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Optimal Risk Control with Non-Cheap Reinsurance

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Abstract

The risk or value process of an insurance company, modelled by a Cramer-Lundberg model, is supposed to be controlled by a reinsurance share, that is a part of the risk is undertaken, but also premium has to be divided. The aim is to control this reinsurance level in way, that the discounted value of the risk process maximizes. First, the process is approximated by a diffusion process, then stochastic control theory is used to find an optimal value function and an optimal control. Non-cheap reinsurance and a bankruptcy value are also considered.
In the last part of the thesis Monte-Carlo simulation is used to calculate examples and verify the solution.

Keywords:
Stochastic Control Process, Cramer-Lundberg Model, Hamilton-Jacobi-Bellman Equation, Non-cheap proportional Reinsurance, Monte-Carlo Simulation

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Chapter 1

Introduction

In this paper I consider an insurance company which protects itself from ruin with proportional reinsurance. This company (the cedent) has to pay a certain amount of the insurance premium of each of its costumers to the reinsurance company, but in return can expect a certain fraction of each claim to be paid by the reinsurance company. If the safety loading of both insurance and reinsurance premium are the same, which means the fraction of the premium the cedent has to pass and the fraction of the claim the reinsurer covers are the same, we call it *cheap reinsurance*. Usually, the safety loading of the insurer is higher. This means that there is an extra reinsurance premium added to the premium of the orignial costumer. This is called *non-cheap reinsurance*, which I will consider in this paper since it is much more common.
Let us assume that the insurance company only writes ordinary insurance policies, where the costumer (or policyholder) has to pay premiums regularly, while he can expect the company to pay a positive amount of cash when a claim occurs. For example this can be health, fire or car accidents insurances. I assume that both the size of the claims U_i can be modelled by a positive random variable with given distribution and the number of occurrences $N(t)$ up to a given time t. So we can describe the amount the insurance company has to pay as a postive risk sum

$$S_{N_t} := \sum_{i=1}^{N_t} U_i.$$

Together with the regularly paid premiums p and p_2 for the reinsurance company and the retention level a, I am able to formulate the risk model for the wealth of the insurance company. It is a rather standard model in collective risk theory, the so called *Cramer-Lundberg Model*

$$R(t) = (p - (1-a)p_2)t - a\sum_{i=1}^{N_t} U_i. \tag{1.1}$$

Only the fraction a of the risk sum has to be paid, while on the other hand the premium $(1-a)p_2$ has to be paid to the reinsurer. Therefore $0 \leq a \leq 1$. Later we will also introduce an initial capital x of the insurance company and a bankruptcy value P when the insurance company hits ruin, i.e. $R(t) \leq P$ for the first $t > 0$. Since normally insurance companies are still valuable when hitting ruin, by bankruptcy I speak of the

state of zero surplus. For example this value can describe non-liquid assets. But P can also be negative, which means the company is fined for going bankrupt.

In other papers, this or a similar risk process is optimized in a way to minimize the probability of ruin. But there are no conditions on the wealth or reserve of the insurance company, just to keep it positive. This models are used for maximizing the expected future dividends to shareholders.
The model used in this paper is different. Its setting aims to control the proportional reinsurance in a way to keep the reserve on a high level. The goal is to optimize the risk process by applying a certain policy π to the retention level a^π. To solve this stochastic control problem, we will transform the model into a stochastic differential equation. The original risk process of the reserve of the company (1.1) now formulates

$$dR_t^\pi = (\mu - (1 - a_t^\pi)\lambda)dt + \sigma a^\pi(t)_t dW_t,$$

where W_t is a standard Brownian motion and μ, λ and σ are constants describing the setting. a_t^π has the same restrictions described above for every time t. I introduce a discount factor r and the time of ruin τ. Also take into account the initial capital x and the bankruptcy value P. Let Π denote the set of all admissible control policies. Then the optimal control π^* has to satisfy

$$J_x(\pi^*) = \sup_{\pi \in \Pi} E\left[\int_0^{\tau_\pi} e^{-rt} R_t^\pi dt + e^{-r\tau_\pi} P\right].$$

J is called *value function* or *optimal return function*. I use stochastic control theory to find a solution both to the value function and the corresponding optimal control policy. It turns out that the solution is highly dependent on the bankruptcy value as well as the proportion of the original insurance premium and the reinsurance premium. Many different cases have to be considered.

This paper is organised as followed:

In Chapter 2 the theory to handle the calculation is described. First, I introduce an important verification theorem on how to solve a stochastic control problem. Second, a way is described how to approximate the classical Cramer-Lundberg model by a Brownian motion.
Backed with theory, the final model is described in a mathematical way in Chapter 3 and the solution is derived. Many different cases have to be considered depending on the variables of the model.
In Chapter 4 *Maple 17* is used to implement the optimal solution calculated in the last chapter to give graphical and numerical examples. Also the challenges and difficulties of the implementation are described. In a next step the influence of each variable on the model is studied. Then Monte-Carlo simulations show the efficiency of the calculated optimal control function in comparison with the discrete risk process.
The observations and results of these calculations are described in the conclusion in Chapter 5.
The corresponding code of the implementation is given in the Appendix, as well as a short errata to two of the studied papers.

Chapter 2

Theory

A short introduction to to the theory of dynamic programming and diffusion approximations will be given. Knowledge of Itô processes is presumed.

2.1 Stochastic Control Theory and Dynamic Programming

In this section, I provide a quick overview of stochastic control problems of Markov diffusions via dynamic programming. I start with a simple set-up in finite time to describe the Bellman-principle which is used to describe the *Hamilton-Jacobi-Bellman equation* (HJB), a partial differential equation that can be derived from the control problem. Further a verification theorem for the HJB equation will be proven, which shows that one has to solve this stochastic differential equation to find a solution to an optimisation problem. The model results in a classic stochastic control problem with infinite time horizon, so the last part of the chapter will focus on the verification theorem in infinite time.

Set in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define a filtration $\mathcal{F}_t$ and a standard Brownian motion W_t that is adapted to $\mathcal{F}_t$, while $t \in [0, T]$ with terminal time T. We only consider the one dimensional case. The standard stochastic control problem, as defined in Saß [10], consists of the following five components:

Definition 2.1 (*Stochastic control problem*)

- A *control policy* π_t is an $\mathcal{F}$ -progressively measurable process with values in a certain set $U \subseteq \mathbb{R}$.
- The *controlled process* X_t^π is an Itô process with adapted coefficients b and σ, given by

$$dX_t^\pi = b(t, X_t^\pi, \pi_t)dt + \sigma(t, X_t^\pi, \pi_t)dW_t, \tag{2.1}$$

  where

$$X_0^\pi = x_0$$

  is the initial value. The superscripted π shows the dependency of the process on the control policy.

- For each policy π and regular functions ψ and Ψ, we consider the *performance functional* or *value function*

$$J(t,x,\pi) = E\left[\int_t^T \psi(s,X_s,\pi_s)ds + \Psi(T,X_T)|X_t = x\right]. \tag{2.2}$$

  Later, conditions set to ψ will ensure the solvability of the controlled value function.

- Π is the *set of all admissible policies* π such that there exists a unique solution of (2.1) and the value function (2.2) is well defined. $\pi \in \Pi$ is called *admissible* if these restrictions hold.

- The *optimal value function* is defined as

$$V(t,x) = \sup_{\pi\in\Pi} J(t,x,\pi). \tag{2.3}$$

The objective is to find this optimal function V and simultaneously an optimal control policy $\pi^* \in \Pi$ such that $V(0,x) = J(0,x,\pi^*)$.

To solve problem (2.1) we consider the Bellman-principle found by Richard Bellman in the 1950s. This takes into account that if one executes an optimal policy from time t up to time t_1, this is globally optimal if one acts optimal after t_1 as well. This principle is expressed in (2.4) and has to be proved in the following.
For simplicity the notation $E_{tx}[Y] := E[Y|X_t = x]$ is used.

Let V be the solution to the problem (2.1) and π_t the possible control policies. Following assumptions have to be made: Let $b(t,x,\pi), \sigma(t,x,\pi)$ be continuous differentiable and polynomially growing for all $t \geq 0$, and $x, \pi \in \mathbb{R}$. A function F is called polynomially growing if there exists a constant M, such that $\|F(x)\|^2 \leq M(1+|x|^2)$ for all $x \in \mathbb{R}$. The admissible control policy π is a progressively measurable process with $E[\int_0^t \|\pi(s)\|^2 ds] < \infty$.
Following the Bellman-principle we write for $t_1 > t$

$$V(t,X_t) = \sup_{\pi\in\Pi} E_{tx}\left[\int_t^{t_1} \psi(s,X_s,\pi_s)ds + V(t_1,X_{t_1})\right]. \tag{2.4}$$

Applying Itô's differential rule to $V(t_1,X_{t_1})$ we get

$$\begin{aligned} V(t_1,X_{t_1}) = V(t,X_t) &+ \int_t^{t_1} (V_t(s,X_s) + V_x(s,X_s)b(s,X_s,\pi_s) \\ &+ \frac{1}{2}\sigma^2(s,X_s,\pi_s)V_{xx}(s,X_s))ds + \int_t^{t_1} V_x(s,X_s)\sigma(s,X_s,\pi_s)dW_s. \end{aligned} \tag{2.5}$$

V_t, V_x, V_{xx} describe the first and second derivatives for t and x respectively. Because of the assumptions we made the last stochastic integral is bounded and therefore a martingale,

so the expected value of it equals 0. Inserting (2.5) into (2.4) and subtracting $V(t, X_t)$ on both sides we get

$$0 = \sup_{\pi \in \Pi} E_{tx} \left[\int_t^{t_1} \psi(s, X_s, \pi_s) + A^\pi(V(s, X_s)) ds \right], \tag{2.6}$$

where the operator $A^\pi(.)$ is defined by

$$A^\pi(V(t, X_t)) := V_t(t, X_t) + V_x(t, X_t) b(t, X_t, \pi_t) + \frac{1}{2} \sigma^2(t, X_t, \pi_t) V_{xx}(t, X_t). \tag{2.7}$$

Dividing (2.6) through $(t_1 - t)$, let $t \to t_1$ and taking expectations we get

$$0 = \sup_{\pi \in \Pi} (\psi(t, X_t, \pi) + A^\pi V(t, X_t)). \tag{2.8}$$

Equation (2.8) is called the *Hamilton-Jacobi-Bellman equation*. It shows that the optimal value function is a solution to a nonlinear differential equation.

Now, starting with a solution to the HJB equation, how do you get the value function? First, find a maximizing π^*. Second, solve the parabolic partial differential equation that you get when putting π^* into the HJB equation. Third, proof if π^* is admissible. Then V^* is the value function and a^* is the optimal control, but only if V^* meets certain conditions. To characterize these conditions we formulate a verification theorem. It states that under a certain set of onditions a solution to to the HJB equation corresponds with the value function.

Theorem 2.2 (*Verification Theorem with finite time horizon*) Using the definitions described at the beginning of this section: Suppose σ and continuous ψ are polynomially growing with $\|\sigma(t, x, \pi)\|^2 \leq M_\sigma(1 + |x|^2 + |\pi|^2)$ and $\|\psi(t, x, \pi)\|^2 \leq M_\psi(1 + |x|^2 + |\pi|^2)$ for some $M_\sigma, M_\psi > 0$ and all $t \geq 0, x \in \mathbb{R}, \pi \in U$.

(i) If there exists a polynomially growing function $f \in C^2(0, \infty)$, which is a solution to the HJB equation (2.8) and fulfills the boundary condition

$$f(T, X_T) = \Psi(T, X_T),$$

then

$$f(t, X_t, \pi) \geq J(t, X_t, \pi)$$

for any admissible control process π and given initial data $X_0 = x$.

(ii) If π^* is the admissible optimal control, that maximizes the right hand side of (2.8), then

$$f(t, X_t, \pi^*) = J(t, X_t, \pi^*) = V(t, X_t).$$

PROOF (*Theorem 2.2*) Let τ_n be a sequence of an increasing $\mathcal{F}_t$-stopping time with $t \leq \tau_n \leq T$ for any fixed $t \in [0,T]$ defined by

$$\tau_n := \inf(s > t \mid |X_s - X_t| = n) \wedge T.$$

The HJB equation for any $t \leq s \leq \tau_n$ becomes

$$0 \geq (\psi(s, X_s, \pi_{\tau_s}) + A^{\pi_s} f(s, X_s)). \tag{2.9}$$

Itô's differential rule states

$$f(\tau_n, X_{\tau_n}) = f(t, X_t) + \int_t^{\tau_n} A^{\pi} f(s, X_s) ds + \int_t^{\tau_n} f_x(s, X_s) \sigma(s, X_s, \pi_s) dW_s,$$

where π_s is an admissible control and the operator A^{π} is the one defined in (2.7). Because $f \in C^2$ and X_s is bounded, the stochastic part of the integral is a martingale. When we insert into the term for J up to time τ_n and take expectations of the last term we obtain

$$\begin{aligned} &E_{tx}\left[\int_t^{\tau_n} \psi(s, X_s, \pi_s) ds + f(\tau_n, X_{\tau_n})\right] \\ &\quad = f(t, X_t) + E_{tx}\left[\int_t^{\tau_n} \psi(s, X_s, \pi_s) + A^{\pi_s} f(\tau_s, X_{\tau_s}) ds\right] \leq f(t, X_t). \end{aligned} \tag{2.10}$$

For the inequality (2.9) is used.
As $\lim_{n\to\infty} \tau_n = T$ and from the growth conditions to ψ and f we get

$$\left|\int_t^{\tau_n} \psi(s, X_s, \pi_s) ds + f(\tau_n, X_{\tau_n}, \pi_{\tau_n})\right| \leq C_\psi \int_t^T \left(1 + \|X_s\|^2 + \|\pi_s\|^2\right) ds + C_f(1 + \|X_T\|^2).$$

Therefore, with dominated convergence,

$$\lim_{n\to\infty} E_{tx}\left[\int_t^{\tau_n} \psi(s, X_s, \pi_s) ds + f(\tau_n, X_{\tau_n})\right] = J(t, X_t, \pi_t).$$

Inserting this in the first part of (2.10) we proved (i).
Inequality (2.9) is an equality if the maximizing control π^* is used. Then, also (2.10) is an equality which immediately proves (ii). □

Since the control model described in the next chapter deals with an infinite time horizon, I formulate an expansion to the model. From now on we assume, that the coefficients b, σ do not depend on the time t anymore. Let $X_0 = x$. The controlled process X and the operator A are now of the form

$$\begin{aligned} X_t^{\pi} &= b(X_t, \pi_t) dt + \sigma(X_t, \pi_t) dW_t \\ A^{\pi}(V(X_t)) &:= V_x(X_t) b(X_t, \pi_t) + \frac{1}{2} \sigma^2(X_t, \pi_t) V_{xx}(X_t). \end{aligned} \tag{2.11}$$

As integration's limit we denote the stopping time $\mathcal{T}$ as the exit time of X from a given value region O of X in $\mathbb{R}$. If $X(s) \in O$ for all $s \geq 0$ we define $\mathcal{T} = \infty$.
We introduce a discount factor $r > 0$ and obtain a new discounted value function

$$J(x, \pi) = E_{0x}\left[\int_0^{\mathcal{T}} e^{-rs} \psi(X_s, \pi_s) ds + \chi_{(\mathcal{T}<\infty)} e^{-r\mathcal{T}} \Psi(\mathcal{T}, X_{\mathcal{T}})\right],$$

where χ is the indicator function of the event $\mathcal{T} < \infty$.
Let β, σ, ψ be of polynomial growth as before and Π denotes the set of all progressively measurable admissible control strategies.

Theorem 2.3 (*Verification Theorem with infinite time horizon*) With the restrictions formulated before:

(i) If there exists a polynomially growing function $f \in C^2(0, \infty)$ which is a solution to the (new) HJB equation

$$0 = \sup_{\pi \in \Pi} \left(\psi(X_t, \pi) + A^\pi f(t, X_t) - r f(t, X_t) \right) \tag{2.12}$$

with given initial data $X_0 = x$ and boundary data $f(X_\mathcal{T}) = \Psi(X_\mathcal{T})$, then $f(X_t, \pi) \geq J(X_t, \pi)$ for any admissible control process π and

$$\liminf_{t_1 \to \infty} \left(e^{-rt_1} E_x[\chi_{(\mathcal{T} \geq t_1)} f(X_{t_1})] \right) \geq 0. \tag{2.13}$$

(ii) Let π^* be the admissible optimal control, that maximizes the right hand side of (2.12), then $f(X_t, \pi^*) = J(X_t, \pi^*) = V(X_t)$ and

$$\lim_{t_1 \to \infty} \left(e^{-rt_1} E_x[\chi_{(\mathcal{T} \geq t_1)} f(X_{t_1})] \right) = 0. \tag{2.14}$$

Proof (*Theorem 2.3*) Only a sketch is given, as it is very similar to the proof of Theorem 2.2.
By using the HJB equation with finite time horizon (2.8) for $\tilde{f}(X_t) := e^{-rt} f(X_t)$, we obtain for the old operator (2.7) since f does not depend on t anymore

$$\begin{aligned} A^\pi \tilde{f}(X_t) &= A^\pi (e^{-rt} f(X_t)) \\ &= -r e^{-rt} f(X_t) + e^{-rt} f_x(X_t) b(X_t, \pi_t) + e^{-rt} \frac{1}{2} \sigma^2(X_t, \pi_t) f_{xx}(X_t) \\ &= e^{-rt} \left(-r f(X_t) + f_x(X_t) b(X_t, \pi_t) + \frac{1}{2} \sigma^2(X_t, \pi_t) f_{xx}(X_t) \right). \end{aligned}$$

After plugging the result into the old HJB, dividing through e^{-rt} and defining the new operator A as in (2.11), we get the new HJB equation (2.12).

Again, we define a $\mathcal{F}$-stopping time τ_n with $\tau_n \to \infty$. When applying Itô's differential rule to $\tilde{f}$ we get

$$\begin{aligned} \tilde{f}(X_{\tau_n}) &= e^{-r\tau_n} f(X_{\tau_n}) \\ &= f(x) + \int_0^{\tau_n} e^{-rs} (-r f(X_s) + A^\pi f(X_s)) ds + \int_0^{\tau_n} e^{-rs} f_x(X_s) \sigma(X_s, \pi_s) dW_s, \end{aligned} \tag{2.15}$$

where the stochastic part again is a martingale.

