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Strong Measure Zero Subsets of the Higher Cantor Space

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Introduction

In the late 19th and early 20th century, increasing interest emerged in combining analytical, set-theoretic and topological notions and methods to describe properties of subsets of the real line. Such efforts were fruitful, leading to the advent of modern measure theory (and later, descriptive set theory) as a mathematical discipline, spearheaded by figures such as Borel, Lebesgue, Luzin, Radon, Fréchet and others.

In searching for a useful notion related to being a Lebesgue measure zero set, Borel [Bor19] introduced strong measure zero sets.

Definition. A subset X of the real numbers is strong measure zero iff for any sequence $(\varepsilon_n)_{n\in\omega}$ of positive real numbers there exists a sequence of intervals $(I_n)_{n\in\omega}$ with $\lambda(I_n) \leq \varepsilon_n$ and $X \subseteq \bigcup_{n\in\omega} I_n$.

Clearly, strong measure zero sets are measure zero and every countable set is strong measure zero. Moreoever, it is also easy to see that perfect sets cannot be strong measure zero. It was conjectured by Borel that countability is perhaps the only constraint on strong measure zero sets, giving rise to the Borel Conjecture (BC):

A set is strong measure zero if and only if it is countable.

In 1928, Sierpiński [Sie28] showed that CH implies the existence of uncountable strong measure zero sets (specifically, he showed that any Luzin set is strong measure zero). It was not until after the advent of Cohen's revolutionary technique of forcing that Laver [Lav76] established the relative consistency (and thus independence from ZFC) of BC. As is remarked in [JSW90], Laver's result will turn out to be significant in two ways; it firmly cemented the efficacy of methods from abstract set theory, such as forcing, in discussions of concrete interest to analysis. Secondly, it is the first appearance of forcing with countable support, which would later lead to Shelah's notion of properness.

Over the years, investigations into matters related to strong measure zero sets (such as the interplay between BC and the size of the continuum [JSW90], the dual notion of strongly meager sets [Gol+13] and others) became testament to the fact that Borel's notion was indeed worthy of interest.

For our purposes the most interesting of these is Corazza's proof of the consistency of "a set is strong measure zero iff it has size less than continuum" ([Cor89]) in which he employs an ω_2-length iteration of strongly proper forcings (a notion stronger than "proper + ω^ω-bounding" that includes well-known forcings such as Sacks and Silver), together with a previous result of Miller [Mil83] to construct a model with

"Every set of reals of size continuum can be mapped uniformly continuously onto $[0, 1]$".

We are interested in a version of Borel's Conjecture on higher cardinals κ. The higher Cantor space 2^κ and the higher Baire space κ^κ come equipped with the standard $<\kappa$-box topology; see [FKK14] for basic properties of these spaces. Their elements are called κ-reals, or simply reals. Note that near universally, the assumption $\kappa^{<\kappa} = \kappa$ is made in discussions on the higher Baire space, without which the space exhibits some undesirable topological properties (see [FHK13, §II.2.1.]). Especially in recent years, renewed interest has sparked among set theorists in studying these spaces; a compendium of open questions can be found in [Kho+16].

The following definition is due to Halko [Hal96]:

Definition. Let $X \subseteq 2^\kappa$. We call X strong measure zero iff

$$\forall f \in \kappa^\kappa \, \exists (\eta_i)_{i<\kappa} : \; \big(\forall i < \kappa : \; \eta_i \in 2^{f(i)}\big) \wedge X \subseteq \bigcup_{i<\kappa} [\eta_i].$$

This is a straightforward combinatorial reformulation (here $[\eta]$ is a basic clopen set as defined in the next section) of Borel's definition that is agnostic to the existence of a measure on 2^κ. Let $\mathcal{SN}$ be the collection of all strong measure zero sets; it is easy to see that $\mathcal{SN}$ is a proper, $\leq\kappa$-complete ideal (see also Lemma 6.2) on 2^κ containing all singletons.

The Borel Conjecture on κ (BC(κ)) is the statement "a subset of 2^κ is strong measure zero iff it has cardinality $\leq\kappa$". Strong measure zero sets for κ regular uncountable have been studied in [HS97], where the authors have proven that BC(κ) is false for successor κ satisfying $\kappa^{<\kappa} = \kappa$.

Throughout this paper we shall restrict our attention to κ at least inaccessible, thus in particular $\kappa^{<\kappa} = \kappa$. The question of the consistency of BC(κ) on such κ is still open [Kho+16]. By the results in [Kho+20], every Laver-like tree forcing on κ^κ necessarily adds a κ-Cohen real. Any treatment of the consistency of BC(κ) thus cannot be merely a straightforward adaptation of Laver's results, since adding κ-Cohen reals makes the ground model reals strong measure zero.

We shall give two proofs establishing the relative consistency of

$$\text{ZFC} + |2^\kappa| = \kappa^{++} + \mathcal{SN} = [2^\kappa]^{\leq\kappa^+},$$

the first of which is an adaptation of an iteration found in [GJS93] and requires κ to be strongly unfoldable (a large cardinal property between weakly compact and Ramsey that is consistent with $V = L$). The second, somewhat better, proof only requires κ to be inaccessible and employs the same iteration by establishing minimality of the respective forcing extension, following the approach of Corazza [Cor89].

The content of this thesis is based on the paper "Strong measure zero sets on 2^κ for κ inaccessible" by Johannes Schürz [Sch19].

I would like to thank my advisor Martin Goldstern and my co-author Johannes Schürz for their invaluable help in the creation of this thesis.

Notation and Basic Definitions

Let us make some preliminary remarks.

The higher Cantor space 2^κ is equipped with the standard $<\kappa$-box topology, whose base consists of the basic clopen sets

$$[\eta] := \{b : \eta \lhd b\}$$

for $\eta \in 2^{<\kappa}$; for the higher Baire space κ^κ the topology is defined analogously. The relation $\eta \lhd \nu$ denotes the extension relation for sequences, i.e. $\eta = \nu \restriction i$ for some $i \leq \mathrm{dom}(\nu)$. The relation $\eta \perp \nu$ denotes incompatibility, i.e. $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$.

A (κ-) tree is a subset of $\kappa^{<\kappa}$ closed under initial segments.

Let $T \subseteq \kappa^{<\kappa}$ be a tree and $\eta \in T$. Then we define the following notions:

- A $b \in \kappa^\kappa$ is a *branch* of T iff $b \restriction i \in T$ for all $i < \kappa$. Let $[T]$ denote the set of all branches of T.
- Denote by $\mathrm{succ}_T(\eta)$ the set of immediate successors of η in T. Call η a *splitting node* of T iff $|\mathrm{succ}_T(\eta)| > 1$. Denote the set of all splitting nodes of T as $\mathrm{split}(T)$. We will only consider trees in which every node has a successor.
- T is *perfect* iff $[T]$ contains no isolated points or, equivalently, above every $\eta \in T$ there is a ν such that $\eta \lhd \nu$ and $\nu \in \mathrm{split}(T)$. Note that for $\kappa \neq \omega$ this is not equivalent to $[T]$ being homeomorphic to 2^κ.
- The *height* $\mathrm{ht}_T(\eta)$ of a node η is the order type of the set $\{\nu \precneqq \eta : \nu \in \mathrm{split}(T)\}$. Additionally, for $i < \kappa$, define

  $$\mathrm{split}_i(T) := \{\eta \in \mathrm{split}(T) : \mathrm{ht}_T(\eta) = i\}.$$

  Perfect trees on regular κ (in particular conditions $p \in PT_f$ as defined in the next section) contain nodes of any height less than κ.
- As usual, the set of branches of a tree is a closed set and every closed set S can be represented as the set of branches of the tree $T = \{b \restriction i : i < \kappa \wedge b \in S\}$. However, it may be the case that this tree T necessarily contains *dying branches*, i.e. T might contain an increasing sequence $(\eta_i)_{i<\lambda}$ with $\lambda < \kappa$ whose limit $\bigcup_{i<\lambda} \eta_i$ is not an element of T [1]. This phenomenon is unique to the κ-case and has no ω-equivalent.

  We say T (or $[T]$) is *superclosed* iff this does not happen, meaning that whenever $\lambda < \kappa$ is a limit ordinal and $\eta \in \kappa^\lambda$, then $\eta \in T \Leftrightarrow \forall i < \lambda : \eta \restriction i \in T$.

We shall attempt to, wherever feasible, adhere to certain self-imposed notational conventions. In this vein, the letters i, j, k, ℓ will generally refer to ordinals $<\kappa$; δ, λ to limit ordinals $\leq\kappa$ and $\alpha, \beta, \gamma, \zeta$ to ordinals $\leq\kappa^{++}$. The letters p, q, s, t denote conditions while η, ν, ρ are elements of $\kappa^{<\kappa}$. The pair F, i will always fulfil $F \in [\alpha]^{<\kappa}, i < \kappa$, where $\alpha \leq \kappa^{++}$ is either explicitly given or clear from context.

[1] Consider for example the closed set $2^\kappa \setminus [\eta]$, where $\eta \in 2^\omega$.

1 Perfect Tree Forcing

We are interested in a particular forcing consisting of $<\kappa$-splitting perfect trees whose splitting is bounded by an $f \in \kappa^\kappa$ with $f(i) \geq 2$ for all $i < \kappa$.

Definition 1.1. Let $p \in PT_f$ iff

(S1) $p \subseteq \kappa^{<\kappa}$ is a nonempty tree

(S2) p is perfect

(S3) $\forall \eta \in p \, \forall i \in \operatorname{dom}(\eta): \ \eta(i) < f(i)$

(S4) p has full splitting: $\forall \eta \in p\colon |\operatorname{succ}_p(\eta)| = 1 \vee \operatorname{succ}_p(\eta) = \{\eta^\frown j\colon j < f(\operatorname{dom} \eta)\}$

(S5) p is superclosed

(S6) splitting is continuous: If $\lambda < \kappa$ is a limit, then
$\forall \eta \in \kappa^\lambda \cap p: \ \{\nu \trianglelefteq \eta\colon \nu \in \operatorname{split}(p)\}$ is unbounded in $\eta \Rightarrow \eta \in \operatorname{split}(p)$

The significance of (S4) and (S6) lies in ensuring $<\kappa$-closure of the forcing (see Lemma 1.6). The axioms (S4) and (S5) guarantee that for all $\eta \in p$ we have

$$[\eta] \cap [p] \neq \emptyset,$$

i.e. there is a branch of p going through η. Under the other axioms, (S2) + (S6) is equivalent to the following statement: whenever $b \in [p]$ is a branch of p, then

$$\{i < \kappa : b \restriction i \in \operatorname{split}(p)\}$$

is a club subset of κ.

For $f \equiv 2$ we have a κ-version of Sacks forcing, first studied by Kanamori [Kan80]. An overview of variants of familiar forcing notions on higher cardinals can be found in [FKK14].

The rest of this section is devoted to proving some regularity properties for PT_f, generalized straightforwardly from the classical treatment of similar tree forcings on ω^ω.

Set $q \leq_{PT_f} p$ iff $q \subseteq p$. For a PT_f-generic filter G define the generic real s_G to be the unique real contained in $\bigcap_{p \in G} [p]$.

Fact 1.2. For a condition $p \in PT_f$ the set $\operatorname{split}_i(p)$ is a front in p, that is,

$$\forall b \in [p] : |b \cap \operatorname{split}_i(p)| = 1.$$

Call it the i-th splitting front of p.

Lemma 1.3. Let $i < \kappa$ and $p \in PT_f$ be a condition. Then $|\operatorname{split}_i(p)| < \kappa$.

Proof. We proceed by induction on i:

- $i = 0$: Trivial.

- $i \to i+1$: The map $\eta \mapsto \min\{\nu \lhd \eta : \mathrm{ht}_p(\nu) = i+1\}$ is bijection between $\mathrm{split}_{i+1}(p)$ and $\bigcup_{\eta \in \mathrm{split}_i(p)} \mathrm{succ}_p(\eta)$. By the inductive hypothesis and the fact that p is $<\kappa$-splitting, the latter set has size $< \kappa$.
- λ is a limit: Since every $\eta \in \mathrm{split}_\lambda(p)$ is the limit of a sequence $(\eta_j)_{j<\lambda}$ with $\eta_j \in \mathrm{split}_j(p)$, we have $|\mathrm{split}_\lambda(p)| \leq |\prod_{j<\lambda} \mathrm{split}_j(p)| < \kappa$ by the inaccessibility of κ.

□

Definition 1.4. Let $(\mathcal{P}, \leq_\mathcal{P})$ be a forcing notion and $(\leq_i)_{i<\kappa}$ a sequence of reflexive and transitive binary relations on $\mathcal{P}$ such that

$$\forall j < i < \kappa \colon (\leq_i) \subseteq (\leq_j) \subseteq (\leq_\mathcal{P}).$$

Then

1. $(p_j)_{j<\delta}$ is a *fusion sequence* of length $\delta \leq \kappa$ iff $\forall j < k < \delta : p_k \leq_j p_j$.
2. $\mathcal{P}$ has *Property B* iff
   - $(\mathcal{P}, \leq_\mathcal{P})$ is $<\kappa$-closed.
   - Whenever $(p_j)_{j<\delta}, \delta \leq \kappa$ is a fusion sequence in $\mathcal{P}$, then there exists a *fusion limit* q with $\forall j < \delta : q \leq_j p_j$.
   - If A is a maximal antichain, $p \in \mathcal{P}$ and $i < \kappa$, then there exists a $q \leq_i p$ such that $A \restriction q := \{r \in A \colon r \parallel q\}$ has size $<\kappa$, where $\parallel$ means compatible.

Note that by weakening the third requirement to $|A \restriction q| \leq \kappa$, we get a κ-version of Baumgartner's Axiom A. Property B is thus a variant of Axiom A combined with the notion of being κ^κ-bounding [BJ95, Def. 7.2.C]; it is well-known from the countable context that many standard tree forcings, such as Sacks and Silver forcing, have this property.

Lemma 1.5. Property B implies κ^κ-bounding.

Proof. Assume $p \Vdash \dot{g} \in \kappa^\kappa$ and $\dot{g}(i)$ is decided by an antichain A_{i+1}. Construct a fusion sequence $(q_i)_{i<\kappa}$ below p by setting $q_0 := p$ and finding a $q_{i+1} \leq_i q_i$ with $|A_{i+1} \restriction q_{i+1}| < \kappa$ in successor steps. In limit steps λ, set q_λ to be a fusion limit of $(q_i)_{i<\lambda}$. The fusion limit q_κ of the whole sequence will force $q_\kappa \Vdash \dot{g} \leq \check{h}$ for some h in the ground model. □

Lemma 1.6. PT_f is $<\kappa$-closed.

