Predicting Capillary Ripples and Nonlinear Squire–Taylor Modes by Viscous–Inviscid Interaction past a Trailing Edge

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Acknowledgements & relevant papers

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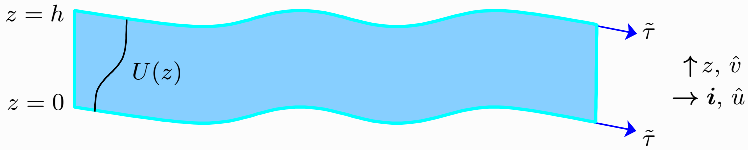
Publications

- Scheichl, Bowles & Pasias (JFM, **850**, 2018 & **926**, 2021)
- Scheichl, Bowles & Pasias (JFM, submitted soon – we talk about this)

Overview

- Motivation: local stability of non-interactive planar sheet
- Interaction problem
- Analytical & numerical treatment of individual flow regimes
- Capillary choking
- Far-downstream (WKBJ) asymptotics
- Context: (axi)symmetric flow through channel/pipe exit
- Achievements & outlook

Planar waves on a fluid (liquid) sheet



Linearise to obtain a Rayleigh problem, $c = c(k, U(z))$

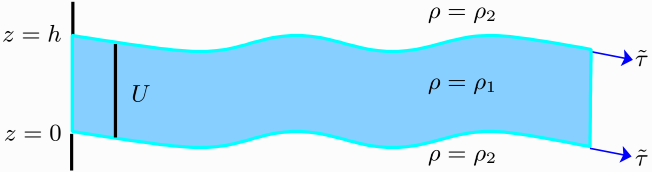
$$\boldsymbol{u} = U(z)\boldsymbol{i} + \hat{\boldsymbol{u}}, \quad (\hat{u}, \hat{v}) = (\psi'(z), -ik\psi(z))\exp(\mathrm{i}k(x - ct))$$

$$(U(z) - c)\left(\psi''(z) - k^2\psi(z)\right) + U''(z)\psi(z) = 0$$

inviscid interface conditions, surface tension $\tilde{\tau}$

This is a planar version of the Rayleigh-Plateau problem for droplet formation from a cylindrical stream of fluid.

Planar waves on a fluid sheet — uniform flow: Squire modes (1953)



Linearise to obtain a Rayleigh problem, $c = c(k, U)$

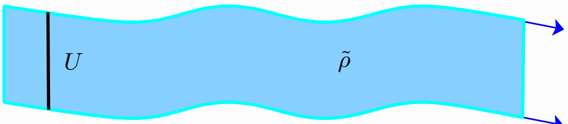
$$\boldsymbol{u} = U\boldsymbol{i} + \hat{\boldsymbol{u}}, \quad (\hat{u}, \hat{v}) = (\psi'(z), -ik\psi(z))\exp(\mathrm{i}k(x - ct))$$

$$(U - c)\left(\psi''(z) - k^2\psi(z)\right) = 0$$

Squire modes: $U(z) = U$, consider flow in air.

Instability possible if $T < 1$, depending on ρ_1/ρ_2.

Planar waves on a fluid sheet — uniform flow: Taylor modes (1959)



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$$\boldsymbol{u} = U\boldsymbol{i} + \hat{\boldsymbol{u}}, \quad (\hat{u}, \hat{v}) = (\psi'(z), -ik\psi(z)) \exp(ik(x - ct))$$

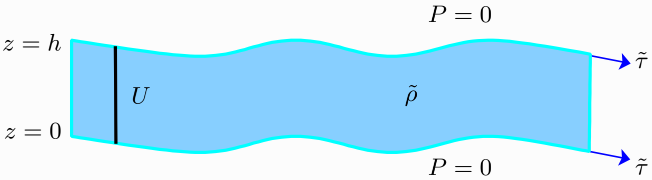
$$(U - c)\left(\psi''(z) - k^2\psi(z)\right) = 0$$

$$-(U - c)^2\psi'(z) \pm \frac{\tilde{\tau}k^2}{\tilde{\rho}}\psi(z) = 0 \quad \text{on} \quad z = 0, h$$

Taylor modes: $U(z) = U$, neglect air.

Neutral waves

Planar waves on a fluid sheet — uniform flow: Taylor modes (1959)

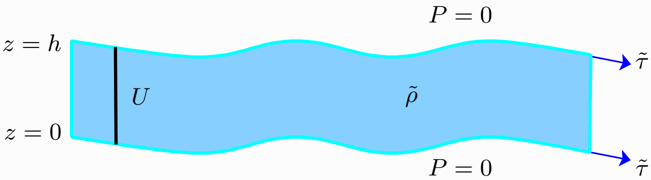


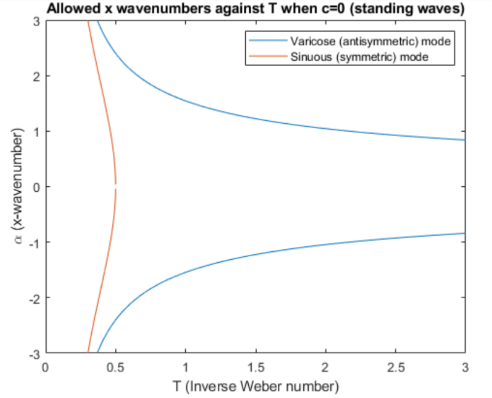
$$(c - U)^2 = \frac{\tilde{\tau}}{\tilde{\rho} h} kh \times \begin{cases} \coth(kh/2) & \text{sinuous modes} \\ \tanh(kh/2) & \text{varicose modes} \end{cases}$$

Anomalous dispersion: Drazin & Reid (1981, chap. 1)

Stability criteria for any $U(z)$: cf. Yih (1972)

Stationary planar waves on a fluid sheet — uniform flow

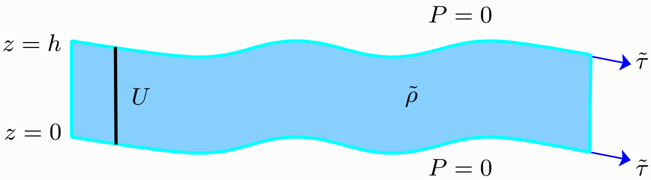


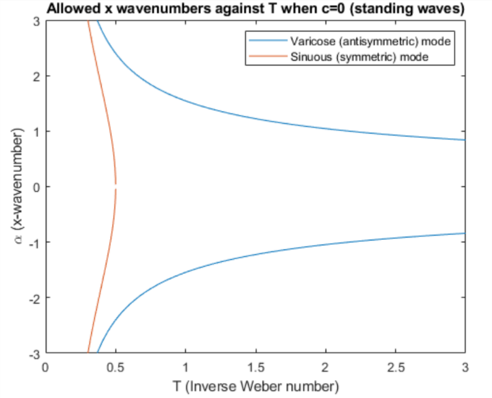


Stationary waves, $c = 0$, $(kh) = (kh)(T)$

$$1 = T(kh) \times \begin{cases} \coth(kh/2) & \text{sinuous modes} \\ \tanh(kh/2) & \text{varicose modes} \end{cases}$$

$$T = \frac{\tilde{\tau}}{U^2 \tilde{\rho} h}$$





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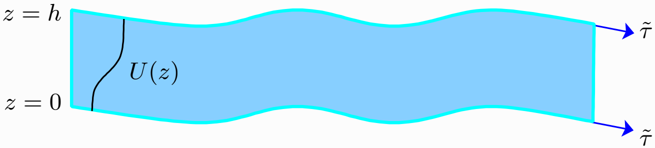
$$T = \frac{\tilde{\tau}}{U^2 \tilde{\rho} h}$$

Long waves: $kh \to 0$

$$\begin{cases} T \to 1/2- & \text{sinuous modes} \\ T \sim 2/(kh)^2 & \text{varicose modes} \end{cases}$$



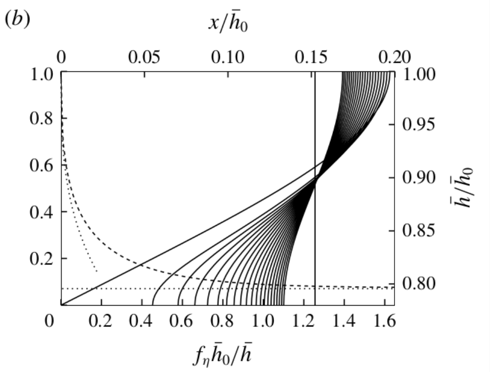
Profile evolution on $O(Re)$-length scale

$$\bar{\psi}_z \bar{\psi}_{zx} - \bar{\psi}_x \bar{\psi}_{zz} = \bar{\psi}_{zzz},$$

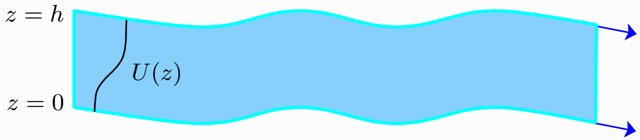
$$z = 0 : \bar{\psi} = \bar{\psi}_{zz} = 0,$$

$$z = \bar{h}(x) : \bar{\psi} - 1 = \bar{\psi}_{zz} = 0.$$

plate edge $x = 0$: Watson's profile (1964)



