.15	2.86	0.09
20	0.41	0.06	0.29	1.63	0.13
25	0.58	0.10	0.24	1.56	0.17
30	0.65	0.17	0.20	1.85	0.21
35	0.67	0.21	0.24	1.77	0.25
40	0.74	0.33	0.37	2.08	0.30
45	0.73	0.40	0.44	2.00	0.34
50	0.79	0.48	0.49	2.69	0.38
55	0.89	0.59	0.63	2.27	0.42
60	1.08	0.73	0.75	2.43	0.47

Table 4.29: Coefficients of variation, computed with respect to the base case, for some time steps t.

t	$CV(DB_t)$	$CV(LP_t)$	$CV(SF_t)$	$CV(F_t)$	$CV(B_t^{-1})$
0	NaN	NaN	NaN	-0.00	0.00
5	0.02	0.00	0.00	1.46	0.01
10	0.07	0.02	0.02	1.68	0.05
15	0.16	0.05	0.06	2.86	0.09
20	0.25	0.09	0.10	1.63	0.13
25	0.30	0.13	0.14	1.56	0.17
30	0.36	0.18	0.19	1.85	0.21
35	0.42	0.24	0.25	1.77	0.25
40	0.52	0.36	0.38	2.08	0.30
45	0.58	0.45	0.47	2.00	0.34
50	0.66	0.54	0.55	2.69	0.38
55	0.81	0.69	0.72	2.27	0.42
60	1.02	0.89	0.90	2.43	0.47

Table 4.30: Coefficients of variation, computed with respect to the interest curve increased by 2%, for some time steps t.

5 Analytical and numerical calculation of the future discretionary benefits

After introducing and validating the assumptions from [GH22] in Chapter 4, a lower and an upper bound for the FDB and consequently the mean of the two bounds as an estimator $\widehat{FDB}$ can be computed. This will be outlined in Section 5.1. Subsequently, the FDB, the lower bound $\widehat{LB}$, the upper bound $\widehat{UB}$ and the estimator $\widehat{FDB}$ are computed with our ALM model in R with 1000 interest rate scenarios simulated respectively with *VolSwi2, VolSwi25, VolSwi4* and *VolSwi6* (introduced in Section 3.2). Furthermore, the effect of increasing the interest curve by 2% is tested with 1000 interest rate scenarios simulated with *VolSwi4*. The outcomes are analysed and compared in Section 5.2.

5.1 Analytical lower and upper bounds for the future discretionary benefits

As all assumptions from [GH22] were verified in Chapter 4, the quantities I, II and III, which were determined in Section 2.2, can indeed be estimated. Gach and Hochgerner found approximations $\widehat{I}$ and $\widehat{II}$ for I and II and derived a lower and an upper bound $\widehat{III}_{lb}$ and $\widehat{III}_{ub}$ for III in their paper [GH22]. This will be illustrated below.

In (2.8), I was defined as $I = \mathbb{E}\Big[B_T^{-1}\big(DB_T + SF_T + gph \cdot (UG_T + V_T + DB_T^{\leq 0})\big)\Big]$. It can be estimated by using Assumption 1 and the assessment that by Assumption 1

$$LP_T = V_T + DB_T^{\leq 0} + DB_T = 0$$

holds. Since V_t, DB_t and $DB_t^{\leq 0}$ are all greater than or equal to 0 at every time step t and $SF_T = UG_T = 0$, it follows that

$$\widehat{I} := I = 0.$$

II was determined as $II = (1 - gph) \cdot \mathbb{E}[\sum_{t=2}^{T} B_t^{-1} sg_t^*]$ in (2.9). With the help of Assumption 2, Assumption 3, Assumption 5 and Assumption 10 it can be estimated as $II \leq \widehat{II}$ with

$$\widehat{II} := (1 - gph) \cdot \sum_{t=2}^{T} \gamma_t \cdot \sigma_t \cdot P(0,t) \cdot l_{t-1}^h \cdot LP_0.$$