Proof. If $(p_i)_{i<\delta}$ with $\delta < \kappa$ is a decreasing sequence, set $q := \bigcap_{i<\delta} p_i$. We check that q is a condition; only (S2) is nontrivial. Note that by (S4) every node in q has a direct successor.

Let thus $\eta \in q$. For some $b \in [q]$ with $\eta \lhd b$ (recall that by (S4) + (S5) such a b exists) consider the sets

$$C_i := \{j < \kappa : b \restriction j \in \mathrm{split}(p_i)\}.$$

By (S2) and (S6), C_i is a club subset of κ. Thus $\bigcap_{i<\delta} C_i$ is a club and yields a $\nu \lhd b$ with $\nu \in \text{split}(q)$ and $\eta \lhd \nu$. □

Remark 1.7. Clearly, the intersection $\bigcap_{i<\delta} p_i$ in the previous lemma is simultaneously also the greatest lower bound of the decreasing sequence $(p_i)_{i<\delta}$, $\delta < \kappa$.

Definition 1.8. For $p, q \in PT_f$, define $q \leq_i p$ iff $q \leq_{PT_f} p$ and $\text{split}_i(p) = \text{split}_i(q)$.

Fact 1.9. The following are equivalent:

1. $q \leq_i p$
2. $q \leq_{PT_f} p$ and $\forall j \leq i : \text{split}_j(p) = \text{split}_j(q)$
3. $q \leq_{PT_f} p$ and $\forall \eta \in p : \text{ht}(\eta) \leq i \Rightarrow \text{succ}_p(\eta) \subseteq q$
4. $q \leq_{PT_f} p$ and $\text{split}_{i+1}(p) \subseteq q$

It remains to prove that equipped with these relations, PT_f has Property B.

Lemma 1.10. For every fusion sequence $(p_j)_{j<\delta}$ of length $\delta \leq \kappa$ in PT_f there exists a q with $\forall j < \delta : q \leq_j p_j$.

Proof. If $\delta < \kappa$, the intersection q from Lemma 1.6 can be seen to also be a fusion limit.

Otherwise once again set $q = \bigcap_{j<\kappa} p_j$ and follow the proof of Lemma 1.6; along a branch $b \in [q]$ again define the sets

$$C_j := \{\ell < \kappa : b \restriction \ell \in \text{split}(p_j)\}.$$

Since $(p_j)_{j<\kappa}$ is a fusion sequence, we observe

$$\bigcap_{j<\kappa} C_j = \Delta_{j<\kappa} C_j$$

and thus $\bigcap_{j<\kappa} C_j$ is also a club by the fact that the club filter is closed under diagonal intersections. □

Before concluding the proof, we first give two definitions which will come in handy later in the iteration context.

Definition 1.11. For a condition $p \in PT_f$ and $\eta \in p$, define $p^{[\eta]} := \{\nu \in p : \nu \lhd \eta \vee \eta \lhd \nu\}$.

One can see easily that $p^{[\eta]}$ is a stronger condition than p and that for any $i < \kappa$ we have $p = \bigcup_{\eta \in \text{split}_i(p)} p^{[\eta]}$.

Definition 1.12. Let $p \in PT_f$ be a condition and $i < \kappa$. We say that a condition $s \in PT_f$ is (p, i)*-determined* iff $s \leq p$ and

$$|s \cap \text{split}_i(p)| = 1.$$

Lemma 1.13. The set of (p, i)-determined conditions is dense below p for all i.

Proof. For any $s \leq p$ we may extend the stem of s in the following way: take any branch $b \in [s] \subseteq [p]$; since we then know $|b \cap \operatorname{split}_i(p)| = 1$, we see that there is a unique ν with $\nu \in b \cap s \cap \operatorname{split}_i(p)$. Then $s^{[\nu]}$ is (p, i)-determined. □

Theorem 1.14. PT_f has Property B.

Proof. It remains to show the antichain condition. To this end, let A be a maximal antichain, $p \in PT_f$ and $i < \kappa$. Enumerate $\operatorname{split}_{i+1}(p)$ as $(\eta_j)_{j<\delta}$ with $\delta < \kappa$. We will decompose p into $|\delta|$-many parts, each of which will be thinned out above the $(i+1)$-th splitting front.

Proceed by finding for each $j < \delta$ a condition $q_j \leq p^{[\eta_j]}$ such that $|A \restriction q_j| = 1$. Set

$$q := \bigcup_{j<\delta} q_j.$$

Then $q \in PT_f$ is a condition with $\operatorname{split}_{i+1}(p) \subseteq q$ and thus $q \leq_i p$. To prove $|A \restriction q| < \kappa$, let $r \in A$ be compatible with q. By the previous lemma we may pick an s_r that is (p, i)-determined with $s_r \leq r, q$ and hence $s_r \cap \operatorname{split}_{i+1}(p) = \{\eta_{j_r}\}$ for some $j_r < \delta$. But since $s_r \leq q$, we can conclude $s_r \leq q_{j_r}$ and thus $r \parallel q_{j_r}$. We have thus found a function from $A \restriction q$ to δ, mapping $r \mapsto j_r$, which is injective (since $|A \restriction q_j| = 1$ for all $j < \delta$). The desired conclusion $|A \restriction q| < \kappa$ follows. □

2 Iteration

The backbone of our forcing construction will consist of an iteration of PT_f forcings. Let therefore $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ be a $\leq\kappa$-supported forcing iteration with

$$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = PT_{f_\alpha}$$

where the sequence $(f_\alpha)_{\alpha<\kappa^{++}}$ is in the ground model and $f_\alpha(i) \geq 2$ for all $i < \kappa$. Set $\mathbb{P} := \mathbb{P}_{\kappa^{++}}$.

As a matter of notation, let $\dot{G}_\alpha$ for $\alpha \leq \kappa^{++}$ denote the canonical $\mathbb{P}_\alpha$-name for a $(V, \mathbb{P}_\alpha)$-generic filter; we know $\Vdash_{\mathbb{P}} \dot{G}_{\kappa^{++}} \restriction \alpha = \dot{G}_\alpha$. We also write $\dot{G}$ for $\dot{G}_{\kappa^{++}}$. Finally, let $\dot{s}_\alpha$ be the canonical name for the α-th generic real.

This section is dedicated to verifying some regularity properties of such iterations. We will observe that

1. $\mathbb{P}$ is $<\kappa$-closed
2. $\mathbb{P}$ does not collapse κ^+
3. if $V \models |2^\kappa| = \kappa^+$, then $\mathbb{P}$ has the κ^{++}-c.c.,

thus in aggregate no cardinals are collapsed when forcing with $\mathbb{P}$.

Fact 2.1. $\mathbb{P}$ is $<\kappa$-closed.

In the countable case, the favoured tool one would look towards in the endeavour of preserving ω_1 is the notion of properness. Finding a satisfactory analogue for higher cardinals is a long-standing open problem (see e.g. [RS13] and [FHZ13]). A relatively straightforward generalization that still enjoys many desirable qualities of properness is the following:

Definition 2.2. A forcing $\mathcal{P}$ is called κ-proper iff for every sufficiently large θ (e.g. $\theta > |2^{\mathcal{P}}|$) and every elementary submodel $M \preccurlyeq H(\theta)$ such that $\mathcal{P} \in M$, $|M| = \kappa$ and ${}^{<\kappa}M \subseteq M$, and every $p \in \mathcal{P} \cap M$, there exists $q \leq_{\mathcal{P}} p$ such that for every dense $D \in M$, $D \cap M$ is predense below q.

Fact 2.3. Forcing notions that are $<\kappa^+$-closed or have the κ^+-c.c. are κ-proper. Furthermore, κ-proper forcing notions do not collapse κ^+.

Further details on κ-properness can be found in [FKK14].

Unfortunately, in stark contrast to the classical setting, there is no preservation theorem for κ-properness in iterations (see [Ros18] for an iteration of κ^+-c.c. forcings whose ω-limit collapses κ^+). Our strategy for ensuring κ-properness is to verify an iteration version of Property B. Similar to fusion with countable support, in such cases the correct tool is the following notion:

Definition 2.4. For $\zeta \leq \kappa^{++}$ let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta : \alpha \leq \zeta, \beta < \zeta \rangle$ be a $\leq\kappa$-support iteration with

$$\forall \alpha < \zeta \colon \Vdash_\alpha \text{“ } \dot{Q}_\alpha \text{ has Property B ”}.$$

Let $F \in [\zeta]^{<\kappa}$ and $i < \kappa$. We define $q \leq_{F,i} p$ iff

$$q \leq_{\mathcal{P}_\zeta} p \text{ and } \forall \beta \in F\colon q \restriction \beta \Vdash_\beta q(\beta) \leq_i^{\dot{\mathcal{Q}}_\beta} p(\beta).$$

Then

1. A sequence $\langle p_i, F_i : i < \delta \rangle$ of length $\delta \leq \kappa$ is called a fusion sequence iff
   - $\forall j < k < \delta : p_k \leq_{F_j,j} p_j$
   - The F_j are increasing and, if $\delta = \kappa$, then $\bigcup_{j<\delta} \operatorname{supp}(p_j) \subseteq \bigcup_{j<\delta} F_j$.
2. We say that $\mathcal{P}_\zeta$ has *Property B** iff
   - For every fusion sequence $\langle p_i, F_i : i < \delta \rangle$, $\delta \leq \kappa$ there exists a fusion limit q with $\forall j < \delta : q \leq_{F_j,j} p_j$.
   - For every maximal antichain A, every $p \in \mathcal{P}_\zeta$ and every $F \in [\zeta]^{<\kappa}, i < \kappa$ there exists a $q \leq_{F,i} p$ such that $|A \restriction q| < \kappa$.

Hence for iterations we consider fusion sequences pointwise, with the added caveat of being able to delay fusion arbitrarily long in each coordinate. In practice, the auxiliary sets F_j will almost always be defined by a bookkeeping argument relative to the p_j.

Fact 2.5. Property B* implies κ-properness.

In the definition of Property B*, only the antichain condition is nontrivial. In fact, for such iterations of Property B forcings, fusion limits always exist.

Lemma 2.6. With notation from the previous definition, every fusion sequence $\langle p_i, F_i : i < \delta \rangle$, $\delta \leq \kappa$ in $\mathcal{P}_\zeta$ has a fusion limit q.

Proof. We construct q inductively such that $\mathcal{P}_\alpha \ni q \restriction \alpha$ is a fusion limit of $\langle p_i \restriction \alpha, F_i \cap \alpha : i < \delta \rangle$ for each $\alpha \leq \zeta$.

Assume $q \restriction \alpha$ has been defined for $\alpha < \zeta$. To define $q(\alpha)$, distinguish three cases:

- $\alpha \in \bigcup_{j<\delta} \operatorname{supp}(p_j) \wedge \alpha \in \bigcup_{j<\delta} F_j$: Find $j^*(\alpha)$ minimal such that $\alpha \in F_{j^*(\alpha)}$. Now $q \restriction \alpha \Vdash$ "$(p_j(\alpha))_{j \geq j^*(\alpha)}$ is a fusion sequence", so let $q(\alpha)$ be a fusion limit of that sequence.
- $\alpha \in \bigcup_{j<\delta} \operatorname{supp}(p_j) \wedge \alpha \notin \bigcup_{j<\delta} F_j$: Note that this case may only occur for $\delta < \kappa$, thus we may use $<\kappa$-closure of $\dot{\mathcal{Q}}_\alpha$ to construct $q(\alpha)$ from $(p_j(\alpha))_{j<\delta}$.
- $\alpha \notin \bigcup_{j<\delta} \operatorname{supp}(p_j)$: Set $q(\alpha) := \mathbb{1}_{\dot{\mathcal{Q}}_\alpha}$.

To see that $q \restriction \lambda \in \mathcal{P}_\lambda$ for limit λ, merely note $\operatorname{supp}(q \restriction \lambda) \subseteq \bigcup_{i<\delta} \operatorname{supp}(p_i \restriction \lambda)$. □

Remark 2.7. Note that the forcings $\dot{\mathcal{Q}}_\alpha = PT_{f_\alpha}$ fulfil $<\kappa$-closure and the existence of fusion limits in a particularly strong way: in either case, a canonical weakest lower bound/fusion limit exists. Thus by following the above proof and choosing these canonical conditions, we can see that an iteration of PT_f forcings also fulfils a stronger fusion condition: for every fusion sequence there exists a canonical, weakest fusion limit.

Some work remains to prove the antichain condition for $\mathbb{P}_\zeta$, which we do in a rather ad hoc manner by induction on ζ. On the way we will introduce some notation that will also come in handy later.

First off, let us define the iteration version of Definition 1.12 and the corresponding density lemma.

Definition 2.8. Let $\zeta \leq \kappa^{++}, p \in \mathbb{P}_\zeta, F \in [\zeta]^{<\kappa}$ and $i < \kappa$. We say a condition $s \in \mathbb{P}_\zeta$ is (p, F, i)-*determined* following $g \in \prod_{\beta \in F} \kappa^{<\kappa}$ iff $s \leq_{\mathbb{P}_\zeta} p$ and

$$\forall \beta \in F \, \exists \eta_\beta \in \kappa^{<\kappa} :$$
$$s \restriction \beta \Vdash s(\beta) \cap \operatorname{split}_i(p(\beta)) = \{\check{\eta_\beta}\} \wedge \operatorname{succ}_{s(\beta)}(\eta_\beta) = \{g(\check{\beta})\}.$$

We say a condition s is (p, F, i)-determined iff it is (p, F, i)-determined following some (unique) g.

The function g prescribes the choices s makes at the i-th splitting front of p; it is completely determined by s.

Lemma 2.9. The set of (p, F, i)-determined conditions is dense below $p \in \mathbb{P}_\zeta$ for all p, F, i and the set of (p, F, i)-determined conditions following g is open for all p, F, i, g.

Proof. Enumerate F as an increasing sequence $(\beta_j)_{j<\delta}$ with $\delta < \kappa$. For a $q \leq p$ we will inductively construct a decreasing sequence $(s_j)_{j<\delta}$ below q and a $\subseteq$-increasing sequence $(g_j)_{j<\delta}$ with $g_j \in \prod_{\beta \in F \cap \beta_j} \kappa^{<\kappa}$ such that s_j is $(p, F \cap \beta_j, i)$-determined following g_j.