Similar to proof (2.2) for any $\tau_n < \infty$ we take expectations from J up to τ_n and obtain

$$E_{0x}\left[\int_0^{\tau_n \wedge \mathcal{T}} e^{-rs}\psi(X_s, \pi_s)ds + e^{-r\tau_n}\chi_{(\mathcal{T}\geq\tau_n)}f(X_{\tau_n}) + e^{-r\mathcal{T}}\chi_{(\mathcal{T}<\tau_n)}\Psi(X_{\mathcal{T}})\right] \leq f(x).$$

For the inequality (2.15) is plugged into the last term and (2.12) is used. For the maximizing π^* equality holds. We further proceed analogously to the last proof, where the terms (2.13) and (2.14) arise from $n \to \infty$ for τ_n. $\square$

2.2 Diffusion Approximation of a Risk Process in the CLM

To model the wealth of an insurance company we use a standard model of collective risk theory, the so-called *Cramer-Lundberg Model*. This process is also referred to as *the renewal model*. The following is a standard definition from Embrechts et al. [2].

Definition 2.4 (*Cramer-Lundberg Model*)

- The claim size is given by the process $\{U_i; i = 1, 2, \dots\}$. The positive claims U_i are independent, identically distributed random variables with finite expected value μ and variance σ^2.
- The claims occur at the random instants of time

  $$0 < T_1 < T_2 < \dots \quad a.s.$$

- $N(t) = N_t$ denotes the number of claims up to time t

  $$N_t = \sup(n \geq 1 : T_n \leq t), \qquad t \geq 0,$$

  where by convention $\sup(\varnothing) = 0$.
- The inter-arrival times Y_t are independent, identically exponentially distributed with finite mean $\frac{1}{\lambda}$.

  $$Y_1 = T_1, \quad Y_k = T_k - T_{k-1}, \qquad k = 2, 3, \dots$$

- The sequences $\{U_i\}$ and $\{Y_i\}$ are independent of each other.

As a consequence of this definition N_t follows a homogenous Poisson process with intensity $\lambda > 0$.

$$\mathbb{P}(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, 2, \dots$$

Following the definition of the CLM, we can state the stochastic risk sum S_t that the insurance company has to pay up to time t:

$$S(t) := \begin{cases} \sum_{i=1}^{N_t} U_i, & N_t > 0 \\ 0, & N_t = 0 \end{cases} \tag{2.16}$$

Next, we consider an initial risk reserve $x > 0$ and the regularly paid premium p. Let $R(t) = R_t$ be the corresponding risk reserve of the insurance company. We obtain the risk process of the CLM

$$R_t = x + pt - S_{N_t} \tag{2.17}$$

as the wealth of the company. When $R_T \leq 0$ for some T, we call this point in time the ruin or bankruptcy time of the company.

For studies of this risk process it is helpful to formulate it as a diffusion process to use the results about control theory from the last chapter. In the following I will describe the approach of Iglehart [5] in a simplified way and will use the same notation. He applies the theory of weak convergence of probability measures on function spaces to show that this positive risk sum can be approximated by a diffusion process. By constructing a certain sequence of risk processes $\{R_n(t)\}$ he aims to find an approximation of the distribution of R_t. It can be shown, that these processes converge weakly to a Brownian motion with drift.

For consistent notation I start with some basic definitions of convergence in metric spaces.

In the setting of a metric space S with metric ρ let $\mathcal{S}$ be the class of Borel sets. For two probability measures P_n and P on $\mathcal{S}$ we say that P_n *converges weakly to* P as $n \to \infty$ if

$$\lim_{n\to\infty} \int_S f dP_n = \int_S f dP$$

for every bounded, real valued, continuous function f on S. We write

$$P_n \Rightarrow P.$$

Let $(\Omega, \mathcal{B}, \mathcal{P})$ be a probability space and X be a random variable into S. The probability measure $P = \mathcal{P}X^{-1}$ is the distribution from X on $(S, \mathcal{S})$. Let $\{X_n\}$ be a sequence of random variables on S. If the distribution of P_n of X_n converges weakly to the distribution of X ($P_n \Rightarrow P$), we say that X_n *converges in distribution to* X. For convenience we write as well

$$X_n \Rightarrow X.$$

Let X_n, Y_n be two different random elements of S. Since S is separable, $\rho(X_n, Y_n)$ is a random variable as well. If

$$P(\rho(X_n, Y_n) \geq \epsilon) \to 0$$

for each $\epsilon > 0$, we say that $\rho(X_n, Y_n)$ *converges in probability* and write

$$X_n \overset{P}{\sim} Y_n.$$

Now let S be the space $C(0, \infty)$ of of all continuous, real-valued functions with the uniform metric ρ. As a preparation to the needed case, we first build a special random function Y_n similar to a continuous version of the risk process

$$Y_n(t, \omega) := \frac{S^{(n)}_{[nt]}(\omega)}{\sqrt{n}\sigma} + (nt - [nt]) \frac{X^{(n)}_{[nt]+1}(\omega)}{\sqrt{n}\sigma}, \tag{2.18}$$

where $X^{(n)}_1, \ldots, X^{(n)}_n$ is a triangular array of independent and identically distributed random variables for each $n = 1, 2, \ldots$ with $E[X^{(n)}_i] = 0$, $Var[X^{(n)}_i] = \sigma^2_n > 0$. The functions

$S_i^{(n)}$ are defined like before as $S_0^{(n)} = 0$ and $S_k^{(k)} = X_1^{(k)} + \cdots + X_k^{(k)}$.

Prokhorov [9] stated in 1956 in a more generalised functional central limit theorem that this process converges in probability to a standard Brownian motion. I will state the simplified version of this theorem without proof.

Theorem 2.5 (*Prokhorov [9] theorem 3.1*) With the notation made before assume that $\sigma_n^2 \to \sigma^2 > 0$ and that $E[(X_i^{(n)})^{2+\epsilon}]$ is bounded in n for some ϵ. Then

$$Y_n \Rightarrow W,$$

where W is a standard Brownian motion with $W(0) = 0$.

Since the original risk process is not continuous it is not in the space $C(0, \infty)$. Therefore we need to focus on the space $D(0, \infty)$ of all real-valued functions $X(t)$ that are right-continuous and have left limits (abbreviated in french 'càdlàg'): for every $t \in (0, \infty)$ the limits $X(t-)$ and $X(t+)$ exist and additionally $X(t+) = X(t)$.

Skorohod[1] introduced his so-called J_1 topology on $D(0, 1)$ to make this space become a complete separable metric space. Billingsley[1] described a metric in 1968 to form such a topology: Let λ denote a strictly increasing, continuous mapping of $D(0, 1)$ onto itself and Λ be the class of all such functions. Then for $\lambda \in \Lambda$ one has $\lambda(0) = 0$ and $\lambda(1) = 1$. For simplicity I will write λt for $\lambda(t)$. Let

$$\|\lambda\| := \sup_{s \neq t} \left| \log \frac{\lambda t - \lambda s}{t - s} \right|.$$

The metric $d(x, y)$ for $x, y \in D$ is defined for an existing $\lambda \in \Lambda$ as

$$d(x, y) := \inf \{\epsilon > 0 : \|\lambda\| \leq \epsilon \text{ and } \rho(x(t) - y(\lambda t)) \leq \epsilon\}.$$

We can think of λ as a new timescale and the conditions in the definition of the new metric as an restriction of the time deformation. This metric generates the Skorohod J_1 topology on $D(0, 1)$, which coincides with the uniform topology when being restricted to $C(0, 1)$. A sequence $\{x_n\}$ converges to an element x in J_1 if there exists a sequence of continous mappings $\lambda_n \in \Lambda$ such that $x_n \circ \lambda_n \overset{P}{\sim} x$ and $\lambda_n \overset{P}{\sim} e$, where $e(t) = t$.

Stone [11] developed a topology using this metric to expand this property in a first step to the space $D(0, N)$ and subsequently to $D(0, \infty)$. It essentially requires convergence for each metric d_N for each $N > 0$.
A good introduction to the function space D, often used for applied probability theory, and the corresponding metric and topology can be found in Iglehart[6] Chapter 2.

Liggett and Rosèn stated that the property of weak convergence can be connected in the two function spaces C and D.

[1] A. V. Skorohod, Limit theorems for stochastic processes. (1956)

Theorem 2.6 (*Ligget and Rosèn*) $\{X_n\}$ is sequence of random functions in (D, d), $\{Y_n\}$ a sequence of random functions in (C, ρ), and X a random function in (C, ρ).

$X_n \Rightarrow X$ in (D, d) if and only if
$Y_n \Rightarrow X$ in (C, ρ).

As a next step let us now define the process X_n in (D, ρ) as

$$X_n(t, \omega) := \frac{S^{(n)}_{[nt]}(\omega)}{\sqrt{n}\sigma}. \tag{2.19}$$

Since $X_n \overset{P}{\sim} Y_n$ with Y_n as defined in (2.18), we link the results of Theorem 2.5 and Theorem 2.6 and obtain

$$X_n \Rightarrow W.$$

However, the risk process R_t (2.17) we are interested in, differs from (2.19) as we are interested in sums of random number of random variables.
To follow Iglehart's notation now define $\eta_i := Y_i$ as the time between any occurence of each claim U_i and U_{i-1}. Hence $\{\eta_n; n = 1, 2, \dots\}$ form a sequence of positive, identically distributed random variables. Let $E(\eta_i) = \frac{1}{\lambda} > 0$. Now define $N(t)$ as the number of claims or renewals up to time $t \geq 0$ as

$$N(t) := \max\left\{k : \sum_{i=1}^{k} \eta_i \leq t\right\}, \tag{2.20}$$

where $N(t) = 0$ if $\eta_1 > t$. As a random function, $N(n\cdot)$ is in the space D as well for any $n = 1, 2, \dots$. Let Λ denote the constant-valued function $\Lambda(t) = \lambda t$. Following a functional central limit theorem connected with renewal theory of Billingsley[1] (Theorem 14.6. p. 154) we conclude from the proof that

$$\frac{N(n\cdot)}{n} \Rightarrow \Lambda. \tag{2.21}$$

Billingley also stated that weak convergence of random functions like (2.19) in D is still valid after certain random modifications of the time scale. Since (2.20) is such a time-scaling function in D that converges to a constant, this theory can be used.

Theorem 2.7 Let

$$Z_n(t, \omega) := \frac{S^{(n)}_{N(nt)}(\omega)}{\sqrt{n}\sigma}.$$

Then

$$Z_n \Rightarrow W \circ \Lambda.$$

The last goal is to construct a sequence of risk processes $\{R_n(t)\}$ from the original process (2.17) and use the stated theory to show that they converge weakly. This is done by compressing the original time scale t by $\frac{1}{n}$.

Let $x_n, p_n > 0$ denote the initial risk reserve and the regularly risk premium respectively for the n-th process. $\{U_i^{(n)}\}$ is the sequence of independent, identically distributed claims with $E[U_i^{(n)}] = \mu_n > 0$ and $\sigma^2(U_i^{(n)}) = \sigma_n^2 > 0$ that occur at the jumps of a renewal process. $N(t)$ is defined as before in (2.20) as the sum of these jumps up to time t. We get

$$R_n(t) = x_n + p_n n t - S_{N(nt)},$$

and are able to state the final result of this section.

Theorem 2.8 (*Iglehart*) If $x_n := x\sqrt{n} + o(\sqrt{n})$, $p_n := \frac{p}{\sqrt{n}} + o(\frac{1}{\sqrt{n}})$, $\mu_n := \frac{\mu}{\sqrt{n}} + o(\frac{1}{\sqrt{n}})$, $\sigma_n^2 \to \sigma^2 > 0$ and $E[(X_i^{(n)})^{2+\epsilon}]$ is bounded in n for some $\epsilon > 0$, then

$$\frac{R_n}{\sqrt{n}} \Rightarrow x + \Gamma + \sigma\sqrt{\lambda}W,$$

where the constant-valued function Γ is defined as $\Gamma(t) := (p - \mu\lambda)t$.

Proof (*Theorem 2.8*) After standardising the stochastic term of R_n using the expected value of the risk sum, we can apply Theorem 2.7 to obtain

$$\frac{S_{N(n\cdot)}^{(n)} - \mu_n N(n\cdot)}{\sqrt{n}} \Rightarrow \sigma W \circ \Lambda.$$

From (2.21) we know that

$$\mu_n \frac{N(n\cdot)}{n} \Rightarrow M,$$

where we define $M(t) := \lambda\mu t$. A continuous mapping theorem of the theory of weak convergence states, that weak convergence holds after a continuous, measurable mapping of the metric space into itself. So we can change signs to get the actual negative part of R_n.
Furthermore we know that if two sequences of random functions converge in probability and one converges weakly to a random function, the other one does too. That is a special case of Theorem 2.6 on the same space. Therefore we can standardise directly with M.

$$-\frac{S_{N(n\cdot)}^{(n)}}{\sqrt{n}} + M \Rightarrow \sigma W \circ \Lambda.$$

Considering the other terms of R_n we get the result

$$\frac{R_n}{\sqrt{n}} \Rightarrow x + \Gamma + \sigma W \circ \Lambda.$$

For Gaussian processes it is sufficient to show that the mean and covariance are the same to prove that they are identical. Since $W \circ \Lambda$ has the same distribution as $\sqrt{\lambda}W$, the proof is complete. $\square$

When the risk process is approximated in such a way, the parameters of the Brownian motion μ and σ can be determined by matching the first two moments of the original process R_n and the Wiener process.
The approximation gets better the greater n gets and therefore the higher t of $R_n(t)$ is. But from Theorem 2.8 we see that the starting value x also has a great impact on the quality of the approximation.

Chapter 3

Formulated Problem and Solution

We will now start to describe the control problem in detail and formulate the corresponding stochastic control problem we will have to solve. We begin with a classical *Cramer-Lundberg Model* (CLM) as defined in (2.4), which uses a compound Poisson process to describe the risk process of an insurance company

$$R(t) = x + pt - \sum_{i=1}^{N_t} U_i.$$

Here, the initial position x of the risk process corresponds to the initial capital of the insurance company $x \in [0, \infty)$, p is the premium rate, U_i is the size of each claim i and N_t corresponds to the number of claims in the time interval $[0, t]$. The claims U_i are i.i.d. random variables with positive finite first an second moments m and s^2. Furthermore they are independent of N_t, which is a Poisson process with intensity β. The event of the process $R(t)$ hitting zero or a given bankruptcy value P is called *ruin*. When this happens, we call the time T bankruptcy or ruin time, with $T = \inf\{t : R(t) \leq 0\}$ or $T = \inf\{t : R(t) \leq P\}$, respectively.

In the simplest case of proportional reinsurance, the company introduces the retention level $a \in [0, 1]$. $a = 1$ means taking no reinsurance and covering all the risk yourself, while $a = 0$ means covering no risk at all and pass it to the reinsurance company. Hence, on the one hand the insurance company only insures a fraction of each claim $a \cdot U_i$, but earns only a fraction of the premium $a \cdot p$ on the other hand. The process of surplus becomes

$$R(t) = x + apt - a \sum_{i=1}^{N_t} U_i.$$

Because there is no extra fee added for the reinsurance premium rate, this model is called *cheap* reinsurance. It is described in [12], where a similar model is considered.
Usually, reinsurance premium rate is higher than the insurance premium rate, so we introduce the premium rate p_2 with $p_2 > p$. The earned premium rate for the initial company now reduces to $p - (1 - a)p_2$. This model is called *non-cheap* reinsurance. Now

the risk process formulates like this:

$$R(t) = x + (p - (1-a)p_2)t - a\sum_{i=1}^{N_t} U_i. \tag{3.1}$$

Our goal is of course to control the retention level a, so that the process $R(t)$ does not hit ruin.

In order to use dynamic programming to find a closed form solution of this problem, we need to transform (3.1) into a diffusion process. According to Section 2.2 it can be approximated by a Brownian motion with drift, where we match expected value and variation. For simplicity we write R_t instead of $R(t)$. Since the expected value of (3.1) by Walds formula equals

$$E[R_t] = (p - (1-a)p_2)t - am\beta t,$$

we can write the drift of the diffusion process as $\mu - (1-a)\lambda$, where $\mu = p - \beta m$ and $\lambda = p_2 - \beta m$. The variance of the poisson process equals

$$\sigma^2(R_t) = \beta t m^2 + \beta t s^2.$$

Therefore we are able to set the diffusion for the transformed process as $a\sigma$, with $\sigma^2 = \beta(s^2 + m^2)$. $p = p_2$ and therefore $\lambda = \mu$ would be the case of cheap reinsurance.

To find an exact solution I want to formulate this diffusion model in a mathematical way. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we name a standard Brownian motion W_t adapted to the filtration $\mathcal{F}_t$. The diffusion risk process under the control policy π with initial capital x is given by

$$\begin{aligned} dR_t^\pi &= (\mu - (1 - a_t^\pi)\lambda)dt + \sigma a^\pi(t)_t dW_t, \qquad \text{with} \\ R_0^\pi &= x. \end{aligned} \tag{3.2}$$

The control process a_t^π describes the amount reinsured of the claims and therefore satisfies $0 \leq a_t^\pi \leq 1$ for all $t \geq 0$. We choose a_t^π to be $\mathcal{F}_t$ measurable. By Π we denote the set of all the admissible controls strategies.
Similar to the Cramer-Lundberg model we define a ruin time by

$$\tau_\pi = \inf\{t : R_t^\pi \leq 0\}.$$

Furthermore, with a given admissible policy π, a discount factor r and value of bankruptcy P we define the return function by

$$J_x(\pi) = E\left[\int_0^{\tau_\pi} e^{-rt} R_t^\pi dt + e^{-r\tau_\pi} P\right]$$

Here x refers to the initial capital of the risk process as in (3.2).

So in our case the previously defined stochastic control problem (2.1) formulates as follows.