$f(x, \eta) := \bar{\psi}(x, z), \quad \eta := z/\bar{h}(x), \quad h_0 := h(0)$



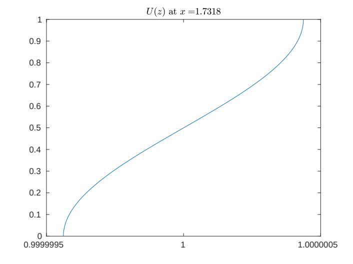
$$(U(z) - c)\left(\psi''(z) - k^2\psi(z)\right) + U''(z)\psi(z) = 0$$

$$(U - c)^2\psi'(z) = \pm\frac{\tilde{\tau}k^2}{\tilde{\rho}}\psi(z) \quad \text{on} \quad z = 0, h$$

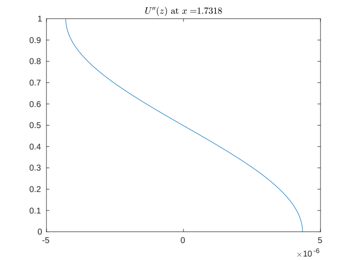
Note: $U'(0) = U'(h) = 0$

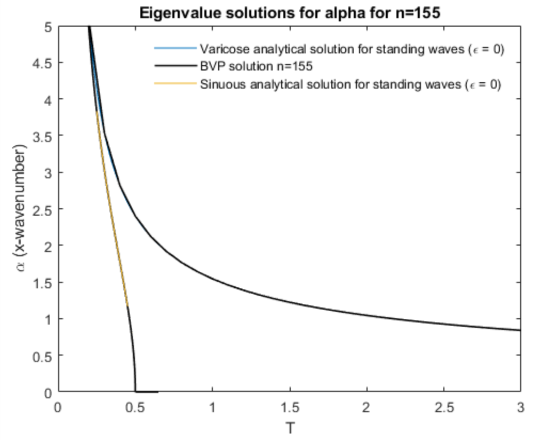
Stationary waves at $x = 1.7318$

$U(z)$



$U''(z)$

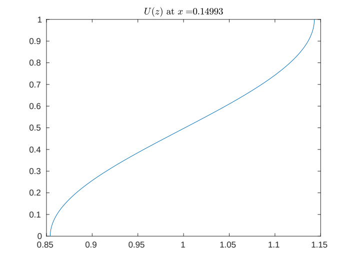




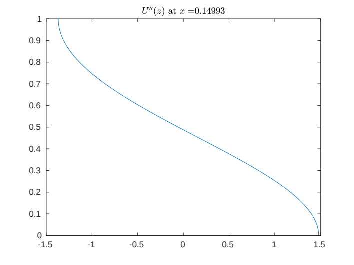
($\epsilon = 0$ is uniform profile)

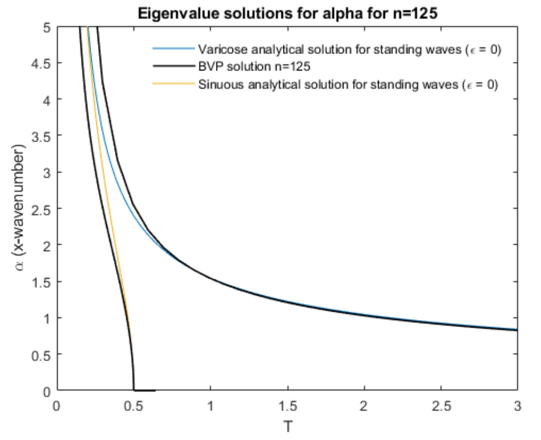
Stationary waves at $x = 0.14993$

$U(z)$



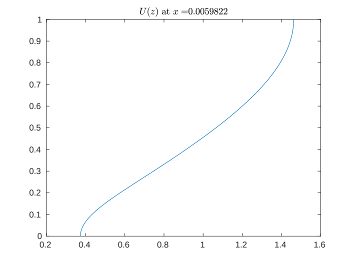
$U''(z)$



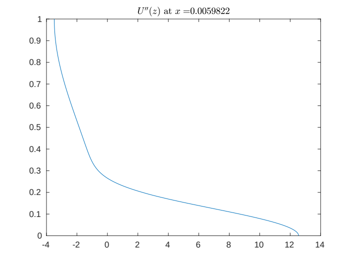


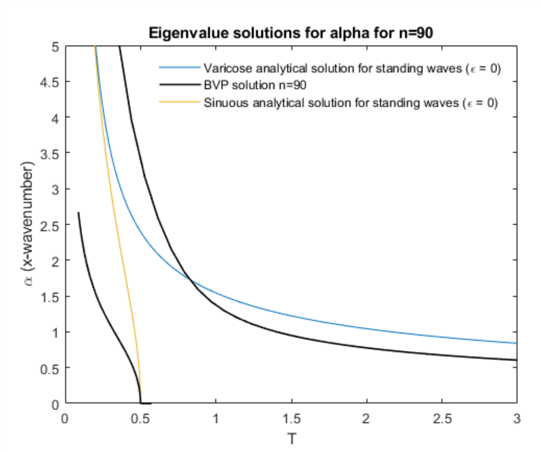
Stationary waves at $x = 0.005982$

$U(z)$



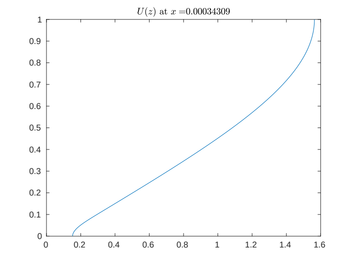
$U''(z)$



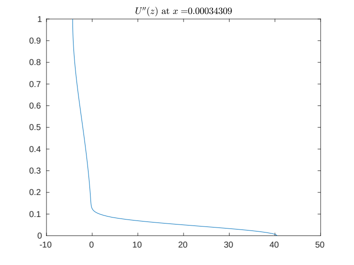


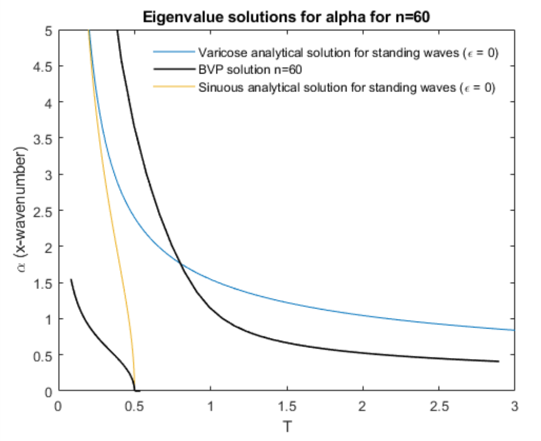
Stationary waves at $x = 0.0003431$

$U(z)$



$U''(z)$





Long-wave analysis of stationary waves

Normalise the Rayleigh problem to:

$$U\left(\psi'' - k^2\psi\right) + U''\psi = 0,$$

$$U^2\psi' = \pm\bar{T}k^2\psi(z) \quad \text{on} \quad z = 0, 1 \qquad \psi(1) = 1, \quad \bar{T} = \frac{\tilde{\tau}}{\tilde{U}^2\tilde{\rho}h}, \quad \tilde{U} = Q/h$$

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If $\psi=U(z)S(z)/U(h)$ then

$$[U^2S']'=k^2U^2S, \quad S(1)=1, \qquad U^2S'=\pm k^2\bar{T}S \quad \text{on} \quad z=0,1$$

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Integrate:

$$\bar{T}(1 + S(0)) = \int_0^1 U^2S\,\mathrm{d}z$$

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$$S(z) = 1 + k^2 \int_1^z \frac{\mathrm{d}t}{U^2(t)} \left[\bar{T} + \int_1^t U^2(v)S(v)\,\mathrm{d}v\right]$$

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$$\bar{T}\left(2 - k^2\int_0^1 \frac{\mathrm{d}t}{U^2(t)}\left[\bar{T} + \int_1^t U^2(v)S(v)\,\mathrm{d}v\right]\right) = \int_0^1 U^2S\,\mathrm{d}z$$

Long-wave analysis of stationary waves

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$$\bar{T}(1 + S(0)) = \int_0^1 U^2 S\,\mathrm{d}t$$

Or:

$$\bar{T}\left(2 - k^2 \int_0^1 \frac{\mathrm{d}t}{U^2(t)} \left[\bar{T} + \int_1^t U^2(v) S(v)\,\mathrm{d}v\right]\right) = \int_0^1 U^2 S\,\mathrm{d}z$$

$k \to 0,\ \bar{T} = O(1)$ — sinuous mode

$$S(z) = 1 + k^2 \int_1^z \frac{\mathrm{d}t}{U^2(t)} \left[\bar{T} + \int_1^t U^2(v) S(v)\,\mathrm{d}v \right]$$

$$\bar{T} \left(2 - k^2 \int_0^1 \frac{\mathrm{d}t}{U^2(t)} \left[\bar{T} + \int_1^t U^2(v) S(v)\,\mathrm{d}v \right] \right) = \int_0^1 U^2 S\,\mathrm{d}z$$