- $j = 0$: Set $s_0 := q$.
- $j \to j + 1$: Since $s_j \restriction \beta_j \Vdash s_j(\beta_j) \leq_{\dot{\mathbb{Q}}_{\beta_j}} p(\beta_j)$, we may use Lemma 1.13 to find $\mathbb{P}_{\beta_j}$-names $\dot{t}, \dot{\eta}_{\beta_j}, \dot{\nu}_{\beta_j}$ with

  $$s_j \restriction \beta_j \Vdash \dot{t} \in \mathbb{Q}_{\beta_j} \wedge \dot{t} \leq_{\dot{\mathbb{Q}}_{\beta_j}} s_j(\beta_j)$$

  and

  $$s_j \restriction \beta_j \Vdash \dot{t} \cap \operatorname{split}_i(p) = \{\dot{\eta}_{\beta_j}\} \wedge \operatorname{succ}_{\dot{t}}(\dot{\eta}_{\beta_j}) = \{\dot{\nu}_{\beta_j}\}.$$

  Find a stronger condition $r \leq s_j \restriction \beta_j$ that decides the names $\dot{\eta}_{\beta_j}, \dot{\nu}_{\beta_j}$ as $\eta_{\beta_j}, \nu_{\beta_j}$. Define $s_{j+1} := r^\frown t^\frown (s_j \restriction (\beta_j, \zeta))$ and $g_{j+1} := g_j \cup \{(\beta_j, \nu_{\beta_j})\}$.
- $\lambda < \delta$ is a limit: By $<\kappa$-closure we can find a lower bound s_λ of the sequence $(s_\ell)_{\ell<\lambda}$. Define $g_\lambda := \bigcup_{\ell<\lambda} g_\ell$.

If s_δ is a lower bound of $(s_j)_{j<\delta}$ (which again exists by $<\kappa$-closure) and $g_\delta = \bigcup_{j<\delta} g_j$, then one can easily see that $s_\delta \leq q$ is (p, F, i)-determined following g_δ.

Lastly, if s is (p, F, i)-determined following g, then clearly any $s' \leq s$ is as well. □

Fact 2.10. If $p' \leq_{F,i} p$ and $s \leq p'$, then s is (p, F, i)-determined iff it is (p', F, i)-determined.

Suppose now that $q \leq_{PT_f} p$. The extension of p to q may be undertaken in two steps by interpolating on the $\leq_i$ relation. In the first step, we thin out as much as is necessary from p, but only in its 'upper regions' - say, above the $(i+1)$-th splitting front - yielding an interpolating condition $p^{(q)}$ with $p^{(q)} \leq_i p$ (above nodes not present in q, p may be left untouched in the extension to $p^{(q)}$). In the second step, nodes are removed from $p^{(q)}$, but only near the base of the tree, such that whenever $\eta \in p^{(q)} \setminus q$, then there is already some initial segment $\nu \lhd \eta$ with $\nu \in p^{(q)} \setminus q$ and $\mathrm{ht}_{p^{(q)}}(\nu) \leq i+1$. We thus have

$$q \leq p^{(q)} \leq_i p.$$

This motivates the next lemma.

Lemma 2.11 (Interpolation)**.** Let $p \in \mathbb{P}_\zeta$ and s be (p, F, i)-determined following $g \in \prod_{\beta \in F} \kappa^{<\kappa}$ for some $F \in [\zeta]^{<\kappa}, i < \kappa$. Then there exists a condition $p^{(s)} \leq_{F,i} p$ with

- $s \leq_{\mathbb{P}_\zeta} p^{(s)} \leq_{F,i} p$ and
- for all (p, F, i)-determined conditions s' following g, whenever $s' \leq_{\mathbb{P}_\zeta} p^{(s)}$, then already $s' \leq_{\mathbb{P}_\zeta} s$.

Proof. Construct $p^{(s)}$ by induction such that for each $\alpha \leq \zeta$ we have $p^{(s)} \restriction \alpha \in \mathbb{P}_\alpha$ and $p^{(s)} \restriction \alpha \leq_{F \cap \alpha, i} p \restriction \alpha$.

Assume $p^{(s)} \restriction \alpha$ has been defined; to define $p^{(s)}(\alpha)$, there are two cases to distinguish:

- If $\alpha \notin F$, set $p^{(s)}(\alpha) := \begin{cases} s(\alpha) & \text{if } s \restriction \alpha \in \dot{G}_\alpha \\ p(\alpha) & \text{otherwise.} \end{cases}$
- If $\alpha \in F$, set $p^{(s)}(\alpha) := \begin{cases} s(\alpha) \cup (p(\alpha) \setminus p(\alpha)^{[g(\alpha)]}) & \text{if } s \restriction \alpha \in \dot{G}_\alpha \\ p(\alpha) & \text{otherwise.} \end{cases}$

  Note that we have $p^{(s)} \restriction \alpha \Vdash g(\alpha) \in p(\alpha)$ and

$$p^{(s)} \restriction \alpha \Vdash p^{(s)}(\alpha) \leq_i p(\alpha).$$

To see that $p^{(s)} \restriction \lambda \in \mathbb{P}_\lambda$ for λ limit, we note that $\mathrm{supp}(p^{(s)} \restriction \lambda) \subseteq \mathrm{supp}(p \restriction \lambda) \cup \mathrm{supp}(s \restriction \lambda)$. Furthermore, we clearly have $s \leq p^{(s)}$.

It remains to check the second requirement. Take some (p, F, i)-determined s' following g with $s' \leq p^{(s)}$. Assume inductively that $s' \restriction \alpha \leq s \restriction \alpha$. The case $\alpha \notin F$ is trivial, we may restrict our attention to the case $\alpha \in F$. Then we have $s' \restriction \alpha \Vdash s'(\alpha) \leq_{\mathbb{Q}_\alpha} p^{(s)}(\alpha) = s(\alpha) \cup (p(\alpha) \setminus p(\alpha)^{[g(\alpha)]})$. But then we already have $s' \restriction \alpha \Vdash s'(\alpha) \leq_{\mathbb{Q}_\alpha} s(\alpha)$. In conclusion, $s' \leq s$, which finishes the proof of the lemma. □

Remark 2.12. The above construction yields the following observation: not only is $p^{(s)}$ an interpolant for p, s, F and i, but we even have that $p^{(s)} \restriction \alpha$ is an interpolant for $p \restriction \alpha, s \restriction \alpha, F \cap \alpha$ and i for any $\alpha < \zeta$.

In the next lemma, we show that under certain conditions, the forcing $\mathbb{P}_\zeta$ admits least upper bounds of the form

$$\bigvee_{\substack{s \leq q, \\ s \text{ is } (q,F,i)-\text{determined following } g}} s.$$

Lemma 2.13. Let $p \in \mathbb{P}_\zeta$ and s be (p, F, i)-determined following $g \in \prod_{\beta \in F} \kappa^{<\kappa}$. Then for every $q \leq_{F,i} p^{(s)}$ there exists an $\tilde{s} \leq q, s$ that is (q, F, i)-determined following g such that for every $s' \leq q$, if s' is (q, F, i)-determined following g, then $s' \leq \tilde{s}$. In other words, $\tilde{s}$ is the weakest (q, F, i)-determined condition following g.

Proof. Construct $\tilde{s}$ by induction such that for all $\alpha \leq \zeta$ we have $\tilde{s} \restriction \alpha \in \mathbb{P}_\alpha$, $\tilde{s} \restriction \alpha \leq q \restriction \alpha$ and $\tilde{s} \restriction \alpha$ is $(q \restriction \alpha, F \cap \alpha, i)$-determined following $g \restriction \alpha$.

Assume $\tilde{s} \restriction \alpha$ has been defined; define $\tilde{s}(\alpha)$ as

$$\tilde{s}(\alpha) := \begin{cases} q(\alpha)^{[g(\alpha)]} & \text{if } \alpha \in F \\ q(\alpha) & \text{otherwise.} \end{cases}$$

If $\alpha \notin F$, there is nothing to prove. For $\alpha \in F$, observe that since $\tilde{s} \restriction \alpha \leq q \restriction \alpha$ is $(q \restriction \alpha, F \cap \alpha, i)$-determined following $g \restriction \alpha$ and $q \leq_{F,i} p^{(s)} \leq_{F,i} p$, so by the above remark we can conclude $\tilde{s} \restriction \alpha \leq s \restriction \alpha$. But $s \restriction \alpha \Vdash g(\alpha) \in \text{split}_i(p(\alpha))$ and $q \restriction \alpha \Vdash \text{split}_i(p(\alpha)) = \text{split}_i(q(\alpha))$, hence $\tilde{s}(\alpha)$ is well-defined. The other two properties follow easily.

If λ is a limit, then we have $\text{supp}(\tilde{s} \restriction \lambda) \subseteq \text{supp}(q) \cup F$, hence $\tilde{s} \restriction \lambda \in \mathbb{P}_\lambda$ is a condition.

Knowing $\tilde{s}$ to be well-defined, one can easily see that for each $s' \leq q$ that is (q, F, i)-determined following g we have $s' \leq \tilde{s}$. □

Fact 2.14. $(\mathbb{P}_\zeta, \leq_{F,i})$ is $<\kappa$-closed for all ζ, F, i.

Let us now introduce two auxiliary "boundedness" properties a $\mathbb{P}_\zeta$ condition may exhibit.

Definition 2.15. We say a condition $p \in \mathbb{P}_\zeta$ is (F, i)-*bounded* for $F \in [\zeta]^{<\kappa}, i < \kappa$ iff there exists a $\mu < \kappa$ with

$$\forall \beta \in F: \ p \restriction \beta \Vdash \text{split}_i(p(\beta)) \subseteq \mu^{<\mu}.$$

Fact 2.16. If $p \in \mathbb{P}_\zeta$ is (F, i)-bounded and $p' \leq_{F,i} p$, then p' is as well.

Definition 2.17. Let $\zeta \leq \kappa^{++}, p \in \mathbb{P}_\zeta, F \in [\zeta]^{<\kappa}$ and $i < \kappa$. Take furthermore a $D \subseteq \mathbb{P}_\zeta$ that is open dense below p. We say p is (D, F, i)-*complete* iff there exists a $C \subseteq \prod_{\beta \in F} \kappa^{<\kappa}, |C| < \kappa$ and a family $(s_g)_{g \in C}$ in D such that

a) s_g is (p, F, i)-determined following g for all $g \in C$

b) whenever $s \leq p$ is (p, F, i)-determined following a function g and $s \in D$, then $g \in C$ and $s \leq s_g$

Fact 2.18. If $p \in \mathbb{P}_\zeta$ is (D, F, i)-complete as witnessed by $(s_g)_{g\in C}$, then $(s_g)_{g\in C}$ is a maximal antichain below p.

Lemma 2.19. Let $p' \leq_{F,i} p$ be $\mathbb{P}_\zeta$-conditions such that p is (D, F, i)-complete and p' is (D', F, i)-complete and let the antichains $(s_g)_{g\in C}$ and $(s'_g)_{g\in C'}$ witness this. Then $C' \subseteq C$. If in addition $D' \subseteq D$, then we even have $s'_g \leq s_g$ for each $g \in C'$.

Proof. Assume that $g \in C'$ and find a $t \leq s'_g$ with $t \in D$. Then $t \leq p$ is (p, F, i)-determined following g by Fact 2.10 and thus $g \in C$ and $t \leq s_g$ by the second requirement in the definition of completeness. If $D' \subseteq D$, we may take $t = s'_g$ and get $s'_g \leq s_g$. □

In particular we know that the set C in the definition of completeness is completely determined by p. Complete conditions are also going to be playing a major role later in Lemma 5.1.

Our strategy for proving Property B* for all $\mathbb{P}_\zeta, \zeta \leq \kappa^{++}$ is by the equivalence of the following four statements:

a(ζ): $\mathbb{P}_\alpha$ has Property B* for each $\alpha < \zeta$.

b(ζ): The set of (F, i)-bounded conditions is $\leq_{F,i}$-dense in $\mathbb{P}_\alpha$ for all $\alpha \leq \zeta$, $F \in [\alpha]^{<\kappa}$ and $i < \kappa$.

c(ζ): The set of (D, F, i)-complete conditions is $\leq_{F,i}$-dense in $\mathbb{P}_\zeta$ for all F, i and open dense $D \subseteq \mathbb{P}_\zeta$.

d(ζ): $\mathbb{P}_\zeta$ has Property B*.

The implication a(ζ) ⇒ b(ζ) is Lemma 2.21, b(ζ) ⇒ c(ζ) is Lemma 2.22 and c(ζ) ⇒ d(ζ) is Lemma 2.23. Thus a(ζ) ⇒ d(ζ) establishes an induction by which Property B* is verified for all $\mathbb{P}_\zeta$.

Corollary 2.20. $\mathbb{P}_\zeta$ has Property B* for all $\zeta \leq \kappa^{++}$.

Lemma 2.21. Let $\zeta \leq \kappa^{++}$ and assume $\mathbb{P}_\alpha$ has Property B* for each $\alpha < \zeta$. Take $\alpha \leq \zeta$, $p \in \mathbb{P}_\alpha$, $F \in [\alpha]^{<\kappa}$ and $i < \kappa$. Then there is a condition $q \leq_{F,i} p$ that is (F, i)-bounded.

Proof. We proceed by induction on $\alpha \leq \zeta$.

- $\alpha = 1$: Trivial by the inaccessibility of κ.
- $\alpha \to \alpha + 1$: Since $\mathbb{P}_\alpha$ is $<\kappa$-closed, κ remains inaccessible in $V^{\mathbb{P}_\alpha}$. Thus

$$\Vdash_{P_\alpha} \forall \beta \in F\, \exists \mu_\beta < \kappa : \ \mathrm{split}_i(p(\beta)) \subseteq \mu_\beta^{<\mu_\beta}$$

  and considering $\sup_{\beta \in F} \mu_\beta$ we can find a name $\dot\mu$ for an ordinal less than κ with

$$\Vdash_{P_\alpha} \forall \beta \in F : \ \mathrm{split}_i(p(\beta)) \subseteq \dot\mu^{<\dot\mu}.$$

  Let now $A \subseteq \mathbb{P}_\alpha$ be a maximal antichain deciding $\dot\mu$; we may find a $\mathbb{P}_\alpha \ni \hat q \leq_{F\cap\alpha,i} p \restriction \alpha$ with $|A \restriction \hat q| < \kappa$. Thus

$$\hat q \Vdash \dot\mu < \mu_q$$

for some $\mu_q < \kappa$ and therefore

$$\forall \beta \in F: \ \hat{q} \restriction \beta \Vdash \operatorname{split}_i(p(\beta)) \subseteq \mu_q^{<\mu_q}.$$

Setting $q := \hat{q}^\frown p(\alpha)$ and noting that since $\hat{q} \leq_{F\cap\alpha,i} p \restriction \alpha$ we have $q \restriction \beta \Vdash \operatorname{split}(q(\beta)) = \operatorname{split}(p(\beta))$ for all $\beta \in F$, so it follows that q is (F,i)-bounded.