The last quantity $III = (1-gph)\cdot\mathbb{E}\big[\sum_{t=1}^{T} F_{t-1}B_t^{-1}(DB_{t-1}+SF_{t-1})\big]$ was established in (2.10). With Assumption 2, Assumption 4 and Assumption 10, the lower bound

$$\begin{aligned}\widehat{III}_{lb} := (1-gph)\Big(F_0(1+F_0)^{-1}SF_0 + \vartheta\sum_{t=1}^{T-1}\big(P(0,t)-P(0,t+1)\big)\cdot l_{t-1}^h\cdot LP_0\Big)\\ + gph(1-gph)\sum_{t=1}^{T-1}(1-CV_{0,t}^1 CV_t^2)\big(1-P(t,t+1)\big)\cdot\mathcal{O}_t^+\cdot(1+\vartheta)\cdot l_{t-1}^h\cdot LP_0\end{aligned}$$

can be found, Assumption 6 and Assumption 7 help to determine the upper bound

$$\begin{aligned}\widehat{III}_{ub} := {} & (1-gph)\big(1-P(0,T)\big)\cdot SF_0\\ & + gph(1-gph)\sum_{t=1}^{T-1}(1+CV_{0,t}^1\cdot CV_t^2)\big(1-P(t,t+1)\big)\cdot\mathcal{O}_t^+\cdot(1+\vartheta)\cdot l_{t-1}^h\cdot LP_0\\ & + gph(1-gph)\sum_{t=2}^{T-1}\sum_{s=1}^{t-1}\big(1-\nu\cdot(1-l_{t-s}^h)\big)(1+CV_{s,t}^1\cdot CV_s^2)\\ & \qquad\cdot\big(P(s,t)-P(s,t+1)\big)\cdot\mathcal{O}_s^+\cdot(1+\vartheta)\cdot l_{s-1}^h\cdot LP_0\end{aligned}$$

such that $\widehat{III}_{lb}\le III\le\widehat{III}_{ub}$. $CV_{s,t}^1$ and CV_s^2 denote the first and the second coefficient of variation and

$$\mathcal{O}_t^{\pm} = \mathbb{E}\Big[B_t^{-1}\big(F_{t-1}+P(0,t)^{-1}\frac{l_{t-1}^d-l_t^d}{l_{t-1}^h}\frac{UG_0}{(1+\vartheta)LP_0}-\frac{(1-\sigma)\rho_t-\gamma_t}{1+\vartheta}\big)^{\pm}\Big]$$

defines the values of the caplet (corresponding to +) and the floorlet (corresponding to -) at 0 with maturity $t-1$ and payment $\big(F_{t-1}+P(0,t)^{-1}\frac{l_{t-1}^d-l_t^d}{l_{t-1}^h}\frac{UG_0}{(1+\vartheta)LP_0}-\frac{(1-\sigma)\rho_t-\gamma_t}{1+\vartheta}\big)$ at the settlement date t. All quantities in $\mathcal{O}_t^{\pm}$ except F_{t-1} are deterministic and the expected value can be computed with the Black formula [Bla76]. The resulting formulas for the lower bound $\widehat{III_{lb}}$ and the upper bound $\widehat{III_{ub}}$ are thus analytic.

Lastly, the cost of guarantees COG, which was defined as $COG=\mathbb{E}[\sum_{t=1}^{T}B_t^{-1}gs_t^-]$, can be simplified to

$$\widehat{COG} := \mathbb{E}[\sum_{t=1}^{T}B_t^{-1}\widehat{gs}_t^-] = \sum_{t=1}^{T}\mathcal{O}_t^-(1+\vartheta)\cdot l_{t-1}^h\cdot LP_0,$$

where the only difference is that gs_t^- was replaced by $\widehat{gs}_t^-$.

With these estimates, a lower and an upper bound $\widehat{LB}$ and $\widehat{UB}$ for the FDB can be determined. The FDB were defined as

$$FDB = SF_0 + gph \cdot (LP_0 + UG_0 - GB) + gph \cdot COG - I - II - III$$

in (2.7) and we get

$$\widehat{LB} \leq FDB \leq \widehat{UB} \tag{5.1}$$

with

$$\widehat{LB} := SF_0 + gph \cdot (LP_0 + UG_0 - GB) - \widehat{II} - \widehat{III}_{ub}$$

and

$$\widehat{UB} := SF_0 + gph \cdot (LP_0 + UG_0 - GB) + gph \cdot \widehat{COG} - \widehat{III}_{lb}.$$

[GH22] states that: If the difference $\widehat{UB} - \widehat{LB}$ is sufficiently small, then

$$\widehat{FDB} = \frac{\widehat{LB} + \widehat{UB}}{2}$$

may be used as an estimator for the FDB.