$S = 1 + \cdots, \quad 2\bar{T} = J = \int_0^1 U^2\,\mathrm{d}z,$ or $\dfrac{\tilde{\tau}}{\tilde{\rho}\int_0^{\tilde{h}} \tilde{u}^2\,\mathrm{d}\tilde{z}} = \dfrac{\tilde{\tau}}{\tilde{J}} = T = \dfrac{1}{2}$

$k \to 0,\ \bar{T} = \hat{T}/k^2$ — varicose mode

$$S(z) = 1 + k^2 \int_1^z \frac{\mathrm{d}t}{U^2(t)} \left[\frac{\hat{T}}{k^2} + \int_1^t U^2(v) S(v)\,\mathrm{d}v \right]$$

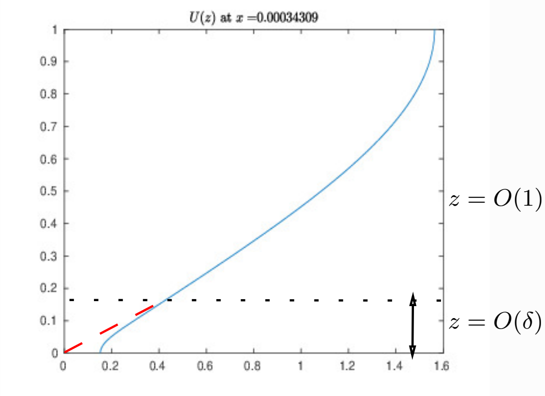
$$\frac{\hat{T}}{k^2} \left(2 - k^2 \int_0^1 \frac{\mathrm{d}t}{U^2(t)} \left[\frac{\hat{T}}{k^2} + \int_1^t U^2(v) S(v)\,\mathrm{d}v \right] \right) = \int_0^1 U^2 S\,\mathrm{d}z$$

$$\hat{T} \left(2 - \hat{T} \int_0^1 \frac{\mathrm{d}z}{U^2} \right) = k^2 \left[\int_0^1 U^2 S\,\mathrm{d}z + \hat{T} \int_0^1 \frac{\mathrm{d}z}{U^2(z)} \int_1^z U^2(t) S(t)\,\mathrm{d}t \right]$$

$$S = 1 + \hat{T} \int_1^z \frac{\mathrm{d}t}{U^2(t)} + \cdots, \qquad \hat{T} = \frac{2}{\int_0^1 U^{-2}\,\mathrm{d}z},$$

note: $S(0) = -1$

Long waves: $U(0) \ll 1$



$$U = \delta\, U_\delta(\eta), \quad \eta = z/\delta$$

$$k^2 \int_0^1 \frac{\mathrm{d}z}{U^2(z)} \sim \frac{k^2}{\delta} \int_0^\infty \frac{\mathrm{d}\eta}{U_\delta^2(\eta)} = k^{*2} I_0$$

$$I_0 = \int_0^\infty \frac{\mathrm{d}\eta}{U_\delta^2(\eta)}$$

$\delta = k^2/k^{*2} \ll 1, \quad k^* = O(1)$

Long waves: $U(0) \ll 1$, cf. interaction theory

$$S(z) = 1 + k^2 \int_1^z \frac{\mathrm{d}t}{U^2(t)} \left[\bar{T} + \int_1^t U^2(v) S(v)\, \mathrm{d}v\right]$$

$$\bar{T}(1 + S(0)) = \int_0^1 U^2 S\, \mathrm{d}t = J_S$$

$z = O(1)$: $\quad S = 1 + O(k^2), \quad J_S \sim J_1 = J$

So: $\quad \bar{T}(1 + S(0)) = J_S \sim J$

$z = O(\delta)$, put $\eta = 0$: $\quad S(0) = 1 + k^{*2}\left(-I_0\left\{\bar{T} - J_S\right\}\right)$

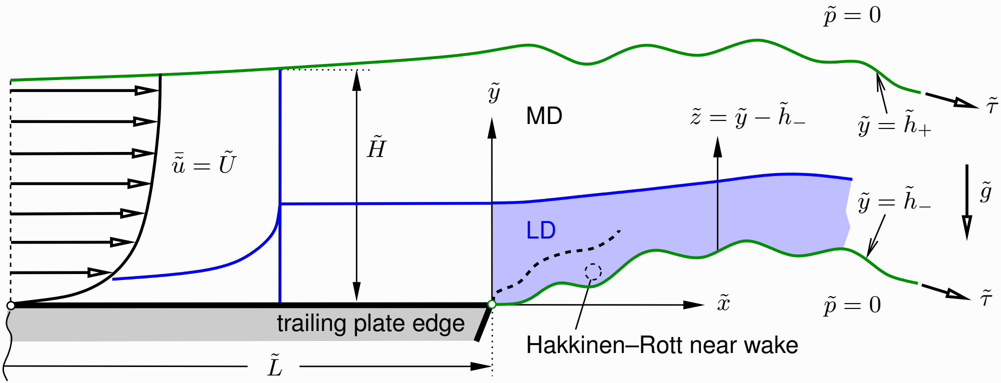
So: $\quad \bar{T}[2 - k^{*2} I_0(\bar{T} - J)] = J$

$$1 = \sigma k^{*2} I_0 J |T - 1|$$

$$\sigma = \frac{\mathrm{sgn}(T-1)T}{2T-1}, \quad T = \bar{T}/J, \quad I_0 = \int_0^\infty \frac{\mathrm{d}\eta}{U_\delta^2(\eta)}$$

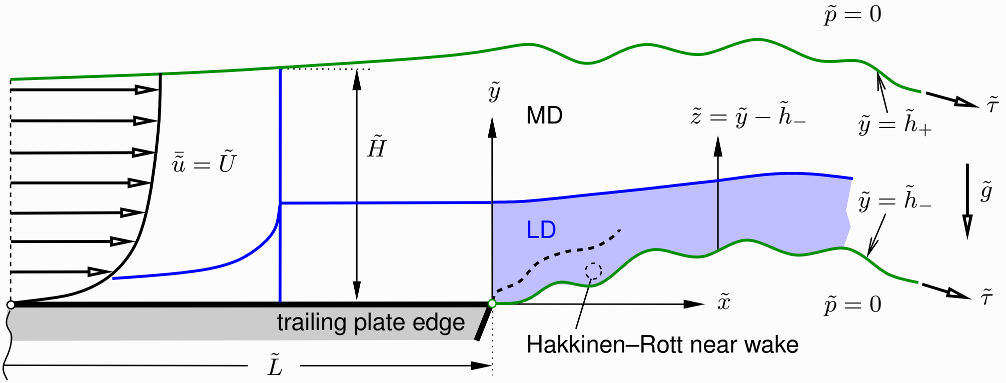
Interaction theory

Motivation: asymptotic theory of developed (real) flow negotiating trailing edge, classical 2D steady supercritical overfall of liquid layer



Topic: interactive LD limit downstream of edge

Basic scalings: high-Re shear-layer balance



Incident base flow: $\tilde{Q}$, adjustment length $\tilde{L}$

$$\tilde{Q} = \tilde{U}\tilde{H}, \quad \tilde{U}^2/\tilde{L} = \tilde{\nu}\tilde{U}/\tilde{H}^2 \quad \Rightarrow \quad \tilde{H}/\tilde{L} = \tilde{\nu}/\tilde{Q} = Re^{-1} \to 0$$

Interactive LD scales: $x = \tilde{x}/\tilde{L} = O(Re^{-6/7}), \;\; z = \tilde{z}/\tilde{H} = O(Re^{-2/7})$

Watson's base flow above edge

$$\frac{\tilde{u}}{\tilde{U}} \sim u_0(z), \quad \frac{\tilde{h}_+}{\tilde{H}} \sim h_0 = \frac{\pi}{\sqrt{3}}, \quad \lambda_0 = u_0''(0) \approx 0.6930, \quad J_0 = \int_0^{h_0} u_0^2(z)\,\mathrm{d}z = \lambda_0$$

Least-degenerate interactive limit: 2 control groups

$$T = \frac{\tilde{\tau}}{\tilde{\rho}\tilde{U}^2\tilde{H}J_0} = O(1), \quad G = \frac{\tilde{g}\tilde{H}}{\tilde{U}^2}\frac{h_0}{(\lambda_0^6\,\epsilon^4)^{1/7}} = O(1)$$

$$\epsilon = (|T-1|J)^{1/2}/Re \to 0$$

LD, leading order

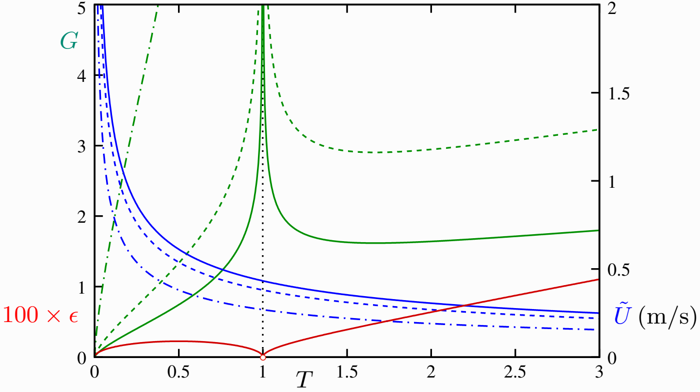
$$X = (\lambda_0^5/\epsilon^6)^{1/7}x, \quad Z = (\lambda_0^4/\epsilon^2)^{1/7}z$$

$$\frac{\tilde{\psi}}{\tilde{Q}} \sim \frac{\epsilon^{4/7}}{\lambda_0^{1/7}}\Psi(X,Z), \quad \frac{\tilde{p}}{\tilde{\rho}\tilde{U}^2} \sim \frac{\epsilon^{4/7}}{\lambda_0^{6/7}}P(X), \quad \left(\frac{\tilde{h}_-}{\tilde{H}}, \frac{\tilde{h}_+}{\tilde{H}} - h_0\right) \sim \frac{\epsilon^{2/7}}{\lambda_0^{4/7}}[H_-(X), H_+(X)]$$