- $\lambda \leq \zeta$ is a limit: By $<\kappa$-closure of $(\mathbb{P}_\lambda, \leq_{F,i})$ and the inductive assumption, we can construct a $\leq_{F,i}$-decreasing sequence $(q_\beta)_{\beta\in F}$ in $\mathbb{P}_\lambda$ with the following properties:
  - $\forall \beta \in F \, \forall \beta' \in F \cap \beta: \ q_\beta \leq_{F,i} q'_\beta \leq_{F,i} p$
  - $\forall \beta \in F \, \exists \mu_\beta < \kappa \, \forall \beta' \in F \cap (\beta+1): \ q_\beta \restriction \beta' \Vdash_{\mathbb{P}_{\beta'}} \operatorname{split}_i(q_\beta(\beta')) \subseteq \mu_\beta^{<\mu_\beta}$.

  Using $<\kappa$-closure of $(\mathbb{P}_\lambda, \leq_{F,i})$, set q to a $\leq_{F,i}$-lower bound of $(q_\beta)_{\beta\in F}$ and $\mu := \sup_{\beta\in F} \mu_\beta$. Now $q \leq_{F,i} p$ and

$$\forall \beta \in F: \ q \restriction \beta \Vdash \operatorname{split}_i(q(\beta)) \subseteq \mu^{<\mu}.$$

□

Lemma 2.22. Let $\zeta \leq \kappa^{++}, F \in [\zeta]^{<\kappa}, i < \kappa$ and suppose $p \in \mathbb{P}_\zeta$ is (F,i)-bounded. Let furthermore $D \subseteq \mathbb{P}_\zeta$ be open dense below p. Then there is a $q \leq_{F,i} p$ which is (D,F,i)-complete.

Proof. By assumption p is (F,i)-bounded, hence we can find a μ such that

$$\forall \beta \in F: \ p \restriction \beta \Vdash \operatorname{split}_i(p(\beta)) \subseteq \mu^{<\mu}.$$

Our strategy is to consider all possible choices a (p,F,i)-determined condition might make at the i-th splitting front of p and then interpolate on the witnesses of such choices. Since we have a uniform bound μ on the respective splitting fronts, this will require us to only iterate through $<\kappa$ many possibilities. Set $\tilde{\mu}_\beta := \sup_{j\leq\mu} f_\beta(j)$ and consider the set

$$\tilde{C} := \prod_{\beta\in F} \tilde{\mu}_\beta^{\leq\mu}.$$

Whenever s is (p,F,i)-determined following some g, then $g \in \tilde{C}$. Enumerate $\tilde{C}$ as $(g_{j+1})_{j<\delta}$ with $\delta < \kappa$. We now construct a $\leq_{F,i}$-decreasing sequence $(t_j)_{j<\delta}$:

- $j = 0$: Set $t_0 := p$.
- $j \to j+1$: If there exists an $s \in D, s \leq t_j$ that is (p,F,i)-determined following g_{j+1}, take an arbitrary such condition and call it $\tilde{s}_{g_{j+1}}$. Set $t_{j+1} := t_j^{(\tilde{s}_{g_{j+1}})}$. If there is no such s, simply set $t_{j+1} := t_j$. In any case we have $t_{j+1} \leq_{F,i} t_j$.
- λ is a limit: Set t_λ to a $\leq_{F,i}$-lower bound of $(t_j)_{j<\lambda}$.

Set q to a $\leq_{F,i}$-lower bound of $(t_j)_{j<\delta}$. We know $q \leq_{F,i} p$. Now let

$$C := \left\{g \in \tilde{C} : \ \tilde{s}_g \text{ exists}\right\},$$

i.e. C is the set of all g_{j+1} for which a witness was found in the inductive step $j \to j+1$. We have $|C| < \kappa$. Finally, for each $g = g_{j+1} \in C$ apply Lemma 2.13 to $p = t_j$, $s = \tilde{s}_{g_{j+1}}$ and $q = q$ to construct the condition s_g. We have $s_g \in D$ since $s_g \leq \tilde{s}_g \in D$ and D is open.

We verify that q is (D, F, i)-complete, witnessed by $(s_g)_{g \in C}$. The first condition in the definition of completeness follows by construction. The second follows immediately from Lemma 2.13 by noting that if $s \leq q$ is (q, F, i)-determined following g, then $g = g_{j+1}$ for some $j < \delta$, and thus a witness was found in the inductive step $j \to j+1$ and $g \in C$. □

Lemma 2.23. If the set of (D, F, i)-complete conditions is $\leq_{F,i}$-dense in $\mathbb{P}_\zeta$ for all F, i and $D \subseteq \mathbb{P}_\zeta$ open dense, then $\mathbb{P}_\zeta$ has Property B*.

Proof. We have seen in Lemma 2.6 that the fusion condition is always fulfilled. We will now prove that $\mathbb{P}_\zeta$ fulfils the antichain condition: let $A \subseteq \mathbb{P}_\zeta$ be a maximal antichain, $p \in \mathbb{P}_\zeta$, $F \in [\zeta]^{<\kappa}$ and $i < \kappa$. Find a $q \leq_{F,i} p$ that is (D, F, i)-complete, where

$$D = \{s : \ |A \restriction s| = 1\}$$

and let $(s_g)_{g \in C}$ witness this. Since $(s_g)_{g \in C}$ is a maximal antichain below q by Fact 2.18, it is easy to see that

$$A \restriction q \subseteq \{r \in A : \ \exists g \in C : \ A \restriction s_g = \{r\}\}$$

and thus $|A \restriction q| \leq |C| < \kappa$. □

From this point onward, assume that

$$V \models |2^\kappa| = \kappa^+.$$

From among our stated goals at the beginning of this section, only one remains to be verified; our interest now turns to the κ^{++}-chain condition:

Theorem 2.24. $\mathbb{P}$ has the κ^{++}-c.c.

This will follow easily from Lemma 2.28 once we have proven that each $\mathbb{P}_\alpha$ for $\alpha < \kappa^{++}$ has a dense subset of size κ^+.

For the purposes of the next definition, for each $\alpha < \kappa^{++}$ fix a $\mathbb{P}_\alpha$-name $\dot{c}_\alpha$ for a bijection $c_\alpha : (PT_{f_\alpha})^{V^{\mathbb{P}_\alpha}} \to (\mathcal{P}(\kappa))^{V^{\mathbb{P}_\alpha}}$ such that $c_\alpha(\mathbb{1}_{PT_{f_\alpha}}) = \emptyset$. In particular, there is a canonical embedding $H_\alpha \hookrightarrow H_{\alpha'}$ for $\alpha < \alpha'$ (see below).

Definition 2.25. Let $\alpha < \kappa^{++}$.

- A $\mathbb{P}_\alpha$-name $\dot\tau$ for a subset of κ is α-good if $\dot\tau$ is a nice name of the form

$$\dot\tau = \{\{j\} \times A_j : j < \kappa\},$$

where $A_j \subseteq H_\alpha$ and $|A_j| \leq \kappa$ for all $j < \kappa$.

- A condition $p \in \mathbb{P}_\alpha$ is in H_α iff $p \restriction \beta \in H_\beta$ for each $\beta < \alpha$ and, if $\alpha = \beta + 1$ is a successor, additionally either
  - there is a β-good name $\dot\tau$ such that $p \restriction \beta \Vdash_\beta \dot c_\beta(p(\beta)) = \dot\tau$

  or

  - there exists a fusion sequence $\langle (p_i, F_i) : i < \delta\rangle$ of length $\delta \leq \kappa$ consisting of H_α conditions such that p is its canonical fusion limit (see Remark 2.7).

Remark 2.26. H_α-conditions and α-good names appeared first as H_κ-$\mathbb{P}_\alpha$-names in [BGS18] and are themselves a straightforward generalization of hereditarily countable names as introduced in [She98].

Lemma 2.27. *For every $\alpha < \kappa^{++}$, $F \in [\alpha]^{<\kappa}$ and $i < \kappa$, H_α is $\leq_{F,i}$-dense in $\mathbb{P}_\alpha$ and $|H_\alpha| = \kappa^+$.*

Proof. We prove the statements by induction on α.

- $\alpha = 1$: We have $H_1 = \mathbb{P}_1$ and $|\mathbb{P}_1| = |PT_{f_0}| = \kappa^+$.
- $\alpha \to \alpha+1$: Let $p \in \mathbb{P}_{\alpha+1}$, $F \in [\alpha+1]^{<\kappa}$ and $i < \kappa$. Using the inductive hypothesis, we may assume $p \restriction \alpha \Vdash_\alpha \dot c_\alpha(p(\alpha)) = \{\{j\} \times A_j : j < \kappa\}$ with $A_j \subseteq H_\alpha \subseteq \mathbb{P}_\alpha$ for all $j < \kappa$. Additionally using Property B*, construct a fusion sequence $\langle q_j, F_j : j < \kappa\rangle$ with
  - $q_0 \leq_{F\cap\alpha,i} p \restriction \alpha$,
  - $\forall \delta < \kappa\ \forall j < \delta : q_\delta \leq_{F_j,i+j} q_j$ and $F \cap \alpha \subseteq F_j$ [2],
  - $\forall j < \kappa : q_j \in H_\alpha$ and $|A_j \restriction q_j| < \kappa$.

  and let q_κ be its fusion limit. By induction on β we have $q_\kappa \restriction \beta \in H_\beta$ for all $\beta \leq \alpha$. By the third property of the fusion sequence,

  $$\dot\tau = \{\{j\} \times (A_j \restriction q_\kappa) : j < \kappa\}$$

  is an α-good name and $q_\kappa \Vdash_\alpha \dot c_\alpha(p(\alpha)) = \dot\tau$, thus $H_{\alpha+1} \ni (q_\kappa {}^\frown p(\alpha)) \leq_{F,i} p$.

  Since $|H_\alpha| = \kappa^+$ and there are only $|\kappa^+ \times (\kappa^+)^\kappa| = \kappa^+$-many good names for reals, we get $|H_{\alpha+1}| = \kappa^+$ by standard arguments.
- γ is a limit: If $\mathrm{cf}(\gamma) = \kappa^+$, density is trivial and $|H_\gamma| \leq |\bigcup_{\beta<\gamma} H_\beta| \leq \kappa^+$.

  Assume $\mathrm{cf}(\gamma) = \delta \leq \kappa$ and let furthermore $p \in \mathbb{P}_\gamma$, $F \in [\gamma]^{<\kappa}$ and $i < \kappa$ be given. For a cofinal sequence $(\beta_j)_{j<\delta}$ once again construct a fusion sequence $\langle q_j, F_j : j < \delta\rangle$ with

[2] Use a bookkeeping argument to construct the F_j.

- $\forall j < \delta : q_j \in H_\gamma$,
- $\forall j < \delta : q_j \restriction \beta_j \leq_{F_j \cap \beta_j, i+j} p \restriction \beta_j$ and $F \cap \beta_j \subseteq F_j$
- $\forall j < \delta \, \forall \ell < j : q_j \leq_{F_\ell, i+\ell} q_\ell$,

where the F_j are constructed using a bookkeeping argument. Set q_δ to be the fusion limit; then we have $H_\gamma \ni q_\delta \leq_{F,i} p$. Lastly, $|H_\gamma| \leq \left|\prod_{j<\delta} H_{\beta_j}\right| \leq \kappa^+$.

□

Lemma 2.28. Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta : \alpha \leq \zeta, \beta < \zeta \rangle$ be an iteration such that

$$\forall \alpha < \zeta : \; P_\alpha \text{ has the } \theta\text{-c.c.},$$

where θ is a regular uncountable cardinal. If $\mathbb{P}_\zeta$ is a direct limit and, additionally, either $\mathrm{cf}(\zeta) \neq \theta$ or the set $\{\alpha < \zeta : \; \mathbb{P}_\alpha \text{ is a direct limit}\}$ is stationary, then $\mathbb{P}_\zeta$ has the θ-c.c.

Proof. See [Jec03, Theorem 16.30]. □

Proof of Theorem 2.24. By Lemma 2.27, each $\mathbb{P}_\alpha$ has a dense subset of size $\leq\kappa^+$ and therefore satisfies the κ^{++}-c.c.; our desired conclusion thus follows easily from Lemma 2.28 and by noting that the set $\{\alpha < \kappa^{++} : \mathrm{cf}(\alpha) = \kappa^+\}$ is stationary in κ^{++}. □

As we have remarked at the beginning of this section, we get the following corollary:

Corollary 2.29. Forcing with $\mathbb{P}_\alpha, \alpha \leq \kappa^{++}$ does not collapse cardinals.

Lemma 2.30. We have

- If $\alpha < \kappa^{++}$, then $V^{\mathbb{P}_\alpha} \models |2^\kappa| = \kappa^+$.
- If $\mathrm{cof}(\alpha) > \kappa$, then $V^{\mathbb{P}_\alpha} \models 2^\kappa = \bigcup_{\beta<\alpha}(2^\kappa \cap V^{\mathbb{P}_\beta})$.
- $V^{\mathbb{P}} \models |2^\kappa| = \kappa^{++}$.

Proof. Suppose $\alpha < \kappa^{++}$. Let $\dot{\tau}$ be a $\mathbb{P}_\alpha$-name and $p \in \mathbb{P}_\alpha$ force $\dot{\tau}$ to be a subset of κ. Without loss of generality assume $\dot{\tau} = \{\{j\} \times A_j : j < \kappa\}$ is a nice name with $A_j \subseteq H_\alpha$ for all $j < \kappa$. Just like in the previous lemma, construct a fusion sequence $\langle q_j, F_j : j < \kappa \rangle$ below p with $|A_j \restriction q_j| < \kappa$ for all $j < \kappa$. The fusion limit q_κ forces $\dot{\tau}$ to be equal to an α-good name, of which there are only κ^+-many. If we additionally assume $\mathrm{cf}(\alpha) > \kappa$, then q_κ forces $\dot{\tau}$ to be equal to a $\mathbb{P}_\gamma$-name for some $\gamma < \alpha$. The first two statements thus follow by a density argument.

