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Preface

In this thesis we study solutions of the Abstract Cauchy Problem

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t > 0, \\ u(0) = x_0, \end{cases}$$

with $u : [0, +\infty) \to X$ for some Banach space X, where A is a linear operator mapping X to itself. It is basic knowledge that, if A is bounded, the solution has the form

$$u(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} f(s)\ ds$$

for reasonably smooth inhomogeneities f. If A is unbounded, it is natural to expect solutions of similar form, especially if A is the infinitesimal generator of a strongly continuous semigroup.

The question arises, what kind of operators A admit unique solutions of the Abstract Cauchy Problem and what kind of properties regarding smoothness can one expect of this solutions. In the best case, the functions u, u' and Au have the same regularity properties as the inhomogeneity f. We want to characterise operators A, such that for every f there is a unique solution such that u' and Au are 'as regular' as f. In this case, we say that A is maximally regular.

In this thesis we study the maximal regularity problem for inhomogeneities $f \in L^p\big((0, +\infty); X\big)$, $p \in (1, +\infty)$. In the twentieth century, many advances were made, for example by Dore, [7], or de Simon, [5]. In recent years, maximal regularity has been tackled from different angles, namely closedness of sums of operators, [8], Fourier transforms, [16] and the theory of singular integrals, [2].

However, the exact definition of maximal regularity is rather ambiguous in the literature. De Simon ([5]) required u' and Au to be of same regularity as f, while Dore ([7]) additionally required u to have the same regularity properties as f. In this thesis we will work with De Simon's definition of maximal regularity while introducing the notion of strict maximal regularity, which, additionally to u' and Au, requires u to be as regular as f. We will study relations between these two notions of maximal regularity. We will see that it is necessary for A to generate an analytic semigroup and, if the underlying space happens to be a Hilbert space, this condition is also sufficient.

Chapter 1 is concerned with basic results regarding differential calculus and unbounded operators, for which we mainly rely on [17] and [19].

In Chapter 2 we introduce the Bochner Integral, the Banach space-valued analogue to the Lebesgue Integral for complex-valued functions, and Banach space-valued versions of Lebesgue- and Sobolev spaces, where we use the notations as in [10] and [3]. We will state results, which are well-known for complex-valued functions, such as the Dominated Convergence Theorem for Banach Space-valued functions and the Fundamental Theorem of Calculus. Furthermore, we introduce the Riemann Integral and complex Path Integrals for Banach space-valued functions and build connections between the Bochner- and the Riemann Integral as in the complex-valued case; see [10] and [17].

Chapter 3 is concerned with basic observations about strongly continuous semigroups as shown for example in [19]. Moreover, we will introduce analytic semigroups, i.e. strongly continuous semigroups which are analytically extendable to an open subset of the right complex half-plane. We will prove that this is equivalent to A being sectorial and the semigroup being differentiable with respect to the operator norm; see [1].

In Chapter 4 we will study existence and uniqueness of solutions of the Abstract Cauchy Problem. We will start by examining the homogeneous problem, i.e. $f \equiv 0$, and show that for existence and uniqueness it is necessary and sufficient for A to generate a strongly continuous semigroup, as seen in [1]. For the inhomogeneous problem, many different notions of weak solutions were introduced in the literature, for example by Ball ([4]) and de Simon ([5]), while many others simply worked with functions that solve the problem almost everywhere; see for example [6]. In order to avoid this ambiguity we define weak solutions in the sense of weak differentiability and prove uniqueness in the case that A is the infinitesimal generator of a differentiable semigroup.

Finally, Chapter 5 is concerned with maximal regularity of the Abstract Cauchy Problem and its implications. Often the abstract Cauchy problem in connection with maximal regularity was only considered on $(0, +\infty)$. Because of that, we will introduce maximal regularity on bounded as well as on unbounded intervals and prove that the latter implies the former, as shown by Dore [7]. In the last two sections we will show that maximal L^p-regularity is independent of p and that in Hilbert spaces maximal regularity is equivalent to A being the infinitesimal generator of an analytic semigroup; see [7] and [5]. While many authors diligently dealt with the estimate

$$\|Au\|_{L^p} + \|u'\|_{L^p} \leq C \|f\|_{L^p} ,$$

where u denotes the mild solution of the Cauchy Problem, i.e.

$$u(t) = \int_{(0,t)} T(t-s)f(s) \, ds$$

($T(t)$ being the semigroup generated by A), the requirement that the Cauchy Problem has a unique weak solution was left out. We will tackle this gap by carefully dealing with existence and uniqueness of weak solutions.

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Notations

We will list symbols and notations that will be frequently used throughout this thesis.

- By $\mathbb{C}$ we will denote the set of all complex numbers, by $\mathbb{R}$ the set of all real numbers and $\mathbb{N} := \{1, 2, \dots\}$ will be the set of all positive integers.
- $\mathbb{1}_A$ will denote the characteristic function of some set A defined in some underlying set Ω, i.e. $\mathbb{1}_A : \Omega \to \{0, 1\}$,

$$\mathbb{1}_A(\omega) := \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

- We write $\|\cdot\|_X$ for the norm of some normed space X. If the underlying space X is obvious from the context, we will simply write $\|\cdot\|$.
- Given a metric space (M, d), $\varepsilon > 0$ and $x \in M$, we write $U_\varepsilon^M(x) := \{y \in M : d(x, y) < \varepsilon\}$ and $K_\varepsilon^M(x) := \{y \in M : d(x, y) \leq \varepsilon\}$. If the underlying space M is clear from the context we will often write $U_\varepsilon(x) := U_\varepsilon^M(x)$ and $K_\varepsilon(x) := K_\varepsilon^M(x)$.
- Given two Banach spaces X and Y, $L_b(X, Y)$ stands for the set of all bounded linear operators $T : X \to Y$ provided with the operatornorm $\|\cdot\|_{L_b(X,Y)}$. If $X = Y$, we simply write $L_b(X) := L_b(X, X)$.
- We write X' for the topological dual space of some normed space X, i.e. $X' := L_b(X, \mathbb{C})$.
- Given two Banach spaces X and Y as well as $T \in L_b(X, Y)$, we write $T' \in L_b(Y', X')$ for the adjoint of T defined by $T'(\varphi)x := \varphi(Tx)$ for all $\varphi \in Y'$ and all $x \in X$.

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Chapter 1

Basic Principles and Initial Remarks

In order to properly state and study the abstract Cauchy Problem, we will gather information regarding differentiation in Banach spaces and (unbounded) linear operators.

1.1 Differentiation and Analyticity in Banach Spaces

In this section we introduce the notion of differentiability as well as analyticity in Banach spaces and state useful, well-known results. Throughout this section, X denotes a Banach space.

1.1.1 Definition. Let $-\infty \leq a < b \leq +\infty$ and $f : (a, b) \to X$ a function. f is called *differentiable in* $t_0 \in (a, b)$, if the limit

$$\lim_{h\to 0} \frac{1}{h}\big(f(t+h) - f(t)\big)$$

exists in X. If $-\infty \leq a < b < +\infty$ and $f : (a, b] \to X$, we call f differentiable at b, if the one sided limit

$$\lim_{h\to 0^-} \frac{1}{h}\big(f(b+h) - f(b)\big)$$

exists in X. For functions defined on intervals of the form $[a, b)$ or $[a, b]$ we define differentiability accordingly. If $f : I \to X$ is differentiable at t for every $t \in I$, where $I \subseteq \mathbb{R}$ is an interval (note that I can have the form (a, b), $(a, b]$, $[a, b)$ or $[a, b]$ for $-\infty \leq a < b \leq +\infty$), we say that f is *differentiable* on I and define the *derivative* $f' : I \to X$ by

$$\frac{d}{dt}f(t) := f'(t) := \lim_{h\to 0} \frac{1}{h}\big(f(t+h) - f(t)\big),$$

where we take one sided limits if $t \in \{a, b\}$. If f' is continuous, we call f *continuously differentiable*. For $k \in \mathbb{N}$ and $f^{(1)} := f'$, we inductively say that f is k *times continuously differentiable*, if $f^{(k-1)}$ is continuously differentiable.

Moreover, we introduce the following spaces.

- $C(I;X) := \{f : I \to X : f \text{ continuous}\}$,
- $C^k(I;X) := \{f : I \to X : f \text{ is } k \text{ times continuously differentiable}\}$ for $k \in \mathbb{N}$,
- $C^\infty(I;X) := \bigcap_{k\in\mathbb{N}} C^k(I;X)$ and
- $C_{00}^\infty(I;X) := \{f \in C^\infty(I;X) : \operatorname{supp}(f) \text{ is compact in } I\}$, where $\operatorname{supp}(f) := \overline{\{t \in I : f(t) \neq 0\}}$ (closure within I) is called the *support* of a function f.

We state some properties regarding to differentiation in Banach spaces. Their proofs can be found in [17], Fakta 9.3.13.

1.1.2 Proposition. Let $I \subseteq \mathbb{R}$ be an interval and Y, Z additional Banach spaces. The following assertions hold true.

a) Given functions $f, g : I \to X$ and $\lambda \in \mathbb{C}$, if f and g are differentiable at $t_0 \in I$, so is $t \mapsto f(t) + \lambda g(t)$ and $(f + \lambda g)'(t_0) = f'(t_0) + \lambda g'(t_0)$.

b) If $f : I \to X$ is differentiable at $t_0 \in I$, it is continuous at t_0.

c) If $S : I \to L_b(X,Y)$ and $T : I \to L_b(Y,Z)$ are differentiable at $t_0 \in I$, then $t \mapsto T(t)S(t)$ also is differentiable at t_0 and $(TS)'(t_0) = T'(t_0)S(t_0) + T(t_0)S'(t_0)$.

d) If $S : I \to L_b(X,Y)$ is differentiable at $t_0 \in I$, also $t \mapsto S(t)x$ is differentiable satisfying $\big(S(\cdot)x\big)'(t_0) = S'(t_0)x$ for every $x \in X$.

e) Given $A \in L_b(X,Y)$ and $f : I \to X$, which is differentiable at $t_0 \in I$, $t \mapsto Af(t)$ also is differentiable at t_0 satisfying $\big(Af(\cdot)\big)'(t_0) = Af'(t_0)$.

f) Let $J \subseteq \mathbb{R}$ be an additional interval and $\varphi : J \to I$ a function that is differentiable at $t_0 \in J$. If $f : I \to X$ is differentiable at $\varphi(t_0)$, then $f \circ \varphi$ is differentiable at t_0 and $(f \circ \varphi)'(t_0) = \varphi'(t_0)f'(\varphi(t_0))$.

Furthermore, we state a generalized version of the product rule in Banach spaces, see Lemma 5.2.11 in [19].

1.1.3 Lemma. Let X, Y be Banach spaces and $[a,b] \subseteq \mathbb{R}$ an interval. Given functions $f : [a,b] \to X$ and $T : [a,b] \to L_b(X,Y)$ with the properties

- there is a constant $C > 0$ such that $\|T(t)\| \leq C$ for all $t \in [a,b]$,
- f is differentiable at $t_0 \in [a,b]$,
- $t \mapsto T(t)f'(t_0)$ is continuous at t_0 and
- $t \mapsto T(t)f(t_0)$ is differentiable at t_0,

$g : [a,b] \to Y$, defined by $g(t) := T(t)f(t)$, is differentiable at t_0 and satisfies

$$g'(t_0) = \big(T(\cdot)f(t_0)\big)'(t_0) + T(t_0)f'(t_0).$$

Similar to differentiation in real intervals, we introduce the notion of complex differentiability and analytic functions with values in a Banach space.

1.1.4 Definition. Let X be a Banach space, $G \subseteq \mathbb{C}$ an open set and $f : G \to X$ a function. f is called *complex differentiable* at $z_0 \in G$ if the limit

$$f'(z_0) := \lim_{z \to z_0} \frac{1}{z - z_0}\big(f(z) - f(z_0)\big)$$

exists in X. If f is complex differentiable at z for all $z \in G$, f is called *analytic* in G.

Analogous results as in Proposition 1.1.2 also hold for complex differentiable functions, as shown in [17], Section 11.6. Moreover, we have the following lemma. Its proof can be found for example in [17], Corollary 11.6.17.

1.1.5 Lemma. Let $w \in \mathbb{C}$, $(a_n)_{n \in \mathbb{N}}$ a sequence in X and $R > 0$ the radius of convergence of the power series

$$\sum_{n=0}^{\infty} z^n a_n.$$

Then, the function

$$z \mapsto \sum_{n=0}^{\infty} (z - w)^n a_n$$

is analytic in $U_R(w)$.

Lastly, we state the Identity Theorem for analytic, Banach space-valued functions. It follows easily from the complex-valued case.

1.1.6 Proposition. If two analytic functions $f, g : G \to X$ coincide on a set $D \subseteq G$, which has an accumulation point in G, then $f(z) = g(z)$ for all $z \in G$.

Proof. Given an analytic function $f : G \to X$ and $\varphi \in X'$, the function $\tilde{f} := \varphi \circ f : G \to \mathbb{C}$ is analytic since for $z, z_0 \in G$

$$\frac{\tilde{f}(z) - \tilde{f}(z_0)}{z - z_0} = \varphi\Big(\frac{1}{z - z_0}\big(f(z) - f(z_0)\big)\Big) \xrightarrow{z \to z_0} \varphi\big(f'(z_0)\big).$$

Let $f, g : G \to X$ be analytic functions which coincide on a set $D \subseteq G$, which has an accumulation point in G and $\varphi \in X'$. Then the analytic functions $\varphi \circ f$ and $\varphi \circ g$ coincide on D. According to the Identity Theorem for complex-valued functions, [9], Theorem V.3.13, $\varphi\big(f(z)\big) = \varphi\big(g(z)\big)$ for all $z \in G$. Since $\varphi \in X'$ was arbitrarily chosen and X' acts point separating on X, as shown for example in [12], Corollary 5.2.7, $f(z) = g(z)$ for all $z \in G$.

$\square$

1.2 Linear Operators

In the present section we gather well-known results about unbounded operators and their resolvents. By X and Y we denote two Banach spaces.

1.2.1 Definition. Let $D \subseteq X$ be a linear subspace.

- A linear map $A : D \to Y$ defined on D is called an *(unbounded) linear operator* and denoted by $A : D \subseteq X \to Y$. We call $D(A) := D$ the domain of A.
- A is called *densely defined* if D is dense in X.
- A is called *closed* if the graph $\{(x, Ax) : x \in D\}$ is closed in $X \times Y$ with respect to the product topology.
- If $B : E \subseteq X \to Y$ is another operator, we call B an *extension* of A, $D \subseteq E$ and $Ax = Bx$ for all $x \in D$, which is equivalent to the inclusion $A \subseteq B$.

For two unbounded operators $A : D \subseteq X \to Y$, $B : E \subseteq X \to Y$ and $\lambda \in \mathbb{C}$ we define

- $D(A + B) := D \cap E$ and $(A + B)x := Ax + Bx$ for $x \in D(A + B)$,
- $D(\lambda A) := D$ and $(\lambda A)x := \lambda Ax$ for $x \in D$ and,
- if $A : D \to Y$ is injective, $D(A^{-1}) := \operatorname{ran}(A)$ and $A^{-1}x := Cx$, where C is the inverse of $A : D \to \operatorname{ran}(A)$.

Lastly, if Z is an additional Banach space and $C : F \subseteq Y \to Z$, then we define

$$D(CA) := \{x \in D : \ Ax \in F\}$$

and $(CA)x := C(Ax)$ for $x \in D(CA)$.

1.2.2 Proposition. Let Z be an additional Banach space, $A : D(A) \subseteq X \to Y$, $B \in L_b(Z, X)$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then the following assertions hold true.

a) If A is densely defined, also λA is densely defined.

b) If A is closed, also λA is closed.

c) If A is closed, also AB is closed. If in addition $\operatorname{ran}(B) \subseteq D(A)$, then AB is a bounded operator.

d) $(\mu I - A)\big(D(A^n)\big) = D(A^{n-1})$ for all $n \in \mathbb{N}$ and every $\mu \in \rho(A)$.

Proof. a): $D(A) = D(\lambda A)$ implies $X = \overline{D(A)} = \overline{D(\lambda A)}$.

b): Given $\lambda \neq 0$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(A)$ satisfying $\lim_{n\to+\infty}(x_n, \lambda Ax_n) = (x, y)$. We obtain $\lim_{n\to+\infty} Ax_n = \frac{1}{\lambda}y$ and by the closedness of A in turn $Ax = \frac{1}{\lambda}y$, meaning $(x, y) = (x, \lambda Ax) \in \operatorname{graph}(\lambda A)$.

c): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $D(AB)$ satisfying $\lim_{n\to+\infty}(x_n, ABx_n) = (x, y)$. $B \in L_b(Z, X)$ yields $\lim_{n\to+\infty} Bx_n = Bx$. Hence, $\lim_{n\to+\infty}\big(Bx_n, A(Bx_n)\big) = (Bx, y)$. Because of the closedness of A, $Bx \in D(A)$ and $ABx = A(Bx) = y$. Therefore, AB is closed. If in addition $\operatorname{ran}(B) \subseteq D(A)$, then $D(AB) = X$. Since AB is closed, we can employ the Closed Graph Theorem, Theorem 4.4.2 in [12], and obtain $AB \in L_b(Z, X)$.

d): Clearly, $(\mu I - A)\big(D(A^n)\big) \subseteq D(A^{n-1})$. In order to show the converse inclusion, let $y \in D(A^{n-1})$ and set $x := R(\mu, A)y$. Because of $A^{n-1}x = R(\mu, A)A^{n-1}y \in D(A)$, we obtain $x \in D(A^n)$ and, in turn, $y = (\mu I - A)x \in (\mu I - A)\big(D(A^n)\big)$.

□

1.2.3 Definition. Let $A : D \subseteq X \to X$ be an unbounded operator. We define the *resolvent set* $\rho(A)$ as the set of all $\lambda \in \mathbb{C}$ such that there exists a bounded operator $R(\lambda, A) \in L_b(X)$, called the *resolvent*, satisfying $(\lambda I - A)R(\lambda, A) = I$ and $R(\lambda, A)(\lambda I - A) \subseteq I$.

The following proposition gathers well-known results about resolvents. Its proofs can be found for example in [19], Section 4.2.

1.2.4 Proposition. Let $A : D \subseteq X \to X$ be an operator. The following assertions hold true.

a) For $\lambda \in \rho(A)$ and $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \frac{1}{\|R(\lambda,A)\|}$ we have $\mu \in \rho(A)$ and

$$R(\mu, A) = \sum_{n=0}^{\infty} (\lambda - \mu)^n R(\lambda, A)^{n+1}.$$

In particular, $\rho(A)$ is an open subset of $\mathbb{C}$.

b) $R(\cdot, A) : \rho(A) \to L_b(X)$ is analytic and

$$\frac{d^n}{d\lambda^n} R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}$$

for any $n \in \mathbb{N}$.

c) For $\lambda, \mu \in \rho(A)$ we have

$$R(\lambda, A) - R(\mu, A) = (\lambda - \mu)R(\lambda, A)R(\mu, A).$$

d) If $\rho(A)$ is non-empty, A is closed.

e) If A is bounded, $\rho(A) \supseteq \Big(K_{\|A\|}(0)\Big)^c$.

Lastly, we introduce the graph norm and show that the domain of a closed operator equipped with this norm forms a Banach space.

1.2.5 Lemma. If $A : D(A) \subseteq X \to Y$ is a closed linear operator, then the space $Z := \big(D(A), \|\cdot\|_G\big)$ is a Banach space and $A \in L_b(Z, X)$. Here $\|\cdot\|_G$ is the graph norm on $D(A)$ satisfying $\|x\|_G := \|x\|_X + \|Ax\|_Y$.

Proof. It is well-known, that $\|(x, y)\|_1 := \|x\|_X + \|y\|_Y$ defines a norm on $X \times Y$, such that with $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ also $(X \times Y, \|\cdot\|_1)$ is a Banach space. Since the graph of A is by assumption a closed subspace of $X \times Y$, $(\text{graph}(A), \|\cdot\|_1 |_{\text{graph}(A)})$ is a Banach space; see [17], Lemma 9.1.6. As $\|x\|_G = \|(x, Ax)\|_1$, also Y is a Banach space.

$$\|Ax\| \leq \|x\| + \|Ax\| = \|x\|_G$$

for every $x \in X$ shows that $A \in L_b(Z, X)$.

□

Chapter 2

Integration in Banach Spaces

To build up the theory of analytic semigroups, we state some results concerning Integration Theory on Banach spaces that will be used throughout this thesis.

2.1 The Riemann Integral

In the present section we define the Riemann integral for Banach space-valued functions and examine its relation to differentiation.

2.1.1 Definition. Given a compact interval $[a, b] \subseteq \mathbb{R}$ with $-\infty < a < b < +\infty$, we call $R := \big((t_j)_{j=0}^{n(R)}, (\alpha_j)_{j=1}^{n(R)}\big)$ a *Riemann partition* of the interval $[a, b]$, if $a = t_0 < \cdots < t_{n(R)} = b$ and $\alpha_j \in [t_{j-1}, t_j]$ for all $j \in \{1, \ldots, n(R)\}$. We call

$$|R| := \max\{t_j - t_{j-1} : j = 1, \ldots, n(R)\}$$

the *fineness* of R and introduce the relation $R_1 \lesssim R_2 :\Leftrightarrow |R_1| \geq |R_2|$. By $\mathcal{R}$ we denote the set of all Riemann partitions of $[a, b]$ and note that $(\mathcal{R}, \lesssim)$ forms a directed set. Let $f : [a, b] \to X$ a bounded function. If the net of Riemann sums

$$\Big(\sum_{j=0}^{n(R)} (t_j - t_{j-1}) f(\alpha_j)\Big)_{R \in \mathcal{R}}$$

converges in X, we call f *Riemann integrable* over $[a, b]$ and define

$$\int_a^b f(t)\, dt := \lim_{R \in \mathcal{R}} \sum_{j=0}^{n(R)} (t_j - t_{j-1}) f(\alpha_j).$$

Let $[a, b) \subseteq \mathbb{R}$ with $-\infty < a < b \leq +\infty$. We call a (possibly unbounded) function $f : [a, b) \to X$ *improperly Riemann integrable* over $[a, b)$ if f is Riemann integrable over $[a, \beta]$ for any $\beta \in [a, b)$ and the limit

$$\lim_{\beta \to b^-} \int_a^\beta f(t)\, dt$$

exists in X. In this case we define

$$\int_a^b f(s)\ ds := \lim_{\beta \to b^-} \int_a^\beta f(s)\ ds.$$

We often omit the term 'improper' and simply call f Riemann integrable. Furthermore, we call f *absolutely improperly Riemann integrable*, or simply absolutely integrable, if $t \mapsto \|f(t)\|$ is improperly Riemann integrable, in particular

$$\int_a^b \|f(t)\|\, dt < +\infty.$$

For $f : (a, b] \to X$ we define the integral accordingly.

We say that a function $f : (a, b) \to X$ is improperly Riemann integrable, if f is improperly Riemann integrable over $(a, c]$ and $[c, b)$ for some $c \in (a, b)$ and define

$$\int_a^b f(t)\ dt := \int_a^c f(t)\ dt + \int_c^b f(t)\ dt.$$

By Fakta 9.3.17 in [17] this definition is independent of the choice of $c \in (a, b)$ and we have

$$\int_a^b f(t)\ dt = \lim_{\substack{\alpha \to a^+ \\ \beta \to b^-}} \int_\alpha^\beta f(t)\ dt.$$

The proof of the following result can be found in [17], Section 9.3.

2.1.2 Proposition. Let $[a, b) \subseteq \mathbb{R}$ be an interval with $-\infty < a < b \leq +\infty$ and $f : [a, b) \to X$ be a function. If f is absolutely integrable over $[a, b)$ and Riemann integrable over $[a, \beta]$ for any $\beta \in [a, b)$, then f is improperly Riemann integrable. In this case,

$$\left\| \int_a^b f(t)\ dt \right\| \leq \int_a^b \|f(t)\|\, dt < +\infty.$$

In particular, if there is an absolutely integrable function $g : [a, b) \to [0, +\infty)$, which satisfies $\|f(t)\| \leq g(t)$ for every $t \in [c, b)$ for some $c \in [a, b)$ and $t \mapsto \|f(t)\|$ is Riemann integrable over $[a, \beta]$ for any $\beta \in [a, b)$, then f is absolutely integrable. Analogous statements hold true for functions defined on intervals of the form $(a, b]$ or (a, b).

The properties of the Banach space-valued Riemann integral are mostly the same as in the real-valued case. Proofs of the following results can be found for example in [17], Section 9.3.

2.1.3 Proposition. Given an interval $I \subseteq \mathbb{R}$ of the form $[a, b], (a, b], [a, b)$ or (a, b) for $-\infty \leq a < b \leq +\infty$, the following assertions hold true.

a) For Riemann integrable $f, g : I \to X$ and $\alpha, \beta \in \mathbb{C}$ also $\alpha f + \beta g$ is Riemann integrable over I satisfying

$$\int_a^b \big(\alpha f(t) + \beta g(t)\big)\ dt = \alpha \int_a^b f(t)\ dt + \beta \int_a^b g(t)\ dt.$$

b) Let $-\infty < a < b < +\infty$, $f : [a,b] \to X$ and $t \mapsto \|f(t)\|$ be Riemann integrable. If we define its supremum norm by $\|f\|_\infty := \sup\{\|f(t)\| : t \in [a,b]\}$, then

$$\left\| \int_a^b f(t)\ dt \right\| \leq \int_a^b \|f(t)\|\ dt \leq \|f\|_\infty (b-a).$$

c) Let Y be an additional Banach space and $T \in L_b(X,Y)$. If $f : I \to X$ is Riemann integrable, also $t \mapsto Tf(t)$ is Riemann integrable and

$$\int_a^b Tf(t)\ dt = T\Big(\int_a^b f(t)\ dt\Big).$$

d) For $-\infty < a < b < +\infty$ every continuous $f : [a,b] \to X$ is Riemann integrable over $[a,b]$.

e) If $f : I \to X$ is integrable over I and $J \subseteq I$ is a real interval, then f is Riemann integrable over J. Furthermore, for $\beta \in I$ we have

$$\int_a^b f(t)\ dt = \int_a^\beta f(t)\ dt + \int_\beta^b f(t)\ dt.$$

f) Given a Riemann integrable $f : I \to X$, the function $F : I \to X$ defined by

$$F(t) := \int_a^t f(s)\ ds$$

is continuous. If f is continuous at $t_0 \in I$, then F is differentiable at t_0 with $F'(t_0) = f(t_0)$.

g) Let $-\infty < a < b < +\infty$ and $f, F : [a,b] \to X$ be continuous. If, in addition, F is continuously differentiable on (a,b) with $F'(t) = f(t)$ for $t \in (a,b)$, then

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

2.2 Complex Analysis on Banach spaces

In order to study complex, Banach space-valued functions, we need to introduce Banach space-valued path integrals. Throughout the present section X will denote a Banach space and G a domain, i.e. an open, non-empty and connected set contained in $\mathbb{C}$.

2.2.1 Definition. Given an interval $I \subseteq \mathbb{R}$, $a := \inf I$, $b := \sup I$ with $-\infty \leq a < b \leq +\infty$ (note that I can have the forms $(a,b), [a,b), (a,b], [a,b]$) and a continuously differentiable path $\gamma : I \to G$ we call a function $f : G \to X$ *integrable along* γ, if $t \mapsto \gamma'(t)(f \circ \gamma)(t)$ is (improperly) Riemann integrable over I. In this case we define

$$\int_\gamma f(z)\ dz := \int_a^b \gamma'(t)(f \circ \gamma)(t)\ dt.$$

Furthermore, two paths $\gamma_1, \gamma_2 : [c,d] \to \mathbb{C}$ satisfying $\gamma_1(c) = \gamma_2(c)$, $\gamma_1(d) = \gamma_2(d)$ are called *homotopic*, if there is a continuous map $h : [0,1] \times [c,d] \to \mathbb{C}$ with the properties

a) $h(0,t) = \gamma_1(t)$ for all $t \in [a,b]$,

b) $h(1,t) = \gamma_2(t)$ for all $t \in [a,b]$,

c) $h(s,a) = \gamma_1(a) = \gamma_2(a)$ for all $s \in [0,1]$ and

d) $h(s,b) = \gamma_1(b) = \gamma_2(b)$ for all $s \in [0,1]$.

Lastly we define the composition of two paths $\gamma_1 : [a,b] \to \mathbb{C}, \gamma_2 : [c,d] \to \mathbb{C}$ with $\gamma_1(b) = \gamma_2(c)$ as

$$\gamma_1\gamma_2 : [0,1] \to \mathbb{C},\ (\gamma_1\gamma_2)(t) := \gamma_1\big(2(b-a)t + a\big)$$

for $t \in [0, \frac{1}{2}]$ and

$$(\gamma_1\gamma_2)(t) := \gamma_2\big(2(d-c)t + 2c - d\big)$$

for $t \in [\frac{1}{2}, 1]$ and the inverse of a path $\gamma : [a,b] \to X$ as

$$\gamma^- : [a,b] \to X,\ \gamma^-(t) = \gamma(a+b-t).$$

2.2.2 Remark. From Proposition 1.6.8, a), in [11] we know that being homotopic in G is an equivalence relation.

From Theorem 2.1.2 we conclude that if for a continuously differentiable path $\gamma : (a,b) \to G$ with $-\infty \leq a < b \leq +\infty$ the function $f : G \to X$ is integrable along $\gamma|_{[\alpha,\beta]}$ for any $a < \alpha < \beta < b$ and $(a,b) \ni t \mapsto \gamma'(t)(f \circ \gamma)(t)$ is absolutely integrable, then f is integrable along γ.

We can expand the definition of the path integral to continuous and piecewise continuously differentiable paths $\gamma : [a,b] \to \mathbb{C}$ with $-\infty < a < b < +\infty$. Let $a = t_0 < t_1 < \cdots < t_n = b$ be such that $\gamma|_{[t_{j-1},t_j]}$ is continuously differentiable for any $j = 1, \ldots, n$. We say that a function $f : G \to X$ is integrable along γ, if f is integrable along $\gamma\big|_{[t_{j-1},t_j]}$ for all $j = 1, \ldots, n$ and define

$$\int_\gamma f(z)\ dz := \sum_{j=1}^{n} \int_{\gamma|_{(t_{j-1},t_j)}} f(z)\ dz.$$

Since every path $\gamma : [a,b] \to \mathbb{C}$ can be reparametrized via an affine bijection $\varphi : [0,1] \to [a,b]$ into a path $\delta := \gamma \circ \varphi : [0,1] \to \mathbb{C}$ defined on $[0,1]$, we can say that two paths $\gamma_1 : [a,b] \to \mathbb{C}, \gamma_2 : [c,d] \to \mathbb{C}$ satisfying $\gamma_1(a) = \gamma_2(c)$ and $\gamma_1(b) = \gamma_2(d)$ are homotopic, if their reparametrizations defined on $[0,1]$ are homotopic.

2.2.3 Proposition. Let $\gamma_1, \gamma_2 : [0,1] \to G$ be two continuous and piecewise continuously differentiable paths satisfying $\gamma_1(0) = \gamma_1(1) = \gamma_2(0) = \gamma_2(1) =: z_0$. If G is star-shaped, i.e. there is $w_0 \in G$ such that $sw_0 + (1-s)z \in G$ for every $t \in [0,1]$ and $z \in G$, then γ_1 and γ_2 are homotopic in G.

Proof. The function $h_1 : [0,1] \times [0,1] \to G$ defined by

$$h_1(s,t) := \begin{cases} z_0 & \text{for } t \in [0, \frac{s}{3}], \\ \gamma_1(\frac{3t-s}{3-2s}) & \text{for } t \in [\frac{s}{3}, 1-\frac{s}{3}], \\ z_0 & \text{for } t \in [1-\frac{s}{3}, 1], \end{cases}$$

is continuous and satisfies $\operatorname{ran}(h_1) = \operatorname{ran}(\gamma_1) \subseteq G$. Therefore, h_1 is a homotopy between γ_1 and $\gamma_3 : [0,1] \to G$ defined by

$$\gamma_3(t) = \begin{cases} z_0 & \text{for } t \in [0, \frac{1}{3}], \\ \gamma_1(3t-1) & \text{for } t \in [\frac{1}{3}, \frac{2}{3}], \\ z_0 & \text{for } [\frac{2}{3}, 1]. \end{cases}$$

Since G is star-shaped, we find $w_0 \in G$ such that $sw_0 + (1-s)z \in G$ for any $s \in [0,1]$ and $z \in G$. Define $h_2 : [0,1] \times [0,1] \to G$ by

$$h_2(s,t) := \begin{cases} (1-s)z_0 + s\big((1-3t)z_0 + 3tw_0\big) & \text{for } t \in [0, \frac{1}{3}], \\ (1-s)\gamma_3(t) + sw_0 & \text{for } t \in [\frac{1}{3}, \frac{2}{3}], \\ (1-s)z_0 + s\big(3(1-t)w_0 + (3t-2)z_0\big) & \text{for } t \in [\frac{2}{3}, 1]. \end{cases}$$

G being star-shaped together with $\operatorname{ran}(\gamma_3) \subseteq G$ implies $\operatorname{ran}(h_2) \subseteq G$. Since h_2 is continuous, it is a homotopy between γ_3 and $\gamma_4 : [0,1] \to G$, were

$$\gamma_4(t) = \begin{cases} (1-3t)z_0 + 3tw_0 & \text{for } t \in [0, \frac{1}{3}], \\ w_0 & \text{for } t \in [\frac{1}{3}, \frac{2}{3}], \\ 3(1-t)w_0 + (3t-2)z_0 & \text{for } [\frac{2}{3}, 1]. \end{cases}$$

Lastly, we define $h_3(s,t) := (1-s)\gamma_4(t) + sz_0$, which only consists of lines connecting z_0 and w_0. Therefore, $\operatorname{ran}(h_3) \subseteq G$ and, since h_3 is continuous, h_3 is a homotopy between γ_4 and the constant path z_0. Since being homotopic in G is an equivalence relation, γ_1 is homotopic to the constant path z_0. Repeating the argument with γ_2, we see that γ_1 and γ_2 are homotopic to the constant path z_0 and hence homotopic to each other.

□

The proofs of the subsequent results can be found in [17], Fakta 11.2.3, Proposition 11.6.8 and Corollary 11.8.14.

2.2.4 Proposition. The following assertions hold true.

a) Let $\gamma_1 : [a,b] \to \mathbb{C}, \gamma_2 : [c,d] \to \mathbb{C}$ be two continuous and piecewise continuously differentiable paths with $\gamma_1(b) = \gamma_2(c)$. If $f : G \to X$ is integrable along γ_1 and γ_2, then f is integrable along $\gamma_1\gamma_2$ and

$$\int_{\gamma_1\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.$$

b) Given a continuous and piecewise continuously differentiable path $\gamma : [a, b] \to X$ and a function $f : G \to X$, which is integrable along γ, f is also integrable along γ^- and

$$\int_{\gamma^-} f(z)\ dz = -\int_\gamma f(z)\ dz.$$

The following two theorems will be heavily used throughout this thesis. Their proofs can be found in [17], Corollaries 11.6.10 and 11.6.12.

2.2.5 Theorem (Cauchy's Integral Theorem)**.** *Let $f : G \to X$ be analytic, $-\infty < a < b < +\infty$ as well as $-\infty < c < d < +\infty$, and $\gamma_1 : [a, b] \to G, \gamma_2 : [c, d] \to G$ be continuous and piecewise continuously differentiable paths satisfying $\gamma_1(a) = \gamma_2(c)$ as well as $\gamma_1(b) = \gamma_2(d)$. If γ_1 and γ_2 are homotopic, then*

$$\int_{\gamma_1} f(z)\ dz = \int_{\gamma_2} f(z)\ dz.$$

In particular, if a closed curve $\gamma : [a, b] \to G$, that is $\gamma(a) = \gamma(b) =: z_0$, is homotopic to the constant path $t \mapsto z_0$, then

$$\int_\gamma f(z)\ dz = 0.$$

2.2.6 Theorem (Cauchy's Integral Formula)**.** *Given an analytic function $f : G \to X$, $z_0 \in G$ and $r > 0$ such that $K_r(z_0) \subseteq G$, it holds that*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{n+1}}\ dz$$

for all $n \in \mathbb{N} \cup \{0\}$ and any $w \in U_r(z_0)$, where $\gamma : [0, 2\pi] \to G$, $\gamma(t) = z_0 + re^{it}$, or γ is a path which is homotopic to $z_0 + re^{it}$ in $G \setminus \{z_0\}$. In particular, analytic functions are infinitely often complex differentiable.

2.3 The Bochner Integral

In this section we define the Bochner integral for Banach space-valued functions on a measure space and state some of its properties and study its relation to the Riemann Integral. Throughout the present section X denotes a Banach space and $(\Omega, \mathcal{A}, \mu)$ will be a σ-finite complete measure space, meaning a set Ω together with a σ-algebra $\mathcal{A}$ on Ω and a measure $\mu : \mathcal{A} \to [0, +\infty]$ with the properties that

- there are countably many sets $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \mathcal{A}$ and $\mu(A_n) < +\infty$ for all $n \in \mathbb{N}$, such that
$$\Omega = \bigcup_{n\in\mathbb{N}} A_n,$$
- if $A \subseteq N$ with $N \in \mathcal{A}$ and $\mu(N) = 0$, $A \in \mathcal{A}$.

We are going to employ the following notions.

- We say that a statement $\mathcal{S}(\omega)$ holds for *almost every* $\omega \in \Omega$, if there exists a *null set* $N \subseteq \Omega$, i.e. $N \in \mathcal{A}$ with $\mu(N) = 0$, such that $\mathcal{S}(\omega)$ holds true for all $\omega \in \Omega \setminus N$.
- A function $f : \Omega \to X$ is called *simple*, if $f(\Omega)$ is finite, $f^{-1}(\{x\}) \in \mathcal{A}$ for any $x \in X$ and $\mu\big(f^{-1}(X \setminus \{0\})\big) < +\infty$.
- We say that a function $f : \Omega \to X$ is *measurable*, if there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions with $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for almost every $\omega \in \Omega$.

We state basic properties of measurable functions. Their proofs can be found in [10], Section X.1.

2.3.1 Proposition. The following assertions hold true.

a) If $f, g : \Omega \to X$ are simple and $z \in \mathbb{C}$, $f + zg$ is also simple. The same is true for measurable functions.

b) Every simple function $f : \Omega \to X$ has a unique representation

$$f(\omega) = \sum_{j=1}^{n} \mathbb{1}_{A_j}(\omega) x_j$$

with $n \in \mathbb{N}$, pairwise disjoint sets $A_j \in \mathcal{A}$, $j = 1, \dots, n$ and pairwise distinct elements $x_j \in X$, $j = 1, \dots, n$. This representation is called the *normal form* of f. In particular, also $\omega \mapsto \|f(\omega)\|$ is a simple, real-valued function with

$$\|f(\omega)\| = \sum_{j=1}^{n} \mathbb{1}_{A_j}(\omega) \, \|x_j\| \,.$$

c) A function $f : \Omega \to X$ is measurable if and only if $f^{-1}(U) \in \mathcal{A}$ for any open subset $U \subseteq X$ and there is a null set $N \in \mathcal{A}$ such that $f(\Omega \setminus N)$ is separable, which means the existence of a countable set $D \subseteq f(\Omega \setminus N)$, which is dense in $f(\Omega \setminus N)$.

d) Given a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $f_n : \Omega \to X$, $n \in \mathbb{N}$, which converges almost everywhere to a function $f : \Omega \to X$, f is also measurable.

e) If $f : \Omega \to X$ is measurable, so is $\omega \mapsto \|f(\omega)\|$.

2.3.2 Example. Let $n \in \mathbb{N}$, $\Omega = U$ an open subset of $\mathbb{R}^n$, denote by $\mathcal{B}^n|_U$ the Borel σ-algebra generated by the collection of all open subsets of U and by $\lambda^n|_U : \mathcal{B}^n|_U \to [0, +\infty]$ the n-dimensional Lebesgue measure on U. We define

$$\mathcal{A} := \{A \subseteq U : \text{there are sets } N, B \in \mathcal{B}^n|_U \text{ with } B \subseteq A \subseteq B \cup N \text{ and } \lambda^n|_U(N) = 0\}$$

and $\mu : \mathcal{A} \to [0, +\infty]$ by $\mu(A) = \lambda^n|_U(B)$ for $A \in \mathcal{A}$, $N, B \in \mathcal{B}|_U^n$, $B \subseteq A \subseteq B \cup N$ and $\lambda^n|_U(N) = 0$. Then, $(\Omega, \mathcal{A}, \mu)$ is a σ-finite, complete measure space. We want to employ Proposition 2.3.1, *c*), to prove that a continuous function $f : U \to X$ is measurable. To that end, we note that since f is continuous, $f^{-1}(O)$ is open in U and therefore belongs to $\mathcal{B}|_U$ for any open set $O \subseteq X$. Let $x \in f(U)$, $t \in U$ such that $f(t) = x$ and $\varepsilon > 0$.

Since f is continuous, there is a constant $\delta > 0$ such that $\|f(t) - f(s)\| < \varepsilon$ whenever $s \in U$ with $\|t - s\|_2 < \delta$. Since $\mathbb{Q}^n \cap U$ is dense in U, there exists a $q \in \mathbb{Q}^n \cap U$ satisfying $\|t - q\| < \delta$ and in turn

$$\|f(t) - f(q)\| < \varepsilon.$$

$f(\mathbb{Q}^n \cap U)$ being countable yields the separability of $f(U)$ and, in consequence, the measurability of f.

2.3.3 Definition. Given a simple function $f : \Omega \to X$ represented in its unique normal form

$$f(\omega) = \sum_{j=1}^{n} \mathbb{1}_{A_j}(\omega) x_j$$

as in Proposition 2.3.1, a), we define

$$\int_\Omega f(\omega)\, d\mu(\omega) := \sum_{j=1}^{n} \mu(A_j) x_j.$$

Since every measurable function f is the limit of a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions, it is natural to define the integral of f as the limit of the sequence

$$\left(\int_\Omega f_n(\omega)\, d\mu(\omega)\right)_{n\in\mathbb{N}},$$

if it exists and does not depend on the concrete sequence $(f_n)_{n\in\mathbb{N}}$.

The next result makes sure the integral is well-defined for special class of measurable functions. Its proof can be found in [10], Corollary X.2.7.

2.3.4 Proposition. Given a measurable function $f : \Omega \to X$ and two sequences $(f_{1,n})_{n\in\mathbb{N}}, (f_{2,n})_{n\in\mathbb{N}}$ of simple functions with $\lim_{n\to\infty} f_{j,n}(\omega) = f(\omega)$ for almost every $\omega \in \Omega$ and $j = 1, 2$ as well as the property that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\int_\Omega \|f_{j,n}(\omega) - f_{j,m}(\omega)\|\, d\mu(\omega) < \varepsilon$$

whenever $n, m \geq N$ for $j = 1, 2$, the sequences

$$\left(\int_\Omega f_{1,n}(\omega)\, d\mu(\omega)\right)_{n\in\mathbb{N}}$$

and

$$\left(\int_\Omega f_{2,n}(\omega)\, d\mu(\omega)\right)_{n\in\mathbb{N}}$$

converge in X to the same limit.

The previous proposition justifies the following definition.

2.3.5 Definition. Let $f : \Omega \to X$ be measurable. We call f *integrable*, if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions with $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for almost every $\omega \in \Omega$ as well as the property that for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\int_\Omega \|f_n(\omega) - f_m(\omega)\| \, d\mu(\omega) < \varepsilon$$

whenever $n, m \geq N$. In this case we define

$$\int_\Omega f(\omega) \; d\mu(\omega) := \lim_{n\to\infty} \int_\Omega f_n(\omega) \; d\mu(\omega).$$

Obviously, if an integrable function happens to be simple, the integral coincides with the definition of the integral of simple functions, Definition 2.3.3.

The properties of the Banach space-valued integral are mostly the same as in the real-valued case. The proofs of the following statements can be found in [10], Theorem X.2.11, Lemma X.2.13, Corollary X.2.16 and Theorem X.3.14.

2.3.6 Proposition. The following assertions hold true.

a) For integrable $f, g : \Omega \to X$ and $\alpha, \beta \in \mathbb{C}$ also $\alpha f + \beta g$ is integrable satisfying

$$\int_\Omega \big(\alpha f(\omega) + \beta g(\omega)\big) \; d\mu(\omega) = \alpha \int_\Omega f(\omega) \; d\mu(\omega) + \beta \int_\Omega g(\omega) \; d\mu(\omega).$$

b) A measurable function $f : \Omega \to X$ is integrable if and only if

$$\int_\Omega \|f(\omega)\| \; d\mu(\omega) < +\infty.$$

In this case

$$\left\| \int_\Omega f(\omega) \; d\mu(\omega) \right\| \leq \int_\Omega \|f(\omega)\| \, d\mu(\omega).$$

c) Let Y be an additional Banach space and $T \in L_b(X, Y)$. If $f : \Omega \to X$ is integrable, then $\omega \mapsto Tf(\omega)$ is also integrable and

$$\int_\Omega Tf(\omega) \; d\mu(\omega) = T\Big(\int_\Omega f(\omega) \; d\mu(\omega)\Big).$$

d) Given an integrable function $f : \Omega \to X$ and $A \in \mathcal{A}$, $\mathbb{1}_A f$ is integrable.

e) If a function $f : \Omega \to X$ vanishes almost everywhere, f is integrable and

$$\int_\Omega f(\omega) \; d\mu(\omega) = 0.$$

f) Given a measurable function $f : \Omega \to X$ satisfying $\|f(\omega)\| \leq g(\omega)$ for almost every $\omega \in \Omega$ and some integrable $g : \Omega \to \mathbb{R}$, f is integrable and

$$\int_\Omega \|f(\omega)\| \, d\mu(\omega) \leq \int_\Omega g(\omega) \; d\mu(\omega).$$

The most versatile result will be the following Banach space version of the Dominated Convergence Theorem, see [10], Theorem X.3.12.

2.3.7 Theorem (Dominated Convergence Theorem)**.** *Given a sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ defined on Ω with values in X, which converges almost everywhere to a function $f : \Omega \to X$, meaning $f_n(\omega) \xrightarrow{n\to+\infty} f(\omega)$ for almost every $\omega \in \Omega$ and an integrable function $g : \Omega \to \mathbb{R}$ satisfying $\|f_n(\omega)\| \leq g(\omega)$ for almost every $\omega \in \Omega$ and every $n \in \mathbb{N}$, then f and f_n, $n \in \mathbb{N}$ are integrable and*

$$\lim_{n\to+\infty} \int_\Omega f_n(\omega)\ d\mu(\omega) = \int_\Omega f(\omega)\ d\mu(\omega)$$

as well as

$$\lim_{n\to+\infty} \int_\Omega \|f_n(\omega) - f(\omega)\|\, d\mu(\omega) = 0.$$

Using the Dominated Convergence Theorem, one can prove results regarding parameter integrals. The proof of the following results can be found in [10], Theorems X.3.17 and X.3.18.

2.3.8 Proposition. Let (M, d) a metric space. Given a function $f : \Omega \times M \to X$ and $m_0 \in M$ with the properties that

a) $\omega \mapsto f(\omega, m)$ is integrable for all $m \in M$,

b) $m \mapsto f(\omega, m)$ is continuous at m_0 for almost every $\omega \in \Omega$ and

c) there exists a constant $\varepsilon > 0$ and a real-valued, non-negative, integrable function $g : \Omega \to \mathbb{R}$ such that $\|f(\omega, m)\| \leq g(\omega)$ for all $m \in U_\varepsilon(m_0)$ and almost every $\omega \in \Omega$,

the function

$$F : M \to X,\ m \mapsto \int_\Omega f(\omega, m)\ d\mu(\omega)$$

is continuous at m_0.

2.3.9 Proposition. Let $I \subseteq \mathbb{R}$ be an interval, $t_0 \in I$ and $f : \Omega \times I \to X$ a function with the properties

a) $\omega \mapsto f(\omega, t)$ is integrable for all $t \in I$,

b) $t \mapsto f(\omega, t)$ is differentiable in t_0 for almost every $\omega \in \Omega$ and

c) there exists a constant $\varepsilon > 0$ and a real-valued, non-negative, integrable function $g : \Omega \to \mathbb{R}$ such that $\frac{1}{|t-t_0|}\|f(\omega, t) - f(\omega, t_0)\| \leq g(\omega)$ for every $t \in \big((t_0 - \varepsilon, t_0 + \varepsilon) \cap I\big) \setminus \{t_0\}$ and almost every $\omega \in \Omega$.

Then the function

$$F : I \to X,\ F(t) := \int_\Omega f(\omega, t)\ d\mu(\omega)$$

is differentiable in t_0 with

$$F'(t_0) = \int_\Omega \frac{\partial f}{\partial t}(\omega, t_0)\ d\mu(\omega).$$

The properties b) and c) are fulfilled if b*) there exists a constant $\varepsilon > 0$ and a real-valued, non-negative, integrable function $g : \Omega \to \mathbb{R}$ such that $t \mapsto f(\omega, t)$ is differentiable on $(t_0 - \varepsilon, t_0 + \varepsilon) \cap I$ for almost every $\omega \in \Omega$ and the derviative satisfies $\left\|\frac{\partial f}{\partial t}(\omega, t)\right\| \leq g(\omega)$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and any $\omega \in \Omega$ for which the derivative exists.

2.3.10 Proposition. Let $(\Omega, \mathcal{A}, \mu)$ be a σ-finite, complete measure space. Given a function $f : \Omega \times G \to X$ with the properties that

a) $\omega \mapsto f(\omega, z)$ is integrable for all $z \in G$,

b) $z \mapsto f(\omega, z)$ is analytic for almost every $\omega \in \Omega$ and

c) given a compact subset $K \subseteq G$, there is a real-valued, non-negative integrable function $g_K : \Omega \to \mathbb{R}$ such that $\|f(\omega, z)\| \leq g_K(\omega)$ for all $z \in K$ and almost every $\omega \in \Omega$,

the function

$$F : G \to X,\ F(z) := \int_\Omega f(\omega, z)\ d\mu(\omega)$$

is analytic and $\omega \mapsto \frac{d^n}{dz^n} f(\omega, z)$ is integrable for all $n \in \mathbb{N}$ and all $z \in G$ satisfying

$$\int_\Omega \frac{d^n}{dz^n} f(\omega, z)\ d\mu(\omega) = F^{(n)}(z).$$

Proof. Let $z_0 \in G$ and $r > 0$ be such that $K_{2r}(z_0) \subseteq G$. Defining $\gamma : [0, 2\pi] \to G$, $\gamma(s) := z_0 + 2re^{is}$, there is a null set N_1 such that we can represent f in the form

$$f(\omega, w) = \frac{1}{2\pi i} \int_\gamma \frac{f(\omega, z)}{z - w}\ dz$$

for every $\omega \in \Omega \setminus N_1$ and $w \in U_{2r}(z_0)$, see Theorem 2.2.6. Fix $w \in U_r(z_0)$ and for $v \in U_r(z_0) \setminus \{w\}$ define $h_v : \Omega \to X$ as

$$\begin{aligned} h_v(\omega) &:= \frac{1}{w - v}\big(f(\omega, w) - f(\omega, v)\big) = \frac{1}{2\pi i(w - v)}\Big(\int_\gamma \frac{f(\omega, z)}{z - w}\ dz - \int_\gamma \frac{f(\omega, z)}{z - v}\ dz\Big) \\ &= \frac{1}{2\pi i(w - v)} \int_\gamma \frac{wf(\omega, z) - vf(\omega, z)}{(z - w)(z - v)}\ dz = \frac{1}{2\pi i} \int_\gamma \frac{f(\omega, z)}{(z - w)(z - v)}\ dz. \end{aligned}$$

for $\omega \in \Omega \setminus N_1$ and $h_v(\omega) = 0$ for $\omega \in N_1$. According to property *c)*, there is a function $g : \Omega \to \mathbb{R}$ and a null set N_2 such that $\|f(\omega, v)\| \leq g(\omega)$ for all $v \in K_{2r}(z_0)$ and every $\omega \in \Omega \setminus N_2$. It follows that the function h_v satisfies the estimate

$$\begin{aligned} \|h_v(\omega)\| &= \frac{1}{2\pi} \left\| \int_0^{2\pi} \frac{f(\omega, z_0 + 2re^{is})}{(z_0 - w + 2re^{is})(z_0 - v + 2re^{is})}\ ds \right\| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\|f(\omega, z_0 + 2re^{is})\|}{|(z_0 - w + 2re^{is})(z_0 - v + 2re^{is})|}\ ds \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{g(\omega)}{r^2}\ ds = \frac{g(\omega)}{r^2} \end{aligned}$$

for all $\omega \in \Omega \setminus (N_1 \cup N_2)$. Since

$$\lim_{v \to w} h_v(\omega) = \lim_{v \to w} \frac{1}{w - v}\big(f(\omega, w) - f(\omega, v)\big) = \frac{d}{dz} f(\omega, w)$$

for all $\omega \in \Omega \setminus N_1$, $N_1 \cup N_2$ is again a null set and the function $\omega \mapsto \frac{g(\omega)}{r^2}$ is integrable, we can use Theorem 2.3.7 to conclude that also $\frac{d}{dz} f(\omega, w)$ is integrable and

$$\begin{aligned}\lim_{v \to w} \frac{1}{w - v}\big(F(w) - F(v)\big) &= \lim_{v \to w} \int_\Omega \frac{1}{w - v}\big(f(\omega, w) - f(\omega, v)\big)\ d\mu(\omega) \\ &= \lim_{v \to w} \int_\Omega h_v(\omega)\ d\mu(\omega) = \int_\Omega \frac{d}{dz} f(\omega, w)\ d\mu(\omega),\end{aligned}$$

which implies that F is complex differentiable in w and

$$F'(w) = \int_\Omega \frac{d}{dz} f(\omega, w)\ d\mu(\omega).$$

Since w was arbitrary, F is analytic in $U_r(z_0)$. Finally, since z_0 was arbitrary, we conclude that F is analytic in G. Let $n \in \mathbb{N}$, $K \subseteq G$ compact and suppose that $\omega \mapsto \frac{d^n}{dz^n} f(\omega, z)$ is integrable as well as

$$F^{(n)}(z) = \int_\Omega \frac{d^n}{dz^n} f(\omega, z)\ dz$$

for all $z \in G$. For every $z \in K$ there is $r_z > 0$ such that $K_{2r_z}(z) \subseteq G$. We obtain

$$K \subseteq \bigcup_{z \in K} U_{r_z}(z),$$

which by the compactness of K implies the existence of $z_1, \ldots, z_m \in K$ such that

$$K \subseteq \bigcup_{j=1}^m U_{r_{z_j}}(z_j).$$

By assumption c), there exist null sets $N_1, \ldots, N_m \subseteq \Omega$ and integrable functions $g_j : \Omega \to \mathbb{R}$ such that $\|f(\omega, z)\| \le g_j(\omega)$ for all $z \in K_{2r_{z_j}}(z_j)$ and every $\omega \in \Omega \setminus N_j$ for $j = 1, \ldots, m$. We define $r := \min\{r_{z_j} : j = 1, \ldots, m\}$ and $g_K : \Omega \to \mathbb{R}$ by

$$g_K(\omega) := \frac{2n!}{r^n} \max_{j=1,\ldots,m} g_j(\omega).$$

which is integrable by Corollary IV.3.5 in [13]. Let $N_0 \subseteq \Omega$ be a null set such that $z \mapsto f(\omega, z)$ is analytic on G for every $\omega \in \Omega \setminus N_0$ and define

$$N := N_0 \cup \bigcup_{j=1}^m N_j,$$

which is again a null set. Let $w \in K$, $j \in \{1, \dots, m\}$ such that $w \in U_{r_{z_j}}(z_j)$, $\omega \in \Omega \setminus N$ and define $\gamma : [0, 2\pi] \to G$, $\gamma(t) := z_j + 2r_{z_j}e^{it}$. By Theorem 2.2.6 we have

$$\begin{aligned}\left\|\frac{d^n}{dz^n}f(w,\omega)\right\| &= \frac{n!}{2\pi}\left\|\int_\gamma \frac{f(\omega,z)}{(z-w)^{n+1}}\,dz\right\| = \frac{n!}{2\pi}\left\|\int_0^{2\pi}\frac{2r_{z_j}ie^{it}f\big(\omega,\gamma(t)\big)}{(z_j+2r_{z_j}e^{it}-w)^{n+1}}\,dt\right\| \\ &\le \frac{2r_{z_j}n!}{2\pi}\int_0^{2\pi}\frac{\big\|f\big(\omega,\gamma(t)\big)\big\|}{|z_j+2r_{z_j}e^{it}-w|^{n+1}}\,dt \le \frac{2r_{z_j}n!}{2\pi}\int_0^{2\pi}\frac{g_j(\omega)}{r_{z_j}^{n+1}}\,dt \\ &\le \frac{2n!}{r^n}g_j(\omega) \le g_K(\omega)\end{aligned}$$

Applying the already proven to the function $\frac{d^n}{dz^n}f$, we see that $\omega \mapsto \frac{d^n}{dz^n}f(\omega, z)$ is integrable and, by assumption,

$$\int_\Omega \frac{d^{n+1}}{dz^{n+1}}f(\omega,z)\,d\mu(\omega) = \frac{d}{dz}\int_\Omega \frac{d^n}{dz^n}f(\omega,z)\,d\mu(\omega) = \frac{d}{dz}F^{(n)}(z) = F^{(n+1)}(z).$$

□

Another important result is the Theorem of Fubini-Tonelli, which allows us to exchange the order of integration. In order to state this Theorem we introduce the notion of product measures, see Theorem V.1.5 in [13].

2.3.11 Proposition. If $(\Psi, \mathcal{B}, \nu)$ is an additional σ-finite complete measure space, we define $\mathcal{A} \otimes \mathcal{B}$ as the σ-algebra generated by

$$\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}.$$

There is a unique measure $\mu \otimes \nu : \mathcal{A} \otimes \mathcal{B} \to [0, +\infty]$ satisfying

$$\mu \otimes \nu(A \times B) = \mu(A)\nu(B)$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

2.3.12 Theorem (Fubini-Tonelli)**.** *Let $(\Psi, \mathcal{B}, \nu)$ be an additional σ-finite complete measure space and define*

$\mathcal{C} := \{A \subseteq \Omega \times \Psi :$ *there are sets* $N, B \in \mathcal{A} \otimes \mathcal{B}$ *with* $B \subseteq A \subseteq B \cup N$ *and* $\mu \otimes \nu(N) = 0\}$.

We extend $\mu \otimes \nu$ to $\mathcal{C}$ by setting $\mu \otimes \nu(A) = \mu \otimes \nu(B)$, if $A \subseteq \Omega \times \Psi$, $B, N \in \mathcal{A} \otimes \mathcal{B}$ with $\mu \otimes \nu(N) = 0$ and $B \subseteq A \subseteq B \cup N$. If $f : \Omega \times \Psi \to X$ is measurable with respect to $\big(\Omega \times \Psi, \mathcal{C}, \mu \otimes \nu\big)$ and

$$\int_\Omega \Big(\int_\Psi \|f(\omega,\psi)\|\,d\nu(\psi)\Big) d\mu(\omega) < +\infty$$

or

$$\int_\Psi \Big(\int_\Omega \|f(\omega,\psi)\|\,d\mu(\omega)\Big) d\nu(\psi) < +\infty,$$

the functions f,

$$\omega \mapsto \int_\Psi f(\omega,\psi)\ d\nu(\psi)$$

defined almost everywhere on Ω *as well as*

$$\psi \mapsto \int_\Omega f(\omega,\psi)\ d\mu(\omega)$$

defined almost everywhere on Ψ *are integrable and*

$$\begin{aligned}\int_{\Omega\times\Psi} f(\omega,\psi)\ d(\mu\otimes\nu)(\omega,\psi) &= \int_\Omega\Big(\int_\Psi f(\omega,\psi)\ d\nu(\psi)\Big)d\mu(\omega)\\ &= \int_\Psi\Big(\int_\Omega f(\omega,\psi)\ d\mu(\omega)\Big)d\nu(\psi).\end{aligned}$$

Its proof can be found for example in [3], 2.9.

Lastly, we state and prove a theorem regarding Bochner integrals and closed operators.

2.3.13 Proposition. Let Y be an additional Banach space and $A : D(A) \subseteq X \to Y$ be a closed operator. Given an integrable function $f : \Omega \to X$, such that $f(\omega) \in D(A)$ for almost every $\omega \in \Omega$ and such that $Af : \Omega \to Y$ is integrable, we have

$$\int_\Omega f(\omega)\ d\mu(\omega) \in D(A)$$

and

$$A\Big(\int_\Omega f(\omega)\ d\mu(\omega)\Big) = \int_\Omega Af(\omega)\ d\mu(\omega).$$

Proof. The function

$$\tilde{f}(\omega) := \begin{cases} f(\omega), & f(\omega) \in D(A)\\ 0, & f(\omega) \notin D(A)\end{cases}$$

satisfies $\tilde{f}(\Omega) \subseteq D(A)$ and $\tilde{f}(\omega) = f(\omega)$ for almost every $\omega \in \Omega$. Hence, by Proposition 2.3.6, e), $\tilde{f}$ and $\omega \mapsto A\tilde{f}(\omega)$ are integrable. It is well-known, that $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} := \max\{\|x\|_X, \|y\|_Y\}$, constitutes a Banach space; see [17], Example 9.1.9. We want to prove that $g : \Omega \to X \times Y$ defined by $g(\omega) = \big(\tilde{f}(\omega), A\tilde{f}(\omega)\big)$ is measurable. To that end, let $\varphi : \Omega \to X$ and $\psi : \Omega \to Y$ be simple functions represented in their normal form

$$\varphi(\omega) = \sum_{k=1}^{n} \mathbb{1}_{A_k}(\omega)x_k$$

and

$$\psi(\omega) = \sum_{k=1}^{m} \mathbb{1}_{B_k}(\omega)y_k.$$

$\iota : \Omega \to X \times Y$ defined by $\iota(\omega) := \big(\varphi(\omega), \psi(\omega)\big)$ satisfies

$$\iota(\omega) = \sum_{k=1}^{n} \mathbb{1}_{A_k}(\omega)(x_k, 0) + \sum_{k=1}^{m} \mathbb{1}_{B_k}(\omega)(0, y_k),$$

and, hence, constitutes also a simple function.