- The control policy a_t^π is adapted to the filtration $\mathcal{F}_t$ with values in $[0,1]$.
- The controlled process R_t^π is described by

$$\begin{aligned} &dR_t^\pi = \mu a_t^\pi dt + \sigma a_t^\pi dW_t, \qquad \text{with} \\ &R_0^\pi = x. \end{aligned}$$

- The associated performance functional is

$$J_x(\pi) = E\left[\int_0^{\tau_\pi} e^{-rt} R_t^\pi dt + e^{-r\tau_\pi} P\right].$$

- All admissible policies are denoted by Π.
- The optimal value function is

$$V(x) = \sup_{\pi \in \Pi} J_x(\pi).$$

Our goal is to find this optimal return function and the corresponding optimal control π^*, so that $J_x(\pi^*) = \sup_{\pi\in\Pi} J_x(\pi)$. The optimal control process $a^{\pi^*}(R_t) = a^*$ is called *optimal feedback control*. It describes the value of the optimal retention level a for a given surplus of the risk process R_t.

3.1 HJB Equation

With the theory stated in Chapter 2.1 we are able to formulate the associated HJB equation that we have to solve. When $\tilde{}$ denotes the notation in the last chapter, we set $\tilde{\psi} := x, \tilde{b} := \mu - (1-a)\lambda$ and $\tilde{\sigma^2} := \sigma^2 a^2$. We obtain

$$\max_{a\in[0,1]} \left[\frac{\sigma^2 a^2}{2} f''(x) + (\mu - (1-a)\lambda) f'(x) - r f(x) + x\right] = 0 \tag{3.3}$$

with the boundary conditions

$$\begin{aligned} &f(0) = P, \qquad \text{and} \\ &\limsup_{x\to\infty} \frac{|f(x)|}{x} < \infty. \end{aligned} \tag{3.4}$$

According to (2.3) we have to solve this nonlinear ordinary differential equation to find the optimal return function V and the optimal feedback control a^*. This is done in the following chapter.

3.2 Solution to the HJB equation

From here on we assume that $p_2 > p$, i.e. $\lambda > \mu$ to consider only the more realistc case of non-cheap reinsurance. $p = p_2$ and therefore $\lambda = \mu$ would be the case of cheap reinsurance.
To find a solution to this problem we use an heuristic approach. We assume that we found a solution f to the HJB equation (3.3) and use its characteristics to get an exact definition of the value function and the optimal feedback control. We have to verify all the characteristics of f and all the assumptions we will make once the solution is found.

We start with an measurable interval $O = [0, x_1)$ for a fixed $x_1 \in [0, \infty)$ and assume that we found a solution f, such that $0 < a^*(x) < 1$ for every $x \in O$. We further assume that f is strictly concave on O.
Now, to find the maximum in (3.3) in O we differentiate the left hand side with respect to a to find a^*. As an extremum it should equal 0

$$\sigma^2 a f''(x) + \lambda f'(x) = 0,$$

which transforms to

$$a(x) = -\frac{\lambda f'(x)}{\sigma^2 f''(x)}, \qquad x \in O. \tag{3.5}$$

Now we substitute the last term into (3.3) and see that f satisfies

$$-\frac{\lambda^2 f'(x)^2}{2\sigma^2 f''(x)} + (\mu - \lambda) f'(x) - r f(x) + x = 0. \tag{3.6}$$

By our assumption f is strictly concave, so we are able to use a transformation to simplify the following steps. Since $f''(x)$ is negative and $a(x)$ is positive, we obtain from (3.5) that $f'(x)$ is strictly positive and therefore decreasing. Therefore, since $f(x)$ is increasing, $-ln(f(x))$ is strictly increasing and there exists a function $X(.)$ such that $-\ln(f'(X(z))) = z$. Also

$$\begin{aligned} f'(X(z)) &= e^{-z}, \\ f''(X(z)) &= \frac{-e^{-z}}{X'(z)}. \end{aligned} \tag{3.7}$$

This transformation is also used in Presman and Sethi[8] among others.

Defining B such that $X(B) = 0$ and $f'(0) = e^{-B}$, we get $X : [B, \infty) \to [0, \infty)$. After substituting $x = X(z)$ into (3.6) and using (3.7) we obtain

$$\begin{aligned} &-\frac{(\lambda e^{-z})^2}{2\sigma^2 \frac{-e^{-z}}{X'(z)}} + (\mu - \lambda) e^{-z} - r f(x) + x \\ &= \frac{\lambda^2}{2\sigma^2} X'(z) e^{-z} + (\mu - \lambda) e^{-z} - r f(X(z)) + X(z) = 0. \end{aligned} \tag{3.8}$$

By differentiating the last term with respect to z, applying (3.7) once again we get

$$\frac{\lambda^2}{2\sigma^2}X''(z)e^{-z} - \frac{\lambda^2}{2\sigma^2}X'(z)e^{-z} - (\mu-\lambda)e^{-z} - re^{-z}X'(z) + X'(z) = 0.$$

By multiplying this expression by $c := \frac{2\sigma^2}{\lambda^2}$ and e^{-z} we can rewrite (3.8) as

$$X''(z) - (1 + cr - ce^z)X'(z) - c(\mu-\lambda) = 0 \tag{3.9}$$

We solve this ODE with the technique of *variation of constants*. First, we easily find a solution to the corresponding homogenous equation

$$X''(z) - (1 + cr - ce^z)X'(z) = 0$$

and get

$$X_h'(z) = k_1 e^{[(1+cr)z - ce^z]}, \tag{3.10}$$

where k_1 is a constant.
To find a particular solution to the non-homogeneous equation, we use the approach $X_p'(z) = d(z)X_h'$ with the differentiable function $d(z)$. Now we plug this term into (3.9) to obtain $d'(z)$.

$$\begin{aligned} &X_p''(z) - (1 + cr - ce^z)X_p'(z) - c(\mu-\lambda) \\ &\quad = d'(z)e^{[(1+cr)z-ce^z]} + d(z)e^{[(1+cr)z-ce^z]}((1+cr) - ce^z) \\ &\quad - (1 + cr - ce^z)d(z)e^{[(1+cr)z-ce^z]} - c(\mu-\lambda) = 0. \end{aligned}$$

When we integrate

$$d'(z) = c(\mu-\lambda)e^{-[(1+cr)z-ce^z]}$$

we obtain the solution for $X_p'(z)$. The final result is

$$\begin{aligned} X'(z) &= X_h' + X_p' \\ &= c(\mu-\lambda)e^{[(1+cr)z-ce^z]}\int_B^z e^{[-(1+cr)y+ce^y]}dy + k_1 e^{[(1+cr)z-ce^z]}. \end{aligned} \tag{3.11}$$

For simplification we will use the density function $g(.)$ of a Gamma distribution with parameters $(cr+1, 1/c)$:

$$g(x) = \frac{c^{cr+1}}{\Gamma(cr+1)}x^{cr}e^{-cx}, \tag{3.12}$$

where Γ is the Gamma function

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx.$$

We plug $g(e^z)$ into (3.11) to get

$$X'(z) = c(\mu - \lambda)e^z g(e^z) \int_B^z \frac{1}{e^y g(e^y))} dy + k_1 \frac{\Gamma(cr+1)}{c^{cr+1}} e^z g(e^z).$$

We set $k_2 := k_1 \frac{\Gamma(cr+1)}{c^{cr+1}}$ and use the substitution $u = e^y$ to define

$$H(z) := \int_{e^B}^z \frac{1}{u^2 g(u)} du, \qquad \text{for } z > e^B. \tag{3.13}$$

The simplified version of (3.11) is

$$X'(z) = c(\mu - \lambda)H(e^z)e(z)g(e^z) + k_2 e^z g(e^z).$$

By integrating this term we obtain a solution of (3.9) for $z > B$:

$$X(z) = \int_B^z [c(\mu - \lambda)H(e^y)e^y g(e^y) + k_2 e^y g(e^y)]dy + k_3 = K(e^y) + k_3,$$

where we use the same substitution as in (3.13) to define

$$K(z) := \int_{e^B}^z [c(\mu - \lambda)H(u)g(u) + k_2 g(u)]du, \qquad \text{for } z > e^B. \tag{3.14}$$

Since by our definition $X(B) = 0$, we get $k_3 = 0$, and therefore $X(z) = K(e^z)$.
$X(z)$ and and e^z are both monotone, so $K(.)$ is invertible. Let K^{-1} denote the inverse of K. By our definition of $X(z)$ to formulate (3.7) and the substitution $X(z) = x$ we get

$$-\ln(f'(X(z))) = z = ln(K^{-1}(x)),$$

which results in

$$\begin{aligned} f'(x) &= \frac{1}{K^{-1}(x)}, \\ f''(x) &= -\frac{1}{k(K^{-1}(x))(K^{-1}(x))^2}. \end{aligned} \tag{3.15}$$

Here

$$k(z) = c(\mu - \lambda)H(z)g(z) + k_2 g(z) \tag{3.16}$$

is the derivative of $K(z)$.
When we insert (3.15) into (3.5) we get a first result of the optimal feedback control on the interval $O = [0, x_1)$

$$a(x) = \frac{\lambda}{\sigma^2} k(K^{-1}(x))K^{-1}(x). \tag{3.17}$$

We also find a first solution for the value function on O by integrating (3.15) an using the boundary condition at zero

$$f(x) = \int_0^x \frac{1}{K^{-1}(y)} dy + P, \qquad \text{for } x \in O.$$

We assumed that $0 < a(x) < 1$ on O. As a next step we also assume that the border of the interval x_1 exists and $a(x_1) = 1$. Of course $a(x) = 1$ for every $x > x_1$, because it represents the retention level of the insurance company. Setting $a = 1$ in (3.3) we get

$$\frac{\sigma^2}{2} f''(x) + \mu f'(x) - r f(x) + x = 0. \tag{3.18}$$

To solve this ODE of second order with constant coefficients, we first consider the characteristic polynomial of the homogenous part (See [7]).

$$f^2 + \frac{2\mu}{\sigma^2} f - \frac{2r}{\sigma^2} = 0 \tag{3.19}$$

The fundamental set of solutions for the homogenous part of (3.18) is given by

$$f = c_1 e^{f_1 x} + c_2 e^{f_2 x},$$

where f_1, f_2 are the solutions of (3.19) and c_1, c_2 are constants. The solutions are

$$f_{1,2} = \frac{-\frac{2\mu}{\sigma^2} \pm \sqrt{\frac{4\mu^2}{\sigma^4} + 4\frac{2r}{\sigma^2}}}{2} = -\frac{\mu \pm \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}.$$

Using the condition (3.4) that f should not grow too fast for $x \to \infty$ I only use the negative solution and define

$$\theta_- := \frac{-\mu - \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}. \tag{3.20}$$

We get

$$f_h(x) = k_4 e^{\theta_- x},$$

where k_4 is a constant.
As before in (3.11) we use *variation of constants* to get a particular solution the non-homogeneous equation. Because of the special form of the source term I use the approach $f_p(x) = q_1 x + q_0$ with constant coefficients c_1, c_0. After plugging this term into (3.18) we get

$$\mu q_1 - r(q_1 x + q_0) + x = 0.$$

To satisfy this equation we get the two conditions

$$\mu q_1 - r q_0 = 0 \qquad \text{and} \tag{3.21}$$
$$-r q_1 x + x = 0, \tag{3.22}$$

which leads to the particular solution

$$f_p(x) = \frac{\mu}{r^2} + \frac{x}{r}.$$

The solution to $f(x)$ is given by

$$f(x) = f_p(x) + f_h(x) = \frac{\mu}{r^2} + \frac{x}{r} + k_4 e^{\theta_- x}, \qquad \text{for } x \geq x_1.$$

Now, we can summarize our suggested solution

$$f(x) = \begin{cases} \int_0^x \frac{1}{K^{-1}(y)} dy + P, & 0 \leq x < x_1 \\ \frac{\mu}{r^2} + \frac{x}{r} + k_4 e^{\theta_- x}, & x \geq x_1 \end{cases} \tag{3.23}$$

We still have to determine the values of the constants k_2, B, k_4 and x_1, before we have to prove that all the assumptions hold that we did at the beginning.

We use the principle of *smooth fit*. Let f_1 denote the function f for $x < x_1$ and f_2 is f for $x \geq x_1$. Now some of the constants are chosen, so that the first an second derivative of the value function f_1 and f_2 become equal in the point x_1.
After differentiating (3.23) we get

$$\begin{aligned} f_2'(x_1) &= \frac{1}{r} + k_4 \theta_- e^{\theta_- x_1}, \\ f_2''(x_1) &= k_4 \theta_- e^{\theta_- x_1} \end{aligned}$$

and from (3.15)

$$f_1'(x_1) = \frac{1}{K^{-1}(x_1)}.$$

For the second derivative of f_1 we don't use (3.15) but the special case of the control process a in x_1 from (3.5). So from

$$a(x_1) = 1 = -\frac{\lambda f_1'(x_1)}{\sigma^2 f_1''(x_1)}$$

we get

$$f_1''(x_1) = -\frac{\lambda}{\sigma^2} f_1'(x_1).$$

We obtain the following two continuity equations

$$\begin{aligned} &\frac{1}{K^{-1}(x_1)} = \frac{1}{r} + k_4 \theta_- e^{\theta_- x_1}, \qquad \text{and} \\ &-\frac{\lambda}{\sigma^2} \frac{1}{K^{-1}(x_1)} = k_4 \theta_- e^{\theta_- x_1}. \end{aligned} \tag{3.24}$$

We solve this these equations for x_1 and k_4 and can state the following as a first result

$$K^{-1}(x_1) = r\left(1 + \frac{\lambda}{\sigma^2\theta_-}\right), \tag{3.25}$$

$$k_4 e^{\theta_- x_1} = \frac{-\lambda}{r(\sigma^2\theta_-^2 + \lambda\theta_-)}. \tag{3.26}$$

Since $f(x_1) > 0$ and therefore $K^{-1}(x_1) > 0$, from (3.25) and the definition of θ_- (3.20) we get the request $\lambda < \mu + \sqrt{\mu^2 + 2r\sigma^2}$. So we can solve (3.25) and (3.26) explicitly

$$\begin{aligned} x_1 &= K\left(r\left(1 + \frac{\lambda}{\sigma^2\theta_-}\right)\right), \\ k_4 &= \frac{-\lambda}{r(\sigma^2\theta_-^2 + \lambda\theta_-)} e^{-\theta_- x_1}. \end{aligned} \tag{3.27}$$

By the solution of $a(x)$ on O, see (3.17), in the point x_1 and using the form of $k(.)$ in (3.16) we get

$$a(x_1) = 1 = \frac{\lambda}{\sigma^2} K^{-1}(x_1)\left(c(\mu - \lambda)H(K^{-1}(x_1))g(K^{-1}(x_1)) + k_2 g(K^{-1}(x_1))\right).$$

So

$$k_2 = \frac{\sigma^2}{\lambda K^{-1}(x_1) g(K^{-1}(x_1))} - c(\mu - \lambda)H(K^{-1}(x_1)). \tag{3.28}$$

Since $K^{-1}(x_1)$ is positive and $\lambda > \mu$ we know that $k_2, x_1 > 0$ as well.
The updated version of (3.23) with $K(.)$ given by (3.14) and θ_- given by (3.20) is

$$f(x) = \begin{cases} \int_0^x \frac{1}{K^{-1}(y)} dy + P, & 0 \le x < x_1, \\ \frac{\mu}{r^2} + \frac{x}{r} + \frac{-\lambda}{r(\sigma^2\theta_-^2 + \lambda\theta_-)} e^{\theta_-(x - x_1)}, & x \ge x_1. \end{cases}$$

Finding a solution to e^B gets more complicated, as it is used in the definition of $K(.)$ and $H(.)$. It will be necessary to make an additional assumption, as this is also needed to formulate the solution for the maximizing function of the HJB equation (3.3), the optimal feedback control.

