

# Spiele, Modaloperatoren und analytische Beweise in nichtklassischen Logiken

DISSERTATION

zur Erlangung des akademischen Grades

**Doktor der Technischen Wissenschaften**

eingereicht von

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# Games, Modalities and Analytic Proofs in Nonclassical Logics

DISSERTATION

submitted in partial fulfillment of the requirements for the degree of

**Doktor der Technischen Wissenschaften**

by

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Registration Number 1249371

to the Faculty of Informatics

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# Erklärung zur Verfassung der Arbeit

Timo Lang, MSc

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Wien, 19. April 2021

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Ich durfte mich glücklich schätzen diese Arbeit in einem Umfeld voll sympathischer und hilfsbereiter Kollegen anzufertigen, die stets dafür sorgten dass auch der soziale Aspekt der Wissenschaft nicht zu kurz kommt. Insbesondere (aber nicht ausschließlich!) seien hier meine jetzigen und ehemaligen Kollegen aus dem vierten Stock genannt: Alexandra, Anela, Björn, David, Francesca, Francesco, Kees, Matteo, Maya, Michael, Paolo, Revantha, Robert, Tim und Tiziano. Auch dem Team des Doktoratsprogramms LogiCS gebührt mein Dank für die Schaffung und Instandhaltung einer ungemein angenehmen Arbeitsumgebung. Bei Jan und Katrin bedanke ich mich für ihre Freundschaft, und bei Jan darüber hinaus für das Korrekturlesen von Teilen dieser Arbeit.

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# Acknowledgements

Let me start by thanking Chris Fermüller for his uncomplicated, passionate and inspiring supervision during the preparation of this thesis. His high standards with regard to the conception and presentation of mathematical ideas have been a great incentive for my work. I also want to thank Agata Ciabattoni for her collaboration and support.

I was happy to be surrounded by many wonderful and supportive colleagues at work who always made sure that the social aspect of science was not neglected. In particular, a big thanks goes out to my current and former colleagues from the 4th floor: Alexandra, Anela, Björn, David, Francesca, Francesco, Kees, Matteo, Maya, Michael, Paolo, Revantha, Robert, Tim and Tiziano. I am also indebted to the team of the doctoral program LogiCS for creating and sustaining such a pleasant working environment. Jan and Katrin must be thanked for their friendship, and Jan in particular for proofreading parts of this thesis.

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# Kurzfassung

Die vorliegende Dissertation behandelt drei Themen aus der Beweistheorie nichtklassischer Logiken.

Zuerst untersuchen wir Logiken die sich mittels analytischer Hypersequenzenkalküle darstellen lassen. Wir beschreiben eine Projektion schnittfreier Hypersequenzbeweise auf den zugrundeliegenden Sequenzenkalkül, aus der sich verschiedene Verschärfungen des Deduktionstheorems ableiten lassen.

Im zweiten Teil betrachten wir einen spieltheoretisch motivierten Sequenzenkalkül, in dem die Verwendung bestimmter Regeln mit Kosten verbunden ist. Die damit verbundene erweiterte Ausdrucksstärke lässt sich durch eine Beschriftung der Kalkülregeln charakterisieren. Über den so erhaltenen beschrifteten Sequenzenkalkül beweisen wir einige syntaktische Ergebnisse.

Der abschließende dritte Teil behandelt eine deontische Modallogik, die wir mittels syntaktischer Übersetzungen ihrer Beweise untersuchen. Insbesondere zeigen wir, dass sich ein wesentliches Fragment der Logik auf die ihr unterliegende klassische Modallogik zurückführen lässt.



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# Abstract

The present thesis deals with three different topics in the proof theory of nonclassical logics.

We first investigate logics which are presented as analytic hypersequent calculi. Using a projection of cutfree hypersequent proofs onto proofs in the sequent calculus, we obtain various strengthenings of the deduction theorem.

In the second part we develop a sequent calculus with a game-theoretic underpinning. By stipulating that the use of certain rules triggers costs, we gain expressivity which in turn can be captured by a suitable labelling of the proof rules. We show some syntactic results about the thus obtained labelled sequent calculus.

The concluding third part employs the method of provability-preserving syntactic translations to study a deontic modal logic which extends classical modal logic. Our main result is that a substantial fragment of the deontic logic can be reduced to the underlying classical modal logic.



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# Introduction

This thesis combines three different streams of research which have been set forth in the following publications:

1. Agata Ciabattoni, Timo Lang, and Revantha Ramanayake. Bounded sequent calculi for non-classical logics via hypersequents. In *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, pages 94–110. Springer, 2019
2. Timo Lang, Carlos Olarte, Elaine Pimentel, and Christian G Fermüller. A game model for proofs with costs. In *International Conference on Automated Reasoning with Analytic Tableaux and Related Methods*, pages 241–258. Springer, 2019
3. Timo Lang. A reduction in violation logic. In Fenrong Liu, Alessandra Marra, Paul Portner, and Frederik Van De Putte, editors, *Deontic Logic and Normative Systems: 15th International Conference (DEON2020/2021, Munich)*. College Publications, 2020

Although their content is quite different, they share an object of study—certain nonclassical propositional logics—and the method by which it is studied. This is the method of structural proof theory.

Let me say a few words about the origins of these works. When I started my PhD at TU Wien in 2016, one of the projects my supervisor Chris Fermüller and I had in our mind was the development of some sort of a systematic underpinning of substructural logics with game-theoretic ideas. After some initial unpublished work on this, I got the impression that a ‘perfect match’ between game-theoretic and semantic ideas and the existing calculi was not achievable: Either one would have to compromise on the perspicuousness of the semantics, or one would have to change the logics. The first option did not interest me, and I found out that the second one had already been worked out by others, most notably by Japaridze in his *computability logic*. Whether or not this initial assessment of mine was correct, the project retreated into the background. Yet it never completely disappeared, and when in 2018 Elaine Pimentel and Carlos Olarte spent a sabbatical at TU Wien, a lot of ideas and questions about games resurfaced in the course of our discussions on subexponential linear logic. Out of these sessions grew the idea of having a labelled calculus for subexponential linear logic, where the labels should

denote the costs of a proof as measured by the use of the dereliction rule. The task for Chris and me in this was to provide an adequate game-theoretic formalization of costs in proofs. Our findings were reported in the above TABLEAUX paper. Afterwards I started working on an extended journal version of the material, but had to suspend the writing at some point because the completion of this dissertation became more important. Hopefully the journal version will see the light of day somewhen in 2021.

From 2018 on I started to work in parallel with Agata Ciabattoni and Revantha Ramanayake on the proof theory of hypersequents. Both Agata's and Revantha's work is characterized by the aim of developing *uniform* methods in proof theory, that is, methods which address not single problems in isolation but rather large classes of logics or calculi in one swoop. This aim is set against the trend of the last decades to introduce more and more specialized logics and proof systems, thus creating a vast and cluttered landscape of proof-theoretic methods. One of the frameworks where such general methods have been obtained is the hypersequent calculus. My idea was to investigate which kind of cuts have to be introduced in order to translate a cutfree hypersequent proof back to the sequent calculus. This idea was initially fleshed out for Gödel logic, but it was clear from the start that there is a more general procedure behind it. We eventually managed to generalize our results to any extension of the hypersequent calculus by so-called structural analytic hypersequent rules. A journal version of this work (which is closer to the presentation of the material in this thesis) has been submitted by the time of writing.

In the winter term of 2019 Guido Governatori came to Vienna and gave an introductory course to deontic logic that I occasionally attended. I did not have much exposure to deontic logic before then, although I was aware that some of my colleagues were working in that area. The course had a strong emphasis on Governatori's own work, and one of the systems discussed involved a kind of 'violation logic' intended to model reasoning in contrary-to-duty scenarios. While being rather oblivious to the philosophical content of this logic and deontic logic in general (a situation which has changed since then), I thought I had a useful proof-theoretic remark to make. Out of this remark, which is about a translation of violation logic into a weaker underlying logic, grew a paper which was then submitted to the DEON conference in deontic logic.

**On the differences to published work.** Concerning the relation between the work as presented here on the one hand and published work on the other hand, the following can be said. Chapter 4 is essentially the same as [36]. The only notable change is that a semantic proof (Theorem 4.29) has been replaced by a syntactic proof. Chapter 2 is very close to a journal submission which is not yet published. Its main difference to the preceding [18] is a broader perspective in which the results are presented and interpreted. The technical content however remained the same. Finally, Chapter 3 is quite different from [37]: The game-theoretic semantics, which were still somewhat informal in [37], have been completely formalized, therefore also requiring a new completeness proof (Theorem 3.7). Furthermore, there is a completely new cut elimination theorem (Theorem 3.19).

**How this thesis is structured.** Chapter 1 of this thesis attempts to wrap up material which is common to all of the subsequent chapters.

The chapters

**Chapter 2** *Bounding Axiomatic Systems via Hypersequents*

**Chapter 3** *Games and SELL*

**Chapter 4** *A Reduction in Violation Logic*

constitute the main body of the thesis. Each chapter comes with its own introduction and is concluded by a section on open questions, which is written in an informal style. The chapters can be read independently. They are ordered (I believe) by decreasing difficulty of content.



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# CHAPTER 1

## Preliminaries

This chapter sums up some of the background material which is required in all of the main chapters of the thesis. It is therefore essentially an introduction to the proof theory of various propositional logics with an emphasis on the three types of calculi which we shall encounter repeatedly later on: Hilbert-style systems, the sequent calculus, and the hypersequent calculus.

There is no original research in this chapter.

### 1.1 Syntax

This thesis deals with propositional logics. The syntax of these logics can be determined by laying down a set  $\mathcal{L}$  of logical connectives, each of which comes with an associated non-negative integer arity. Connectives of arity 0 are termed logical constants. The set  $\mathcal{L}$  is called the *signature* of the logic.

Let us fix an infinite set  $\mathcal{V}$  of *variables*, whose elements will typically be denoted by lowercase latin letters ( $p, q, r, \dots$ ). The notion of an  $\mathcal{L}$ -*formula* is inductively defined by the following two clauses:

1. Every variable in  $\mathcal{V}$  and every constant in  $\mathcal{L}$  is an  $\mathcal{L}$ -formula;
2. If  $A_1, \dots, A_n$  are  $\mathcal{L}$ -formulas and  $\diamond \in \mathcal{L}$  is a connective of arity  $n > 0$ , then also  $(A_1 \diamond \dots \diamond A_n)$  is an  $\mathcal{L}$ -formula.

Henceforth we will often suppress the prefix ' $\mathcal{L}$ -' if the signature is clear from context.

For unary connectives we use infix notation  $\diamond A$ . Formulas will be denoted by uppercase Latin letters ( $A, B, C, \dots$ ). When writing down formulas, we often omit some of the parentheses ( $,$ ). The set of all subformulas of  $A$  is denoted  $\text{subf}(A)$ .

Assuming that  $\mathcal{L}$  contains an implication connective  $\rightarrow$ , the notion of positive and negative subformula is defined as follows: An occurrence of a subformula  $A$  in  $F$  is called *positive* if it occurs within the premise of an implicational subformula an even number (including zero) of times; otherwise, the occurrence is called *negative*.

## 1.2 Consequence and Proof

We take the basic notion of logic to be that of consequence. A *consequence relation* in the language  $\mathcal{L}$  is a relation

$$\Gamma \vdash A$$

between finite sets  $\Gamma$  of  $\mathcal{L}$ -formulas (called *assumptions*) and single  $\mathcal{L}$ -formulas  $A$  (called *conclusion*).<sup>1</sup> We read  $\Gamma \vdash A$  as ‘ $A$  follows from  $\Gamma$ ’. If  $\Gamma = \emptyset$ , we write  $\vdash A$  and say that  $A$  is a *theorem* of  $\vdash$ .

There are two principal ways of arriving at a consequence relation.

*Deductively:* By specifying a notion of *proof from assumptions*. The consequence relation is then: ‘There is a proof of  $A$  from the assumptions in  $\Gamma$ .’

*Semantically:* By specifying a class of algebraic structures, and a notion of *truth* of a formula in a structure. The consequence relation is then: ‘In every structure where all formulas from  $\Gamma$  are true, it is also the case that  $A$  is true.’

We will almost exclusively deal with the deductive approach to logic.

A proof system (or calculus) establishes the rules for writing proofs. In the following sections we present some proof systems that will be featured in the main chapters of the thesis. Calculi for propositional intuitionistic and classical logic serve as main examples throughout.

## 1.3 Axiomatic Systems

A traditional way of presenting a logic is by *axiomatic systems*, also called *Hilbert-style* systems. Such a system **hIL** for propositional intuitionistic logic is pictured in Figure 1.1. The signature employed here contains the binary connectives  $\rightarrow$ ,  $\wedge$  and  $\vee$  as well as the constant  $\perp$  (‘falsum’). Negation may be introduced as  $\neg A := A \rightarrow \perp$ .

Axiomatic systems consist of a long list of *axioms* and a short list of *rules*. Often the only rule is Modus Ponens:

$$\frac{A \rightarrow B \quad A}{B} \text{ (MP)}$$

<sup>1</sup>In the context of algebraic logic, it is common to require further structural properties of a consequence relation (substitution invariance, closure under cut, ...). But since we do not aim at general results about consequence relations the given plain definition is sufficient.

<i>Axioms:</i>	
$A \rightarrow (B \rightarrow A)$	(1)
$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$	(2)
$A \wedge B \rightarrow A$	(3)
$A \wedge B \rightarrow B$	(4)
$A \rightarrow (B \rightarrow A \wedge B)$	(5)
$A \rightarrow A \vee B$	(6)
$B \rightarrow A \vee B$	(7)
$(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$	(8)
$\perp \rightarrow A$	(9)
 <i>Rules:</i>	
$\frac{A \quad A \rightarrow B}{B} \text{ (MP)}$	

Figure 1.1: An axiomatic system for propositional intuitionistic logic.

The axioms and rules in a Hilbert-style system are to be read as schemas: Any substitution of  $A$ ,  $B$  and  $C$  in Figure 1.1 by formulas in the aforementioned signature constitutes an instance of the axiom or rule.

Formally a notion of *proof from assumptions* can then be defined as follows:<sup>2</sup>

#### Definition 1.1

Let  $\Gamma \cup \{A\}$  be a set of formulas. An **hIL**-proof of  $A$  from  $\Gamma$  is a binary tree of formulas rooted in  $A$  obeying the following conditions:

1. The leaves of the tree are formulas from  $\Gamma$  or instances of the axioms of **hIL**.
2. Every internal node, together with its two child<sup>3</sup> nodes, forms an instance of the rule (MP).

We write  $\Gamma \vdash_{\text{hIL}} A$  if there is an **hIL**-proof of  $A$  from  $\Gamma$ .

<sup>2</sup>It is also possible, and maybe more common, to define Hilbert-style proofs as *sequences* of formulas instead of trees. The tree representation has the advantage that assumptions, being leaves of the tree, are encoded explicitly into the proof structure. This will come in handy for the proof systems in Chapter 4.

<sup>3</sup>Trees grow from the root upwards, and child nodes are the nodes immediately above a node.

**Example 1.2**

The following is a **hIL**-proof of  $A \vee B \rightarrow B$  from the assumption  $\neg A$ :

$$\frac{\frac{\text{instance of axiom (8)} \quad (A \rightarrow B) \rightarrow ((B \rightarrow B) \rightarrow (A \vee B \rightarrow B)) \quad \begin{array}{c} \vdots \\ \delta \end{array} \quad A \rightarrow B}{(B \rightarrow B) \rightarrow (A \vee B \rightarrow B)} \quad \begin{array}{c} \vdots \\ \delta' \end{array} \quad B \rightarrow B}{A \vee B \rightarrow B}$$

where  $\delta$  is

$$\frac{\frac{\text{instance of axiom (2)} \quad (A \rightarrow (\perp \rightarrow B)) \rightarrow ((A \rightarrow \perp) \rightarrow (A \rightarrow B))}{(A \rightarrow \perp) \rightarrow (A \rightarrow B)} \quad \frac{\frac{\text{instance of axiom (1)} \quad (\perp \rightarrow B) \rightarrow (A \rightarrow (\perp \rightarrow B)) \quad \text{instance of axiom (9)} \quad (\perp \rightarrow B)}{A \rightarrow (\perp \rightarrow B)}}{\text{assumption} \quad A \rightarrow \perp}}{A \rightarrow B}$$

and  $\delta'$  is

$$\frac{\frac{\text{instance of axiom (1)} \quad B \rightarrow ((B \rightarrow B) \rightarrow B)}{B \rightarrow ((B \rightarrow B) \rightarrow B)} \quad \frac{\text{instance of axiom (2)} \quad (B \rightarrow ((B \rightarrow B) \rightarrow B)) \rightarrow ((B \rightarrow (B \rightarrow B)) \rightarrow (B \rightarrow B))}{(B \rightarrow (B \rightarrow B)) \rightarrow (B \rightarrow B)} \quad \text{instance of axiom (1)} \quad B \rightarrow (B \rightarrow B)}{B \rightarrow B}$$



An important property of **hIL** is the *deduction theorem*:

**Proposition 1.3** (deduction theorem for **hIL**)

$$\Gamma \cup \{A_1, \dots, A_n\} \vdash_{\mathbf{hIL}} B \iff \Gamma \vdash_{\mathbf{hIL}} A_1 \wedge \dots \wedge A_n \rightarrow B.$$

The deduction theorem states that the logical connective  $\rightarrow$  internally reflects the notion of consequence  $\vdash_{\mathbf{hIL}}$ , and moreover  $\wedge$  reflects the combination of assumptions. As a special case, we have

$$\{A_1, \dots, A_n\} \vdash_{\mathbf{hIL}} B \iff \vdash_{\mathbf{hIL}} A_1 \wedge \dots \wedge A_n \rightarrow B$$

which implies that the whole consequence relation  $\vdash_{\mathbf{hIL}}$  can be recovered from its set of theorems. The deduction theorem also shows the admissibility of *hypothetical reasoning* in **hIL**: In order to prove an implication  $A \rightarrow B$ , we may temporarily assume the truth of  $A$  and proceed by proving  $B$ .

A Hilbert-style system **hCL** for classical propositional logic is obtained by adding the *law of excluded middle*  $A \vee \neg A$  to the axioms of **hIL**.

## 1.4 Natural Deduction

*Natural Deduction* was introduced by Gerhard Gentzen [21] in an attempt to model more closely the reasoning task of mathematicians than axiomatic systems do. While Natural Deduction does not play a role in the main part of this thesis, it is fundamental for the understanding of the sequent calculus (see Section 1.6).

Unlike Hilbert-style systems, Natural Deduction prioritizes rules over axiom. In typical Natural Deduction Systems, every logical connective  $\diamond$  has an *introduction rule* ('I-rule'), describing how a formula with main connective  $\diamond$  can be derived, and an *elimination rule* ('E-rule'), describing how a formula with main connective  $\diamond$  can be used in a proof. For example, here are the introduction and elimination rules for  $\wedge$ :

$$\frac{A \quad B}{A \wedge B} (\wedge_I) \quad \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B} (\wedge_E)$$

The main novelty of Natural Deduction however is the inclusion of hypothetical reasoning, as expressed in the introduction rule for  $\rightarrow$ :

$$\frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B} (\rightarrow_I)$$

That is, in order to derive  $A \rightarrow B$  we can temporarily assume the truth of  $A$ . Then, once we have succeeded in deriving  $B$ , we can 'close' all occurrences of the assumption  $A$

$$\begin{array}{c}
 \frac{A \quad B}{A \wedge B} (\wedge_I) \quad \frac{A \wedge B}{A} \quad \frac{A \wedge B}{B} (\wedge_E) \\
 \\
 \frac{A}{A \vee B} \quad \frac{B}{A \vee B} (\vee_I) \quad \frac{A \vee B}{C} \quad \frac{\begin{array}{c} [A] \\ \vdots \\ C \end{array} \quad \begin{array}{c} [B] \\ \vdots \\ C \end{array}}{C} (\vee_E) \\
 \\
 \frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B} (\rightarrow_I) \quad \frac{A \rightarrow B \quad A}{B} (\rightarrow_E) \\
 \\
 \frac{}{\perp} (\perp_E)
 \end{array}$$

Figure 1.2: The Natural Deduction calculus **NI** for intuitionistic logic

above  $B$  (as noted by the square brackets  $[\cdot]$ ) and conclude  $A \rightarrow B$ . An assumption which is not closed in a proof is called an *open assumption*.

By including the rule  $(\rightarrow_I)$ , Natural Deduction makes explicit a reasoning step which is featured only implicitly (as the deduction theorem) in Hilbert-style systems. Figure 1.2 pictures a complete Natural Deduction system **NI** for intuitionistic propositional logic.

We again define proofs from assumptions.

#### Definition 1.4

Let  $\Gamma \cup \{A\}$  be a set of formulas. An **NI**-proof of  $A$  from  $\Gamma$  is a tree of formulas rooted in  $A$  obeying the following:

1. Each open assumption at a leaf of the tree is contained in  $\Gamma$ .
2. Every internal node, together with its child node(s), forms an instance of a rule of **NI**.

We write  $\Gamma \vdash_{\text{NI}} A$  if there is an **NI**-proof of  $A$  from  $\Gamma$ .

Due to the use of hypothetical reasoning, a Natural Deduction proof of a theorem will typically contain formulas which are not theorems themselves. This is in contrast to Hilbert-style systems, where proofs of theorems are composed of theorems only.

**Example 1.5**

The following is an **NI**-proof of  $A \vee B \rightarrow B$  from the assumption  $\neg A (= A \rightarrow \perp)$ .

$$\frac{\frac{\frac{[A] \quad A \rightarrow \perp}{\perp} (\rightarrow E)}{B} (\perp E)}{A \vee B \rightarrow B} (\rightarrow E) \quad \frac{[B]}{B} (\vee E)$$

The corresponding Hilbert-style proof was presented in Example 1.2.

**Proposition 1.6** (Gentzen 1935)

The consequence relations  $\vdash_{\text{HIL}}$  and  $\vdash_{\text{NI}}$  coincide.

A natural deduction system **NK** for classical logic is obtained by adding to **NI** the rule of double negation elimination:

$$\frac{\neg\neg A}{A}$$

## 1.5 Logics as Sets of Theorems

Before proceeding with our presentation of proof systems, we now take a small detour to establish some notions that will be useful later on.

In Section 1.2 we have introduced logics as consequence relations, but there is also a fruitful definition of a logic as a set of formulas. In order to state this definition, we first formalize the notion of a substitution, which is customarily done as follows. An  $\mathcal{L}$ -*substitution* is a mapping from variables to  $\mathcal{L}$ -formulas. Any  $\mathcal{L}$ -substitution  $\sigma$  can be lifted to a mapping between  $\mathcal{L}$ -formulas: For this one simultaneously replaces all variables in a formula by their image under  $\sigma$ . This lifted mapping is also denoted by  $\sigma$ , and any formula of the shape  $\sigma(A)$  is called a *substitution instance* of  $A$ . Similarly,  $\sigma$  can be lifted to sets of formulas, rule instances and so on.

We will sometimes write a formula as  $A(p)$  to denote that it contains the variable  $p$ , and then subsequently  $A(B)$  will denote the instance  $\sigma(A)$  where  $\sigma(p) = B$  and  $\sigma(q) = q$  for  $q \neq p$ . In plain English:  $A(B)$  arises from  $A(p)$  by replacing all occurrences of  $p$  in  $A$  by the formula  $B$ .

Now let  $\mathcal{L}$  be a signature containing the implication connective  $\rightarrow$ . We say that a set  $L$  of  $\mathcal{L}$ -formulas is *closed under Modus Ponens* if whenever  $A \in L$  and  $A \rightarrow B \in L$ , we also have  $B \in L$ . The set  $L$  is *closed under substitution* if  $A \in L$  implies  $\sigma(A) \in L$  for any  $\mathcal{L}$ -substitution  $\sigma$ .

A set of  $\mathcal{L}$ -formulas is then called a *logic* if it is closed under Modus Ponens and substitution. Furthermore it is called *consistent* if it is not the set of all  $\mathcal{L}$ -formulas.

Introducing  $\text{Thm}(\mathcal{C})$  as a notation for the set of theorems of a consequence relation  $\mathcal{C}$ , we can now observe that  $\text{Thm}(\mathbf{hIL})$ ,  $\text{Thm}(\mathbf{NK})$  and so on are (consistent) logics in the above sense. Closure under Modus Ponens is immediate, as Modus Ponens is explicitly included as a rule of the calculi. Closure under substitution follows from the fact that axioms and rules are presented in a schematic way: That is, substitution instances of axioms and rules are again instances of the same axiom or rule, and this property carries over to theorems.

We define

$$\begin{array}{ll} \mathbf{IL} := \text{Thm}(\mathbf{hIL}) & \text{intuitionistic logic} \\ \mathbf{CL} := \text{Thm}(\mathbf{hCL}) & \text{classical logic} \end{array}$$

Every logic  $\mathbf{IL} \subsetneq \mathbf{L} \subsetneq \mathbf{CL}$  is called an *intermediate logic*.

Let us say that  $\mathcal{C}$  is a calculus for the logic  $\mathbf{L}$  if  $\text{Thm}(\mathcal{C}) = \mathbf{L}$ . Hence  $\mathbf{hIL}$  and  $\mathbf{LI}$  are calculi for intuitionistic logic, and  $\mathbf{hCL}$  and  $\mathbf{LK}$  are calculi for classical logic.

Given a logic  $\mathbf{L}$  and an additional set  $\mathcal{A}$  of  $\mathcal{L}$ -formulas, the *axiomatic extension*  $\mathbf{L} + \mathcal{A}$  is the smallest logic containing  $\mathbf{L} \cup \{\mathcal{A}\}$ . Similarly,  $\mathcal{C} + \mathcal{X}$  denotes the extension of the calculus  $\mathcal{C}$  by the axioms or rules in  $\mathcal{X}$ . With the ‘+’-operator overloaded in that way, we can state equations such as

$$\text{Thm}(\mathbf{hIL} + \mathcal{A} \vee \neg\mathcal{A}) = \mathbf{CL} = \mathbf{IL} + \mathcal{p} \vee \neg\mathcal{p}.$$

From now on, we will use the term ‘logic’ interchangeably for sets of theorems and for consequence relations. This slight abuse of language will always be backed up by some form of a deduction theorem, implying that the consequence relation can be recovered from its set of theorems.

## 1.6 The Sequent Calculus

Informally speaking, an *analytic proof* of a theorem is a proof which contains only concepts already expressed in the theorem itself.<sup>4</sup> This chimes with the idea of an *analytic proposition* in philosophy: A proposition which already contains the means of establishing its own truth, and is therefore independent of external data [48].

The prosaic formulation of analyticity in proof theory is by means of the *subformula property*: A proof of a theorem is analytic if all formulas occurring in it are subformulas

<sup>4</sup>This use of the term ‘analytic’ presumably goes back to Bernard Bolzano’s 1817 treatise [11] which presented a proof of the intermediate value theorem that does not depend on geometric intuition. Herein Bolzano was motivated by the idea of methodological purity: A result in pure analysis should not require methods from geometry, which back then was considered a branch of applied mathematics.

of the theorem. A proof system is then called analytic if all of its theorems admit an analytic proof.

It is not easy to judge at first sight whether a system like **NI** is analytic in this sense, due to rules such as Modus Ponens

$$\frac{A \rightarrow B \quad A}{B}$$

which in principle allow to introduce completely unrelated formulas  $B$  into a proof of  $A$ .

In his landmark article [21], now considered the birth of structural proof theory, Gerhard Gentzen showed in 1935 how analyticity can be obtained by proof-theoretic means. Since the Natural Deduction calculus seemed inadequate for his endeavour, he first invented the *sequent calculus*, which is essentially a meta-calculus for the notation and manipulation of Natural Deduction proofs. Instead of formulas it builds on *sequents*

$$A_1, \dots, A_n \Rightarrow B.$$

The left hand side of such a sequent is called the *antecedent* and consists of a (possibly empty) list of formulas. The right hand side, called the *succedent*, is either a single formula or empty. The intended reading of a sequent  $A_1, \dots, A_n \Rightarrow B$  is that  $B$  follows from assumptions  $A_1$  through  $A_n$ . The sequent  $A_1, \dots, A_n \Rightarrow$  indicates that the antecedent is contradictory.

Let us use uppercase greek letters  $\Gamma, \Delta, \Sigma$  to denote lists of formulas. The letter  $\Pi$  will be reserved for such a list of size at most 1.

The axioms (0-ary rules) of the sequent calculus are called *initial sequents*. Among those we always have the identity axiom  $A \Rightarrow A$  (' $A$  follows from the assumption  $A$ ').

The handling of assumptions is made explicit in the sequent calculus by means of the *structural rules*

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi} (e_l) \quad \frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi} (c_l) \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi} (w_l)$$

called (left) *exchange*, *contraction* and *weakening* respectively. They state that it neither matters in which order ( $e_l$ ) or how often ( $c_l$ ) an assumption is used, nor whether it is used at all ( $w_l$ ).

For each connective the sequent calculus has a left rule and a right rule, describing how the connective can be introduced in the antecedent or succedent. Herein the right rules are direct translations of Natural Deduction rules. For example,

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} (\rightarrow_r) \quad \text{corresponds to} \quad \frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \rightarrow B} (\rightarrow_l),$$

*Structural rules:*

$$\frac{}{\overline{A \Rightarrow A}} \text{ (id)}$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi} \text{ (e}_l\text{)} \quad \frac{\Gamma, A, A \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi} \text{ (c}_l\text{)} \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi} \text{ (w}_l\text{)} \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma \Rightarrow A} \text{ (w}_r\text{)}$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

*Logical rules:*

$$\frac{}{\overline{\perp \Rightarrow \Pi}} \text{ (\perp)}$$

$$\frac{\Gamma, A \Rightarrow \Pi \quad \Gamma, B \Rightarrow \Pi}{\Gamma, A \wedge B \Rightarrow \Pi} \text{ (\wedge}_l\text{)} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \text{ (\wedge}_r\text{)}$$

$$\frac{\Gamma, A \Rightarrow \Pi \quad \Gamma, B \Rightarrow \Pi}{\Gamma, A \vee B \Rightarrow \Pi} \text{ (\vee}_l\text{)} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \text{ (\vee}_r\text{)}$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, B \Rightarrow \Pi}{\Gamma, \Delta, A \rightarrow B \Rightarrow \Pi} \text{ (\rightarrow}_l\text{)} \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \text{ (\rightarrow}_r\text{)}$$

Figure 1.3: The sequent calculus LI

which is the rule of inferring  $A \rightarrow B$  from  $B$  and closing the assumption  $A$  (that is, removing it from the antecedent). In a logical rule such as  $(\rightarrow_r)$ ,  $A \rightarrow B$  is called the *principal formula* and  $A, B$  the *auxiliary formulas*.  $\Gamma$  is called the *context*.

The left rules as well as the rule

$$\frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

correspond to certain simple transformations of Natural Deduction proofs: In the case of (cut), we have a Natural Deduction proof of  $\Pi$  from the assumptions  $\Delta \cup \{A\}$  in which all leaves  $A$  get replaced by their derivation from  $\Gamma$ .

The complete sequent calculus LI for intuitionistic logic is pictured in Figure 1.3.

**Definition 1.7**

Let  $G \cup \{S\}$  be a set of sequents. An LI-proof of  $S$  from  $G$  is a tree of sequents rooted in  $S$  obeying the following properties:

1. Each leaf is either an initial sequent, or contained in  $G$ ;

- Every internal node, together with its child node(s), forms an instance of a rule of **LI**.

We write  $G \vdash_{\mathbf{LI}} S$  if there is an **LI**-proof of  $S$  from  $G$ . We say that a formula  $A$  is provable in **LI** if the sequent  $\Rightarrow A$  is provable in **LI**, and the set  $\text{Thm}(\mathbf{LI})$  comprises all such formulas  $A$ .

That  $\text{Thm}(\mathbf{LI})$  is closed under Modus Ponens is witnessed by the following derivation of  $\Rightarrow B$  from  $\Rightarrow A$  and  $\Rightarrow A \rightarrow B$  (which uses the cut rule twice):

$$\frac{\Rightarrow A \quad \frac{\frac{\overline{A \Rightarrow A} \text{ (id)} \quad \frac{\overline{B \Rightarrow B} \text{ (id)}}{A, A \rightarrow B \Rightarrow B} (\rightarrow_l)}{A \Rightarrow B} \text{ (cut)}}{\Rightarrow B} \text{ (cut)}}{\Rightarrow B} \text{ (cut)}$$

The relationship between sequents, proofs in **NI** and theorems of intuitionistic logic is explained by the following proposition [21, 22]:

### Proposition 1.8

The following are equivalent:

- $B$  is provable from assumptions  $A_1, \dots, A_n$  in **NI**.
- The sequent  $A_1, \dots, A_n \Rightarrow B$  is provable in **LI**.
- $A_1 \wedge \dots \wedge A_n \rightarrow B \in \mathbf{IL}$ .

In light of this, the formula  $A_1 \wedge \dots \wedge A_n \rightarrow B$  is sometimes called the *formula interpretation* of the sequent  $A_1, \dots, A_n \Rightarrow B$ .

### Example 1.9

The following is a proof of the sequent  $\neg A \Rightarrow A \vee B \rightarrow B$  in **LI**.

$$\frac{\frac{\overline{A \Rightarrow A} \text{ (id)} \quad \frac{\overline{\perp \Rightarrow B} \text{ (}\perp\text{)}}{\neg A, A \Rightarrow B} (\rightarrow_l)}{\neg A, A \vee B \Rightarrow B} (\vee_l) \quad \frac{\overline{B \Rightarrow B} \text{ (id)}}{\neg A, B \Rightarrow B} \text{ (w}_l\text{)}}{\neg A, A \vee B \Rightarrow B} \text{ (}\vee\text{)}}{\neg A \Rightarrow A \vee B \rightarrow B} (\rightarrow_r)$$

The Natural Deduction proof was presented in Example 1.5.

Crucially, all rules of the sequent calculus with the exception of (cut) are analytic in the sense that their premises contain only subformulas of their conclusion. In other words, in the sequent calculus all non-analytic proof steps have been pushed into the cut rule.

Gentzen then famously showed the following *cut elimination theorem* [21]:

**Theorem** (cut elimination theorem, Gentzen 1935)

Every sequent provable in **LI** has a proof without the rule (cut).

See [21] or [53] for the proof. It comes in the form of a concrete cut reduction algorithm which repeatedly replaces cuts in a proof by simpler cuts. Here a simpler cut is either a cut on a less complex formula, or a cut on the same formula but higher up in the proof. The reduction steps depend on how the cut appears in the proof and therefore involve an extensive case distinction. It is shown that the algorithm terminates, and the resulting elementary cuts can be removed directly.

From the cut elimination theorem it follows that every theorem  $A \in \mathbf{IL}$  has an **LI**-proof containing only subformulas of  $A$ , and hence, the calculus **LI** is analytic.

By the same method a cut elimination theorem can be established for numerous other sequent calculi. In particular the theorem holds for the sequent calculus **LK** for classical propositional logic (to be discussed below) as well as for the first-order versions of **LI** and **LK**. Beyond its philosophical interest, the cut elimination theorem has manifold applications, among which we only mention decision procedures for **IL** and **CL** and proofs of the interpolation property (for these and other applications, see [53]).

Let us now discuss the sequent calculus **LK** for classical logic. Gentzen observed that classical reasoning can be accommodated in the sequent calculus in a purely structural manner by allowing lists of formulas in the succedent, which are then interpreted as disjunctions.

A *multi-conclusion sequent* is of the form

$$A_1, \dots, A_n \Rightarrow B_1, \dots, B_m$$

and corresponds to the statement: ‘From  $A_1$  through  $A_n$ ,  $B_1 \vee \dots \vee B_m$  follows.’

The logical rules of the (multi-conclusion) sequent calculus **LK** for classical logic are just the rules of **LI**, but with an additional list  $\Sigma$  of formulas in the succedent. For example the rule  $(\wedge_r)$  now reads

$$\frac{\Gamma \Rightarrow A, \Sigma \quad \Gamma \Rightarrow B, \Sigma}{\Gamma \Rightarrow A \wedge B, \Sigma} (\wedge_r)$$

On top of that, **LK** has the structural rules of exchange and contraction in the succedent:

$$\frac{\Gamma \Rightarrow \wedge, A, B, \Sigma}{\Gamma \Rightarrow \wedge, B, A, \Sigma} (e_r) \quad \frac{\Gamma \Rightarrow A, A, \Sigma}{\Gamma \Rightarrow A, \Sigma} (c_r)$$

The full system is pictured in Figure 1.4.



*Structural rules:*

$$\frac{}{\overline{A \Rightarrow A}} \text{ (id)}$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Sigma}{\Gamma, B, A, \Delta \Rightarrow \Sigma} \text{ (e}_l\text{)} \quad \frac{\Gamma \Rightarrow A, B, \Sigma}{\Gamma \Rightarrow B, A, \Sigma} \text{ (e}_r\text{)} \quad \frac{\Gamma, A, A \Rightarrow \Sigma}{\Gamma, A \Rightarrow \Sigma} \text{ (c}_l\text{)} \quad \frac{\Gamma \Rightarrow A, \Sigma}{\Gamma \Rightarrow A, A, \Sigma} \text{ (c}_r\text{)}$$

$$\frac{\Gamma \Rightarrow \Sigma}{\Gamma, A \Rightarrow \Sigma} \text{ (w}_l\text{)} \quad \frac{\Gamma \Rightarrow \Sigma}{\Gamma \Rightarrow A, \Sigma} \text{ (w}_r\text{)}$$

$$\frac{\Gamma \Rightarrow A, \Sigma \quad \Delta, A \Rightarrow \Lambda}{\Gamma, \Delta \Rightarrow \Sigma, \Lambda} \text{ (cut)}$$

*Logical rules:*

$$\frac{}{\perp \Rightarrow \Sigma} \text{ (\perp)} \quad \frac{}{\Gamma \Rightarrow \top} \text{ (\top)}$$

$$\frac{\Gamma, A \Rightarrow \Sigma}{\Gamma, A \wedge B \Rightarrow \Sigma} \quad \frac{\Gamma, B \Rightarrow \Sigma}{\Gamma, A \wedge B \Rightarrow \Sigma} \text{ (\wedge}_l\text{)} \quad \frac{\Gamma \Rightarrow A, \Sigma \quad \Gamma \Rightarrow B, \Sigma}{\Gamma \Rightarrow A \wedge B, \Sigma} \text{ (\wedge}_r\text{)}$$

$$\frac{\Gamma, A \Rightarrow \Sigma \quad \Gamma, B \Rightarrow \Sigma}{\Gamma, A \vee B \Rightarrow \Sigma} \text{ (\vee}_l\text{)} \quad \frac{\Gamma \Rightarrow A, \Sigma}{\Gamma \Rightarrow A \vee B, \Sigma} \quad \frac{\Gamma \Rightarrow B, \Sigma}{\Gamma \Rightarrow A \vee B, \Sigma} \text{ (\vee}_r\text{)}$$

$$\frac{\Gamma \Rightarrow A, \Sigma \quad \Delta, B \Rightarrow \Pi}{\Gamma, \Delta, A \rightarrow B \Rightarrow \Sigma, \Pi} \text{ (\rightarrow}_l\text{)} \quad \frac{\Gamma, A \Rightarrow B, \Sigma}{\Gamma \Rightarrow A \rightarrow B, \Sigma} \text{ (\rightarrow}_r\text{)}$$

Figure 1.4: The sequent calculus LK

The multi-succedent version of every rule of **LI** *with the exception of*  $(\rightarrow_r)$  remains sound for intuitionistic logic if the succedent is interpreted disjunctively. Now as for  $(\rightarrow_r)$ , its multi-succedent instance

$$\frac{A \Rightarrow B, C}{\Rightarrow A \rightarrow B, C} \text{ (\rightarrow}_r\text{)}$$

has the formula interpretation

$$X := (A \rightarrow (B \vee C)) \rightarrow ((A \rightarrow B) \vee C)$$

which is not valid in intuitionistic logic; indeed, one can check that  $\text{IL} + X = \text{CL}$ . From this it follows that **LK** is a calculus for classical logic.

## 1.7 $\text{FL}_e^\perp$ and Extensions

To comprehend the effect of structural rules in the sequent calculus, it is useful to study systems lacking some or all of these rules. The resulting logics are called *substructural logics*.

A basic system is  $\text{FL}_e^\perp$  (*Full Lambek Calculus<sup>5</sup> with exchange and additive constants*) which is, roughly speaking, obtained by dropping the weakening rules ( $w_l$ ), ( $w_r$ ) as well as contraction ( $c_l$ ) from **LI**.  $\text{FL}_{e_w}^\perp$  is then obtained by reinstating both left and right weakening, and  $\text{FL}_{e_c}^\perp$  by adding left contraction. In all three systems the cut elimination theorem holds [46].

In **LI** it does not matter if the conjunction rule ( $\wedge_r$ ) is formulated as

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \quad \text{or} \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \wedge B}$$

since both versions are inter-derivable using contraction and weakening. The first format, which we subscribed to in Figure 1.3, is called *additive* or *context-sharing*, and the second one is called *multiplicative* or *context-splitting*. In  $\text{FL}_e^\perp$  the inter-derivability of both versions breaks down, or in other words, the additive rule for  $\wedge$  defines a different connective than the multiplicative rule. It is common to study substructural logics in an extended signature which accommodates both connectives with additive rules and connectives with multiplicative rules.

We will keep the symbol  $\wedge$  for the additive rule. The ‘multiplicative conjunction’, also called *fusion*, will be denoted<sup>6</sup> by  $*$  and its left and right rules are

$$\frac{\Gamma, A, B \Rightarrow \Pi}{\Gamma, A * B \Rightarrow \Pi} (*_l) \quad \text{and} \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A * B} (*_r).$$

Furthermore one has ‘context-free’ versions 0 and 1 of the logical constants:

$$\frac{}{0 \Rightarrow} (0_l) \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0} (0_r)$$

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma, 1 \Rightarrow \Pi} (1_l) \quad \frac{}{\Rightarrow 1} (1_r)$$

The complete system  $\text{FL}_e^\perp$  is presented in Figure 1.5. A consequence relation  $\vdash_{\text{FL}_e^\perp}$  is defined as in Definition 1.7.

In  $\text{FL}_{e_w}^\perp$  we can prove  $0 \equiv \perp$  and  $1 \equiv \top$ . It is therefore customary to investigate extensions of  $\text{FL}_{e_w}^\perp$  in the reduced signature containing only  $\perp$  and  $\top$  as constants.

Given a set  $\mathcal{A}$  of formulas, the *axiomatic extension*  $\text{FL}_e^\perp + \mathcal{A}$  of  $\text{FL}_e^\perp$  by  $\mathcal{A}$  is the sequent calculus obtained by adding the initial sequent  $\Rightarrow A$  for every substitution instance  $A$  of a formula in  $\mathcal{A}$ . Then  $\text{FL}_e^\perp + \mathcal{A}$  is a calculus for the logic  $\text{Thm}(\text{FL}_e^\perp) + \mathcal{A}$ , albeit it never satisfies the cut elimination theorem except in trivial cases. The following is the substructural version of the deduction theorem, called the *local deduction theorem* [20]:

<sup>5</sup> $\text{FL}_e^\perp$  is named after Joachim Lambek, who introduced a basic calculus **FL** in the context of formal grammar theory [35]. **FL** is similar to  $\text{FL}_e^\perp$  but does not include additive constants  $\top$  and  $\perp$  and has no exchange rule. Furthermore, due to the lack of exchange, **FL** has two implication connectives  $\backslash$  and  $/$ .

<sup>6</sup>In the literature on linear logic (see Section 1.8) it is more common to write  $\otimes$  for multiplicative and  $\&$  for additive conjunction.