Flow over edge in lab (H_2O, standard conditions): vary $\tilde{H}_0 = h_0 \tilde{H}$, $\tilde{U}$

$$\tilde{H}_r = \left(\frac{h_0^9 \, \lambda_0^6}{J_0^3} \, \frac{\tilde{\nu}^4 \, \tilde{\tau}^5}{\tilde{g}^7 \, \tilde{\rho}^5} \right)^{1/16} \approx 0.774 \, \mathrm{mm} \ (!), \quad \tilde{U}_r = \left(\frac{h_0^7}{\lambda_0^6 \, J_0^{13}} \, \frac{\tilde{g}^7 \, \tilde{\tau}^{11}}{\tilde{\nu}^4 \, \tilde{\rho}^{11}} \right)^{1/32} \approx 0.433 \, \frac{\mathrm{m}}{\mathrm{s}}$$

$$G = \left(\frac{\tilde{H}_0}{\tilde{H}_r} \right)^{16/7} \frac{T^{5/7}}{|T-1|^{2/7}}, \quad \frac{\tilde{U}}{\tilde{U}_r} = \left(\frac{\tilde{H}_r}{\tilde{H}_0} \, \frac{1}{T} \right)^{1/2}$$



$T = 1$:
interaction invalid, choking

$\tilde{H}_0 = \tilde{H}_r$ (—)

$\tilde{H}_0 = 1 \, \mathrm{mm}$ (– – –)

$\tilde{H}_0 = 2 \, \mathrm{mm}$ (–·–·–)

Interaction problem: $X \geq 0$, jet-type P/A law

$$\Psi_Z \Psi_{ZX} - \Psi_X \Psi_{ZZ} = -P'(X) + \Psi_{ZZZ}$$

$$X > 0, \quad Z = 0: \quad \Psi = \Psi_{ZZ} = 0$$

$$\Psi_{ZZ}(X, \infty) = 1, \quad A(X) = \lim_{z \to \infty}(\Psi_Z - Z)$$

$$P = C(G + SA''), \quad C = T/(2T - 1), \quad S = \mathrm{sgn}(T - 1)$$

$$\Psi(0+, Z) = \Psi(0-, Z), \quad A'(0+) = A'(0-), \quad A''(0+) = -SG \;\Leftarrow\; P(0) = 0$$

Classifcation: streamline curvature vs. capillarity

$$P' = \sigma A''', \quad \sigma = SC \begin{cases} > 0 & (0 < T < 1/2 \text{ or } T > 1) \\ < 0 & (1/2 < T < 1) \\ = \mp\infty & (T = 1/2\pm) \\ = \pm 1 & (T = 1\pm) \\ = 0 & (T = 1) \end{cases}$$

- > 0 $(0 < T < 1/2$ or $T > 1)$ ... stabilising feedback: waves
- < 0 $(1/2 < T < 1)$ ... compressive/expansive
- $= \mp\infty$ $(T = 1/2\pm)$ ... choking (cf. linear waves)
- $= \pm 1$ $(T = 1\pm)$ ... regular limits
- $= 0$ $(T = 1)$ ... choking (excluded)

Elevations of free streamlines

Laplace pressure & interaction law

$$P = TH''_{-}/|T-1| = C(G + SA''), \quad C = T/(2T-1), \quad S = \mathrm{sgn}(T-1)$$

No slip to free slip: small-scale (Navier–Stokes) analysis

$$X = 0+: \quad H_{-} = H'_{-} = H''_{-} = 0 \quad \Rightarrow \quad P(0) = 0$$

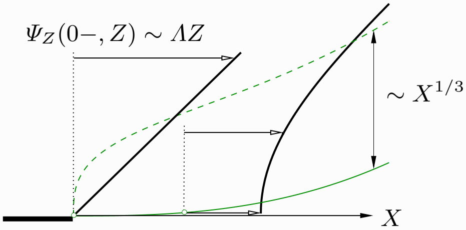
Thus,

$$H_{\pm}(X) = \underbrace{(1-C)[SGX^2/2 - A'(0)X - A(0)]}_{\text{parabola:}} - \underbrace{\begin{cases} CA(X) \\ (C-1)A(X) \end{cases}}$$

$0 < T < 1/2$:	classical downfall,	$H_{\pm}$ wavy, in phase	$(X \gg 1)$
$1 < T$:	'upfall',	$H_{\pm}$ wavy, in antiphase	

Well-posedness

Hakkinen–Rott near wake



$$0 < X \ll \big(T/|2T-1|\big)^{3/7}, \quad \eta = Z(\Lambda/X)^{1/3}:$$

$$[\Psi, P] \sim X^{2/3}[\Lambda^{1/3}F(\eta), \Lambda^{4/3}\Pi], \quad \Pi \approx 0.61334$$

$$\eta \to \infty: \quad F' = \eta + \mathrm{EST}$$

grants

$$A''(X) + SG = O(X^{2/3}), \quad H''_- = \frac{|1-T|}{T}\,P$$

Downstream marching well-posed up to flow reversal

F not perturbed by eigensolutions (Scheichl 2023)

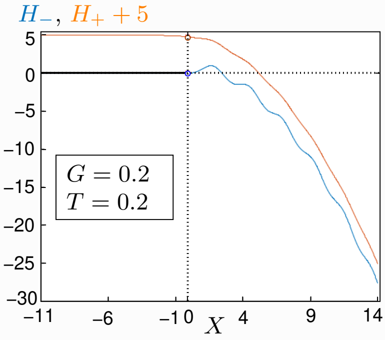
Condensed interaction

$$G = 0, \; T = 1/2: \quad [\Psi, P] \equiv X^{2/3}[\Lambda^{1/3}F(\eta), \Lambda^{4/3}\Pi]$$

Numerical results (G. Pasias)

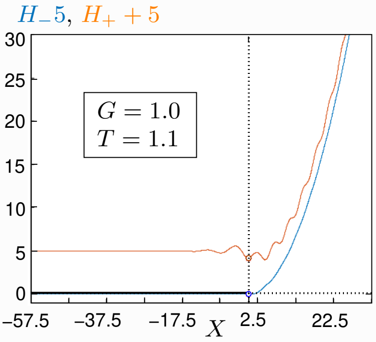
Sinuous / supercritical:

no waves upstream



Varicose / subcritical:

waves upstream

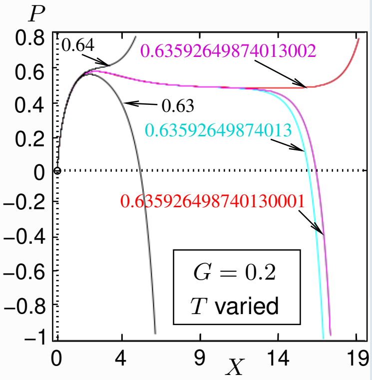


Bifurcation:

compressive flow reversal

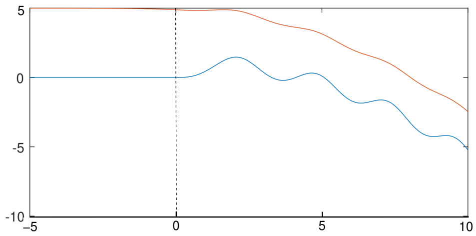
vs.

expansive blow-up



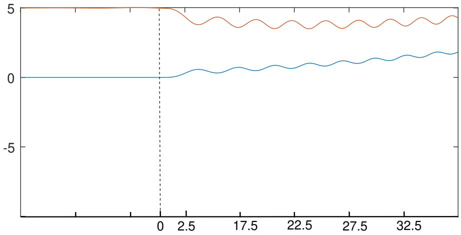
Numerical results (G. Pasias)

Sinuous / ‘flapping’



$G = 0.1,\ T = 0.2$

Varicose / ‘sausage-type’



$G = 0.01,\ T = 2.0$

Choking of a capillary wave & non-wavy breakdown: $T \sim 1/2$

$$P = \frac{T}{2T-1}\left[G + \mathrm{sgn}(T-1)A''\right]$$

Least-degenerate distinguished limit near condensed interaction

$$\hat{G} = \alpha^5 G = O(1), \quad \alpha = (4|T-1/2|)^{-1/7} \to \infty$$

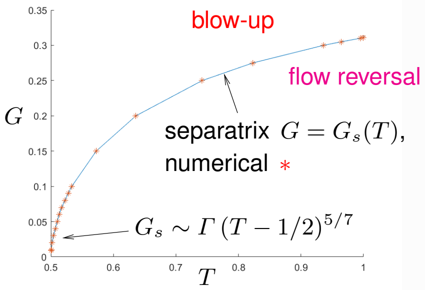
$$[X, Z, \Psi, A, P] \sim \left[\alpha^3 \hat{X}, \alpha\hat{Z}, \alpha^2\hat{\Psi}, \alpha\hat{A}, \alpha^2\hat{P}\right]$$