The last point follows immediately from the previous two. □

For $\alpha < \kappa^{++}$ we can define in $V^{\mathbb{P}_\alpha}$ the tail iteration $\mathbb{P}_{\alpha,\kappa^{++}}$ as the limit of the $\leq\kappa$-support iteration $\langle \tilde{\mathbb{P}}_\gamma, \dot{\tilde{\mathbb{Q}}}_\beta : \gamma \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ where $\Vdash_{\tilde{\mathbb{P}}_\gamma} \dot{\tilde{\mathbb{Q}}}_\gamma = \dot{\mathbb{Q}}_{\alpha+\gamma}$. It follows from standard proper forcing arguments that $\mathbb{P} \simeq \mathbb{P}_\alpha \star \mathbb{P}/_{\dot{G}_\alpha}$ is dense in $\mathbb{P}_\alpha \star \mathbb{P}_{\alpha,\kappa^{++}}$.

3 First Proof

We are now equipped to present the first proof of the relative consistency of

$$\mathrm{ZFC} + |2^\kappa| = \kappa^{++} + \mathcal{SN} = [2^\kappa]^{\leq\kappa^+}.$$

Starting with a model of $|2^\kappa| = \kappa^+$, we consider a $\leq\kappa$-supported forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ with

$$\forall \alpha < \kappa^{++} : \Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = PT_{f_\alpha},$$

where each increasing $f \in \kappa^\kappa \cap V$ appears as an f_α cofinally often. Set $\mathbb{P} := \mathbb{P}_{\kappa^{++}}$. By Lemma 2.30 we see $V^{\mathbb{P}} \models |2^\kappa| = \kappa^{++}$.

By a density argument, the α-th generic real $\dot{s}_\alpha$ will encode a covering of the ground model reals satisfying the 'challenge' f_α. For this argument it is sufficient that only f_α from some dominating family appear cofinally often; from the perspective of some intermediate model $V^{\mathbb{P}_\alpha}$, the tail forcing $\mathbb{P}_{\alpha,\kappa^{++}}$ fulfils this criterion. Hence the observation can be extended to the set of reals appearing already in some $V^{\mathbb{P}_\alpha}$; the following theorem formalizes this.

Theorem 3.1. $V^{\mathbb{P}} \models \forall \alpha < \kappa^{++} : 2^\kappa \cap V^{\mathbb{P}_\alpha} \in \mathcal{SN}$.

Proof. Take $\alpha < \kappa^{++}$ and $f \in \kappa^\kappa$. Since $\mathbb{P}$ is κ^κ-bounding, we find an $h \in \kappa^\kappa \cap V$ with $f \leq h$ and $\beta > \alpha$ with $f_\beta(i) = |2^{h(i)}|$ for all $i < \kappa$. In V we may construct bijections $c_\gamma : |2^\gamma| \to 2^\gamma$ for $\gamma < \kappa$.

Working now in $V^{\mathbb{P}_\beta}$, recall that $2^{<\kappa} \cap V = 2^{<\kappa} \cap V^{\mathbb{P}_\beta}$, thus we can define the function $\dot{\sigma}(i) = c_{h(i)}(\dot{s}_\beta(i))$. For $x \in 2^\kappa \cap V^{\mathbb{P}_\alpha}$ the set

$$D_x := \{p \in \mathbb{Q}_\beta : \ \exists i < \kappa : p \Vdash \dot{\sigma}(i) = x \restriction h(i)\}$$

is dense; in fact, it is easy to see that for any $p \in \mathbb{Q}_\beta$ and $\eta \in \mathrm{split}(p)$, $j = \mathrm{dom}(\eta)$ we have $p^{[\eta^\frown c_{h(j)}^{-1}(x \restriction h(j))]} \in D_x$. Hence $(\sigma(i))_{i<\kappa}$ provides the required covering for the challenge f and $2^\kappa \cap V^{\mathbb{P}_\alpha} \in \mathcal{SN}$ follows. □

If $V^{\mathbb{P}} \models X \subseteq 2^\kappa, |X| \leq \kappa^+$, then by the κ^{++}-c.c., X already appears at some intermediate stage $V^{\mathbb{P}_\alpha}$. We thus get one direction of our desired result by the previous theorem.

Theorem 3.2. $V^{\mathbb{P}} \models [2^\kappa]^{\leq\kappa^+} \subseteq \mathcal{SN}$.

In order to lift the arguments appearing in [GJS93], we require additional large cardinal assumptions on κ. A priori it is sufficient for our purposes for κ to merely be weakly compact, since the only occasion at which a property stronger than inaccessibility is utilized is a crucial invocation of the tree property in Lemma 3.5. However, the aforementioned lemma is invoked not only in V, but also at intermediate stages $V^{\mathbb{P}_\alpha}$; it might be the case that weak compactness of κ is by that point destroyed.

The following large cardinal property was introduced by Villaveces [Vil96, Definition 4]:

Definition 3.3. Let θ be an ordinal. We say an inaccessible cardinal κ is *θ-strongly unfoldable* iff for all transitive models M of ZF^- (ZF without the Power Set Axiom) such that $|M| = \kappa, \kappa \in M$ and ${}^{<\kappa}M \subseteq M$ there exists a transitive model N with $V_\theta \cup \{\theta\} \subseteq N$ and an elementary $j : M \to N$ with critical point κ and $j(\kappa) \geq \theta$.

Furthermore, call κ strongly unfoldable iff it is θ-strongly unfoldable for all θ.

Strongly unfoldable cardinals are weakly compact and are downwards absolute to L [Vil96]. Villaveces also observed that Ramsey cardinals are strongly unfoldable in L (though they may fail to be such in V). The consistency strength of a strongly unfoldable cardinal thus slots between a weakly compact and Ramsey cardinal, with it being a conservative enough strengthening of weak compactness as to still be consistent with $V = L$.

Of interest to us is a preservation theorem by Johnstone [Joh08].

Theorem 3.4 (Johnstone [Joh08])**.** For any κ strongly unfoldable there is a forcing extension in which the strong unfoldability of κ is indestructible under $<\kappa$-closed, κ-proper forcing notions.

We stress that the full strength of strong unfoldability is not used in our proof; we merely require it in order to make the weak compactness of κ indestructible by the forcings $\mathbb{P}_\alpha$.

For a strongly unfoldable κ, after forcing indestructibility using Johnstone's theorem, we may collapse a potentially blown up 2^κ back to κ^+ with a $<\kappa^+$-closed forcing [3]. Throughout this section we may therefore assume

$$V \models \text{“}|2^\kappa| = \kappa^+ + \text{ the strong unfoldability of } \kappa \text{ is indestructible under } <\kappa\text{-closed, } \kappa\text{-proper forcing notions”}.$$

We now set out to prove $V^{\mathbb{P}} \models \mathcal{SN} \subseteq [2^\kappa]^{\leq \kappa^+}$.

The statement of the next two lemmas takes place in $V^{\mathbb{P}_\alpha}$. Recall that $\mathbb{P}_{\alpha,\kappa^{++}}$ denotes the tail forcing.

Lemma 3.5. Let $\alpha < \kappa^{++}$ be an ordinal. Let furthermore $\dot\tau$ be a $\mathbb{P}_{\alpha,\kappa^{++}}$-name for a real in 2^κ, $p \in \mathbb{P}_{\alpha,\kappa^{++}}$ a condition, $i < \kappa$ and $F \in [\kappa^{++}]^{<\kappa}$. Assume $p \Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} \dot\tau \notin V^{\mathbb{P}_\alpha}$. Then there exists a $\delta < \kappa$ such that

$$\forall \eta \in 2^\delta \, \exists q \leq_{F,i} p : \; q \Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} \eta \not\subseteq \dot\tau.$$

We will write $\delta_{p,F,i}$ for the least such δ.

Proof. Suppose not. Then we can find $\alpha, \dot\tau, F, i$ and p with

$$\forall \delta < \kappa \, \exists \eta_\delta \in 2^\delta : \; \neg(\exists q \leq_{F,i} p : \; q \Vdash \eta \not\subseteq \dot\tau).$$

[3] $<\kappa^+$-closed forcings and two-step iterations of κ-proper forcings are κ-proper.

Set $T := \{\eta_\delta \restriction \ell : \delta < \kappa \wedge \ell \leq \delta\}$. By virtue of the preparation of κ,

$$V^{\mathbb{P}_\alpha} \models \kappa \text{ is weakly compact}$$

and therefore, since T is a $<\kappa$-splitting tree of height κ, it has a branch b^* in $V^{\mathbb{P}_\alpha}$. Since $p \Vdash \dot\tau \notin V^{\mathbb{P}_\alpha}$, there is a name $\dot\ell$ for an ordinal less than κ such that $p \Vdash \dot\tau \restriction \dot\ell \neq b^* \restriction \dot\ell$. As $\mathbb{P}_{\alpha,\kappa^{++}}$ satisfies Property B*, there is a $q \leq_{F,i} p$ and $\ell^* < \kappa$ with $q \Vdash \dot\ell < \ell^*$.

Since $b^* \restriction \ell^* \in T$, there is a $\delta \geq \ell^*$ such that $b^* \restriction \ell^* = \eta_\delta \restriction \ell^*$. But this means $q \Vdash \dot\tau \restriction \ell^* \neq b^* \restriction \ell^* = \eta_\delta \restriction \ell^*$ and therefore $q \Vdash \eta_\delta \nsubseteq \dot\tau$, a contradiction. □

In the following we refer to pointwise (everywhere) domination $\leq$ and not just the eventually dominating relation. For a $<\kappa$-closed, κ^κ-bounding forcing, the ground model κ-reals form a pointwise dominating family.

Definition 3.6. Let $D \subseteq \kappa^\kappa$ be a dominating family. We say that H has index D iff $H = \{h_f : f \in D\}$ and $\forall i < \kappa \colon h_f(i) \in 2^{f(i)}$.

Fact 3.7.

$$X \in \mathcal{SN} \Leftrightarrow \forall D \,\text{dominating}\ \exists H \,\text{with index}\, D \colon X \subseteq \bigcap_{f \in D} \bigcup_{\alpha < \kappa} [h_f(\alpha)].$$

Lemma 3.8. Let $D \in V$ be a dominating family, $\alpha < \kappa^{++}$ and $H \in V^{\mathbb{P}_\alpha}$ have index D. Let furthermore $\dot\tau$ be a name for an element of 2^κ with $\Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} \dot\tau \notin V^{\mathbb{P}_\alpha}$. Then we have

$$\Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} \dot\tau \notin \bigcap_{f \in D} \bigcup_{i < \kappa} [h_f(i)].$$

Proof. We prove the claim with a density argument, let therefore $p \in \mathbb{P}_{\alpha,\kappa^{++}}$ be arbitrary. Within $V^{\mathbb{P}_\alpha}$ we will construct an increasing sequence $(\delta_i)_{i<\kappa}$ of ordinals less than κ. On the tree

$$T := \{g \in \prod_{j \leq i} 2^{\delta_j} : i < \kappa\}$$

we shall construct a mapping $\mathfrak{q} : T \to \mathbb{P}_{\alpha,\kappa^{++}}$ and a sequence of increasing sets $(F_i)_{i<\kappa}$ with $F_i \in [\alpha, \kappa^{++}]^{<\kappa}$ such that whenever $b \in \prod_{j<\kappa} 2^{\delta_j}$ is a branch of T in $V^{\mathbb{P}_\alpha}$, then

$$\langle \mathfrak{q}(b \restriction i), F_i : \ i < \kappa \rangle$$

is a fusion sequence below p. Each condition $\mathfrak{q}(g)$ will carry some information about an increasingly long initial segment of $\dot\tau$. More specifically, we want to ensure that for all $i < \kappa$ and $g \in \prod_{j \leq i} 2^{\delta_j}$ we have

$$\mathfrak{q}(g) \Vdash g(i) \nsubseteq \dot\tau.$$

We define $\mathfrak{q}(g)$ for $g \in \prod_{j \leq i} 2^{\delta_j}$ by induction in i.

- $i = 0$: By Lemma 3.5 we can find a δ_0 and $\mathfrak{q}(s_0) \leq p$ for every $\eta_0 \in 2^{\delta_0}$ such that $\mathfrak{q}(\eta_0) \Vdash \eta_0 \nsubseteq \dot\tau$. Set $F_0 = \emptyset$.

- $i \to i+1$: Assume $\mathfrak{q}(g)$ is defined for every $g \in \prod_{j \leq i} 2^{\delta_j}$. Using Lemma 3.5 we can again define $\delta_{i+1} := \sup\{\delta_{\mathfrak{q}(g),F_i,i} : g \in \prod_{j\leq i} 2^{\delta_j}\}$ and for every $g \in \prod_{j \leq i} 2^{\delta_j}, \eta_{i+1} \in 2^{\delta_{i+1}}$ find a condition $\mathfrak{q}(g^\frown \eta_{i+1}) \leq_{F_i,i} \mathfrak{q}(g)$ with
$$\mathfrak{q}(g^\frown \eta_{i+1}) \Vdash \eta_{i+1} \not\subseteq \dot{\tau}.$$
Use a bookkeeping argument to define F_{i+1}.
- $\lambda < \kappa$ is a limit: By construction, for every $h \in \prod_{j<\lambda} 2^{\delta_j}$ the sequence $(\mathfrak{q}(h \restriction j))_{j<\lambda}$ is a fusion sequence. Set $\mathfrak{q}(h)$ to be a fusion limit of said sequence, $F_\lambda := \bigcup_{j<\lambda} F_j$ and $\delta_\lambda := \sup\{\delta_{\mathfrak{q}(h),F_\lambda,\lambda} : h \in \prod_{j<\lambda} 2^{\delta_j}\}$. Once again using Lemma 3.5 we can find $\mathfrak{q}(h^\frown \eta_\lambda) \leq_{F_\lambda,\lambda} \mathfrak{q}(h)$ for every $\eta_\lambda \in 2^\lambda$. Note that since $\mathfrak{q}(h^\frown \eta_\lambda) \leq_{F_\lambda,\lambda} \mathfrak{q}(h) \leq_{F_j,j} \mathfrak{q}(h \restriction j)$ for every $j < \lambda$, we still have
$$\mathfrak{q}(h^\frown \eta_\lambda) \leq_{F_j,j} \mathfrak{q}(h \restriction j).$$

This concludes the construction of $\mathfrak{q}$. Let now $f \in D$ dominate the function $i \mapsto \delta_i$ and set $\eta_i := h_f(i) \restriction \delta_i$. Now $(\mathfrak{q}(\langle \eta_0, \eta_1, \dots, \eta_j \rangle))_{j<\kappa}$ is a fusion sequence and has a fusion limit q_κ. It follows that
$$q_\kappa \Vdash \eta_i \not\subseteq \dot{\tau}$$
for each $i < \kappa$ and therefore $q_\kappa \Vdash \dot{\tau} \notin \bigcap_{f\in D} \bigcup_{i<\kappa} [h_f(i)]$. Thus the set of conditions that force $\dot{\tau} \notin \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)]$ is dense in $\mathbb{P}_{\alpha,\kappa^{++}}$. □

We see that every intermediate model $V^{\mathbb{P}_\alpha}$ believes that a set X which contains a real appearing in a later model will never be strong measure zero with respect to any test conducted in $V^{\mathbb{P}_\alpha}$. This essentially gives us our theorem.