*Structural rules:*

$$\frac{}{\overline{A \Rightarrow A}} \text{ (id)}$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi} \text{ (e}_l\text{)} \quad \frac{\Gamma \Rightarrow A \quad \Delta, A \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

*Logical rules:*

$$\frac{}{\perp \Rightarrow \Pi} \text{ (\perp)} \quad \frac{}{\Gamma \Rightarrow \top} \text{ (\top)}$$

$$\frac{}{0 \Rightarrow} \text{ (0}_l\text{)} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0} \text{ (0}_r\text{)} \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma, 1 \Rightarrow \Pi} \text{ (1}_l\text{)} \quad \frac{}{\Rightarrow 1} \text{ (1}_r\text{)}$$

$$\frac{\Gamma, A \Rightarrow \Pi}{\Gamma, A \wedge B \Rightarrow \Pi} \quad \frac{\Gamma, B \Rightarrow \Pi}{\Gamma, A \wedge B \Rightarrow \Pi} \text{ (\wedge}_l\text{)} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \text{ (\wedge}_r\text{)}$$

$$\frac{\Gamma, A, B \Rightarrow \Pi}{\Gamma, A * B \Rightarrow \Pi} \text{ (*}_l\text{)} \quad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A * B} \text{ (*}_r\text{)}$$

$$\frac{\Gamma, A \Rightarrow \Pi \quad \Gamma, B \Rightarrow \Pi}{\Gamma, A \vee B \Rightarrow \Pi} \text{ (\vee}_l\text{)} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \text{ (\vee}_r\text{)}$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, B \Rightarrow \Pi}{\Gamma, \Delta, A \rightarrow B \Rightarrow \Pi} \text{ (\rightarrow}_l\text{)} \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \text{ (\rightarrow}_r\text{)}$$

Figure 1.5: The sequent calculus  $\mathbf{FL}_e^\perp$

**Theorem 1.10** (local deduction theorem)

The following are equivalent:

1.  $\vdash_{\mathbf{FL}_e^\perp + \mathcal{A}' \cup \mathcal{A}} F$ .
2.  $\vdash_{\mathbf{FL}_e^\perp + \mathcal{A}'} (A_1 \wedge 1) * \dots * (A_n \wedge 1) \rightarrow F$  for some instances  $A_1, \dots, A_n$  of axioms in  $\mathcal{A}$ .

In this thesis, the term *substructural logic* stands for every logic  $\text{Thm}(\mathbf{FL}_e^\perp) \subseteq L \subsetneq \mathbf{IL}$ .<sup>7</sup>

We close this section with an important convention. All sequent systems occurring in this thesis contain the exchange rule ( $e_l$ ). It follows that we do not lose expressivity when formalizing the antecedents of sequents as *multisets* instead of lists. For this we only have to change the way we read an antecedent: In the multiset interpretation, the

<sup>7</sup>To make the second inclusion formally precise,  $\mathbf{IL}$  can be considered as a logic in the signature of substructural logic, where  $*$ ,  $0$ ,  $1$ ,  $\top$  conflate with  $\wedge$ ,  $\perp$ ,  $\neg\perp$ ,  $\neg\perp$ .

comma ‘,’ is interpreted as a *multiset union*. Likewise, uppercase Greek letters  $\Gamma, \Delta, \Sigma \dots$  are interpreted as multisets instead of lists.

Henceforth we assume that all sequent calculi are formalized using multisets.

## 1.8 Linear Logic

*Linear logic* [23] was invented by Jean-Yves Girard in the late 80ies. It can be seen as a hybrid system situated in between substructural and fully structural logic. We will focus here on its single-conclusion variant with weakening, which is called *affine intuitionistic linear logic*.

The basic idea is to start with a substructural system such as  $\text{FL}_{ew}^\perp$ , and then reintroduce contraction in a controlled way. For this the logical signature is augmented by a unary operator ! called ‘bang’. Formulas prefixed with ! are called *unbounded* and it is stipulated that contraction may only be applied to unbounded formulas. That is, we add to  $\text{FL}_{ew}^\perp$  the rule

$$\frac{\Gamma, !A, !A \Rightarrow \Pi}{\Gamma, !A \Rightarrow \Pi} (!c).$$

To use an unbounded formula, we can simply strip off the bang. This is the *dereliction rule*:

$$\frac{\Gamma, A \Rightarrow \Pi}{\Gamma, !A \Rightarrow \Pi} (dr)$$

Finally, there is a right rule for !

$$\frac{! \Gamma \Rightarrow A}{! \Gamma \Rightarrow !A} (pr)$$

called *promotion*. Here ! $\Gamma$  denotes a multiset of unbounded formulas.

The extension of  $\text{FL}_{ew}^\perp$  by the rules (!c), (dr) and (pr) is called **aILL**.

For our purposes, it will be more convenient to work with a simple variant of **aILL**, which is discussed next.

The point of (!c) is that unbounded formulas do not vanish in a proof. We can achieve the same effect by modifying the multiplicative rules, so that unbounded formulas in the conclusion may be copied into both premises. For example we can take the following variant of the ( $*_r$ ) rule:

$$\frac{! \Omega, \Gamma \Rightarrow A \quad ! \Omega, \Delta \Rightarrow B}{! \Omega, \Gamma, \Delta \Rightarrow A * B}$$

If we additionally change the dereliction rule to

$$\frac{\Gamma, !A, A \Rightarrow \Pi}{\Gamma, !A \Rightarrow \Pi} (dr)$$

then an explicit contraction becomes redundant. The full system  $\mathbf{aILL}^*$ , which underlies the work in Chapter 3, is pictured in Figure 1.6. Our calculus  $\mathbf{aILL}^*$  is a minor variant of Andreoli's *dyadic system* for linear logic [3].

### Proposition 1.11

There are proof translations between  $\mathbf{aILL}$  and  $\mathbf{aILL}^*$  which preserve the endsequent and do not introduce cuts. In particular,  $\mathbf{aILL}$  and  $\mathbf{aILL}^*$  derive the same sequents.

*Proof (outline).* Derivations in  $\mathbf{aILL}^*$  can be simulated by  $\mathbf{aILL}$  in a straightforward manner, using (!c) in the simulation of multiplicative rules and the modified (dr) rule. For the converse direction, call a sequent  $T$  a *contraction* of  $S$  if  $T$  arises from the sequent  $S$  by removing some (including zero) but not all occurrences of an unbounded formula from the antecedent. For example,  $!A, B \Rightarrow \Pi$  is a contraction of  $!A, !A, !A, B \Rightarrow \Pi$ . Then by induction on the height of proofs, we can show that whenever  $\mathbf{aILL}$  proves a sequent  $S$ , then *for all* contractions  $T$  of  $S$ ,  $\mathbf{aILL}^*$  proves  $T$ . Neither of the sketched translations introduces cuts.  $\square$

## 1.9 The Hypersequent Calculus

A generalization of the sequent calculus is the *hypersequent calculus*, introduced independently by Avron [6], Mints [42] and Pottinger [47]. A hypersequent is a multiset of sequents written

$$\Gamma_1 \Rightarrow \Pi_1 \mid \dots \mid \Gamma_n \Rightarrow \Pi_n.$$

Each sequent  $\Gamma_i \Rightarrow \Pi_i$  is called a *component* of the hypersequent. The symbol ' $\mid$ ' is interpreted as a disjunction; This shows in the structural rules

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi} \text{ (ew)} \quad \text{and} \quad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi} \text{ (ec)}$$

of *external weakening* and *external contraction*. Here  $G$  is a hypersequent-variable which can be instantiated by a hypersequent. In (ew),  $G$  is required to be nonempty.

Let  $\mathbf{FL}_{e^*}^\perp$  be one of the calculi  $\mathbf{FL}_e^\perp, \mathbf{FL}_{ew}^\perp, \mathbf{FL}_{ec}^\perp$  or  $\mathbf{FL}_{ewc}^\perp \cong \mathbf{LI}$ . Then a corresponding hypersequent calculus  $\mathbf{HFL}_{e^*}^\perp$  is obtained by adding the context variable  $G$  to all rules and initial sequents of  $\mathbf{FL}_{e^*}^\perp$ , and moreover adding the rules (ew) and (ec). So far, the logic has not changed:

### Lemma 1.12

$\mathbf{HFL}_{e^*}^\perp$  proves the same sequents as  $\mathbf{FL}_{e^*}^\perp$ .

*Structural rules:*

$$\frac{}{\overline{A \Rightarrow A}} \text{ (id)}$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi} \text{ (e)} \quad \frac{! \Omega, \Gamma \Rightarrow A \quad ! \Omega, \Delta, A \Rightarrow \Pi}{! \Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi} \text{ (w}_l\text{)} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \text{ (w}_r\text{)}$$

*Logical rules:*

$$\frac{}{\perp \Rightarrow \Pi} \text{ (\perp)} \quad \frac{}{\Gamma \Rightarrow \top} \text{ (\top)}$$

$$\frac{\Gamma, A \Rightarrow \Pi}{\Gamma, A \wedge B \Rightarrow \Pi} \quad \frac{\Gamma, B \Rightarrow \Pi}{\Gamma, A \wedge B \Rightarrow \Pi} \text{ (\wedge}_l\text{)} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \text{ (\wedge}_r\text{)}$$

$$\frac{\Gamma, A, B \Rightarrow \Pi}{\Gamma, A * B \Rightarrow \Pi} \text{ (*}_l\text{)} \quad \frac{! \Omega, \Gamma \Rightarrow A \quad ! \Omega, \Delta \Rightarrow B}{! \Omega, \Gamma, \Delta \Rightarrow A * B} \text{ (*}_r\text{)}$$

$$\frac{\Gamma, A \Rightarrow \Pi \quad \Gamma, B \Rightarrow \Pi}{\Gamma, A \vee B \Rightarrow \Pi} \text{ (\vee}_l\text{)} \quad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \text{ (\vee}_r\text{)}$$

$$\frac{! \Omega, \Gamma \Rightarrow A \quad ! \Omega, \Delta, B \Rightarrow \Pi}{! \Omega, \Gamma, \Delta, A \rightarrow B \Rightarrow \Pi} \text{ (\rightarrow}_l\text{)} \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \text{ (\rightarrow}_r\text{)}$$

$$\frac{\Gamma, !A, A \Rightarrow \Pi}{\Gamma, !A \Rightarrow \Pi} \text{ (dr)} \quad \frac{! \Omega \Rightarrow A}{! \Omega \Rightarrow !A} \text{ (pr)}$$

Figure 1.6: The sequent calculus  $\mathbf{aILL}^*$

*Proof (Outline).* Every  $\mathbf{FL}_{e^*}^\perp$ -proof is also a  $\mathbf{HFL}_{e^*}^\perp$ -proof. Conversely, by induction on the height of proofs one can show the following: If  $\mathbf{HFL}_{e^*}^\perp$  proves a hypersequent  $\Gamma_1 \Rightarrow \Pi_1 \mid \dots \mid \Gamma_n \Rightarrow \Pi_n$ , then there exists  $i \leq n$  such that  $\mathbf{FL}_{e^*}^\perp$  proves  $\Gamma_i \Rightarrow \Pi_i$ . This implies the claim.  $\square$

By adding rules to a hypersequent calculus which take the extended structure into account it is possible to capture logics analytically which do not have a cutfree sequent calculus. For example, the extension of  $\mathbf{HLI}$  (=  $\mathbf{HFL}_{e_{wc}}$ ) by the *communication rule*

$$\frac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} \text{ (com)}$$

yields a calculus for Gödel logic  $\mathbf{G} := \mathbf{IL} + (p \rightarrow q) \vee (q \rightarrow p)$  in which the cut rule is eliminable [5]. In a rule such as (com), the components not contained in the context variable  $G$  are called the *principal components* of the rule.

In Ciabattoni, Terui and Galatos [17] it was established that a large class of substructural and intermediate logics can be captured analytically by hypersequent calculi, and moreover that the necessary hypersequent calculus can be computed from the axiomatization of the logic in question. This result will be heavily used in Chapter 2.

### 1.10 What Has Been Left Out in These Preliminaries

Arguments involving concepts from computational complexity will play a role here and there (mostly in Chapter 2), although they are never central. For the necessary background knowledge we refer the reader to [4].



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# Bounding Axiomatic Systems via Hypersequents

## 2.1 Introduction

The last decades have witnessed an explosion of new logics. Some of these logics come into being in the ivory tower of formal logic, while others arise in fields as diverse as verification and model checking, epistemology, and law.

For several of these new logics it is possible to give a satisfying proof-theoretic treatment in variants of Gentzen's sequent calculus. Yet for a substantial number of others, no reasonable sequent calculus with the subformula property has been found, and moreover there are results indicating that no such calculus exists.<sup>1</sup>

This led proof theorists to look out for more expressive frameworks. Following Gentzen's approach to classical and intuitionistic logic, the general strategy of this task can be summarized as follows: One first tries to construct a proof system for the logic of interest in which the only non-analytic rule is the cut rule (or a suitable generalization thereof). Then in a second step one shows the admissibility of the cut rule, usually by means of a syntactic elimination procedure. In doing so, one obtains the subformula property for the calculus.

One early and successful example of said approach is the discovery of the hypersequent calculus [5]. But the development did not stop there: Nowadays the proof-theoretic landscape contains a multitude of extensions of the sequent calculus, such as labelled sequent systems [31, 43], nested sequents [32, 12] and the display calculus [8, 24].

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<sup>1</sup>For example, some limiting results on the use of sequent calculi in modal logics are contained in [39, Chapter 3.4]. Note that the scope of this and related results is always limited to one particular definition of 'sequent calculus'.



of the axiom of linearity.<sup>2</sup> We now show how to transform  $\delta$  into a proof of  $\text{lin}(\Gamma_1, \Gamma_2) \Rightarrow F$  in the sequent calculus **LI**.

First we create two modified copies of the proof. In each of these, the instance of (com) is traded for an additional formula in the antecedent. The first modified copy looks like this:

$$\begin{array}{c} \dots \tau \\ \dots \pi \\ \frac{\frac{\frac{G \mid \Gamma_2 \Rightarrow \bigwedge \Gamma_2 \quad \frac{G \mid \Sigma_1, \overset{\dots}{\pi_1}}{\Gamma_1} \Rightarrow \Pi_2}{G \mid \Sigma_1, \bigwedge \Gamma_1 \Rightarrow \Pi_1} \text{ some } (\wedge_1)\text{'s and } (c_1)\text{'s}}{G \mid \Sigma_1, \Gamma_2, \bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1 \Rightarrow \Pi_1} (\rightarrow_1)}{G \mid \Sigma_1, \Gamma_2, \bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} \text{ (ew)}}{\bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1 \Rightarrow F} \dots \pi' \end{array}$$

Here  $\pi'$  is like the proof part  $\pi$ , only that the newly added formula  $\bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1$  has been propagated downwards in the proof. Therefore the endsequent is now  $\bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1 \Rightarrow F$  rather than  $\Rightarrow F$ .

In a completely symmetric fashion we obtain the second copy:

$$\begin{array}{c} \dots \tau \\ \dots \pi \\ \frac{\frac{\frac{G \mid \Gamma_1 \Rightarrow \bigwedge \Gamma_1 \quad \frac{G \mid \Sigma_2, \overset{\dots}{\pi_2}}{\Gamma_2} \Rightarrow \Pi_2}{G \mid \Sigma_2, \bigwedge \Gamma_2 \Rightarrow \Pi_2} \text{ some } (\wedge_1)\text{'s and } (c_1)\text{'s}}{G \mid \Sigma_2, \Gamma_1, \bigwedge \Gamma_1 \rightarrow \bigwedge \Gamma_2 \Rightarrow \Pi_2} (\rightarrow_1)}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1, \bigwedge \Gamma_1 \rightarrow \bigwedge \Gamma_2 \Rightarrow \Pi_2} \text{ (ew)}}{\bigwedge \Gamma_1 \rightarrow \bigwedge \Gamma_2 \Rightarrow F} \dots \pi' \end{array}$$

Again  $\pi''$  is a variant of  $\pi$  where  $\bigwedge \Gamma_1 \rightarrow \bigwedge \Gamma_2$  has been added appropriately.

Finally we combine both copies by an application of ( $\vee_1$ ):

$$\frac{(\bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1) \Rightarrow F \quad (\bigwedge \Gamma_1 \rightarrow \bigwedge \Gamma_2) \Rightarrow F}{(\bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1) \vee (\bigwedge \Gamma_1 \rightarrow \bigwedge \Gamma_2) \Rightarrow F} (\vee_1)$$

This proof does not contain (com) any more, and so it can be reduced to a proof in **LI** (see Lemma 1.12).

The crucial observation is now this. Since the cut rule is eliminable in **HLI** + (com), we could assume that the original hypersequent proof  $\delta$  was analytic. Therefore the

<sup>2</sup>Here and henceforth,  $\bigwedge\{A_1, \dots, A_n\} := A_1 \wedge \dots \wedge A_n$  and  $\bigwedge \emptyset := \top$ . There is a slight abuse of notation as the ordering of the formulas  $A_1, \dots, A_n$  in the set  $\{A_1, \dots, A_n\}$  is of course not fixed. However, this will never lead to trouble.

multisets  $\Gamma_1$  and  $\Gamma_2$  consist of subformulas of  $F$ , and consequently the axiom instance  $\text{lin}(\Gamma_1, \Gamma_2)$  is bounded by the theorem  $F$  as follows: It substitutes variables in the linearity axiom by conjunctions of subformulas of  $F$ . Later on, we will call such an axiom instance *set-bounded* with respect to  $F$ . A general result we aim at by the above methods will be:

**Theorem**

$F$  is provable in  $\text{HLI} + (\text{com})$  if there are instances  $\text{lin}_1, \dots, \text{lin}_n$  of the axiom of linearity, all *set-bounded* wrt.  $F$ , such that  $\text{lin}_1, \dots, \text{lin}_n \Rightarrow F$  is provable in  $\text{LI}$ .

An alternative calculus-independent reformulation:

**Theorem**

$F$  is a theorem of Gödel logic if there exist instances  $\text{lin}_1, \dots, \text{lin}_n$  of the axiom of linearity, all *set-bounded* wrt.  $F$ , such that  $(\bigwedge_{i=1}^n \text{lin}_i) \rightarrow F$  is a theorem of  $\text{IL}$ .

The novelty here is the restriction to set-bounded axiom instances: Without this restriction the result stated above would just be a standard deduction theorem, which can be proved by elementary means.

We will see that the sketched method works for a rather large class of logics, namely for all extensions of  $\text{HFL}_e^\perp$  by so-called *analytic structural hypersequent rules*.

Let us now lay out what has to be achieved for the generalization of the proof.

1. A typical hypersequent proof contains multiple instances of hypersequent rules, and the naive method of iterating the rule elimination argument from the example fails. To see this, consider the case that the proof part  $\tau$  in the example contains another instance of  $(\text{com})$ . Since we put together two copies of the proof, this instance will be duplicated, and so termination of a step-by-step elimination of  $(\text{com})$ 's cannot be guaranteed. We overcome this problem by simultaneously eliminating all lowermost hypersequent rules in a single step (Section 2.4).
2. For every analytic structural hypersequent rule  $(r)$ , we have to find a formula which bears the same relation to  $(r)$  as  $\text{lin}$  does to  $(\text{com})$ . We call these formulas *disjunction forms*. The existence of disjunctive forms for all analytic structural hypersequent rules will be established constructively in Section 2.5.

After addressing these issues and proving the main result in Section 2.6, we present a variety of strengthenings and related results in Section 2.7. Here we also mention some work on the 'simple substitution property' by various authors in the late 80ies and early 90ies which is akin to our investigation. Finally in Section 2.8 we show that our method

can be adjusted to give a new proof of Takano’s [52] result stating that the standard sequent calculus for modal logic **S5**—although not cutfree—satisfies the subformula property.

## 2.2 Bounded Axiomatic Extensions

A deduction theorem allows us to relate theorems of an axiomatic extension of a logic  $L$  to theorems in the base logic  $L$ . For example, if  $L$  is Gödel logic  $\mathbf{IL} + \text{lin}$  where  $\text{lin}$  is the axiom of linearity  $(A \rightarrow B) \vee (B \rightarrow A)$  of linearity, then we have

$$F \in \mathbf{IL} + \text{lin} \iff (\text{lin}_1 \wedge \dots \wedge \text{lin}_n \rightarrow F) \in \mathbf{IL}$$

for some substitution instances (♠)  
 $\text{lin}_1, \dots, \text{lin}_n$  of  $\text{lin}$ .

This equivalence suggests the following. What is needed in order to comprehend the axiomatic extension  $\mathbf{IL} + \text{lin}$  is—beyond knowledge of  $\mathbf{IL}$ —only an understanding of which kind of axiom instances are relevant for the proof of a given formula  $F$ .

Unfortunately this ‘only’ is deceitful, as the problem of bounding the axiom instances with respect to  $F$  is a hard one, and without such a bound the equivalence (♠) is of little use: The space of all possible axiom instances is simply too large.

Let us widen our scope again at look at extensions of substructural logics. Here and in the following,  $\text{FL}_{e^*}^\perp$  will denote an extension of  $\text{FL}_e^\perp$  by some analytic structural sequent rules.<sup>3</sup>  $\text{HFL}_{e^*}^\perp$  then denotes the corresponding hypersequent calculus, that is, the hypersequent rules which has all rules of  $\text{FL}_{e^*}^\perp$  and additionally (ew) and (ec).

In the following it will be useful to think of axioms as concrete formulas containing some freely chosen propositional variables instead of the ‘schematic’ formula-variables  $A, B$  etc. For example,  $\text{lin}$  can be the formula  $(p \rightarrow q) \vee (q \rightarrow p)$  where  $p$  and  $q$  are propositional variables. Every instance of the axiom of linearity then arises by replacing  $p$  and  $q$  in  $\text{lin}$  by arbitrary formulas.

Call a *bounding function* any map  $\psi$  which takes as arguments a set of formulas  $\mathcal{A}$  (the axioms) and a formula  $F$  (the potential theorem to be proved), and returns a set  $\psi(\mathcal{A}, F)$  of instances of formulas in  $\mathcal{A}$ . Here are some possible bounding functions, ordered by decreasing specificity (concrete examples follow below):

1. The *variable-analytic assignment*  $\psi_v(\mathcal{A}, F)$  contains all instances of formulas in  $\mathcal{A}$  obtained by substituting variables with variables occurring in  $F$ .

<sup>3</sup>Analytic structural sequent rules will be formally defined only in Definition 2.7. For now, it suffices to think of  $\text{FL}_{e^*}^\perp$  as one of the calculi  $\text{FL}_e^\perp, \text{FL}_{ew}^\perp, \text{FL}_{ec}^\perp$  or  $\text{LI}$ .

2. The *formula-analytic assignment*  $\psi_f(\mathcal{A}, F)$  contains all instances of formulas in  $\mathcal{A}$  obtained by substituting variables with subformulas of  $F$ .
3. The *set-analytic assignment*  $\psi_s(\mathcal{A}, F)$  contains all instances of formulas in  $\mathcal{A}$  obtained by substituting variables with non-repeating<sup>4</sup> fusions of subformulas of  $F$  (including 1 as the empty fusion).
4. The *multiset-analytic assignment*  $\psi_{ms}(\mathcal{A}, F)$  contains all instances of formulas in  $\mathcal{A}$  obtained by substituting variables with fusions of subformulas of  $F$  (including 1 as the empty fusion).
5. The *weakly variable-analytic assignment*  $\psi_{wv}(\mathcal{A}, F)$  contains all instances of formulas from  $\mathcal{A}$  whose variables are among those of  $F$ .
6. The *unrestricted assignment*  $\psi_{any}(\mathcal{A}, F)$  contains all instances of formulas from  $\mathcal{A}$ .

**Example 2.1**

Let  $\mathcal{A} = \{p \rightarrow 1\}$  and  $F = r \wedge q$ . Then

- $\psi_v(\mathcal{A}, F) = \{r \rightarrow 1, q \rightarrow 1\}$
- $\psi_f(\mathcal{A}, F) = \psi_v(\mathcal{A}, F) \cup \{r \wedge q \rightarrow 1\}$
- $\psi_s(\mathcal{A}, F) = \psi_f(\mathcal{A}, F) \cup \{1 \rightarrow 1, r * q \rightarrow 1, (r \wedge q) * r \rightarrow 1, (r \wedge q) * q \rightarrow 1\}$
- $\psi_{ms}(\mathcal{A}, F) = \psi_s(\mathcal{A}, F) \cup \{r * r \rightarrow 1, q * q \rightarrow 1, r * r * q \rightarrow 1, r * q * r \rightarrow 1, \dots\}$
- $\psi_{wv}(\mathcal{A}, F) = \{B \rightarrow 1 \mid \text{The only variables in } B \text{ are } r \text{ and } q\}$
- $\psi_{any}(\mathcal{A}, F) = \{B \rightarrow 1 \mid B \text{ is any formula}\}$

Note that the substitution sets  $\psi_v(\mathcal{A}, F)$ ,  $\psi_f(\mathcal{A}, F)$  and  $\psi_s(\mathcal{A}, F)$  are finite, whereas  $\psi_{ms}(\mathcal{A}, F)$ ,  $\psi_{wv}(\mathcal{A}, F)$  and  $\psi_{any}(\mathcal{A}, F)$  are infinite.

The following definition is fundamental for the whole chapter.

**Definition 2.2** (bounded axiomatic extensions)

Let  $\psi$  be a bounding function and  $\mathcal{A}$  a finite set of formulas. A logic  $L$  is a  $\psi$ -bounded extension of  $\mathbf{FL}_{e^*}^\perp$  by  $\mathcal{A}$  if  $L = \text{Thm}(\mathbf{FL}_{e^*}^\perp) + \mathcal{A}$  and the following deduction theorem

<sup>4</sup>A fusion  $A_1 * \dots * A_n$  is non-repeating if the  $A_i$ 's are pairwise distinct.

holds: For every formula  $F$ ,

$$F \in L \iff \exists A_1, \dots, A_n \in \psi(\mathcal{A}, F) \text{ such that} \\ (A_1 \wedge 1) * \dots * (A_n \wedge 1) \rightarrow F \in \text{Thm}(\mathbf{FL}_{e^*}^\perp).$$

A logic  $L$  is called a  $\psi$ -bounded extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  if for some finite  $\mathcal{A}$ , it is a  $\psi$ -bounded extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  by  $\mathcal{A}$ . For abbreviation, we also say ‘The extension  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  is  $\psi$ -bounded’ instead of ‘ $\text{Thm}(\mathbf{FL}_{e^*}^\perp) + \mathcal{A}$  is a  $\psi$ -bounded extension of  $\mathbf{FL}_{e^*}^\perp$  by  $\mathcal{A}$ ’. Finally, ‘multiset-bounded’ (resp. ‘set-bounded’, ‘formula-bounded’, ‘variable-bounded’) means  $\psi_{\text{ms}}$ -bounded (resp.  $\psi_s$ -bounded,  $\psi_f$ -bounded,  $\psi_v$ -bounded).

We emphasize that in Definition 2.2 the same axiom instance may appear multiple times in the fusion  $(A_1 \wedge 1) * \dots * (A_n \wedge 1)$ . So  $\psi$  acts as a bound on the search space for axiom instances, but not on their multiplicity. Note also that the direction ‘ $\Leftarrow$ ’ of the stated equivalence is always true.

We now present some simple observations on the notion of boundedness.

First, by inspecting definitions we observe that the local deduction theorem of substructural logics (Theorem 1.10) subsumes the statement that every extension  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  is bounded with respect to the unrestricted assignment. Hence:

### Proposition 2.3

Every extension  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  is  $\psi_{\text{any}}$ -bounded.

By plugging in bounding functions  $\psi$  in Definition 2.2 where  $\psi(\mathcal{A}, F) \subseteq \psi_{\text{any}}(\mathcal{A}, F)$ , we therefore obtain various strengthenings of the deduction theorem.

We can go one step further than  $\psi_{\text{any}}$ -boundedness without additional assumptions:

### Proposition 2.4

Every extension  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  is  $\psi_{\text{va}}$ -bounded.

*Proof.* If  $F$  is a theorem of  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$ , then by Proposition 2.3 there are axiom instances  $A_1, \dots, A_n$  such that  $(A_1 \wedge 1) * \dots * (A_n \wedge 1) \rightarrow F$  is a theorem of  $\mathbf{FL}_{e^*}^\perp$ . Since  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  is closed under substitution, we can replace every variable occurring in  $A_1, \dots, A_n$  but not in  $F$  by a constant or by some variable which does occur in  $F$ . This yields a theorem  $(A'_1 \wedge 1) * \dots * (A'_n \wedge 1) \rightarrow F$  of  $\mathbf{FL}_{e^*}^\perp$  which witnesses the  $\psi_{\text{va}}$ -boundedness.  $\square$

The variable renaming trick in the proof of Proposition 2.4 comes up in various contexts and seems to be folklore.

In the presence of contraction, boundedness gives rise to embeddings between logics:

**Proposition 2.5**

Assume that contraction is admissible in  $\mathbf{FL}_{e^*}^\perp$ .

- (i) If  $L$  is a set-bounded extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$ , then the validity problem of  $L$  can be reduced to the validity problem of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  in exponential time.
- (ii) If  $L$  is a formula-bounded extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$ , then the validity problem of  $L$  can be reduced to the validity problem of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  in polynomial time.

*Proof.* In both cases, let  $\mathcal{A}$  be the set of axioms witnessing the boundedness property.

(i) Given a formula  $F$ , let  $A_1(F), \dots, A_n(F)$  be an enumeration of *all* axiom instances in  $\psi_s(\mathcal{A}, F)$  (without repetition). By set-boundedness and contraction, it follows that the following equivalence holds:

$$F \in L \iff (A_1(F) \wedge 1) * \dots * (A_n(F) \wedge 1) \rightarrow F \in \text{Thm}(\mathbf{FL}_{e^*}^\perp)$$

The function  $\pi$  mapping a formula  $F$  to the formula  $(A_1(F) \wedge 1) * \dots * (A_n(F) \wedge 1) \rightarrow F$  is therefore a reduction of  $L$  to  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$ , and the time it takes to compute  $\pi$  roughly corresponds to the size of the set  $\psi_s(\mathcal{A}, F)$ , which in turn is polynomial in the number  $N$  of sets of subformulas of  $F$  (regarding the set  $\mathcal{A}$  as a constant parameter). Since  $N \approx 2^{|F|}$ , we conclude that  $\pi$  is computable in exponential time.

(ii) is similar, but now we observe that the size of  $\psi_f(\mathcal{A}, F)$ —and therefore the time it takes to compute the reduction—is polynomial in  $|F|$ .  $\square$

The *mingle rule*

$$\frac{\Gamma_1, \Sigma \Rightarrow \Pi \quad \Gamma_2, \Sigma \Rightarrow \Pi}{\Gamma_1, \Gamma_2, \Sigma \Rightarrow \Pi} \text{ (m)}$$

is a generalization of the weakening rule. In the presence of contraction and mingle, set-boundedness and multiset-boundedness are the same:

**Proposition 2.6**

Assume that contraction and mingle are admissible in  $\mathbf{FL}_{e^*}^\perp$ . Then any multiset-bounded extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  is also set-bounded.

*Proof.* In the presence of contraction and mingle, we can prove  $B \equiv B * B$ . By a simple induction it follows that

$$\vdash_{\mathbf{FL}_{e^*}^\perp} A(B * B) \equiv A(B)$$

for every formula  $A(p)$ . Hence in any occurrence of a fusion in a formula, multiple occurrences of the same factor can be removed. It follows that any multiset-bounded extension of  $\mathbf{FL}_{e^*}^\perp$  is also set-bounded.  $\square$



## 2.3 Analytic Structural Hypersequent Rules and Disjunction Forms

We now develop a formal notion of a *disjunction form* for an *analytic structural hypersequent rule*. Loosely speaking, a disjunction form is a formula capturing the content of a hypersequent rule, in the way that  $(\bigwedge \Gamma_2 \rightarrow \bigwedge \Gamma_1) \vee (\bigwedge \Gamma_1 \rightarrow \bigwedge \Gamma_2)$  captures (com). Disjunction forms are refinements of the generic interpretations of hypersequent rules (see Definition 6 in [7]).

We first state a definition of analytic structural hypersequent rules (which have been mentioned already a couple of times). The definition is essentially that of the *completed rules* in [17] in a slightly adapted form. We now have to be rigorous about the difference between the schematic presentation of a rule on the one hand, and concrete instances of the rule on the other hand. When we write down a rule such as

$$\frac{G \mid \Gamma, A \Rightarrow \Pi \quad G \mid \Gamma, B \Rightarrow \Pi}{G \mid \Gamma, A \vee B \Rightarrow \Pi} (\vee_1)$$

as part of a calculus, then the occurring symbols  $G, \Gamma, A, B$  and  $\Pi$  are taken as *structure variables* of different types. Here  $G$  is a hypersequent-variable,  $\Gamma$  is a multiset-variable,  $A$  and  $B$  are formula-variables and  $\Pi$  is a variable for a multiset of size  $\leq 1$ . In an instance  $\sigma(\vee_1)$  of the rule (which occurs in a proof), the variables are substituted by concrete objects of the correct type: That is,  $\sigma(\vee_1)$  is of the form

$$\frac{\sigma(G) \mid \sigma(\Gamma), \sigma(A) \Rightarrow \sigma(\Pi) \quad \sigma(G) \mid \sigma(\Gamma), \sigma(B) \Rightarrow \sigma(\Pi)}{\sigma(G) \mid \sigma(\Gamma), \sigma(A) \vee \sigma(B) \Rightarrow \sigma(\Pi)} (\vee_1)$$

where  $\sigma(G)$  is a hypersequent,  $\sigma(\Gamma)$  is a multiset of formulas and so on.

### Definition 2.7 (analytic structural (hyper)sequent rule)

An analytic structural hypersequent rule is a rule of the form

$$\frac{G \mid T_1 \quad \dots \quad G \mid T_l}{G \mid S_1 \mid \dots \mid S_n} (r)$$

where the  $S_i$ 's and  $T_j$ 's are sequents built from structure variables (called the active or principal components of the rule),  $G$  is a hypersequent-variable, and the following conditions are obeyed:

**structurality** The principal components contain only multiset-variables.

**analyticity** Each multiset-variable occurring in a premise component also appears in the conclusion.

**linear conclusion** No multiset-variable occurs more than once in the conclusion.

**separation** No multiset-variable appears both in an antecedent and in a succedent.

**coupling** If some  $S_i$  has a nonempty succedent, that is,  $S_i$  is of the form  $S \Rightarrow \Pi_i$ , then there is a ‘coupled’ multiset-variable  $\Sigma_i \in \mathcal{S}$  which appears exactly in those premises which have  $\Pi_i$  as their succedent, and therein with multiplicity 1.

An analytic structural *sequent* rule is defined similarly: We only remove the hypersequent-variable  $G$  and set  $n = 1$ .

### Example 2.8

The following are three examples of structural analytic hypersequent rules.

- First, we have already seen the communication rule

$$\frac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} \text{ (com)}$$

which yields a calculus for  $\mathbf{G}$  if added to  $\mathbf{HLI}$ . The coupled pairs of variables are  $(\Sigma_1, \Pi_1)$  and  $(\Sigma_2, \Pi_2)$ .

- If added to  $\mathbf{HLI}$ , the rule

$$\frac{G \mid \Gamma, \Delta \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Delta \Rightarrow} \text{ (lq)}$$

yields a calculus for *Jankov logic*  $\mathbf{IL} + \neg A \vee \neg\neg A$  [16]. There are no coupled variables since both succedents of the conclusion are empty.

- For any integers  $k, m \geq 1$ ,

$$\frac{\{G \mid \Gamma_{i_1}, \dots, \Gamma_{i_m}, \Sigma \Rightarrow \Pi\}_{i_1, \dots, i_m \leq n}}{G \mid \Gamma_1, \dots, \Gamma_n, \Sigma \Rightarrow \Pi} \text{ (knot}_m^n\text{)}$$

is a structural analytic hypersequent<sup>5</sup> rule. If added to  $\mathbf{HFL}_e^\perp$ , it characterizes the logic  $\text{Thm}(\mathbf{FL}_e^\perp) + \underbrace{A * \dots * A}_n \rightarrow \underbrace{A * \dots * A}_m$  [28]. The variables  $\Sigma$  and  $\Pi$  are coupled.

We have the following theorem of [17]:

<sup>5</sup>Here the hypersequent structure is not needed, that is, we might as well omit  $G \mid$  and add the resulting analytic structural sequent rule to  $\mathbf{FL}_e^\perp$ .

**Theorem 2.9** (Ciabattoni, Galatos and Terui 2008)

Every extension of  $\mathbf{HFL}_e^\perp$  by analytic structural hypersequent rules satisfies the cut elimination theorem.

Let us call a logic *hyper-amenable* if it is captured by a hypersequent calculus  $\mathbf{HFL}_{e^*}^\perp + \mathcal{R}$  where  $\mathcal{R}$  is a set of analytic structural hypersequent rules. [17] contains a syntactic criterion for an axiomatic extension  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  to be hyper-amenable based on the *substructural hierarchy*: Let  $\mathcal{P}_0 = \mathcal{N}_0 = \mathcal{V}$  and inductively define classes of formulas as follows:

$$\begin{aligned}\mathcal{P}_{n+1} &:= 1 \mid \perp \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} * \mathcal{P}_{n+1} \\ \mathcal{N}_{n+1} &:= 0 \mid \top \mid \mathcal{P}_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1}\end{aligned}$$

Then any axiomatic extension  $\mathbf{FL}_{e^w}^\perp + \mathcal{A}$  where  $\mathcal{A} \subseteq \mathcal{P}_3$  is hyper-amenable [17]. Moreover the analytic structural hypersequent rules witnessing hyper-amenableity can be computed from the set  $\mathcal{A}$ . A similar statement holds for axiomatic extensions of  $\mathbf{FL}_e^\perp$ , but the hierarchy has to be slightly adapted. For our work the details of [17] do not matter much, and the only thing which henceforth will be used is the definition of analytic structural hypersequent rules and Theorem 2.9.

We can now formally state our main result, which will be proved in Section 2.6.

**Theorem**

Every hyper-amenable axiomatic extension of  $\mathbf{Thm}(\mathbf{FL}_{e^*}^\perp)$  is multiset-bounded.

Next, we come to the definition of a disjunction form. We first assign to every multiset-variable  $\Gamma$  a new *propositional* variable  $\hat{\Gamma}$ . Every instantiation  $\sigma$  of the rule (r) can then be extended to a variable substitution  $\hat{\sigma}$  by setting

$$\hat{\sigma}(\hat{\Gamma}) := \odot \sigma(\Gamma)$$

where  $\odot\{A_1, \dots, A_n\} := A_1 * \dots * A_n$  and  $\odot\emptyset := 1$  (as in the case of the  $\wedge$  notation, we implicitly use the fact that  $*$  is associative and commutative). Finally we will use the following notation  $\#$  for adding a formula to the antecedent of a sequent:

$$A\#(\Gamma \Rightarrow \Pi) := (\Gamma, A \Rightarrow \Pi)$$

**Definition 2.10** (Disjunction form)

Let an analytic structural hypersequent rule (r) be given as in Definition 2.7, and let  $\{\Gamma_i \mid i \in I\}$  be an enumeration of all multiset-variables occurring in the active components of (r). The formula  $A_1 \vee \dots \vee A_n$  containing only variables  $\hat{\Gamma}_i$  ( $i \in I$ ) is called a *disjunction form* of (r) over  $\mathbf{FL}_{e^*}^\perp$  if the following are satisfied:

**(soundness)**  $A_1 \vee \dots \vee A_n$  is a theorem of  $\mathbf{HFL}_{e^*}^\perp + (r)$ ;

**(splitting)** For every instantiation  $\sigma$  of  $(r)$  and every  $i \leq n$

$$\{\sigma(T_1), \dots, \sigma(T_i)\} \vdash_{\mathbf{FL}_{e^*}^\perp} \hat{\sigma}(A_i) \# \sigma(S_i);$$

**(weakening)** Each  $A_i$  is weakenable over  $\mathbf{FL}_{e^*}^\perp$ , that is,  $\vdash_{\mathbf{FL}_{e^*}^\perp} A \Rightarrow 1$ .

The (*weakening*) property is a technicality. If  $\mathbf{FL}_{e^*}^\perp$  contains the weakening rule, then every formula is weakenable. Otherwise, examples of weakenable formulas are  $A \wedge 1$  and  $(A \wedge 1) \vee (B \wedge 1)$ . Note that in particular any disjunction form is weakenable.

The (*splitting*) property bears its name because it asserts that every instance of the hypersequent rule  $(r)$  can be split into  $n$  sequent parts. This is best explained by a picture:

$$\frac{\sigma(T_1) \quad \dots \quad \sigma(T_n)}{\sigma(S_1) \mid \dots \mid \sigma(S_n)} (r) \rightsquigarrow \left\{ \begin{array}{c} \{\sigma(T_1), \dots, \sigma(T_i)\} \\ \vdots \\ \mathbf{FL}_{e^*}^\perp \\ \hat{\sigma}(A_i) \# \sigma(S_i) \end{array} , \dots , \left\{ \begin{array}{c} \{\sigma(T_1), \dots, \sigma(T_n)\} \\ \vdots \\ \mathbf{FL}_{e^*}^\perp \\ \hat{\sigma}(A_n) \# \sigma(S_n) \end{array} \right\} \right\}$$

In fact, in all cases of rules that we consider later on it would be possible to also split the set  $\{\sigma(T_1), \dots, \sigma(T_i)\}$  into  $n$  parts, some possibly empty, so that only the premises from the  $i$ -the part are required in the  $\mathbf{FL}_{e^*}^\perp$ -derivation of  $\sigma(S_i) \# \hat{\sigma}(A_i)$ . However, we have no use for this property and therefore omit it.

### Example 2.11

The formula  $(\hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1) \vee (\hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2)$  is a disjunction form of

$$\frac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} (\text{com})$$

over **LI**. (*Soundness*) is witnessed by the **HLI** + (com)-derivation

$$\begin{array}{c} \frac{\hat{\Gamma}_1 \Rightarrow \hat{\Gamma}_1 \quad \hat{\Gamma}_2 \Rightarrow \hat{\Gamma}_2}{\hat{\Gamma}_2 \Rightarrow \hat{\Gamma}_1 \mid \hat{\Gamma}_1 \Rightarrow \hat{\Gamma}_2} (\text{com}) \\ \frac{\hat{\Gamma}_2 \Rightarrow \hat{\Gamma}_1 \mid \hat{\Gamma}_1 \Rightarrow \hat{\Gamma}_2}{\Rightarrow \hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1 \mid \Rightarrow \hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2} (\rightarrow_r) \\ \frac{\Rightarrow \hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1 \mid \Rightarrow \hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2}{\Rightarrow (\hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1) \vee (\hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2)} (\text{ec}), (\vee_r) \end{array}$$

while (*splitting*) is witnessed by the two LI-derivations

$$\frac{\frac{\{A \Rightarrow A\}_{A \in \sigma(\Gamma_2)}}{\sigma(\Gamma_2) \Rightarrow \hat{\sigma}(\hat{\Gamma}_2)} (*_r) \quad \frac{\sigma(\Sigma_1), \sigma(\Gamma_1) \Rightarrow \sigma(\Pi_1)}{\sigma(\Sigma_1), \hat{\sigma}(\hat{\Gamma}_1) \Rightarrow \sigma(\Pi_1)} (*_l)/(1_l)^6}{\sigma(\Sigma_1), \sigma(\Gamma_2), \hat{\sigma}(\hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1) \Rightarrow \sigma(\Pi_1)} (\rightarrow_l)$$

and

$$\frac{\frac{\{A \Rightarrow A\}_{A \in \sigma(\Gamma_1)}}{\sigma(\Gamma_1) \Rightarrow \hat{\sigma}(\hat{\Gamma}_1)} (*_r) \quad \frac{\sigma(\Sigma_2), \sigma(\Gamma_2) \Rightarrow \sigma(\Pi_2)}{\sigma(\Sigma_2), \hat{\sigma}(\hat{\Gamma}_2) \Rightarrow \sigma(\Pi_2)} (*_l)/(1_l)^6}{\sigma(\Sigma_2), \sigma(\Gamma_1), \hat{\sigma}(\hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2) \Rightarrow \sigma(\Pi_2)} (\rightarrow_l)$$

Finally (*weakening*) is trivial because the weakening rule is part of LI.