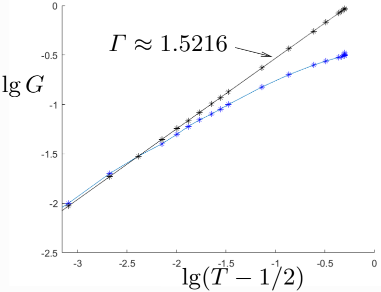
$$\hat{S}\hat{P} = \hat{G} - \hat{A}'', \quad \hat{S} = \mathrm{sgn}(T-1/2)$$

$$\hat{\Psi}(0,\hat{Z}) = \hat{Z}^2/2, \quad \hat{A}'(0) = 0, \quad \hat{A}''(0) = \hat{G}$$

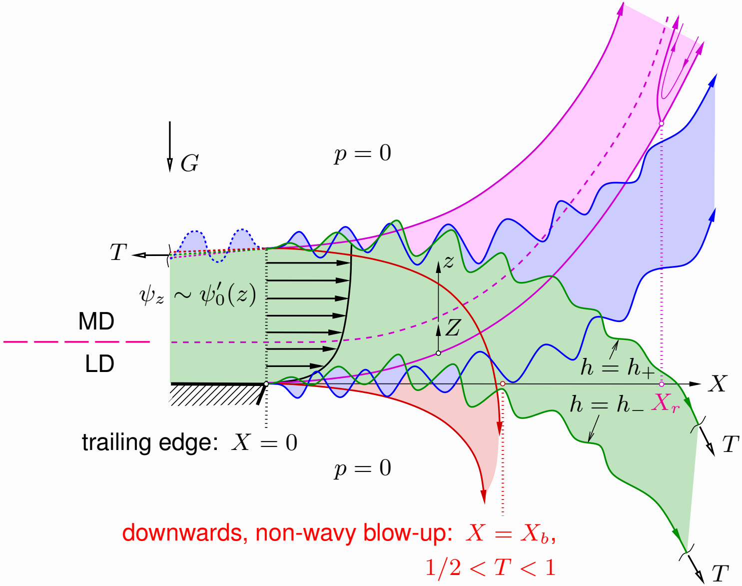
$\hat{S} = -1$: (cnoidal) waves $(\hat{G} \to \infty)$

$\hat{S} = +1$: $4^{5/7}\hat{G}$
- $< \Gamma$: flow reversal $(\hat{X} \to \infty)$
- $= \Gamma$: Goldstein wake $(\hat{X} \to \infty)$
- $> \Gamma$: finite-$\hat{X}$ blow-up





Condensed results: 4 fundamental detached-jet manifestations



upwards, non-wavy
reversed-flow breakdown:
$1/2 < T < 1$

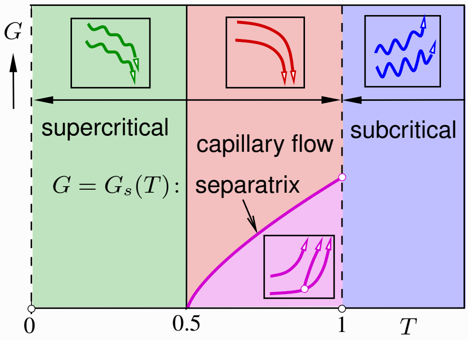
upwards, symmetric (varicose)
modes, phase shift indefinite:
$1 < T$

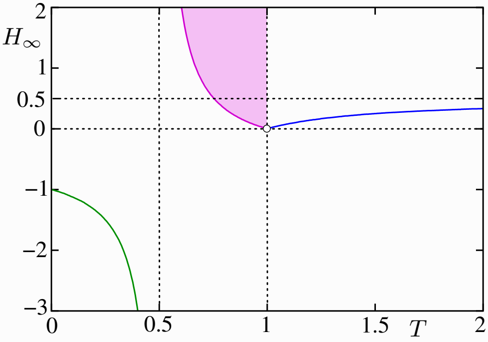
cf. inviscid slender jets
(Keller & Weitz 1957)

downwards, non-wavy blow-up: $X = X_b$,
$1/2 < T < 1$

downwards,
antisymmetric (sinuous) modes,
self-consistent for all X:
$0 < T < 1/2$

Condensed results: 3 manifestations valid for $X < \infty$

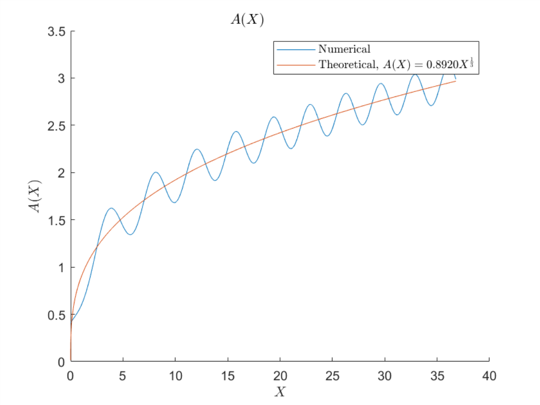


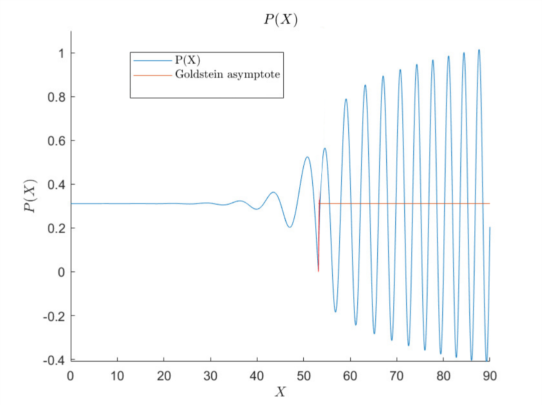


$$A \sim \begin{cases} a_G X^{1/3} & \to +\infty \quad (0 < T < 1/2, \ \ T > 1 \ \text{ or } \ 1/2 < T < 1, \ \ G = G_s) \\ A_\infty X^2/2 & \to -\infty \quad (1/2 < T < 1, \ \ G < G_s) \end{cases}$$

$$H_- \sim H_\infty \frac{GX^2}{2}, \quad H_\infty = \frac{|T-1|}{2T-1} \times \begin{cases} 1 & (0 < T < 1/2, \ \ T > 1) \\ 1 & (1/2 < T < 1, \ \ G = G_s) \\ (1 - A_\infty/G) & (1/2 < T < 1, \ \ G < G_s) \end{cases}$$

WKBJ analysis





WKBJ analysis: $\Psi = X^{1/3}F(X,\eta)$, $\eta = Z/X^{1/3}$, $X \to \infty$

Interaction problem

$$\underbrace{(F_\eta^2 - 2FF_{\eta\eta})/3 - F_{\eta\eta\eta}}_{\text{non-}||,\ \text{viscosity}} + \underbrace{X(F_\eta F_{\eta X} - F_X F_{\eta\eta})}_{\text{`rapid' convection}} = -X^{1/3}P'(X)$$

$$P'(X) = \sigma A'''(X), \quad A(X) = X^{1/3}a(X)$$

$$\eta = 0: \quad F = F_{\eta\eta} = 0, \quad \eta \to \infty: \quad F_\eta = \eta + a + \mathrm{EST}$$

Translational invariance absorbed

$$X \mapsto X - X_0, \quad \eta \mapsto Z/(X - X_0)^{1/3}$$

Global conservation of momentum

$$F = \underbrace{F_m(\eta)}_{\text{mean flow}} + \underbrace{f(X,\eta)}_{\text{`rapid' convection}}, \quad f \to 0$$

WKBJ analysis: $X \to \infty$

Asymptotic hierarchy: algebraic-log decay

$$k(X) = (6k_0/7)X^{1/6} + k_1 X^{\kappa} + o(1) \quad (k_0 > 0,\ \ 1/6 > \kappa > -1), \quad E = \exp[\mathrm{i}k(X)X]$$

$$\{f,\, a - a_m\} = E^0 \Big[\underbrace{X^{2\mu+7/6}\, b_{00}\{f_{00}(\eta),\, 1\}}_{\text{by 'Reynolds stress'}} + \cdots \Big]$$

$$+ E^1 X^{\mu} \Big[b_{10}\{ \underbrace{f_{10}(\eta)}_{\text{eigenfunction}},\, 1\} + \underbrace{X^{\beta}\, b_{11}\{f_{11}(\eta),\, 1\} + X^{-7/6}\, b_{12}\{f_{12}(\eta),\, 1\} + \cdots}_{\text{enforced}} \Big]$$

$$+ E^2 X^{2\mu} \big[b_{20}\{f_{20}(\eta),\, 1\} + \cdots \big] + O(E^3 X^{3\mu}) + c.c. \quad (\mu,\, \beta < 0)$$

$$X^{1/3} P'/\sigma = \cdots - \mathrm{i}k_0^3 E^1 X^{\mu+7/6} \big[b_{10} + b_{11} X^{\beta} + (3b_{10}k_1/k_0)(\kappa+1)X^{\kappa-1/6} + \cdots \big] + \cdots + c.c.$$