We set $\alpha := K^{-1}(x_1)$ since this solution is known from (3.25). $K^{-1}(x)$ exists and is increasing on $[0, x_1)$, if the derivation $k(y) > 0$ for all $e^B < y < \alpha$. Additionally $K^{-1}(0) = e^B$. Furthermore, from the positivity property of K^{-1} and the derivations f_1', f_1'' we see that f is strictly concave on O.
We insert (3.28) into (3.16) and get the inequality

$$\begin{aligned} k(y) &= c(\mu - \lambda)H(y)g(y) + \frac{\sigma^2}{\lambda\alpha g(\alpha)} - c(\mu - \lambda)H(\alpha)g(y) \\ &= c(\lambda - \mu)g(y)(H(\alpha) - H(y)) + \frac{\sigma^2 g(y)}{\lambda\alpha g(\alpha)} > 0 \end{aligned} \tag{3.29}$$

Hence, our solution to a^* is given by

$$a^*(x) = \begin{cases} \frac{\lambda}{\sigma^2} k(K^{-1}(x))K^{-1}(x), & \text{if } 0 \leq x < x_1, \\ 1, & \text{if } x \geq x_1. \end{cases} \tag{3.30}$$

To get a solution to B, we use the boundary condition at 0 and the fact that $f(0+)$ should behave continuous at 0. f has to be a solution to the HJB equation (3.3) for all $x \geq 0$. Therefore we let $f(0+)$ near $f(0)$ in the limit $x \to 0$.
First, we insert (3.29) into (3.30) near 0 and use $K^{-1}(0) = e^B$,

$$a^*(0+) = \frac{\lambda e^B}{\sigma^2} \left(c(\lambda - \mu) g(e^B)(H(\alpha) - H(e^B)) + \frac{\sigma^2 g(e^B)}{\lambda \alpha g(\alpha)} \right).$$

With $H(e^B) = 0$ and the definition of $c = \frac{2\sigma^2}{\lambda^2}$ we get

$$a^*(0+) = \frac{e^B g(e^B)}{\alpha g(\alpha)} + \frac{2(\lambda - \mu)}{\lambda} e^B g(e^B) H(\alpha). \tag{3.31}$$

Second, we use the already rearranged version (3.6) of the HJB equation and the boundary condition $f(0) = P$. We get

$$\frac{\lambda}{2} a^*(0+) f'(0+) + (\mu - \lambda) f'(0+) - rP = 0.$$

Now we insert $f(0) = e^{-B}$ and $a^*(0+)$ from (3.31) and get

$$\frac{\lambda}{2} \left(\frac{g(e^B)}{\alpha g(\alpha)} + \frac{2(\lambda - \mu)}{\lambda} g(e^B) H(\alpha) \right) + (\mu - \lambda) e^{-B} - rP = 0. \tag{3.32}$$

We want to solve this equation for B, which cannot be done in an explicit way. Instead we are going to show that it has a unique solution. To prove that, we split (3.32) into two new defined functions $F(.)$ and $G(.)$ and show that these functions have a unique crossing point in O.
To define these functions we set $y := e^B$, multiply (3.32) with y and use the definition of $H(.)$,

$$\begin{aligned} F(y) &:= \frac{y g(y)}{\alpha g(\alpha)} + \frac{2(\lambda - \mu)}{\lambda} y g(y) \int_y^\alpha \frac{1}{z^2 g(z)} dz, \\ G(y) &:= \frac{2rP}{\lambda} y + \frac{2(\lambda - \mu}{\lambda}. \end{aligned}$$

First, we show that $F'(y)$ has no local minimum on $(0, \alpha)$, which is the new definition set after the transformation of O. To derivate $F(.)$, we first calculate the derivation of $yg(y)$ with the gamma density function $g(.)$ (3.12),

$$\begin{aligned} (yg(y))' &= \left(\frac{c^{cr+1}}{\Gamma(cr+1)} y^{cr+1} e^{-cy} \right)' \\ &= \frac{c^{cr+1}}{\Gamma(cr+1)} \left((cr+1) y^{cr+1} e^{-cy} - c y^{cr+1} e^{-cy} \right) \\ &= g(y)(1 + cr - cy). \end{aligned} \tag{3.33}$$

So we get

$$\begin{aligned} F'(y) &= \frac{1}{y}\left(F(y)(1+rc-cy) - \frac{2(\lambda-\mu)}{\lambda}\right) \qquad \text{and} \\ F''(y) &= \frac{1}{y}\left(F'(y)(rc-cy)\right) - \frac{cF(y)}{y}. \end{aligned} \tag{3.34}$$

When we assume that there exists an extremum $y^* \in (0,\alpha)$ with

$$F''(y^*) = 0, \tag{3.35}$$

then we get from (3.34)

$$\begin{aligned} F'(y^*) &= \frac{cF(y^*)}{rc-cy^*} > 0 \qquad \text{and} \\ F'''(y^*) &= -\frac{2cF'(y^*)}{y^*} < 0. \end{aligned}$$

So the assumed extreme point is a local maximum and therefore F' has no local minimum on $(0,\alpha)$. This is why F is strictly increasing and concave.
Next, we compare the values of F and G at the boundaries 0 and α with each other. For the value of F at 0 we use L'Hospital's rule for $y \to 0$,

$$\begin{aligned} \lim_{y\to 0} F(y) &= \lim_{y\to 0} \frac{\frac{1}{\alpha g(\alpha)} + \frac{2(\lambda-\mu)}{\lambda}\int_y^\alpha \frac{1}{z^2 g(z)}dz}{\frac{1}{yg(y)}} \\ &= \lim_{y\to 0} \frac{\frac{2(\lambda-\mu)}{\lambda}\frac{-1}{y^2 g(y)}}{\frac{-g(y)(1+rc-cy)}{y^2 g(y)^2}} = \frac{2(\lambda-\mu)}{\lambda(1+rc)}, \end{aligned}$$

where (3.33) was used once again.
So

$$F(0) = \frac{2(\lambda-\mu)}{\lambda(1+rc)} < \frac{2(\lambda-\mu)}{\lambda} = G(0).$$

If $F(\alpha) > G(\alpha)$ then (3.32) has a unique solution, because of the shown properties of F and G is linear,

$$F(\alpha) = 1 > G(\alpha) = \frac{2rP}{\lambda}\alpha + \frac{2(\lambda-\mu)}{\lambda}.$$

To ensure this

$$\begin{aligned} G(\alpha) &= \frac{2rP}{\lambda}\alpha + \frac{2(\lambda-\mu)}{\lambda} < 1, \\ P &< \frac{2\mu-\lambda}{2r\alpha}. \end{aligned}$$

We are now able to summarize the result in the different cases of the starting variables μ, σ, P, λ and r. Also, we are going to prove the suggested solutions to verify the assumption we made for deriving them. Depending on the starting variables the value function and the optimal control differ, since x_1 and therefore the domain O does not always exist.

Before we state the results in theorems, we prove that the optimal feedback control $a^*(x)$ we found in (3.17) for the domain $O = [0, x_1)$ fulfills our assumptions as an optimal reinsurance policy.

Lemma 3.1 Let $\mu < \lambda < \mu + \sqrt{\mu^2 + 2r\sigma^2}$. Then

$$a(x) = \frac{\mu}{\sigma^2} k(K^{-1}(x))K^{-1}(x)$$

is increasing and $0 < a(x) < 1$ for all $x \in [0, x_1)$.

Proof Defining $y := K^{-1}(x)$ we have to show

$$a_1(y) = \frac{\mu}{\sigma^2} k(y)y \in (0, 1) \qquad \text{for } y \in [e^B, \alpha)$$

We already showed in (3.29) that $k(y) > 0$ and therefore $a_1(y) > 0$ on $[e^B, \alpha)$.
Similar to calculations before, we show that $a_1(y)$ has no local minimum on the observed domain and use that to show it is increasing as well. First, we use (3.33) and the definitions of $k(.)$ and $H(.)$ to show

$$a_1'(y) = \frac{1}{y}\left(a_1(y)(1 + cr - cy) + \frac{\lambda c}{\sigma^2}(\mu - \lambda)\right) \qquad \text{and}$$
$$a_1''(Y) = \frac{1}{y}\left(a_1'(y)(cr - cy)\right) - \frac{ca_1(y)}{y}.$$

Again we assume that there exists an extremum $y^* \in (e^B, \alpha)$, which fulfills

$$a_1'(y) = 0.$$

Due to

$$a_1''(y) = -\frac{ca_1(y)}{y} < 0,$$

the extremum is a local maximum and so no local minimum exists.
$a_1(.)$ is a continuous function and therefore should fit for $y \to \alpha$. Since we know $a_1(\alpha) = 1$ we set

$$a_1'(\alpha-) = \frac{1}{\alpha}\left((1 + cr - c\alpha) + \frac{\lambda c}{\sigma^2}(\mu - \lambda)\right).$$

When we insert the definition of $c = \frac{2\sigma^2}{\lambda^2}$, $\alpha = r(1 + \frac{\lambda}{\sigma^2\theta_-})$ and $\theta_- = \frac{-\mu - \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$ into the numerator, it transforms to

$$a_1'(\alpha-) = \frac{1}{\alpha}\frac{-1}{\lambda\theta_-}(\lambda\theta_- + 2r - 2\mu\theta_-) =$$
$$= \frac{2\mu(\mu + \sqrt{\mu^2 + 2r\sigma^2}) + 2r\sigma^2 - \lambda(\mu + \sqrt{\mu^2 + 2r\sigma^2})}{\lambda(\mu + \sqrt{\mu^2 + 2r\sigma^2})\alpha}.$$

Using the condition $\lambda < \mu + \sqrt{\mu^2 + 2r\sigma^2}$ we see that

$$a_1'(\alpha-) > \frac{2\mu(\mu + \sqrt{\mu^2 + 2r\sigma^2}) + 2r\sigma^2 - (\mu + \sqrt{\mu^2 + 2r\sigma^2})^2}{\lambda(\mu + \sqrt{\mu^2 + 2r\sigma^2})\alpha} = 0$$

where we set $\lambda = \mu + \sqrt{\mu^2 + 2r\sigma^2}$ in the numerator.
We obtain that $a_1'(y) > 0$ for all $y \in [e^B, \alpha)$, since a_1 has no local minimum and the derivation next to α is positive. Furthermore, we see that $0 < a_1(y) < 1$ for all $y \in [e^B, \alpha)$ from $a_1(\alpha) = 1$. □

We will now state the solution to the HJB equation (3.3) for the different cases of starting variables.

Theorem 3.2 (*Solution to the HJB-equation 1*) Let again be $\mu < \lambda < \mu + \sqrt{\mu^2 + 2r\sigma^2}$.

1. If $P < \frac{2\mu-\lambda}{2r\alpha}$ then the solution to the HJB equation (3.3) with the boundary condition (3.4) is given by

$$f(x) = \begin{cases} \int_0^x \frac{1}{K^{-1}(y)} dy + P, & 0 \le x < x_1, \\ \frac{\mu}{r^2} + \frac{x}{r} + \frac{-\lambda}{r(\sigma^2\theta_-^2 + \lambda\theta_-)} e^{\theta_-(x-x_1)}, & x \ge x_1, \end{cases}$$

   where $K(.)$ is given by (3.14), θ_- is given by (3.20) and x_1 is determined by (3.27), k_2 by (3.28) and e^B is the unique solution to (3.32).

   In this case the optimal feedback control is given by

$$a^*(x) = \begin{cases} \frac{\lambda}{\sigma^2} k(K^{-1}(x)) K^{-1}(x), & \text{if } 0 \le x < x_1, \\ 1, & \text{if } x \ge x_1, \end{cases}$$

   where $k(.)$ is given by (3.29).

2. If $\frac{2\mu-\lambda}{2r\alpha} < P < \frac{\mu}{r^2}$ then the solution is

$$f(x) = \frac{\mu}{r^2} + \frac{x}{r} + (P - \frac{\mu}{r^2}) e^{\theta_- x}, \qquad x \ge 0.$$

   In this case the optimal feedback control is

$$a^*(x) = 1.$$

Proof (*Theorem 3.2*)

1. The construction of f ensures that $f \in C^2((0,\infty))$ and the boundary conditions (3.4) hold. Because $\frac{1}{K^{-1}(x)}$ is decreasing and $\theta_- < 0$, the first derivative of f is polynomially growing, $f \in C_p((0,\infty))$. So we only have to show that our solution satisfies the HJB equation (3.3).

On the interval O, when the right hand side of (3.5) is in $(0,1)$, the HJB equation is equivalent to (3.6). With the restrictions on P we see from Lemma 3.1, that our construction of f ensures that f solves (3.6) and therefore (3.3). Lemma 3.1 also states $a^*(x)$.

If $x > x_1$ then f satisfies

$$\frac{\sigma^2}{2} f''(x) + (\mu) f'(x) - r f(x) + x = 0. \tag{3.36}$$

To prove that f also satisfies (3.3), we have to show that

$$\frac{\sigma^2 a^2}{2} f''(x) + (\mu - (1-a)\lambda) f'(x) - r f(x) + x < 0, \tag{3.37}$$

for every $a < 1$. Combining (3.36) and (3.37) we see that it is sufficient to prove that

$$F(a,x) := \frac{\sigma^2(1-a^2)}{2} f''(x) + \lambda(1-a) f'(x) \geq 0 \qquad \text{for all } a \in [0,1].$$

With the derivations

$$f'(x) = \frac{1}{r} + \frac{-\lambda\theta_-}{r(\sigma^2\theta_-^2 + \lambda\theta_-)} e^{\theta_-(x-x_1)} \qquad \text{and}$$
$$f''(x) = \frac{-\lambda\theta_-}{r(\sigma^2\theta_-^2 + \lambda\theta_-)} e^{\theta_-(x-x_1)},$$

we get

$$e^{-\theta_-(x-x_1)} F(a,x) = \frac{\sigma^2(1-a^2)}{2} \beta\theta_-^2 + \lambda(1-a)\left(\frac{e^{-}\theta_-(x-x_1)}{r} + \beta\theta_- \right), \tag{3.38}$$

where $\beta := \frac{-\lambda}{r\sigma^2\theta_-^2 + \lambda\theta_-}$.

We set

$$G(a,x) := \frac{\sigma^2(1-a^2)}{2} \beta\theta_-^2 + \lambda(1-a)(\frac{1}{r} + \beta\theta_-)$$

as a lower boundary for the right hand side of (3.38). Since we know from (3.24) that $\beta < 0$, we see from the derivatives of G with respect to a

$$G'(a,x) = -\sigma^2 a\beta\theta_-^2 - \lambda a\left(\frac{1}{r} + \beta\theta_- \right),$$
$$G''(a,x) = -\sigma^2\beta\theta_-^2,$$

that $G(.)$ is a convex function, because the second derivative is positive. Therefore its minimum is reached when the first derivative equals 0. From $G'(a^*,x) = 0$ we get

$$a^* = -\frac{\lambda\left(\frac{1}{r} + \beta\theta_-\right)}{\sigma^2\beta\theta_-^2} = 1,$$

when we plug in β. Finally from $F(1,x) = 0$ we obtain that $F(a,x) \geq 0$ for all $a \in [0,1]$.

2. For $\frac{2\mu-\lambda}{2r\alpha} < P < \frac{\mu}{r^2}$ we assume that $a^*(x) = 1$. With the boundary conditions (3.4) we get the constant for the homogenous solution for (3.20). We obtain

$$f(x) = \frac{\mu}{r^2} + \frac{x}{r} + \left(P - \frac{\mu}{r^2}\right) e^{\theta_- x}. \tag{3.39}$$

Obviously $f \in C^2((0,\infty)) \cap C_p((0,\infty))$. We have to show that f satisfies (3.3), which will implicitly also prove our conjecture of $a^*(.)$.

We insert the derivations of f into (3.5) and obtain

$$\begin{aligned} a(x) &= -\frac{\lambda f'(x)}{\sigma^2 f''(x)} = -\frac{\lambda}{\theta}\frac{\frac{1}{r} + (P - \frac{\mu}{r^2})\theta_- e^{\theta_- x}}{(P - \frac{\mu}{r^2})\theta_-^2 e^{\theta_- x}} \\ &= -\frac{\lambda}{\sigma^2 r (P - \frac{\mu}{r^2})\theta_-^2 e^{\theta_- x}} - \frac{\lambda}{\sigma^2 \theta_-}. \end{aligned}$$

So $a(.)$ is an increasing function in x and therefore $a(x) \geq a(0)$ for $x \geq 0$. Since $P \geq \frac{2\lambda-\mu}{2r\alpha}$ we establish the inequality

$$\begin{aligned} a(0) &= -\frac{\lambda}{\sigma^2 r (P - \frac{\mu}{r^2})\theta_-^2} - \frac{\lambda}{\sigma^2 \theta_-} \\ &\geq -\frac{\lambda}{\sigma^2 r (\frac{2\mu-\lambda}{2r\alpha} - \frac{\mu}{r^2})\theta_-^2} - \frac{\lambda}{\sigma^2 \theta_-} = -\frac{\lambda(2\alpha\mu\theta_- + \lambda r\theta_- - 2\mu r\theta_- - 2\alpha r)}{\sigma^2(2\alpha\mu + \lambda r - 2\mu r)\theta_-^2} = 1, \end{aligned}$$

and obtain $a(x) \geq 1$, where the last equality generates when we plug in the definitions of α and θ_-. Therefore $f(.)$ given by (3.39) satisfies (3.3).

□

Theorem 3.3 (*Solution to the HJB-equation 2*) If $\lambda \geq \mu + \sqrt{\mu^2 + 2r\sigma^2}$ and additional $P < \frac{\mu}{r^2}$, the solution to (3.3) is given by

$$f(x) = \frac{\mu}{r^2} + \frac{x}{r} + (P - \frac{\mu}{r^2})e^{\theta_- x}, \qquad x \geq 0,$$

where θ_- is given by (3.20). The optimal feedback control is

$$a^*(x) = 1.$$

Proof (*Theorem 3.3*) As in the proof of Theorem 3.2 2., we get the same solution for f and have to show that it satisfies (3.3). By the same calculations as before we know that

$$a(x) \geq a(0) = -\frac{\lambda}{\sigma^2 r (P - \frac{\mu}{r^2})\theta_-^2} - \frac{\lambda}{\sigma^2 \theta_-}. \tag{3.40}$$

Again we assumed that $a^*(x) = 1$ and now need to show that $a(0) \geq 1$. When we plug the first into (3.40) and transform the inequality we get

$$\frac{\lambda}{\sigma^2\theta_-} + 1 \geq -\frac{\lambda}{\sigma^2 r(P - \frac{\mu}{r^2})\theta_-^2}.$$

In a next step we have to prove that this holds for all $P < \frac{\mu}{r^2}$. Transformed once more we see that

$$\left(P - \frac{\mu}{r^2}\right)(\lambda\theta_- + \sigma^2\theta_-^2) \geq -\frac{\lambda}{r}.$$

By the definition of θ_- the condition $\lambda \geq \mu + \sqrt{\mu^2 + 2r\sigma^2}$ is equivalent to $\lambda \geq -\theta_-\sigma^2$. Therefore left hand side of the last inequality is positive and it holds. This proves the assumption to be correct. □

Theorem 3.4 (*Solution to the HJB-equation 3*) If $P = \frac{\mu}{r^2}$ the solution is given by

$$f(x) = \frac{\mu}{r^2} + \frac{x}{r}, \qquad x \geq 0.$$

The optimal feedback control in this case is

$$a^*(x) = 1.$$

Proof (*Theorem 3.4*) Since f is linear, it is an element of $C^2((0,\infty)) \cap C_p((0,\infty))$. Furthermore f fulfills the boundary conditions. From the derivations of f we get a new HJB equation

$$\max_{a\in[0,1]} \left((\mu - (1-a)\lambda)\frac{1}{r} - r\frac{\mu}{r^2} + \frac{x}{r} + x\right) = 0,$$

where it is easily seen that $a^*(x) = 1$. □

Theorem 3.5 (*Solution to the HJB-equation 4*) Now let $P > \frac{\mu}{r^2}$. If $\lambda \leq \frac{\mu+\sqrt{\mu^2+2r\sigma^2}}{2}$ the solution to (3.3) is given by

$$f(x) = \frac{\mu}{r^2} + \frac{x}{r} + (P - \frac{\mu}{r^2})e^{\theta_- x}, \qquad x \geq 0, \tag{3.41}$$

where θ_- is given by (3.20). The optimal feedback control is

$$a^*(x) = 1.$$

PROOF (*Theorem 3.5*) From the last Theorem 3.4 we conjecture that for P getting even greater, $a^*(x) = 1$ holds as well. With the boundary conditions we are able to formulate f. To show that it satisfies the HJB equation we only consider the part of (3.3) that contains a. It should be at maximum when $a(x) = 1$.