5.2 Comparison of estimated and numerical values

Now that representations of the FDB, $\widehat{LB}$, $\widehat{UB}$ and $\widehat{FDB}$ are known, these values can be calculated with the ALM model in R. This will be done taking account of differently simulated interest rate scenarios.

Interest rate scenarios: VolSwi2

At first, the numerical FDB, the lower bound $\widehat{LB}$, the upper bound $\widehat{UB}$ and the analytical $\widehat{FDB}$ are computed with 1000 interest rate scenarios simulated with *VolSwi2*. *VolSwi2* represents a classical Libor market model (introduced in Section 3.2), the calculated values can be considered in Table 5.1. There, FDB, $\widehat{LB}$, $\widehat{UB}$ and $\widehat{FDB}$ represent the mean value of all simulated scenarios.

FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	$\lvert\delta\rvert$	ϵ	MV_0
1 972 351.9	1 816 994.65	1 596 525.59	2 037 463.71	1.12	1.59	13 894 172

Table 5.1: Mean of FDB, $\widehat{FDB}$, $\widehat{LB}$ and $\widehat{UB}$; δ and ϵ relative to MV_0; all values computed with 1000 interest rate scenarios simulated with *VolSwi2*.

According to [GH22], the estimation of $\widehat{FDB}$ was successful if $|\delta|$ is smaller than ϵ, where $\delta = \widehat{FDB} - FDB$ and $\epsilon = \frac{\widehat{UB}-\widehat{LB}}{2}$. This means that the true value FDB lies within the estimation interval $\widehat{FDB} \pm \epsilon$, δ and ϵ can also be considered relative to MV_0. Therefore the estimation was successful in this case, the estimation error δ is rather small at 1.12% of the initial market value MV_0 and the fact that $\epsilon = 1.59\%$ of the initial market value MV_0 is also quite good.

Interest rate scenarios: VolSwi25

Next, all desired values are computed with 1000 interest rate scenarios simulated with *VolSwi25*, which is a mean-field extension of the LMM (introduced in Section 3.2). *VolSwi25* uses mean-field taming to reduce the variance of the scenarios to make an explosion (significant number of scenarios whose forward rates exceed a predefined threshold) very unlikely. The calculated values can be considered in table 5.2.

FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	$\lvert\delta\rvert$	ϵ	MV_0
1 919 129.9	1 816 994.65	1 596 525.59	2 037 463.71	0.74	1.59	13 894 172

Table 5.2: Mean of FDB, $\widehat{FDB}$, $\widehat{LB}$ and $\widehat{UB}$; δ and ϵ relative to MV_0; all values computed with 1000 interest rate scenarios simulated with *VolSwi25*.

It is very remarkable that the estimation error δ is at 0.74% far below 1% of the initial market value MV_0. In addition, $|\delta|$ is indeed smaller than ϵ and therefore the estimation was successful. The fact that $\epsilon = 1.59\%$ of the initial market value MV_0 is also good.

Interest rate scenarios: VolSwi4

Then the numerical FDB, the lower bound $\widehat{LB}$, the upper bound $\widehat{UB}$ and the analytical $\widehat{FDB}$ are computed with 1000 interest rate scenarios simulated with *VolSwi4*, which is also a mean-field extension of the LMM (introduced in Section 3.2). *VolSwi4* considers an anti-correlation prescription to reduce the probability of blow-ups, this is the framework which was also used to verify the assumptions in Chapter 4. The calculated values can be considered in Table 5.3.

Since $|\delta|$ is smaller than ϵ, this estimation was also successful. The estimation error δ is at 0.79% again far below 1% of the initial market value MV_0, which is striking. The value of ϵ is at 1.59% of the initial market value MV_0 also good.

FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	$\|\delta\|$	ϵ	MV_0
1 926 819.0	1 816 994.65	1 596 525.59	2 037 463.71	0.79	1.59	13 894 172

Table 5.3: Mean of FDB, $\widehat{FDB}$, $\widehat{LB}$ and $\widehat{UB}$; δ and ϵ relative to MV_0; all values computed with 1000 interest rate scenarios simulated with *VolSwi4*.

Interest rate scenarios: VolSwi6

Subsequently the values are computed with 1000 interest rate scenarios simulated with *VolSwi6*. *VolSwi6* is also a mean-field extension of the LMM (introduced in Section 3.2) and uses a decorrelation approach to reduce the probability of blow-ups. The calculated values can be considered in Table 5.4.

FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	$\|\delta\|$	ϵ	MV_0
1 944 176.0	1 816 994.65	1 596 525.59	2 037 463.71	0.92	1.59	13 894 172

Table 5.4: Mean of FDB, $\widehat{FDB}$, $\widehat{LB}$ and $\widehat{UB}$; δ and ϵ relative to MV_0; all values computed with 1000 interest rate scenarios simulated with *VolSwi6*.

It is apparent that the estimation was again successful because $|\delta|$ is smaller than ϵ. The estimation error δ is at 0.92% smaller than 1% of the initial market value MV_0, which is really good. The fact that $\epsilon = 1.59\%$ of the initial market value MV_0 is also good.

Interest rate scenarios: VolSwi4, interest curve increased by 2%

Finally, the FDB, $\widehat{LB}$, $\widehat{UB}$ and $\widehat{FDB}$ are computed with 1000 interest rate scenarios simulated with *VolSwi4* where the interest curve has been increased by 2%. The reason that only scenarios simulated with *VolSwi4* are used for the shifted case is that these scenarios are considered as the most realistic set among the mean field controlled equations. The resulting values can be considered in table 5.5.

Since $|\delta|$ is smaller than ϵ, the estimation was once more successful. The estimation error δ is at 1.22% of the initial market value MV_0 bigger than in the other considered cases, but still small. The value of ϵ is at 1.59% of the initial market value MV_0, which is good.

FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	$\lvert\delta\rvert$	ϵ	MV_0
3 266 434.8	3 435 970.66	3 215 501.60	3 656 439.72	1.22	1.59	13 894 172

Table 5.5: Mean of FDB, $\widehat{FDB}$, $\widehat{LB}$ and $\widehat{UB}$; δ and ϵ relative to MV_0; all values computed with 1000 interest rate scenarios simulated with *VolSwi4* where the interest curve has been increased by2%.

Comparison of the cases studied

It is immediately noticeable that the estimation was successful in all considered cases. The estimation error δ was thereby always smaller than 1.25% of the initial market value MV_0, which is remarkable. The smallest values of δ appeared in the cases where the interest rate scenarios were simulated with *VolSwi25* and *VolSwi4* under consideration of the base case. The value of ϵ stayed at 1.59% of the initial market value MV_0 in all considered cases which is really good. All in all, the estimation works really well.

6 A selective review of existing literature

According to Diehl, Horsky, Reetz and Sass in [DHRS22], stochastic Asset Liability Management models have gained a lot of attention in recent years. The authors have developed an ALM model for a life insurance company selling life insurance products with profit participation in a low interest rate environment and the overall structure of their ALM model is very similar to our ALM model in R. In [DHRS22] the balance sheet model considers stocks, bonds and cash, our model additionally considers property. The bond price is modeled with a Vasiček model and the stock price is modeled with a discretized geometric Brownian motion model. Management targets to determine the numbers of assets held in each time period are taken into account in both models.