## 2.4 Obtaining Multiset-Boundedness

In this section we set out to prove the following: If  $\mathbf{HFL}_{e^*}^\perp + \mathcal{R}$  is an extension of  $\mathbf{HFL}_{e^*}^\perp$  by analytic structural hypersequent rules and every rule in  $\mathcal{R}$  has a disjunction form, then—letting  $\mathcal{A}$  denote the set of disjunction forms—we have  $\text{Thm}(\mathbf{HFL}_{e^*}^\perp + \mathcal{R}) = \text{Thm}(\mathbf{FL}_{e^*}^\perp + \mathcal{A})$  and  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  is multiset-bounded. In the subsequent Section 2.5 we will confirm that indeed *all* analytic structural rules have a disjunction form.

The argument for establishing multiset-boundedness follows the outline in Section 2.1. We discuss two techniques beforehand, so that later on in the main proof we can better focus on the essential ideas.

The first simple technique will be paraphrased by ‘adding a formula, and propagating it downwards’. We start with an  $\mathbf{HFL}_{e^*}^\perp$ -proof from some assumption  $G \mid S$  of a sequent  $T$ :

$$\begin{array}{c} G \mid S \\ \vdots \delta \\ \hline T \end{array}$$

We now want to add a formula  $A$  to the antecedent of  $S$  in the assumption  $G \mid S$  as well as in ‘the appropriate parts’ of  $\delta$  to obtain a proof of  $A\#T$  from the assumption  $G \mid A\#S$ :

$$\begin{array}{c} G \mid A\#S \\ \vdots \delta' \\ \hline A\#T \end{array}$$

To make this work, we will have to require that  $\mathbf{HFL}_{e^*}^\perp$  contains the rule  $(w_l)$ , or if not, that at least the formula  $A$  is weakenable over  $\mathbf{FL}_{e^*}^\perp$ . We sketch a formal presentation of

<sup>6</sup>If  $\sigma(\Gamma_i) = \emptyset$ , then by definition  $\hat{\sigma}(\hat{\Gamma}_i) = 1$  and therefore the rule  $(1_l)$  must be used.

the method. First, one defines the notion of a *successor component* in an  $\mathbf{HFL}_{e^*}^\perp$ -proof as follows: The successor component of a principal component in the premises of a rule instance is the principal component in the conclusion, and the successor component of a side component is the corresponding side component in the conclusion. A component is a *descendant* of another component if they are related by the transitive reflexive closure of the successor relation. Then as a first approximation  $\delta'$  should be like  $\delta$ , but with  $A$  added into the antecedent of all successor components of  $S$ . There is an issue when additive rules are encountered in  $\delta$ , for example:

$$\frac{\begin{array}{c} \vdots \\ H \mid \Gamma \Rightarrow B \end{array} \quad \begin{array}{c} G \mid S \\ \vdots \delta_1 \\ H \mid \Gamma \Rightarrow C \end{array}}{H \mid \Gamma \Rightarrow B \wedge C} (\wedge_r)$$

where  $\Gamma \Rightarrow C$  is the successor component of  $S$ . If  $\mathbf{HFL}_{e^*}^\perp$  contains the weakening rule, we should proceed as follows:

$$\frac{\begin{array}{c} \vdots \\ H \mid \Gamma \Rightarrow B \end{array} \quad \begin{array}{c} G \mid A\#S \\ \vdots \delta'_1 \\ H \mid \Gamma, A \Rightarrow C \end{array}}{H \mid \Gamma, A \Rightarrow B \wedge C} (\wedge_r)$$

In absence of the weakening rule, weakenability of  $A$  permits a similar modification:

$$\frac{\begin{array}{c} A \Rightarrow 1 \\ \frac{H \mid \Gamma \Rightarrow B}{H \mid \Gamma, 1 \Rightarrow B} (1_l) \end{array} \quad \begin{array}{c} G \mid A\#S \\ \vdots \delta'_1 \\ H \mid \Gamma, A \Rightarrow C \end{array}}{\frac{H \mid \Gamma, A \Rightarrow B}{H \mid \Gamma, A \Rightarrow B \wedge C} (\text{cut})} (\wedge_r)$$

Similar simple amendments have to be done when one encounters the rule  $(ec)$ , and when the successor component appears as the side component of a rule in  $\delta$ .

Second, we need a generalization of the rule  $(\vee_1)$  which encompasses instances as the following:

$$\frac{\Gamma, A_1^1, A_1^2 \Rightarrow \Pi \quad \Gamma, A_1^1, A_2^2 \Rightarrow \Pi \quad \Gamma, A_2^1, A_1^2 \Rightarrow \Pi \quad \Gamma, A_2^1, A_2^2 \Rightarrow \Pi}{\Gamma, A_1^1 \vee A_2^1, A_1^2 \vee A_2^2 \Rightarrow \Pi}$$

The general statement and proof goes as follows:

**Lemma 2.12**

Let  $A_1 = A_1^1 \vee \dots \vee A_{n_1}^1, \dots, A_k = A_1^k \vee \dots \vee A_{n_k}^k$  be a list of  $k$  disjunctive formulas, and let  $\Omega$  be the set of functions  $f : \{1, \dots, k\} \rightarrow \{1, 2, \dots\}$  satisfying  $f(i) \leq n_i$ . Then for any sequent  $S$ ,

$$\{\{A_{f(1)}^1, \dots, A_{f(k)}^k\} \# S \mid f \in \Omega\} \vdash_{\text{FL}_e^\perp} \{A_1, \dots, A_k\} \# S.$$

*Proof.* By induction on  $k$ . The base case  $k = 1$  is easy, so assume  $k > 1$ . For any fixed  $f \in \Omega$  we have

$$\{\{A_p^1, A_{f(2)}^2, \dots, A_{f(k)}^k\} \# S \mid 1 \leq p \leq n_1\} \vdash_{\text{FL}_e^\perp} \{A_1, A_{f(2)}^2, \dots, A_{f(k)}^k\} \# S.$$

via  $n_1$ -many applications of  $(\vee_1)$ . Hence to show the Lemma, it suffices to establish that

$$\{\{A_1, A_{f(2)}^2, \dots, A_{f(k)}^k\} \# S \mid f \in \Omega\} \vdash_{\text{FL}_e^\perp} \{A_1, A_2, \dots, A_k\} \# S$$

and this holds by induction hypothesis.  $\square$

As a side remark, the total number  $N$  of  $(\vee_1)$ 's used in the above proof is rather big. An elementary calculation shows

$$N = \sum_{j=1}^k \prod_{i=1}^j n_i.$$

If all  $n_i$ 's are a constant  $n \neq 1$ , this is a geometric expressions with value  $N = \frac{n^{k+1} - n}{n-1}$ .

We now state and prove the main theorem of this chapter.

**Theorem 2.13** (projection theorem)

Let  $\text{HFL}_{e^*}^\perp + \mathcal{R}$  be an extension of  $\text{HFL}_{e^*}^\perp$  by analytic structural hypersequent rules and suppose that every rule in  $\mathcal{R}$  has a disjunction form over  $\text{FL}_{e^*}^\perp$ . Denote this set of disjunction forms by  $\mathcal{A}$ . Then  $\text{Thm}(\text{HFL}_{e^*}^\perp + \mathcal{R}) = \text{Thm}(\text{FL}_{e^*}^\perp) + \mathcal{A}$  and  $\text{FL}_{e^*}^\perp + \mathcal{A}$  is multiset-bounded.

*Proof.* We first argue that it suffices to show the following claim:

( $\heartsuit$ ) Every cutfree  $\text{HFL}_{e^*}^\perp + \mathcal{R}$ -derivation  $\delta$  of  $\Rightarrow F$  can be transformed into a  $\text{FL}_{e^*}^\perp$ -derivation of  $B_1, \dots, B_m \Rightarrow F$  where for all  $i \leq m$ ,  $B_i \in \psi_{m_S}(\mathcal{A}, F)$ .

By Theorem 2.9 we know that every theorem of  $\text{HFL}_{e^*}^\perp + \mathcal{R}$  has a cutfree proof. Together with ( $\heartsuit$ ) this implies the inclusion  $\text{Thm}(\text{HFL}_{e^*}^\perp + \mathcal{R}) \subseteq \text{Thm}(\text{FL}_{e^*}^\perp + \mathcal{A})$ . On the other hand,  $\text{Thm}(\text{HFL}_{e^*}^\perp + \mathcal{R}) \supseteq \text{Thm}(\text{FL}_{e^*}^\perp + \mathcal{A})$  follows from the (*soundness*) property of disjunction forms. Then knowing that  $\text{Thm}(\text{HFL}_{e^*}^\perp + \mathcal{R}) = \text{Thm}(\text{FL}_{e^*}^\perp + \mathcal{A})$ , ( $\heartsuit$ ) describes

the property of multiset-boundedness for the extension  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  (Definition 2.2). Here we do not need the  $\wedge 1$ 's because by the definition of disjunction forms, all the  $B_i$ 's are weakenable.

So let us prove ( $\heartsuit$ ). The general strategy is to repeatedly replace instances of  $\mathcal{R}$ -rules in  $\delta$  by axioms in  $\mathcal{A}$ , using the properties of disjunction forms. Since the elimination of each  $\mathcal{R}$ -rule will entail a duplication of parts of the derivation and hence might introduce new instances of rules in  $\mathcal{R}$ , we eliminate all lowermost  $\mathcal{R}$ -rules in  $\delta$  *simultaneously*. In doing so, we ensure that after each reduction step, the maximal number of  $\mathcal{R}$ -instances on a branch in the proof—henceforth called the  $\mathcal{R}$ -rank of the derivation—decreases, and so the whole procedure terminates.

Let  $\sigma_1(r_1), \dots, \sigma_k(r_k)$  be the lowermost  $\mathcal{R}$ -instances in  $\delta$ . Assume  $\sigma_i(r_i)$  is:

$$\frac{\sigma_i(G) \mid \sigma_i(T_1) \quad \dots \quad \sigma_i(G) \mid \sigma_i(T_{n_i})}{\sigma_i(G) \mid \sigma_i(S_1) \mid \dots \mid \sigma(S_{n_i})} \sigma_i(r_i)$$

By assumption, there is a disjunction form  $A_i = A_1^i \vee \dots \vee A_{n_i}^i \in \mathcal{A}$  for  $r_i$ . Recall that  $A_i$  is built from variables  $\hat{\Gamma}$  where  $\Gamma$  is a multiset-variable in the rule  $r_i$ , and that the substitution  $\hat{\sigma}_i$  is defined by setting  $\hat{\sigma}_i(\hat{\Gamma}) := \odot \sigma_i(\Gamma)$ .

From the subformula property of the derivation  $\delta$  it follows that

(\*) every instance of an  $\mathcal{R}$ -rule instantiates its multiset-variables with a multiset of subformulas of  $F$

and consequently,  $\hat{\sigma}_i(A_i) \in \psi_{ms}(\mathcal{A}, F)$ . By the (*splitting*) property of the disjunction form we can eliminate the instance  $\sigma_i(r_i)$  by introducing a formula  $\hat{\sigma}_i(A_j^i)$  to the antecedent of the  $j$ th principal component in the conclusion ( $j \leq n_i$ ). This gives us  $n_i$  ways of eliminating  $\sigma_i(r_i)$ .

Hence, in order to eliminate all lowermost instances  $\sigma_1(r_1), \dots, \sigma_k(r_k)$ , we have to make  $n_1 \cdot \dots \cdot n_k$  many choices. We may encode every such combined choice by a function in the set

$$\Omega := \{f : \{1, \dots, k\} \rightarrow \mathbb{N} \mid \forall i (f(i) \leq n_i)\}.$$

We fix one  $f \in \Omega$  and formally describe a transformation  $f(\delta)$  of the original proof  $\delta$  which could be paraphrased by ‘for all  $i \leq k$  eliminate  $\sigma_i(r_i)$  by adding  $\hat{\sigma}_i(A_{f(i)}^i)$ ’. Indeed we simultaneously replace all instances  $\sigma_i(r_i)$  in  $\delta$ , for  $i \leq k$ , by

$$\frac{\begin{array}{c} \{\sigma_i(G) \mid \sigma_i(T_1), \dots, \sigma_i(G) \mid \sigma_i(T_{n_i})\} \\ \vdots \\ \mathbf{FL}_{e^*}^\perp \\ \sigma_i(G) \mid \hat{\sigma}_i(A_{f(i)}^i) \# \sigma_i(S_{f(i)}) \end{array}}{\sigma_i(G) \mid \sigma_i(S_1) \mid \dots \mid \hat{\sigma}_i(A_{f(i)}^i) \# \sigma_i(S_{f(i)}) \mid \dots \mid \sigma(S_{n_i})} \text{ (ew)}.$$

Here, the dotted line indicates the  $\mathbf{FL}_{e^*}^\perp$ -derivation guaranteed by the (*splitting*) property; the side hypersequent  $\sigma_i(G)$  is simply appended to all sequents in this derivation. Next,



the formula  $\hat{\sigma}_i(A_{i,f(i)})$  is propagated downwards until the endsequent (for details of this step, see the discussion preceding this proof). This propagation is done simultaneously for all  $i < k$ . We obtain a derivation of

$$\hat{\sigma}_1(A_{f(1)}^1), \dots, \hat{\sigma}_k(A_{f(k)}^k) \Rightarrow F.$$

Call this derivation  $f(\delta)$ , and note that its  $\mathcal{R}$ -rank is smaller than that of  $\delta$ . Recall that each disjunction form has the shape  $A_i = A_1^i \vee \dots \vee A_{n_i}^i$ . To get the proof  $\Omega(\delta)$  of

$$\hat{\sigma}_1(A_1), \dots, \hat{\sigma}_k(A_k) \Rightarrow F$$

we connect all the proofs  $f(\delta)$  for every  $f \in \Omega$  by repeatedly applying the  $(\vee_1)$ -rule to their conclusion (see Lemma 2.12).

We already remarked that  $\hat{\sigma}_i(A_i) \in \psi_{ms}(\mathcal{A}, F)$ . Furthermore,  $\Omega(\delta)$  still satisfies  $(*)$  because no new  $\mathcal{R}$ -instances have been introduced and every  $\mathcal{R}$ -instance of  $\delta$  has either been eliminated or left unchanged. Lastly, the  $\mathcal{R}$ -rank of  $\Omega(\delta)$  equals the maximal  $\mathcal{R}$ -rank of one of the  $f(\delta)$ 's, and therefore is smaller than the  $\mathcal{R}$ -rank of  $\delta$ . It follows that we can repeat the above transformation to eventually obtain a  $\mathcal{R}$ -free derivation  $\Omega^*(\delta)$  of

$$B_1, \dots, B_m \Rightarrow F$$

where each  $B_i \in \psi_{ms}(\mathcal{A}, F)$ . By Lemma 1.12,  $\Omega^*(\delta)$  can be reduced to a proof in  $\mathbf{FL}_{e*}^\perp$ . This concludes the proof of  $(\heartsuit)$ , and hence the proof of the projection theorem.  $\square$

## 2.5 How to Compute Disjunction Forms

Given a hypersequent system  $\mathbf{HFL}_{e*}^\perp + (r)$  and a disjunction form for  $(r)$ , the projection theorem (Theorem 2.13) tells us that the corresponding axiomatic extension is multiset-bounded. In this section we show that such a disjunction form can indeed always be computed from an analytic structural hypersequent rule.

If one is interested only in one specific calculus, one can simply guess a disjunction form instead of using the somewhat cumbersome algorithm presented here. In this way, we have seen for example that  $(\hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1) \vee (\hat{\Gamma} \rightarrow \hat{\Gamma}_2)$  is a disjunction form of  $(\text{com})$  (see Example 2.11), and thus we conclude by the projection theorem that  $\mathbf{LI} + (\text{lin})$  is multiset-bounded.

We have already mentioned that [17] contains an algorithm turning a certain class of axioms  $A$  into equivalent analytic structural rules  $\mathcal{R}_A$ . One can prove that  $A$  is indeed a disjunctive form of its corresponding rule  $\mathcal{R}_A$ . However the proof of this requires quite a thorough understanding of the algorithm  $A \mapsto \mathcal{R}_A$  (which is not easy to invert). We will take a different route here and construct a disjunction form directly from  $\mathcal{R}_A$ , using only the shape of the rule as input. Thereby we avoid going into the details of [17]. Nevertheless, what we are doing can be seen as an ‘inversion’ of the method presented there.

The price we have to pay for this direct approach is that we cannot prove any more that the disjunction form constructed from  $\mathcal{R}_A$  is indeed  $A$ . Instead, we may end up with a syntactically different, but nevertheless equivalent formula  $A'$ .

In this section we occasionally write  $A_{\wedge 1}$  for  $A \wedge 1$  for brevity. Let an analytic structural hypersequent rule  $(r)$  be given.

As the first step, we select one multiset-variable occurrence in the antecedent of each premise of  $(r)$ . In a premise with non-empty succedent, we shall require that the selected variable occurrence be the one which is coupled with the succedent variable. We call any selected variable occurrence in the premises of  $(r)$ , as well as the unique occurrence of the same variable in the conclusion, a *distinguished variable occurrence* and write it in boldface.

Define a *context function*  $\mathcal{C}(\cdot)$  which assigns to each distinguished variable  $\Gamma$  the collection  $\mathcal{C}(\Gamma)$  of all multisets of variables which occur together with the highlighted occurrences of  $\Gamma$  in the antecedent of a premise.

#### Example 2.14

Consider again the rules from Example 2.8.

- In the communication rule, we select the coupled variables in the antecedent as distinguished:

$$\frac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} \text{ (com)}$$

We have  $\mathcal{C}(\Sigma_1) = \{\{\Gamma_1\}\}$  and  $\mathcal{C}(\Sigma_2) = \{\{\Gamma_2\}\}$ .

- In the  $(lq)$  rule

$$\frac{G \mid \Gamma, \Delta \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Delta \Rightarrow} \text{ (lq)}$$

we select  $\Delta$  and have  $\mathcal{C}(\Delta) = \{\{\Gamma\}\}$ ; Alternatively, we could also choose  $\Gamma$  as distinguished.

- In the rule  $(\text{knot}_m^n)$

$$\frac{\{G \mid \Gamma_{i_1}, \dots, \Gamma_{i_m}, \Sigma \Rightarrow \Pi\}_{i_1, \dots, i_m \leq n}}{G \mid \Gamma_1, \dots, \Gamma_n, \Sigma \Rightarrow \Pi} \text{ (knot}_m^n)$$

the coupled variable  $\Sigma$  is selected and we have

$$\mathcal{C}(\Sigma) = \{\{\Gamma_{i_1}, \dots, \Gamma_{i_m}\} \mid i_1, \dots, i_m \leq n\}.$$

We now switch to a rather dense notation for analytic structural hypersequent rules

in which premises are grouped according to the type of distinguished variable they contain:

1. Distinguished variable occurrences which are coupled will be denoted by the letter  $\Sigma$ .
2. Distinguished variable occurrences which are associated to a conclusion component with non-empty succedent will be denoted by the letter  $\Psi$ .
3. Distinguished variable occurrences which are associated to a conclusion component with empty succedent will be denoted by the letter  $\Delta$ .

(We already followed this convention in Example 2.14.) Furthermore, conclusion components are grouped into those with non-empty and those with empty succedent. Writing  $[S_i]_{i \in I}$  for the hypersequent whose components are  $S_i$  for each  $i \in I$ , the resulting notation of an analytic structural hypersequent rule is:

$$\frac{G \mid [\mathcal{C}, \Sigma_i \Rightarrow \Pi_i]_{\substack{i \in I \\ \mathcal{C} \in \mathcal{C}(\Sigma_i)}} \quad G \mid [\mathcal{C}, \Psi \Rightarrow]_{\substack{i \in I, \Psi \in \vec{\Psi}_i \\ \mathcal{C} \in \mathcal{C}(\Psi)}} \quad G \mid [\mathcal{C}, \Delta \Rightarrow]_{\substack{j \in J, \Delta \in \vec{\Delta}_j \\ \mathcal{C} \in \mathcal{C}(\Delta)}}}{G \mid [\mathcal{C}_i, \vec{\Psi}_i, \Sigma_i \Rightarrow \Pi_i]_{i \in I} \mid [\mathcal{C}_j, \vec{\Delta}_j \Rightarrow]_{j \in J}}$$

We call this the *association form*. Some further words of explanation are required.

- $I$  and  $J$  are disjoint index sets;  $I$  lists the principal conclusion components with nonempty succedent, and  $J$  those with empty succedent. Each of  $I, J$  may be empty, but not both at the same time.
- The symbols  $\vec{\Delta}_j$  and  $\vec{\Psi}_i$  denote multisets of distinguished  $\Psi$ - and  $\Delta$ -occurrences respectively. Both can be empty; In particular, there might be a conclusion component without any distinguished variable.
- $\mathcal{C}_i$  and  $\mathcal{C}_j$  are further multisets of non-distinguished variable occurrences (possibly empty).

For a multiset  $\mathcal{S} = \{\Gamma_1, \dots, \Gamma_n\}$  of multiset-variables, let  $\hat{\mathcal{S}}$  denote the multiset  $\{\hat{\Gamma}_1, \dots, \hat{\Gamma}_n\}$  of propositional variables.

**Definition 2.15** (Form(i))

For a rule (r) in association form, let

$$\text{Form}(i) := \odot \hat{\mathcal{C}}_i * \bigodot_{\Psi \in \vec{\Psi}_i} \left( \hat{\Psi} \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot \hat{\mathcal{C}}) \right) \rightarrow \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot \hat{\mathcal{C}}$$

$$\text{Form}(j) := \neg \left( \odot \hat{\mathcal{C}}_j * \bigodot_{\Delta \in \vec{\Delta}_j} \left( \hat{\Delta} \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot \hat{\mathcal{C}}) \right) \right)$$

Finally, let  $\text{Form}(r) := \bigvee_{k \in I \cup J} \text{Form}(k)_{\wedge 1}$ .

We will shortly prove that  $\text{Form}(r)$  is a disjunction form of (r). Note that not all multiset-variables of (r) appear as  $\hat{\Gamma}$  in  $\text{Form}(r)$ . In particular, there are only antecedent variables.

### Example 2.16

We calculate  $\text{Form}(r)$  for the rules from Example 2.14.

- In the case of

$$\frac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} \text{ (com)}$$

We have  $J = \emptyset$  and  $|I| = 2$ , say  $I = \{1, 2\}$ . Moreover  $\mathcal{C}_1 = \{\Gamma_2\}$  and  $\mathcal{C}_2 = \{\Gamma_1\}$  and  $\vec{\Psi}_1 = \vec{\Psi}_2 = \emptyset$  and so we get

$$\text{Form}(1) = \hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1$$

$$\text{Form}(2) = \hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2$$

and therefore  $\text{Form}(\text{com}) = (\hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1)_{\wedge 1} \vee (\hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2)_{\wedge 1}$ .

- In the case of

$$\frac{G \mid \Gamma, \Delta \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Delta \Rightarrow} \text{ (lq)}$$

we have  $I = \emptyset$  and  $|J| = 2$ , say  $J = \{1, 2\}$ . Moreover  $\mathcal{C}_1 = \{\Gamma\}$ ,  $\vec{\Delta}_1 = \emptyset$ ,  $\mathcal{C}_2 = \emptyset$  and  $\vec{\Delta}_2 = \{\Delta\}$  which implies

$$\text{Form}(1) = \neg(\hat{\Gamma} * 1)$$

$$\text{Form}(2) = \neg(1 * (\hat{\Delta} \wedge \neg \hat{\Gamma}))$$

and therefore  $\text{lq} = (\neg(\hat{\Gamma} * 1))_{\wedge 1} \vee (\neg(1 * (\hat{\Delta} \wedge \neg \hat{\Gamma})))_{\wedge 1}$ .

- In the case of

$$\frac{\{G \mid \Gamma_{i_1}, \dots, \Gamma_{i_m}, \Sigma \Rightarrow \Pi\}_{i_1, \dots, i_m \leq n}}{G \mid \Gamma_1, \dots, \Gamma_n, \Sigma \Rightarrow \Pi} \text{ (knot}_m^n\text{)}$$

we have  $J = \emptyset$  and  $|I| = 1$ , say  $I = \{1\}$ . Moreover  $\mathcal{C}_1 = \{\Gamma_{i_1}, \dots, \Gamma_{i_m}\}$  and  $\vec{\Psi}_1 = \emptyset$  from which we get

$$\text{Form}(1) = \hat{\Gamma}_{i_1} * \dots * \hat{\Gamma}_{i_m} * 1 \rightarrow \bigvee_{i_1, \dots, i_m \leq n} \hat{\Gamma}_{i_1} * \dots * \hat{\Gamma}_{i_m}$$

and so  $\text{Form}(\text{knot}_m^n) = (\hat{\Gamma}_{i_1} * \dots * \hat{\Gamma}_{i_m} * 1 \rightarrow \bigvee_{i_1, \dots, i_m \leq n} \hat{\Gamma}_{i_1} * \dots * \hat{\Gamma}_{i_m}) \wedge 1$ .

### Proposition 2.17

$\text{Form}(r)$  is provable in  $\text{HFL}_e^\perp$ .

*Proof.* We explicitly construct a proof of the sequent  $\Rightarrow \text{Form}(r)$  by applying rules backwards (bottom-up). Start by applying (ec) and  $(\vee_r)$  to obtain the hypersequent

$$[\Rightarrow \text{Form}(i) \wedge 1]_{i \in I} \mid [\Rightarrow \text{Form}(j) \wedge 1]_{j \in J}$$

By applying  $(\wedge_r)$  to each component in the above hypersequent, we obtain a number of premises which contain a ‘trivial’ component  $\Rightarrow 1$  and are therefore provable. What remains is the premise

$$[\Rightarrow \text{Form}(i)]_{i \in I} \mid [\Rightarrow \text{Form}(j)]_{j \in J}.$$

Now using the rules  $(\vee_l)$ ,  $(\rightarrow_r)$  and  $(*_l)$ , this premise is transformed into

$$\left[ \hat{\mathcal{C}}_i, \{\hat{\Psi} \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot \hat{\mathcal{C}}) \mid \Psi \in \vec{\Psi}_i\} \Rightarrow \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot \hat{\mathcal{C}} \right]_{i \in I} \quad (\spadesuit)$$

$$\mid \left[ \hat{\mathcal{C}}_j, \{\hat{\Delta} \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot \hat{\mathcal{C}}) \mid \Delta \in \vec{\Delta}_j\} \Rightarrow \right]_{j \in J}.$$

( $\spadesuit$ ) is an instance of the conclusion

$$G \mid [\mathcal{C}_i, \vec{\Psi}_i, \Sigma_i \Rightarrow \Pi_i]_{i \in I} \mid [\mathcal{C}_j, \vec{\Delta}_j \Rightarrow]_{j \in J}$$

of (r), as described by the below substitution  $\sigma$ :

- $\sigma(G) := \emptyset$

- For  $i \in I$ :  $\sigma(\Sigma_i) := \emptyset$   $\sigma(\Pi_i) := \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot \hat{\mathcal{C}}$
- For  $i \in I, \Lambda \in \mathcal{C}_i$ :  $\sigma(\Lambda) := \hat{\Lambda}$
- For  $i \in I, \Psi \in \vec{\Psi}_i$ :  $\sigma(\Psi) := \hat{\Psi} \wedge \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot \hat{\mathcal{C}}$
- For  $j \in J, \Lambda \in \mathcal{C}_j$ :  $\sigma(\Lambda) := \hat{\Lambda}$
- For  $j \in J, \Delta \in \vec{\Delta}_j$ :  $\sigma(\Delta) := \hat{\Delta} \wedge \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot \hat{\mathcal{C}}$

Note that  $\sigma$  is well-defined, as every variable occurs only once in the conclusion of (r). We now apply  $\sigma(r)$  backwards to ( $\spadesuit$ ). To conclude the proof, it suffices to derive each premise of  $\sigma(r)$  in  $\mathbf{HF}L_e^\perp$ .

1. First, for  $i \in I$  and  $\mathcal{C} \in \mathcal{C}(\Sigma_i)$  consider the premise

$$\mathcal{C}, \Sigma_i \Rightarrow \Pi_i$$

of (r). Under  $\sigma$  this becomes

$$\sigma(\mathcal{C}) \Rightarrow \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot \hat{\mathcal{C}}$$

Now apply  $(\bigvee_r)$  backwards to first obtain  $\sigma(\mathcal{C}) \Rightarrow \odot \hat{\mathcal{C}}$ , and then apply  $(*_r)$  to obtain, for every  $\Lambda \in \mathcal{C}$ , the premise

$$\sigma(\Lambda) \Rightarrow \hat{\Lambda}. \quad (2.1)$$

We now make a case distinction on where the variable  $\Lambda$  occurs in the conclusion of (r). That it does occur there, exactly once, is guaranteed by the properties *analyticity* and *linear conclusion* of (r). By the *coupling* property, we can exclude the case that  $\Lambda$  is one of the  $\Sigma_i$ 's. If  $\Lambda$  occurs in some  $\mathcal{C}_{i'}$  or  $\mathcal{C}_j$ , then by definition of  $\sigma$  we have  $\sigma(\Lambda) = \hat{\Lambda}$ . If on the other hand  $\Lambda$  occurs in  $\vec{\Psi}_{i'}$  or  $\vec{\Delta}_j$ , then  $\sigma(\Lambda) = \hat{\Lambda} \wedge \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Lambda)} \odot \hat{\mathcal{C}}$ . Clearly in both cases (2.1) is provable.

2. For  $i \in I, \Psi \in \vec{\Psi}_i$  and  $\mathcal{C} \in \mathcal{C}(\Psi)$ , the premise

$$\mathcal{C}, \Psi \Rightarrow$$

becomes

$$\sigma(\mathcal{C}), \hat{\Psi} \wedge \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot \hat{\mathcal{C}} \Rightarrow$$

under  $\sigma$ , from which one obtains  $\sigma(\mathcal{C}) \Rightarrow \odot \hat{\mathcal{C}}$  by applying backwards  $(\wedge_l), (\neg_l)$  and  $(\bigvee_r)$ , and then by further application of  $(*_r)$  one obtains for every  $\Lambda \in \mathcal{C}$  the sequent

$$\sigma(\Lambda) \Rightarrow \hat{\Lambda}.$$

By the same case distinction as in the first case, this sequent is provable.

3. The provability of the premise  $\mathcal{C}, \Delta \Rightarrow$  where  $j \in J, \Delta \in \vec{\Delta}_j$  and  $\mathcal{C} \in \mathcal{C}(\Delta)$  under  $\sigma$  is shown as in the second case.  $\square$

**Proposition 2.18**

$\text{Form}(r)$  satisfies the (*splitting*) property of disjunction forms (Definition 2.10).

*Proof.* Let  $\sigma(r)$  be an instance of  $(r)$ . We have to show that the  $\sigma$ -instance of any principal conclusion component, appended with the appropriate  $\hat{\sigma}$ -instance of formula  $\text{Form}(r)_{\wedge 1}$  or  $\text{Form}(r)_{\vee 1}$ , is provable from  $\sigma$ -instances of the principal premise components of  $(r)$ .

For  $i \in I$  we first demonstrate the provability of the sequent

$$\sigma(\mathcal{C}_i), \sigma(\vec{\Psi}_i), \sigma(\Sigma_i), \hat{\sigma}(\text{Form}(r)_{\wedge 1}) \Rightarrow \sigma(\Pi_i). \quad (2.2)$$

Recall that for a multiset-variable  $\Gamma$ , the value  $\sigma(\hat{\Gamma})$  is defined as  $\odot\sigma(\Gamma)$ . We therefore have for  $i \in I$

$$\hat{\sigma}(\text{Form}(r)) = \odot\sigma(\mathcal{C}_i) * \odot \left( \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})) \right) \rightarrow \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot\sigma(\mathcal{C})$$

and for  $j \in J$

$$\hat{\sigma}(\text{Form}(r)) = \neg \left( \odot\sigma(\mathcal{C}_j) * \odot \left( \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})) \right) \right).$$

The derivations witnessing the splitting property are pictured in Figures 2.1 and 2.2.  $\square$

$$\begin{array}{c}
 \hat{\delta}(\text{Form}(r)) = \odot\sigma(\mathcal{C}_i) * \odot_{\Psi \in \vec{\Psi}_i} \left( \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})) \right) \rightarrow \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot\sigma(\mathcal{C}) \\
 \\
 \begin{array}{c}
 \text{premise component of } \sigma(r) \\
 \frac{\{\sigma(\Psi), \sigma(\mathcal{C}) \Rightarrow\}_{\Psi \in \vec{\Psi}_i, \mathcal{C} \in \mathcal{C}(\Psi)}}{\{\sigma(\Psi), \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Psi \in \vec{\Psi}_i, \mathcal{C} \in \mathcal{C}(\Psi)}} (*_l) \\
 \frac{\{\sigma(\Psi), \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Psi \in \vec{\Psi}_i, \mathcal{C} \in \mathcal{C}(\Psi)}}{\{\sigma(\Psi), \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Psi \in \vec{\Psi}_i}} (\bigvee_l) \\
 \frac{\{\sigma(\Psi), \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Psi \in \vec{\Psi}_i}}{\{\sigma(\Psi) \Rightarrow \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})\}_{\Psi \in \vec{\Psi}_i}} (\neg_r) \\
 \frac{\{\sigma(\Psi) \Rightarrow \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})\}_{\Psi \in \vec{\Psi}_i}}{\{\sigma(\Psi) \Rightarrow \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C}))\}_{\Psi \in \vec{\Psi}_i}} (\wedge_r) \\
 \frac{\{\sigma(\Psi) \Rightarrow \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C}))\}_{\Psi \in \vec{\Psi}_i}}{\sigma(\vec{\Psi}_i) \Rightarrow \odot_{\Psi \in \vec{\Psi}_i} \left( \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})) \right)} (*_r) \\
 \frac{\sigma(\mathcal{C}_i) \Rightarrow \odot\sigma(\mathcal{C}_i) \quad \sigma(\vec{\Psi}_i) \Rightarrow \odot_{\Psi \in \vec{\Psi}_i} \left( \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})) \right)}{\sigma(\mathcal{C}_i), \sigma(\vec{\Psi}_i) \Rightarrow \odot\sigma(\mathcal{C}_i) * \odot_{\Psi \in \vec{\Psi}_i} \left( \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})) \right)} (*_r) \\
 \frac{\sigma(\mathcal{C}_i), \sigma(\vec{\Psi}_i) \Rightarrow \odot\sigma(\mathcal{C}_i) * \odot_{\Psi \in \vec{\Psi}_i} \left( \odot\sigma(\Psi) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Psi)} \odot\sigma(\mathcal{C})) \right)}{\sigma(\mathcal{C}_i), \sigma(\vec{\Psi}_i), \sigma(\Sigma_i), \hat{\delta}(\text{Form}(r)) \Rightarrow \sigma(\Pi_i)} (\wedge_l) \\
 \frac{\sigma(\mathcal{C}_i), \sigma(\vec{\Psi}_i), \sigma(\Sigma_i), \hat{\delta}(\text{Form}(r)) \wedge 1 \Rightarrow \sigma(\Pi_i)}{\sigma(\mathcal{C}_i), \sigma(\vec{\Psi}_i), \sigma(\Sigma_i), \hat{\delta}(\text{Form}(r)) \Rightarrow \sigma(\Pi_i)} (\wedge_l)
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \text{premise component of } \sigma(r) \\
 \frac{\{\sigma(\Sigma_i), \sigma(\mathcal{C}) \Rightarrow \sigma(\Pi_i)\}_{\mathcal{C} \in \mathcal{C}(\Sigma_i)}}{\{\sigma(\Sigma_i), \odot\sigma(\mathcal{C}) \Rightarrow \sigma(\Pi_i)\}_{\mathcal{C} \in \mathcal{C}(\Sigma_i)}} (*_l) \\
 \frac{\{\sigma(\Sigma_i), \odot\sigma(\mathcal{C}) \Rightarrow \sigma(\Pi_i)\}_{\mathcal{C} \in \mathcal{C}(\Sigma_i)}}{\sigma(\Sigma_i), \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot\sigma(\mathcal{C}) \Rightarrow \sigma(\Pi_i)} (\bigvee_l) \\
 \frac{\sigma(\Sigma_i), \bigvee_{\mathcal{C} \in \mathcal{C}(\Sigma_i)} \odot\sigma(\mathcal{C}) \Rightarrow \sigma(\Pi_i)}{\sigma(\Sigma_i), \sigma(\vec{\Psi}_i), \sigma(\Sigma_i), \hat{\delta}(\text{Form}(r)) \Rightarrow \sigma(\Pi_i)} (\rightarrow_l)
 \end{array}$$

Figure 2.1: The (*splitting*) property for  $i \in I$



$$\begin{array}{c}
 \hat{\delta}(\text{Form}(\mathbf{r})) = \neg \left( \odot\sigma(\mathcal{C}_j) * \odot_{\Delta \in \vec{\Delta}_j} \left( \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})) \right) \right) \\
 \\
 \begin{array}{c}
 \text{premise component of } \sigma(\mathbf{r}) \\
 \frac{\{\sigma(\Delta), \sigma(\mathcal{C}) \Rightarrow\}_{\Delta \in \vec{\Delta}_i, \mathcal{C} \in \mathcal{C}(\Delta)}}{\{\sigma(\Delta), \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Delta \in \vec{\Delta}_i, \mathcal{C} \in \mathcal{C}(\Delta)}} \quad (*_l) \\
 \frac{\{\sigma(\Delta), \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Delta \in \vec{\Delta}_i, \mathcal{C} \in \mathcal{C}(\Delta)}}{\{\sigma(\Delta), \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Delta \in \vec{\Delta}_i}} \quad (\bigvee_l) \\
 \frac{\{\sigma(\Delta), \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C}) \Rightarrow\}_{\Delta \in \vec{\Delta}_i}}{\{\sigma(\Delta) \Rightarrow \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})\}_{\Delta \in \vec{\Delta}_i}} \quad (\neg_r) \\
 \frac{\{\sigma(\Delta) \Rightarrow \neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})\}_{\Delta \in \vec{\Delta}_i}}{\{\sigma(\Delta) \Rightarrow \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C}))\}_{\Delta \in \vec{\Delta}_i}} \quad (\wedge_r) \\
 \frac{\{\sigma(\Delta) \Rightarrow \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C}))\}_{\Delta \in \vec{\Delta}_i}}{\sigma(\vec{\Delta}_i) \Rightarrow \odot_{\Delta \in \vec{\Delta}_i} \left( \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})) \right)} \quad (*_r) \\
 \frac{\sigma(\mathcal{C}_j) \Rightarrow \odot\sigma(\mathcal{C}_j) \quad \sigma(\vec{\Delta}_i) \Rightarrow \odot_{\Delta \in \vec{\Delta}_i} \left( \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})) \right)}{\sigma(\mathcal{C}_j), \sigma(\vec{\Delta}_j) \Rightarrow \odot\sigma(\mathcal{C}_j) * \odot_{\Delta \in \vec{\Delta}_j} \left( \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})) \right)} \quad (*_r) \\
 \frac{\sigma(\mathcal{C}_j), \sigma(\vec{\Delta}_j) \Rightarrow \odot\sigma(\mathcal{C}_j) * \odot_{\Delta \in \vec{\Delta}_j} \left( \odot\sigma(\Delta) \wedge (\neg \bigvee_{\mathcal{C} \in \mathcal{C}(\Delta)} \odot\sigma(\mathcal{C})) \right)}{\sigma(\mathcal{C}_j), \sigma(\vec{\Delta}_j), \hat{\delta}(\text{Form}(\mathbf{r})) \Rightarrow} \quad (\neg_l) \\
 \frac{\sigma(\mathcal{C}_j), \sigma(\vec{\Delta}_j), \hat{\delta}(\text{Form}(\mathbf{r})) \Rightarrow}{\sigma(\mathcal{C}_j), \sigma(\vec{\Delta}_j), \hat{\delta}(\text{Form}(\mathbf{r})) \wedge 1 \Rightarrow} \quad (\wedge_l)
 \end{array}
 \end{array}$$

 Figure 2.2: The (*splitting*) property for  $j \in J$

**Theorem 2.19** (existence of disjunction forms)

For every analytic structural hypersequent rule  $(r)$ , the formula  $\text{Form}(r)$  is a disjunction form of  $(r)$  over  $\text{FL}_e^\perp$ .

*Proof.* In Propositions 2.17 and 2.18 we have established the properties (*soundness*) and (*splitting*) of  $\text{Form}(r)$ . The remaining property (*weakening*) holds because each disjunct of  $\text{Form}(r)$  is of the form  $A \wedge 1$ . So  $\text{Form}(r)$  is a disjunction form of  $(r)$  over  $\text{FL}_e^\perp$ .  $\square$

## 2.6 Putting Everything Together

We now combine the results of Sections 2.4 and 2.5.

**Theorem 2.20**

Every hyper-amenable axiomatic extension of  $\text{Thm}(\text{FL}_{e^*}^\perp)$  is multiset-bounded.

*Proof.* Let  $L$  be a hyper-amenable extension captured by  $\text{HFL}_{e^*}^\perp + \mathcal{R}$ . By Theorem 2.19, each  $(r) \in \mathcal{R}$  possesses a disjunction form  $\text{Form}(r)$ . Hence letting  $\mathcal{A}$  be the set of all formulas  $\text{Form}(r)$  where  $(r) \in \mathcal{R}$ , the projection theorem tells us that  $L = \text{Thm}(\text{FL}_{e^*}^\perp) + \mathcal{A}$  and that  $\text{FL}_{e^*}^\perp + \mathcal{A}$  is multiset-bounded.  $\square$

**Theorem 2.21**

Every hyper-amenable axiomatic extension of  $\text{Thm}(\text{FL}_{ecm}^\perp)$  ( $\text{FL}_{ecm}^\perp = \text{FL}_{ec}^\perp$  plus mingle) is set-bounded.

*Proof.* Follows from Theorem 2.20 and Proposition 2.6.  $\square$

**Theorem 2.22**

Every hyper-amenable axiomatic extension of  $\text{Thm}(\text{FL}_{ecm}^\perp)$  is decidable in doubly exponential time.

*Proof.* By Theorem 2.21 and (the proof of) Proposition 2.5, every hyper-amenable extension  $L$  of  $\text{Thm}(\text{FL}_{ecm}^\perp)$  can be reduced to  $\text{Thm}(\text{FL}_{ecm}^\perp)$ , where the reduction exponentially increases the formula size. Since  $\text{Thm}(\text{FL}_{ecm}^\perp)$  is itself decidable in exponential time (Lemma 2.23 below), this yields the doubly exponential upper bound for  $L$ .  $\square$

We include below a proof sketch of the exponential time upper bound for  $\text{FL}_{\text{ecm}}^\perp$  which we could not find in the literature. The precise complexity of  $\text{FL}_{\text{ecm}}^\perp$  we do not know; But it must be at least PSPACE since intuitionistic logic can be reduced to it.<sup>7</sup>

**Lemma 2.23**

$\text{Thm}(\text{FL}_{\text{ecm}}^\perp)$  is decidable in exponential time.

*Proof (outline).* Due to the presence of contraction and mingle, sequents in  $\text{FL}_{\text{ecm}}^\perp$  can be seen as structures  $\Gamma \Rightarrow \Pi$  where  $\Gamma$  is a *set* of formulas, instead of a multiset. Now given a formula  $F$  to be proved, the size of the collection  $\Omega$  of set-based sequents  $\Gamma \Rightarrow \Pi$  built from subformulas of  $F$  is exponential in the size of  $F$ . So starting from the initial sequents in  $\Omega$ , we can systematically apply rules of  $\text{FL}_{\text{ecm}}^\perp$  to obtain new derivable sequents in  $\Omega$ . After at most  $|\Omega|$  many steps, this process must halt, and then we can check if  $\Rightarrow F$  is contained in the sequents constructed so far.  $\square$

Similarly to Theorem 2.22 we can use the PSPACE-decidability of  $\text{IL}$  to obtain the following:

**Theorem 2.24**

Every hyper-amenable intermediate logic is decidable in exponential space.

## 2.7 More on Boundedness

In this section we compile some further results related to the notion of boundedness in loose order.

### 2.7.1 Sequent Calculi With Bounded Cuts

There is another proof-theoretic account of the projection theorem which is of conceptual interest.

