TU Wien

Towards a Nonrelativistic Approximation of the Dirac Equation in Earth's Gravitational Field

Master thesis

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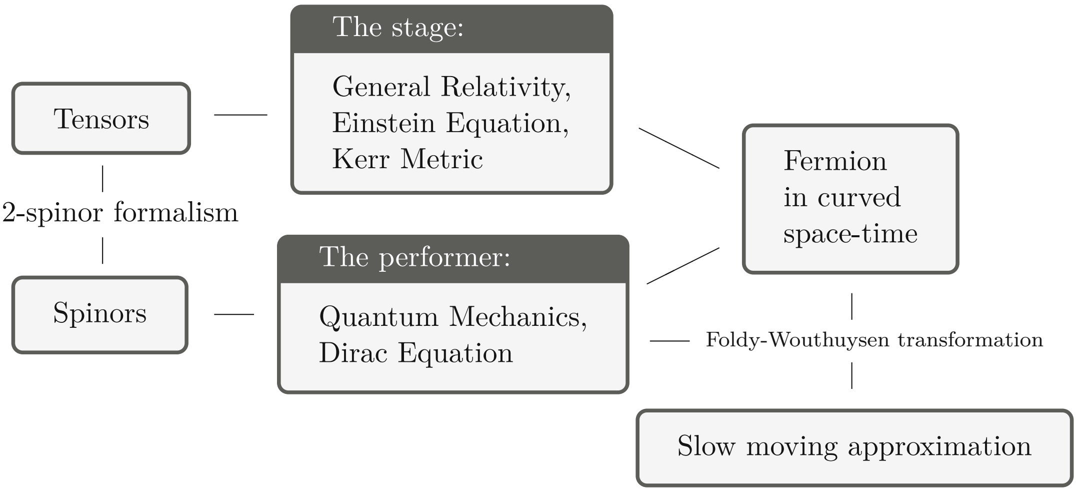
1. Summary

The aim of this thesis is to collect and put into context the complete groundwork required for describing slow-moving uncharged fermions, like neutrons, above earth's surface to obtain the general relativistic corrections that can be employed in experiments like qBOUNCE [4]. For this aim, we collect the necessary parts from a variety of already existing literature, work through them in detail, and arrange them such that they will be easy to use. In section 4 we shortly discuss how the bulk of the thesis might be used to calculate the relativistic corrections in question in subsequent investigations.

In particular we will be establishing some background in general relativity, quantum mechanics, and 2-spinor formalism. We will then discuss, how these techniques could be used to describe a spin-$\frac{1}{2}$ particle over the background of the Kerr-metric and to perform a Foldy-Wouthuysen transformation to obtain a Pauli equation with relativistic corrections. All this provides the necessary background for the search of hypothetical new interactions by using precision measurements.



1.1. Summary

In this section, a brief summary of the sections of this thesis is provided.

1.2. Introduction

This section provides a brief overview of the methods used subsequently. Among these are

- The Dirac equation and the Foldy-Wouthuysen transformation, which provides the nonrelativistic approximation of the Dirac equation [1].
- The tetrad Formalism, which is essential for the description of the Dirac equation in curved space-time [9].

- The 2-spinor formalism, which allows us to treat spinors and tensors on an equal footing [6].
- The Einstein equation and some of it's solutions [9].
- The Schwarzschild solution in particular as a preparation for the derivation of the Kerr solution [9].

1.3. The Kerr metric

This section provides a fully detailed step-by-step derivation of the Kerr metric, following [2].

1.4. Outlook

In this section, we outline prospects for further work and discuss how the techniques of this thesis can be employed to describe the Dirac equation in the background of the Kerr-metric, followed by the application of the Foldy-Wouthuysen transformation in order to obtain the corresponding Pauli equation containing the relativistic corrections in terms of an effective potential.

2. Introduction

This section provides a brief overview of the methods used in later sections. Among these are

- The Dirac equation and the Foldy-Wouthuysen transformation, which provides a nonrelativistic approximation of the Dirac equation [1].
- The tetrad Formalism, which is essential for the description of the Dirac equation in curved space-time [9].
- The 2-spinor formalism, which allows us to treat spinors and tensors on equal footing [6].
- The Einstein equation and some of it's solutions [9].
- The Schwarzschild solution in particular as a preparation for the Kerr solution [9].

2.1. Conventions

Throughout this thesis we will use the metric convention $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$. Except when explicitly stated, the speed of light c and $\hbar$ will be chosen to be 1.

We define commutation and anticommutation of a family of indices $i_j := i_1 i_2 ... i_p$ as

$$A_{\{i_j\}} = \frac{1}{p!} \sum_\sigma A_{\sigma(i_j)}, \qquad A_{[i_j]} = \frac{1}{p!} \sum_\sigma (-1)^\sigma A_{\sigma(i_j)}, \tag{2.1}$$

where σ runs over all permutations.

2.2. Dirac equation

In this section we follow [1] and examine at the Dirac equation.

The Dirac equation (2.2) describes a relativistic spin-$\frac{1}{2}$ particle.

$$\mathrm{i}\hbar\partial_t \Psi = \left[\frac{\hbar c}{\mathrm{i}} \alpha^i \partial_i + \beta m c^2\right] \Psi =: H\Psi, \tag{2.2}$$

where i runs over the spatial indices. To fulfill the relativistic energy-momentum relation (2.3) and for every component to satisfy the Klein-Gordon equation (2.4), α_i and β must fulfill the anticommutation relations (2.5-2.7).

$$E^2 = p^2 c^2 + m^2 c^4 \tag{2.3}$$

$$-\hbar^2 \partial_t^2 \Psi = \left[\hbar^2 c^2 \partial^i \partial_i + m^2 c^4\right] \Psi \tag{2.4}$$

$$\alpha_{\{i}\alpha_{j\}} = \delta_{ij} \tag{2.5}$$

$$\{\alpha_i, \beta\} = 0 \tag{2.6}$$

$$\alpha_i^2 = \beta^2 = 1 \tag{2.7}$$

Those relations imply that α_i and β cannot simply be complex scalars, but they can be represented as matrices. (2.6) and (2.7) are just an extension of (2.5) to include β, thus on an algebraic level, α_i and β are equivalent.

Relation (2.7) tells us that their eigenvalues are ± 1. By multiplying (2.6) by β and taking the trace we get

$$\mathrm{Tr}(\alpha_i) = -\mathrm{Tr}(\beta\alpha_i\beta) = -\mathrm{Tr}(\beta^2\alpha_i) = -\mathrm{Tr}(\alpha_i) = 0, \tag{2.8}$$

where we used the cyclic invariance of the trace. Slight modification shows the same for β. Since the trace is just the sum of the Eigenvalues, they must be an equal number of $+1$ and -1. The dimension of the matrices must therefore be even.

The axioms of quantum mechanics imply that physical observables correspond to Hermitean operators. For the Hamilton operator H to be Hermitean, we need both α and β to be Hermitean matrices. Hermiticity and vanishing trace imply that for dimension 2 they take the form

$$\begin{pmatrix} a & b - \mathrm{i}c \\ b + \mathrm{i}c & -a \end{pmatrix}, \qquad a, b, c \in \mathbb{R}. \tag{2.9}$$

All these matrices can be constructed as a real linear combination of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.10}$$

In four dimensions there is no further matrix that anticommutes with the Pauli matrices. The next even dimension is 4 and indeed there exist 4-dimensional matrices that fulfill relations (2.5 - 2.7). One set of those are

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{2.11}$$

Those matrices are not the only choice that satisfies these requirements. Other representations will be discussed later.

2.2.1. Nonrelativistic approximations

In this section we will investigate the Dirac equation for velocities $v \ll c$. The structure of the α_i and β matrices suggests to decompose the space of 4-component wave functions Ψ into the sum of spaces of 2-component wave functions ψ and χ. For a particle at rest the Dirac equation reads

$$\mathrm{i}\hbar\partial_t\Psi = \beta mc^2\Psi, \tag{2.12}$$

or in the above representation:

$$\mathrm{i}\hbar\partial_t\psi = mc^2\psi, \tag{2.13}$$
$$\mathrm{i}\hbar\partial_t\chi = -mc^2\chi. \tag{2.14}$$

The corresponding solutions are

$$\psi = e^{-i\frac{mc^2}{\hbar}t}\psi_0, \quad \chi = e^{i\frac{mc^2}{\hbar}t}\chi_0, \tag{2.15}$$

where ψ_0 and χ_0 are arbitrary constant 2-spinors. From (2.14) we see that the Dirac equation allows negative energy states.

Next we consider a free slow moving particle. Because the Hamilton operator is time independent, we focus on energy eigenstates

$$\mathrm{i}\hbar\partial_t\Psi = E\Psi, \tag{2.16}$$

with constant E. We assume $E > 0$. In the above representation, the more general Dirac equation takes the form

$$(E - mc^2)\psi = -\mathrm{i}\hbar c\sigma_i\partial_i\chi \tag{2.17}$$

$$(E + mc^2)\chi = -\mathrm{i}\hbar c\sigma_i\partial_i\psi \tag{2.18}$$

As for a slow moving particle $mc^2 \gg p$, (2.18) suggests that in that case, $\chi \ll \psi$. Therefore ψ will be called the large component and χ the small component.

By adding a minimal coupling

$$p_a \mapsto p_a - \frac{e}{c}A_a, \quad a \in \{0, 1, 2, 3\}, \tag{2.19}$$

we can couple a charged Dirac particle to an electromagnetic field. With the definition

$$\Pi_i := p_i - \frac{e}{c}A_i, \quad i \in \{1, 2, 3\}, \tag{2.20}$$

this leads to the Dirac equation

$$\mathrm{i}\hbar\partial_t\tilde{\Psi} = \left[c\alpha_i\Pi_i + \beta mc^2 + e\Phi\right]\tilde{\Psi}, \tag{2.21}$$

where we renamed Ψ to $\tilde{\Psi}$. Because of our slow moving and weak field approximations the majority of the energy will be mc^2. Therefore we can decompose the wave function into a fast oscillating factor and a slowly varying wave function.

$$\tilde{\Psi} = e^{\frac{mc^2}{\hbar}t}\Psi \tag{2.22}$$

Plugging that into the Dirac equation and using the above representation, we get

$$\mathrm{i}\hbar\partial_t\begin{pmatrix}\psi\\ \chi\end{pmatrix} = c\sigma_i\Pi_i\begin{pmatrix}\chi\\ \psi\end{pmatrix} + e\Phi\begin{pmatrix}\psi\\ \chi\end{pmatrix} - 2mc^2\begin{pmatrix}0\\ \chi\end{pmatrix}. \tag{2.23}$$

Looking at the second equation, the left hand side can be neglected because we assume only slow variation of χ. Also since Φ is small and χ is suppressed by a factor $\sim \frac{1}{mc^2}$

from earlier considerations, we can neglect the second term on the right hand side. What is left reduces to

$$\chi = \frac{\sigma_i \Pi_i}{2mc}\psi. \tag{2.24}$$

Plugging this into the first equation it yields

$$\mathrm{i}\hbar\partial_t\psi = \left[\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}}{2m} + e\Phi\right]. \tag{2.25}$$

We can now use the identity

$$\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k \quad \Rightarrow \quad (\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b} + \mathrm{i}\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}), \tag{2.26}$$

to find

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}) = \boldsymbol{\Pi}^2 - \frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\boldsymbol{B}. \tag{2.27}$$

The resulting equation

$$\mathrm{i}\hbar\partial_t\psi = \left[\frac{\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2}{2m} - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B} + e\Phi\right]\psi \tag{2.28}$$

is now recognizable as the famous Pauli equation. As we will later see there is a more systematic way to deduce this equation.

2.2.2. The γ matrices

One of the motivations for the Dirac equation was Lorentz covariance, which is best portrayed by multiplying (2.2) by β from the left and defining

$$\gamma^0 := \beta, \quad \gamma^i := \beta\alpha_i, \quad i \in \{1,2,3\}. \tag{2.29}$$

The free Dirac equation now simply reads

$$\left[\mathrm{i}\hbar\gamma^a\partial_a - mc\right]\Psi = 0. \tag{2.30}$$

The anticommutation relations (2.5 - 2.7) simplify to

$$\gamma_{\{a}\gamma_{b\}} = \eta_{ab}, \tag{2.31}$$

and while γ^0 remains Hermitean, γ^i are anti-Hermitean. Every set of matrices that fulfill these requirements produce a representation of the Dirac equation. As proved in [5] those sets of matrices are connected by unitary similarity transformations.

The γ-matrices for the Dirac representation which we used in the previous sections, are

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{2.32}$$

Another set of valid γ matrices is the Weyl- or chiral representation

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}. \tag{2.33}$$

We will later use this representation to write down the Dirac equation in 2-spinor form, which will also prove Lorentz covariance since 2-spinor equations are inherently Lorentz covariant.

2.2.3. The Foldy-Wouthuysen transformation

For the following outline we rely on [1].

The fact that the nonrelativistic limit of the Dirac equation in the Dirac representation separates the spinor into a large and a small component suggests that if we could find a representation that decouples those components, we can neglect the small component and the nonrelativistic case can be described by an equation for only the large component. That is what the Foldy-Wouthuysen transformation does. The idea is to split the Hamiltonian of the Dirac equation into even terms $\mathscr{E}$ which are of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, and odd terms $\mathscr{O}$ of the form $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$. The goal is to find a unitary transformation $e^{\mathrm{i}S}$ such that only even terms remain.

In this section, operators act on everything that is written after them, with the exception of ∂, which acts only on the right adjacent object. If another operator acts on only the adjacent object, it is denoted by a small arrow, e.g. $\vec{\mathbf{p}}$.

With the transformation

$$\Psi \mapsto \Psi' := e^{\mathrm{i}S}\Psi, \tag{2.34}$$

the Dirac equation reads

$$\mathrm{i}\partial_t \left(e^{-\mathrm{i}S}\Psi'\right) = H\Psi = He^{-\mathrm{i}S}\Psi', \tag{2.35}$$

where

$$H = \mathscr{E} + \mathscr{O} + \beta m. \tag{2.36}$$

This leads to

$$\mathrm{i}\partial_t \Psi' = \underbrace{\left[e^{\mathrm{i}S}\left(H - \mathrm{i}\partial_t\right)e^{-\mathrm{i}S}\right]}_{=:H'}\Psi', \tag{2.37}$$

which will be our starting point for the specific transformations.

2.2.4. The Foldy-Wouthuysen Transformation for the Free Dirac Equation

In the case of a free moving particle our even and odd parts of the Hamilton operator are

$$\mathscr{E} = 0, \quad \mathscr{O} = \alpha \cdot \mathbf{p}. \tag{2.38}$$

We make the Ansatz

$$e^{\mathrm{i}S} = e^{\beta\alpha\cdot\mathbf{p}\theta} = \cos(|\mathbf{p}|\theta) + \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin(|\mathbf{p}|\theta), \tag{2.39}$$

in the sense of the operator valued Taylor expansion of sine and cosine. θ is yet to be determined. The inverse of the operator Q we denote as $\frac{1}{Q}$ and it satisfies $Q\frac{1}{Q} = \frac{1}{Q}Q = \mathbb{1}$ wherever $Q\Psi \neq 0$. The parts of Hilbert space that are annihilated by Q can be ignored here since $\frac{1}{Q}$ will always come in combination with Q. That such an operator exists is shown in appendix A.1.

The transformed Hamilton operator then is

$$\begin{aligned} H' &= \left[\cos(|\mathbf{p}|\theta) + \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin(|\mathbf{p}|\theta)\right](\alpha\cdot\mathbf{p}+\beta m)\left[\cos(|\mathbf{p}|\theta) - \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin(|\mathbf{p}|\theta)\right] && (2.40)\\ &= (\alpha\cdot\mathbf{p}+\beta m)\left[\cos(|\mathbf{p}|\theta) - \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin(|\mathbf{p}|\theta)\right]^2 = (\alpha\cdot\mathbf{p}+\beta m)\, e^{-2\beta\alpha\cdot\mathbf{p}\theta} && (2.41)\\ &= (\alpha\cdot\mathbf{p}+\beta m)\left[\cos(2|\mathbf{p}|\theta) - \frac{\beta\alpha\cdot\mathbf{p}}{|\mathbf{p}|}\sin(2|\mathbf{p}|\theta)\right] && (2.42)\\ &= \alpha\cdot\mathbf{p}\left[\cos(2|\mathbf{p}|\theta) - \frac{m}{|\mathbf{p}|}\sin(2|\mathbf{p}|\theta)\right] + \beta\left[m\cos(2|\mathbf{p}|\theta) + |\mathbf{p}|\sin(2|\mathbf{p}|\theta)\right]. && (2.43) \end{aligned}$$

To cancel the odd term, we have to set

$$\cos(2|\mathbf{p}|\theta) = \frac{m}{|\mathbf{p}|}\sin(2|\mathbf{p}|\theta) \quad\Rightarrow\quad \tan(2|\mathbf{p}|\theta) = \frac{|\mathbf{p}|}{m}. \tag{2.44}$$

This forms a right triangle with angle $2|\mathbf{p}|\theta$, adjacent side m and opposite side $|p|$. The hypotenuse then is $\sqrt{\mathbf{p}^2+m^2}$. We have

$$\cos(2|\mathbf{p}|\theta) = \frac{m}{\sqrt{\mathbf{p}^2+m^2}}, \quad \sin(2|\mathbf{p}|\theta) = \frac{|\mathbf{p}|}{\sqrt{\mathbf{p}^2+m^2}}. \tag{2.45}$$

Inserting that into the Hamilton operator, we get

$$H' = \beta\frac{\mathbf{p}^2+m^2}{\sqrt{\mathbf{p}^2+m^2}} = \beta\sqrt{\mathbf{p}^2+m^2}, \tag{2.46}$$

the square root of the Klein-Gordon equation.

2.2.5. The general Foldy-Wouthuysen transformation

In general it is not so easy to perform the transformation exactly. To approximate, we assume that in the nonrelativistic limit all energies are small compared to m. We then choose S to be proportional to $\frac{1}{m}$ and eliminate the odd terms order by order in $\frac{1}{m}$ to the desired accuracy.