RC: $X\partial_X \sim XE'/E \sim \mathrm{i}k_0 X^{7/6} \quad \square$

WKBJ analysis: $X \to \infty$

$$k(X) = (6k_0/7)X^{1/6} + k_1 X^{\kappa} + o(1) \quad (k_0 > 0,\ \ 1/6 > \kappa > -1), \quad E = \exp[\mathrm{i}k(X)X]$$

$$\{f,\, a - a_m\} = E^0\big[\underbrace{X^{2\mu+7/6}\, b_{00}\{f_{00}(\eta),\, 1\}}_{\text{by `Reynolds stress'}} + \cdots\big]$$

$$+\, E^1 X^{\mu}\big[b_{10}\{\underbrace{f_{10}(\eta)}_{\text{eigenfunction}},\, 1\} + \underbrace{X^{\beta}\, b_{11}\{f_{11}(\eta),\, 1\} + X^{-7/6}\, b_{12}\{f_{12}(\eta),\, 1\} + \cdots}_{\text{enforced}}\big]$$

$$+\, E^2 X^{2\mu}\big[b_{20}\{f_{20}(\eta),\, 1\} + \cdots\big] + O(E^3 X^{3\mu}) + c.c. \quad (\mu,\, \beta < 0)$$

$$X^{1/3}P'/\sigma = \cdots - \mathrm{i}k_0^3 E^1 X^{\mu+7/6}\big[b_{10} + b_{11}X^{\beta} + (3b_{10}k_1/k_0)(\kappa+1)X^{\kappa-1/6} + \cdots\big] + \cdots + c.c.$$

Upstream history fixes amplitudes $b_{jl} \in \mathbb{C}$

We seek $F_m(\eta),\ f_{10}(\eta),\ k_0,\ \mu$

WKBJ analysis: secularity conditions $f'_{jl}(\infty) = 1$

P', k', nonlinear, non-$||$, viscosity $\Rightarrow$ forcing $I_{jl}(\eta)$ of j-th mode, $I_{10} = 0$

- $j > 0,\ l \geq 0$

$$F'_m f'_{jl} - F''_m f_{jl} - \sigma(jk_0)^2 = I_{jl}, \quad f_{jl}(0) = 0 \quad \Rightarrow \quad \frac{f_{jl}(\eta)}{F'_m(\eta)} = \int_0^\eta \frac{\sigma(jk_0)^2 + I_{jl}(t)}{F'^2_m(t)}\,\mathrm{d}t$$

$$1 - \sigma(T)(jk_0)^2 I_0 = \int_0^\infty \frac{I_{jl}(\eta)}{F'^2_m(\eta)}\,\mathrm{d}\eta, \quad I_0 = \int_0^\infty \frac{\mathrm{d}\eta}{F'^2_m(\eta)}$$

WKBJ analysis: $f'_{jl}(\infty) = 1$

P', k', nonlinear, non-$||$, viscosity $\Rightarrow$ forcing $I_{jl}(\eta)$ of j-th mode, $I_{10} = 0$

- $j > 0,\ l \geq 0$

  $$F'_m f'_{jl} - F''_m f_{jl} - \sigma(jk_0)^2 = I_{jl}, \quad f_{jl}(0) = 0 \quad \Rightarrow \quad \frac{f_{jl}(\eta)}{F'_m(\eta)} = \int_0^\eta \frac{\sigma(jk_0)^2 + I_{jl}(t)}{F'^2_m(t)}\,\mathrm{d}t$$

  $$1 - \sigma(T)(jk_0)^2 I_0 = \int_0^\infty \frac{I_{jl}(\eta)}{F'^2_m(\eta)}\,\mathrm{d}\eta, \quad I_0 = \int_0^\infty \frac{\mathrm{d}\eta}{F'^2_m(\eta)}$$

- $j = 1,\ l = 0$

  $$1 = \sigma(T)k_0^2 I_0, \quad \mathrm{Im}\, f_{10} = 0$$

WKBJ analysis: $f'_{jl}(\infty) = 1$

P', k', nonlinear, non-$||$, viscosity $\quad\Rightarrow\quad$ forcing $I_{jl}(\eta)$ of j-th mode, $\quad I_{10} = 0$

- $j > 0,\ l \geq 0$

  $$F'_m f'_{jl} - F''_m f_{jl} - \sigma(jk_0)^2 = I_{jl},\quad f_{jl}(0) = 0 \quad\Rightarrow\quad \frac{f_{jl}(\eta)}{F'_m(\eta)} = \int_0^\eta \frac{\sigma(jk_0)^2 + I_{jl}(t)}{F'^2_m(t)}\,\mathrm{d}t$$

  $$1 - \sigma(T)(jk_0)^2 I_0 = \int_0^\infty \frac{I_{jl}(\eta)}{F'^2_m(\eta)}\,\mathrm{d}\eta,\quad I_0 = \int_0^\infty \frac{\mathrm{d}\eta}{F'^2_m(\eta)}$$

- $j = 1,\ l = 0$

  $$1 = \sigma(T)k_0^2 I_0,\quad \operatorname{Im} f_{10} = 0$$

- $j = l = 0$

  $$(F'^2_m - 2F_m F''_m)/3 - F'''_m \sim \underbrace{2k_0|b_{10}|^2 X^{2\mu+7/6}\operatorname{Im}\big(\overline{f_{10}}f''_{10}\big)}_{\text{dominant Reynolds stress}} = 0\ \forall\,\mu$$

$\Rightarrow\quad b_{00} = 0,\quad U_\delta = F'_m = G'\ \ldots$ Goldstein wake, $a_m \approx 0.89200$

WKBJ analysis: $f'_{jl}(\infty) = 1$

- $j = 0,\ l > 0$: $E^0 X^{\lambda(l)}$ (non-interactive)

  $$(2/3 + \lambda)(G' f'_{0l} - G'' f_{0l}) - 2G f''_{0l}/3 - f'''_{0l} = I_{0l}, \quad f_{0l}(0) = f''_{0l}(0) = 0$$

  $$\eta \to \infty: \quad f'_{0l} = 1 + \mathrm{EST}$$

- $j = 1,\ l = 0$: $E^1 X^{\mu}$

  $$G' f'_{10} - G'' f_{10} - \sigma k_0^2 = 0$$

  $$1 = \sigma(T) k_0^2 I_0, \quad I_0 = \int_0^\infty \frac{\mathrm{d}\eta}{G'^2(\eta)} \approx 0.8525 \quad \Rightarrow \quad k_0(T), \quad \sigma > 0 \quad \square$$

- else

  $$G' f'_{jl} - G'' f_{jl} - \sigma (j k_0)^2 = I_{jl}$$

  $$1 - j^2 = \int_0^\infty \frac{I_{jl}(\eta)}{G'^2(\eta)}\, \mathrm{d}\eta$$

WKBJ analysis: $1 - j^2 = \int_0^\infty (I_{jl}/G'^2)(\eta)\,\mathrm{d}\eta$ (SC)

- $j = 2,\ l = 0:\ \ E^2 X^{2\mu}$

$$I_{20} = (f_{10}f_{10}'' - f_{10}'^2)\,\frac{b_{01}^2}{2\,b_{20}} \quad \Rightarrow \quad \frac{b_{20}}{b_{10}^2} = \frac{1}{6}\int_0^\infty \frac{f_{10}'^2 - f_{10}f_{10}''}{G'^2}\,\mathrm{d}\eta, \quad \mathrm{Im}\, f_{20} = 0$$

- $j = 1,\ l = 1:\ \ E^1 X^{\mu+\beta}$

$$\mathrm{Re}(b_{11}I_{11}/b_{10}) = k_0k_1\underbrace{[3\sigma - (G'f_{10}' - G''f_{10})/k_0^2]}_{2\sigma}(\kappa+1)\,X^{\kappa-1/6-\beta}$$

$$- \underbrace{(b_{20}/b_{10}^2)|b_{10}|^2(f_{20}'f_{10}' - 2f_{20}f_{10}'' + f_{10}f_{20}'')}_{\text{nonlinear feedback}}X^{2\mu-\beta}$$

$$\mathrm{Im}(b_{11}I_{11}/b_{10}) = O(X^{-7/6-\beta})$$

$$\Rightarrow \quad \beta = 2\mu = \kappa - 1/6 > -7/6 \quad \square$$

WKBJ analysis: $1 - j^2 = \int_0^\infty (I_{jl}/G'^2)(\eta)\,\mathrm{d}\eta$ (SC)

- $j = 2,\ l = 0:\ E^2 X^{2\mu}$

$$I_{20} = (f_{10} f_{10}'' - f_{10}'^2)\,\frac{b_{01}^2}{2\,b_{20}} \quad \Rightarrow \quad \frac{b_{20}}{b_{10}^2} = \frac{1}{6}\int_0^\infty \frac{f_{10}'^2 - f_{10} f_{10}''}{G'^2}\,\mathrm{d}\eta, \quad \mathrm{Im}\, f_{20} = 0$$