Let $N_1, N_2 \subseteq \Omega$ be null sets and $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\mathbb{N}}$ be sequences of simple functions satisfying $\varphi_n(\omega) \xrightarrow{n\to+\infty} \tilde{f}(\omega)$ for all $\omega \in \Omega \setminus N_1$ as well as $\psi_n(\omega) \xrightarrow{n\to+\infty} A\tilde{f}(\omega)$ for all $\omega \in \Omega \setminus N_2$. As just shown, $\iota_n : \Omega \to X \times Y$ defined by $\iota_n(\omega) := \big(\varphi_n(\omega), \psi_n(\omega)\big)$ constitute simple functions such that

$$\|\iota_n(\omega) - g(\omega)\|_{X\times Y} = \max\Big\{\Big\|\varphi_n(\omega) - \tilde{f}(\omega)\Big\|_X, \Big\|\psi_n(\omega) - A\tilde{f}(\omega)\Big\|_Y\Big\} \xrightarrow{n\to+\infty} 0,$$

for $\omega \in \Omega \setminus (N_1 \cup N_2)$. In consequence, g is measurable as a function from Ω into $X \times Y$. We want to prove that g is also measurable as a function from Ω to $\operatorname{graph}(A)$. Since $\operatorname{graph}(A)$ is closed in $X \times Y$, it is a Banach space when equipped with

$$\|\cdot\|_{\operatorname{graph}(A)} := \|\cdot\|_{X\times Y}\big|_{\operatorname{graph}(A)};$$

see [17], Lemma 9.1.6. Since $g : \Omega \to \operatorname{graph}(A)$ is measurable in $X \times Y$, $g(\Omega)$ is separable in $X \times Y$ and therefore also in $\operatorname{graph}(A)$ by 2.3.1, c). Any relatively open $O \subseteq \operatorname{graph}(A)$ can be written as $O = U \cap \operatorname{graph}(A)$ for some open $U \subseteq X \times Y$; see [17], Example 12.6.3. From

$$g^{-1}(O) = g^{-1}(U) \in \mathcal{A}$$

we conclude that also $g : \Omega \to \operatorname{graph}(A)$ is measurable. As

$$\begin{aligned}\int_\Omega \|g(\omega)\|_{\operatorname{graph}(A)}\, d\mu(\omega) &= \int_\Omega \max\Big\{\Big\|\tilde{f}(\omega)\Big\|_X, \Big\|A\tilde{f}(\omega)\Big\|_Y\Big\}\, d\mu(\omega) \\ &\le \int_\Omega \Big\|\tilde{f}(\omega)\Big\|\, d\mu(\omega) + \int_\Omega \Big\|A\tilde{f}(\omega)\Big\|\, d\mu(\omega) < +\infty,\end{aligned}$$

g is integrable in $\operatorname{graph}(A)$ according to Proposition 2.3.6, b). The projections $\pi_1 : X \times Y \to X$, $\pi_1(x, y) := x$ and $\pi_2 : X \times Y \to Y$, $\pi_2(x, y) := y$ are bounded linear operators, which implies

$$\pi_1\Big(\int_\Omega g(\omega)\, d\mu(\omega)\Big) = \int_\Omega \pi_1\big(g(\omega)\big)\, d\mu(\omega) = \int_\Omega \tilde{f}(\omega)\, d\mu(\omega) = \int_\Omega f(\omega)\, d\mu(\omega)$$

and

$$\pi_2\Big(\int_\Omega g(\omega)\mu(\omega)\Big) = \int_\Omega Af(\omega)\, d\mu(\omega).$$

Because g is integrable as a function from Ω to $\operatorname{graph}(A)$, we obtain

$$\begin{aligned}\Big(\int_\Omega f(\omega)\, d\mu(\omega), \int_\Omega Af(\omega)\, d\mu(\omega)\Big) &= \Big(\pi_1\Big(\int_\Omega g(\omega)\, d\mu(\omega)\Big), \pi_2\Big(\int_\Omega g(\omega)\, d\mu(\omega)\Big)\Big) \\ &= \int_\Omega g(\omega)\, d\mu(\omega) \in \operatorname{graph}(A)\end{aligned}$$

and in turn

$$A\Big(\int_\Omega f(\omega)\, d\mu(\omega)\Big) = \int_\Omega Af(\omega)\, d\mu(\omega)$$

□

2.4 The Bochner Integral on Borel Sets

We are going to examine properties of the Bochner Integral if the underlying set Ω is a subset of $\mathbb{R}$. Throughout the present section X will denote a Banach space.

2.4.1 Remark. Let $\mathcal{B}$ be the Borel sets in $\mathbb{R}$, i.e. the σ-algebra generated by the collection of all open subsets of $\mathbb{R}$ and $\lambda : \mathcal{B} \to [0, +\infty]$ be the Lebesgue measure. Given $\Omega \in \mathcal{B}$, recall that $\big(\Omega, \mathcal{A}, \lambda|_\Omega\big)$, where

$$\mathcal{A} := \{A \subseteq \Omega : \text{there are sets } N, B \in \mathcal{B}|_\Omega \text{ with } B \subseteq A \subseteq B \cup N \text{ and } \lambda(N) = 0\},$$

$\lambda|_\Omega(A) := \lambda(B)$ for $A \in \mathcal{A}$, $N, B \in \mathcal{B}|_{(a,b)}$, $B \subseteq A \subseteq B \cup N$ and $\lambda(N) = 0$, is a σ-finite complete measure space; see Example 2.3.2. If $f : \Omega \to X$ is integrable with respect to $\lambda|_\Omega$, we will write

$$\int_\Omega f(t)\ d\lambda(t) := \int_\Omega f(t)\ d\lambda|_\Omega(t).$$

We start with two theorems that build a relation between Bochner- and Riemann integrability. Proofs of the following theorems can be found in [10], Theorems X.5.6., X.5.3 and Remark X.5.5.

2.4.2 Theorem. *Given $-\infty < a < b < +\infty$ and $f : [a, b] \to X$ bounded, f is Riemann integrable if and only if f is almost everywhere continuous. In this case, f is integrable and*

$$\int_a^b f(t)\ dt = \int_{(a,b)} f(t)\ d\lambda(t).$$

2.4.3 Theorem. *Given an interval $(a, b) \subseteq \mathbb{R}$ with $-\infty \leq a < b \leq +\infty$ and a function $f : (a, b) \to X$ which is Riemann integrable over $[\alpha, \beta]$ for any $a < \alpha < \beta < b$, f is absolutely integrable if and only if it is integrable. In this case*

$$\int_a^b f(t)\ dt = \int_{(a,b)} f(t)\ d\lambda(t).$$

In particular, if f is continuous, it is integrable if and only if

$$\int_a^b \|f(t)\|\, dt < +\infty.$$

The following result can be seen as the Fundamental Theorem of Calculus for Banach space-valued functions.

2.4.4 Lemma. Let $-\infty < a < b < +\infty$ and $f : [a, b] \to X$ differentiable. If f' is integrable over (a, b), then

$$f(b) - f(a) = \int_{(a,b)} f'(t)\ d\lambda(t).$$

Proof. Let $\varphi \in X'$ and define $g : [a,b] \to \mathbb{R}$, $g(t) := \mathrm{Re}\big((\varphi \circ f)(t)\big)$. Since $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ and φ are bounded linear maps, f is differentiable in $[a,b]$ satisfying

$$g'(t) = \mathrm{Re}\big((\varphi \circ f')(t)\big)$$

for every $t \in [a,b]$, see Proposition 1.1.2, c). Moreover,

$$|g'(t)| \leq \big|\varphi\big(f'(t)\big)\big| \leq \|\varphi\| \, \|f'(t)\| \,, \quad t \in [a,b],$$

implying the integrability of g', see Proposition 2.3.6, f). Hence, from Theorem 7.21 in [21], we obtain

$$\mathrm{Re}\Big(\varphi\big(f(b) - f(a)\big)\Big) = g(b) - g(a) = \int_{(a,b)} g'(t) \; d\lambda(t) = \mathrm{Re}\, \varphi\Big(\int_{(a,b)} f'(t) \; d\lambda(t)\Big).$$

Analogous arguments lead to

$$\mathrm{Im}\Big(\varphi\big(f(b) - f(a)\big)\Big) = \mathrm{Im}\, \varphi\Big(\int_{(a,b)} f'(t) \; dt\Big),$$

which implies

$$\varphi\big(f(b) - f(a)\big) = \varphi\Big(\int_{(a,b)} f'(t) \; d\lambda(t)\Big).$$

Since $\varphi \in X'$ was arbitrary and X' acts point-separating on X (see Theorem 5.2.3 in [12]), we obtain

$$f(b) - f(a) = \int_{(a,b)} f'(t) \; d\lambda(t).$$

□

2.4.5 Proposition. Let $-\infty < a < b < +\infty$, $f : [a,b] \to X$ and $\varphi : [a,b] \to \mathbb{C}$ be continuous and piecewise differentiable, i.e. there are partitions $a = t_0 < \cdots < t_n = b$ and $a = s_0 < \cdots < s_m = b$ such that $f|_{(t_{k-1},t_k)}$ and $\varphi|_{(s_{j-1},s_j)}$ are differentiable for any $k = 1, \dots, n$ and $j = 1, \dots, m$. If the almost everywhere defined functions φ' and f' are integrable over (a,b), then

$$\int_{(a,b)} \varphi'(t) f(t) \; d\lambda(t) = \varphi(b) f(b) - \varphi(a) f(a) - \int_{(a,b)} \varphi(t) f'(t) \; d\lambda(t).$$

Proof. Suppose first that f and φ are differentiable on (a,b). By Proposition 1.1.2, c), we have

$$(\varphi f)'(t) = \varphi'(t) f(t) + \varphi(t) f'(t)$$

and

$$\|(\varphi f)'(t)\| \leq |\varphi'(t)| \, \|f\|_\infty + \|\varphi\|_\infty \, \|f'(t)\| \,, \quad t \in (a,b).$$

Hence, $(\varphi f)'$ is integrable over (a,b). For any $n \in \mathbb{N}$ with $\frac{1}{2}(b-a) > \frac{1}{n}$ we employ Lemma 2.4.4 and obtain that $(\varphi f)'$ is integrable over $(a + \frac{1}{n}, b - \frac{1}{n})$ and

$$\int_{(a+\frac{1}{n}, b-\frac{1}{n})} (\varphi f)'(t) \; d\lambda(t) = \varphi(b - \tfrac{1}{n}) f(b - \tfrac{1}{n}) - \varphi(a + \tfrac{1}{n}) f(b - \tfrac{1}{n}).$$

Defining $g_n : [a,b] \to X$ by $g_n(t) := \mathbb{1}_{[a+\frac{1}{n}, b-\frac{1}{n}]}(t)(\varphi f)'(t)$, we have $g_n(t) \xrightarrow{n\to+\infty} (\varphi f)'(t)$ for every $t \in (a,b)$ and $\|g_n(t)\| \le \|(\varphi f)'(t)\| \le \|(\varphi f)'\|_\infty$ for every $t \in (a,b)$. Theorem 2.3.7 implies

$$\lim_{n\to+\infty} \int_{(a+\frac{1}{n}, b-\frac{1}{n})} (\varphi f)'(t)\ d\lambda(t) = \lim_{n\to+\infty} \int_{(a,b)} g_n(t)\ d\lambda(t) = \int_{(a,b)} (\varphi f)'(t)\ d\lambda(t).$$

Since φ and f are continuous, we obtain

$$\begin{aligned}
\int_{(a,b)} \varphi'(t) f(t)\ d\lambda(t) &= \int_{(a,b)} (\varphi f)'(t)\ d\lambda(t) - \int_{(a,b)} \varphi(t) f'(t)\ d\lambda(t) \\
&= \lim_{n\to+\infty} \int_{(a+\frac{1}{n}, b-\frac{1}{n})} (\varphi f)'(t)\ d\lambda(t) - \int_{(a,b)} \varphi(t) f'(t)\ d\lambda(t) \\
&= \lim_{n\to+\infty} (\varphi f)(b - \tfrac{1}{n}) - (\varphi f)(a + \tfrac{1}{n}) - \int_{(a,b)} \varphi(t) f'(t)\ d\lambda(t) \\
&= \varphi(b) f(b) - \varphi(a) f(a) - \int_{(a,b)} \varphi(t) f'(t)\ d\lambda(t).
\end{aligned}$$

If φ and f are piecewise differentiable, we can assume that there is a partition $a = t_0 < \cdots < t_n = b$ such that $\varphi|_{(t_{k-1}, t_k)}$ and $f|_{(t_{k-1}, t_k)}$ are differentiable for every $k = 1, \ldots, n$. From the first part of the proof we obtain

$$\int_{(t_{k-1}, t_k)} \varphi'(t) f(t)\ d\lambda(t) = \varphi(t_k) f(t_k) - \varphi(t_{k-1}) f(t_{k-1}) - \int_{(t_{k-1}, t_k)} \varphi(t) f'(t)\ d\lambda(t)$$

for every $k = 1, \ldots, n$. Consequently,

$$\begin{aligned}
\int_{(a,b)} \varphi'(t) f(t)\ d\lambda(t) &= \sum_{k=1}^{n} \int_{(t_{k-1}, t_k)} \varphi'(t) f(t)\ d\lambda(t) \\
&= \sum_{k=1}^{n} \varphi(t_k) f(t_k) - \varphi(t_{k-1}) f(t_{k-1}) - \int_{(t_{k-1}, t_k)} \varphi(t) f'(t)\ d\lambda(t) \\
&= \varphi(t_n) f(t_n) - \varphi(t_0) f(t_0) - \int_{(a,b)} \varphi(t) f'(t)\ d\lambda(t) \\
&= \varphi(b) f(b) - \varphi(a) f(a) - \int_{(a,b)} \varphi(t) f'(t)\ d\lambda(t).
\end{aligned}$$

□

The proof of an even more general version of the following result can be found in [10], Theorem X.8.14.

2.4.6 Theorem (Transformation Theorem)**.** *Let $(a,b), (c,d) \subseteq \mathbb{R}$ be two intervals with $-\infty \le a < b \le +\infty$ and $-\infty \le c < d \le +\infty$, $f : (c,d) \to X$ measurable and $\varphi : (a,b) \to (c,d)$ a diffeomorphism, which means that φ is bijective and both φ and φ^{-1} are continuously differentiable. Then, f is integrable if and only if the function $t \mapsto |\varphi'(t)| f\big(\varphi(t)\big)$ defined on (a,b) is integrable. In this case*

$$\int_{(c,d)} f(t)\ d\lambda(t) = \int_{(a,b)} |\varphi'(t)| f\big(\varphi(t)\big)\ d\lambda(t).$$

2.5 Spaces of Integrable Functions

We introduce Bochner spaces, the Banach space-valued analogue of Lebesgue spaces.

Throughout this section $(\Omega, \mathcal{A}, \mu)$ will be a σ-finite complete measure space and X a Banach space.

2.5.1 Definition. Let $p \in [1, +\infty)$. By $L^p(\Omega; \mu; X)$ we denote the space

$$\{f : \Omega \to X \text{ measurable} : \int_\Omega \|f(\omega)\|^p \, d\mu(\omega) < +\infty\}.$$

Furthermore we define $L^\infty(\Omega; \mu; X)$ as the space of all measurable functions $f : \Omega \to X$, such that there exists a constant $C > 0$ with $\|f(\omega)\| \leq C$ for almost every $\omega \in \Omega$. Defining

$$\|f\|_{L^p(\Omega;\mu;X)} := \Big(\int_\Omega \|f(\omega)\|^p \, d\mu(\omega)\Big)^{\frac{1}{p}}$$

for $p \in [1, +\infty)$ and

$$\|f\|_{L^\infty(\Omega;\mu;X)} := \inf\{C > 0 : \|f(\omega)\| \leq C \text{ for almost every } \omega \in \Omega\}$$

the space $\big(L^p(\Omega; \mu; X), \|\cdot\|_{L^p(\Omega;\mu;X)}\big)$ constitutes a Banach space if we identify functions, that differ only on a null set; see [10], Theorem X.4.10. If the spaces Ω and X are clear from the context, we will simply write $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega;\mu;X)}$ for $p \in [1, +\infty]$. If $\Omega \subseteq \mathbb{R}$ is a Borel set, i.e. $\Omega \in \mathcal{B}$, then we write $L^p(\Omega; X)$ instead of $L^p(\Omega; \lambda|_\Omega; X)$.

Furthermore, given $-\infty \leq a < b \leq +\infty$, we define $L^1_{\text{loc}}\big((a, b); X\big)$ as the space of all measurable functions $f : (a, b) \to X$ that are integrable with respect to the Lebesgue measure $\lambda|_{(a,b)}$ over every compact subset of (a, b). By Proposition 2.12 in [3], we have $L^p\big((a, b); X\big) \subseteq L^1_{\text{loc}}\big((a, b); X\big)$ for every $p \in [1, +\infty]$.

We gather properties and basic observations regarding Bochner spaces. Proofs can be found for example in [3], Propositions 2.13, 2.15, Theorem 2.16 and Corollary 2.23 as well as Lemma 1.1.1 in [14].

2.5.2 Proposition. Let $p \in [1, +\infty]$ and $-\infty \leq a < b \leq +\infty$.

a) Given $f \in L^p(\Omega; \mu; X)$ and a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^p(\Omega; \mu; X)$ satisfying $\|f_n - f\|_{L^p} \xrightarrow{n\to+\infty} 0$, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ which converges to f almost everywhere.

b) If $p \in [1, +\infty)$, then $C^\infty_{00}\big((a, b); X\big)$ is densely contained in $L^p\big((a, b); X\big)$.

c) If $f \in L^1_{\text{loc}}\big((a, b); X\big)$ satisfies

$$\int_{(a,b)} \varphi(t) f(t) \, d\lambda(t) = 0$$

for all $\varphi \in C^\infty_{00}\big((a, b); \mathbb{C}\big)$, then $f(t) = 0$ for almost every $t \in (a, b)$. If f is additionally continuous on (a, b), then $f(t) = 0$ for all $t \in (a, b)$.

d) If $f \in L^1_{\text{loc}}\big((a,b);X\big)$, then

$$\lim_{h\to 0^+} \frac{1}{h}\int_{(t,t+h)} f(s)\ d\lambda(s) = \lim_{h\to 0^+} \frac{1}{h}\int_{(t-h,t)} f(s)\ d\lambda(s) = f(t)$$

for almost every $t \in (a,b)$.

e) Let $p \in [1,+\infty)$ and $q \in (1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$. The mapping $\Phi : L^q(\Omega;\mu;X') \to L^p(\Omega;\mu;X)'$ defined by

$$\Phi(f) := \Big(g \mapsto \int_\Omega f(\omega)g(\omega)\ d\mu(\omega)\Big)$$

is linear, bounded and isometric. If X is reflexive, then Φ is also bijective.

f) If $p \in (1,+\infty)$ and X is reflexive, also $L^p\big((a,b);X\big)$ is reflexive.

Next, we state important inequalities of integrable functions, see [10], Theorem X.4.2, Theorem X.7.3 and X.6.21.

2.5.3 Proposition. Let $-\infty \le a < b \le +\infty$ and $-\infty \le c < d \le +\infty$.

a) (Hölder's inequality) Given $p,q \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q}=1$, where $\frac{1}{+\infty} := 0$, and functions $f \in L^p(\Omega;\mu;\mathbb{C})$, $g \in L^q(\Omega;\mu;\mathbb{C})$, we have $fg \in L^1(\Omega;\mathbb{C})$ and

$$\|fg\|_1 \le \|f\|_p \|g\|_q .$$

b) (Young's inequality) Given $p \in [1,\infty]$ and functions $f \in L^p\big(\mathbb{R};\mathbb{C}\big)$ as well as $g \in L_1(\mathbb{R};\mathbb{C})$, $s \mapsto f(t-s)g(s)$ is integrable for almost every $t \in \mathbb{R}$ and

$$\left\| t \mapsto \int_{\mathbb{R}} f(t-s)g(s)\ d\lambda(s) \right\|_p \le \|f\|_p \|g\|_1 .$$

c) (Generalized Minkowski inequality) Given $p \in [1,+\infty)$ and a measurable function $f : (a,b)\times(c,d) \to X$ satisfying

$$\big(s \mapsto f(t,s)\big) \in L^1\big((c,d);X\big)$$

for almost every $t \in (a,b)$ and

$$\int_{(a,b)} \Big(\int_{(c,d)} \|f(t,s)\|\, d\lambda(s)\Big)^p d\lambda(t) < +\infty,$$

the function

$$s \mapsto \int_{(a,b)} f(t,s)\ d\lambda(t)$$

is contained in $L^p\big((c,d);X\big)$ and satisfies

$$\Big(\int_{(c,d)} \Big\| \int_{(a,b)} f(t,s)\ d\lambda(t) \Big\|^p d\lambda(s)\Big)^{\frac{1}{p}} \le \int_{(a,b)} \Big(\int_{(c,d)} \|f(t,s)\|^p\, d\lambda(s)\Big)^{\frac{1}{p}} d\lambda(t)$$

We introduce weak derivatives, a generalization of the notion of classical differentiation.

2.5.4 Definition. Given $-\infty \leq a < b \leq +\infty$, we call a function $f \in L^1_{\text{loc}}\big((a,b);X\big)$ *weakly differentiable*, if there exists a function $g \in L^1_{\text{loc}}\big((a,b);X\big)$ such that

$$\int_{(a,b)} \varphi'(t)f(t)\ d\lambda(t) = -\int_{(a,b)} \varphi(t)g(t)\ d\lambda(t)$$

for any $\varphi \in C^\infty_{00}\big((a,b);\mathbb{C}\big)$. In that case, we call $f' := f^{(1)} := g$ the *weak derivative* of f. Inductively, if $f^{(k-1)}$ is weakly differentiable, we say that f is k times weakly differentiable and define $f^{(k)} := (f^{(k-1)})'$ for $k \in \mathbb{N}$. Furthermore, by $W^{k,p}\big((a,b);X\big)$ we denote the space of all k times weakly differentiable functions $f \in L^p\big((a,b);X\big)$, which satisfy $f^{(m)} \in L^p\big((a,b);X\big)$ for all $m \in \{1,\dots,k\}$.

We state some properties of weakly differentiable functions. Proofs for these statements can be found in [3], Sections 3.1 and 3.2.

2.5.5 Proposition. Given $-\infty \leq a < b \leq +\infty$, the following assertions hold true.

a) The weak derivative is unique up to a null set.

b) If $f, g \in L^1_{\text{loc}}\big((a,b);X\big)$ are weakly differentiable and $\lambda \in \mathbb{C}$, $f + \lambda g$ is weakly differentiable satisfying $(f+\lambda g)' = f' + \lambda g'$.

c) If $f : (a,b) \to X$ is differentiable, then f is weakly differentiable and the (classical) derivative coincides with the weak derivative.

d) If $f \in L^1_{\text{loc}}\big((a,b);X\big)$ is weakly differentiable and $f'(t) = 0$ for almost every $t \in (a,b)$, then there exists an $x \in X$ such that $f(t) = x$ for almost every $t \in (a,b)$.

e) Let $p \in [1,+\infty]$. For every $f \in W^{1,p}\big((a,b);X\big)$ there exists a unique bounded, continuous function $g : (a,b) \to X$ satisfying $f(t) = g(t)$ for almost every $t \in (a,b)$.

2.5.6 Definition. Let $+\infty < a < b < +\infty$. We say that a function is *absolutely continuous*, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for

$$a \leq x_1 \leq y_1 \leq x_2 \leq \dots \leq x_n \leq y_n \leq b$$

satisfying $\sum_{k=1}^{n}(y_k - x_k) < \delta$ we have

$$\sum_{k=1}^{n} \|f(y_k) - f(x_k)\| < \varepsilon.$$

The proof of the next result can be found in [3], Propositions 3.7 and 3.8.

2.5.7 Theorem. *Let $-\infty \leq a < b \leq +\infty$, $p \in [1,+\infty]$ and $f \in L^p\big((a,b);X\big)$. The following statements are equivalent.*

a) $f \in W^{1,p}\big((a,b);X\big)$.

b) *There exists a function* $g \in L^p\big((a,b);X\big)$ *such that*

$$f(t) - f(s) = \int_{(s,t)} g(r)\ d\lambda(r)$$

for every $a < s < t < b$.

c) f *is absolutely continuous on every compact subinterval of* (a,b), f *is differentiable almost everywhere on* (a,b) *satisfying* $f' \in L^p\big((a,b);X\big)$.

If f *satisfies one (and therefore all) of the statements above, the weak derivative coincides with* f' *and* g *almost everywhere.*

With this knowledge we can prove the Fundamental Theorem of Calculus for weakly differentiable functions.

2.5.8 Corollary. If $-\infty < a < b < +\infty$ and $f \in W^{1,p}\big((a,b);X\big)$ is continuously extendable to $[a,b]$, then

$$f(b) - f(a) = \int_{(a,b)} f'(t)\ d\lambda(t).$$

Proof. Let $n \in \mathbb{N}$ with $\frac{1}{n} < \frac{b-a}{2}$ and define $g_n : [a,b] \to X$ by $g_n(t) := \mathbb{1}_{(a+\frac{1}{n},b-\frac{1}{n})} f'(t)$. Clearly, $g_n(t) \xrightarrow{n\to+\infty} f'(t)$ for all $t \in (a,b)$ and $\|g_n(t)\| \leq \|f'(t)\|$. By Theorem 2.3.7 and Theorem 2.5.7

$$\begin{aligned}\int_{(a,b)} f'(t)\ d\lambda(t) &= \lim_{n\to+\infty} \int_{(a,b)} g_n(t)\ d\lambda(t) = \lim_{n\to+\infty} \int_{(a+\frac{1}{n},b-\frac{1}{n})} f'(t)\ d\lambda(t) \\ &= \lim_{n\to+\infty} f(b-\tfrac{1}{n}) - f(a+\tfrac{1}{n}) = f(b) - f(b)\end{aligned}$$

□

2.5.9 Corollary. If $-\infty < a < b < +\infty$, $\varphi \in C^\infty\big([a,b];\mathbb{C}\big)$ and $f \in W^{1,p}\big((a,b);X\big)$ is continuously extendable to $[a,b]$, then $\varphi f \in W^{1,p}\big((a,b);X\big)$ and

$$\int_{(a,b)} \varphi'(t)f(t)\ d\lambda(t) = \varphi(b)f(b) - \varphi(a)f(a) - \int_{(a,b)} \varphi(t)f'(t)\ d\lambda(t).$$

Proof. For $\psi \in C_{00}^\infty\big((a,b);\mathbb{C}\big)$ also $\psi\varphi|_{(a,b)} \in C_{00}^\infty\big((a,b);\mathbb{C}\big)$. By the classical product rule for differentiation

$$\begin{aligned}\int_{(a,b)} \psi'(t)\varphi(t)f(t)\ d\lambda(t) &= \int_{(a,b)} (\psi\varphi)'(t)f(t)\ d\lambda(t) - \int_{(a,b)} \psi(t)\varphi'(t)f(t)\ d\lambda(t) \\ &= -\int_{(a,b)} \psi(t)\varphi(t)f'(t)\ d\lambda(t) - \int_{(a,b)} \psi(t)\varphi'(t)f(t)\ d\lambda(t) \\ &= -\int_{(a,b)} \psi(t)\big(\varphi(t)f'(t) + \varphi'(t)f(t)\big)\ d\lambda(t),\end{aligned}$$

implying that φf is weakly differentiable and $(\varphi f)' = \varphi f' + \varphi' f$. Since $\varphi \in C^\infty\big([a,b];\mathbb{C}\big)$ and $f, f' \in L^p\big((a,b);X\big)$, also $\varphi f, (\varphi f)' \in L^p\big((a,b);X\big)$. Hence, $\varphi f \in W^{1,p}\big((a,b);X\big)$. Corollary 2.5.8 yields

$$\begin{aligned}\int_{(a,b)} \varphi'(t) f(t)\, d\lambda(t) &= \int_{(a,b)} (\varphi f)'(t)\, dt - \int_{(a,b)} \varphi(t) f'(t)\, d\lambda(t) \\ &= \varphi(b)f(b) - \varphi(a)f(a) - \int_{(a,b)} \varphi(t) f'(t)\, d\lambda(t).\end{aligned}$$

□

Chapter 3

Operator Semigroups

One of the most important tools to study abstract Cauchy problems (for short ACP) are operator semigroups. If the operator appearing in the equation happens to be an infinitesimal generator of a strongly continuous semigroup, we will see that the ACP has a unique solution.

3.1 Strongly continuous Semigroups

3.1.1 Definition. Let X be a Banach space and $\big(T(t)\big)_{t\geq 0}$ be a family of bounded linear operators mapping X to itself. $\big(T(t)\big)_{t\geq 0}$ is called a *semigroup*, if

- $T(0) = I$.
- $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.

Furthermore we define $D(A) := \{x \in X : \lim\limits_{t\to 0^+} \frac{T(t)x-x}{t} \text{ exists}\}$ and $A : D(A) \subset X \to X$ by $Ax = \lim\limits_{t\to 0^+} \frac{T(t)x-x}{t}$. This (in general unbounded) linear operator is called the *infinitesimal generator* of the semigroup $\big(T(t)\big)_{t\geq 0}$.
A semigroup is called

- *bounded*, if there exists a constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$,
- *strongly continuous*, if $\lim\limits_{t\to 0^+} T(t)x = x$ for all $x \in X$ and
- *uniformly continuous*, if $\lim\limits_{t\to 0^+} \|T(t) - I\| = 0$.

Obviously, every uniformly continuous semigroup is strongly continuous.

We start by listing simple properties of semigroups and its generators without proofs, which can be found for example in [19], Section 5.2.

3.1.2 Proposition. For a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ with the infinitesimal generator A the following assertions hold true.

a) There exist $M > 0, \omega \in \mathbb{R}$, such that $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

b) The mapping $T(\cdot)x : [0, \infty) \to X$, $t \mapsto T(t)x$, is continuous for any $x \in X$.

c) For $x \in X$

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x \, ds = T(t)x.$$

d) A is closed and densely defined.

e) For $x \in X$ and $t > 0$, $\int_0^t T(s)x \, ds$ is contained in the domain of A and

$$A\Big(\int_0^t T(s)x \, ds\Big) = T(t)x - x.$$

f) If $x \in D(A)$, $T(t)x$ is also included in $D(A)$ for all $t \geq 0$ and

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax.$$

g) For any $\lambda \in \mathbb{C}$, $\big(e^{\lambda t} T(t)\big)_{t \geq 0}$ is a strongly continuous semigroup with infinitesimal generator $A + \lambda I$.

h) For $M > 0$ and $\omega \in \mathbb{R}$ as in *a)* we have $\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) > \omega\} \subseteq \rho(A)$ and for $\mathrm{Re}(\xi) > \omega$ the resolvent satisfies

$$R(\xi, A)x = (\xi I - A)^{-1}x = \int_{(0,+\infty)} e^{-\xi t} T(t)x \, d\lambda(t) = \int_0^{+\infty} e^{-\xi t} T(t)x \, dt \qquad (3.1)$$

for all $x \in X$. In particular,

$$\|R(\xi, A)\| \leq \frac{M}{\mathrm{Re}(\xi) - \omega}.$$

i) For every $t \geq 0$ and $\xi \in \rho(A)$ we have

$$T(t)R(\xi, A) = R(\xi, A)T(t).$$

j) If $A \in L_b(X)$, the semigroup has the form

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

k) If $\big(S(t)\big)_{t \geq 0}$ is an additional strongly continuous semigroup generated by B, then $B \subseteq A$ or $A \subseteq B$ already implies $S(t) = T(t)$ for every $t \geq 0$ and $A = B$.

3.1.3 Corollary. Given a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and its infinitesimal generator A,

$$R(\xi, A)^n x = \frac{(-1)^{n-1}}{(n-1)!}\Big(\frac{d^{n-1}}{d\xi^{n-1}} R(\xi, A)x\Big) = \frac{1}{(n-1)!}\int_{(0,+\infty)} t^{n-1} e^{-\xi t} T(t)x \; d\lambda(t)$$

for any $x \in X$ and $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > \omega$ for $\omega \in \mathbb{R}$ as in Proposition 3.1.2, *a)*.

Proof. Let $K \subseteq \{\xi \in \mathbb{C} : \mathrm{Re}(\lambda) > \omega\}$ be a compact set. The continuous mapping $\mathrm{Re} : K \to (\omega, \infty)$ attains its minimum at a point $z \in K$ and clearly satisfies $\mathrm{Re}(\xi) \geq \mathrm{Re}(z) =: \tau > \omega$ for all $\xi \in K$. For $\xi \in K$, $t \in [0, \infty)$ and $x \in X$ we have

$$\left\|e^{-\xi t} T(t)x\right\| \leq M e^{(\omega - \mathrm{Re}(\xi))t} \|x\| \leq M e^{(\omega-\tau)t} \|x\| =: g_K(t).$$

Since

$$\int_0^\infty |g_K(t)| \; dt = M \|x\| \frac{e^{(\omega-\tau)t}}{\omega-\tau}\Big|_0^\infty = \frac{M\|x\|}{\tau-\omega} < +\infty$$

and since $\xi \mapsto e^{-\xi t} T(t)x$ is analytic, we can use Theorem 2.4.3 and Proposition 2.3.10 as well as Proposition 3.1.2, *g)*, to conclude that $\xi \mapsto R(\xi, A)x$ is analytic in $\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) > \omega\}$ for any $x \in X$ and

$$\frac{d^n}{d\xi^n} R(\xi, A)x = \int_{(0,+\infty)} \frac{d^n}{d\xi^n} e^{-\xi t} T(t)x \; d\lambda(t) = (-1)^n \int_{(0,+\infty)} t^n e^{-\xi t} T(t)x \; d\lambda(t) \qquad (3.2)$$

for any $n \in \mathbb{N}$. By Proposition 1.2.4, b), we have

$$\frac{d^n}{d\xi^n} R(\xi, A) = (-1)^n n! R(\xi, A)^{n+1}. \qquad (3.3)$$

(3.2) and (3.3) together imply

$$\begin{aligned} R(\xi, A)^n x &= \frac{(-1)^{n-1}}{(n-1)!}\Big(\frac{d^{n-1}}{d\xi^{n-1}} R(\xi, A)x\Big) \\ &= \frac{1}{(n-1)!}\int_{(0,+\infty)} t^{n-1} e^{-\xi t} T(t)x \; d\lambda(t). \end{aligned}$$

□

3.1.4 Proposition. If A is the infinitesimal generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$, then $\bigcap_{n=1}^{\infty} D(A^n)$ is dense in X.

Proof. Let $f \in C_{00}^\infty((0, +\infty), \mathbb{C})$. Since $t \mapsto T(t)x$ is continuous and f vanishes outside a compact interval, according to Theorem 2.4.2, the function $t \mapsto f(t)T(t)x$ is integrable and bounded for any $x \in X$. We set

$$x_f = \int_{(0,+\infty)} f(t)T(t)x \; d\lambda(t)$$

for $x \in X$ and $f \in C_{00}^\infty((0,+\infty),\mathbb{C})$. We want to show that $x_f \in D(A)$. To that end, note that since the support of f is compact in $(0,\infty)$, there exists a constant $\delta > 0$ such that $f|_{(0,\delta)} \equiv 0$. For $h \in (0,\delta)$ we have

$$\begin{aligned}\frac{1}{h}(T(h)-I)x_f &= \frac{1}{h}\int_{(0,+\infty)} f(t)\big(T(h)-I\big)T(t)x\, d\lambda(t)\\ &= \frac{1}{h}\int_{(0,+\infty)} f(t)T(h+t)x\, d\lambda(t) - \frac{1}{h}\int_{(0,+\infty)} f(t)T(t)x\, d\lambda(t).\end{aligned}$$

Since the integrand of the first integral is continuous, we can use Theorem 2.4.6 and substitute $s = t+h$. Extending f to a function on $\mathbb{R}$ by $f(t) = 0$ for $t \le 0$ we obtain

$$\begin{aligned}\frac{1}{h}(T(h)-I)x_f &= \frac{1}{h}\int_{(0,+\infty)} f(t)T(h+t)x\, d\lambda(t) - \frac{1}{h}\int_{(0,+\infty)} f(t)T(t)x\, d\lambda(t)\\ &= \frac{1}{h}\int_{(h,+\infty)} f(s-h)T(s)x\, d\lambda(s) - \frac{1}{h}\int_{(0,+\infty)} f(t)T(t)x\, d\lambda(t)\\ &= \frac{1}{h}\int_{(0,+\infty)} f(s-h)T(s)x\, d\lambda(s) - \frac{1}{h}\int_{(0,+\infty)} f(t)T(t)x\, d\lambda(t)\\ &= \int_{(0,+\infty)} \frac{1}{h}\big(f(t-h)-f(t)\big)T(t)x\, d\lambda(t)\\ &= \int_{(a,b+\delta)} \frac{1}{h}\big(f(t-h)-f(t)\big)T(t)x\, d\lambda(t),\end{aligned}$$

where $\operatorname{supp}(f) \subseteq [a,b] \subseteq (0,+\infty)$. By Proposition 2.1.3, *b)* and *g)*, we have

$$\frac{1}{h}|f(t-h)-f(t)| \le \frac{1}{h}\Big|\int_{t-h}^{t} f'(s)\, ds\Big| \le \max_{s\in[a,b]} |f'(s)| =: C < +\infty$$

and for M,ω as in Proposition 3.1.2, *a)*

$$\left\|\frac{1}{h}\big(f(t-h)-f(t)\big)T(t)x\right\| \le CMe^{\omega t}\,\|x\| =: g(t).$$

As a continuous function g is Riemann integrable over $[a,b+\delta]$ and therefore integrable; see Theorem 2.4.3. Since

$$\lim_{h\to 0+}\frac{1}{h}\big(f(t-h)-f(t)\big) = -f'(t),$$

Theorem 2.3.7 yields

$$\begin{aligned}\frac{1}{h}(T(h)-I)x_f = \int_{(a,b+\delta)} \frac{1}{h}\big(f(t-h)-f(t)\big)T(t)x\, d\lambda(t) &\xrightarrow{h\to 0+} -\int_{(a,b+\delta)} f'(t)T(t)x\, d\lambda(t)\\ &= -\int_{(0,+\infty)} f'(t)T(t)x\, d\lambda(t) = -x_{f'},\end{aligned}$$

from which $x_f \in D(A)$ and $Ax_f = -x_{f'}$ follows. Since $f^{(n)} \in C_{00}^\infty\big((0,+\infty),\mathbb{C}\big)$ for any $n \in \mathbb{N}$, by induction we conclude $x_f \in \bigcap_{n=1}^{\infty} D(A^n)$ and

$$A^n x_f = (-1)^n x_{f^{(n)}}.$$

Suppose $Y := \operatorname{span}\big(\{x_f : x \in X, f \in C_{00}^{\infty}((0,+\infty),\mathbb{C})\}\big)$ is not dense in X. According to the Hahn-Banach theorem (see [12], Theorem 5.2.3) there exists $\varphi \in X' \setminus \{0\}$ such that $\varphi(y) = 0$ for all $y \in \overline{Y}$. According to Proposition 2.3.6, *c)*, we obtain

$$0 = \varphi(x_f) = \varphi\Big(\int_0^{+\infty} f(t)T(t)x \, d\lambda(t)\Big) = \int_0^{+\infty} f(t)\varphi\big(T(t)x\big) \, d\lambda(t)$$

for all $x \in X$ and $f \in C_{00}^{\infty}((0,+\infty),\mathbb{C})$. Since $t \mapsto \varphi\big(T(t)x\big)$ is continuous, we can employ Proposition 2.5.2, *c*), to conclude $\varphi\big(T(t)x\big) = 0$ for any $t \geq 0$, $x \in X$ and $\varphi \in X'$. By Proposition 3.1.2, *b*)

$$0 = \varphi\big(T(0)x\big) = \varphi(x)$$

for all $x \in X$ contradicting $\varphi \in X' \setminus \{0\}$. Finally,

$$\overline{\bigcap_{n=1}^{\infty} D(A^n)} \supseteq \overline{Y} = X.$$

□

The probably most important result in the theory of strongly continuous semigroups is the Hille-Yosida Theorem, which states that every closed, densely defined operator on X satisfying certain properties is an infinitesimal generator for a unique strongly continuous semigroup. Its proof can be found in [19], Theorem 5.3.2.

3.1.5 Theorem. *Let A be an unbounded operator on X. This operator is the infinitesimal generator of a unique strongly continuous semigroup if and only if the following assertions hold true.*

- *A is closed and densely defined.*
- *There exist $M > 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subseteq \rho(A)$ and*

$$\|R(\xi, A)^n\| \leq \frac{M}{(\xi - \omega)^n} \tag{3.4}$$

  for all $\xi > \omega$ and $n \in \mathbb{N}$.

If A is the infinitesimal generator of a semigroup $\big(T(t)\big)_{t\geq 0}$, (3.4) holds true for $M > 0$ and $\omega \in \mathbb{R}$ if and only if $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

3.2 The Laplace Transform

In Proposition 3.1.2 we stated that the resolvent $R(\lambda, A)$ of the infinitesimal generator can be computed by integrating $e^{-\xi t}T(t)$. To study semigroups as solutions of Cauchy Problems, we are interested in an integral representation of the semigroup itself inolving its generator. Note that (3.1) means that the Laplace Transform of $t \mapsto T(t)x$ equals $R(\xi, A)x$. We want to prove that we can apply the Inverse Laplace Transform to the resolvent and get the desired result.

3.2.1 Definition. Let $\big(T(t)\big)_{t\geq 0}$ a strongly continuous semigroup, M,ω as in Proposition 3.1.2, *a)*, and A its generator. For $\mu > \max\{\omega, 0\}$ the linear operator

$$A_\mu := \mu A R(\mu, A),$$

defined on $\{x \in X : R(\mu, A)x \in D(A)\} = X$ is called Yosida approximation.

3.2.2 Remark. For $x \in X$ we have

$$\begin{aligned} A_\mu x = \mu A R(\mu,A)x &= \mu A R(\mu,A)x - \mu^2 R(\mu,A)x + \mu^2 R(\mu,A)x \\ &= \mu(A-\mu I)R(\mu,A)x + \mu^2 R(\mu,A)x = \mu^2 R(\mu,A)x - \mu x, \end{aligned}$$

which implies $A_\mu = \mu^2 R(\mu, A) - \mu I \in L_b(X)$ follows. Additionally,

$$\lim_{\mu\to+\infty} \sup_{t\in[a,b]} \left\| e^{A_\mu t}x - T(t)x \right\| = 0$$

for all $x \in X$ and any $0 \leq a < b$. The proof of the latter can be found in [19], Theorem 5.3.2.

For a bounded operator the inversion of the laplace transform is quickly computed.

3.2.3 Lemma. Let $A \in L_b(X)$ and $\tau > \|A\|$. For $t > 0$ we have

$$e^{tA} = \frac{1}{2\pi i}\int_\gamma e^{\xi t} R(\xi, A)\, d\xi,$$

where $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma(s) := \tau + is$ and the integral converges uniformly for t in compact intervals, i.e.

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| e^{tA} - \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}} e^{\xi t} R(\xi, A)\, d\xi \right\| = 0$$

for any $0 \leq a < b$.

Proof. For $\beta = -\alpha = 2\tau$ and $\xi \in U_{\|A\|}(0)$ we have

$$|\xi - \tau| \leq |\xi| + \tau \leq \|A\| + \tau < 2\tau.$$

Hence,

$$U_{\|A\|}(0) \subseteq U_{2\tau}(\tau) = U_{\frac{\beta-\alpha}{2}}\big(\tau + i\tfrac{\alpha+\beta}{2}\big).$$

Given $\tilde\alpha < \alpha \leq -2\tau$, $\tilde\beta > \beta \geq 2\tau$ and $\xi \in U_{\frac{\beta-\alpha}{2}}\big(\tau + i\frac{\alpha+\beta}{2}\big)$, we have

$$|\xi - \tau - i\tfrac{\tilde\alpha+\tilde\beta}{2}| \leq |\xi - \tau - i\tfrac{\alpha+\beta}{2}| + |i\tfrac{\tilde\alpha+\tilde\beta}{2} - i\tfrac{\alpha+\beta}{2}| < \tfrac{\beta-\alpha}{2} + \tfrac{\tilde\beta-\beta}{2} + \tfrac{\alpha-\tilde\alpha}{2} = \tfrac{\tilde\beta-\tilde\alpha}{2},$$

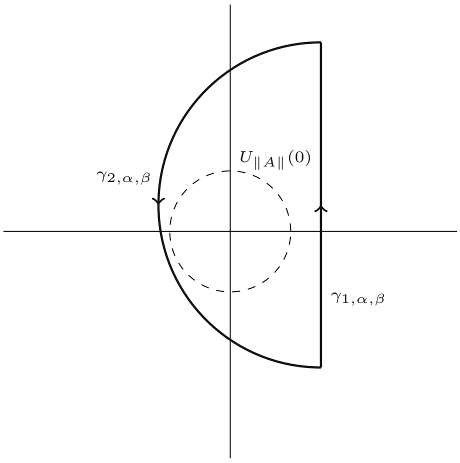
and in turn

$$U_{\frac{\tilde\beta-\tilde\alpha}{2}}\big(\tau + i\tfrac{\tilde\alpha+\tilde\beta}{2}\big) \supseteq U_{\frac{\beta-\alpha}{2}}\big(\tau + i\tfrac{\alpha+\beta}{2}\big).$$

Since $\operatorname{Re}(\xi) \leq |\xi| < \tau$ for every $\xi \in U_{\|A\|}(0)$,

$$U_{\|A\|}(0) \subseteq \{\xi \in U_{\frac{\beta-\alpha}{2}}\big(\tau + i\tfrac{\alpha+\beta}{2}\big) : \operatorname{Re}(\xi) < \tau\}.$$

For $\beta \geq 2\tau$ and $\alpha \leq -2\tau$ we define the paths $\gamma_{1,\alpha,\beta} : [\alpha, \beta] \to \mathbb{C}$, $\gamma_{1,\alpha,\beta}(t) = \tau + is$, and $\gamma_{2,\alpha,\beta} : [\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{C}$, $\gamma_{2,\alpha,\beta}(s) := \tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}e^{is}$, as well as $\gamma_{\alpha,\beta} := \gamma_{1,\alpha,\beta}\gamma_{2,\alpha,\beta}$.



From the first part of the proof we obtain $|\xi| > \|A\|$ for any $\xi \in \operatorname{ran}(\gamma_{\alpha,\beta})$, as soon as $\beta > 2\tau$ and $\alpha < -2\tau$. Moreover,

$$R(\xi, A) = \sum_{n=0}^{\infty} \frac{1}{\xi^{n+1}} A^n \tag{3.5}$$

converges uniformly with respect to the operator norm for $\xi \in \operatorname{ran}(\gamma_{\alpha,\beta})$; see [12], Lemma 6.3.10. Since $\mathbb{C}$ is star-shaped and $\gamma_{\alpha,\beta}$ describes closed curve which is homotopic to the path describing a circle with radius $\frac{\beta-\alpha}{2}$ centered at $\tau + i\frac{\alpha+\beta}{2}$ in $\mathbb{C}$, we can use Cauchy's Integral Formula, Theorem 2.2.6, applied to $\xi \mapsto e^{\xi t}$, which is analytic in $\mathbb{C}$, and obtain

$$\frac{1}{2\pi i} \int_{\gamma_{\alpha,\beta}} \frac{e^{\xi t}}{\xi^{n+1}}\, d\xi = \frac{1}{n!} \frac{d^n}{d\xi^n} e^{\xi t}\Big|_{\xi=0} = \frac{t^n}{n!}.$$

Given $N \in \mathbb{N}$, for $s \in [\alpha, \beta]$ we have

$$\begin{aligned} \left\| \sum_{n=0}^{N} \frac{e^{\gamma_{1,\alpha,\beta}(s)t} \gamma_{1,\alpha,\beta}'(s)}{\gamma_{1,\alpha,\beta}(s)^{n+1}} A^n \right\| &\leq \sum_{n=0}^{N} \left| \frac{i e^{(\tau+is)t}}{(\tau+is)^{n+1}} \right| \|A\|^n = \frac{e^{\tau t}}{|\tau+is|} \sum_{n=0}^{N} \Big(\frac{\|A\|}{|\tau+is|} \Big)^n \\ &\leq \frac{e^{\tau t}}{\tau} \sum_{n=0}^{\infty} \Big(\frac{\|A\|}{\tau} \Big)^n = \frac{e^{\tau t}}{\tau - \|A\|}, \end{aligned}$$

and for $s \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

$$\begin{aligned} \left\| \sum_{n=0}^{N} \frac{e^{\gamma_{2,\alpha,\beta}(s)t} \gamma_{2,\alpha,\beta}'(s)}{\gamma_{2,\alpha,\beta}(s)^{n+1}} A^n \right\| &\leq \sum_{n=0}^{N} \left| \frac{\frac{\beta-\alpha}{2} i e^{is} e^{(\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2} e^{is})t}}{(\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2} e^{is})^{n+1}} \right| \|A\|^n \\ &= \frac{e^{(\tau + \frac{\beta-\alpha}{2}\cos(s))t}}{|\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2} e^{is}|} \sum_{n=0}^{N} \Big(\frac{\|A\|}{|\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2} e^{is}|} \Big)^n \\ &\leq \frac{e^{(\tau + \frac{\beta-\alpha}{2}\cos(s))t}}{|\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2} e^{is}|} \sum_{n=0}^{\infty} \Big(\frac{\|A\|}{|\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2} e^{is}|} \Big)^n \\ &\leq \frac{e^{(\tau + \frac{\beta-\alpha}{2}\cos(s))t}}{|\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2} e^{is}| - \|A\|}. \end{aligned}$$

From $\cos(s) \le 0$ for $s \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ we derive

$$\left\|\sum_{n=0}^{N} \frac{e^{\gamma_{2,\alpha,\beta}(s)t}\gamma_{2,\alpha,\beta}'(s)}{\gamma_{2,\alpha,\beta}(s)^{n+1}} A^n\right\| \le \frac{e^{(\tau+\frac{\beta-\alpha}{2}\cos(s))t}}{|\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}e^{is}| - \|A\|} \le \frac{e^{\tau t}}{|\tau + i\frac{\alpha+\beta}{2}| - \frac{\beta-\alpha}{2} - \|A\|}.$$

Because of

$$\int_\alpha^\beta \frac{e^{\tau t}}{\tau - \|A\|}\, ds = \frac{(\beta-\alpha)e^{\tau t}}{\tau - \|A\|} < +\infty$$

and

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{e^{\tau t}}{|\tau + i\frac{\alpha+\beta}{2}| - \frac{\beta-\alpha}{2} - \|A\|}\, ds = \frac{\pi e^{\tau t}}{|\tau + i\frac{\alpha+\beta}{2}| - \frac{\beta-\alpha}{2} - \|A\|} < +\infty,$$

we can use the Dominated Convergence Theorem, 2.3.7, Proposition 2.1.3, c), and Theorem 2.4.3 and obtain

$$\begin{aligned}
e^{tA} &= \sum_{n=0}^\infty \frac{t^n A^n}{n!} = \sum_{n=0}^\infty \Big(\frac{1}{2\pi i}\int_{\gamma_{\alpha,\beta}} \frac{e^{\xi t}}{\xi^{n+1}}\, d\xi\Big) A^n \\
&= \frac{1}{2\pi i}\sum_{n=0}^\infty \Big(\int_\alpha^\beta \frac{e^{\gamma_{1,\alpha,\beta}(s)t}\gamma_{1,\alpha,\beta}'(s)}{\gamma_{1,\alpha,\beta}(s)^{n+1}} A^n\, ds + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{e^{\gamma_{2,\alpha,\beta}(s)t}\gamma_{2,\alpha,\beta}'(s)}{\gamma_{2,\alpha,\beta}(s)^{n+1}} A^n\, ds\Big) \\
&= \frac{1}{2\pi i}\sum_{n=0}^\infty \Big(\int_{(\alpha,\beta)} \frac{e^{\gamma_{1,\alpha,\beta}(s)t}\gamma_{1,\alpha,\beta}'(s)}{\gamma_{1,\alpha,\beta}(s)^{n+1}} A^n\, d\lambda(s) + \int_{(\frac{\pi}{2},\frac{3\pi}{2})} \frac{e^{\gamma_{2,\alpha,\beta}(s)t}\gamma_{2,\alpha,\beta}'(s)}{\gamma_{2,\alpha,\beta}(s)^{n+1}} A^n\, d\lambda(s)\Big) \\
&= \frac{1}{2\pi i}\int_{(\alpha,\beta)} \sum_{n=0}^\infty \frac{e^{\gamma_{1,\alpha,\beta}(s)t}\gamma_{1,\alpha,\beta}'(s)}{\gamma_{1,\alpha,\beta}(s)^{n+1}} A^n\, d\lambda(s) + \frac{1}{2\pi i}\int_{(\frac{\pi}{2},\frac{3\pi}{2})} \sum_{n=0}^\infty \frac{e^{\gamma_{2,\alpha,\beta}(s)t}\gamma_{2,\alpha,\beta}'(s)}{\gamma_{2,\alpha,\beta}(s)^{n+1}} A^n\, d\lambda(s) \\
&= \frac{1}{2\pi i}\int_\alpha^\beta \sum_{n=0}^\infty \frac{e^{\gamma_{1,\alpha,\beta}(s)t}\gamma_{1,\alpha,\beta}'(s)}{\gamma_{1,\alpha,\beta}(s)^{n+1}} A^n\, ds + \frac{1}{2\pi i}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sum_{n=0}^\infty \frac{e^{\gamma_{2,\alpha,\beta}(s)t}\gamma_{2,\alpha,\beta}'(s)}{\gamma_{2,\alpha,\beta}(s)^{n+1}} A^n\, ds \\
&= \frac{1}{2\pi i}\int_{\gamma_{\alpha,\beta}} e^{\xi t}\sum_{n=0}^\infty \frac{1}{\xi^{n+1}} A^n\, d\xi = \frac{1}{2\pi i}\int_{\gamma_{\alpha,\beta}} e^{\xi t} R(\xi, A)\, d\xi.
\end{aligned}$$

Let $t \in [a,b] \subseteq (0,+\infty)$. Because of (3.5) we have $\|R(\xi,A)\| \le \frac{1}{|\xi|-\|A\|}$, which together with 2.2.4, a), implies

$$\begin{aligned}
\left\|\int_{\gamma_{2,\alpha,\beta}} e^{\xi t} R(\xi,A)\, d\xi\right\| &= \left\|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{\beta-\alpha}{2} i e^{(\tau+i\frac{\alpha+\beta}{2}+\frac{\beta-\alpha}{2}e^{is})t} e^{is} R(\tau + i\tfrac{\alpha+\beta}{2} + \tfrac{\beta-\alpha}{2}e^{is}, A)\, ds\right\| \\
&\le \tfrac{\beta-\alpha}{2} e^{\tau t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\frac{\beta-\alpha}{2}t\cos(s)} \left\|R(\tau + i\tfrac{\alpha+\beta}{2} + \tfrac{\beta-\alpha}{2}e^{is}, A)\right\|\, ds \\
&\le \tfrac{\beta-\alpha}{2} e^{\tau t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{e^{\frac{\beta-\alpha}{2}t\cos(s)}}{|\tau + i\frac{\alpha+\beta}{2} + \frac{\beta-\alpha}{2}e^{is}| - \|A\|}\, ds \\
&\le \frac{\frac{\beta-\alpha}{2}e^{\gamma t}}{\frac{\beta-\alpha}{2} - |\tau + i\frac{\alpha+\beta}{2}| - \|A\|}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\frac{\beta-\alpha}{2}t\cos(s)}\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{e^{\tau t}}{-\alpha-\tau-\|A\|}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{\beta-\alpha}{2} e^{\frac{\beta-\alpha}{2}cos(s)t}\ ds \\
&\leq \frac{e^{\tau t}}{\tau-\|A\|}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \tfrac{\beta-\alpha}{2} e^{\frac{\beta-\alpha}{2}\cos(s)t}\ ds \\
&= \frac{e^{\tau t}}{\tau-\|A\|}\int_{(\frac{\pi}{2},\frac{3\pi}{2})} \tfrac{\beta-\alpha}{2} e^{\frac{\beta-\alpha}{2}\cos(s)t}\ d\lambda(s).
\end{aligned}
$$

For $\frac{\pi}{2} < \varphi < \frac{3\pi}{2}$, $a\cos(\varphi)$ is negative and therefore

$$
\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \tfrac{\beta-\alpha}{2} e^{\frac{\beta-\alpha}{2}a\cos(s)} = 0
$$

almost everywhere on $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Choosing $\beta, -\alpha$ large enough, we can assume $\frac{\beta-\alpha}{2} > \frac{1}{a}$. Since $x \mapsto xe^{xa\cos(s)}$ is decreasing for $x > -\frac{1}{a\cos(s)} > \frac{1}{a}$, we have

$$
|\tfrac{\beta-\alpha}{2} e^{\frac{\beta-\alpha}{2}a\cos(s)}| \leq \tfrac{1}{a} e^{\cos(s)} \leq \frac{1}{a}.
$$

Hence, we can employ Theorem 2.3.7, and obtain

$$
\begin{aligned}
\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \left\| \int_{\gamma_{2,\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi \right\| &\leq \lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \frac{e^{\tau t}}{\tau-\|A\|}\int_{(\frac{\pi}{2},\frac{3\pi}{2})} \tfrac{\beta-\alpha}{2} e^{\frac{\beta-\alpha}{2}\cos(s)t}\ d\lambda(s) \\
&\leq \frac{e^{\tau b}}{\tau-\|A\|}\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\int_{(\frac{\pi}{2},\frac{3\pi}{2})} \tfrac{\beta-\alpha}{2} e^{\frac{\beta-\alpha}{2}a\cos(s)}\ d\lambda(s) \\
&= 0.
\end{aligned}
$$

Note that the last term does not depend on t. According to Proposition 2.2.4, a), we have

$$
\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_{2,\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi &= \frac{1}{2\pi i}\int_{\gamma_{\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi - \frac{1}{2\pi i}\int_{\gamma_{1,\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi \\
&= e^{tA} - \frac{1}{2\pi i}\int_{\gamma_{1,\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi,
\end{aligned}
$$

from which we derive

$$
\sup_{t\in[a,b]} \left\| e^{tA} - \frac{1}{2\pi i}\int_{\gamma_{1,\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi \right\| = \sup_{t\in[a,b]} \left\| \frac{1}{2\pi i}\int_{\gamma_{2,\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi \right\| \xrightarrow[\alpha\to-\infty]{\beta\to+\infty} 0.
$$

Consequently, the integral

$$
\frac{1}{2\pi i}\int_{\gamma} e^{\xi t} R(\xi, A)\ d\xi = \lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \frac{1}{2\pi i}\int_{\gamma_{1,\alpha,\beta}} e^{\xi t} R(\xi, A)\ d\xi
$$

exists and equals e^{tA}.

$\square$

For unbounded A we first need to gather properties of the induced Yosida approximation A_μ.

