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The JIMWLK equation: Fundamentals and weak field expansion

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Axel Polaczek

Matrikelnummer 1127831

ausgeführt am Institut für Theoretische Physik
der Fakultät für Physik der Technischen Universität Wien

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Betreuer: Privatdoz. Dipl.-Ing. Dr.techn. Andreas Ipp

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Die theoretische Beschreibung von Schwerionenkollisionen stellt ein aktives Forschungsgebiet dar. Experimentell werden Schwerionenkollisionen an Anlagen wie beispielsweise dem LHC am europäischen Kernforschungszentrum CERN und dem RHIC am Brookhaven National Laboratory durchgeführt.

Das „Color Glass Condensate“ (deutsch: Farbglas-Kondensat) stellt ein geeignetes Modell dar, um Schwerionen bei hohen Energien zu beschreiben. In diesem Modell erfolgt eine Trennung der Freiheitsgrade nach ihren longitudinalen Impulsen entlang der Bewegungsrichtung des Kerns. Die „harten“ Partonen, welche longitudinale Impulse größer als die der Trennungsskala besitzen, können ausintegriert werden und dann als effektive, klassische Farbladungsdichte aufgefasst werden. Diese wirkt dann als Quellterm für die Dynamik der „weichen“ Freiheitsgrade, welche longitudinale Impulse kleiner als die der Trennungsskala besitzen. In diesem Modell kann der Erwartungswert von Observablen durch die Mittelung über unterschiedliche Ladungskonfigurationen berechnet werden. Die statistische Verteilung der Farbladungsdichten erfolgt über eine Gewichtsfunktion. Bei sehr hohen Energien kann die Gewichtsfunktion durch eine Gauß-Verteilung beschrieben werden, was unter dem Namen McLerran–Venugopalan-Modell bekannt ist. Um Rechnungen bei niedrigeren, experimentell zugänglichen Energien durchzuführen, benötigt man die Gewichtsfunktion für die Energieskala, die von Interesse ist. Wie man durch Integration von weiteren Freiheitsgraden eine Verschiebung der Energieskala, bei der die Trennung in „harte“ und „weiche“ Moden erfolgt, durchführen kann, wird durch die sogenannte JIMWLK-Renomierungsgruppengleichung beschrieben. Diese wurde Ende der 1990er Jahren erstmals formuliert.

In dieser Arbeit leiten wir die JIMWLK-Gleichung im Detail her. Des Weiteren führen wir eine Entwicklung der Gleichung für schwache Felder durch und zeigen explizit, dass dies zur BFKL-Gleichung, welche seit den 1970er Jahren bekannt ist, führt.

The color glass condensate framework is suitable for describing heavy nuclei at high energies. In this framework there is a cutoff scale separating the degrees of freedom according to their longitudinal momentum along the direction of the propagation of the nucleus. The "hard" partons can be integrated out and they contribute to a classical color charge density distribution which acts as a source term for the dynamics of the "soft" partons. A suitable model to describe the gluon distribution at very large energies is the McLerran–Venugopalan model. Passing from one energy scale to a lower energy scale can be accomplished by the JIMWLK renormalization group equation.

We perform a detailed analysis of the JIMWLK equation and its derivation. Furthermore, we perform a weak field expansion of the JIMWLK equation and show that it leads to the BFKL equation explicitly.

Contents

Introduction

The standard model of particle physics is being verified at ever higher energies at the large hadron collider (LHC). The standard model contains all known forces except for gravity, namely the electroweak interaction and the strong interaction. Due to the relative strengths of the coupling constants, the dynamics of particles which are charged under the strong interaction is dominated by the strong interaction. It is well known that the running coupling constant of quantum chromodynamics gets smaller at higher energies – the theory is said to be asymptotically free. However, when studying the dynamics of highly energetic nuclei and hadrons it turns out that the rise of the gluon density is such that even in the regime of weak coupling a perturbative analysis breaks down. In a frame in which the nuclei are ultra-relativistic they appear as thin sheets of color charge propagating along the lightcone directions. In such a frame the gluon correlations can be studied by a classical effective theory which was first proposed by McLerran and Venugopalan [1, 2].

In the picture of an effective theory the irrelevant degrees of freedom above some cutoff scale have already been integrated out. If one wants to go to lower energies, one therefore has to integrate out further degrees of freedom. This was first done by Jalilian-Marian, Kovner, Leonidov and Weigert [3, 4, 5] and the resulting renormalization group equation was referred to as the JKLW equation. In the seminal work [6, 7] Iancu, Leonidov, McLerran and Ferreiro provided a very thorough derivation of the equation and the equation is now known as the JIMWLK[1] equation. The resulting renormalization group equation is of the form of a Fokker–Planck equation. As known from statistical mechanics one can pass to a corresponding Langevin formulation [9, 10] which has been studied numerically in [11].

The theoretical study of heavy ions collisions remains an active area of research. Due to the nonlinear nature of the equations governing the dynamics of the strong force, analytic computations can be carried out only to a very limited degree and one often has to resort to numerical methods. A promising approach is to solve the classical Yang–Mills equations numerically using the colored particle in cell method. This has been carried out for nuclei with finite longitudinal thickness [12, 13, 14]. A major obstacle of this method is the choice of initial conditions. The simulations so far have been limited to using the McLerran–Venugopalan model. A possible improvement to this method is to include first-order quantum corrections by applying the JIMWLK renormalization group equation to obtain refined initial conditions. However, this is not straightforward to do

[1] The acronym is pronounced “gym-walk” [8].

since up to now the coefficients of the JIMWLK equation have only be computed for nuclei of infinitesimal longitudinal thickness.

In Chapter 1 we give an overview of the color glass condensate framework. In particular we will give the solutions to the Yang–Mills equation and discuss the McLerran–Venugopalan model and its extension. In Chapter 2 we present an effective action for the color glass condensate framework and work out the functional derivatives of the terms in great detail. In Chapter 3 we will provide a careful derivation of the JIMWLK equation and discuss its general structure. Finally, in Chapter 4 we perform the weak field expansion of the JIMWLK equation, showing that it leads to the BFKL equation. Generalities and some technical details are relegated to various appendices. Appendix A gives an overview of the notation and conventions which we use throughout. In Appendix B we give an overview of the group theory needed with an emphasis on the gauge group $SU(N)$. Appendix C covers the fundamentals of gauge theories. Appendix D discusses Fourier transforms and generalizations thereof. Finally, Appendix E discusses the propagators needed for our computations.

Chapter 1.

The color glass condensate framework

In this chapter we will provide an overview of the color glass condensate (CGC) framework. Reviews of the topic can be found in [8, 15].

In Section 1.1 we will present solutions to the classical equations of motion for an ultra-relativistic nucleus acting as a source for the gauge fields. In Section 1.2 we will present the McLerran–Venugopalan model. In Section 1.3 we will discuss the quantum extension to the McLerran–Venugopalan model.

1.1. Solution to the field equations

Consider a nucleus moving in the positive z direction, such a nucleus is said to be right-moving. (Conversely, a nucleus moving in the negative z direction is referred to as left-moving.) In a frame in which its velocity is near the speed of light, we can neglect the hadron's rest mass and have in cartesian coordinates

$$P_{\mathrm{c}}^{\mu} \equiv (P^0, P^1, P^2, P^3) = (P, 0, 0, P). \tag{1.1}$$

In lightcone coordinates (see Appendix A) the momentum is given by

$$P^{\mu} \equiv (P^+, P^-, P^1, P^2) = \sqrt{2}P(1, 0, 0, 0) = \delta_+^{\mu}\sqrt{2}P. \tag{1.2}$$

This frame is usually referred to as infinite momentum frame (IMF).

Due to the fact that the hadron is moving very fast in the x^+ direction, the corresponding current J^{μ} can be taken to be only in the x^+ direction, i.e. we neglect any transverse components, $J^{\mu} = \delta_+^{\mu} J^+$. A further assumption we will make is that the current is static by which we mean that it is independent of lightcone-time, i.e. $\partial_+ J^+ = 0$. This is called quenched approximation and is motivated by the effect of time dilation. The current of the hadron therefore is

$$J^{\mu}(x) = \delta_+^{\mu}\rho(\vec{x}), \tag{1.3}$$

where $\vec{x} = (x^-, \boldsymbol{x}_\perp)$.

The equations of motion of Yang–Mills theory are

$$\mathcal{D}_\mu \mathcal{F}^{\mu\nu} = J^\nu, \tag{1.4}$$

where the fields take values in the adjoint representation and $\mathcal{D}_\mu \mathcal{F}^{\mu\nu} = \partial_\mu \mathcal{F}^{\mu\nu} - \mathrm{i}g[\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]$ is the gauge covariant derivative in the adjoint representation. The equations of motion for the individual components are

$$\mathcal{D}_\mu \mathcal{F}^{\mu+}(x) = \rho(\vec{x}), \tag{1.5a}$$
$$\mathcal{D}_\mu \mathcal{F}^{\mu-}(x) = 0, \tag{1.5b}$$
$$\mathcal{D}_\mu \mathcal{F}^{\mu i}(x) = 0. \tag{1.5c}$$

Note that in this section we denote the fields by calligraphic letter to emphasize that they are solutions to the *classical* equations of motion.

1.1.1. Covariant current conservation

In this section we call to attention that the assumption of a static current as a source term for the Yang–Mills equation (Eq. (1.4)) is in general inconsistent. For the situation we are interested in, it is however always possible to find a solution which is compatible with the assumption of a static current. For more details see Section 2.1 in [6].

To see why the assumption of a static current is inconsistent, consider the covariant divergence of the left-hand side of Eq. (1.4)

$$\begin{aligned}
\mathcal{D}_\nu \mathcal{D}_\mu \mathcal{F}^{\mu\nu} &= \mathcal{D}_\nu(\partial_\mu \mathcal{F}^{\mu\nu} - \mathrm{i}g[\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]) \\
&= \partial_\nu(\partial_\mu \mathcal{F}^{\mu\nu} - \mathrm{i}g[\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]) - \mathrm{i}g[\mathcal{A}_\nu, \partial_\mu \mathcal{F}^{\mu\nu} - \mathrm{i}g[\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]] \\
&= -\mathrm{i}g[\partial_\nu \mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] - \mathrm{i}g[\mathcal{A}_\mu, \partial_\nu \mathcal{F}^{\mu\nu}] + \mathrm{i}g[\mathcal{A}_\mu, \partial_\nu \mathcal{F}^{\mu\nu}] + (-\mathrm{i}g)^2[\mathcal{A}_\nu, [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]] \\
&= -\mathrm{i}g[\partial_\nu \mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] + (-\mathrm{i}g)^2[\mathcal{A}_\nu, [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]].
\end{aligned} \tag{1.6}$$

The first term of Eq. (1.6) can be rewritten as

$$[\partial_\nu \mathcal{A}_\mu, \mathcal{F}^{\mu\nu}] = \frac{1}{2}[\partial_\nu \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\nu, \mathcal{F}^{\mu\nu}]. \tag{1.7}$$

The second term of Eq. (1.6) gives

$$\begin{aligned}
[\mathcal{A}_\nu, [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]] &= \frac{1}{2}\left([\mathcal{A}_\nu, [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]] - [\mathcal{A}_\mu, [\mathcal{A}_\nu, \mathcal{F}^{\mu\nu}]]\right) \\
&= \frac{1}{2}\left([\mathcal{A}_\nu, [\mathcal{A}_\mu, \mathcal{F}^{\mu\nu}]] + [\mathcal{A}_\mu, [\mathcal{F}^{\mu\nu}, \mathcal{A}_\nu]]\right) \\
&= -\frac{1}{2}[\mathcal{F}^{\mu\nu}, [\mathcal{A}_\nu, \mathcal{A}_\mu]] \\
&= \frac{1}{2}[[\mathcal{A}_\nu, \mathcal{A}_\mu], \mathcal{F}^{\mu\nu}],
\end{aligned} \tag{1.8}$$

where we used the Jacobi identity. Equation (1.6) therefore is

$$\mathcal{D}_\nu \mathcal{D}_\mu \mathcal{F}^{\mu\nu} = -\mathrm{i}g\frac{1}{2}[\partial_\nu \mathcal{A}_\mu - \partial_\mu \mathcal{A}_\nu - \mathrm{i}g[\mathcal{A}_\nu, \mathcal{A}_\mu], \mathcal{F}^{\mu\nu}] = -\mathrm{i}g\frac{1}{2}[\mathcal{F}_{\nu\mu}, \mathcal{F}^{\mu\nu}] = 0. \tag{1.9}$$

From Eq. (1.4) it then follows that the covariant divergence of the current must vanish, i.e.

$$\mathcal{D}_\mu J^\mu = 0. \tag{1.10}$$

In the case that we assume a purely longitudinal current, $J^\mu = \delta^\mu_+ J^+$, Eq. (1.10) implies that in general the current cannot be static. To see this insert the purely longitudinal current into Eq. (1.10)

$$0 = \mathcal{D}_+ J^+ = \partial_+ J^+ - \mathrm{i}g[\mathcal{A}_+, J^+] = \partial_+ J^+ - \mathrm{i}g[\mathcal{A}^-, J^+], \tag{1.11}$$

where we used $\mathcal{A}_+ = \mathcal{A}^-$ (Eq. (A.10)). As can be checked by an explicit computation, Eq. (1.11) is solved by

$$J^+(x) = W_{x^+,x_0^+}(\vec{x}) J^+(x_0^+, \vec{x}) W^\dagger_{x^+,x_0^+}(\vec{x}), \tag{1.12}$$

where we defined the Wilson line

$$W_{x^+,x_0^+}(\vec{x}) = \mathrm{P}\exp\left(\mathrm{i}g\int_{x_0^+}^{x^+} \mathrm{d}z^+ \, \mathcal{A}^-(z^+, \vec{x})\right). \tag{1.13}$$

Note that the path-ordering ensures that taking the derivative with respect to the endpoint gives

$$\partial_+ W_{x^+,x_0^+}(\vec{x}) = \mathrm{i}g\mathcal{A}^-(x) W_{x^+,x_0^+}(\vec{x}). \tag{1.14}$$

It is usual to take $x_0^+ = -\infty$ and set $J^+(x^+ = -\infty, \vec{x}) = \rho(\vec{x})$. Equation (1.12) then becomes

$$J^+(x) = W_{x^+,-\infty}(\vec{x}) \rho(\vec{x}) W^\dagger_{x^+,-\infty}(\vec{x}). \tag{1.15}$$

Therefore it is inconsistent to assume a static current. However, in the case that we have a solution $\mathcal{A}^- = 0$ the Wilson lines become trivial and the current is independent of x^+ and therefore static. It turns out that the situation of interest here always admits such a solution.

We note that the equation of motion Eq. (1.5a) therefore should in general be written as

$$\mathcal{D}_\mu \mathcal{F}^{\mu+}(x) = W_{x^+,-\infty}(\vec{x}) \rho(\vec{x}) W^\dagger_{x^+,\infty}(\vec{x}). \tag{1.16}$$

It is only for the the case that the classical solution satisfies $\mathcal{A}^- = 0$ that the Wilson lines become trivial and we recover Eq. (1.5a).

1.1.2. Covariant gauge solution

In this section we will present a gauge, namely the covariant gauge, which allows one to solve the Yang–Mills equation of motion explicitly.

We define the covariant gauge condition as follows

$$\partial_\mu \tilde{\mathcal{A}}^\mu = 0. \tag{1.17}$$

We will always denote fields in the covariant gauge with a tilde.

We make the ansatz $\tilde{\mathcal{A}}^\mu(x) = \delta^\mu_+ \tilde{\mathcal{A}}^+(\vec{x})$. This implies in particular that $\mathcal{A}^- = 0$ and $\mathcal{A}^i = 0$. Furthermore we note that the ansatz is compatible with the gauge condition since

$$\partial_\mu \tilde{\mathcal{A}}^\mu = \partial_+ \tilde{\mathcal{A}}^+ = 0. \tag{1.18}$$

The components of the field-strength tensor are

$$\tilde{\mathcal{F}}^{\mu\nu} = \partial^\mu \tilde{\mathcal{A}}^\nu - \partial^\nu \tilde{\mathcal{A}}^\mu - \mathrm{i}g[\tilde{\mathcal{A}}^\mu, \tilde{\mathcal{A}}^\nu] = \delta^\nu_+ \partial^\mu \tilde{\mathcal{A}}^+ - \delta^\mu_+ \partial^\nu \tilde{\mathcal{A}}^+. \tag{1.19}$$

The only non-vanishing components of the field-strength tensor therefore are

$$\tilde{\mathcal{F}}^{i+} = -\tilde{\mathcal{F}}^{+i} = \partial^i \tilde{\mathcal{A}}^+. \tag{1.20}$$

The '+' component of Eq. (1.4) then is

$$\tilde{\rho} = \tilde{\mathcal{D}}_\mu \tilde{\mathcal{F}}^{\mu+} = \tilde{\mathcal{D}}_i \tilde{\mathcal{F}}^{i+} = \partial_i \tilde{\mathcal{F}}^{i+} = \partial_i \partial^i \tilde{\mathcal{A}}^+. \tag{1.21}$$

The equation

$$\boldsymbol{\nabla}_\perp^2 \tilde{\mathcal{A}}^+(\vec{x}) = -\tilde{\rho}(\vec{x}) \tag{1.22}$$

can readily be solved by

$$\tilde{\mathcal{A}}^+(\vec{x}) = -\frac{1}{2\pi} \int \mathrm{d}^2 \boldsymbol{y}_\perp \ln |\boldsymbol{x}_\perp - \boldsymbol{y}_\perp| \tilde{\rho}(x^-, \boldsymbol{y}_\perp). \tag{1.23}$$

We will also write this in the form

$$\tilde{\mathcal{A}}^+(\vec{x}) = \int \mathrm{d}^2 \boldsymbol{y}_\perp \langle \boldsymbol{x}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{y}_\perp \rangle \tilde{\rho}(x^-, \boldsymbol{y}_\perp), \tag{1.24}$$

where we defined the Green's function

$$\langle \boldsymbol{x}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{y}_\perp \rangle = -\frac{1}{2\pi} \ln |\boldsymbol{x}_\perp - \boldsymbol{y}_\perp|, \tag{1.25}$$

where the derivative should be viewed as acting to the left,[1] since (per definition)

$$(-\boldsymbol{\nabla}_\perp^2)_{\boldsymbol{x}_\perp} \langle \boldsymbol{x}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{y}_\perp \rangle = I(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp), \tag{1.26}$$

[1] Note that this is actually irrelevant for the Green's function of the Poisson equation, since $\langle \boldsymbol{x}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{y}_\perp \rangle = \langle \boldsymbol{y}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{x}_\perp \rangle$.

where $I(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp)$ is defined as in Eqs. (A.17). A better way to understand the above expression is by writing the matrix element using the momentum operator, i.e.

$$\langle \boldsymbol{x}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{y}_\perp \rangle \equiv \langle \boldsymbol{x}_\perp | \frac{1}{\boldsymbol{P}_\perp^2} | \boldsymbol{y}_\perp \rangle. \tag{1.27}$$

The main result of this section is that in the covariant gauge Eq. (1.17) we are able to explicitly give the corresponding solution to the Yang–Mills equation for a given charge distribution $\tilde{\rho}$.

1.1.3. Lightcone gauge solution

It is often convenient to work in the lightcone gauge, $\mathcal{A}^+ = 0$. We will reserve labeling the fields without decorators to this particular gauge (unless otherwise explicitly mentioned).

In Appendix C we discuss how one can pass from one gauge to another by means of a gauge transformation in non-abelian gauge theory. We write for the gauge transformation (Eq. (C.23)) from the covariant to the lightcone gauge

$$\mathcal{A}^\mu = U \left(\tilde{\mathcal{A}}^\mu + \frac{\mathrm{i}}{g} \partial^\mu \right) U^\dagger. \tag{1.28}$$

Inserting the gauge condition gives

$$0 = \mathcal{A}^+ = U \left(\tilde{\mathcal{A}}^+ + \frac{\mathrm{i}}{g} \partial^+ \right) U^\dagger \tag{1.29}$$

which leads to the differential equation

$$\partial^+ U^\dagger = \partial_- U^\dagger = \mathrm{i} g \alpha U^\dagger, \tag{1.30}$$

where we used Eq. (A.11) and have written the covariant gauge field now as $\tilde{\mathcal{A}}^+ \equiv \alpha$. Equation (1.30) is solved by

$$U^\dagger(\vec{x}) = \mathrm{P} \exp \left(\mathrm{i} g \int_{x_0^-}^{x^-} \mathrm{d}z^- \, \alpha(z^-, \boldsymbol{x}_\perp) \right). \tag{1.31}$$

Since the field α is hermitian, $\alpha = \alpha^\dagger$, we have for the hermitian conjugate of Eq. (1.31)

$$U(\vec{x}) = \bar{\mathrm{P}} \exp \left(-\mathrm{i} g \int_{x_0^-}^{x^-} \mathrm{d}z^- \, \alpha(z^-, \boldsymbol{x}_\perp) \right), \tag{1.32}$$

where $\bar{\mathrm{P}}$ denotes anti-path-ordering and can be understood by noting that taking the hermitian conjugate reverses the order of the fields.

The non-vanishing field components in the lightcone gauge are then obtained by the gauge transformation (Eq. (1.28)) and read

$$\mathcal{A}^i(\vec{x}) = \frac{\mathrm{i}}{g} U(\vec{x}) \partial^i U^\dagger(\vec{x}). \tag{1.33}$$

The non-vanishing components of the field-strength tensor in the lightcone gauge are

$$\begin{aligned}\mathcal{F}^{i+} &= -\partial^+\mathcal{A}^i \\ &= -\partial_-\mathcal{A}^i \\ &= -\frac{\mathrm{i}}{g}\partial_-(U\partial^i U^\dagger) \\ &= -\frac{\mathrm{i}}{g}(\partial_- U\partial^i U^\dagger + U\partial^i(\partial_- U^\dagger)) \\ &= -\frac{\mathrm{i}}{g}(-\mathrm{i}gU\alpha\partial^i U^\dagger + \mathrm{i}gU\partial^i(\alpha U^\dagger)) \\ &= U\partial^i\alpha U^\dagger. \end{aligned} \tag{1.34}$$

We will now verify explicitly that Eq. (1.33) satisfies the equation of motion

$$\mathcal{D}_\mu\mathcal{F}^{\mu+} = \rho, \tag{1.35}$$

where the lightcone gauge source ρ is related to the covariant gauge source $\tilde{\rho}$ via

$$\rho = U\tilde{\rho}U^\dagger. \tag{1.36}$$

We have

$$\partial_i\mathcal{F}^{i+} = \partial_i U\partial^i\alpha U^\dagger + U\partial_i\partial^i\alpha U^\dagger + U\partial^i\alpha\partial_i U^\dagger \tag{1.37}$$

and

$$[\mathcal{A}_i, \mathcal{F}^{i+}] = [\frac{\mathrm{i}}{g}U\partial_i U^\dagger, U\partial^i\alpha U^\dagger] = \frac{1}{\mathrm{i}g}\left(\partial_i U\partial^i\alpha U^\dagger + U\partial^i\alpha\partial_i U^\dagger\right), \tag{1.38}$$

where we have used $\partial_i UU^\dagger = -U\partial_i U^\dagger$ which follows from $UU^\dagger = I$. The equation of motion (Eq. (1.35)) therefore is fulfilled:

$$\mathcal{D}_\mu\mathcal{F}^{\mu+} = \partial_i\mathcal{F}^{i+} - \mathrm{i}g[\mathcal{A}_i, \mathcal{F}^{i+}] = U\partial_i\partial^i\alpha U^\dagger = U\tilde{\rho}U^\dagger = \rho. \tag{1.39}$$

1.2. The McLerran–Venugopalan model

In the previous section we established a relationship between the classical gauge fields $\mathcal{A}$ and the corresponding color charge density distribution ρ. Given this relation one can choose either as the variable for a model. In the McLerran–Venugopalan (MV) model one usually chooses the charge density distribution ρ as the main variable.

The main idea of the model is that correlation functions of observables can by computed by averaging over all possible charge distributions with an appropriate weight factor. Given observables $O_1[\rho], \ldots, O_n[\rho]$ we have for their correlation function

$$\langle O_1[\rho]\cdots O_n[\rho]\rangle_\rho = \frac{\int[D\rho]W[\rho]O_1[\rho]\cdots O_n[\rho]}{\int[D\rho]W[\rho]}, \tag{1.40}$$

where $W[\rho]$ is the aforementioned weight function. It is often convenient to normalize the weight function such that

$$\int [D\rho] W[\rho] = 1. \tag{1.41}$$

For a normalized weight function the correlation function Eq. (1.40) then simply is

$$\langle O_1[\rho] \cdots O_n[\rho] \rangle_\rho = \int [D\rho] W[\rho] O_1[\rho] \cdots O_n[\rho]. \tag{1.42}$$

In the following we will always assume that the weight function has been normalized according to Eq. (1.41).

In practice it is advantageous to define the following generating functional

$$\begin{aligned} Z[J] &= \int [D\rho] W[\rho] \exp\left(\int \mathrm{d}^3\vec{x} \, \frac{1}{T(\mathrm{R})} \operatorname{tr}[J(\vec{x})\rho(\vec{x})] \right) \\ &= \int [D\rho] W[\rho] \exp\left(\int \mathrm{d}^3\vec{x} \, J^a(\vec{x})\rho_a(\vec{x}) \right), \end{aligned} \tag{1.43}$$

where the fields take values in the representation R and the index $T(\mathrm{R})$ of the representation is defined as in Eq. (B.14). Given the generating functional, the correlation functions (Eq. (1.42)) can be written as

$$\langle O_1[\rho] \cdots O_n[\rho] \rangle_\rho = O_1\left[\frac{\delta}{\delta\rho}\right] \cdots O_n\left[\frac{\delta}{\delta\rho}\right] Z[J]\bigg|_{J=0}. \tag{1.44}$$

1.2.1. Gaussian weight function

In the case of ultra-relativistic nuclei at small x it is reasonable to assume that the values of the color charge density distribution at separate spacetime points are completely uncorrelated and that the charge is on average zero. This amounts to imposing the following correlators [1, 2, 8]

$$\langle \rho_a(\vec{x}) \rangle_\rho = 0, \tag{1.45a}$$

$$\langle \rho_a(\vec{x}) \rho_b(\vec{y}) \rangle_\rho = \mu^2(x^-) \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}), \tag{1.45b}$$

where the longitudinal structure is encoded entirely in $\mu^2(x^-)$.

The correlators of Eqs. (1.45) can be realized by the following weight function

$$\begin{aligned} W[\rho] &= \exp\left(-\frac{1}{2} \int \mathrm{d}^3\vec{x} \, \frac{1}{T(\mathrm{R})} \frac{\operatorname{tr}[\rho(\vec{x})\rho(\vec{x})]}{\mu^2(x^-)} \right) \\ &= \exp\left(-\frac{1}{2} \int \mathrm{d}^3\vec{x} \, \frac{\rho^a(\vec{x})\rho_a(\vec{x})}{\mu^2(x^-)} \right). \end{aligned} \tag{1.46}$$

The generating functional for this weight function is (using a compact notation for integration as described in Appendix A.3)

$$\begin{aligned}
Z[J] &= \int [D\rho] W[\rho] \exp\left(\int_{\vec{x}} J^a(\vec{x}) \rho_a(\vec{x}) \right) \\
&= \int [D\rho] \exp\left(\int_{\vec{x}} -\frac{1}{2\mu^2(x^-)} \rho^a(\vec{x}) \rho_a(\vec{x}) + J^a(\vec{x}) \rho_a(\vec{x}) \right) \\
&= \int [D\rho] \exp\left(\int_{\vec{x}} -\frac{1}{2\mu^2(x^-)} (\rho^a(\vec{x}) - \mu^2(x^-) J^a(\vec{x}))(\rho_a(\vec{x}) - \mu^2(x^-) J_a(\vec{x})) \right. \\
&\qquad\qquad \left. + \mu^2(x^-) J^a(\vec{x}) J_a(\vec{x}) \right) \\
&= \int [D\rho] \exp\left(\int_{\vec{x}} -\frac{1}{2\mu^2(x^-)} \rho^a(\vec{x}) \rho_a(\vec{x}) + \mu^2(x^-) J^a(\vec{x}) J_a(\vec{x}) \right) \\
&= \exp\left(\frac{1}{2} \int_{\vec{x}} \mu^2(x^-) J^a(\vec{x}) J_a(\vec{x}) \right) \int [D\rho] W[\rho] \\
&= \exp\left(\frac{1}{2} \int_{\vec{x}} \mu^2(x^-) J^a(\vec{x}) J_a(\vec{x}) \right),
\end{aligned} \tag{1.47}$$

where we performed a shift in the integration variable and used the normalization condition of Eq. (1.41).

We will now verify that the weight function Eq. (1.46) gives Eqs. (1.45). The one-point correlation function is

$$\begin{aligned}
\langle \rho_a(\vec{x}) \rangle_\rho &= \frac{\delta}{\delta J^a(\vec{x})} Z[J] \Big|_{J=0} \\
&= \frac{\delta}{\delta J^a(\vec{x})} \exp\left(\frac{1}{2} \int_{\vec{z}} \mu^2(z^-) J^c(\vec{z}) J_c(\vec{z}) \right) \Big|_{J=0} \\
&= \frac{1}{2} \int_{\vec{z}} 2\mu^2(z^-) \delta_a^c \delta^{(3)}(\vec{x} - \vec{y}) J_c(\vec{z}) Z[J] \Big|_{J=0} \\
&= \mu^2(x^-) J_a(\vec{x}) Z[J]\big|_{J=0} \\
&= 0.
\end{aligned} \tag{1.48}$$

The two-point correlation function is

$$\begin{aligned}
\langle \rho_a(\vec{x}) \rho_b(\vec{y}) \rangle_\rho &= \frac{\delta}{\delta J^a(\vec{x})} \frac{\delta}{\delta J^b(\vec{y})} Z[J] \Big|_{J=0} \\
&= \frac{\delta}{\delta J^a(\vec{x})} \left(\mu^2(y^-) J_b(\vec{y}) Z[J] \right) \Big|_{J=0} \\
&= \mu^2(x^-) \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}) Z[J] + \mu^2(x^-) \mu^2(y^-) J_a(\vec{x}) J_b(\vec{y}) Z[J] \Big|_{J=0} \\
&= \mu^2(x^-) \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}).
\end{aligned} \tag{1.49}$$

1.3. The quantum extension

The above discussion was entirely classical. It is clear that when one wants to obtain an improved model, one has to take quantum effects into account.