We again start with the Hamilton operator

$$H = \beta m + \mathscr{O} + \mathscr{E}. \tag{2.47}$$

Since $\mathscr{O}$ is proportional to α_i, $\mathscr{E}$ is diagonal in the sense of 2×2 blocks, and the blocks of β are diagonal, we have

$$\beta\mathscr{O} = -\mathscr{O}\beta, \quad \beta\mathscr{E} = \mathscr{E}\beta. \tag{2.48}$$

For an operator Ω,

$$e^{\mathrm{i}\lambda S}\Omega e^{-\mathrm{i}\lambda S} =: F(\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left.\frac{\partial^n F(\tilde{\lambda})}{\partial \tilde{\lambda}^n}\right|_{\tilde{\lambda}=0}, \tag{2.49}$$

where

$$\frac{\partial F(\lambda)}{\partial \lambda} = e^{\mathrm{i}\lambda S}\mathrm{i}[S,\Omega]e^{-\mathrm{i}\lambda S}, \tag{2.50}$$

and thus, with $L := [S, \cdot]$

$$\frac{\partial^n F(\lambda)}{\partial \lambda^n} = e^{\mathrm{i}\lambda S}\mathrm{i}^n \left(L^n\Omega\right) e^{-\mathrm{i}\lambda S}. \tag{2.51}$$

By setting $\lambda = 1$ after plugging into the Taylor series, we get

$$e^{\mathrm{i}S}\Omega e^{-\mathrm{i}S} = \Omega + \mathrm{i}[S,\Omega] - \frac{1}{2}[S,[S,\Omega]] - \frac{\mathrm{i}}{6}[S,[S,[S,\Omega]]] + \dots . \tag{2.52}$$

Now we are prepared to expand the transformed Hamilton operator. We will take into account terms up to order $\frac{1}{m^3}$. Using (2.37), we get

$$\begin{aligned} H' = H + \mathrm{i}[S,H] - \frac{1}{2}[S,[S,H]] - \frac{\mathrm{i}}{6}[S,[S,[S,H]]] + \frac{1}{24}[S,[S,[S,[S,\beta m]]]] \\ -\dot{S} - \mathrm{i}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]]. \end{aligned} \tag{2.53}$$

To order 1 this reads

$$H' = \beta m + \mathscr{E} + \mathscr{O} + \mathrm{i}[S,\beta m]. \tag{2.54}$$

For $\mathscr{O}$ to vanish at this order, we need

$$S = -\mathrm{i}\frac{\beta\mathscr{O}}{2m}. \tag{2.55}$$

Using this with the original Hamiltonian we have

$$\mathrm{i}[S,H] = \frac{1}{2m}[\beta\mathscr{O}, \beta m + \mathscr{E} + \mathscr{O}] = -\mathscr{O} + \frac{\beta}{2m}[\mathscr{O},\mathscr{E}] + \frac{\beta}{m}\mathscr{O}^2, \tag{2.56}$$

$$-\frac{1}{2}[S,[S,H]] = \frac{\mathrm{i}}{2}[S,\mathrm{i}[S,H]] = -\frac{\beta}{2m}\mathscr{O}^2 - \frac{1}{8m^2}[\mathscr{O},[\mathscr{O},\mathscr{E}]] - \frac{1}{2m^2}\mathscr{O}^3, \tag{2.57}$$

$$-\frac{\mathrm{i}}{6}[S,[S,[S,H]]] = \frac{\mathrm{i}}{3}[S,-\frac{1}{2}[S,[S,H]]] = \frac{1}{6m^2}\mathscr{O}^3 - \frac{\beta}{48m^3}[\mathscr{O},[\mathscr{O},[\mathscr{O},\mathscr{E}]]] - \frac{\beta}{6m^3}\mathscr{O}^4, \tag{2.58}$$

$$\frac{1}{24}[S,[S,[S,[S,H]]]] = \frac{\mathrm{i}}{4}[S,-\frac{\mathrm{i}}{6}[S,[S,[S,H]]]] = \frac{\beta}{24m^3}\mathscr{O}^4, \tag{2.59}$$

$$-\dot{S} = \mathrm{i}\frac{\beta\dot{\mathscr{O}}}{2m}, \tag{2.60}$$

$$-\frac{\mathrm{i}}{2}[S,\dot{S}] = -\frac{\mathrm{i}}{8m^2}[\mathscr{O},\dot{\mathscr{O}}], \tag{2.61}$$

$$\frac{1}{6}[S,[S,\dot{S}]] = \frac{\mathrm{i}}{3}[S,-\frac{\mathrm{i}}{2}[S,\dot{S}]] = -\mathrm{i}\frac{\beta}{48m^3}[\mathscr{O},[\mathscr{O},\dot{\mathscr{O}}]]. \tag{2.62}$$

Combining the terms we obtain

$$H' = \beta m + \mathscr{E}' + \mathscr{O}', \tag{2.63}$$

where

$$\mathscr{E}' = \mathscr{E} + \frac{\beta}{2m}\mathscr{O}^2 - \frac{1}{8m^3}\mathscr{O}^4 - \frac{1}{8m^2}[\mathscr{O},[\mathscr{O},\mathscr{E}]] - \mathrm{i}\frac{1}{8m^2}[\mathscr{O},\dot{\mathscr{O}}], \tag{2.64}$$

$$\mathscr{O}' = \frac{\beta}{2m}[\mathscr{O},\mathscr{E}] - \frac{1}{3m^2}\mathscr{O}^3 + \mathrm{i}\frac{\beta}{2m}\dot{\mathscr{O}} - \frac{\beta}{48m^3}[\mathscr{O},[\mathscr{O},[\mathscr{O},\mathscr{E}]]] - \mathrm{i}\frac{\beta}{48m^3}[\mathscr{O},[\mathscr{O},\dot{\mathscr{O}}]]. \tag{2.65}$$

Here, the odd terms only appear to order $\frac{1}{m}$ upwards. We can now repeat this procedure to get rid of the $\frac{1}{m}$ odd terms. The procedure for the second transformation is exactly the same as for the first one. The new exponent is

$$S' = -\mathrm{i}\frac{\beta\mathscr{O}'}{2m}, \tag{2.66}$$

which is now of order $\frac{1}{m^2}$. The terms for the transformed Hamilton operator up to order $\frac{1}{m^3}$ are

$$\mathrm{i}[S',H'] = -\mathscr{O}' + \frac{\beta}{2m}[\mathscr{O}',\mathscr{E}'] + \frac{\beta}{m}(\mathscr{O}')^2, \tag{2.67}$$

$$-\frac{1}{2}[S',[S',H']] = -\frac{\beta}{2m}(\mathscr{O}')^2, \tag{2.68}$$

$$-\dot{S}' = \mathrm{i}\frac{\beta}{2m}\dot{\mathscr{O}}'. \tag{2.69}$$

Therefore we have

$$H'' = \beta m + \mathscr{E}'' + \mathscr{O}'', \tag{2.70}$$

with

$$\mathscr{E}'' = \mathscr{E}' + \frac{\beta}{2m}(\mathscr{O}')^2, \tag{2.71}$$

$$\mathscr{O}'' = \frac{\beta}{2m}[\mathscr{O}', \mathscr{E}'] + \mathrm{i}\frac{\beta}{2m}\dot{\mathscr{O}}'. \tag{2.72}$$

The next transformation goes accordingly. The new exponent S'' is now of order $\frac{1}{m^3}$, which leaves us with the transformed Hamilton operator

$$H''' = \beta m + \mathscr{E}''' + \mathscr{O}''', \tag{2.73}$$

where

$$\mathscr{E}''' = \mathscr{E}'', \tag{2.74}$$

$$\mathscr{O}''' = \frac{\beta}{2m}[\mathscr{O}'', \mathscr{E}''] + \mathrm{i}\frac{\beta}{2m}\dot{\mathscr{O}}''. \tag{2.75}$$

For the final transformation, since the exponent S''' would be of order $\frac{1}{m^4}$, the only terms left are

$$H^{\mathrm{IV}} = \beta m + \mathscr{E}'' \tag{2.76}$$

which decouples the large and small components up to order $\frac{1}{m^3}$. Putting in the lower order even and odd terms, we get

$$\begin{aligned} H^{\mathrm{IV}} = \mathscr{E} + \beta\left(m + \frac{\mathscr{O}^2}{2m}\right) - \frac{\mathscr{O}^4}{8m^3} - \frac{1}{8m^2}\left([\mathscr{O}, [\mathscr{O}, \mathscr{E}]] + \mathrm{i}[\mathscr{O}, \dot{\mathscr{O}}]\right) \\ + \frac{1}{8m^3}\left(\dot{\mathscr{O}}^2 - \mathrm{i}\{[\mathscr{O}, \mathscr{E}], \dot{\mathscr{O}}\} - [\mathscr{O}, \mathscr{E}]^2\right). \end{aligned} \tag{2.77}$$

2.2.6. A charged particle in the electromagnetic field

By minimal substitution we can couple a point charge to the electromagnetic field.

$$H = \beta m + \mathrm{e}\Phi + \alpha \cdot (\mathbf{p} - \mathrm{e}\mathbf{A}) =: \beta m + \mathscr{E} + \mathscr{O} \tag{2.78}$$

The following useful relation can directly be taken from the Pauli matrices:

$$\alpha_i \alpha_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\sigma_k. \tag{2.79}$$

We will only take into account terms up to $(\text{momentum})^3/m^3$ and order e. So the only relevant terms in the Foldy-Wouthuysen transformed Hamilton operator are

$$H^{\mathrm{IV}} = \mathscr{E} + \beta\left(m + \frac{\mathscr{O}^2}{2m}\right) - \frac{\mathscr{O}^4}{8m^3} - \frac{1}{8m^2}\left([\mathscr{O}, [\mathscr{O}, \mathscr{E}] + \mathrm{i}\dot{\mathscr{O}}]\right). \tag{2.80}$$

We can now calculate the relevant terms:

$$\begin{aligned}\mathscr{O}^2 &= \alpha_i(p_i - \mathrm{e}A_i)\alpha_j(p_j - \mathrm{e}A_j)\\ &= (\mathbf{p} - \mathrm{e}\mathbf{A})^2 - \mathrm{i}\sigma_k \mathrm{e}\epsilon_{ijk}\left(\vec{p}_i A_j + A_j p_i + A_i p_j\right)\\ &= (\mathbf{p} - \mathrm{e}\mathbf{A})^2 - \mathrm{e}\sigma\cdot\mathbf{B},\end{aligned} \tag{2.81}$$

$$\mathscr{O}^4 = \mathbf{p}^4, \tag{2.82}$$

$$\dot{\mathscr{O}} = -\mathrm{e}\alpha\cdot\dot{\mathbf{A}}, \tag{2.83}$$

$$[\mathscr{O},\mathscr{E}] = -\mathrm{i}\mathrm{e}\alpha\cdot\nabla\Phi, \tag{2.84}$$

$$[\mathscr{O},\mathscr{E}] + \mathrm{i}\dot{\mathscr{O}} = -\mathrm{i}\mathrm{e}\alpha\cdot\left(\nabla\Phi + \dot{\mathbf{A}}\right) = \mathrm{i}\mathrm{e}\alpha\cdot\mathbf{E}, \tag{2.85}$$

$$\begin{aligned}[][\alpha\cdot\mathbf{p},\alpha\cdot\mathbf{E}] &= -\mathrm{i}\alpha_i\alpha_j\left(\nabla_i E_j + E_j\nabla_i - E_i\nabla_j\right)\\ &= -\mathrm{i}\left(\nabla\cdot\mathbf{E} + \mathrm{i}\sigma\cdot(\nabla\times\mathbf{E}) - 2\mathrm{i}\sigma\cdot(\mathbf{E}\times\nabla)\right),\end{aligned} \tag{2.86}$$

$$[\mathscr{O},[\mathscr{O},\mathscr{E}] + \mathrm{i}\dot{\mathscr{O}}] = \mathrm{e}\left(\nabla\cdot\mathbf{E} + \mathrm{i}\sigma\cdot(\nabla\times\mathbf{E}) - 2\mathrm{i}\sigma\cdot(\mathbf{E}\times\nabla)\right). \tag{2.87}$$

This leads us to the transformed hamilton operator

$$\begin{aligned}H^{\mathrm{IV}} = \beta\left(m + \frac{(\mathbf{p} - \mathrm{e}\mathbf{A})^2}{2m} - \frac{\mathbf{p}^4}{8m^3}\right) + \mathrm{e}\Phi - \beta\frac{\mathrm{e}}{2m}\sigma\cdot\mathbf{B}\\ -\frac{\mathrm{i}e}{8m^2}\sigma\cdot(\nabla\times\mathbf{E}) - \frac{\mathrm{e}}{4m^2}\sigma\cdot(\mathbf{E}\times\mathbf{p}) - \frac{\mathbf{e}}{8m^2}\nabla\cdot\mathbf{E}.\end{aligned} \tag{2.88}$$

2.3. Tetrad formalism

It is sometimes convenient to switch to an orthonormal basis, or tetrad, that moves along in space-time in such a way that it stays orthonormal at every point. Instead of just equipping the manifold with direction and differential structure, the tangent space now becomes its own entity, acting as the fibre to the manifold basis of the tangent bundle. A fibre bundle is a topological space E with a projection $\pi : E \to B$ which maps to a topological space B, called the basis, such that there exists an open set U_P around every point P of B such that the pre-image $\pi^{-1}(U_P)$ is homeomorphic to $U_P \times F$. F is a topological space called the fibre [8]. While in this section it is just another way of looking at a manifold, where we treat the tangent space as the fibre, later on, when we extend tensor calculus to include spinors, this will be essential. For calculations it is useful to employ the algebra of differential forms.

The substance of this section is taken from [9], [6].

2.3.1. Differential Forms

A scalar differential form is a completely antisymmetric $(0, p)$ tensor, where p denotes the number of lower indices. A differential form with p antisymmetric lower indices is also called a p-form. For convenience we will suppress form indices. If that is done, the symbol denoting the differential form will be written in bold face. If the tensor has additional indices, like upper indices or ones that are not antisymmetric, it is called a tensorial differential form. We will call the space of p-forms $\Lambda^p(\mathscr{A})$, where $\mathscr{A}$ denotes the additional tensor indices. There are several possible operations on differential forms that preserve their form-character.

Addition between two p-forms can be directly taken from tensor addition. The result is again a p-form of the same form character.

Multiplication on the other hand is a bit more involved. If only tensor multiplication is applied to two differential forms, the resulting differential form's form indices might no longer be totally antisymmetric. Therefore we need a new multiplication called $\wedge$ which preserves antisymmetry.

$$\begin{aligned} \wedge: \quad & \Lambda^p(\mathscr{A}) \times \Lambda^q(\mathscr{B}) \to \Lambda^{p+q}(\mathscr{A}\mathscr{B}) \\ & \alpha^{\mathscr{A}}_{i_1 \dots i_p} \wedge \beta^{\mathscr{B}}_{i_{p+1} \dots i_{p+q}} = \frac{(p+q)!}{p!q!} \alpha^{\mathscr{A}}_{[i_1 \dots i_p} \beta^{\mathscr{B}}_{i_{p+1} \dots i_{p+q}]} \end{aligned} \tag{2.89}$$

Also differentiation needs to be modified, because the derivative operator brings an index that is not in general antisymmetrized with the rest. Therefore we introduce the exterior derivative d.

$$\begin{aligned} \mathrm{d}: \quad & \Lambda^p(\mathscr{A}) \to \Lambda^{p+1}(\mathscr{A}) \\ & \mathrm{d}\omega^{\mathscr{A}}_{i_1 \dots i_p} = (p+1) \nabla_{[j} \omega^{\mathscr{A}}_{i_1 \dots i_p]} \end{aligned} \tag{2.90}$$

One immediate consequence of this definition is that d^2 acts on tensorial 0-forms via the Riemann tensor, and vanishes on scalar forms. To prove the latter we will use the symmetry of the Christoffel symbol ${\Gamma^d}_{[ab]} = 0$, as well as the symmetry of the partial derivatives $\partial_{[a}\partial_{b]} = 0$.

$$\begin{aligned} \nabla_{[a} \nabla_b \omega_{c]} &= \nabla_{[a} \partial_b \omega_{c]} - \nabla_{[a} {\Gamma^d}_{bc]} \omega_d \\ &= \partial_{[a} \partial_b \omega_{c]} - {\Gamma^d}_{[ab} \partial_{|d|} \omega_{c]} - {\Gamma^d}_{[ac} \partial_{b]} \omega_d = 0 \end{aligned} \tag{2.91}$$

The outer derivative fulfills the weighted Leibniz rule

$$\mathrm{d}(\alpha^{\mathscr{A}} \wedge \beta^{\mathscr{B}}) = \mathrm{d}\alpha^{\mathscr{A}} \wedge \beta^{\mathscr{B}} + (-1)^p \alpha^{\mathscr{A}} \wedge \mathrm{d}\beta^{\mathscr{B}}, \tag{2.92}$$

if $\alpha^{\mathscr{A}}$ is a p-form and $\beta^{\mathscr{B}}$ is an arbitrary differential form, because the index of the exterior derivative in the second term must be translated through $\alpha^{\mathscr{A}}$.

2.3.2. Tetrad Formalism

First we introduce an orthonormal basis of the tangent space we denote by E^a_α. While latin letters are ordinary abstract tensor indices, greek letters from the beginning of the alphabet denote labels of the specific basis vector. We will suppress tensor indices in the further discussion (except where they are needed for clarity) as we did with form indices. Its dual basis will be denoted by $\boldsymbol{e}^\alpha$. They are given different symbols to distinguish them due to the suppressed indices. To express a tensor in the tetrad basis we simply contract it with the respective basis vector.

$$T^{\alpha}{}_{\beta} = T^{a}{}_{b}\, e^{\alpha}_{a} E^{b}_{\beta} \tag{2.93}$$

Because of the suppressed indices we introduce a new symbol for the inner product, or the contraction between suppressed indices.

$$\boldsymbol{\alpha}^{\mathscr{A}} \lrcorner \boldsymbol{\beta}^{\mathscr{B}} := \alpha^{a\mathscr{A}}_{\mathscr{C}}\, \beta^{\mathscr{B}}_{a\mathscr{D}} \tag{2.94}$$

$\mathscr{C}$ and $\mathscr{D}$ represent other suppressed indices. We define the inner product in such a way that the contraction happens between the first contravariant tensor index and the first form index.

From the orthonormality follows

$$g_{ab} E^{a}_{\alpha} E^{b}_{\beta} = \eta_{\alpha\beta} \quad \text{or} \quad \boldsymbol{E}_{\alpha} \lrcorner \boldsymbol{e}^{\beta} = \delta^{\beta}_{\alpha}. \tag{2.95}$$

Similarly the metric can be decomposed into tetrad components:

$$\eta_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} = g_{ab} \quad \text{or} \quad E^{a}_{\alpha} e^{\alpha}_{b} = \delta^{a}_{b}. \tag{2.96}$$

If we differentiate $\boldsymbol{E}_{\alpha}$, the result will still have a vector index which can be decomposed into tetrad components.

$$\mathrm{d}E^{a}_{\alpha} =: \boldsymbol{\omega}^{a}{}_{\alpha} = E^{a}_{\beta} \boldsymbol{\omega}^{\beta}{}_{\alpha}. \tag{2.97}$$

$\boldsymbol{\omega}^{\beta}{}_{\alpha}$ is a matrix of 1-forms and is called connection one-forms.

We can now use the second equation of (2.96) to deduce the equivalent of (2.97) for $\boldsymbol{e}^{\alpha}$ called the first Cartan structure equation. Keep in mind that $\boldsymbol{E}_{\alpha}$ is a vector 0-form and $\boldsymbol{e}^{\alpha}$ is a scalar 1-form.

$$\begin{aligned} 0 = \mathrm{d}(\boldsymbol{E}_{\alpha} \boldsymbol{e}^{\alpha}) &= \mathrm{d}\boldsymbol{E}_{\alpha} \wedge \boldsymbol{e}^{\alpha} + \boldsymbol{E}_{\alpha} \mathrm{d}\boldsymbol{e}^{\alpha} \\ &= \boldsymbol{E}_{\beta} \boldsymbol{\omega}^{\beta}{}_{\alpha} \wedge \boldsymbol{e}^{\alpha} + \boldsymbol{E}_{\alpha} \mathrm{d}\boldsymbol{e}^{\alpha} \\ 0 &= \boldsymbol{E}_{\alpha} \left(\boldsymbol{\omega}^{\alpha}{}_{\beta} \wedge \boldsymbol{e}^{\beta} + \mathrm{d}\boldsymbol{e}^{\alpha} \right) \end{aligned} \tag{2.98}$$

Contracting this relation with $\boldsymbol{e}^{\gamma}$ and renaming indices, we get the first Cartan structure relation,

$$\mathrm{d}\boldsymbol{e}^{\alpha} = -\boldsymbol{\omega}^{\alpha}{}_{\beta} \wedge \boldsymbol{e}^{\beta}. \tag{2.99}$$

Since $\boldsymbol{E}_{\alpha}$ is a 0-form, $\mathrm{d}_{b} E^{a}_{\alpha} = \nabla_{b} E^{a}_{\alpha}$. Using (2.95) we get from (2.97) that

$$\omega^{\alpha}{}_{\beta b} = e^{\alpha}_{a} \nabla_{b} E^{a}_{\beta}. \tag{2.100}$$

If we substitute d for ∇_{b} in (2.98), the calculation is very similar, only that $\wedge$ is replaced by an ordinary tensor product. Eventually one arrives at

$$\nabla_{b} e^{\alpha}_{a} = -\omega^{\alpha}{}_{\beta b} e^{\beta}_{a}. \tag{2.101}$$

Using similar methods as before, and also $\nabla_a g_{bc} = 0$, we get

$$-\omega^{\alpha}{}_{\beta b} = E^a_\beta \nabla_b e^\alpha_a = e_{\beta a} \nabla_b E^{\alpha a} \overset{(2.100)}{=} \omega_\beta{}^\alpha{}_b. \tag{2.102}$$

Therefore

$$\boldsymbol{\omega}^\alpha{}_\beta = -\boldsymbol{\omega}_\beta{}^\alpha. \tag{2.103}$$

Since $\boldsymbol{E}_\alpha$ is a vector 0-form, d^2 acting on it gives the Riemann tensor.

$$\begin{aligned} \boldsymbol{R}^\beta{}_\alpha \boldsymbol{E}_\beta = \mathrm{d}^2 \boldsymbol{E}_\alpha \overset{(2.97)}{=} \mathrm{d}\left(\boldsymbol{E}_\beta \boldsymbol{\omega}^\beta{}_\alpha\right) = \boldsymbol{E}_\gamma \boldsymbol{\omega}^\gamma{}_\beta \wedge \boldsymbol{\omega}^\beta{}_\alpha + \boldsymbol{E}_\beta \mathrm{d}\boldsymbol{\omega}^\beta{}_\alpha \\ 0 = \left(\boldsymbol{\omega}^\beta{}_\gamma \wedge \boldsymbol{\omega}^\gamma{}_\alpha + \mathrm{d}\boldsymbol{\omega}^\beta{}_\alpha - \boldsymbol{R}^\beta{}_\alpha\right) \boldsymbol{E}_\beta \end{aligned} \tag{2.104}$$

Contracting this relation with $\boldsymbol{e}^\delta$ and renaming indices, we get the second Cartan structure equation,

$$\boldsymbol{R}^\alpha{}_\beta = \mathrm{d}\boldsymbol{\omega}^\alpha{}_\beta + \boldsymbol{\omega}^\alpha{}_\gamma \wedge \boldsymbol{\omega}^\gamma{}_\beta. \tag{2.105}$$

$\boldsymbol{R}^\alpha{}_\beta$ is a matrix of 2-forms and is called the Riemann 2-form. The trace of the Riemann 2-form gives us the Ricci 1-form, which corresponds to the Ricci tensor:

$$\boldsymbol{R}_\alpha = \boldsymbol{E}_\beta \lrcorner \boldsymbol{R}^\beta{}_\alpha. \tag{2.106}$$

2.4. 2-Spinors

In this section we will introduce the 2-spinor formalism following [6]. This will allow us to treat spinors and tensors on equal footing and provide the machinery to formulate the Dirac equation in a Kerr background.