3.2.4 Lemma. Let $\big(T(t)\big)_{t\geq 0}$ a strongly continuous semigroup, M, ω as in Proposition 3.1.2, *a)*, and denote by A its generator. For $\mu > \max\{\omega, 0\}$ consider the Yosida approximation A_μ. If $\xi \in \mathbb{C}$ satisfies $\mathrm{Re}(\xi) > \frac{\mu\omega}{\mu-\omega}$, then $\xi \in \rho(A_\mu)$ and

$$R(\xi, A_\mu) = \tfrac{1}{\mu+\xi}(\mu I - A)R\big(\tfrac{\xi\mu}{\xi+\mu}, A\big)$$

as well as

$$\|R(\xi, A_\mu)\| \leq \frac{M}{\mathrm{Re}(\xi) - \frac{\mu\omega}{\mu-\omega}}. \tag{3.6}$$

Proof. Let $\xi = a + ib \in \mathbb{C}$ with $a > \frac{\mu\omega}{\mu-\omega} > -\mu$. We have

$$\begin{aligned}\omega(\mu+a)^2 + \omega b^2 &= (\mu+a)\big((\omega(\mu+a) - \mu a\big) + \mu a(\mu+a) + (\omega-\mu)b^2 + \mu b^2 \\ &= (\mu+a)(\omega-\mu)\big(a - \tfrac{\mu\omega}{\mu-\omega}\big) + \mu a(\mu+a) + (\omega-\mu)b^2 + \mu b^2 \\ &< \mu a(\mu+a) + \mu b^2\end{aligned}$$

which implies

$$\mathrm{Re}\Big(\frac{\mu\xi}{\mu+\xi}\Big) = \mu\frac{\mathrm{Re}(\xi)\big(\mu + \mathrm{Re}(\xi)\big) + \mathrm{Im}(\xi)^2}{\big(\mu + \mathrm{Re}(\xi)\big)^2 + \mathrm{Im}(\xi)^2} > \omega \tag{3.7}$$

and therefore $\frac{\mu\xi}{\mu+\xi} \in \rho(A)$. Given $x \in D(A)$, because of $\operatorname{ran} R(\mu, A) = D(A)$ and $AR(\mu, A)x = R(\mu, A)Ax$ we have

$$\begin{aligned}\tfrac{\mu\xi}{\mu+\xi}x - Ax &= \tfrac{\xi}{\mu+\xi}(\mu I - A)x - \tfrac{\mu}{\mu+\xi}Ax = \tfrac{1}{\mu+\xi}(\mu I - A)(\xi x - \mu R(\mu, A)Ax) \\ &= \tfrac{1}{\mu+\xi}(\mu I - A)(\xi x - \mu AR(\mu, A)x) = \tfrac{1}{\mu+\xi}(\mu I - A)(\xi x - A_\mu x),\end{aligned}$$

which implies

$$(\xi I - A_\mu)x = (\mu+\xi)R(\mu, A)(\tfrac{\mu\xi}{\mu+\xi}x - Ax) = (\mu+\xi)(\tfrac{\mu\xi}{\mu+\xi}I - A)R(\mu, A)x. \tag{3.8}$$

We can apply the operator $\frac{1}{\mu+\xi}(\mu I - A)R(\frac{\mu\xi}{\mu+\xi}, A)$, which is well-defined and bounded by Proposition 1.2.2, *c)*, since $\operatorname{ran} R(\frac{\mu\xi}{\mu+\xi}, A) = D(A)$, to (3.8) and obtain

$$\tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)(\xi I - A_\mu)x = x. \tag{3.9}$$

By the density of $D(A)$ (3.9) holds true for any $x \in X$; see Lemma 12.3.7 in [17]. For $y \in D(A)$ we have

$$\tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)y = \tfrac{1}{\mu+\xi}R(\tfrac{\mu\xi}{\mu+\xi}, A)(\mu I - A)y \in D(A).$$

Substituting $x = \frac{1}{\mu+\xi}(\mu I - A)R(\frac{\mu\xi}{\mu+\xi}, A)y$ into (3.8) we obtain

$$(\xi I - A_\mu)\big(\tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)\big)y = y.$$

Since $y \in D(A)$ was arbitrary and since $\frac{1}{\mu+\xi}(\mu I - A)R(\frac{\mu\xi}{\mu+\xi}, A)(\xi I - A_\mu)$ is bounded, the equation above holds true for any $y \in X$ since $D(A)$ is dense in X. In conclusion, $\xi \in \rho(A_\mu)$ and

$$R(\xi, A_\mu) = \frac{1}{\mu+\xi}(\mu I - A)R(\frac{\xi\mu}{\xi+\mu}, A).$$

For (3.6) note first that A_μ is the infinitesimal generator of the semigroup $(e^{tA_\mu})_{t\geq 0}$; see Proposition 3.1.2, j). Because of Remark 3.2.2, Theorem 3.1.5 and the fact that $e^{B+C} = e^B e^C$ for commuting $B, C \in L_b(X)$, as shown in Example 5.2.15 in [19], we have

$$\begin{aligned}\left\|e^{tA_\mu}\right\| &= \left\|\sum_{n=0}^{\infty} \frac{t^n}{n!}\big(\mu^2 R(\mu, A) - \mu I\big)^n\right\| \leq e^{-\mu t}\sum_{n=0}^{\infty}\frac{t^n\mu^{2n}}{n!}\left\|R(\mu, A)^n\right\| \\ &\leq e^{-\mu t} M \sum_{n=0}^{\infty}\frac{t^n\mu^{2n}}{n!(\mu-\omega)^n} = M\exp\Big(t\big(\tfrac{\mu^2}{\mu-\omega} - \mu\big)\Big) = M\exp\big(\frac{t\mu\omega}{\mu-\omega}\big). \qquad (3.10)\end{aligned}$$

Applying the norm estimate of the Hille-Yosida Theorem 3.1.5 to the semigroup $(e^{tA_\mu})_{t\geq 0}$ we obtain

$$\|R(\xi, A_\mu)\| \leq \frac{M}{\mathrm{Re}(\xi) - \frac{\mu\omega}{\mu-\omega}}.$$

□

3.2.5 Corollary. Let A be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and M, ω as in Proposition 3.1.2, *a*). Given $\mu > \max\{\omega, 0\}$ and $\tau > \frac{\mu\omega}{\mu-\omega}$, we have

$$e^{tA_\mu}x = \frac{1}{2\pi i}\int_\gamma e^{\xi t}R(\xi, A_\mu)x\, d\xi$$

for any $x \in D(A)$, where $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma(t) := \tau + it$. Furthermore, this limit is uniform for $t \geq 0$ in compact intervals, i.e.

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| e^{tA_\mu}x - \int_{\gamma|_{[\alpha,\beta]}} e^{\xi t}R(\xi, A_\mu)x\, d\xi\right\| = 0$$

for any $0 \leq a < b < +\infty$.

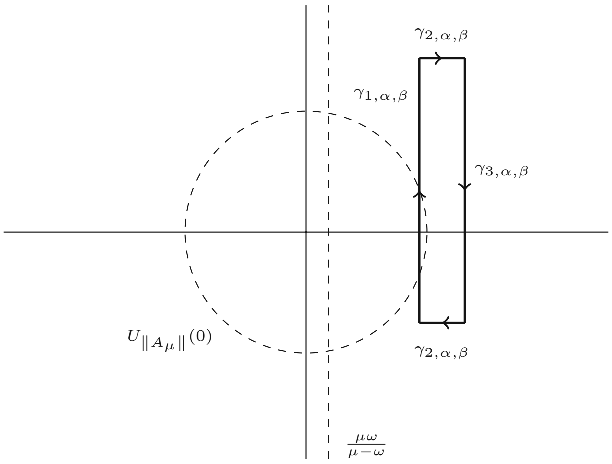
Proof. If $\tau > \|A_\mu\|$, we simply apply Lemma 3.2.3. If this is not the case, choose $\delta > \|A_\mu\| \geq \tau$, $\alpha, \beta \in \mathbb{R}$ with $\alpha < 0 < \beta$ and define the paths

$$\begin{aligned}\gamma_{1,\alpha,\beta} &: [\alpha, \beta] \to \mathbb{C},\ \gamma_{1,\alpha,\beta}(s) := \tau + is, \\ \gamma_{2,\alpha,\beta} &: [\tau, \delta] \to \mathbb{C},\ \gamma_{2,\alpha,\beta}(s) := s + i\beta,\end{aligned}$$

$$\gamma_{3,\alpha,\beta} : [-\beta, -\alpha] \to \mathbb{C},\ \gamma_{3,\alpha,\beta}(s) := \delta - is,$$
$$\gamma_{4,\alpha,\beta} : [-\delta, -\tau] \to \mathbb{C},\ \gamma_{4,\alpha,\beta}(s) := -s - i\alpha$$

as well as

$$\gamma_{\alpha,\beta} := \gamma_{1,\alpha,\beta}\gamma_{2,\alpha,\beta}\gamma_{3,\alpha,\beta}\gamma_{4,\alpha,\beta}.$$



Note that $\gamma_{\alpha,\beta}$ describes a closed curve with $\mathrm{Re}(\xi) \geq \tau > \frac{\mu\omega}{\mu-\omega}$ for any $\xi \in \mathrm{ran}(\gamma_{\alpha,\beta})$. According to Proposition 1.2.4, *a)*, and Lemma 3.2.4, $\xi \mapsto e^{\xi t}R(\xi, A_\mu)$ is analytic in the star-shaped set

$$\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) > \tfrac{\mu\omega}{\mu-\omega}\} \supseteq \mathrm{ran}(\gamma_{\alpha,\beta}).$$

Hence, $\gamma_{\alpha,\beta}$, as a closed curve, is homotopic to the constant path $\tau + i\alpha$ in $\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) > \frac{\mu\omega}{\mu-\omega}\}$; see Proposition 2.2.3. By Cauchy's Integral Theorem 2.2.5

$$\int_{\gamma_{\alpha,\beta}} e^{\xi t}R(\xi, A_\mu)x\ d\xi = 0.$$

Let $0 \leq a < b < +\infty$. In order to show that for $x \in D(A)$

$$\lim_{\beta\to+\infty} \int_{\gamma_{2,\alpha,\beta}} e^{\xi t}R(\xi, A_\mu)x\ d\xi = \lim_{\alpha\to-\infty} \int_{\gamma_{4,\alpha,\beta}} e^{\xi t}R(\xi, A_\mu)x\ d\xi = 0,$$

we derive from Lemma 3.2.4 and (3.7)

$$\begin{aligned}
\left\|\int_{\gamma_{2,\alpha,\beta}} e^{\xi t}R(\xi, A_\mu)x\ d\xi\right\| &= \left\|\int_\tau^\delta ie^{st}e^{it\beta}R(s+i\beta, A_\mu)x\ ds\right\| \\
&\leq \|Ax - \mu x\| \int_\tau^\delta \frac{e^{st}}{|\mu+s+ik|}\left\|R(\tfrac{\mu(s+ik)}{\mu+s+ik}, A)\right\|\ ds \\
&\leq \|A - \mu x\| \int_\tau^\delta \frac{e^{st}}{\sqrt{(\mu+s)^2+\beta^2}} \cdot \frac{M}{\mathrm{Re}(\frac{\mu(s+i\beta)}{\mu+s+i\beta}) - \omega}\ ds \\
&\leq \frac{M\|Ax - \mu x\|}{\beta(\mu-\omega)} \int_\tau^\delta e^{st}\ ds
\end{aligned}$$

$$= \frac{M \|Ax - \mu x\| (e^{\delta t} - e^{\tau t})}{\beta t(\mu - \omega)}.$$

By the continuity of $t \mapsto \frac{e^{\delta t} - e^{\tau t}}{t}$ on $\mathbb{R}$ there exists $C > 0$ such that

$$\frac{e^{\delta t} - e^{\tau t}}{t} \leq C$$

for $t \in [a, b]$. We obtain

$$\sup_{t \in [a,b]} \left\| \int_{\gamma_{2,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi \right\| \leq \frac{CM \|Ax - \mu x\|}{\beta(\mu - \omega)} \xrightarrow{\beta \to +\infty} 0 \tag{3.11}$$

An analogous computation yields

$$\lim_{\alpha \to -\infty} \sup_{t \in [a,b]} \int_{\gamma_{4,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi = 0.$$

By Lemma 3.2.3

$$\lim_{\substack{\alpha \to -\infty \\ \beta \to +\infty}} \sup_{t \in [a,b]} \left\| \frac{1}{2\pi i} \int_{\gamma_{3,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi + e^{tA_\mu} x \right\| = 0, \tag{3.12}$$

which implies

$$\begin{aligned} \int_{\gamma_{1,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi &= \int_{\gamma_{\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi - \int_{\gamma_{2,\alpha,\beta} \gamma_{3,\alpha,\beta} \gamma_{4,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi \\ &= -\int_{\gamma_{2,\alpha,\beta} \gamma_{4,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi - \int_{\gamma_{3,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi \\ &\xrightarrow[\alpha \to -\infty]{\beta \to +\infty} 2\pi i e^{tA_\mu} x. \end{aligned}$$

By (3.11) and (3.12) this limit is uniform in $t \in [a, b]$. Consequently, the integral

$$\frac{1}{2\pi i} \int_\gamma e^{\xi t} R(\xi, A_\mu) x \, d\xi = \lim_{\substack{\alpha \to -\infty \\ \beta \to +\infty}} \frac{1}{2\pi i} \int_{\gamma_{1,\alpha,\beta}} e^{\xi t} R(\xi, A_\mu) x \, d\xi$$

exists and equals $e^{tA_\mu} x$.

$\square$

3.2.6 Theorem. *Let $\big(T(t)\big)_{t \geq 0}$ a strongly continuous semigroup, M, ω as in Proposition 3.1.2, a), and denote by A its infinitesimal generator. Given $\mu > \max\{\omega, 0\}$ and $\xi \in \mathbb{C}$ with* $\operatorname{Re}(\xi) > \frac{\mu\omega}{\mu - \omega}$

$$\lim_{\mu \to +\infty} R(\xi, A_\mu) x = R(\xi, A) x$$

for any $x \in X$. Furthermore this limit is uniform for fixed $\mathrm{Re}(\xi)$ *and* $\mathrm{Im}(\xi)$ *in compact intervals, meaning that for any* $a > \frac{\mu\omega}{\mu-\omega}$ *and* $[\alpha, \beta] \subseteq \mathbb{R}$ *satisfying* $-\infty < \alpha < \beta < +\infty$ *we have*

$$\lim_{\mu \to +\infty} \sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \|R(\xi, A_\mu)x - R(\xi, A)x\| = 0$$

for any $x \in X$.

Proof. Because of $\frac{\mu\omega}{\mu-\omega} > \omega$, we have $\xi \in \rho(A) \cap \rho(A_\mu)$. We want to show the equality

$$R(\xi, A_\mu)x - R(\xi, A)x = \tfrac{1}{\mu+\xi} A^2 R(\tfrac{\mu\xi}{\mu+\xi}, A) R(\xi, A)x$$

for any $x \in D(A^2)$ and $\mu > \max\{\omega, 0\}$. To that end, observe that

$$\tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)\big((\xi I - A)(\mu I - A)\big)R(\mu, A)R(\xi, A)x = \tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)x,$$

$$\tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)\big((\mu+\xi)(\tfrac{\mu\xi}{\mu+\xi} I - A)\big)R(\mu, A)R(\xi, A)x = R(\xi, A)x$$

and

$$(\xi I - A)(\mu I - A)x - (\mu + \xi)(\tfrac{\mu\xi}{\mu+\xi} I - A) = A^2 x.$$

By gathering the three equations above, employing Lemma 3.2.4 and noting that A and $R(\xi, A)$ commute on $D(A)$ we obtain

$$\begin{aligned} R(\xi, A_\mu)x - R(\xi, A)x &= \tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)x - R(\xi, A)x \\ &= \tfrac{1}{\mu+\xi}(\mu I - A)R(\tfrac{\mu\xi}{\mu+\xi}, A)A^2 R(\mu, A)R(\xi, A)x \\ &= \tfrac{1}{\mu+\xi} A^2 R(\tfrac{\mu\xi}{\mu+\xi}, A)R(\xi, A)x. \end{aligned}$$

Given $\mu > \max\{\omega, 0\}$ and $\xi \in \mathbb{C}$ with $a := \mathrm{Re}(\xi) > \frac{\mu\omega}{\mu-\omega}$ and $b := \mathrm{Im}(\xi) \in [\alpha, \beta]$ for fixed $-\infty < \alpha < \beta < +\infty$ we set $\varepsilon_\mu := a - \frac{\mu\omega}{\mu-\omega} > 0$ and obtain

$$\|R(\xi, A)\| \leq \frac{M}{a - \omega} \leq \frac{M}{a - \frac{\mu\omega}{\mu-\omega}} = \frac{M}{\varepsilon_\mu}$$

for all $\mu > \max\{\omega, 0\}$ by Proposition 3.1.2, g). In order to show a similar estimate for $R(\frac{\mu\xi}{\mu+\xi}, A)$, by

$$\mathrm{Re}\Big(\frac{\mu\xi}{\mu+\xi}\Big) = \mu\frac{a(\mu+a) + b^2}{(\mu+a)^2 + b^2}$$

we estimate

$$\begin{aligned} \varepsilon_\mu = a - \frac{\mu\omega}{\mu-\omega} &= \frac{\mu a - \omega a - \mu\omega}{\mu - \omega} = \frac{\mu a - \omega(\mu + a)}{\mu - \omega} = \frac{\mu a(\mu+a) - \omega(\mu+a)^2}{(\mu-\omega)(\mu+a)} \\ &\leq \frac{\mu a(\mu+a) - \omega(\mu+a)^2 + (\mu-\omega)b^2}{(\mu-\omega)(\mu+a)} \\ &= \frac{(\mu+a)^2 + b^2}{(\mu-\omega)(\mu+a)} \cdot \frac{\mu a(\mu+a) + \mu b^2}{(\mu+a)^2 + b^2} - \frac{\omega(\mu+a)^2 + \omega b^2}{(\mu-\omega)(\mu+a)} \end{aligned}$$

$$= \frac{(\mu+a)^2+b^2}{(\mu-\omega)(\mu+a)} \cdot \left(\frac{\mu a(\mu+a)+\mu b^2}{(\mu+a)^2+b^2} - \omega\right) = \frac{(\mu+a)^2+b^2}{(\mu-\omega)(\mu+a)}\left(\mathrm{Re}(\tfrac{\mu\xi}{\mu+\xi}) - \omega\right)$$

We obtain

$$\mathrm{Re}(\tfrac{\mu\xi}{\mu+\xi}) - \omega \geq \frac{(\mu-\omega)(\mu+a)\varepsilon_\mu}{(\mu+a)^2+b^2}$$

for any $\mu > \max\{\omega, 0\}$ and in turn

$$\left\|R(\tfrac{\mu\xi}{\mu+\xi}, A)\right\| \leq \frac{M}{\mathrm{Re}(\tfrac{\mu\xi}{\mu+\xi}) - \omega} \leq \frac{(\mu+a)^2+b^2}{(\mu-\omega)(\mu+a)} \cdot \frac{M}{\varepsilon_\mu}.$$

For $x \in D(A^2)$ and $\mu > \max\{\omega, 0\}$ we have

$$\begin{aligned}
\|R(\xi, A)x - R(\xi, A_\mu)x\| &= \left\|\tfrac{1}{\mu+\xi} A^2 R(\tfrac{\mu\xi}{\mu+\xi}, A) R(\xi, A)x\right\| \\
&\leq \tfrac{1}{|\mu+\xi|} \left\|R(\tfrac{\mu\xi}{\mu+\xi}, A)\right\| \|R(\xi, A)\| \left\|A^2 x\right\| \\
&\leq \frac{(\mu+a)^2+b^2}{(\mu-\omega)(\mu+a)} \cdot \frac{M^2}{|\mu+\xi|\varepsilon_\mu^2} \left\|A^2 x\right\| \\
&\leq \frac{(\mu+a)^2+\beta^2}{(\mu-\omega)(\mu+a)} \cdot \frac{M^2}{(\mu+a)\varepsilon_\mu^2} \left\|A^2 x\right\| \\
&\leq \frac{(\mu+a)^2+\beta^2}{(\mu-\omega)(\mu+a)^2} \cdot \frac{M^2}{\varepsilon_\mu^2} \left\|A^2 x\right\|
\end{aligned}$$

This expression tends to zero for $\mu \to +\infty$ independently from $b \in [\alpha, \beta]$. Lastly, let $\varepsilon > 0$, $x \in X$ and $y \in D(A^2)$ be such that $\|x - y\| < \varepsilon$; see Proposition 3.1.4. Employing Lemma 3.2.4, we obtain

$$\begin{aligned}
\sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \|R(\xi, A_\mu)x - R(\xi, A_\mu)y\| &\leq \sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \|R(\xi, A_\mu)\| \|x - y\| \\
&\leq \sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \frac{\varepsilon M}{\mathrm{Re}(\xi) - \frac{\mu\omega}{\mu-\omega}} = \frac{\varepsilon M}{a - \frac{\mu\omega}{\mu-\omega}}
\end{aligned}$$

as well as

$$\begin{aligned}
\sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \|R(\xi, A)x - R(\xi, A)y\| &\leq \sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \|R(\xi, A)\| \|x - y\| \\
&\leq \sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \frac{\varepsilon M}{\mathrm{Re}(\xi) - \omega} = \frac{\varepsilon M}{a - \omega}.
\end{aligned}$$

Putting these inequalities together yields

$$\begin{aligned}
\sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \|R(\xi, A_\mu)x - R(\xi, A)x\| &\leq \frac{\varepsilon M}{a - \frac{\mu\omega}{\mu-\omega}} + \frac{\varepsilon M}{a - \omega} + \sup_{\substack{\mathrm{Re}(\xi)=a \\ \mathrm{Im}(\xi)\in[\alpha,\beta]}} \|R(\xi, A_\mu)y - R(\xi, A)y\| \\
&\xrightarrow{\mu\to+\infty} \frac{\varepsilon M}{a - \omega} + \frac{\varepsilon M}{a - \omega} = \frac{2\varepsilon M}{a - \omega}.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the desired result holds true.

$\square$

3.2.7 Theorem. *Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup, A its generator and M, ω as on Proposition 3.1.2, a). For $\mu > \max\{\omega, 0\}$, $\tau > \max\{\frac{\mu\omega}{\mu-\omega}, 0\}$ and $x \in D(A)$*

$$\int_0^t e^{sA_\mu}x\ ds = \frac{1}{2\pi i}\int_\gamma \frac{e^{\xi t}}{\xi}R(\xi, A_\mu)x\ d\xi,$$

where $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma(s) = \tau + is$. Furthermore, for any $0 \leq a < b < +\infty$,

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi}R(\xi, A_\mu)x\ d\xi - \int_0^t e^{sA_\mu}x\ ds \right\| = 0.$$

Proof. For $s > 0$ and $x \in D(A)$ the function $\eta \mapsto e^{(\tau+i\eta)s}R(\tau+i\eta, A_\mu)x$ is continuous on $\mathbb{R}$ and, according to Lemma 3.2.4,

$$\left\|e^{(\tau+i\eta)s}R(\tau+i\eta, A_\mu)x\right\| \leq \frac{Me^{\tau s}\left\|x\right\|}{\tau - \frac{\mu\omega}{\mu-\omega}} < +\infty, \tag{3.13}$$

which implies $\eta \mapsto e^{(\tau+i\eta)s}R(\tau+i\eta, A_\mu)x$ is integrable over any bounded interval $[\alpha, \beta]$; see Theorem 2.4.3. We define

$$F_{\alpha,\beta}(s) := \frac{1}{2\pi}\int_{(\alpha,\beta)} e^{(\tau+i\eta)s}R(\tau+i\eta, A_\mu)x\ d\lambda(\eta).$$

For $t > 0$ and $s \in (0, t)$ by (3.13) we have

$$\left\|e^{(\tau+i\eta)s}R(\tau+i\eta, A_\mu)x\right\| \leq \frac{Me^{\tau t}\left\|x\right\|}{\tau - \frac{\mu\omega}{\mu-\omega}}$$

and

$$\int_\alpha^\beta \frac{Me^{\tau t}\left\|x\right\|}{\tau - \frac{\mu\omega}{\mu-\omega}}\ ds = \frac{Me^{\tau t}(\beta-\alpha)\left\|x\right\|}{\tau - \frac{\mu\omega}{\mu-\omega}} < +\infty.$$

Therefore, because of the continuity of $s \mapsto e^{(\tau+i\eta)s}R(\tau+i\eta, A_\mu)x$, we can employ Corollary 2.3.8 to see that $F_{\alpha,\beta} : [0, t] \to X$ is continuous. Together with

$$\begin{aligned}
\int_0^t \left\|F_{\alpha,\beta}(s)\right\|\ ds &\leq \frac{1}{2\pi}\int_0^t \Big(\int_{(\alpha,\beta)} \left\|e^{(\tau+i\eta)s}R(\tau+i\eta, A_\mu)x\right\|\ d\lambda(\eta)\Big)\ ds \\
&= \frac{1}{2\pi}\int_0^t \Big(\int_{(\alpha,\beta)} \frac{Me^{\tau t}\left\|x\right\|}{\tau - \frac{\mu\omega}{\mu-\omega}}\ d\lambda(\eta)\Big)\ ds \\
&= \frac{Mte^{\tau t}(\beta-\alpha)\left\|x\right\|}{2\pi(\tau - \frac{\mu\omega}{\mu-\omega})} < +\infty
\end{aligned}$$

we conclude that $F_{\alpha,\beta}$ is integrable. According to Theorem 2.3.12 we exchange the order of integration and obtain

$$\begin{aligned}\int_0^t F_{\alpha,\beta}(s)\,ds &= \frac{1}{2\pi}\int_{(0,t)}\Big(\int_{(\alpha,\beta)} e^{(\tau+i\eta)s} R(\tau+i\eta,A_\mu)x\,d\lambda(\eta)\Big)\,d\lambda(s)\\ &= \frac{1}{2\pi}\int_{(\alpha,\beta)}\Big(\int_{(0,t)} e^{(\tau+i\eta)s}\,d\lambda(s)\Big) R(\tau+i\eta,A_\mu)x\,d\lambda(\eta)\\ &= \frac{1}{2\pi}\int_{(\alpha,\beta)} \frac{e^{(\tau+i\eta)t}-1}{\tau+i\eta} R(\tau+i\eta,A_\mu)x\,d\lambda(\eta)\\ &= \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi}R(\xi,A_\mu)x\,d\xi - \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}}\frac{1}{\xi}R(\xi,A_\mu)x\,d\xi \end{aligned}\tag{3.14}$$

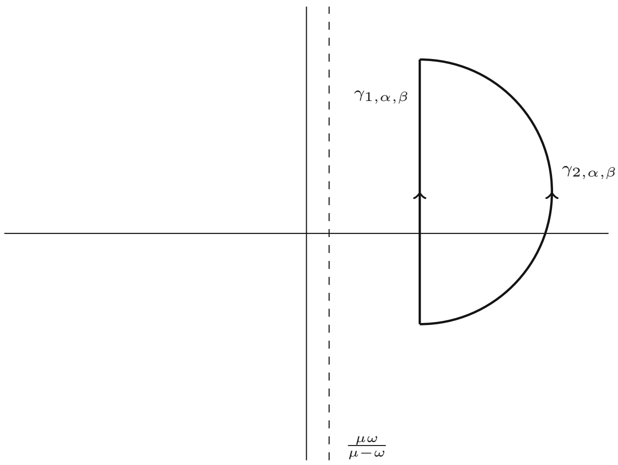
We want to prove that the second integral tends to zero for $\alpha\to-\infty$, $\beta\to+\infty$, i.e.

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\int_\alpha^\beta \frac{1}{\gamma+i\eta}R(\gamma+i\eta,A_\mu)x\,d\eta = 0. \tag{3.15}$$

To that end for $-\infty<\alpha<\beta<+\infty$ we define the paths

$$\begin{aligned}\gamma_{1,\alpha,\beta}&:[\alpha,\beta]\to\mathbb{C},\ \gamma(\eta):=\tau+i\eta,\\ \gamma_{2,\alpha,\beta}&:[-\tfrac{\pi}{2},\tfrac{\pi}{2}]\to\mathbb{C},\ \gamma(\eta):=\tau+i\frac{\alpha+\beta}{2}+\frac{\beta-\alpha}{2}e^{-i\eta}\end{aligned}$$

and $\gamma_{\alpha,\beta}:=\gamma_{1,\alpha,\beta}\gamma_{2,\alpha,\beta}$.



Since $\gamma_{\alpha,\beta}$ is a closed curve in the star-shaped set $\{\xi\in\mathbb{C}:\mathrm{Re}(\xi)>\omega\}$ and $\xi\mapsto\frac{1}{\xi}R(\xi,A_\mu)x$ is analytic in $\{\xi\in\mathbb{C}:\mathrm{Re}(\xi)>\omega\}$,

$$\int_{\gamma_{\alpha,\beta}}\frac{1}{\xi}R(\xi,A_\mu)x\,d\xi=0$$

by Proposition 2.2.3 and Theorem 2.2.5. According to Lemma 3.2.4 we have

$$\left\|\frac{\gamma'_{2,\alpha,\beta}(\eta)}{\gamma_{2,\alpha,\beta}(\eta)}R(\gamma_{2,\alpha,\beta}(\eta),A_\mu)x\right\| = \frac{|-\frac{\beta-\alpha}{2}ie^{-i\eta}|}{|\tau+i\frac{\alpha+\beta}{2}+\frac{\beta-\alpha}{2}e^{-i\eta}|}\left\|R(\tau+i\tfrac{\alpha+\beta}{2}+\tfrac{\beta-\alpha}{2}e^{-i\eta},A_\mu)x\right\|$$

$$\leq \frac{\beta-\alpha}{2|\tau+i\frac{\alpha+\beta}{2}+\frac{\beta-\alpha}{2}e^{-i\eta}|}\left\|R(\tau+i\tfrac{\alpha+\beta}{2}+\tfrac{\beta-\alpha}{2}e^{-i\eta},A_\mu)\right\|\|x\|$$
$$\leq \frac{(\beta-\alpha)M\|x\|}{2\big(\tau+\frac{\beta-\alpha}{2}\cos(\eta)\big)\big(\tau+\frac{\beta-\alpha}{2}\cos(\eta)-\frac{\mu\omega}{\mu-\omega}\big)} =: g_{\alpha,\beta}(\eta)$$

if we assume $\alpha < -\tau < \tau < \beta$. Note that $g_{\alpha,\beta}$ is continuous,

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} g_{\alpha,\beta}(\eta) = 0$$

for any $\eta \in (-\frac{\pi}{2},\frac{\pi}{2})$ and, since $\cos(\eta)\geq 0$,

$$g_{\alpha,\beta}(\eta) = \frac{(\beta-\alpha)M\|x\|}{2\big(\tau+\frac{\beta-\alpha}{2}\cos(\eta)\big)\big(\tau+\frac{\beta-\alpha}{2}\cos(\eta)-\frac{\mu\omega}{\mu-\omega}\big)} \leq \frac{M\|x\|}{\tau(\tau-\frac{\mu\omega}{\mu-\omega})}.$$

Because of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{M\|x\|}{\tau(\tau-\frac{\mu\omega}{\mu-\omega})}\,d\eta = \frac{\pi M\|x\|}{\tau(\tau-\frac{\mu\omega}{\mu-\omega})} < +\infty$$

we can employ Theorem 2.3.7, obtaining

$$\left\|\int_{\gamma_{2,\alpha,\beta}} \frac{1}{\xi}R(\xi,A_\mu)x\,d\xi\right\| = \left\|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\gamma_{2,\alpha,\beta}'(\eta)}{\gamma_{2,\alpha,\beta}(\eta)}R(\gamma_{2,\alpha,\beta}(\eta),A_\mu)x\,d\eta\right\|$$
$$\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\|\frac{\gamma_{2,\alpha,\beta}'(\eta)}{\gamma_{2,\alpha,\beta}(\eta)}R(\gamma_{2,\alpha,\beta}(\eta),A_\mu)x\right\|d\eta$$
$$\leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_{\alpha,\beta}(\eta)\,d\eta \xrightarrow[\alpha\to-\infty]{\beta\to+\infty} 0,$$

and in turn

$$\int_\alpha^\beta \frac{1}{\tau+i\eta}R(\tau+i\eta,A_\mu)x\,d\eta = \frac{1}{i}\int_{\gamma_{1,\alpha,\beta}} \frac{1}{\xi}R(\xi,A_\mu)x\,d\xi$$
$$= \frac{1}{i}\int_{\gamma_{\alpha,\beta}} \frac{1}{\xi}R(\xi,A_\mu)x\,d\xi - \frac{1}{i}\int_{\gamma_{2,\alpha,\beta}} \frac{1}{\xi}R(\xi,A_\mu)x\,d\xi$$
$$= -\frac{1}{i}\int_{\gamma_{2,\alpha,\beta}} \frac{1}{\xi}R(\xi,A_\mu)x\,d\xi \xrightarrow[\alpha\to-\infty]{\beta\to+\infty} 0.$$

In our next step, we want to show that $\xi \mapsto \frac{e^{\xi t}}{\xi}R(\xi,A_\mu)x$ is integrable along $\gamma(s) = \tau + is$ for any $t > 0$ and

$$\lim_{t\to0+}\int_\gamma \frac{e^{\xi t}}{\xi}R(\xi,A_\mu)x\,d\xi = \int_\gamma \frac{1}{\xi}R(\xi,A_\mu)x\,d\xi.$$

To that end assume $t \in (0,C]$ and observe

$$\|\xi R(\xi,A_\mu)x\| \leq \|R(\xi,A_\mu)(\xi I - A_\mu)x\| + \|R(\xi,A_\mu)A_\mu x\|$$

$$= \|x\| + \|R(\xi, A_\mu)A_\mu x\| \le \|x\| + \frac{M\,\|A_\mu x\|}{\operatorname{Re}(\xi) - \frac{\mu\omega}{\mu-\omega}}.$$

We obtain

$$\left\| \frac{e^{\gamma(\eta)t}\gamma'(\eta)}{\gamma(\eta)} R(\gamma(\eta), A_\mu)x \right\| = \left\| \frac{ie^{\tau t + it\eta}}{\tau + i\eta} R(\tau + i\eta, A_\mu)x \right\| = \frac{e^{\tau t}}{|\tau + i\eta|} \|R(\tau + i\eta, A_\mu)x\|$$
$$\le \frac{e^{\tau t}\left(\|x\| + \frac{M\|A_\mu x\|}{\tau - \frac{\mu\omega}{\mu-\omega}}\right)}{|\tau + i\eta|^2} \le \frac{e^{\tau C}\left(\|x\| + \frac{M\|A_\mu x\|}{\tau - \frac{\mu\omega}{\mu-\omega}}\right)}{|\tau + i\eta|^2}.$$

As

$$\int_{-\infty}^{+\infty} \frac{e^{\tau C}\left(\|x\| + \frac{M\|A_\mu x\|}{\tau - \frac{\mu\omega}{\mu-\omega}}\right)}{|\tau + i\eta|^2}\, ds = e^{\tau C}\left(\|x\| + \frac{M\,\|A_\mu x\|}{\tau - \frac{\mu\omega}{\mu-\omega}}\right) \int_{-\infty}^{+\infty} \frac{1}{\tau^2 + s^2}\, ds$$
$$= \frac{\pi e^{\tau C}\left(\|x\| + \frac{M\|A_\mu x\|}{\tau - \frac{\mu\omega}{\mu-\omega}}\right)}{\tau} < +\infty \tag{3.16}$$

we can employ Theorem 2.3.7, to conclude that $\xi \mapsto \frac{e^{\xi t}}{\xi} R(\xi, A_\mu)x$ is integrable along γ and

$$\lim_{t\to 0+} \int_\gamma \frac{e^{\xi t}}{\xi} R(\xi, A_\mu)x\; d\xi = \lim_{t\to 0^+} \int_{-\infty}^{+\infty} \frac{e^{\gamma(\eta)t}\gamma'(\eta)}{\gamma(\eta)} R(\gamma(\eta), A_\mu)x\; d\eta$$
$$= \int_{-\infty}^{+\infty} \frac{\gamma'(\eta)}{\gamma(\eta)} R(\gamma(\eta), A_\mu)x\; d\eta$$
$$= \int_\gamma \frac{1}{\xi} R(\xi, A_\mu)x\; d\xi.$$

By (3.15) the last integral vanishes. Hence, given $\varepsilon > 0$ we find $\delta \in (0, \varepsilon)$ such that

$$\left\| \frac{1}{2\pi i} \int_\gamma \frac{e^{\xi\delta}}{\xi} R(\xi, A_\mu)x\; d\xi \right\| < \varepsilon.$$

We fix $0 \le a < b < +\infty$ with $\delta < b$ and employ Corollary 3.2.5 to see that

$$\lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \sup_{s\in[\delta,b]} \left\| F_{\alpha,\beta}(s) - e^{sA_\mu}x \right\| = 0,$$

which implies

$$\sup_{\substack{t\in[a,b] \\ t\ge\delta}} \left\| \int_\delta^t F_{\alpha,\beta}(s)\; ds - \int_\delta^t e^{sA_\mu}x\; ds \right\| \le \sup_{\substack{t\in[a,b] \\ t\ge\delta}} \int_\delta^t \left\| F_{\alpha,\beta}(s) - e^{sA_\mu}x \right\|\; dt$$
$$\le \sup_{\substack{t\in[a,b] \\ t\ge\delta}} \left((t-\delta) \sup_{s\in[\delta,t]} \left\| F_{\alpha,\beta}(s) - e^{sA_\mu}x \right\| \right)$$
$$\le b \sup_{s\in[\delta,b]} \left\| F_{\alpha,\beta}(s) - e^{sA_\mu}x \right\| \xrightarrow[\alpha\to-\infty]{\beta\to+\infty} 0.$$

Together with

$$\left\|\int_0^\delta e^{sA_\mu}x\ ds\right\| \le \int_0^\delta e^{s\|A_\mu\|}\ ds \le \delta e^{\delta\|A_\mu\|} \le \varepsilon e^{\varepsilon\|A_\mu\|}$$

and

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\left\|\int_0^\delta F_{\alpha,\beta}(s)\ ds\right\| = \lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\left\|\frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}}\frac{e^{\xi\delta}}{\xi}R(\xi,A_\mu)x\ d\xi - \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}}\frac{1}{\xi}R(\xi,A_\mu)x\ d\xi\right\|$$
$$= \left\|\frac{1}{2\pi i}\int_\gamma\frac{e^{\xi\delta}}{\xi}R(\xi,A_\mu)x\ d\xi\right\| < \varepsilon$$

we conclude

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\sup_{t\in[a,b]}\left\|\int_0^t F_{\alpha,\beta}(s)\ ds - \int_0^t e^{sA_\mu}x\ ds\right\| < \varepsilon e^{\varepsilon\|A_\mu\|} + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\sup_{t\in[a,b]}\left\|\int_0^t F_{\alpha,\beta}(s)\ ds - \int_0^t e^{sA_\mu}x\ ds\right\| = 0.$$

Furthermore, from

$$\frac{1}{2\pi i}\int_{\gamma_{1,\alpha,\beta}}\frac{e^{\xi t}}{\xi}R(\xi,A_\mu)x\ d\xi = \frac{1}{2\pi}\int_\alpha^\beta\frac{e^{(\tau+i\eta)t}}{\tau+i\eta}R(\tau+i\eta,A_\mu)x\ d\eta$$
$$= \frac{1}{2\pi}\int_{(\alpha,\beta)}\frac{e^{(\tau+i\eta)t}}{\tau+i\eta}R(\tau+i\eta,A_\mu)x\ d\lambda(\eta)$$

together with (3.14) we infer

$$\int_0^t F_{\alpha,\beta}(s)\ ds - \frac{1}{2\pi i}\int_{\gamma_{1,\alpha,\beta}}\frac{e^{\xi t}}{\xi}R(\xi,A_\mu)x\ d\xi = \frac{1}{2\pi}\int_{(\alpha,\beta)}\frac{1}{\tau+i\eta}R(\tau+i\eta,A_\mu)x\ d\lambda(\eta)$$
$$= \frac{1}{2\pi}\int_\alpha^\beta\frac{1}{\tau+i\eta}R(\tau+i\eta,A_\mu)x\ d\eta.$$

Note that the last term does not depend on t and converges to zero for $\alpha\to-\infty$, $\beta\to+\infty$. Combining the previous two estimates yield

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}}\sup_{t\in[a,b]}\left\|\int_0^t e^{sA_\mu}x\ ds - \frac{1}{2\pi i}\int_{\gamma_{1,\alpha,\beta}}\frac{e^{\xi t}}{\xi}R(\xi,A_\mu)x\ d\xi\right\| = 0.$$

□

3.2.8 Corollary. Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup, A its generator and M,ω as on Proposition 3.1.2, a). For $\tau > \max\{\omega, 0\}$ and $x \in D(A)$

$$\int_0^t T(s)x\ ds = \frac{1}{2\pi i}\int_\gamma \frac{e^{\xi t}}{\xi} R(\xi, A)x\ d\xi,$$

where $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma(\eta) = \tau + i\eta$. Furthermore, for any $0 \leq a < b < +\infty$

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi} R(\xi, A)x\ d\xi - \int_0^t T(s)x\ ds \right\| = 0.$$

Proof. Let $\varepsilon > 0$, $t \in [a, b]$, $0 \leq a < b < +\infty$, and $\mu_0 > \tau$, such that $\tau > \frac{\mu\omega}{\mu-\omega}$ for every $\mu > \mu_0$ and $x \in D(A)$. For $\alpha < -\tau$ and $\beta > \tau$ we define the path $\gamma_{\alpha,\beta} : [\alpha, \beta] \to \mathbb{C}$, $\gamma_{\alpha,\beta}(\eta) = \tau + i\eta$. By Lemma 3.2.4 and Theorem 3.1.5 for $\xi := \tau + i\eta$, $\eta \in \mathbb{R}$, we have

$$\begin{aligned}
\|\xi R(\xi, A_\mu)x\| &\leq \|(\xi I - A_\mu)R(\xi, A_\mu)x\| + \|A_\mu R(\xi, A_\mu)x\| \\
&\leq \|x\| + \|R(\xi, A_\mu)\|\, \|A_\mu x\| \\
&\leq \|x\| + \|\mu A R(\mu, A)x\| \frac{M}{\tau - \frac{\mu\omega}{\mu-\omega}} \\
&\leq \frac{\mu M\, \|Ax\|}{(\mu-\omega)(\tau - \frac{\mu\omega}{\mu-\omega})} = \frac{M\, \|Ax\|}{(1-\frac{\omega}{\mu})(\tau - \frac{\mu\omega}{\mu-\omega})}
\end{aligned}$$

for any $\mu > \mu_0$. As

$$\lim_{\mu\to+\infty} \frac{M\, \|Ax\|}{(1-\frac{\omega}{\mu})(\tau - \frac{\mu\omega}{\mu-\omega})} = \frac{M\, \|Ax\|}{\tau - \omega},$$

there is a constant $C > 0$, which neither depends on μ nor on ξ, such that

$$\|R(\xi, A_\mu)x\| \leq \frac{1}{|\xi|} \cdot \frac{M\, \|Ax\|}{(1-\frac{\omega}{\mu})(\tau - \frac{\mu\omega}{\mu-\omega})} \leq \frac{C}{|\xi|}.$$

We derive

$$\left\| \frac{\gamma'(\eta)}{\gamma(\eta)} R(\gamma(\eta), A_\mu) \right\| = \left\| \frac{i}{\tau + i\eta} R(\tau + i\eta, A_\mu) \right\| \leq \frac{C}{|\tau + i\eta|^2}$$

where

$$\int_{-\infty}^{+\infty} \frac{C}{|\tau + i\eta|^2}\ d\eta = C\int_{-\infty}^{+\infty} \frac{1}{\tau^2 + \eta^2}\ d\eta = \frac{\pi C}{\tau} < +\infty.$$

By Theorem 3.2.6 we have $R(\lambda, A_\mu)x \xrightarrow{\mu\to+\infty} R(\lambda, A)x$. Therefore, the requirements of Theorem 2.3.7 are fulfilled and we obtain

$$\begin{aligned}
\lim_{\mu\to+\infty} \int_\gamma \frac{1}{\lambda} R(\lambda, A_\mu)x\ d\lambda &= \lim_{\mu\to+\infty} \int_{-\infty}^{+\infty} \frac{1}{\tau + i\eta} R(\tau + i\eta, A_\mu)x\ d\eta \\
&= \lim_{\mu\to+\infty} \int_{\mathbb{R}} \frac{1}{\tau + i\eta} R(\tau + i\eta, A_\mu)x\ d\lambda(\eta) \\
&= \int_{\mathbb{R}} \frac{1}{\tau + i\eta} R(\tau + i\eta, A)x\ d\lambda(\eta)
\end{aligned}$$

$$\begin{aligned}&= \int_{-\infty}^{+\infty} \frac{1}{\tau + i\eta} R(\tau + i\eta, A)x \; d\eta \\ &= \int_\gamma \frac{1}{\xi} R(\xi, A)x \; d\xi\end{aligned}$$

as well as

$$\begin{aligned}\left\| \int_\gamma \frac{e^{\xi t}}{\xi} \big(R(\xi, A_\mu) - R(\xi, A)\big)x \; d\xi \right\| &= \left\| \int_{-\infty}^{+\infty} \frac{i e^{\gamma(\eta)t} \gamma'(\eta)}{\gamma(\eta)} \big(R(\gamma(\eta), A_\mu) - R(\gamma(\eta), A)\big)x \; d\eta \right\| \\ &\leq e^{\tau t} \int_{\mathbb{R}} \left\| \frac{\gamma'(\eta)}{\gamma(\eta)} \big(R(\gamma(\eta), A_\mu) - R(\gamma(\eta), A)\big)x \right\| \; d\lambda(\eta) \\ &\leq e^{\tau b} \int_{\mathbb{R}} \left\| \tfrac{\gamma'(\eta)}{\gamma(\eta)} \big(R(\gamma(\eta), A_\mu) - R(\gamma(\eta), A)\big)x \right\| \; d\lambda(\eta) \\ &\xrightarrow{\mu \to +\infty} 0\end{aligned}$$

We conclude

$$\lim_{\mu \to +\infty} \sup_{t \in [a,b]} \left\| \int_\gamma \frac{e^{\xi t}}{\xi} \big(R(\xi, A_\mu) - R(\xi, A)\big)x \; d\xi \right\| = 0.$$

Since $s \mapsto T(s)x$ is continuous in $[0, t]$, it is Riemann integrable; see Proposition 3.1.2, b). Furthermore, according to Remark 3.2.2

$$\begin{aligned}\sup_{t \in [a,b]} \left\| \int_0^t e^{sA_\mu} x \; ds - \int_0^t T(s)x \; ds \right\| &\leq \sup_{t \in [a,b]} \int_0^t \left\| e^{sA_\mu} x - T(s)x \right\| ds \\ &\leq \sup_{t \in [a,b]} \left(t \sup_{s \in [0,t]} \left\| e^{sA_\mu} x - T(s)x \right\| \right) \\ &= b \sup_{s \in [0,b]} \left\| e^{sA_\mu} x - T(s)x \right\| \xrightarrow{\mu \to +\infty} 0.\end{aligned}$$

If $\varepsilon > 0$ and $\mu_1 > \mu_0$ are such that

$$\sup_{t \in [a,b]} \int_{-\infty}^{+\infty} \left\| \frac{e^{(\tau + i\eta)t}}{\tau + i\eta} \big(R(\tau + i\eta, A_\mu) - R(\tau + i\eta, A)\big)x \right\| d\eta < \frac{\varepsilon}{3}$$

and

$$\sup_{t \in [a,b]} \left\| \int_0^t e^{sA_\mu} x \; ds - \int_0^t T(s)x \; ds \right\| < \frac{\varepsilon}{3}$$

for any $\mu > \mu_1$, then

$$\begin{aligned}\left\| \int_{\gamma_{\alpha,\beta}} \frac{e^{\xi t}}{\xi} \big(R(\xi, A_\mu) - R(\xi, A)\big)x \; d\xi \right\| &= \left\| \int_\alpha^\beta \frac{i e^{\gamma(\eta)t}}{\gamma(\eta)} \big(R(\gamma(\eta), A_\mu) - R(\gamma(\eta), A)\big)x \; d\eta \right\| \\ &\leq \int_\alpha^\beta \left\| \frac{e^{\gamma(\eta)t}}{\gamma(\eta)} \big(R(\gamma(\eta), A_\mu) - R(\gamma(\eta), A)\big)x \right\| d\eta \\ &\leq \int_{-\infty}^{+\infty} \left\| \frac{e^{\gamma(\eta)t}}{\gamma(\eta)} \big(R(\gamma(\eta), A_\mu) - R(\gamma(\eta), A)\big)x \right\| d\eta < \frac{\varepsilon}{3}\end{aligned}$$

for any $\alpha < -\tau$ and $\beta > \tau$. Let $\mu > \mu_1$. By Theorem 3.2.7 there are constants $\alpha_0 < -\tau$ and $\beta_0 > \tau$ such that

$$\sup_{t\in[a,b]} \left\| \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi} R(\xi, A_\mu)x \, d\xi - \int_0^t e^{sA_\mu} x \, ds \right\| < \frac{\varepsilon}{3}$$

for any $\alpha < \alpha_0$ and $\beta > \beta_0$. Combining the previous inequalities we obtain for $\alpha < \alpha_0$ and $\beta > \beta_0$

$$\sup_{t\in[a,b]} \left\| \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi} R(\xi, A)x \, d\xi - \int_0^t T(s)x \, ds \right\| < \varepsilon.$$

□

We can prove the final result of this section.

3.2.9 Corollary. Let $\big(T(t)\big)_{t\geq 0}$ a strongly continuous semigroup, A its generator and M, ω as in Proposition 3.1.2, a). For $\tau > \max\{\omega, 0\}$ and $x \in D(A^2)$

$$T(t)x = \frac{1}{2\pi i} \int_\gamma e^{\xi t} R(\xi, A)x \, d\xi,$$

where $\gamma : \mathbb{R} \to \mathbb{C}$, $\gamma(\eta) = \tau + i\eta$ and

$$\lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| T(t)x - \int_{\gamma|_{[\alpha,\beta]}} e^{\xi t} R(\xi, A)x \, d\xi \right\|$$

for any $0 \leq a < b < +\infty$.

Proof. First, consider the bounded operator $B := 0$. Clearly, $\rho(B) = \mathbb{C} \setminus \{0\}$, $R(\xi, B) = \frac{1}{\xi} I$ for $\xi \neq 0$ and $\|B\| = 0 < \tau$. Therefore, employing Theorem 3.2.3, we obtain

$$x = \sum_{n=0}^{\infty} \frac{t^n}{n!} B^n x = e^{tB} x = \frac{1}{2\pi i} \int_\gamma \frac{e^{\xi t}}{\xi} x \, d\xi$$

as well as

$$\lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| x - \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi} x \, d\xi \right\| = 0. \tag{3.17}$$

The mappings $s \mapsto T(s)x$ and $s \mapsto AT(t)x = T(t)Ax$ are continuous in $[0, t]$. Therefore, by Corollary 2.3.13, 3.1.2, f), 2.1.3, g), and Corollary 3.2.8

$$T(t)x - x = \int_0^t T(s)Ax \, ds = \frac{1}{2\pi i} \int_\gamma \frac{e^{\xi t}}{\xi} R(\xi, A)Ax \, d\xi. \tag{3.18}$$

Furthermore,

$$\frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} e^{\xi t} R(\xi, A)x - \frac{e^{\xi t}}{\xi} x \, d\xi = \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} e^{\xi t} \big(R(\xi, A)x - \tfrac{1}{\xi} x\big) \, d\xi$$

$$= \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} e^{\xi t} R(\xi, A)(x - (\xi I - A)\tfrac{1}{\xi}x)\, d\xi$$

$$= \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi} R(\xi, A)Ax\, d\xi. \tag{3.19}$$

Since $Ax \in D(A)$, we can employ Corollary 3.2.8 to obtain the integrability of $\xi \mapsto e^{\xi t}R(\xi, A)$ along γ and

$$\frac{1}{2\pi i} \int_\gamma e^{\xi t} R(\xi, A)x\, d\xi = \frac{1}{2\pi i} \int_\gamma \frac{e^{\xi t}}{\xi} R(\xi, A)Ax\, d\xi + \frac{1}{2\pi i} \int_\gamma \frac{e^{\xi t}}{\xi} x\, d\xi$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{e^{\xi t}}{\xi} R(\xi, A)Ax\, d\xi + x$$

Corollary 3.2.8 also yields

$$\lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| \int_0^t T(s)Ax\, ds - \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} \frac{e^{\xi t}}{\xi} R(\xi, A)Ax\, d\xi \right\| = 0,$$

which by (3.17), (3.18), (3.19) implies

$$\lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \sup_{t\in[a,b]} \left\| T(t)x - \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} e^{\xi t} R(\xi, A)x\, d\xi \right\| = 0.$$

□

3.3 Analytic Semigroups

Recall the fact that for any $z \in \mathbb{C} \setminus \{0\}$ there is a unique pair $(r, \varphi) \in (0, +\infty) \times [-\pi, \pi)$ such that $z = re^{i\varphi}$. We call $\arg(z) := \varphi$ the *argument* of z. The argument can be computed by

$$\arg(a + ib) = \begin{cases} \arctan(\frac{b}{a}) & \text{for } a > 0, \\ \arctan(\frac{b}{a}) + \pi & \text{for } a < 0, b > 0, \\ \arctan(\frac{b}{a}) - \pi & \text{for } a < 0, b < 0, \\ \frac{\pi}{2} & \text{for } a = 0, b > 0, \\ -\frac{\pi}{2} & \text{for } a = 0, b < 0, \\ -\pi & \text{for } a < 0, b = 0. \end{cases} \tag{3.20}$$

Throughout this section X denotes a Banach space.

3.3.1 Definition. Let $t_0 \geq 0$ and $\big(T(t)\big)_{t\geq 0}$ a strongly continuous semigroup. $\big(T(t)\big)_{t\geq 0}$ is called *differentiable for* $t > t_0$ if $t \mapsto T(t)x$ is differentiable on $(t_0, +\infty)$ for any $x \in X$. If $t_0 = 0$ we call $\big(T(t)\big)_{t\geq 0}$ a *differentiable* semigroup.
Furthermore, for $\varphi \in (0, \pi)$ we call

$$\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \varphi\}$$

a *sector* in the complex plane and set $\Sigma_\varphi^0 := \Sigma_\varphi \cup \{0\}$. For $\varphi \in (0, \frac{\pi}{2})$ we call a family of bounded operators $\big(T(z)\big)_{z\in\Sigma_\varphi^0}$ on X an *analytic semigroup of angle* φ, if

- $T(0) = I.$,
- $T(z+w) = T(z)T(w)$ for all $z, w \in \Sigma_\varphi^0$,
- $z \mapsto T(z)$ is analytic in Σ_φ,
- $\lim\limits_{\substack{z\in\Sigma_\varphi \\ z\to 0}} T(z)x = x$ for all $x \in X$.

An analytic semigroup is called *bounded*, if

$$\sup_{z\in\Sigma_\theta} \|T(z)\| < +\infty$$

for all $\theta < \varphi$.

Clearly, for an analytic semigroup $\big(T(z)\big)_{z\in\Sigma_\varphi}$ the semigroup $\big(T(t)\big)_{t\geq 0}$ is a differentiable semigroup. We define the generator of $\big(T(z)\big)_{z\in\Sigma_\varphi}$ as the generator of $\big(T(t)\big)_{t\geq 0}$.

We want to characterize generators of analytic semigroups and, in the course of that, what kind of properties a strongly continuous semigroup has to have, in order to be extentable to an analytic semigroup. To that end, we introduce the class of *sectorial* operators.

3.3.2 Definition. An operator $A : D \subseteq X \to X$ is called *sectorial of angle* $\delta \in (0, \frac{\pi}{2})$, if A is densely defined,

$$\Sigma_{\frac{\pi}{2}+\delta} \subseteq \rho(A)$$

and

$$\sup_{\xi\in\Sigma_{\frac{\pi}{2}+\eta}} \|\xi R(\xi, A)\| < +\infty$$

for all $\eta \in (0, \delta)$.

Note that, given a sectorial operator A, $\rho(A) \neq \emptyset$ causes A to be closed; see Proposition 1.2.4, *d*).

3.3.3 Theorem. *If A is a sectorial operator of angle δ with the additional property that $0 \in \rho(A)$ and $\frac{\pi}{2} < \theta < \frac{\pi}{2} + \delta$, then A is the infinitesimal generator of a bounded and strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ satisfying*

$$T(t) = \frac{1}{2\pi i} \int_\gamma e^{\xi t} R(\xi, A)\, d\xi,$$

where $\gamma : \mathbb{R} \to \mathbb{C}$ is defined by $\gamma(s) := -se^{-i\theta}$ for $s \in (-\infty, 0]$ and $\gamma(s) := se^{i\theta}$ for $s \in (0, +\infty)$. Furthermore, for any $\varepsilon > 0$

$$\lim_{\substack{\alpha\to-\infty \\ \beta\to+\infty}} \sup_{t\geq\varepsilon} \left\| T(t) - \frac{1}{2\pi i} \int_{\gamma|_{[\alpha,\beta]}} e^{\xi t} R(\xi, A)\, d\xi \right\| = 0.$$

Proof. Because A is sectorial,

$$\|R(\xi, A)\| \leq \frac{M_\theta}{|\xi|}$$

for some M_θ and all $\xi \in \Sigma_\theta$. Hence, for $s \in (-\infty, -1]$

$$\left\|e^{\gamma(s)t}\gamma'(s)R(\gamma(s), A)\right\| = \left\|-e^{-se^{-i\theta}t}e^{-i\theta}R(-se^{-i\theta}, A)\right\| = e^{-s\cos(\theta)t}\left\|R(-se^{-i\theta}, A)\right\|$$
$$\leq \frac{M_\theta e^{-s\cos(\theta)t}}{|s|} \leq M_\theta e^{-s\cos(\theta)t}.$$

$\theta \in (\frac{\pi}{2}, \pi)$ implies $\cos(\theta) < 0$ and, in turn,

$$\int_{-\infty}^{-1} M_\theta e^{-s\cos(\theta)t}\, ds = -M_\theta \frac{e^{-s\cos(\theta)t}}{\cos(\theta)t}\Big|_{-\infty}^{-1} = M_\theta \frac{e^{\cos(\theta)}}{|\cos(\theta)|t} < +\infty.$$

By the continuity of $s \mapsto e^{\gamma(s)t}\gamma'(s)R(\gamma(s), A)$ on $[-1, 0]$, it is Riemann integrable. Therefore, by Proposition 2.1.2 $\xi \mapsto e^{\xi t}R(\xi, A)$ is integrable along $\gamma|_{(-\infty,0]}$. An analogous computation yields the integrability of $\xi \mapsto e^{\xi t}R(\xi, A)$ along $\gamma|_{[0,+\infty)}$. We define $S : [0, +\infty) \to L_b(X)$ by

$$S(t) := \frac{1}{2\pi i}\int_\gamma e^{\xi t}R(\xi, A)\, d\xi$$

for $t > 0$ and $S(0) := I$ as well as the paths $\gamma_\alpha : (-\infty, \alpha] \to \mathbb{C}$, $\gamma_\alpha(s) := -se^{-i\theta}$, $\gamma_\beta : [\beta, +\infty) \to \mathbb{C}$, $\gamma_\beta(s) := se^{i\theta}$ for $\alpha < 0 < \beta$. Given $\varepsilon > 0$ and $t \geq \varepsilon$,

$$\left\|\int_{\gamma_\alpha} e^{\xi t}R(\xi, A)\, d\xi\right\| = \left\|\int_{-\infty}^{\alpha} e^{-se^{-i\theta}t}(-e^{i\theta})R(-se^{-i\theta}, A)\, ds\right\|$$
$$\leq \int_{-\infty}^{\alpha} e^{-s\cos(\theta)t}\left\|R(-se^{-i\theta}, A)\right\|\, ds$$
$$\leq M_\theta \int_{-\infty}^{\alpha} \frac{e^{-s\cos(\theta)t}}{-s}\, ds \leq \frac{M_\theta}{|\alpha|}\int_{-\infty}^{\alpha} e^{-s\cos(\theta)t}\, ds$$
$$= \frac{M_\theta e^{-\alpha\cos(\theta)t}}{|\cos(\theta)|t} \leq \frac{M_\theta e^{-\varepsilon\alpha\cos(\theta)}}{\varepsilon|\cos(\theta)|} \xrightarrow{\alpha\to-\infty} 0.$$

We obtain

$$\lim_{\alpha\to-\infty}\sup_{t\geq\varepsilon}\left\|\int_{\gamma_\alpha} e^{\xi t}R(\xi, A)\, d\xi\right\| = 0$$

and analogously

$$\lim_{\beta\to+\infty}\sup_{t\geq\varepsilon}\left\|\int_{\gamma_\beta} e^{\xi t}R(\xi, A)\, d\xi\right\| = 0.$$

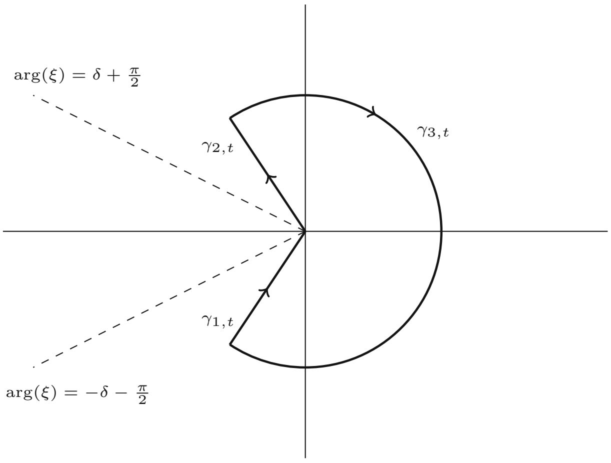
From

$$S(t) - \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}} e^{\xi t}R(\xi, A)\, d\xi = \frac{1}{2\pi i}\Big(\int_{\gamma_\alpha} e^{\xi t}R(\xi, A)\, d\xi + \int_{\gamma_\beta} e^{\xi t}R(\xi, A)\, d\xi\Big)$$

we conclude

$$\lim_{\substack{\alpha\to-\infty\\ \beta\to+\infty}} \sup_{t\geq\varepsilon} \left\| S(t) - \frac{1}{2\pi i}\int_{\gamma|_{[\alpha,\beta]}} e^{\xi t}R(\xi,A)\,d\xi \right\| = 0.$$

It remains to show that $\big(S(t)\big)_{t\geq 0}$ is a bounded and strongly continuous semigroup generated by A. For fixed $t>0$ we define the paths $\gamma_{1,t} : [-\frac{1}{t}, 0] \to \mathbb{C}$, by $\gamma_{1,t}(s) := -se^{-i\theta}$, $\gamma_{2,t} : [0, \frac{1}{t}] \to \mathbb{C}$ by $\gamma_{2,t}(s) := se^{i\theta}$ and $\gamma_{3,t} : [-\theta, \theta] \to \mathbb{C}$ by $\gamma_{3,t}(s) := \frac{1}{t}e^{-is}$.