The MV model should be seen as being valid at a given energy scale. A question of great importance is how one can pass from one energy scale to another. The procedure of how to do this will be discussed in Chapter 3.

The full quantum theory for hadronic interactions would be given by the following generating functional in path integral form

$$Z[J,\eta,\bar{\eta}] = \int [DA][D\Psi][D\overline{\Psi}] \mathrm{e}^{\mathrm{i}S_{\mathrm{QCD}}[A,\Psi,\overline{\Psi}]+\mathrm{i}\int \mathrm{d}^4x (J_a^\mu(x)A_\mu^a(x)+\bar{\eta}(x)\Psi(x)+\overline{\Psi}(x)\eta(x))}, \quad (1.50)$$

where the action is given by

$$S_{\mathrm{QCD}}[A,\Psi,\overline{\Psi}] = \int \mathrm{d}^4x \left(\sum_f \overline{\Psi}_f (\mathrm{i}\not{D} - m_f)\Psi_f - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a \right), \quad (1.51)$$

where the explicit sum is taken over the different quark flavors.

As mentioned above, we are now only interested in the dynamics of the fields at a given scale. In particular we will be interested only in the case of small x, corresponding to the fraction of hadronic longitudinal momentum the individual partons carry. Therefore for a right-moving nucleus we will perform the separation of scales in the longitudinal momenta p^+. The "hard" degrees of freedom with $|p^+| > \Lambda^+$ corresponding to the valence quarks and high-momentum gluons thus can be integrated out and we arrive at an effective description with generating functional

$$Z[J] = \int [D\rho] W_{\Lambda^+}[\rho] \frac{\int^{\Lambda^+}[DA] \mathrm{e}^{\mathrm{i}S_{\mathrm{CGC}}[A,\rho]+\mathrm{i}\int \mathrm{d}^4x\, J_a^\mu(x)A_\mu^a(x)}}{\int^{\Lambda^+}[DA] \mathrm{e}^{\mathrm{i}S_{\mathrm{CGC}}[A,\rho]}}, \quad (1.52)$$

where the upper boundary Λ^+ of the integral reminds us that the fields A above the scale Λ^+ have been integrated out and only the fields with $|p^+| < \Lambda^+$ contribute to the path integral. The integration of the "hard" energy scales has been carried out explicitly in [16]. Note that the exact form of the effective action $S_{\mathrm{CGC}}[A,\rho]$ is not important at this point. We will present the effective action in Chapter 2.

It is important to note that the generating functional is not of the form

$$\frac{\int [D\rho] W_{\Lambda^+}[\rho] \int^{\Lambda^+}[DA] \mathrm{e}^{\mathrm{i}S_{\mathrm{CGC}}[A,\rho]+\mathrm{i}\int \mathrm{d}^4x\, J_a^\mu(x)A_\mu^a(x)}}{\int [D\rho] W_{\Lambda^+}[\rho] \int^{\Lambda^+}[DA] \mathrm{e}^{\mathrm{i}S_{\mathrm{CGC}}[A,\rho]}} \quad (1.53)$$

which would be the correct expression if both A and ρ were the dynamical degrees of freedom. The correct interpretation is that the classical color charge distributions ρ enter as stochastic variables, whereas the "soft" gluons are truly dynamical — it is only for a fixed color charge distribution (corresponding to the "hard" partons of the nucleus) that the quantum average over the "soft" field is performed.

The physical picture is that the "hard" partons have high longitudinal momentum and thus a small (lightcone) energy, whereas the "soft" degrees of freedom have small longitudinal momenta and therefore a large (lightcone) energy. Due to the uncertainty principle the (lightcone) time scales of the "hard" partons are therefore much larger than for the "soft" partons. Therefore when studying the dynamics of the "soft" partons the "hard" partons remain fixed. This is reminiscent of a spin glass, explaining the "glass" in CGC. [15]

If we are interested in the dynamics of only the gluons with momenta $|p^+| \lesssim \Lambda^+$ the correlators of operators at that scale can be written as

$$\langle O_1[\mathcal{A}] \dots O_n[\mathcal{A}] \rangle_{\Lambda^+} = \int [D\rho] W_{\Lambda^+}[\rho] O_1[\mathcal{A}] \dots O_n[\mathcal{A}], \tag{1.54}$$

where the classical gauge field solution is related to the classical color charge distribution, i.e. $\mathcal{A} = \mathcal{A}[\rho]$, as discussed in Section 1.1.

If one wants to take quantum effects into account one has to compute

$$\langle O_1[A] \dots O_n[A] \rangle = \int [D\rho] W_{\Lambda^+}[\rho] \frac{\int^{\Lambda^+} [DA] \mathrm{e}^{\mathrm{i}S_{\mathrm{CGC}}[A,\rho]} O_1[A] \dots O_n[A]}{\int^{\Lambda^+} [DA] \mathrm{e}^{\mathrm{i}S_{\mathrm{CGC}}[A,\rho]}}, \tag{1.55}$$

where now $A = \mathcal{A} + \Delta A$ and ΔA has only longitudinal momentum modes with $|p^+| < \Lambda^+$.

Chapter 2.

Effective action

As detailed in Section 1.3 integration of the "hard" modes gives an effective theory valid at some specific energy scale. We will now present the effective action.

The effective action is given by

$$S_{\mathrm{CGC}}[A,\rho] = S_{\mathrm{YM}}[A] + S_W[A^-,\rho], \tag{2.1}$$

where S_{YM} is the Yang–Mills action and S_W is a term introducing the classical color charge distribution ρ which couples eikonally to the field component A^- which can be understood immediately by comparing with the coupling of external sources in Maxwell theory via a term $J \cdot A$.

A desideratum of the effective action is that it reproduces the equation of motion

$$\frac{\delta S}{\delta A} = \text{ classical equation of motion .} \tag{2.2}$$

Equation (1.16) then implies that for the Wilson line piece of the action we must have

$$\frac{\delta S_W[A^-,\rho]}{\delta A_a^-(x)} = -\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left[W_{x^+,-\infty}(\vec{x})\rho(\vec{x})W^\dagger_{x^+,-\infty}(\vec{x})T^a\right]. \tag{2.3}$$

We will see below in Section 2.2.2 that it is actually impossible to construct an action satisfying Eq. (2.3). The solution to the dilemma is passing to the complex time Schwinger–Keldysh formulation. The careful analysis in [6] reveals, however, that to the order of interest one can find an action which gives a sufficiently similar equation of motion without resorting to the complex time contour.

In this chapter we will take all the fields to take values in the adjoint representation A (see Appendix B).

2.1. Yang–Mills term

The Yang–Mills action is given by

$$S_{\mathrm{YM}}[A] = -\frac{1}{4}\int \mathrm{d}^4x\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] = -\frac{1}{4}\int \mathrm{d}^4x\,F^a_{\mu\nu}F^{\mu\nu}_a, \tag{2.4}$$

where we used $F^{\mu\nu} = F^{\mu\nu}_a T^a$ and $\mathrm{tr}[T^a T^b] = T(\mathrm{A})\delta^{ab}$. The field-strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu] \tag{2.5}$$

and in terms of the algebra components

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^a{}_{bc} A^b_\mu A^c_\nu = \delta^{\alpha\beta}_{\mu\nu}\left(\partial_\alpha A^a_\beta + \frac{1}{2} g f^a{}_{bc} A^b_\alpha A^c_\beta\right), \tag{2.6}$$

where we have defined $\delta^{\alpha\beta}_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu$. Note that due to the anti-symmetry in both the spacetime and algebra indices the last term is now symmetric in A.

The gauge covariant derivative acting on fields in the adjoint representation is given by $D_\mu X = \partial_\mu X - \mathrm{i}g[A_\mu, X]$ or in components of the Lie algebra

$$(D_\mu X)_a = \partial_\mu X_a - \mathrm{i}g[A_\mu, X]_a = \partial_\mu X_a + g f_{abc} A^b_\mu X^c, \tag{2.7}$$

where we used Eq. (B.12). We can therefore write the covariant derivative acting in the adjoint representation as multiplication by a matrix, i.e. $(D_\mu X)_a = (D_\mu)_a{}^b X_b$, where

$$(D_\mu)_a{}^b = \delta^b_a \partial_\mu + g f_{acb} A^c_\mu. \tag{2.8}$$

In the following sections we will have derivatives acting on different variables and it is important to keep track on which variable the derivatives act. We therefore will make frequent use of the compact notation defined in Eq. (A.2).

2.1.1. Functional derivatives of the field-strength tensor

The first functional derivative of the field-strength tensor is

$$\begin{aligned}
\frac{\delta}{\delta A^a_\mu(x)} F^e_{\alpha\beta}(u) &= \frac{\delta}{\delta A^a_\mu(x)}\left[\delta^{\gamma\delta}_{\alpha\beta}\left(\partial^u_\gamma A^e_\delta(u) + \frac{1}{2} g f^e{}_{cd} A^c_\gamma(u) A^d_\delta(u)\right)\right] \\
&= \delta^{\gamma\delta}_{\alpha\beta}\left(\delta^\mu_\delta \delta^e_a \partial^u_\gamma \delta^{(4)}(u-x) + g f^e{}_{cd} A^c_\gamma(u) \delta^\mu_\delta \delta^d_a \delta^{(4)}(u-x)\right) \\
&= \delta^{\gamma\mu}_{\alpha\beta}\left(\delta^e_a \partial^u_\gamma + g f^e{}_{ca} A^c_\gamma(u)\right)\delta^{(4)}(u-x) \\
&= \delta^{\gamma\mu}_{\alpha\beta}\left(-\delta^e_a \partial^x_\gamma - g f_{ac}{}^e A^c_\gamma(x)\right)\delta^{(4)}(u-x) \\
&= \delta^{\mu\gamma}_{\alpha\beta}\left(D^x_\gamma\right)_a{}^e \delta^{(4)}(u-x),
\end{aligned} \tag{2.9}$$

where we made use of the anti-symmetry of the generalized Kronecker delta and the structure constants and furthermore used

$$(\partial_x + \partial_y)\delta(x-y) = 0. \tag{2.10}$$

The second derivative is

$$\frac{\delta^2}{\delta A^a_\mu(x)\delta A^b_\nu(y)} F^e_{\alpha\beta}(u) = \delta^{\mu\nu}_{\alpha\beta} g f_{ab}{}^e \delta^{(4)}(u-x)\delta^{(4)}(u-y). \tag{2.11}$$

All higher derivatives vanish.

2.1.2. Functional derivatives of the action

In this section we will make use of the condensed notation $1 \equiv (\mu, a, x)$, $2 \equiv (\nu, b, y)$, etc. Furthermore we write for the action

$$S_{\mathrm{YM}}[A] = -\frac{1}{4}\int_u F^2, \tag{2.12}$$

where all indices and arguments have been suppressed and the notation highlights the symmetry in F.

First derivative

The first derivative of the action is

$$\delta_1 S_{\mathrm{YM}}[A] = -\frac{1}{2}\int_u F\delta_1 F. \tag{2.13}$$

Reinserting all the indices and arguments and using Eq. (2.9) we obtain

$$\begin{aligned}
\frac{\delta S_{\mathrm{YM}}[A]}{\delta A^a_\mu(x)} &= -\frac{1}{2}\int_u F_e^{\alpha\beta}(u)\frac{\delta}{\delta A^a_\mu(x)}F^e_{\alpha\beta}(u) \\
&= -\frac{1}{2}\int_u F_e^{\alpha\beta}(u)\delta^{\mu\gamma}_{\alpha\beta}(D^x_\gamma)_a{}^e\delta^{(4)}(u-x) \\
&= -(D^x_\gamma)_a{}^e F_e^{\mu\gamma}(x) \\
&= (D^x_\gamma)_a{}^e F_e^{\gamma\mu}(x) \\
&= (D^x_\gamma F^{\gamma\mu}(x))_a.
\end{aligned} \tag{2.14}$$

Second derivative

The second derivative is

$$\delta_1\delta_2 S_{\mathrm{YM}}[A] = -\frac{1}{2}\int_u \left(F\delta_1\delta_2 F + \delta_1 F\delta_2 F\right). \tag{2.15}$$

Using Eq. (2.11) we have for the first term

$$\begin{aligned}
&-\frac{1}{2}\int_u F_e^{\alpha\beta}(u)\frac{\delta^2}{\delta A^a_\mu(x)\delta A^b_\nu(y)}F^e_{\alpha\beta}(u) \\
&= -\frac{1}{2}\int_u F_e^{\alpha\beta}(u)\delta^{\mu\nu}_{\alpha\beta}f_{ab}{}^e\delta^{(4)}(u-x)\delta^{(4)}(u-y) \\
&= -gf_{ab}{}^e F_e^{\mu\nu}(x)\delta^{(4)}(x-y).
\end{aligned} \tag{2.16}$$

Before we compute the second term we note an important identity for the gauge covariant derivative acting on a delta function:

$$\begin{aligned}
(D^x_\mu)_{ab}\delta^{(4)}(x-y) &= (\partial^x_\mu\delta_{ab} + gf_{acb}\,A^c_\mu(x))\delta^{(4)}(x-y) \\
&= (-\partial^y_\mu\delta_{ab} - gf_{bca}\,A^c_\mu(y))\delta^{(4)}(x-y) \\
&= -(D^y_\mu)_{ba}\delta^{(4)}(x-y).
\end{aligned} \tag{2.17}$$

Using Eqs. (2.9) and (2.17) gives for the second term of Eq. (2.15)

$$\begin{aligned}
&-\frac{1}{2}\int_u \frac{\delta}{\delta A^a_\mu(x)} F_e^{\alpha\beta}(u) \frac{\delta}{\delta A^b_\nu(y)} F^e_{\alpha\beta}(u) \\
&= -\frac{1}{2}\int_u g^{\alpha\kappa} g^{\beta\lambda}\delta_{ef} \frac{\delta}{\delta A^a_\mu(x)} F^e_{\alpha\beta}(u) \frac{\delta}{\delta A^b_\nu(y)} F^f_{\kappa\lambda}(u) \\
&= -\frac{1}{2}\int_u g^{\alpha\kappa} g^{\beta\lambda}\delta_{ef} \delta^{\mu\gamma}_{\alpha\beta} (D^x_\gamma)_a{}^e \delta^{(4)}(u-x) \delta^{\nu\delta}_{\kappa\lambda} (D^y_\delta)_b{}^f \delta^{(4)}(u-y) \\
&= -\int_u \delta^{\mu\gamma}{}_{,}{}^{\nu\delta} \delta_{ef} (D^x_\gamma)_a{}^e \delta^{(4)}(u-x) (D^y_\delta)_b{}^f \delta^{(4)}(u-y) \\
&= -\delta^{\mu\gamma}{}_{,}{}^{\nu\delta} (D^x_\gamma)_a{}^e (D^y_\delta)_{be} \delta^{(4)}(x-y) \\
&= \delta^{\mu\gamma}{}_{,}{}^{\nu\delta} (D^x_\gamma)_a{}^e (D^x_\delta)_{eb} \delta^{(4)}(x-y) \\
&= \delta^{\mu\gamma}{}_{,}{}^{\nu\delta} (D^x_\gamma D^x_\delta)_{ab} \delta^{(4)}(x-y) \\
&= (g^{\mu\nu} g^{\gamma\delta} - g^{\mu\delta} g^{\nu\gamma}) (D^x_\gamma D^x_\delta)_{ab} \delta^{(4)}(x-y) \\
&= (g^{\mu\nu} (D^2_x)_{ab} - (D^\nu_x D^\mu_x)_{ab}) \delta^{(4)}(x-y) \\
&= (g^{\mu\nu} (D^2_x)_{ab} - (D^\mu_x D^\nu_x)_{ab} - [D^\nu_x, D^\mu_x]_{ab}) \delta^{(4)}(x-y) \\
&= (g^{\mu\nu} (D^2_x)_{ab} - (D^\mu_x D^\nu_x)_{ab} + g f_{ab}{}^c F^{\nu\mu}_c(x)) \delta^{(4)}(x-y) \\
&= (g^{\mu\nu} (D^2_x)_{ab} - (D^\mu_x D^\nu_x)_{ab} - g f_{ab}{}^c F^{\mu\nu}_c(x)) \delta^{(4)}(x-y),
\end{aligned} \tag{2.18}$$

where we introduced the notation

$$\delta^{\alpha\beta}_{\gamma\delta} g_{\alpha\mu} = \delta_\mu{}^\beta{}_{,\gamma\delta} \tag{2.19}$$

and generalizations thereof and used the identity

$$[D^\mu, D^\nu]_{ab} = -g f_{ab}{}^c F^{\mu\nu}_c. \tag{2.20}$$

Putting everything together we have for the second derivative

$$\frac{\delta^2 S_{\mathrm{YM}}[A]}{\delta A^a_\mu(x) \delta A^b_\nu(y)} = \left(g^{\mu\nu} (D^2_x)_{ab} - (D^\mu_x D^\nu_x)_{ab} - 2 g f_{ab}{}^c F^{\mu\nu}_c(x) \right) \delta^{(4)}(x-y). \tag{2.21}$$

Third derivative

The third derivative is

$$\begin{aligned}
\delta_1\delta_2\delta_3 S_{\mathrm{YM}}[A] = &-\frac{1}{2}\int_u (\delta_1\delta_2 F \delta_3 F + \delta_2\delta_3 F \delta_1 F + \delta_3\delta_1 F \delta_2 F) \\
= &-\frac{1}{2}\int_u (\delta_1\delta_2 F \delta_3 F + 2 \text{ cyclic permutations}).
\end{aligned} \tag{2.22}$$

The first term in Eq. (2.22) is with all indices and arguments made explicit

$$\begin{aligned}
&-\frac{1}{2}\int_u \frac{\delta^2}{\delta A^a_\mu(x)\delta A^b_\nu(y)} F^e_{\alpha\beta}(u) g^{\alpha\kappa} g^{\beta\lambda} \delta_{ef} \frac{\delta}{\delta A^c_\rho(z)} F^f_{\kappa\lambda}(u) \\
&= -\frac{1}{2}\int_u \delta^{\mu\nu}_{\alpha\beta} g f_{ab}{}^e \delta^{(4)}(u-x)\delta^{(4)}(u-y) g^{\alpha\kappa} g^{\beta\lambda} \delta_{ef} \delta^{\rho\gamma}_{\kappa\lambda} (D^z_\gamma)_c{}^f \delta^{(4)}(u-z) \\
&= -\int_u \delta^{\mu\nu}{}_{,}{}^{\rho\gamma} g f_{ab}{}^e \delta^{(4)}(u-x)\delta^{(4)}(u-y)(D^z_\gamma)_{ce}\delta^{(4)}(u-z) \\
&= -\delta^{\mu\nu}{}_{,}{}^{\rho\gamma} g f_{ab}{}^e (D^z_\gamma)_{ce}\delta^{(4)}(x-z)\delta^{(4)}(x-y) \\
&= \delta^{\mu\nu}{}_{,}{}^{\rho\gamma} g f_{ab}{}^e (D^x_\gamma)_{ec}\delta^{(4)}(x-z)\delta^{(4)}(x-y).
\end{aligned} \tag{2.23}$$

Adding the cyclic permutations Eq. (2.22) becomes

$$\begin{aligned}
\frac{\delta^3 S_{\mathrm{YM}}[A]}{\delta A^a_\mu(x)\delta A^b_\nu(y)\delta A^c_\rho(z)} =& \delta^{\mu\nu}{}_{,}{}^{\rho\gamma} g f_{ab}{}^e (D^x_\gamma)_{ec}\delta^{(4)}(x-z)\delta^{(4)}(x-y) \\
&+ \delta^{\nu\rho}{}_{,}{}^{\mu\gamma} g f_{bc}{}^e (D^y_\gamma)_{ea}\delta^{(4)}(y-x)\delta^{(4)}(y-z) \\
&+ \delta^{\rho\mu}{}_{,}{}^{\nu\gamma} g f_{ca}{}^e (D^z_\gamma)_{eb}\delta^{(4)}(z-y)\delta^{(4)}(z-x).
\end{aligned} \tag{2.24}$$

Fourth derivative

The fourth derivative is

$$\begin{aligned}
\delta_1\delta_2\delta_3\delta_4 S_{\mathrm{YM}}[A] =& -\frac{1}{2}\int_u (\delta_1\delta_2 F \delta_3\delta_4 F + \delta_1\delta_3 F \delta_1\delta_4 F + \delta_1\delta_4 F \delta_2\delta_3 F) \\
=& -\frac{1}{2}\int_u (\delta_1\delta_2 F \delta_3\delta_4 F + (2\leftrightarrow 3) + (3\leftrightarrow 4)).
\end{aligned} \tag{2.25}$$

Using Eq. (2.11) and expanding all the indices and arguments the first term of Eq. (2.25) becomes

$$\begin{aligned}
&-\frac{1}{2}\int_u \frac{\delta^2}{\delta A^a_\mu(x)\delta A^b_\nu(y)} F^e_{\alpha\beta}(u) g^{\alpha\kappa} g^{\beta\lambda} \delta_{ef} \frac{\delta^2}{\delta A^c_\rho(z)\delta A^d_\sigma(w)} F^f_{\kappa\lambda}(u) \\
&= -\frac{1}{2}\int_u \delta^{\mu\nu}_{\alpha\beta} f_{ab}{}^e \delta^{(4)}(u-x)\delta^{(4)}(u-y) g^{\alpha\kappa} g^{\beta\lambda} \delta_{ef} \delta^{\rho\sigma}_{\kappa\lambda} f_{cd}{}^f \delta^{(4)}(u-z)\delta^{(4)}(u-w) \\
&= -\int_u \delta^{\mu\nu}{}_{,}{}^{\rho\sigma} g f_{ab}{}^e \delta^{(4)}(u-x)\delta^{(4)}(u-y)\delta_{ef} g f_{cd}{}^f \delta^{(4)}(u-z)\delta^{(4)}(u-w) \\
&= -\delta^{\mu\nu}{}_{,}{}^{\rho\sigma} g f_{ab}{}^e g f_{cde}\, \delta^{(4)}(x-y)\delta^{(4)}(x-z)\delta^{(4)}(x-w).
\end{aligned} \tag{2.26}$$

Adding all terms together we get for Eq. (2.25)

$$\begin{aligned}
&\frac{\delta^4 S_{\mathrm{YM}}[A]}{\delta A^a_\mu(x)\delta A^b_\nu(y)\delta A^c_\rho(z)\delta A^d_\sigma(w)} \\
&= -g^2 \left[\delta^{\mu\nu}{}_{,}{}^{\rho\sigma} f_{ab}{}^e f_{cde} + \delta^{\mu\rho}{}_{,}{}^{\nu\sigma} f_{ac}{}^e f_{bde} + \delta^{\mu\sigma}{}_{,}{}^{\nu\rho} f_{ad}{}^e f_{bce}\right] \\
&\quad \times \delta^{(4)}(x-y)\delta^{(4)}(x-z)\delta^{(4)}(x-w).
\end{aligned} \tag{2.27}$$

2.2. Wilson line term

In this section we will give examples for possible terms resulting in the desired coupling of the gluons to the classical color charge density distribution ρ.

The main ingredient for the source term of the action is the Wilson line

$$W[A^-](\vec{u}) \equiv W_{\infty,-\infty}[A^-](\vec{u}), \tag{2.28}$$

where

$$W_{x^+,y^+}[A^-](\vec{u}) = \mathrm{P}\exp\left[\mathrm{i}g\int_{y^+}^{x^+}\mathrm{d}z^+\,A^-(z^+,\vec{u})\right]. \tag{2.29}$$

2.2.1. Functional derivatives of the Wilson line

In this subsection we will consider the functional derivatives of the Wilson line. At the end of this subsection we state the results evaluated at the solution of the classical equations of motion (in the lightcone gauge $A^+ = 0$), $A^\mu = \delta^\mu_i \mathcal{A}^i$.

First derivative

The first derivative of the Wilson line is

$$\frac{\delta W(\vec{u})}{\delta A^-_a(x)} = (\mathrm{i}g)W_{\infty,x^+}(\vec{u})T^a W_{x^+,-\infty}(\vec{u})\delta^{(3)}(\vec{u}-\vec{x}) \tag{2.30}$$

Second derivative

The second derivative is

$$\begin{aligned}\frac{\delta^2 W(\vec{u})}{\delta A^-_a(x)\delta A^-_b(y)} =&(\mathrm{i}g)^2\big\{\theta(x^+-y^+)W_{\infty,x^+}(\vec{u})T^a W_{x^+,y^+}(\vec{u})T^b W_{x^+,-\infty}(\vec{u})\\ &+\theta(y^+-x^+)W_{\infty,y^+}(\vec{u})T^b W_{y^+,x^+}(\vec{u})T^a W_{y^+,-\infty}(\vec{u})\big\}\\ &\times\delta^{(3)}(\vec{u}-\vec{x})\delta^{(3)}(\vec{u}-\vec{y})\end{aligned} \tag{2.31}$$

Third derivative

The third derivative is

$$\frac{\delta^3 W(\vec{u})}{\delta A^-_a(x)\delta A^-_b(y)\delta A^-_c(z)}$$

$$
\begin{aligned}
= (\mathrm{i}g)^3 \big\{ &\theta(x^+ - y^+)\theta(y^+ - z^+) w_{\infty,x^+}(\vec{u}) T^a w_{x^+,y^+}(\vec{u}) T^b w_{y^+,z^+}(\vec{u}) T^c w_{z^+,-\infty}(\vec{u}) \\
&+ \theta(y^+ - z^+)\theta(z^+ - x^+) w_{\infty,y^+}(\vec{u}) T^b w_{y^+,z^+}(\vec{u}) T^c w_{z^+,x^+}(\vec{u}) T^a w_{x^+,-\infty}(\vec{u}) \\
&+ \theta(z^+ - x^+)\theta(x^+ - y^+) w_{\infty,z^+}(\vec{u}) T^c w_{z^+,x^+}(\vec{u}) T^a w_{x^+,y^+}(\vec{u}) T^b w_{y^+,-\infty}(\vec{u}) \\
&+ \theta(x^+ - z^+)\theta(z^+ - y^+) w_{\infty,x^+}(\vec{u}) T^a w_{x^+,z^+}(\vec{u}) T^c w_{z^+,y^+}(\vec{u}) T^b w_{y^+,-\infty}(\vec{u}) \\
&+ \theta(y^+ - x^+)\theta(x^+ - z^+) w_{\infty,y^+}(\vec{u}) T^b w_{y^+,x^+}(\vec{u}) T^a w_{x^+,z^+}(\vec{u}) T^c w_{z^+,-\infty}(\vec{u}) \\
&+ \theta(z^+ - y^+)\theta(y^+ - x^+) w_{\infty,z^+}(\vec{u}) T^c w_{z^+,y^+}(\vec{u}) T^b w_{y^+,x^+}(\vec{u}) T^a w_{x^+,-\infty}(\vec{u}) \big\} \\
&\times \delta^{(3)}(\vec{u} - \vec{x}) \delta^{(3)}(\vec{u} - \vec{y}) \delta^{(3)}(\vec{u} - \vec{z})
\end{aligned}
\tag{2.32}
$$

Evaluation on classical solutions

Evaluated on the classical solutions of the equations of motion the Wilson lines become trivial. We therefore have

$$
\left.\frac{\delta W(\vec{u})}{\delta A_a^-(x)}\right|_{A=\mathcal{A}} = (\mathrm{i}g) T^a \delta^{(3)}(\vec{u} - \vec{x}), \tag{2.33a}
$$

$$
\begin{aligned}
\left.\frac{\delta^2 W(\vec{u})}{\delta A_a^-(x) \delta A_b^-(y)}\right|_{A=\mathcal{A}} = &(\mathrm{i}g)^2 \Big\{ \theta(x^+ - y^+) T^a T^b + \theta(y^+ - x^+) T^b T^a \Big\} \\
&\times \delta^{(3)}(\vec{u} - \vec{x}) \delta^{(3)}(\vec{u} - \vec{y}),
\end{aligned}
\tag{2.33b}
$$

$$
\begin{aligned}
\left.\frac{\delta^3 W(\vec{u})}{\delta A_a^-(x) \delta A_b^-(y) \delta A_c^-(z)}\right|_{A=\mathcal{A}} = &(\mathrm{i}g)^3 \big\{ \theta(x^+ - y^+)\theta(y^+ - z^+) T^a T^b T^c \\
&+ \theta(y^+ - z^+)\theta(z^+ - x^+) T^b T^c T^a \\
&+ \theta(z^+ - x^+)\theta(x^+ - y^+) T^c T^a T^b \\
&+ \theta(x^+ - z^+)\theta(z^+ - y^+) T^a T^c T^b \\
&+ \theta(y^+ - x^+)\theta(x^+ - z^+) T^b T^a T^c \\
&+ \theta(z^+ - y^+)\theta(y^+ - x^+) T^c T^b T^a \big\} \\
&\times \delta^{(3)}(\vec{u} - \vec{x}) \delta^{(3)}(\vec{u} - \vec{y}) \delta^{(3)}(\vec{u} - \vec{z}).
\end{aligned}
\tag{2.33c}
$$

2.2.2. Non-integrability of the equation of motion

Having established the functional derivatives of Wilson lines, we are now able to proceed showing that it is actually impossible to find an action such that Eq. (2.2) holds.

Recall that the Wilson line term should satisfy (Eq. (2.3))

$$
\frac{\delta S_W[A^-, \rho]}{\delta A_a^-(x)} = -\frac{1}{T(\mathrm{A})} \operatorname{tr}\Big[W_{x^+,-\infty}(\vec{x}) \rho(\vec{x}) W^\dagger_{x^+,-\infty}(\vec{x}) T^a \Big] \tag{2.34}
$$

if we want Eq. (2.2) to hold. Taking a second functional derivative would give

$$\begin{aligned}\left.\frac{\delta^2 S_W[A^-,\rho]}{\delta A_b^-(y)\delta A_a^-(x)}\right|_{A=\mathcal{A}} &= -\mathrm{i}g\theta(x^+-y^+)\delta^{(3)}(\vec{x}-\vec{y})\frac{1}{T(\mathrm{A})}\\ &\quad\times \mathrm{tr}[W_{x^+,y^+}(\vec{x})T^b W_{y^+,-\infty}(\vec{x})\rho(\vec{x})W^\dagger_{x^+,\infty}(\vec{x})T^a\\ &\qquad - W_{x^+,-\infty}(\vec{x})\rho(\vec{x})W^\dagger_{x^+,y^+}(\vec{x})T^b W^\dagger_{y^+,\infty}(\vec{x})T^a]_{A=\mathcal{A}}\\ &= -\mathrm{i}g\theta(x^+-y^+)\delta^{(3)}(\vec{x}-\vec{y})\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left[T^b\rho(\vec{x})T^a-\rho(\vec{x})T^bT^a\right]\\ &= -\mathrm{i}g\theta(x^+-y^+)\delta^{(3)}(\vec{x}-\vec{y})\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left[[T^a,T^b]\rho(\vec{x})\right]\\ &= \mathrm{i}g\theta(x^+-y^+)\delta^{(3)}(\vec{x}-\vec{y})\frac{1}{T(\mathrm{A})}\rho^{ab}(\vec{x}).\end{aligned} \tag{2.35}$$

Clearly, this is not symmetric in the derivatives, i.e.

$$\left.\frac{\delta^2 S_W[A^-,\rho]}{\delta A_a^-(x)\delta A_b^-(y)}\right|_{A=\mathcal{A}} \neq \left.\frac{\delta^2 S_W[A^-,\rho]}{\delta A_b^-(y)\delta A_a^-(x)}\right|_{A=\mathcal{A}}. \tag{2.36}$$

Therefore there is no action for which we can have Eq. (2.2).