Theorem 3.9. $V^{\mathbb{P}} \models \mathcal{SN} = [2^\kappa]^{\leq \kappa^+}$.

Proof. The $\supseteq$-direction is Theorem 3.2. For the other direction, let $X \in V^{\mathbb{P}}$ be of size κ^{++} and D be a dominating family in $V^{\mathbb{P}}$ which lies in V. We will show that there is no $H \in V^{\mathbb{P}}$ with index D such that
$$X \subseteq \bigcap_{f\in D} \bigcup_{i<\kappa} [h_f(i)],$$
hence X is not strong measure zero by Fact 3.7. Towards a contradiction, note that since D appears in V, such an H can have cardinality at most κ^+. Since $\mathbb{P}$ fulfils the κ^{++}-c.c., we know H must already appear in some $V^{\mathbb{P}_\alpha}$. But $|X| = \kappa^{++}$, thus there must be an $x \in X$ with $x \notin V^{\mathbb{P}_\alpha}$. Let $\dot{x}$ be a $\mathbb{P}_{\alpha,\kappa^{++}}$-name such that
$$\Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} \dot{x} \in \dot{X} \wedge \dot{x} \notin V^{\mathbb{P}_\alpha};$$
then by Lemma 3.8 we have
$$\Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} \dot{x} \notin \bigcap_{f\in D} \bigcup_{i<\kappa} [h_f(i)],$$
and X is not strong measure zero, a contradiction. □

4 Coding of Continuous Functions

For the reader's convenience we collect some selected facts about the coding of continuous functions that are going to find use in the next section.

Throughout this section, every tree T is assumed to be a tree on $2^{<\kappa}$.

Definition 4.1. Let T be a tree and $(T_\eta)_{\eta\in 2^{<\kappa}}$ a family of trees. Then $\langle T, (T_\eta)_{\eta\in 2^{<\kappa}}\rangle$ is a code for a continuous function iff

1. if $\eta_1 \perp \eta_2$, then $[T_{\eta_1}] \cap [T_{\eta_2}] = \emptyset$,
2. if $\eta_1 \lhd \eta_2$, then $[T_{\eta_2}] \subseteq [T_{\eta_1}]$
3. $\bigcup_{\eta\in 2^i}[T_\eta] = [T]$ for each $i < \kappa$.

Theorem 4.2. If $\mathcal{P}$ is a $<\kappa$-closed forcing notion, then $\mathbf{\Pi}^1_1$-absoluteness holds between V and $V^{\mathcal{P}}$, i.e. $(V^V_{\kappa+1}, V^V_\kappa, \in) \prec_{\Pi_1} (V^{(V^{\mathcal{P}})}_{\kappa+1}, V^{(V^{\mathcal{P}})}_\kappa, \in)$ [4].

Proof. See [FKK14]. □

Lemma 4.3. Let $\langle T, (T_\eta)_{\eta\in 2^{<\kappa}}\rangle$ be a code. Then there exists a unique continuous function $g_{\langle T,(T_\eta)_{\eta\in 2^{<\kappa}}\rangle} : [T] \to 2^\kappa$ such that

$$g^{-1}_{\langle T,(T_\eta)_{\eta\in 2^{<\kappa}}\rangle}([\eta]) = [T_\eta]$$

for all $\eta \in 2^{<\kappa}$.

Proof. If we set $g(y) := \bigcup\{\eta \in 2^{<\kappa} : y \in [T_\eta]\}$, then it is easy to see that $g : [T] \to 2^\kappa$ is a well-defined continuous function and $g^{-1}([\eta]) = [T_\eta]$ for all $\eta \in 2^{<\kappa}$. Since $([\eta])_{\eta\in 2^{<\kappa}}$ forms a clopen basis of 2^κ, uniqueness is given. □

On the other hand, if $g : Y \to 2^\kappa$ is a continuous function where $Y \subseteq 2^\kappa$ is closed, then $\langle T, (T_\eta)_{\eta\in 2^{<\kappa}}\rangle$ is a code for g, where T_η are trees with $[T_\eta] = g^{-1}([\eta])$ and $[T] = Y$.

Definition 4.4. For codes c, c' define $c \preccurlyeq c' :\Leftrightarrow g_c \subseteq g_{c'}$.

Clearly $\preccurlyeq$ is reflexive and transitive.

Definition 4.5. A function $g : Y \to Z$ with $Y, Z \subseteq 2^\kappa$ is uniformly continuous iff

$$\forall i < \kappa\, \exists j(i) < \kappa\, \forall x \in Y : \; g''([x \restriction j(i)] \cap Y) \subseteq [g(x) \restriction i] \cap Z.$$

The map $i \mapsto j(i)$ is the *modulus of continuity* of g.

Fact 4.6. The following statements are $\mathbf{\Pi}^1_1$ and therefore absolute for $<\kappa$-closed forcing extensions:

- c is a code for a continuous function

[4] Note that $V^V_\kappa = V^{(V^{\mathcal{P}})}_\kappa$.

- "$[T] = [T']$" for trees T, T'
- "$c \preccurlyeq c'$" for codes c, c'
- "$\mathrm{ran}(g_c) \subseteq [T]$" for a code c and a tree T
- g_c is uniformly continuous with modulus of continuity $i \mapsto j(i)$

It is easy to prove that if $c \in V$ is a code and $\mathcal{P}$ a $<\kappa$-closed forcing notion, then $(g_c)^{V^{\mathcal{P}}}$ extends $(g_c)^V$.

Let now $Y \subseteq 2^\kappa$ be closed and $g : Y \to 2^\kappa$ be continuous. The above thus yields a method to continuously and uniquely extend g to $\tilde{g} : Y^{(V^{\mathcal{P}})} \to (2^\kappa)^{(V^{\mathcal{P}})}$. To do so, take an arbitrary tree T such that $[T] = Y$, then choose a code c for g as a function from $[T]$ to 2^κ and evaluate c in $V^{\mathcal{P}}$. By Fact 4.6 the function $\tilde{g} = (g_c)^{V^{\mathcal{P}}}$ is an extension of g. Furthermore, $\tilde{g}$ is independent of the chosen code c, since the statement $c \preccurlyeq c'$ is $\mathbf{\Pi}^1_1$ and thus absolute. Lastly, we note that $\tilde{g}$ is the unique extension of g, since $[T]^V$ is dense in $[T]^{V^{\mathcal{P}}}$.

By $<\kappa$-closure, $2^{<\kappa} \cap V = 2^{<\kappa} \cap V^{\mathcal{P}}$ and thus total functions g extend to total functions $\tilde{g}$.

In the future we will not be making a notational distinction between g and $\tilde{g}$.

5 Second Proof

In this section we will construct a model in which every $X \subseteq 2^\kappa$ of size $|2^\kappa|$ can be uniformly continuously mapped onto 2^κ. The construction closely follows Corazza's approach [Cor89].

We will consider the same forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ with $\leq\kappa$-support as in the previous section. Additionally, we also choose $\dot{\mathbb{Q}}_\alpha$ to be κ-Sacks forcing (i.e. $f_\alpha \equiv 2$) for $\alpha = 0$ and for α with cofinality κ^+. We still assume $V \models |2^\kappa| = \kappa^+$, but κ is only required to be inaccessible this time.

Since the forcing iteration is identical to the one in the previous section, Theorem 3.2 holds and thus

$$V^{\mathbb{P}} \models [2^\kappa]^{\leq \kappa^+} \subseteq \mathcal{SN}.$$

The other direction of the proof hinges on a technical lemma.

Lemma 5.1. Let $p \in \mathbb{P}, F \in [\kappa^{++}]^{<\kappa}, i < \kappa$, $Y \in [2^\kappa]^{<\kappa}$ and a $\mathbb{P}$-name $\dot{\tau}$ be given such that p forces $\dot{\tau} \in 2^\kappa$ and $\dot{\tau} \notin V$. Then we may find an $X \in [2^\kappa]^{<\kappa}$ and a sequence $(q_j)_{j<\kappa}$ of conditions below p such that

- $\forall j_1 < j_2 < \kappa : \; q_{j_2} \leq_{F,i} q_{j_1} \leq_{F,i} p$,
- $\forall j < \kappa : \; q_j \Vdash \exists x \in \check{X} : \; \dot{\tau} \restriction j = x \restriction j$ and
- $X \cap Y = \emptyset$.

Proof. If necessary, we may strengthen p twice in the following manner:

- Firstly, since $|Y| < \kappa$ and $p \Vdash \dot{\tau} \notin \check{Y}$, we may find a name $\dot{\ell}$ for an ordinal less than κ such that
$$p \Vdash \forall y \in \check{Y} : \; \dot{\tau} \restriction \dot{\ell} \neq y \restriction \dot{\ell}.$$
Property B* enables us to find a $p' \leq_{F,i} p$ and $\ell^* < \kappa$ with
$$\forall y \in Y : p' \Vdash \; \dot{\tau} \restriction \ell^* \neq y \restriction \ell^*$$
by restricting a maximal antichain deciding $\dot{\ell}$.
- Secondly, we can find a $p'' \leq_{F,i} p'$ that is (F, i)-bounded (see Definition 2.15).

So without loss of generality assume that p already has both these properties. We construct the sequence $(q_j)_{j<\kappa}$ inductively:

- $j = 0$: Set $q_0 := p$.
- $j \to j + 1$: Since
$$D_{j+1} := \{r \leq q_j : r \text{ decides } \dot{\tau} \restriction (j + 1)\}$$
is open dense below q_j, we may apply Lemma 2.22 to q_j, F, i and D_{j+1} [5] to get q_{j+1} and $(s_g^{j+1})_{g \in C_{j+1}}$, where q_{j+1} is (D_{j+1}, F, i)-complete as witnessed by $(s_g^{j+1})_{g \in C_{j+1}}$. Note that we have $q_{j+1} \leq_{F,i} q_j \leq_{F,i} p$.

[5] q_j is (F, i)-bounded by Fact 2.16.

- λ is a limit: Find a $\leq_{F,i}$-lower bound $\tilde{q}_\lambda$ of $(q_\ell)_{\ell<\lambda}$. Just as in the successor step, apply Lemma 2.22 to $\tilde{q}_\lambda, F, i$ and
$$D_\lambda := \{r \leq \tilde{q}_\lambda : r \text{ decides } \dot{\tau} \restriction \lambda\}$$
to get q_λ and $(s^\lambda_g)_{g \in C_\lambda}$.

By Lemma 2.19 we know that $(C_j)_{j<\kappa}$ is a decreasing sequence of non-empty sets smaller than κ; as such, the sequence is eventually constant. Write j^* for the index at which this happens.

Now define
$$X := \{x \in 2^\kappa : \exists g \in C_{j^*}\, \forall j < \kappa :\ s^j_g \Vdash \dot{\tau} \restriction j = x \restriction j\}$$

For $g \in C_{j^*}$ the sequence $(s^j_g)_{j<\kappa}$ is decreasing by Lemma 2.19. Hence each $g \in C_{j^*}$ successfully interprets $\dot{\tau}$ as some (unique) $x \in X$, i.e.
$$\forall g \in C_{j^*}\, \exists x \in X\, \forall j < \kappa :\ s^j_g \Vdash \dot{\tau} \restriction j = x \restriction j.$$

Since $\forall y \in Y :\ p \Vdash \dot{\tau} \restriction \ell^* \neq y \restriction \ell^*$, we know that $X \cap Y = \emptyset$.

Suppose now that $j \geq j^*$ and $s \leq q_j$. Then s is compatible with s^j_g for some $g \in C_j = C_{j^*}$ and we can find a $t \leq s, s^j_g$. But then $\exists x \in X :\ t \Vdash \dot{\tau} \restriction j = x \restriction j$, so we can conclude
$$q_j \Vdash \exists x \in \check{X} :\ \dot{\tau} \restriction j = x \restriction j.$$

Since $|Y| < \kappa$, we can easily modify X such that it remains disjoint from Y and
$$q_j \Vdash \exists x \in \check{X} :\ \dot{\tau} \restriction j = x \restriction j$$
holds for $j < j^*$ as well. □

We are now preparing to show that every new real $\dot{\tau}^G \in V^{\mathbb{P}}$ can be mapped onto the first Sacks real $\dot{s}_0$ via a continuous ground model function. In what follows we shall slightly abuse notation; for $p \in \mathbb{P}$ and a node $\eta \in p(0)$ denote by $p^{[\eta]}$ the condition that satisfies $p^{[\eta]}(0) = p(0)^{[\eta]}$ and $p^{[\eta]}(\beta) = p(\beta)$ for $\beta > 0$.

Lemma 5.2. Let $p \in \mathbb{P}, F \in [\kappa^{++}]^{<\kappa}$ and $i, \ell < \kappa$. Let furthermore a $\mathbb{P}$-name $\dot{\tau}$ be given such that p forces $\dot{\tau} \in 2^\kappa$ and $\dot{\tau} \notin V$. Then we can find a $q \leq_{F,i} p$, an $\ell^* > \ell$ and a family $(A_\eta)_{\eta \in \mathrm{split}_i(p(0))}$ of non-empty, clopen sets with

- $A_\eta = \bigcup_{\nu \in S_\eta} [\nu]$ for some $S_\eta \subseteq 2^{\ell^*}$
- if $\eta_1 \perp \eta_2$, then $A_{\eta_1} \cap A_{\eta_2} = \emptyset$ and
- $q^{[\eta]} \Vdash \dot{\tau} \in A_\eta$.