$$\frac{\sigma^2 a^2}{2} f''(x) + \lambda a f'(x) = \left(\frac{\sigma^2 a^2}{2} \left(P - \frac{\mu}{r^2} \right) \theta_-^2 + \lambda a \left(\frac{1}{r} e^{-\theta_- x} + \left(P - \frac{\mu}{r^2} \right) \theta_- \right) \right) e^{\theta_- x}$$

The last term is a quadratic polynomial in a which is convex, since the coefficient of a^2 is positive. Therefore the maximum in the interval $[0, 1]$ can only be obtained at a boundary point. At the point $a = 0$ the value of the function is 0. On the other hand at $a = 1$ because of the conditions on P and λ

$$\begin{aligned} \frac{\sigma^2}{2} f''(x) + \lambda f'(x) &= \left(\frac{\sigma^2}{2} \left(P - \frac{\mu}{r^2} \right) \theta_-^2 + \lambda \left(\frac{1}{r} e^{-\theta_- x} + \left(P - \frac{\mu}{r^2} \right) \theta_- \right) \right) e^{\theta_- x} \\ &\leq \left(P - \frac{\mu}{r^2} \right) \left(\frac{\sigma^2 \theta_-^2}{2} + \lambda \theta_- \right) e^{\theta_- x} \leq 0. \end{aligned}$$

This shows that

$$\arg \max_{0 \geq a \geq 1} \left(\frac{\sigma^2 a^2}{2} f''(x) + \lambda a f'(x) \right) = 1,$$

which verifies (3.3). □

REMARK 3.6 When $P > \frac{\mu}{r^2}$ and $\lambda > \frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{2}$ finding a solution is not ensured. But from the last theorem (3.5) it is seen, that when the starting variables P, μ, σ^2, r and x satisfy

$$\left(P - \frac{\mu}{r^2} \right) \left(\frac{\sigma^2 \theta_-^2}{2} + \lambda \theta_- \right) e^{\theta_- x} + \frac{\lambda}{r} \geq 0,$$

the solution is (3.41). Otherwise we cannot use the HJB equation to find a solution, but we still know that the optimal feedback control is

$$a^*(x) = 1.$$

REMARK 3.7 The special cases for $P = 0$ are already considered in the stated Theorems 3.2 and 3.3. They coinside with the results of Højgaard and Taksar [4], where no bankruptcy value was considered at all.
First, in the case of $0 = P < \frac{2\mu - \lambda}{2r\alpha}$, i.e. $\lambda < 2\mu$, the condition can be simplified to

$$\mu < \lambda < \min(2\mu, \mu + \sqrt{\mu^2 + 2r\sigma^2}) = 2\mu.$$

The solution to both the value function and the optimal control can be seen in Theorem 3.2 1.
Second, when $\frac{2\mu - \lambda}{2r\alpha} < P = 0 < \frac{\mu}{r^2}$, i.e. $\lambda \geq 2\mu$, Theorem 3.2 2 and Theorem 3.3 show the solutions.

Chapter 4

Analysis

This chapter focuses on the interpretation of the found solutions for the optimal return function and the control process. Maple 17 is used to model the process and provide graphical illustrations. In the first part of this chapter implementation challenges are described. The corresponding code is given in the Appendix.
Later, I focus on each variable in an economic way. Illustrations are given to show the influence of different values of the used variables.
In the last part of the chapter numerical calculations are done to check if the calculated control process is truly optimal. I use Monte-Carlo simulations for the surplus process for different distributions of the claim process.

4.1 Maple 17 Implementation

Using the *statistics* and *finance* package in *Maple 17*, most of the auxiliary functions of the solution can easily be implemented for a start. For example there are functions for the Gamma function as used in (3.12) (as probability function of the Gamma distribution: `PDF(GammaDistribution(.,.),.)`) or a numerical solve function (`fsolve`) for finding the solution for e^B in (3.32).

But when it comes to the complicated solution as stated in Lemma 3.2, the easiest implementation either takes a very long time or fails at all. As there is no closed form solution to the optimal return function $f(x)$ for $x < x_1$, many succesive (solving-)functions have to be called. Because of Maple trying to evaluate an equation first before solving it for an output parameter, the (numerical) solving operation can easily fail as there are too many unknown variables. This especially occurs, when successive solving-functions are called. Also using different input variable names at the declaration of a function can effect these successive functions to abort.

Some of these problems can be eliminated with the right nomenclature and by forcing every subfunction to be solved first before being evaluated in the next function. But especially the latter can lead to very long calculation time when a function is required to be evaluated at many points and each of this points requires more numerical evaluations of $g(.)$ and $H(.)$.

Calculating the inverse of $K(.)$ (3.14), which is defined as a definite integral, is such a problem. $K(.)$ has to be numerically evaluated at many points to find the value for $K^{-1}(.)$, where each point requires a numerical calculation

$$K(z) := \int_{e^B}^{z} [c(\mu - \lambda)H(u)g(u) + k_2 g(u)]du, \qquad \text{for } z > e^B.$$

With a different approach of differential equations and the corresponding implemented Maple functions, calculation time can be saved dramatically.
First, we consider the differential of the process $K(.)$

$$\frac{dK}{dz} = c(\mu - \lambda)H(z)g(z) + k_2 g(z),$$

with $K(e^B) = 0$.
We have shown before that the inverse exists, therefore we can consider $z(.)$ as a function of K as well. It satisfies the new differential equation

$$\frac{dz(K)}{dK} = \frac{1}{c(\mu - \lambda)H(z)g(z) + k_2 g(z)}, \tag{4.1}$$

with the initial condition

$$z(0) = e^B. \tag{4.2}$$

Using the Maple function `diff`, equation (4.1) can be implemented and then solved with the calling sequence `dsolve` for solving ordinary differential equations numerically considering the initial condition.
This works great for the cases of cheap reinsurance, when the denominator simplifies to $k_2 g(z)$. By replacing $K^{-1}(.)$ in the piecewise solution of the optimal return function with the solution of (4.1), significant computation time was saved in some of the following examples.

Unfortunately, in the case of non-cheap reinsurance Maple cannot solve the integral of $H(.)$ originally defined in (3.13)

$$H(z) := \int_{e^B}^{z} \frac{1}{u^2 g(u)} du, \qquad \text{for } z > e^B$$

in the numerical evaluation. But if $H(z(K))$ is differentiated for K, the solution gets numerically solvable,

$$\frac{dH(z)}{dK} = \frac{\frac{dz}{dK}}{z^2 g(z)}.$$

In a next step I differentiate (4.1) a second time to obtain a second order differential equation

$$\frac{d^2z}{dK^2} = \frac{\frac{dz}{dK}\left(c\lambda\frac{dH}{dK}g(z) - c\mu\frac{dH}{dK}g(z) + c\lambda H(z)\frac{dg}{dK} - c\mu H(z)\frac{dg}{dK} - k_2\frac{dg}{dK}\right)}{g(z)^2(c\lambda H(z) - c\mu H(z) - k_2)^2}. \tag{4.3}$$

To eliminate $H(z)$ in this equation, I solve (4.1) for it to get

$$H(z) = \frac{\frac{dz}{dK}g(z)k_2 - 1}{\frac{dz}{dK}cg(z)(\lambda - \mu)}. \tag{4.4}$$

Inserting the values of $H(.)$ and its differential into (4.3), I obtain

$$\frac{d^2z}{dK^2} = \frac{(c\lambda(\frac{dz}{dK})^2g(z) - c\mu(\frac{dz}{dK})^2g(z) - z^2\frac{dg}{dK})(\frac{dz}{dK})^2}{z^2g(z)}. \tag{4.5}$$

Since we know that $H(e^B) = 0$, we get another initial condition for this new simplified ODE (4.5) by evaluating (4.4) at this point

$$\frac{dz}{dK}(0) = \frac{1}{k_2g(e^B)}. \tag{4.6}$$

Maple still has problems evaluating equation (4.5) using `dsolve` because of the differential of $g(z(K))$,

$$\frac{dg(z)}{dK} = -\frac{\left(\frac{dz}{dK}\right)c^2e^{-zc}(-r+z)(zc)^{cr}}{z\Gamma(cr+1)}.$$

Now, as soon as this differential is inserted into (4.5), this this simplified ODE can be solved using `dsolve` with the two initial conditions (4.2) and (4.6) as a replacement for calculating $K^{-1}(.)$.

It has to be noted that this kind of numerical evaluation for non-cheap reinsurance models also leads to calculation errors.

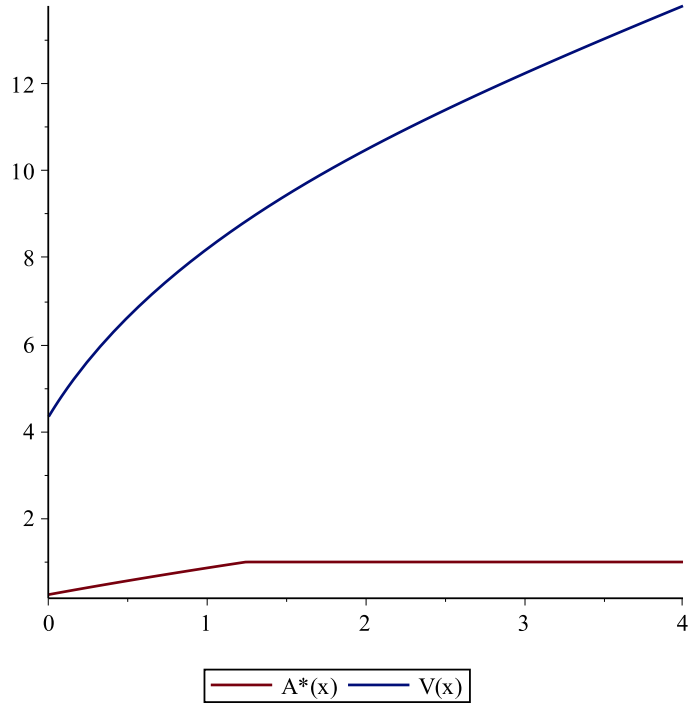

Figure 4.1: Value function for $\mu = 4, \lambda = 4, \sigma = \sqrt{10}, r = 0.7$ and $P = 4.33$

4.2 Economic Analysis

Here, the influence of each variable on the control process and the optimal return function are studied.

As a first introduction we look at an example from Taksar [12] with $\mu = 4, \lambda = 4, \sigma = \sqrt{10}, r = 0.7$ and $P = 4.33$. This is the case of cheap reinsurance as μ equals λ. The solution (3.2) is used for the calculation. Figure 4.1 shows the corresponding optimal control process and the value function for each capital x from 1 to 4. The point x_1, where the optimal control a^* changes to 1, is clearly seen.

Dependency on P

With non-cheap reinsurance as discussed in this paper, many different cases can occur. In a first step I discuss the dependency on the bankruptcy value P.
As it is clearly seen, the solution of the optimal return function and the control process are highly dependent on P. The more we know about the financial situation of the insurance company at the time of bankruptcy for the examined product line or class of risks, the more we know about the reinsurance strategy and the optimal return function.
In Theorem 3.4 we saw that $a^*(x) = 1$ for $x \geq 0$ if $P = \frac{\mu}{r^2}$, i.e. the insurer is not going to reinsure at all and bears all the risk himself but also retains all the potential profit. The suggestion, that if P gets even greater no reinsurance is concluded as well, was shown in Theorem 3.5 and Remark 3.6. But we can also look at this in a more reality relevant expected value perspective.

If P is high ($P \geq \frac{\mu}{r^2}$) we assume that the reinsurance factor is $a^*(x) = 1$ for $x \geq 0$ and donate this strategy as π_1. Let π_2 be the optimal strategy in this case. In an expected value approach we get

$$\begin{aligned} E\left(\int_0^{\tau_{\pi_1}} e^{-rt} R^{\pi_1} dt + e^{-r\tau_{\pi_1}} P\right) =& E\left(\int_0^{\tau_{\pi_1}} e^{-rt} R^{\pi_1} dt + e^{-r\tau_{\pi_1}} \frac{\mu}{r^2}\right) + E\left(e^{-r\tau_{\pi_1}} \left(P - \frac{\mu}{r^2}\right)\right) \\ \leq& E\left(\int_0^{\tau_{\pi_2}} e^{-rt} R^{\pi_2} dt + e^{-r\tau_{\pi_2}} \frac{\mu}{r^2}\right) + E\left(e^{-r\tau_{\pi_2}} \left(P - \frac{\mu}{r^2}\right)\right) \\ \leq& E\left(\int_0^{\tau_{\pi_1}} e^{-rt} R^{\pi_1} dt + e^{-r\tau_{\pi_1}} \frac{\mu}{r^2}\right) + E\left(e^{-r\tau_{\pi_2}} \left(P - \frac{\mu}{r^2}\right)\right) \end{aligned}$$

The last inequality holds because we know that if $P = \frac{\mu}{r^2}$ the optimal reinsurance strategy is π_1, i.e. to not reinsure at all. Therefore the focus is on $P - \frac{\mu}{r^2}$. From the inequalities we get

$$E\left(e^{-r\tau_{\pi_1}} \left(P - \frac{\mu}{r^2}\right)\right) \leq E\left(e^{-r\tau_{\pi_2}} \left(P - \frac{\mu}{r^2}\right)\right), \tag{4.7}$$

which is only possible if $\tau_{\pi_2} \leq \tau_{\pi_1}$. If the optimal strategy π_2 does not equal π_1, it means that some reinsurance is taken. But (4.7) shows that this would speed up the time of ruin of the insurance company, which is in contrast to the objective of reinsurance, decreasing the risk. Therefore we see that the insurance company would not reinsure and $\pi_2 = \pi_1$ for $P \geq \frac{\mu}{r^2}$.
On the other hand for very small values of $P \to 0$ the amount of reinsurance to be taken depends on the relationsship of μ and λ. This was described in Remark 3.7. In the paper by Taksar and Hunderup [12] only cheap reinsurance was considered. That is why for bankruptcy values $P \leq 0$ the probability of hitting ruin equals 0, because no risk is taken at all. In the case of non-cheap reinsurance this statement is not valid anymore, since ruin would also be hit because of the higher reinsurance premium.

In the next step I focus on the optimal return function. Plotting the function using Maple, the influence of P is clearly seen. While other papers (e.g. Højgaard and Taksar [4]) set $P = 0$, the return functions were all concave. I our case, the optimal return function can be of every type. It can be convex, linear or concave, depending on the bankruptcy value P. This is seen in Figure 4.2, where we see three calculations of $V(x)$ for different values of P.
We also see in this Figure, that in a certain point the value function for $P \geq \frac{\mu}{r^2}$ is not monotone and increasing anymore. Setting the first derivative of solution (3.41) of Theorem 3.5 zero, we see that if

$$P > \frac{\mu}{r^2} - \frac{1}{r\theta_-}$$

V is decreasing in a neighbourhood of 0. Economically this means that because of a high value of P, immediate ruin gains a higher value for the company than running business. Since this is not an option in this model, because of the discount factor r the present value of future profits and the discounted bankruptcy value P are lower than the current or starting value x.

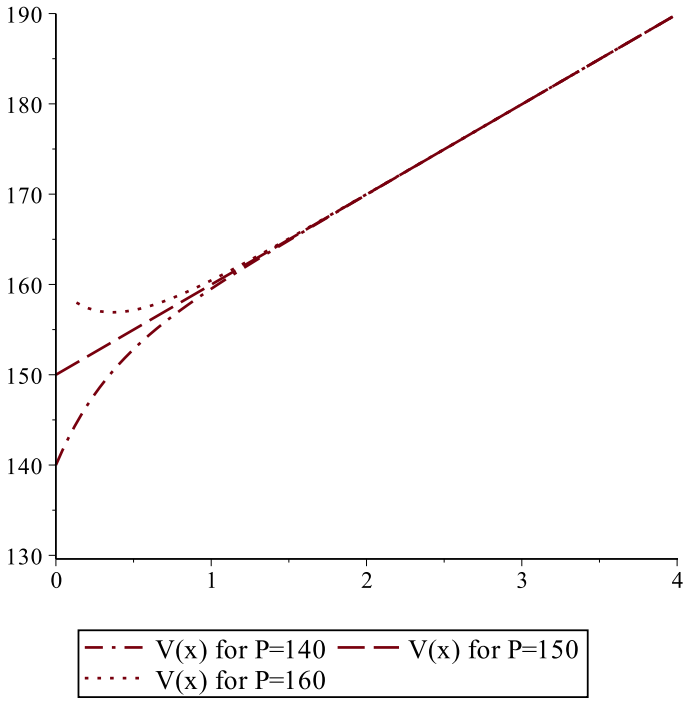

Figure 4.2: Value function for $\mu = 1.5, \lambda = 3, \sigma = 1, r = 0.1$ and $P = \{140, 150, 160\}$

In all cases the optimal control is $a^* = 1$. When $P = 160$ there is no explicit solution given for $x \geq 0$. $V(x)$ can only be plotted for the domain where

$$\left(P - \frac{\mu}{r^2}\right)\left(\frac{\sigma^2\theta_-^2}{2} + \lambda\theta_-\right)e^{\theta_- x} + \frac{\lambda}{r} \geq 0,$$

see Remark 3.6.
In Figure 4.2 it is clearly seen, that the influence of P on the value function is stronger, the smaller the actual or starting value x is. For a company with a huge initial capital the bankruptcy value rarely affects the control policy and the corresponding optimal return. The greater x gets, the more similar the different versions of $V(x)$ get.

Dependency on σ

If σ converges to 0, we expect from the definition of the risk process R_t the value function to approach to the linear function $\frac{\mu}{r^2} + \frac{x}{r}$. Obviously, in this case no reinsurance is needed as it is a non-stochastic function.
In the case where $P \geq \frac{\mu}{r^2}$ no reinsurance is going to be taken at all, but the smaller σ gets, the faster $V(x)$ converges to this linear function. An interpretation of the strong convexity in such a case would be, that the bankruptcy value P is higher than the expected return, which was also discussed at the dependency on P. This is seen in Figure 4.3. This also occurs because of the approximation as a diffusion process, as we will see in the next Section 4.3. For simplicity, again the case considering cheap reinsurance is used.
The more risk has to be taken when offering insurance policies, i.e. the greater σ gets, the more reinsurance is taken for small initial capital x. This can be seen in Figure 4.4.