2.2.3. Standard term

The most common source term for the CGC effective action reads [6]

$$S_W[A^-,\rho] = -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{u}\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}(W[A^-](\vec{u})\rho(\vec{u})). \tag{2.37}$$

First derivative

The first derivative reads

$$\begin{aligned}\left.\frac{\delta S_W[A^-]}{\delta A_a^-(x)}\right|_{A=\mathcal{A}} &= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{u}\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left(\left.\frac{\delta W[A^-](\vec{u})}{\delta A_a^-(x)}\right|_{A=\mathcal{A}}\rho(\vec{u})\right)\\ &= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{u}\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left(\mathrm{i}gT^a\delta^{(3)}(\vec{u}-\vec{x})\rho(\vec{u})\right)\\ &= -\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left(T^a\rho(\vec{x})\right).\end{aligned} \tag{2.38}$$

Second derivative

The second derivative reads

$$\begin{aligned}&\left.\frac{\delta^2 S_W[A^-]}{\delta A_a^-(x)\delta A_b^-(y)}\right|_{A=\mathcal{A}}\\ &= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{u}\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left(\left.\frac{\delta^2 W[A^-](\vec{u})}{\delta A_a^-(x)\delta A_b^-(y)}\right|_{A=\mathcal{A}}\rho(\vec{u})\right)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{u}\,\frac{1}{T(\mathrm{A})}\mathrm{tr}\left((\mathrm{i}g)^2\left\{\theta(x^+-y^+)T^aT^b+\theta(y^+-x^+)T^bT^a\right\}\right.\\
&\qquad\qquad\left.\times\,\delta^{(3)}(\vec{u}-\vec{x})\delta^{(3)}(\vec{u}-\vec{y})\rho(\vec{u})\right)\\
&= -(\mathrm{i}g)\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left(\left\{\theta(x^+-y^+)T^aT^b+\theta(y^+-x^+)T^bT^a\right\}\rho(\vec{x})\right)\delta^{(3)}(\vec{x}-\vec{y})\\
&= -(\mathrm{i}g)\frac{\mathrm{i}}{2}\left\{\theta(x^+-y^+)f^{abc}+\theta(y^+-x^+)f^{bac}\right\}\rho_c(\vec{x})\delta^{(3)}(\vec{x}-\vec{y})\\
&= \frac{\mathrm{i}g}{2}\left\{\theta(x^+-y^+)-\theta(y^+-x^+)\right\}(-\mathrm{i}f^{abc}\rho_c(\vec{x}))\delta^{(3)}(\vec{x}-\vec{y})\\
&= \frac{\mathrm{i}g}{2}\left\{\theta(x^+-y^+)-\theta(y^+-x^+)\right\}\rho^{ab}(\vec{x})\delta^{(3)}(\vec{x}-\vec{y})\\
&= \frac{\mathrm{i}g}{2}\left\{\theta(x^+-y^+)-\theta(y^+-x^+)\right\}(-\mathrm{i}f^{abc}\rho_c(\vec{x}))\delta^{(3)}(\vec{x}-\vec{y})\\
&= \frac{\mathrm{i}g}{2}\epsilon(x^+-y^+)\rho^{ab}(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}),
\end{aligned}
\tag{2.39}
$$

where we used Eqs. (B.26) and (B.27) and defined the sign function

$$
\epsilon(x)=\theta(x)-\theta(-x). \tag{2.40}
$$

Third derivative

The third derivative reads

$$
\begin{aligned}
&\left.\frac{\delta^3 S_W[A^-]}{\delta A_a^-(x)\delta A_b^-(y)\delta A_c^-(z)}\right|_{A=\mathcal{A}}\\
&= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{u}\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}\left(\left.\frac{\delta^3 W[A^-](\vec{u})}{\delta A_a^-(x)\delta A_b^-(y)\delta A_c^-(z)}\right|_{A=\mathcal{A}}\rho(\vec{u})\right)\\
&= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{u}\,\frac{1}{T(\mathrm{A})}\mathrm{tr}[(\mathrm{i}g)^3\big\{\theta(x^+-y^+)\theta(y^+-z^+)T^aT^bT^c\\
&\qquad\qquad+\theta(y^+-z^+)\theta(z^+-x^+)T^bT^cT^a\\
&\qquad\qquad+\theta(z^+-x^+)\theta(x^+-y^+)T^cT^aT^b\\
&\qquad\qquad+\theta(x^+-z^+)\theta(z^+-y^+)T^aT^cT^b\\
&\qquad\qquad+\theta(y^+-x^+)\theta(x^+-z^+)T^bT^aT^c\\
&\qquad\qquad+\theta(z^+-y^+)\theta(y^+-x^+)T^cT^bT^a\big\}\\
&\qquad\qquad\times\,\delta^{(3)}(\vec{u}-\vec{x})\delta^{(3)}(\vec{u}-\vec{y})\delta^{(3)}(\vec{u}-\vec{z})\rho(\vec{u})]
\end{aligned}
$$

$$
\begin{aligned}
= -(\mathrm{i}g)^2 \frac{1}{T(\mathrm{A})} \operatorname{tr}[\big\{ & \theta(x^+ - y^+)\theta(y^+ - z^+) T^a T^b T^c \\
& + \theta(y^+ - z^+)\theta(z^+ - x^+) T^b T^c T^a \\
& + \theta(z^+ - x^+)\theta(x^+ - y^+) T^c T^a T^b \\
& + \theta(x^+ - z^+)\theta(z^+ - y^+) T^a T^c T^b \\
& + \theta(y^+ - x^+)\theta(x^+ - z^+) T^b T^a T^c \\
& + \theta(z^+ - y^+)\theta(y^+ - x^+) T^c T^b T^a \big\} \rho(\vec{x})] \\
\times \delta^{(3)}(\vec{x} - \vec{y}) \delta^{(3)}(\vec{y} - \vec{z}) &
\end{aligned}
\tag{2.41}
$$

2.2.4. Logarithm term

A less popular choice is a term containing a logarithm [17]

$$
S_{\ln W}[A^-, \rho] = -\frac{1}{\mathrm{i}g} \int \mathrm{d}^3\vec{u}\, \frac{1}{T(\mathrm{A})} \operatorname{tr}(\ln\left(W[A^-](\vec{u})\right)\rho(\vec{u})). \tag{2.42}
$$

This term was later studied by Fukushima who derived it by explicitly integrating out high momentum quarks and gluons [16] and derived the JIMWLK hamiltonian using it [18]. In light of the explicit derivation, it is tempting to conclude that this term is the more physical one.

Derivatives of the logarithm

The logarithm can be conveniently written as

$$
\ln W = \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)} \left(F^{-1}(W;\mu) - I \right), \tag{2.43}
$$

where

$$
F(W;\mu) = I - \mu(I - W). \tag{2.44}
$$

Using

$$
\delta F^{-1} = -F^{-1}\delta F F^{-1} = (-\mu) F^{-1} \delta W F^{-1} \tag{2.45}
$$

we have for the first functional derivative

$$
\delta_1 \ln(W) = \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)} (-\mu) F^{-1} \delta_1 W F^{-1}, \tag{2.46}
$$

where we introduced a condensed notation $1 \equiv (a, x)$ which we will use judiciously in the following. The second functional derivative is

$$
\begin{aligned}
& \delta_1\delta_2 \ln W \\
& = \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)} (-\mu) \left\{ \delta_2 F^{-1} \delta_1 W F^{-1} + F^{-1}\delta_1\delta_2 W F^{-1} + F^{-1}\delta_1 W \delta_2 F^{-1} \right\} \\
& = \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)} \left\{ (-\mu)^2 \left[F^{-1}\delta_1 W F^{-1} \delta_2 W F^{-1} + (1 \leftrightarrow 2) \right] + (-\mu) F^{-1}\delta_1\delta_2 W F^{-1} \right\},
\end{aligned}
\tag{2.47}
$$

where we have brought the terms into an order which suggests the general pattern. The third functional derivative is

$$\begin{aligned}\delta_1\delta_2\delta_3 \ln W = \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)} \Big\{ &(-\mu)^3 \left[F^{-1}\delta_1 W F^{-1}\delta_2 W F^{-1}\delta_3 W F^{-1} + 5 \text{ permutations}\right] \\ &+(-\mu)^2 \big[\left(F^{-1}\delta_1\delta_2 W F^{-1}\delta_3 W F^{-1} + 2 \text{ cyclic}\right) \\ &\qquad + \left(F^{-1}\delta_1 W F^{-1}\delta_2\delta_3 W F^{-1} + 2 \text{ cyclic}\right) \big] \\ &+(-\mu) F^{-1}\delta_1\delta_2\delta_3 W F^{-1} \Big\}.\end{aligned} \tag{2.48}$$

Next we note that when evaluating the above derivatives on the classical solution we have from Eq. (2.44)

$$F|_{A=\mathcal{A}} = I. \tag{2.49}$$

Using this the integrals can be carried out and we obtain

$$\delta_1 \ln(W)|_{A=\mathcal{A}} = \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)}(-\mu)\delta_1 W = \delta_1 W, \tag{2.50a}$$

$$\begin{aligned}\delta_1\delta_2 \ln W|_{A=\mathcal{A}} &= \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)} \left\{(-\mu)^2 \left[\delta_1 W \delta_2 W + (1 \leftrightarrow 2)\right] + (-\mu)\delta_1\delta_2 W\right\} \\ &= -\frac{1}{2}\left[\delta_1 W \delta_2 W + (1 \leftrightarrow 2)\right] + \delta_1\delta_2 W,\end{aligned} \tag{2.50b}$$

$$\begin{aligned}\delta_1\delta_2\delta_3 \ln W|_{A=\mathcal{A}} = \int_0^1 \mathrm{d}\mu\, \frac{1}{(-\mu)} \{ &(-\mu)^3 \left[\delta_1 W \delta_2 W \delta_3 W + 5 \text{ permutations}\right] \\ &+ (-\mu)^2 \big[\left(\delta_1\delta_2 W \delta_3 W + 2 \text{ cyclic}\right) \\ &\qquad + \left(\delta_1 W \delta_2\delta_3 W + 2 \text{ cyclic}\right)\big] \\ &+ (-\mu)\delta_1\delta_2\delta_3 W \} \\ = \frac{1}{3}&\left[\delta_1 W \delta_2 W \delta_3 W + 5 \text{ permutations}\right] \\ &- \frac{1}{2}\left[\left(\delta_1\delta_2 W \delta_3 W + 2 \text{ cyclic}\right) + \left(\delta_1 W \delta_2\delta_3 W + 2 \text{ cyclic}\right)\right] \\ &+ \delta_1\delta_2\delta_3 W.\end{aligned} \tag{2.50c}$$

We note that the first and second derivative are the same as for the standard term. The third derivative contains the same term as the standard term, which is encoded in the three derivatives acting on the same Wilson line. However, there are many more ($3! + 2 \times 3 \times 2! = 18$) additional terms from the cases in which the derivatives act on different Wilson lines.

2.2.5. Gauge invariance

Under a gauge transformation the Wilson line (Eq. (C.40)) and color charge density distribution transform as follows

$$W(\vec{x}) \to W'(\vec{x}) = U(\infty, \vec{x}) W(\vec{x}) U^\dagger(-\infty, \vec{x}), \tag{2.51a}$$

$$\rho(\vec{x}) \to \rho'(\vec{x}) = U(-\infty, \vec{x})\rho(\vec{x})U^\dagger(-\infty, \vec{x}), \tag{2.51b}$$

where we assumed that the color charge density is defined at $x^+ = -\infty$.

Inserting this into the standard term action (Eq. (2.37)) gives[1]

$$\begin{aligned} S'_W[A^-, \rho] &= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{x}\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}[U(\infty, \vec{x})W(\vec{x})U^\dagger(-\infty, \vec{x})U(-\infty, \vec{x})\rho(\vec{x})U^\dagger(-\infty, \vec{x})] \\ &= -\frac{1}{\mathrm{i}g}\int \mathrm{d}^3\vec{x}\,\frac{1}{T(\mathrm{A})}\,\mathrm{tr}[W(\vec{x})\rho(\vec{x})U^\dagger(-\infty, \vec{x})U(\infty, \vec{x})]. \end{aligned} \tag{2.52}$$

We see therefore that the action is only invariant under "periodic" gauge transformations which satisfy

$$U(\infty, \vec{x}) = U(-\infty, \vec{x}). \tag{2.53}$$

[1] The argument is the same for the logarithm term action (Eq. (2.42)) as can be seen by noting that for any function $f(W'(\vec{x})) = U(\infty, \vec{x})f(W(\vec{x}))U^\dagger(-\infty, \vec{x})$.

Chapter 3.

The JIMWLK equation

3.1. Derivation of the JIMWLK equation

In this section we follow the original derivation of the JIMWLK equation found in [4]. Alternative derivations can be found in [18, 19]. A discussion on the relation to the BRST symmetry can be found in [20].

3.1.1. Renormalization group procedure

In Section 1.3 we argued that we can introduce a cutoff $\Lambda \equiv \Lambda^+$ in the magnitude of longitudinal momenta p^+ and that those degrees of freedom with longitudinal momenta greater than that cutoff scale can be integrated out to give an effective theory at the cutoff scale with generating functional

$$Z[J] = \int [D\rho] W_\Lambda[\rho] \int^{\Lambda} [DA] \mathrm{e}^{\mathrm{i}S[A,\rho] + \mathrm{i}\int \mathrm{d}^4x\, J^\mu_a(x) A^a_\mu(x)}, \tag{3.1}$$

where we now denote the effective action merely by S and we have normalized the path measure, eliminating the need for the normalizing factor as in Eq. (1.52). It is clear that the cutoff scale is arbitrary. Assuming that we know the effective action $S[A, \rho]$ for a certain cutoff Λ, it is natural to ask what would happen if we chose a lower cutoff $\Lambda' < \Lambda$. Following this question will lead us to a renormalization group equation (RGE) for the statistical weight function $W_\Lambda[\rho]$.

In order to obtain the RGE one has to integrate out some degrees of freedom. To do so we introduce a new momentum scale Λ'. To that end we separate the gluon field into the following contributions

$$A = \mathcal{A} + \delta A + a, \tag{3.2}$$

where $\mathcal{A} = \mathcal{A}[\rho]$ is the solution to the classical equations of motion for a given charge density ρ, δA are the soft gluons with longitudinal momenta $|p^+| < \Lambda'$ and a are the semi-fast gluons with momenta $\Lambda' < |p^+| < \Lambda$. It is instructive to write the soft and semi-hard gluon field formally as a Fourier integral

$$\delta A(x) = \int_{|p^+|<\Lambda'} \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}px} \delta A(p), \tag{3.3a}$$

$$a(x) = \int_{\Lambda<|p^+|<\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}px} a(p). \tag{3.3b}$$

An important feature of the lightcone gauge is that it is compatible with this separation of scales. To see why this is the case, recall that there is some residual gauge freedom in the lightcone gauge. The gauge transformation Eq. (C.23) allows transformations within the lightcone gauge, where the gauge group element satisfies

$$\partial^+ U = \partial_- U = \frac{\partial}{\partial x^-} U = 0. \tag{3.4}$$

In momentum space this relation reads (p^+ is the conjugate momentum of x^-)

$$U(p) = U(p^-, \boldsymbol{p}_\perp) I(p^+), \tag{3.5}$$

where we defined $I(p^+) = 2\pi\delta(p^+)$. In momentum space the gauge transformation Eq. (C.23) reads[1]

$$A'_\mu(p) = (U * (A_\mu * U^\dagger))(p) + \frac{1}{g}(U * (\mathrm{id}_\mu \cdot U^\dagger))(p), \tag{3.6}$$

where we defined the vector valued identity function $\mathrm{id}_\mu(p) = p_\mu$. The convolution of a function with the (rescaled) Dirac delta leaves a function invariant, i.e.

$$(f * I)(p^+) = \int \frac{\mathrm{d}p'^+}{2\pi} f(p'^+) I(p^+ - p'^+) = \int \frac{\mathrm{d}p'^+}{2\pi} f(p'^+) 2\pi\delta(p^+ - p'^+) = f(p^+). \tag{3.7}$$

Therefore the dependence of the gauge fields on the longitudinal momentum p^+ remains unchanged under residual gauge transformations.

It is the contribution of the semi-fast gluons a which needs to be integrated out in order to obtain the effective theory at the new scale Λ'. The integral measure in Eq. (3.1) then is

$$\int^{\Lambda} [DA] \equiv \int^{\Lambda'} [D\delta A] \int_{\Lambda'}^{\Lambda} [Da]. \tag{3.8}$$

In order to make progress we assume that the effective theory at the new scale Λ' has functionally the same form as the original theory at the old scale Λ. The only difference between the two momentum scales Λ and Λ' will turn out to be that the weight functions differ. Explicitly we have

$$\int [D\rho] W_\Lambda[\rho] \int^{\Lambda'} [D\delta A] \int_{\Lambda'}^{\Lambda} [Da] \mathrm{e}^{\mathrm{i}S[\mathcal{A}+\delta A+a,\rho]} = \int [D\rho] W_{\Lambda'}[\rho] \int^{\Lambda'} [D\delta A] \mathrm{e}^{\mathrm{i}S[\mathcal{A}+\delta A,\rho]}. \tag{3.9}$$

[1] We have defined here the Fourier transform of the gauge transformation for $U^\dagger$, i.e. $U^\dagger(x) = \int_p \mathrm{e}^{-\mathrm{i}px} U^\dagger(p)$.

3.1.2. Expansion of the action

In this section we will expand the action around the classical solutions plus soft gluon contributions, i.e. around $A = \mathcal{A} + \delta A$. In a next step we will further expand the terms interacting with the semi-fast gluons in powers of the soft gluon fields δA and retain only the linear terms which will give us the quantum corrections to the soft current.

The expansion reads (we omit the ρ-dependence for now)

$$\begin{aligned} S[\mathcal{A}+\delta A+a] &= S[\mathcal{A}+\delta A] + a_u \left.\frac{\delta S[A]}{\delta A_u}\right|_{A=\mathcal{A}+\delta A} + \frac{1}{2}a_u a_v \left.\frac{\delta^2 S[A]}{\delta A_u \delta A_v}\right|_{A=\mathcal{A}+\delta A} + O(a^3) \\ &= S[\mathcal{A}+\delta A] + a_u \left(\frac{\delta S[\mathcal{A}]}{\delta \mathcal{A}_u} + \delta A_x \frac{\delta^2 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_x} + O(\delta A^2)\right) \\ &\quad + \frac{1}{2}a_u a_v \left(\frac{\delta^2 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_v} + \delta A_x \frac{\delta^3 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_v \delta \mathcal{A}_x} + O(\delta A^2)\right) + O(a^3) \\ &= S[\mathcal{A}+\delta A] + \delta A_x \left(a_u \frac{\delta^2 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_x} + \frac{1}{2}a_u a_v \frac{\delta^3 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_v \delta \mathcal{A}_x}\right) \\ &\quad + \frac{1}{2}a_u a_v \frac{\delta^2 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_v} + \cdots, \end{aligned} \tag{3.10}$$

where we used a condensed notation where x, y, u, v denote spacetime arguments, Minkowski indices and color indices and it is understood that repeated indices are summed or integrated over, i.e.

$$A^u B_u \equiv \int \mathrm{d}^4 u \, A^\mu_a(u) B^a_\mu(u). \tag{3.11}$$

We also used the fact that $\mathcal{A}$ is the solution to the equations of motion, i.e.

$$\left.\frac{\delta S[A]}{\delta A_u}\right|_{A=\mathcal{A}} = 0. \tag{3.12}$$

It would be incorrect to claim that Eq. (3.10) is valid up to some order in δA. Indeed, the term $S[\mathcal{A}+\delta A]$ contains terms which are quartic in δA. Nevertheless, the form stated in Eq. (3.10) is what we are interested in. The first term would naturally appear as the action at the new scale Λ' and the quantum corrections can be summarized as the soft fields coupling to a soft current

$$\delta J^x = -\left.\frac{\delta S[A]}{\delta A_x}\right|_{A=\mathcal{A}+a} = -a_u \frac{\delta^2 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_x} - \frac{1}{2}a_u a_v \frac{\delta^3 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_v \delta \mathcal{A}_x} \tag{3.13}$$

given in the second term. The last term defines the propagator of the semi-fast gluons in the classical background field

$$(G^{-1}[\rho])^{uv} \equiv \frac{\delta^2 S[\mathcal{A}]}{\delta \mathcal{A}_u \delta \mathcal{A}_v}. \tag{3.14}$$

With the above definitions the expansion of the action (Eq. (3.10)) reads

$$S[\mathcal{A}+\delta A+a] = S[\mathcal{A}+\delta A] - \delta A_x \delta J^x + \frac{1}{2} a_u (G^{-1}[\rho])^{uv} a_v. \tag{3.15}$$

If we assume that the gauge field corresponding to the soft gluons is weak, we can linearize the Wilson line

$$W_{\infty,-\infty}[\delta A^-](\vec{x}) \approx I + \mathrm{i}g \int_{-\infty}^{\infty} \mathrm{d}x^+ \, \delta A^-(x^+, \vec{x}). \tag{3.16}$$

Inserting this into the effective action with the standard form Eq. (2.37) for the Wilson line term gives

$$\begin{aligned} S[\mathcal{A}+\delta A] &= S_{\mathrm{YM}}[\mathcal{A}+\delta A] + S_W[\mathcal{A}^- + \delta A^-] \\ &= S_{\mathrm{YM}}[\mathcal{A}+\delta A] - \int \mathrm{d}^4 x \, \frac{1}{T(\mathrm{A})} \operatorname{tr}(\delta A^-(x)\rho(\vec{x})) \\ &= S_{\mathrm{YM}}[\mathcal{A}+\delta A] - \delta A^-_x \rho^x, \end{aligned} \tag{3.17}$$

where the index x stands for the spacetime arguments and the color indices and we have used that $\operatorname{tr}\rho = 0$.

Another approximation that we will make, is that the only component of the soft current contributing to the action is $\delta J^+ \equiv \delta\rho$. This is the *eikonal approximation*. Physically this means that the scattering of the soft gluons off the induced current is recoilless — the momenta of the partons contributing to the induced current do not obtain a transverse component from interactions with the soft fields. Therefore we can write for the coupling term

$$\delta A_x \delta J^x \approx \delta A^-_x \delta\rho^x, \tag{3.18}$$

where the meaning of the index x has changed on the right-hand side, no longer including Minkowski indices.

Putting everything together we arrive at the following expansion of the action

$$S[\mathcal{A}+\delta A+a] = S_{\mathrm{YM}}[\mathcal{A}[\rho]+\delta A] - \delta A^-_x(\rho^x + \delta\rho^x[a,\rho]) + \frac{1}{2} a_u (G^{-1}[\rho])^{uv} a_v, \tag{3.19}$$

where we have made all functional interdependences explicit.

3.1.3. Integrating over the semi-fast gluons

The next step is to insert a factor of one in the path integral by means of a (functional) delta function

$$1 = \int [D\rho'] \delta(\rho' - \rho - \delta\rho[a,\rho]) \tag{3.20}$$

resulting in

$$\begin{aligned}
&\int [D\rho] W_\Lambda[\rho] \int^{\Lambda'} [D\delta A] \int_{\Lambda'}^{\Lambda} [Da] \mathrm{e}^{\mathrm{i}S[\mathcal{A}+\delta A+a,\rho]} \\
&= \int [D\rho] W_\Lambda[\rho] \int^{\Lambda'} [D\delta A] \int_{\Lambda'}^{\Lambda} [Da] \\
&\quad \times \int [D\rho'] \delta(\rho' - \rho - \delta\rho[a,\rho]) \mathrm{e}^{\mathrm{i}(S_{\mathrm{YM}}[\mathcal{A}[\rho]+\delta A] - \delta A^-(\rho+\delta\rho[a,\rho]) + \frac{1}{2}a(G^{-1}[\rho])a)} \\
&= \int [D\rho'] \int^{\Lambda'} [D\delta A] \mathrm{e}^{\mathrm{i}(S_{\mathrm{YM}}[\mathcal{A}[\rho']+\delta A] - \delta A^- \rho')} \\
&\quad \times \int [D\rho] W_\Lambda[\rho] \int_{\Lambda'}^{\Lambda} [Da] \delta(\rho' - \rho - \delta\rho[a,\rho]) \mathrm{e}^{\frac{\mathrm{i}}{2}aG^{-1}[\rho]a}.
\end{aligned} \tag{3.21}$$

where we used that in leading order in α_s we can write $\mathcal{A}[\rho] = \mathcal{A}[\rho + \delta\rho]$.

Next we will use a (functional) Taylor expansion in orders of $\delta\rho$. When looking at the explicit expressions for $\delta\rho$ (Section 3.2), one sees that both linear and quadratic terms in $\delta\rho$ contribute to the leading order in the coupling constant g.[2] Therefore we need to expand to quadratic order in $\delta\rho$.

We further introduce the following symbols for the correlators of the induced color charge density distribution

$$\sigma_x = \langle \delta\rho_x \rangle_a, \tag{3.22a}$$

$$\chi_{xy} = \langle \delta\rho_x \delta\rho_y \rangle_a, \tag{3.22b}$$

where we defined the a-average as

$$\langle O[a] \rangle_a \equiv \int_{\Lambda'}^{\Lambda} [Da] \mathrm{e}^{\frac{\mathrm{i}}{2}aG^{-1}[\rho]a} O[a]. \tag{3.23}$$

Performing a functional Taylor expansion for the delta function

$$\begin{aligned}
&\delta(\rho + \delta\rho[a,\rho] - \rho') \\
&= \left(1 + \int_x \delta\rho_a(x) \frac{\delta}{\delta\rho_a(x)} + \frac{1}{2} \int_{x,y} \delta\rho_a(x) \delta\rho_b(y) \frac{\delta^2}{\delta\rho_a(x)\delta\rho_b(y)} + O((\delta\rho)^3)\right) \delta(\rho - \rho') \\
&\equiv \left(1 + \delta\rho_x \frac{\delta}{\delta\rho_x} + \frac{1}{2} \delta\rho_x \delta\rho_y \frac{\delta^2}{\delta\rho_x \delta\rho_y} + O((\delta\rho)^3)\right) \delta(\rho - \rho')
\end{aligned} \tag{3.24}$$

[2] The lowest-order contribution from the linear term vanishes.

we have

$$\begin{aligned}
&\int [D\rho] W_\Lambda[\rho] \int_{\Lambda'}^{\Lambda} [Da] \mathrm{e}^{\frac{\mathrm{i}}{2} a G^{-1}[\rho] a} \delta(\rho + \delta\rho[a,\rho] - \rho') \\
&= \int [D\rho] W_\Lambda[\rho] \int_{\Lambda'}^{\Lambda} [Da] \mathrm{e}^{\frac{\mathrm{i}}{2} a G^{-1}[\rho] a} \left(1 + \delta\rho_x \frac{\delta}{\delta\rho_x} + \frac{1}{2}\delta\rho_x \delta\rho_y \frac{\delta^2}{\delta\rho_x \delta\rho_y} + \cdots\right) \delta(\rho - \rho') \\
&= \int [D\rho] W_\Lambda[\rho] \left(1 + \sigma_x \frac{\delta}{\delta\rho_x} + \frac{1}{2}\chi_{xy} \frac{\delta^2}{\delta\rho_x \delta\rho_y} + \cdots\right) \delta(\rho - \rho') \\
&= \left(1 - \frac{\delta}{\delta\rho_x}\sigma_x + \frac{1}{2}\frac{\delta^2}{\delta\rho_x \delta\rho_y}\chi_{xy} + \cdots\right) W_\Lambda[\rho'],
\end{aligned} \tag{3.25}$$

where the ellipses denote terms of cubic and higher order in $\delta\rho$ and in the last expression the functional derivatives act on everything to the right of them. Comparison with Eq. (3.9) then gives (to leading order)

$$W_{\Lambda'}[\rho] - W_\Lambda[\rho] = -\frac{\delta}{\delta\rho_x}(\sigma_x W_\Lambda[\rho]) + \frac{1}{2}\frac{\delta^2}{\delta\rho_x \delta\rho_y}(\chi_{xy} W_\Lambda[\rho]). \tag{3.26}$$

This is indeed the JIMWLK renormalization group equation.

3.1.4. Differential representation

Consider the right-moving nucleus having total (longitudinal) momentum P^+. After choosing an arbitrary cutoff scale Λ one defines the small x variable as $x = \Lambda/P^+$ and x can be viewed as the typical fraction of (longitudinal) momentum carried by a parton.

For a step in the RG procedure one must choose a lower momentum cutoff scale $\Lambda' = b\Lambda$. The choice must satisfy the conditions (i) $b \ll 1$ such that the new scale is significantly lower than the previous one and (ii) $\alpha_\mathrm{s} \ln\frac{1}{b} < 1$ so that one can perform a perturbative analysis in this quantity.

Next we introduce the rapidity variable $\tau = \ln\frac{P^+}{\Lambda} = \ln\frac{1}{x}$ and $\tau' = \ln\frac{P^+}{\Lambda'} = \ln\frac{1}{x'} = \tau + \Delta\tau$, where $\Delta\tau = \ln\frac{1}{b}$. Note that as the cutoff is lowered, the rapidity increases.