Proof. Enumerate $\text{split}_i(p(0))$ as $(\eta_k)_{k<\delta}$ with $\delta < \kappa$. We inductively construct sequences $((t^k_j)_{j<\kappa})_{k<\delta}$ and a sequence of sets $(X_k)_{k<\delta}$: assuming that X_m has been constructed for $m < k$, apply Lemma 5.1 to $p^{[\eta_k]}$ and $Y := \bigcup_{m<k} X_m$ to get a sequence of conditions $(t^k_j)_{j<\kappa}$ and a set X_k. Let $\ell^* > \ell$ be an ordinal large enough such that whenever $j_1 \neq j_2$ for $j_1, j_2 < \delta$ and $x_1 \in X_{j_1}, x_2 \in X_{j_2}$ then $x_1 \restriction \ell^* \neq x_2 \restriction \ell^*$. This is possible, since the $(X_k)_{k<\delta}$ are disjoint and of size less than κ. This allows us to define

$$A_{\eta_k} := \bigcup_{x \in X_k} [x \restriction \ell^*].$$

Now we glue the conditions $t^k_{\ell^*}$ together in the following way: Set

$$q(0) := \bigcup_{k<\delta} t^k_{\ell^*}(0)$$

and for $\beta > 0$

$$q(\beta) := \begin{cases} t^k_{\ell^*}(\beta) & \text{if } t^k_{\ell^*} \restriction \beta \in \dot{G}_\beta \\ \mathbb{1}_{\dot{\mathbb{Q}}_\beta} & \text{otherwise.} \end{cases}$$

It remains to remark that by induction on β, we can see that $(t^k_{\ell^*} \restriction \beta)_{k<\delta}$ is a maximal antichain below $q \restriction \beta$. Therefore, since $\text{split}_i(p(0)) = \{\eta_k : k < \delta\}$ and by Lemma 5.1 we have $t^k_{\ell^*} \leq_{F,i} p^{[\eta_k]}$ for each $k < \delta$, we can conclude $q \restriction \beta \leq_{F\cap\beta,i} p \restriction \beta$ for all $\beta \leq \kappa^{++}$.

To see the last claim, only note that $q^{[\eta]} = t^k_{\ell^*}$ for some $k < \delta$, therefore by Lemma 5.1 we have $t^k_{\ell^*} \Vdash \exists x \in X_k : \dot{\tau} \restriction \ell^* = x \restriction \ell^*$ and thus

$$q^{[\eta]} \Vdash \dot{\tau} \in A_\eta$$

by definition of A_η. □

Remark 5.3. Without loss of generality, we may choose the A_η in the previous lemma to be minimal in the following sense: for each $\nu \in 2^{\ell^*}$ we have $\nu \in S_\eta$ iff there exists a condition $t \leq q^{[\eta]}$ such that $t \Vdash \dot{\tau} \in [\nu]$.

Lemma 5.4. Let $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{\tau}$ be given such that p forces $\dot{\tau} \in 2^\kappa$ and $p \Vdash \dot{\tau} \notin V$. Then there exists a $q \leq p$, a sequence $(\ell^*(i))_{i<\kappa}$ and a family $(A_\eta)_{\eta\in\text{split}(q(0))}$ such that $A_\eta \subseteq 2^\kappa$ are non-empty, clopen and:

- if $\eta \in \text{split}_i(q(0))$, then $A_\eta = \bigcup_{\nu\in S_\eta} [\nu]$ for some $S_\eta \subseteq 2^{\ell^*(i)}$
- if $\eta_1 \perp \eta_2$, then $A_{\eta_1} \cap A_{\eta_2} = \emptyset$,
- if $\eta_1 \lhd \eta_2$, then $A_{\eta_2} \subseteq A_{\eta_1}$ and
- $q^{[\eta]} \Vdash \dot{\tau} \in A_\eta$.

Proof. We shall construct a fusion sequence $\langle q_i, F_i : i < \kappa \rangle$ and a strictly increasing sequence $(\ell^*(i))_{i<\kappa}$ of ordinals less than κ such that q_{i+1} has the required properties for $(A_\eta)_{\eta\in\text{split}_i(q_i(0))}$.

- $i = 0$: Set $q_0 := p$ and $F_0 := \{0\}$.
- $i \to i+1$: Applying Lemma 5.2 to q_i, F_i, i and $\sup_{j<i} \ell^*(j)$ yields a $\tilde{q} \leq_{F_i,i} q_i$, an ordinal $\ell^*(i)$ and a family $(A^i_\eta)_{\eta \in \mathrm{split}_i(q_i(0))}$. Set $q_{i+1} := \tilde{q}$. Define F_{i+1} with a bookkeeping argument.
- λ is a limit: Set q_λ to a fusion limit of $\langle q_j, F_j : j < \lambda \rangle$ and $F_\lambda := \bigcup_{j<\lambda} F_j$.

Let now q_κ be a fusion limit of the sequence $\langle q_i, F_i : i < \kappa \rangle$ and

$$A_\eta := A^{i(\eta)}_\eta,$$

where $i(\eta)$ is the unique i with $\eta \in \mathrm{split}_i(q_\kappa(0)) = \mathrm{split}_i(q_i(0))$. We claim q_κ has the properties we are looking for:

- The first property holds by Lemma 5.2.
- If we assume the $A^{i(\eta)}_\eta$ have been chosen minimal in each step as in Remark 5.3, then the second property holds. To see this, take $\nu \lhd \eta$ and $\eta' \in S_\eta$, where S_η is as stated in Lemma 5.2. By Remark 5.3 there is a condition $t \leq q^{[\eta]}_{i(\eta)+1}$ such that $t \Vdash \dot{\tau} \in [\eta']$. But then $t \leq q^{[\eta]}_{i(\eta)+1} \leq q^{[\nu]}_{i(\nu)+1}$, and thus $\eta' \upharpoonright \ell^*(i(\nu)) \in S_\nu$. Hence $A_\eta \subseteq A_\nu$.
- To see the third property, let $\eta \in \mathrm{split}(q_\kappa(0))$. Then we have $q^{[\eta]}_\kappa \leq q^{[\eta]}_{i(\eta)}$ and therefore

  $$q^{[\eta]}_\kappa \Vdash \dot{\tau} \in A_\eta,$$

  as desired.

□

The following lemma substitutes in for Tietze's Extension Theorem from the countable case in [Cor89]. Recall the notion of superclosure (page 6) and uniform continuity (Definition 4.5).

Lemma 5.5. Let $Y, Z \subseteq 2^\kappa$, where Y is closed and Z is superclosed, and let $g : Y \to Z$ be uniformly continuous. Then g can be extended to a uniformly continuous function $\tilde{g} : 2^\kappa \to Z$ with the same modulus of continuity as g.

Proof. The open set $2^\kappa \setminus Y$ can be be written as a union of basic open sets $\bigcup_{i<\lambda}[\nu_i]$ with $\lambda \leq \kappa, \nu_i \in 2^{\delta_i}$ such that the ν_i are minimal, i.e.

$$\forall j < \delta_i : \ [\nu_i \upharpoonright j] \cap Y \neq \emptyset.$$

In particular the sets $[\nu_i]$ are pairwise disjoint. We will define $\tilde{g}$ to be constant on each $[\nu_i]$.

For $i < \lambda$ define

$$S(i) := \{\eta \in 2^{<\kappa} : \exists j < \delta_i : g''([\nu_i \upharpoonright j] \cap Y) \subseteq [\eta] \cap Z\}.$$

Clearly $S(i)$ consists of pairwise $\lhd$-compatible elements; furthermore, for each $\eta \in S(i)$ we have $[\eta] \cap Z \neq \emptyset$. Since Z is superclosed [6], we have $Z \cap [\bigcup S(i)] \neq \emptyset$. We may thus set $\tilde{g} \upharpoonright [\nu_i]$ to be constant with an arbitrary, fixed value from $Z \cap [\bigcup S(i)]$.

It remains to check that $\tilde{g} : 2^\kappa \to Z$ is uniformly continuous with the same modulus of continuity as g. To this end, let $i < \kappa$ and $x \in 2^\kappa$. Consider $y \in [x \upharpoonright j(i)]$.

- If $x \in Y$, the interesting case is $y \notin Y$, hence $y \in [\nu_\ell]$ for some ℓ. But then $j(i) < \delta_\ell$ and
$$g''([x \upharpoonright j(i)] \cap Y) \subseteq [g(x) \upharpoonright i] \cap Z,$$
hence by definition $g(x) \upharpoonright i \in S(\ell)$ and thus $\tilde{g}(y) \in [\bigcup S(\ell)] \cap Z \subseteq [\tilde{g}(x) \upharpoonright i] \cap Z$.
- On the other hand, if $x \notin Y$, then x is in $[\nu_\ell]$ for some ℓ. Now either $[x \upharpoonright j(i)] \cap Y = \emptyset$, in which case $\tilde{g}$ is constant on $[x \upharpoonright j(i)]$ and therefore $\tilde{g}(y) = \tilde{g}(x) \in [\tilde{g}(x) \upharpoonright i] \cap Z$, or $[x \upharpoonright j(i)] \cap Y \neq \emptyset, j(i) < \delta_\ell$ and $S(\ell)$ contains a sequence η of length i (namely $g(x') \upharpoonright i$ for some $x' \in [x \upharpoonright j(i)] \cap Y$) and thus $\tilde{g}(y) \in [\eta] \cap Z = [\tilde{g}(x) \upharpoonright i] \cap Z$.

□

A natural question the inquisitive reader might pose is the validity of Lemma 5.5 in case of the additional "artificial" assumption of superclosure being dropped. Indeed, the statement no longer holds; in [LS15] the authors observe, for instance, that the closed subset Y of 2^κ consisting of all sequences with finitely many zeroes is not a retract of 2^κ (and thus the identity $Y \to Y$ cannot be extended to a continuous function on 2^κ).

Theorem 5.6. Let $p \in \mathbb{P}$ force $\dot{\tau} \in 2^\kappa$ and $\dot{\tau} \notin V$. Then there exists a $q \leq p$ and a uniformly continuous function $f^* : 2^\kappa \to [q(0)]$ in V such that

$$q \Vdash f^*(\dot{\tau}) = \dot{s}_0,$$

where $\dot{s}_0$ denotes the first Sacks real.

Proof. Lemma 5.4 yields a condition $q \leq p$, a sequence $(\ell^*(i))_{i<\kappa}$ and a family $(A_\eta)_{\eta \in \mathrm{split}(q(0))}$ of clopen sets. This family codes [7] a continuous function

$$f : Y \to [q(0)]$$
$$y \mapsto \bigcup \{\eta : y \in A_\eta\}$$

defined on the closed set $Y = \bigcap_{i<\kappa} \bigcup_{\eta \in \mathrm{split}_i(q(0))} A_\eta$.

[6] If $|S(i)| = \kappa$, then $[\bigcup S(i)]$ is not defined, so work with $\{\bigcup S(i)\}$ instead.

[7] To avoid abuse of notation, we could also define

$$A'_\eta := \begin{cases} A_\nu, \text{ where } \nu = \min\{\rho \in \mathrm{split}(q(0)) : \eta \lhd \rho\} & \text{for } \eta \in q(0) \\ \emptyset & \text{for } \eta \notin q(0) \end{cases}$$

and use the code $\langle T, (T_\eta)_{\eta \in 2^{<\kappa}} \rangle$, where $[T] = Y$ and $[T_\eta] = A'_\eta \cap Y$.

We claim that f is in fact uniformly continuous. To see this, let $i < \kappa$ and $y \in Y$. Choose η such that $y \in A_\eta$ and $\eta \in \operatorname{split}_i(q(0))$. Recall that A_η is of the form (see Lemma 5.4)

$$A_\eta = \bigcup_{\nu \in S_\eta} [\nu].$$

with $S_\eta \subseteq 2^{\ell^*(i)}$. Therefore we have

$$f''([y \restriction \ell^*(i)]) \subseteq [\eta] \subseteq f(y) \restriction i$$

for all $y \in Y$, since $i \subseteq \operatorname{dom}(\eta)$.

Since the set $[q(0)]$ is superclosed, we can apply Lemma 5.5 and extend f to a uniformly continuous function $f^* : 2^\kappa \to [q(0)]$. Lastly, we have

$$q^{[\eta]} \Vdash \dot{\tau} \in A_\eta \subseteq (f^*)^{-1}([\eta])$$

for each $\eta \in \operatorname{split}(q(0))$ and thus

$$q \Vdash f^*(\dot{\tau}) = \dot{s}_0.$$

□

As in the classical case, every κ-Sacks condition can be decomposed into 2^κ-many κ-Sacks conditions in a continuous way. The last auxiliary result we require formalizes this:

Lemma 5.7. Let $p \in \mathbb{P}$ be a condition and recall that $p(0) \subseteq 2^{<\kappa}$. Then there exists a uniformly continuous $g^* : [p(0)] \to 2^\kappa$ and for each $x \in 2^\kappa \cap V$ a condition $q_x \leq p$ such that

$$q_x \Vdash x = g^*(\dot{s}_0).$$

Proof. First we construct a function $e = (e_1, e_2) : p(0) \to 2^{<\kappa} \times 2^{<\kappa}$ with the following properties:

- e is continuous and monotone increasing
- $e(\emptyset) = (\emptyset, \emptyset)$
- if $\eta \notin \operatorname{split}(p(0))$, then $e(\eta^\frown i) = e(\eta)$
- if $\eta \in \operatorname{split}_j(p(0))$ and
  - j is a successor, then $e(\eta^\frown i) = (e_1(\eta)^\frown i, e_2(\eta))$
  - j is a limit, then $e(\eta^\frown i) = (e_1(\eta), e_2(\eta)^\frown i)$.

Define $\hat{g} : [p(0)] \to 2^\kappa \times 2^\kappa$ as $\hat{g}(b) = \bigcup_{i<\kappa} e(b \restriction i)$. Since $[p(0)]$ is perfect, $\hat{g}$ is well-defined. Moreover, $\hat{g}$ maps the clopen basis sets $([\eta])_{\eta \in \operatorname{split}(p(0))}$ to a clopen basis of $2^\kappa \times 2^\kappa$, hence it is a homeomorphism.

For $x \in 2^\kappa$ now set $q_x(0) := \{\eta \in 2^{<\kappa} :\ \exists y \in \hat{g}^{-1}(\{x\} \times 2^\kappa) :\ \eta \lhd y\}$ and $q_x(\beta) = p(\beta)$ for $\beta > 0$. We claim that q_x is a condition; it is sufficient to check that $q_x(0)$ is. We check (S2), (S5) and (S6); the rest is left as an exercise for the reader.

- (S2): Since $\hat{g}$ is a homeomorphism, it follows that $\hat{g}^{-1}(\{x\} \times 2^\kappa)$ is a perfect set.
- (S5): Let $(\eta_j)_{j<\delta}$ with $\eta_j \in q_x(0)$ be a strictly increasing sequence of length $\delta < \kappa$. Set $\eta := \bigcup_{j<\delta} \eta_j$. It easily follows that $\nu \in q_x(0) \Leftrightarrow x \in [e_1(\nu)]$. As $e(\eta) = \bigcup_{j<\delta} e(\eta_j)$ we see that $x \in [e_1(\eta)]$, hence $\eta \in q_x(0)$.
- (S6): Let $(\eta_j)_{j<\delta}$ be a strictly increasing sequence of length $<\kappa$ such that $\eta_j \in \mathrm{split}(q_x(0))$. Again, set $\eta := \bigcup_{j<\delta} \eta_j$. It follows that $\eta \in \mathrm{split}_\lambda(p(0))$ for some limit λ. But as $x \in [e_1(\eta)]$ and $e_1(\eta) = e_1(\eta^\frown i)$, we have $\eta^\frown i \in q_x(0)$ for $i = 0, 1$, hence $\eta \in \mathrm{split}(q_x(0))$.