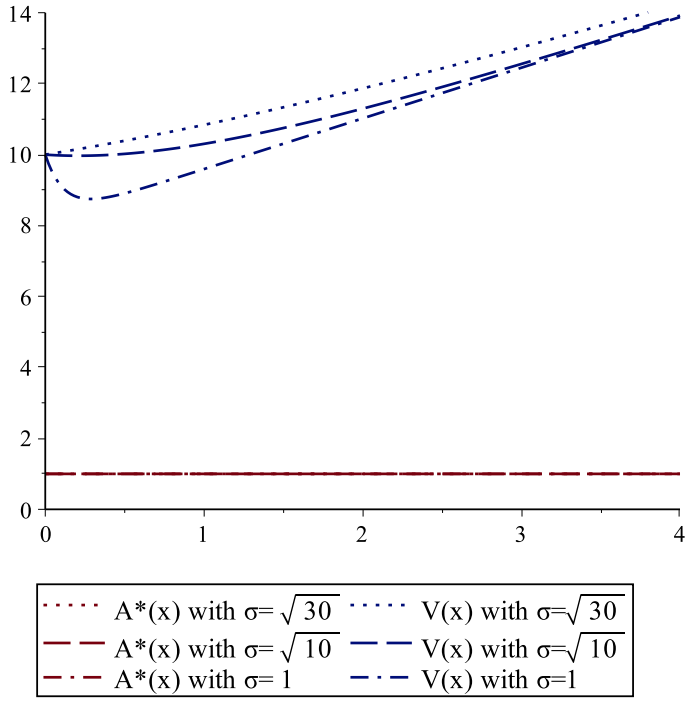

Figure 4.3: Value function for $\mu = \lambda = 4, r = 0.7, P = 10$ and $\sigma = \{\sqrt{30}, \sqrt{10}, 1\}$

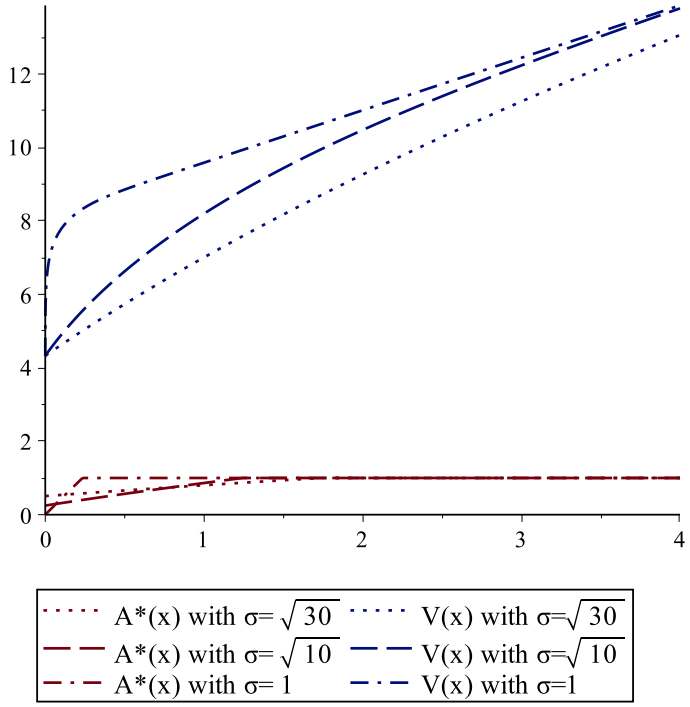

Figure 4.4: Value function for $\mu = \lambda = 4, r = 0.7, P = 4.33$ and $\sigma = \{\sqrt{30}, \sqrt{10}, 1\}$

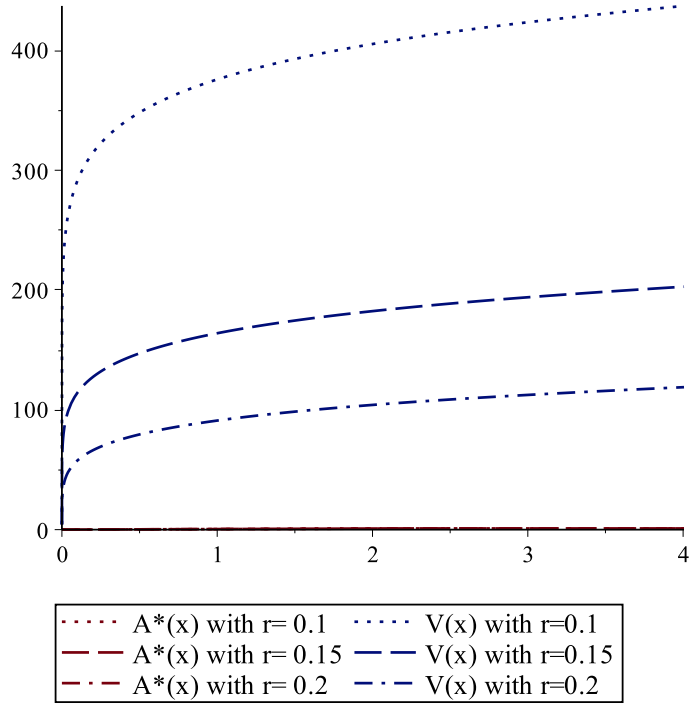

Figure 4.5: Value function for $\mu = \lambda = 4, P = 4.33, \sigma = \sqrt{10}$ and $r = \{0.1, 0.15, 0.2\}$

Dependency on r

The discount factor r describes the importance of future payments in our model. As seen in the different solutions, the optimal return function tends to converge to $f(x) = \frac{\mu}{r^2} + \frac{x}{r}$. The smaller r gets, the greater optimal return is calculated for any given starting capital x. This property intensifies even more when the volatility σ decreases. A small σ makes ruin in the future more unlikely, therefore the process gains value very fast. The higher x is, the more unlikely ruin gets. Figure 4.5 shows this for different values of r.

Dependency on μ and λ

The influence on the value function of μ and λ, or particularly between the original premium rate p and the reinsurance premium rate p_2, is mainly dependent on μ. Recapitulate, that $\mu = p - \beta m$ and $\lambda = p_2 - \beta m$. The ratio of P and μ describes if reinsurance is needed in the first place. But even if $P < \frac{\mu}{r^2}$, reinsurance is only concluded if $\lambda < 2\mu$, which is seen in Theorem 3.2. The more expensive the reinsurance premium gets, the less likely reinsurance is going to be used. Transformed into the original premium rates, reinsurance could only be necessary if $p_2 < 2p - \beta m$, where βm describes the amount of expected claim size up to time $t = 1$.

4.3 Numerical Calculations

In this section I compare the calculated optimal solution to numerical illustrations of the original process of the used model. I use Maple 17 to describe the Cramer-Lundberg-Model of the risk process (3.1). Different assumptions on the underlying distributions are taken, which were not relevant for the theoretical solution. Monte Carlo simulation is used to test different outcomes of the discrete return function. By applying custom reinsurance strategies I am able to compare the corresponding return function to the above result of the optimal return function and study the efficiency of the optimal control.

The risk process and the stated solution depend on the two assumptions of the claim size and claim occurence distribution. Since we studied in the setting of the Cramer-Lundber-Model, the aggregated claim size model depends on a compound Poisson process. As stated before the number of claims up to time t is Poisson distributed.

$$\mathbb{P}(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, 2, \dots$$

Other discrete distributions of the claim number could be negativ-binomial, logarithmic, geometric or binomial. As a way of deciding if the Poisson distribution fits to the observed number or another distribution would fit more, one can use the *index of dispersion* or *variance-to-mean ratio (VMR)*. It is defined as the ration between variance σ^2 and mean μ

$$VMR = \frac{\sigma^2}{\mu}.$$

For the Poisson distribution $VMR = 1$ applies, as it has equal variance and mean. If $VMR < 1$ the binomial distribution would be a guess for the number of claims, if $VMR > 1$ the negative binomial distribution fits more.
Since finding a solution for the optimal value function takes the Poisson distribution as a starting point, in the following it will be set fixed as claim occurence distribution.

The presets for claim size distribution are not that strict. Since the CLM does not depend on a certain distribution, studies are done for different claim distributions. These distributions can be for example

- Gamma distribution
- Exponential distribution (Special case of gamma distribution as a rough model for small claims in household, third-party vehicle insurance or automobile insurance)
- Pareto, Burr, Log-Gamma or Log-Normal distribution (heavy-tailed distributions for problems where large claims may occur)
- Inverse Gaussian distribution

The following studies will focus on differences how these distributions influence the model. For better comparison the same expected value E and variation σ^2 for all different types

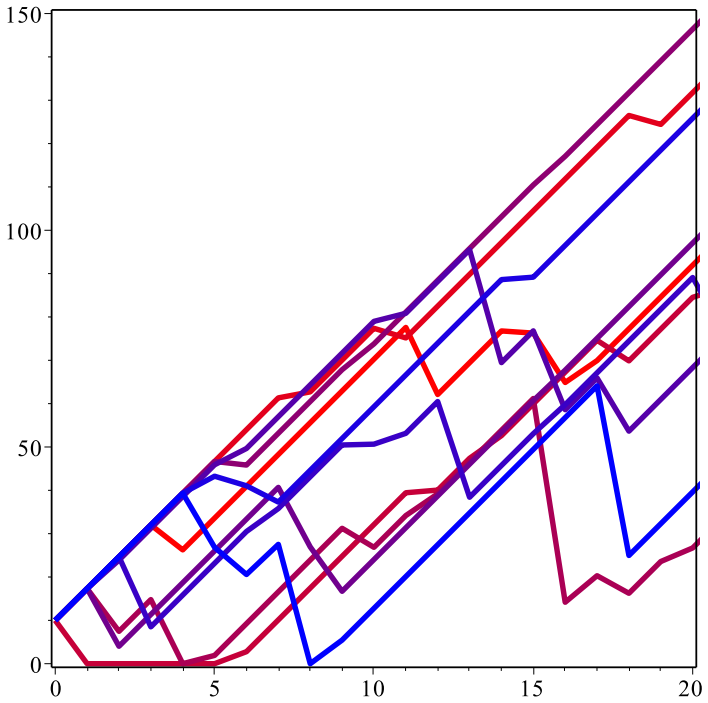

Figure 4.6: 10 simulations of the risk process R_t. $p_1 = p_2 = 7.33$, $m = 10$, $\beta = 0.33$ and $x = 10$.

of distributions is used.

First, I will use exponential distribution to describe the way to model the Monte Carlo simulation of the risk process.

Parameters are set as in simple example calculations before. Premium costs in the calculated solution are $\mu = \lambda = 4$. Therefore, with the notation introduced in Chapter 3, premium costs in CLM result in $p_1 = p_2 = 7.33$. Claim distribution is set with mean $m = 10$, the claim number with intensity $\beta = 0.33$. Thus, the second moment of the exponential distribution equals $s^2 = 200$.

The original discrete risk process $R(t)$ (3.1) is modelled as a function in Maple. Figure 4.6 shows 10 simulations of $R(t)$ starting at $x = 10$ with no reinsurance and a minimisation at 0. The first time that any of these sample paths hits 0 is the corresponding ruin time τ_i for simulation i $(i = 1, \ldots, 10)$. In our simplified example, up to time 20 only 3 simulations hit ruin.

As a next step I model the corresponding value function, to compare the result with the calculated solution. First the stopping time τ_i for each simulation is calculated, then the cash value at time $t = 0$ is computed by discounting the values of $R_i(t)$ up to τ_i with a given discount factor r. At the time of ruin the bankruptcy value P is added if $\tau_i \neq \infty$ and $\tau_i \leq N$, where N is the limit of t for simulation purposes.
Figure 4.7 shows the net values of the 10 simulations of Figure 4.6. The simulation path

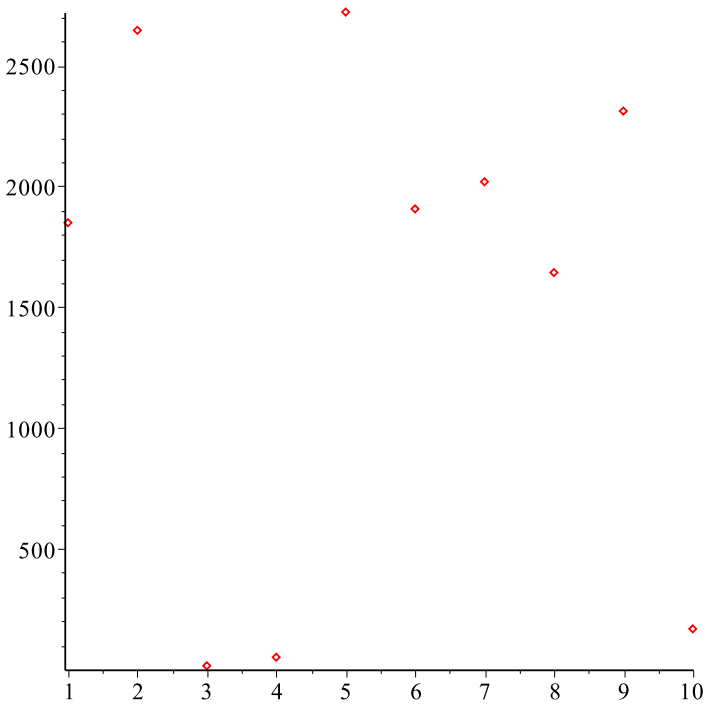

Figure 4.7: Net values of 10 Simulations of the risk process R_t. $p_1 = p_2 = 7.33$, $m = 10$, $\beta = 0.33$, $x = 10$, $P = 4.33$ and $r = 0.05$.

has been extended for t to $N = 1000$, which one can argue that $\tau_i = \infty$ if t hits N, as with the given discount factor $r = 0.05$ values of $R(t)$ get neglectable.

As the last step, the mean of the simulated sample value functions describe the value of the return function at $x = 10$, in this case $J_{10} = 1532.30$.

Repeating these steps, values for all positive starting points x can be calculated. Figure 4.8 shows a pointplot of the value or return function J for $x = 0, \ldots, 20$ at timesteps of 0.1. For each value of x 500 simulations were made to calculate the mean.
By definition ruin is hit immediately at the starting point $x = 0$, therefore at this point the simulation equals $P = 4.33$. With the given parameters, the probability of hitting ruin is very small. That is why for each postitive x the mean of the Monte Carlo simulation converges to a value starting around 1200, where future profits get neglectable because of the high discount rate.

It has to be remembered, that in this simulation no reinsurance is taken at all, thus $a = 1$. Figure 4.9 shows the calculated optimal control function a^* for the given parameters.

The easiest implementation of a reinsurance share is to adjust the control level in the Monte Carlo simulation analogous a^* for each starting point x and keep it at the same level for the ongoing simulation of R_t in time. This way a great level of reinsurance is taken for small values of x. As it is not changed, the insurance company only gains a small amount of profit, which can be seen in Figure 4.10.

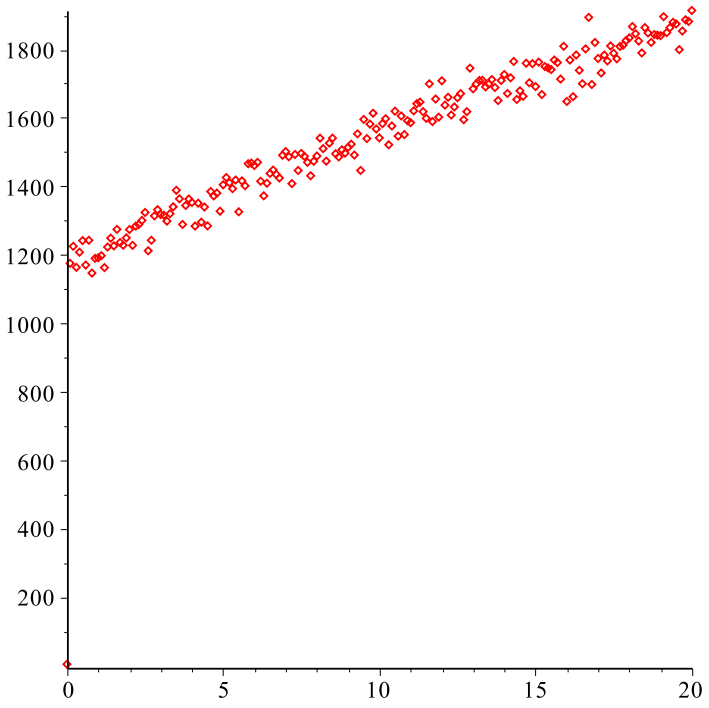

Figure 4.8: Simulation of the value function J without reinsurance for $x = 0, \ldots, 20$. $p_1 = p_2 = 7.33$, $m = 10$, $\beta = 0.33$, $x = 10$, $P = 4.33$ and $r = 0.05$.

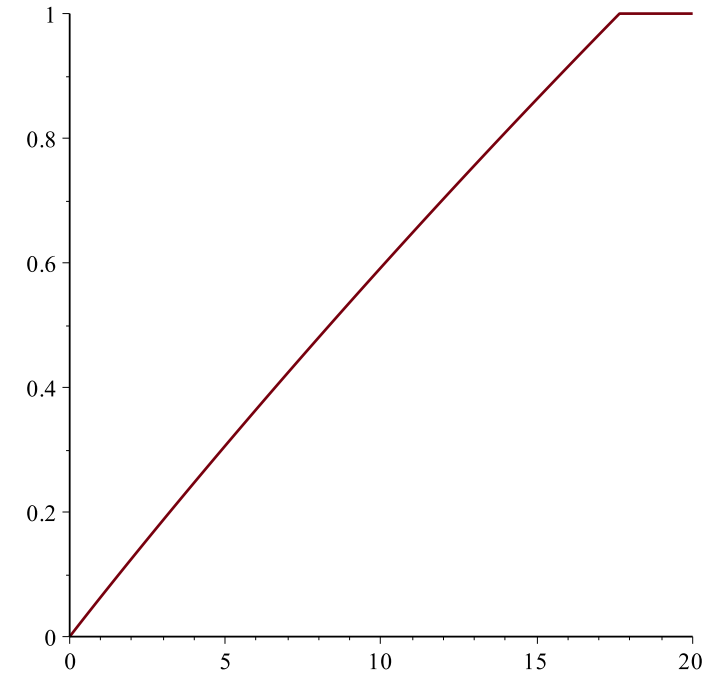

Figure 4.9: Optimal control a^* for the optimal return function of Figure 4.8.