Equation (3.26) then becomes

$$\frac{W_{\tau'}[\rho] - W_\tau[\rho]}{\Delta\tau} = \frac{1}{\Delta\tau}\left\{-\frac{\delta}{\delta\rho_x}(\sigma_x W_\tau[\rho]) + \frac{1}{2}\frac{\delta^2}{\delta\rho_x \delta\rho_y}(\chi_{xy} W_\tau[\rho])\right\} \tag{3.27}$$

and taking the limit $\Delta\tau \to 0$ we can write

$$\frac{\partial}{\partial\tau} W_\tau[\rho] = \frac{1}{\Delta\tau}\left\{-\frac{\delta}{\delta\rho_x}(\sigma_x W_\tau[\rho]) + \frac{1}{2}\frac{\delta^2}{\delta\rho_x \delta\rho_y}(\chi_{xy} W_\tau[\rho])\right\}. \tag{3.28}$$

Note that the factor $\Delta\tau$ is often absorbed into the definition of the coefficients σ and χ in the literature. See for example [6, 7].

The factor of $1/\Delta\tau$ appearing in Eq. (3.26) might seem troubling, since the quantity $\Delta\tau$ depends on the way in which we choose the cutoff. In the case when one integrates out the longitudinal modes, one finds that σ and χ are proportional to $\Delta\tau$, thus cancelling the offending factor.

3.2. Induced current

As explained in Section 3.1 the expansion of the action around the field configuration $A = \mathcal{A} + \delta A$ leads to a term describing the coupling of the soft modes δA coupling to an induced current $\delta J^+ = \delta\rho$ (Eq. (3.13)) which reads

$$\delta\rho^x = -\left.\frac{\delta S[A,\rho]}{\delta A^-_x}\right|_{A=\mathcal{A}+a} = -\left(\frac{\delta^2 S[\mathcal{A},\rho]}{\delta\mathcal{A}^-_x\delta\mathcal{A}_u}a_u + \frac{1}{2}\frac{\delta^3 S[\mathcal{A},\rho]}{\delta\mathcal{A}^-_x\delta\mathcal{A}_u\delta\mathcal{A}_v}a_u a_v\right), \tag{3.29}$$

where the repeated index u denotes

$$A_u B^u \equiv \int_u A_a(u)B^a(u) \tag{3.30}$$

and the repeated index x denotes (and similarly for y)

$$A_x B^x \equiv \int_x A^\mu_a(x)B^a_\mu(x). \tag{3.31}$$

We will further write

$$\delta\rho \equiv \delta\rho^{(1)} + \delta\rho^{(2)}, \tag{3.32}$$

where the superscript denotes the order of the semi-fast gluons, and

$$\delta\rho^{(i)} \equiv \delta\rho^{(i)}_{\mathrm{YM}} + \delta\rho^{(i)}_W, \quad i = 1,2, \tag{3.33}$$

where the contributions from the two terms S_{YM} and S_W are separated.

We will assume that all the fields are in the lightcone (LC) gauge $A^+ = 0$. In the LC gauge the solutions to the classical equations of motion $\delta S_{\mathrm{YM}}[\mathcal{A}]/\delta\mathcal{A}_\mu(x) = 0$ take the form $\mathcal{A}^\mu(x) = \delta^\mu_i\mathcal{A}^i(\vec{x})$.

3.2.1. Separation of scales

Recall that the soft gluons have momenta $|p^+| \lesssim b\Lambda^+$ and the semi-fast gluons have momenta $b\Lambda^+ < |p^+| < \Lambda^+$. The term in the effective action (Eq. (3.19)) which describes the coupling of the induced current to the soft gluons can be written in momentum space as

$$\begin{aligned}
-\int_x \frac{1}{T(\mathrm{A})}\,\mathrm{tr}[\delta A^-(x)\delta\rho(x)] &= -\int_x \frac{1}{T(\mathrm{A})}\int_{p,q}\mathrm{tr}[\delta A^-(p)\delta\rho(q)]\mathrm{e}^{-\mathrm{i}(p+q)x} \\
&= -\int_{p,q}\frac{1}{T(\mathrm{A})}\,\mathrm{tr}[\delta A^-(p)\delta\rho(q)]I(p,-q) \\
&= -\int_p \frac{1}{T(\mathrm{A})}\,\mathrm{tr}[\delta A^-(p)\delta\rho(-p)],
\end{aligned} \tag{3.34}$$

where $I(p,q)$ is the identity in momentum space defined in Eq. (D.40b). From this it can be seen that only contributions to $\delta\rho$ with non-vanishing momentum support in the region of soft modes need to be kept, since the soft gluons satisfy (per definition)

$$\delta A(p) = \vartheta(b\Lambda^+ - |p^+|)\delta A(p). \tag{3.35}$$

In particular this means that terms containing only semi-fast gluons can be dropped when writing down the induced color charge density. To see this, note that the semi-fast gluons satisfy (per definition)

$$a(p) = \vartheta(|p^+| - b\Lambda^+)\vartheta(\Lambda^+ - |p^+|)a(p) \tag{3.36}$$

and therefore

$$\delta A(p)a(p) = 0. \tag{3.37}$$

3.2.2. First order contributions

Yang–Mills term

The second functional derivative of the Yang–Mills actions is given by Eq. (2.21)

$$\frac{\delta^2 S_{\mathrm{YM}}[\mathcal{A}]}{\delta\mathcal{A}^a_\mu(x)\delta\mathcal{A}^b_\nu(y)} = \left(g^{\mu\nu}\mathcal{D}_x^2 - \mathcal{D}_x^\mu\mathcal{D}_x^\nu - 2\mathrm{i}g\mathcal{F}^{\mu\nu}(x)\right)_{ab}\delta^{(4)}(x-y) \tag{3.38}$$

we have

$$\begin{aligned}
(\delta\rho^{(1)}_{\mathrm{YM}})_a(x) &= \int_y \frac{\delta^2 S_{\mathrm{YM}}[\mathcal{A}]}{\delta\mathcal{A}^a_+(x)\delta\mathcal{A}^b_\nu(y)} a^b_\nu(y) \\
&= -\int_y \left(g^{+\nu}\mathcal{D}_x^2 - \mathcal{D}_x^+\mathcal{D}_x^\nu - 2\mathrm{i}g\mathcal{F}^{+\nu}(x)\right)_{ab}\delta^{(4)}(x-y)a^b_\nu(y) \\
&= -\left(g^{+\nu}\mathcal{D}_x^2 - \mathcal{D}_x^+\mathcal{D}_x^\nu - 2\mathrm{i}g\mathcal{F}^{+\nu}(x)\right)_{ab} a^b_\nu(x) \\
&= \left(\mathcal{D}_x^+\mathcal{D}_x^\nu + 2\mathrm{i}g\mathcal{F}^{+\nu}(x)\right) a^b_\nu(x) \\
&= (\mathcal{D}_x^+\mathcal{D}_x^+)_{ab}a^b_+(x) + (\mathcal{D}_x^+\mathcal{D}_x^i)_{ab}a^b_i(x) + 2\mathrm{i}g\mathcal{F}^{+i}_{ab}(x)a^b_i(x) \\
&= \delta_{ab}\partial^+\partial^+ a^b_+(x) + \partial^+(\mathcal{D}^i_x)_{ab}a^b_i(x) + 2\mathrm{i}g\mathcal{F}^{+i}_{ab}(x)a^b_i(x) \\
&= \partial_-^2 a^-_a(x) + \partial_-(\mathcal{D}^i_x)_{ab}a^b_i(x) + 2\mathrm{i}g\mathcal{F}^{+i}_{ab}(x)a^b_i(x) \\
&= \partial_-^2 a^-_a(x) + \partial_-\partial_i a^i_a(x) - f_{ab}{}^c\partial_-(\mathcal{A}^i_c(x)a^b_i(x)) + 2\mathrm{i}g\mathcal{F}^{+i}_{ab}(x)a^b_i(x),
\end{aligned} \tag{3.39}$$

where we made repeated use of the lightcone gauge condition $A^+ = 0$. Following the reasoning of Section 3.2.1 we note that the first two terms may be dropped as they only contain contributions from the semi-fast gluons and therefore the corresponding terms in the effective action would vanish when performing the momentum integral. The separation of scales can be exploited further. Inserting into Eq. (3.34) we have for the third term

$$-\mathrm{i}f^a{}_b{}^c\int_{p,q}\delta A^-_a(p)p^+\mathcal{A}^i_c(p-q)a^b_i(q) \tag{3.40}$$

and for the fourth term

$$\mathrm{i}f^a{}_b{}^c\int_{p,q}\delta A^-_a(p)(p^+ - q^+)\mathcal{A}^i_c(p-q)a^b_i(q), \tag{3.41}$$

where we used $\mathcal{F}^{+i}_{ab} = -\mathrm{i}f_{ab}{}^c\partial^+\mathcal{A}^i_c$. Recalling that the soft gluons δA and semi-fast gluons a have longitudinal momenta $|p^+| < b\Lambda^+$ and $b\Lambda^+ < |p^+| < \Lambda^+$ and that

furthermore $b \ll 1$ we realize that the dominant contribution must come from Eq. (3.41). The contribution of Eq. (3.40) actually cancels part of Eq. (3.41). However, the part which is canceled does not contribute to leading order. We will therefore drop the term corresponding to Eq. (3.40) entirely.

Thus the first order contribution to the induced charge of the Yang–Mills action is

$$(\delta\rho^{(1)}_{\mathrm{YM}})_a(x) = 2\mathrm{i}g\mathcal{F}^{+i}_{ab}(x)a^b_i(x). \tag{3.42}$$

Wilson line term

The second functional derivative of the Wilson line term as a matrix in the adjoint representation is given by Eq. (2.39)

$$\frac{\delta^2 S_W[\mathcal{A}^-]}{\delta\mathcal{A}_\mu(x)\delta\mathcal{A}_\nu(y)} = \delta^\mu_+\delta^\nu_+\frac{\mathrm{i}g}{2}\epsilon(x^+ - y^+)\rho(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}), \tag{3.43}$$

where $\epsilon(x) = \theta(x) - \theta(-x)$ is the sign function as defined in Eq. (2.40). The induced current therefore is

$$\begin{aligned}\delta\rho^{(1)}_W(x) &= -\int_y \frac{\delta^2 S_W[\mathcal{A}]}{\delta\mathcal{A}_+(x)\delta\mathcal{A}_\nu(y)}a_\nu(y)\\ &= -\int_y \frac{\mathrm{i}g}{2}\epsilon(x^+ - y^+)\rho(\vec{x})\delta^{(3)}(\vec{x}-\vec{y})a_+(y)\\ &= -\frac{\mathrm{i}g}{2}\rho(\vec{x})\int_{y^+}\epsilon(x^+ - y^+)a_+(y^+,\vec{x}),\end{aligned} \tag{3.44}$$

where now $a(x) \equiv a(x^+, \vec{x})$.

Sometimes the sign function is written in the following way (see Appendix D)

$$\epsilon(x^+ - y^+) = 2\mathrm{i}\langle x^+|\,\mathrm{PV}\,\frac{1}{P^-}|y^+\rangle. \tag{3.45}$$

3.2.3. Second order terms

Yang–Mills term

The third functional derivative of the Yang–Mills action is given by Eq. (2.24)

$$\begin{aligned}\frac{\delta^3 S_{\mathrm{YM}}[\mathcal{A}]}{\delta\mathcal{A}^a_\mu(x)\delta\mathcal{A}^b_\nu(y)\delta\mathcal{A}^c_\rho(z)} =& \delta^{\mu\nu}_{\ ,}{}^{\rho\gamma} g f_{ab}{}^e(\mathcal{D}^x_\gamma)_{ec}\delta^{(4)}(x-z)\delta^{(4)}(x-y)\\ &+ \delta^{\nu\rho}_{\ ,}{}^{\mu\gamma} g f_{bc}{}^e(\mathcal{D}^y_\gamma)_{ea}\delta^{(4)}(y-x)\delta^{(4)}(y-z)\\ &+ \delta^{\rho\mu}_{\ ,}{}^{\nu\gamma} g f_{ca}{}^e(\mathcal{D}^z_\gamma)_{eb}\delta^{(4)}(z-y)\delta^{(4)}(z-x).\end{aligned} \tag{3.46}$$

The induced current therefore is

$$
\begin{aligned}
(\delta\rho^{(2)}_{\mathrm{YM}})_a(x) &= -\frac{1}{2}\int_{y,z}\frac{\delta^3 S_{\mathrm{YM}}[\mathcal{A}]}{\delta\mathcal{A}^a_+(x)\delta\mathcal{A}^b_\nu(y)\delta\mathcal{A}^c_\rho(z)}a^b_\nu(y)a^c_\rho(z)\\
&= -\frac{1}{2}\int_{y,z}[\delta^{+\nu}{}_{,}{}^{\rho\gamma}gf_{ab}{}^e(\mathcal{D}^x_\gamma)_{ec}\delta^{(4)}(x-z)\delta^{(4)}(x-y)\\
&\qquad + \delta^{\nu\rho}{}_{,}{}^{+\gamma}gf_{bc}{}^e(\mathcal{D}^y_\gamma)_{ea}\delta^{(4)}(y-x)\delta^{(4)}(y-z)\\
&\qquad + \delta^{\rho+}{}_{,}{}^{\nu\gamma}gf_{ca}{}^e(\mathcal{D}^z_\gamma)_{eb}\delta^{(4)}(z-y)\delta^{(4)}(z-x)]a^b_\nu(y)a^c_\rho(z)\\
&= -\frac{g}{2}[\delta^{+\nu}{}_{,}{}^{\rho\gamma}f_{ab}{}^e(\mathcal{D}_\gamma)_{ec}a^c_\rho(x)a^b_\nu(x) - \delta^{\nu\rho}{}_{,}{}^{+\gamma}f_{bc}{}^e(\mathcal{D}_\gamma)_{ae}a^b_\nu(x)a^c_\rho(x)\\
&\qquad + \delta^{\rho+}{}_{,}{}^{\nu\gamma}f_{ca}{}^e(\mathcal{D}_\gamma)_{eb}a^b_\nu(x)a^c_\rho(x)]\\
&= -\frac{g}{2}[(g^{+\rho}g^{\nu\gamma} - g^{+\gamma}g^{\nu\rho})f_{ab}{}^e(\mathcal{D}_\gamma)_{ec}a^c_\rho(x)a^b_\nu(x)\\
&\qquad - (g^{\nu+}g^{\rho\gamma} - g^{\nu\gamma}g^{\rho+})f_{bc}{}^e(\mathcal{D}_\gamma)_{ae}a^b_\nu(x)a^c_\rho(x)\\
&\qquad + (g^{\rho\nu}g^{+\gamma} - g^{\rho\gamma}g^{+\nu})f_{ca}{}^e(\mathcal{D}_\gamma)_{eb}a^b_\nu(x)a^c_\rho(x)]\\
&= -\frac{g}{2}\left[-g^{+\gamma}g^{\nu\rho}f_{ab}{}^e(\mathcal{D}_\gamma)_{ec}a^c_\rho(x)a^b_\nu(x) + g^{\rho\nu}g^{+\gamma}f_{ca}{}^e(\mathcal{D}_\gamma)_{eb}a^b_\nu(x)a^c_\rho(x)\right]\\
&= -\frac{g}{2}g^{\nu\rho}\left[-f_{abc}\mathcal{D}^+a^c_\rho(x)a^b_\nu(x) + f_{cab}\mathcal{D}^+a^b_\nu(x)a^c_\rho(x)\right]\\
&= -\frac{g}{2}g^{\nu\rho}\left[-f_{abc}\partial^+a^c_\rho(x)a^b_\nu(x) + f_{cab}\partial^+a^b_\nu(x)a^c_\rho(x)\right]\\
&= gg^{\nu\rho}f_{abc}\partial^+a^c_\rho(x)a^b_\nu(x)\\
&= -g\delta^{ij}f_{abc}\partial^+a^c_i(x)a^b_j(x).
\end{aligned}
\tag{3.47}
$$

Wilson line term

The third functional derivative of the (standard) Wilson line term is given by Eq. (2.41)

$$
\begin{aligned}
&\frac{\delta^3 S_W[\mathcal{A}^-]}{\delta\mathcal{A}^a_\mu(x)\delta\mathcal{A}^b_\nu(y)\delta\mathcal{A}^c_\rho(z)}\\
&= \delta^\mu_+\delta^\nu_+\delta^\rho_+g^2\frac{1}{T(\mathrm{A})}\mathrm{tr}[\{\theta(x^+-y^+)\theta(y^+-z^+)T_aT_bT_c\\
&\qquad\qquad + \theta(y^+-z^+)\theta(z^+-x^+)T_bT_cT_a\\
&\qquad\qquad + \theta(z^+-x^+)\theta(x^+-y^+)T_cT_aT_b\\
&\qquad\qquad + \theta(x^+-z^+)\theta(z^+-y^+)T_aT_cT_b\\
&\qquad\qquad + \theta(y^+-x^+)\theta(x^+-z^+)T_bT_aT_c\\
&\qquad\qquad + \theta(z^+-y^+)\theta(y^+-x^+)T_cT_bT_a\}\rho(\vec{x})]\\
&\quad\times\delta^{(3)}(\vec{x}-\vec{y})\delta^{(3)}(\vec{y}-\vec{z}).
\end{aligned}
\tag{3.48}
$$

The corresponding induced current therefore is

$$\begin{aligned}
&(\delta\rho_W^{(2)})_a(x) \\
&= -\frac{1}{2}\int_{y,z} \frac{\delta^3 S_{\mathrm{YM}}[\mathcal{A}]}{\delta\mathcal{A}_+^a(x)\delta\mathcal{A}_\nu^b(y)\delta\mathcal{A}_\rho^c(z)} a_\nu^b(y)a_\rho^c(z) \\
&= -\int_{y^+,z^+} g^2\frac{1}{T(\mathrm{A})}\mathrm{tr}[\{\theta(x^+-y^+)\theta(y^+-z^+)T_aT_bT_c \\
&\qquad\qquad + \theta(y^+-z^+)\theta(z^+-x^+)T_bT_cT_a \\
&\qquad\qquad + \theta(z^+-x^+)\theta(x^+-y^+)T_cT_aT_b\}\rho(\vec{x})] \\
&\qquad \times a_+^b(y^+,\vec{x})a_+^c(z^+,\vec{x}).
\end{aligned} \tag{3.49}$$

where we used the fact that the third functional derivative in Eq. (3.48) is symmetric under permutations of the order in which the derivatives are taken to halve the number of terms. Note that in [6, 7] the generators are ordered according to what we would have obtained when choosing an anti-path-ordering prescription for the Wilson line.

3.2.4. Overview

In this section we collect our results for further reference. The induced charge density $\delta\rho = \delta\rho^{(1)} + \delta\rho^{(2)}$ has the contributions

$$(\delta\rho_{\mathrm{YM}}^{(1)})_a(x) = 2\mathrm{i}g\mathcal{F}_{ab}^{+i}(\vec{x})a_i^b(x), \tag{3.50a}$$

$$(\delta\rho_W^{(1)})_a(x) = -\frac{\mathrm{i}g}{2}\rho_{ab}(\vec{x})\int_{y^+}\epsilon(x^+-y^+)a_+^b(y^+,\vec{x}), \tag{3.50b}$$

$$(\delta\rho_{\mathrm{YM}}^{(2)})_a(x) = -g\delta^{ij}f_{abc}\partial^+a_i^c(x)a_j^b(x), \tag{3.50c}$$

$$\begin{aligned}
(\delta\rho_W^{(2)})_a(x) = -\int_{y^+,z^+} & g^2\frac{1}{T(\mathrm{A})}\mathrm{tr}[\{\theta(x^+-y^+)\theta(y^+-z^+)T_aT_bT_c \\
&+ \theta(y^+-z^+)\theta(z^+-x^+)T_bT_cT_a \\
&+ \theta(z^+-x^+)\theta(x^+-y^+)T_cT_aT_b\}\rho(\vec{x})] \\
&\times a_+^b(y^+,\vec{x})a_+^c(z^+,\vec{x}).
\end{aligned} \tag{3.50d}$$

3.3. Coefficients of the JIMWLK equation

In this section we will discuss the structure of the coefficients which enter the JIMWLK renormalization group equation (Eq. (3.26)).

The time-ordered two-point function of the semi-fast gluons a gives the propagator of the semi-fast gluons in the background field of the color charge density distribution ρ

$$\langle \mathrm{T}a(x)a(y)\rangle = \mathrm{i}G(x,y), \tag{3.51}$$

where we note that in this section we will denote the a-average defined in Eq. (3.23) without the subscript 'a'.

We further note that the one-point function of the semi-fast gluons vanishes, i.e.

$$\langle a(x)\rangle = 0. \tag{3.52}$$

In Eq. (3.51) the time-ordered correlation function is considered. Due to the staticity of the problem, it suffices to compute the equal-time correlation functions. The correct way to take the equal-time limit which is consistent with the choice of boundary conditions $\mathcal{A}(x^+ = -\infty, \vec{x}) = 0$ is [6]

$$G(\vec{x}, \vec{y}) = G(x^+, \vec{x}; y^+ = x^+ + \epsilon, \vec{x}). \tag{3.53}$$

3.3.1. The coefficient σ

The coefficient σ is the one-point function of the induced charge density $\delta\rho$

$$\sigma(x) = \langle \delta\rho(x)\rangle = \langle \delta\rho^{(2)}(x)\rangle. \tag{3.54}$$

where we have used Eq. (3.52) and recalled our definition in Eq. (3.32). To facilitate computation we split σ as

$$\sigma(x) = \sigma_1(x) + \sigma_2(x), \tag{3.55a}$$
$$\sigma_1(x) \equiv \langle \delta\rho^{(2)}_{\mathrm{YM}}(x)\rangle, \tag{3.55b}$$
$$\sigma_2(x) \equiv \langle \delta\rho^{(2)}_{W}(x)\rangle. \tag{3.55c}$$

The contribution in Eq. (3.55b) from the Yang–Mills action is using Eq. (3.47)

$$\begin{aligned}(\sigma_1)_a(x) &= g f_{abc}\delta^{ij}\langle \partial_x^+ a_i^b(x) a_j^c(x)\rangle \\ &= \frac{1}{2} g f_{abc}\delta^{ij}(\partial_x^+ - \partial_y^+)\langle a_i^b(x) a_j^c(y)\rangle|_{y=x} \\ &= \frac{1}{2}\mathrm{i} g f_{abc}\delta^{ij}(\partial_x^+ - \partial_y^+) G_{ij}^{bc}(x,y)|_{y=x},\end{aligned} \tag{3.56}$$

where we chose a more symmetric way for the derivatives to act.

Inserting the induced current $\delta\rho^{(2)}_W$ (Eq. (3.49)) into Eq. (3.55c) gives

$$\begin{aligned}(\sigma_2)_a(x) = -\mathrm{i} g^2 \rho^d(\vec{x}) \int_{y^+,z^+} \frac{1}{T(\mathrm{A})} \mathrm{tr}[&\theta(x^+ - y^+)\theta(y^+ - z^+) T_a T_b T_c T_d \\ &+ \theta(y^+ - z^+)\theta(z^+ - x^+) T_b T_c T_a T_d \\ &+ \theta(z^+ - x^+)\theta(x^+ - y^+) T_c T_a T_b T_d] \\ &\times G^{bc}_{++}(y^+, \vec{x}; z^+, \vec{x}).\end{aligned} \tag{3.57}$$

3.3.2. The coefficient χ

To leading order the coefficient χ

$$\chi(x,y) = \langle \delta\rho(x)\delta\rho(y)\rangle \tag{3.58}$$

has only contributions from the linear induced charge densities, i.e.

$$\chi(x,y) \approx \langle \delta\rho^{(1)}(x)\delta\rho^{(1)}(y)\rangle. \tag{3.59}$$

This is because terms involving contributions from $\delta\rho^{(2)}$ would be higher order in g. This corresponds to considering only tree-level diagrams. See the discussion in [6] below their Eq. (4.49).

To make the calculation tractable we split χ into the following parts

$$\chi_1(x,y) = \langle \delta\rho^{(1)}_{\mathrm{YM}}(x)\delta\rho^{(1)}_{\mathrm{YM}}(y)\rangle, \tag{3.60a}$$
$$\chi_2(x,y) = \langle \delta\rho^{(1)}_{\mathrm{YM}}(x)\delta\rho^{(1)}_{W}(y)\rangle + \langle \delta\rho^{(1)}_{W}(x)\delta\rho^{(1)}_{\mathrm{YM}}(y)\rangle, \tag{3.60b}$$
$$\chi_3(x,y) = \langle \delta\rho^{(1)}_{W}(x)\delta\rho^{(1)}_{W}(y)\rangle. \tag{3.60c}$$

For Eq. (3.60a) we have

$$\begin{aligned}(\chi_1)_{ab}(x,y) &= (2\mathrm{i}g)^2 \mathcal{F}^{+i}_{ac}(\vec{x})\mathcal{F}^{+j}_{bd}(\vec{y})\langle a^c_i(x)a^d_j(y)\rangle \\ &= -4\mathrm{i}g^2 \mathcal{F}^{+i}_{ac}(\vec{x})\mathcal{F}^{+j}_{bd}(\vec{y})G^{cd}_{ij}(x,y) \\ &= 4\mathrm{i}g^2 \mathcal{F}^{+i}_{ac}(\vec{x})G^{cd}_{ij}(x,y)\mathcal{F}^{+j}_{db}(\vec{y}).\end{aligned} \tag{3.61}$$

We further split the term χ_2 (Eq. (3.60b)) into

$$\chi_2(x,y) = \chi_{21}(x,y) + \chi_{22}(x,y), \tag{3.62a}$$
$$\chi_{21}(x,y) = \langle \delta\rho^{(1)}_{\mathrm{YM}}(x)\delta\rho^{(1)}_{W}(y)\rangle, \tag{3.62b}$$
$$\chi_{22}(x,y) = \langle \delta\rho^{(1)}_{W}(x)\delta\rho^{(1)}_{\mathrm{YM}}(y)\rangle. \tag{3.62c}$$

Equation (3.62b) is

$$\begin{aligned}(\chi_{21})_{ab}(x,y) &= \langle (\delta\rho^{(1)}_{\mathrm{YM}})_a(x)(\delta\rho^{(1)}_{W})_b(y)\rangle \\ &= (2\mathrm{i}g)\left(-\frac{\mathrm{i}g}{2}\right)\mathcal{F}^{+i}_{ac}(\vec{x})\rho_{bd}(\vec{y})\int_{z^+}\epsilon(y^+-z^+)\langle a^c_i(x)a^d_+(z^+,\vec{y})\rangle \\ &= \mathrm{i}g^2\mathcal{F}^{+i}_{ac}(\vec{x})\rho_{bd}(\vec{y})\int_{z^+}\epsilon(y^+-z^+)G^{cd}_{i+}(x^+,\vec{x};z^+,\vec{y}) \\ &= -\mathrm{i}g^2\mathcal{F}^{+i}_{ac}(\vec{x})\rho_{bd}(\vec{y})\int_{z^+}G^{cd}_{i+}(x^+,\vec{x};z^+,\vec{y})\epsilon(z^+-y^+) \\ &= -\mathrm{i}g^2\mathcal{F}^{+i}_{ac}(\vec{x})\rho_{bd}(\vec{y})\langle x|G^{cd}_{i+}\,\epsilon\,|y\rangle \\ &= \mathrm{i}g^2\mathcal{F}^{+i}_{ac}(\vec{x})\langle x|G^{cd}_{i+}\,\epsilon\,|y\rangle\rho_{db}(\vec{y}),\end{aligned} \tag{3.63}$$

where we used $\epsilon(x) = -\epsilon(-x)$ and the identification of the sign function as a matrix element discussed in Appendix D.1.3. For Eq. (3.62c) we have

$$\begin{aligned}(\chi_{22})_{ab}(x,y) &= \langle (\delta\rho_W^{(1)})_a(x)(\delta\rho_{\mathrm{YM}}^{(1)})_b(y)\rangle \\ &= \left(-\frac{\mathrm{i}g}{2}\right)(2\mathrm{i}g)\rho_{ac}(\vec{x})\mathcal{F}_{bd}^{+i}(\vec{y})\int_{z^+}\epsilon(x^+-z^+)\langle a_+^c(z^+,\vec{x})a_i^d(y)\rangle \\ &= \mathrm{i}g^2\rho_{ac}(\vec{x})\mathcal{F}_{bd}^{+i}(\vec{y})\int_{z^+}\epsilon(x^+-z^+)G_{+i}^{cd}(z^+,\vec{x};y^+,\vec{y}) \\ &= \mathrm{i}g^2\rho_{ac}(\vec{x})\mathcal{F}_{bd}^{+i}(\vec{y})\langle x|\,\epsilon\,G_{+i}^{cd}|y\rangle \\ &= -\mathrm{i}g^2\rho_{ac}(\vec{x})\langle x|\,\epsilon\,G_{+i}^{cd}|y\rangle\mathcal{F}_{db}^{+i}(\vec{y}).\end{aligned} \tag{3.64}$$

Equation (3.60c) reads

$$\begin{aligned}(\chi_3)_{ab}(x,y) &= \langle (\delta\rho_W^{(1)})_a(x)(\delta\rho_W^{(1)})_b(y)\rangle \\ &= \left(-\frac{\mathrm{i}g}{2}\right)^2\rho_{ac}(\vec{x})\rho_{bd}(\vec{y})\int_{z^+,w^+}\epsilon(x^+-z^+)\epsilon(y^+-w^+)\langle a_+^c(z^+,\vec{x})a_+^d(w^+,\vec{y})\rangle \\ &= -\mathrm{i}\frac{g^2}{4}\rho_{ac}(\vec{x})\rho_{bd}(\vec{y})\int_{z^+,w^+}\epsilon(x^+-z^+)\epsilon(y^+-w^+)G_{++}^{cd}(z^+,\vec{x};w^+,\vec{y}) \\ &= \mathrm{i}\frac{g^2}{4}\rho_{ac}(\vec{x})\rho_{bd}(\vec{y})\int_{z^+,w^+}\epsilon(x^+-z^+)G_{++}^{cd}(z^+,\vec{x};w^+,\vec{y})\epsilon(w^+-y^+) \\ &= \mathrm{i}\frac{g^2}{4}\rho_{ac}(\vec{x})\rho_{bd}(\vec{y})\langle x|\,\epsilon\,G_{++}^{cd}\,\epsilon\,|y\rangle \\ &= -\mathrm{i}\frac{g^2}{4}\rho_{ac}(\vec{x})\langle x|\,\epsilon\,G_{++}^{cd}\,\epsilon\,|y\rangle\rho_{db}(\vec{y}).\end{aligned} \tag{3.65}$$

3.4. Separation of scales

In [6] the following separation of scales is advocated. This will be important when discussing the gauge transformation from the lightcone to the covariant gauge in Section 3.5.