The ALM model of Diehl, Horsky, Reetz and Sass and our ALM model both consider model points instead of individual contracts to approximate a real life insurance portfolio. While our ALM model directly simulates model points (and not individual policyholders) or receives them from an insurance company, [DHRS22] provides an explicit approach for the generation of model points: the insured collective is grouped according to gender, the current age and the exit age of the policyholders. The formed groups are called cohorts or model points and one representative policyholder of each cohort is selected randomly. The number of generated cohorts depends on the size and the heterogeneity of the initial insurance portfolio. In contrast to our ALM model, the model in [DHRS22] also considers new business in each time period. When new policyholders occur, they are also grouped into cohorts and those new cohorts are then merged with existing ones to avoid an increase of the number of cohorts. Mortality and surrender effects are also taken into account, but only at the end of each time period. This leads to changes in the sizes of the cohorts during the simulation period. Mortality is simulated by using cohort life tables and the surrender probabilities are simulated with an exponential distribution, our ALM model simulates those quantities by using life and surrender tables.

The resulting ALM model is then used to perform simulation studies to investigate the long-term stability of a life insurer's balance sheet. For this purpose, a run-off approach is compared with a scenario which also allows new business. Furthermore, two different investment strategies are compared in the case of stationary new business and the robustness of those strategies is tested in a market which allows crashes in stock and bond markets. Finally, a sensitivity analysis is performed.

The paper *Asset-liability management for long-term insurance business* [ABE+18] discusses some challenges insurance companies have to face in accordance with Asset

Liability Management for long-term insurance business, some of their arguments will be outlined below.

One of the most important processes in regulating and monitoring insurance businesses is valuation, which includes risk capital calculations, providing a fair value assessment for asset and liability portfolios and reserving. Thereby the market consistent valuation of liabilities could be difficult as it has to be derived from models. The corresponding cash flows can be grouped into three categories: the components that can be perfectly replicated by existing financial instruments, the components that can be perfectly replicated by future cash flows and the remaining components that cannot be replicated by existing and future financial instruments. The market-consistent value then consists of the costs of producing those cash flows and is thus the sum of the best estimate and a risk margin, as there is a certain risk that the costs for replicating some of the cash flows will change in the future. The resulting value strongly depends on assumptions on future management actions, models for the capital markets and the cash flows, and the policyholder behaviour. Therefore different insurance companies will have different resulting values as those depend on the used model. As future management actions, market features, product design and policyholder behaviour are not the same in different insurance companies, it does not seem reasonable to use one general framework. Nevertheless, more transparency on the models and on the assumptions would be good to be able to compare the results of different insurance companies and to be more transparent on uncertainties and risks provided in the near future. However, the formula of [GH22] shows that an estimation of a range of possible values is feasible.

As mentioned above, the Asset Liability Management relies on models and a few issues that arise when creating a model were also discussed in [ABE+18]. The first one is that an optimal investment strategy is usually determined with an economic model and an objective function that is maximized. The problem is that this optimal strategy might be extremely sensitive to the underlying model assumptions, therefore the strategy could be optimal in one model but perform badly in the real world. One possibility to solve this is to find a strategy that performs well under multiple models.

The second important issue is the determination of management rules. As those parameters might have more impact on the final outputs than other ones, a sensitivity analysis should be performed and the top management should know how the model reacts to different management rules. Thereby it is quite interesting that Swiss regulators have observed very heterogeneous management rules in the supervised companies and a standardization is to be discussed.

In addition to those issues, a risk-free interest curve has to be constructed and the volatility surface for yield curves has to be chosen. Furthermore, there should be a balance between the simplicity and the accuracy of the model and what-if scenarios should be performed.

El Karoui, Loisel, Prigent and Vedani discuss in *Market inconsistencies of the market-*

consistent European life insurance economic valuations: pitfalls and practical solutions [VELP17] some of the major risk sources that are included in the current regulatory life insurance valuation scheme.

One of the mentioned risk sources is that insurance companies have to forecast yield curves in the economic valuation framework and thereby mistakes in the scheme which is used to calibrate interest rate models can lead to a disconnection to fair pricing. Furthermore the risk-neutral models might lead to unrealistic projected trajectories, which might have an unpredictable impact on the way long-term risks are taken into account in valuation. On this topic, a model-free calibration procedure is explained in [VELP17] and the effects on economic scenarios are shown graphically.

Besides that, the actuarial market-consistency criterion implies very volatile valuations. The criterion is directly affected by market movements and depends on the calibration of the chosen model. Insurance companies should beware that December 31st is not the ideal date to calibrate their model with market data because the markets are known to provide inexact and highly volatile prices in December as a consequence of the accounting closing date. One possibility to compute more stable values is to use averaged calibration sets, for example from the whole month October. Tests in [VELP17] showed that this is a good approach and that the difference in the observed values is significant.