Note that if spinors are mentioned without further context, it will always refer to 2-spinors and not four component Dirac spinors. However, as we will see later, Dirac spinors can be expressed as a pair of two-component spin-vectors.

2.4.1. Spin vectors and spin transformations from the light cone

Since one can construct a basis of Minkowski space entirely from null vectors, it seems natural to formulate an algebra based on the properties of the light cone.

Our goal is to conveniently parametrize null directions. For this aim we take Minkowski space $\mathbb{M}$ with coordinates (T, X, Y, Z) and origin $O = (0, 0, 0, 0)$ and look at null rays through O passing through the 3-plane $T = -1$, which we will parametrize by (x, y, z). These null rays trace out a unit sphere at $T = -1$ which is called the celestial sphere S^-. All light that reaches an observer at O in one time unit is encoded on that sphere. This creates a bijective map from the set of light rays through O and the celestial sphere.

A sphere can be parametrized via stereographical projection onto $\mathbb{R}^2$. If instead we use $\mathbb{C}$ as projection plane, the sphere is called Riemann sphere. We will perform this projection in a way such that the complex plane corresponds to $z = 0$. Through the north pole we draw rays which intersect both the sphere and the complex plane, and identify those two intersection points (Fig. 1). Thus the complex number ζ parametrizes the sphere without the north pole as

$$\zeta = \frac{x + \mathrm{i}y}{1 - z}. \tag{2.107}$$

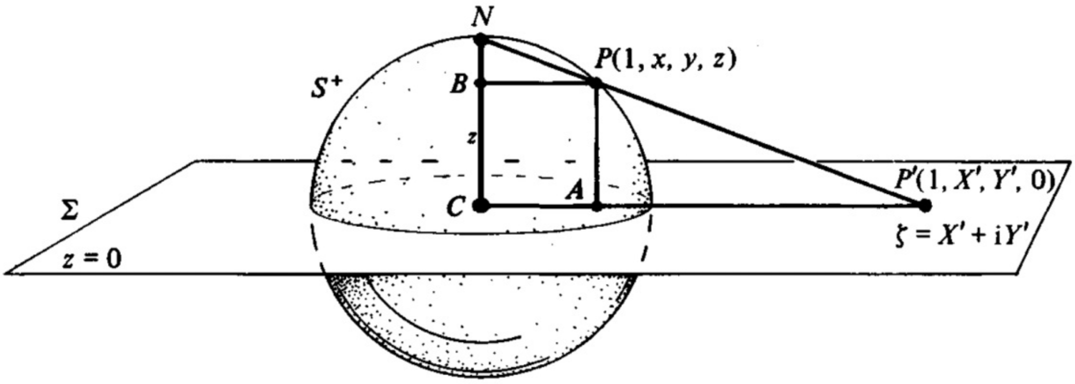


Figure 1: Stereographic projection of S^+ to the complex plane. The same can be done for S^-. This figure was taken from [6].

The inverse relations are

$$x = \frac{\zeta + \bar{\zeta}}{\zeta\bar{\zeta} + 1}, \quad y = \frac{\zeta - \bar{\zeta}}{\mathrm{i}(\zeta\bar{\zeta} + 1)} \quad z = \frac{\zeta\bar{\zeta} - 1}{\zeta\bar{\zeta} + 1}. \tag{2.108}$$

To include the north pole, which would be at a point $\zeta = \infty$, in our coordinates, we parametrize S^- as a complex projective line using homogeneous coordinates

$$(\xi, \eta) : \quad \zeta = \frac{\xi}{\eta}, \qquad (\lambda\xi, \lambda\eta) = (\xi, \eta) \quad \forall \lambda \in \mathbb{C}. \tag{2.109}$$

The north pole is then represented by $(1, 0)$. The vectors tracing out the sphere are then represented by

$$x = \frac{\xi\bar{\eta} + \eta\bar{\xi}}{\xi\bar{\xi} + \eta\bar{\eta}}, \quad y = \frac{\xi\bar{\eta} - \eta\bar{\xi}}{i(\xi\bar{\xi} + \eta\bar{\eta})}, \quad z = \frac{\xi\bar{\xi} - \eta\bar{\eta}}{\xi\bar{\xi} + \eta\bar{\eta}}. \tag{2.110}$$

Those are null vectors $(1, x, y, z)$ with $x^2 + y^2 + z^2 = 1$. We can scale them by the positive real number $\frac{1}{\sqrt{2}}(\xi\bar{\xi} + \eta\bar{\eta})$ to obtain any null vector in $\mathbb{M}$. (The factor $\frac{1}{\sqrt{2}}$ is chosen for later convenience.) The coordinates of these null vectors then are

$$T = \frac{1}{\sqrt{2}}(\xi\bar{\xi} + \eta\bar{\eta}), \quad X = \frac{1}{\sqrt{2}}(\xi\bar{\eta} + \eta\bar{\xi}), \quad Y = \frac{1}{\mathrm{i}\sqrt{2}}(\xi\bar{\eta} - \eta\bar{\xi}), \quad Z = \frac{1}{\sqrt{2}}(\xi\bar{\xi} - \eta\bar{\eta}). \tag{2.111}$$

While these vectors are no longer invariant under arbitrary rescaling $(\xi, \eta) \mapsto (\lambda\xi, \lambda\eta)$, there remains a phase invariance $(\xi, \eta) \mapsto (e^{\mathrm{i}\phi}\xi, e^{\mathrm{i}\phi}\eta)$. Since the light cone spans $\mathbb{M}$, the coordinates of any vector can be given by (2.111). Because (2.111) are real for any complex numbers ξ and η, a regular linear transformation

$$\xi \mapsto \alpha\xi + \beta\eta, \tag{2.112}$$
$$\eta \mapsto \gamma\xi + \delta\eta, \quad \{\alpha, \beta, \gamma, \delta\} \in \mathbb{C}, \tag{2.113}$$

induces a regular real linear transformation on $\mathbb{M}$. The regularity condition on (2.112) is

$$\alpha\delta - \beta\gamma \neq 0. \tag{2.114}$$

For ζ this results in a Möbius transformation

$$\zeta \mapsto \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}. \tag{2.115}$$

We will now impose the unimodularity condition which does not constrict the transformation of ζ:

$$\alpha\delta - \beta\gamma = 1. \tag{2.116}$$

Such transformations are called spin transformations and can be expressed as a spin matrix

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det(\mathbf{A}) = 1, \tag{2.117}$$

acting on $(\xi, \eta) \in \mathbb{C}^2$. Using succession as multiplication, the group of Spin transformations are isomorphic to the abstract group $SL(2, \mathbb{C})$.

From (2.111) we observe[1]

$$\frac{1}{\sqrt{2}}\mathbf{X} := \frac{1}{\sqrt{2}} \begin{pmatrix} T + Z & X + \mathrm{i}Y \\ X - \mathrm{i}Y & T - Z \end{pmatrix} = \begin{pmatrix} \xi\bar{\xi} & \xi\bar{\eta} \\ \eta\bar{\xi} & \eta\bar{\eta} \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \bar{\xi} & \bar{\eta} \end{pmatrix}. \tag{2.118}$$

Again, we can interpret $\mathbf{X}$ as coordinates for not only a null vector, but an arbitrary element of $\mathbb{M}$. A spin transformation therefore acts on a vector in $\mathbb{M}$ as

$$\mathbf{X} \mapsto \mathbf{A}\mathbf{X}\mathbf{A}^\dagger. \tag{2.119}$$

Hermitean conjugation results in the complex conjugation of the determinant, which is 1 in our case. That means that the determinant of $\mathbf{X}$ is invariant under spin transformations. But

$$\det(\mathbf{X}) = T^2 - X^2 - Y^2 - Z^2, \tag{2.120}$$

which means that a spin transformation on $\mathbb{C}^2$ induces a Lorentz transformation on $\mathbb{M}$. Suppose two spin transformations $\mathbf{A}$ and $\mathbf{B}$ induce the same Lorentz transformation. Then

$$\mathbf{A}\mathbf{X}\mathbf{A}^\dagger = \mathbf{B}\mathbf{X}\mathbf{B}^\dagger, \tag{2.121}$$

[1] In some literature on quantum field theory, other conventions are used. If we map $\zeta \mapsto \bar{\zeta}$ in the construction (2.107), we get $y \mapsto -y$ and $Y \mapsto -Y$, respectively. The matrix $\mathbf{X}$ will then, instead of $\mathbf{X} = \mathbb{1}T + \bar{\sigma} \cdot \boldsymbol{X}$, read $\mathbf{X} = \mathbb{1}T + \boldsymbol{\sigma} \cdot \boldsymbol{X}$, as is in agreement with [7].

and

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}\mathbf{X}\mathbf{B}^{\dagger}(\mathbf{A}^{\dagger})^{-1} = \mathbf{A}^{-1}\mathbf{B}\mathbf{X}(\mathbf{A}^{-1}\mathbf{B})^{\dagger}. \tag{2.122}$$

Therefore the spin transformation $\mathbf{A}^{-1}\mathbf{B}$ must induce the identity map on $\mathbb{M}$. Earlier we saw that the only transformations that leave arbitrary vectors invariant are phase transformations

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto e^{\mathrm{i}\phi} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \tag{2.123}$$

restricting the transformation to

$$\mathbf{A}^{-1}\mathbf{B} = e^{\mathrm{i}\phi}\mathbb{1} \overset{\det=1}{\Rightarrow} \mathbf{A}^{-1}\mathbf{B} = \pm\mathbb{1} \Rightarrow \mathbf{A} = \pm\mathbf{B}. \tag{2.124}$$

Thus one Lorentz transformation is induced by two spin transformations with mutually opposite sign.

As proven in [6]:

Corollary 1. *Every spin transformation induces a unique proper orthochronous Lorentz transformation. Conversely every proper orthochronous Lorentz transformation is induced by exactly two spin transformations, one being the negative of the other.*

This 2 to 1 epimorphism shows, that $SL(2,\mathbb{C})$ is the double cover of $\Lambda^{+\uparrow}$. In fact, $SL(2,\mathbb{C})$ is simply connected and therefore the universal cover of $\Lambda^{+\uparrow}$.

Those elements on $\mathbb{C}^2$ the components of which are ξ and η, on which the spin transformations act, are called spin vectors.

2.4.2. Spinors in space-time

As we have seen in the last section, we can describe null vectors as a product of a spin-vector and its complex conjugate (2.118). We will now use this to define a spinor calculus similar to how it can be done for tensors, which will incorporate the usual tensor calculus. For that we introduce a vector bundle with fibre $\mathbb{C}^2$ over space-time, the elements of the fibre will be called spin-vectors. The abstract space of spin vectors will be called $\mathfrak{S}^A$, its complex conjugate space $\mathfrak{S}^{A'}$, and their dual spaces with lowered indices. We define a general spinor as a multilinear map $\mathfrak{S}^{A_1} \times \ldots \times \mathfrak{S}_{B_1} \times \ldots \times \mathfrak{S}^{C_1'} \times \ldots \times \mathfrak{S}_{D_1'} \times \ldots \to \mathbb{C}$. The operation of complex conjugation is defined as an involutory isomorphism $\mathfrak{S}^{\mathscr{A}} \to \mathfrak{S}^{\mathscr{A}'}$ where $\mathscr{A} \in \{A, A'\}$. Elements of the tangent space are repesented by elements of $\mathfrak{S}^{AA'}$. For convenience, as long as abstract indices are used, pairs of complex conjugated spinor indices will be used interchangeably with tensor indices $\mathfrak{S}^{AA'} \leftrightarrow \mathfrak{S}^a$.

It would be advantageous to find an inner product $\mathfrak{S}^A \times \mathfrak{S}^B \to \mathbb{C}$ which is invariant under spin transformations, just like the Minkowski metric is invariant under Lorentz transformations. Consider two spin vectors κ^A and ω^A. The inner product

$$\langle \boldsymbol{\kappa}, \boldsymbol{\omega} \rangle := \kappa^0\omega^1 - \kappa^1\omega^0 \tag{2.125}$$

fulfills the required condition. To show this, consider a spin transformation

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1. \tag{2.126}$$

Then

$$\tilde{\kappa}^0\tilde{\omega}^1 - \tilde{\kappa}^1\tilde{\omega}^0 = \begin{vmatrix} \tilde{\kappa}^0 & \tilde{\omega}^0 \\ \tilde{\kappa}^1 & \tilde{\omega}^1 \end{vmatrix} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \begin{vmatrix} \kappa^0 & \omega^0 \\ \kappa^1 & \omega^1 \end{vmatrix} = (\alpha\delta - \beta\gamma)(\kappa^0\omega^1 - \kappa^1\omega^0) = \kappa^0\omega^1 - \kappa^1\omega^0. \tag{2.127}$$

This product is realized by the antisymmetric ϵ-spinor

$$\varepsilon_{AB} = -\varepsilon_{BA} \tag{2.128}$$

by defining

$$\langle \boldsymbol{\kappa}, \boldsymbol{\omega} \rangle =: \varepsilon_{AB}\kappa^A\omega^B. \tag{2.129}$$

With similar considerations we find the analogues for the complex conjugate and dual spaces

$$\varepsilon_{A'B'} = -\varepsilon_{B'A'}, \quad \varepsilon^{AB} = -\varepsilon^{BA}, \quad \varepsilon^{A'B'} = -\varepsilon^{B'A'}. \tag{2.130}$$

The antisymmetry portrays the importance of the order of indices within primed or unprimed indices. The permutation between these sets has no effect. The ϵ-spinors are defined such that

$$\varepsilon_{AB}\varepsilon^{AB} = \varepsilon_{A'B'}\varepsilon^{A'B'} = 2. \tag{2.131}$$

We can use the ϵ-spinor to map between the spinor spaces and their duals

$$\varepsilon_{AB}\kappa^B = -\kappa_A, \quad \varepsilon_{A'B'}\kappa^{B'} = -\kappa_{A'}, \quad \varepsilon^{AB}\kappa_B = \kappa^A, \quad \varepsilon^{A'B'}\kappa_{B'} = \kappa^{A'}. \tag{2.132}$$

This leads to

$$\kappa_A\omega^A = \kappa_0\omega^0 + \kappa_1\omega^1 \overset{(2.125)}{=} \kappa^0\omega^1 - \kappa^1\omega^0, \tag{2.133}$$

$$\kappa_0 = -\kappa^1, \quad \kappa_1 = \kappa^0. \tag{2.134}$$

Summarizing we find

$$\begin{aligned} \varepsilon_{AB}\kappa^A\omega^B &= \kappa_A\omega^B = -\kappa^A\omega_B = \varepsilon^{AB}\kappa_A\omega_B \\ \varepsilon_{A'B'}\kappa^{A'}\omega^{B'} &= \kappa_{A'}\omega^{B'} = -\kappa^{A'}\omega_{B'} = \varepsilon^{A'B'}\kappa_{A'}\omega_{B'} \end{aligned} \tag{2.135}$$

Collecting the statements above, $\varepsilon_{AB}\varepsilon_{A'B'}$ translates to a tensor of the tangent space. Since ε_{AB} and $\varepsilon_{A'B'}$ are invariant under spin transformations, which translate to Lorentz transformations in tangent space, the tensor they represent is invariant under Lorentz transformations. Thus

$$\varepsilon_{AB}\varepsilon_{A'B'} = \eta_{ab}. \tag{2.136}$$

2.4.3. Symmetric and antisymmetric spinors

A spinor is called symmetric [antisymmetric] if it is symmetric [antisymmetric] in all of it's indices. Because of the antisymmety of the ε-spinor, the trace of a pair of symmetric indices vanishes. Since spin space is only 2-dimensional, a pair of antisymmetric indices has only one independent component. This component is the trace over the respective indices.

$$\phi_{\mathscr{D}[AB]} = \frac{1}{2}\phi_{\mathscr{D}C}{}^{C}\varepsilon_{AB} \tag{2.137}$$

This can be shown by assuming following identity for arbitrary spinors κ, ω, and τ, which can be easily checked by switching to components.

$$\kappa_A\omega^A\tau^B + \omega_A\tau^A\kappa^B + \tau_A\kappa^A\omega^B = 0 \tag{2.138}$$

This can be turned into an identity for ε-spinors which when contracted with two components of a spinor leads to the desired result.

$$\kappa^A\omega^B\tau^X\left(\varepsilon_{AB}\varepsilon_X{}^D + \varepsilon_{BX}\varepsilon_A{}^D + \varepsilon_{XA}\varepsilon_B{}^D\right) = 0 \tag{2.139}$$

$$\varepsilon_{AB}\varepsilon_X{}^D + \varepsilon_{BX}\varepsilon_A{}^D + \varepsilon_{XA}\varepsilon_B{}^D = 0 \qquad \Big| \cdot \varepsilon^{CX} \tag{2.140}$$

$$\varepsilon_A{}^C\varepsilon_B{}^D - \varepsilon_B{}^C\varepsilon_A{}^D = \varepsilon_{AB}\varepsilon^{CD} \qquad \Big| \cdot \Phi_{\mathscr{D}CD} \tag{2.141}$$

$$\Phi_{\mathscr{D}AB} - \Phi_{\mathscr{D}BA} = \Phi_{\mathscr{D}C}{}^{C}\varepsilon_{AB} \tag{2.142}$$

Identity (2.137) already suggests that the antisymmetric parts of spinors do not hold more information than a scalar which is trivially a symmetric spinor. Indeed every spinor can be decomposed into outer products between symmetric spinors and ε-spinors. This is shown in appendix B.1.1.