Due to the uncertainty principle we have $x^-p^+ \gtrsim 1$ and we will assume that this is actually an equality in what follows. Since the "hard" modes have longitudinal momenta $|p^+| > \Lambda^+$ those modes satisfy

$$|x^-| < \frac{1}{\Lambda^+}. \tag{3.66}$$

The semi-fast gluons have longitudinal momenta in the momentum strip $b\Lambda^+ < |p^+| < \Lambda^+$. Therefore they have support in

$$\frac{1}{b\Lambda^+} > |x^-| > \frac{1}{\Lambda^+}. \tag{3.67}$$

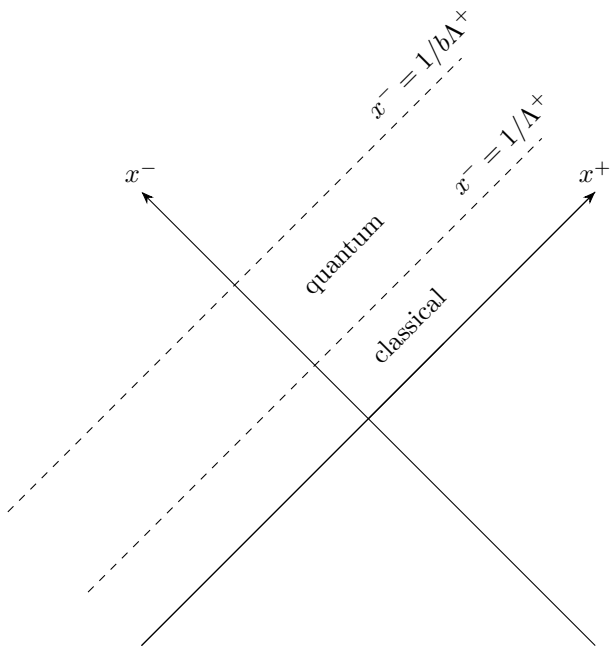

Figure 3.1.: A spacetime diagram illustrating the separation of scales for a right-moving nucleus. As explained in the main text, we assume that the fields for support only for $x^+ > 0$.

Another key argument is that to be consistent with the retarded boundary conditions of the classical gauge field solution presented in Section 1.1 all the fields have only support at positive values of x^-.

A spacetime diagram for the separation of scales of a right-moving nucleus is given in Fig. 3.1.

3.5. Transforming to the covariant gauge

As explained in Section 1.1 the classical solution to the Yang–Mills equation is known only in terms of the color charge density distribution in the covariant gauge $\tilde{\rho}$. In this section we will discuss how the results we have obtained so far in the lightcone gauge can be transformed to the covariant gauge.

The reason why this gauge transformation is not straightforward to do is because the transformation depends on the solution and therefore the color charge density distribution itself. As we have seen the color charge density distribution gets a correction $\delta\rho$ due to quantum effects.

At the scale Λ^+ we have the following set of equations

$$\tilde{\rho}(\vec{x}) = U^\dagger(\vec{x})\rho(\vec{x})U(\vec{x}) \tag{3.68a}$$

$$\mathcal{A}^i[\tilde{\rho}](\vec{x}) = \frac{\mathrm{i}}{g}U(\vec{x})\partial^i U^\dagger(\vec{x}), \tag{3.68b}$$

$$U^\dagger(\vec{x}) = \mathrm{P}\exp\left(\mathrm{i}g\int_{-\infty}^{x^-}\mathrm{d}y^-\,\alpha(y^-,\boldsymbol{x}_\perp)\right), \tag{3.68c}$$

$$-\boldsymbol{\nabla}_\perp^2\alpha(\vec{x}) = \tilde{\rho}(\vec{x}). \tag{3.68d}$$

At the scale $b\Lambda^+$ we will denote the fields in the covariant gauge with a bar. This is because the gauge transformation from the lightcone gauge to the covariant gauge is now different. The set of equations at this scale are

$$\bar{\rho}(\vec{x}) + \delta\bar{\rho}(\vec{x}) = \bar{U}^\dagger(\vec{x})(\rho+\delta\rho)(\vec{x})\bar{U}(\vec{x}) \tag{3.69a}$$

$$\mathcal{A}^i[\bar{\rho}+\delta\bar{\rho}](\vec{x}) = \frac{\mathrm{i}}{g}\bar{U}(\vec{x})\partial^i\bar{U}^\dagger(\vec{x}), \tag{3.69b}$$

$$\bar{U}^\dagger(\vec{x}) = \mathrm{P}\exp\left(\mathrm{i}g\int_{-\infty}^{x^-}\mathrm{d}y^-\,\bar{\alpha}(y^-,\boldsymbol{x}_\perp)\right), \tag{3.69c}$$

$$-\boldsymbol{\nabla}_\perp^2\bar{\alpha}(\vec{x}) = \bar{\rho}(\vec{x}) + \delta\bar{\rho}(\vec{x}). \tag{3.69d}$$

We now make the ansatz that the classical gauge field can be written as $\bar{\alpha} = \alpha + \delta\alpha$, where the field $\delta\alpha$ corresponds to the quantum corrections and therefore has support only at $x^- > 1/\Lambda^+$ (see Section 3.4). Equation (3.69c) can then be written as

$$\bar{U}^\dagger(\vec{x}) = \delta U^\dagger(\vec{x})U(\vec{x}), \tag{3.70}$$

where

$$\delta U^\dagger(\vec{x}) = \mathrm{P}\exp\left(\mathrm{i}g\int_{1/\Lambda^+}^{x^-}\mathrm{d}y^-\,\delta\alpha(y^-,\boldsymbol{x}_\perp)\right). \tag{3.71}$$

The color charge density distribution then is

$$\bar{\rho} = \bar{U}^\dagger\rho\bar{U} = U^\dagger\rho U = \tilde{\rho}, \tag{3.72}$$

where we used the fact that the color charge density distribution ρ has support only at $x^- < 1/\Lambda^+$. Therefore the above ansatz for $\bar{\alpha}$ is consistent since Eq. (3.68d) is satisfied. We now have for the gauge field $\delta\alpha$

$$-\boldsymbol{\nabla}_\perp^2\delta\alpha(\vec{x}) = \delta\bar{\rho}(\vec{x}) \tag{3.73}$$

with the solution given analogously to Eq. (1.24)

$$\delta\alpha(\vec{x}) = \int\mathrm{d}^2\boldsymbol{y}_\perp\langle\boldsymbol{x}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{y}_\perp\rangle\delta\bar{\rho}(x^-,\boldsymbol{y}_\perp). \tag{3.74}$$

Performing a gauge transformation of Eq. (3.26) we obtain the JIMWLK renormalization group equation in the covariant gauge

$$W_{\Lambda'}[\bar{\rho}] - W_\Lambda[\bar{\rho}] = -\frac{\delta}{\delta\bar{\rho}_x}(\bar{\sigma}_x W_\Lambda[\bar{\rho}]) + \frac{1}{2}\frac{\delta^2}{\delta\bar{\rho}_x\delta\bar{\rho}_y}(\bar{\chi}_{xy}W_\Lambda[\bar{\rho}]), \tag{3.75}$$

where we used the crucial assumption that the weight functional W_Λ is form invariant under gauge transformations. Note that due to Eq. (3.72) we could also have written $\tilde{\rho}$ instead of $\bar{\rho}$.

The coefficients appearing in Eq. (3.75) are given by

$$\bar{\sigma}_a(\vec{x}) = \langle \delta\bar{\rho}_a(\vec{x})\rangle, \tag{3.76a}$$
$$\bar{\chi}_{ab}(\vec{x}) = \langle \delta\bar{\rho}_a(\vec{x})\delta\bar{\rho}_b(\vec{y})\rangle. \tag{3.76b}$$

The correction to the covariant gauge source is related to the lightcone gauge correction via

$$\delta\bar{\rho} = \bar{U}^\dagger\delta\rho\bar{U} = \delta U^\dagger U^\dagger\delta\rho U\delta U = \delta U^\dagger\delta\tilde{\rho}\delta U. \tag{3.77}$$

Expanding the quantum piece of the gauge transformation $\delta U^\dagger$ we have

$$\delta U^\dagger(\vec{x}) = I + \mathrm{i}g\int_{1/\Lambda^+}^{x^-} \mathrm{d}y^-\,\delta\alpha(y^-,\boldsymbol{x}_\perp) + O(\delta\alpha^2). \tag{3.78}$$

Therefore we have for $\delta\bar{\rho}$

$$\begin{aligned}\delta\bar{\rho}_a(\vec{x}) &= \delta\tilde{\rho}_a(\vec{x}) + \mathrm{i}g\int_{1/\Lambda^+}^{x^-}\mathrm{d}y^-[\delta\alpha(y^-,\boldsymbol{x}_\perp),\delta\tilde{\rho}(\vec{x})]_a \\ &= \delta\tilde{\rho}_a(\vec{x}) - gf_{abc}\int_{1/\Lambda^+}^{x^-}\mathrm{d}y^-\,\delta\alpha^b(y^-,\boldsymbol{x}_\perp)\delta\tilde{\rho}^c(\vec{x}) \\ &= \delta\tilde{\rho}_a(\vec{x}) + gf_{abc}\int_{1/\Lambda^+}^{x^-}\mathrm{d}y^-\int\mathrm{d}^2\boldsymbol{y}_\perp\,\delta\tilde{\rho}^b(\vec{x})\delta\tilde{\rho}^c(\vec{y})\langle\boldsymbol{x}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{y}_\perp\rangle,\end{aligned} \tag{3.79}$$

where we used the antisymmetry of the structure constants.

Using Eq. (3.72) we then have for the coefficients of the JIMWLK equation to the order of interest

$$\bar{\sigma}_a(\vec{x}) = \tilde{\sigma}_a(\vec{x}) - \delta\bar{\sigma}_a(\vec{x}), \tag{3.80a}$$
$$\bar{\chi}_{ab}(\vec{x},\vec{y}) = \tilde{\chi}_{ab}(\vec{x},\vec{y}), \tag{3.80b}$$

where $\tilde{\sigma}$ and $\tilde{\chi}$ are given by

$$\tilde{\sigma}_a(\vec{x}) = \langle\delta\tilde{\rho}_a(\vec{x})\rangle, \tag{3.81a}$$
$$\tilde{\chi}_{ab}(\vec{x},\vec{y}) = \langle\delta\tilde{\rho}_a(\vec{x})\delta\tilde{\rho}_b(\vec{y})\rangle. \tag{3.81b}$$

and we defined

$$\delta\bar{\sigma}_a(\vec{x}) = -gf_a{}^{bc}\int_{1/\Lambda^+}^{x^-}\mathrm{d}y^-\int\mathrm{d}^2\boldsymbol{y}_\perp\,\tilde{\chi}_{bc}(\vec{x},\vec{y})\langle\boldsymbol{x}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{y}_\perp\rangle. \tag{3.82}$$

Using Eq. (C.56) we have for Eqs. (3.81)

$$\tilde{\sigma}_a(\vec{x}) = U^\dagger_{ab}(\vec{x})\tilde{\sigma}^b(\vec{x}), \tag{3.83a}$$
$$\tilde{\chi}_{ab}(\vec{x},\vec{y}) = U^\dagger_{ac}(\vec{x})\tilde{\chi}^{cd}(\vec{x})U_{db}(\vec{y}). \tag{3.83b}$$

3.6. Representation in terms of the gauge field

It is sometimes convenient to represent the JIMWLK equation with the gauge field α as the independent variable rather than $\tilde{\rho}$.

The solution of Eq. (3.68d) is given by Eq. (1.24)[3]

$$\alpha_b(\vec{z}) = \int \mathrm{d}^2\boldsymbol{u}_\perp \langle \boldsymbol{z}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{u}_\perp \rangle \tilde{\rho}_b(z^-, \boldsymbol{u}_\perp). \tag{3.84}$$

Using the functional derivative

$$\begin{aligned}\frac{\delta \alpha_b(\vec{z})}{\delta \tilde{\rho}_a(\vec{x})} &= \int \mathrm{d}^2\boldsymbol{u}_\perp \langle \boldsymbol{z}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{u}_\perp \rangle \delta_b^a \delta(z^- - x^-) \delta^{(2)}(\boldsymbol{u}_\perp - \boldsymbol{x}_\perp) \\ &= \langle \boldsymbol{z}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{x}_\perp \rangle \delta_b^a \delta(z^- - x^-).\end{aligned} \tag{3.85}$$

Using this we can write the chain rule as

$$\begin{aligned}\frac{\delta}{\delta \tilde{\rho}_a(\vec{x})} &= \int_{\vec{z}} \frac{\delta \alpha_b(\vec{z})}{\delta \tilde{\rho}_a(\vec{x})} \frac{\delta}{\delta \alpha_b(\vec{z})} \\ &= \int_{\vec{z}} \langle \boldsymbol{z}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{x}_\perp \rangle \delta_b^a \delta(z^- - x^-) \frac{\delta}{\delta \alpha_b(\vec{z})} \\ &= \int_{\boldsymbol{z}_\perp} \langle \boldsymbol{z}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{x}_\perp \rangle \frac{\delta}{\delta \alpha_a(x^-, \boldsymbol{z}_\perp)}\end{aligned} \tag{3.86}$$

Consider now the term in Eq. (3.75) with one derivative. Changing the independent field from $\tilde{\rho}$ to α gives[4]

$$\begin{aligned}\frac{\delta}{\delta \tilde{\rho}_x}(\bar{\sigma}_x W_\Lambda[\tilde{\rho}]) &\equiv \int_{\vec{x}} \frac{\delta}{\delta \tilde{\rho}_a(\vec{x})} (\bar{\sigma}_a(\vec{x}) W_\Lambda[\tilde{\rho}]) \\ &= \int_{\vec{x}, \boldsymbol{z}_\perp} \langle \boldsymbol{z}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{x}_\perp \rangle \frac{\delta}{\delta \alpha_a(x^-, \boldsymbol{z}_\perp)} (\bar{\sigma}_a(\vec{x}) W_\Lambda[\alpha]) \\ &= \int_{\vec{x}, \boldsymbol{z}_\perp} \langle \boldsymbol{x}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{z}_\perp \rangle \frac{\delta}{\delta \alpha_a(\vec{x})} (\bar{\sigma}_a(x^-, \boldsymbol{z}_\perp) W_\Lambda[\alpha]) \\ &= \int_{\vec{x}} \frac{\delta}{\delta \alpha_a(\vec{x})} (\nu_a(x^-, \boldsymbol{z}_\perp) W_\Lambda[\alpha]) \\ &\equiv \frac{\delta}{\delta \alpha_x} (\nu_x W_\Lambda[\alpha]),\end{aligned} \tag{3.87}$$

where we denoted $W_\Lambda[\alpha] \equiv W_\Lambda[\tilde{\rho}[\alpha]]$ and defined

$$\nu_a(\vec{x}) = \int_{\boldsymbol{z}_\perp} \langle \boldsymbol{x}_\perp | \frac{1}{-\boldsymbol{\nabla}_\perp^2} | \boldsymbol{z}_\perp \rangle \bar{\sigma}_a(x^-, \boldsymbol{z}_\perp) \tag{3.88}$$

[3] Recall that we have defined $\tilde{\mathcal{A}}^+ \equiv \alpha$.

[4] Here and in the following we will use Eq. (3.72) and write $\tilde{\rho}$ instead of $\bar{\rho}$.

Consider next the term in Eq. (3.75) with two derivatives. The change of fields gives

$$\begin{aligned}
&\frac{\delta^2}{\delta\tilde{\rho}_x\delta\tilde{\rho}_y}(\bar{\chi}_{xy}W_\Lambda[\tilde{\rho}]) \\
&\equiv \int_{\vec{x},\vec{y}} \frac{\delta^2}{\delta\tilde{\rho}_a(\vec{x})\delta\tilde{\rho}_b(\vec{y})}(\bar{\chi}_{ab}(\vec{x},\vec{y})W_\Lambda[\tilde{\rho}]) \\
&= \int_{\vec{x},\vec{y},\boldsymbol{z}_\perp,\boldsymbol{w}_\perp} \langle \boldsymbol{z}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{x}_\perp\rangle\langle \boldsymbol{w}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{y}_\perp\rangle \frac{\delta^2}{\delta\alpha_a(x^-,\boldsymbol{z}_\perp)\delta\alpha_b(y^-,\boldsymbol{w}_\perp)}(\bar{\chi}_{ab}(\vec{x},\vec{y})W_\Lambda[\alpha]) \\
&= \int_{\vec{x},\vec{y},\boldsymbol{z}_\perp,\boldsymbol{w}_\perp} \langle \boldsymbol{x}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{z}_\perp\rangle\langle \boldsymbol{y}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{w}_\perp\rangle \frac{\delta^2}{\delta\alpha_a(\vec{x})\delta\alpha_b(\vec{y})}(\bar{\chi}_{ab}(x^-,\boldsymbol{z}_\perp;y^-,\boldsymbol{w}_\perp)W_\Lambda[\alpha]) \\
&= \int_{\vec{x},\vec{y}} \frac{\delta^2}{\delta\alpha_a(\vec{x})\delta\alpha_b(\vec{y})}(\eta_{ab}(\vec{x},\vec{y})W_\Lambda[\alpha]) \\
&\equiv \frac{\delta^2}{\delta\alpha_x\delta\alpha_y}(\eta_{xy}W_\Lambda[\alpha]),
\end{aligned} \tag{3.89}$$

where we defined

$$\eta_{ab}(\vec{x},\vec{y}) = \int_{\boldsymbol{z}_\perp,\boldsymbol{w}_\perp} \langle \boldsymbol{x}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{z}_\perp\rangle\langle \boldsymbol{y}_\perp|\frac{1}{-\boldsymbol{\nabla}_\perp^2}|\boldsymbol{w}_\perp\rangle\bar{\chi}_{ab}(x^-,\boldsymbol{z}_\perp;y^-,\boldsymbol{w}_\perp). \tag{3.90}$$

Using the definitions Eqs. (3.88) and (3.90) we therefore can write for the JIMWLK equation

$$W_{\Lambda'}[\alpha] - W_\Lambda[\alpha] = -\frac{\delta}{\delta\alpha_x}(\nu_x W_\Lambda[\alpha]) + \frac{1}{2}\frac{\delta^2}{\delta\alpha_x\delta\alpha_y}(\eta_{xy}W_\Lambda[\alpha]). \tag{3.91}$$

Chapter 4.

Weak field limit

In the case where the color charge distribution is weak, one can perform a perturbative analysis in powers of ρ. In this chapter we assume that this is the case.

It is well-known that the weak field limit of the JIMWLK equation gives the BFKL equation [3, 6]. We will derive this result explicitly.

4.1. Choice of gauge

Our discussion has been mainly in the lightcone gauge $A^+ = 0$. However, we noted that generally we only know the solution to the Yang–Mills equation in the covariant gauge $\partial_\mu \tilde{A}^\mu = 0$. We obtained the solution to the field equation then by the gauge transformation given in Eq. (1.31)

$$U^\dagger(\vec{x}) = \mathrm{P} \exp\left(\mathrm{i}g \int_{x_0^-}^{x^-} \mathrm{d}z^- \, \alpha(z^-, \boldsymbol{x}_\perp) \right). \tag{4.1}$$

We can expand the exponential according to

$$U^\dagger(\vec{x}) = I + \mathrm{i}g \int_{x_0^-}^{x^-} \mathrm{d}z^- \, \alpha(z^-, \boldsymbol{x}_\perp) + O(\tilde{\rho}^2) = I + O(\tilde{\rho}). \tag{4.2}$$

The important observation is now that the color charge distributions themselves are related by a gauge transformation

$$\rho = U \tilde{\rho} U^\dagger. \tag{4.3}$$

Expanding the exponentials of the gauge transformations we get

$$\rho = \tilde{\rho} + O(\tilde{\rho}^2). \tag{4.4}$$

In the weak field limit we therefore do know the solution to the Yang–Mills equation explicitly. It is given by

$$\mathcal{A}^i(\vec{x}) = -\int_{x_0^-}^{x^-} \mathrm{d}z^- \, \partial^i \alpha(z^-, \boldsymbol{x}_\perp), \tag{4.5}$$

where now

$$\alpha(\vec{x}) = -\frac{1}{\boldsymbol{\nabla}_\perp^2}\rho(\vec{x}). \tag{4.6}$$

Using Eq. (1.23) the classical solution can also be written as

$$\begin{aligned}\mathcal{A}^i(\vec{x}) &= \frac{1}{2\pi}\partial^i \int_{x_0^-}^{x^-} \mathrm{d}y^- \int \mathrm{d}^2\boldsymbol{y}_\perp \ln|\boldsymbol{x}_\perp - \boldsymbol{y}_\perp|\rho(\vec{y}) \\ &= \frac{1}{2\pi}\int_{x_0^-}^{x^-} \mathrm{d}y^- \int \mathrm{d}^2\boldsymbol{y}_\perp \frac{(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)^i}{(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)^2}\rho(\vec{y}).\end{aligned} \tag{4.7}$$

4.2. Weak field limit of σ

In Appendix E.2 we present a perturbative expansion of the gluon propagator. In this section we will formally write for the propagator

$$G = G_0 + \delta G_{\mathrm{YM}} + \delta G_W, \tag{4.8}$$

where it is understood that the terms δG_{YM} and δG_W correspond to the corrections linear in the color charge density distribution ρ.

Note that in the weak field limit we do not need to compute the additional term $\delta\bar{\sigma}$ which arises from a gauge transformation (see Section 3.5). This can be understood by explicitly noting that it is of higher order in ρ or by simply recalling that in the leading order of the weak field limit the gauge transformation can be taken to be trivial since otherwise they would lead to higher order terms in ρ.

4.2.1. Weak field limit of σ_1

Using Eq. (4.8) we have for σ_1 from Eq. (3.57)

$$\begin{aligned}(\sigma_1)_a(x) &= \frac{1}{2}\mathrm{i}g f_{abc}\delta^{ij}(\partial_x^+ - \partial_y^+)G_{ij}^{bc}(x,y)|_{y=x} \\ &= \frac{1}{2}\mathrm{i}g f_{abc}\delta^{ij}(\partial_x^+ - \partial_y^+)\left((G_0)_{ij}^{bc}(x,y) + (\delta G_{\mathrm{YM}})_{ij}^{bc}(x,y) + (\delta G_W)_{ij}^{bc}(x,y)\right)|_{y=x} \\ &= \frac{1}{2}\mathrm{i}g f_{abc}\delta^{ij}(\partial_x^+ - \partial_y^+)(\delta G_W)_{ij}^{bc}(x,y)|_{y=x},\end{aligned} \tag{4.9}$$

where the free part does not contribute because it is diagonal in the color indices, i.e. $G_0^{ab} = \delta^{ab}G_0$, and the Yang–Mills correction (corresponding to the gluon self-energy)

does not have a logarithmic enhancement [3]. Going to momentum space we have

$$\begin{aligned}
&(\sigma_1)_a(x)\\
&= \frac{1}{2} g f_{abc} \delta^{ij} \int_{p,q} (p^+ + q^+) \mathrm{e}^{-\mathrm{i}(p-q)x} (\delta G_W)^{bc}_{ij}(p,q)\\
&= \frac{1}{2} g f_{abc} \delta^{ij} \int_{p,q} \mathrm{e}^{-\mathrm{i}(p-q)x} (p^+ + q^+) (\delta G_W)^{bc}_{ij}(p,q)\\
&= -\frac{1}{2} g f_{abc} \delta^{ij} \int_{p,q} \mathrm{e}^{-\mathrm{i}(p-q)x} (p^+ + q^+) (G_0)^{bd}_{i\mu}(p) \Gamma^{\mu\nu}_{de}(p,q) (G_0)^{ce}_{\nu j}(q)\\
&= -\frac{1}{2} g f_{abc} \delta_{ij} \int_{q} \mathrm{e}^{-\mathrm{i}(p-q)x} (p^+ + q^+) G_0^{i-}(p) \Gamma^{bc}_{--}(p,q) G_0^{-j}(q)\\
&= -\frac{1}{2} g f_{abc} \delta_{ij} \int_{p,q} \mathrm{e}^{-\mathrm{i}(p-q)x} (p^+ + q^+) \frac{p^i}{p^+} \frac{1}{p^2 + \mathrm{i}\epsilon} \left(-g \,\mathrm{PV}\, \frac{1}{p^-} \rho^{bc}(\vec{p} - \vec{q}) I(p^-, q^-) \right)\\
&\qquad\qquad \times \frac{q^j}{q^+} \frac{1}{q^2 + \mathrm{i}\epsilon}\\
&= \frac{1}{2} g^2 f_{abc} \int_{p^-} \int_{\vec{p},\vec{q}} \mathrm{e}^{-\mathrm{i}(\vec{p}-\vec{q})\cdot\vec{x}} \rho^{bc}(\vec{p} - \vec{q}) \boldsymbol{p}_\perp \cdot \boldsymbol{q}_\perp \frac{p^+ + q^+}{p^+ q^+}\\
&\qquad\qquad \times \frac{1}{2p^+p^- - \boldsymbol{p}_\perp^2 + \mathrm{i}\epsilon} \,\mathrm{PV}\, \frac{1}{p^-} \frac{1}{2q^+p^- - \boldsymbol{q}_\perp^2 + \mathrm{i}\epsilon},
\end{aligned} \tag{4.10}$$

where $\delta G_W = -G_0 \Gamma G_0$ and $\Gamma^{ab}_{\mu\nu}(p)$ is given in Eq. (E.28). The p^- integral can be done by closing the contour in the upper half plane

$$\begin{aligned}
&\int_{p^-} \frac{1}{2p^+p^- - \boldsymbol{p}_\perp^2 + \mathrm{i}\epsilon} \,\mathrm{PV}\, \frac{1}{p^-} \frac{1}{2q^+p^- - \boldsymbol{q}_\perp^2 + \mathrm{i}\epsilon}\\
&= \frac{2\pi\mathrm{i}}{2\pi} \left(\frac{1}{2} \frac{1}{\boldsymbol{p}_\perp^2 \boldsymbol{q}_\perp^2} + \vartheta(-p^+) \frac{p^+}{\boldsymbol{p}_\perp^2 (q^+ \boldsymbol{p}_\perp^2 - p^+ \boldsymbol{q}_\perp^2)} + \vartheta(-q^+) \frac{q^+}{\boldsymbol{q}_\perp^2 (p^+ \boldsymbol{q}_\perp^2 - q^+ \boldsymbol{p}_\perp^2)} \right).
\end{aligned} \tag{4.11}$$

Therefore

$$\begin{aligned}
\sigma_1(\vec{x}) = \frac{1}{2} g^2 T(\mathrm{A}) \int_{\vec{p},\vec{q}} \mathrm{e}^{-\mathrm{i}(\vec{p}-\vec{q})\cdot\vec{x}} \rho(\vec{p} - \vec{q}) \boldsymbol{p}_\perp \cdot \boldsymbol{q}_\perp \frac{p^+ + q^+}{p^+ q^+}\\
\times \left(\frac{1}{2} \frac{1}{\boldsymbol{p}_\perp^2 \boldsymbol{q}_\perp^2} + \vartheta(-p^+) \frac{p^+}{\boldsymbol{p}_\perp^2 (q^+ \boldsymbol{p}_\perp^2 - p^+ \boldsymbol{q}_\perp^2)} \right.\\
\left. + \vartheta(-q^+) \frac{q^+}{\boldsymbol{q}_\perp^2 (p^+ \boldsymbol{q}_\perp^2 - q^+ \boldsymbol{p}_\perp^2)} \right),
\end{aligned} \tag{4.12}$$

where we used $f_{abc} \rho^{bc} = -\mathrm{i} T(\mathrm{A}) \rho_a$. We note that the virtual correction only depends on the spatial coordinates $\vec{x}$ without having to take the equal time limit "by hand".

4.2.2. Weak field limit of σ_2

In the weak field limit we can replace the full propagator with the free propagator. We then have

$$\begin{aligned}(\sigma_2)_a(x) = -\mathrm{i}g^2\rho^d(\vec{x})\int_{y^+,z^+}\frac{1}{T(\mathrm{A})}\mathrm{tr}[\{&\theta(x^+-y^+)\theta(y^+-z^+)T_aT_bT_cT_d\\ &+\theta(y^+-z^+)\theta(z^+-x^+)T_bT_cT_aT_d\\ &+\theta(z^+-x^+)\theta(x^+-y^+)T_cT_aT_bT_d\}]\\ &\times\delta^{bc}G_0^{--}(y^+-z^+,\vec{0})\\ = -\mathrm{i}g^2T(\mathrm{A})\rho_a(\vec{x})\int_{y^+,z^+}&\{\theta(x^+-y^+)\theta(y^+-z^+)+\theta(y^+-z^+)\theta(z^+-x^+)\\ &+\frac{1}{2}\theta(z^+-x^+)\theta(x^+-y^+)\}\\ &\times G_0^{--}(y^+-z^+,\vec{0}),\end{aligned} \tag{4.13}$$

where the traces over the color matrices simplify as follows:

$$\frac{1}{T(\mathrm{A})}\,\mathrm{tr}[T_aT_bT_cT_d]\delta^{bc} = \frac{1}{T(\mathrm{A})}\,\mathrm{tr}[T_aC(\mathrm{A})I_{\mathrm{A}}T_d] = \mathrm{tr}[T_aT_d] = T(\mathrm{A})\delta_{ad}, \tag{4.14}$$

$$\frac{1}{T(\mathrm{A})}\,\mathrm{tr}[T_bT_cT_aT_d]\delta^{bc} = \frac{1}{T(\mathrm{A})}\,\mathrm{tr}[C(\mathrm{A})I_{\mathrm{A}}T_aT_d] = \mathrm{tr}[T_aT_d] = T(\mathrm{A})\delta_{ad}, \tag{4.15}$$

$$\begin{aligned}\frac{1}{T(\mathrm{A})}\,\mathrm{tr}[T_cT_aT_bT_d]\delta^{bc} &= \frac{1}{T(\mathrm{A})}\,\mathrm{tr}[T_aT_cT_bT_d+[T^b,T_a]T_bT_d]\\ &= T(\mathrm{A})\delta_{ad}+\frac{1}{T(\mathrm{A})}\mathrm{i}f^b{}_a{}^e\,\mathrm{tr}[T_eT_bT_d]\\ &= T(\mathrm{A})\delta_{ad}+\frac{1}{T(\mathrm{A})}\mathrm{i}f^b{}_a{}^e\frac{\mathrm{i}}{2}T(\mathrm{A})f_{ebf}\delta^f_d\\ &= T(\mathrm{A})\delta_{ad}-\frac{1}{2}f^b{}_a{}^ef_{ebd}\\ &= T(\mathrm{A})\delta_{ad}-\frac{1}{2}T(\mathrm{A})\delta_{ad}\\ &= \frac{1}{2}T(\mathrm{A})\delta_{ad}.\end{aligned} \tag{4.16}$$

The integrand in Eq. (4.13) is symmetric[1] under the exchange of y^+ and z^+ and the terms in braces can be thus written as

$$\begin{aligned}&\theta(x^+-y^+)\theta(y^+-z^+)+\theta(y^+-z^+)\theta(z^+-x^+)+\frac{1}{2}\theta(z^+-x^+)\theta(x^+-y^+)\\ &\simeq\theta(x^+-z^+)\theta(z^+-y^+)+\theta(z^+-y^+)\theta(y^+-x^+)+\frac{1}{2}\theta(z^+-x^+)\theta(x^+-y^+)\end{aligned}$$

[1] Here it is essential that the propagator is the Feynman propagator such that $G(x) = G(-x)$.

$$
\begin{aligned}
&= \theta(x^+ - z^+)\theta(z^+ - y^+) + \theta(z^+ - y^+)\theta(y^+ - x^+) + \theta(z^+ - x^+)\theta(x^+ - y^+) \\
&\quad - \frac{1}{2}\theta(z^+ - x^+)\theta(x^+ - y^+) \\
&= \theta(z^+ - y^+) - \frac{1}{2}\theta(z^+ - x^+)\theta(x^+ - y^+),
\end{aligned} \tag{4.17}
$$

where by $\simeq$ we denote an equality up to swaps of y^+ and z^+ and the last equality can be understood by noting that the first line of the expression before covers exactly the three possible cases of the relation that x^+ can have with respect to y^+ and z^+, i.e. either it is greater than or less than both or lies between the two.