Moreover, the calibration of financial models which are then used to valuate the economic balance sheet can also be an issue as the model is usually used to valuate all of the insurer's life insurance products and then the calibration results in only one valuation probability measure. The estimated values are different for each insurance company as all models are different and cannot be compared. This is a problem as comparability is very important for efficient valuation. The authors suggest a consideration of a more locally defined market-consistency criterion and different calibrations of a model such that there exist different risk neutral probability measures for different life insurance liability valuations.

7 Conclusions

This thesis analyzed a new method to calculate the FDB approximately but much faster than usual. After introducing a closed formula for the FDB, the assumptions which Gach and Hochgerner needed to be able to estimate the FDB in [GH22] were validated. This was done with the help of an Asset Liability Management model in R, all assumptions were tested with 1000 interest rate scenarios (*VolSwi4*) and then additionally with 1000 interest rate scenarios (*VolSwi4*) where the initial interest curve has been increased by 2%.

The results of our analysis were very good because Assumption 1, Assumption 2, Assumption 4, Assumption 5, Assumption 6, Assumption 7, Assumption 8 and Assumption 9 were fulfilled for both considered interest curves. Assumption 3 could not be fully verified, however, the most important estimation for which Assumption 3 is needed in the paper [GH22] still works. Thus it is not important for the application of Assumption 3 in the paper [GH22] that the assumption is not fulfilled exactly as formulated.

The second part of Assumption 9 was a little tricky to check because the corresponding equation could not be solved for the variable we were looking for. So a regression analysis was done, the results were very good for both of the considered interest curves. This is particularly remarkable because this assumption relies on a deterministic approximation of a stochastic quantity which was derived by rather crude heuristic arguments.

Assumption 10 was also not fulfilled as formulated, but the result also works when applying the results in the paper [GH22].

Overall, all assumptions are at least fulfilled to the extent that the application of the assumptions in [GH22] works.

After verifying the assumptions it was possible to calculate a lower and an upper bound for the FDB and consequently to estimate $\widehat{FDB}$. Subsequently, FDB and $\widehat{FDB}$ were calculated in differently simulated interest rate scenarios. It was very remarkable that the estimation was successful in all considered cases and that the estimation error was always very small compared to the initial market value MV_0.

All in all, our results lead to the conclusion that the analytical calculation of the FDB, $\widehat{FDB}$, works very well as all assumptions needed to estimate the $\widehat{FDB}$ are correct enough for the application in [GH22] and the estimation was successful in all considered cases.

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Appendix: List of symbols

The content of the Appendix has in slightly adapted form been taken from [GH22] as the same notation was used.

Symbol	Meaning	Definition and relations
A		
$\mathcal{A}_t$	set of assets, excluding cash, at time t	-
B		
B_t	bank account at time t	$B_t = \prod_{j=0}^{t-1}(1+F_j)$
BE	best estimate	$BE = \mathbb{E}\big[\sum_{t=1}^{T} B_t^{-1}(gbf_t + gbf_t^{\leq 0} + ph_t + co_t - pr_t)\big]$
bd_t	total bonus declaration to DB_t at time t	$bd_t = \nu_t \cdot ph_t^* + \eta_t \cdot SF_{t-1}$
BV_t	book value of the asset portfolio at time t	$BV_t = \sum_{a\in\mathcal{A}_t} BV_t^a + C_t$
BV_t^a	book value of an asset a	-
C		
C_t	amount of cash held by the company at time t	-
cf_t^a	cash flow of asset a at time t	-
co_t	cost cash flows at time t	-
COG	cost of guarantees	$COG = \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} gs_t^-]$
$CV_{s,t}^1$	first coefficient of variation	$CV_{s,t}^1 = CV\big[D(s,t) - D(s,t+1)\big]$
CV_s^2	second coefficient of variation	$CV_s^2 = CV[B_s^{-1} gs_s^+]$
χ_t	surrender fee factor at time t	-
$\chi_t^{\leq 0}$	surrender fee factor at time t for $DB_t^{\leq 0}$	-
D		
$D(t,s)$	discount factor from s to $t < s$	$D(t,s) = \prod_{j=t}^{s-1}(1+F_j)^{-1} = B_t B_s^{-1}$
DB_t	declared bonuses after valuation time	$DB_t = \sum_{x\in\mathcal{X}_t} DB_t^x$ $= DB_{t-1} + \eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* - ph_t - sg_t^*$, $DB_0 = 0$
DB_t^x	declared bonuses after valuation time of model point x at time t	-