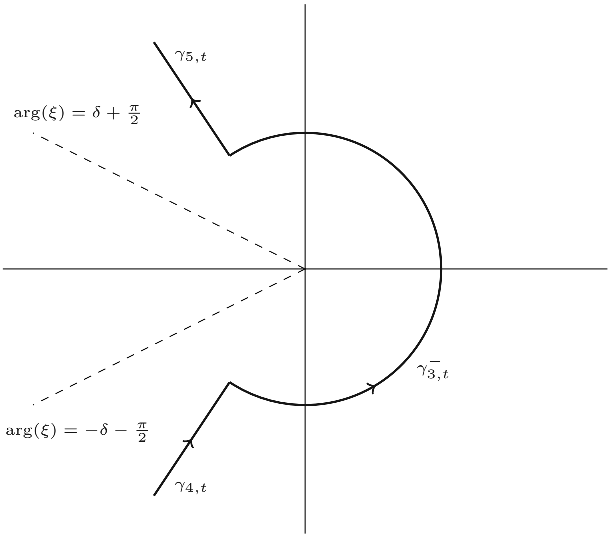
Since $\xi \mapsto e^{\xi t}R(\xi,A)$ is analytic on $\rho(A) \supseteq \Sigma_{\frac{\pi}{2}+\delta} \cup U_\eta(0)$ for sufficiently small $\eta > 0$ and $\gamma_{1,t}\gamma_{2,t}\gamma_{3,t}$ is a closed curve in the star-shaped set $\Sigma_{\frac{\pi}{2}+\delta} \cup U_\eta(0)$, we can employ Proposition 2.2.3 and Theorem 2.2.5 and obtain

$$\int_{\gamma_{1,t}\gamma_{2,t}\gamma_{3,t}} e^{\xi t}R(\xi,A)\,d\xi = 0.$$

Defining $\gamma_{4,t} := \gamma|_{(-\infty,-\frac{1}{t}]}$ and $\gamma_{5,t} := \gamma_{[\frac{1}{t},+\infty)}$ we derive

$$\begin{aligned} S(t) &= \frac{1}{2\pi}\int_\gamma e^{\xi t}R(\xi,A)\,d\xi = \frac{1}{2\pi i}\int_{\gamma_{4,t}\gamma_{5,t}} e^{\xi t}R(\xi,A)\,d\xi + \frac{1}{2\pi i}\int_{\gamma_{1,t}\gamma_{2,t}} e^{\xi t}R(\xi,A)\,d\xi \\ &= \frac{1}{2\pi i}\int_{\gamma_{4,t}\gamma_{5,t}} e^{\xi t}R(\xi,A)\,d\xi - \frac{1}{2\pi i}\int_{\gamma_{3,t}} e^{\xi t}R(\xi,A)\,d\xi, = \frac{1}{2\pi i}\int_{\gamma_t} e^{\xi t}R(\xi,A)\,d\xi, \end{aligned}$$

where $\gamma_t := \gamma_{4,t}\bar{\gamma_{3,t}}\gamma_{5,t}$.



We want to show that there is a constant $C > 0$ such that $\|S(t)\| \leq C$ for all $t \geq 0$. To that end we estimate

$$\left\| \int_{\gamma_{4,t}} e^{\xi t} R(\xi, A) \, d\xi \right\| \leq M_\theta \int_{-\infty}^{-\frac{1}{t}} \frac{e^{-s\cos(\theta)t}}{s} \, ds \leq M_\theta t \int_{-\infty}^{-\frac{1}{t}} e^{-s\cos(\theta)t} \, ds$$
$$= M_\theta t \cdot \frac{e^{\frac{1}{t}|\cos(\theta)|t}}{\cos(\theta)t} = M_\theta \cdot \frac{e^{\cos(\theta)}}{|\cos(\theta)|} < +\infty.$$

Note that the last term does not depend on t. An analogous computation yields the same estimate for the integral along $\gamma_{5,t}$. Finally,

$$\left\| \int_{\gamma_{3,t}^-} e^{\xi t} R(\xi, A) \, d\xi \right\| = \left\| \int_{-\theta}^{\theta} i\tfrac{1}{t} e^{is} e^{\frac{1}{t}e^{is}t} R(\tfrac{1}{t}e^{is}, A) \, ds \right\| \leq \frac{1}{t} \int_{-\theta}^{\theta} e^{\cos(\theta)} \left\| R(\tfrac{1}{t}e^{is}, A) \right\| \, ds$$
$$\leq \frac{1}{t} \int_{-\theta}^{\theta} t M_\theta e^{\cos(s)} \, ds \leq 2\theta M_\theta < +\infty.$$

We define $C := \max\{-\frac{M_\theta e^{\cos(\theta)}}{2\pi\cos(\theta)}, \frac{\theta M_\theta}{\pi}, 1\} > 0$ and obtain

$$\|S(t)\| \leq C$$

for all $t \geq 0$.
In order to show that $\big(S(t)\big)_{t\geq 0}$ is a strongly continuous semigroup generated by A, we want to employ Theorem 3.1.5. To that end we first prove

$$R(\mu, A) = \int_0^{+\infty} e^{-\mu t} S(t) \, dt$$

for all $\mu > 0$. By $\|e^{-\mu t} S(t)\| \leq C e^{-\mu t}$ and

$$\int_0^{+\infty} C e^{-\mu t} \, dt = -C \frac{e^{-\mu t}}{\mu} \bigg|_0^{+\infty} = \frac{C}{\mu} < +\infty,$$

$t \mapsto e^{-\mu t} S(t)$ is Riemann integrable and by the definition S

$$\int_0^{+\infty} e^{-\mu t} S(t)\ dt = \frac{1}{2\pi i} \int_0^{+\infty} \int_\gamma e^{(\xi-\mu)t} R(\xi, A)\ d\xi dt.$$

for $\mu > 0$. In order to change the order of integration, we set

$$f(t,s) := e^{(s\cos(\theta)-\mu)t} \left\| R(se^{i\theta}, A) \right\|, \quad s,t \in [0,+\infty),$$

and note that

$$\left\| e^{(\gamma(s)-\mu)t} \gamma'(s) R(\gamma(s), A) \right\| = \left\| e^{(se^{i\theta}-\mu)t} (e^{i\theta}) R(se^{i\theta}, A) \right\| = f(t,s).$$

For $t \geq 0$ and $s > 0$ we have

$$f(t,s) \leq \frac{M_\theta}{s} e^{(s\cos(\theta)-\mu)t}.$$

As

$$\begin{aligned}
\int_1^{+\infty} \int_0^1 \frac{M_\theta}{s} e^{(s\cos(\theta)-\mu)t}\ dt\ ds &= M_\theta \int_1^{+\infty} \frac{e^{(s\cos(\theta)-\mu)t}}{s(s\cos(\theta)-\mu)} \Big|_0^1\ ds \\
&= M_\theta \int_1^{+\infty} \Big(\frac{e^{s\cos(\theta)-\mu}}{s(s\cos(\theta)-\mu)} - \frac{1}{s(s\cos(\theta)-\mu)} \Big)\ ds \\
&= M_\theta \int_1^{+\infty} \Big(\frac{1}{s(\mu - s\cos(\theta))} - \frac{e^{s\cos(\theta)-\mu}}{s(\mu - s\cos(\theta))} \Big)\ ds \\
&\leq M_\theta \int_1^{+\infty} \frac{1}{s(\mu - s\cos(\theta))}\ ds \\
&= \frac{M_\theta}{\mu} \ln\Big(\frac{s}{\mu - s\cos(\theta)} \Big) \Big|_1^{+\infty} \\
&= \frac{M_\theta}{\mu} \ln(1 - \tfrac{\mu}{\cos(\theta)}) < +\infty,
\end{aligned}$$

we can apply Theorems 2.3.12 and 2.4.3 and obtain

$$\int_1^{+\infty} \Big(\int_0^1 e^{(\gamma(s)-\mu)t} \gamma'(s) R(\gamma(s), A)\ dt \Big)\ ds = \int_0^1 \Big(\int_1^{+\infty} e^{(\gamma(s)-\mu)t} \gamma'(s) R(\gamma(s), A)\ ds \Big)\ dt.$$

By continuity there exists a constant $C > 0$ such that

$$f(t,s) \leq C, \quad s,t \in [0,1].$$

Consequently,

$$\int_0^1 \Big(\int_0^1 e^{(\gamma(s)-\mu)t} \gamma'(s) R(\gamma(s), A)\ dt \Big)\ ds = \int_0^1 \Big(\int_0^1 e^{(\gamma(s)-\mu)t} \gamma'(s) R(\gamma(s), A)\ ds \Big)\ dt.$$

For $t, s > 1$ we conclude from

$$f(t,s) \leq \frac{M_\theta}{s} e^{(s\cos(\theta)-\mu)t} \leq M_\theta e^{(s\cos(\theta)-\mu)t}$$

that

$$\int_1^{+\infty}\int_1^{+\infty} M_\theta e^{(s\cos(\theta)-\mu)t}\ ds\ dt = M_\theta \int_1^{+\infty} e^{-\mu t}\frac{e^{s\cos(\theta)t}}{s\cos(\theta)}\Big|_1^{+\infty}\ dt$$
$$= M_\theta \int_1^{+\infty} \frac{e^{(\cos(\theta)-\mu)t}}{-\cos(\theta)}\ dt$$
$$= M_\theta \frac{e^{(\cos(\theta)-\mu)t}}{\cos(\theta)(\mu-\cos(\theta))}\Big|_1^{+\infty}$$
$$= \frac{M_\theta e^{\cos(\theta)-\mu}}{\cos(\theta)(\cos(\theta)-\mu)} < +\infty.$$

Again by Theorem 2.3.12

$$\int_1^{+\infty}\Big(\int_1^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ dt\Big)\ ds = \int_1^{+\infty}\Big(\int_1^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ ds\Big)\ dt.$$

Lastly, for $t > 1$ and $s \in [0,1]$, by continuity, there is a constant $K > 0$ satisfying $\big\|R(se^{i\theta},A)\big\| \le K$ for any $s \in [0,1]$. Hence,

$$f(t,s) \le Ke^{(s\cos(\theta)-\mu)t} \le Ke^{-\mu t}.$$

From

$$\int_1^{+\infty}\int_0^1 Ke^{-\mu t}\ ds\ dt = K\int_1^{+\infty} e^{-\mu t}\ dt = \frac{Ke^{-\mu}}{\mu} < +\infty$$

we derive

$$\int_0^1\Big(\int_1^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ dt\Big)\ ds = \int_1^{+\infty}\Big(\int_0^1 e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ ds\Big)\ dt.$$

Altogether we obtain

$$\int_0^{+\infty}\Big(\int_0^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ dt\Big)\ ds = \int_0^{+\infty}\Big(\int_0^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ ds\Big)\ dt.$$

By analogous computations for $s \le 0$ we obtain

$$\int_{-\infty}^0\Big(\int_0^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ dt\Big)\ ds = \int_0^{+\infty}\Big(\int_{-\infty}^0 e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ ds\Big)\ dt.$$

Hence,

$$\int_0^{+\infty} e^{-\mu t}S(t)\ dt = \frac{1}{2\pi i}\int_0^{+\infty}\int_\gamma e^{(\xi-\mu)t}R(\xi,A)\ d\xi\ dt$$
$$= \frac{1}{2\pi i}\int_0^{+\infty}\int_{-\infty}^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ ds\ dt$$
$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\int_0^{+\infty} e^{(\gamma(s)-\mu)t}\gamma'(s)R(\gamma(s),A)\ dt\ ds$$

$$= \frac{1}{2\pi i}\int_\gamma \Big(\int_0^{+\infty} e^{(\xi-\mu)t}\ dt\Big) R(\xi, A)\ d\xi$$
$$= \frac{1}{2\pi i}\int_\gamma \frac{1}{\mu-\xi} R(\xi, A)\ d\xi$$

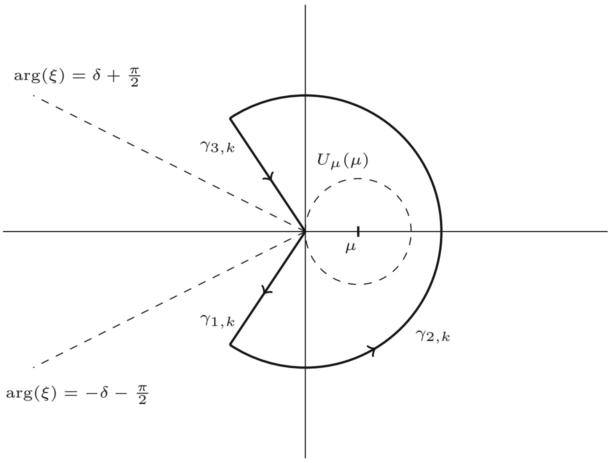
Let $k > 2\mu$ and define

$$\gamma_{1,k} : [0, k] \to \mathbb{C},\ \ \gamma_{1,k}(s) = se^{-i\theta},$$
$$\gamma_{2,k} : [-\theta, \theta] \to \mathbb{C},\ \ \gamma_{2,k}(s) := ke^{is}$$

and

$$\gamma_{3,k} : [-k, 0] \to \mathbb{C},\ \ \gamma_{3,k}(s) := -se^{i\theta}$$

as well as the closed curve $\gamma_k := \gamma_{1,k}\gamma_{2,k}\gamma_{3,k}$.



$h(r, s) := (1 - r)\gamma_k(s) + r(\mu + \mu e^{i\pi(2s-1)})$ constitutes a homotopy bewtween γ_k and $s \mapsto \mu + \mu e^{i\pi(2s-1)}$, $s \in [0, 1]$, in $\rho(A) \setminus \{\mu\}$. Since $\xi \mapsto R(\xi, A)$ is analytic in $\rho(A) \supseteq \operatorname{ran}(\gamma_k)$, we can employ Theorem 2.2.6 and obtain

$$R(\mu, A) = \frac{1}{2\pi i}\int_{\gamma_k} \frac{1}{\xi-\mu} R(\xi, A)\ d\xi.$$

Furthermore, since A is sectorial,

$$\Big\|\frac{\gamma_{2,k}'(s)}{\gamma_{2,k}(s)-\mu} R(\gamma_{2,k}(s), A)\Big\| = \Big\|\frac{ike^{is}}{ke^{is}-\mu} R(ke^{is}, A)\Big\| = \frac{k}{|ke^{is}-\mu|}\big\|R(ke^{is}, A)\big\|$$
$$\le \frac{k}{k-\mu}\big\|R(ke^{is}, A)\big\| \le \frac{kM_\theta}{k(k-\mu)},$$

and

$$\int_{-\theta}^{\theta} \frac{kM_\theta}{k(k-\mu)}\ ds = \frac{2\theta kM_\theta}{k(k-\mu)} \xrightarrow{k\to+\infty} 0,$$

from which we conclude

$$\lim_{k\to+\infty}\int_{\gamma_{2,k}} \frac{1}{\xi-\mu} R(\xi, A)\ d\xi = 0.$$

Taking the limit $k \to +\infty$ in

$$\frac{1}{2\pi i}\int_{\gamma_{3,k}\gamma_{1,k}} \frac{1}{\xi-\mu}R(\xi,A)\ d\xi = -\frac{1}{2\pi i}\int_{\gamma_{2,k}} \frac{1}{\xi-\mu}R(\xi,A)\ d\xi + \frac{1}{2\pi i}\int_{\gamma_k} \frac{1}{\xi-\mu}R(\xi,A)\ d\xi$$
$$= -\frac{1}{2\pi i}\int_{\gamma_{2,k}} \frac{1}{\xi-\mu}R(\xi,A)\ d\xi + R(\mu,A)$$

we obtain

$$\int_0^{+\infty} e^{-\mu t}S(t)\ dt = -\frac{1}{2\pi i}\int_{\gamma} \frac{1}{\xi-\mu}R(\xi,A)\ d\xi = \lim_{k\to+\infty}\frac{1}{2\pi i}\int_{\gamma_{3,k}\gamma_{1,k}} \frac{1}{\xi-\mu}R(\xi,A)\ d\xi$$
$$= R(\mu,A) - \lim_{k\to+\infty}\frac{1}{2\pi i}\int_{\gamma_{2,k}} \frac{1}{\mu-\xi}R(\xi,A)\ d\xi = R(\mu,A).$$

$e^{-\mu t}S(t)$ is absolutely integrable and therefore integrable for any $\mu > 0$; see Theorem 2.4.3. Furthermore, $\frac{d}{d\mu}\big(e^{-\mu t}S(t)\big) = -te^{-\mu t}S(t)$,

$$\left\|\frac{d}{d\mu}e^{-\mu t}S(t)\right\| \leq tCe^{-\mu t}$$

and

$$\int_0^{+\infty} tCe^{-\mu t}\ dt = \frac{C}{\mu^2} < +\infty.$$

Hence, by Proposition 2.3.9

$$\frac{d}{d\mu}R(\mu,A) = \frac{d}{d\mu}\int_0^{+\infty} e^{-\mu t}S(t)\ dt = -\int_0^{+\infty} te^{-\mu t}S(t)\ dt.$$

Repeating this argument for $t^n e^{-\mu t}S(t)$ yields

$$\frac{d^n}{d\mu^n}R(\mu,A) = (-1)^n\int_0^{+\infty} t^n e^{-\mu t}S(t)\ dt.$$

for any $n \in \mathbb{N}$. By Proposition 1.2.4, b), we have

$$\frac{d^n}{d\mu^n}R(\mu,A) = (-1)^n n! R(\mu,A)^{n+1}$$

for $n \in \mathbb{N}$. Hence,

$$R(\mu,A)^n = -\frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\mu t}S(t)\ dt,$$

and in turn

$$\|R(\mu,A)^n\| \leq \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\mu t}\,\|S(t)\|\,dt \leq \frac{C}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\mu t}\ dt$$
$$= \frac{C}{(n-1)!}\cdot\frac{(n-1)!}{\mu^n} = \frac{C}{\mu^n}.$$

By Theorem 3.1.5, A generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ satisfying $\|T(t)\| \leq C$ for any $t \geq 0$. It remains to show $S(t) = T(t)$ for $t \geq 0$. To that end let $x \in D(A^2)$ and $t \in [a,b]$, $0 \leq a < b < +\infty$. By Theorem 3.2.9 we can write $T(t)x$ as

$$T(t)x = \frac{1}{2\pi i} \int_{\tilde{\gamma}} e^{\xi t} R(\xi, A)x \; d\xi,$$

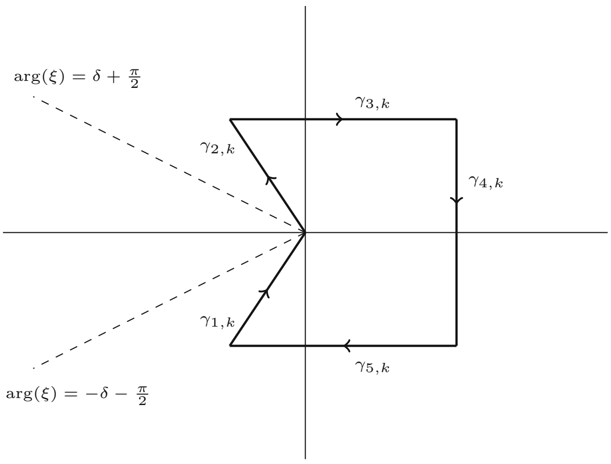
where $\tilde{\gamma} : \mathbb{R} \to \mathbb{C}$, $\tilde{\gamma}(s) := \tau + is$ for some $\tau > 0$. For $k > \tau$ we define

$$\begin{aligned}
&\gamma_{1,k} : [-k, 0] \to \mathbb{C}, \; \gamma_{1,k}(s) := -se^{-i\theta}, \\
&\gamma_{2,k} : [0, k] \to \mathbb{C}, \; \gamma_{2,k}(s) := se^{i\theta}, \\
&\gamma_{3,k} : [0, 1] \to \mathbb{C}, \; \gamma_{3,k}(s) := ke^{i\theta} + s\big(\tau - k\cos(\theta)\big), \\
&\gamma_{4,k} : [-k\sin(\theta), k\sin(\theta)] \to \mathbb{C}, \; \gamma_{4,k}(s) := \tau - is
\end{aligned}$$

and

$$\gamma_{5,k} : [0,1] \to \mathbb{C}, \; \gamma_{5,k}(s) := \tau - ik\sin(\theta) + s(k\cos(\theta) - \tau)$$

as well as $\gamma_k := \gamma_{1,k}\gamma_{2,k}\gamma_{3,k}\gamma_{4,k}\gamma_{5,k}$.



Since $\xi \mapsto e^{\xi t} R(\xi)x$ is analytic in the star-shaped set $\Sigma_{\delta + \frac{\pi}{2}} \cup U_\eta(0) \subseteq \rho(A)$ we can employ Proposition 2.2.3 and Theorem 2.2.5, so that

$$\int_{\gamma_k} e^{\xi t} R(\xi, A)x \; d\xi = 0.$$

We want to prove

$$\lim_{k \to +\infty} \int_{\gamma_{3,k}} e^{\xi t} R(\xi, A) \; d\xi = \lim_{k \to +\infty} \int_{\gamma_{5,k}} e^{\xi t} R(\xi, A) \; d\xi = 0.$$

To that end, note that $\cos(\theta) < 0$, $\sin(\theta) > 0$ and consider

$$\begin{aligned}
\left\| e^{\gamma_{3,k}(s)t} \gamma_{3,k}'(s) R(\gamma_{3,k}(s), A) \right\| &= \left\| e^{(ke^{i\theta} + s(\tau - k\cos(\theta)))t} \big(\tau - k\cos(\theta)\big) R(\gamma_{3,k}(s), A) \right\| \\
&\leq e^{(k\cos(\theta) + s(\tau - k\cos(\theta)))t} \big(\tau - k\cos(\theta)\big) \, \|R\gamma_{3,k}(s), A)\|
\end{aligned}$$

$$\le \frac{M_\theta e^{(k\cos(\theta)+s(\tau-k\cos(\theta)))t}\big(\tau-k\cos(\theta)\big)}{|\gamma_{3,k}(s)|}$$
$$\le \frac{M_\theta e^{(k\cos(\theta)+s(\tau-k\cos(\theta)))t}\big(\tau-k\cos(\theta)\big)}{|ke^{i\theta}+s\big(\tau-k\cos(\theta)\big)|}$$
$$\le \frac{M_\theta e^{(k\cos(\theta)+s(\tau-k\cos(\theta)))t}\big(\tau-k\cos(\theta)\big)}{k\sin(\theta)}$$

as well as

$$\int_0^1 e^{s(\tau-k\cos(\theta))t}\ ds = \frac{e^{(\tau-k\cos(\theta))t}-1}{\big(\tau-k\cos(\theta)\big)t}.$$

We obtain

$$\int_0^1 \frac{M_\theta e^{(k\cos(\theta)+s(\tau-k\cos(\theta)))t}\big(\tau-k\cos(\theta)\big)}{k\sin(\theta)}\ ds = \frac{M_\theta\big(e^{\tau t}-e^{k\cos(\theta)t}\big)}{k\sin(\theta)t} \xrightarrow{k\to+\infty} 0$$

and, in consequence,

$$\begin{aligned}\left\|\int_{\gamma_{3,k}} e^{\xi t}R(\xi,A)\ d\xi\right\| &= \left\|\int_0^1 e^{\gamma_{3,k}(s)t}\gamma_{3,k}'(s)R(\gamma_{3,k}(s),A)\ ds\right\| \\ &\le \int_0^1 \left\|e^{\gamma_{3,k}(s)t}\gamma_{3,k}'(s)R(\gamma_{3,k}(s),A)\right\| ds \\ &\le \int_0^1 \frac{M_\theta e^{(k\cos(\theta)+s(\tau-k\cos(\theta)))t}\big(\tau-k\cos(\theta)\big)}{k\sin(\theta)}\ ds \xrightarrow{k\to 0^+} 0.\end{aligned}$$

In a similar fashion it can be shown that

$$\lim_{k\to+\infty}\left\|\int_{\gamma_{5,k}} e^{\xi t}R(\xi,A)\ d\xi\right\| = 0.$$

Consequently,

$$\begin{aligned}S(t)x &= \frac{1}{2\pi i}\int_\gamma e^{\xi t}R(\xi,A)x\ d\xi = \frac{1}{2\pi i}\lim_{k\to+\infty}\int_{\gamma_{1,k}\gamma_{2,k}} e^{\xi t}R(\xi,A)x\ d\xi \\ &= \lim_{k\to+\infty}\frac{1}{2\pi i}\Big(\int_{\gamma_k} e^{\xi t}R(\xi,A)x\ d\xi - \int_{\gamma_{3,k}\gamma_{5,k}} e^{\xi t}R(\xi,A)x\ d\xi - \int_{\gamma_{4,k}} e^{\xi t}R(\xi,A)x\ d\xi\Big) \\ &= -\frac{1}{2\pi i}\lim_{k\to+\infty}\int_{\gamma_{4,k}} e^{\xi t}R(\xi,A)x\ d\xi = \frac{1}{2\pi i}\lim_{k\to+\infty}\int_{\tilde\gamma|_{[-k\sin(\theta),k\sin(\theta)]}} e^{\xi t}R(\xi,A)x\ d\xi \\ &= \frac{1}{2\pi i}\int_{\tilde\gamma} e^{\xi t}R(\xi,A)x\ d\xi = T(t)x\end{aligned}$$

for any $t>0$ and $x\in D(A^2)$. Because of $T(0)=I=S(0)$ and $\overline{D(A^2)}=X$ we finally have $T=S$.

$\square$

The goal of the present section is to find necessary and sufficient conditions under which a strongly continuous semigroup can be extended to an analytic semigroup; see Theorem 3.3.5. Before we can prove this result, we state and prove the following lemma regarding differentiable semigroups.

3.3.4 Lemma. Let $\big(T(t)\big)_{t\geq 0}$ a differentiable semigroup and A its generator. Then, $T:(0,+\infty)\to L_b(X)$ is infinitely many times differentiable and

$$T^{(n)}(t) = A^n T(t) = \big(AT(\tfrac{t}{n})\big)^n = \big(T'(\tfrac{t}{n})\big)^n. \tag{3.21}$$

Proof. First we will prove by induction that $A^nT(t)\in L_b(X)$ as well as ran $A^{n-1}T(t)\subseteq D(A)$ for every $t>0$ and that $A^{n-1}T:(0,+\infty)\to L_b(X)$ is continuous for all $n\in\mathbb{N}$. For $n=1$, because $\big(T(t)\big)_{t\geq 0}$ is differentiable,

$$\frac{1}{h}\big(T(h)T(t)x - T(t)x\big) = \frac{1}{h}\big(T(t+h)x - T(t)x\big) \xrightarrow{h\to 0^+} \big(T(\cdot)x\big)'(t).$$

Hence, $T(t)x\in D(A)$ for all $t>0$ and $x\in X$. Therefore, $AT(t)$ is defined on X and, because A is closed and $T(t)$ is bounded, by Proposition 1.2.2, c) belongs to $L_b(X)$. Let $x\in X$ and $s,t>0$ satisfying $s<t$ as well as $|s-t|\leq 1$. Since $t\mapsto T(t)x$ is continuous, A is closed and $T(t)x\in D(A)$ for every $t>0$, we can employ Proposition 3.1.2, e) and obtain

$$\begin{aligned}\|T(t)x - T(s)x\| &= \left\|A\Big(\int_s^t T(r)x\,dr\Big)\right\| = \left\|A\Big(\int_s^t T(s)T(r-s)x\,dr\Big)\right\| \\ &= \left\|AT(s)\Big(\int_s^t T(r-s)x\,dr\Big)\right\| \leq \sup_{r\in[0,1]}\|T(r)\|\,(t-s)\,\|AT(s)\|\,\|x\|,\end{aligned}$$

which implies

$$\|T(t)-T(s)\| \leq \sup_{r\in[0,1]}\|T(r)\|\,(t-s)\,\|AT(s)\|.$$

Since by Proposition 3.1.2, a) we have $\sup_{r\in[0,1]}\|T(r)\|<+\infty$, $T:(0,+\infty)\to L_b(X)$ is continuous.

Let $n\in\mathbb{N}$ and assume that $A^nT(t)\in L_b(X)$ and ran $A^{n-1}T(t)\subseteq D(A)$ for every $t>0$ as well as that $t\mapsto A^{n-1}T(t)$ is continuous. Let $0<s<t$, $x\in X$, and set $y:=A^nT(s)x$. From

$$\begin{aligned}\frac{1}{h}\big(T(h)A^nT(t)x - A^nT(t)x\big) &= \frac{1}{h}\big(T(h)T(t-s)A^nT(s)x - T(t-s)A^nT(s)x\big) \\ &= \frac{1}{h}\big(T(t-s+h)y - T(t-s)y\big) \xrightarrow{h\to 0^+} \big(T(\cdot)y\big)'(t-s)\end{aligned}$$

we conclude $A^nT(t)x\in D(A)$. Since $x\in X$ was arbitrary, ran $A^nT(t)\subseteq D(A)$. Again by Proposition 1.2.2, c), the operator $A^{n+1}T(t) = A\big(A^nT(t)\big)$ is bounded. Since for $0<s<t<s+1$

$$\|A^nT(t)x - A^nT(s)x\| = \big\|A^n\big(T(t)x - T(s)x\big)\big\| = \left\|A^{n+1}\Big(\int_s^t T(r)x\,dr\Big)\right\|$$

$$
\begin{aligned}
&= \left\| A^{n+1}\Big(\int_s^t T(s)T(r-s)x \, dr \Big) \right\| \\
&= \left\| A^{n+1}T(s)\Big(\int_s^t T(r-s)x \, dr \Big) \right\| \\
&\leq \sup_{r\in[0,1]} \|T(r)\| \, (t-s) \left\| A^{n+1}T(s) \right\| \|x\| ,
\end{aligned}
$$

$(0,+\infty) \ni t \mapsto A^n T(t) \in L_b(X)$ is continuous.

The proof of (3.21) uses induction. Let $0 < s < t_0$. Since $t \mapsto AT(t)$ is continuous at t_0, the mapping $F : [s,+\infty) \to L_b(X)$, defined by

$$F(t) := \int_s^t AT(r) \, dr + T(s),$$

is differentiable at t_0 and $F'(t_0) = AT(t_0)$; see Proposition 2.1.3, f). From the continuity of $t \mapsto T(t)$ and $t \mapsto AT(t)$ on $[s,+\infty)$ we conclude that both maps are Riemann integrable. Hence, we can employ Corollary 2.3.13 and Proposition 3.1.2, e) and obtain

$$T(t)x = A\Big(\int_s^t T(r)x \, dr \Big) + T(s)x = \int_s^t AT(r)x \, dr + T(s)x = F(t)x$$

for any $x \in X$, which implies $F(t) = T(t)$ for every $t > 0$ and that $t \mapsto T(t)$ is differentiable at t_0 satisfying $T'(t_0) = F'(t_0) = AT(t_0)$. Since $t_0 > 0$ was arbitrary, T is differentiable on $(0,+\infty)$. Let $n \in \mathbb{N}$ and assume that (3.21) holds true. Let $0 < s < t_0$. Since $t \mapsto T(t)$ and $t \mapsto A^{n+1}T(t)$ are Riemann integrable, see Proposition 2.1.3, d), we can employ Corollary 2.3.13 and Proposition 3.1.2, e), and obtain

$$T^{(n)}(t) = A^n T(t) = A^{n+1}\Big(\int_s^t T(r) \, dr \Big) + A^n T(s) = \int_s^t A^{n+1} T(r) \, dr + A^n T(s).$$

By Proposition 2.1.3, f), we obtain that $t \mapsto T^{(n)}(t)$ is differentiable at t_0 satisfying $T^{(n+1)}(t_0) = (T^{(n)})'(t_0) = A^{n+1}T(t_0)$. Since $t_0 > 0$ was arbitrary, we have $T^{(n+1)} = A^n T$. Lastly, by noting that A and $T(t)$ commute and using the semigroup property of T, we see that

$$A^n T(t) = \big(AT(\tfrac{t}{n}) \big)^n = \big(T'(\tfrac{t}{n}) \big)^n .$$

□

Now we are able to prove the central result of the present section.

3.3.5 Theorem. *If $\big(T(t)\big)_{t\geq 0}$ is a strongly continuous semigroup and A is its generator, then the following statements are equivalent.*

a) There exists an angle $\varphi \in (0, \frac{\pi}{2})$, such that $\big(T(t)\big)_{t\geq 0}$ can be extended to a bounded analytic semigroup of angle φ.

b) $\big(T(t)\big)_{t\geq 0}$ is bounded and there exists a constant $C > 0$, such that for all $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > 0$ and $\mathrm{Im}(\xi) \neq 0$ the resolvent satisfies

$$\|R(\xi, A)\| \leq \frac{C}{|\,\mathrm{Im}(\xi)|}.$$

c) $\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) > 0\} \subseteq \rho(A)$ *and there exists a constant* $C > 0$ *such that for all* $\xi \in \mathbb{C}$ *with* $\mathrm{Re}(\xi) > 0$ *the resolvent satisfies*

$$\|R(\xi, A)\| \leq \frac{C}{|\xi|}.$$

d) A *is sectorial.*

e) $\big(T(t)\big)_{t\geq 0}$ *is a differentiable semigroup and there exists a constant* C *such that*

$$\|AT(t)\| \leq \frac{C}{t} \tag{3.22}$$

for all $t > 0$.

Proof. a) $\Rightarrow$ *b).* Fix $\theta \in (0, \varphi)$ and $a > 0$. Let M, ω as in Proposition 3.1.2, a). Since $\big(T(t)\big)_{t\geq 0}$ is bounded, we have $\omega = 0$ and according to Proposition 3.1.2, h), in turn, $\{\xi \in \mathbb{C} : \mathrm{Re}(\xi) > 0\} \subseteq \rho(A)$. By Proposition 3.1.2, h)

$$R(\xi, A)x = \int_0^{+\infty} e^{-\xi t} T(t)x \; dt$$

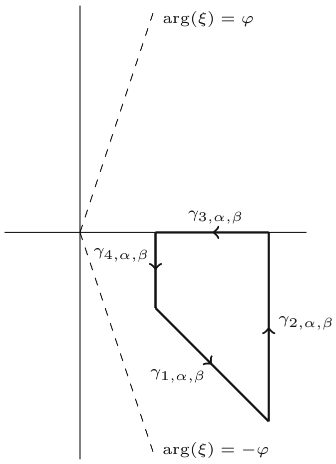
for any $x \in X$ and $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > 0$. Let $0 < \alpha < \beta$ and define the paths

$$\gamma_{1,\alpha,\beta} : [\alpha, \beta] \to \mathbb{C}, \; \gamma_{1,\alpha,\beta}(s) := se^{-i\theta},$$
$$\gamma_{2,\alpha,\beta} : [0, \beta \sin(\theta)] \to \mathbb{C}, \; \gamma_{2,\alpha,\beta}(s) := \beta e^{-i\theta} + is,$$
$$\gamma_{3,\alpha,\beta} : [-\beta \cos(\theta), -\alpha \cos(\theta)] \to \mathbb{C}, \; \gamma_{3,\alpha,\beta}(s) := -s$$

and

$$\gamma_{4,\alpha,\beta} : [-\alpha \sin(\theta), 0] \to \mathbb{C}, \; \gamma_{4,\alpha,\beta}(s) := \alpha e^{-i\theta} - is$$

as well as $\gamma_{\alpha,\beta} := \gamma_{1,\alpha,\beta}\gamma_{2,\alpha,\beta}\gamma_{3,\alpha,\beta}\gamma_{4,\alpha,\beta}$.



Since $z \mapsto e^{-\xi z}T(z)x$ is analytic in the star-shaped set $\Sigma_\varphi \supseteq \operatorname{ran}(\gamma_{\alpha,\beta})$, we can apply Proposition 2.2.3 and Theorem 2.2.5 and obtain

$$\int_{\gamma_{\alpha,\beta}} e^{-\xi z}T(z)x \; dz = 0.$$

Since $\big(T(z)\big)_{z\in\Sigma_\varphi^0}$ is bounded, we have $\|T(z)\| \leq M_\theta$ for some $M_\theta > 0$ and all $z \in \overline{\Sigma_\theta} \supseteq \operatorname{ran}(\gamma_{\alpha,\beta})$. We set $a := \operatorname{Re}(\xi)(>0)$, $b := \operatorname{Im}(\xi)$ and assume first that $b > 0$. From $\cos(\theta), \sin(\theta) > 0$ we conclude

$$\begin{aligned}\left\|e^{-\xi\gamma_{2,\alpha,\beta}(s)}\gamma_{2,\alpha,\beta}'(s)T\big(\gamma_{2,\alpha,\beta}(s)\big)x\right\| &= \left\|ie^{-\xi(\beta e^{-i\theta}+is)}T(\beta e^{-i\theta}+is)x\right\| \\ &\leq e^{-(a\beta\cos(\theta)+b\beta\sin(\theta)-bs)}\left\|T(\beta e^{-i\theta}+is)\right\|\|x\| \\ &\leq M_\theta e^{-(a\beta\cos(\theta)+b\beta\sin(\theta)-bs)}\|x\|\end{aligned}$$

and

$$\begin{aligned}\int_0^{\beta\sin(\theta)} M_\theta e^{-(a\beta\cos(\theta)+b\beta\sin(\theta)-bs)}\|x\| \; ds &= M_\theta e^{-(a\beta\cos(\theta)+b\beta\sin(\theta))}\|x\| \int_0^{\beta\sin(\theta)} e^{bs}\; ds \\ &= M_\theta e^{-(a\beta\cos(\theta)+b\beta\sin(\theta))}\|x\| \left(\frac{e^{b\beta\sin(\theta)}-1}{b}\right) \\ &= \frac{M_\theta\|x\|}{b}\big(e^{-a\beta\cos(\theta)} - e^{-(a\beta\cos(\theta)+b\beta\cos(\theta))}\big) \\ &\xrightarrow{\beta\to+\infty} 0,\end{aligned}$$

which implies

$$\lim_{\beta\to+\infty}\int_{\gamma_{2,\alpha,\beta}} e^{-\xi z}T(z)x \; dz = 0.$$

From

$$\begin{aligned}\left\|e^{-\xi\gamma_{4,\alpha,\beta}(s)}\gamma_{4,\alpha,\beta}'(s)T\big(\gamma_{4,\alpha,\beta}(s)\big)x\right\| &= \left\|-ie^{-\xi(\alpha e^{-i\theta}-is)}T(\alpha e^{-i\theta}-is)x\right\| \\ &\leq e^{-(a\alpha\cos(\theta)+b\alpha\sin(\theta)+bs)}\left\|T(\alpha e^{-i\theta}-is)\right\|\|x\| \\ &\leq M_\theta e^{-(a\alpha\cos(\theta)+b\alpha\sin(\theta)+bs)}\|x\|\end{aligned}$$

we infer

$$\begin{aligned}\left\|\int_{\gamma_{4,\alpha,\beta}} e^{-\xi z}T(z)\; dz\right\| &\leq \int_{-\alpha\sin(\theta)}^{0} M_\theta e^{-(a\alpha\cos(\theta)+b\alpha\sin(\theta)+bs)}\|x\| \; ds \\ &= M_\theta e^{-(a\alpha\cos(\theta)+b\alpha\sin(\theta))}\|x\| \int_{-\alpha\sin(\theta)}^{0} e^{-bs}\; ds \\ &= M_\theta e^{-(a\alpha\cos(\theta)+b\alpha\sin(\theta))}\|x\| \left(\frac{e^{b\alpha\sin(\theta)}-1}{b}\right) \\ &= \frac{M_\theta\|x\|}{b}\big(e^{-a\alpha\cos(\theta)} - e^{-(a\alpha\cos(\theta)+b\alpha\cos(\theta))}\big) \\ &\xrightarrow{\alpha\to0^+} 0.\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\gamma_{1,\alpha,\beta}} e^{-\xi z} T(z)x \, dz &= \int_{\gamma_{\alpha,\beta}} e^{-\xi z} T(z)x \, dz - \int_{\gamma_{2,\alpha,\beta}\gamma_{3,\alpha,\beta}\gamma_{4,\alpha,\beta}} e^{-\xi z} T(z)x \, dz \\
&= -\int_{\gamma_{2,\alpha,\beta}\gamma_{4,\alpha,\beta}} e^{-\xi z} T(z)x \, dz - \int_{\gamma_{3,\alpha,\beta}} e^{-\xi z} T(z)x \, dz \\
&\xrightarrow[\alpha \to 0^+]{\beta \to +\infty} \int_0^{+\infty} e^{-\xi t} T(t)x \, dt = R(\xi, A)x
\end{aligned}$$

and we conclude that $z \mapsto e^{-\xi z} T(z)x$ is integrable along $\gamma : [0, +\infty) \to \mathbb{C}$, $\gamma(s) := se^{-i\theta}$, so that

$$R(\xi, A)x = \int_\gamma e^{-\xi z} T(z)x \, dz.$$

As $a\cos(\theta) + b\sin(\theta) > 0$

$$\begin{aligned}
\|R(\xi, A)x\| &= \left\| \int_\gamma e^{-\xi z} T(z)x \, dz \right\| = \left\| \int_0^\infty e^{-\xi s e^{-i\theta}} T(se^{-i\theta}) \, ds \right\| \\
&\leq \int_0^\infty e^{-s(a\cos(\theta) + b\sin(\theta))} \left\| T(se^{-i\theta})x \right\| \, ds \\
&\leq M_\theta \|x\| \int_0^\infty e^{-s(a\cos(\theta) + b\sin(\theta))} \, ds \\
&= \frac{M_\theta \|x\|}{a\cos(\theta) + b\sin(\theta)}.
\end{aligned}$$

With $C := \frac{M_\theta}{\sin(\theta)} > 0$ we obtain from $a\cos(\theta) > 0$

$$\|R(\xi, A)x\| \leq \frac{M_\theta \|x\|}{a\cos(\theta) + b\sin(\theta)} \leq \frac{C}{b} \|x\| = \frac{C}{|\operatorname{Im}(\xi)|} \|x\|.$$

By analogous computation and integrating over the ray $se^{i\theta}$, $s > 0$, also in the case $\operatorname{Im}(\xi) = b < 0$ we obtain

$$\|R(\xi, A)x\| \leq \frac{C}{|\operatorname{Im}(\xi)|} \|x\|.$$

Clearly, if $\big(T(z)\big)_{z \in \Sigma_\varphi}$ is bounded, also $\big(T(t)\big)_{t \geq 0}$ is.

b) $\Rightarrow$ *c)*. Let $C > 0$ be as in *b)* and note that since $\|T(t)\| \leq M$ for some $M > 0$ and all $t \geq 0$, Theorem 3.1.5 yields

$$\|R(\xi, A)\| \leq \frac{M}{\operatorname{Re}(\xi)}$$

for every $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) > 0$. Defining $K := 2\max\{C, M\}$ leads to

$$\frac{|\xi|}{K} \leq \frac{\operatorname{Re}(\xi)}{2M} + \frac{|\operatorname{Im}(\xi)|}{2C} \leq \frac{1}{\|R(\xi, A)\|}$$

for all $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) > 0$.

c) ⇒ d). Let $C > 0$ be as in *c*), $q \in (0,1)$ and define $\delta := \arctan(\frac{q}{C}) \in (0, \frac{\pi}{2})$. We want to prove $\Sigma_{\frac{\pi}{2}+\delta} \subseteq \rho(A)$ and $\|R(\xi, A)\| \leq \frac{D}{|\xi|}$ for all $\xi \in \Sigma_{\frac{\pi}{2}+\delta}$ and some $D > 0$. Let $\xi = a + ib \in \Sigma_{\frac{\pi}{2}+\delta}$. Since the right half plane is by assumption contained in $\rho(A)$, we have $\xi \in \rho(A)$ and

$$\|R(\xi, A)\| \leq \frac{C}{|\xi|}$$

in the case $a > 0$. For $a < 0$ we have $b \neq 0$ and according to (3.20)

$$|\arg(\xi)| = \pi - \arctan\big(\tfrac{|b|}{|a|}\big),$$

which implies

$$-\arctan\big(\tfrac{|b|}{|a|}\big) < \delta - \frac{\pi}{2}.$$

We derive

$$-\frac{|b|}{|a|} < \tan\big(\delta - \tfrac{\pi}{2}\big) = -\cot(\delta) = -\frac{C}{q},$$

and $\frac{|a|}{|b|} < \frac{q}{C}$. This inequality also holds true if $a = 0$. For $\varepsilon \in (0, \frac{q|b|}{C} - |a|)$ we define $\mu := \varepsilon + ib \in \rho(A)$. By Proposition 1.2.4, *a*)

$$|\xi - \mu| = |a - \varepsilon| = |b|\frac{|a-\varepsilon|}{|b|} \leq |b|\Big(\frac{|a|}{|b|} + \frac{\varepsilon}{|b|}\Big) < \frac{q|b|}{C} \leq \frac{q|\mu|}{C} \leq \frac{q}{\|R(\mu, A)\|}$$

implies $\xi \in \rho(A)$ and

$$R(\xi, A) = \sum_{n=0}^{\infty} (\mu - \xi)^n R(\mu, A)^{n+1}.$$

Consequently,

$$\|R(\xi, A)\| \leq \sum_{n=0}^{\infty} |\xi - \mu|^n \, \|R(\mu, A)\|^{n+1} \leq \|R(\mu, A)\| \sum_{n=0}^{\infty} q^n \leq \frac{C}{(1-q)|\mu|} \leq \frac{C}{(1-q)|b|}.$$

Since

$$C^2|\xi|^2 = C^2a^2 + C^2b^2 < q^2b^2 + C^2b^2 = (q^2 + C^2)b^2$$

we have

$$\|R(\xi, A)\| \leq \frac{C}{(1-q)|b|} < \frac{\sqrt{q^2 + C^2}}{(1-q)|\xi|},$$

which yields the sectoriality of A.

d) ⇒ e). Let $\varepsilon > 0$ and define $S(t) := e^{-\varepsilon t}T(t)$. By Proposition 3.1.2, *g*), $\big(S(t)\big)_{t\geq 0}$ is a strongly continuous semigroup generated by $B := A - \varepsilon I$ and, in turn, $R(\xi, B) = R(\xi + \varepsilon, A)$ for any $\xi \in \rho(B) = \{\mu - \varepsilon : \mu \in \rho(A)\}$. We want to show that B is sectorial. Clearly, $\overline{D(B)} = \overline{D(A)} = X$. For $\Sigma_{\frac{\pi}{2}+\delta} \subseteq \rho(B) = \rho(A) - \varepsilon$ or equivalently $\varepsilon + \Sigma_{\frac{\pi}{2}+\delta} \subseteq \rho(A)$ it is enough to verify $\varepsilon + \Sigma_{\frac{\pi}{2}+\delta} \subseteq \Sigma_{\frac{\pi}{2}+\delta}$. Let $\xi := a + ib \in \Sigma_{\frac{\pi}{2}+\delta}$. We consider three distinct cases.
Case $a + \varepsilon < 0$:

$$|\arg(\xi + \varepsilon)| = \pi - \arctan\big(\tfrac{|b|}{|a+\varepsilon|}\big) = \pi + \arctan\big(\tfrac{|b|}{a+\varepsilon}\big)$$

$$< \pi + \arctan(\tfrac{|b|}{a}) = |\arg(\xi)| < \frac{\pi}{2} + \delta.$$

Case $a + \varepsilon = 0$, which is only possible for $b \neq 0$:

$$|\arg(\xi + \varepsilon)| = \frac{\pi}{2} < \frac{\pi}{2} + \delta.$$

Case $a + \varepsilon > 0$:

$$|\arg(\xi + \varepsilon)| = \arctan(\tfrac{|b|}{|a+\varepsilon|}) < \frac{\pi}{2} < \frac{\pi}{2} + \delta.$$

Let $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta)$ and $\xi \in \Sigma_\theta$. Since $\mu \mapsto |\arg(\mu)|$ is continuous on $\{\mu \in \mathbb{C} : \mathrm{Re}(\mu) < 0\}$ we have $|\arg(z)| > \frac{\pi}{2} + \delta$ for all $z \in U_r(-\varepsilon)$ and some sufficiently small $r > 0$. Because of $\xi \in \Sigma_\theta \subseteq U_r(-\varepsilon)^c$ we obtain

$$\frac{|\xi|}{|\xi + \varepsilon|} \leq \frac{|\xi + \varepsilon| + \varepsilon}{|\xi + \varepsilon|} = 1 + \frac{\varepsilon}{|\xi + \varepsilon|} \leq 1 + \frac{\varepsilon}{r}.$$

Since A is sectorial and $\xi + \varepsilon \in \Sigma_\theta$, there exists a constant $M_\theta > 0$, which does not depend on $\xi + \varepsilon$, such that

$$\|R(\xi + \varepsilon, A)\| \leq \frac{M_\theta}{|\xi + \varepsilon|}.$$

From

$$\|R(\xi, B)\| = \|R(\xi + \varepsilon, A)\| \leq \frac{M_\theta}{|\xi + \varepsilon|} = \frac{|\xi|}{|\xi + \varepsilon|} \cdot \frac{M_\theta}{|\xi|} \leq \frac{M_\theta(1 + \frac{\varepsilon}{r})}{|\xi|}$$

we conclude that B is sectorial. Furthermore, $\varepsilon \in \Sigma_{\frac{\pi}{2}+\delta} \subseteq \rho(A)$ yields

$$0 = \varepsilon - \varepsilon \in \{\mu - \varepsilon : \mu \in \rho(A)\} = \rho(B).$$

Hence, we can apply Theorem 3.3.3 and obtain

$$S(t) = \frac{1}{2\pi i} \int_\gamma e^{\xi t} R(\xi, B)\, d\xi$$

for any $t > 0$, where $\gamma(s) := -se^{-i\theta}$ for $s \leq 0$ and $\gamma(s) := se^{i\theta}$ for $s > 0$ with some $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta)$. Let $x \in X$ and define $f(t, s) := e^{\gamma(s)t}\gamma'(s)R(\gamma(s), A)x$. Because B is sectorial, there exists a constant $M_\theta > 0$ such that

$$\|R(\xi, B)\| \leq \frac{M_\theta}{|\xi|}$$

for any $\xi \in \Sigma_\theta$. For $s \leq -1$ we have

$$\begin{aligned}\|f(t, s)\| &= \left\| e^{-se^{-i\theta}t}(-e^{-i\theta})R(-se^{-i\theta}, B)x \right\| = e^{-s\cos(\theta)t} \left\| R(-se^{-i\theta}, B)x \right\| \\ &\leq e^{-s\cos(\theta)t} \frac{M_\theta \|x\|}{|s|} \leq M_\theta e^{-s\cos(\theta)t} \|x\|,\end{aligned}$$

where because of $\cos(\theta) < 0$

$$\int_{-\infty}^{-1} M_\theta e^{-s\cos(\theta)t} \|x\|\ ds = M_\theta \|x\| \int_{-\infty}^{-1} e^{-s\cos(\theta)t}\ ds = \frac{M_\theta e^{\cos(\theta)t} \|x\|}{|\cos(\theta)|t}.$$

In consequence $s \mapsto f(t,s)$ is absolutely Riemann integrable over $(-\infty,-1)$, and in turn integrable; see Theorem 2.4.3. Analogous arguments lead to the integrability of $s \mapsto f(t,s)$ over $(1,+\infty)$. Since $s \mapsto \|f(t,s)\|$ is continuous, it is integrable over $[-1,1]$. We conclude that $s \mapsto f(t,s)$ is integrable over $\mathbb{R}$ for every $t>0$. Furthermore, $\frac{d}{dt}f(t,s) = \gamma(s)e^{\gamma(s)t}\gamma'(s)R(\gamma(s),B)x$. For $t_0>0$, $0<\varepsilon<t_0$, $t\in(t_0-\varepsilon,t_0+\varepsilon)$ and $s<0$ we estimate

$$\begin{aligned}\left\|\frac{d}{dt}f(t,s)\right\| &= \left\|(-se^{-i\theta})e^{-se^{-i\theta}t}(-e^{-i\theta})R(-se^{-i\theta},B)x\right\| \\ &= |s|e^{-s\cos(\theta)t}\left\|R(-se^{-i\theta},B)\right\|\|x\| \\ &\leq |s|e^{-s\cos(\theta)t}\frac{M_\theta\|x\|}{|s|} = M_\theta e^{-s\cos(\theta)t}\|x\| \\ &\leq M_\theta e^{-s\cos(\theta)(t_0-\varepsilon)}\|x\|.\end{aligned}$$

Note that the last term does not depend on t and

$$\int_{-\infty}^0 M_\theta e^{-s\cos(\theta)(t_0-\varepsilon)}\|x\|\ ds = M_\theta\|x\|\int_{-\infty}^0 e^{-s\cos(\theta)(t_0-\varepsilon)}\ ds = \frac{M_\theta\|x\|}{|\cos(\theta)|(t_0-\varepsilon)} < +\infty.$$

We obtain a similar result for $s>0$. Hence, by Proposition 2.3.9

$$\begin{aligned}S(t)x &= \frac{1}{2\pi i}\int_\gamma e^{\xi t}R(\xi,B)x\ d\xi = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} e^{\gamma(s)t}\gamma'(s)R(\gamma(s),B)x\ ds \\ &= \frac{1}{2\pi i}\int_{\mathbb{R}} e^{\gamma(s)t}\gamma'(s)R(\gamma(s),B)x\ d\lambda(s)\end{aligned}$$

is differentiable at t_0 and

$$\big(S(\cdot)x\big)'(t_0) = \frac{1}{2\pi i}\int_{\mathbb{R}}\gamma(s)e^{\gamma(s)t}\gamma'(s)R(\gamma(s),B)x\ d\lambda(s) = \frac{1}{2\pi i}\int_\gamma \xi e^{\xi t}R(\xi,B)x\ d\xi.$$

Consequently, also $t\mapsto T(t)x = e^{\varepsilon t}S(t)x$ is differentiable in t_0 satisfying

$$\big(T(\cdot)x\big)'(t_0) = \varepsilon e^{\varepsilon t_0}S(t_0)x + e^{\varepsilon t_0}\big(S(\cdot)x\big)'(t_0)$$

Since $t_0>0$ and $x\in X$ were arbitrary, $\big(S(t)\big)_{t\geq 0}$ and $\big(T(t)\big)_{t\geq 0}$ are differentiable semigroups. We employ Lemma 3.3.4 and obtain

$$\begin{aligned}\|BS(t)x\| = \|S'(t)x\| = \left\|\big(S(\cdot)x\big)'(t)\right\| &= \left\|\frac{1}{2\pi i}\int_\gamma \xi e^{\xi t}R(\xi,B)x\ d\xi\right\| \\ &= \frac{1}{2\pi}\left\|\int_{-\infty}^{+\infty}\gamma(s)e^{\gamma(s)t}\gamma'(s)R(\gamma(s),B)x\ ds\right\| \\ &\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\gamma(s)e^{\gamma(s)t}\gamma'(s)|\,\|R(\gamma(s),B)\|\,\|x\|\ ds \\ &\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\gamma(s)e^{\gamma(s)t}\gamma'(s)|\frac{M_\theta\|x\|}{|\gamma(s)|}\ ds \\ &\leq \frac{M_\theta\|x\|}{2\pi}\Big(\int_{-\infty}^0 e^{-s\cos(\theta)t}\ ds + \int_0^{+\infty} e^{s\cos(\theta)t}\ ds\Big) = \frac{M_\theta\|x\|}{\pi|\cos(\theta)|t}.\end{aligned}$$

Setting $C := \dfrac{M_\theta}{\pi\cos(\theta)}$ we conclude

$$\|BS(t)\| \leq \frac{C}{t}.$$

Note that C does not depend on ε. Since B is sectorial, $\big(S(t)\big)_{t\geq 0}$ is bounded according to Theorem 3.3.3, i.e. $\|S(t)\| \leq M$ for all $t \geq 0$ and some $M > 0$. In consequence,

$$\|AT(t)\| = \left\|e^{\varepsilon t}(B+\varepsilon I)S(t)\right\| \leq e^{\varepsilon t}\|BS(t)\| + \varepsilon e^{\varepsilon t}\|S(t)\| \leq e^{\varepsilon t}\frac{C}{t} + \varepsilon e^{\varepsilon t}M.$$

Since $\varepsilon > 0$ was arbitrary, we have

$$\|AT(t)\| \leq \frac{C}{t}.$$

e) $\Rightarrow$ *a)*. Clearly, in (3.22) we may assume that $C > e^{-1}$. According to Lemma 3.3.4, $T : (0, +\infty) \to L_b(X)$ is infinitely often differentiable and satisfies

$$\left\|T^{(n)}(t)\right\| = \left\|\big(AT(\tfrac{t}{n})\big)^n\right\| \leq \left\|\big(AT(\tfrac{t}{n})\big)\right\|^n \leq \frac{C^n n^n}{t^n} \leq \frac{C^n e^n n!}{t^n} \tag{3.23}$$

for any $t > 0$. Here we used the fact that $n^n \leq e^n n!$ for all $n \in \mathbb{N}$. Hence, given $z \in \mathbb{C}$,

$$\frac{\left\|T^{(n)}(t)\right\| |z-t|^n}{n!} \leq \frac{C^n e^n n! |z-t|^n}{t^n n!} = \left(\frac{Ce|z-t|}{t}\right)^n$$

implying absolute convergence of

$$\left(\sum_{n=0}^{N} \frac{(z-t)^n}{n!} T^{(n)}(t)\right)_{N\in\mathbb{N}}$$

for $|z-t| < \frac{t}{Ce}$. Due to Lemma 1.1.5 for $t > 0$ the function $S_t : U^{\mathbb{C}}_{\frac{t}{Ce}}(t) \to L_b(X)$ defined by

$$S_t(z) := \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} T^{(n)}(t)$$

is analytic. Given $t > 0$ and $p \in (0,1)$, we want to prove that $S_t(s) = T(s)$ for any $s \in [t, t + \frac{pt}{Ce}]$. To that end, consider the remainder of the Taylor polynomial and observe

$$\begin{aligned}
\left\|\int_t^s \frac{(s-r)^n}{n!} T^{(n+1)}(r)\, dr\right\| &\leq \int_t^s \frac{(s-r)^n}{n!} \left\|T^{(n+1)}(r)\right\| dr \\
&\leq \frac{(s-t)^n}{n!} \int_t^s \left\|T^{(n+1)}(r)\right\| dr \\
&= \frac{(s-t)^n}{n!} \int_t^s \frac{C^{n+1} e^{n+1} (n+1)!}{r^{n+1}}\, dr \\
&\leq \frac{(s-t)^n C^{n+1} e^{n+1} (n+1)!}{n!} \int_t^s \frac{1}{t^{n+1}}\, dr
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(s-t)^{n+1}C^{n+1}e^{n+1}(n+1)}{t^{n+1}} \\
&\le \frac{p^{n+1}t^{n+1}C^{n+1}e^{n+1}(n+1)}{t^{n+1}C^{n+1}e^{n+1}} \\
&= (n+1)p^{n+1} \xrightarrow{n\to+\infty} 0.
\end{aligned}
$$

By Taylor's Theorem for Banach space-valued functions (see for example Fakta 9.3.17 in [17])

$$
T(s) = \sum_{n=0}^{\infty} \frac{(s-t)^n}{n!} T^{(n)}(t) = S_t(s), \;\; s \in [t, \tfrac{pt}{Ce}]. \tag{3.24}
$$

For $0 < t < t'$ satisfying $G := U^{\mathbb{C}}_{\frac{t}{eC}}(t) \cap U^{\mathbb{C}}_{\frac{t'}{eC}}(t') \neq \emptyset$ we have $t' - \frac{t'}{Ce} < t + \frac{t}{Ce}$. We want to show that $S_t(z) = S_{t'}(z)$ for $z \in G$. For that aim define the sequence $(t_n)_{n\in\mathbb{N}}$ recursively by $t_0 := t$ and

$$
t_n := \frac{1}{2}\Big(t' - \frac{t'}{Ce} + t_{n-1} + \frac{t_{n-1}}{Ce}\Big), \;\; n \ge 1.
$$

In order to show that $(t_n)_{n\in\mathbb{N}}$ is monotone and bounded, we observe that $t_0 = t < t'$ and

$$
\begin{aligned}
t_1 &= \frac{1}{2}\Big(t' - \frac{t'}{Ce} + t + \frac{t}{Ce}\Big) = \frac{1}{2}\big((1-\tfrac{1}{Ce})t' - (1-\tfrac{1}{Ce})t + 2t\big) \\
&= \frac{1}{2}(1-\tfrac{1}{Ce})(t'-t) + t > t = t_0.
\end{aligned}
$$

Assuming $t_n < t'$ and $t_{n-1} < t_n$ we obtain

$$
t_{n+1} = \frac{1}{2}\Big(t' - \frac{t'}{Ce} + t_n + \frac{t_n}{Ce}\Big) < \frac{1}{2}\Big(t' - \frac{t'}{Ce} + t' + \frac{t'}{Ce}\Big) = t'
$$

and

$$
t_n = \frac{1}{2}\Big(t' - \frac{t'}{Ce} + t_{n-1} + \frac{t_{n-1}}{Ce}\Big) < \frac{1}{2}\Big(t' - \frac{t'}{Ce} + t_n + \frac{t_n}{Ce}\Big) = t_{n+1}.
$$

The limit $\lim_{n\to+\infty} t_n = \tilde{t}$ satisfies

$$
\tilde{t} = \frac{1}{2}\Big(t' - \frac{t'}{Ce} + \tilde{t} + \frac{\tilde{t}}{Ce}\Big),
$$

which is possible only for $\tilde{t} = t'$. As $t_n + \frac{t_n}{Ce} \xrightarrow{n\to+\infty} t' + \frac{t'}{Ce} > t'$ there exists a $N \in \mathbb{N}$ such that $t_N + \frac{t_N}{Ce} > t'$ and hence $t' \in U_{\frac{t_N}{Ce}}(t_N)$. From

$$
\begin{aligned}
t_n - t_{n-1} &= \frac{1}{2}(t' - \tfrac{t'}{Ce} + t_{n-1} + \tfrac{t_{n-1}}{Ce}) - t_{n-1} \\
&= \frac{1}{2}(t' - \tfrac{t'}{Ce} + t_{n-1} + \tfrac{t_{n-1}}{Ce}) - (1+\tfrac{1}{Ce})t_{n-1} + \tfrac{t_{n-1}}{Ce} \\
&= \frac{1}{2}(t' - \tfrac{t'}{Ce}) - \frac{1}{2}(1+\tfrac{1}{Ce})t_{n-1} + \tfrac{t_{n-1}}{Ce} \\
&< \frac{1}{2}(t' - \tfrac{t'}{Ce}) - \frac{1}{2}(1+\tfrac{1}{Ce})t + \tfrac{t_{n-1}}{Ce}
\end{aligned}
$$

$$= \frac{1}{2}\big(t' - \tfrac{t'}{Ce} - (t + \tfrac{t}{Ce})\big) + \tfrac{t_{n-1}}{Ce} < \tfrac{t_{n-1}}{Ce}$$

we derive $t_n \in U_{\frac{t_{n-1}}{Ce}}(t_{n-1}) \cap U_{\frac{t_n}{Ce}}(t_n)$. Let $s \in [t_n, t_{n-1} + \frac{t_{n-1}}{Ce}) = [t_{n-1}, t_{n-1} + \frac{t_{n-1}}{Ce}) \cap [t_n, t_n + \frac{t_n}{Ce})$ and choose $p \in (0,1)$ such that $s \in [t_{n-1}, t_{n-1} + \frac{pt_{n-1}}{Ce}] \cap [t_n, t_n + \frac{pt_n}{Ce}]$. According to (3.24) we have

$$S_{t_{n-1}}(s) = T(s) = S_{t_n}(s).$$

Since $[t_n, t_{n-1} + \frac{t_{n-1}}{Ce})$ clearly has an accumulation point in $U_{\frac{t_{n-1}}{Ce}}(t_{n-1}) \cap U_{\frac{t_n}{Ce}}(t_n)$, we can employ Proposition 1.1.6 to see that $S_{t_{n-1}}(z) = S_{t_n}(z)$ for every $z \in U_{\frac{t_{n-1}}{Ce}}(t_{n-1}) \cap U_{\frac{t_n}{Ce}}(t_n)$. By the same argument we derive from $[t', t_N + \frac{t_N}{Ce}) \neq \emptyset$ that S_{t_N} and $S_{t'}$ coincide on $U_{\frac{t_N}{Ce}}(t_N) \cap U_{\frac{t'}{Ce}}(t')$. Hence, by $S_t^{t'}(s) := S_{t_n}(s)$ for $s \in U_{\frac{t_n}{Ce}}(t_n)$, $n = 0, \dots N$, as well as $S_t^{t'}(s) = S_{t'}(s)$ for $s \in U_{\frac{t'}{Ce}}(t')$, we get a well-defined and analytic mapping

$$S_t^{t'} : \bigcup_{n=0}^{N} U_{\frac{t_n}{Ce}}(t_n) \cup U_{\frac{t'}{Ce}}(t') \to L_b(X).$$

Consequently, $S_t(z) = S_t^{t'}(z) = S_{t'}(z)$ for any $z \in U_{\frac{t}{Ce}}(t) \cap U_{\frac{t'}{Ce}}(t')$. Since t and t' were arbitrary,

$$S : \{z \in \mathbb{C} : |z - t| < \frac{t}{eC} \text{ for some } t > 0\} \to L_b(X)$$

with

$$S(z) := S_t(z) = \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} T^{(n)}(t)$$

if $z \in U_{\frac{t}{Ce}}(t)$ for some $t > 0$, is well defined and analytic. Since $S_t(s) = T(s)$ for any $s \in [t, t + \frac{t}{2Ce})$ and every $t > 0$, $S(s) = T(s)$ for every $s > 0$. Therefore S is an analytic extension of $T|_{(0,+\infty)}$.