- $j = 1,\ l = 1:\ E^1 X^{\mu+\beta}$

$$\mathrm{Re}(b_{11} I_{11}/b_{10}) = 2\sigma k_0 k_1 (\kappa + 1) - (b_{20}/b_{10}^2)|b_{10}|^2 (f_{20}' f_{10}' - 2 f_{20} f_{10}'' + f_{10} f_{20}'')$$

$$\mathrm{Im}(b_{11} I_{11}/b_{10}) \equiv 0$$

$$\beta = 2\mu = \kappa - 1/6 > -7/6,$$

$$\text{(SC)} \quad \Rightarrow \quad k_1 \propto |b_{10}|^2$$

WKBJ analysis: $1 - j^2 = \int_0^\infty (I_{jl}/G'^2)(\eta)\,\mathrm{d}\eta$ (SC)

- $j = 1,\ l = 2:\ \ E^1 X^{\mu - 7/6}$

$$\mathrm{Im}(b_{12} I_{12}/b_{10}) = \underbrace{\left[(2/3 + \mu)\overbrace{(G' f_{10}' - G'' f_{10})}^{\sigma k_0^2} - 2/3\, G f_{10}'' - f_{10}'''\right]}_{\text{non-||, viscosity}} / k_0 - \sigma k_0 (3\mu + 3/2)$$

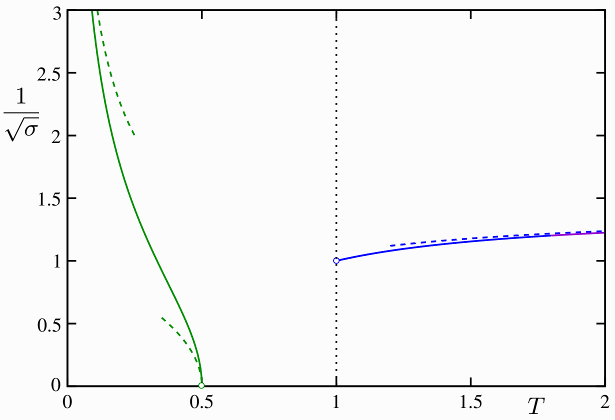
$$= -(2/3\, G f_{10}'' + f_{10}''')/k_0 - \sigma k_0 (5/6 + 2\mu)$$

(SC) $\Rightarrow\ \ \mu = -\dfrac{5}{12} - \dfrac{\Omega}{2} > -\dfrac{7}{12}, \quad \Omega = \displaystyle\int_0^\infty \frac{2/3\, G f_{10}'' + f_{10}'''}{G'^2}\,\mathrm{d}\eta \approx 0.1074 \quad \square$

$$\beta = -\frac{5}{6} - \Omega, \quad \kappa = -\frac{2}{3} - \Omega$$

WKBJ analysis: recovers linear long-wave limit

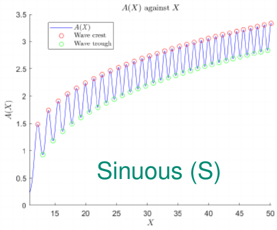
$$k_0 = \frac{1}{\sqrt{I_0\,\sigma(T)}}, \qquad I_0 = \int_0^\infty \frac{\mathrm{d}\eta}{G'^2(\eta)}$$



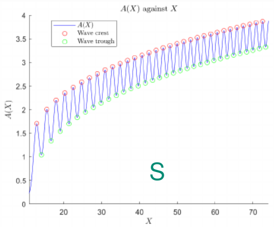
○ choking

WKBJ analysis (G. Pasias)

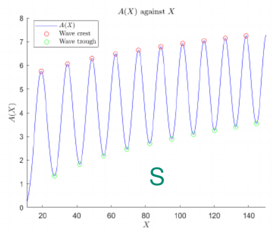
Numerical trends confirm $b_{10} \to \infty,\ k_0 \to 0$ as $T \to 1/2-$ (choking)



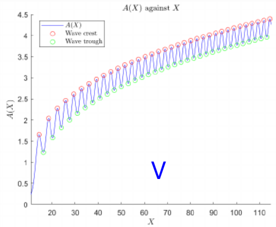
(a) $G = 0.2,\ T = 0.1$



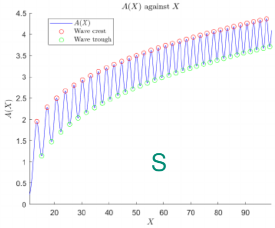
(b) $G = 0.2,\ T = 0.2$



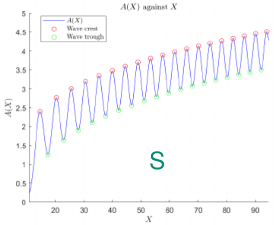
(e) $G = 0.2,\ T = 0.49$



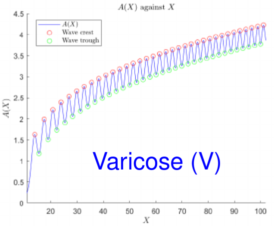
(f) $G = 0.2,\ T = 1.1$



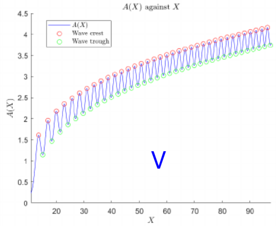
(c) $G = 0.2,\ T = 0.3$



(d) $G = 0.2,\ T = 0.4$



(g) $G = 0.2,\ T = 2$



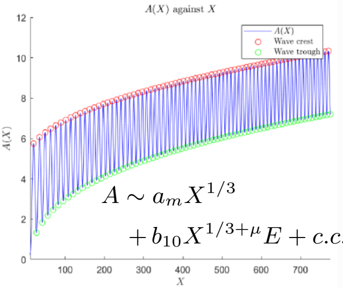
(h) $G = 0.2,\ T = 5$

WKBJ analysis (G. Pasias)

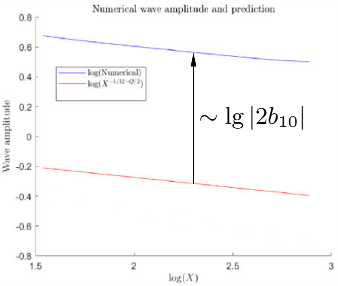
$\mu = -5/12 - \Omega/2, \quad \Omega \approx 0.1074$

$G = 0.2,$

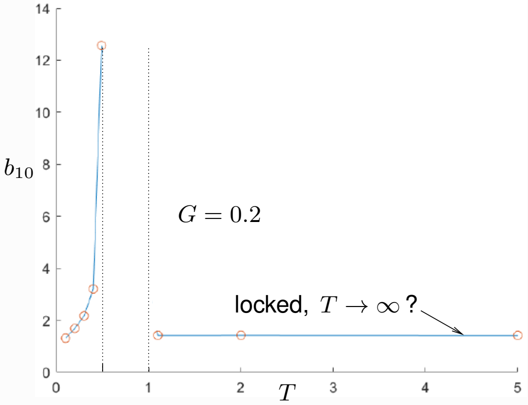
$T = 0.49$



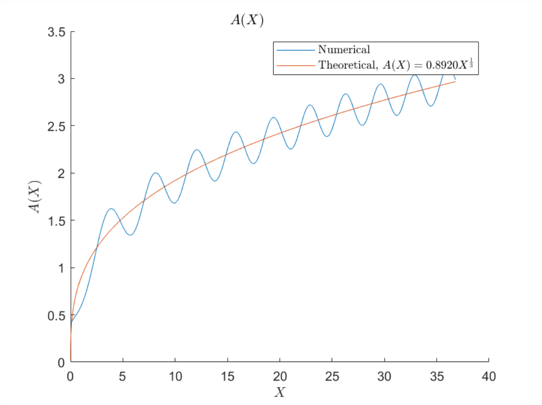
✓

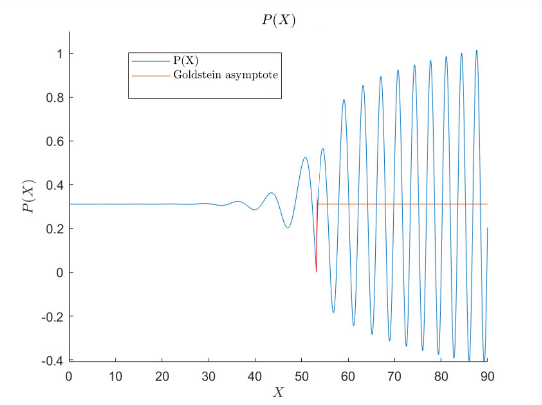


WKBJ don't unveal $b_{10}(G, T)\ldots$



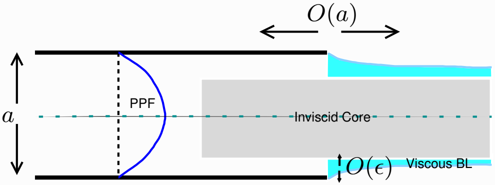
WKBJ analysis (G. Pasias)





$$X \to \infty: \quad A \sim a_m X^{1/3} + b_{10} X^{-1/12-\Omega/2} E(X) + c.c., \quad P \sim -(b_{10}/I_0) X^{1/4-\Omega/2} E(X) + c.c. \,!$$

Flow through channel exit $(G = 0,$ S. Harris)



$$\epsilon = Re^{-1/3}, \quad Re = U_{\text{max}}\, a/\nu$$

No surface tension: Tillett (1968)

Flow symmetric, gravity in flow direction.