Recall that  $\text{FL}_{\text{e}^*}^\perp + \mathcal{A}$  is the sequent calculus obtained by adding  $\Rightarrow A$  as an initial sequent for every instance  $A$  of an axiom from  $\mathcal{A}$ . We already remarked that  $\text{FL}_{\text{e}^*}^\perp + \mathcal{A}$  never satisfies the cut elimination theorem but in trivial cases. Let us call a cut in  $\text{FL}_{\text{e}^*}^\perp + \mathcal{A}$  *multiset-bounded* if it is of the form

$$\frac{\Rightarrow A \quad \Gamma, A \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} \text{ (cut)}$$

where  $A \in \psi_{\text{ms}}(\mathcal{A}, \Gamma \Rightarrow \Pi)$ , that is,  $A$  arises by substituting variables in some axiom from  $\mathcal{A}$  with fusions of subformula occurring in  $\Gamma \Rightarrow \Pi$ . Furthermore, call  $\text{FL}_{\text{e}^*}^\perp + \mathcal{A}$  a

<sup>7</sup>The reduction replaces every subformula  $A$  of a formula  $F$  by  $A \wedge 1$ .

*multiset-bounded sequent calculus* if every theorem of  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  has a proof in which all cuts are multiset-bounded, and initial sequents  $\Rightarrow A$  from  $\mathcal{A}$  occur only as left premises of such cuts.

**Theorem 2.25**

Every hyper-amenable axiomatic extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  has a multiset-bounded sequent calculus.

*Proof.* Let  $L$  be a hyper-amenable axiomatic extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$ , and let  $\mathcal{A}$  be the set of disjunction forms for the hypersequent rules capturing  $L$ . By the (proof of the) projection theorem, a formula  $F$  is a theorem of  $L$  if and only if for some  $A_1, \dots, A_n \in \psi_{ms}(\mathcal{A}, F)$ , the formula  $A_1 * \dots * A_n \rightarrow F$  is a theorem of  $\mathbf{FL}_{e^*}^\perp$ . The latter is equivalent to the provability of the sequent  $A_1, \dots, A_n \Rightarrow F$  in  $\mathbf{FL}_{e^*}^\perp$ . Now extend such a proof using cuts on  $A_1, \dots, A_n$  to obtain  $\Rightarrow F$ . Since  $A_i \in \psi_{ms}(\mathcal{A}, F)$ , each such cut is multiset-bounded. From this it follows that  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$  is a multiset-bounded sequent calculus for  $L$ .  $\square$

In other words, while cut elimination fails in  $\mathbf{FL}_{e^*}^\perp + \mathcal{A}$ , often a reasonably strong *cut reduction* is possible. We will come back to this point in the section on open questions.

### 2.7.2 A Polarity Restriction

In the proof of the projection theorem we have used the following fact: In a cutfree derivation of  $\Rightarrow F$  in  $\mathbf{HFL}_{e^*}^\perp + \mathcal{R}$  the multiset-variables of every  $\mathcal{R}$ -instance are instantiated by a multiset of subformulas of  $F$ . We can strengthen this statement by taking the polarity of formulas into account as follows. By a simple induction, one sees that every cutfree derivation of  $\Rightarrow F$  in  $\mathbf{HFL}_{e^*}^\perp + \mathcal{R}$  has the property that antecedents contain only negative<sup>8</sup> subformulas of  $F$ , whereas succedents contain only positive subformulas of  $F$ .

It follows that in each analytic structural rule instance in such a proof multiset-variables in the antecedent are instantiated with fusions of negative subformulas of  $F$ .

Let us introduce a new bounding function  $\psi_{ms}^-$  which maps  $(\mathcal{A}, F)$  to all instances of  $\mathcal{A}$ -axioms where variables are substituted by fusions of *negative* subformulas of  $F$ . Keeping in mind the fact that the disjunction form  $\text{Form}(r)$  contains only multiset-variables  $\bar{\Gamma}$  where  $\Gamma$  occurs in the antecedents of  $(r)$ , we may observe that the proof of the projection theorem establishes the following slightly stronger claim:

**Theorem 2.26**

Every hyper-amenable extension of  $\text{Thm}(\mathbf{FL}_{e^*}^\perp)$  is  $\psi_{ms}^-$ -bounded.

A similar strengthening of Theorem 2.21 holds as well.

<sup>8</sup>For the definition of negative and positive subformula occurrences, see Section 1.1.

### 2.7.3 Formula Boundedness from Single-Formula Variants

Theorem 2.21 establishes set-boundedness for a large class of logics above  $\text{FL}_{ecw}^\perp$ . We now turn to the stronger property of formula-boundedness. The discussion in this section is confined to extensions of  $\text{FL}_{ecw}^\perp = \text{LI}$ , that is, to intermediate logics.

One way of obtaining formula-boundedness for hyper-amenable intermediate logics is based on the observation that we can sometimes replace hypersequent rules by *single-formula variants* thereof. Compare for example

$$\frac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} \text{ (com)}$$

with

$$\frac{G \mid \Sigma_1, A_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, A_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, A_2 \Rightarrow \Pi_1 \mid \Sigma_2, A_1 \Rightarrow \Pi_2} \text{ (scom)},$$

or

$$\frac{G \mid \Delta, \Gamma \Rightarrow}{G \mid \Delta \Rightarrow \mid \Gamma \Rightarrow} \text{ (lq)} \quad \text{with} \quad \frac{G \mid A, \Gamma \Rightarrow}{G \mid A \Rightarrow \mid \Gamma \Rightarrow} \text{ (slq)}.$$

The latter rule (lq) can be simulated by (slq) as follows ( $\Delta = \{A_1, \dots, A_n\}$ ):

$$\frac{\frac{G \mid A_1, \dots, A_n, \Gamma \Rightarrow}{G \mid A_1 \Rightarrow \mid A_2, \dots, A_n, \Gamma \Rightarrow} \text{ (slq)} \quad \vdots \text{ (slq)}}{G \mid A_1 \Rightarrow \mid \dots \mid A_n \Rightarrow \mid \Gamma \Rightarrow} \text{ (w}_1\text{)} \quad \frac{\quad}{G \mid A_1, \dots, A_n \Rightarrow \mid \dots \mid A_1, \dots, A_n \Rightarrow \mid \Gamma \Rightarrow} \text{ (ec)}$$

Note that (cut) is not used. It follows that  $\text{LI} + (\text{slq})$  and  $\text{LI} + (\text{lq})$  prove the same theorems, and that the cut rule is admissible in  $\text{LI} + (\text{slq})$ .

Something similar (but more complicated) works for the rule (com):

#### Proposition 2.27

The rule

$$\frac{G \mid \Sigma_1, \Gamma_1 \Rightarrow \Pi_1 \quad G \mid \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{G \mid \Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} \text{ (com)}$$

is cutfree derivable in  $\text{HLI} + (\text{scom})$ .

*Proof.* By induction on  $n := |\Gamma_1| + |\Gamma_2|$ . We will omit the context variable  $G$  since it does not play a role. If  $\Gamma_1 = \emptyset$  or  $\Gamma_2 = \emptyset$ , then the instance of (com) can be derived using weakenings. Let us therefore assume that  $\Gamma_1$  and  $\Gamma_2$  are non-empty, that is,  $\Gamma_1 = \Gamma'_1 \cup \{A_1\}$  and  $\Gamma_2 = \Gamma'_2 \cup \{A_2\}$ . For the induction step, we proceed as follows (principal formulas in (com) and (scom) are underlined):

$$\frac{\frac{\Sigma_1, \Gamma'_1, A_1 \Rightarrow \Pi_1 \quad \Sigma_2, \Gamma_2 \Rightarrow \Pi_2}{\Sigma_1, \Gamma_2, A_1 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma'_1 \Rightarrow \Pi_2} \text{I.H.} \quad \frac{\Sigma_2, \Gamma'_2, A_2 \Rightarrow \Pi_1 \quad \Sigma_1, \Gamma_1 \Rightarrow \Pi_1}{\Sigma_2, \Gamma_1, A_2 \Rightarrow \Pi_2 \mid \Sigma_1, \Gamma'_2 \Rightarrow \Pi_1} \text{I.H.}}{\frac{\Sigma_1, \Gamma_2, A_2 \Rightarrow \Pi_2 \mid \Sigma_2, \Gamma'_1 \Rightarrow \Pi_2 \mid \Sigma_2, \Gamma_1, A_1 \Rightarrow \Pi_2 \mid \Sigma_1, \Gamma'_2 \Rightarrow \Pi_1}{\Sigma_1, \Gamma_2 \Rightarrow \Pi_2 \mid \Sigma_2, \Gamma'_1 \Rightarrow \Pi_2 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2 \mid \Sigma_1, \Gamma'_2 \Rightarrow \Pi_1} (c_1)}{\Sigma_1, \Gamma_2 \Rightarrow \Pi_1 \mid \Sigma_2, \Gamma_1 \Rightarrow \Pi_2} (w_1), (ec)$$

For the instance of  $(c_1)$  in the proof above, note that  $A_1 \in \Gamma_1$  and  $A_2 \in \Gamma_2$ .  $\square$

As a side remark, the number of  $(scom)$ 's needed to simulate one  $(com)$  with the above strategy is exponential in  $|\Gamma_1| + |\Gamma_2|$ .

The point of all of this is the following: It is easy to see that the formula  $(A_2 \rightarrow A_1) \vee (A_1 \rightarrow A_2)$  can act as a disjunction form of  $(scom)$  in the same way that  $(\hat{\Gamma}_2 \rightarrow \hat{\Gamma}_1) \vee (\hat{\Gamma}_1 \rightarrow \hat{\Gamma}_2)$  acts as a disjunction form of  $(com)$ . So if we run the argument in Section 2.4 starting from cutfree proofs in  $\mathbf{HLI} + (scom)$  instead of proofs in  $\mathbf{HLI} + (com)$ , we obtain a procedure to transform hypersequent proofs of  $\Rightarrow F$  into  $\mathbf{LI}$ -proofs of

$$(A_2^1 \rightarrow A_1^1) \vee (A_1^1 \rightarrow A_2^1), \dots, (A_2^n \rightarrow A_1^n) \vee (A_1^n \rightarrow A_2^n) \Rightarrow F$$

where each  $A_j^i$  is now a subformula of  $F$  instead of a conjunction of subformulas of  $F$ . Hence the method applied to single-formula variants yields formula-boundedness:

**Proposition 2.28**

The extensions  $\mathbf{LI} + (\text{lin})$  and  $\mathbf{LI} + (\text{lq})$  are formula-bounded.

Note that in both cases the single-formula variant still contains multiset-variables. What is important is that we can associate a disjunction form which consists of formula-variables only.

Unfortunately we do not know of any general criterion telling us when a structural rule can be replaced by a single-formula version. We may observe that the analogue of Proposition 2.27 fails in the substructural setting:

**Proposition 2.29**

The rule  $(com)$  is not admissible in the cutfree fragment of  $\mathbf{HFL}_{e_w}^\perp + (scom)$ .

*Proof.* It is easy to check that the formula  $F = (p \rightarrow q * r) \vee (q \rightarrow (r \rightarrow p))$  is derivable in  $\mathbf{HFL}_{e_w}^\perp + (com)$ . But there is no cutfree proof of  $F$  in  $\mathbf{HFL}_{e_w}^\perp + (scom)$ . Indeed, let us assume towards a contradiction that such a cutfree proof  $\delta$  exists, and assume furthermore without loss of generality that the number of  $(*_r)$ 's is minimal in  $\delta$ . There must be at least one  $(*_r)$ , because otherwise we could replace  $q * r$  everywhere in  $\delta$  by

some new variable  $t$ , yielding a proof of the invalid  $(p \rightarrow t) \vee (q \rightarrow (r \rightarrow p))$ . Now focus on the active conclusion component  $\Gamma \Rightarrow \Pi$  of one instance of  $(*_r)$  in  $\delta$ . By analyticity, we must have  $\Pi = \{q * r\}$ . If we follow the path of the component  $\Gamma \Rightarrow \Pi$  down to the endsequent  $\Rightarrow F$ , we can observe that we must encounter the component  $p \Rightarrow q * r$  at some point. Now the only rules which can be applied on this path are  $(w_l)$ ,  $(ec)$ ,  $(ew)$  and  $(scom)$ . Crucially, due to analyticity the principal formulas in  $(scom)$  must be negative subformulas of  $F$ , and those are only atoms:  $p, q, r$ . From this it follows that  $\Gamma$  is either empty or consists of a single variable only. In any case, since  $(*_r)$  is a multiplicative rule, one active premise component of the focused instance of  $(*_r)$  will have an empty antecedent. Without loss of generality, let us assume it is the left premise. Then the instance of  $(*_r)$  looks like this:

$$\frac{\begin{array}{c} \vdots \delta_1 \\ G \mid \Rightarrow q \end{array} \quad \begin{array}{c} \vdots \delta_2 \\ G \mid \Gamma \Rightarrow r \end{array}}{G \mid \Gamma \Rightarrow q * r} (*_r)$$

$$\begin{array}{c} \vdots \\ \Rightarrow F \end{array}$$

Since no rule is applicable to  $\Rightarrow q$  in  $\delta_1$ , the component must have been introduced by  $(ew)$ . But then we can remove it altogether from  $\delta_1$ , thus creating a proof with one less  $(*_r)$ :

$$\frac{\begin{array}{c} \vdots \delta'_1 \\ G \end{array}}{G \mid \Gamma \Rightarrow q * r} (ew)$$

$$\begin{array}{c} \vdots \\ \Rightarrow F \end{array}$$

This contradicts the minimality assumption on  $\delta$ . □

#### 2.7.4 Formula Boundedness via Propagation Properties

Here we discuss another way of obtaining formula-boundedness from set-boundedness.

Let  $A(B/p)$  denote the formula obtained from  $A$  by substituting every occurrence of the propositional variable  $p$  with the formula  $B$ .

##### Definition 2.30

A formula  $A$  has the  $\Omega$ -*propagation property* for a set  $\Omega$  of binary connectives if for variables  $p, q, r$  and every  $\diamond \in \Omega$ ,

$$\vdash_{LI} A(q/p), A(r/p) \Rightarrow A(q \diamond r/p).$$

Note that this condition is trivially satisfied if  $p$  does not occur in  $A$ .

**Lemma 2.31**

Let  $\text{LI} + \mathcal{A}$  be a set-bounded extension. If every  $A \in \mathcal{A}$  has the  $\{\wedge\}$ -propagation property, then  $\text{LI} + \mathcal{A}$  is formula-bounded.

*Proof.* Suppose that  $\vdash_{\text{LI}+\mathcal{A}} F$ . By set-boundedness, there exist  $A_1, \dots, A_n$  such that  $\vdash_{\text{LI}} A_1, \dots, A_n \Rightarrow F$  and each  $A_i$  is a set-bounded instance, that is, a substitution of the propositional variables of some formula in  $\mathcal{A}$  by conjunctions of subformulas of  $F$ . By repeatedly using the  $\{\wedge\}$ -propagation property, some sequent  $\vdash_{\text{LI}} A'_1, \dots, A'_m \Rightarrow F$  is derivable such that each  $A'_i$  is a substitution of the propositional variables of some formula in  $\mathcal{A}$  by subformulas of  $F$ .  $\square$

Showing that an axiom has the  $\{\wedge\}$ -propagation property can be tedious in practice, as many checks need to be done. Consider for example the  $\text{Bc}_k$  axiom (Kripke models with  $k$  world)  $p_0 \vee (p_0 \rightarrow p_1) \vee \dots \vee (p_0 \wedge \dots \wedge p_{k-1} \rightarrow p_k)$ . In the remainder of this section we introduce a sufficient criterion for  $\{\wedge\}$ -propagation which is easier to check.

Let  $U_p$  denote the set of formulas that possess the  $\{\wedge\}$ -propagation property with respect to the variable  $p$ :

$$U_p = \{A \mid \forall q, r \in \mathcal{V} \vdash_{\text{LI}} A(q/p), A(r/p) \Rightarrow A(q \wedge r/p)\}$$

We then have the following.

**Lemma 2.32**

If  $A \in U_p$  for every variable  $p$  occurring in  $A$ , then  $A$  has the  $\{\wedge\}$ -propagation property.

We now fix one variable  $p \in \mathcal{V}$  and write  $A(B)$  for  $A(B/p)$ . Define the following sets of formulas:

$$\begin{aligned} U_p^* &= \{A \mid \forall q, r \in \mathcal{V}. \vdash_{\text{LI}} A(q) \Rightarrow A(q \wedge r)\} \\ D_p &= \{A \mid \forall q, r \in \mathcal{V}. \vdash_{\text{LI}} A(q \wedge r) \Rightarrow A(q) \wedge A(r)\} \\ N_p &= \{A \mid p \text{ occurs only negatively in } A\} \end{aligned}$$

As a mnemonic,  $U$  stands for ‘upwards propagation’ (going from simple instances up to conjunctive instances),  $U^*$  for ‘strong upwards propagation’,  $D$  stands for ‘downwards propagation’ (going from conjunctive instances down to simple instances), and of course  $N$  stands for negative. For sets  $M, N$  of formulas and a binary connective  $\diamond$ , define  $M \diamond N$  to be the set  $\{A \diamond B \mid A \in M, B \in N\}$ .



**Lemma 2.33**

The following holds:

1.  $p \in \mathcal{U}_p$
2. If  $p$  does not occur in  $A$ , then  $A \in \mathcal{U}_p^* \cap \mathcal{U}_p \cap \mathcal{D}_p$
3.  $\mathcal{N}_p \subseteq \mathcal{U}_p^*$
4.  $\mathcal{D}_p \rightarrow \mathcal{U}_p \subseteq \mathcal{U}_p$
5.  $\mathcal{U}_p^* \vee \mathcal{U}_p \subseteq \mathcal{U}_p$

*Proof.* (1) Since  $\vdash_{\text{LI}} q, r \Rightarrow q \wedge r$ . (2) holds trivially. (3) Let  $A \in \mathcal{N}_p$ , and let  $\delta$  be the standard proof of  $A(q) \Rightarrow A(r)$ . Start constructing a proof  $\delta'$  of  $A(q) \Rightarrow A(q \wedge r)$  bottom up by imitating the proof steps in  $\delta$ . Whenever the formula  $q \wedge r$  appears isolated in  $\delta'$ , then this is in an antecedent because  $A \in \mathcal{N}_p$ . We can thus apply a cut with  $q \wedge r \Rightarrow q$  to (again, reading the proof bottom up) replace  $q \wedge r$  with  $q$ . Then copy the remaining steps of  $\delta$ . (4) Let  $A \in \mathcal{D}_p$ ,  $B \in \mathcal{U}_p$  and consider the following derivation showing that  $A \rightarrow B \in \mathcal{U}_p$ :

$$\frac{\frac{\text{since } A \in \mathcal{D}_p \quad A(q \wedge r) \Rightarrow A(r) \quad \text{since } B \in \mathcal{U}_p \quad B(q), B(r) \Rightarrow B(q \wedge r)}{A(q \wedge r) \Rightarrow A(q) \quad B(q), (A \rightarrow B)(r), A(q \wedge r) \Rightarrow B(q \wedge r)} (\rightarrow_1)}{(A \rightarrow B)(q), (A \rightarrow B)(r), A(q \wedge r) \Rightarrow B(q \wedge r)} (\rightarrow_1)}{(A \rightarrow B)(q), (A \rightarrow B)(r) \Rightarrow (A \rightarrow B)(q \wedge r)} (\rightarrow_r)$$

(5) Let  $A \in \mathcal{U}_p^*$  and  $B \in \mathcal{U}_p$ . Start constructing a proof of  $(A \vee B)(q), (A \vee B)(r) \Rightarrow (A \vee B)(q \wedge r)$  by bottom-up applying the rule  $(\vee_1)$  twice. We then have to check provability of the following four sequents:

- (i)  $A(q), A(r) \Rightarrow (A \vee B)(q \wedge r)$
- (ii)  $A(q), B(r) \Rightarrow (A \vee B)(q \wedge r)$
- (iii)  $B(q), A(r) \Rightarrow (A \vee B)(q \wedge r)$
- (iv)  $B(q), B(r) \Rightarrow (A \vee B)(q \wedge r)$

Now (i)–(iii) are provable since  $A \in \mathcal{U}_p^*$ , and (iv) is provable since  $B \in \mathcal{U}_p$ . □

Taken together, Lemmas 2.32 and 2.33 provide a convenient sufficient condition for the  $\{\wedge\}$ -propagation property. Here are two examples.

**Example 2.34**

The  $Bc_k$  axiom  $p_0 \vee (p_0 \rightarrow p_1) \vee \dots \vee (p_0 \wedge \dots \wedge p_{k-1} \rightarrow p_k)$  enjoys the  $\{\wedge\}$ -propagation property: Let  $i \leq k$  and consider the disjunct  $(p_0 \wedge \dots \wedge p_{i-1}) \rightarrow p_i$ . Its premise does not contain  $p_i$  and so by Lemma 2.33(2) it belongs to  $D_p$ . Its conclusion belongs to  $U_p$  by Lemma 2.33(1). So by Lemma 2.33(4), the disjunct is in  $U_p$ . Since  $p$  occurs only negatively in the remainder of the axiom, this remainder is in  $U_p^*$  by Lemma 2.33(3) and so the whole axiom  $Bc_k$  is in  $U_p$  by Lemma 2.33(5). By Lemma 2.32 we conclude that  $Bc_k$  satisfies  $\wedge$ -propagation.

**Example 2.35**

Consider the linearity axiom  $lin = (p \rightarrow q) \vee (q \rightarrow p)$ . To show that  $lin \in U_p$ , we observe that  $(p \rightarrow q) \in U_p^*$  by Lemma 2.33(3) and  $(q \rightarrow p) \in U_p$  by Lemmas 2.33(1,2,4). So  $lin \in U_p$  by Lemma 2.33(5). By symmetry  $lin \in U_q$ , and so by Lemma 2.32,  $lin$  has the  $\{\wedge\}$ -propagation property. From this and Lemma 2.31 we obtain that again the standard axiomatization for Gödel logic  $LI + (p \rightarrow q) \vee (q \rightarrow p)$  is formula-bounded (we already established this by a different method in Section 2.7.3).

**2.7.5 Variable-Boundedness for Intermediate Logics**

We finally come to variable-boundedness, the strongest boundedness condition on axiomatic extensions that we considered. It became known to us during our research that variable-boundedness for intermediate logics was investigated under the name *simple substitution property* in [30] and subsequently in the series of papers [49, 50, 51]. In this short section we report on some of the results in the area (suitably adapted to our notation). There are no results of our own.

We first remark that boundedness properties may depend on a specific axiomatisation: Alternative axiomatisation of the same logic may not have the same property. For example, as an extension of  $IL$ , Gödel logic  $G$  is not variable-bounded with respect to the standard axiomatization  $(p \rightarrow q) \vee (q \rightarrow p)$ . A simple counterexample is the formula  $\neg p \vee \neg\neg p$  which is a theorem of Gödel logic, but clearly  $LI$  does not prove  $(p \rightarrow p) \vee (p \rightarrow p) \Rightarrow \neg p \vee \neg\neg p$ . However, as shown in [49], a variable-bounded axiomatisation of Gödel logic is  $LI + (p \rightarrow q) \vee ((p \rightarrow q) \rightarrow p) + (\neg p \vee \neg\neg p)$ .

A simple sufficient criterion for variable-boundedness is presented in [30] using a propagation property. We reformulate the proof there to our setting.

**Lemma 2.36**

The extension  $LI + \mathcal{A}$  is variable-bounded if every  $A \in \mathcal{A}$  has the  $\{\wedge, \vee, \rightarrow\}$ -

propagation property.

*Proof.* If  $\vdash_{\mathbf{LI}+\mathcal{A}} (\Rightarrow F)$  for some formula  $F$ , then  $\vdash_{\mathbf{LI}} A_1, \dots, A_n \Rightarrow F$  for some instances  $A_i$  of axioms in  $\mathcal{A}$ . By repeatedly applying the  $\{\wedge, \vee, \rightarrow\}$ -propagation property of the axioms, we can replace the list  $A_1, \dots, A_n$  of assumptions by a list of atomic instances  $A'_1, \dots, A'_m$ . If any variable occurring in some  $A'_i$  does not occur in  $F$ , we can uniformly rename such a variable with one that does occur in  $F$  (this is the same argument as in the proof of Proposition 2.4). We obtain a proof of  $A''_1, \dots, A''_m \Rightarrow F$  where each  $A''_i \in \psi_v(\mathcal{A}, F)$ .  $\square$

Note that in contrast to Lemma 2.31, the proof of Lemma 2.36 does not need any additional assumptions on  $\mathbf{LI} + \mathcal{A}$  such as set-boundedness. This is because of the variable-renaming trick which does not have an analogue in the set-bounded setting. Lemma 2.36 is used in [30] to show that  $\mathbf{CL} = \mathbf{IL} + (p \vee \neg p)$  and the logic  $\mathbf{LQ} = \mathbf{LI} + (\neg p \vee \neg\neg p)$  are variable-bounded.

Let  $\mathcal{V}(A)$  denote the set of variables occurring in  $A$ . A logic  $L$  has the *Craig interpolation property* if  $A \rightarrow B \in L$  implies the existence of a formula  $I$  with  $(A \rightarrow I) \wedge (I \rightarrow B) \in L$  and  $\mathcal{V}(I) \subseteq \mathcal{V}(A) \cap \mathcal{V}(B)$ . It is well known that both  $\mathbf{IL}$  and  $\mathbf{CL}$  possess the Craig interpolation property [53].

We reproduce below the elegant argument from [49, attributed to H. Ono] where the Craig interpolation property for  $\mathbf{LQ}$  is inferred from interpolation in  $\mathbf{LI}$  using variable boundedness.

#### Theorem 2.37

$\mathbf{LQ}$  has the Craig interpolation property.

*Proof.* For a formula  $X$ , let  $A_X$  denote the conjunction of all formulas  $\neg q \vee \neg\neg q$  such that  $q \in \mathcal{V}(X)$ . If  $\vdash_{\mathbf{LQ}} B \rightarrow C$ , then by variable-boundedness we know  $\vdash_{\mathbf{LI}} A_{B \rightarrow C} \rightarrow (B \rightarrow C)$ . Since  $\neg q \vee \neg\neg q$  has only one variable, every conjunct in  $A_B \wedge A_C$  appears in  $A_{B \rightarrow C}$  and *vice versa*. Thus  $\vdash_{\mathbf{LI}} A_{B \rightarrow C} \leftrightarrow A_B \wedge A_C$ , and hence  $\vdash_{\mathbf{LI}} (A_B \wedge A_C) \rightarrow (B \rightarrow C)$ . It follows that  $\vdash_{\mathbf{LI}} (A_B \wedge B) \rightarrow (A_C \rightarrow C)$ . By the interpolation property of  $\mathbf{LI}$ , there is a formula  $I$  such that  $\mathcal{V}(I) \subseteq \mathcal{V}(A) \cap \mathcal{V}(B)$  and  $\vdash_{\mathbf{LI}} ((A_B \wedge B) \rightarrow I) \wedge (I \rightarrow (A_C \rightarrow C))$ . Then the latter formula is also provable in  $\mathbf{LQ}$ . Since  $A_B, A_C \in \mathbf{LQ}$ , it follows that  $\vdash_{\mathbf{LQ}} (B \rightarrow I) \wedge (I \rightarrow C)$ . Hence  $I$  is a Craig interpolant of  $A \rightarrow B$  in  $\mathbf{LQ}$ .  $\square$

In [50] it is shown that classical logic and  $\mathbf{LQ}$  are the only consistent variable-bounded logics over  $\mathbf{LI}$  that are axiomatisable by a single-variable formula. The same paper shows that all finite-valued Gödel logics are variable-bounded. Using algebraic methods these results have been generalized in [51] to establish necessary and sufficient criterion for variable-boundedness for intermediate logics on a *finite slice* (see also [29]).

## 2.8 An Application in Modal Logic

In this section we present a new proof of Takano's result [52] that the standard sequent calculus for modal logic **S5** is analytic. We derive this by translating cutfree hypersequent proofs to sequent proofs, observing that the only cuts which need to be introduced are analytic. The method is very similar to the proof of the projection theorem (Theorem 2.13).

### 2.8.1 Sequent Calculi for Some Modal Logics

In addition to the connectives of classical logic, the signature of modal logic contains a unary connective  $\Box$  (called 'box'). In defining modal logics as sets of formulas (see Section 1.5), we not only require closure under substitution instances and Modus Ponens, but also under the *Necessitation Rule*

$$\frac{A}{\Box A} \text{ (N)}.$$

The modal logics **K**, **S4** and **S5** are axiomatized as follows:

$$\begin{aligned} \mathbf{K} &= \mathbf{CL} + \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \\ \mathbf{S4} &= \mathbf{K} + \Box A \rightarrow A + \Box A \rightarrow \Box \Box A \\ \mathbf{S5} &= \mathbf{S4} + \neg \Box \neg A \rightarrow \Box \neg \Box \neg A \end{aligned}$$

For more information on these systems see [15].

A sequent calculus **seqK** for modal logic **K** is obtained by adding to the sequent calculus **LK** for classical logic the rule

$$\frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} \text{ (K)}$$

where  $\Box \Gamma := \{\Box A \mid A \in \Gamma\}$ .

To capture **S4** one adds to **LK** the rules

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \text{ (T)} \quad \text{and} \quad \frac{\Box \Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A} \text{ (4)}$$

Let us call the resulting calculus **seqS4**. Finally, a sequent calculus **seqS5** for **S5** is obtained by replacing the rule (4) in **seqS4** with the more liberal

$$\frac{\Box \Gamma \Rightarrow A, \Box \Delta}{\Box \Gamma \Rightarrow \Box A, \Box \Delta} \text{ (5)}.$$

While the cut rule is admissible in **seqK** and **seqS4** [44], the same does not hold for **seqS5**. As observed in [45], the cut in the following inference cannot be eliminated:

$$\frac{\frac{\frac{\Box p \Rightarrow \Box p}{\Rightarrow \Box p, \neg \Box p} (\neg_r)}{\Rightarrow \Box p, \Box \neg \Box p} (5) \quad \frac{p \Rightarrow p}{\Box p \Rightarrow p} (\top)}{\Rightarrow \Box \neg \Box p, p} (\text{cut}) \quad (\spadesuit)$$

To the present day no cutfree ‘standard’ sequent system for **S5** is known.<sup>9</sup>

If however one is willing to go beyond the sequent calculus, a number of cutfree proof systems for **S5** are on offer. We mention here a hypersequent calculus due to Kurokawa [34] which will be useful for our purposes. This system, which shall be called **HS5** here, extends a hypersequent version of **seqS4** with the rule

$$\frac{G \mid \Box \Sigma, \Gamma \Rightarrow \Delta}{G \mid \Box \Sigma \Rightarrow \mid \Gamma \Rightarrow \Delta} (\text{ms})$$

(the ms stands for ‘modalized splitting’).

**Theorem 2.38** (Kurokawa 2014)

**HS5** is sound and complete for **S5**, and the cut rule is admissible in **HS5**.

As illustration we present a cutfree **HS5**-proof of the sequent  $\Rightarrow \Box \neg \Box p, p$  which served as the counterexample to cut elimination in **seqS5**.

$$\frac{\frac{\frac{\frac{p \Rightarrow p}{\Box p \Rightarrow p} (\top)}{\Box p \Rightarrow \mid \Rightarrow p} (\text{ms})}{\Rightarrow \neg \Box p \mid \Rightarrow p} (\neg_r)}{\Rightarrow \Box \neg \Box p \mid \Rightarrow p} (4) \quad \frac{\Rightarrow \Box \neg \Box p \mid \Rightarrow p}{\Rightarrow \Box \neg \Box p, p \mid \Rightarrow \Box \neg \Box p, p} (w_1)}{\Rightarrow \Box \neg \Box p, p} (\text{ec})$$

## 2.8.2 Analyticity of **S5**

Going back to the derivation ( $\spadesuit$ ) of  $\Rightarrow \Box \neg \Box p, p$  in **seqS5**, we see that the cut formula  $\Box p$  is a subformula of the conclusion of the cut. Such a cut is called an *analytic cut*. If all cuts in a derivation are analytic—as it is the case in ( $\spadesuit$ )—then all formulas in the derivation are subformulas of the endsequent.

<sup>9</sup>This apparent intractability of **S5** should not be over-interpreted: In terms of computational complexity the coNP-complete **S5** looks tame as compared to most other modal logics, which tend to be PSPACE-complete.

In 1992 Takano proved that the sequent calculus  $\text{seqS5}$ , although lacking cut elimination, has the *analytic cut property*: Every theorem of  $\text{S5}$  has a derivation in  $\text{seqS5}$  containing only analytic cuts, and therefore satisfying the subformula property. Takano established the analytic cut property by means of an intricate global transformation of  $\text{seqS5}$ -proofs which replaces arbitrary cuts by analytic ones.

In Section 2.4, we have derived analyticity properties similar to the analytic cut property from the existence of analytic hypersequent proofs. The question arises whether a result such as Takano's can also be inferred in a similar way. We answer this in the affirmative: The analytic cut property of  $\text{SeqS5}$  may also be derived from cut-freeness of the hypersequent calculus  $\text{HS5}$ , using essentially the same method as in Section 2.4. That is, we will start from an analytic hypersequent proof in  $\text{HS5}$  and remove instances of  $(\text{ms})$  until a sequent derivation is obtained. Unlike before, instances of  $(\text{ms})$  will now be traded with analytic cuts instead of bounded axiom instances.

**Theorem 2.39**

$\text{seqS5}$  has the analytic cut property.

*Proof.* Let  $\Gamma_0 \Rightarrow \Delta_0$  be a sequent provable in  $\text{seqS5}$ . By Theorem 2.38,  $\Gamma_0 \Rightarrow \Delta_0$  has a cutfree derivation  $\delta$  in the hypersequent calculus  $\text{HS5}$ . We first observe that instances of  $(\text{ms})$  in  $\delta$  can be replaced by single-formula instances (see also Section 2.7.3)

$$\frac{G \mid \Box B, \Gamma \Rightarrow \Delta}{G \mid \Box B \Rightarrow \mid \Gamma \Rightarrow \Delta} (\text{sms}),$$

following the idea exposed in the following derivation:

$$\frac{\frac{\frac{\frac{\Box B_1, \Box B_2, \Gamma \Rightarrow \Delta}{\Box B_1 \Rightarrow \mid \Box B_2, \Gamma \Rightarrow \Delta} (\text{sms})}{\Box B_1 \Rightarrow \mid \Box B_2 \Rightarrow \mid \Gamma \Rightarrow \Delta} (\text{sms})}{\Box B_1, \Box B_2 \Rightarrow \mid \Box B_1, \Box B_2 \Rightarrow \mid \Gamma \Rightarrow \Delta} (\text{w}_1)}{\Box B_1, \Box B_2 \Rightarrow \mid \Gamma \Rightarrow \Delta} (\text{ec})$$

Let us therefore assume that all instances of  $(\text{ms})$  in  $\delta$  are of the form  $(\text{sms})$ . For simplicity, we will furthermore assume that there is only one such instance of  $(\text{sms})$ . The elimination of multiple hypersequent rules can be performed following the same strategy as in the proof of Theorem 2.13. So let us picture the instance of  $(\text{sms})$  in the proof of  $\Gamma_0 \Rightarrow \Delta_0$  as follows:

$$\begin{array}{c} \vdots \rho \\ \frac{G \mid \Box B, \Gamma \Rightarrow \Delta}{G \mid \Box B \Rightarrow \mid \Gamma \Rightarrow \Delta} (\text{ms}) \\ \tau \quad \eta \\ \Gamma_0 \Rightarrow \Delta_0 \end{array}$$

We create two modified copies of the proof where (sms) is eliminated, namely

$$\begin{array}{c} \vdots \tau \\ \dots \\ \Gamma_0 \Rightarrow \Box B, \Delta_0 \end{array} \quad \frac{\Box B \Rightarrow \Box B}{G \mid \Box B \Rightarrow \Box B \mid \Gamma \Rightarrow \Delta} \text{ (ew)} \quad \begin{array}{c} \vdots \eta' \\ \dots \end{array}$$

and

$$\begin{array}{c} \vdots \tau \\ \dots \\ \Gamma_0, \Box B \Rightarrow \Delta_0 \end{array} \quad \frac{\begin{array}{c} \vdots \rho \\ \dots \\ G \mid \Box B, \Gamma \Rightarrow \Delta \end{array}}{G \mid \Box B \Rightarrow \mid \Box B, \Gamma \Rightarrow \Delta} \text{ (ew)} \quad \begin{array}{c} \vdots \eta'' \\ \dots \end{array}$$

Call the first and second copy  $\delta_1$  and  $\delta_2$  respectively. In  $\delta_1$ , the rule (sms) has been eliminated by the addition of the formula  $\Box B$  into the succedent. The additional formula in the succedent is then propagated downwards  $\eta'$ , so that we obtain a proof of  $\Gamma_0 \Rightarrow \Box B, \Delta_0$ . Crucially, by this addition of  $\Box B$  instances of (4) in  $\eta$  become instances of (5) in  $\eta'$ , that is,

$$\frac{G' \mid \Box \Gamma' \Rightarrow A}{G' \mid \Box \Gamma' \Rightarrow \Box A} \text{ (4)} \quad \text{becomes} \quad \frac{G' \mid \Box \Gamma' \Rightarrow A, \Box B}{G' \mid \Box \Gamma' \Rightarrow \Box A, \Box B} \text{ (5)}.$$

Let us now turn to  $\delta_2$ . Here (sms) has been eliminated by the addition of  $\Box B$  into the antecedent, and the additional formula is then propagated downwards  $\eta''$  to obtain a proof of  $\Gamma_0, \Box B \Rightarrow \Delta_0$ . Note that instances of (4) in  $\delta$  remain sound in  $\eta''$  because the formula that we add in the antecedent is boxed:

$$\frac{G' \mid \Box \Gamma' \Rightarrow A}{G' \mid \Box \Gamma' \Rightarrow \Box A} \text{ (4)} \quad \text{becomes} \quad \frac{G' \mid \Box \Gamma', \Box B \Rightarrow A}{G' \mid \Box \Gamma', \Box B \Rightarrow \Box A} \text{ (4)}.$$

We can now combine  $\delta_1$  and  $\delta_2$  using a cut on  $\Box B$ :

$$\frac{\begin{array}{c} \vdots \delta_1 \\ \dots \\ \Gamma_0 \Rightarrow \Box B, \Delta_0 \end{array} \quad \begin{array}{c} \vdots \delta_2 \\ \dots \\ \Gamma_0, \Box B \Rightarrow \Delta_0 \end{array}}{\frac{\Gamma_0, \Gamma_0 \Rightarrow \Delta_0, \Delta_0}{\Gamma_0 \Rightarrow \Delta_0} \text{ (cut)}} \text{ (c}_l\text{), (c}_r\text{)}$$

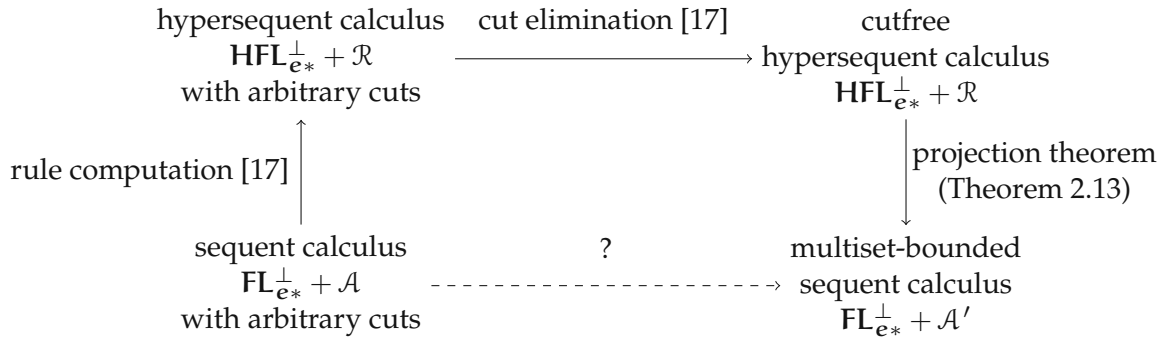
The resulting proof  $\delta'$  does not contain the rule (sms) any more, and we may therefore assume that  $\delta$  contains only sequents (this is similar to Lemma 1.12). Hence  $\delta'$  is a proof in **SeqS5**. Furthermore, since the cut formula  $\Box B$  was taken from the analytic hypersequent proof  $\delta$ , it is a subformula of  $\Gamma_0 \Rightarrow \Pi_0$ , and therefore the cut in  $\delta'$  is analytic.  $\square$

Inspecting the proof, we can say a little bit more about the cut formulas which might be needed in a **seqS5**-proof: They are *boxed* subformulas and occur *negatively* in the endsequent.

## 2.9 Open Questions

**Separating various forms of boundedness.** So far we have no results establishing that a certain logic or calculus *lacks* a boundedness property, and so in particular, we have not separated the various forms of boundedness. Such results could potentially be obtained using complexity considerations, building on the results in Sections 2.2 and 2.6. For example, it follows from Proposition 2.5 that an intermediate logic with computational complexity worse than PSPACE cannot be formula-bounded (since a formula-bounded intermediate logic has a polynomial reduction to IL, which is itself PSPACE-complete).

**Finding a direct argument for boundedness.** The boundedness result we have established for various axiomatic extensions crucially depend on the existence of an analytic hypersequent calculus for the logic in question. Adopting the proof-theoretic view of Section 2.7.1, the whole strategy can be described by the following diagram:



The dotted arrow in the diagram tells us that we have found a way to reduce arbitrary cuts in  $\text{FL}_{e^*}^\perp + \mathcal{A}$  to multiset-bounded cuts. Is there a direct method of obtaining this result? It seems clear that such a direct method still needs to incorporate a cut elimination-style argument, so one should not expect a simplistic answer. It would be interesting to know in particular which arguments a direct method uses that go beyond the cut elimination theorem of  $\text{FL}_{e^*}^\perp$ .