We want to prove $\Sigma_\varphi \subseteq \{z \in \mathbb{C} : |z - t| < \frac{t}{eC}$ for some $t > 0\}$ for $\varphi := \arctan(\frac{1}{Ce})$. In fact, $z := a + ib \in \Sigma_\varphi$ implies $a > 0$ and

$$\arctan(\tfrac{1}{eC}) > |\arg(z)| = |\arctan(\tfrac{b}{a})| = \arctan(\tfrac{|b|}{a}),$$

from which we derive $\frac{|b|}{a} < \frac{1}{eC}$. For $t := a > 0$ we have

$$|z - t| = |b| = \frac{a|b|}{a} < \frac{a}{eC} = \frac{t}{eC}.$$

We set $S(0) := I$ and want to show that $\big(S(z)\big)_{z \in \Sigma_\varphi^0}$ is a semigroup. Let $z, w \in \Sigma_\varphi$ and $t, s > 0$ such that $|z - t| < \frac{t}{Ce}$ and $|w - s| < \frac{s}{Ce}$. We obtain

$$|(z + w) - (t + s)| \leq |z - t| + |w - s| < \frac{t + s}{Ce}$$

and, using the Binomial Theorem, Cauchy's Product Formula and Lemma 3.3.4,

$$S(z + w) = \sum_{n=0}^{\infty} \frac{\big(z + w - (t + s)\big)^n}{n!} T^{(n)}(t + s)$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{k=0}^{n} \binom{n}{k} (z-t)^k (w-s)^{n-k}\Big) A^n T(t+s) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{k=0}^{n} \frac{n!(z-t)^k (w-s)^{n-k}}{k!(n-k)!}\Big) A^n T(t+s) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(z-t)^k}{k!} \frac{(w-s)^{n-k}}{(n-k)!} A^k T(t) A^{n-k} T(s) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Big(\frac{(z-t)^k}{k!} T^{(k)}(t)\Big)\Big(\frac{(w-s)^{n-k}}{(n-k)!} T^{(n-k)}(s)\Big) \\
&= \Big(\sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} T^{(n)}(t)\Big) \cdot \Big(\sum_{n=0}^{\infty} \frac{(w-s)^n}{n!} T^{(n)}(s)\Big) \\
&= S(z)S(w).
\end{aligned}
$$

In order to show that $\big(S(z)\big)_{z\in\Sigma_\varphi^0}$ is bounded, let $\theta < \varphi = \arctan(\frac{1}{Ce})$ and choose $p < 1$ be such that $\theta = \arctan(\frac{p}{eC})$. Given $z := a + ib \in \overline{\Sigma_\theta}$, we obtain $a > 0$ and

$$\arctan(\tfrac{|b|}{a}) = |\arctan(\tfrac{b}{a})| = |\arg(z)| \leq \arctan(\tfrac{p}{Ce}),$$

which implies

$$\frac{|b|}{a} \leq \frac{p}{Ce}.$$

For $t := a$ we have

$$|z-t| = |b| \leq \frac{pa}{Ce} = \frac{pt}{Ce} < \frac{t}{Ce}$$

and, in turn

$$
\begin{aligned}
\|S(z)\| &= \Big\| \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} T^{(n)}(t) \Big\| \leq \sum_{n=0}^{\infty} \frac{|z-t|^n}{n!} \big\|T^{(n)}(t)\big\| \\
&\leq \sum_{n=0}^{\infty} \frac{p^n t^n}{e^n C^n n!} \cdot \frac{e^n C^n n!}{t^n} = \sum_{n=0}^{\infty} p^n = \frac{1}{1-p} < +\infty,
\end{aligned}
$$

which proves boundedness.
Finally we show strong continuity of S. Let $\varepsilon > 0$ and $x \in X$. Since $\big(T(t)\big)_{t\geq 0}$ is strongly continuous, we find $t_0 > 0$ such that

$$\|S(t)x - x\| = \|T(t)x - x\| < \varepsilon$$

for any $t \leq t_0$. On the other hand, because $z \mapsto S(z + t_0)$ is analytic in $z = 0$, there exists $r > 0$ such that

$$\|S(z + t_0) - S(t_0)\| < \varepsilon$$

for $|z| < r$ and $z \in \Sigma_\varphi$. If $z \in \Sigma_\varphi$ satisfies $|\arg(z)| \leq \arctan(\frac{p}{Ce})$ for some $p \in (0, 1)$ and $|z| < r$, then

$$
\begin{aligned}
\|S(z)x - x\| &\leq \|S(z)x - S(z+t_0)x\| + \|S(z+t_0)x - S(t_0)x\| + \|S(t_0)x - x\| \\
&< \|S(z)\|\,\|T(t_0)x - x\| + \varepsilon\,\|x\| + \varepsilon \\
&< (\tfrac{1}{1-p} + \|x\| + 1)\varepsilon.
\end{aligned}
$$

□

3.3.6 Corollary. Let $\big(T(t)\big)_{t\geq 0}$ be a strongly continuous semigroup, A its generator and M, ω as in Proposition 3.1.2, *a*). The following statements are equivalent.

a) There exists an angle $\varphi \in (0, \frac{\pi}{2})$, such that $\big(T(t)\big)_{t\geq 0}$ can be extended to an analytic semigroup of angle φ.

b) There exists a constant $C > 0$, such that for all $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) > \omega$ and $\operatorname{Im}(\xi) \neq 0$ the resolvent satisfies

$$\|R(\xi, A)\| \leq \frac{C}{|\operatorname{Im}(\xi)|}.$$

c) There exists a constant $C > 0$, such that for all $\xi \in \mathbb{C}$ with $\operatorname{Re}(\xi) > \omega$ the resolvent satisfies

$$\|R(\xi, A)\| \leq \frac{C}{|\xi - \omega|}.$$

d) $A - \omega I$ is sectorial.

e) $\big(T(t)\big)_{t\geq 0}$ is a differentiable semigroup and for every $\varepsilon > 0$ there exists a constant C_ε, such that

$$\|AT(t)\| \leq \frac{C_\varepsilon e^{(\omega+\varepsilon)t}}{t}$$

for all $t > 0$.

Proof.: Define $S(t) := e^{-\omega t}T(t)$. Proposition 3.1.2, *g*), identifies $\big(S(t)\big)_{t\geq 0}$ as a strongly continuous semigroup generated by $B := A - \omega I$ satisfying

$$\|S(t)\| = e^{-\omega t}\,\|T(t)\| \leq M e^{-\omega t} e^{\omega t} = M, \quad t \geq 0.$$

a) $\Rightarrow$ *b)*. For $0 < \theta < \varphi$ and all $x \in X$ the mapping $z \mapsto T(z)x$ is continuous on the compact set $\{z \in \overline{\Sigma_\theta} : \operatorname{Re}(z) \leq 1\}$ implying the existence of a constant $K_x > 0$, such that $\|T(z)x\| \leq K_x$ for all $z \in \overline{\Sigma_\theta}$ with $\operatorname{Re}(z) \leq 1$. By the Principle of uniform boundedness (Corollary 4.2.2 in [12]) there exists a constant $K > 0$, such that $\|T(z)\| \leq K$ for all $z \in \overline{\Sigma_\theta}$ with $\operatorname{Re}(z) \leq 1$. Let $z \in \overline{\Sigma_\theta}$ and $n \in \mathbb{N}$ be such that $\operatorname{Re}(z) \in [n-1, n)$. We obtain

$$\|T(z)\| = \left\|T(n \cdot \tfrac{z}{n})\right\| \leq \left\|T(\tfrac{z}{n})\right\|^n \leq K^n \leq K \cdot K^{\operatorname{Re}(z)} = K e^{\ln(K)\operatorname{Re}(z)}.$$

Assuming $\omega \geq \ln(K)$ we have $\|S(z)\| = e^{-\omega \operatorname{Re}(z)}\,\|T(z)\| \leq K$ for all $z \in \overline{\Sigma_\theta}$. Hence, $\big(S(z)\big)_{z\in\Sigma_\theta}$ is a bounded analytic semigroup. By Theorem 3.3.5 for $\xi \in \mathbb{C}$ satisfying $\operatorname{Re}(\xi) > \omega$ and $\operatorname{Im}(\xi) \neq 0$ we have

$$\|R(\xi, A)\| = \|R(\xi - \omega, B)\| \leq \frac{C}{|\operatorname{Im}(\xi - \omega)|} = \frac{C}{|\operatorname{Im}(\xi)|}.$$

b) $\Rightarrow$ *c)*. According to Theorem 3.1.5 $\|R(\xi, A)\| \leq \frac{M}{\mathrm{Re}(\xi)-\omega}$ for every $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > \omega$. If $C > 0$ is as in *b*), then we define $K := 2\max\{C, M\}$ and obtain

$$\frac{|\xi - \omega|}{K} \leq \frac{\mathrm{Re}(\xi) - \omega + |\operatorname{Im}(\xi)|}{K} \leq \frac{\mathrm{Re}(\xi) - \omega}{2M} + \frac{|\operatorname{Im}(\xi)|}{2C} \leq \frac{1}{\|R(\xi, A)\|}$$

for every $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > \omega$ and $\operatorname{Im}(\xi) \neq 0$. If $\operatorname{Im}(\xi) = 0$, we have

$$\|R(\xi, A)\| \leq \frac{M}{\mathrm{Re}(\xi) - \omega} = \frac{M}{|\xi - \omega|} \leq \frac{K}{|\xi - \omega|}.$$

c) $\Rightarrow$ *d)*. For $\xi \in \mathbb{C}$ satisfying $\mathrm{Re}(\xi) > 0$ we obtain

$$\|R(\xi, B)\| = \|R(\xi + \omega, A)\| \leq \frac{C}{|\xi + \omega - \omega|} = \frac{C}{|\xi|}.$$

Consequently, $B = A - \omega I$ is sectorial according to Theorem 3.3.5.

d) $\Rightarrow$ *e)*. Since $B = A - \omega I$ is sectorial, we obtain that $\big(S(t)\big)_{t\geq 0}$ is differentiable and

$$\|BS(t)\| \leq \frac{C}{t}, \ t > 0.$$

Since $(0, +\infty) \ni t \mapsto S(t)x \in X$ is differentiable for $x \in X$, also $T(t)x = e^{\omega t}S(t)x$ is differentiable and satisfies

$$\big(T(\cdot)x\big)'(t) = \omega e^{\omega t}S(t)x + e^{\omega t}\big(S(\cdot)x\big)'(t).$$

$\big(S(t)\big)_{t\geq 0}$ being bounded yields

$$\|AT(t)\| = \left\|e^{\omega t}(B + \omega I)S(t)\right\| \leq e^{\omega t}\|BS(t)\| + \omega e^{\omega t}\|S(t)\| \leq e^{\omega t}\frac{C + \omega M t}{t}.$$

Given $\varepsilon > 0$, by a standard argument the right hand side is less or equal $\frac{C_\varepsilon e^{(\omega+\varepsilon)t}}{t}$ for a sufficiently large C_ε.

e) $\Rightarrow$ *a)*. Given $\varepsilon > 0$, according to Proposition 3.1.2, *g*), $R(t) := e^{-(\omega+\varepsilon)t}T(t)$ defines a strongly continuous semigroup generated by $A - (\omega + \varepsilon)I$. Let $x \in X$ and $t > 0$. Since $\big(T(t)\big)_{t\geq 0}$ is differentiable, also $\big(R(t)\big)_{t\geq 0}$ is differentiable and satisfies

$$\big(R(\cdot)x\big)'(t) = -(\omega + \varepsilon)e^{-(\omega+\varepsilon)t}T(t)x + e^{-(\omega+\varepsilon)t}\big(T(\cdot)x\big)'(t)$$

for every $x \in X$. Moreover, by assumption

$$\begin{aligned}\left\|\big(A - (\omega + \varepsilon)I\big)R(t)\right\| &\leq e^{-(\omega+\varepsilon)t}\|AT(t)\| + e^{-(\omega+\varepsilon)t}(\omega + \varepsilon)\|T(t)\| \\ &\leq \frac{C_\varepsilon}{t} + Me^{-\varepsilon t}(\omega + \varepsilon) < \frac{L}{t}\end{aligned}$$

for all $t > 0$ and a sufficiently large $L > 0$. By Theorem 3.3.5 $\big(R(t)\big)_{t\geq 0}$ can be extended to a bounded analytic semigroup. Since $z \mapsto e^{(\omega+\varepsilon)z}$ is analytic in $\mathbb{C}$, also $t \mapsto T(t) = e^{(\omega+\varepsilon)t}R(t)$ can be extended to an analytic mapping.

□

Chapter 4

The Abstract Cauchy Problem

Let X be a Banach space. We will use the theory about semigroups developed in Chapter 3 to study the Abstract Cauchy Problem: Find $u : J_0 \to X$ such that

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in J, \\ u(0) = x_0, \end{cases}$$

where either $J = (0, t_0]$ for some $t_0 > 0$ or $J = (0, +\infty)$, $J_0 = J \cup \{0\}$, $A : D(A) \subseteq X \to X$ is linear, $f : J \to X$ and $x_0 \in X$. We will use the just employed notations throughout the present chapter.

At the beginning we will deal with the homogeneous problem $f \equiv 0$.

4.1 The homogeneous Problem

Given X, $x_0 \in X$, $A : D(A) \subseteq X \to X$ as above we consider the homogeneous Cauchy Problem

$$\begin{cases} u'(t) = Au(t), & t \in J, \\ u(0) = x_0. \end{cases} \tag{4.1}$$

4.1.1 Definition. We say that $u : J_0 \to X$ is a solution of (4.1), if $u(t) \in D(A)$ for all $t \in J$, $u \in C(J_0; X)$, $u|_J \in C^1(J; X)$ and u solves (4.1).

We want to show that, if A generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$ and x_0 is contained in $D(A)$, problem (4.1) has a unique solution u which satisfies $u(t) = T(t)x_0$ for $t \in J_0$. Regarding to uniqueness we need two lemmata.

4.1.2 Lemma. If $u : [0, t_0] \to X$ is continuous for a fixed $t_0 > 0$, such that

$$\sup_{n\in\mathbb{N}} \left\| \int_0^{t_0} e^{ns} u(s)\, ds \right\| < +\infty,$$

then $u(t) = 0$ for all $t \in [0, t_0]$.

Proof. Let $\varphi \in X'$ be an element of the topological dual space of X and let $f : [0, t_0] \to \mathbb{C}$ be defined by $f(t) := \varphi(u(t))$. By assumption there is a constant $C > 0$ such that

$$\left\| \int_0^{t_0} e^{ns} f(s)\ ds \right\| = \left\| \varphi\Big(\int_0^{t_0} e^{ns} u(s)\ ds \Big) \right\| \leq C \left\| \varphi \right\|$$

for all $n \in \mathbb{N}$. Since $u : [0, t_0] \to X$ is continuous, we also have $\|u(t)\| \leq K$ for some $K > 0$ and all $t \in [0, t_0]$. For $s \in \mathbb{R}$ consider the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kns} = 1 - \sum_{k=0}^{\infty} \frac{(-e^{ns})^k}{k!} = 1 - e^{-e^{ns}}.$$

For fixed $t \in [0, t_0)$ the sequence of functions

$$g_N(s) = \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k!} e^{kn(t+s-t_0)} f(s)$$

satisfies $g_N(s) \xrightarrow{N \to +\infty} (1 - e^{-e^{n(t+s-t_0)}}) f(s)$ and

$$\|g_N(s)\| \leq \sum_{k=1}^{N} \frac{1}{k!} e^{kn(t+s-t_0)} \|f(s)\| \leq \sum_{k=1}^{\infty} \frac{1}{k!} e^{kn(t+s-t_0)} \|\varphi\| \|u(s)\| \leq K \|\varphi\| e^{e^{n(t+s-t_0)}}.$$

Since the map $s \mapsto K \|\varphi\| e^{e^{n(t+s-t_0)}}$ is continuous, it is integrable over $[0, t_0]$. We employ Theorem 2.3.7 and obtain

$$\begin{aligned}
\left\| \int_0^{t_0} (1 - e^{-e^{n(t+s-t_0)}}) \varphi(s)\ ds \right\| &= \left\| \int_0^{t_0} \lim_{N \to +\infty} g_N(s)\ ds \right\| = \lim_{N \to +\infty} \left\| \int_0^{t_0} g_N(s)\ ds \right\| \\
&= \lim_{N \to +\infty} \left\| \sum_{k=1}^{N} \frac{(-1)^{k-1} e^{kn(t-t_0)}}{k!} \int_0^{t_0} e^{kns} f(s)\ ds \right\| \\
&\leq \sum_{k=1}^{\infty} \frac{e^{kn(t-t_0)}}{k!} \left\| \int_0^{t_0} e^{kns} f(s)\ ds \right\| \\
&= C \|\varphi\| (e^{e^{n(t-t_0)}} - 1),
\end{aligned}$$

which tends to zero as $n \to +\infty$. Since

$$\left\| (1 - e^{-e^{n(t+s-t_0)}}) f(s) \right\| \leq \|f(s)\| \leq K \|\varphi\|$$

for every $n \in \mathbb{N}$ and

$$\lim_{n \to +\infty} (1 - e^{-e^{n(t+s-t_0)}}) f(s) = f(s)$$

for $s > t_0 - t$ as well as

$$\lim_{n \to +\infty} (1 - e^{-e^{n(t+s-t_0)}}) f(s) = 0$$

for $s < t_0 - t$, again by Theorem 2.3.7

$$\int_{t_0 - t}^{t_0} f(s)\ ds = \lim_{n \to \infty} \int_0^{t_0} (1 - e^{-e^{n(t+s-t_0)}}) f(s)\ ds = 0.$$

Defining

$$F(t) := \int_{t_0-t}^{t_0} f(s)\ ds, \quad t \in [0, t_0),$$

we obtain $F \equiv 0$. Since f is continuous, we can employ Proposition 2.1.3, f), and see that F is differentiable satisfying

$$f(t) = F'(t) = 0$$

for every $t \in [0, t_0)$. By continuity also $f(t_0) = 0$. Hence, $\varphi\big(u(t)\big) = 0$ for any $t \in [0, t_0]$ and any $\varphi \in X'$. The Hahn-Banach Theorem, Theorem 5.2.3 in [12], yields $u(t) = 0$ for any $t \in [0, t_0]$.

□

4.1.3 Lemma. Let $t_0 > 0$, $J := (0, t_0]$, $A : D(A) \subseteq X \to X$ be linear and densely defined. Given $\xi_0 \in \mathbb{R}$ such that

$$\{\xi \in \mathbb{R} : \xi \geq \xi_0\} \subseteq \rho(A)$$

and

$$\sup_{\xi \geq \xi_0} \|R(\xi, A)x\| < +\infty \tag{4.2}$$

for any $x \in D(A)$, the homogeneous Cauchy Problem (4.1) with arbitrary $x_0 \in X$ has at most one solution in the sense of Definition 4.1.1.

Proof. If u_1 and u_2 are two solutions of (4.1), then $u := u_1 - u_2$ satisfies

$$\begin{cases} u'(t) = Au(t), & t \in (0, t_0], \\ u(0) = 0. \end{cases}$$

In order to show that $u \equiv 0$ on $[0, t_0]$, we assume for the moment that $\xi_0 = 0$ and consider the mapping $v : [0, +\infty) \to X$ defined by $v(\xi) := R(\xi, A)u(t_0)$. Employing (4.2) and the fact that $u_1(t), u_2(t) \in D(A)$ for all $t \in (0, t_0]$, there exists a constant $C > 0$, such that

$$\|v()\| \leq C, \quad \geq 0.$$

Clearly, $s \mapsto e^{(t_0-s)}$ is integrable over $[0, t_0]$. Since $u \in C([0, t_0]; X)$ and $u|_{(0,t_0]} \in C^1\big((0, t_0]; X\big)$, by Theorem 2.4.2 and Proposition 2.4.5 we have

$$\begin{aligned} \int_{\alpha}^{t_0} e^{(t_0-s)} Au(s)\ ds &= \int_{(\alpha,t_0)} e^{(t_0-s)} u'(s)\ d\lambda(s) \\ &= u(t_0) - e^{(t_0-\alpha)} u(\alpha) + \int_{(\alpha,t_0)} e^{(t_0-s)} u(s)\ d\lambda(s) \\ &= u(t_0) - e^{(t_0-\alpha)} u(\alpha) + \int_{\alpha}^{t_0} e^{(t_0-s)} u(s)\ ds \end{aligned}$$

$$\xrightarrow{\alpha\to 0^+} u(t_0) + \int_0^{t_0} e^{\xi(t_0-s)} u(s)\ ds.$$

Hence, $s \mapsto e^{\xi(t_0-s)}Au(s)$ is improperly integrable over $(0,t_0]$ satisfying

$$\int_0^{t_0} e^{\xi(t_0-s)}(\xi I - A)u(s)\ ds = -u(t_0).$$

Consequently, by Proposition 2.1.3, c),

$$\begin{aligned} v(\xi) = R(\xi,A)u(t_0) &= -R(\xi,A)\int_0^{t_0} e^{\xi(t_0-s)}(\xi I - A)u(s)\ ds \\ &= -\int_0^{t_0} e^{\xi(t_0-s)}R(\xi,A)(\xi I - A)u(s)\ ds = -\int_0^{t_0} e^{\xi(t_0-s)}u(s)\ ds. \end{aligned}$$

Employing Theorem 2.4.6 and substituting $s \mapsto t_0 - s$ leads to

$$v(\xi) = -\int_{(0,t_0)} e^{\xi(t_0-s)}u(s)\ d\lambda(s) = -\int_{(0,t_0)} e^{\xi s}u(t_0-s)\ d\lambda(s) = -\int_0^{t_0} e^{\xi s}u(t_0-s)\ ds$$

which implies

$$\left\| \int_0^{t_0} e^{nt}u(t_0-t)\ dt \right\| = \|v(n)\| \leq C$$

for all $n \in \mathbb{N}$. According to the previous lemma, $u(t_0-t) = 0$ for all $t \in [0,t_0]$. This concludes the proof for $\xi_0 = 0$.

For arbitrary $\xi_0 \in \mathbb{R}$ we define $B := A - \xi_0 I$, $v_j(t) := e^{-\xi_0 t}u_j(t)$, $j = 1,2$, and conclude

$$\begin{cases} v'(t) = Bv(t), & t \in (0,t_0], \\ u(0) = x_0. \end{cases}$$

Because of

$$\rho(B) = \{\xi - \xi_0 : \xi \in \rho(A)\} \supseteq [0,+\infty)$$

by the first part of the proof we obtain $v_1(t) = v_2(t)$ for every $t \in [0,t_0]$ implying $u_1 \equiv u_2$.

□

4.1.4 Corollary. If A generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$, then (4.1) with arbitrary $x_0 \in X$ has at most one solution.

Proof. Suppose first that $J = (0,t_0]$ for some $t_0 > 0$. Let M, ω as in Proposition 3.1.2, a), and set $\xi_0 := \omega + 1$. According to Proposition 3.1.2, h), we have

$$\|R(\xi,A)x\| \leq \frac{M\,\|x\|}{\xi - \omega} \leq M\,\|x\|$$

for every $\xi \geq \xi_0$ and $x \in X$. According to Lemma 4.1.3, (4.1) has at most one solution. In case $J = (0,+\infty)$, let u_1, u_2 be two solutions of (4.1). Since $u_1|_{[0,t_0]}$ and $u_2|_{[0,t_0]}$ solve the homogeneous Cauchy Problem on $(0,t_0]$ for every t_0, we obtain $u_1(t) = u_2(t)$, $t \geq 0$, by the first part of the proof.

□

4.1.5 Theorem. *Let $r > 0$ and $A : D(A) \subseteq X \to X$ be linear and densely defined with $\rho(A) \neq \emptyset$. The homogeneous Cauchy Problem* (4.1) *on $J := (0, t_0]$ has a unique solution for all $x_0 \in D(A)$ and $t_0 \in (0, r]$, which is continuously differentiable on $[0, t_0]$, if and only if A generates a strongly continuous semigroup $\big(T(t)\big)_{t \geq 0}$. In this case the unique solution is given by $u(t) = T(t)x_0$.*

Proof. Suppose first that A generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$. Let $x_0 \in D(A)$ and define $u : J_0 \to X$ by $u(t) := T(t)x_0$. By Proposition 3.1.2, f), u is continuously differentiable on $[0, t_0]$ satisfying

$$u'(t) = \frac{d}{dt}T(t)x_0 = AT(t)x_0 = Au(t)$$

as well as $u(0) = T(0)x_0 = Ix_0 = x_0$. Together with Corollary 4.1.4 we conclude that u is the unique solution of (4.1).

For the converse let u_{x_0} be the unique solution of (4.1) with initial value x_0, which is continuously differentiable on $[0, t_0]$. Because of $\rho(A) \neq \emptyset$ the operator A is closed; see Proposition 1.2.4, d). By Lemma 1.2.5 $Y := \big(D(A), \|\cdot\|_G\big)$ constitutes a Banach space. According to Example 9.1.9 in [17], $C([0, t_0]; Y)$ is a Banach space when equipped with the norm

$$\|v\|_\infty := \sup_{t\in[0,t_0]} \|v(t)\|_G = \sup_{t\in[0,t_0]} \big(\|v(t)\| + \|Av(t)\|\big).$$

Furthermore, we define $T : Y \to C([0, t_0], Y)$ by $Tx := u_x$. Note that for all $x \in D(A)$ the solution u_x exists and is unique as well as continuously differentiable as a function from $[0, t_0]$ to X, meaning the maps $t \mapsto u_x(t)$ and $t \mapsto u_x'(t)$ are continuous on $[0, t_0]$. Since $u_x'(t) = Au_x(t)$ for every $t \in (0, t_0]$, Au_x is continuous on $(0, t_0]$. Because of

$$\lim_{t\to 0^+} Au_x(t) = \lim_{t\to 0^+} u_x'(t) = u_x'(0)$$

and

$$\lim_{t\to 0^+} u_x(t) = u_x(0) = x$$

the closedness of A implies $Au_x(t) \xrightarrow{t\to 0^+} u_x'(0) = Ax = Au_x(0)$ verifying the continuity of Au_x on $[0, t_0]$. Hence, $u_x \in C([0, t_0]; Y)$ for any $x \in D(A)$. We split up the remaining proof into parts.

Step 1: We want to show that $T \in L_b\big(Y, C([0, t_0], Y)\big)$. Given $x, y \in D(A)$ and $\xi \in \mathbb{C}$, $u := u_x + \xi u_y$ satisfies

$$u'(t) = u_x'(t) + \xi u_y'(t) = Au_x + \xi Au_y = A(u_x + \xi u_y) = Au$$

as well as $u(0) = u_x(0) + \xi u_y(0) = x + \xi y$. Since u is continuously differentiable in $[0, t_0]$ and $u(t) \in D(A)$ for any $t \in [0, t_0]$, the uniqueness of solutions implies

$$T(x + \xi y) = u_{x+\xi y} = u = u_x + \xi u_y = Tx + \xi Ty.$$

In order to verify T's boundedness, we prove that T is closed and employ the Closed Graph Theorem; see Theorem 4.4.2 in [12]. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Y satisfying $\lim_{n\to\infty} \|x_n - x\|_G = 0$ and $\lim_{n\to\infty} \|Tx_n - v\|_\infty = 0$ for some $x \in D(A)$ and $v \in C([0,t_0];Y)$. Since u_{x_n} and Au_{x_n} are continuous in $[0,t_0]$, they are Riemann integrable; see Proposition 2.1.3, d). We employ Proposition 2.1.3, g), Corollary 2.3.13 and derive for $t \in [0,t_0]$

$$(Tx_n)(t) - x_n = u_{x_n}(t) - u_{x_n}(0) = \int_0^t u'_{x_n}(s)\ ds = \int_0^t Au_{x_n}(s)\ ds = A\Big(\int_0^t u_{x_n}(s)\ ds\Big).$$

Hence,

$$\lim_{n\to+\infty} A\Big(\int_0^t u_{x_n}(s)\ ds\Big) = v(t) - x.$$

Furthermore,

$$\Big\|\int_0^t u_{x_n}(s)\ ds - \int_0^t v(s)\ ds\Big\| \le \int_0^t \|u_{x_n}(s) - v(s)\|\ ds \le t \sup_{s\in[0,t]} \|u_{x_n}(s) - v(s)\|$$
$$\le t_0 \|Tx_n - v\|_\infty \xrightarrow{n\to+\infty} 0,$$

see Proposition 2.1.3, b). By the closedness of A

$$\int_0^t v(s)\ ds \in D(A)$$

and

$$A\Big(\int_0^t v(s)\ ds\Big) = v(t) - x.$$

$v \in C([0,t_0];Y)$ yields continuity of v and Av. Hence, they are Riemann integrable and

$$\int_0^t Av(s)\ ds = A\Big(\int_0^t v(s)\ ds\Big) = v(t) - x;$$

see Corollary 2.3.13. Because Av is continuous, we can employ Proposition 2.1.3, g), and obtain that v is differentiable satisfying

$$v'(t) = Av(t)$$

for every $t \in [0,t_0]$. Together with

$$v(0) = \lim_{n\to\infty} u_{x_n}(0) = \lim_{n\to\infty} x_n = x$$

the assumed uniqueness of solutions implies $Tx = u_x = v$. Hence, the graph of T is closed.

For $t \in [0,t_0]$ the linear mapping $T(t) : Y \to Y$, defined by $T(t)x = (Tx)(t) = u_x(t)$ satisfies $\|T(t)\|_G \le \|T\|$ and therefore belongs to $L_b(Y)$.

Step 2: We are going to verify the semigroup property for $T(\cdot)$. Fix $s \in [0, t_0)$, $x \in D(A)$ and consider the mappings $v_1, v_2 : [0, t_0 - s] \to Y$ defined by

$$v_1(t) := T(t)T(s)x = u_{u_x(s)}(t), \quad v_2(t) := T(t+s)x = u_x(t+s).$$

Their derivatives satisfy

$$v_1'(t) = u_{u_x(s)}'(t) = Au_{u_x(s)}(t) = Av_1(t)$$

and

$$v_2'(t) = u_x'(t+s) = Au_x(t+s) = Av_2(t).$$

Since solutions are unique,

$$v_1(0) = u_{u_x(s)}(0) = u_x(s) = v_2(0)$$

implies $T(t)T(s)x = v_1(t) = v_2(t) = T(t+s)x$ for all $t \in [0, t_0 - s]$. Moreover, $T(0)x = u_x(0) = x$ for all $x \in D(A)$. Hence, $T(0) = I_Y$ and $T(t+s) = T(t)T(s)$ on Y for all $t, s \in [0, t_0]$ with $t + s \leq t_0$. In order to show that the operators $\big(T(t)\big)_{t\in[0,t_0]}$ commute with each other, let $t, s \in [0, t_0]$ with $s < t$. In the case $s + t \leq t_0$ we have $T(t)T(s) = T(t+s) = T(s)T(t)$. Otherwise, if $n \in \mathbb{N}$ is chosen such that $s + \frac{t}{n} \leq t_0$, then

$$T(t)T(s) = T(n\tfrac{t}{n})T(s) = T(\tfrac{t}{n})^n T(s) = T(s)T(\tfrac{t}{n})^n = T(s)T(t).$$

We extend $T(t)$ to $[0, +\infty)$ by

$$T(t) := T(t - nt_0)T(t_0)^n$$

for $t \in (nt_0, (n+1)t_0]$ and $n \in \mathbb{N}$. In order to prove the semigroup property on all of $[0, +\infty)$, let $t, s \geq 0$. For $s = 0$ or $t = 0$ clearly $T(t+s) = T(t)T(s)$. Otherwise let $n, m \in \mathbb{N} \cup \{0\}$ be such that $t \in (nt_0, (n+1)t_0]$ and $s \in (mt_0, (m+1)t_0]$ implying $t + s \in ((n+m)t_0, (n+m+2)t_0]$. In the case $t + s \in ((n+m)t_0, (n+m+1)t_0]$ we obtain

$$T(t+s) = T\big((t+s)-(n+m)t_0\big)T(t_0)^{n+m} = T(t-nt_0)T(t_0)^n T(s-mt_0)T(t_0)^m = T(t)T(s).$$

For $t + s \in ((n+m+1)t_0, (n+m+2)t_0]$ choosing $p, q \in [0, 1]$ with $p + q = 1$, such that $t \geq (n+p)t_0$ and $s \geq (m+q)t_0$, we obtain

$$\begin{aligned} T(t+s) &= T\big((t+s) - (n+m+1)t_0\big)T(t_0)^{n+m+1} \\ &= T(t-(n+p)t_0)T(pt_0)T(s-(m+q)t_0)T(qt_0)T(t_0)^n T(t_0)^m \\ &= T(t-nt_0)T(t_0)^n T(s-mt_0)T(t_0)^m = T(t)T(s). \end{aligned}$$

Step 3: From $(Tx)(0) = T(0)x = x$, $x \in Y$, we conclude $\|T\| \geq 1$. We define $\omega := \frac{\ln(\|T\|)}{t_0}$, $\tilde{M} := \|T\|$ and obtain for $t \in (nt_0, (n+1)t_0]$

$$\|T(t)\|_G = \|T(t-nt_0)T(t_0)^n\|_G \leq \tilde{M}^{n+1} \leq \tilde{M}^{\frac{t}{t_0}+1} = \tilde{M}\exp(\tfrac{\ln(\tilde{M})t}{t_0}) = \tilde{M}e^{\omega t}.$$

Step 4: In order to show that $T(t)$ and A commute on $D(A^2)$, let $x \in D(A^2)$ and define

$$v(t) := x + \int_0^t u_{Ax}(s)\, ds.$$

$v : [0, t_0] \to X$ is well-defined since $Ax \in D(A)$ and u_{Ax} is continuous and therefore Riemann integrable over every subinterval of $[0, t_0]$; see Proposition 2.1.3, *d*). By Proposition 2.1.3, *f*), v is differentiable and satisfies

$$v'(t) = u_{Ax}(t).$$

Noting that u_{Ax} and Au_{Ax} are continuous and therefore Riemann integrable, we can employ Corollary 2.3.13, (4.1) and Proposition 2.1.3, *g*), and obtain

$$v'(t) = u_{Ax}(t) = Ax + \int_0^t Au_{Ax}(s)\, ds = A\Big(x + \int_0^t u_{Ax}(s)\, ds\Big) = Av(t).$$

By the uniqueness of solutions $v(0) = x$ yields $v = u_x$. Hence,

$$T(t)Ax = u_{Ax}(t) = v'(t) = u_x'(t) = Au_x(t) = AT(t)x.$$

Step 5: We want to extend $T(t)$ to the whole space X for every $t \geq 0$. Let $x \in D(A)$ and $\xi \in \rho(A)$. For $y := R(\xi, A)x$ we have $Ay = \xi y - x \in D(A)$ implying $y \in D(A^2)$. Moreover, by Lemma 1.2.5 the generator $A : Y \to X$ is bounded. Hence $\|(\xi I - A)z\| \leq C\, \|z\|_G$ for every $z \in D(A)$ and some $C \geq 1$. Together with Step 4 we obtain

$$\begin{aligned}
\|T(t)x\| &= \|T(t)(\xi I - A)y\| = \|(\xi I - A)T(t)y\| \\
&\leq C\, \|T(t)y\|_G \leq \tilde{M}Ce^{\omega t}\, \|y\|_G \\
&= \tilde{M}Ce^{\omega t}\big(\|R(\xi, A)x\| + \|R(\xi, A)Ax\|\big) \\
&= \tilde{M}Ce^{\omega t}\big(\|R(\xi, A)x\| + \|(\xi R(\xi, A) - I)x\|\big) \\
&\leq \tilde{M}Ce^{\omega t}\big(\|R(\xi, A)\| + |\xi|\, \|R(\xi, A)\| + 1\big)\, \|x\|\, .
\end{aligned}$$

Defining $M := \tilde{M}C\big(\|R(\xi, A)\| + |\xi|\, \|R(\xi, A)\| + 1\big) \geq 1$ by $D(A)$'s density in X we can extend $T(t)$ to a bounded operator from X to X satisfying

$$\|T(t)\| \leq Me^{\omega t};$$

see Theorem 1.1.1 in [12]. Given $t, s \geq 0$, the operators $T(t + s)$ and $T(t)T(s)$ are continuous and coincide on $D(A)$. Since $D(A)$ is densely contained in X, $T(t + s) \equiv T(t)T(s)$ on X.

Step 6: In order to prove that $\big(T(t)\big)_{t\geq 0}$ is a strongly continuous semigroup, it remains to show that $t \mapsto T(t)x$ is continuous at 0 for all $x \in X$. We already know, $T(\cdot)x = u_x : [0, t_0] \to X$ is continuous for $x \in D(A)$. In order to show the strong continuity on the whole space X, let $\varepsilon > 0$, $x \in X$ and choose $y \in D(A)$, such that

$\|x-y\| < \varepsilon$. Moreover, let $h \in (0,1)$ be such that $\|T(t)y - y\| < \varepsilon$ for all $t \in (0,h)$. We obtain

$$\begin{aligned}\|T(t)x - x\| &\leq \|T(t)x - T(t)y\| + \|T(t)y - y\| + \|y - x\| \\ &< Me^{\omega t}\|x-y\| + \|T(t)y - y\| + \varepsilon \\ &< (Me^{\omega} + 2)\varepsilon,\end{aligned}$$

implying $\lim_{t\to 0^+} T(t)x = x$ for all $x \in X$.

Step 7: It remains to show that A is the infinitesimal generator of $\big(T(t)\big)_{t\geq 0}$. Let $\tilde{A}$ be the generator of $\big(T(t)\big)_{t\geq 0}$. For $x \in D(A)$ we have

$$\frac{d}{dt}\Big|_{t=0} T(t)x = \frac{d}{dt}\Big|_{t=0} u_x(t) = Au_x(0) = Ax,$$

implying $A \subseteq \tilde{A}$. For the converse let $x \in D(A^2)$ and $\xi > \omega$. Employing Proposition 3.1.2, h), we see that $t \mapsto e^{-\xi t}T(t)$ and $t \mapsto A\big(e^{-\xi t}T(t)x\big) = e^{-\xi t}T(t)Ax$ are Riemann integrable over $[0,+\infty)$. According to Proposition 3.1.2, h), we have

$$R(\xi,\tilde{A})x = \int_0^{+\infty} e^{-\xi t}T(t)x\,dt$$

which by Corollary 2.3.13 belongs to $D(A)$, and

$$AR(\xi,\tilde{A})x = A\Big(\int_0^{+\infty} e^{-\xi t}T(t)x\,dt\Big) = \int_0^{+\infty} e^{-\xi t}T(t)Ax\,dt = R(\xi,\tilde{A})Ax.$$

Since $\tilde{A}$ commutes with its resolvents, we obtain

$$AR(\xi,\tilde{A})x = R(\xi,\tilde{A})Ax = R(\xi,\tilde{A})\tilde{A}x = \tilde{A}R(\xi,\tilde{A})x.$$

By Proposition 1.2.2, d), $(\xi I - A)\big(D(A^2)\big) = D(A)$, implying $R(\xi,A)\big(D(A)\big) = D(A^2)$. As

$$\overline{D(A^2)} = \overline{R(\xi,A)\big(D(A)\big)} \supseteq R(\xi,A)\big(\overline{D(A)}\big) = R(\xi,A)(X) = D(A),$$

$D(A^2)$ is dense in X. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$ a sequence contained in $D(A^2)$ satisfying $\lim_{n\to+\infty} x_n = x$. Since $\operatorname{ran} R(\xi,\tilde{A}) = D(\tilde{A})$ and $\tilde{A}$ is closed, $\tilde{A}R(\xi,\tilde{A})$ is a bounded operator; see Proposition 1.2.2, c). We conclude

$$\lim_{n\to+\infty} AR(\xi,\tilde{A})x_n = \lim_{n\to+\infty} \tilde{A}R(\xi,\tilde{A})x_n = \tilde{A}R(\xi,\tilde{A})x$$

as well as

$$\lim_{n\to+\infty} R(\xi,\tilde{A})x_n = R(\xi,\tilde{A})x.$$

A being closed implies $R(\xi,\tilde{A})x \in D(A)$. Since $x \in X$ was arbitrary, we obtain

$$D(\tilde{A}) = \operatorname{ran} R(\xi,\tilde{A}) \subseteq D(A)$$

and in turn $A = \tilde{A}$.

$\square$

4.1.6 Corollary. Let $A : D(A) \subseteq X \to X$ be linear and densely defined satisfying $\rho(A) \neq \emptyset$. The homogeneous Cauchy Problem

$$\begin{cases} u'(t) = Au(t), & t \in (0, +\infty), \\ u(0) = x_0, \end{cases} \tag{4.3}$$

has a unique solution for all $x_0 \in D(A)$, which is continuously differentiable on $[0, +\infty)$, if and only if A generates a strongly continuous semigroup $\big(T(t)\big)_{t \geq 0}$. In this case the solution has the form $u(t) = T(t)x_0$.

Proof. If $x_0 \in D(A)$ and A is the infinitesimal generator of a strongly continuous semigroup $\big(T(t)\big)_{t \geq 0}$, then $t \mapsto T(t)x_0$ satisfies $T(0)x_0 = x_0$ and is differentiable with

$$\frac{d}{dt} T(t)x_0 = AT(t)x_0 = T(t)Ax_0$$

for every $t \geq 0$; see Proposition 3.1.2, f). By Corollary 4.1.4 $t \mapsto T(t)x_0$ is the unique solution of (4.3).

Conversely, suppose that for every $x_0 \in D(A)$ there is a unique continuously differentiable solution $u_{x_0} : [0, +\infty) \to X$ of (4.3). Let $t_0 > 0$ and $x_0 \in D(A)$. Clearly, $u_{x_0}|_{[0,t_0]}$ is a continuously differentiable solution of (4.1) on $(0, t_0]$. Let $v : [0, t_0] \to X$ be another continuously differentiable solution of (4.1) on $(0, t_0]$ and define $w : [0, +\infty) \to X$ by

$$w(t) := \begin{cases} v(t), & t \in [0, t_0], \\ u_{v(t_0)}(t - t_0), & t > t_0. \end{cases}$$

By assumption $v(t_0) \in D(A)$. Hence, w is well-defined and continuous. Moreover, $w(t) \in D(A)$ for all $t \in [0, +\infty)$ and w is continuously differentiable on $[0, +\infty) \setminus \{t_0\}$ satisfying $w'(t) = Aw(t)$ for all $t \in (0, +\infty) \setminus \{t_0\}$. In order to show that w is differentiable at t_0, note that $u_{v(t_0)}$ is differentiable at 0 from the right and v is differentiable at t_0 from the left, implying

$$\frac{1}{h}\big(w(t_0 + h) - w(t_0)\big) = \frac{1}{h}\big(u_{v(t_0)}(h) - u_{v(t_0)}(0)\big) \xrightarrow{h \to 0^+} u'_{v(t_0)}(0)$$

as well as

$$\frac{1}{h}\big(w(t_0 - h) - w(t_0)\big) = \frac{1}{h}\big(v(t_0 - h) - v(t_0)\big) \xrightarrow{h \to 0^+} v'(t_0) = Av(t_0).$$

Since $u_{v(t_0)}$ is continuously differentiable at 0, we have $u_{v(t_0)}(h) \xrightarrow{h \to 0^+} u_{v(t_0)}(0) = v(t_0)$ and $Au_{v(t_0)}(h) = u'_{v(t_0)}(h) \xrightarrow{h \to 0^+} u'_{v(t_0)}(0)$. By the closedness of A

$$\lim_{h \to 0^+} \frac{1}{h}\big(w(t_0 + h) - w(t_0)\big) = u'_{v(t_0)}(0) = Av(t_0) = \lim_{h \to 0^+} \frac{1}{h}\big(w(t_0 - h) - w(t_0)\big).$$

Consequently, w is differentiable at t_0 satisfying $w'(t_0) = Av(t_0) = Aw(t_0)$. We conclude that w is a solution of (4.3), which is continuously differentiable on $[0, +\infty)$. By assumption $w \equiv u_{x_0}$, in particular $v(t) = u_{x_0}(t)$ for all $t \in [0, t_0]$. Hence, we have shown that for every $x_0 \in D(A)$ there is a unique continuously differentiable solution of (4.1) on $(0, t_0]$. By Theorem 4.1.5 A is the infinitesimal generator of a strongly continuous semigroup. Lastly, according to Theorem 4.1.5, $u_{x_0}|_{[0,t_0]} \equiv \big(T(\cdot)x_0\big)|_{[0,t_0]}$ for every $t_0 > 0$ implying $u_{x_0}(t) = T(t)x_0$ for every $t \geq 0$.

□

If A generates a semigroup with stronger properties, existence and uniqueness prevails for all initial values in X.

4.1.7 Corollary. If A generates a differentiable semigroup $\big(T(t)\big)_{t\geq 0}$, the Cauchy Problem (4.1) has a unique solution for all $x_0 \in X$.

Proof. According to Lemma 3.3.4, $t \mapsto T(t)$ is differentiable for $t > 0$ with

$$\frac{d}{dt}T(t) = AT(t).$$

Hence, also $(0, +\infty) \ni t \mapsto T(t)x_0$ is differentiable for all $x_0 \in X$ with

$$\frac{d}{dt}T(t)x_0 = AT(t)x_0.$$

Together with $T(0)x_0 = x_0$ this shows that $[0, +\infty) \ni t \mapsto T(t)x_0$ is a solution of (4.1). By Corollary 4.1.4, this solution is unique.

□

Of course, because analytic semigroups are differentiable, the corollary above also holds true if A generates an analytic semigroup; see Corollary 3.3.6.

4.2 The inhomogeneous Problem

Throughout the present section, we assume that $A : D(A) \subseteq X \to X$ generates a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$. In the previous section we studied the homogenous problem. Here we consider the inhomogeneous Cauchy Problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in J, \\ u(0) = x_0, \end{cases}. \tag{4.4}$$

where either $J = (0, t_0]$ or $J = (0, +\infty)$. Moreover, as in Section 4.1 we set $J_0 := J \cup \{0\}$. In general we cannot expect a continuously differentiable solution of (4.4). Therefore, we will introduce notions of solvability, which do not include continuous differentiability.

4.2.1 Definition. Let $u : J_0 \to X$ be a function.

a) u is called a classical solution of (4.4), if $u \in C(J_0; X)$, $u|_J \in C^1(J; X)$ with $u(t) \in D(A)$ for $t \in J$ and u solves (4.4).

b) u is called a weak solution of (4.4), if u is continuous and weakly differentiable in the sense of Definition 2.5.4, such that $u(t) \in D(A)$ for almost every $t \in J$, and $u(0) = x_0$ as well as $u'(t) = Au(t) + f(t)$ for almost every $t \in J$.

We want to prove that every classical solution has a particular form. To that end, we need two lemmata.

4.2.2 Lemma. Let $t \in J$ and $u : J_0 \to X$ be a continuous function, which satisfies $u(0) = x_0$. Then the function $v_t : [0, t] \to X$ defined by $v_t(s) = T(t - s)u(s)$ is continuous. Moreover, if u is differentiable at some $s_0 \in (0, t]$, $u(s_0) \in D(A)$ and $u'(s_0) = Au(s_0) + f(s_0)$, then v_t is differentiable at s_0 and $v_t'(s_0) = T(t - s_0)f(s_0)$.

Proof. Let M, ω be as in Proposition 3.1.2, $a)$ and $s \in [0, t]$. Given a sequence $(s_n)_{n\in\mathbb{N}}$ in $[0, t]$ converging to s, by the continuity of $r \mapsto T(t - r)u(s)$ we obtain

$$\begin{aligned}
\|v_t(s) - v_t(s_n)\| &= \|T(t - s)u(s) - T(t - s_n)u(s_n)\| \\
&\le \|T(t - s)u(s) - T(t - s_n)u(s)\| + \|T(t - s_n)u(s) - T(t - s_n)u(s_n)\| \\
&\le \|T(t - s)u(s) - T(t - s_n)u(s)\| + Me^{\omega(t-s_n)} \|u(s) - u(s_n)\| \\
&\le \|T(t - s)u(s) - T(t - s_n)u(s)\| + Me^{\omega t} \|u(s) - u(s_n)\| \xrightarrow{n\to+\infty} 0.
\end{aligned}$$

Hence, v_t is continuous. Given $s_0 \in (0, t]$ such that u is differentiable at s_0, $u(s_0) \in D(A)$ and $u'(s_0) = Au(s_0) + f(s_0)$, u and $s \mapsto T(t - s)u(s_0)$ are differentiable at s_0 since $u(s_0) \in D(A)$; see Proposition 3.1.2, $f)$. By Proposition 3.1.2, $b)$, $s \mapsto T(t - s)u'(s_0)$ is continuous at s_0. As $\|T(t - s_0)\| \le Me^{\omega(t-s_0)} \le Me^{\omega t}$, we can employ Lemma 1.1.3 and conclude that v_t is differentiable at s_0 and satisfies

$$\begin{aligned}
v'(s_0) &= \big(T(t - \cdot)u(s_0)\big)'(s_0) + T(t - s_0)u'(s_0) \\
&= -AT(t - s_0)u(s_0) + T(t - s_0)\big(Au(s_0) + f(s_0)\big) = T(t - s_0)f(s_0).
\end{aligned}$$

Here A and $T(t - s_0)$ commute in $D(A)$ according to Proposition 3.1.2, $f)$.

□

4.2.3 Lemma. If $f \in L^1\big((0, t); X\big)$ for all $t \in J$, then

$$\big(s \mapsto T(t - s)f(s)\big) \in L^1\big((0, t); X\big)$$

for all $t \in J$ and

$$t \mapsto \int_{(0,t)} T(t - s)f(s)\, d\lambda(s)$$

is continuous in J_0.

Proof. Let $t \in J$ be fixed. By Proposition 2.5.2 b), there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of functions contained in $C_{00}^\infty\big((0,t);X\big)$ with $\|\varphi_n - f\|_{L^1((0,t);X)} \xrightarrow{n\to+\infty} 0$. We employ Proposition 2.5.2, a), and find a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ such that $\varphi_{n_k}(s) \xrightarrow{k\to+\infty} f(s)$ for almost every $s \in (0,t)$. By Proposition 3.1.2, b), $T(t-s)\varphi_{n_k}(s) \xrightarrow{k\to+\infty} T(t-s)f(s)$ for almost every $s \in (0,t)$. Since $s \mapsto T(t-s)\varphi_{n_k}(s)$ is continuous for all $k \in \mathbb{N}$, $s \mapsto T(t-s)f(s)$ is measurable; see Proposition 2.3.1, d) and Example 2.3.2. If M, ω are as in Proposition 3.1.2, a), then

$$\int_{(0,t)} \|T(t-s)f(s)\| \; d\lambda(s) \le \int_{(0,t)} Me^{\omega(t-s)} \|f(s)\| \; d\lambda(s) \le Me^{\omega t} \|f\|_{L^1((0,t);X)} < +\infty.$$

Hence, $s \mapsto T(t-s)f(s)$ is integrable over $(0,t)$; see Proposition 2.3.6, b). Let $t_0 \in J_0$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in J_0 satisfying $t_n \xrightarrow{n\to+\infty} t_0$. In consequence, there exists a constant $C \in J$ such that $t_n \le C$ for all $n \in \mathbb{N}$. Define $g_n : [0,C] \to X$ by

$$g_n(s) := \mathbb{1}_{[0,t_n]}(s)T(t_n - s)f(s).$$

Let $s \in [0,t_0)$, $\varepsilon > 0$ and $N \in \mathbb{N}$ be such that $t_n > s$ and

$$\|T(t_n - s)f(s) - T(t_0 - s)f(s)\| < \varepsilon$$

for all $n \ge N$. We obtain $\|g_n(s) - T(t_0 - s)f(s)\| < \varepsilon$ for every $n \ge N$, and in turn $g_n(s) \xrightarrow{n\to+\infty} T(t_0 - s)f(s)$ for every $s \in [0,t_0)$. For $s \in (t_0, C]$, let $N \in \mathbb{N}$ be such that $t_n < s$ for every $n \ge N$. We obtain $g_n(s) = 0$ for every $n \ge N$. Hence, $(g_n)_{n\in\mathbb{N}}$ converges almost everywhere to the the function $g : [0,C] \to X$ defined by

$$g(s) := \begin{cases} T(t_0 - s)f(s), & s \in [0,t_0], \\ 0, & s \in (t_0, C]. \end{cases}$$

Furthermore,

$$\|g_n(s)\| \le \mathbb{1}_{[0,t_n]}(s) \|T(t_n - s)\| \|f(s)\| \le \mathbb{1}_{[0,t_n]}(s)Me^{\omega(t_n - s)} \|f(s)\| \le Me^{\omega(C-s)} \|f(s)\|.$$

Since $s \mapsto Me^{\omega C} \|f(s)\|$ is integrable over $(0,C)$, we can employ Theorem 2.3.7 and obtain

$$\begin{aligned} \lim_{n\to+\infty} \int_{(0,t_n)} T(t_n - s)f(s) \; d\lambda(s) &= \lim_{n\to+\infty} \int_{(0,C)} g_n(s) \; d\lambda(s) \\ &= \int_{(0,C)} g(s) \; ds = \int_{(0,t_0)} T(t_0 - s)f(s) \; d\lambda(s) \end{aligned}$$

verifying that

$$t \mapsto \int_{(0,t)} T(t-s)f(s) \; d\lambda(s)$$

is continuous at any $t_0 \in J_0$.

□

4.2.4 Remark. Let $t \in J$, $p \in (1, +\infty]$ and $f \in L^p(J;X)$. Given $q \in [1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, we employ Hölder's inequality, Proposition 2.5.3, a), and obtain

$$\int_{(0,t)} \|f(s)\| \; d\lambda(s) \leq t^{\frac{1}{q}} \|f\|_{L^p} < +\infty,$$

which implies $f \in L^1\big((0,t);X\big)$. If $f \in L^1(J;X)$, clearly $f \in L^1\big((0,t);X\big)$ for all $t \in J$. By Lemma 4.2.3 we obtain that $u : J_0 \to X$ defined by

$$u(t) = \int_{(0,t)} T(t-s)f(s) \; d\lambda(s)$$

is well-defined and continuous for all $f \in L^p(J;X)$, $p \in [1, +\infty]$.

4.2.5 Definition. If $f \in L^p(J;X)$ for some $p \in [1, +\infty]$, we call the function $u : J_0 \to X$ defined by

$$u(t) := T(t)x_0 + \int_{(0,t)} T(t-s)f(s) \; d\lambda(s)$$

the mild solution of the inhomogeneous Cauchy Problem (4.4).

By Lemma 4.2.3 and Remark 4.2.4 the mild solution u in Definition 4.2.5 is well-defined and continuous. From Proposition 2.5.5, c), we know that every classical solution is a weak solution. As our next step, we want to prove that every classical solution is also a mild solution.

4.2.6 Proposition. Let $p \in [1, +\infty]$ and $f \in L^p(J;X)$. Every classical solution of (4.4) coincides with the mild solution of (4.4). In particular, there is at most one classical solution of (4.4).

Proof. Let $t \in J$, $u : J_0 \to X$ be a classical solution of (4.4) and define $v_t : [0,t] \to X$ by $v_t(s) = T(t-s)u(s)$. From Lemma 4.2.2 and 4.2.3 we know that $v_t'(s) = T(t-s)f(s)$ for every $s \in (0,t]$ and $v_t' \in L^1\big((0,t);X\big)$, which implies

$$v_t(t) = v_t(\varepsilon) + \int_{(\varepsilon,t)} T(t-s)f(s) \; d\lambda(s)$$

for every $\varepsilon > 0$, see Lemma 2.4.4. Define $f_\varepsilon : [0,t] \to X$ by $f_\varepsilon(s) := \mathbb{1}_{[\varepsilon,t]}(s)T(t-s)f(s)$. Clearly, $f_\varepsilon(s) \xrightarrow{\varepsilon \to 0^+} T(t-s)f(s)$ for any $s \in (0,t]$. Since $\|f_\varepsilon(s)\| \leq \|T(t-s)f(s)\|$ for every $s \in [0,t]$, we employ Theorem 2.3.7 and derive

$$\lim_{\varepsilon \to 0^+} \int_{(\varepsilon,t)} T(t-s)f(s) \; d\lambda(s) = \lim_{\varepsilon \to 0^+} \int_{(0,t)} f_\varepsilon(s) \; d\lambda(s) = \int_{(0,t)} T(t-s)f(s) \; d\lambda(s).$$

By Lemma 4.2.2, v_t is continuous at 0. Therefore,

$$v_t(t) = \lim_{\varepsilon \to 0^+} \Big(v_t(\varepsilon) + \int_{(\varepsilon,t)} T(t-s)f(s)\; d\lambda(s) \Big) = v_t(0) + \int_{(0,t)} T(t-s)f(s)\; d\lambda(s)$$

and because of $v_t(0) = T(t)x_0$ and $v_t(t) = u(t)$

$$u(t) = T(t)x_0 + \int_{(0,t)} T(t-s)f(s)\; d\lambda(s).$$

□

If A generates a semigroup with stronger properties, we obtain uniqueness of weak solutions.

4.2.7 Proposition. Let $p \in [1, +\infty]$, $f \in L^p(J; X)$. If A generates a differentiable semigroup $\big(T(t)\big)_{t \geq 0}$, every weak solution of (4.4) coincides with the mild solution of (4.4). In particular, there is at most one weak solution of (4.4).