DB_t^-	account of declared bonuses before bonus declaration at time t	-
$DB_t^{\leq 0}$	declared bonuses up to and including valuation time	$DB_t^{\leq 0} = \sum_{x \in \mathcal{X}_t} (DB_t^{\leq 0})^x$
$(DB_t^{\leq 0})^x$	declared bonuses up to and including valuation time of model point x at time t	-
Δf_t	increment of f_t	$\Delta f_t = f_t - f_{t-1}$
E		
η_t	fraction of declaration of SF_{t-1} to DB_t	-
F		
F_t	simple one year forward rate between t and $t+1$	-
FC_t [1]	free capital at time t	$FC_t = BV_t - L_t$
FDB	value of future discretionary benefits	$FDB = \mathbb{E}[\sum_{t=1}^{T} B_t^{-1} ph_t]$
G		
GB	value of guaranteed benefits	$GB = BE - FDB$
gbf_t	guaranteed benefits at time t	$gbf_t = \sum_{x \in \mathcal{X}_t} gbf_t^x$
gbf_t^x	guaranteed benefits generated by model point x at time t	-
$gbf_t^{\leq 0}$	cash flows due to $DB_{t-1}^{\leq 0}$	$gbf_t^{\leq 0} = \sum_{x \in \mathcal{X}_t} (gbf_t^{\leq 0})^x$
$(gbf_t^{\leq 0})^x$	cash flows due to $(DB_{t-1}^{\leq 0})^x$	-
gph	policyholder share in gross surplus	-
gs_t	gross surplus at time t	$gs_t = sh_t + ph_t^* + tax_t = ROA_t - \Delta V_t - \Delta DB_t^{\leq 0} - DB_t^- + DB_{t-1} + pr_t - gbf_t - gbf_t^{\leq 0} - ph_t - co_t$
$\widehat{gs}_t$	simplified model of gs_t	$\widehat{gs}_t := l_{t-1}^h \cdot LP_0 \cdot (1+\vartheta) \cdot \big(F_{t-1} + P(0,t)^{-1} \frac{l_{t-1}^d - l_t^d}{l_{t-1}^h} \frac{UG_0}{LP_0 \cdot (1+\vartheta)} + \frac{\gamma_t - \rho_t \cdot (1-\sigma)}{1+\vartheta}\big)$
gsh	shareholder share in gross surplus	-
$gtax$	tax paid on gross surplus at time t	-
γ_t	fraction of technical gains	$\gamma_t = \frac{tg_t + \chi_t^{\leq 0} DB_{t-1}^{\leq 0} + \chi_t DB_{t-1}}{LP_{t-1}}$
H		
h	half of life assurance provisions	$\mathbb{E}[LP_h] = LP_0/2$
I		

[1] We assume without loss of generality that $FC_t = 0$ because the return on free capital is not shared with the policyholders and does therefore not contribute to the FDB which we are interested in.

J

K

L

L_t	book value of liabilities at time t	$L_t = LP_t + SF_t$
LP_t	life assurance provision at time t	$LP_t = V_t + DB_t^{\leq 0} + DB_t$

M

μ_k^t	fraction of bonus declarations from time k paid out (or kept - as surrender fee) at time t	-
MV_t	market value of the portfolio at time t	$MV_t = \sum_{a\in\mathcal{A}_t} MV_t^a + C_t$
MV_t^a	market value of asset a at time t	$MV_{t-1}^a = (1+F_{t-1})^{-1}\big(\mathbb{E}[cf_t^a \mid \mathcal{F}_{t-1}] + \mathbb{E}[MV_t^a \mid \mathcal{F}_{t-1}]\big)$