2.4.4. The spin dyad

As hinted at the beginning of this chapter, we can introduce an analogue (and also an extension) to the tetrad in tangent space for spin space. This basis is called spin dyad or spin frame. This dyad is defined by the basis spin vectors o^A and ι^A, as well as their orthonormality condition

$$o_A\iota^A = 1, \quad o_{A'}\iota^{A'} = 1. \tag{2.143}$$

Alongside this dyad there emerges naturally a complex null tetrad

$$l^a := o^A o^{A'}, \quad n^a := \iota^A\iota^{A'}, \quad m^a := o^A\iota^{A'}, \quad \bar{m}^a = \iota^A o^{A'}, \tag{2.144}$$

where

$$l^a n_a = 1, \quad m^a\bar{m}_a = -1. \tag{2.145}$$

As can be easily checked, it is possible to obtain the standard orthonormal tetrad by

$$\begin{aligned} t^a &= \frac{1}{\sqrt{2}}\left(l^a + n^a\right), \quad & x^a &= \frac{1}{\sqrt{2}}\left(m^a + \bar{m}^a\right), \\ y^a &= \frac{\mathrm{i}}{\sqrt{2}}\left(m^a - \bar{m}^a\right), \quad & z^a &= \frac{1}{\sqrt{2}}\left(l^a - n^a\right). \end{aligned} \tag{2.146}$$

In terms of components we can write a spin vector as

$$\kappa^A = \kappa^0 o^A + \kappa^1 \iota^A. \tag{2.147}$$

That gives us

$$\kappa^0 = -\iota_A \kappa^A, \quad \kappa^1 = o_A \kappa^A. \tag{2.148}$$

From (2.134) we get

$$\kappa_0 = -\kappa^1 = o^A \kappa_A, \quad \kappa_1 = \kappa^0 = \iota^A \kappa_A. \tag{2.149}$$

We can write the ε-spinor in terms of the spin frame

$$\varepsilon_{AB} = o_A \iota_B - \iota_A o_B, \tag{2.150}$$

which gives us

$$\varepsilon_{0B} = o_B, \quad \varepsilon_{1B} = \iota_B, \quad \varepsilon^{0B} = \iota^B, \quad \varepsilon^{1B} = -o^B. \tag{2.151}$$

Spinor components will be symbolized by bold face capital letters. The spin dyad is therefore $\varepsilon_{\mathbf{A}}{}^{B}$ or $\varepsilon_{B}{}^{\mathbf{A}}$, respectively.

2.4.5. Covariant derivative

The isomorphism between the tangent vectors and real $\left[\begin{smallmatrix}1 & 1\\ 0 & 0\end{smallmatrix}\right]$ spinors associates to each covariant derivative ∇_a a spinor valued derivative $\nabla_{A'A}$. This derivative is only defined for real directions and acting on real scalars yet. The first step will therefore be to extend its definition to complex directions and scalars.
Consider U^a and V^a to be real vector fields and f and g to be real scalar fields. Then

$$U^a \nabla_a (f + \mathrm{i}g) := U^a \nabla_a (f) + \mathrm{i} U^a \nabla_a (g) \tag{2.152}$$
$$(U^a + \mathrm{i}V^a) \nabla_a (f) := U^a \nabla_a (f) + \mathrm{i} V^a \nabla_a (f) \tag{2.153}$$

Now we can define the covariant derivative for arbitrary spinors.

Definition 1. *A spinor covariant derivative is a map*

$$\nabla_{AA'} : \mathfrak{S}^B \to \mathfrak{S}^B_{AA'} \tag{2.154}$$

satisfying, for each $\xi^B,\ \eta^B \in \mathfrak{S}^B,\ f \in \mathfrak{S}$,

$$\nabla_{AA'}(\xi^B + \eta^B) = \nabla_{AA'}\xi^B + \nabla_{AA'}\eta^B \tag{2.155}$$
$$\nabla_{AA'}(f\xi^B) = f\nabla_{AA'}\xi^B + \xi^B \nabla_{AA'} f. \tag{2.156}$$

This definition can be extended for $\nabla_{AA'}$ also acting on $\mathfrak{S}_B$ by requiring the Leibniz rule.

$$(\nabla_{AA'}\alpha_B)\xi^B := \nabla_{AA'}(\alpha_B \xi^B) - \alpha_B \nabla_{AA'}\xi^B \tag{2.157}$$

The extension to $\nabla_{AA'}$ acting on $\mathfrak{S}^{B'}$ and $\mathfrak{S}_{B'}$ follows from complex conjugation.

$$\nabla_{AA'}\zeta^{B'} := \overline{\nabla_{AA'}\bar{\zeta}^B}, \qquad \nabla_{AA'}\omega_{B'} := \overline{\nabla_{AA'}\bar{\omega}_B} \tag{2.158}$$

We can now extend $\nabla_{AA'}$ to act on a general spinor, again requiring that the Leibniz rule holds.

$$\begin{aligned}(\nabla_{AA'}\chi^{B\dots C'}{}_{D\dots E'})\beta_B\dots\gamma_{C'}\delta^D\dots\eta^{E'} &:= \nabla_{AA'}(\chi^{B\dots C'}{}_{D\dots E'}\beta_B\dots\eta^{E'}) \\ -\chi^{B\dots C'}{}_{D\dots E'}(\nabla_{AA'}\beta_B)\dots\eta^{E'} &- \dots - \chi^{B\dots C'}{}_{D\dots E'}\beta_B\dots(\nabla_{AA'}\eta^{E'})\end{aligned} \tag{2.159}$$

The derivative constructed like that is unique if we require vanishing torsion and that it annihilates ε_{AB}.

2.4.6. Spin coefficients

As we defined the connection one-form as the bundle connection of tangent space

$$\nabla_b E^a_{\mathbf{a}} = E^a_{\mathbf{b}}\omega^{\mathbf{b}}{}_{\mathbf{a}b}, \tag{2.160}$$

we will define the spin coefficients as a connection on the spin bundle

$$\begin{aligned}\nabla_{\mathbf{AA'}}\varepsilon_{\mathbf{B}}{}^A &=: \Gamma_{\mathbf{AA'B}}{}^{\mathbf{C}}\varepsilon_{\mathbf{C}}{}^A, \\ \nabla_{\mathbf{AA'}}\varepsilon_A{}^{\mathbf{B}} &= -\Gamma_{\mathbf{AA'C}}{}^{\mathbf{B}}\varepsilon_A{}^{\mathbf{C}}.\end{aligned} \tag{2.161}$$

Because ε_{AB} is constant we get

$$0 = \nabla_{\mathbf{AA'}}\varepsilon_{BC} = \nabla_{\mathbf{AA'}}\left(\varepsilon_{\mathbf{B}}{}^A\varepsilon_{A\mathbf{C}}\right) = \varepsilon_{A\mathbf{C}}\nabla_{\mathbf{AA'}}\varepsilon_{\mathbf{B}}{}^A - \varepsilon_{\mathbf{B}A}\nabla_{\mathbf{AA'}}\varepsilon^A{}_{\mathbf{C}} = \Gamma_{\mathbf{AA'BC}} - \Gamma_{\mathbf{AA'CB}}, \tag{2.162}$$

the symmetry of the spin coefficients

$$\Gamma_{\mathbf{AA'BC}} = \Gamma_{\mathbf{AA'CB}}. \tag{2.163}$$

We can now establish the relationship between the connection one-forms and the spin coefficients. Note that in the conventions we use, where we derive the tetrad directly from the spin basis, the Infeld-van der Waerden symbols $g_{\mathbf{a}}{}^{\mathbf{AA'}}$ which describe the isomorphism between spinors and tensors are constant.

$$\begin{aligned}\omega^{\mathbf{a}}{}_{\mathbf{b}i} &= e^{\mathbf{a}}_a\nabla_i E^a_{\mathbf{b}} \\ &= e^{\mathbf{a}}_a\nabla_i\left(g_{\mathbf{b}}{}^{\mathbf{AA'}}\varepsilon_{\mathbf{A}}{}^A\varepsilon_{\mathbf{A'}}{}^{A'}\right) \\ &= e^{\mathbf{a}}_a g_{\mathbf{b}}{}^{\mathbf{AA'}}\left(\varepsilon_{\mathbf{A'}}{}^{A'}\nabla_i\varepsilon_{\mathbf{A}}{}^A + \varepsilon_{\mathbf{A}}{}^A\nabla_i\varepsilon_{\mathbf{A'}}{}^{A'}\right) \\ &= g_{\mathbf{BB'}}{}^{\mathbf{a}}\varepsilon_A{}^{\mathbf{B}}\varepsilon_{A'}{}^{\mathbf{B'}}\left(\varepsilon_{\mathbf{A'}}{}^{A'}\Gamma_{i\mathbf{A}}{}^{\mathbf{C}}\varepsilon_{\mathbf{C}}{}^A + \varepsilon_{\mathbf{A}}{}^A\bar{\Gamma}_{i\mathbf{A'}}{}^{\mathbf{C'}}\varepsilon_{\mathbf{C'}}{}^{A'}\right)g_{\mathbf{b}}{}^{\mathbf{AA'}} \\ &= g_{\mathbf{BB'}}{}^{\mathbf{a}}g_{\mathbf{b}}{}^{\mathbf{AA'}}\left(\varepsilon_{\mathbf{A'}}{}^{\mathbf{B'}}\Gamma_{i\mathbf{A}}{}^{\mathbf{B}} + \varepsilon_{\mathbf{A}}{}^{\mathbf{B}}\Gamma_{i\mathbf{A'}}{}^{\mathbf{B'}}\right)\end{aligned} \tag{2.164}$$

Thus

$$\omega^{\mathbf{CC'}}{}_{\mathbf{DD'}i} = \varepsilon_{\mathbf{D'}}{}^{\mathbf{C'}}\Gamma_{i\mathbf{D}}{}^{\mathbf{C}} + \varepsilon_{\mathbf{D}}{}^{\mathbf{C}}\bar{\Gamma}_{i\mathbf{D'}}{}^{\mathbf{C'}}. \tag{2.165}$$

We get the reverse relations by contracting $\mathbf{D'}$ with $\mathbf{C'}$ or $\mathbf{D}$ with $\mathbf{C}$, respectively.

$$\begin{aligned}\Gamma_{i\mathbf{D}}{}^{\mathbf{C}} &= \frac{1}{2}\omega^{\mathbf{CC'}}{}_{\mathbf{DC'}i} \\ \bar{\Gamma}_{i\mathbf{D'}}{}^{\mathbf{C'}} &= \frac{1}{2}\omega^{\mathbf{CC'}}{}_{\mathbf{CD'}i}\end{aligned} \tag{2.166}$$

2.5. Null surfaces

The orthogonal surface (if it exists) to a null vector field u^a is called a null hypersurface. A null surface is a submanifold of a null hypersurface.

Proposition 1. *The vanishing of the scalar product between two null vectors u^a and v^a is equivalent to them being proportional to each other.*

$$u_a v^a = 0 \quad \Leftrightarrow \quad v^a = \alpha u^a \tag{2.167}$$

Proof. Consider two nonzero null vectors u^a and v^a that point in distinct directions. Then their sum $(u^a + v^a)$ will no longer be null.

$$0 \neq (u_a + v_a)(u^a + v^a) = u_a u^a + 2u_a v^a + v_a v^a = 2u_a v^a \tag{2.168}$$

The negation of that proves the "$\Rightarrow$" direction. The inverse is clear by the definition of a null vector. □

This proposition provides that u^a is the only null vector (up to proportionality) tangent to the null hypersurface.

Proposition 2. *The metric for a null surface is degenerate, i.e. its determinand vanishes.*

Proof. For a non-singular coordinate transformation the vanishing of the metric determinand is coordinate independent. That means if we can show it in one coordinate frame it is true for all coordinate frames. Consider a point P on the null surface. We can choose a coordinate frame where the metric at P has the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{2.169}$$

or

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.170}$$

where the columns and rows are in the order (v, u, x^1, x^2).

If M is a manifold with a metric and S is a submanifold of M, then M induces a metric on S in the sense that vectors in $T(S)$ correspond to vectors in $T(M)$ by embedding S in M. Since u is tangent to the null hypersurface and v is not, the induced metric is

$$g^{(S)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.171}$$

This metric is clearly degenerate and every null surface must be tangent to u so its metric will also be degenerate. Since we did not specify the point P, this is true for the whole null surface. □

2.6. The Einstein equation

For the following exposure we rely on [9].

The Einstein equation relates space-time geometry to the energy-momentum tensor of the present matter.

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \tag{2.172}$$

$R_{ab} = R^{x}{}_{axb}$ is the Ricci tensor which is the trace of the Riemmann tensor. $R = R^{a}{}_{a}$ is the curvature scalar which is the trace of the Ricci tensor. Together they form the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$, which is the trace reversal of the Ricci tensor. Λ is the cosmological constant, and T_{ab} is the energy-momentum tensor of matter.

The vacuum Einstein equation refers to (2.172), but with $T_{ab} = 0$:

$$R_{ab} = \Lambda g_{ab}. \tag{2.173}$$

2.7. Some solutions of the Einstein equation

During the history of general relativity, several analytic solutions to the Einstein equation were found. We will take a short look at some selected solutions which describe the space-times produced by spherical objects. The Schwarzschild and Kerr solution will be treated more fully in the next sections.

All solutions discussed below contain black holes, which are regions in space-time from which no future timelike curve can reach infinity. Black holes are bounded by a null surface called the event horizon.

2.7.1. The Schwarzschild solution

The Schwarzschild solution was the first nontrivial analytic solution of the Einstein equation to be found. It assumes the following symmetries:

- **vacuum:** Space-time is void of matter, $T_{ab} = 0$, and a cosmological constant is excluded ($\Lambda = 0$).
- **static:** There exists a timelike Killing vector field (stationary), the orthogonal spaces of which are the tangent spaces of a spacelike submanifold (hypersurface orthogonal).
- **spherically symmetric:** The isometry group has a subgroup the orbits of which are geometrical spheres S^2.

That is what we expect for a space-time outside a spherical uncharged massive object at rest. The line element for the Schwarzschid solution is

$$\mathrm{d}s^2 = f dt^2 - \frac{1}{f}\mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2, \quad f = 1 - \frac{2M}{r}, \tag{2.174}$$

where t is the Killing time, r is the area radius of the spacelike S^2 submanifolds, $d\Omega^2$ is the line element for a unit S^2, and $M := Gm$ is the Mass parameter, where m is the mass of the object.

This metric describes a black hole with its event horizon at $r = 2M$. The singularity at the horizon is only due to the chosen coordinates. In the later discussion we will circumvent it by transforming into Kruskal coordinates. The singularity at $r = 0$ is a real curvature singularity. Notice that, passing the horizon, ∂_t and ∂_r switch their roles as timelike and spacelike vectors, respectively.

If we take the Newtonian limit and identify the gravitational potential with $\frac{1}{2}g_{tt}$, we see that it has the same behaviour as Newton's gravitational potential $\Phi = -\frac{M}{r}$.

2.7.2. The Reissner-Nordström solution

For this we still consider a spherically symmetric massive object at rest, but with charge $\sqrt{\frac{4\pi\epsilon_0}{G}}Q$ in SI-units. Therefore we assume the following symmetries:

- **electrovacuum:** The only physical fields present are the electromagnetic field and the gravitational field. We also exclude a cosmological constant ($\Lambda = 0$).
- **static:** There exists a timelike Killing vector field (stationary), the orthogonal spaces of which are the tangent spaces of a spacelike submanifold (hypersurface orthogonal).
- **spherically symmetric:** The isometry group has a subgroup the orbits of which are geometrical spheres S^2.

The resulting line element is

$$ds^2 = f dt^2 - \frac{1}{f}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \quad f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{2.175}$$

This again describes a black hole, but now with two horizons $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The first horizon r_+ coincides with the Schwarzschild horizon for $Q = 0$, whereas the second horizon r_- coincides with the singularity at the horizon for $Q = 0$. For the other extreme $Q = M$, the two horizons coincide at $r = M$. For $Q > M$ there is no horizon and the singularity is "naked", i.e. not inside a black hole.

2.7.3. The Kerr solution

For this we consider a massive uncharged rotating object in vacuum. Due to the rotation we expect space-time to no longer be spherically symmetric, but axially symmetric. We assume following symmetries:

- **vacuum:** Space-time is void of matter, $T_{ab} = 0$. We also exclude a cosmological constant ($\Lambda = 0$).
- **stationary:** There exists a timelike Killing vector field.
- **axially symmetric:** The isometry group has a subgroup the orbits of which are geometrical circles S^1.

The resulting line element is

$$\mathrm{d}s^2 = \frac{\rho^2\Delta}{\Sigma^2}\mathrm{d}t^2 - \frac{\Sigma^2\sin^2\theta}{\rho^2}\left(\mathrm{d}\phi - \frac{2aMr}{\Sigma^2}\mathrm{d}t\right)^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2, \tag{2.176}$$

where

$$\Delta = r^2 - 2Mr + a^2, \tag{2.177}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{2.178}$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \tag{2.179}$$

and a and M are constant parameters. The curvature singularity here is a ring in the equatorial plane. Outside of the black hole there is a region called the ergosphere, where the g_{tt} changes its sign. This means that there is no stationary observer in this region. Rotational energy can be extracted from the black hole via the Penrose process by dropping matter into the ergosphere and splitting it in a way such that one part escapes, and the infalling part has negative energy.

2.8. The Schwarzschild Solution

Other than the conditions listed in section 2.7.1, as an additional requirement we assume that the direction of the static Killing vector is unique.

We can use the second and third requirement to restrain the metric to a special form that makes it easier to calculate. The general form of the line element is

$$\mathrm{d}s^2 = g_{ab}(x^\mu)\mathrm{d}x^a\mathrm{d}x^b. \tag{2.180}$$

For our space-time to be stationary means, that we can choose coordinates in such a way that g_{ab} does not depend on the coordinate in Killing time direction (t). Also hypersurface orthogonality enables us to choose our other three basis vectors in the tangent space of the spacelike orthogonal surface. Therefore our timelike Killing vector is always orthogonal to the rest of the basis and we get

$$\mathrm{d}s^2 = f(x^\rho)^2\mathrm{d}t^2 + h_{ij}(x^\rho)\mathrm{d}x^i\mathrm{d}x^j. \tag{2.181}$$

We chose f^2, because we expect the time to stay at a positive length. Note that we can always rescale a Killing vector by a constant and it still fulfills the Killing equation

$$\nabla_{(a}\xi_{b)} = 0. \tag{2.182}$$

Therefore f can without hesitation be replaced by cf, where c is constant.

Because we assume that the static Killing vector is unique modulo rescaling, there can be no isometry that changes its direction. This means that the orbits of the spherically symmetric isometry lie entirely insinde the spacelike surfaces Σ orthogonal to the static Killing vector. The metric of a geometrical sphere is

$$r^2\mathrm{d}\Omega^2, \quad \text{where} \quad \mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2, \tag{2.183}$$

in the usual angular coordinates θ and ϕ. r^2 is called the area radius and measures the area A of the sphere as

$$r := \sqrt{\frac{A}{4\pi}}. \tag{2.184}$$

We then choose r as a coordinate to select those spheres by carrying them along $\nabla_a r$. This provides $\nabla_a r$ to be orthogonal to the spheres. We see here, that this coordinate representation breaks down at points where $\nabla_a r = 0$. For the spatial metric we then get

$$h_{ij}(r,\theta,\phi)\mathrm{d}x^i\mathrm{d}x^j = -g(r)^2\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{2.185}$$

f and g do not depend on θ and ϕ because of the spherical symmetry.

Putting everything together we get the metric for static spherically symmetric spacetimes:

$$\mathrm{d}s^2 = f(r)^2\mathrm{d}t^2 - g(r)^2\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{2.186}$$

2.8.1. The derivation of the Schwarzschild solution using the tetrad formalism

We will now use the tetrad formalism from section 2.3 and the Einstein equation (2.172) to calculate f and g. First we need to use (2.96) to decompose the metric into tetrad components

$$\eta_{\alpha\beta}e^\alpha_a e^\beta_b = g_{ab}. \tag{2.187}$$

That provides the basis covectors $\boldsymbol{e}^\alpha$. From the orthogonality condition (2.95) we get the corresponding basis vectors $\boldsymbol{E}_\alpha$.