$$\Psi_Z \Psi_{ZX} - \Psi_X \Psi_{ZZ} = \Psi_{ZZZ}$$

$$X > 0, \;\; Z = 0: \quad \Psi = \Psi_{ZZ} = 0$$

$$\Psi_{ZZ}(X, \infty) = 2, \;\; A(X) = \lim_{z\to\infty}(\Psi_Z - 2Z)$$

$$\Psi(0+, Z) = \Psi(0-, Z) = Z^2$$

Symmetry imposes no net displacement from BL

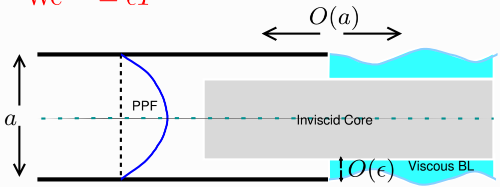
No upstream influence

$$H = A$$

$H \sim X^{1/3}$ Goldstein wake

Flow through channel exit $(G = 0,$ S. Harris)

$We^{-1} = \epsilon T$



$\epsilon = Re^{-1/3}, \quad Re = U_{\text{max}}\, a/\nu$

$P = T A_{XX}$

$$\Psi_Z \Psi_{ZX} - \Psi_X \Psi_{ZZ} = -P_X + \Psi_{ZZZ}$$

$$X > 0, \quad Z = 0: \quad \Psi = \Psi_{ZZ} = 0$$

$$\Psi_{ZZ}(X, \infty) = 2, \; A(X) = \lim_{z\to\infty}(\Psi_Z - 2Z)$$

$$\Psi(0+, Z) = \Psi(0-, Z) = Z^2$$

- No upstream influence
- $H = A$
- T can be scaled out:

$$[X, \Psi, H, P] \sim [T^{3/7}, T^{2/7}, T^{1/7}, T^{2/7}]$$

Flow through channel exit $(G = 0,$ S. Harris)

Surface Displacement $H(x)$
for $\gamma = 1$, $dx = 0.01$

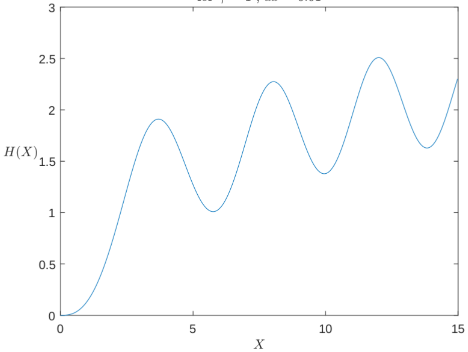
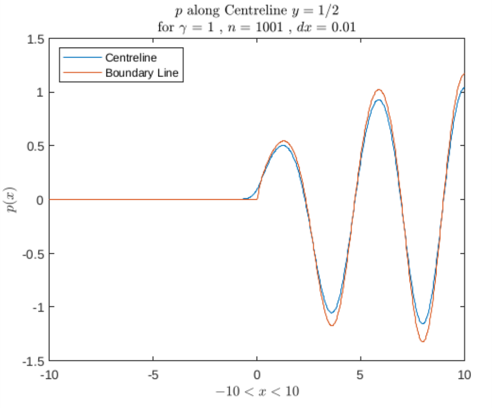


Figure 9

- Waves with Goldstein wake emerging on average

Flow through channel exit $(G = 0,$ S. Harris)



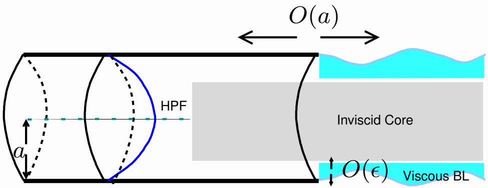
Tillett's core-flow/Rayleigh problem

$$y(1-y)\left[v_{xx} + v_{yy}\right] + 2v = 0$$
$$v\left(x, \tfrac{1}{2}\right) = 0$$
$$v(x,0) = -\tfrac{1}{2}p_x$$

$$\psi_x = -v$$
$$p = 2(1-2y)\psi - 2y(1-y)\psi_y$$

- Weak linear upstream influence through core.

Flow through pipe exit $(G = 0,$ S. Harris)



$$P = T\,(A + A_{XX})$$

$$\epsilon = Re^{-1/3}, \quad Re = U_{\text{max}} a/\nu$$

$$p \sim \text{We}^{-1}(\boldsymbol{\nabla}\cdot\boldsymbol{n})$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{n} = \frac{-R''(x)}{(1+R'^2)^{3/2}} + \frac{1}{R\sqrt{1+R'^2}}$$

$$R(x) = 1 - \epsilon H(x), \quad \text{We}^{-1} = \frac{\tilde{\tau}}{\tilde{\rho} U_{\text{max}}^2 a} = \epsilon T$$

$$p \sim \epsilon^2 T\,(H + H'')$$

$+H$: cf. subcritical hydrostatic layer

$$\Psi_Z \Psi_{ZX} - \Psi_X \Psi_{ZZ} = -P_X + \Psi_{ZZZ}$$

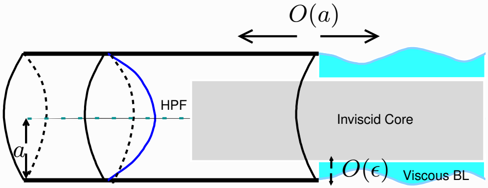
$$X > 0, \quad Z = 0: \quad \Psi = \Psi_{ZZ} = 0$$

$$\Psi_{ZZ}(X,\infty) = 2, \; A(X) = \lim_{z\to\infty}(\Psi_Z - 2Z)$$

$$\Psi(0+,Z) = \Psi(0-,Z) = Z^2$$

$$H = A$$

Flow through pipe exit $(G = 0,$ S. Harris)



$$\epsilon = Re^{-1/3}, \quad Re = U_{\mathsf{max}} a/\nu$$

$$p \sim \mathrm{We}^{-1}(\boldsymbol{\nabla} \cdot \boldsymbol{n})$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{n} = \frac{-R''(x)}{(1+R'^2)^{3/2}} + \frac{1}{R\sqrt{1+R'^2}}$$

$$R(x) = 1 - \epsilon H(x), \quad \mathrm{We}^{-1} = \frac{\tilde{\tau}}{\tilde{\rho} U_{\mathsf{max}}^2 a} = \epsilon T$$

$$p \sim \epsilon^2 T \left(H + H'' \right)$$

$$P = T\left(A + A_{XX}\right)$$

$$\Psi_Z \Psi_{ZX} - \Psi_X \Psi_{ZZ} = -P_X + \Psi_{ZZZ}$$

$$X > 0, \quad Z = 0: \quad \Psi = \Psi_{ZZ} = 0$$

$$\Psi_{ZZ}(X, \infty) = 2, \; A(X) = \lim_{z \to \infty} (\Psi_Z - 2Z)$$

$$\Psi(0+, Z) = \Psi(0-, Z) = Z^2$$

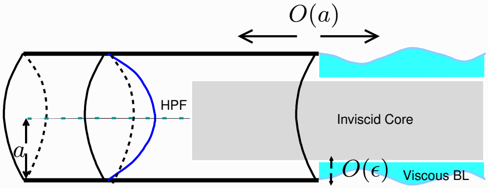
$$H = A$$

$$T \to 0$$

$$[X, H] \sim [T^{3/7}, T^{1/7}]$$

$$P = T^{5/7} A + A_{XX} \sim A_{XX}$$

Flow through pipe exit ($G=0$, S. Harris)



$$\epsilon = Re^{-1/3}, \quad Re = U_{\text{max}} a/\nu$$

$$p \sim \mathrm{We}^{-1}(\boldsymbol{\nabla}\cdot\boldsymbol{n})$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{n} = \frac{-R''(x)}{(1+R'^2)^{3/2}} + \frac{1}{R\sqrt{1+R'^2}}$$

$$R(x) = 1-\epsilon H(x), \quad \mathrm{We}^{-1} = \frac{\tilde{\tau}}{\tilde{\rho} U_{\text{max}}^2 a} = \epsilon T$$

$$p \sim \epsilon^2 T\,(H + H'')$$

$$P = T\,(A + A_{XX})$$

$$\Psi_Z \Psi_{ZX} - \Psi_X \Psi_{ZZ} = -P_X + \Psi_{ZZZ}$$

$$X > 0, \quad Z = 0: \quad \Psi = \Psi_{ZZ} = 0$$

$$\Psi_{ZZ}(X,\infty) = 2, \; A(X) = \lim_{z\to\infty}(\Psi_Z - 2Z)$$

$$\Psi(0+,Z) = \Psi(0-,Z) = Z^2$$

$$H = A$$

$$T \to \infty$$

$$[X, H] \sim [T^3, T]$$

$$P = A + T^{-7} A_{XX} \sim A$$

Achievements & further outlook

Core results

- Self-consistent theory of developed film having just passed plate edge
- Flow regimes of surprisingly rich physics identified
- Capillary ripples: nonlinear extension
- Choking: $T \sim 1/2$, $T \sim 1$
- Breakdowns by flow reversal or blow-up

To-dos

- Regularise: breakdowns, $T \sim 1$
- Unsteadiness & stability
- Symmetry-breaking effects in exit problem
- Careful experiment desirable!

Thank you for attention!