Proof. Let $t \in J$, u be a weak solution of (4.4) and define $v_t : [0, t] \to X$ by $v_t(s) := T(t-s)u(s)$. Since $u'(s) = Au(s) + f(s)$ for every $s \in J \setminus N$ with some null set $N \subseteq J$, $v_t'(s) = T(t-s)f(s)$ for every $s \in (0, t] \setminus N$; see Lemma 4.2.2. We show that v_t is absolutely continuous on every compact subinterval of $(0, t)$. Let M, ω be as in Proposition 3.1.2, a). Since u is continuous, there exists a constant $K > 0$ such that $\|u(s)\| \leq K$ for all $s \in [0, t]$. Given $r, s \in (0, t)$ we have

$$\begin{aligned} \|v_t(r) - v_t(s)\| &= \|T(t-r)u(r) - T(t-s)u(s)\| \\ &\leq \|T(t-r)u(r) - T(t-r)u(s)\| + \|T(t-r)u(s) - T(t-s)u(s)\| \\ &\leq \|T(t-r)\|\, \|u(r) - u(s)\| + \|T(t-r) - T(t-s)\|\, \|u(s)\| \\ &\leq Me^{\omega(t-s)}\, \|u(r) - u(s)\| + K\, \|T(t-r) - T(t-s)\| \\ &\leq Me^{\omega t}\, \|u(r) - u(s)\| + K\, \|T(t-r) - T(t-s)\|\,. \end{aligned} \tag{4.5}$$

Let $\varepsilon > 0$ and $0 < a < b < t$. By Theorem 2.5.7 u is absolutely continuous on $[a, b]$, which implies that there exists a constant $\tilde{\delta} > 0$ such that

$$\sum_{k=1}^{n} \|u(s_k) - u(r_k)\| < \frac{\varepsilon e^{-\omega t}}{2M}$$

for any collection $a \leq r_1 \leq s_1 \leq r_2 \leq s_2 \leq \cdots \leq r_n \leq s_n \leq b$ satisfying

$$\sum_{k=1}^{n} (s_k - r_k) < \tilde{\delta}.$$

Define

$$\delta := \min\Big\{ \tilde{\delta}, \frac{\varepsilon(t-b)e^{-(\omega+1)t}}{2CK} \Big\} > 0$$

and let $a \leq r_1 \leq s_1 \leq r_2 \leq s_2 \leq \cdots \leq r_n \leq s_n \leq b$ be a collection satisfying

$$\sum_{k=1}^{n}(s_k - r_k) < \delta.$$

By Lemma 3.3.4 T is continuously differentiable on $(0, +\infty]$ with $T' = AT$, which implies that there exists a constant $C > 0$ such that $\|AT(r)\| \leq C$ for every $r \in [t-b, t-a]$. By Lemma 2.4.4

$$\begin{aligned}\sum_{k=1}^{n}\|T(t-s_k) - T(t-r_k)\| &= \sum_{k=0}^{n}\left\|\int_{(t-s_k,t-r_k)} AT(\xi)\ d\lambda(\xi)\right\| = \sum_{k=0}^{n}\left\|\int_{t-s_k}^{t-r_k} AT(\xi)\ d\xi\right\| \\ &\leq \sum_{k=0}^{n}\int_{t-s_k}^{t-r_k}\|AT(\xi)\|\ d\xi \leq \sum_{k=0}^{n}\int_{t-s_k}^{t-r_k} C\ d\xi \\ &= C\sum_{k=1}^{n}(s_k - r_k) < \frac{\delta C e^{(\omega+1)t}}{t-b} < \frac{\varepsilon}{2K}\end{aligned}$$

which by (4.5) leads to

$$\begin{aligned}\sum_{k=0}^{n}\|v_t(s_k) - v_t(r_k)\| &\leq Me^{\omega t}\sum_{k=0}^{n}\|u(s_k) - u(r_k)\| + K\sum_{k=0}^{n}\|T(t-r_k) - T(t-s_k)\| \\ &< Me^{\omega t}\frac{\varepsilon e^{-\omega t}}{2M} + K\frac{\varepsilon}{2K} = \varepsilon.\end{aligned}$$

We obtain that v_t is absolutely continuous on $[a, b]$. By Theorem 2.5.7 together with $v_t' \in L^1\big((0,t);X\big)$ we conclude $v_t \in W^{1,1}\big((0,t);X\big)$. By Lemma 4.2.2 v_t is continuous on all of $[0, t]$. Consequently, by Corollary 2.5.8,

$$v_t(t) - v_t(0) = \int_{(0,t)} T(t-r)f(r)\ d\lambda(r) = v_t(t) - v_t(0) = u(t) - T(t)x_0$$

and, in turn,

$$u(t) = v_t(t) = v_t(0) + \int_{(0,t)} T(t-s)f(s)\ d\lambda(s) = T(t)x_0 + \int_{(0,t)} T(t-s)f(s)\ d\lambda(s).$$

□

We will finish the present chapter with conditions which ensure the mild solution to be a classical solution.

4.2.8 Theorem. *Let $p \in [1, +\infty]$ and $x_0 \in D(A)$. If $f \in L^p(J;X)$ is continuously differentiable on J_0, then the mild solution is a classical solution.*

Proof. Let $v : J_0 \to X$ be defined by

$$v(t) := \int_{(0,t)} T(t-s)f(s)\ d\lambda(s),$$

which is well-defined and continuous by Lemma 4.2.3. We want to prove that v is differentiable. To that end let $t \in J$, where we assume $t \neq t_0$ in the case $J = (0, t_0]$ for some $t_0 > 0$. By Theorem 2.4.6

$$v(t) = \int_{(0,t)} T(s)f(t-s)\ d\lambda(s).$$

For $h > 0$ with $t + h \in J$ we have

$$\begin{aligned} \frac{1}{h}\big(v(t+h) - v(t)\big) &= \frac{1}{h}\Big(\int_{(0,t+h)} T(s)f(t+h-s)\ d\lambda(s) - \int_{(0,t)} T(s)f(t-s)\ d\lambda(s)\Big) \\ &= \frac{1}{h}\int_{(0,t)} T(s)\big(f(t+h-s) - f(t-s)\big)\ d\lambda(s) + \frac{1}{h}\int_{(t,t+h)} T(s)f(t+h-s)\ d\lambda(s). \end{aligned} \tag{4.6}$$

We will deal with both summands separately. Define $g_h : (0,t) \to X$ by

$$g_h(s) := T(s)\Big(\frac{1}{h}\big(f(t+h-s) - f(t-s)\big)\Big).$$

$T(s) \in L_b(X)$ yields $g_h(s) \xrightarrow{h \to 0^+} T(s)f'(t-s)$. By Proposition 3.1.2, a), there exist M, ω such that $\|T(s)\| \leq Me^{\omega s}$ for all $s \geq 0$ and by continuity of f' on J_0 there exists a constant $C > 0$ such that $\|f'(s)\| \leq C$ for all $s \in (0,t)$. By Proposition 2.1.3, b) and g),

$$\|g_h(s)\| \leq \|T(s)\| \frac{1}{h} \|f(t+h-s) - f(t-s)\| \leq \frac{M}{h} e^{\omega s} \left\| \int_{t-s}^{t+h-s} f'(r)\ dr \right\| \leq CMe^{\omega t}. \tag{4.7}$$

As $\int_{(0,t)} CMe^{\omega t}\ ds = tCMe^{\omega t} < +\infty$, the requirements of Theorem 2.3.7 are fulfilled. Hence,

$$\frac{1}{h}\int_{(0,t)} T(s)\big(f(t+h-s) - f(t-s)\big)\ d\lambda(s) = \int_{(0,t)} g_h(s)\ d\lambda(s) \xrightarrow{h \to 0^+} \int_{(0,t)} T(s)f'(t-s)\ d\lambda(s).$$

We compute the limit of the second addend on the right hand side of (4.6). Since f is continuous at 0, given $\varepsilon > 0$, there exists $h_0 > 0$ such that

$$\|f(h) - f(0)\| < \frac{\varepsilon e^{-\omega t}}{2M}$$

for all $h \in (0, h_0)$. By Propositon 3.1.2, b), we find $h_1 > 0$, such that

$$\|T(t)f(0) - T(s)f(0)\| < \frac{\varepsilon}{2}$$

for all $s \in (t - h_1, t + h_1)$. For $h \in (0, \min\{h_0, h_1\})$ and $s \in (t, t+h)$ we obtain

$$\begin{aligned} \|T(s)f(t+h-s) - T(t)f(0)\| &= \|T(s)f(t+h-s) - T(s)f(0) + T(s)f(0) - T(t)f(0)\| \\ &\leq Me^{\omega s}\|f(t+h-s) - f(0)\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Consequently,

$$
\begin{aligned}
&\left\|\frac{1}{h}\int_{(t,t+h)} T(s)f(t+h-s)\,d\lambda(s)-T(t)f(0)\right\| \\
&\qquad\qquad\qquad\qquad = \left\|\frac{1}{h}\int_{(t,t+h)} T(s)f(t+h-s)-f(t)\,d\lambda(s)\right\| \\
&\qquad\qquad\qquad\qquad \le \frac{1}{h}\int_{(t,t+h)} \|T(s)f(t+h-s)-T(t)f(0)\|\,\,d\lambda(s) \\
&\qquad\qquad\qquad\qquad < \frac{1}{h}\int_{(t,t+h)} \varepsilon\,d\lambda(s)=\varepsilon,
\end{aligned}
$$

implying

$$
\lim_{h\to 0^+}\frac{1}{h}\int_{(t,t+h)} T(s)f(t+h-s)\,d\lambda(s)=T(t)f(0). \tag{4.8}
$$

By (4.6) we therefore conclude that v is differentiable from the right and

$$
\lim_{h\to 0^+}\frac{1}{h}\big(v(t+h)-v(t)\big)=\int_{(0,t)} T(s)f'(t-s)\,d\lambda(s)+T(t)f(0).
$$

In order to show that v is also differentiable from the left, let $t\in J$, $h\in(0,t)$ and compute

$$
\begin{aligned}
\frac{1}{h}\big(v(t)-v(t-h)\big) &= \frac{1}{h}\Big(\int_{(0,t)} T(s)f(t-s)\,d\lambda(s)-\int_{(0,t-h)} T(s)f(t-h-s)\,d\lambda(s)\Big) \\
&= \frac{1}{h}\int_{(0,t-h)} T(s)\big(f(t-s)-f(t-h-s)\big)\,d\lambda(s)+\frac{1}{h}\int_{(t-h,t)} T(s)f(t-s)\,d\lambda(s).
\end{aligned}
$$

We define $f_h:(0,t)\to X$ by

$$
f_h(s):=\mathbb{1}_{(0,t-h)}(s)T(s)\Big(\frac{1}{h}\big(f(t-s)-f(t-h-s)\big)\Big).
$$

Let $s\in(0,t)$ and $h>0$ with $t-h>s$. We obtain

$$
f_h(s)=T(s)\Big(\frac{1}{h}\big(f(t-s)-f(t-h-s)\big)\Big)\xrightarrow{h\to 0^+} T(s)f'(t-s)
$$

By similar arguments as in (4.7) we have $\|f_h(s)\|\le CMe^{\omega t}$. Again we can employ Theorem 2.3.7 and obtain

$$
\lim_{h\to 0^+}\frac{1}{h}\int_{(0,t-h)} T(s)\big(f(t-s)-f(t-h-s)\big)\,d\lambda(s)=\int_{(0,t)} T(s)f'(t-s)\,d\lambda(s).
$$

Analogous to (4.8), we obtain

$$
\lim_{h\to 0^+}\frac{1}{h}\int_{(t-h,t)} T(s)f(t-s)\,d\lambda(s)=T(t)f(0).
$$

In consequence, v is differentiable on J satisfying

$$v'(t) = \int_{(0,t)} T(s)f'(t-s)\; d\lambda(s) + T(t)f(0) = \int_{(0,t)} T(t-s)f'(s)\; d\lambda(s) + T(t)f(0)$$

by Theorem 2.4.6. As f' is continuous on J_0, it is integrable over $(0,t)$ for every $t \in J$; see Theorem 2.4.2. By Lemma 4.2.2 and Proposition 3.1.2 b), we obtain that v' is continuous.

For $t \in J$, where we assume $t \neq t_0$ in the case $J = (0, t_0]$ for some $t_0 > 0$, and $h > 0$ such that $t + h \in J$ we compute

$$\begin{aligned} v(t+h) - v(t) &= \int_{(0,t+h)} T(t+h-s)f(s)\; d\lambda(s) - \int_{(0,t)} T(t-s)f(s)\; d\lambda(s) \\ &= \big(T(h) - I\big)\Big(\int_{(0,t)} T(t-s)f(s)\; d\lambda(s)\Big) + \int_{(t,t+h)} T(t+h-s)f(s)\; d\lambda(s) \\ &= \big(T(h)v(t) - v(t)\big) + \int_{(t,t+h)} T(t+h-s)f(s)\; d\lambda(s) \end{aligned}$$

implying

$$\frac{1}{h}\big(T(h)v(t) - v(t)\big) = \frac{1}{h}\big(v(t+h) - v(t)\big) - \int_{(t,t+h)} T(t+h-s)f(s)\; d\lambda(s). \tag{4.9}$$

Let $\varepsilon > 0$ and $h \in (0,1)$ be such that $\|f(t) - f(s)\| < \frac{\varepsilon e^{-\omega}}{2M}$ and $\|T(s)f(t) - f(t)\| < \frac{\varepsilon}{2}$ for every $s \in (t-h, t+h)$. Given $s \in (t, t+h)$, we obtain

$$\begin{aligned} \|T(t+h-s)f(s) - f(t)\| &\leq \big\|T(t+h-s)\big(f(s) - f(t)\big)\big\| + \|T(t+h-s)f(t) - f(t)\| \\ &< Me^{\omega(t+h-s)}\, \|f(s) - f(t)\| + \frac{\varepsilon}{2} < Me^{\omega}\frac{\varepsilon e^{-\omega}}{2M} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and in turn

$$\begin{aligned} \left\|\frac{1}{h}\int_{(t,t+h)} T(t+h-s)f(s)\; d\lambda(s) - f(t)\right\| &\leq \frac{1}{h}\int_{(t,t+h)} \|T(t+h-s)f(s) - f(t)\|\, d\lambda(s) \\ &< \frac{1}{h}\int_{(t,t+h)} \varepsilon\; d\lambda(s) = \varepsilon, \end{aligned}$$

which implies

$$\lim_{h\to 0^+} \frac{1}{h}\int_{(t,t+h)} T(t+h-s)f(s)\; d\lambda(s) = f(t).$$

Consequently, (4.9) yields $v(t) \in D(A)$ and

$$Av(t) = v'(t) - f(t).$$

It remains to prove that if $J = (0, t_0]$ for some $t_0 > 0$, $v(t_0) \in D(A)$ and $v'(t_0) = Av(t_0) + f(t_0)$. Since v is continuous we have $v(t) \xrightarrow{t\to t_0^-} v(t_0)$ and by continuity of u' and f also $Av(t) = v'(t) - f(t) \xrightarrow{t \to t_0^-} v'(t_0) - f(t_0)$. A being closed yields $v(t_0) \in D(A)$ and $Av(t_0) = v'(t_0) - f(t_0)$. By Proposition 3.1.2, f), the function

$$t \mapsto T(t)x_0 + \int_{(0,t)} T(t-s)f(s)\ d\lambda(s) = T(t)x_0 + v(t)$$

is a classical solution of (4.4).

□

Chapter 5

Maximal Regularity

We again study the problem

$$\begin{cases} u'(t) = Au(t) + f(t), & t \in J, \\ u(0) = 0, \end{cases} \tag{5.1}$$

where either $J = (0, t_0]$ for some $t_0 > 0$ or $J = (0, +\infty)$, $f \in L^p(J; X)$ for some $p \in (1, +\infty)$ and A is the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geq 0}$. Furthermore, as before we set $J_0 := J \cup \{0\}$. We will use these notations throughout the present chapter.

We want to find conditions such that u, u' and Au have the same regularity properties as f, i.e. $u, u', Au \in L^p(J; X)$, which leads to the notion of maximal regularity.

We will see that this property is strongly linked to the properties of A and the semigroup $\big(T(t)\big)_{t\geq 0}$, more precisely, it is necessary for $\big(T(t)\big)_{t\geq 0}$ to be extendable to an analytic semigroup in order to achieve maximal regularity for the problem (5.1).

Unfortunately, the reverse statement is not true for all Banach spaces, but we will impose further conditions on X to ensure sufficiency.

5.1 Maximal Regularity and Analytic Semigroups

5.1.1 Definition. Let $p \in (1, +\infty)$. We say that A has the *maximal L^p-regularity property* on J or A is *maximally L^p-regular* on J, if there is a constant $C > 0$ such that for every $f \in L^p(J; X)$ problem (5.1) has a unique weak solution $u : J_0 \to X$ such that u' and Au are contained in $L^p(J; X)$ and

$$\|u'\|_{L^p} + \|Au\|_{L^p} \leq C \|f\|_{L^p} .$$

We say that A has the *strict maximal L^p-regularity property* on J or A is *strictly maximally L^p-regular* on J, if in addition to $u', Au \in L^p(J; X)$ also $u \in L^p(J; X)$ and there exists a constant $C > 0$, such that

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} \leq C \|f\|_{L^p} .$$

5.1.2 Remark. We can weaken the requirements on u in the definition above. For a weak solution u by (5.1) we have

$$\|u'\|_{L^p} \leq \|Au\|_{L^p} + \|f\|_{L^p}$$

as well as

$$\|Au\|_{L^p} \leq \|u'\|_{L^p} + \|f\|_{L^p}.$$

Hence, in order to prove maximal regularity, it suffices to show $\|Au\|_{L^p} \leq C\,\|f\|_{L^p}$ for all $f \in L^p(J;X)$ and some $C > 0$.

First we want to prove that every operator with the maximal regularity property generates an analytic semigroup. To that end we start with a lemma.

5.1.3 Lemma. Let $p \in (1, +\infty)$.

a) There exists a function $\varepsilon : [0,+\infty) \to [0,+\infty)$ satisfying $\lim_{s\to+\infty} \varepsilon(s) = 0$, such that whenever A has the maximal L^p-regularity property on $(0, t_0]$ for some $t_0 > 0$ and $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) \geq \frac{1}{t_0}$, there exist bounded linear operators $R_{\xi,t_0}, P_{\xi,t_0} \in L_b(X)$, that commute with A on $D(A)$ and a constant $K > 0$ satisfying $\|R_{\xi,t_0}\| \leq \frac{1}{|\xi|}K$, $\|P_{\xi,t_0}\| \leq \varepsilon(\mathrm{Re}(\xi)t_0)$ as well as

$$AR_{\xi,t_0}x = P_{\xi,t_0}x - x + \xi R_{\xi,t_0}x, \quad x \in X.$$

b) If $\big(T(t)\big)_{t\geq 0}$ is bounded and A has the strict maximal L^p-regularity property on $(0, +\infty)$, then there exists a constant $L > 0$ and for every $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > 0$ a bounded linear operator $R_\xi \in L_b(X)$, which commutes with A on $D(A)$, satisfying $\|R_\xi\| \leq L$ for every $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > 0$ and

$$AR_\xi x = \xi R_\xi x - x, \quad x \in X.$$

Proof. Let $x \in X$, $t_0 > 0$ and $\xi := a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and $a \geq \frac{1}{t_0}$. We define $f_{\xi,x}(t) := \mathbb{1}_{[0,\frac{1}{a}]}(t)e^{\xi t}x$ and observe that $f_{\xi,x} \in L^p\big((0,+\infty);X\big)$. In fact,

$$\|f_{\xi,x}\|_{L^p((0,t_0];X)} = \|f_{\xi,x}\|_{L^p((0,+\infty);X)} = \left(\frac{e^p - 1}{ap}\right)^{\frac{1}{p}} \|x\|.$$

We want to prove that $u_{\xi,x} : [0, t_0] \to X$ defined by

$$u_{\xi,x}(t) := \int_{(0,t)} T(t-s)f_{\xi,x}(s)\; d\lambda(s)$$

is a weak solution of

$$\begin{cases} u'_{\xi,x}(t) = Au_{\xi,x}(t) + f_{\xi,x}(t), \; t \in (0, t_0], \\ u_{\xi,x}(0) = 0. \end{cases} \tag{5.2}$$

Note that $u_{\xi,x}$ is well-defined and continuous by Lemma 4.2.3. Employing Theorem 4.2.8 yields that $u_{\xi,x}|_{[0,\frac{1}{a}]}$ is a classical solution of

$$\begin{cases} u'_{\xi,x}(t) = Au_{\xi,x}(t) + f_{\xi,x}(t), \ t \in (0, \frac{1}{a}], \\ u_{\xi,x}(0) = 0, \end{cases}$$

implying $u_{\xi,x}|_{(0,\frac{1}{a}]}$ is continuously differentiable and $u_{\xi,x}(t) \in D(A)$ for all $t \in (0, \frac{1}{a}]$. Moreover, for $t \in [0, t_0 - \frac{1}{a}]$ by Proposition 2.3.6, c), we have

$$\begin{aligned} T(t)u_{\xi,x}(\tfrac{1}{a}) &= T(t)\Big(\int_{(0,\frac{1}{a})} T(\tfrac{1}{a} - s)f_{\xi,x}(s)\ d\lambda(s)\Big) = \int_{(0,\frac{1}{a})} T(t + \tfrac{1}{a} - s)f_{\xi,x}(s)\ d\lambda(s) \\ &= \int_{(0,t+\frac{1}{a})} T(t + \tfrac{1}{a} - s)f_{\xi,x}(s)\ d\lambda(s) = u_{\xi,x}(t + \tfrac{1}{a}), \end{aligned}$$

implying $u_{\xi,x}|_{[\frac{1}{a},t_0]}$ is continuously differentiable; see Proposition 3.1.2, f). Furthermore, given $t \in [\frac{1}{a}, t_0]$

$$u'_{\xi,x}(t) = \frac{d}{dt}T(t - \tfrac{1}{a})u_{\xi,x}(\tfrac{1}{a}) = AT(t - \tfrac{1}{a})u_{\xi,x}(\tfrac{1}{a}) = Au_{\xi,x}(t) = Au_{\xi,x}(t) + f_{\xi,x}(t).$$

Given $\varphi \in C_{00}^{\infty}\big((0, t_0); \mathbb{C}\big)$ and $0 < a < b < t_0$ such that $\operatorname{supp}(\varphi) \subseteq [a, b] \subseteq (0, t_0)$, we note that $u'_{\xi,x}$ is continuous and bounded on any compact subset of $(0, t_0] \setminus \{\frac{1}{a}\}$ and, in turn, integrable over (a, b); see Theorem 2.4.2. Employing Proposition 2.4.5 yields

$$\begin{aligned} \int_{(0,t_0)} \varphi'(t)u_{\xi,x}(t)\ d\lambda(t) &= \int_{(a,b)} \varphi'(t)u_{\xi,x}(t)\ d\lambda(t) \\ &= \varphi(b)u_{\xi,x}(b) - \varphi(a)u_{\xi,x}(a) - \int_{(a,b)} \varphi(t)u'_{\xi,x}(t)\ d\lambda(t) \\ &= -\int_{(0,t_0)} \varphi(t)\big(Au_{\xi,x}(t) + f_{\xi,x}(t)\big)\ d\lambda(t). \end{aligned}$$

Together with $u_{\xi,x}(0) = 0$ and $u_{\xi,x}(t) \in D(A)$ for all $t \in (0, t_0]$ we see that $u_{\xi,x}$ is a weak solution of (5.2). By the same arguments, $u_{\xi,x} : [0, +\infty) \to X$ defined by

$$u_{\xi,x}(t) = \int_{(0,t)} T(t - s)f_{\xi,x}(s)\ d\lambda(s)$$

is a weak solution of

$$\begin{cases} u'_{\xi,x}(t) = Au_{\xi,x}(t) + f_{\xi,x}(t), \quad t \in (0, +\infty), \\ u_{\xi,x}(0) = 0, \end{cases}$$

and satisfies $u_{\xi,x}|_{(0,\frac{1}{a}]} \in C^1\big((0, \frac{1}{a}]; X\big)$ as well as $u_{\xi,x}|_{(\frac{1}{a},+\infty)} \in C^1\big((\frac{1}{a}, +\infty); X\big)$.

ad a): For $x \in X$ and $t_0 > 0$ by $u_{\xi,x}$'s continuity we can define $R_{\xi,t_0} : X \to X$ by

$$R_{\xi,t_0}x = a\int_{(0,t_0)} e^{-\xi t}u_{\xi,x}(t)\ d\lambda(t).$$

Since $x \mapsto f_{\xi,x}$ is linear and $u_{\xi,x}$ depends linearly on $f_{\xi,x}$, also $x \mapsto u_{\xi,x}$ and R_{ξ,t_0} are linear. $Au_{\xi,x} \in L^p\big((0,t_0];X\big)$ together with Hölder's inequality, Proposition 2.5.3, a), yields

$$\begin{aligned}\int_{(0,t_0)} \big\|e^{-\xi t}Au_{\xi,x}(t)\big\|\, d\lambda(t) &= \int_{(0,t_0)} |e^{-\xi t}|\, \|Au_{\xi,x}(t)\|\, d\lambda(t)\\ &\leq \big\|t \mapsto e^{-\xi t}\big\|_{L^q}\, \|Au_{\xi,x}\|_{L^p} < +\infty,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. By the same arguments $u'_{\xi,x} \in L^p\big((0,t_0];X\big)$ implies the integrability of $t \mapsto e^{-\xi t}u'_{\xi,x}(t)$. By Proposition 2.3.13

$$\begin{aligned}AR_{\xi,t_0}x &= a\int_{(0,t_0)} e^{-\xi t}Au_{\xi,x}(t)\ d\lambda(t) = a\int_{(0,t_0)} e^{-\xi t}(u'_{\xi,x}(t) - f_{\xi,x}(t))\ d\lambda(t)\\ &= a\int_{(0,t_0)} e^{-\xi t}u'_{\xi,x}(t)\ d\lambda(t) - a\int_{(0,\frac{1}{a})} e^{-\xi t}e^{\xi t}x\ d\lambda(t)\\ &= a\int_{(0,t_0)} e^{-\xi t}u'_{\xi,x}(t)\ d\lambda(t) - x.\end{aligned}$$

$u_{\xi,x}|_{(0,\frac{1}{a}]}$ and $u_{\xi,x}|_{(\frac{1}{a},t_0]}$ being continuously differentiable and Proposition 2.4.5 yield

$$\begin{aligned}a\int_{(0,t_0)} e^{-\xi t}u'_{\xi,x}(t)\ d\lambda(t) &= ae^{-\xi t_0}u_{\xi,x}(t_0) + a\xi\int_{(0,t_0)} e^{-\xi t}u_{\xi,x}(t)\ d\lambda(t)\\ &= ae^{-\xi t_0}u_{\xi,x}(t_0) + \xi R_{\xi,t_0}x\end{aligned}$$

and in turn

$$AR_{\xi,t_0}x = ae^{-\xi t_0}u_{\xi,x}(t_0) + \xi R_{\xi,t_0}x - x.$$

Since $u_{\xi,x}$ is continuous, it is bounded on $[0,t_0]$ and measurable. Consequently,

$$\int_{(0,t_0)} \|u_{\xi,x}(t)\|^p\ d\lambda(t) \leq t_0\big(\sup_{t\in[0,t_0]} \|u_{\xi,x}(t)\|\big)^p < +\infty$$

and $u_{\xi,x} \in W^{1,p}\big((0,t_0);X\big)$. Employing Corollary 2.5.8 yields

$$u_{\xi,x}(t_0) = \int_{(0,t_0)} u'_{\xi,x}(t)\ d\lambda(t).$$

We define the linear operator $P_{\xi,t_0} : X \to X$ by $P_{\xi,t_0}x := ae^{-\xi t_0}u_{\xi,x}(t_0)$. Proposition 2.5.3, a), yields

$$\begin{aligned}\|P_{\xi,t_0}x\| &= |ae^{-\xi t_0}|\, \|u_{\xi,x}(t_0)\| = ae^{-at_0}\left\|\int_{(0,t_0)} u'_{\xi,x}(t)\ d\lambda(t)\right\|\\ &\leq ae^{-at_0}\int_{(0,t_0)} \big\|u'_{\xi,x}(t)\big\|\ d\lambda(t) \leq ae^{-at_0}t_0^{\frac{1}{q}}\, \big\|u'_{\xi,x}\big\|_{L^p}\\ &\leq Cae^{-at_0}t_0^{\frac{1}{q}}\, \|f_{\xi,x}\|_{L^p} = Cae^{-at_0}t_0^{\frac{1}{q}}\left(\frac{e^p-1}{ap}\right)^{\frac{1}{p}}\|x\|\end{aligned}$$

$$= C(at_0)^{\frac{1}{q}} e^{-at_0} \Big(\frac{e^p - 1}{p} \Big)^{\frac{1}{p}} \|x\|,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If we set

$$\varepsilon(s) := C s^{\frac{1}{q}} e^{-s} \Big(\frac{e^p - 1}{p} \Big)^{\frac{1}{p}},$$

then $\|P_{\xi,t_0}\| \le \varepsilon(at_0)$.
Clearly $\varepsilon(s) \xrightarrow{s\to+\infty} 0$, and, by continuity, ε is bounded on $[0,+\infty)$. As $\|u'_{\xi,x}\|_{L^p} \le C \|f_{\xi,x}\|_{L^p}$ and

$$\int_{(0,t_0)} |e^{-at}|^q \; d\lambda(t) = \Big(\frac{1 - e^{-at_0 q}}{aq} \Big) < +\infty,$$

due to Hölder's inequality, Proposition 2.5.3, a), and Proposition 2.4.5,

$$\begin{aligned}
\|R_{\xi,t_0} x\| &= \Big\| a \int_{(0,t_0)} e^{-\xi t} u_{\xi,x}(t) \; d\lambda(t) \Big\| = \Big\| -\frac{ae^{-\xi t_0}}{\xi} u_{\xi,x}(t_0) + \frac{a}{\xi} \int_{(0,t_0)} e^{-\xi t} u'_{\xi,x}(t) \; d\lambda(t) \Big\| \\
&= \Big\| \frac{a}{\xi} \int_{(0,t_0)} e^{-\xi t} u'_{\xi,x}(t) \; d\lambda(t) - \frac{1}{\xi} P_{\xi,t_0} x \Big\| \\
&\le \frac{1}{|\xi|} \|P_{\xi,t_0} x\| + \frac{a}{|\xi|} \int_{(0,t_0)} e^{-at} \|u'_{\xi,x}(t)\| \; d\lambda(t) \\
&\le \frac{1}{|\xi|} \|P_{\xi,t_0}\| \|x\| + \frac{a}{|\xi|} \Big(\frac{1 - e^{-at_0 q}}{aq} \Big)^{\frac{1}{q}} \|u'_{\xi,x}\|_{L^p} \\
&\le \frac{1}{|\xi|} \|P_{\xi,t_0}\| \|x\| + \frac{Ca}{|\xi|} \Big(\frac{1 - e^{-at_0 q}}{aq} \Big)^{\frac{1}{q}} \|f\|_{L^p} \\
&= \frac{1}{|\xi|} \Big(\|P_{\xi,t_0}\| + Ca \big(\frac{1 - e^{-at_0 q}}{aq} \big)^{\frac{1}{q}} \big(\frac{e^p - 1}{ap} \big)^{\frac{1}{p}} \Big) \|x\| \\
&\le \frac{1}{|\xi|} \Big(\varepsilon(at_0) + C \big(\frac{1 - e^{-at_0 q}}{q} \big)^{\frac{1}{q}} \big(\frac{e^p - 1}{p} \big)^{\frac{1}{p}} \Big) \|x\| \le \frac{1}{|\xi|} \underbrace{\Big(\sup_{s \ge 0} \varepsilon(s) + \frac{Ce}{p^{\frac{1}{p}} q^{\frac{1}{q}}} \Big)}_{=:K} \|x\|.
\end{aligned}$$

Hence, $R_{\xi,t_0} \in L_b(X)$ satisfies $\|R_{\xi,t_0}\| \le \frac{1}{|\xi|} K$. It remains to show that A commutes with R_{ξ,t_0} and P_{ξ,t_0}. For $x \in D(A)$ by Lemma 4.2.3 $s \mapsto T(t-s) f_{\xi,Ax}(s)$ is integrable. Employing Proposition 2.3.13 we have

$$\begin{aligned}
u_{\xi,Ax}(t) &= \int_{(0,t)} T(t-s) f_{\xi,Ax}(s) \; d\lambda(s) = \int_{(0,\frac{1}{a})} T(t-s) e^{\xi s} Ax \; d\lambda(s) \\
&= A \int_{(0,t)} T(t-s) f_{\xi,x}(s) \; d\lambda(s) = A u_{\xi,x}(t)
\end{aligned}$$

and in turn

$$R_\xi Ax = a \int_{(0,t_0)} e^{-\xi t} u_{\xi,Ax}(t) \; d\lambda(t) = aA \int_{(0,t_0)} e^{-\xi t} u_{\xi,x}(t) \; d\lambda(t) = AR_\xi x$$

as well as

$$P_\xi Ax = ae^{-\xi t_0}u_{\xi,Ax}(t_0) = A\big(ae^{-\xi t_0}u_{\xi,x}(t_0)\big) = AP_\xi x.$$

ad b): Suppose that $\big(T(t)\big)_{t\geq 0}$ is bounded and A has the strict maximal L^p-regularity property on $(0,+\infty)$. Let $\xi := a+ib \in \mathbb{C}$ with $a,b\in\mathbb{R}$ and $a>0$. Since $f_{\xi,x}\in L^p\big((0,+\infty);X\big)$, there exists a unique weak solution $u_{\xi,x}$ solving

$$\begin{cases} u'_{\xi,x}(t) = Au_{\xi,x}(t) + f_{\xi,x}(t), \quad t\in(0,+\infty),\\ u_{\xi,x}(0)=0,\end{cases}$$

and a constant $C>0$, such that

$$\|u_{\xi,x}\|_{L^p} + \left\|u'_{\xi,x}\right\|_{L^p} + \|Au_{\xi,x}\|_{L^p} \leq C\,\|f_{\xi,x}\|_{L^p}\,.$$

For $q\in(1,+\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\int_{(0,+\infty)} |e^{-\xi t}|^q \; d\lambda(t) = \int_0^{+\infty} e^{-atq}\; d\lambda(t) = \frac{1}{aq} < +\infty.$$

According to Proposition 2.5.3, a), $t\mapsto e^{-\xi t}u_{\xi,x}(t)$ is integrable and we can define $R_\xi : X\to X$ by

$$R_\xi x = a\int_{(0,+\infty)} e^{-\xi t}u_{\xi,x}(t)\; d\lambda(t).$$

By the same arguments as in a), R_ξ is linear. Again by Hölder's inequality, Proposition 2.5.3, a), as well as the strict maximal regularity property

$$\begin{aligned}\|R_\xi x\| &\leq a\int_{(0,+\infty)} |e^{-\xi t}|\,\|u_{\xi,x}(t)\|\; d\lambda(t) \leq \frac{a\,\|u_{\xi,x}\|_{L^p}}{(aq)^{\frac{1}{q}}} \leq \frac{Ca^{\frac{1}{p}}\,\|f_{\xi,x}\|_{L^p}}{q^{\frac{1}{q}}}\\ &\leq \frac{Ca^{\frac{1}{p}}(e^p-1)^{\frac{1}{p}}}{(ap)^{\frac{1}{p}}q^{\frac{1}{q}}}\,\|x\| = \frac{C(e^p-1)^{\frac{1}{p}}}{p^{\frac{1}{p}}q^{\frac{1}{q}}}\,\|x\|\,,\quad x\in X,\end{aligned} \tag{5.3}$$

and in turn

$$\|R_\xi\| \leq \frac{C(e^p-1)^{\frac{1}{p}}}{p^{\frac{1}{p}}q^{\frac{1}{q}}} =: L.$$

Since $u'_{\xi,x}, Au_{\xi,x}\in L^p\big((0,+\infty);X\big)$, $t\mapsto e^{-\xi t}Au_{\xi,x}(t)$ and $t\mapsto e^{-\xi t}u'_{\xi,x}(t)$ are integrable by Hölder's inequality. By Proposition 2.3.13

$$\begin{aligned}AR_\xi x &= a\int_{(0,+\infty)} e^{-\xi t}Au_{\xi,x}(t)\; d\lambda(t) = a\int_{(0,+\infty)} e^{-\xi t}(u'_{\xi,x}(t) - f_{\xi,x}(t))\; d\lambda(t)\\ &= a\int_{(0,+\infty)} e^{-\xi t}u'_{\xi,x}(t)\; d\lambda(t) - a\int_{(0,\frac{1}{a})} e^{-\xi t}e^{\xi t}x\; d\lambda(t)\\ &= a\int_{(0,+\infty)} e^{-\xi t}u'_{\xi,x}(t)\; d\lambda(t) - x.\end{aligned} \tag{5.4}$$

Since $\big((T(t)\big)_{t\geq 0}$ is bounded, there is a constant $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. Hence, for $t \geq \frac{1}{a}$

$$\|u_{\xi,x}(t)\| = \left\| \int_{(0,t)} T(t-s) f_{\xi,x}(s)\ ds \right\| = \left\| \int_{(0,\frac{1}{a})} T(t-s) e^{\xi s} x\ ds \right\| \tag{5.5}$$
$$\leq \int_{(0,\frac{1}{a})} \|T(t-s)\| e^{as} \|x\|\ ds \leq M \|x\| \int_{(0,\frac{1}{a})} e^{as}\ ds = \frac{M\|x\|(e-1)}{a}.$$

For $n \in \mathbb{N}$ we define $g_n : [0,+\infty) \to X$ by $g_n(t) := \mathbb{1}_{[0,n)}(t) e^{-\xi t} u'_{\xi,x}(t)$ as well as $h_n : [0,+\infty) \to X$ by $h_n(t) := \mathbb{1}_{[0,n)} e^{-\xi t} u_{\xi,x}(t)$. Clearly, $g_n(t) \xrightarrow{n\to+\infty} e^{-\xi t} u'_{\xi,x}(t)$ and $h_n(t) \xrightarrow{n\to+\infty} e^{-\xi t} u_{\xi,x}(t)$ as well as $\|g_n(t)\| \leq \left\|e^{-\xi t} u'_{\xi,x}(t)\right\|$ and $\|h_n(t)\| \leq \left\|e^{-\xi t} u_{\xi,x}(t)\right\|$ for all $t \geq 0$. By Proposition 2.4.5 and Theorem 2.3.7 we obtain

$$\begin{aligned}
\int_{(0,+\infty)} e^{-\xi t} u'_{\xi,x}(t)\ d\lambda(t) &= \lim_{n\to+\infty} \int_{(0,+\infty)} g_n(t)\ d\lambda(t) = \lim_{n\to+\infty} \int_{(0,n)} e^{-\xi t} u'_{\xi,x}(t)\ d\lambda(t) \\
&= \lim_{n\to+\infty} e^{-\xi n} u_{\xi,x}(n) + \xi \int_{(0,n)} e^{-\xi t} u_{\xi,x}(t)\ d\lambda(t) \\
&= \lim_{n\to+\infty} e^{-\xi n} u_{\xi,x}(n) + \xi \int_{(0,n)} h_n(t)\ d\lambda(t) \\
&= \xi \int_{(0,+\infty)} e^{-\xi t} u_{\xi,x}(t)\ d\lambda(t) = \frac{\xi}{a} R_\xi x
\end{aligned}$$

because of (5.5) and $\mathrm{Re}(\xi) = a > 0$. From (5.4) we conclude

$$AR_\xi x = a \int_{(0,+\infty)} e^{-\xi t} u'_{\xi,x}(t)\ d\lambda(t) - x = \xi R_\xi x - x.$$

Given $x \in D(A)$, $s \mapsto T(t-s) f_{\xi,Ax}$ is integrable by Lemma 4.2.3. Employing Proposition 2.3.13 yields

$$\begin{aligned}
u_{\xi,Ax}(t) &= \int_{(0,t)} T(t-s) f_{\xi,Ax}(s)\ d\lambda(s) = \int_{(0,\frac{1}{a})} T(t-s) e^{\xi s} Ax\ d\lambda(s) \\
&= A \int_{(0,t)} T(t-s) f_{\xi,x}(s)\ d\lambda(s) = A u_{\xi,x}(t)
\end{aligned}$$

and in turn

$$R_\xi Ax = a \int_0^{+\infty} e^{-\xi t} u_{\xi,Ax}(t)\ d\lambda(t) = aA \int_0^{+\infty} e^{-\xi t} u_{\xi,x}(t)\ d\lambda(t) = AR_\xi x.$$

□

5.1.4 Theorem. *Let $p \in (1,+\infty)$ and $t_0 > 0$. If A has the maximal L^p-regularity property on $(0,t_0]$, then A generates an analytic semigroup.*

Proof. Let $\xi = a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$ and $a \geq \frac{1}{t_0}$. From Lemma 5.1.3, $a)$, we know that there exists a function $\varepsilon : [0, +\infty) \to [0, +\infty)$ as well as bounded linear operators $R_\xi, P_\xi \in L_b(X)$ satisfying $\lim_{s\to+\infty} \varepsilon(s) = 0$ and $\|R_\xi\| \leq \frac{1}{|\xi|}K$, $\|P_\xi\| \leq \varepsilon(at_0)$ for some constant $K > 0$. Moreover,

$$AR_\xi x = P_\xi x - x + \xi R_\xi x, \quad x \in X.$$

Let $\tau > \frac{1}{t_0}$ such that $\|P_\xi\| \leq \varepsilon(at_0) \leq \frac{1}{2}$ for all $\xi \in \mathbb{C}$ with $a > \tau$. Given $a > \tau$, $I - P_\xi$ is invertible and $\|(I - P_\xi)^{-1}\| \leq 2$; see Lemma 6.3.9 in [12]. Hence,

$$(\xi I - A)R_\xi(I - P_\xi)^{-1}x = x, \quad x \in X.$$

Since R_ξ and P_ξ commute with A on $D(A)$, we obtain $R(\xi, A) = R_\xi(I - P_\xi)^{-1}$ and consequently

$$\|R(\xi, A)\| \leq \frac{2K}{|\xi|} \leq \frac{2K}{|\operatorname{Im}(\xi)|}$$

for every $\xi \in \mathbb{C}$ satisfying $\operatorname{Re}(\xi) > \tau$ and $\operatorname{Im}(\xi) \neq 0$. Let M, ω be as in Proposition 3.1.2, $a)$. Assuming w.l.o.g. $\omega \geq \tau$, Corollary 3.3.6 yields that A generates an analytic semigroup.

□

5.1.5 Theorem. *Let $p \in (1, +\infty)$. If A has the maximal L^p-regularity property on $(0, +\infty)$, then it has the maximal L^p-regularity property on $(0, t_0]$ for all $t_0 > 0$.*

Proof. Let $t_0 > 0$, $f \in L^p\big((0, t_0)]; X\big)$ and set $\tilde{f} := \mathbb{1}_{(0,t_0]}f$. Clearly, $\left\|\tilde{f}\right\|_{L^p((0,+\infty);X)} = \|f\|_{L^p((0,t_0];X)}$. Since A is maximally L^p-regular, there is a uniqe weak solution u_∞ of

$$\begin{cases} u_\infty'(t) = Au_\infty(t) + \tilde{f}(t), & t \in (0, +\infty), \\ u_\infty(0) = 0, \end{cases}$$

satisfying

$$\|Au_\infty\|_{L^p((0,+\infty);X)} \leq C\left\|\tilde{f}\right\|_{L^p((0,+\infty);X)}$$

for some $C > 0$. The restriction u of u_∞ to $[0, t_0]$ is a weak solution of (5.1) on $(0, t_0]$ satisfying

$$\|Au\|_{L^p((0,t_0);X)} \leq \|Au_\infty\|_{L^p((0,+\infty);X)} \leq C\left\|\tilde{f}\right\|_{L^p((0,+\infty);X)} = C\,\|f\|_{L^p((0,t_0];X)}.$$

It remains to show that u is unique. Let $\tilde{u} : [0, t_0] \to X$ be another weak solution of (5.1) on $(0, t_0]$ and define $v := u - \tilde{u}$. v is a weak solution of

$$\begin{cases} v'(t) = Av(t), & t \in (0, t_0], \\ v(0) = 0. \end{cases}$$

Let $\tilde{t} \in (0, t_0)$ be such that $v(\tilde{t}) \in D(A)$ (see Definition 4.2.1), and define $\tilde{v} : [0, +\infty) \to X$ by

$$\tilde{v}(t) := \begin{cases} v(t), & t \in [0, \tilde{t}], \\ T(t - \tilde{t})v(\tilde{t}), & t > \tilde{t}. \end{cases}$$

By Proposition 3.1.2, b), $\tilde{v}$ is continuous. $v(\tilde{t}) \in D(A)$ implies $\tilde{v}(t) \in D(A)$ for almost every $t \in (0, +\infty)$ by Proposition 3.1.2, f). Let $\varphi \in C_{00}^{\infty}\big((0, +\infty); \mathbb{C}\big)$ and $0 < a < b < +\infty$ such that $\operatorname{supp}(\varphi) \subseteq [a, b]$ and w.l.o.g. $\tilde{t} \in [a, b]$. Since v is weakly differentiable on $(0, t_0)$, we have $v, Av \in L^1_{\text{loc}}\big((0, t_0); X\big)$ and, in turn, $v \in W^{1,1}\big((a, \tilde{t}); X\big)$. By Corollary 2.5.9

$$\begin{aligned} \int_{(0,\tilde{t})} \varphi'(t)v(t)\; d\lambda(t) &= \int_{(a,\tilde{t})} \varphi'(t)v(t)\; d\lambda(t) \\ &= \varphi(\tilde{t})v(\tilde{t}) - \varphi(a)v(a) - \int_{(a,\tilde{t})} \varphi(t)Av(t)\; d\lambda(t) \\ &= \varphi(\tilde{t})v(\tilde{t}) - \int_{(0,\tilde{t})} \varphi(t)Av(t)\; d\lambda(t). \end{aligned}$$

Proposition 3.1.2, f) and Proposition 2.4.5 yield

$$\begin{aligned} \int_{(\tilde{t},+\infty)} \varphi'(t)T(t-\tilde{t})v(\tilde{t})\; d\lambda(t) &= \int_{(\tilde{t},b)} \varphi'(t)T(t-\tilde{t})v(\tilde{t})\; d\lambda(t) \\ &= \varphi(b)T(b-\tilde{t})v(\tilde{t}) - \varphi(\tilde{t})v(\tilde{t}) - \int_{(\tilde{t},b)} \varphi(t)AT(t-\tilde{t})v(\tilde{t})\; d\lambda(t) \\ &= -\varphi(\tilde{t})v(\tilde{t}) - \int_{(\tilde{t},+\infty)} \varphi(t)AT(t-\tilde{t})v(\tilde{t})\; d\lambda(t). \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{(0,+\infty)} \varphi'(t)\tilde{v}(t)\; d\lambda(t) &= \int_{(0,\tilde{t})} \varphi'(t)v(t)\; d\lambda(t) + \int_{(\tilde{t},+\infty)} \varphi'(t)T(t-\tilde{t})v(\tilde{t})\; d\lambda(t) \\ &= -\int_{(0,\tilde{t})} \varphi(t)\big(Av(t)\big)\; d\lambda(t) - \int_{(\tilde{t},+\infty)} \varphi(t)AT(t-\tilde{t})v(\tilde{t})\; dt\lambda(t) \\ &= -\int_{(0,+\infty)} \varphi(t)\big(A\tilde{v}(t)\big)\; d\lambda(t). \end{aligned}$$

Together with $\tilde{v}(0) = v(0) = 0$ we conclude that $\tilde{v}$ is a weak solution of

$$\begin{cases} \tilde{v}'(t) = A\tilde{v}(t), & t \in [0, +\infty), \\ \tilde{v}(0) = 0. \end{cases}$$

Since A is maximally L^p-regular on $[0, +\infty)$ and the constant zero function is a weak solution of the above problem, $\tilde{v}(t) = 0$ for every $t \in [0, +\infty)$ and therefore $v(t) = 0$ for every $t \in [0, \tilde{t}]$. Since $v(\tilde{t}) \in D(A)$ holds true up to a null set, we conclude that $v(t) = 0$ for almost every $t \in (0, t_0]$. v being continuous yields $v(t) = 0$ and in turn $u(t) = \tilde{u}(t)$ for every $t \in [0, t_0]$.

□

Combining the previous two results yields the following corollary.

5.1.6 Corollary. Let $p \in (1, +\infty)$. If A has the maximal L^p-regularity property on $(0, +\infty)$, then A generates a bounded analytic semigroup.

Proof. By Theorem 5.1.5 A is maximally L^p-regular on $(0, t_0]$ for all $t_0 > 0$. Let $\xi := a + ib \in \mathbb{C}$ with $a, b \in \mathbb{R}$, $a > 0$ and $t_1 > 0$ be such that $a \geq \frac{1}{t_1}$. By Lemma 5.1.3, a), there is a function $\varepsilon : [0, +\infty) \to [0, +\infty)$ as well as bounded linear operators $R_{\xi,t_0} P_{\xi,t_0} \in L_b(X)$ satisfying $\lim_{s \to +\infty} \varepsilon(s) = 0$,

$$AR_{\xi,t_0}x = P_{\xi,t_0}x - x + \xi R_{\xi,t_0}x, \quad x \in X$$

and $\|R_{\xi,t_0}\| \leq \frac{1}{|\xi|}K$, $\|P_{\xi,t_0}\| \leq \varepsilon(at_0)$ for some constant $K > 0$. Let $t_2 > t_1$ be such that $\varepsilon(at_0) \leq \frac{1}{2}$ for all $t_0 \geq t_2$. $t_0 \geq t_2$ yields the invertibility of $I - P_{\xi,t_0}$ with $\|(I - P_{\xi,t_0})^{-1}\| \leq 2$; see Lemma 6.3.9 in [12]. Hence,

$$(\xi I - A)R_{\xi,t_0}(I - P_{\xi,t_0})^{-1}x = x.$$

Since R_{ξ,t_0} and P_{ξ,t_0} commute with A on $D(A)$, we obtain $R(\xi, A) = R_{\xi,t_0}(I - P_{\xi,t_0})^{-1}$ and consequently

$$\|R(\xi, A)\| \leq \frac{2K}{|\xi|}$$

for all $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) = a > 0$. By Theorem 3.3.5 A generates a bounded analytic semigroup.

□

5.2 Strict Maximal Regularity

In the present section we study the relation between strict maximal regularity and maximal regularity.

5.2.1 Proposition. Let $p \in (1, +\infty)$ and $t_0 > 0$. The generator A of a strongly continuous semigroup has the maximal L^p-regularity property on $(0, t_0]$ if and only if A has the strict maximal L^p-regularity property on $(0, t_0]$.

Proof. Clearly, strict maximal regularity implies maximal regularity. Suppose that A is maximally L^p-regular. By Theorem 5.1.4 the operator A is the infinitesimal generator of an analytic semigroup, which implies the unique weak solution $u : [0, t_0] \to X$ of (5.1) is given by

$$u(t) := \int_{(0,t)} T(t - s)f(s) \, d\lambda(s)$$

for every $f \in L^p\big((0,t_0];X\big)$; see Proposition 4.2.7. It suffices to show that $u \in L^p\big((0,t_0);X\big)$ and

$$\|u\|_{L^p} \le C\,\|f\|_{L^p}\,, \quad f \in L^p\big((0,t_0];X\big),$$

for some $C > 0$. By continuity and Theorem 2.4.2, $u \in L^p\big((0,t_0];X\big)$. We will show that the operator $S : L^p\big((0,t_0];X\big) \to L^p\big((0,t_0];X\big)$ defined by

$$(Sf)(t) = \int_{(0,t)} T(t-s)f(s)\; d\lambda(s)$$

is bounded. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^p\big((0,t_0];X\big)$ and $f,u \in L^p\big((0,t_0];X\big)$ with $\|f_n - f\|_{L^p} \xrightarrow{n\to+\infty} 0$ and $\|Sf_n - u\|_{L^p} \xrightarrow{n\to+\infty} 0$. By Proposition 2.5.2, *a*), there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ satisfying $Sf_{n_k}(t) \xrightarrow{k\to+\infty} u(t)$ for almost every $t \in (0,t_0]$. Given $t \in (0,t_0]$, we define $\varphi_t : L^p\big((0,t_0];X\big) \to X$ by

$$\varphi_t(g) := (Sg)(t) = \int_{(0,t)} T(t-s)g(s)\; d\lambda(s).$$

For $0 < s < t < t_0$, $g \in L^p\big((0,t_0];X\big)$ and M,ω as in Proposition 3.1.2, *a*) we have

$$\|T(t-s)g(s)\| \le \|T(t-s)\|\,\|g(s)\| \le Me^{\omega(t-s)}\,\|g(s)\| \le Me^{\omega t_0}\,\|g(s)\|$$

and by Proposition 2.5.3, *a*),

$$\begin{aligned}
\|\varphi_t(g)\| &\le \int_{(0,t)} \|T(t-s)g(s)\|\; d\lambda(s) \le \int_{(0,t)} Me^{\omega t_0}\,\|g(s)\|\, d\lambda(s) \\
&\le \Big(\int_{(0,t)} M^q e^{q\omega t_0}\; d\lambda(s)\Big)^{\frac{1}{q}} \cdot \Big(\int_{(0,t)} \|g(s)\|^p\; d\lambda(s)\Big)^{\frac{1}{p}} \\
&\le t^{\frac{1}{q}} Me^{\omega t_0}\,\|g\|_{L^p} \le t_0^{\frac{1}{q}} Me^{\omega t_0}\,\|g\|_{L^p}\,,
\end{aligned}$$

for $\frac{1}{p} + \frac{1}{q} = 1$. Consequently,

$$u(t) = \lim_{k\to+\infty} (Sf_{n_k})(t) = \lim_{k\to+\infty} \varphi_t(f_{n_k}) = \varphi_t(f) = (Sf)(t)$$

for almost every $t \in (0,t_0]$ implying $u = Sf$ in $L^p\big((0,t_0];X\big)$. By the Closed Graph Theorem, 4.4.2 in [12], S is bounded, which means $\|u\|_{L^p} = \|Sf\|_{L^p} \le \|S\|\,\|f\|_{L^p}$, $f \in L^p\big((0,t_0];X\big)$.

□

5.2.2 Theorem. *Let $p \in (1,+\infty)$. The generator A of a strongly continuous semigroup has the strict maximal L^p-regularity property on $(0,+\infty)$ if and only if A has the maximal L^p-regularity property on $(0,+\infty)$ and $0 \in \rho(A)$. In this case there exist constants $M,\delta > 0$, such that $\|T(t)\| \le Me^{-\delta t}$ for all $t \ge 0$.*

Proof. Suppose first that A has the strict maximal L^p-regularity property. By Lemma 5.1.3, $b)$, there exists a bounded operator $R_\xi \in L_b(X)$ such that $AR_\xi x = \xi R_\xi x - x$, $x \in X$, and $\|R_\xi\| \le L$ for all $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > 0$ and some $L > 0$. Since R_ξ commutes with A, we conclude that $R_\xi = R(\xi, A)$ for all $\xi \in \mathbb{C}$ with $\mathrm{Re}(\xi) > 0$. Let $\varepsilon > 0$ and $\xi, \mu \in \mathbb{C}$ with $\mathrm{Re}(\xi), \mathrm{Re}(\mu) > 0$ and $|\xi|, |\mu| < \frac{\varepsilon}{2L^2}$. We employ Proposition 1.2.4, $b)$ and obtain

$$\begin{aligned}\|R(\xi, A) - R(\mu, A)\| &= \|(\xi - \mu)R(\xi, A)R(\mu, A)\| \le |\xi - \mu| \, \|R(\xi, A)\| \, \|R(\mu, A)\| \\ &\le L^2(|\xi| + |\mu|) < L^2\big(\tfrac{\varepsilon}{2L^2} + \tfrac{\varepsilon}{2L^2}\big) < \varepsilon.\end{aligned}$$

Therefore, $(R(\xi, A))_{\mathrm{Re}(\xi)>0}$ is a Cauchy net in $L_b(X)$ for $\xi \to 0$. Denoting its limit by $R \in L_b(X)$ we obtain for $x \in D(A)$

$$\begin{aligned}R(-Ax) &= \lim_{\xi \to 0} -R(\xi, A)Ax = \lim_{\xi \to 0} R(\xi, A)(\xi I - A)x - \xi R(\xi, A)x \\ &= x - \lim_{\xi \to 0} \xi R(\xi, A)x = x.\end{aligned} \tag{5.6}$$

For $x \in X$ we have $R(\xi, A)x \xrightarrow{\xi \to 0} Rx$ and, in turn,

$$AR(\xi, A)x = (A - \xi I)R(\xi, A)x + \xi R(\xi, A)x = \xi R(\xi, A)x - x \xrightarrow{\xi \to 0} -x.$$

By the closedness of A we obtain $Rx \in D(A)$ and $ARx = -x$ which together with (5.6) implies $R(0, A) = R \in L_b(X)$, and hence $0 \in \rho(A)$.

For the converse suppose that A is maximally L^p-regular and $0 \in \rho(A)$. By Corollary 5.1.6 the generator A is the infinitesimal generator of a bounded analytic semigroup. Since every analytic semigroup is differentiable, given $f \in L^p\big((0, +\infty); X\big)$, by Proposition 4.2.7 the unique weak solution u of (5.1) has the form

$$u(t) = \int_{(0,t)} T(t - s)f(s) \, d\lambda(s).$$

By Theorem 3.3.5 A is sectorial of some angle $\delta \in (0, \frac{\pi}{2})$. Given $\theta \in (\frac{\pi}{2}, \frac{\pi}{2} + \delta)$, due to Theorem 3.3.3

$$T(t) = \frac{1}{2\pi i} \int_\gamma e^{\xi t} R(\xi, A) \, d\xi, \quad t > 0,$$

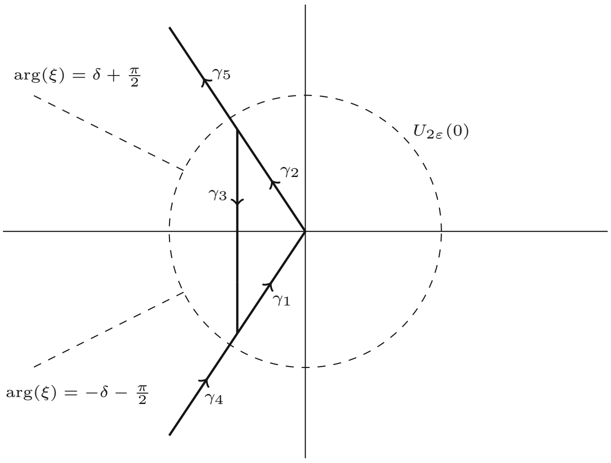
where $\gamma(s) := -se^{-i\theta}$ for $s \le 0$ and $\gamma(s) := se^{i\theta}$ for $s > 0$. Since $\rho(A)$ is open and $0 \in \rho(A)$, there exists $\varepsilon > 0$, such that $U_{2\varepsilon}(0) \subseteq \rho(A)$. We define

$$\begin{aligned}\gamma_1 &: [-\varepsilon, 0] \to \mathbb{C}, \ \gamma_1(s) := -se^{-i\theta}, \\ \gamma_2 &: [0, \varepsilon] \to \mathbb{C}, \ \gamma_2(s) := se^{i\theta}, \\ \gamma_3 &: [-\varepsilon\sin(\theta), \varepsilon\sin(\theta)] \to \mathbb{C}, \ \gamma_3(s) := \varepsilon\cos(\theta) - is, \\ \gamma_4 &: (-\infty, -\varepsilon] \to \mathbb{C}, \ \gamma_4(s) := -se^{-i\theta}\end{aligned}$$

and

$$\gamma_5 : [\varepsilon, +\infty) \to \mathbb{C}, \ \gamma_5(s) := se^{i\theta}$$

as well as $\tilde{\gamma}_1 := \gamma_1\gamma_2\gamma_3$ and $\tilde{\gamma}_2 := \gamma_4\gamma_3^-\gamma_5$.