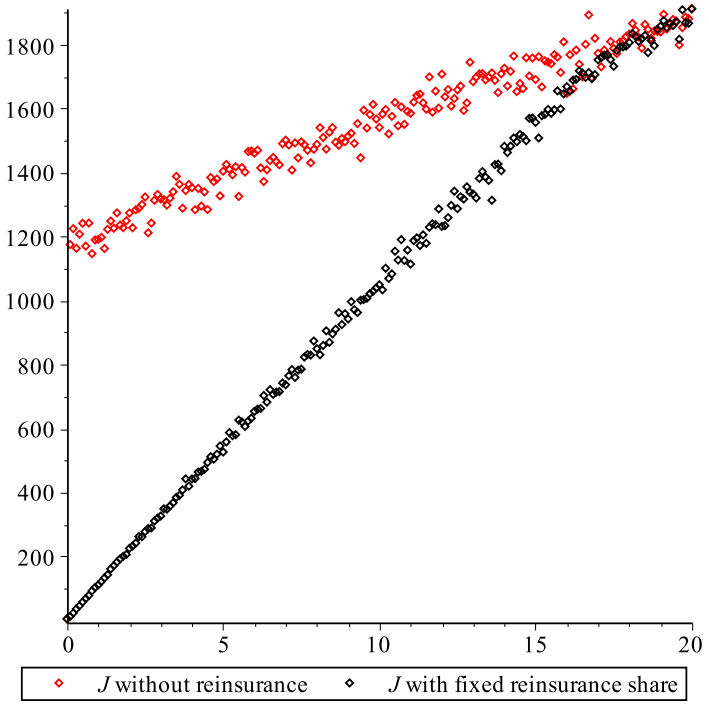

Figure 4.10: Comparison of the simulated value function of Figure 4.8 with a simulated value function with fixed control level.

This is not the intended control mechanism. Therefore the next step of the simulation should take the optimal strategy into account at every time step. This can only be done recursively, as in every simulated path of the risk process $R(t)$ the proportion of reinsurance to be taken depends on the value of $R(t-1)$. This is why it is not possible anymore to model the risk process using a pre-defined Maple function.
In a first step I simulate a sample path of the stochastic sum of claims, which is defined by a Poisson process. 10 simulations of the sum of claims can be seen in Figure 4.11.
Second, I calculate the discrete risk process step by step until it hits 0 or ruin for the first time. This time point is saved as the corresponding ruin time. These calculations take a long time, therefore only 100 simulations are done for each starting point x. Furthermore, it is necessary to clean the internal memory of Maple during the calculations. That is why after every 50 starting points I run the garbage collecting function `gc()`, which deletes all data to which no references are made. This operation ist also quite time consuming.
Figure 4.12 shows the comparison of this recursively controlled function with the simulation of no reinsurance.

It is clearly seen that this recursively controlled value function yields to higher values for small starting values. This is the expected behavior, as it is more likely for small starting values to hit ruin when no reinsurance is taken at all. The higher the starting values get, the more the two functions approach. With the given parameters it is very unlikely to hit ruin with a starting value when no reinsurance is taken at the beginning. That is why to the end of the simulation the two plots of the functions are hard to differentiate.

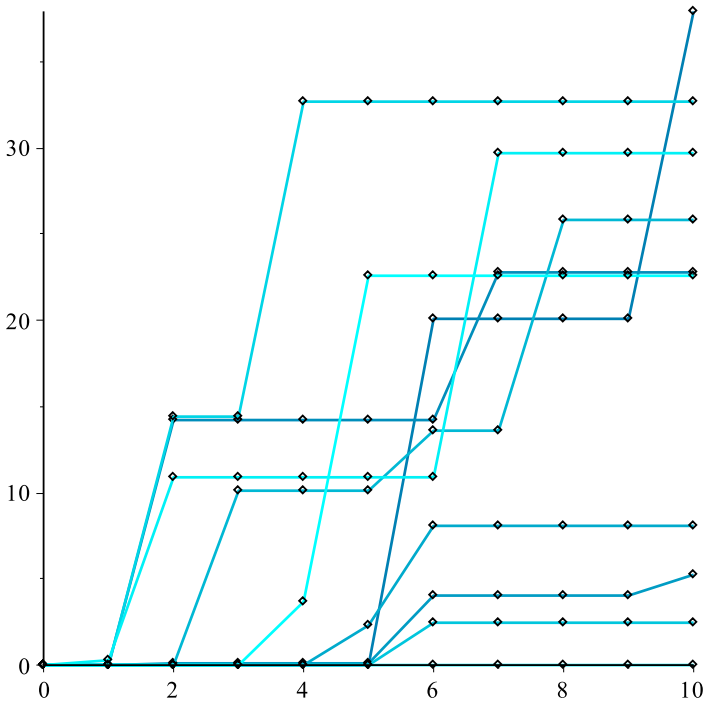

Figure 4.11: 10 Simulations of the claim sum. $m = 10$ and $\beta = 0.33$.

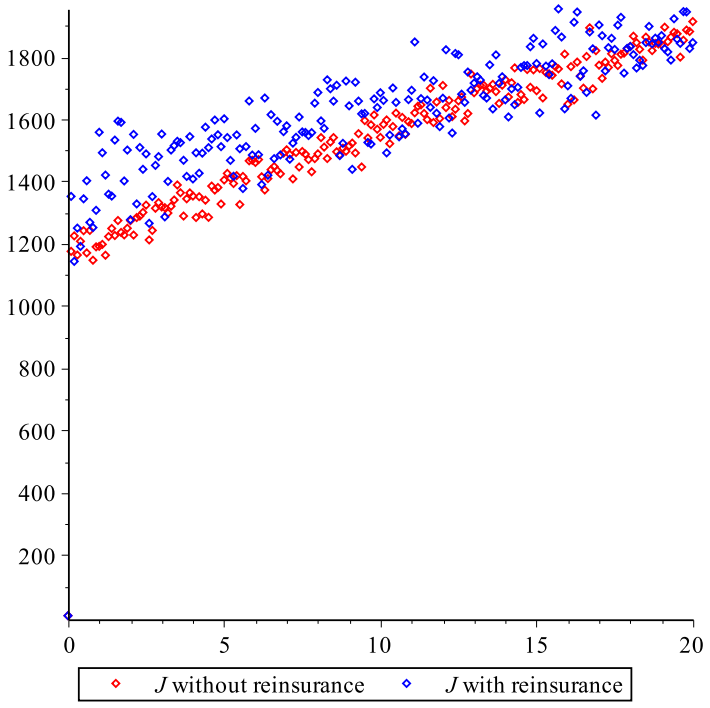

Figure 4.12: Comparison of the simulated value function of Figure 4.8 with a simulated value function with recursive optimal control.

Up to this point all simulations are done with exponential claim distribution. In the next step I will check for major differences if different claim distributions are chosen.
The same calculation steps are repeated with the claim size following a log-normal distribution and an inverse Gaussian distribution. For better comparison, I will choose parameters in a way that they will have the same expected value $m = 10$ and second moment $s^2 = 200$. The intensity of the Poisson process is locked at $\beta = 0.33$.

First, I will focus on log-normal distributed claims.
The first and second moment of the log-normal distribution $\mathcal{LN}(\alpha_{LN}, \beta_{LN})$ are

$$m = e^{\alpha_{LN} + \frac{1}{2}\beta_{LN}^2} \quad \text{and}$$
$$s^2 = e^{2\beta_{LN}^2 + 2\alpha_{LN}}.$$

When the values of m and s^2 are set, the corresponding parameters are calculated as

$$\alpha_{LN} = -\frac{1}{2}\ln\left(\frac{s^2}{m^2}\right) + \ln(m) \quad \text{and}$$
$$\beta_{LN} = \sqrt{\ln\left(\frac{s^2}{m^2}\right)}.$$

In this example the parameters are set as $\alpha_{LN} = 1.9560$ and $\beta_{LN} = 0.8326$. All steps for calculating and simulating a recursively controlled value function are repeated. Figure 4.13 shows the comparison to the previously calculated non-controlled value function. The result is similar as in Figure 4.12 with exponential distributed claims.

Next, these calculations are repeated when the claims are supposed to be invers Gaussian distributed $\mathcal{IG}(\alpha_{IG}, \beta_{IG})$. The first and second moment are

$$m = \alpha_{IG} \quad \text{and}$$
$$s^2 = \frac{\alpha_{IG}^2(\alpha_{IG} + \beta_{IG})}{\beta_{IG}}.$$

As before the parameters are calculated using

$$\alpha_{IG} = m \quad \text{and}$$
$$\beta_{IG} = -\frac{m^3}{m^2 - s^2}$$

to get $\alpha_{IG} = 10$ and $\beta_{IG} = 10$. This simulation yields to the same results as before. The comparison with the non-controlled value function is seen in Figure 4.14.

Now I can compare the results of the Monte-Carlo simulations to the calculated optimal return function V from Section 3.2.

Figure 4.15 shows the different graphs of all simulated value functions with different kinds of claim distributions as well as the continuous optimal return function.

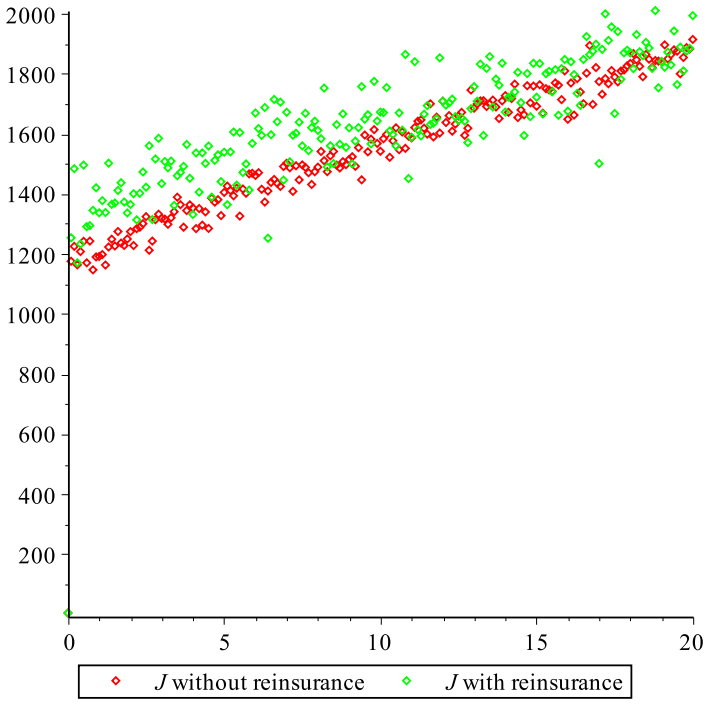

Figure 4.13: Comparison of the simulated value function of Figure 4.8 with a simulated value function with recursive optimal control and log-normal distributed claims.

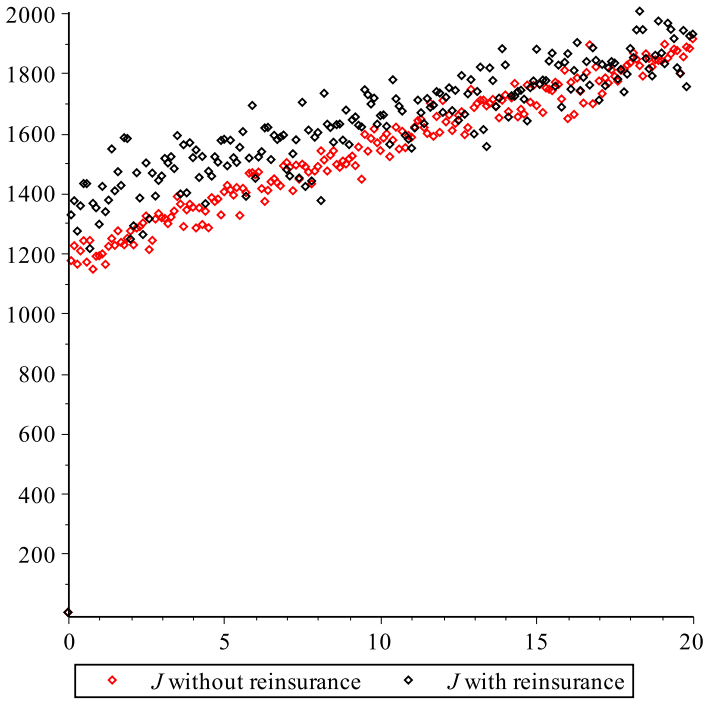

Figure 4.14: Comparison of the simulated value function of Figure 4.8 with a simulated value function with recursive optimal control and inverse Gaussian distributed claims.

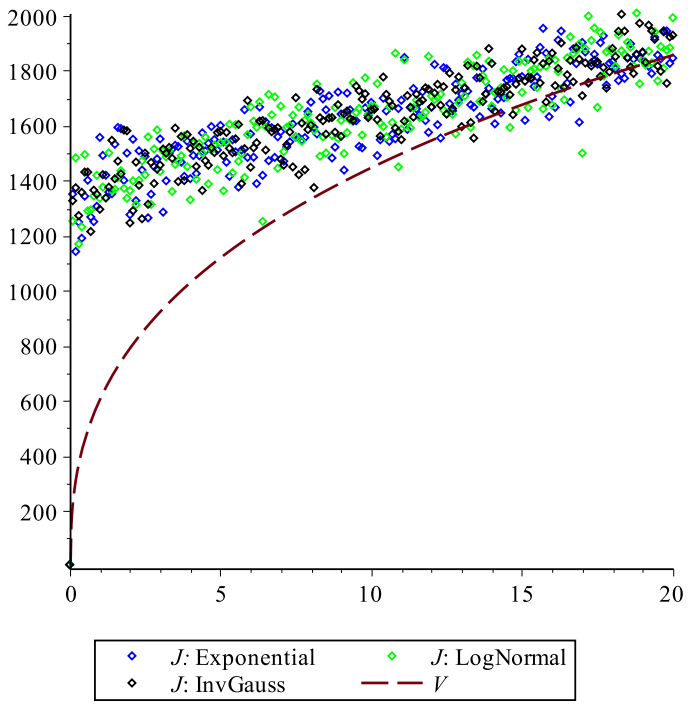

Figure 4.15: Comparison of the simulated value function of Figure 4.8 with the corresponding optimal return function.

First, this shows that all simulations are very similar with no recognizable differences depending on the claim size distribution. This was assumed in the last figures.
The comparison with the optimal return function shows the common starting value, as by definition all plots start at $P = 4.33$ for $x = 0$. For small starting values the simulated functions gain higher values than the continuous function. So one can argue, that the optimal return function V is not optimal. But we know from Section 2.2 that the approximation of the discrete risk process as a diffusion process gets better the higher the starting value x gets. As this is used finding the solution of the optimal return function, a good fit at small values of x is not expected. This is also seen in the graph. The higher the starting value is, the better the calculated optimal solution coincides with the simulation.

It has to be noted, that these examples only considered cheap reinsurance, but the interesting case where a reinsurance share is concluded.

Next, the simulation is also done for the case described in Figure 4.2 with $P = 160$, where no reinsurance is taken at all, but the optimal value function is convex. As before, the parameters of the gamma distribution and the Poisson process are chosen in a way to fit the parameter values of the approximated solution. Figure 4.16 shows the comparison of the optimal return function and the Monte Carlo simulation. As described in the last Section for Figure 4.2, the optimal return function is descending in the neighbourhood of 0. Because the risk of hitting ruin is so low, the bankruptcy value does not have great impact on the wealth of the company when calculating the discounted reserve.
This is immediately seen in the simulation, where the return of the insurance company

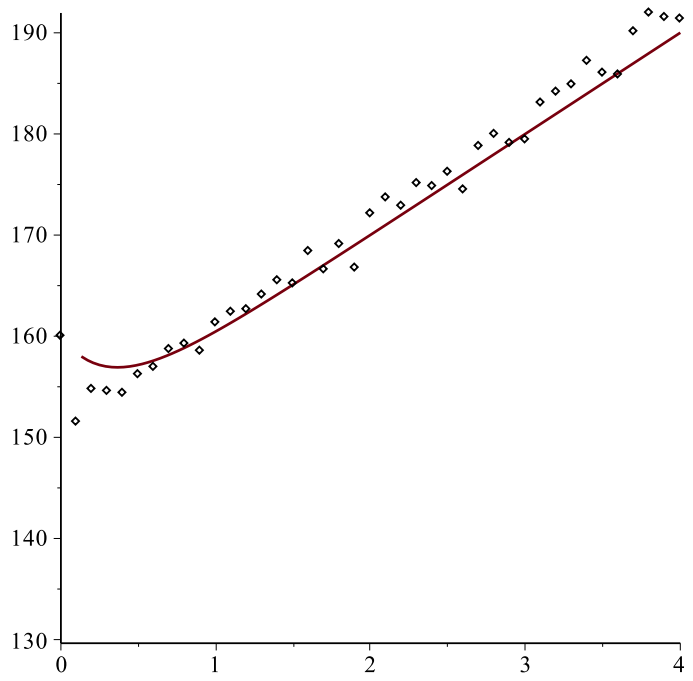

Figure 4.16: Comparison of the optimal return function of Figure 4.2 with $P = 160$ and the corresponding simulated value function.

is increasing for any value starting value $x > 0$. Only at $x = 0$, where ruin is hit by definition, the function and the simulation coincide at the bankruptcy value P. The decreasing part of the return function, which is a result of the approximation, cannot be simulated.