Before we start treating the Schwarzschild metric it is useful for the later calculation to first consider the metric of a geometrical unit-S^2. For this we have the metric

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2. \tag{2.188}$$

We choose our Zweibein $\tilde{\boldsymbol{E}}_\iota$ to be

$$\tilde{\boldsymbol{e}}^\theta = \mathrm{d}\theta, \qquad \tilde{\boldsymbol{E}}_\theta = \partial_\theta, \tag{2.189}$$

$$\tilde{\boldsymbol{e}}^\phi = \sin\theta\mathrm{d}\phi, \qquad \tilde{\boldsymbol{E}}_\phi = \frac{1}{\sin\theta}\partial_\phi. \tag{2.190}$$

By use of the first Cartan structure equation (2.99)

$$\mathrm{d}\tilde{\boldsymbol{e}}^\alpha = -\tilde{\boldsymbol{\omega}}^\alpha{}_\beta \wedge \tilde{\boldsymbol{e}}^\beta, \tag{2.191}$$

we can calculate the connection one-form by comparing it to the outer derivative of (2.190).

$$\mathrm{d}\tilde{\boldsymbol{e}}^\phi = -\tilde{\boldsymbol{\omega}}^\phi{}_\theta \wedge \tilde{\boldsymbol{e}}^\theta = -\tilde{\boldsymbol{\omega}}^\phi{}_\theta \wedge \mathrm{d}\theta \tag{2.192}$$

$$\mathrm{d}\tilde{\boldsymbol{e}}^\phi = \mathrm{d}\sin\theta \wedge \mathrm{d}\phi + \sin\theta\,\underbrace{\mathrm{d}^2\phi}_{=0} = \cos\theta\mathrm{d}\theta \wedge \mathrm{d}\phi = -\cos\theta\mathrm{d}\phi \wedge \mathrm{d}\theta \tag{2.193}$$

Comparing these two, we can read off

$$\tilde{\boldsymbol{\omega}}^{\phi}{}_{\theta} \overset{(2.103)}{=} -\tilde{\boldsymbol{\omega}}^{\theta}{}_{\phi} = \cos\theta \mathrm{d}\phi + a\mathrm{d}\theta. \tag{2.194}$$

The a term must be added because it would not contribute to equation (2.192) since $\mathrm{d}\theta \wedge \mathrm{d}\theta = 0$. To calculate a, we can use the same method on (2.189):

$$\mathrm{d}\tilde{\boldsymbol{e}}^{\theta} = -\tilde{\boldsymbol{\omega}}^{\theta}{}_{\phi} \wedge \tilde{\boldsymbol{e}}^{\phi} = \cos\theta \sin\theta \mathrm{d}\phi \wedge \mathrm{d}\phi + a \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi = a \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi, \tag{2.195}$$

$$\mathrm{d}\tilde{\boldsymbol{e}}^{\theta} = \mathrm{d}^2\theta = 0. \tag{2.196}$$

Thus we see that $a = 0$ and

$$\tilde{\boldsymbol{\omega}}^{\phi}{}_{\theta} \overset{(2.103)}{=} -\tilde{\boldsymbol{\omega}}^{\theta}{}_{\phi} = \cos\theta \mathrm{d}\phi. \tag{2.197}$$

To calculate the Riemann 2-form, we can now use the second Cartan structure equation (2.105)

$$\tilde{\boldsymbol{R}}^{\alpha}{}_{\beta} = \mathrm{d}\tilde{\boldsymbol{\omega}}^{\alpha}{}_{\beta} + \tilde{\boldsymbol{\omega}}^{\alpha}{}_{\gamma} \wedge \tilde{\boldsymbol{\omega}}^{\gamma}{}_{\beta}. \tag{2.198}$$

Since $\mathrm{d}\phi \wedge \mathrm{d}\phi = 0$, the second term on the right hand side does not contribute and we have

$$\tilde{\boldsymbol{R}}^{\phi}{}_{\theta} = \mathrm{d}\tilde{\boldsymbol{\omega}}^{\phi}{}_{\theta} = -\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi = \sin\theta \mathrm{d}\phi \wedge \mathrm{d}\theta = \tilde{\boldsymbol{e}}^{\phi} \wedge \tilde{\boldsymbol{e}}^{\theta}. \tag{2.199}$$

Because of the antisymmetry of the Riemann tensor in the first two indices and the wedge product we get

$$\tilde{\boldsymbol{R}}^{\iota}{}_{\kappa} = \tilde{\boldsymbol{e}}^{\iota} \wedge \tilde{\boldsymbol{e}}^{\kappa}. \tag{2.200}$$

We can now return to our static spherically symmetric metric (2.186), which already has a convenient form for choosing a tetrad basis.

$$\boldsymbol{e}^{t} = f\mathrm{d}t \qquad \boldsymbol{E}_{t} = \frac{1}{f}\partial_{t} \tag{2.201}$$

$$\boldsymbol{e}^{r} = g\mathrm{d}r \qquad \boldsymbol{E}_{r} = \frac{1}{g}\partial_{r} \tag{2.202}$$

$$\boldsymbol{e}^{\iota} = r\tilde{\boldsymbol{e}}^{\iota} \qquad \boldsymbol{E}_{\iota} = \frac{1}{r}\tilde{\boldsymbol{E}}_{\iota} \tag{2.203}$$

$\boldsymbol{E}_{\iota}$ are the directions tangential to the geometric S^2s. As before we can use (2.99) to calculate the connection one-forms. Since f and g both only depend on r, we will use $'$ to denote a derivative with respect to r.

$$\mathrm{d}\boldsymbol{e}^{t} = f'\mathrm{d}r \wedge \mathrm{d}t \overset{(2.99)}{=} -\boldsymbol{\omega}^{t}{}_{r} \wedge \boldsymbol{e}^{r} - \boldsymbol{\omega}^{t}{}_{\iota} \wedge \boldsymbol{e}^{\iota} \tag{2.204}$$

$$f'\mathrm{d}t \wedge \mathrm{d}r = g\boldsymbol{\omega}^{t}{}_{r} \wedge \mathrm{d}r + \boldsymbol{\omega}^{t}{}_{\iota} \wedge \boldsymbol{e}^{\iota} \tag{2.205}$$

This allows us to extract

$$\boldsymbol{\omega}^{t}{}_{r} = \frac{f'}{g}\mathrm{d}t + a\mathrm{d}r, \quad \boldsymbol{\omega}^{t}{}_{\iota} = b\mathrm{d}\boldsymbol{e}^{\iota}. \tag{2.206}$$

$$\mathrm{d}\boldsymbol{e}^{\iota} = \mathrm{d}r \wedge \tilde{\boldsymbol{e}}^{\iota} + r\mathrm{d}\tilde{\boldsymbol{e}}^{\iota} \tag{2.207}$$
$$= -\frac{1}{g}\tilde{\boldsymbol{e}}^{\iota} \wedge \boldsymbol{e}^{r} - \tilde{\boldsymbol{\omega}}^{\iota}{}_{\kappa} \wedge \boldsymbol{e}^{\kappa} \tag{2.208}$$
$$\mathrm{d}\boldsymbol{e}^{\iota} \overset{(2.99)}{=} -\boldsymbol{\omega}^{\iota}{}_{t} \wedge \boldsymbol{e}^{t} - \boldsymbol{\omega}^{\iota}{}_{r} \wedge \boldsymbol{e}^{r} - \boldsymbol{\omega}^{\iota}{}_{\kappa} \wedge \boldsymbol{e}^{\kappa} \tag{2.209}$$

We can therefore read off

$$b = 0, \quad \boldsymbol{\omega}^{\iota}{}_{r} = \frac{1}{g}\tilde{\boldsymbol{e}}^{\iota} + c\mathrm{d}r, \quad \boldsymbol{\omega}^{\iota}{}_{\kappa} = \tilde{\boldsymbol{\omega}}^{\iota}{}_{\kappa} + d\boldsymbol{e}^{\kappa}. \tag{2.210}$$

d can be eliminated by using (2.103):

$$\boldsymbol{\omega}^{\iota}{}_{\kappa} = \tilde{\boldsymbol{\omega}}^{\iota}{}_{\kappa} + d\boldsymbol{e}^{\kappa}, \tag{2.211}$$
$$\boldsymbol{\omega}^{\iota}{}_{\kappa} \overset{(2.103)}{=} -\boldsymbol{\omega}^{\kappa}{}_{\iota} \overset{(2.211)}{=} -\tilde{\boldsymbol{\omega}}^{\kappa}{}_{\iota} - d\boldsymbol{e}^{\iota} \overset{(2.103)}{=} \tilde{\boldsymbol{\omega}}^{\iota}{}_{\kappa} - d\boldsymbol{e}^{\iota}, \tag{2.212}$$

thus

$$d = 0. \tag{2.213}$$

$\mathrm{d}\boldsymbol{e}^{r}$ now fixes the remaining unknown functions a and c.

$$\mathrm{d}\boldsymbol{e}^{r} = g'\mathrm{d}r \wedge \mathrm{d}r = 0 \tag{2.214}$$
$$\overset{(2.99)}{=} -\boldsymbol{\omega}^{r}{}_{t} \wedge \boldsymbol{e}^{t} - \boldsymbol{\omega}^{r}{}_{\iota} \wedge \boldsymbol{e}^{\iota} \tag{2.215}$$
$$\overset{(2.206)(2.210)}{=} -\frac{f'f}{g}\mathrm{d}t \wedge \mathrm{d}t - fa\mathrm{d}r \wedge \mathrm{d}t + \frac{1}{rg}\boldsymbol{e}^{\iota} \wedge \boldsymbol{e}^{\iota} + c\mathrm{d}r \wedge \boldsymbol{e}^{\iota}, \tag{2.216}$$

thus

$$a = 0, \quad c = 0. \tag{2.217}$$

If we define the function $h = \frac{1}{g}$, the collected connection one-forms read

$$\boldsymbol{\omega}^{t}{}_{r} = \boldsymbol{\omega}^{r}{}_{t} = f'h\mathrm{d}t, \tag{2.218}$$
$$\boldsymbol{\omega}^{t}{}_{\iota} = \boldsymbol{\omega}^{\iota}{}_{t} = 0, \tag{2.219}$$
$$\boldsymbol{\omega}^{\iota}{}_{r} = -\boldsymbol{\omega}^{r}{}_{\iota} = h\tilde{\boldsymbol{e}}^{\iota}, \tag{2.220}$$
$$\boldsymbol{\omega}^{\iota}{}_{\kappa} = \tilde{\boldsymbol{\omega}}^{\iota}{}_{\kappa}. \tag{2.221}$$

To calculate the Riemann 2-form we will now use (2.105). We will demonstrate the calculation for the component that is the least straight forward:

$$\begin{aligned}\boldsymbol{R}^{\iota}{}_{\kappa} &= \mathrm{d}\boldsymbol{\omega}^{\iota}{}_{\kappa} + \boldsymbol{\omega}^{\iota}{}_{\rho} \wedge \boldsymbol{\omega}^{\rho}{}_{\kappa} + \boldsymbol{\omega}^{\iota}{}_{r} \wedge \boldsymbol{\omega}^{r}{}_{\kappa} \\ &= \underbrace{\mathrm{d}\tilde{\boldsymbol{\omega}}^{\iota}{}_{\kappa} + \tilde{\boldsymbol{\omega}}^{\iota}{}_{\rho} \wedge \tilde{\boldsymbol{\omega}}^{\rho}{}_{\kappa}}_{=\tilde{\boldsymbol{R}}^{\iota}{}_{\kappa}=\tilde{\boldsymbol{e}}^{\iota}\wedge\tilde{\boldsymbol{e}}^{\kappa}} - h^{2}\tilde{\boldsymbol{e}}^{\iota} \wedge \tilde{\boldsymbol{e}}^{\kappa} \\ &= \left(1 - h^{2}\right)\tilde{\boldsymbol{e}}^{\iota} \wedge \tilde{\boldsymbol{e}}^{\kappa}.\end{aligned} \tag{2.222}$$

Similar calculations for the other components lead to

$$\boldsymbol{R}^r{}_t = \boldsymbol{R}^t{}_r = (f'h)'\,\mathrm{d}r \wedge \mathrm{d}t, \tag{2.223}$$
$$\boldsymbol{R}^t{}_\iota = \boldsymbol{R}^\iota{}_t = -f'h^2\mathrm{d}t \wedge \tilde{\boldsymbol{e}}^\iota, \tag{2.224}$$
$$\boldsymbol{R}^r{}_\iota = -\boldsymbol{R}^\iota{}_r = -h'\mathrm{d}r \wedge \tilde{\boldsymbol{e}}^\iota, \tag{2.225}$$
$$\boldsymbol{R}^\iota{}_\kappa = \left(1 - h^2\right)\tilde{\boldsymbol{e}}^\iota \wedge \tilde{\boldsymbol{e}}^\kappa. \tag{2.226}$$

We can now use (2.106) to obtain the Ricci 1-form. For that, note that

$$\tilde{\boldsymbol{E}}_\iota \lrcorner \tilde{\boldsymbol{e}}^\iota = 2, \tag{2.227}$$

and

$$\tilde{\boldsymbol{E}}_\iota \lrcorner \left(\tilde{\boldsymbol{e}}^\iota \wedge \tilde{\boldsymbol{e}}^\kappa\right) = \left(\tilde{\boldsymbol{E}}^\iota \lrcorner \tilde{\boldsymbol{e}}^\iota\right)\tilde{\boldsymbol{e}}^\kappa - \left(\tilde{\boldsymbol{E}}_\iota \lrcorner \tilde{e}^\kappa\right)\tilde{\boldsymbol{e}}^\iota = 2\tilde{\boldsymbol{e}}^\kappa - \delta^\kappa_\iota \tilde{\boldsymbol{e}}^\iota = \tilde{\boldsymbol{e}}^\kappa. \tag{2.228}$$

With that, we get

$$\boldsymbol{R}_t = \boldsymbol{E}_r \lrcorner \boldsymbol{R}^r{}_t + \boldsymbol{E}_\iota \lrcorner \boldsymbol{R}^\iota{}_t = h\,(f'h)'\,\mathrm{d}t + 2\frac{f'h^2}{r}\mathrm{d}t, \tag{2.229}$$
$$\boldsymbol{R}_r = \boldsymbol{E}_t \lrcorner \boldsymbol{R}^t{}_r + \boldsymbol{E}_\iota \lrcorner \boldsymbol{R}^\iota{}_r = -\frac{(f'h)'}{f}\mathrm{d}r - 2\frac{h'}{r}\mathrm{d}r, \tag{2.230}$$
$$\boldsymbol{R}_\iota = \boldsymbol{E}_t \lrcorner \boldsymbol{R}^t{}_\iota + \boldsymbol{E}_r \lrcorner \boldsymbol{R}^r{}_\iota + \boldsymbol{E}_\kappa \lrcorner \boldsymbol{R}^\kappa{}_\iota = -\frac{f'h^2}{f}\tilde{\boldsymbol{e}}^\iota - hh'\tilde{\boldsymbol{e}}^\iota + \frac{1-h^2}{r}\tilde{\boldsymbol{e}}^\iota. \tag{2.231}$$

The vacuum Einstein equation (2.173) with $\Lambda = 0$ is equivalent to

$$\boldsymbol{R}_\alpha = 0. \tag{2.232}$$

That leads us to the three equations

$$(f'h)' + 2\frac{f'h}{r} = 0 \tag{2.233}$$
$$-(f'h)' - 2\frac{fh'}{r} = 0 \tag{2.234}$$
$$-\frac{f'h^2}{f} - hh' + \frac{1-h^2}{r} = 0 \tag{2.235}$$

Adding the first two equations gives

$$\frac{f'}{f} = \frac{h'}{h} \tag{2.236}$$
$$\ln f = \ln h + \hat{C} \tag{2.237}$$
$$f = \tilde{C}h \tag{2.238}$$

To eliminate the constant, we can rescale the Killing time

$$\tilde{C}\mathrm{d}t \to \mathrm{d}t, \tag{2.239}$$
$$\tilde{C}h \to h, \tag{2.240}$$

so we get

$$f = h = \frac{1}{g}. \tag{2.241}$$

Using that, equation (2.235) becomes

$$0 = -2f'f + \frac{1-f^2}{r}. \tag{2.242}$$

With the substitution $k := (1 - f^2)$, this reads

$$0 = k' + \frac{1}{r}k, \tag{2.243}$$

Which is a linear differential equation and has the solution

$$k = \frac{C}{r} \quad \Rightarrow \quad f^2 = \frac{1}{g^2} = 1 - \frac{C}{r} \tag{2.244}$$

Therefore the Schwarzschild line element is

$$\mathrm{d}s^2 = \left(1 - \frac{C}{r}\right)\mathrm{d}t^2 - \frac{1}{1-\frac{C}{r}}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2. \tag{2.245}$$

Comparing its Newtonian limit with the Newtonian gravitation of a spherically symmetric object determines the mass of this object to be $m = \frac{C}{2G}$. With the definition $M := mG$, we get

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2d\Omega^2. \tag{2.246}$$

As we can see, if we go to large distances ($r \to \infty$), as well as if we set $M = 0$, we arrive at the Minkowski metric. Birkhoff's theorem states that any spherically symmetric vacuum solution of (2.172) (with $\Lambda = 0$) is static and asymptotically flat. That means that we did not actually need "static" as a requirement as it follows from Birkhoff's theorem [2][3]. It also implies that in vacuum (with $\Lambda = 0$), the Schwarzschild solution is the unique spherically symmetric solution.

While the singularity at $r = 0$ is a real curvature singularity, meaning that the curvature diverges as $r \to 0$, the one at $r = 2M$ is a coordinate artefact. We can see that, if employ to null Kruskal-Szekeres coordinates.

$$uv := \left(\frac{r}{2M} - 1\right)e^{\frac{r}{2M}} \tag{2.247}$$

$$\frac{u}{v} := e^{\frac{t}{2M}} \tag{2.248}$$

$$ds^2 = \frac{32M^3}{r}e^{-\frac{r}{2M}}dudv + r^2d\Omega^2 \tag{2.249}$$

3. The Kerr Metric

In this chapter we derive the Kerr metric. The earth, as approximately rotating ball, has symmetries which transfer to the surrounding space-time.

This section relies heavily on [2].

3.1. A stationary axisymmetric Space-time

Stationarity means that there exists a timelike Killing vector field ∂_t. That means that the metric is invariant with respect to infinitesimal changes in the direction of this vector. Axisymmetry on the other hand means that there is an angle φ which leaves the metric invariant. With the other two coordinates being x^2 and x^3,

$$g_{ab} = g_{ab}(x^2, x^3). \tag{3.1}$$

In addition to that we will assume that the source is axisymmetric and rotating with φ. This source is invariant with respect to the transformation $t \to -t$, $\varphi \to -\varphi$. Therefore we also assume the metric to be. All terms which change sign after this transformation must be zero:

$$g_{t2} = g_{t3} = g_{\phi 2} = g_{\phi 3} = 0. \tag{3.2}$$

That leaves us with the block diagonal line element

$$ds^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{\phi\phi}\mathrm{d}\phi^2 + \left[g_{22}(\mathrm{d}x^2)^2 + 2g_{23}\mathrm{d}x^2\mathrm{d}x^3 + g_{33}(\mathrm{d}x^3)^2\right]. \tag{3.3}$$

We can simplify this metric by using [2]:

Theorem 1. *The Riemannian metric*

$$g_{22}(dx^2)^2 + 2g_{23}dx^2dx^3 + g_{33}(dx^3)^2 \tag{3.4}$$

of a two-dimensional space parametrized by x^2 and x^3 can always be brought into the diagonal form

$$ds^2 = \pm e^{2\mu}\left[(dx^2)^2 + (dx^3)^2\right] \tag{3.5}$$

by a coordinate transformation.

Although we could turn the non-Killing part of the metric into a multiple of the unit matrix we will only assume it as diagonal, which leaves us with the choice of an arbitrary gauge function, as we see in the following proposition.