Since $\xi \mapsto e^{\xi t} R(\xi, A)$ is analytic on $\rho(A) \supseteq U_{2\varepsilon}(0) \supseteq \operatorname{ran}(\tilde{\gamma}_1)$, by Theorem 2.2.5

$$\int_{\tilde{\gamma}_1} e^{\xi t} R(\xi, A)\ d\xi = 0$$

for every $t \geq 0$. Since $\xi \mapsto e^{\xi t} R(\xi, A)$ is integrable along γ, it is integrable along γ_j for $j = 1, \ldots 5$. We have

$$\begin{aligned} T(t) &= \frac{1}{2\pi i} \int_{\gamma} e^{\xi t} R(\xi, A)\ d\xi = \frac{1}{2\pi i} \int_{\tilde{\gamma}_1} e^{\xi t} R(\xi, A)\ d\xi + \frac{1}{2\pi i} \int_{\tilde{\gamma}_2} e^{\xi t} R(\xi, A)\ d\xi \\ &= \frac{1}{2\pi i} \int_{\tilde{\gamma}_2} e^{\xi t} R(\xi, A)\ d\xi. \end{aligned} \tag{5.7}$$

If $C > 0$ is such that $\|R(\xi, A)\| \leq \frac{C}{|\xi|}$ for every $\xi \in \overline{\Sigma_\theta}$, then $\cos(\theta) < 0$ yields

$$\begin{aligned} \left\| \int_{\gamma_5} e^{\xi t} R(\xi, A)\ d\xi \right\| &= \left\| \int_{\varepsilon}^{+\infty} e^{ste^{i\theta}} e^{i\theta} R(se^{i\theta}, A)\ dr \right\| \leq \int_{\varepsilon}^{\infty} e^{st\cos(\theta)} \left\| R(re^{i\theta}, A) \right\|\ dr \\ &\leq C \int_{\varepsilon}^{\infty} \frac{e^{rt\cos(\theta)}}{r}\ dr \leq \frac{C}{\varepsilon} \int_{\varepsilon}^{\infty} e^{rt\cos(\theta)}\ dr \\ &= \frac{C}{\varepsilon} \frac{e^{\varepsilon t \cos(\theta)}}{t|\cos(\theta)|} \leq \frac{C}{\varepsilon |\cos(\theta)|} e^{\cos(\theta)\varepsilon t} \end{aligned}$$

for $t \geq 1$. A similar approach for γ_4 leads to

$$\left\| \int_{\gamma_4} e^{\xi t} R(\xi, A)\ d\xi \right\| \leq K e^{\cos(\theta)\varepsilon t}$$

for some $K > 0$ independent of t. Furthermore,

$$\begin{aligned} \left\| \int_{\gamma_3} e^{\xi t} R(\xi, A)\ d\xi \right\| &= \left\| \int_{-\varepsilon \sin(\theta)}^{\varepsilon \sin(\theta)} e^{\varepsilon t \cos(\theta) + ist} R(\varepsilon \cos(\theta) + is, A) i\ ds \right\| \\ &\leq \int_{-\varepsilon \sin(\theta)}^{\varepsilon \sin(\theta)} e^{\varepsilon t \cos(\theta)} \left\| R(\varepsilon \cos(\theta) + is, A) \right\|\ ds \end{aligned}$$

$$\leq Ce^{\cos(\theta)\varepsilon t} \int_{-\varepsilon \sin(\theta)}^{\varepsilon \sin(\theta)} \frac{1}{|\varepsilon \cos(\theta) + is|} \, ds$$
$$\leq \frac{2C\varepsilon \sin(\theta)}{\varepsilon |\cos(\theta)|} e^{\cos(\theta)\varepsilon t} = 2C|\tan(\theta)| e^{\cos(\theta)\varepsilon t}.$$

We define

$$\tilde{M} := \frac{1}{2\pi} \max\Big\{ \frac{C}{\varepsilon |\cos(\theta)|}, K, 2C|\tan(\theta)| \Big\} > 0$$

as well as $\delta := -\cos(\theta)\varepsilon > 0$ and derive from (5.7)

$$\|T(t)\| \leq \left\| \frac{1}{2\pi i} \int_{\tilde{\gamma}_2} e^{\xi t} R(\xi, A) \, d\xi \right\| \leq \tilde{M} e^{-\delta t}, \quad t \geq 1.$$

Since according to Proposition 3.1.2, a), $\|T(t)\|$ is bounded on $[0,1]$, we obtain

$$\|T(t)\| \leq M e^{-\delta t}$$

for all $t \geq 0$ and a sufficiently large $M > 0$. Lastly, by Proposition 3.3.4, $T : (0, +\infty) \to L_b(X)$ is continuous. Together with

$$\int_{(0,+\infty)} \|T(t)\| \ d\lambda(t) \leq M \int_0^{+\infty} e^{-\delta t} \ d\lambda(t) = \frac{M}{\delta} < +\infty$$

we conclude that $T \in L^1\big((0, +\infty); L_b(X)\big)$; see Theorem 2.4.3. Extending T and f to $(-\infty, 0)$ by $T(t) := 0$ and $f(t) := 0$ we can employ Young's inequality, Proposition 2.5.3, b), and obtain

$$\int_{(0,+\infty)} \|u(t)\|^p \, d\lambda(t) \leq \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \|T(t-s)\| \, \|f(s)\| \, ds \Big)^p d\lambda(t)$$
$$\leq \Big(\int_{\mathbb{R}} \|T(t)\| \, d\lambda(t) \Big)^p \cdot \Big(\int_{\mathbb{R}} \|f(t)\|^p \, d\lambda(t) \Big) \leq \frac{M^p}{\delta^p} \|f\|_{L^p}^p ,$$

which implies $\|u\|_{L^p} \leq \frac{M}{\delta} \|f\|_{L^p}$.

□

Under certain conditions we can conclude strict maximal regularity on $(0, +\infty)$ from maximal regularity on bounded intervals.

5.2.3 Theorem. *Let $p \in (1, +\infty)$ and $t_0 > 0$. If there exist constants $M, \delta > 0$ such that $\|T(t)\| \leq M e^{-\delta t}$ for all $t \geq 0$ and A has the maximal L^p-regularity property on $(0, t_0]$, then A has the strict maximal regularity property on $(0, +\infty)$.*

Proof. By Theorem 5.1.4 the operator A is the infinitesimal generator of an analytic semigroup $\big(T(z)\big)_{z \in \Sigma_\varphi}$ of angle $\varphi \in (0, \frac{\pi}{2})$. By Theorem 3.3.6, $\big(T(t)\big)_{t \geq 0}$ is differentiable and there exists a constant $C > 0$ such that $\|AT(t)\| \leq \frac{Ce^{(-\delta+\delta)t}}{t} \leq \frac{C}{t}$ for all $t > 0$. Given $t > 0$ and $\varepsilon \in (0, t)$, according to Proposition 3.3.4 we have

$$\|AT(t)\| \leq \|T(t - \varepsilon)\| \, \|AT(\varepsilon)\| \leq M e^{-\delta(t-\varepsilon)} \, \|AT(\varepsilon)\| .$$

Hence, $AT \in L^q\big((\varepsilon,+\infty);L_b(X)\big)$ for all $\varepsilon>0$ and $q\in[1,+\infty]$.

Let $f\in L^p\big((0,+\infty);X\big)$ and denote by u the mild solution of (5.1) on $(0,+\infty)$, i.e

$$u(t) := \int_{(0,t)} T(t-s)f(s)\ d\lambda(s). \tag{5.8}$$

By Lemma 4.2.3 u is well-defined and continuous. We want to prove that $u(t)\in D(A)$ for almost every $t>0$. By Proposition 4.2.7, the unique solution of (5.1) on $(0,t_0]$ has the form

$$u_{t_0}(t) = \int_{(0,t)} T(t-s)f(s)\ d\lambda(s), \tag{5.9}$$

which implies $u(t)=u_{t_0}(t)\in D(A)$ for almost every $t\in(0,t_0]$. For $t>t_0$, there exists $n\in\mathbb{N}$, such that $t\in(nt_0,(n+1)t_0]$ and by Theorem 2.4.6

$$\begin{aligned} u(t) &= \int_{(0,t)} T(t-s)f(s)\ d\lambda(s) = \int_{(0,nt_0)} T(t-s)f(s)\ d\lambda(s) + \int_{(nt_0,t)} T(t-s)f(s)\ d\lambda(s) \\ &= \int_{(t-nt_0,t)} T(s)f(t-s)\ d\lambda(s) + \int_{(0,t-nt_0)} T(t-nt_0-s)f(s+nt_0)\ d\lambda(s). \end{aligned} \tag{5.10}$$

In order to show that both integrals above are contained in $D(A)$, we note that $t\mapsto T(t)$ is continuous on $(0,+\infty)$ and $\|T(t)\|\le Me^{-\delta t}$, implying $T\in L^q\big((0,+\infty);X\big)$. By Proposition 2.5.3, *a)* $s\mapsto T(s)f(t-s)$ and $s\mapsto T(t-nt_0-s)f(s+nt_0)$ are integrable over $(t-nt_0,t)$ and $(0,t-nt_0)$, respectively. Let $q\in(1,+\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $\varepsilon\in(0,t-nt_0)$. Since $AT\in L^q\big((\varepsilon,+\infty);L_b(X)\big)$, by Hölder's inequality, Proposition 2.5.3, *a)*

$$\int_{(t-nt_0,t)} \|AT(s)f(t-s)\|\ \ d\lambda(s) \le \|AT\|_{L^q((\varepsilon,+\infty);L_b(X))}\,\|f\|_{L^p((0,+\infty;X)} < +\infty.$$

Hence, by Proposition 2.3.13

$$\int_{(t-nt_0,t)} T(s)f(t-s)\ d\lambda(s)\in D(A).$$

The function $g(s):=f(s+nt_0)$, $s\in(0,t_0]$ belongs to $L^p\big((0,t_0];X\big)$. If v denotes the unique weak (and therefore mild) solution of

$$\begin{cases} v'(s) = Av(s)+g(s), & s\in(0,t_0], \\ v(0)=0, \end{cases}$$

then by Proposition 4.2.7

$$\int_{(0,t-nt_0)} T(t-nt_0-s)f(s+nt_0)\ d\lambda(s) = \int_{(0,t-nt_0)} T(t-nt_0-s)g(s)\ d\lambda(s) = v(t-nt_0).$$

Since $v(s)\in D(A)$ for almost every $s\in(0,t_0]$, $u(t)\in D(A)$ for almost every $t>0$; see (5.10)

We show that $\|u\|_{L^p((0,+\infty);X)} \le K \|f\|_{L^p((0,+\infty);X)}$ for some $K > 0$ independent of f. Since $\|T(t)\| \le Me^{-\delta t}$ for all $t \ge 0$, $T \in L^1\big((0,+\infty); L_b(X)\big)$. We extend T and f to $(-\infty, 0)$ by $T(t) = 0$, $f(t) = 0$, apply Young's Convolution Inequality, Proposition 2.5.3, b), and get

$$\int_{(0,+\infty)} \|u(t)\|^p \, d\lambda(t) = \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} T(t-s)f(s) \, d\lambda(s) \right\|^p d\lambda(t)$$
$$\le \|f\|^p_{L^p((0,+\infty);X)} \underbrace{\|T\|^p_{L^1((0,+\infty);L_b(X))}}_{=:K^p} .$$

We show that there is a constant $L > 0$ such that $\|Au\|_{L^p((0,+\infty);X)} \le L \|f\|_{L^p((0,+\infty);X)}$ and start with

$$\|Au\|^p_{L^p((0,+\infty);X)} = \int_{(0,t_0)} \|Au(t)\|^p \, d\lambda(t) + \int_{(t_0,+\infty)} \|Au(t)\|^p \, d\lambda(t). \tag{5.11}$$

Since A is maximally regular on $(0, t_0]$, there exists a constant $D > 0$, such that considering (5.9)

$$\int_{(0,t_0)} \|Au(t)\|^p \, d\lambda(t) = \int_{(0,t_0)} \|Au_{t_0}(t)\|^p \, d\lambda(t) = \|Au_{t_0}\|^p_{L^p((0,t_0];X)}$$
$$\le D^p \|f\|^p_{L^p((0,t_0];X)} \le D^p \|f\|^p_{L^p((0,+\infty);X)} .$$

From $\|x+y\|^p \le 2^{p-1}(\|x\|^p + \|y\|^p)$ and (5.8) we conclude that

$$\frac{1}{2^{p-1}} \int_{(t_0,+\infty)} \|Au(t)\|^p \, dt$$

is less or equal to

$$\underbrace{\left\| A\Big(\int_{(0,\cdot-t_0)} T(\cdot - s)f(s) \, ds\Big) \right\|^p_{L^p((t_0,+\infty);X)}}_{=:I_1} + \underbrace{\left\| A\Big(\int_{(\cdot-t_0,\cdot)} T(\cdot - s)f(s) \, ds\Big) \right\|^p_{L^p((t_0,+\infty);X)}}_{=:I_2} .$$

Let $q \in (1,+\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $t > t_0$. We saw in the beginning of the proof that $AT \in L^q\big((t_0,t); L_b(X)\big)$. By Proposition 2.5.3, a), applied to $\mathbb{1}_{(0,+\infty)} \|f(t-\cdot)\|$ and $\mathbb{1}_{(t_0,t)} \|AT\|$

$$\int_{(t_0,t)} \|AT(s)f(t-s)\| \ \, d\lambda(s) \le \|AT\|_{L^q((t_0,t);X)} \|f\|_{L^p((0,+\infty);X)} ,$$

which implies that $s \mapsto AT(s)f(t-s)$ is integrable over (t_0,t) for every $t \in (t_0,+\infty)$. Employing Proposition 2.3.13 yields

$$A\Big(\int_{(t_0,t)} T(s)f(t-s) \, d\lambda(s)\Big) = \int_{(t_0,t)} AT(s)f(t-s) \, d\lambda(s).$$

By Theorem 2.4.6, Proposition 2.5.3, c), we have

$$\begin{aligned}
I_1 &= \int_{(t_0,+\infty)} \left\| A\Big(\int_{(t_0,t)} T(s)f(t-s)\ d\lambda(s)\Big)\right\|^p d\lambda(t) \\
&= \int_{(t_0,+\infty)} \left\| \int_{(t_0,t)} AT(s)f(t-s)\ d\lambda(s)\right\|^p d\lambda(t) \\
&\leq \int_{(t_0,+\infty)} \Big(\int_{(t_0,t)} \|AT(s)f(t-s)\|\ \ d\lambda(s)\Big)^p d\lambda(t) \\
&\leq \int_{(t_0,+\infty)} \Big(\int_{(t_0,+\infty)} \|AT(s)f(t-s)\|\ \ d\lambda(s)\Big)^p d\lambda(t) \\
&\leq \Big(\int_{(t_0,+\infty)} \Big(\int_{(t_0,+\infty)} \|AT(s)f(t-s)\|^p\ \ d\lambda(t)\Big)^{\frac{1}{p}} d\lambda(s)\Big)^p \\
&\leq \Big(\int_{(t_0,+\infty)} \|AT(s)\|\, \|f\|_{L^p((0,+\infty);X)}\ \ d\lambda(s)\Big)^p = \|AT\|^p_{L^1((t_0,+\infty);L_b(X))}\, \|f\|^p_{L^p((0,+\infty);X)}\,.
\end{aligned}$$

Defining

$$J_{1,n} := \int_{(nt_0,(n+1)t_0)} \left\| A\Big(\int_{(t-t_0,nt_0)} T(t-s)f(s)\ d\lambda(s)\Big)\right\|^p d\lambda(t)$$

and

$$J_{2,n} := \int_{(nt_0,(n+1)t_0)} \left\| A\Big(\int_{(nt_0,t)} T(t-s)f(s)\ d\lambda(s)\Big)\right\|^p d\lambda(t)$$

for $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} \int_{(nt_0,(n+1)t_0)} \left\| A\Big(\int_{(t-t_0,t)} T(t-s)f(s)\ d\lambda(s)\Big)\right\|^p d\lambda(t) \\
&\leq 2^{p-1} \sum_{n=1}^{\infty} (J_{1,n} + J_{2,n}).
\end{aligned}$$

For $t \in (0,t_0)$ and $n \in \mathbb{N}$ by Proposition 2.5.3, a)

$$\int_{(0,t_0-t)} \|AT(t+s)f(nt_0-s)\|\ \ d\lambda(s) \leq \|AT\|_{L^q((t,t_0);X)}\, \|f\|_{L^p((0,+\infty);X)}\,.$$

Hence, by Proposition 2.3.13

$$A\Big(\int_{(0,t_0-t)} T(t+s)f(nt_0-s)\ d\lambda(s)\Big) = \int_{(0,t_0-t)} AT(t+s)f(nt_0-s)\ d\lambda(s).$$

Substituting $s \mapsto nt_0 - s$ and $t \mapsto t + nt_0$ together with the estimate $\|AT(t)\| \leq \frac{C}{t}$, $t > 0$, yields

$$\begin{aligned}
J_{1,n} &= \int_{(nt_0,(n+1)t_0)} \left\| A\Big(\int_{(t-t_0,nt_0)} T(t-s)f(s)\ d\lambda(s)\Big)\right\|^p d\lambda(t) \\
&= \int_{(nt_0,(n+1)t_0)} \left\| A\Big(\int_{(0,(n+1)t_0-t)} T(t+s-nt_0)f(nt_0-s)\ d\lambda(s)\Big)\right\|^p\ d\lambda(t)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{(0,t_0)} \left\| A\Big(\int_{(0,t_0-t)} T(t+s) f(nt_0 - s)\ d\lambda(s) \Big) \right\|^p d\lambda(t) \\
&= \int_{(0,t_0)} \left\| \int_{(0,t_0-t)} AT(t+s) f(nt_0 - s)\ d\lambda(s) \right\|^p d\lambda(t) \\
&\le \int_{(0,t_0)} \Big(\int_{(0,t_0-t)} \| AT(t+s) f(nt_0 - s) \|\ \ d\lambda(s) \Big)^p d\lambda(t) \\
&\le \int_{(0,t_0)} \Big(\int_{(0,t_0-t)} \frac{C\, \| f(nt_0 - s) \|}{t+s}\ d\lambda(s) \Big)^p d\lambda(t) \\
&\le C^p \int_{(0,t_0)} \Big(\int_{(0,t_0)} \frac{\| f(nt_0 - s) \|}{t+s}\ d\lambda(s) \Big)^p d\lambda(t).
\end{aligned}
$$

For fixed $t > 0$ we substitute $s \mapsto st$ and obtain

$$
\begin{aligned}
\int_{(0,t_0)} \frac{\| f(nt_0 - s) \|}{t+s}\ d\lambda(s) &= \int_{(0,+\infty)} \frac{\mathbb{1}_{(0,t_0)}(s)\, \| f(nt_0 - s) \|}{t+s}\ d\lambda(s) \\
&= \int_{(0,+\infty)} \frac{\mathbb{1}_{(0,t_0)}(st)\, \| f(nt_0 - st) \|}{1+s}\ d\lambda(s).
\end{aligned}
$$

By Proposition 2.5.3, $c)$ and substituting $t \mapsto \frac{t}{s}$

$$
\begin{aligned}
J_{1,n} &= C^p \int_{(0,t_0)} \Big(\int_{(0,+\infty)} \frac{\mathbb{1}_{(0,t_0)}(st)\, \| f(nt_0 - st) \|}{1+s}\ d\lambda(s) \Big)^p\ d\lambda(t) \\
&\le C^p \Big(\int_{(0,+\infty)} \Big(\int_{(0,t_0)} \frac{\mathbb{1}_{(0,t_0)}(st)\, \| f(nt_0 - st) \|^p}{(1+s)^p}\ d\lambda(t) \Big)^{\frac{1}{p}}\ d\lambda(s) \Big)^p \\
&\le C^p \Big(\int_{(0,+\infty)} \Big(\int_{(0,st_0)} \frac{\mathbb{1}_{(0,t_0)}(t)\, \| f(nt_0 - t) \|^p}{s(1+s)^p}\ d\lambda(t) \Big)^{\frac{1}{p}}\ d\lambda(s) \Big)^p \\
&\le C^p \, \| f \|^p_{L^p(((n-1)t_0, nt_0);X)} \Big(\int_{(0,+\infty)} \frac{1}{(1+s) s^{\frac{1}{p}}}\ d\lambda(s) \Big)^p.
\end{aligned}
$$

We choose $r_1 := \frac{p+1}{2} > 1$ and $r_2 \in (1, +\infty)$ such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$. By Hölder's inequality, Proposition 2.5.3, $a)$,

$$
\int_{(0,1)} \frac{1}{(1+s) s^{\frac{1}{p}}}\ d\lambda(s) \le \Big(\int_{(0,1)} \frac{1}{(1+s)^{r_2}}\ d\lambda(s) \Big)^{\frac{1}{r_2}} \Big(\int_{(0,1)} \frac{1}{s^{\frac{p+1}{2p}}}\ d\lambda(s) \Big)^{\frac{2}{p+1}} \le \frac{2p}{p-1}.
$$

Together with

$$
\int_{(1,+\infty)} \frac{1}{(1+s) s^{\frac{1}{p}}}\ d\lambda(s) \le \int_{(1,+\infty)} \frac{1}{s^{1+\frac{1}{p}}}\ d\lambda(s) = p
$$

we obtain $J_{1,n} \le C^p (\frac{2p}{p-1} + p)^p \, \| f \|^p_{L^p(((n-1)t_0, nt_0);X)}$ and, in consequence,

$$
\sum_{n=1}^{\infty} J_{1,n} = C^p (\tfrac{2p}{p-1} + p)^p \sum_{n=1}^{\infty} \| f \|^p_{L^p(((n-1)t_0, nt_0);X)} = C^p (\tfrac{2p}{p-1} + p)^p \, \| f \|^p_{L^p((0,+\infty);X)} \, .
$$

In order to estimate $J_{2,n}$, $n \in \mathbb{N}$, we substitute $s \mapsto s + nt_0$, $t \mapsto t + nt_0$ and derive from A being maximally L^p-regular on $(0, t_0]$

$$\begin{aligned}
J_{2,n} &= \int_{(nt_0,(n+1)t_0)} \left\| A\Big(\int_{(nt_0,t)} T(t-s) f(s) \; d\lambda(s) \Big) \right\|^p d\lambda(t) \\
&= \int_{(nt_0,(n+1)t_0)} \left\| A\Big(\int_{(0,t-nt_0)} T(t-s-nt_0) f(s+nt_0) \; d\lambda(s) \Big) \right\|^p d\lambda(t) \\
&= \int_{(0,t_0)} \left\| A\big(\int_{(0,t)} T(t-s) f(s+nt_0) \; d\lambda(s) \big) \right\|^p d\lambda(t) \\
&\leq D^p \left\| f \right\|^p_{L^p((nt_0,(n+1)t_0);X)} .
\end{aligned}$$

We obtain

$$\sum_{n=1}^{\infty} J_{2,n} \leq K^p \sum_{n=1}^{\infty} \left\| f \right\|^p_{L^p((nt_0,(n+1)t_0);X)} \leq K^p \left\| f \right\|^p_{L^p((0,+\infty);X)} .$$

Starting with (5.11) the derived estimates imply

$$\left\| Au \right\|_{L^p} \leq \big(2^{p-1} \left\| AT \right\|^p_{L^1((t_0,+\infty);X)} + (1 + 2^{2p-2}) D^p + 2^{2p-2} C^p (\tfrac{2p}{p-1} + p)^p \big)^{\frac{1}{p}} \left\| f \right\|_{L^p} .$$

It remains to show that u is a weak solution of (5.1) on $(0, +\infty)$. To that end, we verify $u \in W^{1,p}\big((nt_0, (n+1)t_0); X\big)$ for every $n \in \mathbb{N} \cup \{0\}$. Define $v_n : (0, t_0) \to X$ by

$$v_n(t) := T(t) u(nt_0) + \int_{(0,t)} T(t-s) f(s+nt_0) \; d\lambda(s).$$

Because A has the maximal L^p-regularity property on $(0, t_0]$ and $s \mapsto f(s + nt_0) \in L^p\big((0, t_0]; X\big)$, according to Proposition 4.2.7 the unique weak solution u_n of

$$\begin{cases} u_n'(t) = Au_n(t) + f(t + nt_0), & t \in (0, t_0], \\ u_n(0) = 0 \end{cases}$$

is given by

$$u_n(t) = \int_{(0,t)} T(t-s) f(s+nt_0) \; d\lambda(s).$$

By Theorem 3.3.6 and Proposition 3.3.4 $t \mapsto T(t)u(nt_0)$ is differentiable. Employing Proposition 2.5.5, b) and c), yields that $v_n = T(\cdot)u(nt_0) + u_n$ is weakly differentiable and $v_n'(t) = Av_n(t) + f(t + nt_0)$. Moreover,

$$T(t)u(nt_0) = T(t) \int_{(0,nt_0)} T(nt_0 - s) f(s) \; d\lambda(s) = \int_{(0,nt_0)} T(t + nt_0 - s) f(s) \; d\lambda(s)$$

and substituting $s \mapsto s - nt_0$ yields

$$u_n(t) = \int_{(0,t)} T(t-s) f(s+nt_0) \; d\lambda(s) = \int_{(nt_0,(t+nt_0)} T(t + nt_0 - s) f(s) \; d\lambda(s).$$

We obtain

$$v_n(t) = \int_{(0,t+nt_0)} T(t + nt_0 - s)f(s)\, d\lambda(s) = u(t + nt_0)$$

and, in turn, the weak differentiability of u on $(nt_0, (n+1)t_0)$ and

$$u'(t) = v_n'(t - nt_0) = Av_n(t - nt_0) + f(t) = Au(t) + f(t).$$

$Au, f \in L^p\big((nt_0, (n+1)t_0); X\big)$ yields $u' \in L^p\big((nt_0, (n+1)t_0); X\big)$ and, together with $u \in L^p\big((0, +\infty); X\big)$, $u \in W^{1,p}\big((nt_0, (n+1)t_0); X\big)$. For $\varphi \in C_{00}^\infty\big((0, +\infty); \mathbb{C}\big)$ also $(\varphi u)|_{(nt_0,(n+1)t_0)} \in W^{1,p}\big((nt_0, (n+1)t_0); X\big)$ and

$$\int_{(nt_0,(n+1)t_0)} \varphi'(t)u(t)\, d\lambda(t) = (\varphi u)\big((n+1)t_0\big) - (\varphi u)(nt_0) - \int_{(nt_0,(n+1)t_0)} \varphi(t)u'(t)\, d\lambda(t)$$

according to Corollary 2.5.9. Consequently,

$$\begin{aligned}
\int_{(0,+\infty)} \varphi'(t)u(t)\, d\lambda(t) &= \sum_{n=0}^{\infty} \int_{(nt_0,(n+1)t_0)} \varphi'(t)u(t)\, d\lambda(t) \\
&= \sum_{n=0}^{\infty} \Big((\varphi u)\big((n+1)t_0\big) - (\varphi u)(nt_0) \Big) - \int_{(nt_0,(n+1)t_0)} \varphi(t)u'(t)\, d\lambda(t) \\
&= -\sum_{n=0}^{\infty} \int_{(nt_0,(n+1)t_0)} \varphi(t)\big(Au(t) + f(t)\big)\, d\lambda(t) \\
&= -\int_{(0,+\infty)} \varphi(t)\big(Au(t) + f(t)\big)\, d\lambda(t).
\end{aligned}$$

Since $\varphi \in C_{00}^\infty\big((0, +\infty); \mathbb{C}\big)$ was arbitrary, we conclude that u is weakly differentiable on $(0, +\infty)$ with $u'(t) = Au(t) + f(t)$ for almost every $t \in (0, +\infty)$. Together with the continuity of u (Lemma 4.2.3) and the fact that $u(t) \in D(A)$ for almost every $t \in (0, +\infty)$ we deduce that u is a weak solution of (5.1). Since A is the infinitesimal generator of an analytic semigroup, according to Proposition 4.2.7 this weak solution is unique.

□

5.2.4 Lemma. Let $p \in (1, +\infty)$ and $t_0 > 0$. If A has the maximal L^p-regularity property on $(0, t_0]$, then $A + \xi I$ has the maximal L^p-regularity property on $(0, t_0]$ for all $\xi \in \mathbb{C}$.

Proof. Let $g \in L^p\big((0, t_0]; X\big)$ and let u be the mild solution of (5.1) for $f(t) := e^{-\xi t} g(t)$, i.e.

$$u(t) = \int_{(0,t)} T(t - s)f(s)\, d\lambda(s).$$

By Lemma 4.2.3 u is continuous. According to Theorem 5.1.4 A is the generator of an analytic and according to Corollary 3.3.6 differentiable semigroup. A being maximally regular and Proposition 4.2.7 yield that u a is the unique weak solution of (5.1). It is easy to see that $f \in L^p\big((0,t_0];X\big)$ and that $v:[0,t_0]\to X$ defined by $v(t):=e^{\xi t}u(t)$ is continuous as well as $v(t)\in D(A)$ for almost every $t\in(0,t_0]$. Let $\varphi\in C_{00}^\infty\big((0,t_0);X\big)$. Since u is weakly differentiable,

$$\begin{aligned}\int_{(0,t_0)} \varphi'(t)v(t)\ d\lambda(t) &= \int_{(0,t_0)} \big(e^{\xi t}\varphi'(t)+\xi e^{\xi t}\varphi(t)\big)u(t)\ d\lambda(t) - \xi\int_{(0,t_0)} e^{\xi t}\varphi(t)u(t)\ d\lambda(t)\\ &= \int_{(0,t_0)} \big(e^{\xi t}\varphi(t)\big)'u(t)\ d\lambda(t) - \xi\int_{(0,t_0)} e^{\xi t}\varphi(t)u(t)\ d\lambda(t)\\ &= -\int_{(0,t_0)} e^{\xi t}\varphi(t)u'(t)\ d\lambda(t) - \xi\int_{(0,t_0)} e^{\xi t}\varphi(t)u(t)\ d\lambda(t)\\ &= -\int_{(0,t_0)} \varphi(t)\big(e^{\xi t}u'(t)+\xi e^{\xi t}u(t)\big)\ d\lambda(t).\end{aligned}$$

Hence, v is weakly differentiable and

$$v'(t)=e^{\xi t}u'(t)+\xi e^{\xi t}u(t)=e^{\xi t}Au(t)+e^{\xi t}f(t)+\xi v(t)=(A+\xi I)v(t)+g(t).$$

Consequently, v is a weak solution of

$$\begin{cases} v'(t)=(A+\xi I)v(t)+g(t), & t\in(0,t_0],\\ v(0)=0.\end{cases}$$

By Proposition 3.1.2, g), $A+\xi I$ is the infinitesimal generator of the strongly continuous semigroup $\big(e^{\xi t}T(t)\big)_{t\geq 0}$. By Theorem 5.1.4 $\big(T(t)\big)_{t\geq 0}$ can be extended to an analytic semigroup. $z\mapsto e^{\xi z}$ being analytic on $\mathbb{C}$ the same is true for $\big(e^{\xi t}T(t)\big)_{t\geq 0}$. According to Proposition 4.2.7

$$v(t)=\int_{(0,t)} e^{\xi(t-s)}T(t-s)g(s)\ d\lambda(s),\quad t\in[0,t_0].$$

for every $t\in[0,t_0]$. By Proposition 5.2.1, there exists a constant $C>0$ such that

$$\|u\|_{L^p((0,t_0);X)}+\|Au\|_{L^p((0,t_0);X)}\leq C\,\|f\|_{L^p((0,t_0);X)}\,.$$

We set $C_1:=\max_{t\in[0,t_0]}|e^{\xi t}|$, $C_2:=\max_{t\in[0,t_0]}|e^{-\xi t}|$ and derive

$$\begin{aligned}\|Av\|^p_{L^p((0,t_0);X)} &= \int_{(0,t_0)} |e^{p\xi t}|\,\|Au(t)\|^p\ d\lambda(t)\leq C_1^p\,\|Au\|^p_{L^p((0,t_0);X)}\leq C_1^pC^p\,\|f\|^p_{L^p((0,t_0);X)}\\ &= C_1^pC^p\int_{(0,t_0)} |e^{-p\xi t}|\,\|g(t)\|^p\ d\lambda(t)\leq C_1^pC_2^pC^p\,\|g\|^p_{L^p((0,t_0);X)}\end{aligned}$$

as well as

$$\|v\|^p_{L^p((0,t_0);X)}=\int_{(0,t_0)} |e^{p\xi t}|\,\|u(t)\|^p\ d\lambda(t)\leq C_1^pC^p\,\|f\|^p_{L^p((0,t_0);X)}\leq C_1^pC_2^pC^p\,\|g\|^p_{L^p((0,t_0);X)}\,.$$

Consequently,

$$\|(A+\xi I)v\|_{L^p((0,t_0);X)}\leq\|Av\|_{L^p((0,t_0);X)}+|\xi|\,\|v\|_{L^p((0,t_0);X)}\leq C_1C_2C(1+|\xi|)\,\|g\|_{L^p((0,t_0);X)}\,.$$

□

5.2.5 Corollary. Let $p \in (1, +\infty)$ and $t_0 > 0$. If A has the maximal L^p-regularity property on $(0, t_0]$, A has the maximal L^p-regularity property on $(0, t_1]$ for every $t_1 > 0$.

Proof. Let M, ω as in Proposition 3.1.2, $a)$ and $\delta > 0$. By Lemma 5.2.4 $A - (\omega + \delta)I$ is maximally L^p-regular and by Proposition 3.1.2, $g)$, $A - (\omega + \delta)I$ generates the semigroup $\left(e^{-(\omega+\delta)t}T(t)\right)_{t \geq 0}$. Because of

$$\left\| e^{-(\omega+\delta)t}T(t) \right\| \leq Me^{-\delta t}, \quad t \in [0, +\infty)$$

we conclude from Theorem 5.2.3 that $A - (\omega + \delta)I$ has the strict maximal L^p-regularity property on $(0, +\infty)$. By Theorem 5.1.5, $A - (\omega + \delta)I$ has the maximal L^p-regularity property on $(0, t_1]$ for every $t_1 > 0$ and so does A; see Lemma 5.2.4.

□

5.3 Independence of p

Our next goal is to prove that the maximal L^p-regularity property is independent of p. To that end we need two preliminary lemmata.

5.3.1 Lemma (Calderón-Zygmund decomposition)**.** Let $f \in L^1\big([0, +\infty); X\big)$, $\varepsilon > 0$ and denote by λ the Lebesgue measure on $[0, +\infty)$. Then there exist a sequence $(I_n)_{n\in\mathbb{N}}$ of closed intervals with $|I_n \cap I_m| \leq 1$ for $n \neq m$ and a sequence $(h_n)_{n\in N}$ in $L^1\big([0, +\infty); X\big)$ as well as $g \in L^1\big([0, +\infty); X\big)$ satisfying

a) $f = g + \sum_{n\in\mathbb{N}} h_n$,

b) $\|g\|_{L^1} + \sum_{n\in\mathbb{N}} \|h_n\|_{L^1} \leq 3\,\|f\|_{L^1}$,

c) $\|g(t)\| \leq 2\varepsilon$ for almost every $t \geq 0$,

d) $\operatorname{supp}(h_n) \subseteq I_n$ for all $n \in \mathbb{N}$,

e) $\int_{(0,+\infty)} h_n(t)\, d\lambda(t) = 0$ for all $n \in \mathbb{N}$ and

f) $\sum_{n\in\mathbb{N}} |I_n| \leq \frac{\|f\|_{L^1}}{\varepsilon}$.

Proof. For $f = 0$ the statement is obvious by taking $g := f$, $h_n := 0$, $I_n = \emptyset$, $n \in \mathbb{N}$. For $f \in L^1\big([0, +\infty); X\big) \setminus \{0\}$ we set $S_1 := \mathbb{N}$ and decompose $[0, +\infty)$ into closed intervals $J_{0,k}$, $k \in \mathbb{N}$, of length $\frac{1}{\varepsilon}\|f\|_{L^1}$, such as

$$J_{0,k} = [(k-1)\tfrac{1}{\varepsilon}\|f\|_{L^1}, k\tfrac{1}{\varepsilon}\|f\|_{L^1}].$$

Furthermore, we decompose each of these intervals into two smaller intervals of length $\frac{1}{2\varepsilon}\|f\|_{L^1}$, whose intersection contains at most one element, which we denote by $J_{1,k}$, $k \in \mathbb{N} = S_1$. If $K_1 \subseteq \mathbb{N}$ denotes the set of all $k \in \mathbb{N}$ with

$$\frac{2\varepsilon}{\|f\|_{L^1}} \int_{J_{1,k}} \|f(t)\| \ d\lambda(t) = \frac{1}{\lambda(J_{1,k})} \int_{J_{1,k}} \|f(t)\| \ d\lambda(t) > \varepsilon,$$

then

$$|K_1| = \sum_{k\in K_1} 1 < \sum_{k\in K_1} \frac{2}{\|f\|_{L^1}} \int_{J_{1,k}} \|f(t)\| \ d\lambda(t) \leq \sum_{k=1}^{\infty} \frac{2}{\|f\|_{L^1}} \int_{J_{0,k}} \|f(t)\| \ d\lambda(t) = 2,$$

which implies $N_1 := \mathbb{N} \setminus K_1 \neq \emptyset$. Assume that for $m > 1$ we have defined S_{m-1}, $K_{m-1}, N_{m-1} \subseteq S_{m-1}$ with $K_{m-1} \cup N_{m-1} = S_{m-1}$, $|N_{m-1}| = |S_{m-1}| = +\infty$ and $(J_{m-1,k})_{k\in S_{m-1}}$ with $\lambda(J_{m-1,k}) = \frac{1}{2^{m-1}\varepsilon}\|f\|_{L^1}$ as well as

$$\frac{2^{m-1}\varepsilon}{\|f\|_{L^1}} \int_{J_{m-1,k}} \|f(t)\| \ d\lambda(t) = \frac{1}{\lambda(J_{m-1,k})} \int_{J_{m-1,k}} \|f(t)\| \ d\lambda(t) > \varepsilon,$$

if and only if $k \in K_{m-1}$. We decompose each of the intervals $J_{m-1,k}$, $k \in N_{m-1}$, into two closed intervals of length $\frac{1}{2^m\varepsilon}\|f\|_{L^1}$, whose intersection contains at most one element. We call these intervals $J_{m,k}$, $k \in S_m$, where $S_m \subseteq \mathbb{N}$ is a set of indices with $|S_m| = 2|N_{m-1}| = +\infty$. We obtain

$$\bigcup_{k\in N_{m-1}} J_{m-1,k} = \bigcup_{k\in S_m} J_{m,k} \tag{5.12}$$

By K_m we denote the set of all $k \in S_m$ such that

$$\frac{2^m\varepsilon}{\|f\|_{L^1}} \int_{J_{m,k}} \|f(t)\| \ d\lambda(t) = \frac{1}{\lambda(J_{m,k})} \int_{J_{m,k}} \|f(t)\| \ d\lambda(t) > \varepsilon. \tag{5.13}$$

Since

$$|K_m| = \sum_{k\in K_m} 1 < \sum_{k\in K_m} \frac{2^m}{\|f\|_{L^1}} \int_{J_{m,k}} \|f(t)\| \ d\lambda(t) \leq \sum_{k=1}^{\infty} \frac{2^m}{\|f\|_{L^1}} \int_{J_{0,k}} \|f(t)\| \ d\lambda(t) = 2^m,$$

K_m has at most $2^m - 1$ elements and, in consequence, $N_m := S_m \setminus K_m$ is infinite. Hence, we inductively constructed sequences $(S_m)_{m\in\mathbb{N}}$, $(K_m)_{m\in\mathbb{N}}$, $(N_m)_{m\in\mathbb{N}}$ with $K_m \cap N_m = \emptyset$ and $K_m \cup N_m = S_m$, $m \in \mathbb{N}$, as well as a sequence of intervals $(J_{m,k})_{m\in\mathbb{N},k\in S_m}$ with $\lambda(J_{m,k}) = \frac{1}{2^m\varepsilon}\|f\|_{L^1}$. For $n, m \in \mathbb{N}$, $k \in K_m$, $l \in K_n$ with $(m,k) \neq (n,l)$ we have $|J_{m,k} \cap J_{n,l}| \leq 1$ and

$$\frac{1}{\lambda(J_{m,k})} \int_{J_{m,k}} \|f(t)\| \ d\lambda(t) > \varepsilon$$

for all $m \in \mathbb{N}$, $k \in K_m$ as well as

$$\frac{1}{\lambda(J_{m,k})} \int_{J_{m,k}} \|f(t)\| \ d\lambda(t) \leq \varepsilon$$

for all $m \in \mathbb{N}$, $k \in N_m$. Furthermore, for every $m \geq 2$ and $k \in S_m$ there exists $l \in N_{m-1}$ such that $J_{m,k} \subseteq J_{m-1,l}$.

For $m \in \mathbb{N}$ and $k \in K_m$ we denote by $J^{\text{o}}_{m,k}$ the interior of $J_{m,k}$ (i.e. $J^{\text{o}}_{m,k} = J_{m,k} \setminus \{\inf J_{m,k}, \sup J_{m,k}\}$) and define

$$h_{m,k} := \mathbb{1}_{J^{\text{o}}_{m,k}}\Big(f - \frac{1}{\lambda(J_{m,k})}\int_{J_{m,k}} f(t)\; d\lambda(t)\Big).$$

We note that since $|J_{m,k} \cap J_{n,l}| \leq 1$ for $m,n \in \mathbb{N}$, $k \in K_m$, $l \in K_n$ with $(m,k) \neq (n,l)$, the sets $(J^{\text{o}}_{m,k})_{m\in\mathbb{N},k\in K_m}$ are pairwise disjoint. Consequently, for any $t \geq 0$

$$h(t) := \sum_{m\in\mathbb{N}}\sum_{m\in K_m} h_{m,k}(t)$$

has at most one non-vanishing addend. We define

$$g := f - \sum_{m\in\mathbb{N}}\sum_{m\in K_m} h_{m,k}.$$

Clearly,

$$f = g + \sum_{m\in\mathbb{N}}\sum_{m\in K_m} h_{m,k},$$

$\operatorname{supp}(h_{m,k}) \subseteq J_{m,k}$ and

$$\int_{[0,+\infty)} h_{m,k}(t)\; d\lambda(t) = \int_{J_{m,k}} f(s)\; d\lambda(s) - \int_{J_{m,k}}\Big(\frac{1}{\lambda(J_{m,k})}\int_{J_{m,k}} f(t)\; d\lambda(t)\Big)\; d\lambda(s) = 0$$

We want to prove that $\|g(t)\| \leq 2\varepsilon$ for almost every $t \geq 0$. To that end, let

$$t \in [0,+\infty) \setminus \bigcup_{m\in\mathbb{N}}\bigcup_{k\in K_m} J_{m,k} = \bigcap_{m\in\mathbb{N}}\Big([0,+\infty) \setminus \bigcup_{k\in K_m} J_{m,k}\Big).$$

We want to show by induction that for each $m \in \mathbb{N}$ there exists a $k_m \in N_m$ satisfying $t \in J_{m,k_m}$. For $m = 1$ we have

$$t \in [0,+\infty) \setminus \Big(\bigcup_{k\in K_1} J_{1,k}\Big) \subseteq \bigcup_{k\in N_1} J_{1,k}.$$

Hence, $t \in J_{1,k_1}$ for some $k_1 \in N_1$. Assume that $t \in J_{m-1,k_{m-1}}$ for some $k_{m-1} \in N_{m-1}$. $K_m \cap N_m = \emptyset$, $K_m \cup N_m = S_m$ and (5.12) yield

$$\begin{aligned} t \in [0,+\infty) \setminus \Big(\bigcup_{k\in K_m} J_{m,k}\Big) &\subseteq \bigcup_{k\in N_m} J_{m,k} \cup \Big([0,+\infty) \setminus \Big(\bigcup_{k\in S_m} J_{m,k}\Big)\Big) \\ &= \bigcup_{k\in N_m} J_{m,k} \cup \Big([0,+\infty) \setminus \Big(\bigcup_{l\in N_{m-1}} J_{m-1,l}\Big)\Big). \end{aligned} \tag{5.14}$$

$t \in J_{m-1,k_{m-1}}$ yields

$$t \in \bigcup_{l\in N_{m-1}} J_{m-1,l}$$

and, by (5.14)

$$t \in \bigcup_{k \in N_m} J_{m,k}.$$

Hence, $t \in J_{m,k_m}$ for some $k_m \in N_m$. We define $p_m := \frac{1}{2}(\sup J_{m,k_m} + \inf J_{m,k_m})$ and

$$\delta_m := \begin{cases} \sup J_{m,k_m} - t, & t \leq p_m, \\ \inf J_{m,k_m} - t, & t > p_m, \end{cases}$$

$m \in \mathbb{N}$. Clearly, $|\delta_m| \in [\frac{1}{2}\lambda(J_{m,k_m}), \lambda(J_{m,k_m})]$. For $m \in \mathbb{N}$ we obtain in the case $t \leq p_m$

$$\frac{1}{|\delta_m|} \int_{(t,t+\delta_m)} \|f(s)\| \ d\lambda(s) \leq \frac{2}{\lambda(J_{m,k_m})} \int_{J_{m,k_m}} \|f(s)\| \ d\lambda(s) \leq 2\varepsilon. \tag{5.15}$$

as well as

$$\frac{1}{|\delta_m|} \int_{(t+\delta_m,t)} \|f(s)\| \ d\lambda(s) \leq \frac{2}{\lambda(J_{m,k_m})} \int_{J_{m,k_m}} \|f(s)\| \ d\lambda(s) \leq 2\varepsilon. \tag{5.16}$$

in the case $t > p_m$. Since $|\delta_m| \in [\frac{1}{2}\lambda(J_{m,k_m}), \lambda(J_{m,k_m})]$ and $\lambda(J_{m,k_m}) = \frac{1}{2^m\varepsilon} \|f\|_{L^1}$ $\delta_m \xrightarrow{m \to +\infty} 0$ and by Proposition 2.5.2, d), $\|g(t)\| = \|f(t)\|$ is the limit of the sequence defined by

$$\frac{1}{|\delta_m|} \int_{(t,t+\delta_m)} \|f(s)\| \ d\lambda(s)$$

for $m \in \mathbb{N}$ with $\delta_m > 0$ and

$$\frac{1}{|\delta_m|} \int_{(t+\delta_m,t)} \|f(s)\| \ d\lambda(s),$$

else, for almost every $t \in [0, +\infty) \setminus \bigcup_{m \in \mathbb{N}} \bigcup_{k \in K_m} J_{m,k}$. Together with (5.15) and (5.16) we conclude that $\|g(t)\| \leq 2\varepsilon$ for almost every $t \in [0, +\infty) \setminus \bigcup_{m \in \mathbb{N}} \bigcup_{k \in K_m} J_{m,k}$. If $t \in J^\circ_{m,k}$ for some $m \in \mathbb{N}$ and $k \in K_m$, there exists $l \in N_{m-1}$, such that $t \in J^\circ_{m,k} \subseteq J_{m-1,l}$. Because of

$$\frac{1}{\lambda(J_{m-1,l})} \int_{J_{m-1,l}} \|f(s)\| \ d\lambda(s) \leq \varepsilon$$

and Proposition 2.3.6, b), we have

$$\begin{aligned} \|g(t)\| = \|f(t) - h_{m,k}(t)\| &= \left\| \frac{1}{\lambda(J_{m,k})} \int_{J_{m,k}} f(s) \ d\lambda(s) \right\| \leq \frac{1}{\lambda(J_{m,k})} \int_{J_{m,k}} \|f(s)\| \ d\lambda(s) \\ &\leq \frac{\lambda(J_{m-1,l})}{\lambda(J_{m,k})} \frac{1}{\lambda(J_{m-1,l})} \int_{J_{m-1,l}} \|f(s)\| \ d\lambda(s) \leq \frac{\lambda(J_{m-1,l})}{\lambda(J_{m,k})} \varepsilon = \frac{2^m \varepsilon \|f\|_{L^1}}{2^{m-1}\varepsilon \|f\|_{L^1}} \varepsilon = 2\varepsilon, \end{aligned}$$

implying $\|g(t)\| \leq 2\varepsilon$ for almost every $t \in \bigcup_{m \in \mathbb{N}} \bigcup_{k \in K_m} J_{m,k}$ since

$$\{\sup J_{m,k} : m \in \mathbb{N}, k \in K_m\} \cup \{\inf J_{m,k} : m \in \mathbb{N}, k \in K_m\}$$

is countable and therefore a null set.

From
$$|J_{m,k}| \leq \frac{1}{\varepsilon} \int_{J_{m,k}} \|f(s)\| \ d\lambda(s), \quad m \in \mathbb{N}, k \in K_m,$$
we infer
$$\sum_{m\in\mathbb{N}} \sum_{k\in K_m} |J_{m,k}| \leq \frac{1}{\varepsilon} \sum_{m\in\mathbb{N}} \sum_{k\in K_m} \int_{J_{m,k}} \|f(s)\| \ d\lambda(s) \leq \frac{\|f\|_{L^1}}{\varepsilon}.$$
We set $J := \bigcup\limits_{m\in\mathbb{N}} \bigcup\limits_{k\in K_m} J^{\circ}_{m,k}$ and obtain
$$\begin{aligned}
\|g\|_{L^1} &= \int_{[0,+\infty)} \left\| f(t) - \mathbb{1}_J(t) f(t) + \sum_{m\in\mathbb{N}} \sum_{k\in K_m} \frac{\mathbb{1}_{J^{\circ}_{m,k}}(t)}{\lambda(J_{m,k})} \int_{J_{m,k}} f(s) \ d\lambda(s) \right\| \ d\lambda(t) \\
&\leq \int_{[0,+\infty)\setminus J} \|f(t)\| \ d\lambda(t) + \int_{[0,+\infty)} \sum_{m\in\mathbb{N}} \sum_{k\in K_m} \frac{\mathbb{1}_{J^{\circ}_{m,k}}(t)}{\lambda(J_{m,k})} \Big(\int_{J_{m,k}} \|f(s)\| \ d\lambda(s) \Big) \ d\lambda(t) \\
&\leq \int_{[0,+\infty)\setminus J} \|f(t)\| \ d\lambda(t) + \sum_{m\in\mathbb{N}} \sum_{k\in K_m} \int_{J^{\circ}_{m,k}} \frac{1}{\lambda(J_{m,k})} \Big(\int_{J_{m,k}} \|f(s)\| \ d\lambda(s) \Big) \ d\lambda(t) \\
&\leq \int_{[0,+\infty)\setminus J} \|f(t)\| \ d\lambda(t) + \sum_{m\in\mathbb{N}} \sum_{k\in K_m} \int_{J_{m,k}} \|f(s)\| \ d\lambda(s) = \|f\|_{L^1}.
\end{aligned}$$
Moreover,
$$\|h_{m,k}\|_{L^1} \leq \int_{J_{m,k}} \|f(t)\| + \frac{1}{\lambda(J_{m,k})} \Big(\int_{J_{m,k}} \|f(s)\| \ d\lambda(s) \Big) \ d\lambda(t) = 2 \int_{J_{m,k}} \|f(t)\| \ d\lambda(t)$$
for any $m \in \mathbb{N}$ and $k \in K_m$. Consequently,
$$\|g\|_{L^1} + \sum_{m\in\mathbb{N}} \sum_{k\in K_m} \|h_{m,k}\|_{L^1} \leq 3 \|f\|_{L^1}.$$
Since $\bigcup\limits_{m\in\mathbb{N}} K_m$ is countable, we can rearrange $(J_{m,k})_{m\in\mathbb{N},k\in K_m}$ into a sequence $(I_n)_{n\in\mathbb{N}}$. In the same manner, we rearrange $(h_{m,k})_{m\in\mathbb{N},k\in K_m}$ into $(h_n)_{n\in\mathbb{N}}$. By the previous reasoning g, $(h_n)_{n\in\mathbb{N}}$ and $(I_n)_{n\in\mathbb{N}}$ fulfill $a)$ - $f)$.

□

5.3.2 Lemma (Marcinkiewicz Interpolation Theorem)**.** Let $q, r \in [1, +\infty)$ with $q < r$, $S : L^q\big((0,+\infty); X\big) \cap L^r\big((0,+\infty); X\big) \to \{f : (0,+\infty) \to X \mid f \text{ measurable}\}$ be a linear operator, such that there exists a constant $C > 0$ with
$$\lambda(\{t > 0 : \|Sf(t)\| > \xi\}) \leq \frac{C \|f\|^q_{L^q}}{\xi^q}$$
and
$$\lambda(\{t > 0 : \|Sf(t)\| > \xi\}) \leq \frac{C \|f\|^r_{L^r}}{\xi^r}$$

for all $f \in L^q\big((0,+\infty);X\big) \cap L^r\big((0,+\infty);X\big)$ and $\xi > 0$. Then for every $p \in (q,r)$ we can extend S to a bounded linear operator from $L^p\big((0,+\infty);X\big)$ into $L^p\big((0,+\infty);X\big)$.

Proof. For a measurable function $f : (0,+\infty) \to X$ we define $m_f : [0,+\infty) \to [0,+\infty)$ by $m_f(\xi) := \lambda(\{t > 0 : \|f(t)\| > \xi\})$. Let $f \in L^q\big((0,+\infty);X\big) \cap L^r\big((0,+\infty);X\big)$ and $p \in (q,r)$. Because of

$$\begin{aligned}\int_{(0,+\infty)} \|f(t)\|^p \, d\lambda(t) &= \int_{\{t>0:\|f(t)\|\le 1\}} \|f(t)\|^p \, d\lambda(t) + \int_{\{t>0:\|f(t)\|>1\}} \|f(t)\|^p \, d\lambda(t)\\ &\le \int_{\{t>0:\|f(t)\|\le 1\}} \|f(t)\|^q \, d\lambda(t) + \int_{\{t>0:\|f(t)\|>1\}} \|f(t)\|^r \, d\lambda(t)\\ &\le \|f\|_{L^q}^q + \|f\|_{L^r}^r < +\infty\end{aligned}$$

$f \in L^p\big((0,+\infty);X\big)$. Let $\xi > 0$ and define $g_\xi := \mathbb{1}_{[t>0:\|f(t)\|>\xi]} f$, $h_\xi := f - g_\xi$. Since $g_\xi, h_\xi \in L^q\big((0,+\infty);X\big) \cap L^r\big((0,+\infty);X\big)$, by assumption

$$m_{Sg_\xi}(\kappa) \le \frac{C\,\|g_\xi\|_{L^q}^q}{\kappa^q}, \ m_{Sh_\xi}(\kappa) \le \frac{C\,\|h_\xi\|_{L^r}^r}{\kappa^r}$$

for all $\kappa > 0$.

$$\begin{aligned}\{t>0 : \|Sf(t)\| > \xi\} &\subseteq \{t > 0 : \|Sg_\xi(t)\| + \|Sh_\xi(t)\| > \xi\}\\ &\subseteq \{t>0 : \|Sg_\xi(t)\| > \tfrac{\xi}{2}\} \cup \{t>0 : \|Sh_\xi(t)\| > \tfrac{\xi}{2}\},\end{aligned}$$

implies

$$m_{Sf}(\xi) \le m_{Sg_\xi}(\tfrac{\xi}{2}) + m_{Sh_\xi}(\tfrac{\xi}{2}) \le \frac{2^q C\,\|g_\xi\|_{L^q}^q}{\xi^q} + \frac{2^r C\,\|h_\xi\|_{L^r}^r}{\xi^r}.$$

By Fubini's Theorem for non-negative functions, as shown in [13], Theorem V.2.1, and Lemma 2.4.4 we obtain

$$\begin{aligned}\|Sf\|_{L^p}^p &= \int_{(0,+\infty)} \|Sf(t)\|^p \, d\lambda(t) = \int_{(0,+\infty)} \Big(\int_{(0,\|Sf(t)\|)} p\xi^{p-1} \, d\lambda(\xi)\Big) \, d\lambda(t)\\ &= p\int_{(0,+\infty)}\int_{(0,+\infty)} \mathbb{1}_{(0,\|Sf(t)\|)}(\xi)\xi^{p-1} \, d\lambda(\xi) \, d\lambda(t)\\ &= p\int_{(0,+\infty)} \xi^{p-1}\Big(\int_{(0,+\infty)} \mathbb{1}_{(0,\|Sf(t)\|)}(\xi) \, d\lambda(t)\Big) \, d\lambda(\xi)\\ &= p\int_{(0,+\infty)} \xi^{p-1}\Big(\int_{\{t>0:\|Sf(t)\|>\xi\}} 1 \, d\lambda(t)\Big) \, d\lambda(\xi)\\ &= p\int_{(0,+\infty)} \xi^{p-1} m_{Sf}(\xi) \, d\lambda(\xi)\\ &\le p\int_{(0,+\infty)} \frac{2^q C\xi^{p-1}\,\|g_\xi\|_{L^q}^q}{\xi^q} \, d\lambda(\xi) + p\int_{(0,+\infty)} \frac{2^r C\xi^{p-1}\,\|h_\xi\|_{L^r}^r}{\xi^r} \, d\lambda(\xi).\end{aligned}$$

Again by Fubini's Theorem for non-negative functions

$$\int_{(0,+\infty)} \frac{\xi^{p-1}\,\|g_\xi\|_{L^q}^q}{\xi^q} \, d\lambda(\xi) = \int_{(0,+\infty)} \xi^{p-q-1}\Big(\int_{(0,+\infty)} \mathbb{1}_{[s>0:\|f(s)\|>\xi]}(t)\,\|f(t)\|^q \, d\lambda(t)\Big) \, d\lambda(\xi)$$

$$= \int_{(0,+\infty)} \|f(t)\|^q \Big(\int_{(0,+\infty)} \mathbb{1}_{(0,\|f(t)\|)}(\xi) \xi^{p-q-1} \, d\lambda(\xi) \Big) \, d\lambda(t)$$
$$= \int_{(0,+\infty)} \|f(t)\|^q \Big(\int_{(0,\|f(t)\|)} \xi^{p-q-1} \, d\lambda(\xi) \Big) \, d\lambda(t)$$
$$= \int_{(0,+\infty)} \|f(t)\|^q \, \frac{\|f(t)\|^{p-q}}{p-q} \, d\lambda(t) = \frac{\|f\|_{L^p}^p}{p-q}.$$

An analogous computation leads to

$$\int_{(0,+\infty)} \frac{\xi^{p-1} \, \|h_\xi\|_{L^r}^r}{\xi^r} \, d\lambda(\xi) = \frac{\|f\|_{L^p}^p}{r-p}.$$

Consequently,

$$\|Sf\|_{L^p} \leq \big(Cp(\tfrac{2^q}{p-q} + \tfrac{2^r}{r-p})\big)^{\frac{1}{p}} \, \|f\|_{L^p} .$$

Since

$$C_{00}^\infty\big((0,+\infty);X\big) \subseteq L^q\big((0,+\infty);X\big) \cap L^r\big((0,+\infty);X\big) \subseteq L^p\big((0,+\infty);X\big),$$

$L^q\big((0,+\infty);X\big) \cap L^r\big((0,+\infty);X\big)$ is densely contained in $L^p\big((0,+\infty);X\big)$; see Proposition 2.5.2, *b*). Therefore, we can extend S to $L^p\big((0,+\infty);X\big)$ satisfying

$$\|Sf\|_{L^p} \leq \big(Cp(\tfrac{2^q}{p-q} + \tfrac{2^r}{r-p})\big)^{\frac{1}{p}} \, \|f\|_{L^p}$$

for every $f \in L^p\big((0,+\infty);X\big)$; see [12], Theorem 1.1.1.

□

5.3.3 Lemma. Let X be reflexive, $p \in (1,+\infty)$ and $K : \mathbb{R} \to L_b(X)$ be a measurable function such that $K|_{\mathbb{R}\setminus\{0\}} \in L^1_{loc}\big(\mathbb{R}\setminus\{0\}; L_b(X)\big)$ as well as

$$\int_{\mathbb{R}\setminus[-2|s|,2|s|]} \|K(t-s) - K(t)\| \, d\lambda(t) \leq C$$

for all $s \in \mathbb{R}\setminus\{0\}$ and some $C > 0$. If $S \in L_b\big(L^p\big((0,+\infty);X\big)\big)$ satisfies

$$(Sf)(t) = \int_{(0,+\infty)} K(t-s) f(s) \, d\lambda(s) \tag{5.17}$$

for all $f \in L^p\big((0,+\infty);X\big)$ with compact support and almost every $t \in \mathbb{R}\setminus[\inf\operatorname{supp}(f), \sup\operatorname{supp}(f)]$, then for any $q \in (1,+\infty)$ the operator $S|_{L^1((0,+\infty);X)\cap L^p((0,+\infty);X)\cap L^q((0,+\infty);X)}$ can be extended boundedly to an operator $S_q \in L^q\big((0,+\infty);X\big)$.