The first integral is

$$
\int_{y^+,z^+} G_0^{--}(y^+ - z^+, \vec{0})\theta(z^+ - y^+) = \int_{y^+,z^+} G_0^{--}(z^+ - y^+, \vec{0})\theta(z^+ - y^+), \tag{4.18}
$$

where we used the symmetry of the Feynman propagator. Introducing new coordinates $u^+ = \frac{1}{2}(y^+ + z^+)$ and $v^+ = z^+ - y^+$ we get

$$
\begin{aligned}
\int_{u^+,v^+} G_0^{--}(v^+, \vec{0})\theta(v^+) &= \left(\int_{u^+}\right) \int_{v^+} G_0^{--}(v^+, \vec{0})\theta(v^+) \\
&= \left(\int_{u^+}\right) \int_0^\infty \mathrm{d}v^+\, G_0^{--}(v^+, \vec{0}) \\
&= \left(\int_{u^+}\right) \frac{1}{2}\int_{-\infty}^\infty \mathrm{d}v^+\, G_0^{--}(v^+, \vec{0}) \\
&= \left(\int_{u^+}\right) \frac{1}{2}\int_{v^+}\int_q \mathrm{e}^{-iq^- v^+} G_0^{--}(q) \\
&= \left(\int_{u^+}\right) \frac{1}{2}\int_q I(q^-, 0)\frac{2q^-}{q^+}\frac{1}{q^2 + \mathrm{i}\epsilon} \\
&= \left(\int_{u^+}\right) \times 0.
\end{aligned} \tag{4.19}
$$

The contribution therefore vanishes.

The second integral is

$$
\begin{aligned}
&\int_{y^+,z^+} G_0^{--}(y^+ - z^+, \vec{0})\theta(z^+ - x^+)\theta(x^+ - y^+) \\
&= \int_{y^+,z^+}\int_q \mathrm{e}^{-\mathrm{i}q^+(y^+ - z^+)} G_0^{--}(q)\int_{r^-} \mathrm{e}^{-\mathrm{i}r^-(z^+ - x^+)}\theta(r^-)\int_{s^-} \mathrm{e}^{-\mathrm{i}s^-(x^+ - y^+)}\theta(s^-) \\
&= \int_{q,r^-,s^-} I(q^+, r^+)I(q^+, s^+)\mathrm{e}^{-\mathrm{i}(r^- - s^-)x^+} G_0^{--}(q)\theta(r^-)\theta(s^-) \\
&= \int_q G_0^{--}(q)\theta(q^-)\theta(q^-) \\
&= \int_q \frac{2q^-}{q^+}\frac{1}{q^2 + \mathrm{i}\epsilon}\frac{\mathrm{i}}{q^- + \mathrm{i}\epsilon}\frac{\mathrm{i}}{q^- + \mathrm{i}\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
&= -2\mathrm{i}\int_{q^+}\frac{1}{q^+}\vartheta(-q^+)\int_{\boldsymbol{q}_\perp}\frac{1}{\boldsymbol{q}_\perp^2}\\
&= \mathrm{i}\frac{1}{\pi}\ln\frac{1}{b}\int_{\boldsymbol{q}_\perp}\frac{1}{\boldsymbol{q}_\perp^2}
\end{aligned}
\tag{4.20}
$$

With this and the additional factor $-\frac{1}{2}$ from Eq. (4.17) we finally have

$$
\begin{aligned}
\sigma_2(\vec{x}) &= -\mathrm{i}g^2T(\mathrm{A})\rho(\vec{x})\left(-\frac{1}{2}\right)\mathrm{i}\frac{1}{\pi}\ln\frac{1}{b}\int_{\boldsymbol{q}_\perp}\frac{1}{\boldsymbol{q}_\perp^2}\\
&= -\frac{g^2}{2\pi}T(\mathrm{A})\rho(\vec{x})\ln\frac{1}{b}\int_{\boldsymbol{q}_\perp}\frac{1}{\boldsymbol{q}_\perp^2}.
\end{aligned}
\tag{4.21}
$$

4.2.3. Ultra thin limit

For sources without longitudinal extent $\rho(\vec{x}) = \delta(x^-)\rho(\boldsymbol{x}_\perp)$ all the dynamics take place in the transverse plane. One then defines

$$
\sigma_1(\boldsymbol{x}_\perp) \equiv \int_{x^-}\sigma_1(\vec{x}). \tag{4.22}
$$

The x^- integration over Eq. (4.12) gives

$$
\int_{x^-}\mathrm{e}^{-\mathrm{i}(p^+-q^+)x^-} = I(p^+,q^+) \tag{4.23}
$$

and therefore

$$
\begin{aligned}
&\sigma_1(\boldsymbol{x}_\perp)\\
&= g^2T(\mathrm{A})\int_{p^+}\int_{\boldsymbol{p}_\perp,\boldsymbol{q}_\perp}\mathrm{e}^{\mathrm{i}(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{x}_\perp}\rho(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp\frac{1}{p^+}\\
&\qquad\qquad\times\left(\frac{1}{2}\frac{1}{\boldsymbol{p}_\perp^2\boldsymbol{q}_\perp^2}+\vartheta(-p^+)\frac{1}{\boldsymbol{p}_\perp^2(\boldsymbol{p}_\perp^2-\boldsymbol{q}_\perp^2)}+\vartheta(-p^+)\frac{1}{\boldsymbol{q}_\perp^2(\boldsymbol{q}_\perp^2-\boldsymbol{p}_\perp^2)}\right)\\
&= g^2T(\mathrm{A})\int_{p^+}\int_{\boldsymbol{p}_\perp,\boldsymbol{q}_\perp}\mathrm{e}^{\mathrm{i}(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{x}_\perp}\rho(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp\frac{1}{p^+}\frac{1}{\boldsymbol{p}_\perp^2\boldsymbol{q}_\perp^2}\left(\frac{1}{2}-\vartheta(-p^+)\right)\\
&= \frac{1}{2}g^2T(\mathrm{A})\int_{p^+}\frac{\operatorname{sgn}p^+}{p^+}\int_{\boldsymbol{p}_\perp,\boldsymbol{q}_\perp}\mathrm{e}^{\mathrm{i}(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{x}_\perp}\rho(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp}{\boldsymbol{p}_\perp^2\boldsymbol{q}_\perp^2}\\
&= \frac{1}{2}g^2T(\mathrm{A})\frac{1}{\pi}\ln\frac{1}{b}\int_{\boldsymbol{p}_\perp,\boldsymbol{q}_\perp}\mathrm{e}^{\mathrm{i}(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{x}_\perp}\rho(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp}{\boldsymbol{p}_\perp^2\boldsymbol{q}_\perp^2},\\
&= \frac{g^2}{2\pi}T(\mathrm{A})\ln\frac{1}{b}\int_{\boldsymbol{p}_\perp,\boldsymbol{q}_\perp}\mathrm{e}^{\mathrm{i}(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\cdot\boldsymbol{x}_\perp}\rho(\boldsymbol{p}_\perp-\boldsymbol{q}_\perp)\frac{\boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp}{\boldsymbol{p}_\perp^2\boldsymbol{q}_\perp^2}\\
&= \frac{g^2}{2\pi}T(\mathrm{A})\ln\frac{1}{b}\int_{\boldsymbol{p}_\perp,\boldsymbol{q}_\perp}\mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\rho(\boldsymbol{p}_\perp)\frac{(\boldsymbol{p}_\perp+\boldsymbol{q}_\perp)\cdot\boldsymbol{q}_\perp}{(\boldsymbol{p}_\perp+\boldsymbol{q}_\perp)^2\boldsymbol{q}_\perp^2}\\
&= \frac{g^2}{2\pi}T(\mathrm{A})\ln\frac{1}{b}\int_{\boldsymbol{p}_\perp,\boldsymbol{q}_\perp}\mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp}\rho(\boldsymbol{p}_\perp)\left[-\frac{\boldsymbol{p}_\perp^2}{2}\frac{1}{\boldsymbol{q}_\perp^2(\boldsymbol{q}_\perp-\boldsymbol{p}_\perp)^2}+\frac{1}{\boldsymbol{q}_\perp^2}\right],
\end{aligned}
\tag{4.24}
$$

where we used $\frac{1}{2} - \vartheta(p) = \frac{1}{2}\operatorname{sgn} p$ and various shifts in both $\boldsymbol{p}_\perp$ and $\boldsymbol{q}_\perp$ have been performed.

4.3. Weak field limit of χ

In this Section we will discuss the weak field limit of χ.

We will be working exclusively in the lightcone gauge and impose the momentum strip restriction on the longitudinal momenta p^+. The additional poles appearing in the lightcone gauge propagators (see Appendix E.1) therefore need no pole prescription.

Since we assume that the classical charge distribution ρ is weak, we can replace the propagators appearing in Eqs. (3.61) and (3.63) to (3.65) by the free propagator, because the expressions are then still quadratic in ρ. Note that the non-vanishing component of the field-strength tensor, $\mathcal{F}^{+i}$, is linearly related to ρ. The free propagator is diagonal in the color indices, i.e.

$$(G_0)^{ab}_{\mu\nu} = \delta^{ab}(G_0)_{\mu\nu}. \tag{4.25}$$

Equations (3.61) and (3.63) to (3.65) then become

$$\chi_1(x,y) = 4\mathrm{i}g^2\mathcal{F}^{+i}(\vec{x})\mathcal{F}^{+j}(\vec{y})(G_0)_{ij}(x,y), \tag{4.26a}$$
$$\chi_{21}(x,y) = \mathrm{i}g^2\mathcal{F}^{+i}(\vec{x})\rho(\vec{y})\langle x|(G_0)_{i+}\,\epsilon\,|y\rangle, \tag{4.26b}$$
$$\chi_{22}(x,y) = -\mathrm{i}g^2\rho(\vec{x})\mathcal{F}^{+i}(\vec{y})\langle x|\,\epsilon(G_0)_{+i}|y\rangle, \tag{4.26c}$$
$$\chi_3(x,y) = -\mathrm{i}\frac{g^2}{4}\rho(\vec{x})\rho(\vec{y})\langle x|\,\epsilon(G_0)_{++}\,\epsilon\,|y\rangle. \tag{4.26d}$$

For later convenience we rewrite this as

$$\chi_1(x,y) = 4\mathrm{i}g^2\mathcal{F}_{-i}(\vec{x})\mathcal{F}_{-j}(\vec{y})G_0^{ij}(x,y), \tag{4.27a}$$
$$\chi_{21}(x,y) = \mathrm{i}g^2\mathcal{F}_{-i}(\vec{x})\rho(\vec{y})\langle x|G_0^{i-}\,\epsilon\,|y\rangle, \tag{4.27b}$$
$$\chi_{22}(x,y) = -\mathrm{i}g^2\rho(\vec{x})\mathcal{F}_{-i}(\vec{y})\langle x|\,\epsilon\,G_0^{-i}|y\rangle, \tag{4.27c}$$
$$\chi_3(x,y) = -\mathrm{i}\frac{g^2}{4}\rho(\vec{x})\rho(\vec{y})\langle x|\,\epsilon\,G_0^{--}\,\epsilon\,|y\rangle. \tag{4.27d}$$

4.3.1. Computing the matrix elements

In this section we will compute the matrix elements of the lightcone gauge propagator (see Appendix E.1.1). Recalling the equal-time prescription of Eq. (3.53) we have

$$\chi(\vec{x},\vec{y}) = \chi(x^+,\vec{x};y^+ = x^+ + \epsilon,\vec{y}). \tag{4.28}$$

Recalling that

$$G_0^{ij} = \delta^{ij}G_0 \tag{4.29}$$

the matrix element for χ_1 is

$$\begin{aligned}\langle x|G_0|y\rangle &= \int_{p^-} \mathrm{e}^{\mathrm{i}p^-\epsilon} \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{p^2+\mathrm{i}\epsilon} \\ &= \int_{p^-} \mathrm{e}^{\mathrm{i}p^-\epsilon} \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{2p^+} \frac{1}{p^- - (\frac{\boldsymbol{p}_\perp^2}{2p^+} - \mathrm{i}\epsilon\varepsilon(p^+))} \\ &= \frac{\mathrm{i}}{2} \int_{p^+} \mathrm{e}^{-\mathrm{i}p^+(x^- - y^-)} \frac{\vartheta(-p^+)}{p^+} \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)} \\ &= \frac{\mathrm{i}}{4\pi} F_1(x^- - y^-) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)}, \end{aligned} \tag{4.30}$$

where we have defined the function

$$F_1(x^-) = 2\pi \int_{p^+} \mathrm{e}^{-\mathrm{i}p^+x^-} \frac{\vartheta(-p^+)}{p^+} = \int_{-\Lambda^+}^{-b\Lambda^+} \mathrm{d}p^+ \, \mathrm{e}^{-\mathrm{i}p^+x^-} \frac{1}{p^+}. \tag{4.31}$$

The free propagator is symmetric and diagonal in momentum space. Therefore we have

$$\langle x|G_0^{i-}\epsilon|y\rangle = \langle x|\epsilon G_0^{-i}|y\rangle. \tag{4.32}$$

The matrix element for χ_2 then is

$$\begin{aligned}\langle x|G_0^{i-}\epsilon|y\rangle &= \int_{p^-} \mathrm{e}^{\mathrm{i}p^-\epsilon} \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{p^i}{p^+} \frac{1}{p^2+\mathrm{i}\epsilon} \,\mathrm{PV}\, \frac{2\mathrm{i}}{p^-} \\ &= 2\mathrm{i} \int_{p^-} \mathrm{e}^{\mathrm{i}p^-\epsilon} \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{p^i}{p^+} \frac{1}{2p^+} \frac{1}{p^- - (\frac{\boldsymbol{p}_\perp^2}{2p^+} - \mathrm{i}\epsilon\varepsilon(p^+))} \,\mathrm{PV}\, \frac{1}{p^-} \\ &= 2\mathrm{i} \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{p^i}{p^+} \frac{1}{2p^+} \mathrm{i} \left(\frac{1}{2} \frac{1}{-\boldsymbol{p}_\perp^2} + \frac{2p^+}{\boldsymbol{p}_\perp^2} \frac{1}{2p^+} \vartheta(-p^+) \right) \\ &= -2 \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{p^i}{p^+} \frac{1}{\boldsymbol{p}_\perp^2} \frac{1}{2} (-\varepsilon(p^+)) \\ &= \int_{p^+} \mathrm{e}^{-\mathrm{i}p^+(x^- - y^-)} \frac{\varepsilon(p^+)}{p^+} \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)} \frac{p^i}{\boldsymbol{p}_\perp^2} \\ &= \frac{1}{2\pi} F_2(x^- - y^-) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)} \frac{p^i}{\boldsymbol{p}_\perp^2}, \end{aligned} \tag{4.33}$$

where we have defined the function

$$\begin{aligned} F_2(x^-) &= 2\pi \int_{p^+} \mathrm{e}^{-\mathrm{i}p^+x^-} \frac{\varepsilon(p^+)}{p^+} \\ &= \left(\int_{b\Lambda^+}^{\Lambda^+} - \int_{-\Lambda^+}^{-b\Lambda^+} \right) \mathrm{d}p^+ \, \mathrm{e}^{-\mathrm{i}p^+x^-} \frac{1}{p^+}. \end{aligned} \tag{4.34}$$

The matrix element for χ_3 is

$$\begin{aligned}
\langle x|\epsilon G_0^{--}\epsilon|y\rangle &= \int_{p^-} \mathrm{e}^{\mathrm{i}p^-\epsilon} \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{2p^-}{p^+} \frac{1}{p^2+\mathrm{i}\epsilon} \left(\mathrm{PV}\,\frac{2\mathrm{i}}{p^-}\right)^2 \\
&= -8\int_{p^-} \mathrm{e}^{\mathrm{i}p^-\epsilon} \int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{p^-}{p^+}\frac{1}{2p^+} \frac{1}{p^- - (\frac{\boldsymbol{p}_\perp^2}{2p^+} - \mathrm{i}\epsilon\varepsilon(p^+))} \left(\mathrm{PV}\,\frac{1}{p^-}\right)^2 \\
&= -8\mathrm{i}\int_{\vec{p}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} \frac{1}{p^+}\frac{\boldsymbol{p}_\perp^2}{2p^+}\frac{1}{2p^+}\left(\frac{2p^+}{\boldsymbol{p}_\perp^2}\right)^2 \vartheta(-p^+) \\
&= -8\mathrm{i}\int_{p^+} \mathrm{e}^{-\mathrm{i}p^+(x^- - y^-)} \frac{\vartheta(-p^+)}{p^+} \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} \frac{1}{\boldsymbol{p}_\perp^2} \\
&= -\frac{4\mathrm{i}}{\pi} F_3(x^- - y^-) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} \frac{1}{\boldsymbol{p}_\perp^2},
\end{aligned} \tag{4.35}$$

where we have defined the function

$$F_3(x^-) = 2\pi \int_{p^+} \mathrm{e}^{-\mathrm{i}p^+x^-} \frac{\vartheta(-p^+)}{p^+} \tag{4.36}$$

which we recognize to be the same as $F_1(x^-)$.

Inserting the matrix elements into Eqs. (4.27) we have

$$\chi_1(\vec{x},\vec{y}) = -\frac{g^2}{\pi} F_1(x^- - y^-)\mathcal{F}_{-i}(\vec{x})\mathcal{F}_{-j}(\vec{y}) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)}, \tag{4.37a}$$

$$\chi_{21}(\vec{x},\vec{y}) = \frac{g^2}{2\pi} F_2(x^- - y^-)\mathcal{F}_{-i}(\vec{x})\rho(\vec{y}) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} \frac{\mathrm{i}p^i}{\boldsymbol{p}_\perp^2}, \tag{4.37b}$$

$$\chi_{22}(\vec{x},\vec{y}) = -\frac{g^2}{2\pi} F_2(x^- - y^-)\rho(\vec{x})\mathcal{F}_{-i}(\vec{y}) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} \frac{\mathrm{i}p^i}{\boldsymbol{p}_\perp^2}, \tag{4.37c}$$

$$\chi_3(\vec{x},\vec{y}) = -\frac{g^2}{\pi} F_1(x^- - y^-)\rho(\vec{x})\rho(\vec{y}) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot(\boldsymbol{x}_\perp-\boldsymbol{y}_\perp)} \frac{1}{\boldsymbol{p}_\perp^2}. \tag{4.37d}$$

4.3.2. Ultra thin limit

The limit of very thin nuclei $\rho(\vec{x}) \approx \delta(x^-)\rho(\boldsymbol{x}_\perp)$ can be recovered from the above expressions by letting $x^-, y^- \to 0$.

The functions F_1 and F_2 have the following limits

$$\lim_{x^-\to 0} F_1(x^-) = \int_{-\Lambda^+}^{-b\Lambda^+} \mathrm{d}p^+ \frac{1}{p^+} = -\ln\frac{1}{b}, \tag{4.38a}$$

$$\lim_{x^-\to 0} F_2(x^-) = \left(\int_{b\Lambda^+}^{\Lambda^+} - \int_{-\Lambda^+}^{-b\Lambda^+}\right) \mathrm{d}p^+ \frac{1}{p^+} = 2\ln\frac{1}{b}. \tag{4.38b}$$

By performing the integral over the longitudinal coordinate x^-

$$\chi(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = \int_{x^-, y^-} \chi(\vec{x}, \vec{y}) \tag{4.39}$$

we recover the well-known results (see e.g. Section 5 of [6])

$$\chi_1(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = \frac{g^2}{\pi} \ln \frac{1}{b} \mathcal{F}_{-i}(\boldsymbol{x}_\perp) \mathcal{F}_{-j}(\boldsymbol{y}_\perp) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp \cdot (\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)}, \tag{4.40a}$$

$$\chi_{21}(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = \frac{g^2}{\pi} \ln \frac{1}{b} \mathcal{F}_{-i}(\boldsymbol{x}_\perp) \rho(\boldsymbol{y}_\perp) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp \cdot (\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)} \frac{\mathrm{i}p^i}{\boldsymbol{p}_\perp^2}, \tag{4.40b}$$

$$\chi_{22}(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = -\frac{g^2}{\pi} \ln \frac{1}{b} \rho(\boldsymbol{x}_\perp) \mathcal{F}_{-i}(\boldsymbol{y}_\perp) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp \cdot (\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)} \frac{\mathrm{i}p^i}{\boldsymbol{p}_\perp^2}, \tag{4.40c}$$

$$\chi_3(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp) = \frac{g^2}{\pi} \ln \frac{1}{b} \rho(\boldsymbol{x}_\perp) \rho(\boldsymbol{y}_\perp) \int_{\boldsymbol{p}_\perp} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp \cdot (\boldsymbol{x}_\perp - \boldsymbol{y}_\perp)} \frac{1}{\boldsymbol{p}_\perp^2}. \tag{4.40d}$$

Conclusion & Outlook

In this thesis we gave an overview of the color glass condensate framework characterized by an effective action which allows one to compute observables at a given energy scale. The question of how one passes from one energy scale to the next is answered by the JIMWLK renormalization group equation. Finally, we performed a weak field expansion of the JIMWLK equation and saw that it corresponds to leading order to the BFKL equation.

As discussed in the introduction, it is an interesting problem to extend the JIMWLK equation to describe nuclei with finite longitudinal thickness. In particular one feature which should be recovered from such an equation, is the growing longitudinal extent when going to lower energies. This can be understood by recalling the uncertainty principle. At higher longitudinal momentum, the partons will be more strongly localized in the longitudinal direction.

The JIMWLK equation as presented in Chapter 3 gives only information to leading logarithmic accuracy in $\alpha_{\mathrm{s}} \ln \frac{1}{b}$. The longitudinal structure would only contribute to subleading terms. To see why consider for example the crude model of a gaussian longitudinal structure as in

$$\rho(\vec{x}) = f(x^-)\rho(\boldsymbol{x}_\perp), \quad f(x^-) = \frac{1}{\sqrt{2\pi d^2}} \mathrm{e}^{-\frac{1}{2d^2}(x^-)^2}.$$

The typical integral appearing (in the weak field limit) would then be

$$\int_{x^-} f(x^-) \int_{b\Lambda^+}^{\Lambda^+} \mathrm{d}p^+ \, \frac{\mathrm{e}^{\mathrm{i}p^+x^-}}{p^+} = \int_{b\Lambda^+}^{\Lambda^+} \mathrm{d}p^+ \, \frac{\mathrm{e}^{-\frac{1}{2}d^2(p^+)^2}}{p^+} = \ln \frac{1}{b} + O(d^2).$$

This shows that the effects of the longitudinal structure would not contribute to the leading logarithmic enhancement.

Nevertheless, it is still an interesting question how effects of the longitudinal structure could be incorporated into a renormalization group equation. In particular, as explained above, a lowering of the cutoff should correspond to an increase in the longitudinal thickness of the nucleus.

When deriving the coefficients σ and χ for the JIMWLK renormalization group equation Eq. (3.26) a crucial ingredient is the nonperturbative background field propagator describing the propagation of gluons in the gauge field generated by the classical color charge distribution ρ to all orders. The propagator used in [6, 7] however does not capture the longitudinal structure of the nuclei. The nuclei are considered to be infinitely

thin. This, they argue, is because the semi-fast gluons are localized at larger x^- and therefore they "see" the classical color charge density as an ultra thin sheet of color charge (see also the discussion in Section 3.4).

Therefore an important step in capturing the longitudinal structure of nuclei would be to extend the background field propagator to the case of nuclei with a finite longitudinal thickness.

Another interesting question is whether there exists a relation between the coefficients σ and χ as in the case where one has performed the integration over the longitudinal coordinates. In that case one can check by explicit calculation that the following identity holds [10]

$$\sigma_a(\boldsymbol{x}_\perp) = \frac{1}{2}\frac{\delta}{\delta\rho_b(\boldsymbol{y}_\perp)}\chi_{ab}(\boldsymbol{x}_\perp, \boldsymbol{y}_\perp).$$

It is unclear if this identity is satisfied for the case where the longitudinal structure has not been integrated out.

Appendix A.

Notation and conventions

In this appendix we summarize our notation and conventions.

A.1. Derivatives

Derivatives always act only on the object immediately to its right unless explicitly stated otherwise. As an example consider

$$\partial_\mu f g \equiv (\partial_\mu f) g. \tag{A.1}$$

When there are multiple variables involved, it can be useful to attach a label to the derivative symbol to specify with respect to which coordinate the derivative is to be taken. We use the following shorthand notation

$$\partial^x_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{A.2}$$

A.2. Metric and lightcone coordinates

For the metric signature we choose that the Minkowski metric in cartesian coordinates

$$(x^0, x^1, x^2, x^3) \equiv (x^0, \boldsymbol{x}) \tag{A.3}$$

takes the following form[1]

$$(g^{\mathrm{c}}_{\mu\nu}) = \mathrm{diag}(+, -, -, -). \tag{A.4}$$

We introduce lightcone coordinates by the following coordinate transformation

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3). \tag{A.5}$$

For lightcone coordinates we write (note the change of ordering)

$$(x^+, x^-, x^1, x^2) \equiv (x^+, x^-, \boldsymbol{x}_\perp) \equiv (x^+, x^-, x^i) \equiv (x^+, \vec{x}), \tag{A.6}$$

[1] Since we will be almost exclusively working with lightcone coordinates, quantities referring to the cartesian coordinates will be labeled by 'c'.

where we have grouped the components as

$$\boldsymbol{x}_\perp \equiv (x^1, x^2), \tag{A.7a}$$
$$\vec{x} \equiv (x^-, \boldsymbol{x}_\perp). \tag{A.7b}$$

We will also use this notation for volume elements

$$\mathrm{d}^2\boldsymbol{x}_\perp \equiv \mathrm{d}x^1\,\mathrm{d}x^2\,, \tag{A.8a}$$
$$\mathrm{d}^3\vec{x} \equiv \mathrm{d}x^-\,\mathrm{d}^2\boldsymbol{x}_\perp\,. \tag{A.8b}$$

In lightcone coordinates the metric reads

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.9}$$

Note that the distinction between upper and lower indices is therefore crucial for the '+' and '−' components. For example, given the contravariant vector $V^\mu = (V^+, V^-, \boldsymbol{V}_\perp)$ the corresponding covariant vector is $V_\mu = g_{\mu\nu}V^\nu$ and the covariant '+' component is

$$V_+ = g_{+\mu}V^\mu = g_{+-}V^- = V^- \tag{A.10}$$

and likewise the covariant '−' component is

$$V_- = V^+. \tag{A.11}$$

The inner product of two vectors is given by

$$U \cdot V = U^\mu V_\mu = U^+V^- + U^-V^+ - \boldsymbol{U}_\perp \cdot \boldsymbol{V}_\perp, \tag{A.12}$$

where $U^\mu = (U^+, U^-, \boldsymbol{U}_\perp)$ and $V^\mu = (U^+, U^-, \boldsymbol{U}_\perp)$.

A.2.1. A caveat

It is common practice when passing from a covariant expression to an expression where one has performed a "time+space" split to treat all the spatial coordinates as belonging to a euclidean metric and no longer distinguishing between upper and lower indices. We will *not* do this. Indices are always meant with respect to the Minkowski metric. However, we will often write the following

$$U^iV_i = U^iV^jg_{ij} = -U^iV^j\delta_{ij} \equiv -\boldsymbol{U}_\perp \cdot \boldsymbol{V}_\perp \tag{A.13}$$

A.3. Integration

We define the following abbreviations for one-dimensional integrals

$$\int_x \equiv \int \mathrm{d}x\,, \tag{A.14a}$$

$$\int_p \equiv \int \frac{\mathrm{d}p}{2\pi}. \tag{A.14b}$$

which are readily generalized to the D-dimensional case as

$$\int_x \equiv \int \mathrm{d}^D x\,, \tag{A.15a}$$

$$\int_p \equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D}. \tag{A.15b}$$

A.4. Position and momentum basis

We define our (D-dimensional) position and momentum bases such that

$$I = \int_x |x\rangle\langle x| \equiv \int \mathrm{d}^D x |x\rangle\langle x|, \tag{A.16a}$$

$$I = \int_p |p\rangle\langle p| \equiv \int \frac{\mathrm{d}^D p}{(2\pi)^D} |p\rangle\langle p|. \tag{A.16b}$$

For the inner products we have

$$\langle x|y\rangle = \delta^{(D)}(x-y) \equiv I(x,y), \tag{A.17a}$$

$$\langle p|q\rangle = (2\pi)^D \delta^{(D)}(x-y) \equiv I(p,q), \tag{A.17b}$$

$$\langle x|p\rangle = \mathrm{e}^{-\mathrm{i}px}. \tag{A.17c}$$

Appendix B.

$SU(N)$ Group theory

In this appendix we give a brief introduction to Lie groups and their representations. We will restrict our focus to the special unitary groups since the gauge symmetry of strong interactions is of this type. For more details see [21].