Proposition 3. *Assume a 2-dimensional manifold with a metric with line element*

$$ds^2 = e^{2\mu_1}\left((dx^3)^2 + (dx^4)^2\right). \tag{3.6}$$

Then there exists a coordinate transformation turning the line element into

$$ds^2 = e^{2\mu_1}(dx^1)^2 + e^{2\mu_2}(dx^2)^2 \tag{3.7}$$

for an arbitrary gauge function $f = \mu_1 - \mu_2$.

Proof. Consider the coordinate transformation

$$\begin{aligned}
\mathrm{d}x^1 &:= \frac{1}{\sqrt{2}}\left(\mathrm{d}x^3 - \mathrm{d}x^4\right) \quad \Rightarrow \quad (\mathrm{d}x^1)^2 = \frac{1}{2}\left((\mathrm{d}x^3)^2 - 2\mathrm{d}x^3\mathrm{d}x^4 + (\mathrm{d}x^4)^2\right), \\
\mathrm{d}x^2 &:= \frac{e^f}{\sqrt{2}}\left(\mathrm{d}x^3 + \mathrm{d}x^4\right) \quad \Rightarrow \quad (\mathrm{d}x^2)^2 = \frac{e^{2f}}{2}\left((\mathrm{d}x^3)^2 + 2\mathrm{d}x^3\mathrm{d}x^4 + (\mathrm{d}x^4)^2\right),
\end{aligned} \tag{3.8}$$

f being an arbitrary function. Multiplying the second square with e^{-2f} and adding them together we get

$$(\mathrm{d}x^1)^2 + e^{-2f}(\mathrm{d}x^2)^2 = (\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2. \tag{3.9}$$

Plugging that into (3.6) we get

$$\mathrm{d}s^2 = e^{2\mu_1}(\mathrm{d}x^1)^2 + e^{2(\mu_1 - f)}(\mathrm{d}x^2)^2 \overset{f=:\mu_1-\mu_2}{=} e^{2\mu_1}(\mathrm{d}x^1)^2 + e^{2\mu_2}(\mathrm{d}x^2)^2. \tag{3.10}$$

□

Therefore we can start with the line element (3.7) and fix the gauge function $f = \mu_1 - \mu_2$ accordingly.

With the help of this theorem the line element for our entire manifold now has the form

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{\phi\phi}\mathrm{d}\phi^2 - e^{2\mu_2}(\mathrm{d}x^2)^2 - e^{2\mu_3}(\mathrm{d}x^3)^2. \tag{3.11}$$

We can now take the (t, ϕ)-part of the metric and express it in terms of new functions

$$g_{tt} =: e^{2\nu} - \omega^2 e^{2\psi}, \quad g_{t\phi} =: \omega e^{2\psi}, \quad g_{\phi\phi} =: -e^{2\psi}, \tag{3.12}$$

where, because of the Lorentzian metric signature, $e^{2\nu} > \omega^2 e^{2\psi}$ must be fulfilled. With these choices we can factor the line element into its final form

$$\mathrm{d}s^2 = e^{2\nu}\mathrm{d}t^2 - e^{2\psi}\left(\mathrm{d}\phi - \omega\mathrm{d}t\right)^2 - e^{2\mu_2}(\mathrm{d}x^2)^2 - e^{2\mu_3}(\mathrm{d}x^3)^2. \tag{3.13}$$

3.2. Derivation of the Kerr Metric

In the last section we derived a convenient general form of the metric for the space-time outside of rotating axisymmetric bodies. We will now use the Cartan structure relations and the Einstein equation to calculate its components.

Similar to the Schwarzschild case we first choose an orthonormal tetrad frame

$$\boldsymbol{e}^t := e^{\nu}\mathrm{d}t, \qquad \boldsymbol{E}_t := e^{-\nu}\left(\partial_t + \omega\partial_\phi\right), \tag{3.14}$$

$$\boldsymbol{e}^\phi := e^{\psi}\left(\mathrm{d}\phi - \omega\mathrm{d}t\right), \qquad \boldsymbol{E}_\phi := e^{-\psi}\partial_\phi, \tag{3.15}$$

$$\boldsymbol{e}^i := e^{\mu_i}\mathrm{d}x^i, \qquad \boldsymbol{E}_i := e^{-\mu_i}\partial_i, \tag{3.16}$$

where $i \in \{2, 3\}$. For this derivation we will write sums over the indices 2 and 3 explicitly to avoid confusion.

First, we will use Cartan's first structure relation (2.99) to calculate the connection one-forms. We will then employ them in Cartan's second structure relation (2.105) to obtain the Riemann 2-form, the contraction of which we can use in Einstein's equation.

3.2.1. The connection one-forms

In this section we will calculate the connection one-forms for an axisymmetric stationary vacuum. We will make the most general ansatz for the connection one-forms and constrain them step by step until they are fully determined. For now they are

$$\boldsymbol{\omega}^t{}_\phi = \boldsymbol{\omega}^\phi{}_t = a_t \mathrm{d}t + a_\phi \mathrm{d}\phi + \sum_i a_i \mathrm{d}x^i, \tag{3.17}$$

$$\boldsymbol{\omega}^t{}_i = \boldsymbol{\omega}^i{}_t = b_{t_i} \mathrm{d}t + b_{\phi_i} \mathrm{d}\phi + \sum_j b_{j_i} \mathrm{d}x^j, \tag{3.18}$$

$$\boldsymbol{\omega}^\phi{}_i = -\boldsymbol{\omega}^i{}_\phi = c_{t_i} \mathrm{d}t + c_{\phi_i} \mathrm{d}\phi + \sum_j c_{j_i} \mathrm{d}x^j, \tag{3.19}$$

$$\boldsymbol{\omega}^i{}_j = -\boldsymbol{\omega}^j{}_i = f_{t_{ij}} \mathrm{d}t + f_{\phi_{ij}} \mathrm{d}\phi + \sum_k f_{k_{ij}} \mathrm{d}x^k, \tag{3.20}$$

where the indexed a, b, c, and h are functions which are to be determined. We begin by using Cartan's first structure relation on $\boldsymbol{e}^t$,

$$\mathrm{d}\boldsymbol{e}^t = \sum_i \partial_i \nu e^\nu \mathrm{d}x^i \wedge \mathrm{d}t \overset{(2.99)}{=} \underbrace{-e^\psi \boldsymbol{\omega}^t{}_\phi \wedge (\mathrm{d}\phi - \omega \mathrm{d}t)}_{A} \underbrace{- \sum_i e^{\mu_i} \boldsymbol{\omega}^t{}_i \wedge \mathrm{d}x^i}_{B}. \tag{3.21}$$

Using the connection one-forms gives

$$A = -e^\psi a_t \mathrm{d}t \wedge \mathrm{d}\phi + e^\psi a_\phi \omega \mathrm{d}\phi \wedge \mathrm{d}t - e^\psi \sum_i a_i \mathrm{d}x^i \wedge \mathrm{d}\phi + e^\psi \sum_i a_i \omega \mathrm{d}x^i \wedge \mathrm{d}t, \tag{3.22}$$

$$B = \sum_i \left[-e^{\mu_i} b_{t_i} \mathrm{d}t \wedge \mathrm{d}x^i - e^{\mu_i} b_{\phi_i} \mathrm{d}\phi \wedge \mathrm{d}x^i - e^{\mu_i} \sum_j b_{j_i} \mathrm{d}x^j \wedge \mathrm{d}x^i \right]. \tag{3.23}$$

By comparing the different coefficients we get the restrictive conditions

$$\overset{\mathrm{d}t \wedge \mathrm{d}\phi}{\Rightarrow} \qquad 0 = -e^\psi a_t - e^\psi a_\phi \omega, \tag{3.24}$$

$$\overset{\mathrm{d}x^i \wedge \mathrm{d}\phi}{\Rightarrow} \qquad 0 = -e^\psi a_i + e^{\mu_i} b_{\phi_i}, \tag{3.25}$$

$$\overset{\mathrm{d}x^i \wedge \mathrm{d}t}{\Rightarrow} \qquad \partial_i \nu e^\nu = e^\psi a_i \omega + e^{\mu_i} b_{t_i}, \tag{3.26}$$

$$\overset{\mathrm{d}x^j \wedge \mathrm{d}x^i}{\Rightarrow} \qquad 0 = \sum_{ij} e^{\mu_i} b_{j_i} \mathrm{d}x^j \wedge \mathrm{d}x^i. \tag{3.27}$$

From the first three equations we can read off

$$a_t \overset{(3.24)}{=} -a_\phi \omega, \tag{3.28}$$

$$b_{\phi_i} \overset{(3.25)}{=} e^{\psi - \mu_i} a_i, \tag{3.29}$$

$$b_{t_i} \overset{(3.26)}{=} e^{-\mu_i} \left(e^\nu \partial_i \nu - e^\psi \omega a_i \right). \tag{3.30}$$

Condition (3.27) we cannot treat without the sum, because $\mathrm{d}x^i$ and $\mathrm{d}x^j$ are not necessarily independent. Therefore we write out the sum explicitly:

$$0 = -e^{\mu_2} b_{3_2} \mathrm{d}x^3 \wedge \mathrm{d}x^2 - e^{\mu_3} b_{2_3} \mathrm{d}x^2 \wedge \mathrm{d}x^3 = \left[e^{\mu_3} b_{2_3} - e^{\mu_2} b_{3_2}\right] \mathrm{d}x^3 \wedge \mathrm{d}x^2. \tag{3.31}$$

Thus we get

$$b_{3_2} = e^{\mu_3 - \mu_2} b_{2_3}. \tag{3.32}$$

After these restrictions, the relevant connection one-forms are of the form

$$\boldsymbol{\omega}^t{}_\phi = a_\phi \left(\mathrm{d}\phi - \omega \mathrm{d}t\right) + \sum_i a_i \mathrm{d}x^i, \tag{3.33}$$

$$\boldsymbol{\omega}^t{}_i = e^{\nu - \mu_i} \partial_i \nu \mathrm{d}t + e^{\psi - \mu_i} a_i \left(\mathrm{d}\phi - \omega \mathrm{d}t\right) + \sum_j b_{j_i} \mathrm{d}x^j \quad \text{with} \quad b_{3_2} = e^{\mu_3 - \mu_2} b_{2_3}. \tag{3.34}$$

For the next step we consider $\boldsymbol{e}^\phi$.

$$\begin{aligned} \mathrm{d}\boldsymbol{e}^\phi &= \sum_i \left[e^\psi \partial_i \psi \mathrm{d}x^i \wedge \mathrm{d}\phi - e^\psi \left(\partial_i \psi \omega + \partial_i \omega\right) \mathrm{d}x^i \wedge \mathrm{d}t\right] \\ &= \underbrace{-e^\nu \boldsymbol{\omega}^\phi{}_t \wedge \mathrm{d}t}_{A} \underbrace{- \sum_i e^{\mu_i} \boldsymbol{\omega}^\phi{}_i \wedge \mathrm{d}x^i}_{B} \end{aligned} \tag{3.35}$$

Using the connection one-forms with the previous restrictions we get

$$A = -e^\nu a_\phi \mathrm{d}\phi \wedge \mathrm{d}t - e^\nu \sum_i a_i \mathrm{d}x^i \wedge \mathrm{d}t \tag{3.36}$$

$$B = \sum_i \left[-e^{\mu_i} c_{t_i} \mathrm{d}t \wedge \mathrm{d}x^i - e^{\mu_i} c_{\phi_i} \mathrm{d}\phi \wedge \mathrm{d}x^i - e^{\mu_i} \sum_j c_{j_i} \mathrm{d}x^j \wedge \mathrm{d}x^i\right]. \tag{3.37}$$

The restrictive conditions are

$$\overset{\mathrm{d}\phi \wedge \mathrm{d}t}{\Rightarrow} \qquad 0 = -e^\nu a_\phi, \tag{3.38}$$

$$\overset{\mathrm{d}x^i \wedge \mathrm{d}\phi}{\Rightarrow} \qquad e^\psi \partial_i \psi = e^{\mu_i} c_{\phi_i}, \tag{3.39}$$

$$\overset{\mathrm{d}x^i \wedge \mathrm{d}t}{\Rightarrow} \qquad -e^\psi \left(\partial_i \psi \omega + \partial_i \omega\right) = -e^\nu a_i + e^{\mu_i} c_{t_i}, \tag{3.40}$$

$$\overset{\mathrm{d}x^j \wedge \mathrm{d}x^i}{\Rightarrow} \qquad 0 = -\sum_{ij} e^{\mu_i} c_{j_i} \mathrm{d}x^j \wedge \mathrm{d}x^i. \tag{3.41}$$

The four restrictions arise analogous to the previous calculations.

$$a_\phi \overset{(3.38)}{=} 0 \tag{3.42}$$

$$c_{\phi_i} \overset{(3.39)}{=} e^{\psi - \mu_i} \partial_i \psi \tag{3.43}$$

$$c_{t_i} \overset{(3.40)}{=} e^{-\mu_i} \left(e^\nu a_i - e^\psi \partial_i \omega\right) - e^{\psi - \mu_i} \partial_i \psi \omega \tag{3.44}$$

$$c_{3_2} \overset{(3.41)}{=} e^{\mu_3 - \mu_2} c_{2_3} \tag{3.45}$$

At the current stage, the restricted connection one-forms are

$$\boldsymbol{\omega}^{t}{}_{\phi} = \sum_i a_i \mathrm{d}x^i, \tag{3.46}$$

$$\begin{aligned} \boldsymbol{\omega}^{t}{}_{i} &= e^{\nu-\mu_i}\partial_i\nu \mathrm{d}t + e^{\psi-\mu_i} a_i \left(\mathrm{d}\phi - \omega \mathrm{d}t\right) + \sum_j b_{j_i} \mathrm{d}x^j \\ &\qquad \text{with} \quad b_{3_2} = e^{\mu_3-\mu_2} b_{2_3}, \end{aligned} \tag{3.47}$$

$$\begin{aligned} \boldsymbol{\omega}^{\phi}{}_{i} &= e^{-\mu_i}\left(e^{\nu}a_i - e^{\psi}\partial_i\omega\right)\mathrm{d}t + e^{\psi-\mu_i}\partial_i\Psi\left(\mathrm{d}\phi - \omega\mathrm{d}t\right) + \sum_j c_{j_i}\mathrm{d}x^j \\ &\qquad \text{with} \quad c_{3_2} = e^{\mu_3-\mu_2} c_{2_3}. \end{aligned} \tag{3.48}$$

The calculation for the last basis one-form proceeds just as for the previous ones.

$$\mathrm{d}\boldsymbol{e}^i = \sum_j e^{\mu_i}\partial_j\mu_i \mathrm{d}x^j \wedge \mathrm{d}x^i = \underbrace{-e^{\nu}\boldsymbol{\omega}^{i}{}_{t} \wedge \mathrm{d}t}_{A} \underbrace{-e^{\psi}\boldsymbol{\omega}^{i}{}_{\phi} \wedge \left(\mathrm{d}\phi - \omega\mathrm{d}t\right)}_{B} \underbrace{- \sum_j e^{\mu_j}\boldsymbol{\omega}^{i}{}_{j} \wedge \mathrm{d}x^j}_{C} \tag{3.49}$$

Using the connection one-forms gives

$$A = -e^{\psi+\nu-\mu_i} a_i \mathrm{d}\phi \wedge \mathrm{d}t - e^{\nu}\sum_j b_{j_i}\mathrm{d}x^j \wedge \mathrm{d}t, \tag{3.50}$$

$$B = e^{\psi-\mu_i}\left(e^{\nu}a_i - e^{\psi}\partial_i\omega\right)\mathrm{d}t \wedge \mathrm{d}\phi + e^{\psi}\sum_j c_{j_i}\mathrm{d}x^j \wedge \mathrm{d}\phi - e^{\psi}\omega\sum_j c_{j_i}\mathrm{d}x^j \wedge \mathrm{d}t, \tag{3.51}$$

$$C = \sum_j \left[-e^{\mu_j} f_{t_{ij}}\mathrm{d}t \wedge \mathrm{d}x^j - e^{\mu_j} f_{\phi_{ij}}\mathrm{d}\phi \wedge \mathrm{d}x^j - e^{\mu_j}\sum_k f_{k_{ij}}\mathrm{d}x^k \wedge \mathrm{d}x^j\right]. \tag{3.52}$$

The resulting restrictive conditions are

$$\overset{\mathrm{d}\phi\wedge\mathrm{d}t}{\Rightarrow} \qquad 0 = -e^{\psi+\nu-\mu_i}a_i - e^{\psi+\nu-\mu_i}a_i + e^{2\psi-\mu_i}\partial_i\omega, \tag{3.53}$$

$$\overset{\mathrm{d}x^j\wedge\mathrm{d}t}{\Rightarrow} \qquad 0 = -e^{\nu}b_{j_i} - e^{\psi}\omega c_{j_i} + e^{\mu_j} f_{t_{ij}}, \tag{3.54}$$

$$\overset{\mathrm{d}x^j\wedge\mathrm{d}\phi}{\Rightarrow} \qquad 0 = e^{\psi}c_{j_i} + e^{\mu_j} f_{\phi_{ij}}, \tag{3.55}$$

$$\overset{\mathrm{d}x^j\wedge\mathrm{d}x^i}{\Rightarrow} \qquad \sum_j e^{\mu_i}\partial_j\mu_i \mathrm{d}x^j \wedge \mathrm{d}x^i = -\sum_{jk} e^{\mu_j} f_{k_{ij}}\mathrm{d}x^k \wedge \mathrm{d}x^j. \tag{3.56}$$

The first three equations give

$$a_i \overset{(3.53)}{=} \frac{1}{2}e^{\psi-\nu}\partial_i\omega, \tag{3.57}$$

$$f_{t_{ij}} \overset{(3.54)}{=} e^{-\mu_j}\left(e^{\nu}b_{j_i} + e^{\psi}\omega c_{j_i}\right), \tag{3.58}$$

$$f_{\phi_{ij}} \overset{(3.55)}{=} -e^{\psi-\mu_j}c_{j_i}. \tag{3.59}$$

For equation (3.56) we can choose values for i to get

$$i = 2: \qquad \partial_3\mu_2 e^{\mu_2} \mathrm{d}x^3 \wedge \mathrm{d}x^2 = -e^{\mu_3} f_{2_{23}} \mathrm{d}x^2 \wedge \mathrm{d}x^3, \tag{3.60}$$

$$i = 3: \qquad \partial_2\mu_3 e^{\mu_3} \mathrm{d}x^2 \wedge \mathrm{d}x^3 = -e^{\mu_2} f_{3_{32}} \mathrm{d}x^3 \wedge \mathrm{d}x^2, \tag{3.61}$$

which give us

$$f_{2_{23}} \overset{(3.60)}{=} e^{\mu_2-\mu_3}\partial_3\mu_2, \tag{3.62}$$

$$f_{3_{32}} \overset{(3.61)}{=} e^{\mu_3-\mu_2}\partial_2\mu_3. \tag{3.63}$$

To assign the last unknown functions we can exploit the connection one-form's antisymmetry

$$\boldsymbol{\omega}^i{}_j = -\boldsymbol{\omega}^j{}_i. \tag{3.64}$$

First, we realize that

$$f_{\phi_{ii}} = 0, \quad f_{t_{ii}} = 0 \quad f_{k_{ii}} = 0, \tag{3.65}$$

which leads us to

$$c_{i_i} \overset{(3.59)}{=} 0, \quad b_{i_i} \overset{(3.58)}{=} 0. \tag{3.66}$$