As an illustration of what could be such a direct method, let us sketch how a single arbitrary cut on a linearity axiom can be reduced to a multiset-bounded cut, using the cut elimination theorem of LI. This is not yet a complete method, as among other things we have to assume that there is only a single cut (further restrictions will be state in the course of the outline). Anyway, assume that a proof ends like this:

$$\frac{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A) \quad (A \rightarrow B) \vee (B \rightarrow A) \Rightarrow F}{\Rightarrow F} \text{ (cut)}$$



where  $\delta$  is an LI-proof and neither the formula  $A$  nor  $B$  is necessarily composed of subformulas of  $F$ . We want to replace the cut on  $(A \rightarrow B) \vee (B \rightarrow A)$  by a multiset-bounded cut and proceed as follows. First, we can assume without loss of generality that the lowermost inference in  $\delta$  is  $(\vee_1)$ , so that the proof is of the form

$$\frac{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A) \quad \frac{\begin{array}{c} \vdots \delta_1 \\ A \rightarrow B \Rightarrow F \end{array} \quad \begin{array}{c} \vdots \delta_2 \\ B \rightarrow A \Rightarrow F \end{array}}{(A \rightarrow B) \vee (B \rightarrow A) \Rightarrow F} (\vee_1)}{\Rightarrow F} (\text{cut})$$

We now trace back the formula  $A \rightarrow B$  in  $\delta_1$  until it is principal. For simplicity, assume that this happens exactly once, and write  $\delta_1$  as

$$\frac{\begin{array}{c} \vdots \rho_1 \\ \Gamma_1 \Rightarrow A \end{array} \quad \begin{array}{c} \vdots \nu_1 \\ \Delta_1, B \Rightarrow \Pi_1 \end{array}}{\Gamma_1, \Delta_1, A \rightarrow B \Rightarrow \Pi_1} (\rightarrow_1)}{\begin{array}{c} \vdots \theta_1 \\ A \rightarrow B \Rightarrow F \end{array}}$$

Let us furthermore assume that no contraction is applied to  $A \rightarrow B$  in  $\theta_1$ . Then by analyticity of  $\delta_1$ ,  $\Gamma_1$  and  $\Delta_1$  contain only subformulas of  $F$ . Find a similar representation of  $\delta_2$  as

$$\frac{\begin{array}{c} \vdots \rho_2 \\ \Gamma_2 \Rightarrow B \end{array} \quad \begin{array}{c} \vdots \nu_2 \\ \Delta_2, A \Rightarrow \Pi_2 \end{array}}{\Gamma_2, \Delta_2, B \rightarrow A \Rightarrow \Pi_2} (\rightarrow_1)}{\begin{array}{c} \vdots \theta_2 \\ B \rightarrow A \Rightarrow F \end{array}}$$

Observe now that it is possible to perform a cut between the pairs of proofs  $(\rho_1, \nu_2)$  and  $(\rho_2, \nu_1)$ . By cut admissibility in LI we therefore obtain proofs  $\rho_1 \times \nu_2$  of  $\Delta_2, \Gamma_1 \Rightarrow \Pi_1$  and  $\rho_2 \times \nu_1$  of  $\Delta_1, \Gamma_2 \Rightarrow \Pi_2$ . From this we can transform  $\delta_1$  into the following proof  $\delta'_1$ :

$$\frac{\begin{array}{c} \vdots \rho_2 \times \nu_1 \\ \Delta_1, \Gamma_2 \Rightarrow \Pi_2 \end{array} \quad \begin{array}{c} \vdots \rho_1 \times \nu_2 \\ \Delta_2, \Gamma_1 \Rightarrow \Pi_1 \end{array}}{\Gamma_1, \Delta_1, \Delta_2, \Gamma_2 \Rightarrow \Pi_1} (\wedge_1)}{\frac{\Gamma_1, \Delta_1, \Delta_2, \Gamma_2 \Rightarrow \Pi_1}{\Gamma_1, \Delta_1, \wedge \Gamma_1 \rightarrow \wedge \Gamma_2 \Rightarrow \Pi_1} (\rightarrow_1)} (\wedge_1)}{\begin{array}{c} \vdots \theta_1 \\ \wedge \Gamma_1 \rightarrow \wedge \Gamma_2 \Rightarrow F \end{array}}$$

In a similar way we obtain a proof  $\delta'_2$  of  $\wedge \Gamma_2 \rightarrow \wedge \Gamma_1 \Rightarrow F$ , and then we arrive at the desired proof with a multiset-bounded cut:

$$\frac{\Rightarrow (\wedge \Gamma_1 \rightarrow \wedge \Gamma_2) \vee (\wedge \Gamma_2 \rightarrow \wedge \Gamma_1) \quad \frac{\begin{array}{c} \vdots \delta'_1 \\ \wedge \Gamma_1 \rightarrow \wedge \Gamma_2 \Rightarrow F \end{array} \quad \begin{array}{c} \vdots \delta'_2 \\ \wedge \Gamma_2 \rightarrow \wedge \Gamma_1 \Rightarrow F \end{array}}{(\wedge \Gamma_1 \rightarrow \wedge \Gamma_2) \vee (\wedge \Gamma_2 \rightarrow \wedge \Gamma_1) \Rightarrow F} (\vee_1)}{\Rightarrow F} (\text{cut})$$

**Other calculi.** The diagram above suggests a correspondence

$$\begin{array}{ccc} \text{cut elimination in} & & \text{reduction to} \\ \text{hypersequent calculus} & \cong & \text{multiset-bounded cuts} \\ & & \text{in sequent calculus} \end{array}$$

An obvious way to proceed from the results presented here is to investigate more expressive proof systems. Take for example the nested sequent calculus. How can we translate cutfree proofs in the nested sequent calculus to the sequent calculus, and which kind of cuts have to be introduced?

$$\begin{array}{ccc} \text{cut elimination in} & & \text{reduction to} \\ \text{nested sequent calculus} & \cong & ? \\ & & \text{in sequent calculus} \end{array}$$

One can speculate that the necessary cuts are more complicated than the multiset-bounded cuts introduced in exchange for the hypersequent structure. Even more speculatively, it might be possible to introduce a hierarchy of proof systems, where the calculi are compared by the cuts which are ‘hidden’ in their structure.

**Proof complexity.** The proof transformation involved in the projection theorem leads to an exponential blow-up in proof size. We do not know if this blow-up is necessary, or if it can be avoided by a better method. More generally, it would be interesting to know the difference between a cutfree hypersequent system and a sequent system with bounded cuts in terms of their proof size.

As a concrete example consider the standard hypersequent system  $\mathbf{HLI} + (\text{com})$  for Gödel logic  $\mathbf{G}$  and compare it to a sequent system  $\mathbf{seqG}$  extending  $\mathbf{LI}$  with the rule

$$\frac{\Gamma, A \rightarrow B \Rightarrow \Pi \quad \Gamma, B \rightarrow A \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} (\text{com})'$$

where both  $A$  and  $B$  are subformulas of  $\Gamma \Rightarrow \Pi$ . Using the results of this chapter, it can be seen that  $\mathbf{seqG}$  is complete for  $\mathbf{G}$ . Are there polynomial simulations between the proofs in  $\mathbf{seqG}$  and cutfree proofs in  $\mathbf{HLI} + (\text{com})$ ?<sup>10</sup>

<sup>10</sup>One reason to believe that  $\mathbf{seqG}$  is more efficient is that the rule  $(\text{com})'$  allows for a form of *deep inference*, as  $A$  and  $B$  need not be immediate subformulas of the lower sequent. This shows in proofs of theorems such as

$$\underbrace{((A \rightarrow B) \wedge \dots \wedge (A \rightarrow B))}_n \vee \underbrace{((B \rightarrow A) \wedge \dots \wedge (B \rightarrow A))}_n.$$

Here a  $\mathbf{seqG}$ -proof takes about  $2n$  inferences, whereas  $\mathbf{HLI} + (\text{com})$  needs roughly  $n^2$  steps. The problem in  $\mathbf{HLI} + (\text{com})$  is that the (redundant)  $\wedge$ 's have to be decomposed *before* the communication rule can be applied.

# Provability Games and SELL

## 3.1 Introduction

In this chapter we describe a game semantics for proof search in subexponential linear logic. This is not to be seen as a fully-fledged semantics for linear logic as in [10] or [2], but rather as an extension of the game-theoretic view on proof search: The view that there is a player (the *proponent*) who tries to demonstrate provability of a sequent by cleverly applying rules of the calculus. According to this view a proof is nothing but a winning strategy for proponent in the ‘provability game’.

Our semantics extends the game-theoretic view by two ingredients.

First, the various subtasks which arise in the course of proof search are combined in two different ways. Premises of multiplicative rules simply aggregate tasks. Premises of additive rules on the other hand will be taken as *alternative conjunctions* of tasks: Another element of the game (which can be seen as an opponent, or nature, or even chance) decides which task is to be executed, but the proponent has no control over this choice. So to accomplish an alternative conjunction of tasks, one must be prepared to accomplish each individual task *in principle*; factually however, only one task will be executed. This distinction between two types of conjunctions pays homage to the motivation between the additive/multiplicative distinction in linear logic already expressed in Girard’s seminal article [23].

Second, we will stipulate that the use of the dereliction rule will result in costs for proponent, as indicated by the label of the exponent of the principal formula. This allows us to not only talk about the existence of proofs (that is, winning strategies), but also their associated costs.

The interplay of both features gives rise to a rather expressive framework. After laying down the game-theoretic foundations, we show how this framework can be syntactically

captured by using labelled sequents. We then conclude with some proof-theoretic results on the labelled sequent system, including a restricted cut elimination theorem.

## 3.2 Two Perspectives on Proof Search

Let us discuss informally the notion of a *task*. A task can be pretty much everything, from reading a book over buying groceries to constructing a proof in the sequent calculus.

We employ the notion of a *stage* as an abstract representation of an environment in which tasks are carried out. Tasks are accomplished by extending the current stage to some possibly different successor stage. For example, in buying groceries I transition to a stage in which I am richer in groceries, but also slightly poorer and a bit older than in the initial stage.

Sets of tasks can be joined into a single combined task in various ways. We are interested in the following two combinations of tasks, which are best described by the conditions under which they can be accomplished.

- The *cumulative conjunction* of  $t_1$  and  $t_2$  can be accomplished at a stage  $s$  if  $s$  can be extended to a stage  $s'$  where both  $t_1$  and  $t_2$  are accomplished.
- The *alternative conjunction* of  $t_1$  and  $t_2$  can be accomplished at a stage  $s$  if  $s$  can be extended both to a stage  $s'$  where  $t_1$  is accomplished, and to a stage  $s''$  where  $t_2$  is accomplished.

Expressed as a task, the cumulative conjunction is simply ‘doing both  $t_1$  and  $t_2$ ’. The alternative conjunction is slightly more complex: We may think of it as the task of tossing a coin and then doing  $t_1$  if the result is heads, and doing  $t_2$  if the result is tails. So in order to accomplish the alternative conjunction of  $t_1$  and  $t_2$  at some stage we must be prepared to accomplish both  $t_1$  and  $t_2$ , even though only one task will be executed in the end.

Consider now the aforementioned task of proving a sequent in the sequent calculus. In this context, the presence of a rule such as

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} (\wedge_r)$$

tells us that the task of proving  $\Gamma \Rightarrow A \wedge B$  can be reduced to the combination of two tasks: Proving  $\Gamma \Rightarrow A$ , and proving  $\Gamma \Rightarrow B$ . This combination of provability is, in its most natural interpretation, a cumulative conjunction: We really write down both subproofs of  $\Gamma \Rightarrow A$  and  $\Gamma \Rightarrow B$  to get a complete proof.

From these considerations we can arrive at a dynamic, or game-theoretic, view of proof search. The ‘game’ is just the game a student of proof theory plays when she is trying to prove a sequent  $\Gamma \Rightarrow \Pi$ . She has a given choice of rules, and needs to apply the rules

(bottom-up) strategically to arrive at the axioms. Whenever a binary rule is invoked, a new task is created. The student might try to first prove the left premise of the rule and then work on the right premise, or do it the other way round, or even jump between the proofs of the left and the right premise at will. In this way, provability tasks keep accumulating. If the student eventually manages to solve all open tasks – that is, reduce them to initial sequents – she has completed the proof (and won the provability game).

Can we also interpret the two premises  $\Gamma \Rightarrow A$  and  $\Gamma \Rightarrow B$  as an alternative conjunction of tasks? The answer is yes, and a good corresponding picture is this: The student is challenged to prove  $\Gamma \Rightarrow \Pi$  in an exam, but now for time constraints she does not have to write down the complete proof. Instead, every time she wants to apply a binary rule, the examiner will tell her which of the premises she should provide a proof for (and the other premise will be discarded). The combination of the premise tasks is now an *alternative conjunction*, where the role of the coin is taken by the examiner. Speaking in the language of proof search, only one branch of the proof search tree is expanded in the examination scenario, but crucially, the student does not know in advance which branch it is.

It is clear that by sheer luck (or incompetence of the examiner), the student might be able to ‘prove’ sequents in the examination scenario which are not actually valid. But there is no way to always win this game—that is, to have a winning strategy—unless the sequent is valid.

Thus, as long as we are only interested in the accomplishability of a provability task, it does not matter whether the branching in rules is interpreted as alternative or cumulative conjunction of tasks. In fact, we may even interpret some branchings in proofs as alternative and others as cumulative combinations.

This conflation of concepts will always appear when considering only tasks which do not bring about a change of the state, that is, tasks without *side effects*. There is simply no difference between being theoretically able to accomplish a task and actually doing it if the task has no effect.

For a change of scenario, assume now that the student is given the task to prove the sequent  $(A \wedge C), (B \wedge C) \Rightarrow A \wedge B$  in LI with the additional constraint that she may use the rule  $(\wedge_l)$  only once. Now the provability task has side effects: Using  $(\wedge_l)$  once prevents the student from using  $(\wedge_l)$  again. In this scenario, the proof

$$\frac{\frac{A, (B \wedge C) \Rightarrow A}{(A \wedge C), (B \wedge C) \Rightarrow A} (\wedge_l) \quad \frac{(A \wedge C), B \Rightarrow B}{(A \wedge C), (B \wedge C) \Rightarrow B} (\wedge_l)}{(A \wedge C), (B \wedge C) \Rightarrow A \wedge B} (\wedge_r)$$

denotes a valid strategy for the student *only if* the branching is interpreted as the alternative conjunction of tasks – in the cumulative interpretation,  $(\wedge_l)$  will inevitably be used twice. In the examination scenario the student is fine because only one of the two branches will be expanded, and therefore only one  $(\wedge_l)$  will be used.

It is not always possible in the examination scenario to tell how often the student will use a certain rule, even if her strategy is fixed in advance. For example, if she proves the sequent

$$(A \wedge C), ((B \wedge D) \wedge C) \Rightarrow A \wedge B$$

according to the strategy

$$\frac{\frac{A, ((B \wedge D) \wedge C) \Rightarrow A}{(A \wedge C), ((B \wedge D) \wedge C) \Rightarrow A} (\wedge_l) \quad \frac{\frac{(A \wedge C), B \Rightarrow B}{(A \wedge C), (B \wedge D) \Rightarrow B} (\wedge_l)}{(A \wedge C), ((B \wedge D) \wedge C) \Rightarrow B} (\wedge_r)}{(A \wedge C), ((B \wedge D) \wedge C) \Rightarrow A \wedge B} (\wedge_r)$$

she might have to use  $(\wedge_l)$  either once or twice, depending on whether the examiner demands to see the left or the right branch to be expanded. What can be stated with certainty is the upper bound on the uses of  $(\wedge_l)$  (here 2).

We thus see that it is the presence of side effects which fills the cumulative/alternative distinction with life. In the present chapter, we want to investigate the task of proof search in a sequent calculus where some branchings are interpreted as alternative and others as cumulative conjunctions of tasks; and furthermore, the application of certain rules has side effects.

The sequent calculus we base our investigation on is a variant of linear logic which goes by the name *affine intuitionistic subexponential linear logic* [19]. This calculus has a *dereliction rule* ( $dr^\gamma$ ) which is parametric over a ‘subexponential’  $\gamma$ , which will be a non-negative real number in our case. The side effect we want to model is then: Every time the rule ( $dr^\gamma$ ) is used, costs of  $\gamma$  have to be paid.

### 3.3 Subexponential Linear Logic

Affine intuitionistic subexponential linear logic is an extension of **aILL** (see Section 1.8) by *subexponentials*. In subexponential linear logic the exponential ‘!’ is replaced by a family of labelled exponentials ‘!’ $^\gamma$ ’ where the label  $\gamma$  is taken from some partially ordered set. We will limit ourselves to the case  $\gamma \in \mathbb{R}^+$ , where  $\mathbb{R}^+$  is the set of non-negative real numbers together with their natural ordering. Furthermore we will build our calculus on top of the variant **aILL**<sup>\*</sup> of **aILL** (see Section 1.8).

The calculus **aSELL**( $\mathbb{R}^+$ ) has a standard dereliction rule

$$\frac{\Gamma, !^\gamma A, A \Rightarrow \Pi}{\Gamma, !^\gamma A \Rightarrow \Pi} (dr^\gamma)$$

for each labelled exponential, while the promotion rule in **aSELL**( $\mathbb{R}^+$ ) becomes

$$\frac{\Omega \geq \gamma \Rightarrow A}{\Omega \geq \gamma \Rightarrow !^\gamma A} (pr)$$

where  $\Omega^{\geq \gamma} := \{!^{\delta}B \in \Omega \mid \gamma \leq \delta\}$ . All other logical rules are as in  $\mathbf{aILL}^*$ , and the complete system is pictured in Figure 3.1. Here the schematic variable  $!\Omega$  denotes any multiset of unbounded formulas, irrespective of their labelling.  $\Gamma, \Delta$  and  $\Pi$  may contain bounded and unbounded formulas alike.

Note two deviations from the presentation of  $\mathbf{aILL}^*$  in Section 1.8: First, we admit only  $p \Rightarrow p$  as an initial sequent where  $p$  is a variable, instead of the more general  $A \Rightarrow A$  for arbitrary formulas  $A$ . Second, we do not include the cut rule.

*Initial sequents:*

$$\overline{p \Rightarrow p} \text{ (id)} \quad \overline{\perp \Rightarrow \Pi} \text{ } (\perp) \quad \overline{\Gamma \Rightarrow \top} \text{ } (\top)$$

*Structural rules:*

$$\frac{\Gamma, A, B, \Delta \Rightarrow \Pi}{\Gamma, B, A, \Delta \Rightarrow \Pi} \text{ (e)} \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma, A \Rightarrow \Pi} \quad \frac{\Gamma \Rightarrow \Pi}{\Gamma \Rightarrow \Pi} \text{ (w)}$$

*Propositional rules:*

$$\frac{\Gamma, A \Rightarrow \Pi \quad \Gamma, B \Rightarrow \Pi}{\Gamma, A \vee B \Rightarrow \Pi} \text{ } (\vee_l) \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \text{ } (\vee_r) \text{ } (i = 1, 2)$$

$$\frac{\Gamma, A_i \Rightarrow \Pi}{\Gamma, A_1 \wedge A_2 \Rightarrow \Pi} \text{ } (\wedge_l) \text{ } (i = 1, 2) \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \text{ } (\wedge_r)$$

$$\frac{\Gamma, A, B \Rightarrow \Pi}{\Gamma, A * B \Rightarrow \Pi} \text{ } (*_l) \quad \frac{!\Omega, \Gamma \Rightarrow A \quad !\Omega, \Delta \Rightarrow B}{!\Omega, \Gamma, \Delta \Rightarrow A * B} \text{ } (*_r)$$

$$\frac{!\Omega, \Gamma \Rightarrow A \quad !\Omega, \Delta, B \Rightarrow \Pi}{!\Omega, \Gamma, \Delta, A \rightarrow B \Rightarrow \Pi} \text{ } (\rightarrow_l) \quad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \text{ } (\rightarrow_r)$$

*Exponential rules:*

$$\frac{\Gamma, !^{\gamma}A, A \Rightarrow \Pi}{\Gamma, !^{\gamma}A \Rightarrow \Pi} \text{ } (dr^{\gamma}) \quad \frac{!\geq \gamma \Omega \Rightarrow A}{!\geq \gamma \Omega \Rightarrow !^{\gamma}A} \text{ } (pr)$$

Figure 3.1: The calculus  $\mathbf{aSELL}(\mathbb{R}^+)$

### Theorem 3.1

The cut rule

$$\frac{!\Omega, \Gamma \Rightarrow A \quad !\Omega, \Delta, A \Rightarrow \Pi}{!\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

is admissible in  $\mathbf{aSELL}(\mathbb{R}^+)$ .

*Proof.* Follows from the cut elimination theorem in [19] and Proposition 1.11.  $\square$

### 3.4 A Formal Task Semantics

As mentioned before, we want to extend proof search in  $\mathbf{aSELL}(\mathbb{R}^+)$  by the following: A distinction between cumulative and alternative conjunction of provability tasks, and a notion of side effects.

Let us first speak about the side effects. Inspecting the rules of  $\mathbf{aSELL}(\mathbb{R}^+)$ , we see that the structure of  $\mathbb{R}^+$  is only referred to in the right rule (promotion) for the subexponentials, but not in the left rule

$$\frac{\Gamma, !^\gamma A, A \Rightarrow \Pi}{\Gamma, !^\gamma A \Rightarrow \Pi} (\text{dr}^\gamma).$$

To add a layer of expressivity, we will stipulate that an application of dereliction ( $\text{dr}^\gamma$ ) comes with a side effect, namely the paying of ‘costs’  $\gamma$ . More intuitively, we think of an application of ( $\text{dr}^\gamma$ ) as the ‘unboxing’ of the formula  $A$  (for further use), where the subexponential index  $\gamma$  indicates the costs of this unboxing.

Concerning the difference between cumulative and alternative tasks, we split the branching rules of  $\mathbf{aSELL}(\mathbb{R}^+)$  into two groups:

**alternative** :  $(\wedge_r), (\vee_l)$

**cumulative** :  $(*_r), (\rightarrow_l)$

The choice is not random: it follows the distinction between additive and multiplicative rules and their intuitive *resource interpretation* in linear logic. In a well-known example, Girard [23] described two possible meanings of being able to ‘buy a pack of Marlboros and buy a pack of Camels’: In one meaning, both packs have to be bought, while in the other either pack has to be bought (but not both). Corresponding to this distinction Girard motivated two different format of rules for logical conjunction. Writing  $*$  for the first and  $\wedge$  for the second interpretation of ‘both’, they are:

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A * B} (*)_r \quad \text{and} \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} (\wedge)_r$$

The *multiplicative format* in  $(*)_r$  demands a splitting of the context  $\Gamma \cup \Delta$  into two parts as intuitively both  $A$  and  $B$  have to be obtained, and so a decision must be made which premise is used for which formula. No such splitting is necessary in the *additive format* of the rule  $(\wedge)_r$ , as both  $A$  and  $B$  have to be obtainable only in principle, but not at the same time. Closing the circle and coming back to our notation, the two premises of  $(*)_r$  are seen as a cumulative conjunction of tasks, whereas the two premises of  $(\wedge)_r$  are seen as an alternative conjunction of tasks.

We are now in a position to state a formal semantics for proof search in  $\mathbf{aSELL}(\mathbb{R}^+)$  with the aforementioned extensions.



**Definition 3.2** (multitask semantics)

A *multitask* is a non-empty multiset of  $\mathbf{aSELL}(\mathbb{R}^+)$ -sequents, interpreted as a cumulative conjunction of provability tasks. A sequent  $\Gamma \Rightarrow \Pi$  is identified with the multitask  $\{\Gamma \Rightarrow \Pi\}$ . A multitask is *solved* if all sequents in it are initial sequents. A *reduction step* on a multitask  $\Sigma$  involves the following moves by two players **Prop** and **Opp**, after which a *successor multitask*  $\Sigma'$  is reached:

1. **Prop** chooses a sequent occurrence  $S \in \Sigma$  and a rule instance  $(r)$  of  $\mathbf{aSELL}(\mathbb{R}^+)$  such that  $S$  is the conclusion of  $(r)$ . If  $(r) = (dr^\gamma)$ , then **Prop** has to pay  $\gamma$ .
2. Writing  $\Sigma = \{S\} \cup \Pi$ , the successor multitask  $\Sigma'$  is determined as follows:
  - If  $(r)$  is a unary rule with premise  $S'$ , then  $\Sigma' = \{S'\} \cup \Pi$ .
  - If  $(r)$  is a cumulative rule with premises  $S_1$  and  $S_2$ , then  $\Sigma' = \{S_1, S_2\} \cup \Pi$ .
  - If  $(r)$  is an alternative rule with premises  $S_1$  and  $S_2$ , then player **Opp** chooses whether  $\Sigma' = \{S_1\} \cup \Pi$  or  $\Sigma' = \{S_2\} \cup \Pi$ .
3. Unless  $\Sigma'$  is solved, the steps (1)-(3) are repeated.

Given  $\alpha \in \mathbb{R}^+$  and a multitask  $\Sigma$ , we let  $\models^\alpha \Sigma$  denote the fact that **Prop** can move in the reduction steps so that no matter how **Opp** plays a solved multitask will be obtained eventually, and **Prop**'s total costs never exceed  $\alpha$ . A strategy witnessing this will be called an  $\alpha$ -*bounded winning strategy*. Write  $\models \Sigma$  if  $\models^\alpha \Sigma$  for some  $\alpha \in \mathbb{R}^+$ .

For example, the following equivalences are easy to check:

$$\begin{aligned} \models^\alpha (!^\gamma p \Rightarrow p * p) &\iff \alpha \geq 2 \cdot \gamma \\ \models^\alpha (!^\gamma p \Rightarrow p \wedge p) &\iff \alpha \geq \gamma \end{aligned}$$

Forgetting about the label in  $\models^\alpha$ , it is not difficult to show the following correspondence:

**Theorem 3.3** (unlabelled completeness)

For any  $\mathbf{aSELL}(\mathbb{R}^+)$ -sequent  $S$ ,

$$\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S \iff \models S.$$

Since a stronger correspondence will be established later on (Theorem 3.4 Theorem 3.7), we omit the proof at this point.

### 3.5 $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$

We now want to extend Theorem 3.3 to the notion  $\models^\alpha$ , which calls for a syntactic counterpart to the  $\models^\alpha$ -relation. In the present section we develop a calculus  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  of labelled sequents

$$\alpha \parallel \Gamma \Rightarrow \Pi$$

where  $\alpha \in \mathbb{R}^+$ . The rules of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  will be chosen such that

$$\models^\alpha S \iff \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S.$$

The first rule of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  is *label weakening*

$$\frac{\alpha \parallel \Gamma \Rightarrow \Pi}{\beta \parallel \Gamma \Rightarrow \Pi} \text{ (l-weak)}$$

which has the side condition  $\alpha \leq \beta$ . (l-weak) expresses the upwards monotonicity of  $\models^\alpha$ .

All further rules of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  are now obtained by a suitable labelling of the corresponding rules of  $\mathbf{aSELL}(\mathbb{R}^+)$ .

For an initial sequent  $S$  of  $\mathbf{aSELL}(\mathbb{R}^+)$ , the task  $S$  can be immediately accomplished and no costs are paid. Consequently we admit  $\alpha \parallel S$  as initial sequents of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  where  $\alpha \in \mathbb{R}^+$  is arbitrary.<sup>1</sup>

The labelled version of (dr)

$$\frac{\alpha \parallel \Gamma, !^\gamma A, A \Rightarrow \Pi}{\alpha + \gamma \parallel \Gamma, !^\gamma A \Rightarrow \Pi} \text{ (dr)}$$

expresses that whenever the task  $\Gamma, !^\gamma A, A \Rightarrow \Pi$  can be accomplished with costs not exceeding  $\alpha$ , then the task  $\Gamma, A \Rightarrow \Pi$  can be accomplished with costs not exceeding  $\alpha + \gamma$ . This is undoubtedly true, as the latter task can be reduced to the former one by a single application of  $(\text{dr}^\gamma)$ , which costs  $\gamma$ .

Let us turn now to branching rules. Assume first that the labelled sequents

$$\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \text{and} \quad \beta \parallel !\Omega, \Delta \Rightarrow B$$

are valid, which means that the task  $!\Omega, \Gamma \Rightarrow A$  can be accomplished with costs not exceeding  $\alpha$ , and  $!\Omega, \Delta \Rightarrow B$  can be accomplished with costs not exceeding  $\beta$ . Doing both tasks after another, the costs do not exceed  $\alpha + \beta$ . Thus our labelling of  $(*_r)$  is the following:

$$\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \beta \parallel !\Omega, \Delta \Rightarrow B}{\alpha + \beta \parallel !\Omega, \Gamma, \Delta \Rightarrow A * B} \text{ } (*_r)$$

<sup>1</sup>Recall that  $\alpha$  is only supposed to denote an *upper bound* on the costs.

Assume now that the labelled sequents

$$\alpha \parallel \Gamma \Rightarrow A \quad \text{and} \quad \beta \parallel \Gamma \Rightarrow B$$

are valid. For which  $\gamma$  do we then know that  $\gamma \parallel \Gamma \Rightarrow A \wedge B$  is valid? By applying  $(\wedge_r)$ , we know that the task  $\Gamma \Rightarrow A \wedge B$  can be reduced to the additive conjunction of the tasks  $\Gamma \Rightarrow A$  and  $\Gamma \Rightarrow B$ . Since only one of these tasks is to be executed, any  $\gamma$  which is an upper bound to the costs in both tasks will be an upper bound for the additive combination. Therefore the labelling of  $(\wedge_r)$  is as follows:

$$\frac{\alpha \parallel \Gamma \Rightarrow A \quad \beta \parallel \Gamma \Rightarrow B}{\max\{\alpha, \beta\} \parallel \Gamma \Rightarrow A \wedge B} (\wedge_r)$$

In all unary rules different from dereliction, no costs are paid and no additional tasks are created. Hence any upper bound for the cost of the premise also serves as an upper bound for the costs of the conclusion. For example, the labelling of  $(\rightarrow_r)$  is:

$$\frac{\alpha \parallel \Gamma, A \Rightarrow B}{\alpha \parallel \Gamma \Rightarrow A \rightarrow B} (\rightarrow_r)$$

To sum up, the labelling of  $\mathbf{aSELL}(\mathbb{R}^+)$ -rules goes about as follows:

rule	premise(s) label(s)	conclusion label
initial sequent	-	$\alpha$ (any)
unary rule $\neq$ (dr)	$\alpha$	$\alpha$
(dr) with principal formula $! \gamma A$	$\alpha$	$\alpha + \gamma$
binary, multiplicative	$\alpha, \beta$	$\alpha + \beta$
binary, additive	$\alpha, \beta$	$\max\{\alpha, \beta\}$

The complete labelled system, to which we will refer to as  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ , is pictured in Figure 3.2.

From the preceding discussion, the soundness of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  with respect to the task semantics is clear.

**Theorem 3.4** (soundness of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ )

If  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel \Gamma \Rightarrow \Pi$ , then  $\models^\alpha \Gamma \Rightarrow \Pi$ .

### The Completeness Theorem

Since the task semantics is by definition very close to  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ , the completeness proof for  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  is easier than comparable proofs for other logics. Nevertheless one complication arises, and this complication is of conceptual interest.

*Initial sequents:*

$$\frac{}{\alpha \parallel p \Rightarrow p} \text{ (id)} \quad \frac{}{\alpha \parallel \perp \Rightarrow \Pi} \text{ (\perp)} \quad \frac{}{\alpha \parallel \Gamma \Rightarrow \top} \text{ (\top)}$$

*Structural rules:*

$$\frac{\alpha \parallel \Gamma, A, B, \Delta \Rightarrow \Pi}{\alpha \parallel \Gamma, B, A, \Delta \Rightarrow \Pi} \text{ (e)} \quad \frac{\alpha \parallel \Gamma \Rightarrow \Pi}{\alpha \parallel \Gamma, A \Rightarrow \Pi} \quad \frac{\alpha \parallel \Gamma \Rightarrow}{\alpha \parallel \Gamma \Rightarrow \Pi} \text{ (w)}$$

*Label weakening:*

$$\frac{\alpha \parallel \Gamma \Rightarrow \Pi}{\beta \parallel \Gamma \Rightarrow \Pi} \text{ (l-weak), where } \alpha \leq \beta$$

*Propositional rules:*

$$\frac{\alpha \parallel \Gamma, A \Rightarrow \Pi \quad \beta \parallel \Gamma, B \Rightarrow \Pi}{\max\{\alpha, \beta\} \parallel \Gamma, A \vee B \Rightarrow \Pi} \text{ (\vee_l)} \quad \frac{\alpha \parallel \Gamma \Rightarrow A_i}{\alpha \parallel \Gamma \Rightarrow A_1 \vee A_2} \text{ (\vee_r) (i = 1, 2)}$$

$$\frac{\alpha \parallel \Gamma, A_i \Rightarrow \Pi}{\alpha \parallel \Gamma, A_1 \wedge A_2 \Rightarrow \Pi} \text{ (\wedge_l) (i = 1, 2)} \quad \frac{\alpha \parallel \Gamma \Rightarrow A \quad \beta \parallel \Gamma \Rightarrow B}{\max\{\alpha + \beta\} \parallel \Gamma \Rightarrow A \wedge B} \text{ (\wedge_r)}$$

$$\frac{\alpha \parallel \Gamma, A, B \Rightarrow \Pi}{\alpha \parallel \Gamma, A * B \Rightarrow \Pi} \text{ (*_l)} \quad \frac{\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \beta \parallel !\Omega, \Delta \Rightarrow B}{\alpha + \beta \parallel !\Omega, \Gamma, \Delta \Rightarrow A * B} \text{ (*_r)}$$

$$\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \beta \parallel !\Omega, \Delta, B \Rightarrow \Pi}{\alpha + \beta \parallel !\Omega, \Gamma, \Delta, A \rightarrow B \Rightarrow \Pi} \text{ (\rightarrow_l)} \quad \frac{\alpha \parallel \Gamma, A \Rightarrow B}{\alpha \parallel \Gamma \Rightarrow A \rightarrow B} \text{ (\rightarrow_r)}$$

*Exponential rules:*

$$\frac{\alpha \parallel \Gamma, !^\gamma A, A \Rightarrow \Pi}{\alpha + \gamma \parallel \Gamma, !^\gamma A \Rightarrow \Pi} \text{ (dr}^\gamma\text{)} \quad \frac{\alpha \parallel !^{\geq \gamma} \Omega \Rightarrow A}{\alpha \parallel !^{\geq \gamma} \Omega \Rightarrow !^\gamma A} \text{ (pr)}$$

 Figure 3.2: The labelled calculus  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ 

As one would expect, the completeness proof proceeds by induction on the maximal number of steps (the *height*) of an  $\alpha$ -bounded winning strategy  $\sigma$  for **Prop**. In particular, we will have to consider the case that the task is  $\Rightarrow A * B$  and that **Prop**'s first step is to choose the rule  $(*_r)$ . In the task semantics, the successor multitask is then  $\{\Rightarrow A, \Rightarrow B\}$ . For this multitask,  $\sigma$  must therefore contain a 'substrategy'  $\sigma'$ , and this strategy will still be  $\alpha$ -bounded. In order to apply the induction hypothesis we need to establish the following:

1. There is a pair  $\sigma_1, \sigma_2$  of winning strategies for  $\Rightarrow A$  and  $\Rightarrow B$  respectively.
2. The height of  $\sigma_1, \sigma_2$  is smaller than the height of  $\sigma$ .
3. The sum of the cost bounds for  $\sigma_1, \sigma_2$  does not exceed the bound for  $\sigma$ .

Given (1)-(3), we obtain by induction hypothesis proofs of  $\alpha_1 \parallel \Rightarrow A$  and  $\alpha_2 \parallel \Rightarrow B$  for some numbers  $\alpha_1, \alpha_2 \in \mathbb{R}^+$  satisfying  $\alpha_1 + \alpha_2 \leq \alpha$ . Then by applying the labelled rule  $(*_r)$  we obtain  $\alpha_1 + \alpha_2 \parallel \Rightarrow A * B$ , and from a further (l-weak), we arrive at the desired sequent  $\alpha \parallel \Rightarrow A * B$ .

Concerning (1), the problem is that in general there is *no* reason to believe that  $\sigma'$  encodes in an obvious way a pair of strategies in  $\Rightarrow A$  and  $\Rightarrow B$  respectively: Since we have left it completely open how **Prop** organizes her strategies in multitasks, the moves she makes in one subtask may depend on the moves in the other subtask. These naturally arising interdependencies impede a trivial extraction of the desired substrategies  $\sigma_1, \sigma_2$ .

This complication could have been altogether avoided by postulating that **Prop**'s strategies in different subtasks have to be *independent*. But this solution is somewhat artificial, and apart from that, it is unnecessary. Instead we will prove below that it is indeed possible to extract independent substrategies for  $\Rightarrow A$  and  $\Rightarrow B$  from *any* strategy in  $\{\Rightarrow A, \Rightarrow B\}$  by using a slightly more involved argument. Furthermore, we show that such strategies can be extracted in a way that the sum of their cost bounds does not exceed the cost bound on the strategy for  $\{\Rightarrow A, \Rightarrow B\}$ .

The key observation is this. Given a strategy  $\sigma'$  for **Prop** in  $\{\Rightarrow A, \Rightarrow B\}$ , every strategy  $\pi_B$  for **Opp** in  $\Rightarrow B$  induces a **Prop**-strategy  $\sigma(\pi_B)$  in  $\Rightarrow A$  as follows: **Prop** starts moving according to  $\sigma'$  *as if* she were solving  $\{\Rightarrow A, \Rightarrow B\}$  instead of  $\Rightarrow A$ . However, only the moves in  $\Rightarrow A$  are recorded. Whenever a move of **Opp** in  $\Rightarrow B$  is required, **Prop** simulates this move herself using the fixed strategy  $\pi_B$ . In a similar manner, we can get a **Prop**-strategy  $\sigma(\pi_A)$  in  $\Rightarrow B$  from any **Opp**-strategy  $\pi_A$  in  $\Rightarrow A$ .<sup>2</sup>

The last thing we will have to check concerns the costs. We will see that by choosing the **Opp**-strategies  $\pi_B$  and  $\pi_A$  so that the bounds  $\alpha_1, \alpha_2$  of the induced **Prop**-strategies  $\sigma(\pi_B)$  and  $\sigma(\pi_A)$  are minimal, their sum  $\alpha_1 + \alpha_2$  does not exceed  $\alpha$ .

To make the argument in the preceding paragraphs precise, we introduce a formal representation of a **Prop**-strategy  $\sigma$  as a finite tree. The root of this tree is the (multi)task to be solved, and its edges denote moves by either player. A branching node corresponds to a choice of **Opp**, and its child nodes are the outcomes of all possible choices of **Opp**. No branching occurs at points where **Prop** is to make a decision, since such a decision is fixed by the strategy  $\sigma$ . The height of  $\sigma$  can then be defined as the height of its associated tree (which is the length of a longest branch in it). In what follows, we will tacitly identify  $\sigma$  with its tree representation.

<sup>2</sup>A similar idea appears in the 'parallel composition and hiding' interpretation of the cut rule in game semantics, see [1].

As an example, Figure 3.3 pictures the tree representation of a strategy  $\sigma_0$  for the multitask  $\{(!^1p, !^2p \Rightarrow p), (r \wedge s \Rightarrow r \wedge s)\}$ . We will refer to  $\{!^1p, !^2p \Rightarrow p\}$  and  $\{r \wedge s \Rightarrow r \wedge s\}$  as the first and second subtask respectively.

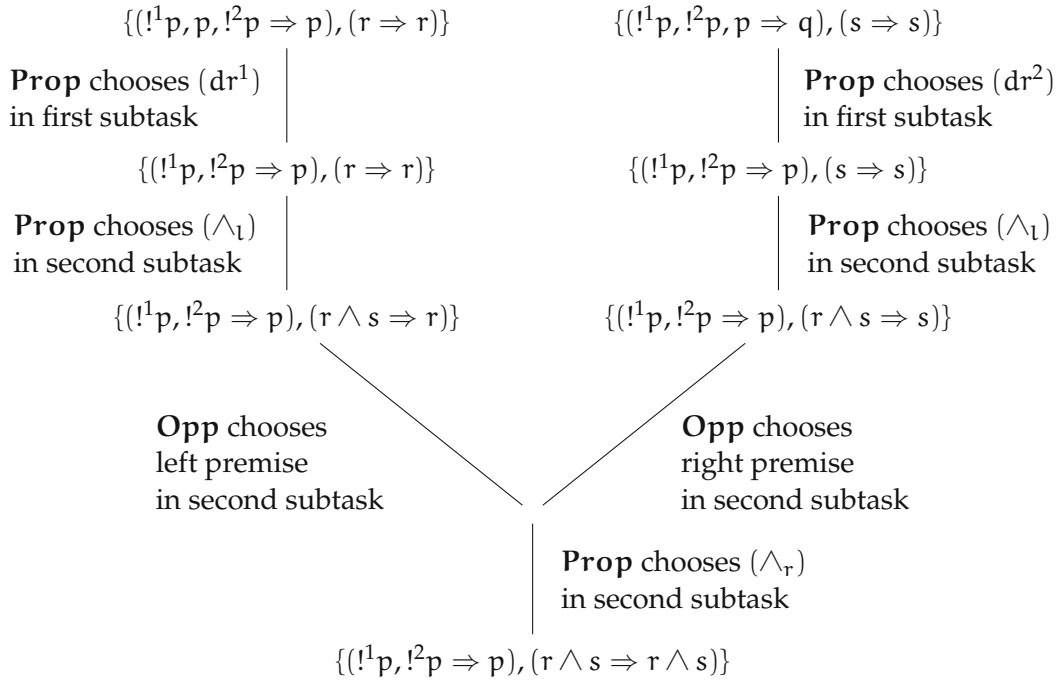
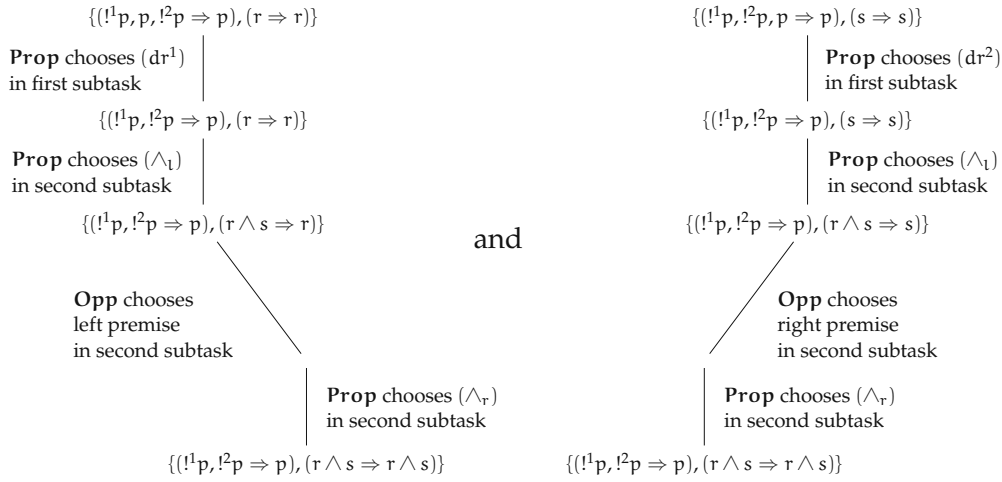


Figure 3.3: A winning strategy for **Prop** in the multitask  $\{(!^1p, !^2p \Rightarrow p), (r \wedge s \Rightarrow r \wedge s)\}$ .

We will now define some additional notions specific to strategies  $\sigma$  in multitasks  $\{S_1, S_2\}$  with two components, and use the strategy  $\sigma_0$  of Figure 3.3 as a running example throughout.

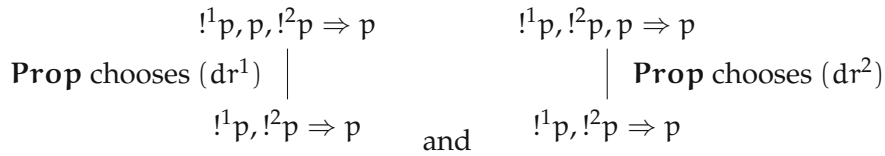
For  $i \in \{1, 2\}$ , the set  $\text{idet}(\sigma)$  of *i-determinations* is the set of all subtrees of  $\sigma$  which arise by pruning one outgoing branch from each branching node in  $\sigma$  where the outgoing edges are moves of **Opp** in  $S_i$ . Intuitively, a tree  $T \in \text{idet}(\sigma)$  arises from  $\sigma$  by fixing a strategy of **Opp** in  $S_i$ . Note that higher up in the strategy tree, the subtask  $S_i$  may again be subdivided into several new tasks. We count moves in all these subtasks as moves in  $S_i$ .

In the example of  $\sigma_0$ , there is no splitting pertaining to a move of **Opp** in  $S_1$ , and so  $\sigma_0$  has only one 1-determination which is  $\sigma_0$  itself. But  $\sigma_0$  has two 2-determinations:

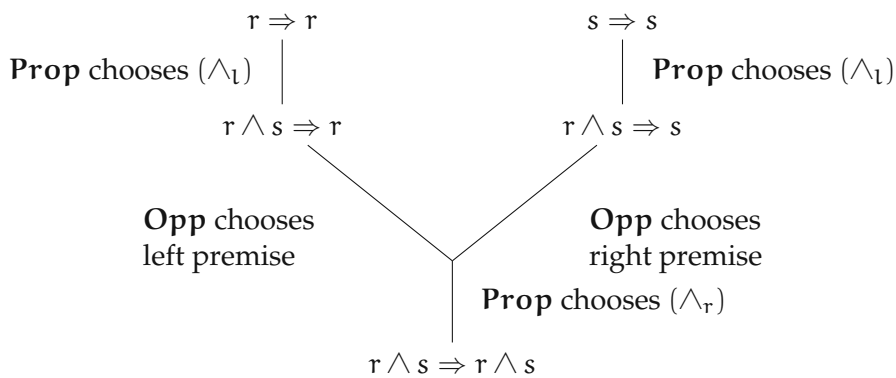


For  $i \in \{0, 1\}$  let  $\bar{i}$  be the unique element of  $\{0, 1\} \setminus \{i\}$ . If we define an  $i$ -*reduct*  $\text{ired}(T)$  of a determination  $T \in \bar{i}\text{det}(\sigma)$  by removing each edge corresponding to a move in the subtask  $S_{\bar{i}}$  (and glueing together the remaining parts appropriately), we obtain a tree representation of a strategy for **Prop** in  $S_i$ .