B.1. Generators of the group

Consider a complex N-dimensional vector space with inner product

$$(\phi, \chi) = \phi^\dagger \chi. \tag{B.1}$$

Unitary transformations are the transformations U leaving the scalar product invariant, i.e. for $\phi \to \phi' = U\phi$ we have

$$\phi^\dagger \phi \to \phi'^\dagger \phi' = (U\phi)^\dagger U\phi = \phi^\dagger U^\dagger U \phi \stackrel{!}{=} \phi^\dagger \phi. \tag{B.2}$$

Therefore unitary transformations satisfy

$$U^\dagger U = I \tag{B.3}$$

and therefore $U^\dagger = U^{-1}$. The transformations satisfying Eq. (B.3) are said to be elements of the unitary group $U(N)$. Note that Eq. (B.3) is unchanged by the transformation $U \to \mathrm{e}^{\mathrm{i}\alpha} U$, where α is a real parameter. Requiring that the transformation has unit determinant,

$$\det U = 1, \tag{B.4}$$

fixes this phase. Unitary transformations with unit determinant are called special unitary transformations and are elements of the special unitary group $SU(N)$.

Consider next an infinitesimal transformation

$$U = I + \mathrm{i}\theta + O(\theta^2). \tag{B.5}$$

From Eq. (B.3) we have

$$I \stackrel{!}{=} U^\dagger U = (I - \mathrm{i}\theta^\dagger)(I + \mathrm{i}\theta) = I + \mathrm{i}(\theta - \theta^\dagger) + O(\theta^2) \tag{B.6}$$

and therefore we conclude that the matrix θ must be hermitian. Equation (B.4) gives

$$\begin{aligned}
1 &\overset{!}{=} \det U \\
&= \frac{1}{N!}\epsilon^{i_1\ldots i_N}\epsilon_{j_1\ldots j_N}U_{i_1}{}^{j_1}\ldots U_{i_N}{}^{j_N} \\
&= \frac{1}{N!}\epsilon^{i_1\ldots i_N}\epsilon_{j_1\ldots j_N}(\delta^{j_1}_{i_1}+\mathrm{i}\theta_{i_1}{}^{j_1})\ldots(\delta^{j_N}_{i_N}+\mathrm{i}\theta_{i_N}{}^{j_N})+O(\theta^2) \\
&= \frac{1}{N!}\epsilon^{i_1\ldots i_N}\epsilon_{j_1\ldots j_N}(\delta^{j_1}_{i_1}\ldots\delta^{j_N}_{i_N}+N\mathrm{i}\delta^{j_1}_{i_1}\ldots\delta^{j_{N-1}}_{i_{N-1}}\theta_{i_N}{}^{j_N})+O(\theta^2) \\
&= 1+\mathrm{i}\,\mathrm{tr}\,\theta+O(\theta^2)
\end{aligned} \tag{B.7}$$

and therefore the matrix θ must be traceless.

Since the transformations form a group, consecutive transformations must be a transformation too. By computing

$$U'^{\dagger}U^{\dagger}U'U=(I-\mathrm{i}\theta')(I-\mathrm{i}\theta)(I+\mathrm{i}\theta')(I+\mathrm{i}\theta)=I+[\theta,\theta']+O(\theta^2)+O(\theta'^2) \tag{B.8}$$

we deduce that there must be some θ'' such that

$$\mathrm{i}\theta''=[\theta,\theta']. \tag{B.9}$$

We now introduce a set of basis matrices such that we can write

$$\theta=\theta_a t^a. \tag{B.10}$$

We call this set of basis matrices the generators of the group. From Eq. (B.9) we then deduce that

$$[t^a,t^b]=\mathrm{i}f^{ab}{}_c\,t^c, \tag{B.11}$$

where the coefficients f_{abc} are called structure constants. One can show that the structure constants tensor f^{abc} is totally antisymmetric and all its components are real. Since it is a rank 3 tensor, this means that it is symmetric under cyclic permutations.

For algebra elements $X=X_a t^a$ and $Y=Y_a t^a$ we know from Eq. (B.11) that their commutator gives again an element of the algebra, i.e. $[X,Y]=[X,Y]_a t^a$. From Eq. (B.11) it is also easy to see that the components are

$$[X,Y]_a=\mathrm{i}f_{abc}X^bY^c. \tag{B.12}$$

The above discussion featured transformations which take the form of complex $N\times N$ matrices. Given the structure constants as in Eq. (B.11) a representation R of the group is any set of $D(\mathrm{R})\times D(\mathrm{R})$ matrices that satisfy

$$[t^a_{\mathrm{R}},t^b_{\mathrm{R}}]=\mathrm{i}f^{ab}{}_c\,t^c_{\mathrm{R}}. \tag{B.13}$$

The special case $D(\mathrm{R})=N$ is referred to as fundamental or defining representation and will be denoted as N.

The index $T(\mathrm{R})$ of a representation is defined as

$$\mathrm{tr}(t^a_\mathrm{R} t^b_\mathrm{R}) = T(\mathrm{R})\delta^{ab}. \tag{B.14}$$

The Casimir $C(\mathrm{R})$ of a representation R is defined as

$$\delta_{ab} t^a_\mathrm{R} t^b_\mathrm{R} = C(\mathrm{R}) I_\mathrm{R}, \tag{B.15}$$

where I_R is the identity matrix in the representation R.

Summing over the algebra indices in Eq. (B.14) and using Eq. (B.15) gives

$$C(\mathrm{R})D(\mathrm{R}) = T(\mathrm{R})D(\mathrm{A}), \tag{B.16}$$

where A is the adjoint representation which is further discussed in Appendix B.4.

The anomaly coefficient $A(\mathrm{R})$ of a representation is defined as

$$A(\mathrm{R})d^{abc} = \frac{1}{2}\,\mathrm{tr}\left(t^a_\mathrm{R}\left\{t^b_\mathrm{R}, t^c_\mathrm{R}\right\}\right), \tag{B.17}$$

where we note that the constants d^{abc} are totally symmetric. Note that Eqs. (B.13) and (B.14) can be combined to give

$$T(\mathrm{R})f^{abc} = -\mathrm{i}\,\mathrm{tr}\left(t^a_\mathrm{R}[t^b_\mathrm{R}, t^c_\mathrm{R}]\right). \tag{B.18}$$

Combining Eqs. (B.17) and (B.18) gives

$$\begin{aligned}
\mathrm{tr}(t^a_\mathrm{R} t^b_\mathrm{R} X) &= \frac{1}{2}\,\mathrm{tr}(\{t^a_\mathrm{R}, t^b_\mathrm{R}\}X + [t^a_\mathrm{R}, t^b_\mathrm{R}]X) \\
&= \frac{1}{2}\,\mathrm{tr}(\{t^a_\mathrm{R}, t^b_\mathrm{R}\}t^c_\mathrm{R} + [t^a_\mathrm{R}, t^b_\mathrm{R}]t^c_\mathrm{R})X_c \\
&= \left(A(\mathrm{R})d^{abc} + \frac{\mathrm{i}}{2}T(\mathrm{R})f^{abc}\right)X_c.
\end{aligned} \tag{B.19}$$

B.2. Complex conjugate representation

By taking the complex conjugate of Eq. (B.13) one sees that the matrices $-(t^a_\mathrm{R})^*$ also satisfy the commutation relation Eq. (B.13) and therefore they also form a representation.

If $t^a_\mathrm{R} = -(t^a_\mathrm{R})^*$ or if there exists a unitary matrix U such that the transformed matrices $U^{-1}t^a_\mathrm{R}U$ satisfy $t^a_\mathrm{R} = -(t^a_\mathrm{R})^*$ the representation is called real.

If there exists a unitary matrix V such that $-(t^a_\mathrm{R})^* = V^{-1}t^a_\mathrm{R}V$ the representation is called pseudoreal.

Representations that are neither real nor pseudoreal specify the complex conjugate representation $\overline{\mathrm{R}}$

$$t^a_{\overline{\mathrm{R}}} = -(t^a_\mathrm{R})^* = -(t^a_\mathrm{R})^\mathrm{T}. \tag{B.20}$$

Using the invariance of the trace under cyclic permutations and transpositions we see that the anomaly coefficient defined in Eq. (B.17) satisfies

$$A(\mathrm{R}) = -A(\overline{\mathrm{R}}). \tag{B.21}$$

B.3. Fundamental representation

Since the generators of $SU(N)$ in the fundamental representation N form a basis for the hermitian and traceless matrices of dimension $D(\mathrm{N}) \times D(\mathrm{N}) = N \times N$ they satisfy the completeness relation

$$\delta_{ab}(t^a_{\mathrm{N}})_i{}^j(t^b_{\mathrm{N}})_k{}^l = T(\mathrm{N})\left(\delta^l_i\delta^j_k - \frac{1}{D(\mathrm{N})}\delta^j_i\delta^l_k\right). \tag{B.22}$$

From the completeness the following identity follows

$$\frac{1}{T(\mathrm{N})}\operatorname{tr}(Xt^a_{\mathrm{N}})\frac{1}{T(\mathrm{N})}\operatorname{tr}(Yt^b_{\mathrm{N}}) = \frac{1}{T(\mathrm{N})}\left(\operatorname{tr}(XY) - \frac{1}{D(\mathrm{N})}\operatorname{tr}(X)\operatorname{tr}(Y)\right), \tag{B.23}$$

where X and Y are, of course, also given in the fundamental representation N.

The fundamental representation has

$$D(\mathrm{N}) = N, \quad T(\mathrm{N}) = \frac{1}{2}, \quad C(\mathrm{N}) = \frac{N^2-1}{2N}. \tag{B.24}$$

B.4. Adjoint representation

Given a representation for which the generators satisfy Eq. (B.13) the matrices defined via

$$(T^a)^{bc} \equiv (t^a_{\mathrm{A}})^{bc} = -\mathrm{i}f^{abc} \tag{B.25}$$

define another representation called the adjoint representation. We note here that we exclusively denote the generators of the adjoint representation with T^a.

Since the structure constants are real, the generators of the adjoint representation are purely imaginary and the adjoint representation is real, i.e. $T^a = -(T^a)^*$. From Eq. (B.21) it follows that $A(\mathrm{A}) = 0$. Equation (B.19) becomes

$$\operatorname{tr}(T^aT^bX) = \frac{\mathrm{i}}{2}T(\mathrm{A})f^{abc}X_c. \tag{B.26}$$

Since the structure constants can be viewed as the components of the generators of the adjoint representation we can write for the components of a field in the adjoint representation

$$X^{ab} = (T^c)^{ab}X_c = -\mathrm{i}f^{cab}X_c = -\mathrm{i}f^{abc}X_c. \tag{B.27}$$

Equation (B.26) can then be written as

$$\operatorname{tr}(T^aT^bX) = -\frac{1}{2}T(\mathrm{A})X^{ab}. \tag{B.28}$$

From Eq. (B.16) it follows that for the adjoint representation

$$C(\mathrm{A}) = T(\mathrm{A}). \tag{B.29}$$

The adjoint representation has

$$D(\mathrm{A}) = N^2 - 1, \quad T(\mathrm{A}) = N, \quad C(\mathrm{A}) = N. \tag{B.30}$$

Appendix C.

Gauge theory

Symmetries have proven to be a guiding principle of modern physics. There are several kinds of symmetries. One kind of symmetry expresses the independence of choosing a particular reference frame to describe a physical system. Another kind of symmetry arises when one describes a physical system in a formalism with redundant and therefore unphysical degrees of freedom. It is then possible to find (infinitely) many different, but physically equivalent solutions of the system. These physically equivalent solutions are related by a transformation called a gauge symmetry. This will be the topic of this appendix.

C.1. Abelian case

Consider the following lagrangian

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi, \tag{C.1}$$

where $\phi = \phi(x)$ is a complex scalar field and $\phi^\dagger = \phi^\dagger(x)$ is the complex conjugate field.

Clearly, the lagrangian is invariant under a transformation

$$\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}\alpha}\phi(x), \tag{C.2}$$

where α is a real constant. This is called a *global* gauge symmetry.

One can view the factor $\mathrm{e}^{\mathrm{i}\alpha}$ as a unitary matrix acting on the field ϕ which takes values in the singlet representation of the group $U(1)$. This point of view will become more relevant in the following section.

The next step is to consider the case in which the gauge parameter is no longer a constant, but rather a field itself, i.e. $\alpha = \alpha(x)$. The transformation now reads

$$\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}\alpha(x)}\phi(x). \tag{C.3}$$

The lagrangian of Eq. (C.1) is not invariant under this transformation due to the derivatives acting on $\alpha(x)$. A solution to overcome this problem is to introduce a new vector field $A_\mu(x)$ via a process called minimal substitution

$$\partial_\mu \phi \to D_\mu \phi \equiv (\partial_\mu - \mathrm{i}gA_\mu)\phi. \tag{C.4}$$

We will refer to the new vector field $A_\mu(x)$ as gauge field. The new lagrangian now reads

$$\mathcal{L} = (D^\mu\phi)^\dagger D_\mu\phi - m^2\phi^\dagger\phi. \tag{C.5}$$

The transformation property of the field $A_\mu(x)$ under Eq. (C.3) can be worked out by requiring the invariance of Eq. (C.3). In particular, the requirement that

$$D'_\mu\phi' \stackrel{!}{=} \mathrm{e}^{\mathrm{i}\alpha} D_\mu\phi \tag{C.6}$$

would yield Eq. (C.5) gauge invariant. Inserting Eq. (C.4) into Eq. (C.6) gives

$$(\partial_\mu - \mathrm{i}gA'_\mu)(\mathrm{e}^{\mathrm{i}\alpha}\phi) = \mathrm{e}^{\mathrm{i}\alpha}(\partial_\mu\phi + \mathrm{i}\partial_\mu\alpha\phi - \mathrm{i}gA'_\mu) \stackrel{!}{=} \mathrm{e}^{\mathrm{i}\alpha}(\partial_\mu\phi - \mathrm{i}gA_\mu) \tag{C.7}$$

and therefore the following transformation property of $A_\mu(x)$ under Eq. (C.3)

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{g}\partial_\mu\alpha(x). \tag{C.8}$$

In the abelian case one defines a field-strength tensor according to

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{C.9}$$

The field-strength tensor is invariant under local gauge transformations

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \frac{1}{g}\partial_\nu\alpha) - \partial_\nu(A_\mu + \frac{1}{g}\partial_\mu\alpha) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}. \tag{C.10}$$

The dynamics of the gauge field can then be described by the Maxwell lagrangian

$$\mathcal{L}_\mathrm{M} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{C.11}$$

C.2. Non-abelian case

Consider the following lagrangian

$$\mathcal{L} = \partial^\mu\phi^\dagger\partial_\mu\phi - m^2\phi^\dagger\phi, \tag{C.12}$$

where now $\phi = \phi(x)$ is a complex scalar field taking values in some representation R of the special unitary group $SU(N)$ and $\phi^\dagger = \phi^\dagger(x)$ is the hermitian conjugate field. For our purposes it suffices to regard the field $\phi(x)$ as some $D(\mathrm{R})$-dimensional multiplet

$$\phi(x) \equiv \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_{D(\mathrm{R})}(x) \end{pmatrix}. \tag{C.13}$$

Equation (C.12) is now invariant under global gauge transformations

$$\phi(x) \to \phi'(x) = U\phi(x), \tag{C.14}$$

where U is a constant group element satisfying

$$U^\dagger U = I. \tag{C.15}$$

In index notation the transformation in Eq. (C.14) reads

$$\phi_i(x) \to \phi_i'(x) = U_i{}^j \phi_j(x). \tag{C.16}$$

As in the abelian case we introduce a new field $A_\mu(x)$ via the process of minimal substitution

$$\partial_\mu \phi \to D_\mu \phi \equiv (\partial_\mu - \mathrm{i}gA_\mu)\phi \tag{C.17}$$

with the crucial new feature that the gauge field takes values in the Lie algebra of the representation R

$$A^\mu(x) = A^\mu_a(x) t^a_{\mathrm{R}}. \tag{C.18}$$

The new lagrangian then reads

$$\mathcal{L} = (D^\mu \phi)^\dagger D_\mu \phi - m^2 \phi^\dagger \phi. \tag{C.19}$$

For Eq. (C.19) to be invariant under the local gauge transformation

$$\phi(x) \to \phi'(x) = U(x)\phi(x) \tag{C.20}$$

we now require the gauge field to transform in such a way that

$$D'_\mu \phi' \stackrel{!}{=} U D_\mu \phi. \tag{C.21}$$

Expanding gives

$$D'_\mu \phi' = (\partial_\mu - \mathrm{i}gA'_\mu)(U\phi) = \partial_\mu U\phi + U\partial_\mu\phi - \mathrm{i}gA'_\mu U \stackrel{!}{=} U D_\mu \phi = U\partial_\mu\phi - \mathrm{i}gUA_\mu\phi \tag{C.22}$$

and therefore the transformation property

$$A_\mu(x) \to A'_\mu(x) = U(x)\left(A_\mu(x) + \frac{\mathrm{i}}{g}\partial_\mu\right)U^\dagger(x), \tag{C.23}$$

where we used $\partial_\mu U U^\dagger = -U\partial_\mu U^\dagger$. In the non-abelian case the field-strength tensor is defined as

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \mathrm{i}g[A_\mu(x), A_\nu(x)]. \tag{C.24}$$

Inserting the transformation of the gauge field Eq. (C.23) gives

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \mathrm{i}gA'_\mu A'_\nu - (\mu \leftrightarrow \nu). \tag{C.25}$$

The first term in Eq. (C.25) is

$$\begin{aligned} \partial_\mu A'_\nu &= \partial_\mu[U(A_\nu - \frac{1}{\mathrm{i}g}\partial_\nu)U^\dagger] \\ &= \partial_\mu U A_\nu U^\dagger + U\partial_\mu A_\nu U^\dagger + UA_\nu\partial_\mu U^\dagger - \frac{1}{\mathrm{i}g}\left(\partial_\mu U\partial_\nu + U\partial_\mu\partial_\nu\right)U^\dagger. \end{aligned} \tag{C.26}$$

The second term in Eq. (C.25) is

$$\begin{aligned}
-\mathrm{i}gA'_\mu A'_\nu &= -\mathrm{i}gU(A_\mu - \frac{1}{\mathrm{i}g}\partial_\mu)U^\dagger U(A_\nu - \frac{1}{\mathrm{i}g}\partial_\nu)U^\dagger \\
&= -\mathrm{i}gUA_\mu A_\nu U^\dagger + (U\partial_\mu U^\dagger U A_\nu U^\dagger + UA_\mu \partial_\nu U^\dagger) - \frac{1}{\mathrm{i}g}U\partial_\mu U^\dagger U\partial_\nu U^\dagger \\
&= -\mathrm{i}gUA_\mu A_\nu U^\dagger + (-\partial_\mu U A_\nu U^\dagger + UA_\mu \partial_\nu U^\dagger) + \frac{1}{\mathrm{i}g}\partial_\mu U\partial_\nu U^\dagger, \qquad \text{(C.27)}
\end{aligned}$$

where we used $U\partial_\mu U^\dagger = -\partial_\mu U U^\dagger$. Combining Eqs. (C.26) and (C.27) gives

$$\partial_\mu A'_\nu - \mathrm{i}gA'_\mu A'_\nu = U(\partial_\mu A_\nu - \mathrm{i}gA_\mu A_\nu)U^\dagger + U(A_\nu\partial_\mu + A_\mu\partial_\nu)U^\dagger - \frac{1}{\mathrm{i}g}U\partial_\mu\partial_\nu U^\dagger. \qquad \text{(C.28)}$$

Therefore the transformation of Eq. (C.25) becomes

$$F'_{\mu\nu} = U(\partial_\mu A_\nu - \partial_\mu A_\nu - \mathrm{i}g[A_\mu, A_\nu])U^\dagger = UF_{\mu\nu}U^\dagger. \qquad \text{(C.29)}$$

The field-strength tensor is gauge covariant. Note that this contrasts the abelian case, where the field-strength tensor is invariant under gauge transformations — a result which can be obtained from Eq. (C.29) by noting that in the abelian case the gauge field and the transformation matrix commute.

The dynamics of the non-abelian gauge fields can be derived from the Yang–Mills lagrangian

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\frac{1}{T(\mathrm{R})}\,\mathrm{tr}[F^{\mu\nu}F_{\mu\nu}] = -\frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu}. \qquad \text{(C.30)}$$

C.3. Wilson lines

Given a path

$$P = \{z(t) \mid z(0) = x, z(1) = y, 0 \le t \le 1\} \qquad \text{(C.31)}$$

we define a Wilson line

$$\begin{aligned}
U_P(y,x) &= \mathrm{P}\exp\left\{\mathrm{i}g\int_0^1 \mathrm{d}t\,\frac{\mathrm{d}z^\mu}{\mathrm{d}t}A_\mu(z)\right\} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{P}\int_0^1 \mathrm{d}t_1\cdots\int_0^1 \mathrm{d}t_n\,\frac{\mathrm{d}z^{\mu_1}}{\mathrm{d}t_1}A_{\mu_1}(z_1)\ldots\frac{\mathrm{d}z^{\mu_n}}{\mathrm{d}t_n}A_{\mu_n}(z_n) \\
&= \sum_{n=0}^{\infty}\int_{1\ge t_1>\cdots>t_n\ge 0}\mathrm{d}t_1\cdots\mathrm{d}t_n\,\frac{\mathrm{d}z^{\mu_1}}{\mathrm{d}t_1}A_{\mu_1}(z_1)\ldots\frac{\mathrm{d}z^{\mu_n}}{\mathrm{d}t_n}A_{\mu_n}(z_n), \qquad \text{(C.32)}
\end{aligned}$$

where P is the path-ordering symbol denoting that the fields are to be ordered in such a way, that the path parameter increases from right to left. The gauge field $A(x) = A_a(x)t^a_{\mathrm{R}}$ can take values in any representation R of the gauge group.

If there is no path, $P = \{z(t)|z(t) = x, 0 \le t \le 1\}$, the Wilson line is trivial

$$U_P(x,x) = I. \qquad \text{(C.33)}$$

Taking the hermitian conjugate corresponds to changing the direction of the path, i.e.

$$U_P^\dagger(y,x) = U_P(x,y). \tag{C.34}$$

Given a point $p \in P$ such that p separates P into P_1 and P_2 (with obvious endpoint identifications) we have

$$U_P(y,x) = U_{P_2}(y,p)U_{P_1}(p,x). \tag{C.35}$$

The functional derivative of the Wilson line with respect to the gauge field gives [22]

$$\frac{\delta}{\delta A_a^\mu(p)} U_P(y,x) = \mathrm{i}g \int_0^1 \mathrm{d}t\, U_P(y,z(t))\dot{z}_\mu(t) t_{\mathrm{R}}^a U_P(z(t),x)\delta^{(4)}(z(t)-p). \tag{C.36}$$

It is obvious that the functional derivative is non-vanishing only in the case that the spacetime point p at which the derivative is taken lies on the curve P. A special case of this identity is used in Section 2.2

To obtain the transformation properties of Wilson lines, note that any Wilson line can be decomposed into infinitesimal Wilson lines using Eq. (C.35). An infinitesimal Wilson line can be written as

$$U(x+\epsilon,x) = I + \mathrm{i}g\epsilon^\mu A_\mu(x) + O(\epsilon^2). \tag{C.37}$$

Performing a gauge transformation

$$A_\mu(x) \to A'_\mu(x) = \Omega(x)\left(A_\mu(x) + \frac{\mathrm{i}}{g}\partial_\mu\right)\Omega^\dagger(x) \tag{C.38}$$

gives

$$\begin{aligned} U(x+\epsilon,x) \to U'(x+\epsilon,x) & \\ &= I + \mathrm{i}g\epsilon^\mu \Omega(x)A_\mu(x)\Omega^\dagger(x) - \Omega(x)\epsilon^\mu\partial_\mu\Omega^\dagger(x) + O(\epsilon^2) \\ &= I + \mathrm{i}g\epsilon^\mu \Omega(x)A_\mu(x)\Omega^\dagger(x) + \epsilon^\mu\partial_\mu\Omega(x)\Omega^\dagger(x) + O(\epsilon^2) \\ &= \Omega(x+\epsilon)\left(I + \mathrm{i}g\epsilon^\mu A_\mu(x)\right)\Omega^\dagger(x) + O(\epsilon^2) \\ &= \Omega(x+\epsilon)U(x+\epsilon,x)\Omega^\dagger(x) + O(\epsilon^2) \end{aligned} \tag{C.39}$$

which leads to the transformation property

$$U_P(y,x) \to U'_P(y,x) = \Omega(y)U_P(y,x)\Omega^\dagger(x). \tag{C.40}$$

C.3.1. Wilson lines in different representations

Let $A = A^a t_a^{\mathrm{R}} \equiv A^a t_a$ and $\mathcal{A} = A^a t_a^{\mathrm{A}} \equiv A^a T_a$ denote the gauge field in an arbitrary and the adjoint representation, respectively. Furthermore let I and $\mathcal{I}$ denote the identities in the fundamental and adjoint representation, respectively.

Firstly, we will prove that

$$[t^a, (A)^n] = [(\mathcal{A} + A\mathcal{I})^n - (A)^n\mathcal{I}]^a{}_b\, t^b. \tag{C.41}$$

For $n = 1$ we have

$$[t^a, A] = [t^a, A_c t^c] = A_c \mathrm{i} f^{acb} t_b = (\mathcal{A})^a{}_b t^b. \tag{C.42}$$

Assuming that the Eq. (C.41) is true for n we have

$$\begin{aligned}
[t^a, (A)^{n+1}] &= (A)^n [t^a, A] + [t^a, (A)^n] A \\
&= (A)^n (\mathcal{A})^a{}_b t^b + [(\mathcal{A} + A\mathcal{I})^n - (A)^n \mathcal{I}]^a{}_b \, t^b A \\
&= (A)^n (\mathcal{A})^a{}_b t^b + [(\mathcal{A} + A\mathcal{I})^n - (A)^n \mathcal{I}]^a{}_b \, [\mathcal{A} + A\mathcal{I}]^b{}_c \, t^c \\
&= \left[(\mathcal{A} + A\mathcal{I})^{n+1} - (A)^{n+1} \mathcal{I}\right]^a{}_b \, t^b.
\end{aligned} \tag{C.43}$$

Using induction we therefore have proved Eq. (C.41).

Now consider a function $f(A) = \sum_n f_n A^n$. From Eq. (C.41) we have

$$t^a f(A) = [t^a, f(A)] + f(A) t^a = f(\mathcal{A} + A\mathcal{I})^a{}_b t^b. \tag{C.44}$$

The special case $f(A) = \exp \mathrm{i} A$ gives

$$t^a \mathrm{e}^{\mathrm{i}A} = [\mathrm{e}^{\mathrm{i}(\mathcal{A} + A\mathcal{I})}]^a{}_b t^b = \mathrm{e}^{\mathrm{i}A} [\mathrm{e}^{\mathrm{i}\mathcal{A}}]^a{}_b t^b. \tag{C.45}$$

Next we will need to generalize our above results a bit. Firstly, we will prove that

$$[t^a, A_1 \cdots A_n] = [(\mathcal{A}_1 + A_1 \mathcal{I}) \cdots (\mathcal{A}_n + A_n \mathcal{I}) - A_1 \cdots A_n \mathcal{I}]^a{}_b t^b. \tag{C.46}$$

The case $n = 1$ gives nothing new. Consider now that the expression is true for n and consider

$$\begin{aligned}
&[t^a, A_1 \cdots A_{n+1}] \\
&= A_1 \cdots A_n [t^a, A_{n+1}] + [t^a, A_1 \cdots A_n] A_{n+1} \\
&= A_1 \cdots A_n (\mathcal{A}_{n+1})^a{}_b t^b + [(\mathcal{A}_1 + A_1 \mathcal{I}) \cdots (\mathcal{A}_n + A_n \mathcal{I}) - A_1 \cdots A_n \mathcal{I}]^a{}_b t^b A_{n+1} \\
&= A_1 \cdots A_n (\mathcal{A}_{n+1})^a{}_b t^b \\
&\quad + [(\mathcal{A}_1 + A_1 \mathcal{I}) \cdots (\mathcal{A}_n + A_n \mathcal{I}) - A_1 \cdots A_n \mathcal{I}]^a{}_b [A_{n+1} \mathcal{I} + \mathcal{A}_{n+1}]^b{}_c t^c \\
&= [(\mathcal{A}_1 + A_1 \mathcal{I}) \cdots (\mathcal{A}_{n+1} + A_{n+1} \mathcal{I}) - A_1 \cdots A_{n+1} \mathcal{I}]^a{}_b t^b.
\end{aligned} \tag{C.47}$$

Using induction we therefore have proven Eq. (C.46).

Let $V = \mathrm{P} \exp \mathrm{i} g \int \mathrm{d}z \cdot A$ and $U = \mathrm{P} \exp \mathrm{i} g \int \mathrm{d}z \cdot \mathcal{A}$ denote Wilson lines along the same path in an arbitary and the adjoint representation, respectively. We formally write

$$V = \mathrm{P} \mathrm{e}^{\mathrm{i}A} = \sum_n V_n A_1 \cdots A_n. \tag{C.48}$$

Using Eq. (C.46) we have

$$\begin{aligned}
t^a V &= [t^a, V] + V t^a \\
&= \sum_n V_n [t^a, A_1 \cdots A_n] + V t^a \\
&= \sum_n V_n [(\mathcal{A}_1 + A_1 \mathcal{I}) \cdots (\mathcal{A}_n + A_n \mathcal{I}) - A_1 \cdots A_n \mathcal{I}]^a{}_b t^b + V t^a \\
&= \mathrm{P} [\mathrm{e}^{\mathrm{i}(\mathcal{A} + A\mathcal{I})}]^a{}_b t^b \\
&= V U^a{}_b t^b,
\end{aligned} \tag{C.49}$$

where in the last step we used that $[\mathcal{A}, \mathcal{I}] = 0$ and the fact that the "splitting" of the fundamental and adjoint representation fields can be performed without spoiling the ordering among the fields in the same representation. We finally arrive at the important identity [23]

$$V^\dagger t^a V = U^a{}_b t^b. \tag{C.50}$$

A special case is that the arbitrary representation is chosen to be the adjoint representation, R = A, which gives

$$U^\dagger T^a U = U^a{}_b T^b. \tag{C.51}$$

Given a field $X = X_a t^a$ in a representation R consider conjugate field

$$X' = V^\dagger X V. \tag{C.52}$$

Inserting the components and using Eq. (C.50) we have

$$X' = X'_a t^a = V^\dagger X_b T^b V = X_b U^b{}_a t^a. \tag{C.53}$$

and therefore

$$X'_a = X_b U^b{}_a. \tag{C.54}$$

Using the hermiticity and the fact that the generators of the adjoint representation are purely imaginary we have

$$U_{ab} = U^\dagger_{ba}. \tag{C.55}$$

Equation (C.54) can therefore also be written as

$$X'_a = U^\dagger{}_a{}^b X_b. \tag{C.56}$$

Appendix D.