N

ν	bonus declaration bond	$\exists\ 0<\nu<1$ s.t. $\eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* \geq \nu \cdot ph_t^*$ $\forall\ 1 \leq t \leq T$
ν_t	declaration fraction of ph_t^*	-

O

$\mathcal{O}_s^+$	value at 0 of the caplet with maturity $s-1$	$\mathcal{O}_s^+ := \mathbb{E}\Big[B_s^{-1}\big(F_{s-1} - \frac{(1-\sigma)\rho_s-\gamma_s}{1+\vartheta} + P(0,s)^{-1}\frac{l_{s-1}^d-l_s^d}{l_{s-1}^h}\frac{UG_0}{(1+\vartheta)LP_0}\big)^+\Big]$
$\mathcal{O}_s^-$	value at 0 of the floorlet with maturity $s-1$	$\mathcal{O}_s^- := \mathbb{E}\Big[B_s^{-1}\big(F_{s-1} - \frac{(1-\sigma)\rho_s-\gamma_s}{1+\vartheta} + P(0,s)^{-1}\frac{l_{s-1}^d-l_s^d}{l_{s-1}^h}\frac{UG_0}{(1+\vartheta)LP_0}\big)^-\Big]$

P

$P(t,s)$	value of a zero coupon bond, with nominal of 1 at s, at time t	$P(t,s) = \mathbb{E}[D(t,s)]$
PH^*	time value of the accounting flows ph_t^*	$PH^* = \mathbb{E}[\sum_{t=1}^T B_t^{-1} ph_t^*]$
ph_t	amount of discretionary benefits paid out at time t	$ph_t = \sum_{x\in\mathcal{X}_t} ph_t^x$
ph_t^x	cash flows due to DB_{t-1}^x	-
ph_t^*	policyholder accounting flow at time t	$ph_t^* = gph \cdot gs_t^+$
pr_t	premium payments at time t	-

Q

R

ρ_t	average technical interest rate at time $t-1$	-
ROA_t	book value return at time t	$ROA_t = \sum_{a\in\mathcal{A}_{t-1}} ROA_t^a + F_{t-1}C_{t-1}$

ROA_t^a	book value return of asset a at time t	$ROA_t^a = cf_t^a + \Delta BV_t^a$
S		
SF_t	surplus fund at time t	$SF_t = BV_t - LP_t$ $= SF_{t-1} + (1-\nu_t)ph_t^* - \eta_t SF_{t-1}$
sg_t^*	surrender fee at time t	$sg_t^* = \chi_t \cdot DB_{t-1},\ \ 0 \leq \chi_t \leq 1$
sh_t	shareholder cash flow at time t	$sh_t = gsh \cdot gs_t^+ - gs_t^-$
σ	total declared bonuses fraction	$\mathbb{E}[DB_t^{\leq 0} + DB_t] = \sigma \cdot \mathbb{E}[LP_t]$ $0 \leq \sigma \leq 1$ fixed
T		
T	projection horizon	-
TAX	time value of tax	$TAX = \mathbb{E}[\sum_{t=1}^T B_t^{-1} tax_t]$
tax_t	tax cash flow at time t	$tax_t = gtax \cdot gs_t^+$
TDB_t	total declared bonuses at time t	$TDB_t = DB_t^{\leq 0} + DB_t$
tg_t	technical gains at time t	-
ϑ	surplus fund fraction	$\mathbb{E}[SF_t] = \vartheta \cdot \mathbb{E}[LP_t]$
U		
UG_t	unrealized gains at time t	$UG_t = MV_t - BV_t$
UG_t^a	unrealized gains of asset a at time t	$UG_t^a = MV_t^a - BV_t^a$
V		
V_t	mathematical reserves at time t	$V_t = \sum_{x \in \mathcal{X}_t} V_t^x$
V_t^x	mathematical reserve of model point x at time t	-
VIF	value of in-force business	$VIF = \mathbb{E}[\sum_{t=1}^T B_t^{-1} sh_t]$
W		
X		
$\mathcal{X}_t$	set of model points active at time t	-
Y		
Z		