For  $\sigma_0$ , the 1-reducts of the two 2-determinations are



which are both **Prop**-strategies in the first subtask  $!^1p, !^2p \Rightarrow p$ . The 2-reduct of (the 1-determination of)  $\sigma_0$  is



which is a **Prop**-strategy in the second subtask  $r \wedge s \Rightarrow r \wedge s$ .

Let  $\text{cost}(\sigma)$  denote the maximal sum of costs (in any subgame) arising in a branch of  $\sigma$ . In other words,  $\text{cost}(\sigma)$  is the smallest  $\alpha \in \mathbb{R}^+$  such that  $\sigma$  is  $\alpha$ -bounded. Define

$$c_i(\sigma) = \min_{T \in \bar{\text{idet}}(\sigma)} \text{cost}(\text{ired}(T)).$$

An *optimal  $\sigma$ -induced  $i$ -strategy* is then a strategy  $\text{ired}(T)$  where  $T \in \bar{\text{idet}}(\sigma)$  witnesses the value of  $c_i(\sigma)$ , that is,  $\text{cost}(\text{ired}(T)) = c_i(\sigma)$ .

In our example, the optimal  $\sigma_0$ -induced 1-strategy is

$$\begin{array}{c} !^1p, p, !^2p \Rightarrow p \\ \text{Prop chooses } (dr^1) \mid \\ !^1p, !^2p \Rightarrow p \end{array}$$

and therefore  $c_1(\sigma) = 1$ . The optimal  $\sigma_0$ -induced 2-strategy is the 2-reduct of  $\sigma_0$  (see above), and since no derelictions occur,  $c_2(\sigma) = 0$ .

**Lemma 3.5**

For every **Prop**-strategy  $\sigma$  in a multitask  $\{S_1, S_2\}$ ,

$$c_1(\sigma) + c_2(\sigma) \leq \text{cost}(\sigma).$$

Before coming to the proof of this lemma, let us first state a corollary which will be key to the completeness proof.

**Corollary 3.6** (splitting lemma)

Let  $\sigma$  be an  $\alpha$ -bounded strategy for **Prop** in the multitask  $\{S_1, S_2\}$ . Then for  $i \in \{1, 2\}$  there is a strategy  $\sigma_i$  and  $\alpha_i \in \mathbb{R}^+$  satisfying the following:

1.  $\sigma_i$  is an  $\alpha_i$ -bounded strategy for **Prop** in  $S_i$ .
2.  $h(\sigma_i) \leq h(\sigma)$
3.  $\alpha_1 + \alpha_2 \leq \alpha$

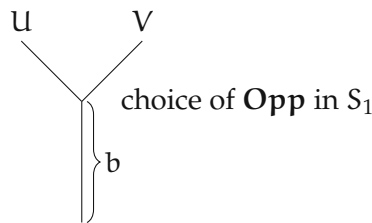
*Proof of the corollary.* For  $\sigma_i$  we take any optimal  $\sigma$ -induced  $i$ -strategy, and then set  $\alpha_i := c(\sigma_i) = c_i(\sigma)$ . By Lemma 3.5 we have  $\alpha_1 + \alpha_2 \leq \alpha$ . Finally the statement about the height of  $\sigma_1$  and  $\sigma_2$  follows from the fact that both strategies arise by pruning the tree  $\sigma$ .  $\square$



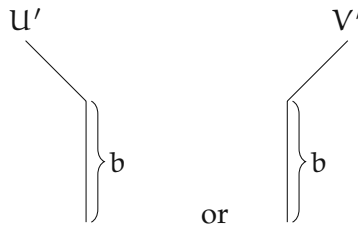
*Proof of Lemma 3.5.* We show the inequality by induction on the number of branchings in the tree representation of  $\sigma$ . If there is none, then  $1\text{det}(\sigma) = 2\text{det}(\sigma) = \{\sigma\}$  and the statement becomes

$$\text{cost}(1\text{red}(\sigma)) + \text{cost}(2\text{red}(\sigma)) \leq \text{cost}(\sigma)$$

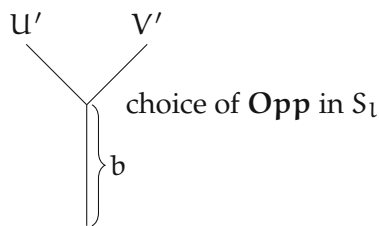
which is true because  $\sigma$  is a branch, and thus we actually have equality. Otherwise, let  $b$  be the branch of  $\sigma$  connecting the root to the first branching node, and let  $U$  and  $V$  the two subtrees stemming from that node. Both subtrees have lesser height than  $\sigma$ . Assume furthermore, without loss of generality, that the branching corresponds to a choice of  $\mathbf{Opp}$  in  $S_1$ . So  $\sigma$  looks like this:



We make the following observations: Every 1-determination of  $T$  is of the form



where  $U' \in 1\text{det}(U)$  and  $V' \in 1\text{det}(V)$ , and every 2-determination of  $T$  is of the form



where  $U' \in 2\text{det}(U)$  and  $V' \in 2\text{det}(V)$ . Consequently we have:

$$\begin{aligned} c_1(T) &= \min\{c_1(U), c_1(V)\} + \text{cost}(1\text{red}(b)) \\ c_2(T) &= \max\{c_2(U), c_2(V)\} + \text{cost}(2\text{red}(b)) \end{aligned}$$

Assume, again without loss of generality, that  $c_2(T) = c_2(U) + \text{cost}(2\text{red}(b))$ . Then we have

$$\begin{aligned}
 c_1(T) + c_2(T) &= \min\{c_1(U), c_1(V)\} + c_2(U) + \text{cost}(1\text{red}(b)) + \text{cost}(2\text{red}(b)) \\
 &= \min\{c_1(U), c_1(V)\} + c_2(U) + \text{cost}(b) \\
 &\leq c_1(U) + c_2(U) + \text{cost}(b) \\
 &\stackrel{\text{I.H.}}{\leq} c(U) + \text{cost}(b) \\
 &\leq \text{cost}(T). \quad \square
 \end{aligned}$$

We now have all necessary tools to prove the completeness theorem.

**Theorem 3.7** (completeness of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ )

For any sequent  $S$ ,  $\models^\alpha S$  implies  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S$ .

*Proof.* By induction on the height of an  $\alpha$ -bounded **Prop**-strategy  $\sigma$  for  $S$ .

The base case is that the task  $S$  is immediately accomplished, which holds only if  $S$  is an initial sequent of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ . But then  $\alpha \parallel S$  is an initial sequent of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ , and so we have  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S$ .

For the other cases, let us proceed by case distinction on the first move of **Prop** in  $S$  according to  $\sigma$ .

1. If **Prop**'s first move is to pick a unary rule instance  $(r) \neq (dr^\gamma)$  of  $\mathbf{aSELL}(\mathbb{R}^+)$  with premise  $S'$ , then  $\sigma$  must contain a strategy for the subtask  $S'$ . This strategy has lesser height and must also be  $\alpha$ -bounded because no costs arise by the choice of  $(r)$ . By induction hypothesis, we obtain  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S'$ , and then by applying  $(r)$  we obtain  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S$ .
2. If **Prop**'s first move is to pick the rule  $(dr^\gamma)$  with premise  $S'$ ,  $\sigma$  must contain a strategy for the subtask  $S'$ . This strategy will be of lesser height and  $(\alpha - \gamma)$ -bounded because of the cost  $\gamma$  arising by the choice of  $(dr^\gamma)$ . By induction hypothesis, we obtain  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} (\alpha - \gamma) \parallel S'$  and then  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S$  by applying  $(dr^\gamma)$ .
3. Assume **Prop**'s first move is to pick a binary additive rule instance  $(r)$  of  $\mathbf{aSELL}(\mathbb{R}^+)$  with premises  $S_1$  and  $S_2$ . By the definition of the task semantics, **Prop**'s task switches to either  $S_1$  or  $S_2$  in the subsequent round (as chosen by **Opp**). Since  $\sigma$  is  $\alpha$ -bounded, neither task can result in costs  $> \alpha$  for **Prop** if she follows the

strategy  $\sigma$  in the subtasks. So we have  $\models^\alpha S_1$  and  $\models^\alpha S_2$ , and so by induction hypothesis  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S_1$  and  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S_2$ . Finally we obtain  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \max\{\alpha, \alpha\} \parallel S$  by applying (r), and of course  $\max\{\alpha, \alpha\} = \alpha$ .<sup>3</sup>

4. Assume **Prop**'s first move is to pick a binary additive rule instance (r) of  $\mathbf{aSELL}(\mathbb{R}^+)$  with premises  $S_1$  and  $S_2$ . Then  $\sigma$  must contain a strategy  $\sigma'$  of lesser height for the subsequent multitask  $\{S_1, S_2\}$ , which is still  $\alpha$ -bounded. By the splitting lemma (Corollary 3.6), for  $i \in \{1, 2\}$  there exists an  $\alpha_i$ -bounded strategy  $\sigma_i$  for  $S_i$  such that  $h(\alpha_i) \leq h(\sigma') < h(\sigma)$  and such that  $\alpha_1 + \alpha_2 \leq \alpha$ . By the induction hypothesis, we obtain proofs of  $\alpha_1 \parallel S_1$  and  $\alpha_2 \parallel S_2$ , which can then be combined into a proof of  $\alpha_1 + \alpha_2 \parallel S$  using (r), and then by applying (l-weak) (if necessary) we obtain  $\alpha \parallel S$ .

□

We close this section by presenting two example applications of the labelled system.

### Example 3.8

Consider the following well-known riddle:

You have white and black socks in a drawer in a completely dark room. How many socks do you have to take out blindly to be sure of having a matching pair?

We can model the matching pair by the disjunction  $M := (w * w) \vee (b * b)$ , and the act of drawing a random sock by the formula  $!(w \vee b)$ . The above question then becomes: For which  $n$  is

$$n \parallel !(w \vee b) \Rightarrow (w * w) \vee (b * b)$$

provable in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ ?

The following  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -proof sketch shows that  $n = 3$  drawings suffice:

<sup>3</sup>This part of the completeness proof suggests that instead of computing  $\max$ , we could have required equality of the labels in the premises of additive rule, i.e., employ a rule format like

$$\frac{\alpha \parallel \Gamma \Rightarrow A \quad \alpha \parallel \Delta \Rightarrow B}{\alpha \parallel \Gamma \Rightarrow A \wedge B} (\wedge_r)$$

for additive rules. This is indeed possible, as both formats are equivalent in the presence of (l-weak).

$$\begin{array}{c}
 \vdots \\
 \frac{0 \parallel !^1(w \vee b), w, w \Rightarrow w * w}{0 \parallel !^1(w \vee b), w, w \Rightarrow M} (\vee_r) \\
 \frac{\frac{0 \parallel !^1(w \vee b), w, b, w \Rightarrow w * w}{0 \parallel !^1(w \vee b), w, b, w \Rightarrow M} (\vee_r) \quad \frac{0 \parallel !^1(w \vee b), b, b, w \Rightarrow b * b}{0 \parallel !^1(w \vee b), b, b, w \Rightarrow M} (\vee_r)}{0 \parallel !^1(w \vee b), w \vee b, b, w \Rightarrow M} (\vee_l) \\
 \frac{0 \parallel !^1(w \vee b), w \vee b, b, w \Rightarrow M}{1 \parallel !^1(w \vee b), b, w \Rightarrow M} (\text{dr}^1) \\
 \frac{1 \parallel !^1(w \vee b), w \vee b, w \Rightarrow M}{2 \parallel !^1(w \vee b), w \Rightarrow M} (\vee_l) \\
 \frac{2 \parallel !^1(w \vee b), w \vee b \Rightarrow M}{3 \parallel !^1(w \vee b) \Rightarrow M} (\text{dr}^1) \\
 \vdots
 \end{array}$$

**Example 3.9** (labelled transition systems)

An  $\mathbb{R}^+$ -labelled transition system  $\mathcal{T} = (S, T)$  is a set  $S$  of states together with a set  $T$  of transitions

$$s \xrightarrow{\gamma} t$$

where  $s, t \in S$  and  $\gamma \in \mathbb{R}^+$ . We interpret  $\gamma$  as the time it takes the system to move from state  $s$  to state  $t$ . Identifying each  $s \in S$  with a propositional variable, we can assign to  $\mathcal{T}$  the set of formulas

$$\Omega(\mathcal{T}) := \{!^\gamma(s \rightarrow t) \mid (s \xrightarrow{\gamma} t) \in T\}$$

Then the timed reachability problem in  $\mathcal{T}$  is naturally encoded in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  in the following way:

$$t \text{ is reachable from } s \text{ in time } \leq \alpha \iff \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel \Omega(\mathcal{T}) \cup \{s\} \Rightarrow t$$

In fact the proof of  $\alpha \parallel \Omega(\mathcal{T}) \cup \{s\} \Rightarrow t$  is nothing but a description of the path from  $s$  to  $t$ . This correspondence can be extended to transition systems containing more complicated transitions. For example,

$$!^\gamma(s \rightarrow t_1 \vee t_2)$$

is a transition which takes time  $\gamma$  and leads non-deterministically either to state  $t_1$  or  $t_2$ .

### 3.6 Proof Theory of $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$

We now turn to some syntactical observations on  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ .

Define the *skeleton* of a  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -proof  $\delta$  as the result of removing all labels and all instances of (l-weak). Then the skeleton of  $\delta$  is an  $\mathbf{aSELL}(\mathbb{R}^+)$ -proof. Conversely, given a  $\mathbf{aSELL}(\mathbb{R}^+)$ -proof  $\delta$  of the sequent  $S$  we define its *frugal labelling* as follows: Add the label 0 to all initial sequents in  $\delta$  and then propagate the labels downward in the proof

according to the labelling rules of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ . This yields a  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -proof of  $\alpha \parallel S$  for some  $\alpha \in \mathbb{R}^+$ .

Hence:

**Proposition 3.10**

$$\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S \iff \exists \alpha \in \mathbb{R}^+ \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S.$$

By the completeness theorem, Proposition 3.10 is equivalent to

$$\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S \iff \exists \alpha \in \mathbb{R}^+ \models^\alpha \{S\}$$

which by definition of  $\models$  means

$$\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S \iff \models \{S\}$$

and this is exactly the ‘unlabelled completeness theorem’ that we already mentioned, but did not prove (Theorem 3.3).

**Lemma 3.11**

For any sequent  $S$  such that  $\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S$ , the set  $\{\alpha \mid \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S\}$  has a minimum.

*Proof.* By Proposition 3.10, the set  $\Lambda := \{\alpha \mid \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S\}$  is non-empty. But since there are usually infinitely many different proofs of  $S$ , it may not be immediately obvious that  $\Lambda$  takes a minimum. But we can argue as follows. Given any  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -proof  $\delta$  of  $\alpha \parallel S$ , the frugal labelling of the skeleton of  $\delta$  is an  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -proof of  $\alpha' \parallel S$ , and it is clear that  $\alpha' \leq \alpha$ . So in order to calculate the infimum of  $\Lambda$ , it suffices to consider proofs which arise as frugal labellings of  $\mathbf{aSELL}(\mathbb{R}^+)$ -proofs. All such proofs have the property that the label of their endsequent is a linear combination with integer coefficients of the finitely many reals  $\gamma$  which appear as supexponentials  $!^\gamma A$  in  $S$ . It is not difficult to see that the set of all such linear combinations is nowhere dense in the reals, and therefore  $\Lambda$ , being a subset of it, must take a minimal value.  $\square$

We are thus justified in stating the following definition.

**Definition 3.12**

For any sequent  $S$  such that  $\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S$ , we let

$$\text{cost}(S) := \min\{\alpha \mid \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S\}.$$

Note however that we have not described a way to compute the function  $S \mapsto \text{cost}(S)$ , and in fact we do not know how to achieve this. This problem will be included in the section on open questions.

**Proposition 3.13**

The rule (l-weak) is admissible in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ .

*Proof.* By a simple induction on the height of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -proofs. Let us just consider one case for illustration: Assume that a proof ends with

$$\frac{\alpha \parallel \Gamma, !^\gamma A, A \Rightarrow \Pi}{\alpha + \gamma \parallel \Gamma, !^\gamma A \Rightarrow \Pi} \text{ (dr}^\gamma\text{)}$$

We want to show that for a given  $\beta \geq \alpha + \gamma$ , the sequent  $\beta \parallel \Gamma, !^\gamma A \Rightarrow \Pi$  is derivable. Indeed, we have  $\beta - \gamma \geq \alpha$ , and from applying the induction hypothesis to the subderivation  $\delta$  with endsequent  $\alpha \parallel \Gamma, !^\gamma A, A \Rightarrow \Pi$  we can conclude that  $(\beta - \gamma) \parallel \Gamma, !^\gamma A, A \Rightarrow \Pi$  is derivable. Then by applying (dr) with principle formula  $!^\gamma A$ , the sequent  $\beta \parallel \Gamma, !^\gamma A \Rightarrow \Pi$  follows.  $\square$

By Theorem 3.1 the *cut rule*

$$\frac{!\Omega, \Gamma \Rightarrow A \quad !\Omega, \Delta, A \Rightarrow \Pi}{!\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

is admissible in  $\mathbf{aSELL}(\mathbb{R}^+)$ . In the remainder of this section we try to prove admissibility of labelled versions of (cut) in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ . To start with, note that if  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  proves

$$\alpha \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi \quad \text{and} \quad \beta \parallel !\Omega, \Delta, A \Rightarrow \Pi,$$

then (by Proposition 3.10)  $\mathbf{aSELL}(\mathbb{R}^+)$  proves  $!\Omega, \Gamma, \Delta \Rightarrow \Pi$  and  $!\Omega, \Delta, A \Rightarrow \Pi$ , and so by cut admissibility  $\mathbf{aSELL}(\mathbb{R}^+)$  also proves the sequent  $!\Omega, \Gamma, \Delta \Rightarrow \Pi$ . This in turns implies, again by Proposition 3.10, that  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  proves  $\alpha \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi$  for *some*  $\alpha$ , for example  $\alpha = \text{cost}(!\Omega, \Gamma, \Delta \Rightarrow \Pi)$ . For this reason the following labelled cut rule

$$\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \beta \parallel !\Omega, \Delta, A \Rightarrow \Pi}{\text{cost}(!\Omega, \Gamma, \Delta \Rightarrow \Pi) \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

is admissible in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ . But this is only of limited interest because we do not know how to calculate  $\text{cost}(!\Omega, \Gamma, \Delta \Rightarrow \Pi)$ . A more informative result for a restricted class of cuts is the following:

**Theorem 3.14** (simple cut elimination)

For any  $A$  not containing  $!$ , the following cut rule is admissible in  $\mathcal{L}\alpha\text{SELL}(\mathbb{R}^+)$ :

$$\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \beta \parallel !\Omega, \Delta, A \Rightarrow \Pi}{\alpha + \beta \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

*Proof.* We follow a standard cut-reduction and observe that it is compatible with the proposed labelling. In fact, since  $A$  does not contain  $!$ , the cut-reduction strategy is essentially that of  $\text{FL}_{\text{ew}}^\perp$ .

To reduce the number of cases we have to consider, we introduce the notion of a *maximal instance* of a multiplicative rule: This is an instance where all unbounded formulas in the antecedent of the conclusion are copied to both premises. For example, an instance of (cut) as in the statement of the theorem is maximal if no further unbounded formula occurs in  $\Gamma \cup \Delta$ . This restriction is harmless in the presence of  $(w_1)$ , and we will henceforth assume that all multiplicative rule instances are maximal.

Define as usual the *degree* of a cut as the number of connectives in the cut formula, and its *rank* as the sum of the heights of its subproofs. For technical reasons related to the use of maximal instances, we do not count  $(w_1)$ 's in the computation of the rank.

It suffices to eliminate a single cut. If the cut is on a variable  $p$  and its left premise is the initial sequent  $\alpha \parallel p \Rightarrow p$ , then the cut can be replaced by (l-weak). In all other cases, at least one of the reduction steps below is applicable, and after the reduction all occurring cuts are simpler in the following sense: They either have a lower degree than the original cut, or the same degree but a lower rank. It follows that the reduction methods terminates.

1. Consider first the case where the cut formula is principal in both premises. Here the cut can be replaced by one or two cuts of lower degree, depending on the principal connective of the cut formula.
  - a) If  $A = A_1 \wedge A_2$ , the proof looks like this:

$$\frac{\frac{\frac{\alpha_1 \parallel !\Omega, \Gamma \Rightarrow A_1 \quad \alpha_2 \parallel !\Omega, \Gamma \Rightarrow A_2}{\max\{\alpha_1, \alpha_2\} \parallel !\Omega, \Gamma \Rightarrow A_1 \wedge A_2} (\wedge_r)}{\max\{\alpha_1, \alpha_2\} + \beta \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} (\wedge_l)}{\beta \parallel !\Omega, \Delta, A_1 \wedge A_2 \Rightarrow \Pi} (\text{cut})$$

This cut is reduced as follows:

$$\frac{\frac{\alpha_1 \parallel !\Omega, \Gamma \Rightarrow A_1 \quad \beta \parallel !\Omega, \Delta, A_1 \Rightarrow \Pi}{\alpha_1 + \beta \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} (\text{cut})}{\max\{\alpha_1, \alpha_2\} + \beta \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} (\text{l-weak})$$

b) If  $A = A_1 * A_2$ , the proof looks like this:

$$\frac{\frac{\frac{\alpha_1 \parallel !\Omega, \Gamma_1 \Rightarrow A_1}{\alpha_1 + \alpha_2 \parallel !\Omega, \Gamma_1, \Gamma_2 \Rightarrow A_1 * A_2} \quad \frac{\alpha_2 \parallel !\Omega, \Gamma_2 \Rightarrow A_2}{\alpha_1 + \alpha_2 \parallel !\Omega, \Gamma_1, \Gamma_2 \Rightarrow A_1 * A_2} \quad (*_r) \quad \frac{\beta \parallel !\Omega, \Delta, A_1, A_2 \Rightarrow \Pi}{\beta \parallel !\Omega, \Delta, A_1 * A_2 \Rightarrow \Pi} \quad (*_l)}{\alpha_1 + \alpha_2 + \beta \parallel !\Omega, \Gamma_1, \Gamma_2, \Delta \Rightarrow \Pi} \quad (\text{cut})$$

Here we use the assumption on maximality of multiplicative rules: Otherwise, we would not know that the multiset  $!\Omega$  which is copied to both premises of (cut) is the same multiset which is copied to both premises of  $(*_r)$ . That being said, the cut can be reduced as follows:

$$\frac{\frac{\alpha_2 \parallel !\Omega, \Gamma_2 \Rightarrow A_2}{\alpha_1 + \alpha_2 + \beta \parallel !\Omega, \Gamma_1, \Gamma_2, \Delta \Rightarrow \Pi} \quad \frac{\frac{\alpha_1 \parallel !\Omega, \Gamma_1, [A_2] \Rightarrow A_1}{\alpha_1 + \beta \parallel !\Omega, \Gamma_1, A_2, \Delta \Rightarrow \Pi} \quad \frac{\beta \parallel !\Omega, \Delta, A_1, A_2 \Rightarrow \Pi}{\alpha_1 + \beta \parallel !\Omega, \Gamma_1, A_2, \Delta \Rightarrow \Pi} \quad (\text{cut})}{\alpha_1 + \alpha_2 + \beta \parallel !\Omega, \Gamma_1, \Gamma_2, \Delta \Rightarrow \Pi} \quad (\text{cut})$$

If  $A_2$  is unbounded, it must be copied to the left premise of the upper cut to retain maximality. In this case,  $A_2$  should be introduced immediately above by  $(w_l)$ . This optional introduction of an unbounded formula via weakening will be indicated by the notation  $[A_2]$  here and in the following reduction steps.

The remaining principal cases are similar.

2. Assume now that  $A$  is not principal in the left premise of (cut).

a) If the lowermost rule in the left premise of (cut) is  $(dr^\gamma)$ , then the proof looks like this:

$$\frac{\frac{\frac{\alpha' \parallel !\Omega, !^\gamma B, B, \Gamma \Rightarrow A}{\alpha' + \gamma \parallel !\Omega, !^\gamma B, \Gamma \Rightarrow A} \quad (dr^\gamma) \quad \frac{\beta \parallel !\Omega, !^\gamma B, \Delta, A \Rightarrow \Pi}{\alpha' + \gamma + \beta \parallel !\Omega, !^\gamma B, \Gamma, \Delta \Rightarrow \Pi} \quad (\text{cut})}{\alpha' + \gamma + \beta \parallel !\Omega, !^\gamma B, \Gamma, \Delta \Rightarrow \Pi} \quad (\text{cut})$$

The cut reduction is as follows:

$$\frac{\frac{\alpha' \parallel !\Omega, !^\gamma B, B, \Gamma \Rightarrow A}{\alpha' + \beta \parallel !\Omega, !^\gamma B, B, \Gamma, \Delta \Rightarrow \Pi} \quad \frac{\beta \parallel !\Omega, !^\gamma B, [B], \Delta, A \Rightarrow \Pi}{\alpha' + \beta \parallel !\Omega, !^\gamma B, B, \Gamma, \Delta \Rightarrow \Pi} \quad (\text{cut})}{\alpha' + \gamma + \beta \parallel !\Omega, !^\gamma B, \Gamma, \Delta \Rightarrow \Pi} \quad (dr^\gamma)$$

Here it is important that we do not count  $(w_l)$ 's in the computation of the rank, as otherwise the above proof might not be smaller in case  $B$  has to be introduced by weakening above the upper right premise.



- b) If the lowermost inference in the left premise of (cut) is  $(\vee_1)$ , then the proof looks like this:

$$\frac{\frac{\frac{\alpha_1 \parallel !\Omega, \Gamma, B_1 \Rightarrow A \quad \alpha_2 \parallel !\Omega, \Gamma, B_2 \Rightarrow A}{\max\{\alpha_1, \alpha_2\} \parallel !\Omega, \Gamma, B_1 \vee B_2 \Rightarrow A} (\wedge_r) \quad \beta \parallel !\Omega, \Delta, A \Rightarrow \Pi}{\max\{\alpha_1, \alpha_2\} + \beta \parallel !\Omega, \Gamma, B_1 \vee B_2, \Delta \Rightarrow \Pi} (\text{cut})$$

The cut can be reduced as follows:

$$\frac{\frac{\alpha_1 \parallel !\Omega, \Gamma, B_1 \Rightarrow A \quad \beta \parallel !\Omega, \Delta, [B_1, ]A \Rightarrow \Pi}{\alpha_1 + \beta \parallel !\Omega, \Gamma, B_1, \Delta \Rightarrow \Pi} (\text{cut}) \quad \frac{\alpha_2 \parallel !\Omega, \Gamma, B_2 \Rightarrow A \quad \beta \parallel !\Omega, \Delta, [B_2, ]A \Rightarrow \Pi}{\alpha_2 + \beta \parallel !\Omega, \Gamma, B_2, \Delta \Rightarrow \Pi} (\text{cut})}{\max\{\alpha_1 + \beta, \alpha_2 + \beta\} \parallel !\Omega, \Gamma, B_1 \vee B_2, \Delta \Rightarrow \Pi} (\vee_1)$$

Note that  $\max\{\alpha_1 + \beta, \alpha_2 + \beta\} = \max\{\alpha_1, \alpha_2\} + \beta$ .

- c) If the lowermost inference in the left premise is  $(\rightarrow_1)$ , then the proof looks like this:

$$\frac{\frac{\frac{\alpha_1 \parallel !\Omega, \Gamma_1 \Rightarrow B_1 \quad \alpha_2 \parallel !\Omega, \Gamma_2, B_2 \Rightarrow A}{\alpha_1 + \alpha_2 \parallel !\Omega, \Gamma_1, \Gamma_2, B_1 \rightarrow B_2 \Rightarrow A} (\rightarrow_1) \quad \beta \parallel !\Omega, \Delta, A \Rightarrow \Pi}{\alpha_1 + \alpha_2 + \beta \parallel !\Omega, \Gamma_1, \Gamma_2, B_1 \rightarrow B_2, \Delta \Rightarrow \Pi} (\text{cut})$$

The cut can be reduced as follows:

$$\frac{\frac{\alpha_1 \parallel !\Omega, \Gamma_1 \Rightarrow B_1 \quad \frac{\alpha_2 \parallel !\Omega, \Gamma_2, B_2 \Rightarrow A \quad \beta \parallel !\Omega, \Delta, [B_2, ]A \Rightarrow \Pi}{\alpha_2 + \beta \parallel !\Omega, \Gamma_2, B_2 \Rightarrow \Pi} (\text{cut})}{\alpha_1 + \alpha_2 + \beta \parallel !\Omega, \Gamma_1, \Gamma_2, B_1 \rightarrow B_2, \Delta \Rightarrow \Pi} (\rightarrow_1)$$

The other cases are similar. Note that when we shift the cut above  $(w_1)$ , due to our way of counting the rank we must actually shift the cut above a maximal sequence of  $(w_1)$ 's and do one further reduction step in order to obtain a smaller cut.

3. Assume now that  $A$  is not principal in the right premise of (cut).

- a) Assume that the lowermost inference in the right subproof of (cut) is  $(*_r)$ . Since  $A$  is bounded, it will appear only in one premise of  $(*_r)$ . Assume it is in the left premise. Then the proof looks like this:

$$\frac{\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \frac{\beta_1 \parallel !\Omega, A, \Delta_1 \Rightarrow B_1 \quad \beta_2 \parallel !\Omega, \Delta_2 \Rightarrow B_2}{\beta_1 + \beta_2 \parallel !\Omega, \Delta_1, \Delta_2, A \Rightarrow B_1 * B_2} (*_r)}{\alpha + \beta_1 + \beta_2 \parallel !\Omega, \Gamma, \Delta_1, \Delta_2 \Rightarrow B_1 * B_2} (\text{cut})$$

The cut is reduced as follows:

$$\frac{\frac{\alpha \parallel \vdots \! \Omega, \Gamma \Rightarrow A \quad \beta_1 \parallel \vdots \! \Omega, \Delta_1 \Rightarrow B_1}{\alpha + \beta_1 \parallel \! \Omega, \Gamma, \Delta_1 \Rightarrow B_1} \text{ (cut)} \quad \beta_2 \parallel \vdots \! \Omega, \Delta_2 \Rightarrow B_2}{\alpha + \beta_1 + \beta_2 \parallel \! \Omega, \Gamma, \Delta_1, \Delta_2 \Rightarrow B_1 * B_2} (*_r)$$

The other cases are similar. Note that there is no subcase where the lowermost inference in the right subproof of (cut) is (pr), as this is ruled out by the bound-  
edness of  $A$ .  $\square$

One reason that the proof of Theorem 3.14 goes through is that the labels behave in a way similar to the contexts. This can be seen as an a posteriori justification for identifying multiplicative rules with cumulative tasks and additive rules with alternative tasks. To be a bit more formal, define operations on multisets of formulas as follows:

$$\Gamma_1 \uplus \Gamma_2 := \text{multiset union of } \Gamma_1 \text{ and } \Gamma_2$$

$$\Gamma_1 \sqcup \Gamma_2 := \text{the smallest multiset containing } \Gamma_1 \text{ and } \Gamma_2 \text{ as submultisets}$$

Noting first that in the presence of ( $w_l$ ), ( $\wedge_r$ ) can be replaced by its variant

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma \sqcup \Delta \Rightarrow A \wedge B} (\wedge'_r)$$

we have the following labelled versions of the multiplicative ( $*_r$ ) and the additive ( $\wedge'_r$ ):

$$\frac{\alpha_1 \parallel \Gamma \Rightarrow A \quad \alpha_2 \parallel \Delta \Rightarrow B}{\alpha_1 + \alpha_2 \parallel \Gamma \uplus \Delta \Rightarrow A * B} (*_r) \quad \text{and} \quad \frac{\alpha_1 \parallel \Gamma \Rightarrow A \quad \alpha_2 \parallel \Delta \Rightarrow B}{\max\{\alpha_1, \alpha_2\} \parallel \Gamma \sqcup \Delta \Rightarrow A \wedge B} (\wedge'_r)$$

Hence there is an analogy in the pairs  $+/\uplus$ , and  $\max/\sqcup$ . Similarly, the fact that the cut rule in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  is labelled  $\alpha + \beta$  (as opposed to  $\max\{\alpha, \beta\}$ ) corresponds to the fact that (cut) is a multiplicative rule.

The labelling of (cut) proposed in Theorem 3.14 is minimal. This we can observe from the following example:  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  proves

$$\gamma \parallel !^\gamma p \Rightarrow p \quad \text{and} \quad \delta \parallel p, !^\delta (p \rightarrow q) \Rightarrow q$$

and letting  $S$  denote their cut conclusion  $!^\gamma p, !^\delta (p \rightarrow q) \Rightarrow q$  we have

$$\text{cost}(S) = \gamma + \delta.$$

#### Example 3.15 (labelled transition systems, part 2)

Consider again the encoding of labelled transition systems from Example 3.9. By Theorem 3.14, the rule

$$\frac{\alpha \parallel \Omega(\mathcal{J}), s \Rightarrow t \quad \beta \parallel \Omega(\mathcal{J}), t \Rightarrow u}{\alpha + \beta \parallel \Omega(\mathcal{J}), s \Rightarrow u} \text{ (cut)}$$

is admissible in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ . In the semantics of labelled transition systems, Theorem 3.14 expresses a transitivity property: If  $t$  is reachable from  $s$  in time  $\leq \alpha$  (as witnessed by some path  $\rho_1$ ) and  $u$  is reachable from  $t$  in time  $\leq \beta$  (as witnessed by some path  $\rho_2$ ), then  $u$  is reachable from  $s$  in time  $\leq \alpha + \beta$ . By inspecting the cut elimination procedure, one can observe that the cutfree proof of  $\alpha + \beta \parallel \Omega(\mathcal{T}), s \Rightarrow u$  will describe the concatenation of the paths  $\rho_1$  and  $\rho_2$ .

If  $A$  contained negative occurrences of  $!$ , there would be another principal case to be taken care of in the cut reduction. Consider for example the cut

$$\frac{\frac{\alpha \parallel \Rightarrow B}{\alpha \parallel \Rightarrow !\gamma B} \text{ (pr)} \quad \frac{\beta \parallel \Delta, !\gamma B, B \Rightarrow \Pi}{\beta + \gamma \parallel \Delta, !\gamma B \Rightarrow \Pi} \text{ (dr)}}{\alpha + \beta + \gamma \parallel \Delta \Rightarrow \Pi} \text{ (cut)}$$

This is usually reduced to

$$\frac{\frac{\alpha \parallel \Rightarrow B}{\alpha \parallel \Rightarrow B} \quad \frac{\alpha \parallel \Rightarrow !\gamma B \quad \beta \parallel \Delta, !\gamma B, B \Rightarrow \Pi}{\alpha + \beta \parallel \Delta, B \Rightarrow \Pi} \text{ (cut)}}{2\alpha + \beta \parallel \Delta \Rightarrow \Pi} \text{ (cut)}$$

where the upper cut has smaller rank, and the lower cut has smaller degree than the original cut. But this does not work in the labelled setting, since the label  $2\alpha + \beta$  is not necessarily smaller than the original label  $\alpha + \beta + \gamma$ . As it turns out, this problem goes deeper than the particular choice of labelling:

**Theorem 3.16** (No simple cut labelling)

There is *no* function  $f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that the cut rule

$$\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow A \quad \beta \parallel !\Omega, \Delta, A \Rightarrow \Pi}{f(\alpha, \beta) \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

is admissible in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  for arbitrary  $A$ .

*Proof.* Let  $p, q$  be different variables and let  $A^{*n}$  denote the  $n$ -fold multiplicative conjunction of  $A$ . Consider the sequents

$$! \frac{1}{k} p \Rightarrow ! \frac{1}{k} p^{*(k \cdot \alpha)} \quad \text{and} \quad ! \frac{1}{k} p^{*(k \cdot \alpha)} \Rightarrow p^{*(k \cdot k \cdot \alpha \cdot \beta)},$$

where  $k, \alpha$  and  $\beta$  are non-zero natural numbers. They have costs  $\alpha$  and  $\beta$  respectively. Their cut conclusion  $S$  is the sequent

$$! \frac{1}{k} p \Rightarrow p^{*(k \cdot k \cdot \alpha \cdot \beta)},$$

and  $\text{cost}(S) = k \cdot \alpha \cdot \beta$ . Since every function  $f$  in  $\alpha, \beta$  (the labels of the premises of cut) is dominated by  $k \cdot \alpha \cdot \beta$  when choosing  $k$  large enough, there cannot be a sound labelling of the cut rule which depends only on  $\alpha$  and  $\beta$ .  $\square$

Theorem 3.16 tells us that in order to find an admissible labelled cut rule, we must either

1. restrict the shape of the cut formula, and/or
2. allow the labelling function  $f$  to take more information of the premises into account than just their labels.

Theorem 3.14 is an instance of the first approach, where the cut formula was required to be  $!$ -free. We will now consider another, and less limiting, syntactic restriction on the cut formula.

**Definition 3.17** (simply exp-labelled)

A formula of the form  $!^\gamma A$  is *simply exp-labelled* if  $\gamma \neq 0$  and no further  $!$  appears in  $A$ .

The cut formula used as a counterexample in the proof of Theorem 3.16 is simply exp-labelled, and therefore we cannot expect to find an admissible cut rule for all simply exp-labelled cut formulas where the labelling only takes the labels of the premises into account. If we however lift the second requirement and also use the information of the label  $\gamma$  in the simply ex-labelled formula  $!^\gamma A$ , we do succeed.

First, one preliminary lemma.

**Lemma 3.18**

If  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel \Gamma, !^\gamma A \Rightarrow \Pi$  for some  $\alpha < \gamma$ , then  $\vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel \Gamma \Rightarrow \Pi$ .

*Proof.* Let  $\delta$  be a  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -proof of  $\alpha \parallel \Gamma, !^\gamma A \Rightarrow \Pi$  where  $\alpha < \gamma$ . Then every label in  $\delta$  is smaller than  $\gamma$ , and so  $!^\gamma A$  can never be principal in an application of  $(\text{dr}^\gamma)$ . Furthermore, since  $!^\gamma A$  is not atomic, it cannot appear in an initial sequent.<sup>4</sup> It follows that we can simply remove the denoted occurrence of  $!^\gamma A$ , as well as all its ancestors and applications of  $(w_\perp)$  stemming from them, from the proof  $\delta$ .  $\square$

**Theorem 3.19**

For any simply exp-labelled formula  $!^\gamma A$ , the following cut rule is admissible

<sup>4</sup>Recall that the initial sequents of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  are  $\alpha \parallel p \Rightarrow p$ .

in  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ :

$$\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow !^\gamma A \quad \beta \parallel !\Omega, \Delta, !^\gamma A \Rightarrow \Pi}{f(\alpha, \beta, \gamma) \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

where  $f(\alpha, \beta, \gamma) = \beta + \lfloor \beta/\gamma \rfloor \cdot \alpha$ .

Here  $\lfloor \xi \rfloor$  denotes the largest integer smaller or equal than  $\xi \in \mathbb{R}^+$ . The intuition for this labelling is as follows. Assume for simplicity that all rules in the right subproof of (cut) are multiplicative. Since the costs of this subproof are bounded by  $\beta$ , the cut formula  $!^\gamma A$  can be principal in an instance of  $(\text{dr}^\gamma)$  at most  $\lfloor \beta/\gamma \rfloor$  many times in it. Each of these times, the usual cut reduction will create a cut on  $A$  with the left subproof. Since  $A$  does not contain  $!$  by assumption, each of these  $\leq \lfloor \beta/\gamma \rfloor$  cuts can be eliminated resulting in additional costs of  $\alpha$  by Theorem 3.14, and so the total cost of the elimination is no more than  $\lfloor \beta/\gamma \rfloor \cdot \alpha$ . These costs are then added to the cost  $\beta$  of the original proof.

Note also that the proposed labelling of cut is consistent with the counterexample used in the proof of Theorem 3.16: For non-zero natural numbers  $\alpha, \beta$  and  $k$  we have

$$\frac{\alpha \parallel !^{\frac{1}{k}} p \Rightarrow !^{\frac{1}{k}} p^{*(k \cdot \alpha)} \quad \beta \parallel !^{\frac{1}{k}} p^{*(k \cdot \alpha)} \Rightarrow p^{*(k \cdot k \cdot \alpha \cdot \beta)}}{\beta + \lfloor \beta / (\frac{1}{k}) \rfloor \cdot \alpha \parallel !^{\frac{1}{k}} p \Rightarrow p^{*(k \cdot k \cdot \alpha \cdot \beta)}} \text{ (cut)}$$

and

$$\beta + \lfloor \beta / (\frac{1}{k}) \rfloor \cdot \alpha = \beta + k \cdot \beta \cdot \alpha \geq k \cdot \alpha \cdot \beta = \text{cost}(!^{\frac{1}{k}} p \Rightarrow p^{*(k \cdot k \cdot \alpha \cdot \beta)}).$$

*Proof of Theorem 3.19.* It suffices to consider proofs which contain only one such cut. As in the proof of Theorem 3.14, we assume that all multiplicative rules are maximal. The proof proceeds by induction on the rank of the cut, that is the sum of the heights of its subproofs. As in Theorem 3.14, we do not count  $(w_1)$ 's in the computation of the rank.

We first isolate two cases in which the cut can be removed immediately. If the cut formula was introduced via  $(w_1)$  immediately above the right premise of cut

$$\frac{\alpha \parallel !\Omega, \Gamma \Rightarrow !^\gamma A \quad \frac{\beta \parallel !\Omega, \Delta \Rightarrow \Pi}{\beta \parallel !\Omega, \Delta, !^\gamma A \Rightarrow \Pi} (w_1)}{f(\alpha, \beta, \gamma) \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

then the cut can be removed as follows:

$$\frac{\beta \parallel !\Omega, \Delta \Rightarrow \Pi}{\beta \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} (w_1)}{f(\alpha, \beta, \gamma) \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (l-weak)}$$

Here we use  $\beta \leq f(\alpha, \beta, \gamma)$ . A similar argument applies if the cut formula is introduced above the left premise of cut via  $(w_r)$ .

Second, we can remove any cut where the labelling obeys  $\beta < \gamma$  using Lemma 3.18:

$$\frac{\begin{array}{c} \vdots \\ \alpha \parallel !\Omega, \Gamma \Rightarrow !^\gamma A \end{array} \quad \begin{array}{c} \vdots \\ \beta \parallel !\Omega, \Delta, !^\gamma A \Rightarrow \Pi \end{array}}{f(\alpha, \beta, \gamma) \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (cut)}$$

is replaced by

$$\frac{\begin{array}{c} \vdots \\ \beta \parallel !\Omega, \Delta \Rightarrow \Pi \end{array} \text{ (w}_l\text{)}}{\beta \parallel !\Omega, \Gamma, \Delta \Rightarrow \Pi} \text{ (l-weak)}$$

Hence we can from now on assume that  $\beta \geq \gamma$ , and therefore  $\lfloor \beta/\gamma \rfloor \geq 1$ . In all other cases we now show how a cut can be replaced by one or two cuts of lower rank. Any sequence of such reduction steps eventually leads to one of the two cases above where the cut is finally removed. The reduction steps are all standard, but additionally we always have to make sure that the label is not increased after the reduction.