For $i \neq j$ we get

$$f_{\phi_{ij}} = -f_{\phi_{ji}} \qquad \overset{(3.59),i\mapsto 2}{\Rightarrow} \qquad -e^{\psi-\mu_3}c_{3_2} = e^{\psi-\mu_2}c_{2_3} \tag{3.67}$$

$$\Rightarrow \qquad e^{\mu_3-\mu_2}c_{2_3} \overset{(3.45)}{=} c_{3_2} \overset{(3.67)}{=} -e^{\mu_3-\mu_2}c_{2_3} \tag{3.68}$$

$$\Rightarrow \qquad c_{2_3} = c_{3_2} = 0, \tag{3.69}$$

$$f_{t_{ij}} = -f_{t_{ji}} \qquad \overset{(3.58),i\mapsto 2}{\Rightarrow} \qquad e^{\nu-\mu_3}b_{3_2} = -e^{\nu-\mu_2}b_{2_3} \tag{3.70}$$

$$\Rightarrow \qquad e^{\mu_3-\mu_2}b_{2_3} \overset{(3.32)}{=} b_{3_2} \overset{(3.70)}{=} -e^{\mu_3-\mu_2}b_{2_3} \tag{3.71}$$

$$\Rightarrow \qquad b_{2_3} = b_{3_2} = 0. \tag{3.72}$$

That, with the use of (3.58) and (3.59), gives

$$f_{t_{ij}} = 0, \quad f_{\phi_{ij}} = 0. \tag{3.73}$$

For the non-Killing coordinate coefficients the antisymmetry leads to

$$f_{k_{ij}} = -f_{k_{ji}}. \tag{3.74}$$

Conclusively, the connection one-forms then take the form

$$\boldsymbol{\omega}^t{}_\phi = \boldsymbol{\omega}^\phi{}_t = \frac{1}{2}e^{\psi-\nu}\sum_i \partial_i\omega \mathrm{d}x^i, \tag{3.75}$$

$$\boldsymbol{\omega}^t{}_i = \boldsymbol{\omega}^i{}_t = e^{\nu-\mu_i}\partial_i\nu \mathrm{d}t + \frac{1}{2}e^{2\psi-\nu-\mu_i}\partial_i\omega\left(\mathrm{d}\phi - \omega\mathrm{d}t\right), \tag{3.76}$$

$$\boldsymbol{\omega}^\phi{}_i = -\boldsymbol{\omega}^i{}_\phi = -\frac{1}{2}e^{\psi-\mu_i}\partial_i\omega\mathrm{d}t + e^{\psi-\mu_i}\partial_i\psi\left(\mathrm{d}\phi - \omega\mathrm{d}t\right), \tag{3.77}$$

$$\boldsymbol{\omega}^i{}_j = -\boldsymbol{\omega}^j{}_i = e^{\mu_i-\mu_j}\partial_j\mu_i\mathrm{d}x^i - e^{\mu_j-\mu_i}\partial_i\mu_j\mathrm{d}x^j. \tag{3.78}$$

3.2.2. The Riemann 2-form

From the previously obtained connection one-forms we can now use Cartan's second structure relation (2.105) to calculate the Riemann 2-form.

$$\boldsymbol{R}^{\alpha}{}_{\beta} = \mathrm{d}\boldsymbol{\omega}^{\alpha}{}_{\beta} + \boldsymbol{\omega}^{\alpha}{}_{\gamma} \wedge \boldsymbol{\omega}^{\gamma}{}_{\beta} \tag{3.79}$$

For the component $\boldsymbol{R}^{t}{}_{\phi}$, the relevant parts are

$$\mathrm{d}\boldsymbol{\omega}^{t}{}_{\phi} = \frac{1}{2}\sum_{ij} e^{\psi-\nu}\left(\partial_j(\psi-\nu)\partial_i\omega + \partial_j\partial_i\omega\right)\mathrm{d}x^j \wedge \mathrm{d}x^i, \tag{3.80}$$

$$\sum_i \boldsymbol{\omega}^{t}{}_{i} \wedge \boldsymbol{\omega}^{i}{}_{\phi} = -\sum_i e^{\psi-\nu-2\mu_i}\left(e^{2\nu}\partial_i\nu\partial_i\psi + \frac{1}{4}e^{2\psi}(\partial_i\omega)^2\right)\mathrm{d}t \wedge \mathrm{d}\phi. \tag{3.81}$$

Adding them up yields

$$\boldsymbol{R}^{t}{}_{\phi} = e^{\psi-\nu}\sum_i\left[-e^{-2\mu_i}\left(e^{2\nu}\partial_i\nu\partial_i\psi + \frac{1}{4}e^{2\psi}(\partial_i\omega)^2\right)\mathrm{d}t\wedge\mathrm{d}\phi + \frac{1}{2}\sum_j \partial_j(\psi-\nu)\partial_i\omega\mathrm{d}x^j \wedge \mathrm{d}x^i\right]. \tag{3.82}$$

The $\partial_i\partial_j\omega$ term does not contribute because of its symmetry and the contraction with the antisymmetric $\mathrm{d}x^j \wedge \mathrm{d}x^i$.
For $\boldsymbol{R}^{t}{}_{i}$ we need

$$\mathrm{d}\boldsymbol{\omega}^{t}{}_{i} = e^{-\nu-\mu_i}\sum_j\left[\left(e^{2\nu}\left(\partial_j(\nu-\mu_i)\partial_i\nu + \partial_j\partial_i\nu\right) - \frac{1}{2}e^{2\psi}\partial_i\omega\partial_j\omega\right)\mathrm{d}x^j\wedge\mathrm{d}t\right. \tag{3.83}$$

$$\left. + \frac{1}{2}e^{2\psi}\left(\partial_j(2\psi-\nu-\mu_i)\partial_i\omega + \partial_j\partial_i\omega\right)\mathrm{d}x^j \wedge (\mathrm{d}\phi - \omega\mathrm{d}t)\right], \tag{3.84}$$

$$\boldsymbol{\omega}^{t}{}_{\phi} \wedge \boldsymbol{\omega}^{\phi}{}_{i} = e^{2\psi-\nu-\mu_i}\frac{1}{2}\sum_j\left[-\frac{1}{2}\partial_i\omega\partial_j\omega\mathrm{d}x^j\wedge\mathrm{d}t + \partial_i\psi\partial_j\omega\mathrm{d}x^j\wedge(\mathrm{d}\phi-\omega\mathrm{d}t)\right], \tag{3.85}$$

$$\sum_j \boldsymbol{\omega}^{t}{}_{j}\wedge\boldsymbol{\omega}^{j}{}_{i} = e^{-\nu-\mu_i}\sum_j\left[e^{2\nu}\partial_i\mu_j\partial_j\nu\mathrm{d}t\wedge\mathrm{d}x^j - \frac{1}{2}e^{2\psi}\partial_i\mu_j\partial_j\omega\mathrm{d}x^j\wedge(\mathrm{d}\phi-\omega\mathrm{d}t)\right. \tag{3.86}$$

$$\left. - e^{2(\nu+\mu_i-\mu_j)}\partial_j\nu\partial_j\mu_i\mathrm{d}t\wedge\mathrm{d}x^i + \frac{1}{2}e^{2(\psi+\mu_i-\mu_j)}\partial_j\omega\partial_j\mu_i\mathrm{d}x^i\wedge(\mathrm{d}\phi-\omega\mathrm{d}t)\right]. \tag{3.87}$$

Written together, those read

$$\boldsymbol{R}^t{}_i = e^{-\nu-\mu_i}\sum_j\Bigg[\Big[e^{2\nu}\left(\partial_i\partial_j\nu + \partial_i\nu\partial_j\nu - (\partial_i\nu\partial_j\mu_i + \partial_j\nu\partial_i\mu_j)\right) - \frac{3}{4}e^{2\psi}\partial_i\omega\partial_j\omega\Big]\,\mathrm{d}x^j$$
$$+ e^{2(\nu+\mu_i-\mu_j)}\partial_j\nu\partial_j\mu_i\mathrm{d}x^i\Bigg]\wedge\mathrm{d}t$$
$$+\frac{1}{2}e^{2\psi-\nu-\mu_i}\sum_j\Bigg[\Big[\partial_i\partial_j\omega + \partial_i\omega\partial_j(\psi-\nu) + (\partial_j\psi\partial_i\omega + \partial_i\psi\partial_j\omega) - (\partial_j\mu_i\partial_i\omega + \partial_i\mu_j\partial_j\omega)\Big]\mathrm{d}x^j$$
$$+ e^{2(\mu_i-\mu_j)}\partial_j\nu\partial_j\mu_i\mathrm{d}x^i\Bigg]\wedge(\mathrm{d}\phi-\omega\mathrm{d}t). \tag{3.88}$$

The relevant terms for $\boldsymbol{R}^\phi{}_i$ are

$$\begin{aligned}\mathrm{d}\boldsymbol{\omega}^\phi{}_i = \frac{1}{2}e^{\psi-\mu_i}\sum_j\Big[&-\partial_i\psi\partial_j\omega + \partial_j\mu_i\partial_i\omega - \partial_j\partial_i\omega \\ &- (\partial_i\psi\partial_j\omega + \partial_j\psi\partial_i\omega)\Big]\mathrm{d}x^j\wedge\mathrm{d}t \\ &+ e^{\psi-\mu_i}\sum_j\left[\partial_j(\psi-\mu_i)\partial_i\psi + \partial_j\partial_i\psi\right]\mathrm{d}x^j\wedge(\mathrm{d}\phi-\omega\mathrm{d}t),\end{aligned} \tag{3.89}$$

$$\boldsymbol{\omega}^\phi{}_t\wedge\boldsymbol{\omega}^t{}_i = \frac{1}{2}e^{\psi-\mu_i}\sum_j\left[\partial_i\nu\partial_j\omega\mathrm{d}x^j\wedge\mathrm{d}t + \frac{1}{2}e^{2(\psi-\nu)}\partial_i\omega\partial_j\omega\mathrm{d}x^j\wedge(\mathrm{d}\phi-\omega\mathrm{d}t)\right], \tag{3.90}$$

$$\begin{aligned}\sum_j\boldsymbol{\omega}^\phi{}_j\wedge\boldsymbol{\omega}^j{}_i = e^{\psi-\mu_i}\sum_j\Big[&-\frac{1}{2}\partial_j\omega\partial_i\mu_j\mathrm{d}t\wedge\mathrm{d}x^j + \partial_j\psi\partial_i\mu_j(\mathrm{d}\phi-\omega\mathrm{d}t)\wedge\mathrm{d}x^j \\ &+ e^{2(\mu_i-\mu_j)}\left[\frac{1}{2}\partial_j\omega\partial_j\mu_i\mathrm{d}t\wedge\mathrm{d}x^i - \partial_j\psi\partial_j\mu_i(\mathrm{d}\phi-\omega\mathrm{d}t)\wedge\mathrm{d}x^i\right]\Big].\end{aligned} \tag{3.91}$$

Collecting them gives

$$\boldsymbol{R}^\phi{}_i = \frac{1}{2}e^{\psi-\mu_i}\sum_j\Bigg[\Big[\partial_i(\nu-\psi)\partial_j\omega - \partial_i\partial_j\omega + (\partial_i\omega\partial_j\mu_i + \partial_j\omega\partial_i\mu_j) - (\partial_i\psi\partial_j\omega + \partial_j\psi\partial_i\omega)\Big]\mathrm{d}x^j$$
$$- e^{2(\mu_i-\mu_j)}\partial_j\omega\partial_j\mu_i\mathrm{d}x^i\Bigg]\wedge\mathrm{d}t$$
$$+ e^{\psi-\mu_i}\sum_j\Bigg[\Big[\partial_i\partial_j\psi + \partial_i\psi\partial_j\psi - (\partial_i\psi\partial_j\mu_i + \partial_j\psi\partial_i\mu_j) + \frac{1}{4}e^{2(\psi-\nu)}\partial_i\omega\partial_j\omega\Big]\,\mathrm{d}x^j$$
$$+ e^{2(\mu_i-\mu_j)}\partial_j\psi\partial_j\mu_i\mathrm{d}x^i\Bigg]\wedge(\mathrm{d}\phi-\omega\mathrm{d}t). \tag{3.92}$$

To calculate $\boldsymbol{R}^i{}_j$, we need the terms

$$\begin{aligned}
\mathrm{d}\boldsymbol{\omega}^i{}_j &= e^{\mu_i-\mu_j}\sum_k \left[\partial_k(\mu_i-\mu_j)\partial_j\mu_i+\partial_k\partial_j\mu_i\right]\mathrm{d}x^k\wedge \mathrm{d}x^i \\
&\quad - e^{\mu_j-\mu_i}\sum_k \left[\partial_k(\mu_j-\mu_i)\partial_i\mu_j+\partial_k\partial_i\mu_j\right]\mathrm{d}x^k\wedge \mathrm{d}x^j,
\end{aligned} \tag{3.93}$$

$$\boldsymbol{\omega}^i{}_t\wedge\boldsymbol{\omega}^t{}_j = \frac{1}{2}e^{2\psi-\mu_i-\mu_j}\left(\partial_i\nu\partial_j\omega-\partial_j\nu\partial_i\omega\right)\mathrm{d}t\wedge\mathrm{d}\phi, \tag{3.94}$$

$$\boldsymbol{\omega}^i{}_\phi\wedge\boldsymbol{\omega}^\phi{}_j = \frac{1}{2}e^{2\psi-\mu_i-\mu_j}\left(\partial_i\omega\partial_j\psi-\partial_j\omega\partial_i\psi\right)\mathrm{d}t\wedge\mathrm{d}\phi, \tag{3.95}$$

$$\begin{aligned}
\sum_k\boldsymbol{\omega}^i{}_k\wedge\boldsymbol{\omega}^k{}_j &= \sum_k\Big[e^{\mu_i-\mu_j}\partial_k\mu_i\partial_j\mu_k\mathrm{d}x^i\wedge\mathrm{d}x^k - e^{\mu_j-\mu_i}\partial_k\mu_j\partial_i\mu_k\mathrm{d}x^j\wedge\mathrm{d}x^k \\
&\quad - e^{\mu_i+\mu_j-2\mu_k}\partial_k\mu_i\partial_k\mu_j\mathrm{d}x^i\wedge\mathrm{d}x^j\Big].
\end{aligned} \tag{3.96}$$

Adding those gives us

$$\begin{aligned}
\boldsymbol{R}^i{}_j &= \frac{1}{2}e^{2\psi-\mu_i-\mu_j}\left[\partial_j\omega\partial_i(\nu-\psi)-\partial_i\omega\partial_j(\nu-\psi)\right]\mathrm{d}t\wedge\mathrm{d}\phi \\
&\quad - \left[\sum_k e^{\mu_i+\mu_j-2\mu_k}\partial_k\mu_i\partial_k\mu_j\right]\mathrm{d}x^i\wedge\mathrm{d}x^j \\
&\quad + e^{\mu_i-\mu_j}\sum_k\left[\partial_k\partial_j\mu_i+\partial_k(\mu_i-\mu_j)\partial_j\mu_i-\partial_k\mu_i\partial_j\mu_k\right]\mathrm{d}x^k\wedge\mathrm{d}x^i \\
&\quad - e^{\mu_j-\mu_i}\sum_k\left[\partial_k\partial_i\mu_j+\partial_k(\mu_j-\mu_i)\partial_i\mu_j-\partial_k\mu_j\partial_i\mu_k\right]\mathrm{d}x^k\wedge\mathrm{d}x^j.
\end{aligned} \tag{3.97}$$

Collecting the previous results, the Riemann 2-form components are

$$\begin{aligned}
\boldsymbol{R}^t{}_\phi &= -\sum_i e^{-2\mu_i}\left(\partial_i\nu\partial_i\psi+\frac{1}{4}e^{2(\psi-\nu)}(\partial_i\omega)^2\right)\boldsymbol{e}^t\wedge\boldsymbol{e}^\phi \\
&\quad + \sum_{ij}e^{\psi-\nu-\mu_i-\mu_j}\partial_j(\psi-\nu)\partial_i\omega\boldsymbol{e}^j\wedge\boldsymbol{e}^i,
\end{aligned} \tag{3.98}$$

$$\begin{aligned}
\boldsymbol{R}^t{}_i &= \sum_j e^{-\mu_i-\mu_j}\Bigg[\left[(\partial_i\partial_j\nu+\partial_i\nu\partial_j\nu-(\partial_i\nu\partial_j\mu_i+\partial_j\nu\partial_i\mu_j))-\frac{3}{4}e^{2(\psi-\nu)}\partial_i\omega\partial_j\omega\right]\boldsymbol{e}^j \\
&\qquad + e^{\mu_i-\mu_j}\partial_j\nu\partial_j\mu_i\boldsymbol{e}^i\Bigg]\wedge\boldsymbol{e}^t \\
&\quad + \frac{1}{2}\sum_j e^{\psi-\nu-\mu_i-\mu_j}\Bigg[\Big[\partial_i\partial_j\omega+\partial_i\omega\partial_j(\psi-\nu)+(\partial_j\psi\partial_i\omega+\partial_i\psi\partial_j\omega)-(\partial_j\mu_i\partial_i\omega+\partial_i\mu_j\partial_j\omega)\Big]\boldsymbol{e}^j \\
&\qquad + e^{\mu_i-\mu_j}\partial_j\omega\partial_j\mu_i\boldsymbol{e}^i\Bigg]\wedge\boldsymbol{e}^\phi,
\end{aligned} \tag{3.99}$$

$$
\begin{aligned}
\boldsymbol{R}^{\phi}{}_{i} = \frac{1}{2}\sum_j e^{\psi-\nu-\mu_i-\mu_j}\Bigg[\Big[\partial_i(\nu-\psi)\partial_j\omega - \partial_i\partial_j\omega + (\partial_i\omega\partial_j\mu_i + \partial_j\omega\partial_i\mu_j) - (\partial_i\psi\partial_j\omega + \partial_j\psi\partial_i\omega)\Big]\boldsymbol{e}^j \\
- e^{\mu_i-\mu_j}\partial_j\omega\partial_j\mu_i\boldsymbol{e}^i\Bigg] \wedge \boldsymbol{e}^t \\
+ \sum_j e^{-\mu_i-\mu_j}\Bigg[\Big[\partial_i\partial_j\psi + \partial_i\psi\partial_j\psi - (\partial_i\psi\partial_j\mu_i + \partial_j\psi\partial_i\mu_j) + \frac{1}{4}e^{2(\psi-\nu)}\partial_i\omega\partial_j\omega\Big]\boldsymbol{e}^j \\
+ e^{\mu_i-\mu_j}\partial_j\psi\partial_j\mu_i\boldsymbol{e}^i\Bigg] \wedge \boldsymbol{e}^{\phi}.
\end{aligned}
\tag{3.100}
$$

$$
\begin{aligned}
\boldsymbol{R}^i{}_j = &\frac{1}{2}e^{\psi-\nu-\mu_i-\mu_j}\left[\partial_j\omega\partial_i(\nu-\psi) - \partial_i\omega\partial_j(\nu-\psi)\right]\boldsymbol{e}^t \wedge \boldsymbol{e}^{\phi} \\
&- \left[\sum_k e^{-2\mu_k}\partial_k\mu_i\partial_k\mu_j\right]\boldsymbol{e}^i \wedge \boldsymbol{e}^j \\
&+ \sum_k e^{-\mu_j-\mu_k}\left[\partial_k\partial_j\mu_i + \partial_k(\mu_i-\mu_j)\partial_j\mu_i - \partial_k\mu_i\partial_j\mu_k\right]\boldsymbol{e}^k \wedge \boldsymbol{e}^i \\
&- \sum_k e^{-\mu_i-\mu_k}\left[\partial_k\partial_i\mu_j + \partial_k(\mu_j-\mu_i)\partial_i\mu_j - \partial_k\mu_j\partial_i\mu_k\right]\boldsymbol{e}^k \wedge \boldsymbol{e}^j.
\end{aligned}
\tag{3.101}
$$

3.2.3. The Ricci 1-form

Since we have the Riemann 2-form we can now contract it with the tetrad basis to obtain the Ricci 1-form (2.106).

$$
\boldsymbol{R}_{\alpha} = \sum_{\beta} \boldsymbol{E}_{\beta} \lrcorner \boldsymbol{R}^{\beta}{}_{\alpha}
\tag{3.102}
$$