Chapter 5

Conclusion

The objective of this paper is to maximize the total discounted reserve of an insurance company including a bankruptcy value by controlling the reinsurance factor. Non-cheap reinsurance is considered, that means the reinsurance premium λ can be higher than the premium μ. By approximating the value process with a diffusion process a solution was found. Depending on the variables, especially the bankruptcy value P and the proportion between μ and λ, different cases have to be considered.

Theorems 3.2, 3.3, 3.4 and 3.5 state the results of the optimal risk control and also give a solution of the optimal return function.
In most of the cases no reinsurance is taken at all, be it because the bankruptcy value when hitting ruin is too big or the reinsurance premium is too high. In these cases the optimal return functions can be of different types, either concave, linear or convex. The influence of each parameter on the solution is discussed in detail in Section 4.2.

In the second part of this paper I studied whether the theoretical solution of optimal control has positive impact on any 'real' Monte Carlo simulation.
Because there is no closed-form solution of the optimal control, a relatively fast implementation is derived using ordinary differential equations. Building on that, a recursive Monte Carlo simulation is calculated that takes the reinsurance share depending on the current wealth of the company into account at every time step. This model is then compared with different calculated optimal solutions of Section 3.2.

The biggest differences between the calculated optimal value function and the simulated risk processes occur for small starting values x, while they tend to coincide for large x. This is an assumed behavior because of the approximation in Section 2.2 that was used deriving the solution.
On the other hand the positive neighbourhood of 0 is also the area of interest, where the risk to hit ruin is the highest as the wealth of the insurance company is still very small. In certain cases ($x < x_1$, depending on P, μ and λ) reinsurance has to be taken, which is generally not the case for high starting values.
One can argue, that the calculation of the optimal control also depends on the approximation as a diffusion process and therefore the optimal value function. This is why the solution for controlling the model at small starting values is questionable.

But as the comparison with the Monte-Carlo simulation showed, using the optimal control strategy on the discrete risk model gains a positive effect. This is clearly seen in Figure 4.12 and the following. So although the statement of the optimal return function is not valid for small starting values, the calculated control can be used.

Appendix

Maple Code

Maple 17 was used for all calculations and generated figures. Here the corresponding code is given.
The following packages were loaded to use build-in commands.

```
with(Statistics); with(plots); with(finance); with(ListTools);
```

The value function V and optimal control a^* are defined for all cases. As a start, the following code was used as a straight-forward implementation using given Maple functions. As described in Section 4.1, numerical calculation of K^{-1} consumes a lot of time.

```
if lambda < mu + sqrt(mu^2 + 2*r*sigma^2) and P < (2*mu - lambda
    )/(2*r*alpha) then
    g := X -> PDF(GammaDistribution(1/c, c*r+1), X):
    expB := fsolve(lambda/2*(g(x)/(alpha*g(alpha))+2*(lambda-mu)
        /lambda * g(x) * int(1/(y^2 * g(y)),y = x .. alpha))*x +(
        mu-lambda) -r*P*x=0, x = alpha/2):
    H := Z -> int(1/(y^2 * g(y)),y = expB .. Z):
    K := X -> int(c * (mu - lambda) * H(y) * g(y) + k2 * g(y), y
         = expB .. X):
    K_inv := Y -> fsolve(K(X) = Y, X=1):

    k2 := sigma^2/(mu*alpha*g(alpha)) - c*(mu-lambda)*H(alpha):
    x1 := K(alpha):

    k := X -> c*(mu - lambda) * H(X) * g(X) + k2 * g(X):

    A_ast := X -> piecewise(X < 0, 0, X < x1, lambda/sigma^2 * k
        (K_inv(X)) * K_inv(X),1);
    V := X -> piecewise(X < 0, 0, X < x1, int(1/K_inv, 0..X )+P,
         X/r+mu/r^2 -lambda/(r*(sigma^2 * theta^2 +lambda* theta)
        ) * exp(theta*(X-x1)));

elif P < mu/(r^2) then
    A_ast := X -> 1:
    V := X -> X/r + mu/r^2 + (P - mu/r^2) * exp(theta*X);
```

```
elif P = mu/(r^2) then
    A_ast := X -> 1:
    V := X -> X/r + mu/r^2;

elif P > mu/r^2 and lambda <= (mu + sqrt(mu^2 + 2*r*sigma^2))/2
   then
    A_ast := X -> 1:
    V := X -> X/r + mu/r^2 + (P - mu/r^2) * exp(theta*X);

else
    A_ast := X -> 1:
    V := X -> piecewise((P-mu/r^2)*((sigma^2*theta^2)/2 + lambda
       *theta)* exp(theta*X) + lambda/r >= 0,X/r + mu/r^2 + (P -
        mu/r^2) * exp(theta*X));

end if:
```

Next, the solving method using ODEs as described in Section 4.1 is used. A case differentiation has to be done for cheap and non-cheap reinsurance.
This definition of V and a^* is saved as external file `solution_wu_new.txt` that is later read by Maple for easier calculations.

```
if lambda < mu + sqrt(mu^2 + 2*r*sigma^2) and P < (2*mu - lambda
   )/(2*r*alpha) then
    g := X1 -> evalf(PDF(GammaDistribution(1/c, c*r+1), X1)):
    expB := fsolve(lambda/2*(g(x)/(alpha*g(alpha))+2*(lambda-mu)
       /lambda * g(x) * int(1/(y^2 * g(y)),y = x .. alpha))*x +(
       mu-lambda) -r*P*x=0, x = alpha/2):
    H := Z -> evalf(Int(1/(y1^2 * g(y1)),y1 = expB .. Z)):
    K := X2 -> evalf(Int(c * (mu - lambda) * H(y) * g(y) + k2 *
       g(y), y = expB .. X2)):

    k2 := sigma^2/(lambda*alpha*g(alpha)) - c*(mu-lambda)*H(
       alpha):
    x1 := evalf(K(alpha)):

    if mu = lambda then
      ode := diff(x(F), F) = 1/(k2*g(x(F))):
      soln := dsolve({ode, x(0) = expB}, x(F), numeric, output=
         listprocedure):
    else
      ode := diff(x(F), F, F) = (c*(D(x))(F)^2*lambda*g(x(F))-c
         *(D(x))(F)^2*mu*g(x(F))-x(F)^2*piecewise(x(F) < 0., 0,
         -1.*(D(x))(F)*c^2*exp(-1.*x(F)*c)*(-r+x(F))*(x(F)*c)^(c
         *r)/(x(F)*GAMMA(c*r+1.))))*(D(x))(F)^2/(g(x(F))*x(F)^2)
         :
      soln := dsolve({ode, x(0) = expB, (D(x))(0) = 1/(k2*g(expB
         ))}, x(F), numeric, output = listprocedure):
```

```
    end if:
    soln1 := eval(x(F), soln):

    k := X4 -> c*(mu - lambda) * H(X4) * g(X4) + k2 * g(X4):

    A_ast := X5 -> piecewise(X5 < 0, 0, X5 < x1, lambda/sigma^2
        * k(soln1(X5)) * soln1(X5),1);
    V := X6 -> piecewise(X6 < 0, 0, X6 < x1, Int(1/soln1(X3), X3
        =0..X6 )+P, X6/r+mu/r^2 -lambda/(r*(sigma^2 * theta^2 +
        lambda* theta)) * exp(theta*(X6-x1)));

elif P < mu/(r^2) then
    A_ast := X -> 1:
    V := X -> X/r + mu/r^2 + (P - mu/r^2) * exp(theta*X);

elif P = mu/(r^2) then
    A_ast := X -> 1:
    V := X -> X/r + mu/r^2;

elif P > mu/r^2 and lambda <= (mu + sqrt(mu^2 + 2*r*sigma^2))/2
   then
    A_ast := X -> 1:
    V := X -> X/r + mu/r^2 + (P - mu/r^2) * exp(theta*X);

else
    A_ast := X -> 1:
    V := X -> piecewise((P-mu/r^2)*((sigma^2*theta^2)/2 + lambda
        *theta) * exp(theta*X) + lambda/r >= 0,X/r + mu/r^2 + (P
        - mu/r^2) * exp(theta*X));

end if:
```

To generate the Figures the following code was used. This example generated Figure 4.1.

```
with(Statistics); with(plots); with(finance);
mu := 4; lambda := 4; sigma := sqrt(10); r := .7; P := 4.33;

c := 2*sigma^2/lambda^2:
theta := (-mu-sqrt(2*r*sigma^2 + mu^2))/sigma^2:
alpha := r*(1+lambda/(theta*sigma^2)):

read "solution_wu_new.txt";
plot([A_ast, V], 0 .. 4, resolution = 50, numpoints = 50);
```

As a start for Monte Carlo simulation, the discrete risk process is modeled assuming exponential claim distribution. The result is shown in Figure 4.6.

```
mu := 1.5; lambda := 3; sigma := 1; r := .1; P := 140;
```

```
beta := 1/(m^2+s2); a := 1;
U := RandomVariable(LogNormal(1, 1)):
S := PoissonProcess(beta, U):
m := evalf(Moment(U, 1)); s2 := evalf(Moment(U, 2));
p1 := beta*m+mu; p2 := beta*m+lambda; sigma := sqrt(beta*(m^2+s2
    ));
Rd := proc (t) options operator, arrow; piecewise(t <= 0, x, max
    (x+(p1-(1-a)*p2)*t-a*S(t), 0)) end proc
PathPlot(simulations, markers = false, color = red .. blue,
    thickness = 3, gridlines = false, axes = BOXED);
```

To model the return function, Monte Carlo simulation is used. The simulation is saved to the file `mc_wu.txt`.

```
simulations := SamplePath(Rd(t), t = 0 .. N, timesteps = N,
    replications = rep):
pres_value := Vector(rep):
stoptime := Vector(rep):

for count1 from 1 to rep do
  # if stoptime = inf, NULL is assigned
  stoptime(count1) := SelectFirst(proc (r) options operator,
      arrow; evalb(r = 0) end proc, convert(simulations(count1, 1
      .. N+1), list), output = indices);

count2:=0:
  for count2 from 1 to min(stoptime(count1), N+1) do
    pres_value(count1) := pres_value(count1) + evalf(exp(-r*(
        count2-1)) * simulations(count1,count2)):
  end do:
  if type(stoptime(count1),'integer') and stoptime(count1) <= N
      then
    pres_value(count1) := pres_value(count1) + evalf(exp(-r*(
        count2-2)) * P):
  end if
end do:
```

This simulation is called for every starting point $x = 0, \ldots, 20$. The result is shown in Figure 4.8.

```
N := 50; rep := 100;
exp_value := []
for run from 0 by 0.1 to 20 do
  x := run; read "mc_wu.txt";
  exp_value := [op(exp_value), [run, Mean(pres_value)]]:
end do:

plot1 := pointplot(exp_value, color = red)
```

```
display(plot1);
```

This simulation is repeated with the control parameter being dependent on the starting value x, as seen in Figure 4.10.

```
Rd := proc (t) options operator, arrow; piecewise(t <= 0, x, max
   (x+(p1-(1-A_ast(x))*p2)*t-A_ast(x)*S(t), 0)) end proc;

exp_value_ast := [];
for run from 0 by .1 to 20 do
  x := run; read "mc_wu.txt";
  exp_value_ast := [op(exp_value_ast), [run, Mean(pres_value)]]
end do;

plot4 := pointplot(exp_value_ast, color = black);
display(plot1, plot4);
```

For recursive modelling of the risk process, a new routine has to be written. It is saved to `mc_wu_rec.txt`.

```
Ssimulations := SamplePath(S(t), t = 0 .. N, timesteps = N,
   replications = rep):

simulationsR:=0*Ssimulations:
pres_value := Vector(rep,0):
stoptime := Vector(rep,N+1): #standard: Stoptime = max

for count1 from 1 to rep do
  simulationsR(count1,1):=x;
  if simulationsR(count1,1)=0 then stoptime(count1) := 1: else
    for count3 from 2 to (N+1) do
      simulationsR(count1,count3) := max(x+ (p1 -(1-A_ast(
         simulationsR(count1, count3-1)))*p2) *(count3-1) -A_ast
         (simulationsR(count1, count3-1))*Ssimulations(count1,
         count3), 0);
      if simulationsR(count1,count3)=0 then stoptime(count1) :=
         count3: break end if:
    end do;
  end if:

  count2:=0:
  for count2 from 1 to min(stoptime(count1), N+1) do
    pres_value(count1) := pres_value(count1) + evalf(exp(-r*(
       count2-1)) * simulationsR(count1,count2)):
  end do:
  if type(stoptime(count1),'integer') and stoptime(count1) <= N
     then
    pres_value(count1) := pres_value(count1) + evalf(exp(-r*(
```

```
        count2-2)) * P):
    end if
```

The following code was used to generate the recursive simulation with exponentially distributed claims. The simulation is split into four parts, where each result is saved in an external file. This way, I can continue with already calculated means if Maple freezes because of the calculations taking too long. The result can be seen in Figure 4.12.

```
with(Statistics); with(plots); with(finance); with(ListTools);
beta := 1/3; P := 4.33; r := 0.5e-1; p1 := 7.+1/3; p2 := 7.+1/3;
m := 10; s2 := 200;
U := RandomVariable(Gamma(alpha1, beta1));

parameters := solve({m = evalf(Moment(U, 1)), s2 = evalf(Moment(
   U, 2))}, {alpha1, beta1});
assign(parameters);
U := RandomVariable(Gamma(alpha1, beta1));
S := PoissonProcess(beta, U);
mu := -beta*m+p1; lambda := -beta*m+p2; sigma := sqrt(beta*(m^2+
   s2));

Rd := proc (t) options operator, arrow; piecewise(t <= 0, x, max
   (x+(p1-(1-a)*p2)*t-a*S(t), 0)) end proc;
N := 1000; rep := 100;
read "solution_wu_new.txt";

a := 1;
exp_value := []; for run from 0 by .1 to 20 do x := run; read "
   mc_wu.txt"; exp_value := [op(exp_value), [run, Mean(
   pres_value)]] end do;
plot1 := pointplot(exp_value, color = red);

save exp_value, "exp_value";
gc();
exp_value_ast := [];

for run from 0 by .1 to 5 do x := run; read "mc_wu_rec.txt";
   exp_value_ast := [op(exp_value_ast), [run, Mean(pres_value)]]
    end do;
save exp_value_ast, "exp_value_ast_gamma";
gc();
for run from 5.1 by .1 to 10 do x := run; read "mc_wu_rec.txt";
   exp_value_ast := [op(exp_value_ast), [run, Mean(pres_value)]]
    end do;
save exp_value_ast, "exp_value_ast_gamma";
gc();
for run from 10.1 by .1 to 15 do x := run; read "mc_wu_rec.txt";
    exp_value_ast := [op(exp_value_ast), [run, Mean(pres_value)
```

```
    ]] end do;
save exp_value_ast, "exp_value_ast_gamma";
gc();
for run from 15.1 by .1 to 20 do x := run; read "mc_wu_rec.txt";
    exp_value_ast := [op(exp_value_ast), [run, Mean(pres_value)
    ]] end do;
save exp_value_ast, "exp_value_ast_gamma";
read "exp_value_ast_gamma";
plot2 := pointplot(exp_value_ast);

display(plot1,plot2);
```

The following was used to simulate an existing optimal return function, the result can be seen in 4.16.

```
mu := 1.5; lambda := 3; sigma := 1; r := .1; P := 160;

read "solution_wu_new.txt";
plot1 := plot([V], 0 .. 4, 130 .. 190, resolution = 100,
    numpoints = 100, discont = true);

m := 1; beta := 1/3; s2 := solve(sigma = sqrt(beta*(m^2+s2)), s2
    );
p1 := beta*m+mu; p2 := beta*m+lambda;

U := RandomVariable(Gamma(alpha1, beta1));
parameters := solve({m = evalf(Moment(U, 1)), s2 = evalf(Moment(
    U, 2))}, {alpha1, beta1});
assign(parameters);
U := RandomVariable(Gamma(alpha1, beta1));
S := PoissonProcess(beta, U);

N := 1000; rep := 200;
exp_value_ast := [];

for run from 0 by .1 to 2 do x := run; read "mc_wu_rec.txt";
    exp_value_ast := [op(exp_value_ast), [run, Mean(pres_value)]]
    end do;
gc();

plot2 := pointplot(exp_value_ast);

display(plot1, plot2);
```

Short Errata to studied Papers

Taksar and Hunderup (2007) [12]

Studying controlled diffusion models with proportional reinsurance strongly set up on the paper by M. Taksar and C. L. Hunderup *The influence of bankruptcy value on optimal risk control for diffusion models with proportional reinsurance* [12]. It has a very similar set up, but only deals with the case of cheap reinsurance, i.e. in the notation of chapter 3 $\lambda = \mu$.
Some typos and mistakes were found in this paper which are collected in the following list.

- p. 313, theorem 3.1.:
$$f(x) \geq V(x).$$
- p. 314, theorem 3.1.:
Let $A^*(x)$ be the maximizer of the *left* hand side ...
- p. 314, before (3.8):
... we can find $A^*(x)$ differentiating the *left* hand side of (3.1) ...
- p. 315, before (3.15):
$$A(y) = \frac{\mu k_1}{\sigma^2} y g(y)$$
- p. 315, after (3.15):
Since $A(-k_2) = 0$, ...
- p. 316, top line:
$$F(x) = \frac{\mu^2}{2\sigma^2} k_1 g\left(G^{-1}\left(\frac{x+k_2}{k_1}\right)\right)$$
- p. 319, Proof 2.:
$$A(x) = \ldots = -\frac{\mu}{\sigma^2 \gamma \theta_-^2 (P - \frac{\mu}{\gamma^2}) e^{\theta_- x}} - \frac{\mu}{\sigma^2 \theta_-}.$$

Wu, Wu and Zhou (2011) [13]

Also the paper *Optimal risk control policies for diffusion models with non-cheap proportional reinsurance and bankruptcy value* by M. Wu, R. Wu and A. Zhou [13] was studied intensely, which expanded the control model to non-cheap reinsurance.

- p. 905, last paragraph:
... ; by Theorem 2, when $0 = \frac{2\mu - \lambda}{2r\alpha} \leq P = 140 < \frac{\mu}{r^2}$, ...
It also should be noted that the optimal return function cannot be described on the whole interval in this case. For details see the description to Figure 4.2 in this paper.

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