1. First consider the main case in which the cut formula is principal in both premises. We show how this cut can be shifted upwards in the proof and thereby decreased in rank. So assume the cut looks like this:

$$\frac{\begin{array}{c} \vdots \\ \alpha \parallel !^{\geq \gamma} \Omega \Rightarrow A \end{array} \text{ (pr)} \quad \begin{array}{c} \vdots \\ \beta' \parallel !^{\geq \gamma} \Omega, \Delta, !^\gamma A, A \Rightarrow \Pi \end{array} \text{ (dr)}}{\alpha \parallel !^{\geq \gamma} \Omega \Rightarrow !^\gamma A \quad \beta' + \gamma \parallel !^{\geq \gamma} \Omega, \Delta, !^\gamma A \Rightarrow \Pi} \text{ (cut)}$$

$$f(\alpha, \beta' + \gamma, \gamma) \parallel !^{\geq \gamma} \Omega, \Delta \Rightarrow \Pi$$

We transform this cut to

$$\frac{\begin{array}{c} \vdots \\ \alpha \parallel !^{\geq \gamma} \Omega \Rightarrow A \end{array} \quad \frac{\begin{array}{c} \vdots \\ \alpha \parallel !^{\geq \gamma} \Omega \Rightarrow !^\gamma A \end{array} \text{ (pr)} \quad \begin{array}{c} \vdots \\ \beta' \parallel !^{\geq \gamma} \Omega, \Delta, !^\gamma A, A \Rightarrow \Pi \end{array} \text{ (dr)}}{f(\alpha, \beta', \gamma) \parallel !^{\geq \gamma} \Omega, \Delta, A \Rightarrow \Pi} \text{ (cut)}}{\alpha \parallel !^{\geq \gamma} \Omega \Rightarrow A \quad f(\alpha, \beta', \gamma) \parallel !^{\geq \gamma} \Omega, \Delta \Rightarrow \Pi} \text{ (cut)}$$

$$f(\alpha, \beta', \gamma) + \alpha \parallel !^{\geq \gamma} \Omega, \Delta \Rightarrow \Pi$$

$$f(\alpha, \beta' + \gamma, \gamma) \parallel !^{\geq \gamma} \Omega, \Delta \Rightarrow \Pi \text{ (l-weak)}$$

Note that the upper cut has decreased rank and is therefore admissible by induction hypothesis. The lower cut is admissible by Theorem 3.14, as  $A$  is  $!$ -free. We also have to check the soundness of the rule  $(l\text{-weak})$ : For this, we first observe that  $\lfloor (\beta' + \gamma)/\gamma \rfloor = \lfloor \beta'/\gamma + 1 \rfloor = \lfloor \beta'/\gamma \rfloor + 1$ . Then:

$$\begin{aligned} f(\alpha, \beta', \gamma) + \alpha &= \beta' + \lfloor \beta'/\gamma \rfloor \cdot \alpha + \alpha = \beta' + (\lfloor \beta'/\gamma \rfloor + 1) \cdot \alpha \\ &\leq \beta' + \gamma + \lfloor (\beta' + \gamma)/\gamma \rfloor \cdot \alpha = f(\alpha, \beta' + \gamma, \gamma) \end{aligned}$$

2. Next we argue that a cut can be shifted above the *left* premise of cut as long as the cut formula is not principal in the left premise.

a) *Lifting cut above* ( $\text{dr}^\xi$ ) *in left premise* The cut then looks like this:

$$\frac{\frac{\alpha' \parallel \vdots \!|\Omega, \Gamma, !^\xi B, B \Rightarrow !^\gamma A}{\alpha' + \xi \parallel \!|\Omega, \Gamma, !^\xi B \Rightarrow !^\gamma A} \text{ (dr}^\xi) \quad \beta \parallel \vdots \!|\Omega, !^\xi B, \Delta, !^\gamma A \Rightarrow \Pi}{f(\alpha' + \xi, \beta, \gamma) \parallel \!|\Omega, \Gamma, !^\xi B, \Delta \Rightarrow \Pi} \text{ (cut)}$$

We transform this cut to:

$$\frac{\frac{\alpha' \parallel \vdots \!|\Omega, \Gamma, !^\xi B, B \Rightarrow !^\gamma A \quad \beta \parallel \vdots \!|\Omega, !^\xi B, \Delta, !^\gamma A \Rightarrow \Pi}{f(\alpha', \beta, \gamma) \parallel \!|\Omega, \Gamma, !^\xi B, B, \Delta \Rightarrow \Pi} \text{ (cut)}}{\frac{f(\alpha', \beta, \gamma) \parallel \!|\Omega, \Gamma, !^\xi B, B, \Delta \Rightarrow \Pi}{f(\alpha' + \xi, \beta, \gamma) \parallel \!|\Omega, \Gamma, !^\xi B, \Delta \Rightarrow \Pi} \text{ (l-weak)}} \text{ (dr}^\xi)$$

Note that

$$f(\alpha', \beta, \gamma) + \xi = \beta + \lfloor \beta/\gamma \rfloor \cdot \alpha' + \xi \leq \beta + \lfloor \beta/\gamma \rfloor \cdot (\alpha' + \xi) = f(\alpha' + \xi, \beta, \gamma).$$

Here we use the assumption that  $\beta \geq \gamma$ .

b) *Shifting cut above* ( $\rightarrow_1$ ) *in the left premise* The cut then looks like this:

$$\frac{\frac{\alpha' \parallel \!|\Omega, \Gamma_1 \Rightarrow B_1 \quad \alpha'' \parallel \!|\Omega, \Gamma_2, B_2 \Rightarrow !^\gamma A}{\alpha' + \alpha'' \parallel \!|\Omega, \Gamma_1, \Gamma_2, B_1 \rightarrow B_2 \Rightarrow !^\gamma A} (\rightarrow_1) \quad \beta \parallel \!|\Omega, \Delta, !^\gamma A \Rightarrow \Pi}{f(\alpha' + \alpha'', \beta, \gamma) \parallel \!|\Omega, \Gamma_1, \Gamma_2, B_1 \rightarrow B_2, \Delta \Rightarrow \Pi} \text{ (cut)}$$

We can lift this cut upwards as follows:<sup>5</sup>

$$\frac{\frac{\alpha' \parallel \!|\Omega, \Gamma_1 \Rightarrow B_1 \quad \frac{\alpha'' \parallel \!|\Omega, \Gamma_2, B_2 \Rightarrow !^\gamma A \quad \beta \parallel \!|\Omega, [B_2, ]\Delta, !^\gamma A \Rightarrow \Pi}{f(\alpha'', \beta, \gamma) \parallel \!|\Omega, \Gamma_2, B_2, \Delta \Rightarrow \Pi} \text{ (cut)}}{\alpha' + f(\alpha'', \beta, \gamma) \parallel \!|\Omega, \Gamma_1, \Gamma_2, B_1 \rightarrow B_2, \Delta \Rightarrow \Pi} (\rightarrow_1)}{f(\alpha' + \alpha'', \beta, \gamma) \parallel \!|\Omega, \Gamma_1, \Gamma_2, B_1 \rightarrow B_2, \Delta \Rightarrow \Pi} \text{ (l-weak)}$$

For the soundness of the lowermost inference (l-weak), we calculate

$$\begin{aligned} \alpha' + f(\alpha'', \beta, \gamma) &= \alpha' + \beta + \lfloor \beta/\gamma \rfloor \cdot \alpha'' \\ &\leq \beta + \lfloor \beta/\gamma \rfloor \cdot (\alpha' + \alpha'') \\ &= f(\alpha' + \alpha'', \beta, \gamma) \end{aligned}$$

using  $\lfloor \beta/\gamma \rfloor \geq 1$  in the second line.

<sup>5</sup>For the  $\lfloor \cdot \rfloor$ -notation, see the proof of Theorem 3.14.

As mentioned before, reduction steps as in (a) and (b) are standard and the only new thing is the computation of the labelling. We therefore switch now to a more economic notation of the argument which abstracts away from the logical structure of the sequents and focuses only on the labels. In this notation, the argument in (b) above will be written as:

$$\frac{\frac{\alpha' \quad \alpha''}{\alpha' + \alpha''} (\rightarrow_1) \quad \beta}{f(\alpha' + \alpha'', \beta, \gamma)} (\text{cut}) \rightsquigarrow \frac{\alpha' \quad \frac{\alpha'' \quad \beta}{f(\alpha'', \beta, \gamma)} (\text{cut})}{\alpha' + f(\alpha'', \beta, \gamma)} (\rightarrow_1)$$

$$\begin{aligned} \alpha' + f(\alpha'', \beta, \gamma) &= \alpha' + \beta + \lfloor \beta/\gamma \rfloor \cdot \alpha'' \\ &\leq \beta + \lfloor \beta/\gamma \rfloor \cdot (\alpha' + \alpha'') \\ &= f(\alpha' + \alpha'', \beta, \gamma) \end{aligned}$$

So the figure is a schematic representation of the cut reduction, and it is followed by a computation in which the label after and before the reduction are compared. We now discuss the remaining cases using this new notation.

a) *Shifting cut above* ( $\vee_1$ ) *in the left premise*

$$\frac{\frac{\alpha \quad \alpha''}{\max\{\alpha', \alpha''\}} (\vee_1) \quad \beta}{f(\max\{\alpha', \alpha''\}, \beta, \gamma)} (\text{cut}) \rightsquigarrow \frac{\frac{\alpha' \quad \beta}{f(\alpha', \beta, \gamma)} (\text{cut}) \quad \frac{\alpha'' \quad \beta}{f(\alpha'', \beta, \gamma)} (\text{cut})}{\max\{f(\alpha', \beta, \gamma), f(\alpha'', \beta, \gamma)\}} (\vee_1)$$

$$\begin{aligned} \max\{f(\alpha', \beta, \gamma), f(\alpha'', \beta, \gamma)\} &= \max\{\beta + \lfloor \beta/\gamma \rfloor \cdot \alpha', \beta + \lfloor \beta/\gamma \rfloor \cdot \alpha''\} \\ &= \beta + \lfloor \beta/\gamma \rfloor \cdot \max\{\alpha', \alpha''\} \\ &= f(\max\{\alpha', \alpha''\}, \beta, \gamma) \end{aligned}$$

b) *Shifting the cut above* ( $r \in \{(\wedge_1), (*_1), (w_1)\}$ ) *in the left premise*

$$\frac{\frac{\alpha}{\alpha} (r) \quad \beta}{f(\alpha, \beta, \gamma)} (\text{cut}) \rightsquigarrow \frac{\frac{\alpha \quad \beta}{f(\alpha, \beta, \gamma)} (\text{cut})}{f(\alpha, \beta, \gamma)} (r)$$

Note that as in the proof of Theorem 3.14, we have to shift a cut above a whole sequence of  $(w_1)$ 's and one further rule in order to obtain a reduction in rank.

3. We may now assume that the cut rule is principal in the left premise. Recall that the case where the cut rule is principal in an instance of  $(w_r)$  was treated in the very beginning. So, in the only remaining case, the cut formula must be principal in an instance of  $(pr)$ , which implies in particular that the antecedent of the left premise of cut consists of unbounded formulas. If the cut formula is also principal in the right premise, we are in the main case which was discussed above. Otherwise, one of the following reductions applies:



a) *Shifting the cut above  $(dr^\xi)$  in the right premise*

$$\frac{\alpha \frac{\beta'}{\beta' + \xi} (dr^\xi)}{f(\alpha, \beta' + \xi, \gamma)} (\text{cut}) \rightsquigarrow \frac{\frac{\alpha \beta'}{f(\alpha, \beta', \gamma)} (\text{cut})}{f(\alpha, \beta', \gamma) + \xi} (dr^\xi)$$

$$\begin{aligned} f(\alpha, \beta', \gamma) + \xi &= \beta' + \lfloor \beta' / \gamma \rfloor \cdot \alpha + \xi \\ &\leq \beta' + \lfloor (\beta' + \xi) / \gamma \rfloor \cdot \alpha + \xi \\ &= f(\alpha, \beta' + \xi, \gamma) \end{aligned}$$

b) *Shifting the cut above  $r \in \{(*_r), (\rightarrow_l)\}$  in the right premise*

$$\frac{\alpha \frac{\beta' \beta''}{\beta' + \beta''} (r)}{f(\alpha, \beta' + \beta'', \gamma)} (\text{cut}) \rightsquigarrow \frac{\frac{\alpha \beta'}{f(\alpha, \beta', \gamma)} (\text{cut}) \frac{\alpha \beta''}{f(\alpha, \beta'', \gamma)} (\text{cut})}{f(\alpha, \beta', \gamma) + f(\alpha, \beta'', \gamma)} (r)$$

$$\begin{aligned} f(\alpha, \beta', \gamma) + f(\alpha, \beta'', \gamma) &= \beta' + \lfloor \beta' / \gamma \rfloor \cdot \alpha + \beta'' + \lfloor \beta'' / \gamma \rfloor \cdot \alpha \\ &= \beta' + \beta'' + (\lfloor \beta' / \gamma \rfloor + \lfloor \beta'' / \gamma \rfloor) \cdot \alpha \\ &\leq \beta' + \beta'' + \lfloor (\beta' + \beta'') / \gamma \rfloor \cdot \alpha \\ &= f(\alpha, \beta' + \beta'', \gamma) \end{aligned}$$

c) *Shifting the cut above  $r \in \{(\vee_l), (\wedge_r)\}$  in the right premise*

$$\frac{\alpha \frac{\beta' \beta''}{\max\{\beta', \beta''\}} (r)}{f(\alpha, \max\{\beta', \beta''\}, \gamma)} (\text{cut}) \rightsquigarrow \frac{\frac{\alpha \beta'}{f(\alpha, \beta', \gamma)} (\text{cut}) \frac{\alpha \beta''}{f(\alpha, \beta'', \gamma)} (\text{cut})}{\max\{f(\alpha, \beta', \gamma), f(\alpha, \beta'', \gamma)\}} (r)$$

$$\begin{aligned} \max\{f(\alpha, \beta', \gamma), f(\alpha, \beta'', \gamma)\} &= \max\{\beta' + \lfloor \beta' / \gamma \rfloor \cdot \alpha, \beta'' + \lfloor \beta'' / \gamma \rfloor \cdot \alpha\} \\ &= \max\{\beta', \beta''\} + \lfloor \max\{\beta', \beta''\} / \gamma \rfloor \cdot \alpha \\ &= f(\alpha, \max\{\beta', \beta''\}, \gamma) \end{aligned}$$

d) *Shifting the cut above  $(r) \in \{(*_l), (\rightarrow_r), (\vee_r), (\wedge_l), (\text{pr}), (\text{w}_l), (\text{w}_r)\}$  (all unary rule different from dereliction) in the right premise*

$$\frac{\alpha \frac{\beta}{\beta} (r)}{f(\alpha, \beta, \gamma)} (\text{cut}) \rightsquigarrow \frac{\frac{\alpha \beta}{f(\alpha, \beta, \gamma)} (\text{cut})}{f(\alpha, \beta, \gamma)} (r)$$

□

### 3.7 Open Questions

**Decidability and complexity.** We have not discussed any issues related to the decidability and computational complexity of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ . If we restrict our attention to labelled sequents in which no exponent 0 occurs, then the label of the sequent puts a bound on the uses of contractions which may occur in a proof. Since in this context the contraction rule is the only obstruction to a ‘brute-force’ proof search, the decision problem can be solved by exhaustively generating all possible proofs. If on the other hand we allow the exponent 0, we do have unlimited contraction and so matters are more complicated. A reason to believe in the decidability of  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  is the fact that its underlying logic  $\mathbf{aILL}$  is decidable [33] (unlike intuitionistic linear logic without weakening [40]).

**Embeddings.** Another question is that of embeddings between  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$  and  $\mathbf{aSELL}(\mathbb{R}^+)$ . By Proposition 3.10 we have the non-constructive equivalence

$$\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S \iff \exists \alpha \in \mathbb{R}^+ \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} \alpha \parallel S.$$

If it is possible to find a computable upper bound  $F(S)$  for the cost of a sequent  $S$ , then this equivalence can be extended to a partial computable reduction:

$$\vdash_{\mathbf{aSELL}(\mathbb{R}^+)} S \iff \vdash_{\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)} F(S) \parallel S$$

(The reduction is partial because not every possible label occurs on the right.) Such an upper bound function would already follow from a computable upper bound to the length of cutfree  $\mathbf{aSELL}(\mathbb{R}^+)$ -proofs. Given the latter upper bound, we obtain a *very* rough estimate for  $\text{cost}(S)$  by assuming that every step in a maximal-length proof is dereliction with the highest exponential occurring in  $S$ .

**Extending the cut elimination theorem.** It is conceivable that Theorem 3.19 can be extended to cover cuts on arbitrary formulas by a suitable ‘iteration’ of the ideas involved there. Assign a degree to  $\mathcal{L}\mathbf{aSELL}(\mathbb{R}^+)$ -formulas by counting their maximal nesting of subexponentials. Then Theorem 3.14 is the cut elimination theorem for formulas of degree 0, which tells us that a correct labelling of the cut rule in this case is  $f^0(\alpha, \beta, \gamma) = \alpha + \beta$ . From this we inferred the cut elimination theorem for formulas of degree 1 (Theorem 3.19) by estimating how many degree 0 cuts are involved in eliminating a degree 1 cut, and this estimate led us to the labelling function

$$\begin{aligned} f^1(\alpha, \beta, \gamma) &= f^0(\alpha, f^0(\alpha, \dots f^0(\alpha, \beta, \gamma) \dots, \gamma), \gamma) \quad (\lfloor \beta/\gamma \rfloor\text{-many iterations}) \\ &= \beta + \lfloor \beta/\gamma \rfloor \cdot \alpha. \end{aligned}$$

In a nutshell,  $f^1$  is a  $(\beta, \gamma)$ -bounded iteration of  $f^0$ . By the same argument, the labelling function  $f^2$  for cuts on degree 2 formulas should be a bounded iteration of  $f_1$ , and continuing like this we might obtain a whole hierarchy of labelling functions  $f^0, f^1, f^2, f^3, \dots$  for cuts on formulas of degree 1, 2, 3 and so on. The danger of course is that the  $f^k$ 's become so difficult to describe that they are of little use.

# A Reduction in Violation Logic

## 4.1 Introduction

This chapter contains a collection of proof-theoretic remarks on a family of deontic logics called *violation logic* as introduced by Governatori, Rotolo et. al [27, 13, 25, 26, 14].

*Deontic logic* is the formal study of expressions such as

‘It is obligatory that  $A$ ’ and ‘It is permitted that  $A$ ’.

Starting with the seminal work of von Wright [55], various authors have modelled obligations and permissions by extending a suitable base logic, often classical propositional logic, using *modal operators*. The above sentences are then formulated as  $OA$  and  $PA$ ; Furthermore permission is often considered a derived connective  $PA := \neg O\neg A$ . This makes deontic logic a sub-branch of modal logic. The question of course is which axioms to postulate for  $O$ .

One logic derived from von Wright’s work rose to prominence as *standard deontic logic* **SDL**. A Hilbert-style system is pictured in Figure 4.1. Seen as a modal logic, **SDL** coincides with **KD**, the logic of serial Kripke frames [15]. The seriality axiom  $OA \rightarrow \neg O\neg A$  is interpreted deontically as the absence of conflicting obligations.

It was soon observed that **SDL** and related systems allow for the derivation of seemingly counter-intuitive statements when provided with a set of ‘real world’ deontic assumptions. Examples of such failures have become known under the name *deontic paradoxes*, and they have been a driving force in the development of new deontic logics ever since. A list of some famous paradoxes can be found in [41].

What is important for our purpose are the problems arising when modelling *contrary-to-duty* (CTD) reasoning, that is, reasoning in the presence of contradicting obligations. An

<i>Axioms:</i>		
	All theorems of <b>CL</b>	(CL)
	$O(A \rightarrow B) \rightarrow (OA \rightarrow OB)$	(O-K)
	$OA \rightarrow \neg O\neg A$	(O-D)
<i>Rules:</i>		
	$\frac{A \quad A \rightarrow B}{B} \text{ (MP)}$	
	$\frac{A}{OA} \text{ (O-NEC)}$	

Figure 4.1: The standard deontic logic **SDL**.

appropriate handling of CTD is nowadays considered a benchmark of a useful deontic logic. **SDL** fails here in the strongest possible sense, as contradictory obligations always lead to a *logical* contradiction by virtue of the (O-D) axiom.

It is a common approach for deontic conflict resolution to assign different precedence levels to obligations. A system in this spirit are the *violation logics* introduced by Governatori, Rotolo et al. in a series of papers. These logics extend classical modal logic E (which features the operator O, for ‘obligation’) by an additional operator  $\otimes$  with the intended meaning that [25, p.1]

‘[t]he interpretation of a chain like  $a \otimes b \otimes c$  is that  $a$  is obligatory, but if it is violated (i.e.,  $\neg a$  holds), then  $b$  is the new obligation (and  $b$  compensates for the violation of  $a$ ); again, if the obligation of  $b$  is violated as well, then  $c$  is obligatory [...]’

For these so-called  $\otimes$ -chains a variety of rules and axioms are proposed, resulting in a number of different systems of violation logic. One therefore has two levels of obligations, one stemming from the  $\otimes$ -chains, and the other one from the O modality of the underlying logic E. As the authors put it in [25, p.2] regarding their semantics for the  $\otimes$ -operator,

‘We [...] split the treatment of  $\otimes$ -chains and obligations; the intuition is that chains are the generators of obligations and permissions [...]’

This mirrors a common distinction between *descriptive norms* and *prescriptive norms*. While the descriptive norms simply describe which norms are active according to some

code of law, the prescriptive norms subsume all obligations which arise from them for an agent.

In this chapter we investigate the role of  $\otimes$ -chains as generators of obligations using proof theoretic methods. Our main result is that  $\otimes$ -chains can be replaced by formulas in the underlying logic  $E$  which generate exactly the same obligations. This yields a reduction of a large fragment of violation logic to the base logic  $E$ . As a consequence, tools available for  $E$  – such as neighbourhood semantics on the model theoretic side, or cutfree Gentzen systems on the proof theoretic side – can be used to study violation logics. We establish coNP-completeness of the ‘translatable’ fragment of violation logic and close with some remarks on the choice of axioms for  $\otimes$ -chains. While the chief part of our results is of a technical nature, we occasionally hint at philosophical ramifications.

Our arguments take place exclusively in the original Hilbert-style presentation of violation logic, which gives this chapter quite a different flavour than the work in the other chapters (which is more in the spirit of Gentzen-style proof theory). The method we use is that of provability-preserving translations.

## 4.2 Preliminaries

### Classical Modal Logic

The deontic logic underlying the treatment of  $\otimes$ -chains is given by axiomatic extensions of the classical non-normal modal logic  $E$  [15]. The signature of  $E$  contains the following connectives:

$$\perp \text{ (constant), } \wedge, \rightarrow \text{ (binary) and } O \text{ (unary)}$$

Any formula in this signature will be called a *basic deontic formula*, and a basic deontic formula without  $O$  will be called *classical*. Additional connectives are defined as abbreviations:  $\neg A := A \rightarrow \perp$ ,  $\top := \neg \perp$  and  $A \equiv B := (A \rightarrow B) \wedge (B \rightarrow A)$ .

The logic  $E$  is defined to be the smallest logic of deontic formulas containing  $CL$  and closed under the rules

$$\frac{A \quad A \rightarrow B}{B} \text{ (MP)} \quad \text{and} \quad \frac{A \equiv B}{OA \equiv OB} \text{ (O-RE)}.$$

To be more precise, we define the following notion of proofs from assumptions in an axiomatic extension of  $E$ . Here a set  $\Gamma$  will play the role of local assumptions, whereas a set  $\Delta$  plays the role of additional axioms.

#### Definition 4.1

Let  $\Delta \cup \Gamma \cup \{A\}$  be a set of basic deontic formulas. An  $(E + \Delta)$ -proof of  $A$  from  $\Gamma$  is a

tree of basic deontic formulas rooted in  $A$  which obeys the following properties:

1. Every leaf of the tree is either a formula from  $\Gamma$  or a substitution instance of formulas from  $\mathbf{CL} \cup \Delta$ . Leaves of the former type are called *local assumptions*.<sup>1</sup>
2. Every internal node of the tree together with its child node(s) forms an instance of (MP) or (O-RE).
3. (locality condition) No instance of (O-RE) appears below a local assumption.

We write  $\Gamma \vdash_{E+\Delta} A$ , and say that  $A$  is derivable from  $\Gamma$  in  $E + \Delta$ , if there is a  $(E + \Delta)$ -proof of  $A$  from  $\Gamma$ .

The locality condition reflects the well-known fact that modal rules such as (O-RE) should not be applied to local assumptions in modal logic, see the chapter on proof theory in [9]. With proofs from assumptions defined in this way, the deduction theorem holds in its usual formulation:

**Proposition 4.2** (deduction theorem)

$$\Gamma \cup \{B\} \vdash_{E+\Delta} A \iff \Gamma \vdash_{E+\Delta} B \rightarrow A.$$

*Proof.* See [9]. □

We will also use the following generalization of (O-RE):

**Proposition 4.3** (uniform substitution)

For any formula  $C(p)$ , the following rule is admissible in  $E + \Delta$ :

$$\frac{A \equiv B}{C(A) \equiv C(B)} \text{ (O-RE')}$$

*Proof.* See [9]. □

We now review the notion of neighbourhood models, which form the standard semantics of classical modal logics. A *neighbourhood model*  $\mathcal{W} = \langle W, \mathcal{N}, V \rangle$  is composed of the following elements:

- A nonempty set  $W$  of worlds

<sup>1</sup>More precisely, we call local assumptions only those leaves which are not at the same time instances of formulas from  $\mathbf{CL} \cup \Delta$  or classical theorems.

- A neighbourhood function  $\mathcal{N} : \mathcal{W} \rightarrow \mathcal{P}(\mathcal{P}(\mathcal{W}))$
- A valuation function  $V : \mathcal{V} \rightarrow \mathcal{P}(\mathcal{W})$

Given a neighbourhood model  $\mathcal{W}$ , we can define the notion  $\langle \mathcal{W}, w \rangle \models A$  of truth at a world  $w \in \mathcal{W}$  by induction on the formula  $A$ :  $\langle \mathcal{W}, w \rangle \not\models \perp$ ,  $\langle \mathcal{W}, w \rangle \models a \Leftrightarrow w \in V(a)$ ,  $\langle \mathcal{W}, w \rangle \models A \wedge B \Leftrightarrow \langle \mathcal{W}, w \rangle \models A$  and  $\langle \mathcal{W}, w \rangle \models B$ ,  $\langle \mathcal{W}, w \rangle \models A \rightarrow B \Leftrightarrow \langle \mathcal{W}, w \rangle \not\models A$  or  $\langle \mathcal{W}, w \rangle \models B$ , and finally

$$\langle \mathcal{W}, w \rangle \models OA \quad : \Leftrightarrow \quad [A]_{\mathcal{W}} \in \mathcal{N}(w)$$

where  $[A]_{\mathcal{W}} := \{w \in \mathcal{W} \mid \langle \mathcal{W}, w \rangle \models A\}$ . The part  $\mathcal{F}_{\mathcal{W}} = \langle \mathcal{W}, \mathcal{N} \rangle$  of a neighbourhood model  $\mathcal{W}$  is called a *neighbourhood frame*, and conversely  $\langle \mathcal{W}, \mathcal{N}, V \rangle$  is called a neighbourhood model *based on*  $\mathcal{F}$ . Truth on a frame is defined as follows:  $\mathcal{F} \models A$  iff for all models  $\mathcal{W}$  based on  $\mathcal{F}$  and all worlds  $w \in \mathcal{W}$ ,  $\langle \mathcal{W}, w \rangle \models A$ . For a set  $\Gamma \cup \Delta \cup \{A\}$  of basic deontic formulas, we define the semantic consequence relation  $\Gamma \models_{\Delta} A$  as follows:

$$\Gamma \models_{\Delta} A \quad : \Leftrightarrow \quad \text{for all neighbourhood models } \mathcal{W} \text{ and } w \in \mathcal{W}, \text{ if } \mathcal{F}_{\mathcal{W}} \models \bigwedge \Delta \text{ and } \langle \mathcal{W}, w \rangle \models \bigwedge \Gamma, \text{ then } \langle \mathcal{W}, w \rangle \models A.$$

**Proposition 4.4** (soundness and completeness)

$$\Gamma \models_{\Delta} A \Leftrightarrow \Gamma \vdash_{E+\Delta} A.$$

*Proof.* This follows from the strong completeness theorem for E with respect to neighbourhood models (see [15]) and the deduction theorem (Proposition 4.2).  $\square$

Local assumptions in  $E + \Delta$  therefore correspond to truths at a certain world.

## Violation Logics

We now review our main object of study, the violation logics of Governatori, Rotolo et al. They were originally introduced in [27] and then developed in a series of subsequent articles. On the syntactic level, they extend axiomatic extensions of E by an operator  $\otimes$ , which comes in any<sup>2</sup> arity  $n > 0$ . A formula

$$A_1 \otimes A_2 \otimes A_3 \otimes \dots \otimes A_n$$

<sup>2</sup>A pedantic way to handle the indefinite arity of  $\otimes$  is to consider a family  $\otimes^n$  of  $n$ -ary connectives, for each  $n$ ; or to consider a unary and a binary connective, declaring all longer chains to be nestings thereof (although this makes the subsequent definition of the *nesting condition* cumbersome). But there is also nothing wrong with allowing connectives of indefinite arity.

is meant to model a chain of obligations and corresponding compensations:  $A_1$  is obligatory, but if  $A_1$  is violated, then the new (secondary) obligation is  $A_2$ ; the fulfillment of  $A_2$  compensates the violation of  $A_1$ ; if however  $A_2$  is violated as well, then there is a new (ternary) obligation  $A_3$ , and so on.

**Example 4.5**

Consider three propositional variables  $w, p$  and  $f$  with meaning  $w$ ='it is the weekend',  $p$ ='parking downtown' and  $f$ ='paying a fine'. Then the intended meaning of the formula

$$A_{Ex} = w \rightarrow (\neg p) \otimes f$$

taken from [26] is: *On weekends it is forbidden to park downtown; but if one does so, one has to pay a fine.* The formula  $A_{Ex}$  will serve as a running example throughout this chapter.

A formula of violation logic (henceforth just called a formula) is any formula built from the above signature subject to the following *nesting condition*: No pair of operators from  $\{O, \otimes\}$  appears nested in  $A$ .<sup>3</sup> For example,  $\neg(Oa \wedge (b \otimes c \otimes d))$  is a formula of violation logic, whereas  $\neg O(a \wedge (b \otimes c))$  is not. For any  $n > 0$  a formula of the form  $A_1 \otimes \dots \otimes A_n$  is called a  $\otimes$ -chain. Due to the nesting condition, every formula  $A_i$  occurring in a  $\otimes$ -chain is classical. Chains of length 1 are written in prefix notation  $\otimes A$ .

Concerning rules and axioms for  $\otimes$ , the aforementioned articles follow a modular approach and present numerous possible choices instead of a designated 'standard system'. For the sake of our investigation, it is however more convenient to fix a framework. We remark already here that our results apply to different systems as well, an observation which will be made precise later on (Corollary 4.24).

That being said, we will have the following two rules for  $\otimes$ :

$$\frac{A \equiv B}{\nu \otimes A \otimes \nu' \equiv \nu \otimes B \otimes \nu'} \quad (\otimes\text{-RE})$$

$$\frac{A \equiv B}{\nu \otimes A \otimes \nu' \otimes B \otimes \nu'' \equiv \nu \otimes A \otimes \nu' \otimes \nu''} \quad (\otimes\text{-contraction})$$

Here, a string such as  $\nu \otimes A \otimes \nu'$  stands symbolically for a  $\otimes$ -chain containing the (classical) formula  $A$  at some position. It is allowed that  $\nu$  or  $\nu'$  are empty, so that  $A$  is the first or last element of the chain. The rule ( $\otimes$ -RE) is the generalization of (O-RE) to the language of violation logic, and ( $\otimes$ -contraction) is a principle of redundancy elimination.

<sup>3</sup>The problem of nested formulas is not technical, but rather the lack of a satisfactory interpretation of such expressions. See the discussion preceding Definition 2.1 in [25].



As standard set of axioms, we take the following set  $\Sigma$  of formulas:

$$\begin{aligned} A_1 \otimes \dots \otimes A_n \wedge \bigwedge_{i=1}^k \neg A_i &\rightarrow OA_{k+1} && \text{(O-detachment)} \\ A_1 \otimes \dots \otimes A_n \otimes A_{n+1} &\rightarrow A_1 \otimes \dots \otimes A_n && \text{(\otimes-shortening)} \\ A_1 \otimes \dots \otimes A_{n+1} \wedge \neg A_1 &\rightarrow A_2 \otimes \dots \otimes A_{n+1} && \text{(\otimes-detachment)} \end{aligned}$$

In these axioms we have  $n \geq 1$  and  $0 \leq k < n$ .<sup>4</sup> The axiom (O-detachment) captures the intended meaning of  $\otimes$ -chains as descriptions of compensatory obligations: If the first  $k$  obligations expressed in a  $\otimes$ -chain  $A_1 \otimes \dots \otimes A_k \otimes A_{k+1} \otimes \dots \otimes a_n$  have been violated, then the next obligation  $A_{k+1}$  comes into effect.

On top of these axioms and rules, we allow a background theory  $\Delta$  of basic deontic axioms. We refer the reader to [25] for an extensive discussion of this and related systems.

All in all, a (*standard*) *violation logic* therefore consists of the following parts:

1. The set  $\Sigma$  of all previously discussed rules and axioms for the  $\otimes$ -chains: ( $\otimes$ -RE), ( $\otimes$ -contraction), (O-detachment), ( $\otimes$ -shortening) and ( $\otimes$ -detachment).
2. An additional set of basic deontic axioms  $\Delta$ . Of this, we only require that  $E + \Delta$  is consistent.

The resulting Hilbert system is pictured in Figure 4.2. The term ‘standard’ shall refer to the set  $\Sigma$ , which is fixed. As an example, the logic  $D^\otimes$  from [25] is the standard violation logic with  $\Delta = \{OA \rightarrow \neg O\neg A\}$ .

We again define a notion of derivations from assumptions.<sup>5</sup>

#### Definition 4.6

Let  $\Delta$  be a set of basic deontic formulas, and  $\Gamma \cup \{A\}$  a set of arbitrary formulas. A  $(V_\Sigma + \Delta)$ -*proof of A from  $\Gamma$*  is a tree of formulas rooted in  $A$  obeying the following:

1. Every leaf is either a substitution instance of formulas from

$$CL \cup \Delta \cup \{(\text{O-detachment}), (\otimes\text{-shortening}), (\otimes\text{-detachment})\},$$

or if not, a formula from  $\Gamma$ . The latter type of leaves are called *local assumptions*.

<sup>4</sup>By the usual convention  $\bigwedge \emptyset := \top$  on empty conjunctions, it follows that  $A_1 \otimes \dots \otimes A_n \wedge \top \rightarrow OA_1$  is an instance of (O-detachment).

<sup>5</sup>The literature on violation logic leaves this definition implicit.

*Axioms:*

All theorems of CL	(CL)
All theorems of $\Delta$	( $\Delta$ )
$A_1 \otimes \dots \otimes A_n \wedge \bigwedge_{i=1}^k \neg A_i \rightarrow OA_{k+1}$	(O-detachment)
$A_1 \otimes \dots \otimes A_n \otimes A_{n+1} \rightarrow A_1 \otimes \dots \otimes A_n$	( $\otimes$ -shortening)
$A_1 \otimes \dots \otimes A_{n+1} \wedge \neg A_1 \rightarrow A_2 \otimes \dots \otimes A_{n+1}$	( $\otimes$ -detachment)

*Rules:*

$\frac{A \rightarrow B \quad A}{B}$ (MP)
$\frac{A \equiv B}{OA \equiv OB}$ (O-RE)
$\frac{A \equiv B}{v \otimes A \otimes v' \equiv v \otimes B \otimes v'}$ ( $\otimes$ -RE)
$\frac{A \equiv B}{v \otimes A \otimes v' \otimes B \otimes v'' \equiv v \otimes A \otimes v' \otimes v''}$ ( $\otimes$ -contraction)

Figure 4.2: Violation logic  $V_\Sigma + \Delta$

2. Every internal node, together with its child node(s), forms an instance of (MP), (O-RE), ( $\otimes$ -RE) or ( $\otimes$ -contraction)
3. (locality condition) No instance of (O-RE), ( $\otimes$ -RE) or ( $\otimes$ -contraction) appears below a local assumption.

We write  $\Gamma \vdash_{V_\Sigma + \Delta} A$ , and say that  $A$  is derivable from  $\Gamma$  in  $V_\Sigma + \Delta$ , if there is a  $(V_\Sigma + \Delta)$ -proof of  $A$  from  $\Gamma$ .

**Proposition 4.7** (deduction theorem)

$$\Gamma \cup \{B\} \vdash_{V_\Sigma + \Delta} A \iff \Gamma \vdash_{V_\Sigma + \Delta} B \rightarrow A.$$

*Proof.* By induction on the height of proofs. □

The deduction theorem equips us with the following mode of inference in violation logic: If we can prove  $A$  from assumption  $B$  *without using rules (O-RE), ( $\otimes$ -RE) or ( $\otimes$ -contraction) below the assumption  $B$* , then we can infer  $B \rightarrow A$ .

#### Example 4.8

We look again at the formula  $A_{E_x}$  from Example 4.5. The following proof shows that  $\{A_{E_x}, w, p\} \vdash_{V_\Sigma} \text{Of}$ , which means in plain English: Parking downtown on a weekend leads to the obligation of paying a fine.

$$\begin{array}{c}
 \begin{array}{c}
 \text{(local assumption)} \\
 w \\
 \hline
 (\neg p) \otimes f
 \end{array}
 \quad
 \begin{array}{c}
 \text{(local assumption)} \\
 w \rightarrow (\neg p) \otimes f \\
 \hline
 (\neg p) \otimes f
 \end{array}
 \quad
 \begin{array}{c}
 \text{(local assumption)} \\
 p \\
 \hline
 \neg p
 \end{array}
 \\
 \hline
 \neg p \wedge ((\neg p) \otimes f)
 \end{array}
 \quad
 \begin{array}{c}
 \text{(instance of O-detachment)} \\
 \neg p \wedge ((\neg p) \otimes f) \rightarrow \text{Of} \\
 \hline
 \text{Of}
 \end{array}
 \quad
 \text{(MP)}$$

A double line abbreviates some steps of ‘classical reasoning’, that is, the use of classical theorems and (MP). Since none of the rules (O-RE), ( $\otimes$ -RE) or ( $\otimes$ -contraction) are applied in the proof above, we may for example also conclude that  $\{A_{E_x}, w\} \vdash_{V_\Sigma} p \rightarrow \text{Of}$ .

The article [25] contains a completeness proof of violation logic with respect to *sequence semantics* which are a straightforward generalization of neighbourhood semantics. We include the definitions for illustration only, as we are not going to use sequence semantics at all – in fact, we uphold the thesis that the proof-theoretic approach is more perspicuous than the semantic account here.

A sequence model extends a neighbourhood model  $\mathcal{W} = \langle W, \mathcal{N}, V \rangle$  by a function  $\mathcal{C}$  which maps each world  $w$  to a set  $\mathcal{C}_w$  of finite non-empty sequences  $\langle X_1, \dots, X_n \rangle$  of sets of worlds.  $\mathcal{C}_w$  obeys the following closure conditions:

1. If  $\langle X_1, \dots, X_n \rangle \in \mathcal{C}_w$  and  $w \notin X_1 \cup \dots \cup X_k$  for some  $0 \leq k < n$ , then  $X_{k+1} \in \mathcal{N}(w)$  and  $\langle X_{k+1}, \dots, X_n \rangle \in \mathcal{C}_w$ .
2. If  $\langle X_1, \dots, X_n \rangle \in \mathcal{C}_w$  and  $n > 1$ , then  $\langle X_1, \dots, X_{n-1} \rangle \in \mathcal{C}_w$ .
3. Let  $L \in \mathcal{C}_w$  be a list in which a set of worlds  $X$  occurs at a certain position. Then  $\mathcal{C}_w$  must contain also all lists arising from removing or introducing copies of  $X$  at a later position in  $L$ .

Note how these closure conditions mimic (O/ $\otimes$ -detachment), ( $\otimes$ -shortening) and ( $\otimes$ -contraction). The satisfaction clauses of the standard neighbourhood semantics are then extended by setting  $\langle \mathcal{W}, \mathcal{C}, w \rangle \models A_1 \otimes \dots \otimes A_n \Leftrightarrow \langle [A_1]_{\mathcal{W}}, \dots, [A_n]_{\mathcal{W}} \rangle \in \mathcal{C}_w$ . Finally a notion of semantic validity is defined as follows:

$$\models_{\Delta} A \quad : \Leftrightarrow \quad \text{for all sequence model models } \langle \mathcal{W}, \mathcal{C} \rangle \text{ and } w \in \mathcal{W}, \\
 \text{if } \mathcal{F}_{\mathcal{W}} \models \bigwedge \Delta, \text{ then } \langle \mathcal{W}, \mathcal{C}, w \rangle \models A.$$

**Theorem 4.9** (Governatori, Olivieri, Calardo and Rotolo 2016)

$$\models_{\Delta} A \iff \vdash_{V_{\Sigma} + \Delta} A.$$

*Proof.* See [25]. □

### 4.3 The $\perp$ -Translation

Let us fix some standard violation logic  $V_{\Sigma} + \Delta$ .

We start our proof-theoretic investigation with a simple but effective observation. If we take the rules and axioms for  $\otimes$ -chains in  $\Sigma$  and replace all occurring chains by the logical constant  $\perp$ , we obtain the following schemata. Note that the premises of ( $\otimes$ -RE) and ( $\otimes$ -contraction) are classical formulas due to the nesting condition, and hence they are left unchanged by the  $\perp$ -translation; the same holds for formulas occurring within  $\otimes$ -chains.

$$\begin{array}{l} \frac{A \equiv B}{\perp \equiv \perp} \quad (\otimes\text{-RE})^{\perp} \\ \frac{A \equiv B}{\perp \equiv \perp} \quad (\otimes\text{-contraction})^{\perp} \\ \perp \wedge \bigwedge_{i=1}^k \neg A_i \rightarrow \text{OA}_{k+1} \quad (\text{O-detachment})^{\perp} \\ \perp \rightarrow \perp \quad (\otimes\text{-shortening})^{\perp} \\ \perp \wedge \neg A_1 \rightarrow \perp \quad (\otimes\text{-detachment})^{\perp} \end{array}$$

Clearly these rules and axioms are all sound in E (in fact, in classical logic). This means that it is possible to interpret  $\otimes$ -chains as logical contradictions, or in other words: It is not possible to prove in violation logic anything which depends on the soundness of  $\otimes$ -chains.

To be more formal, let us denote by  $(\cdot)^{\perp}$  the syntactic translation which replaces all  $\otimes$ -chains in a formula or rule instance by the constant  $\perp$ . Then from the above observation, we obtain the following:

**Theorem 4.10**

$$\text{If } \vdash_{V_{\Sigma} + \Delta} A, \text{ then } \vdash_{E + \Delta} A^{\perp}.$$

*Proof.* Let  $\delta$  be a proof witnessing  $\vdash_{V_{\Sigma+\Delta}} A$ , and let  $\delta^\perp$  be the tree of formulas arising from  $\delta$  by applying the  $\perp$ -translation everywhere. We will see that up to minor modifications,  $\delta^\perp$  is proof of  $A^\perp$ .

Consider the various nodes  $B$  in  $\delta$ . If a leaf node  $B$  is an instance of a classical theorem or an axiom of  $\Delta$ , then  $B^\perp$  is, albeit not necessarily the same formula, still an instance of the same theorem or axiom. If a leaf node  $B$  is an instance of one of the axioms in  $\Sigma$  for  $\otimes$ -chains, then  $B^\perp$  is an instance of a classical theorem – see the translated schemes above. If an internal node  $B$  is the conclusion of an instance of ( $\otimes$ -RE) or ( $\otimes$ -contraction), then  $B^\perp$  is the formula  $\perp \equiv \perp$  (see above) which is a classical theorem; in this case, we can cut the proof tree above  $B^\perp$ . If  $B$  appears in  $\delta$  as an instance  $C/B$  of (O-RE), then also  $C^\perp/B^\perp$  is an instance of (O-RE), as  $(\cdot)^\perp$  commutes with O and  $\equiv$ . Finally, if  $B$  appears as an instance  $C, C \rightarrow B/B$  of (MP), then also  $C^\perp, (C \rightarrow B)^\perp/B^\perp$  is an instance of (MP) because  $(C \rightarrow B)^\perp = C^\perp \rightarrow B^\perp$ .