Clearly $q_x \leq p$. Now set $g^* = \pi_1 \circ \hat{g}$, where π_1 is the projection onto the first coordinate. Then g^* is uniformly continuous with modulus of continuity

$$i \mapsto j(i) := \sup\{\mathrm{dom}(\nu) : \nu \in \mathrm{split}_{|i|^+}(p(0))\}.$$

Finally, we have $q_x \Vdash x = g^*(\dot{s}_0)$ by the definition of $q_x(0)$ and the absoluteness (see Fact 4.6) of the statement

$$\mathrm{ran}(g^* \restriction [q_x(0)]) \subseteq \{x\}.$$

□

Theorem 5.8. In $V^{\mathbb{P}}$, every subset X of 2^κ of size κ^{++} can be uniformly continuously mapped onto 2^κ.

Proof. Assume that $\dot{X}$ is a $\mathbb{P}$-name for a subset of 2^κ such that

$$\Vdash_{\mathbb{P}} \forall h \text{ uniformly continuous function } \exists y \in 2^\kappa : \; y \notin h''X.$$

We will show $\exists \alpha^* < \kappa^{++} : \Vdash_{\mathbb{P}} \dot{X} \subseteq V^{\mathbb{P}_{\alpha^*}}$, thus $\Vdash_{\mathbb{P}} |\dot{X}| \leq \kappa^+$.

By our assumption on $\dot{X}$ and $\mathbb{P}$ satisfying the κ^{++}-c.c. we get

$$\forall \alpha < \kappa^{++} \, \forall \dot{h} \; \mathbb{P}_\alpha\text{-name for a uniformly continuous function}$$
$$\exists \beta < \kappa^{++}, \beta \geq \alpha \, \exists y \; \mathbb{P}_\beta\text{-name for a real} : \Vdash_{\mathbb{P}} \dot{y} \notin \dot{h}''\dot{X}.$$

To increase legibility, let the ellipsis $(\dots)$ denote the four quantifications in the above statement. By interpreting the name $\dot{X}$ partially in the intermediate model $V^{\mathbb{P}_\beta}$, i.e. by identifying $\dot{X}$ with a canonical $\mathbb{P}_\beta$-name for a $\mathbb{P}_{\beta,\kappa^{++}}$-name, we get

$$(\dots) : \Vdash_{P_\beta} \Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} \dot{y} \notin \dot{h}''\dot{X}.$$

Keep in mind that $\dot{y}, \dot{h}$ are both $\mathbb{P}_\beta$-names, since $\beta \geq \alpha$.

Without loss of generality assume that the function $\alpha \mapsto \beta(\alpha)$ maps to the minimal β for which the statement holds. Observe that, crucially, since every continuous function $h : 2^\kappa \to 2^\kappa$ can be coded by an element of 2^κ (see Section 4), no new functions of the kind appear at stages of cofinality $>\kappa$ (Lemma 2.30); therefore we can easily find a fixed point of the function $\alpha \mapsto \beta(\alpha)$ with cofinality κ^+. Call it α^*. For α^* we thus know that

$$V^{\mathbb{P}_{\alpha^*}} \models \forall h \text{ uniformly continuous } \exists y \in 2^\kappa : \Vdash_{\mathbb{P}_{\alpha^*,\kappa^{++}}} y \notin h''\dot{X}.$$

For the remainder of this proof we will be working within $V^{\mathbb{P}_{\alpha^*}}$. We wish to show $V^{\mathbb{P}_{\alpha^*}} \models \Vdash_{\mathbb{P}_{\alpha^*,\kappa^{++}}} \dot{X} \subseteq V^{\mathbb{P}_{\alpha^*}}$.

Let thus $p \in \mathbb{P}_{\alpha^*,\kappa^{++}}$ and $\dot{\tau}$ be a $\mathbb{P}_{\alpha^*,\kappa^{++}}$-name such that p forces $\dot{\tau} \in 2^\kappa$ and $\dot{\tau} \notin V^{\mathbb{P}_{\alpha^*}}$. Theorem 5.6 applied within $V^{\mathbb{P}_{\alpha^*}}$ (recall that the tail iteration $\mathbb{P}_{\alpha*,\kappa^{++}}$ has the same structure as the full iteration) yields a $q \leq p$ and a uniformly continuous function $f^* : 2^\kappa \to [q(0)]$. Likewise, Lemma 5.7 applied to q gives us a uniformly continuous function $g^* : [q(0)] \to 2^\kappa$ and conditions $(q_x)_{x \in 2^\kappa \cap V^{\mathbb{P}_{\alpha^*}}}$.

Now let $x \in 2^\kappa \cap V^{\mathbb{P}_{\alpha^*}}$ be arbitrary. By construction we have $q_x \Vdash x = (g^* \circ f^*)(\dot{\tau})$. For the uniformly continuous function $(g^* \circ f^*) : 2^\kappa \to 2^\kappa$ we can by our assumption on α^* find a $y \in 2^\kappa \cap V^{\mathbb{P}_{\alpha^*}}$ with $V^{\mathbb{P}_{\alpha^*}} \models \Vdash_{\mathbb{P}_{\alpha^*,\kappa^{++}}} y \notin (g^* \circ f^*)'' \dot{X}$. The condition q_y thus forces $\dot{\tau} \notin \dot{X}$. Since $\dot{\tau}$ and p were arbitrary, we may conclude

$$V^{\mathbb{P}_{\alpha^*}} \models \Vdash_{\mathbb{P}_{\alpha^*,\kappa^{++}}} \dot{X} \subseteq V^{\mathbb{P}_{\alpha^*}}.$$

Thus we have shown $\Vdash_{\mathbb{P}_{\alpha^*}} \Vdash_{\mathbb{P}_{\alpha^*,\kappa^{++}}} \dot{X} \subseteq V^{\mathbb{P}_{\alpha^*}}$, which finishes the proof. □

It is easy to see that the uniformly continuous image of a strong measure zero set remains strong measure zero; thus we have shown

$$V^{\mathbb{P}} \models \mathcal{SN} \subseteq [2^\kappa]^{\leq \kappa^+}.$$

Corollary 5.9. $V^{\mathbb{P}} \models \mathcal{SN} = [2^\kappa]^{\leq \kappa^+}$.

6 Stationary Strong Measure Zero

Finally, let us take a look at the following definition, introduced by Halko [Hal96]:

Definition 6.1. A set $X \subseteq 2^\kappa$ is called *stationary strong measure zero* iff

$$\forall f \in \kappa^\kappa \, \exists (\eta_i)_{i<\kappa} : \; (\forall i < \kappa : \; \eta_i \in 2^{f(i)}) \wedge X \subseteq \bigcap_{cl \subseteq \kappa \text{ club}} \bigcup_{i \in cl} [\eta_i].$$

So a set X is stationary strong measure zero iff we can find coverings that cover every point of X stationarily often. To motivate why this definition might be of interest, observe that even for regular strong measure zero sets, we can always find coverings that cover each point at least unboundedly often:

Lemma 6.2. Let $X \subseteq 2^\kappa$ be strong measure zero. Then

$$\forall f \in \kappa^\kappa : \; \exists (\eta_i)_{i<\kappa} : \; (\forall i < \kappa : \; \eta_i \in 2^{f(i)}) \wedge X \subseteq \bigcap_{j<\kappa} \bigcup_{i \geq j} [\eta_i].$$

Proof. Partition κ into sets $(U_i)_{i<\kappa}$, where each U_i has size κ. For a challenge $f \in \kappa^\kappa$ and every $i < \kappa$ we can find coverings $(\eta^i_j)_{j \in U_i}$ that satisfy the challenge $(f(j))_{j \in U_i}$. But now $(\eta^i_j)_{j \in U_i, i<\kappa}$ has the property we are looking for. □

In the Corazza-type model from Section 5, the notions of strong measure zero and stationary strong measure zero coincide.

Theorem 6.3. $V^{\mathbb{P}} \models \forall X \subseteq 2^\kappa : \; X \in \mathcal{SN} \Leftrightarrow X$ is stationary strong measure zero.

Proof. Modify the argument in Theorem 3.1 to show

$$V^{\mathbb{P}} \models \forall \alpha < \kappa^{++} : 2^\kappa \cap V^{\mathbb{P}_\alpha} \text{ is stationary strong measure zero}$$

by instead showing the set

$$D_{x,cl} := \{p \in \mathbb{Q}_\beta : \exists i \in cl : \; p \Vdash \dot{\sigma}(i) = x \restriction h(i)\}$$

to be dense for every $x \in V^{\mathbb{P}_\alpha}$ and every ground model club $cl \subseteq \kappa$, where $\dot{\sigma}$ is as defined in Theorem 3.1. As $\mathbb{P}$ is κ^κ-bounding, every club $cl \in V^{\mathbb{P}}$ contains a ground model club cl', thus this is sufficient. To see that $D_{x,cl}$ is dense, merely note that for any $p \in \mathbb{P}$ and $b \in [p] \cap V^{\mathbb{P}_\beta}$, the set

$$\{j < \kappa : b \restriction j \in \operatorname{split}(p)\}$$

is a club and thus intersects cl. □

On the other hand, it follows from $|2^\kappa| = \kappa^+$ that there is a strong measure zero set which is not stationary strong measure zero.

Theorem 6.4. Under $|2^\kappa| = \kappa^+$ there exists an $X \in \mathcal{SN}$ that is not stationary strong measure zero.

Proof. First off, let us enumerate all strictly increasing functions in κ^κ as $(f_\alpha)_{\alpha<\kappa^+}$ and likewise enumerate the set

$$\mathcal{S} := \{\sigma \in (2^{<\kappa})^\kappa : \forall i < \kappa : \ \mathrm{dom}(\sigma(i)) = i + 1\}$$

as $(\sigma_\alpha)_{\alpha<\kappa^+}$.

We shall inductively construct three sequences $(x_\alpha)_{\alpha<\kappa^+}$, $(\tau_\alpha)_{\alpha<\kappa^+}$ and $(cl_\alpha)_{\alpha<\kappa^+}$ with the following properties:

a) $\forall\alpha < \kappa^+ : \ x_\alpha \in 2^\kappa, \tau_\alpha \in (2^{<\kappa})^\kappa$ and cl_α is a club subset of κ

b) $\forall\alpha < \kappa^+\,\forall i < \kappa : \ \mathrm{dom}(\tau_\alpha(i)) = f_\alpha(i)$

c) $\forall\alpha < \kappa^+\,\forall i < \kappa : \ \bigcup_{j\geq i}[\tau_\alpha(j)]$ is open dense

d) $\forall\alpha < \kappa^+\,\forall\beta \leq \alpha : \ x_\beta \in \bigcup_{i<\kappa}[\tau_\alpha(i)]$

e) $\forall\beta < \kappa^+\,\forall\alpha < \beta : \ x_\beta \in \bigcup_{i<\kappa}[\tau_\alpha(i)]$

f) $\forall\alpha < \kappa^+ : \ x_\alpha \notin \bigcup_{i\in cl_\alpha}[\sigma_\alpha(i)]$

Setting $X = \{x_\alpha : \alpha < \kappa^+\}$ yields a strong measure zero set (by b), d) and e)). However, X is not stationary strong measure zero, since for the challenge $g : i \mapsto i + 1$ property f) ensures

$$\forall\sigma \in \mathcal{S}\,\exists x \in X\,\exists cl \text{ club } : \ x \notin \bigcup_{i\in cl}[\sigma(i)].$$

Suppose now, inductively, that $(x_\alpha)_{\alpha<\gamma}$, $(\tau_\alpha)_{\alpha<\gamma}$ and $(cl_\alpha)_{\alpha<\gamma}$ have been constructed for $\gamma < \kappa^+$. We wish to define x_γ, τ_γ and cl_γ. To this end, reindex $(x_\alpha)_{\alpha<\gamma}$ and $(\tau_\alpha)_{\alpha<\gamma}$ as $(\tilde{x}_{i+1})_{i<\kappa}$, $(\tilde{\tau}_{i+1})_{i<\kappa}$ [8] and inductively construct x_γ and cl_γ:

- $j = 0$: Set $cl^0_\gamma := 0$ and $x^0_\gamma := \langle 1 - \sigma_\gamma(0)(0)\rangle$.
- $j \to j + 1$: Since by assumption $\ell \mapsto \mathrm{dom}(\tilde{\tau}_{j+1}(\ell))$ is strictly increasing and $\bigcup_{\ell\geq\ell^*}[\tilde{\tau}_{j+1}(\ell)]$ is open dense for all $\ell^* < \kappa$, we can find an $\ell > cl^j_\gamma$ with $x^j_\gamma \lhd \tilde{\tau}_{j+1}(\ell)$. Set $cl^{j+1}_\gamma := \mathrm{dom}(\tilde{\tau}_{j+1}(\ell))$ and $x^{j+1}_\gamma := \tilde{\tau}_{j+1}(\ell)^\frown(1 - \sigma_\gamma(cl^{j+1}_\gamma)(cl^{j+1}_\gamma))$.
- λ is a limit: Set $cl^\lambda_\gamma := \sup_{j<\lambda} cl^j_\gamma$ and $x^\lambda_\gamma := (\bigcup_{j<\lambda} x^j_\gamma)^\frown(1 - \sigma_\lambda(cl^\lambda_\gamma)(cl^\lambda_\gamma))$.

Now set $x_\gamma := \bigcup_{j<\kappa} x^j_\gamma$ and $cl_\gamma := \{cl^j_\gamma : j < \kappa\}$. In the construction we have ensured $x_\gamma \notin \bigcup_{j\in cl_\gamma}[\sigma_\gamma(j)]$ and $x_\gamma \in \bigcup_{j<\kappa}[\tilde{\tau}_{i+1}(j)]$ for all $i < \kappa$. Finally, construct τ_γ such that b), c) and d) holds. □

[8] If $\gamma < \kappa$, use some x and τ multiple times. For $\gamma = 0$ pick x_0 and cl_0 arbitrarily such that $x_0 \notin \bigcup_{i\in cl_0}[\sigma_0(i)]$.

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