Fourier transformation

We adopt the convention of differentiating between a function and its Fourier transform merely by the argument names. We will use the letters x, y, z, w to denote spatial coordinates and the momenta will be denoted by the letters p, q, r, s.

Auxiliary functions, such as the Heaviside theta and the sign function, which do not have a natural space (spacetime or energy-momentum) attached to them will be distinguished by different symbols.

D.1. One-argument Fourier transformation

Given a function $f(t)$ the relations between it and its Fourier transform $f(E)$ are given by

$$f(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} \mathrm{e}^{-\mathrm{i}Et} f(E), \tag{D.1a}$$

$$f(E) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}Et} f(t). \tag{D.1b}$$

For functions $f(\boldsymbol{x})$ the relations between it and its Fourier transform $f(\boldsymbol{p})$ are given by

$$f(\boldsymbol{x}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} f(\boldsymbol{p}), \tag{D.2a}$$

$$f(\boldsymbol{p}) = \int_{-\infty}^{\infty} \mathrm{d}^3\boldsymbol{x}\, \mathrm{e}^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{p}} f(\boldsymbol{x}). \tag{D.2b}$$

For functions $f(x)$ the relations between it and its Fourier transform $f(p)$ are given by

$$f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}px} f(p), \tag{D.3a}$$

$$f(E) = \int_{-\infty}^{\infty} \mathrm{d}^4 x\, \mathrm{e}^{\mathrm{i}px} f(x), \tag{D.3b}$$

where $x \equiv (t, \boldsymbol{x})$, $p \equiv (E, \boldsymbol{p})$ and $px \equiv Et - \boldsymbol{p}\cdot\boldsymbol{x}$.

In lightcone coordinates we have $px \equiv p^+x^- + p^-x^+ - \boldsymbol{p}_\perp \cdot \boldsymbol{x}_\perp$. Comparison with Eq. (D.3) immediately gives the following natural relations

$$f(x^+) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p^-}{2\pi} \mathrm{e}^{-\mathrm{i}p^-x^+} f(p^-), \tag{D.4a}$$

$$f(p^-) = \int_{-\infty}^{\infty} \mathrm{d}x^+ \, \mathrm{e}^{\mathrm{i}p^-x^+} f(x^+), \tag{D.4b}$$

$$f(x^-) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p^+}{2\pi} \mathrm{e}^{-\mathrm{i}p^+x^-} f(p^-), \tag{D.5a}$$

$$f(p^+) = \int_{-\infty}^{\infty} \mathrm{d}x^- \, \mathrm{e}^{-\mathrm{i}p^+x^-} f(x^-), \tag{D.5b}$$

$$f(\boldsymbol{x}_\perp) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^2\boldsymbol{p}_\perp}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp} f(\boldsymbol{p}_\perp), \tag{D.6a}$$

$$f(\boldsymbol{p}_\perp) = \int_{-\infty}^{\infty} \mathrm{d}^2\boldsymbol{x}_\perp \, \mathrm{e}^{-\mathrm{i}\boldsymbol{x}_\perp\cdot\boldsymbol{p}_\perp} f(\boldsymbol{x}_\perp). \tag{D.6b}$$

D.1.1. Heaviside theta

In this section we will discuss the Fourier transforms of the Heaviside theta function.

Spacetime

For spacetime arguments we define

$$\theta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}. \tag{D.7}$$

For time- and lightlike coordinates (t, x^+, x^-) the Fourier transform of the Heaviside theta is given by

$$\theta(E) = \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}Et}\theta(t) = \int_{0}^{\infty} \mathrm{d}t \, \mathrm{e}^{\mathrm{i}Et} = \left.\frac{\mathrm{e}^{\mathrm{i}Et-\epsilon t}}{\mathrm{i}(E+\mathrm{i}\epsilon)}\right|_{t=0}^{\infty} = \frac{\mathrm{i}}{E+\mathrm{i}\epsilon}. \tag{D.8}$$

For spacelike coordinates ($\boldsymbol{x}$, $\boldsymbol{x}_\perp$) the Fourier transform of the Heaviside theta is

$$\theta(p) = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}px}\theta(x) = \int_{0}^{\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}px} = \left.\frac{\mathrm{e}^{-\mathrm{i}px-\epsilon x}}{-\mathrm{i}(p-\mathrm{i}\epsilon)}\right|_{x=0}^{\infty} = \frac{-\mathrm{i}}{p-\mathrm{i}\epsilon}. \tag{D.9}$$

Momentum space

Since we distinguish a function and its Fourier transform merely by its argument we need to define the momentum space Heaviside theta function

$$\vartheta(p) = \begin{cases} 0 & p < 0 \\ \frac{1}{2} & p = 0 \\ 1 & p > 0 \end{cases}. \tag{D.10}$$

For energies and longitudinal momenta (E, p^+, p^-) the Fourier transform of the Heaviside theta is given by

$$\vartheta(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}E}{2\pi} \mathrm{e}^{-\mathrm{i}Et} \vartheta(E) = \int_0^{\infty} \frac{\mathrm{d}E}{2\pi} \mathrm{e}^{-\mathrm{i}Et} = \frac{1}{2\pi} \left. \frac{\mathrm{e}^{-\mathrm{i}Et-\epsilon t}}{-\mathrm{i}(t-\mathrm{i}\epsilon)} \right|_{E=0}^{\infty} = \frac{1}{2\pi} \frac{-\mathrm{i}}{t-\mathrm{i}\epsilon}. \tag{D.11}$$

For spacelike momenta ($\boldsymbol{p}$, $\boldsymbol{p}_\perp$) the Fourier transform is

$$\vartheta(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}px} \vartheta(p) = \int_0^{\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}px} = \frac{1}{2\pi} \left. \frac{\mathrm{e}^{\mathrm{i}px-\epsilon x}}{\mathrm{i}(x+\mathrm{i}\epsilon)} \right|_{x=0}^{\infty} = \frac{1}{2\pi} \frac{\mathrm{i}}{x+\mathrm{i}\epsilon}. \tag{D.12}$$

D.1.2. Sign function

In this section we will discuss the Fourier transforms of the sign functions. It turns out that they are related to the principal value defined as

$$\mathrm{PV}\,\frac{1}{x} = \frac{1}{2}\left(\frac{1}{x+\mathrm{i}\epsilon} + \frac{1}{x-\mathrm{i}\epsilon}\right). \tag{D.13}$$

Spacetime

For spacetime arguments we have defined in Eq. (2.40)

$$\epsilon(x) \equiv \operatorname{sgn} x \equiv \theta(x) - \theta(-x), \tag{D.14}$$

where x is any of the spacetime coordinates.

For time- and lightlike coordinates (t, x^+, x^-) the Fourier transform of the sign function is given by

$$\epsilon(E) = \theta(E) - \theta(-E) = \frac{\mathrm{i}}{E+\mathrm{i}\epsilon} - \frac{\mathrm{i}}{-E+\mathrm{i}\epsilon} = 2\mathrm{i}\,\mathrm{PV}\,\frac{1}{E}. \tag{D.15}$$

For spacelike coordinates ($\boldsymbol{x}$, $\boldsymbol{x}_\perp$) the Fourier transform of the sign function is

$$\epsilon(p) = \theta(p) - \theta(-p) = \frac{-\mathrm{i}}{p-\mathrm{i}\epsilon} - \frac{-\mathrm{i}}{-p-\mathrm{i}\epsilon} = -2\mathrm{i}\,\mathrm{PV}\,\frac{1}{p}. \tag{D.16}$$

Momentum space

The momentum space sign function is defined as

$$\varepsilon(p) \equiv \operatorname{sgn} p \equiv \vartheta(p) - \vartheta(-p), \tag{D.17}$$

where p may be any of the momentum coordinates.

For energies and longitudinal momenta (E, p^+, p^-) the Fourier transform of the Heaviside theta is given by

$$\varepsilon(t) = \vartheta(t) - \vartheta(-t) = \frac{1}{2\pi}\left(\frac{-\mathrm{i}}{t - \mathrm{i}\epsilon} - \frac{-\mathrm{i}}{-t - \mathrm{i}\epsilon}\right) = \frac{-2\mathrm{i}}{2\pi}\,\mathrm{PV}\,\frac{1}{t} \tag{D.18}$$

For spacelike momenta ($\boldsymbol{p}$, $\boldsymbol{p}_\perp$) the Fourier transform is

$$\varepsilon(x) = \vartheta(x) - \vartheta(-x) = \frac{1}{2\pi}\left(\frac{\mathrm{i}}{x + \mathrm{i}\epsilon} - \frac{\mathrm{i}}{-x + \mathrm{i}\epsilon}\right) = \frac{2\mathrm{i}}{2\pi}\,\mathrm{PV}\,\frac{1}{t} \tag{D.19}$$

D.1.3. Sign function as matrix element

With the above relations it is natural to interpret the sign function as a matrix element of an operator. Consider for example

$$\epsilon(x-y) = \int_p \mathrm{e}^{\mathrm{i}p(x-y)}\epsilon(p) = \int_p \langle x|p\rangle\langle p|y\rangle = \int_p \langle x|\,\mathrm{PV}\,\frac{-2\mathrm{i}}{P}|p\rangle\langle p|y\rangle = \langle x|\,\mathrm{PV}\,\frac{-2\mathrm{i}}{P}|y\rangle. \tag{D.20}$$

We therefore may identify the following operator

$$\epsilon \equiv \mathrm{PV}\,\frac{-2\mathrm{i}}{P} \tag{D.21}$$

and can therefore write

$$\epsilon(x-y) \equiv \langle x|\,\epsilon\,|y\rangle. \tag{D.22}$$

It is sometimes convenient to extend the matrix element notation to more than one component of the coordinates. For instance, we might want to express the sign function for lightcone time arguments as follows

$$\langle x|\,\epsilon\,|y\rangle \equiv \langle x^+|\,\epsilon\,|y^+\rangle\langle \vec{x}|\vec{y}\rangle = \epsilon(x^+ - y^+)I(\vec{x},\vec{y}), \tag{D.23}$$

where $I(\vec{x},\vec{y})$ is defined as in Eqs. (A.17).

D.2. Two-argument Fourier transformation

In this appendix we will discuss a generalized notion of the Fourier transformation for two-argument functions.

D.2.1. One-dimensional case

Conventions

A multiplication of two two-argument functions F and G is defined as

$$(FG)(x,y) = \int_z F(x,z)G(z,y) \tag{D.24}$$

and similarly

$$(FG)(p,q) = \int_r F(p,r)G(r,q). \tag{D.25}$$

The identity functions for position and momentum space are, respectively

$$I(x,y) = \delta(x-y) = \int_p \mathrm{e}^{\mathrm{i}p(x-y)}, \tag{D.26a}$$

$$I(p,q) = (2\pi)\delta(p-q) = \int_x \mathrm{e}^{-\mathrm{i}(p-q)x}. \tag{D.26b}$$

In order to simplify notation and facilitate calculations it may prove helpful to "promote" a one-argument function $f(x)$ two a two-argument function by defining

$$F(x,y) = f(x)I(x,y). \tag{D.27}$$

Fourier transform

Given a two-argument function $F(x,y)$ we define its Fourier transform via[1]

$$F(p,q) = \int_{x,y} F(x,y)\mathrm{e}^{-\mathrm{i}px+\mathrm{i}qy} \tag{D.28}$$

which is equivalent to

$$F(x,y) = \int_{p,q} F(p,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy}. \tag{D.29}$$

The transformation of a product of functions can be obtained by considering

$$\begin{aligned}
(AB)(x,y) &= \int_z A(x,z)B(z,y) \\
&= \int_z \int_{p,r} A(p,r)\mathrm{e}^{\mathrm{i}px-\mathrm{i}rz} \int_{s,q} B(s,q)\mathrm{e}^{\mathrm{i}sz-\mathrm{i}qy} \\
&= \int_{p,q,r,s} A(p,r)B(s,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy} \int_z \mathrm{e}^{\mathrm{i}(r-s)z} \\
&= \int_{p,q,r,s} A(p,r)B(s,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy} I(r,s)
\end{aligned}$$

[1] The choice of signs in the exponent corresponds to the conventional interpretation of x as a spatial coordinate.

$$= \int_{p,q} \int_r A(p,r)B(r,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy}$$
$$= \int_{p,q} (AB)(p,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy}. \tag{D.30}$$

We conclude that the Fourier transform of a product of functions is the product of the Fourier transforms.

Useful transforms

Consider the function

$$A(x,y) = \partial_x F(x,y). \tag{D.31}$$

Its transform can be obtained by considering

$$\begin{aligned} A(x,y) &= \partial_x F(x,y) \\ &= \partial_x \int_{p,q} F(p,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy} \\ &= \int_{p,q} \mathrm{i}pF(p,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy} \\ &= \int_{p,q} \mathrm{i}pF(p,q)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy}. \end{aligned} \tag{D.32}$$

We conclude that

$$A(p,q) = \mathrm{i}pF(p,q). \tag{D.33}$$

Using the promotion of one-argument functions Eq. (D.27)

$$F(x,y) = f(x)I(x,y) \tag{D.34}$$

we have for its Fourier transform

$$F(p,q) = \int_{x,y} F(x,y)\mathrm{e}^{-\mathrm{i}px+\mathrm{i}qy} = \int_{x,y} f(x)I(x,y)\mathrm{e}^{-\mathrm{i}px+\mathrm{i}qy} = \int_x f(x)\mathrm{e}^{-\mathrm{i}(p-q)x} = f(p-q), \tag{D.35}$$

where the one-argument function $f(x)$ has the one-argument Fourier transform $f(p) = \int_x f(x)\mathrm{e}^{-\mathrm{i}px}$.

Next, consider the function

$$B(x,y) = f(x)G(x,y) = \int_z f(x)I(x,z)G(z,y) = \int_z F(x,z)G(z,y) = (FG)(x,y). \tag{D.36}$$

Its Fourier transform is, using Eq. (D.30),

$$B(p,q) = (FG)(p,q) = \int_r F(p,r)G(r,q) = \int_r f(p-r)G(r,q). \tag{D.37}$$

D.2.2. D-dimensional case

For the D-dimensional ($D = d+1$) case we define the two-argument Fourier transform as

$$F(p,q) = \int_{x,y} F(x,y) \mathrm{e}^{\mathrm{i}px - \mathrm{i}qy}, \tag{D.38}$$

where we call attention to the apparent change of signs in the exponent, which can be understood by noting that

$$px \equiv p^\mu x^\nu g_{\mu\nu} \tag{D.39}$$

and recalling Footnote 1. The identity functions for position and momentum space are, respectively

$$I(x,y) = \delta^{(D)}(x-y) = \int_p \mathrm{e}^{-\mathrm{i}p(x-y)}, \tag{D.40a}$$

$$I(p,q) = (2\pi)^D \delta^{(D)}(p-q) = \int_x \mathrm{e}^{\mathrm{i}(p-q)x}. \tag{D.40b}$$

With our choices of signs the transform of

$$A(x,y) = \partial^x_\mu I(x,y) \tag{D.41}$$

is given by

$$A(p,q) = -\mathrm{i}p_\mu I(p,q). \tag{D.42}$$

Appendix E.

Propagators

E.1. Axial gauge gluon propagator

Consider the lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{gf}}, \tag{E.1}$$

where we have for the free lagrangian

$$\mathcal{L}_0 = \frac{1}{2} A^\mu (g_{\mu\nu}\partial^2 - \partial_\mu \partial_\nu) A^\nu \tag{E.2}$$

and the gauge-fixing lagrangian is given by

$$\mathcal{L}_{\text{gf}} = \frac{1}{2}\xi^{-1} A^\mu n_\mu n_\nu A^\nu. \tag{E.3}$$

Note that the gauge condition $n_\mu A^\mu = 0$ can be obtained by taking the limit $\xi \to 0$. From the action $S[A] = \int \mathrm{d}^4x\, \mathcal{L}$ we get the inverse propagator

$$(G_0^{-1})_{\mu\nu}(x,y) = \frac{\delta^2 S[A]}{\delta A^\mu(x)\delta A^\nu(y)} = (g_{\mu\nu}\partial_x^2 - \partial_\mu^x \partial_\nu^x + \xi^{-1} n_\mu n_\nu)\delta^{(4)}(x-y). \tag{E.4}$$

In momentum space we therefore have

$$(G_0^{-1})_{\mu\nu}(p) = -g_{\mu\nu}p^2 + p_\mu p_\nu + \xi^{-1} n_\mu n_\nu. \tag{E.5}$$

For the propagator we make the following ansatz

$$G_0^{\mu\nu}(p) = -\frac{1}{p^2}\left[\alpha g^{\mu\nu} + \beta p^\mu p^\nu + \gamma(p^\mu n^\nu + n^\mu p^\nu) + \delta n^\mu n^\nu\right]. \tag{E.6}$$

Requiring that $(G_0^{-1})_{\mu\nu}(p) G_0^{\nu\rho}(p) = \delta_\mu^\rho$ gives

$$\alpha = 1, \tag{E.7a}$$

$$\beta = \frac{n^2 - \xi p^2}{(n\cdot p)^2}, \tag{E.7b}$$

$$\gamma = -\frac{1}{(n\cdot p)}, \tag{E.7c}$$

$$\delta = 0. \tag{E.7d}$$

Therefore the propagator for the fields A_μ is

$$G_0^{\mu\nu}(p) = -\frac{1}{p^2}\left(g^{\mu\nu} + \frac{n^2-\xi p^2}{(n\cdot p)^2}p^\mu p^\nu - \frac{1}{(n\cdot p)}(p^\mu n^\nu + n^\mu p^\nu)\right). \tag{E.8}$$

Taking the limit $\xi \to 0$ gives

$$G_0^{\mu\nu}(p) = -\frac{1}{p^2}\left(g^{\mu\nu} + \frac{n^2}{(n\cdot p)^2}p^\mu p^\nu - \frac{1}{(n\cdot p)}(p^\mu n^\nu + n^\mu p^\nu)\right). \tag{E.9}$$

For the special case that $n^2 = 0$, i.e. that the vector is lightlike, we get for the propagator

$$G_0^{\mu\nu}(p) = -\frac{1}{p^2}\left(g^{\mu\nu} - \frac{1}{(n\cdot p)}(p^\mu n^\nu + n^\mu p^\nu)\right). \tag{E.10}$$

E.1.1. Lightcone gauge propagator

The lightcone gauge $A^+ = 0$ is an axial gauge, where the unit normal vector is given by

$$n^\mu = \delta^\mu_-. \tag{E.11}$$

The propagator then has components

$$G_0^{\mu\nu}(p) = X^{\mu\nu}(p)G_0(p), \tag{E.12}$$

where G_0 is the scalar propagator

$$G_0(p) = \frac{1}{p^2 + \mathrm{i}\epsilon} \tag{E.13}$$

and the coefficients are

$$X^{--}(p) = \frac{2p^-}{p^+}, \qquad X^{-i}(p) = X^{i-}(p) = \frac{p^i}{p^+}, \qquad X^{ij}(p) = \delta^{ij}. \tag{E.14}$$

E.2. Perturbative gluon propagator

In this section we will derive the background field propagator for the color glass condensate via a perturbative expansion. This is useful when studying the weak field limit, since then the expansion can be truncated at the desired order.

The inverse propagator of the semi-fast gluons in the presence of a background field

$$G^{-1}_{\mu\nu}(x,y) = \left.\frac{\delta^2 S[A,\rho]}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{A=\mathcal{A}} \tag{E.15}$$

takes the form (with spacetime indices suppressed)

$$G^{-1} = G_0^{-1} + \delta(G^{-1}) = G_0^{-1}(I + G_0\delta(G^{-1})), \tag{E.16}$$

where G_0 denotes the propagator for gluons in the vacuum (no background field). This expression can easily be inverted

$$G = (I + G_0\delta(G^{-1}))^{-1}G_0 = \sum_{n=0}^{\infty}(-1)^n(G_0\delta(G^{-1}))^n G_0. \tag{E.17}$$

In the weak field approximation this series can be truncated in such a way that only terms up to some order in the field-strength and charge density contribute. Note that $\delta(G^{-1})$ itself is not linear in the perturbative quantities.

E.2.1. Yang–Mills term

From Eq. (2.21) we have for the second functional derivative of the Yang–Mills term evaluated on the classical field solution as a matrix in the adjoint representation[1]

$$\left.\frac{\delta^2 S_{\mathrm{YM}}[A]}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{A=\mathcal{A}} = \left(g_{\mu\nu}\mathcal{D}^2 - \mathcal{D}_\mu\mathcal{D}_\nu - 2\mathrm{i}g\mathcal{F}_{\mu\nu}\right)_x \delta^{(4)}(x-y), \tag{E.18}$$

where the subscript x means that the derivatives are to be taken with respect to x and all the fields have x as their argument.

The individual terms are

$$\mathcal{D}^2 = (\partial_\rho + (-\mathrm{i}g)\mathcal{A}_\rho)(\partial^\rho + (-\mathrm{i}g)\mathcal{A}^\rho) = \partial^2 + (-\mathrm{i}g)(\partial\cdot\mathcal{A} + 2\mathcal{A}\cdot\partial) + (-\mathrm{i}g)^2\mathcal{A}^2, \tag{E.19a}$$

$$\begin{aligned}\mathcal{D}_\mu\mathcal{D}_\nu &= (\partial_\mu + (-\mathrm{i}g)\mathcal{A}_\mu)(\partial_\nu + (-\mathrm{i}g)\mathcal{A}_\nu) \\ &= \partial_\mu\partial_\nu + (-\mathrm{i}g)(\partial_\mu\mathcal{A}_\nu + \mathcal{A}_\nu\partial_\mu + \mathcal{A}_\mu\partial_\nu) + (-\mathrm{i}g)^2\mathcal{A}_\mu\mathcal{A}_\nu,\end{aligned} \tag{E.19b}$$

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + (-\mathrm{i}g)[\mathcal{A}_\mu, \mathcal{A}_\nu]. \tag{E.19c}$$

Combining all the terms gives

$$\begin{aligned}&g_{\mu\nu}\mathcal{D}^2 - \mathcal{D}_\mu\mathcal{D}_\nu - 2\mathrm{i}g\mathcal{F}_{\mu\nu} \\ &= g_{\mu\nu}\left[\partial^2 + (-\mathrm{i}g)(\partial\cdot\mathcal{A} + 2\mathcal{A}\cdot\partial) + (-\mathrm{i}g)^2\mathcal{A}^2\right] \\ &\quad - \left[\partial_\mu\partial_\nu + (-\mathrm{i}g)(\partial_\mu\mathcal{A}_\nu + \mathcal{A}_\nu\partial_\mu + \mathcal{A}_\mu\partial_\nu) + (-\mathrm{i}g)^2\mathcal{A}_\mu\mathcal{A}_\nu\right] \\ &\quad + 2(-\mathrm{i}g)\left[\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + (-\mathrm{i}g)[\mathcal{A}_\mu, \mathcal{A}_\nu]\right] \\ &= g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu \\ &\quad + (-\mathrm{i}g)\left[g_{\mu\nu}(\partial\cdot\mathcal{A} + 2\mathcal{A}\cdot\partial) + \partial_\mu\mathcal{A}_\nu - 2\partial_\nu\mathcal{A}_\mu - \mathcal{A}_\nu\partial_\mu - \mathcal{A}_\mu\partial_\nu\right] \\ &\quad + (-\mathrm{i}g)^2\left[g_{\mu\nu}\mathcal{A}^2 + \mathcal{A}_\mu\mathcal{A}_\nu - 2\mathcal{A}_\nu\mathcal{A}_\mu\right].\end{aligned} \tag{E.20}$$

[1] To obtain the given expression use Eq. (B.27).

By noticing that the first part is the inverse free gluon propagator, i.e.

$$(G_0^{-1})_{\mu\nu}(x,y) = (g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)_x \delta^{(4)}(x-y) \tag{E.21}$$

we can write

$$\left.\frac{\delta^2 S_{\mathrm{YM}}[A]}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{A=\mathcal{A}} = (G_0^{-1})_{\mu\nu}(x,y) + \Delta_{\mu\nu}(x,y), \tag{E.22}$$

where we have defined

$$(\delta(G^{-1})_{\mathrm{YM}})_{\mu\nu}(x,y) \equiv \Delta_{\mu\nu}(x,y) \equiv \Delta^{(1)}_{\mu\nu}(x,y) + \Delta^{(2)}_{\mu\nu}(x,y), \tag{E.23a}$$

$$\Delta^{(1)}_{\mu\nu}(x,y) = (-\mathrm{i}g)\left[g_{\mu\nu}(\partial\cdot\mathcal{A} + 2\mathcal{A}\cdot\partial) + \partial_\mu\mathcal{A}_\nu - 2\partial_\nu\mathcal{A}_\mu - \mathcal{A}_\nu\partial_\mu - \mathcal{A}_\mu\partial_\nu\right]_x \delta^{(4)}(x-y), \tag{E.23b}$$

$$\Delta^{(2)}_{\mu\nu}(x,y) = (-\mathrm{i}g)^2\left[g_{\mu\nu}\mathcal{A}^2 + \mathcal{A}_\mu\mathcal{A}_\nu - 2\mathcal{A}_\nu\mathcal{A}_\mu\right]_x \delta^{(4)}(x-y), \tag{E.23c}$$

where $\Delta^{(1)}$ and $\Delta^{(2)}$ denote terms linear and quadratic in $\mathcal{A}$, respectively.

Using the Fourier transformation for two-argument functions detailed in Appendix D we have

$$\begin{aligned}
&\Delta^{(1)}_{\mu\nu}(p,q) \\
&= (-\mathrm{i}g)(-\mathrm{i})[g_{\mu\nu}((p-q)\cdot\mathcal{A}(p-q) + 2\mathcal{A}(p-q)\cdot q) + (p-q)_\mu\mathcal{A}_\nu(p-q) \\
&\qquad - 2(p-q)_\nu\mathcal{A}_\mu(p-q) - \mathcal{A}_\nu(p-q)q_\mu - \mathcal{A}_\mu(p-q)q_\nu] \\
&= (-\mathrm{i}g)(-\mathrm{i})\left[g_{\mu\nu}(p+q)\cdot\mathcal{A}(p-q) + (p-2q)_\mu\mathcal{A}_\nu(p-q) + (q-2p)_\nu\mathcal{A}_\mu(p-q)\right],
\end{aligned} \tag{E.24a}$$

$$\Delta^{(2)}_{\mu\nu}(p,q) = (-\mathrm{i}g)^2\left[g_{\mu\nu}\mathcal{A}^\rho * \mathcal{A}_\rho + \mathcal{A}_\mu * \mathcal{A}_\nu - 2\mathcal{A}_\nu * \mathcal{A}_\mu\right](p-q), \tag{E.24b}$$

where the convolution is defined as

$$(f*g)(p) = \int_q f(q)g(p-q). \tag{E.25}$$

E.2.2. Wilson line term

From Eq. (2.39) we have for the second functional derivative of the Wilson line term evaluated at the classical solution

$$\left.\frac{\delta^2 S_W[A^-,\rho]}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{A=\mathcal{A}} = \delta^-_\mu\delta^-_\nu\frac{\mathrm{i}g}{2}\epsilon(x^+-y^+)\rho(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}), \tag{E.26}$$

where $\epsilon(x) = \theta(x) - \theta(-x)$ as defined in Eq. (2.40). This term has no free part and we thus write

$$(\delta(G^{-1})_W)_{\mu\nu}(x,y) \equiv \Gamma_{\mu\nu}(x,y) = \left.\frac{\delta^2 S_W[A^-,\rho]}{\delta A^\mu(x)\delta A^\nu(y)}\right|_{A=\mathcal{A}}. \tag{E.27}$$

The Fourier transform is

$$\begin{aligned}
\Gamma_{\mu\nu}(p,q) &= \int_{x,y} \Gamma_{\mu\nu}(x,y)\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy} \\
&= \int_{x,y} \delta_\mu^- \delta_\nu^- \frac{\mathrm{i}g}{2}\epsilon(x^+-y^+)\rho(\vec{x})\delta^{(3)}(\vec{x}-\vec{y})\mathrm{e}^{\mathrm{i}px-\mathrm{i}qy} \\
&= \delta_\mu^- \delta_\nu^- \frac{\mathrm{i}g}{2}\int_{x^+,y^+} \epsilon(x^+-y^+)\mathrm{e}^{\mathrm{i}p^-x^+-\mathrm{i}q^-y^+}\int_{\vec{x},\vec{y}} \rho(\vec{x})\delta^{(3)}(\vec{x}-\vec{y})\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}-\mathrm{i}\vec{q}\cdot\vec{y}} \\
&= \delta_\mu^- \delta_\nu^- \frac{\mathrm{i}g}{2}\int_{x^+,y^+} \epsilon(x^+)\mathrm{e}^{\mathrm{i}p^-x^+-\mathrm{i}(q^--p^-)y^+}\int_{\vec{x}} \rho(\vec{x})\mathrm{e}^{\mathrm{i}(\vec{p}-\vec{q})\cdot\vec{x}} \\
&= \delta_\mu^- \delta_\nu^- \frac{\mathrm{i}g}{2}\int_{x^+} \epsilon(x^+)\mathrm{e}^{\mathrm{i}p^-x^+} I(p^-,q^-)\rho(\vec{p}-\vec{q}) \\
&= \delta_\mu^- \delta_\nu^- \frac{\mathrm{i}g}{2} 2\mathrm{i}\,\mathrm{PV}\,\frac{1}{p^-} I(p^-,q^-)\rho(\vec{p}-\vec{q}) \\
&= -g\delta_\mu^- \delta_\nu^-\,\mathrm{PV}\,\frac{1}{p^-} I(p^-,q^-)\rho(\vec{p}-\vec{q}),
\end{aligned} \tag{E.28}$$

where we denoted $\vec{p} \equiv (p^+, \boldsymbol{p}_\perp)$ and $\vec{p}\cdot\vec{x} \equiv p^+x^- - \boldsymbol{p}_\perp\cdot\boldsymbol{x}_\perp$ and used

$$\int_x \epsilon(x)\mathrm{e}^{\mathrm{i}px} = 2\mathrm{i}\,\mathrm{PV}\,\frac{1}{p}. \tag{E.29}$$

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