From all of this, it follows that  $\delta^\perp$  is a valid proof after possibly pruning some branches; in fact, it is an  $E + \Delta$  proof as all  $\otimes$ -chains have been eliminated. Since the root of  $\delta^\perp$  is  $A^\perp$ , the statement follows.  $\square$

#### Corollary 4.11

$V_{\Sigma+\Delta}$  is conservative over  $E + \Delta$  (that is, it proves the same basic deontic formulas). In particular,  $V_{\Sigma+\Delta}$  is consistent.

*Proof.* If  $V_{\Sigma+\Delta}$  proves  $A$ , then  $E + \Delta$  proves  $A^\perp$  by Theorem 4.10, and  $A = A^\perp$  if  $A$  does not contain  $\otimes$ -chains.  $\square$

#### Corollary 4.12

Let  $C$  be a  $\otimes$ -chain. Then

- (i)  $\not\vdash_{V_{\Sigma+\Delta}} C$
- (ii) For no satisfiable basic deontic formula  $D$ ,  $\not\vdash_{V_{\Sigma+\Delta}} D \rightarrow C$ .

*Proof.* (i) If  $\vdash_{V_{\Sigma+\Delta}} C$ , then  $\vdash_{E+\Delta} \perp$  by Theorem 4.10, contradicting the consistency of  $E + \Delta$ . (ii) If  $\vdash_{V_{\Sigma+\Delta}} D \rightarrow C$ , then by Theorem 4.10  $\vdash_{E+\Delta} D \rightarrow \perp$ , so  $D$  cannot be satisfiable.  $\square$

## 4.4 A Reduction Theorem

The technical results we are going to present in this section apply to a fragment of violation logic that we will call the *chain negative fragment*.

**Definition 4.13** (chain negative fragment)

We call a formula *chain negative* (resp. chain positive) if all occurrences of  $\otimes$ -chains in it are negative (resp. positive).

The notion of positive and negative subformulas has been defined in Section 1.1. For example, the chain  $A \otimes B$  appears positively in the formulas  $A \otimes B$ ,  $\neg\neg(C \wedge A \otimes B)$  and  $C \rightarrow A \otimes B$ , and negatively in  $\neg(A \otimes B)$ ,  $(A \otimes B) \rightarrow OC$  and  $(A \otimes B) \wedge C \rightarrow OD$ .<sup>6</sup> The simplest non-trivial example of a chain positive formula is a  $\otimes$ -chain. Intuitively, a chain negative formula is a formula in which  $\otimes$ -chains appear *only as assumptions, but not as conclusions*.

As our main result, we will now show that questions about the chain negative fragment of violation logic can be answered without using the machinery of violation logic, but with a suitable reduction to the underlying deontic logic  $E + \Delta$  instead. To this end, we first give a meaning to  $\otimes$ -chains as basic deontic formulas.

**Definition 4.14** ( $\pi$ -translation)

The translation  $\pi$  from  $\otimes$ -chains to basic deontic formulas is defined inductively on the length of chains as follows:

$$(\otimes A)^\pi := OA$$

$$(A_1 \otimes \dots \otimes A_n \otimes A_{n+1})^\pi := (A_1 \otimes \dots \otimes A_n)^\pi \wedge \left( \left( \bigwedge_{i=1}^n \neg A_i \right) \rightarrow OA_{n+1} \right)$$

As an example, we have  $(a \otimes b \otimes c)^\pi = Oa \wedge (\neg a \rightarrow Ob) \wedge (\neg a \wedge \neg b \rightarrow Oc)$ . In the following we will write  $\pi$  in closed form as

$$(A_1 \otimes \dots \otimes A_n)^\pi = \bigwedge_{i=1}^n \left( \left( \bigwedge_{j=1}^{i-1} \neg A_j \right) \rightarrow OA_i \right),$$

where by a harmless abuse of notation, we identify the conjunct  $\top \rightarrow OA_1$  corresponding to the index  $i = 1$  with the formula  $OA_1$ . We extend  $\pi$  to arbitrary formulas by letting it commute with  $\wedge, \rightarrow$  and  $O$ , so that for example

$$(A_{Ex})^\pi = (w \rightarrow (\neg p) \otimes f)^\pi = w \rightarrow O(\neg p) \wedge (\neg\neg p \rightarrow Of).$$

Given a set  $\Gamma$  of formulas,  $\pi(\Gamma)$  denotes  $\{A^\pi \mid A \in \Gamma\}$ .

We point out that the meaning given to  $\otimes$ -chains by the translation  $\pi$  is quite close to the intuitive interpretation of  $\otimes$ -chain from [25], which was already quoted in the introduction and is repeated here for convenience:

<sup>6</sup>Recall that  $\neg A = A \rightarrow \perp$  by definition.

‘[t]he interpretation of a chain like  $a \otimes b \otimes c$  is that  $a$  is obligatory, but if it is violated (i.e.,  $\neg a$  holds), then  $b$  is the new obligation (and  $b$  compensates for the violation of  $a$ ); again, if the obligation of  $b$  is violated as well, then  $c$  is obligatory [...]’

As a first observation, the axioms for  $\otimes$ -chains remain true if translated via  $\pi$ :

**Lemma 4.15** (axiom soundness)

For any axiom  $A \in \Sigma$ ,  $\vdash_E A^\pi$ .

*Proof.* Let us write down the three axioms schemes with their respective  $\pi$ -translations below:

$$\begin{aligned}
 \text{(O-detachment)} \quad & A_1 \otimes \dots \otimes A_n \wedge \bigwedge_{i=1}^k \neg A_i \rightarrow OA_{k+1} \\
 & \bigwedge_{i=1}^n \left( \left( \bigwedge_{j=1}^{i-1} \neg A_j \right) \rightarrow OA_i \right) \wedge \left( \bigwedge_{i=1}^k \neg A_i \right) \rightarrow OA_{k+1} \\
 \text{(\(\otimes\)-shortening)} \quad & A_1 \otimes \dots \otimes A_n \otimes A_{n+1} \rightarrow A_1 \otimes \dots \otimes A_n \\
 & \bigwedge_{i=1}^{n+1} \left( \left( \bigwedge_{j=1}^{i-1} \neg A_j \right) \rightarrow OA_i \right) \rightarrow \bigwedge_{i=1}^n \left( \left( \bigwedge_{j=1}^{i-1} \neg A_j \right) \rightarrow OA_i \right) \\
 \text{(\(\otimes\)-detachment)} \quad & A_1 \otimes \dots \otimes A_{n+1} \wedge \neg A_1 \rightarrow A_2 \otimes \dots \otimes A_{n+1} \\
 & \bigwedge_{i=1}^{n+1} \left( \left( \bigwedge_{j=1}^{i-1} \neg A_j \right) \rightarrow OA_i \right) \wedge \neg A_1 \rightarrow \bigwedge_{i=2}^{n+1} \left( \left( \bigwedge_{j=2}^{i-1} \neg A_j \right) \rightarrow OA_i \right)
 \end{aligned}$$

By inspection the  $\pi$ -translations are provable in  $E$ ; in fact, they are all valid in classical logic.  $\square$

We now want to argue that, in some way,  $A_1 \otimes \dots \otimes A_n$  and its translation  $(A_1 \otimes \dots \otimes A_n)^\pi$  are equivalent. One half of this claim holds in the literal sense:

**Lemma 4.16** (chain soundness)

$\vdash_{V\Sigma} A_1 \otimes \dots \otimes A_n \rightarrow (A_1 \otimes \dots \otimes A_n)^\pi$ .

*Proof.* Let  $1 \leq i \leq n$ . From the assumptions  $A_1 \otimes \dots \otimes A_n$  and  $\bigwedge_{j=1}^{i-1} \neg A_j$ , we can infer  $OA_i$  using the axiom (O-detachment). So by the deduction theorem, we can infer  $\left( \bigwedge_{j=1}^{i-1} \neg A_j \right) \rightarrow OA_i$ . From this and classical reasoning we obtain

$$\bigwedge_{i=1}^n \left( \left( \bigwedge_{j=1}^{i-1} \neg A_j \right) \rightarrow OA_i \right)$$

which is precisely  $(A_1 \otimes \dots \otimes A_n)^\pi$ . □

**Corollary 4.17**

For every chain negative formula  $N$ ,  $\vdash_{V_\Sigma} N^\pi \rightarrow N$ .

*Proof.* By induction on the structure of  $N$ . Simultaneously, one has to prove that

$$\vdash_{V_\Sigma} P \rightarrow P^\pi$$

for chain positive  $P$ . Both statements are trivially true if the formula does not contain  $\otimes$ . Furthermore if  $P$  is a  $\otimes$ -chain, we can use the chain soundness lemma.

As an example for the inductive step, assume that a chain negative formula  $N$  is of the form  $A \rightarrow B$ . Then  $A$  is chain positive and  $B$  is chain negative. By induction hypothesis, we therefore have  $\vdash_{V_\Sigma} A \rightarrow A^\pi$  and  $\vdash_{V_\Sigma} B^\pi \rightarrow B$ . From this and classical reasoning, we obtain

$$\vdash_{V_\Sigma} (A^\pi \rightarrow B^\pi) \rightarrow (A \rightarrow B),$$

which is what we need since  $(A \rightarrow B)^\pi = A^\pi \rightarrow B^\pi$ .

The other cases are similar. We note that the induction step for formulas beginning with  $O$  is trivial since, by the nesting condition, such formulas do not contain the  $\otimes$ -operator. □

The converse to Lemma 4.16 does not hold in general, see Corollary 4.12. Nevertheless, we will see that the deontic formula  $(A_1 \otimes \dots \otimes A_n)^\pi$  is as strong as the  $\otimes$ -chain  $A_1 \otimes \dots \otimes A_n$  when it comes to the derivation of basic deontic formulas: In particular, the obligations arising from  $A_1 \otimes \dots \otimes A_n$  are exactly the obligations arising from  $(A_1 \otimes \dots \otimes A_n)^\pi$ .

This follows from the reduction theorem below, which is our main technical result. We first state and prove the theorem and then discuss its technical and conceptual consequences.

**Theorem 4.18** (reduction theorem for the chain negative fragment)

For any chain negative formula  $N$ ,

$$\vdash_{V_{\Sigma+\Delta}} N \iff \vdash_{E+\Delta} N^\pi.$$



*Proof.* The direction from right to left is easy: If  $\vdash_{E+\Delta} N^\pi$ , then also  $\vdash_{V_\Sigma+\Delta} N^\pi$  since violation logic has all the axioms and rules of E. But then  $\vdash_{V_\Sigma+\Delta} N$  follows from Corollary 4.17, since N is chain negative.

For the direction from left to right, we argue by induction on the height of a proof  $\delta$  witnessing  $\vdash_{V_\Sigma+\Delta} N$ .

1. Assume first that  $\delta$  has height 1, which means that N is an axiom of  $V_\Sigma + \Delta$ .
  - a) If N is a substitution instance of a classical theorem, then  $N^\pi$  is again a substitution instance of the the same classical theorem as  $\pi$  commutes with boolean connectives. Hence  $\vdash_{E+\Delta} N^\pi$ .
  - b) Similarly, if N is a substitution instance of a basic deontic axiom in  $\Delta$ , then  $N^\pi$  is again a substitution instance of the the same axiom in  $\Delta$  since  $\pi$  commutes with boolean connectives and O. Hence  $\vdash_{E+\Delta} N^\pi$ .
  - c) If N is a substitution instance of an axiom in  $\Sigma$ , then  $\vdash_{E+\Delta} N^\pi$  by the axiom soundness lemma (Lemma 4.15).
2. If the last step in  $\delta$  is an instance of (MP)  $A, A \rightarrow B/B$ , then by induction hypothesis  $\vdash_{E+\Delta} B^\pi$  and  $\vdash_{E+\Delta} (A \rightarrow B)^\pi$ . Since  $(A \rightarrow B)^\pi$  equals  $A^\pi \rightarrow B^\pi$ , we can conclude  $\vdash_{E+\Delta} B^\pi$  by applying (MP) in E.
3. If the last step in  $\delta$  is an instance of (O-RE)  $A \equiv B/OA \equiv OB$ , then by induction hypothesis  $\vdash_{E+\Delta} (A \equiv B)^\pi$ . By the nesting condition, A and B must be basic deontic formulas and so  $(A \equiv B)^\pi$  equals  $A \equiv B$ . We can conclude  $\vdash_{E+\Delta} OA \equiv OB$  by applying (O-RE) in E, and this equals  $(OA \equiv OB)^\pi$ .
4. Assume that the last step in  $\delta$  is an inference

$$\frac{A \equiv B}{\nu \otimes A \otimes \nu' \equiv \nu \otimes B \otimes \nu'} (\otimes\text{-RE}).$$

By induction hypothesis  $\vdash_{E+\Delta} (A \equiv B)^\pi$ . Since A and B occur in a  $\otimes$ -chain, they must be classical formulas by the nesting condition, and so the premise  $(A \equiv B)^\pi$  equals  $A \equiv B$ . Now the deontic formula  $(\nu \otimes A \otimes \nu')^\pi$  arises from replacing some occurrences of B in  $(\nu \otimes B \otimes \nu')^\pi$  by the formula A. Hence

$$\frac{A \equiv B}{(\nu \otimes A \otimes \nu')^\pi \equiv (\nu \otimes B \otimes \nu')^\pi}$$

is an instance of uniform substitution (O-RE') (see Lemma 4.3), and so  $\vdash_{E+\Delta} (\nu \otimes A \otimes \nu' \equiv \nu \otimes B \otimes \nu')^\pi$  as desired.

5. Assume that the last step in  $\delta$  is an inference ( $\otimes$ -contraction). We only consider a characteristic case (the general case is similar):

$$\frac{A \equiv B}{X \otimes A \otimes Y \otimes B \otimes Z \equiv X \otimes A \otimes Y \otimes Z} (\otimes\text{-contraction})$$

Again  $A$  and  $B$  must be classical, and so we have  $\vdash_{E+\Delta} A \equiv B$  by induction hypothesis. Now arguing in  $E + \Delta$ , we can use (O-RE') to derive from  $A \equiv B$  the equivalence

$$(X \otimes A \otimes Y \otimes B \otimes Z)^\pi \equiv (X \otimes A \otimes Y \otimes A \otimes Z)^\pi$$

Written verbosely, the formula  $(X \otimes A \otimes Y \otimes A \otimes Z)^\pi$  equals

$$\begin{aligned} OX \wedge (\neg X \rightarrow OA) \wedge (\neg X \wedge \neg A \rightarrow OY) \wedge (\neg X \wedge \neg A \wedge \neg Y \rightarrow OA) \\ \wedge (\neg X \wedge \neg A \wedge \neg Y \wedge \neg A \rightarrow OZ). \end{aligned}$$

By using classical reasoning we see that the fourth conjunct can be omitted since it is implied by the second conjunct. Furthermore, the second  $\neg A$  in the last conjunct can be removed. The above formula is therefore equivalent to

$$OX \wedge (\neg X \rightarrow OA) \wedge (\neg X \wedge \neg A \rightarrow OY) \wedge (\neg X \wedge \neg A \wedge \neg Y \rightarrow OZ)$$

which is precisely  $(X \otimes A \otimes Y \otimes Z)^\pi$ . Hence we have  $\vdash_{E+\Delta} (X \otimes A \otimes Y \otimes B \otimes Z \equiv X \otimes A \otimes Y \otimes Z)^\pi$  as desired.

This concludes the proof of the reduction theorem. □

It is instructive to single out a special case of Theorem 4.18.

**Theorem 4.19** (reduction theorem, special case)

Let  $\Gamma \cup \{D\}$  be a set of basic deontic formulas. Then for any chain positive formula  $P$ , the following are equivalent:

- (i)  $\Gamma \cup \{P\} \vdash_{V_{\Sigma}+\Delta} D$
- (ii)  $\Gamma \cup \{P^\pi\} \vdash_{E+\Delta} D$
- (iii)  $\Gamma \cup \{P^\pi\} \vdash_{V_{\Sigma}+\Delta} D$

In particular, the equivalence holds if  $P$  is a  $\otimes$ -chain.

*Proof.*  $\Gamma \cup \{P\} \vdash_{V_{\Sigma}+\Delta} D$  is equivalent to  $\vdash_{V_{\Sigma}+\Delta} \bigwedge(\Gamma \cup \{P\}) \rightarrow D$  by the deduction theorem. Since  $\bigwedge(\Gamma \cup \{P\}) \rightarrow D$  is chain negative, its provability is equivalent to  $\vdash_{E+\Delta} (\bigwedge(\Gamma \cup \{P\}) \rightarrow D)^\pi$  by the reduction theorem. Now  $(\bigwedge(\Gamma \cup \{P\}) \rightarrow D)^\pi$  equals  $\bigwedge(\Gamma \cup \{P^\pi\}) \rightarrow D$  since neither  $\Gamma$  nor  $D$  contain  $\otimes$ -chains by assumption. So by another application of the deduction theorem, we obtain equivalence with  $\Gamma \cup \{P^\pi\} \vdash_{E+\Delta} D$ . We have thus established (i) $\leftrightarrow$ (ii), and applying (i) $\leftrightarrow$ (ii) to  $P^\pi$  instead of  $P$  yields (ii) $\leftrightarrow$ (iii). □

**Example 4.20**

Recall the formula  $A_{E_x} = w \rightarrow (\neg p) \otimes f$  from Example 4.5. For any set  $\Gamma$  of basic deontic formulas, we may ask whether

$$\{A_{E_x}\} \cup \Gamma \vdash_{V_{\Sigma+\Delta}} \text{Of}$$

holds, that is, whether under the assumption of  $A_{E_x}$  the deontic circumstances expressed in  $\Gamma$  lead to the obligation of paying a fine. By the special case of the reduction theorem, this question is equivalent to asking whether

$$\{A_{E_x}^\pi\} \cup \Gamma \vdash_{E+\Delta} \text{Of}$$

holds, where  $A_{E_x}^\pi = w \rightarrow (O(\neg p) \wedge (\neg\neg p \rightarrow \text{Of}))$ .

Conceptually, of most interest in Theorem 4.19 is the equivalence (i) $\leftrightarrow$ (iii) in the case that  $P$  is a  $\otimes$ -chain  $C$ , and its meaning can then be described as follows:

*Within a context of basic deontic formulas, using a  $\otimes$ -chain  $C$  as an assumption has exactly the same effect as using its translation  $C^\pi$ .*

In other words, as long as we are only interested in the role  $\otimes$ -chains as generators of obligations (under some circumstances described by basic deontic formulas), then we may as well replace all chains by their  $\pi$ -translations.

The questions which are not covered by the reduction theorem are those about the relation between different  $\otimes$ -chains, such as the question when one  $\otimes$ -chain implies another one. We will come back to this in Section 4.6.

An easy example demonstrating that the reduction theorem does not hold for the full language of violation logic is the following. Consider the (chain positive!) formula  $P = (a \otimes b)^\pi \rightarrow a \otimes b$ .  $P$  is not provable in  $V_\Sigma$ ; otherwise so would be

$$P^\perp = (a \otimes b)^\pi \rightarrow \perp = \neg(Oa \wedge (\neg a \rightarrow Ob))$$

by virtue of Theorem 4.10. But this formula is easily seen to be falsifiable on a neighbourhood model. On the other hand,  $P^\pi = (a \otimes b)^\pi \rightarrow (a \otimes b)^\pi$  is obviously a theorem of  $E$ .

## 4.5 Applications of the Reduction Theorem

We can combine the reduction theorem and Theorem 4.10 to demonstrate the undefinability of  $\otimes$ -chains. More specifically, the following corollary tells us that a chain only has a basic deontic definition in the absurd case:

**Corollary 4.21**

For any  $\otimes$ -chain  $C$ , the following are equivalent:

- (i) There exists a basic deontic formula  $D$  such that  $\vdash_{V_\Sigma + \Delta} C \equiv D$ .
- (ii)  $\vdash_{V_\Sigma + \Delta} C \equiv \perp$ .
- (iii)  $\vdash_{E + \Delta} C^\pi \equiv \perp$ .

*Proof.* If  $\vdash_{V_\Sigma + \Delta} C \equiv D$  for a basic deontic formula  $D$ , then by Theorem 4.10 also  $\vdash_{V_\Sigma + \Delta} (D \rightarrow C)^\perp$ , that is,  $\vdash_{V_\Sigma + \Delta} D \rightarrow \perp$ . By the equivalence of  $C$  and  $D$  this entails  $\vdash_{V_\Sigma + \Delta} C \rightarrow \perp$  and therefore  $\vdash_{V_\Sigma + \Delta} C \equiv \perp$ . Hence (i) $\rightarrow$ (ii). Assuming (ii) we can take  $D := \perp$  and also get the other direction. Therefore (i) $\leftrightarrow$ (ii).

On the other hand, (ii) and (iii) are equivalent to  $\vdash_{V_\Sigma + \Delta} C \rightarrow \perp$  and  $\vdash_{E + \Delta} C^\pi \rightarrow \perp$  respectively, and since  $C \rightarrow \perp$  is chain negative, both statements are equivalent by the reduction theorem. So (ii) $\leftrightarrow$ (iii).  $\square$

It depends on the axiomatization  $E + \Delta$  how many chains have a sound  $\pi$ -translation, and are therefore undefinable. Over the basic logic  $E$  every  $\pi$ -translations is satisfiable<sup>7</sup> and consequently no chain is definable. Over  $E + \neg O\perp$ , the chain  $\otimes\perp$  is equivalent to  $\perp$ .

The main point of a reduction as expressed in Theorem 4.18 is that the logic  $E + \Delta$  one reduces to is well studied, and one can transfer results about it back to the ‘new’ logic  $V_\Sigma + \Delta$ . Let us see some examples.

**Corollary 4.22**

The validity problem for the chain negative fragment of the violation logic  $V_\Sigma$  is coNP-complete.

*Proof.* By the reduction theorem,  $\vdash_{V_\Sigma} D$  is equivalent to  $\vdash_E D^\pi$  for a chain negative  $D$ , and the mapping  $D \mapsto D^\pi$  is computable in polynomial (in fact, quadratic) time. Since theoremhood in  $E$  is coNP-decidable ([54], Theorem 3.3), the same therefore holds for  $V_\Sigma$ . On the other hand, the chain negative fragment of  $V_\Sigma$  is a conservative extension of  $CL$ , which is coNP-hard.  $\square$

By the same argument, complexity (or just decidability) results can be obtained for other violation logics  $V_\Sigma + \Delta$ : We only have to know the complexity of the underlying deontic

<sup>7</sup>This follows from the existence of neighbourhood models where all formulas of the form  $OA$  are true everywhere.

logic  $E + \Delta$ . As far as we know, no decidability results for violation logics have been established so far.

It also follows from the reduction theorem that the neighbourhood semantics of classical modal logics provides a complete semantics for the chain negative fragment of violation logic. This semantics is simpler than the sequence semantics proposed in [25, 26].

#### Corollary 4.23

Let  $\Gamma \cup \{D\}$  be a set of deontic formulas. Then for any chain positive formula  $P$ ,  $\Gamma \cup \{P\} \vdash_{V_{\Sigma} + \Delta} D$  if and only if for every neighbourhood model  $\mathcal{W}$  with  $\mathcal{A}_{\mathcal{W}} \models \Delta$  the following is true: For any world  $w \in \mathcal{W}$ , if  $\langle \mathcal{W}, w \rangle \models \bigwedge \Gamma$  and  $\langle \mathcal{W}, w \rangle \models P^{\pi}$ , then  $\langle \mathcal{W}, w \rangle \models D$ .

*Proof.* By the reduction theorem,  $\Gamma \cup \{P\} \vdash_{V_{\Sigma} + \Delta} D$  is equivalent to  $\Gamma \cup \{P^{\pi}\} \vdash_{E + \Delta} D$ , which in turn is equivalent to  $\Gamma \cup \{P^{\pi}\} \models_{\Delta} D$  by Proposition 4.4.  $\square$

So within a context of deontic formulas, having a  $\otimes$ -chain  $C = a \otimes b \otimes c$  as a local assumption amounts to assuming the truth of

$$(a \otimes b \otimes c)^{\pi} = Oa \wedge (\neg a \rightarrow Ob) \wedge (\neg a \wedge \neg b \rightarrow Oc)$$

at a world of a neighbourhood model  $\mathcal{W}$ .

The reduction theorem is formulated relative to violation logics  $V_{\Sigma} + \Delta$  with a fixed axiomatization

$$\Sigma = \{(\otimes\text{-RE}), (\otimes\text{-contraction}), (O\text{-detachment}), (\otimes\text{-shortening}), (\otimes\text{-detachment})\}$$

of  $\otimes$ -chains (whereas the basic deontic axioms  $\Delta$  can be anything). Nevertheless, the proof is modular and can be adapted to ‘non-standard’ violation logics  $V_{\Pi} + \Delta$  where  $\Sigma$  is replaced by another set  $\Pi$  of axioms and rules for  $\otimes$ -chains: We only have to check that the  $\pi$ -translation of all axioms and rules in  $\Pi$  is sound in  $E + \Delta$  and that the chain soundness lemma holds; that is, we must have  $\vdash_{V_{\Pi} + \Delta} C \rightarrow \pi(C)$ . Then the proof of the reduction theorem goes through. Note in particular that the chain soundness lemma holds for any  $\Pi$  which contains (O-detachment).

A consequence of all this is the following observation.

#### Corollary 4.24

Let  $\Pi \neq \Sigma$  be any alternative axiomatization of  $\otimes$ -chains containing at least (O-detachment), and such that the  $\pi$ -translation of every axiom and rule of  $\Pi$  is sound in  $E + \Delta$ . Then the chain negative fragments of  $V_{\Sigma} + \Delta$  and  $V_{\Pi} + \Delta$  coincide.

*Proof.* By the above discussion, the proof of the reduction theorem goes through for  $V_{\Pi} + \Delta$  under the given assumptions. But then  $V_{\Sigma} + \Delta$  and  $V_{\Pi} + \Delta$  have the same characterization of their chain negative fragment (which does not depend on  $\Sigma$  or  $\Pi$ ), namely

$$\vdash_{V_{\Sigma} + \Delta} N \iff \vdash_{E + \Delta} N^{\pi} \iff \vdash_{V_{\Pi} + \Delta} N.$$

□

An immediate consequence of Corollary 4.24 is that the axioms and rules

( $\otimes$ -RE), ( $\otimes$ -contraction), ( $\otimes$ -detachment), ( $\otimes$ -shortening)

are all redundant in the chain negative fragment of  $V_{\Sigma} + \Delta$ . As another consequence, consider the axiom ( $\otimes$ -I)

$$\left( a_1 \otimes \dots \otimes a_n \wedge \left( \bigwedge_{i=1}^n \neg a_i \rightarrow b_1 \otimes \dots \otimes b_m \right) \right) \rightarrow a_1 \otimes \dots \otimes a_n \otimes b_1 \otimes \dots \otimes b_m$$

for creating  $\otimes$ -chains which is considered in [27, 14] but not in [25, 26]. It is easy to see that its  $\pi$ -translation is a theorem of E, and so by Corollary 4.24 its inclusion as an additional axiom has no effect on the chain negative fragment.

The simplest non-standard axiomatization of violation logic extends  $E + \Delta$  with the axiom

$$C \rightarrow \pi(C)$$

for every  $\otimes$ -chain C. The resulting logic has the same chain negative fragment as  $V_{\Sigma} + \Delta$ .

An axiomatization of  $\otimes$ -chains to which the reduction theorem does *not* apply is the one given in [13], where axioms such as  $a \otimes (\neg a) \equiv \top$  are included. Indeed, the  $\pi$ -translation of the latter axiom is  $Oa \wedge (\neg a \rightarrow O\neg a) \equiv \top$ , which does not hold in E.

Another consequence of the reduction theorem is that questions in violation logic can be approached using the proof theory of classical modal logics. For example, [38] presents cutfree Gentzen systems for the logics

$$E, \quad EC = E + OA \wedge OB \rightarrow O(A \wedge B) \quad \text{and} \quad M = E + O(A \wedge B) \rightarrow OA \wedge OB$$

which are called **Eseq**, **ECseq** and **Mseq**, respectively. All three calculi build on LK and then add rules for O as follows.

$$\text{Eseq:} \quad \frac{A \Rightarrow B \quad B \Rightarrow A}{OA \Rightarrow OB}$$

$$\text{ECseq:} \quad \frac{A_1, \dots, A_n \Rightarrow B \quad B \Rightarrow A_1 \quad \dots \quad B \Rightarrow A_n}{OA_1, \dots, OA_n \Rightarrow OB} \quad (\text{for every } n \geq 1)$$

$$\text{Mseq:} \quad \frac{A \Rightarrow B}{OA \Rightarrow OB}$$

**Corollary 4.25**

Let  $\Delta$  be  $\emptyset$ ,  $\{Oa \wedge Ob \rightarrow O(a \wedge b)\}$  or  $\{O(a \wedge b) \rightarrow Oa \wedge Ob\}$ . Then for any chain negative formula  $N$ ,  $\vdash_{V_\Sigma + \Delta} N$  if and only if there is a cutfree proof of  $N^\pi$  in **Eseq**, **ECseq** or **Mseq**, respectively.

**Example 4.26**

Here is a Gentzen-style proof establishing  $\{A_{Ex}, w, p\} \vdash_{V_\Sigma} Of$  by means of the  $\pi$ -translation (compare this to the Hilbert-style proof in Example 4.8):

$$\frac{\frac{\frac{p \Rightarrow p}{p, \neg p \Rightarrow} (\neg_l)}{p \Rightarrow \neg\neg p} (\neg_r) \quad Of \Rightarrow Of}{\neg\neg p \rightarrow Of, p \Rightarrow Of} (\rightarrow_l)}{w \Rightarrow w \quad \frac{O(\neg p) \wedge (\neg\neg p \rightarrow Of), p \Rightarrow Of}{w \rightarrow (O(\neg p) \wedge (\neg\neg p \rightarrow Of)), w, p \Rightarrow Of} (\wedge_l)} (\rightarrow_l)$$

**4.6 More on the Interpretation of  $\otimes$ -Chains**

Arguably, the formalization of many contrary-to-duty reasoning scenarios in the framework of violation logic remains in the chain negative fragment, and therefore in the scope of the reduction theorem. In particular all questions of the form

*Given some situation described by basic deontic formulas, which obligations arise from a collection of  $\otimes$ -chains?*

can be written down as a chain negative formula

$$S \wedge C_1 \wedge \dots \wedge C_n \rightarrow OA.$$

The reduction theorem then suggests that in the chain negative fragment, the meaning of a  $\otimes$ -chain can be identified with its  $\pi$ -translation (assuming, of course, one believes that the meaning of  $\otimes$ -chains is given by their proof-theoretic behaviour). Furthermore, we have seen (Corollary 4.24) that this identification is to some extent independent of the exact axiomatization  $\Sigma$  of  $\otimes$ -chains.

If we move beyond the chain negative fragment, the axiomatization of  $\otimes$ -chains matters more. So let us now consider an arbitrary violation logic  $V_\Pi + \Delta$  where  $\Pi$  is another axiomatization of  $\otimes$ -chains satisfying the premises of Corollary 4.24 and for which therefore the reduction theorem holds ( $\Delta$  is again any set of basic deontic axioms). The pivotal question outside the chain negative fragment is: When are two chains  $C_1$  and  $C_2$  considered equal, that is, when does  $\vdash_{V_\Pi + \Delta} C_1 \equiv C_2$  hold? A good axiomatization  $\Pi$

should give a tangible meaning to the notion of equality between chains. Hence, the question we have to ask is:

When *should* two  $\otimes$ -chains be equal?

Here is one possible proposal. We say that a pair  $C_1$  and  $C_2$  of chains is *deontically equivalent* over  $V_\Pi + \Delta$  if for every deontic formula  $D$ ,

$$\vdash_{V_\Pi + \Delta} C_1 \rightarrow D \iff \vdash_{V_\Pi + \Delta} C_2 \rightarrow D.$$

An intuitive interpretation of deontic equivalence might be this: ‘No matter whether  $C_1$  or  $C_2$  are employed as a law, the arising obligations are the same.’

**Definition 4.27**

The violation logic  $V_\Pi + \Delta$  is *faithful* if it proves  $C_1 \equiv C_2$  for every pair  $C_1$  and  $C_2$  of deontically equivalent chains.

So in a faithful violation logic, the meaning of equality between chains is that of deontic equivalence. From the reduction theorem arises a simple characterization of deontic equivalence:

**Lemma 4.28**

$C_1$  and  $C_2$  are deontically equivalent over  $V_\Pi + \Delta$  iff  $\vdash_{E+\Delta} C_1^\pi \equiv C_2^\pi$ .

*Proof.* Assume that  $C_1$  and  $C_2$  are deontically equivalent. Since  $\vdash_{V_\Pi + \Delta} C_2 \rightarrow C_2^\pi$  by assumption on  $\Pi$ , we also have  $\vdash_{V_\Pi + \Delta} C_1 \rightarrow C_2^\pi$  by deontic equivalence. But then  $\vdash_{E+\Delta} C_1^\pi \rightarrow C_2^\pi$  by the reduction theorem. By a symmetric argument,  $\vdash_{E+\Delta} C_2^\pi \rightarrow C_1^\pi$  and so  $C_1^\pi$  and  $C_2^\pi$  are provably equivalent.

Conversely, if  $\vdash_{E+\Delta} C_1^\pi \equiv C_2^\pi$  and  $D$  is a deontic formula implied by  $C_2$ , then  $\vdash_{E+\Delta} C_2^\pi \rightarrow D$  by the reduction theorem, and so  $\vdash_{E+\Delta} C_1^\pi \rightarrow D$  by assumption. Then by another application of the reduction theorem,  $\vdash_{V_\Pi + \Delta} C_1 \rightarrow D$  follows.  $\square$

As a simple example, it is easy to see that every violation logic in which  $\otimes$ -chains are definable by basic deontic formulas is faithful. But it is possible to have faithfulness without having definability of  $\otimes$ -chains: Artificially, such a logic is obtained by adding to  $V_\Sigma$  the rule

$$\frac{C_1^\pi \rightarrow C_2^\pi}{C_1 \rightarrow C_2}.$$

The argument for the undefinability of  $\otimes$ -chains in the resulting logic goes through as in Corollary 4.21.

We can show that the standard axiomatization is not faithful.



**Theorem 4.29**

Assume<sup>8</sup>  $OA \rightarrow \neg A$  is not a theorem of  $E + \Delta$ . Then  $V_{\Sigma} + \Delta$  is not faithful.

*Proof.* Let  $a, b$  be two distinct variables. The counterexample will be the two chains

$$C_1 = a \otimes (\neg a) \quad \text{and} \quad C_2 = a \otimes (\neg a) \otimes b.$$

They have the  $\pi$ -translations  $C_1^{\pi} = Oa \wedge (\neg a \rightarrow O(\neg a))$  and  $C_2^{\pi} = Oa \wedge (\neg a \rightarrow O(\neg a)) \wedge (\neg a \wedge \neg\neg a \rightarrow Ob)$ . Since both formulas are equivalent, we conclude by Lemma 4.28 that  $C_1$  and  $C_2$  are deontically equivalent.

However, while  $C_2 \rightarrow C_1$  is an instance of ( $\otimes$ -shortening),  $V_{\Pi} + \Delta$  fails to prove the converse  $C_1 \rightarrow C_2$ . We show this by employing yet another syntactic translation. Given a chain  $C$ , let  $d(C)$  be the number of all formulas occurring in the chain *up to logical equivalence*. For example, we have  $d((a \vee b) \otimes (\neg a \rightarrow b) \otimes b) = 2$ ,  $d(C_1) = 2$  and  $d(C_2) = 3$ . Now define a translation  $\rho$  as follows:

$$C^{\rho} = \begin{cases} C & \text{if } d(C) \leq 2; \\ \perp & \text{if } d(C) > 2. \end{cases}$$

In words,  $\rho$  declares all chains containing more than two non-equivalent formulas as contradictory. Now lift  $\rho$  to arbitrary formulas, and consider the  $\rho$ -translations of the axioms and rules in  $\Sigma$ :

$$\begin{aligned} (A_1 \otimes \dots \otimes A_n)^{\rho} \wedge \bigwedge_{i=1}^k \neg A_i &\rightarrow OA_{k+1} && \text{(O-detachment)}^{\rho} \\ (A_1 \otimes \dots \otimes A_n \otimes A_{n+1})^{\rho} &\rightarrow (A_1 \otimes \dots \otimes A_n)^{\rho} && \text{(\otimes-shortening)}^{\rho} \\ (A_1 \otimes \dots \otimes A_{n+1})^{\rho} \wedge \neg A_1 &\rightarrow (A_2 \otimes \dots \otimes A_{n+1})^{\rho} && \text{(\otimes-detachment)}^{\rho} \\ \frac{A \equiv B}{(\nu \otimes A \otimes \nu')^{\rho} \equiv (\nu \otimes B \otimes \nu')^{\rho}} &&& \text{(\otimes-RE)}^{\rho} \\ \frac{A \equiv B}{(\nu \otimes A \otimes \nu' \otimes B \otimes \nu'')^{\rho} \equiv (\nu \otimes A \otimes \nu' \otimes \nu'')^{\rho}} &&& \text{(\otimes-contraction)}^{\rho} \end{aligned}$$

One can check that they are all sound in  $V_{\Pi} + \Delta$ . We consider only two cases in detail:

First, look at ( $\otimes$ -shortening) $^{\rho}$ . We make a case distinction on the values of  $d_1 := d(A_1 \otimes \dots \otimes A_n)$  and  $d_2 := d(A_1 \otimes \dots \otimes A_n \otimes A_{n+1})$ . Clearly if  $d_1 > 2$ , then ( $\otimes$ -shortening) $^{\rho}$  is the tautology  $\perp = \perp$ . If both  $d_1 \leq 2$  and  $d_2 \leq 2$ , then  $\rho$  is the identity on both chains

<sup>8</sup>This assumption is harmless, as there is no reasonable deontic logic containing  $OA \rightarrow \neg A$  ('Obligations are never satisfied.').

and so  $(\otimes\text{-shortening})^p$  is just  $(\otimes\text{-shortening})$ . In the last remaining case that  $d_1 = 2$  and  $d_2 = 3$ , the instance  $(\otimes\text{-shortening})^p$  translates to  $\perp \rightarrow A_1 \otimes \dots \otimes A_n$  which is a classical theorem.

Second we consider  $(\otimes\text{-contraction})^p$ . Under the assumption  $A \equiv B$  we have  $d := d(\nu \otimes A \otimes \nu' \otimes B \otimes \nu'') = d(\nu \otimes A \otimes \nu' \otimes \nu'')$ , and so  $(\otimes\text{-contraction})^p$  is either  $(\otimes\text{-contraction})$  if  $d \leq 2$  or the trivially sound rule

$$\frac{A \equiv B}{\perp \equiv \perp}$$

if  $d > 2$ .

The other cases are similar. By a similar proof as for Theorem 4.10, one can then show that for any formula  $A$ ,  $\vdash_{V_{\Pi+\Delta}} A$  implies  $\vdash_{V_{\Pi+\Delta}} A^p$ . It is thus consistent with the standard axiomatization that all chains containing more than two non-equivalent formulas are unsatisfiable.

Assume now towards a contradiction that  $\vdash_{V_{\Sigma+\Delta}} C_1 \rightarrow C_2$ . Then also  $(C_1 \rightarrow C_2)^p = (C_1 \rightarrow \perp)$  is derivable in  $V_{\Sigma} + \Delta$ . By the reduction theorem, this implies that  $\neg C_1^\pi = \neg(Oa \wedge (\neg a \rightarrow O\neg a))$  is derivable in  $E + \Delta$ . The latter formula is equivalent to  $Oa \rightarrow \neg a$ , which we assumed  $E + \Delta$  does not derive.  $\square$

Theorem 4.29 extends to axiomatisations  $V_{\Pi} + \Delta$  given that the  $\rho$ -translations of all axioms and rules in  $\Pi$  are sound in  $V_{\Pi} + \Delta$ .

Earlier on, we already mentioned the axiom  $(\otimes\text{-I})$

$$\left( a_1 \otimes \dots \otimes a_n \wedge \left( \bigwedge_{i=1}^n \neg a_i \rightarrow b_1 \otimes \dots \otimes b_m \right) \right) \rightarrow a_1 \otimes \dots \otimes a_n \otimes b_1 \otimes \dots \otimes b_m.$$

From  $(\otimes\text{-I})$  we can prove  $a \otimes (\neg a) \rightarrow a \otimes (\neg a) \otimes b$ , the implication which was used as a counterexample to faithfulness in Theorem 4.29 (and consequently the  $\rho$ -translation of  $(\otimes\text{-I})$  must be unsound, which can easily be checked as well). This suggests the following question, to which we do not know the answer:

*Is the extension of  $V_{\Sigma}$  by  $(\otimes\text{-I})$  a faithful violation logic?*

Finally, let us comment on the definability of  $\otimes$ -chains. We have seen that  $\otimes$ -chains are undefinable over the standard axiomatization (Corollary 4.21). Nevertheless, given the similarity of chains to their  $\pi$ -translation, it might be of interest to see how large the gap to definability is. As it turns out, the missing link are the axioms

$$(\otimes\text{-I}) \quad \text{and} \quad (O\otimes) : Oa \rightarrow \otimes a.$$

Indeed, given a violation logic  $V_{\Pi} + \Delta$  let us denote by  $V_{\Pi^*} + \Delta$  the system which additionally has the axioms  $(\otimes\text{-I})$  and  $(O\otimes)$ . Then we have the following:

**Theorem 4.30**

The violation logics  $V_{\Pi} + \Delta$  and  $V_{\Pi^*} + \Delta$  coincide on their chain negative fragment, and in  $V_{\Pi^*} + \Delta$  every  $\otimes$ -chain is equivalent to its  $\pi$ -translation. In particular,  $V_{\Pi^*} + \Delta$  is faithful.

*Proof.* The chain negative fragments of both logics coincide since the  $\pi$ -translation of  $(\otimes\text{-I})$  and  $(\text{O}\otimes)$  is sound and so the reduction theorem applies to the logic  $V_{\Pi^*} + \Delta$ . For definability claim, it suffices to show by induction on  $n$  that

$$\vdash_{V_{\Pi^*} + \Delta} (A_1 \otimes \dots \otimes A_n)^\pi \rightarrow A_1 \otimes \dots \otimes A_n.$$

The base case  $n = 1$  is precisely the axiom  $(\text{O}\otimes)$ . For the induction step, we first note that the assumption  $(A_1 \otimes \dots \otimes A_n \otimes A_{n+1})^\pi$  equals

$$(A_1 \otimes \dots \otimes A_n)^\pi \wedge \left( \left( \bigwedge_{i=1}^n \neg A_i \right) \rightarrow \text{O}A_{n+1} \right)$$

by the definition of  $\pi$ . By the induction hypothesis and  $(\text{O}\otimes)$  we can infer

$$A_1 \otimes \dots \otimes A_n \wedge \left( \left( \bigwedge_{i=1}^n \neg A_i \right) \rightarrow \otimes A_{n+1} \right)$$

and then the axiom  $(\otimes\text{-I})$  yields  $A_1 \otimes \dots \otimes A_n \otimes A_{n+1}$  as desired.  $\square$

It is also possible to prove a converse: Every violation logic in which chains are definable by their  $\pi$ -translation validates  $(\otimes\text{-I})$  and  $(\text{O}\otimes)$ . Note however that  $(\text{O}\otimes)$  is to be rejected if one wants to syntactically separate descriptive norms (as generators of obligations) and prescriptive norms, which was one of the aims of the inventors of violation logic; see the short discussion in the introductory section.

## 4.7 Open Questions

Here we mention only one. We think the main challenge for violation logic is to develop an intuition about  $\otimes$ -chains which is firm enough to answer the question when two chains are the same. The results in this chapter show that such an intuition can *not* be established by looking at chains only as generators of obligations, as this role is to some extent independent of the axiomatization of chains (Corollary 4.24), whereas equality between chains is not (see Theorem 4.29 and the remark below). We have proposed one possible precise notion of equality between chains ('faithfulness'). It would be of interest to find further mathematical or philosophical arguments which either attack or support this notion, or to come up with completely new notions of equality.



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