Proof. For a measurable function $f : (0, +\infty) \to X$ we define $m_f : [0, +\infty) \to [0, +\infty)$ by $m_f(\xi) := \lambda(\{t > 0 : \|f(t)\| > \xi\})$. Given $f \in L^p\big((0, +\infty); X\big)$ and $\xi > 0$, we have

$$\xi^p m_{Sf}(\xi) = \xi^p \lambda(\{t > 0 : \|Sf(t)\|^p > \xi^p\}) = \int_{(0,+\infty)} \mathbb{1}_{\{s>0:\|Sf(s)\|^p>\xi^p\}}(t)\xi^p \, d\lambda(t)$$
$$\leq \int_{\{s>0:\|Sf(s)\|^p>\xi^p\}} \|Sf(t)\|^p \, d\lambda(t) \leq \|Sf\|_{L^p}^p \leq \|S\|_{L^p}^p \|f\|_{L^p}^p ,$$

which implies

$$m_{Sf}(\xi) \leq \frac{\|S\|_{L^p}^p \|f\|_{L^p}^p}{\xi^p}$$

for all $\xi > 0$. In order to prove the existence of a constant $M > 0$, such that

$$m_{Sf}(\xi) \leq \frac{M \|f\|_{L^1}^1}{\xi}$$

for all $f \in L^1\big((0, +\infty); X\big) \cap L^p\big((0, +\infty); X\big)$ and $\xi > 0$, let
$f \in L^1\big((0, +\infty); X\big) \cap L^p\big((0, +\infty); X\big)$ and $\varepsilon > 0$. We extend f to $[0, +\infty)$ by setting $f(0) := 0$. By Lemma 5.3.1 we find closed intervals $I_n \subseteq [0, +\infty)$ and functions h_n, $n \in \mathbb{N}$ and g, such that *a*) - *f*) in Lemma 5.3.1 are fulfilled. From

$$\|g\|_{L^p}^p = \int_{(0,+\infty)} \|g(t)\|^p \, d\lambda(t) = \int_{(0,+\infty)} \|g(t)\| \, \|g(t)\|^{p-1} \, d\lambda(t) \leq (2\varepsilon)^{p-1} \|g\|_{L^1} < +\infty$$

we derive $g \in L^p\big((0, +\infty); X\big)$. Consequently, by *c*) in Lemma 5.3.1

$$m_{Sg}(\tfrac{\varepsilon}{2}) \leq \frac{2^p \|S\|_{L^p}^p \|g\|_{L^p}^p}{\varepsilon^p} \leq \frac{2^{2p-1}\varepsilon^{p-1} \|S\|_{L^p}^p \|g\|_{L^1}}{\varepsilon^p}$$
$$= \frac{2^{2p-1} \|S\|_{L^p}^p \|g\|_{L^1}}{\varepsilon} = \frac{2^{2p+1} \|S\|_{L^p}^p \|f\|_{L^1}}{\varepsilon}.$$

Define

$$h(t) := \begin{cases} 0, & t \in \{\inf I_n : n \in \mathbb{N}\} \cup \{\sup I_n : n \in \mathbb{N}\}, \\ \sum_{n=1}^{\infty} h_n(t), & \text{else}, \end{cases}$$

$s_n := \frac{1}{2}(\sup I_n + \inf I_n)$, $J_n := [s_n - \lambda(I_n), s_n + \lambda(I_n)]$ and

$$I := \bigcup_{n\in\mathbb{N}} J_n.$$

Note that, since $|I_n \cap I_m| \leq 1$, the sum

$$\sum_{n=1}^{\infty} h_n(t)$$

contains at most one non-vanishing summand for $t \notin \{\inf I_n : n \in \mathbb{N}\} \cup \{\sup I_n : n \in \mathbb{N}\}$.
It is easy to see that $s_n = \frac{1}{2}(\sup J_n + \inf J_n)$, $I_n \subseteq J_n$ and $\lambda(J_n) = 2\lambda(I_n)$.

$$\{t > 0 : \|Sh(t)\| > \tfrac{\varepsilon}{2}\} = \{t \in I : \|Sh(t)\| > \tfrac{\varepsilon}{2}\} \cup \{t \in (0, +\infty) \setminus I : \|Sh(t)\| > \tfrac{\varepsilon}{2}\}$$

$$\subseteq I \cup \{t \in (0,+\infty) \setminus I : \|Sh(t)\| > \tfrac{\varepsilon}{2}\},$$

yields

$$m_{Sh}(\tfrac{\varepsilon}{2}) \le \lambda(I) + \lambda(\{t \in (0,+\infty) \setminus I : \|Sh(t)\| > \tfrac{\varepsilon}{2}\}). \tag{5.18}$$

By $f)$ in Lemma 5.3.1

$$\lambda(I) \le \sum_{n=1}^{\infty} \lambda(J_n) = 2\sum_{n=1}^{\infty} \lambda(I_n) \le \frac{2\,\|f\|_{L^1}}{\varepsilon}.$$

$f(t) = g(t) + h_n(t)$ for every $t \in I_n$ yields $h_n \in L^p(I_n; X)$. Since $\operatorname{supp}(h_n) \subseteq I_n$ by $d)$ in Lemma 5.3.1 and $f = g + h$ almost everywhere, $h_n \in L^p\big((0,+\infty); X\big)$ as well as $h \in L^p\big((0,+\infty); X\big)$. Since $\{\inf I_n : n \in \mathbb{N}\} \cup \{\sup I_n : n \in \mathbb{N}\}$ is countable and therefore a null set, we have

$$\lim_{N\to+\infty} \sum_{n=1}^{N} h_n(t) = h(t)$$

and

$$\left\|\sum_{n=1}^{N} h_n(t) - h(t)\right\|^p \le \|h(t)\|^p, \quad N \in \mathbb{N},$$

for almost every $t \in (0,+\infty)$. By Theorem 2.3.7

$$\lim_{N\to+\infty} \left\|\sum_{n=1}^{N} h_n - h\right\|_{L^p} = 0.$$

Hence,

$$S(h) = S\Big(\lim_{N\to+\infty} \sum_{n=1}^{N} h_n\Big) = \lim_{N\to+\infty} \sum_{n=1}^{N} S(h_n) = \sum_{n=1}^{\infty} S(h_n).$$

Since this is an equality in $L^p\big((0,+\infty); X\big)$, the functions $S(h)$ and $\sum_{n=1}^{\infty} S(h_n)$ differ only on a null set. Furthermore, $\operatorname{supp}(h_n) \subseteq I_n$ and $t \in (0,+\infty) \setminus I$ implies $t \notin I_n \supseteq [\inf \operatorname{supp}(h_n), \sup \operatorname{supp}(h_n)]$. Consequently, by our assumption (5.17), by Fubini's Theorem for non-negative functions and due to

$$\int_{I_n} h_n(s)\, d\lambda(s) = 0$$

we have

$$\begin{aligned}
\int_{(0,+\infty)\setminus I} \|Sh(t)\|\, d\lambda(t) &= \int_{(0,+\infty)\setminus I} \left\|\sum_{n=1}^{\infty} S(h_n)(t)\right\| d\lambda(t) \le \sum_{n=1}^{\infty} \int_{(0,+\infty)\setminus I} \|Sh_n(t)\|\; d\lambda(t) \\
&= \sum_{n=1}^{\infty} \int_{(0,+\infty)\setminus I} \left\|\int_{I_n} K(t-s) h_n(s)\; d\lambda(s)\right\|\; d\lambda(t) \\
&\le \sum_{n=1}^{\infty} \int_{(0,+\infty)\setminus I} \left\|\int_{I_n} \big(K(t-s) - K(t-s_n)\big) h_n(s)\; d\lambda(s)\right\|\; d\lambda(t)
\end{aligned}$$

$$\leq \sum_{n=1}^{\infty} \int_{(0,+\infty)\setminus I} \int_{I_n} \big\| \big(K(t-s) - K(t-s_n)\big) h_n(s) \big\| \ d\lambda(s)\ d\lambda(t)$$
$$\leq \sum_{n=1}^{\infty} \int_{I_n} \|h_n(s)\| \Big(\int_{(0,+\infty)\setminus I} \|K(t-s) - K(t-s_n)\| \ d\lambda(t) \Big)\ d\lambda(s).$$

For $n \in \mathbb{N}$ and $s \in I_n$ substituting $t \mapsto t + s_n$ leads to

$$\int_{(0,+\infty)\setminus I} \|K(t-s) - K(t-s_n)\| \ d\lambda(t) = \int_{((0,+\infty)\setminus I)-s_n} \big\|K\big(t-(s-s_n)\big) - K(t)\big\| \ d\lambda(t).$$

For $t + s_n \notin I \supseteq J_n$ we have $|t| = |(t+s_n) - s_n| > \frac{1}{2}\lambda(J_n) = \lambda(I_n)$. On the other hand, from $s \in I_n$ we conclude $|s - s_n| \leq \frac{1}{2}\lambda(I_n)$, which implies $|t| > \lambda(I_n) \geq 2|s-s_n|$. By assumption and $b)$ in Lemma 5.3.1 we obtain

$$\int_{(0,+\infty)\setminus I} \|Sh(t)\|\, d\lambda(t) \leq \sum_{n=1}^{\infty} \int_{I_n} \|h_n(s)\| \Big(\int_{(0,+\infty)\setminus I} \|K(t-s) - K(t-s_n)\| \ d\lambda(t) \Big)\ d\lambda(s)$$
$$= \sum_{n=1}^{\infty} \int_{I_n} \|h_n(s)\| \Big(\int_{((0,+\infty)\setminus I)-s_n} \big\|K\big(t-(s-s_n)\big) - K(t)\big\| \ d\lambda(t) \Big)\ d\lambda(s)$$
$$\leq \sum_{n=1}^{\infty} \int_{I_n} \|h_n(s)\| \Big(\int_{\mathbb{R}\setminus[-2|s-s_n|, 2|s-s_n|]} \big\|K\big(t-(s-s_n)\big) - K(t)\big\| \ d\lambda(t) \Big)\ d\lambda(s)$$
$$\leq C \sum_{n=1}^{\infty} \int_{I_n} \|h_n(s)\| \ d\lambda(s) \leq 3C \, \|f\|_{L^1}$$

and, in conclusion,

$$\varepsilon\lambda(\{t \in (0,+\infty)\setminus I : \|Sh(t)\| > \tfrac{\varepsilon}{2}\}) = \int_{\{t\in(0,+\infty)\setminus I : \|Sh(t)\| > \frac{\varepsilon}{2}\}} \varepsilon \ d\lambda(t)$$
$$\leq 2 \int_{\{t\in(0,+\infty)\setminus I : \|Sh(t)\| > \frac{\varepsilon}{2}\}} \|Sh(t)\| \ d\lambda(t)$$
$$\leq 2 \int_{(0,+\infty)\setminus I} \|Sh(t)\| \ d\lambda(t) \leq 6C \, \|f\|_1 .$$

Consequently, (5.18) gives

$$m_{Sh}(\tfrac{\varepsilon}{2}) \leq \frac{(2+6C)\, \|f\|_{L^1}}{\varepsilon},$$

and

$$\{t > 0 : \|Sf(t)\| > \varepsilon\} \subseteq \{t > 0 : \|Sg(t)\| > \tfrac{\varepsilon}{2}\} \cup \{t > 0 : \|Sh(t)\| > \tfrac{\varepsilon}{2}\},$$

implies

$$m_{Sf}(\varepsilon) \leq m_{Sg}(\tfrac{\varepsilon}{2}) + m_{Sh}(\tfrac{\varepsilon}{2}) \leq \frac{(2^{2p+1}L + 6C + 2)\, \|f\|_{L^1}}{\varepsilon}.$$

Since $\varepsilon > 0$ and $f \in L^1\big((0,+\infty);X\big) \cap L^p\big((0,+\infty);X\big)$ were arbitrary and

$$L^1\big((0,+\infty);X\big) \cap L^q\big((0,+\infty);X\big) \cap L^p\big((0,+\infty);X\big) = L^1\big((0,+\infty);X\big) \cap L^p\big((0,+\infty);X\big),$$

by Lemma 5.3.2 for every $q \in (1,p)$ $S|_{L^1((0,+\infty);X)\cap L^q((0,+\infty);X)\cap L^p((0,+\infty);X)}$ can be extended boundedly to $L^q\big((0,+\infty);X\big)$.

Let $S' \in L^p\big((0,+\infty);X\big)' \to L^p\big((0,+\infty);X\big)'$ be the conjugate operator of S, i.e. $S'(\varphi)(f) = \varphi(Sf)$ for every $\varphi \in L^p\big((0,+\infty);X\big)'$ and $f \in L^p\big((0,+\infty);X\big)$, $p' \in (1,+\infty)$ such that $\frac{1}{p}+\frac{1}{p'}=1$. Since X is reflexive, the mapping $\Phi_p : L^{p'}\big((0,+\infty);X'\big) \to L^p\big((0,+\infty);X\big)'$ defined by

$$\Phi_p(\varphi)(f) := \int_{(0,+\infty)} \varphi(t)\big(f(t)\big)\ d\lambda(t)$$

is an isometric isomorphism; see Proposition 2.5.2, e). We define

$$R := \Phi_p^{-1} \circ S' \circ \Phi_p : L^{p'}\big((0,+\infty);X'\big) \to L^{p'}\big((0,+\infty);X'\big)$$

and $L : \mathbb{R}_+ \to L_b(X')$ by $L(t) := \big(K(-t)\big)'$. We want to prove that R and L have the same properties as S and K. By Theorem 6.1.2 in [12], $L_b(Y) \ni T \mapsto T' \in L_b(Y')$ is linear and isometric for every Banach space Y. Consequently, substituting $t \mapsto -t$ for $s \in \mathbb{R}\setminus\{0\}$ we have

$$\begin{aligned}
\int_{\mathbb{R}\setminus[-2|s|,2|s|]} \|L(t-s)-L(t)\|\ d\lambda(t) &= \int_{\mathbb{R}\setminus[-2|s|,2|s|]} \Big\|\big(K(s-t)-K(-t)\big)'\Big\|\ d\lambda(t)\\
&= \int_{\mathbb{R}\setminus[-2|s|,2|s|]} \|K(s-t)-K(-t)\|\ d\lambda(t)\\
&= \int_{\mathbb{R}\setminus[-2|s|,2|s|]} \|K(t+s)-K(t)\|\ d\lambda(t) \le C.
\end{aligned}$$

Let $x \in X$ and $f \in L^{p'}\big((0,+\infty);X'\big)$ with compact support and set $a := \inf \operatorname{supp}(f)$, $b := \sup \operatorname{supp}(f)$. Moreover, let $\varphi_1 \in C_{00}^\infty\big((0,a);\mathbb{C}\big)$ and $\varphi_2 \in C_{00}^\infty\big((b,+\infty);\mathbb{C}\big)$ be extended to the whole half axis by $\varphi_1(t)=0$ for $t \in [a,+\infty)$ and $\varphi_2(t)=0$ for $t \in (0,b]$. $\varphi := \varphi_1+\varphi_2$ belongs to $C_{00}^\infty\big((0,+\infty);\mathbb{C}\big) \subseteq L^p\big((0,+\infty);\mathbb{C}\big)$ satisfying $\operatorname{supp}(\varphi) \subseteq (0,+\infty)\setminus[a,b]$. We obtain

$$\begin{aligned}
\int_{(0,+\infty)} (Rf)(s)\varphi(s)x\ d\lambda(s) &= \Phi_p(Rf)(\varphi(\cdot)x) = (S'\circ\Phi_p)(f)(\varphi(\cdot)x) = \Phi_p(f)\big(S(\varphi(\cdot)x)\big)\\
&= \int_{(0,+\infty)} f(s)\big((S(\varphi(\cdot)x))(s)\big)\ d\lambda(s)\\
&= \int_{(a,b)} f(s)\big((S(\varphi(\cdot)x))(s)\big)\ d\lambda(s)\\
&= \int_{(a,b)} f(s)\Big(\int_{(0,+\infty)} K(s-t)\varphi(t)x\ d\lambda(t)\Big)\ d\lambda(s)\\
&= \int_{(a,b)}\int_{(0,+\infty)} f(s)\big(K(s-t)\varphi(t)x\big)\ d\lambda(t)\ d\lambda(s)\\
&= \int_{(a,b)}\int_{(0,+\infty)} \big(K(s-t)\big)'\big(f(s)\big)\varphi(t)x\ d\lambda(t)\ d\lambda(s)
\end{aligned}$$

$$= \int_{(a,b)} \int_{(0,+\infty)} L(t-s)\big(f(s)\big)\varphi(t)x \; d\lambda(t) \; d\lambda(s).$$

For $t \in \operatorname{supp}(\varphi_1) \cup \operatorname{supp}(\varphi_2)$ and $s \in [a,b]$ we have $t - s \neq 0$, meaning $\{t-s : t \in \operatorname{supp}(\varphi_1) \cup \operatorname{supp}(\varphi_2), s \in [a,b]\}$ is contained in some compact set K with $0 \notin K$.

$$\|f\|_{L^1((a,b)\cup(c,d);X')} = \int_{(a,b)\cup(c,d)} \|f(t)\| \; d\lambda(t) \leq (b-a+d-c)^{\frac{1}{p}} \|f\|_{L^{p'}((0,+\infty);X')} < +\infty$$

implies

$$\begin{aligned}
&\int_{(a,b)} \int_{(0,+\infty)} \big\|L(t-s)\big(f(s)\big)\varphi(t)x\big\| \; d\lambda(t) \; d\lambda(s) \\
&\qquad\qquad \leq \|\varphi\|_\infty \|x\| \int_{(a,b)} \int_{\operatorname{supp}(\varphi)} \|L(t-s)\| \, \|f(s)\| \; d\lambda(t) \; d\lambda(s) \\
&\qquad\qquad \leq \|\varphi\|_\infty \|x\| \, \|L\|_{L^1(K;X)} \|f\|_{L^1((a,b)\cup(c,d);X')} < +\infty.
\end{aligned}$$

By Theorem 2.3.12 we obtain

$$\begin{aligned}
\int_{(0,+\infty)} (Rf)(s)\varphi(s)x \; d\lambda(s) &= \int_{(0,+\infty)} \int_{(a,b)} L(t-s)\big(f(s)\big)\varphi(t)x \; d\lambda(s) \; d\lambda(t) \\
&= \int_{(0,+\infty)} \Big(\int_{(a,b)} L(t-s)\big(f(s)\big) \; d\lambda(s)\Big)\varphi(t)x \; d\lambda(t).
\end{aligned}$$

Subtracting the left side from the right side leads to

$$\int_{(0,+\infty)} \varphi(t)\Big(\int_{(0,+\infty)} L(t-s)\big(f(s)\big) \; d\lambda(s) - (Rf)(t)\Big)x \; d\lambda(t) = 0.$$

Since

$$(0,+\infty) \ni t \mapsto \varphi(t)\Big(\int_{(0,+\infty)} L(t-s)\big(f(s)\big) \; d\lambda(s) - (Rf)(t)\Big) \in X'$$

is integrable, we can employ Proposition 2.3.6, $c)$ and obtain

$$\Big(\int_{(0,+\infty)} \varphi(t)\Big(\int_{(0,+\infty)} L(t-s)\big(f(s)\big) \; d\lambda(s) - (Rf)(t)\Big) \; d\lambda(t)\Big)x = 0$$

for any $x \in X$ implying

$$\int_{(0,+\infty)} \varphi(t)\Big(\int_{(0,+\infty)} L(t-s)\big(f(s)\big) \; d\lambda(s) - (Rf)(t)\Big) \; d\lambda(t) = 0 \in X'.$$

Choosing $\varphi_2 = 0$ we obtain

$$\int_{(0,a)} \varphi_1(t)\Big(\int_{(0,+\infty)} L(t-s)\big(f(s)\big) \; d\lambda(s) - (Rf)(t)\Big) \; d\lambda(t) = 0$$

for every $\varphi_1 \in C_{00}^\infty\big((0,a);\mathbb{C}\big)$. Proposition 2.5.2, c) implies

$$(Rf)(t) = \int_{(0,+\infty)} L(t-s)\big(f(s)\big)\, d\lambda(s)$$

for almost every $t \in (0,a)$. An analogous argument shows

$$(Rf)(t) = \int_{(0,+\infty)} L(t-s)\big(f(s)\big)\, d\lambda(s)$$

for almost every $t \in (b,+\infty)$. Applying the first part of the proof to R, we conclude that $R|_{L^1((0,+\infty);X')\cap L^{q'}((0,+\infty);X')\cap L^{p'}((0,+\infty);X')}$ is boundedly extendable to an operator $R_{q'} \in L_b\big(L^{q'}\big((0,+\infty);X'\big)\big)$ for all $1 < q' < p'$.

Fix $q \in (p,+\infty)$ and let $q' \in (1,p')$ satisfy $\frac{1}{q} + \frac{1}{q'} = 1$ and let $\Phi_q : L^{q'}\big((0,+\infty);X'\big) \to L^q\big((0,+\infty);X\big)'$ be the isometric isomorphism as in Proposition 2.5.2, e). By Lemma 6.1.3 in [12] $\Phi_q' : L^q\big((0,+\infty);X\big)'' \to L^{q'}\big((0,+\infty);X'\big)'$ also is an isometric isomorphism. Since X is reflexive, so is $L^q\big(0,+\infty);X\big)$ by Proposition 2.5.2, f), and the mapping $\iota : L^q\big((0,+\infty);X\big) \to L^q\big((0,+\infty);X\big)''$ defined by $\iota(f)(\varphi) := \varphi(f)$ is an isometric isomorphism. Let $f \in L^1\big((0,+\infty);X\big) \cap L^p\big((0,+\infty);X\big) \cap L^q\big((0,+\infty);X\big)$ and define

$$g := (\iota^{-1} \circ (\Phi_q^{-1})' \circ R_{q'}' \circ \Phi_q' \circ \iota)(f) \in L^q\big((0,+\infty);X\big),$$

where $R_{q'}' \in L_b\big(L^{q'}\big((0,+\infty);X'\big)'\big)$ is the conjugate operator of $R_{q'} \in L_b\big(L^{q'}\big((0,+\infty);X'\big)\big)$. For $\varphi \in L^{q'}\big((0,+\infty);X'\big)$ we have

$$\Phi_q'\big(\iota(f)\big)(\varphi) = \iota(f)\big(\Phi_q(\varphi)\big) = \Phi_q(\varphi)(f) = \int_{(0,+\infty)} \varphi(t)\big(f(t)\big)\, d\lambda(t),$$

and, in consequence,

$$(R_{q'}' \circ \Phi_q' \circ \iota)(f)(\varphi) = \Phi_q'\big(\iota(f)\big)(R_{q'}\varphi) = \int_{(0,+\infty)} (R_{q'}\varphi)(t)\big(f(t)\big)\, d\lambda(t).$$

Moreover,

$$\Phi_q'\big(\iota(g)\big)(\varphi) = \int_{(0,+\infty)} \varphi(t)\big(g(t)\big)\, d\lambda(t),$$

implying

$$\Phi_q(R_{q'}\varphi)(f) = \int_{(0,+\infty)} (R_{q'}\varphi)(t)\big(f(t)\big)\, d\lambda(t) = \int_{(0,+\infty)} \varphi(t)\big(g(t)\big)\, d\lambda(t) = \Phi_q(\varphi)(g)$$

for all $\varphi \in L^{q'}\big((0,+\infty);X'\big)$. For $\varphi \in L^1\big((0,+\infty);X'\big) \cap L^{q'}\big((0,+\infty);X'\big) \cap L^{p'}\big((0,+\infty);X'\big)$ we obtain $R_{q'}\varphi = (\Phi_p^{-1} \circ S' \circ \Phi_p)(\varphi) \in L^{p'}\big((0,+\infty);X'\big)$ and hence

$$\int_{(0,+\infty)} \varphi(t)\big(g(t)\big)\, d\lambda(t) = \Phi_q(\varphi)(g) = \Phi_q(R_{q'}\varphi)(f) = \int_{(0,+\infty)} (R_{q'}\varphi)(t)\big(f(t)\big)\, d\lambda(t)$$

$$= \Phi_p(R_{q'}\varphi)(f) = S'\big(\Phi_p(\varphi)\big)(f) = \Phi_p(\varphi)(Sf)$$
$$= \int_{(0,+\infty)} \varphi(t)\big((Sf)(t)\big)\, d\lambda(t).$$

For $\psi \in C_{00}^{\infty}\big((0,+\infty);\mathbb{C}\big)$ and $x' \in X'$ we have $\psi(\cdot)x' \in L^1\big((0,+\infty);X'\big) \cap L^{p'}\big((0,+\infty);X'\big)$. By Hölder's inequality, Proposition 2.5.3, *a*), $\psi(g - Sf)$ is integrable and we have

$$0 = \int_{(0,+\infty)} \psi(t)x'\big(g(t) - (Sf)(t)\big)\, d\lambda(t) = x'\Big(\int_{(0,+\infty)} \psi(t)\big(g(t) - (Sf)(t)\big)\, d\lambda(t)\Big).$$

By Corollary 5.2.7 in [12], X' acts point separating on X, implying

$$\int_{(0,+\infty)} \psi(t)\big(g(t) - (Sf)(t)\big)\, d\lambda(t) = 0.$$

Since $\psi \in C_{00}^{\infty}\big((0,+\infty);\mathbb{C}\big)$ was arbitrary, $g(t) = (Sf)(t)$ fo almost every $t \in (0,+\infty)$; see Proposition 2.5.2, *c*). We obtain $Sf = g \in L^q\big((0,+\infty);X\big)$ and

$$\begin{aligned}
\|Sf\|_{L^q((0,+\infty);X)} &= \|g\|_{L^q((0,+\infty);X)} = \big\|(\iota^{-1} \circ (\Phi'_{sr})^{-1} \circ R'_{q'} \circ \Phi'_{sr} \circ \iota)(f)\big\|_{L^q((0,+\infty);X)} \\
&= \big\|(R'_{q'} \circ \Phi'_{sr} \circ \iota)(f)\big\|_{L^{q'}((0,+\infty);X')'} \leq \big\|R'_{q'}\big\| \, \|(\Phi'_{sr} \circ \iota)(f)\|_{L^{q'}((0,+\infty);X')'} \\
&= \|R_{q'}\| \, \|f\|_{L^q((0,+\infty);X)} \, .
\end{aligned}$$

By Theorem 1.1.1 in [12] we can extend $S|_{L^1((0,+\infty);X)\cap L^p((0,+\infty);X)\cap L^q((0,+\infty);X)}$ to an operator $S_q \in L_b\big(L^q\big((0,+\infty);X\big)\big)$.

□

5.3.4 Theorem. *If X is reflexive and A has the maximal L^p-regularity property on $(0,+\infty)$ for one $p \in (1,+\infty)$, then A has the maximal L^q-regularity property on $(0,+\infty)$ for all $q \in (1,+\infty)$.*

Proof. Since A is maximally L^p-regular, for every $f \in L^p\big((0,+\infty);X\big)$ there exists a unique weak solution $u_f : [0,+\infty) \to X$ of (5.1). According to Definition 4.2.1, $u_f(t) \in D(A)$ for almost every $t \in (0,+\infty)$. Hence, we can define the linear operator S by

$$(Sf)(t) := \begin{cases} Au_f(t), & u_f(t) \in D(A), \\ 0, & \text{else}. \end{cases}$$

Moreover, A being maximally L^p-regular yields

$$\|Au_f\|_{L^p((0,+\infty);X)} \leq D\, \|f\|_{L^p((0,+\infty);X)}\, , \quad f \in L^p\big((0,+\infty);X\big),$$

for some $D > 0$, implying $\|Sf\|_{L^p((0,+\infty);X)} \leq D\, \|f\|_{L^p((0,+\infty);X)}$ and, consequently, $S \in L_b\big(L^p\big((0,+\infty);X\big)\big)$. By Corollary 5.1.6, A is the infinitesimal generator of an analytic and therefore differentiable semigroup (Proposition 3.3.4) and by Proposition 4.2.7

$$u_f(t) = \int_{(0,t)} T(t-s)f(s)\ d\lambda(s), \quad t \in [0,+\infty),$$

which leads to

$$(Sf)(t) = A\Big(\int_{(0,t)} T(t-s)f(s)\ d\lambda(s)\Big), \quad f \in L^p\big((0,+\infty);X\big)$$

for every $t > 0$, such that $u_f(t) \in D(A)$.

By Proposition 3.3.4, the mapping $K : \mathbb{R}\setminus\{0\} \to L_b(X)$, $K := \mathbb{1}_{(0,+\infty)}AT$ is well-defined. We are going to verify the assumptions of Lemma 5.3.3. K is continuous on $\mathbb{R}\setminus\{0\}$ by Proposition 3.3.4, implying $K \in L^1_{\text{loc}}(\mathbb{R}\setminus\{0\};X)$ by Theorem 2.4.2. By Proposition 3.3.4 AT is differentiable with $(AT)'(t) = A^2T(t)$ for every $t > 0$. By Theorem 3.3.5 there is a constant $C > 0$ such that

$$\|AT(t)\| \le \frac{C}{t},\ t > 0.$$

Employing Lemma 2.4.4 given $s \in \mathbb{R}\setminus\{0\}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}\setminus[-2|s|,2|s|]} \|K(t-s) - K(t)\|\ d\lambda(t) \\
&\qquad = \int_{\mathbb{R}\setminus[-2|s|,2|s|]} \Big\|\int_{(t-s,t)} A^2T(r)\ d\lambda(r)\Big\|\ d\lambda(t) \\
&\qquad \le \int_{\mathbb{R}\setminus[-2|s|,2|s|]} \Big|\int_{(t-s,t)} \|AT(\tfrac{r}{2})\|\,\|AT(\tfrac{r}{2})\|\ d\lambda(r)\Big|\ d\lambda(t) \\
&\qquad \le 4C^2 \int_{\mathbb{R}\setminus[-2|s|,2|s|]} \Big|\int_{(t-s,t)} \frac{1}{r^2}\ d\lambda(r)\Big|\ d\lambda(t) \\
&\qquad = 4C^2 \int_{\mathbb{R}\setminus[-2|s|,2|s|]} \Big|\frac{1}{t-s} - \frac{1}{t}\Big|\ d\lambda(t) \\
&\qquad = 4C^2 \int_{(2|s|,+\infty)} \Big|\frac{1}{t} - \frac{1}{t+s}\Big| + \Big|\frac{1}{t-s} - \frac{1}{t}\Big|\ d\lambda(t)
\end{aligned}$$

For $s > 0$ we have

$$\begin{aligned}
\int_{(2s,+\infty)} \frac{1}{t} - \frac{1}{t+s} + \frac{1}{t-s} - \frac{1}{t}\ d\lambda(t) &= \int_{2s}^{\infty} \frac{1}{t-s} - \frac{1}{t+s}\ dt \\
&= \ln(t-s) - \ln(t+s)\Big|_{2s}^{\infty} = \ln(3)
\end{aligned}$$

and for $s < 0$

$$\begin{aligned}
\int_{(-2s,+\infty)} \frac{1}{t+s} - \frac{1}{t} + \frac{1}{t} - \frac{1}{t-s}\ d\lambda(t) &= \int_{-2s}^{\infty} \frac{1}{t+s} - \frac{1}{t-s}\ dt \\
&= \ln(t+s) - \ln(t-s)\Big|_{-2s}^{\infty} = \ln(3).
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}\setminus[-2|s|,2|s|]} \|K(t-s)-K(t)\| \, d\lambda(t) \leq 4K^2\ln(3).$$

Lastly, let $f \in L^p\big((0,+\infty);X\big)$ have compact support and $t \in (0,+\infty)\setminus[\inf\operatorname{supp}(f),\sup\operatorname{supp}(f)]$. By Lemma 4.2.3 $s \mapsto T(t-s)f(s)$ is integrable over $(0,t)$. If $t > \sup\operatorname{supp}(f)$, then

$$\begin{aligned}\int_{(0,t)} \|AT(t-s)f(s)\| \, d\lambda(s) &= \int_{(\inf\operatorname{supp}(f),\sup\operatorname{supp}(f))} \|AT(t-s)f(s)\| \, d\lambda(s)\\ &\leq \frac{K}{t-\sup\operatorname{supp}(f)} \|f\|_{L^1((0,+\infty);X)}.\end{aligned}$$

We employ Proposition 2.3.13 and obtain

$$\begin{aligned}(Sf)(t) = Au(t) &= A\Big(\int_{(0,t)} T(t-s)f(s) \, d\lambda(s)\Big)\\ &= \int_{(0,t)} AT(t-s)f(s)\, d\lambda(s) = \int_{(0,+\infty)} K(t-s)f(s)\, d\lambda(s).\end{aligned}$$

For $t < \inf\operatorname{supp}(f)$ we have

$$\begin{aligned}(Sf)(t) &= A\Big(\int_{(0,t)} T(t-s)f(s)\, d\lambda(s)\Big) = 0\\ &= \int_{(0,t)} AT(t-s)f(s)\, d\lambda(s) = \int_{(0,+\infty)} K(t-s)f(s)\, d\lambda(s).\end{aligned}$$

Hence, we can apply Lemma 5.3.3 and obtain that for any $q \in (1,+\infty)$ we can boundedly extend $S|_{L^1((0,+\infty);X)\cap L^p((0,+\infty);X)\cap L^q((0,+\infty);X)}$ to an operator $S_q \in L_b\big(L^q(\mathbb{R}_+;X)\big)$.

We show that for every $f \in L^q\big((0,+\infty);X\big)$, $q \in (1,+\infty)$, there is a unique weak solution of (5.1). By Proposition 4.2.7, it suffices to show that the mild solution

$$u(t) = \int_{(0,t)} T(t-s)f(s)\, d\lambda(s)$$

is a weak solution. By Proposition 2.5.2, $b)$ we have $\|\varphi_n - f\|_{L^q} \xrightarrow{n\to+\infty} 0$ for a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_{00}^\infty\big((0,+\infty);X\big) \subseteq L^p\big((0,+\infty);X\big)$. Since $\varphi_n \in L^p\big((0,+\infty);X\big)$, by assumption there is a unique weak solution u_n satisfying $u_n' \in L^p\big((0,+\infty);X\big)$,

$$\begin{cases} u_n'(t) = Au_n(t) + \varphi_n(t), \quad t \in (0,+\infty),\\ u_n(0) = 0,\end{cases}$$

and

$$u_n(t) = \int_{(0,t)} T(t-s)\varphi_n(s)\, d\lambda(s),\ t \geq 0.$$

Let M, ω as in Proposition 3.1.2, $a)$, $t > 0$ and $r \in (1, +\infty)$ such that $\frac{1}{q} + \frac{1}{r} = 1$. By Proposition 2.5.3, $a)$

$$\begin{aligned} \|u_n(t) - u(t)\| &\le \int_{(0,t)} \|T(t-s)\| \, \|\varphi_n(s) - f(s)\| \; d\lambda(s) \\ &\le M \int_{(0,t)} e^{\omega(t-s)} \|\varphi_n(s) - f(s)\| \; d\lambda(s) \\ &\le M e^{\omega t} (1 - e^{-\omega r t})^{\frac{1}{r}} \|\varphi_n - f\|_{L^q} \xrightarrow{n \to +\infty} 0. \end{aligned}$$

Moreover,

$$\|Au_n - S_q f\|_{L^q} = \|S_q \varphi_n - S_q f\|_{L^q} \le \|S_q\| \, \|\varphi_n - f\|_{L^q} \xrightarrow{n \to +\infty} 0, \tag{5.19}$$

which by Proposition 2.5.2, $a)$ implies that there is a subsequence $(Au_{n_k})_{n \in \mathbb{N}}$ satisfying $Au_{n_k}(t) \xrightarrow{k \to +\infty} (S_q f)(t)$ for almost every $t > 0$. Since A is closed, $u(t) \in D(A)$ for almost every $t > 0$ and $Au(t) = (S_q f)(t)$ for every $t > 0$, for which $u(t) \in D(A)$ holds true. We obtain

$$\|Au\|_{L^q} = \|S_q f\|_{L^q} \le \|S_q\| \, \|f\|_{L^q} .$$

Given $0 < c < d < +\infty$, we have $u_n \in W^{1,p}\big((c,d); X\big)$ and

$$u_n(d) - u_n(c) = \int_{(c,d)} Au_n(t) + \varphi_n(t) \; d\lambda(t)$$

by Corollary 2.5.8.

$$\begin{aligned} \left\| \int_{(c,d)} \varphi_n(t) \; d\lambda(t) - \int_{(c,d)} f(t) \; d\lambda(t) \right\| &\le \int_{(c,d)} \|\varphi_n - f\| \; d\lambda(t) \\ &\le (d-c)^{\frac{1}{r}} \|\varphi_n - f\|_{L^q} \xrightarrow{n \to +\infty} 0, \end{aligned}$$

together with (5.19) implies

$$\begin{aligned} u(d) - u(c) &= \lim_{n \to +\infty} u_n(d) - u_n(c) \\ &= \lim_{n \to +\infty} \int_{(c,d)} Au_n(t) + \varphi_n(t) \; d\lambda(t) = \int_{(c,d)} Au(t) + f(t) \; d\lambda(t). \end{aligned}$$

By Theorem 2.5.7 we obtain $u \in W^{1,q}\big((c,d); X\big)$ and $u' = Au + f$. Since $0 < c < d < +\infty$ were arbitrarily chosen, u is weakly differentiable and hence a weak solution of (5.1).

□

5.3.5 Corollary. Let X be reflexive and $p \in (1, +\infty)$. If A has the strict maximal L^p-regularity property on $(0, +\infty)$, then A has the strict maximal L^q-property on $(0, +\infty)$ for all $q \in (1, +\infty)$.

Proof. By Theorem 5.2.2, A is maximally L^p-regular, $0 \in \rho(A)$ and $\|T(t)\| \leq Me^{-\delta t}$, $t \geq 0$, for some $M, \delta > 0$. By Theorem 5.3.4 A is maximally L^q-regular for all $q \in (1, +\infty)$. It remains to show that for every $q \in (1, +\infty)$ there exists a constant $C_q > 0$, such that the mild solution $u : [0, +\infty) \to X$ defined by

$$u(t) := \int_{(0,t)} T(t-s)f(s)\; d\lambda(s)$$

satisfies $\|u\|_{L^q} \leq C_q \|f\|_{L^q}$ for every $f \in L^q\big((0, +\infty); X\big)$. Let $f \in L^q\big((0, +\infty); X\big)$ and extend T as well as f to the real line by $T(t) := 0$ and $f(t) := 0$ for $t < 0$. We employ Young's inequality, Proposition 2.5.3, b), and obtain

$$\begin{aligned}
\|u\|_{L^q} &= \Big(\int_{(0,+\infty)} \Big\| \int_{(0,t)} T(t-s)f(s)\; d\lambda(s) \Big\|^q \; d\lambda(t)\Big)^{\frac{1}{q}} \\
&\leq \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} Me^{-\delta(t-s)} \|f(s)\| \; d\lambda(s)\Big)^q \; d\lambda(t)\Big)^{\frac{1}{q}} \\
&\leq \big\|t \mapsto Me^{-\delta t}\big\|_{L^1} \|f\|_{L^q} = \frac{M}{\delta} \|f\|_{L^q} .
\end{aligned}$$

□

5.3.6 Corollary. Let X be reflexive, $p \in (1, +\infty)$ and $t_0 > 0$. If A is maximally L^p-regular on $(0, t_0]$, A is maximally L^q-regular on $(0, t_0]$ for every $q \in (1, +\infty)$.

Proof. Let M, ω as in Proposition 3.1.2, a) and $\delta > 0$. By Lemma 5.2.4, $B := A - (\omega + \delta)I$ is maximally L^p-regular and by Proposition 3.1.2, g), B is the infinitesimal generator of $\big(e^{-(\omega+\delta)t}T(t)\big)_{t \geq 0}$. Since $\big\|e^{-(\omega+\delta)t}T(t)\big\| \leq Me^{-\delta t}$ for every $t \geq 0$, we can employ Theorem 5.2.3 and obtain that B has the strict maximal L^p-regularity property on $(0, +\infty)$. By Corollary 5.3.5, B has the strict maximal L^q-regularity property for all $q \in (1, +\infty)$. By Theorem 5.1.5, B is maximally L^q-regular on $(0, t_0]$ for every $q \in (1, +\infty)$ and by Lemma 5.2.4 the same holds true for $A = B + (\omega + \delta)I$.

□

5.4 Maximal Regularity in Hilbert Spaces

We saw that for A to be maximally regular, it is necessary for A to generate an analytic semigroup. If the underlying space happens to be a Hilbert space, this condition is also sufficient. Throughout the present section X denotes a Banach space and H a Hilbert space.

Our main tool here will be the Fourier Transform for Banach space-valued functions. We state Plancherel's Theorem for Hilbert space-valued functions. Its proof can be found in [15], Proposition 4.1.

5.4.1 Definition. Let $t \in \mathbb{R}$ and $f \in L^1(\mathbb{R}; X)$. We call the function $\mathcal{F}(f)$ defined by

$$\mathcal{F}(f)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-its} f(s)\ d\lambda(s)$$

the Fourier transform of f and

$$\overline{\mathcal{F}}(f)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{its} f(s)\ d\lambda(s)$$

the adjoint Fourier transform.

$\|e^{\pm its} f(s)\| = \|f(s)\|$ implies the integrability of $s \mapsto e^{\pm its} f(s)$ and the inequalities $\|\mathcal{F}(f)(t)\|, \|\overline{\mathcal{F}}(f)(t)\| \le \|f\|_{L^1}$, $t \in \mathbb{R}$.

5.4.2 Theorem (Plancherel)**.** *There exists a unique linear, isometric and bijective operator $U : L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H)$, which satisfies*

$$Uf = \mathcal{F}(f)$$

for every $f \in L^1(\mathbb{R}; H) \cap L^2(\mathbb{R}; H)$. Moreover, $U\big(U(f)\big)(t) = f(-t)$ for every $f \in L^2(\mathbb{R}; H)$ and almost every $t \in \mathbb{R}$ as well as

$$\int_{\mathbb{R}} \big((Uf)(t), (Ug)(t)\big)\ d\lambda(t) = \int_{\mathbb{R}} \big(f(t), g(t)\big)\ d\lambda(t)$$

for every $f, g \in L^2(\mathbb{R}; H)$.

Recall the fact that the space $D(A)$ equipped with the graph norm $\|x\|_G := \|x\| + \|Ax\|$ forms a Banach space; see Lemma 1.2.5.

5.4.3 Lemma. Let $p \in [1, +\infty)$ and $-\infty \le a < b \le +\infty$. If $\big(T(t)\big)_{t\ge 0}$ is a differentiable semigroup, the space $C_{00}^\infty\big((a,b); D(A)\big)$ is densely contained in $L^p\big((a,b); X\big)$.

Proof. Let $f \in C_{00}^\infty\big((a,b); X\big)$ and define $f_n := T(\frac{1}{n})f$, $n \in \mathbb{N}$. By Proposition 3.3.4 $f_n(t) \in D(A)$ for every $t \in (a,b)$ and $AT(\frac{1}{k}) \in L_b(X)$ for every $k \in \mathbb{N}$. By Proposition 1.1.2, c), we have $f_n^{(m)}(t) = T(\frac{1}{n}) f^{(m)}(t)$ and $(Af_n)^{(m)}(t) = AT(\frac{1}{n}) f^{(m)}(t)$ for all $t \in (a,b)$ and $m \in \mathbb{N} \cup \{0\}$. Given $h \in \mathbb{R}$ such that $t, t+h \in (a,b)$ and $m \in \mathbb{N}$ we obtain

$$\left\| \frac{1}{h}\big(f_n^{(m-1)}(t+h) - f_n^{(m-1)}(t)\big) - f_n^{(m)}(t) \right\| \xrightarrow{h\to 0} 0$$

as well as

$$\left\| \frac{1}{h}\big(Af_n^{(m-1)}(t+h) - Af_n^{(m-1)}(t)\big) - Af_n^{(m)}(t) \right\| \xrightarrow{h\to 0} 0,$$

implying

$$\left\| \frac{1}{h}\big(f_n^{(m-1)}(t+h) - f_n^{(m-1)}(t)\big) - f_n^{(m)}(t) \right\|_G \xrightarrow{h\to 0} 0,$$

which means $f_n \in C^\infty\big((a,b);D(A)\big)$. Together with $\operatorname{supp}(f_n) = \operatorname{supp}(f)$ we conclude that $f_n \in C_{00}^\infty\big((a,b);D(A)\big)$ for all $n \in \mathbb{N}$. By Proposition 3.1.2, b), $\|f_n(t) - f(t)\|^p \xrightarrow{n\to+\infty} 0$ for every $t \in (a,b)$. For M, ω as in Proposition 3.1.2, a) with w.l.o.g. $\omega \geq 0$ we have

$$\begin{aligned}\|f_n(t) - f(t)\|^p &= \big\|\big(T(\tfrac{1}{n}) - I\big)f(t)\big\|^p \leq \big(\big\|T(\tfrac{1}{n})\big\| + 1\big)^p \|f(t)\|^p \\ &\leq (e^{\frac{\omega}{n}} + 1)^p \|f(t)\|^p \leq (e^\omega + 1)^p \|f(t)\|^p .\end{aligned}$$

We conclude from Theorem 2.3.7

$$\int_{(a,b)} \|f_n(t) - f(t)\|^p \; d\lambda(t) \xrightarrow{n\to+\infty} 0$$

and, in turn, $\|f_n - f\|_{L^p((a,b);X)} \xrightarrow{n\to+\infty} 0$. Since by Proposition 2.5.2 $C_{00}^\infty\big((a,b);X\big)$ is dense in $L^p\big((a,b);X\big)$,

$$L^p\big((a,b);X\big) \supseteq \overline{C_{00}^\infty\big((a,b);D(A)\big)} \supseteq \overline{C_{00}^\infty\big((a,b);X\big)} = L^p\big((a,b);X\big).$$

$\square$

5.4.4 Theorem. *If $A : D(A) \subseteq H \to H$ is the infinitesimal generator of a bounded analytic semigroup, then A has the maximal L^p-regularity property on $(0,+\infty)$ for all $p \in (1,+\infty)$.*

Proof. Let $\varphi \in C_{00}^\infty\big((0,+\infty);D(A)\big)$. By Lemma 4.2.3 $v : [0,+\infty) \to H$ defined by

$$v(t) := \int_0^t T(t-s)\varphi(s) \; d\lambda(s)$$

is well-defined and continuous. Because $A\varphi$ is continuous, also $s \mapsto AT(t-s)\varphi(s) = T(t-s)A\varphi(s)$ is integrable. We employ Proposition 2.3.13 and obtain $v(t) \in D(A)$ as well as

$$Av(t) = \int_{(0,t)} AT(t-s)\varphi(s) \; d\lambda(s) = \int_{(0,t)} T(t-s)A\varphi(s) \; d\lambda(s).$$

We want to prove that $Av \in L^2\big((0,+\infty);H\big)$. By Theorem 3.3.5 there exists a constant $C > 0$ such that

$$\|AT(t)\| \leq \frac{C}{t}, \quad t > 0$$

Choosing $0 < a < b < +\infty$ and $M, K > 0$, such that $\operatorname{supp}(\varphi) \subseteq [a,b]$, $\|A\varphi(s)\| \leq K$ for every $s \geq 0$ and $\|T(t)\| \leq M$ for all $t \geq 0$, we obtain

$$\begin{aligned}\int_{(0,b+1)} \|Av(t)\|^2 \; d\lambda(t) &\leq \int_{(0,b+1)} \Big(\int_{(0,t)} \|T(t-s)\| \, \|A\varphi(s)\| \; d\lambda(s)\Big)^2 \; d\lambda(t) \\ &\leq (b+1)^3 K^2 M^2 < +\infty\end{aligned}$$

and

$$\int_{(b+1,+\infty)} \|Av(t)\|^2 \, d\lambda(t) \leq \int_{(b+1,+\infty)} \Big(\int_{(a,b)} \|AT(t-s)\| \, \|\varphi(s)\| \, d\lambda(s) \Big)^2 \, d\lambda(t)$$
$$\leq C^2 \, \|\varphi\|^2_{L^1((0,+\infty);X)} \int_{(b+1,+\infty)} \frac{1}{(t-b)^2} \, d\lambda(t)$$
$$= C^2 \, \|\varphi\|^2_{L^1((0,+\infty);X)} < +\infty \tag{5.20}$$

We extend Av, T and φ to $\mathbb{R}$ by $Av(t) = 0$, $T(t) = 0$ for $t < 0$ and $\varphi(s) = 0$ for $s \leq 0$. For $n \in \mathbb{N}$ we define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(t) := \mathbb{1}_{[0,+\infty)}(t) e^{-\frac{t}{n}}$. $f_n \in L^2(\mathbb{R};\mathbb{R})$ together with (5.20) yields $f_n Av \in L^1(\mathbb{R}; H)$ and

$$\mathcal{F}(f_n Av)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itr} f_n(t) Av(t) \, d\lambda(t)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itr} f_n(t) \int_{(0,t)} AT(t-s)\varphi(s) \, d\lambda(s) \, d\lambda(t)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-itr} f_n(t) T(t-s) A\varphi(s) \, d\lambda(s) \, d\lambda(t), \quad r \in \mathbb{R}$$

Because of

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left\| e^{-itr} f_n(t) T(t-s) A\varphi(s) \right\| \, d\lambda(s) \, d\lambda(t) \leq M \int_{(0,+\infty)} \int_{\mathbb{R}} e^{-\frac{t}{n}} \|A\varphi(s)\| \, d\lambda(s) \, d\lambda(t)$$
$$= Mn \, \|A\varphi\|_{L^1((0,+\infty);H)} < +\infty,$$

we can apply Theorem 2.3.12 and obtain

$$\mathcal{F}(f_n Av)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-itr} f_n(t) T(t-s) A\varphi(s) \, d\lambda(t) \, d\lambda(s)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(t+s)r} f_n(t+s) T(t) A\varphi(s) \, d\lambda(t) \, d\lambda(s)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{(0,+\infty)} \int_{(-t,+\infty)} e^{-(\frac{1}{n}+ir)(t+s)} T(t) A\varphi(s) \, d\lambda(s) \, d\lambda(t)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{(0,+\infty)} \int_{(0,+\infty)} e^{-(\frac{1}{n}+ir)(t+s)} T(t) A\varphi(s) \, d\lambda(s) \, d\lambda(t)$$
$$= \int_{(0,+\infty)} e^{-(\frac{1}{n}+ir)t} T(t) \Big(\frac{1}{\sqrt{2\pi}} \int_{(0,+\infty)} e^{-isr} e^{-\frac{s}{n}} A\varphi(s) \, d\lambda(s) \Big) \, d\lambda(t)$$
$$= \int_{(0,+\infty)} e^{-(\frac{1}{n}+ir)t} T(t) \mathcal{F}(f_n A\varphi)(r) \, d\lambda(t).$$

Employing Proposition 3.1.2, h), leads to

$$\mathcal{F}(f_n Av)(r) = \int_{(0,+\infty)} e^{-(\frac{1}{n}+ir)t} T(t) \mathcal{F}(f_n A\varphi)(r) \, d\lambda(t) = R(\tfrac{1}{n}+ir, A) \mathcal{F}(f_n A\varphi)(r).$$

Since $f_n\varphi$ and $f_n A\varphi$ are integrable, by Proposition 2.3.13 we obtain $\mathcal{F}(f_n\varphi)(r) \in D(A)$ as well as

$$\mathcal{F}(f_n A\varphi)(r) = \int_{\mathbb{R}} e^{-itr} f_n(t) A\varphi(t) \, d\lambda(t) = A\Big(\int_{\mathbb{R}} e^{-itr} f_n(t) \varphi(t) \, d\lambda(t) \Big) = A\mathcal{F}(f_n\varphi)(r).$$

Let $U : L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H)$ be the isometric isomorphism as in Theorem 5.4.2. Since A is sectorial (Theorem 3.3.5), there exists a constant $L > 0$ such that

$$\begin{aligned}\left\|AR(\tfrac{1}{n} + ir, A)\right\| &= \left\|\big((\tfrac{1}{n} + ir)I - A\big)R(ir, A) - (\tfrac{1}{n} + ir)R(\tfrac{1}{n} + ir, A)\right\| \\ &\leq \|I\| + \left\|(\tfrac{1}{n} + ir)R(\tfrac{1}{n} + ir, A)\right\| \leq 1 + L\end{aligned}$$

for all $n \in \mathbb{N}$ and $r \in \mathbb{R}$. We obtain

$$\begin{aligned}\|f_n Av\|_{L^2(\mathbb{R};H)} &= \|U(f_n Av)\|_{L^2(\mathbb{R};H)} = \|\mathcal{F}(f_n Av)\|_{L^2(\mathbb{R};H)} = \left\|AR(\tfrac{1}{n} + i\cdot, A)\mathcal{F}(f_n\varphi)\right\|_{L^2(\mathbb{R};H)} \\ &\leq \Big(\int_{\mathbb{R}} \left\|AR(\tfrac{1}{n} + ir, A)\right\|^2 \|U(f_n\varphi)(r)\|^2 \; d\lambda(r)\Big)^{\frac{1}{2}} \\ &\leq (1 + L)\Big(\int_{\mathbb{R}} \|U(f_n\varphi)(r)\|^2 \; d\lambda(r)\Big)^{\frac{1}{2}} = (1 + L) \|U(f_n\varphi)\|_{L^2(\mathbb{R};H)} \\ &= (1 + L) \|f_n\varphi\|_{L^2(\mathbb{R};H)} = (1 + L)\Big(\int_{(0,+\infty)} e^{-\frac{2r}{n}} \|\varphi(r)\|^2 \; d\lambda(r)\Big)^{\frac{1}{2}} \\ &\leq (1 + L)\Big(\int_{(0,+\infty)} \|\varphi(r)\|^2 \; d\lambda(r)\Big)^{\frac{1}{2}} = (1 + L) \|\varphi\|_{L^2(\mathbb{R};H)}.\end{aligned}$$

From $\|f_n(t)Av(t) - Av(t)\|^2 \xrightarrow{n\to+\infty} 0$ and $\|f_n(t)Av(t) - Av(t)\|^2 \leq 4 \|Av(t)\|^2$ for all $t \in \mathbb{R}$, we infer $\|f_n Av - Av\|_{L^2(\mathbb{R};H)} \xrightarrow{n\to+\infty} 0$. Consequently,

$$\|Av\|_{L^2((0,+\infty);H)} \leq (1 + L) \|\varphi\|_{L^2((0,+\infty);H)}. \tag{5.21}$$

Let $f \in L^2\big((0, +\infty); H\big)$. By Lemma 5.4.3 there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_{00}^{\infty}\big((0, +\infty); D(A)\big)$ satisfying $\|\varphi_n - f\|_{L^2((0,+\infty);H)} \xrightarrow{n\to+\infty} 0$. By Lemma 4.2.3 $u : [0, +\infty) \to H$ defined by

$$u(t) := \int_{(0,t)} T(t - s)f(s) \; d\lambda(s)$$

and $u_n : [0, +\infty) \to H$, $n \in \mathbb{N}$, defined by

$$u_n(t) := \int_{(0,t)} T(t - s)\varphi_n(s) \; d\lambda(s)$$

are well-defined and continuous. By Hölder's inequality, Proposition 2.5.3, a), for $t \geq 0$

$$\begin{aligned}\|u(t) - u_n(t)\| &= \left\|\int_{(0,t)} T(t - s)\big(f(s) - \varphi_n(s)\big) \; d\lambda(s)\right\| \\ &\leq \int_{(0,t)} \|T(t - s)\| \|f(s) - \varphi_n(s)\| \; d\lambda(s) \\ &\leq \int_{(0,t)} M \|f(s) - \varphi(s)\| \; d\lambda(s) \leq \sqrt{t}M \|f - \varphi_n\|_{L^2((0,+\infty;H)} \xrightarrow{n\to+\infty} 0.\end{aligned}$$

Moreover, by (5.21)

$$\|Au_n - Au_m\|_{L^2((0,+\infty);H)} \leq (1 + L) \|\varphi_n - \varphi_m\|_{L^2((0,+\infty);H)},$$

which implies that $(Au_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2\big((0,+\infty);H\big)$. Hence, $\|Au_n - g\|_{L^2((0,+\infty);H)} \xrightarrow{n\to+\infty} 0$ for some $g \in L^2\big((0,+\infty);H\big)$. By Proposition 2.5.2, a), there exists a subsequence $(Au_{n_k})_{k\in\mathbb{N}}$ such that $Au_{n_k}(t) \xrightarrow{k\to+\infty} g(t)$ for almost every $t > 0$. Since A is closed, we obtain $u(t) \in D(A)$ and $g(t) = Au(t)$ for almost every $t > 0$ implying $\|Au_n - Au\|_{L^2((0,+\infty);H)} \xrightarrow{n\to+\infty} 0$. $\|Au_n\|_{L^2((0,+\infty);H)} \leq (1+L)\,\|\varphi_n\|_{L^2((0,+\infty);H)}$ implies $\|Au\|_{L^2((0,+\infty);H)} \leq (1+L)\,\|f\|_{L^2((0,+\infty);H)}$. It remains to show that u is a weak solution of (5.1). By Theorem 4.2.8 u_n is a classical solution of

$$\begin{cases} u_n'(t) = Au_n(t) + \varphi_n(t), & t > 0,\\ u_n(0) = 0, \end{cases}$$

implying that u_n is continuously differentiable on $(0,+\infty)$. By Lemma 2.4.4

$$u_n(t) - u_n(s) = \int_{(s,t)} u_n'(r)\; d\lambda(r) = \int_{(s,t)} Au_n(r) + \varphi_n(r)\; d\lambda(r)$$

for all $0 < s < t$ and by Proposition 2.5.3, a)

$$\begin{aligned}\left\|\int_{(s,t)} Au_n(r) - Au(r)\; d\lambda(r)\right\| &\leq \int_{(s,t)} \|Au_n(r) - Au(r)\|\;\; d\lambda(r)\\ &\leq \sqrt{t-s}\,\|Au_n - Au\|_{L^2((0,+\infty);H)} \xrightarrow{n\to+\infty} 0.\end{aligned}$$

Analogous arguments lead to

$$\left\|\int_{(s,t)} \varphi_n(r) - f(r)\; dr\right\| \xrightarrow{n\to+\infty} 0.$$

Since $u_n(t) \xrightarrow{n\to+\infty} u(t)$ for every $t > 0$, we obtain

$$u(t) - u(s) = \lim_{n\to+\infty} u_n(t) - u_n(s) = \lim_{n\to+\infty} \int_{(s,t)} Au_n(r) + \varphi_n(r)\; d\lambda(r) = \int_{(s,t)} Au(r) + f(r)\; d\lambda(r).$$

According to Theorem 2.5.7 we have $u \in W^{1,2}\big((s,t);H\big)$ and $u' = Au + f$. Since $0 < s < t < +\infty$ were arbitrarily chosen, u is a weak solution of (5.1) and, because of Proposition 4.2.7, the only one. Consequently, A is maximally L^2-regular. Since Hilbert spaces are reflexive (Corollary 1.11.10 in [20]) by Theorem 5.3.4 A is maximally L^p-regular for every $p \in (1,+\infty)$.

□

5.4.5 Corollary. If $A : D(A) \subseteq H \to H$ is the infinitesimal generator of a bounded analytic semigroup with $0 \in \rho(A)$, then A has the strict maximal L^p-regularity property on $(0,+\infty)$ for all $p \in (1,+\infty)$.

Proof. By Theorem 5.4.4 A is maximally L^p-regular for all $p \in (1,+\infty)$ and according to Theorem 5.2.2 also strictly maximally L^p-regular for all $p \in (1,+\infty)$.

□

5.4.6 Corollary. Let $t_0 > 0$. If $A : D(A) \subseteq H \to H$ is the infinitesimal generator of an analytic semigroup, then A has the maximal L^p-regularity property on $(0, t_0]$ for all $p \in (1, +\infty)$.

Proof. Let M, ω as in Proposition 3.1.2, a) and $\delta > 0$. The operator $B := A - (\omega + \delta)I$ is the infinitesimal generator of the semigroup $\big(S(t)\big)_{t\geq 0}$ defined by $S(t) := e^{-(\omega+\delta)t}T(t)$ by Proposition 3.1.2, g). We have $\|S(t)\| \leq Me^{-\delta t}$ and we can extend $\big(S(t)\big)_{t\geq 0}$ to an analytic semigroup. By Corollary 3.3.6 there exists a constant $C > 0$ such that

$$\|BS(t)\| \leq \frac{Ce^{(-\delta+\delta)t}}{t} = \frac{C}{t}, \quad t > 0.$$

According to Theorem 3.3.5 $\big(S(t)\big)_{t\geq 0}$ is a bounded analytic semigroup. By Theorem 5.4.4 B is maximally L^p-regular on $(0, +\infty)$ for all $p \in (1, +\infty)$ and therefore for also on $(0, t_0]$; see Theorem 5.1.5. Finally, according to Lemma 5.2.4 $A = B + (\omega + \delta)I$ has the maximal L^p-regularity property on $(0, t_0]$ for all $p \in (1, +\infty)$.

□

References

[1] Pazy A. Semigroups of Linear operators and Applications to Partial Differential Equations. Springer Verlag, 1983.

[2] Calderon A.P. and Zygmund A. On the existence of certain singular integrals. Acta Math. 88, p. 85-139, 1952.

[3] W. Arendt, Zacher R., and Kreuter M. Sobolev spaces of vector-valued functions. Master's thesis, Ulm University, 2015.

[4] J. M. Ball. Strongly continuous semigroups, weak solutions, and the variation of constants formula. Proceedings of the American Mathematical Society, Vol. 63, No. 2, 1977.

[5] L. de Simon. Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. Rendiconti del Seminario Matematico della Università di Padova, tome 34 (1964), p.205-223, 1964.

[6] G. Dore. Maximal regularity in lp spaces for an abstract cauchy problem. Advances in Differential Equations, Vol. 5 (1-3), pp. 293-322, 2000.

[7] Dore G. Lp regularity for abstract differential equations. Functional Analysis and Related Topics, 1991, Lecture Notes in Mathematics, vol 1540, 1993.

[8] Dore G. and Venni A. On the closedness of the sum of two closed operators. Math Z 196, 189-201, 1987.

[9] Amann H. and Escher J. Analysis I. Birkhäuser Verlag, 2006.

[10] Amann H. and Escher J. Analysis III. Birkhäuser Verlag, 2008.

[11] Woracek H. Topology. Lecture Notes, 2021.

[12] Woracek H., Kaltenbäck M., and Blümlinger M. Funktionalanalysis I. Lecture Notes, 2023.

[13] Elstrodt J. Maß- und Integrationstheorie. Springer Verlag Berlin Heidelberg, 2011.

[14] Jost J. and Li-Jost X. Calculus of Variations. Cambridge Studies in Advanced Mathematics 64, 1998.

[15] S. Kwapień. Isomorphic characterisations of inner product spaces by orthogonal series with vector valued coefficients. Studia Mathematica, T. XLIV, 1972.

[16] Weis L. Operator-valued fourier multiplier theorems and maximal lp-regularity. Mathematische Annalen 319, p. 735-758, 2001.

[17] Kaltenbäck M. Fundament Analysis. Heldermann Verlag, 2014.

[18] Kaltenbäck M. Analysis III. Lecture Notes, 2018.

[19] Kaltenbäck M. Funktionalanalysis II. Lecture Notes, 2023.

[20] Meise M. and Vogt D. Einführung in die Funktionalanalysis. Verlag Vieweg, 1992.

[21] Rudin W. Real and Complex Analysis. McGraw-Hill Book Company, 1987.