There are two terms relevant for $\boldsymbol{R}_t$, which are

$$
\boldsymbol{E}_{\phi} \lrcorner \boldsymbol{R}^{\phi}{}_{t} = \sum_i e^{-2\mu_i}\left(\partial_i\nu\partial_i\psi + \frac{1}{4}e^{2(\psi-\nu)}(\partial_i\omega)^2\right)\boldsymbol{e}^t,
\tag{3.103}
$$

$$
\begin{aligned}
\sum_i \boldsymbol{E}_i \lrcorner \boldsymbol{R}^i{}_t = &\sum_i\left(e^{-2\mu_i}\left[\partial_i^2\nu + (\partial_i\nu)^2 - 2\partial_i\nu\partial_i\mu_i - \frac{3}{4}e^{2(\psi-\nu)}(\partial_i\omega)^2\right] + \sum_j e^{-2\mu_j}\partial_j\nu\partial_j\mu_i\right)\boldsymbol{e}^t \\
&+ \frac{1}{2}e^{\psi-\nu}\sum_i\left(e^{-2\mu_i}\left[\partial_i^2\omega + \partial_i\omega\partial_i(3\psi-\nu-2\mu_i)\right] + \sum_j e^{-2\mu_j}\partial_j\omega\partial_j\mu_i\right)\boldsymbol{e}^{\phi},
\end{aligned}
\tag{3.104}
$$

yielding

$$\boldsymbol{R}_t = \sum_i \left(e^{-2\mu_i} \left[\partial_i^2 \nu + \partial_i \nu \partial_i (\psi + \nu - 2\mu_i) - \frac{1}{2} e^{2(\psi-\nu)} (\partial_i \omega)^2 \right] + \sum_j e^{-2\mu_j} \partial_j \nu \partial_j \mu_i \right) \boldsymbol{e}^t$$
$$+ \frac{1}{2} e^{\psi-\nu} \sum_i \left(e^{-2\mu_i} \left[\partial_i^2 \omega + \partial_i \omega \partial_i (3\psi - \nu - 2\mu_i) \right] + \sum_j e^{-2\mu_j} \partial_j \omega \partial_j \mu_i \right) \boldsymbol{e}^\phi. \tag{3.105}$$

For $\boldsymbol{R}_\phi$, we need the terms

$$\boldsymbol{E}_t \lrcorner \boldsymbol{R}^t{}_\phi = - \sum_i e^{-2\mu_i} \left(\partial_i \nu \partial_i \psi + \frac{1}{4} e^{2(\psi-\nu)} (\partial_i \omega)^2 \right) \boldsymbol{e}^\phi, \tag{3.106}$$

$$\sum_i \boldsymbol{E}_i \lrcorner \boldsymbol{R}^i{}_\phi = \frac{1}{2} e^{\psi-\nu} \sum_i \left(e^{-2\mu_i} \left[\partial_i^2 \omega + \partial_i \omega \partial_i (3\psi - \nu - 2\mu_i) \right] + \sum_j e^{-2\mu_j} \partial_j \omega \partial_j \mu_i \right) \boldsymbol{e}^t$$
$$- \sum_i \left(e^{-2\mu_i} \left[\partial_i^2 \psi + \partial_i \psi \partial_i (\psi - 2\mu_i) + \frac{1}{4} e^{2(\psi-\nu)} (\partial_i \omega)^2 \right] + \sum_j e^{-2\mu_j} \partial_j \psi \partial_j \mu_i \right) \boldsymbol{e}^\phi, \tag{3.107}$$

to obtain

$$\boldsymbol{R}_\phi = \frac{1}{2} e^{\psi-\nu} \sum_i \left(e^{-2\mu_i} \left[\partial_i^2 \omega + \partial_i \omega \partial_i (3\psi - \nu - 2\mu_i) \right] + \sum_j e^{-2\mu_j} \partial_j \omega \partial_j \mu_i \right) \boldsymbol{e}^t$$
$$- \sum_i \left(e^{-2\mu_i} \left[\partial_i^2 \psi + \partial_i \psi \partial_i (\psi + \nu - 2\mu_i) + \frac{1}{2} e^{2(\psi-\nu)} (\partial_i \omega)^2 \right] + \sum_j e^{-2\mu_j} \partial_j \psi \partial_j \mu_i \right) \boldsymbol{e}^\phi. \tag{3.108}$$

The relevant terms for $\boldsymbol{R}_i$ are

$$\begin{aligned}\boldsymbol{E}_t \lrcorner \boldsymbol{R}^t{}_i = &-\sum_j e^{-\mu_i-\mu_j}\Bigg(\Big[\partial_i\partial_j\nu + \partial_i\nu\partial_j\nu - (\partial_i\nu\partial_j\mu_i + \partial_j\nu\partial_i\mu_j) \\ &-\frac{3}{4}e^{2(\psi-\nu)}\partial_i\omega\partial_j\omega\Big]\boldsymbol{e}^j + e^{\mu_i-\mu_j}\partial_j\nu\partial_j\mu_i\boldsymbol{e}^i\Bigg),\end{aligned} \tag{3.109}$$

$$\begin{aligned}\boldsymbol{E}_\phi \lrcorner \boldsymbol{R}^\phi{}_i = &-\sum_j e^{-\mu_i-\mu_j}\Bigg(\Big[\partial_i\partial_j\psi + \partial_i\psi\partial_j\psi - (\partial_i\psi\partial_j\mu_i + \partial_j\psi\partial_i\mu_j) \\ &+\frac{1}{4}e^{2(\psi-\nu)}\partial_i\omega\partial_j\omega\Big]\boldsymbol{e}^j + e^{\mu_i-\mu_j}\partial_j\psi\partial_j\mu_i\boldsymbol{e}^i\Bigg),\end{aligned} \tag{3.110}$$

$$\begin{aligned}\sum_j \boldsymbol{E}_j \lrcorner \boldsymbol{R}^j{}_i = &\sum_k e^{-2\mu_k}\left((\partial_k\mu_i)^2 - \sum_j \partial_k\mu_i\partial_k\mu_j - \partial_k^2\mu_i - \partial_k\mu_i\partial_k(\mu_i-2\mu_k)\right)\boldsymbol{e}^i \\ &+\sum_k e^{-\mu_i-\mu_k}\Bigg(\partial_i\partial_k(\mu_i+\mu_k) - 2\partial_i\mu_k\partial_k\mu_i \\ &-\sum_j[\partial_i\partial_k\mu_j + \partial_k(\mu_j-\mu_i)\partial_i\mu_j - \partial_k\mu_j\partial_i\mu_k]\Bigg)\boldsymbol{e}^k,\end{aligned} \tag{3.111}$$

and, after renaming some summation indices, they result in

$$\begin{aligned}\boldsymbol{R}_i = &-\sum_j e^{-2\mu_j}\left(\partial_j^2\mu_i + \partial_j\mu_i\partial_j(\psi+\nu-2\mu_j) + \sum_k \partial_j\mu_i\partial_j\mu_k\right)\boldsymbol{e}^i \\ &-\sum_j e^{-\mu_i-\mu_j}\Bigg(\partial_i\partial_j(\psi+\nu) + \partial_i\psi\partial_j\psi + \partial_i\nu\partial_j\nu - [\partial_i(\psi+\nu)\partial_j\mu_i + \partial_j(\psi+\nu)\partial_i\mu_j] \\ &-\frac{1}{2}e^{2(\psi-\nu)}\partial_i\omega\partial_j\omega - \partial_i\partial_j(\mu_i+\mu_j) + 2\partial_j\mu_i\partial_i\mu_j \\ &+\sum_k[\partial_i\partial_j\mu_k + \partial_i\mu_k\partial_j(\mu_k-\mu_i) - \partial_i\mu_j\partial_j\mu_k]\Bigg)\boldsymbol{e}^j\end{aligned} \tag{3.112}$$

3.2.4. The curvature scalar

Even though for a vacuum space-time the Ricci one-form would be sufficient to extract the relevant differential equations, it is in this case more convenient to use the Einstein one-form

$$\boldsymbol{G}^\alpha = \boldsymbol{R}^\alpha - \frac{1}{2}R\boldsymbol{e}^\alpha. \tag{3.113}$$

For that we need the curvature scalar

$$R = \boldsymbol{E}_\alpha \lrcorner \boldsymbol{R}^\alpha. \tag{3.114}$$

$$
\boldsymbol{E}_t \lrcorner \boldsymbol{R}^t = \sum_i e^{-2\mu_i} \left[\partial_i^2 \nu + \partial_i \nu \partial_i (\psi + \nu - 2\mu_i) - \frac{1}{2} e^{2(\psi - \nu)} (\partial_i \omega)^2 + \sum_j \partial_i \nu \partial_i \mu_j \right]
\tag{3.115}
$$

$$
\boldsymbol{E}_\phi \lrcorner \boldsymbol{R}^\phi = \sum_i e^{-2\mu_i} \left[\partial_i^2 \psi + \partial_i \psi \partial_i (\psi + \nu - 2\mu_i) + \frac{1}{2} e^{2(\psi - \nu)} (\partial_i \omega)^2 + \sum_j \partial_i \psi \partial_i \mu_j \right]
\tag{3.116}
$$

$$
\begin{aligned}
\sum_i \boldsymbol{E}_i \lrcorner \boldsymbol{R}^i = {} & \sum_{ij} e^{-2\mu_i} \left[\partial_i^2 \mu_j + \partial_i \mu_j \partial_i (\psi + \nu - 2\mu_i) + \sum_k \partial_i \mu_j \partial_i \mu_k \right] \\
& + \sum_i e^{-2\mu_i} \Big[\partial_i^2 (\psi + \nu) + (\partial_i \psi)^2 + (\partial_i \nu)^2 - 2 \partial_i \mu_i \partial_i (\psi + \nu) \\
& - \frac{1}{2} e^{2(\psi - \nu)} (\partial_i \omega)^2 - 2 \partial_i^2 \mu_i + 2 (\partial_i \mu_i)^2 + \sum_j \Big(\partial_i^2 \mu_j + \partial_i \mu_j \partial_i (\mu_j - 2\mu_i) \Big) \Big]
\end{aligned}
\tag{3.117}
$$

Putting those parts together gives

$$
\begin{aligned}
R = \sum_i e^{-2\mu_i} \Big[& 2 \partial_i^2 (\psi + \nu) + (\partial_i \psi)^2 + (\partial_i \nu)^2 + \partial_i (\psi + \nu) \partial_i (\psi + \nu - 4\mu_i) \\
& - 2 \partial_i^2 \mu_i + 2 (\partial_i \mu_i)^2 - \frac{1}{2} e^{2(\psi - \nu)} (\partial_i \omega)^2 \\
& + \sum_j \Big(\partial_i \mu_j \partial_i (2(\psi + \nu - 2\mu_i) + \mu_j) + 2 \partial_i^2 \mu_j + \sum_k \partial_i \mu_j \partial_i \mu_k \Big) \Big].
\end{aligned}
\tag{3.118}
$$

3.2.5. The differential equations

We get the differential equations for the metric components for vacuum space-time by

$$
\boldsymbol{R}_\alpha = 0, \qquad \boldsymbol{G}^\alpha = 0.
\tag{3.119}
$$

Which one is chosen depends on how convenient the resulting equations are. The tetrad components of the hidden indices will be denoted by parentheses. At this stage we will plug in 2 and 3 for the indices i and j and carry out the summations. The resulting equations are

$$R_{t(t)}=0 \quad \Rightarrow \quad \begin{cases} & e^{-2\mu_2}\left[\partial_2^2\nu+\partial_2\nu\partial_2(\psi+\nu-\mu_2+\mu_3)\right] \\ + & e^{-2\mu_3}\left[\partial_3^2\nu+\partial_3\nu\partial_3(\psi+\nu-\mu_3+\mu_2)\right] \\ = & \frac{1}{2}e^{2(\psi-\nu)}\left[e^{-2\mu_2}(\partial_2\omega)^2+e^{-2\mu_3}(\partial_3\omega)^2\right], \end{cases} \tag{3.120}$$

$$R_{\phi(\phi)}=0 \quad \Rightarrow \quad \begin{cases} & e^{-2\mu_2}\left[\partial_2^2\psi+\partial_2\psi\partial_2(\psi+\nu-\mu_2+\mu_3)\right] \\ + & e^{-2\mu_3}\left[\partial_3^2\psi+\partial_3\psi\partial_3(\psi+\nu-\mu_3+\mu_2)\right] \\ = & -\frac{1}{2}e^{2(\psi-\nu)}\left[e^{-2\mu_2}(\partial_2\omega)^2+e^{-2\mu_3}(\partial_3\omega)^2\right], \end{cases} \tag{3.121}$$

$$R_{t(\phi)}=0 \quad \Rightarrow \quad 0=\partial_2\left(e^{3\psi-\nu-\mu_2+\mu_3}\partial_2\omega\right)+\partial_3\left(e^{3\psi-\nu-\mu_3+\mu_2}\partial_3\omega\right), \tag{3.122}$$

$$R_{2(3)}=0 \quad \Rightarrow \quad \begin{cases} & \partial_2\partial_3(\psi+\nu)-\partial_2(\psi+\nu)\partial_3\mu_2-\partial_3(\psi+\nu)\partial_2\mu_3 \\ + & \partial_2\psi\partial_3\psi+\partial_2\nu\partial_3\nu=\frac{1}{2}e^{2(\psi-\nu)}\partial_2\omega\partial_3\omega, \end{cases} \tag{3.123}$$

$$G^2{}_{(2)}=0 \quad \Rightarrow \quad \begin{cases} & e^{-2\mu_3}\left[\partial_3^2(\psi+\nu)+\partial_3(\psi+\nu)\partial_3(\nu-\mu_3)+(\partial_3\psi)^2\right] \\ + & e^{-2\mu_2}\left[\partial_2\nu\partial_2(\psi+\mu_3)+\partial_2\psi\partial_2\mu_3\right] \\ = & -\frac{1}{4}e^{2(\psi-\nu)}\left[e^{-2\mu_2}(\partial_2\omega)^2-e^{-2\mu_3}(\partial_3\omega)^2\right], \end{cases} \tag{3.124}$$

$$G^3{}_{(3)}=0 \quad \Rightarrow \quad \begin{cases} & e^{-2\mu_2}\left[\partial_2^2(\psi+\nu)+\partial_2(\psi+\nu)\partial_2(\nu-\mu_2)+(\partial_2\psi)^2\right] \\ + & e^{-2\mu_3}\left[\partial_3\nu\partial_3(\psi+\mu_2)+\partial_3\psi\partial_3\mu_2\right] \\ = & \frac{1}{4}e^{2(\psi-\nu)}\left[e^{-2\mu_2}(\partial_2\omega)^2-e^{-2\mu_3}(\partial_3\omega)^2\right]. \end{cases} \tag{3.125}$$

With

$$\beta:=\psi+\nu, \tag{3.126}$$

equations (3.120) and (3.121), when multiplied by $e^{\beta+\mu_2+\mu_3}$ can be written as

$$\partial_2\left(e^{\beta+\mu_3-\mu_2}\partial_2\nu\right)+\partial_3\left(e^{\beta+\mu_2-\mu_3}\partial_3\nu\right)=\frac{1}{2}e^{3\psi-\nu}\left[e^{\mu_3-\mu_2}(\partial_2\omega)^2+e^{\mu_2-\mu_3}(\partial_3\omega)^2\right], \tag{3.127}$$

$$\partial_2\left(e^{\beta+\mu_3-\mu_2}\partial_2\psi\right)+\partial_3\left(e^{\beta+\mu_2-\mu_3}\partial_3\psi\right)=-\frac{1}{2}e^{3\psi-\nu}\left[e^{\mu_3-\mu_2}(\partial_2\omega)^2+e^{\mu_2-\mu_3}(\partial_3\omega)^2\right], \tag{3.128}$$

and their sum and difference, respectively, give

$$\partial_2\left[e^{\mu_3-\mu_2}\partial_2e^{\beta}\right]+\partial_3\left[e^{\mu_2-\mu_3}\partial_3e^{\beta}\right]=0, \tag{3.129}$$

$$\begin{aligned}\partial_2\left[e^{\beta+\mu_3-\mu_2}\partial_2(\psi-\nu)\right]+\partial_3\left[e^{\beta+\mu_2-\mu_3}\partial_3(\psi-\nu)\right]& \\ =-e^{3\psi-\nu}\left[e^{\mu_3-\mu_2}(\partial_2\omega)^2+e^{\mu_2-\mu_3}(\partial_3\omega)^2\right]&.\end{aligned} \tag{3.130}$$

While the sum of (3.124) and (3.125) is the same as (3.129), their difference yields

$$\begin{aligned}&4e^{\mu_3-\mu_2}\left(\partial_2\beta\partial_2\mu_3+\partial_2\psi\partial_2\nu\right)-4e^{\mu_2-\mu_3}\left(\partial_3\beta\partial_3\mu_2+\partial_3\psi\partial_3\nu\right)\\ =&2e^{-\beta}\left[\partial_2\left(e^{\mu_3-\mu_2}\partial_2e^{\beta}\right)-\partial_3\left(e^{\mu_2-\mu_3}\partial_3e^{\beta}\right)\right]\\ &-e^{2(\psi-\nu)}\left[e^{\mu_3-\mu_2}(\partial_2\omega)^2-e^{\mu_2-\mu_3}(\partial_3\omega)^2\right].\end{aligned} \tag{3.131}$$

We will now group the metric coefficients with the following new functions:

$$\Delta := e^{2(\mu_3-\mu_2)}, \qquad \chi := e^{\nu-\psi}. \tag{3.132}$$

With these definitions, the line element takes the form

$$\mathrm{d}s^2 = e^{\beta}\left[\chi \mathrm{d}t^2 - \frac{1}{\chi}(\mathrm{d}\phi - \omega \mathrm{d}t)^2\right] - \frac{e^{\mu_2+\mu_3}}{\sqrt{\Delta}}\left[(\mathrm{d}x^2)^2 + \Delta(\mathrm{d}x^3)^2\right]. \tag{3.133}$$

Remember that $\mu_2 - \mu_3$ is the gauge function, that means we can choose Δ as we wish.

3.2.6. The conjugate metric

If we undertake a coordinate transformation

$$t \mapsto \mathrm{i}\phi, \qquad \phi \mapsto -\mathrm{i}t \tag{3.134}$$

on (3.133), its Killing part transforms as

$$\begin{aligned}
\chi \mathrm{d}t^2 - \frac{1}{\chi}(\mathrm{d}\phi - \omega \mathrm{d}t)^2 \mapsto & -\chi \mathrm{d}\phi^2 + \frac{1}{\chi}\mathrm{d}t^2 + \frac{\omega^2}{\chi}\mathrm{d}\phi^2 + \frac{2\omega}{\chi}\mathrm{d}t\mathrm{d}\phi \\
&= \frac{1}{\chi}\mathrm{d}t^2 - \frac{\chi^2-\omega^2}{\chi}\left(\mathrm{d}\phi^2 - \frac{2\omega}{\chi^2-\omega^2}\mathrm{d}\phi\mathrm{d}t\right) \\
&= \frac{1}{\chi}\mathrm{d}t^2 + \frac{\omega^2}{\chi(\chi^2-\omega^2)}\mathrm{d}t^2 - \frac{\chi^2-\omega^2}{\chi}\left(\mathrm{d}\phi - \frac{\omega}{\chi^2-\omega^2}\mathrm{d}t\right)^2 \\
&= \frac{\chi}{\chi^2-\omega^2}\mathrm{d}t^2 - \frac{\chi^2-\omega^2}{\chi}\left(\mathrm{d}\phi - \frac{\omega}{\chi^2-\omega^2}\mathrm{d}t\right)^2 \\
&=: \tilde{\chi}\mathrm{d}t^2 - \frac{1}{\tilde{\chi}}\left(\mathrm{d}\phi - \tilde{\omega}\mathrm{d}t\right)^2,
\end{aligned} \tag{3.135}$$

where

$$\tilde{\chi} := \frac{\chi}{\chi^2-\omega^2}, \qquad \tilde{\omega} := \frac{\omega}{\chi^2-\omega^2}. \tag{3.136}$$

Thus if the equations are fulfilled by (χ, ω), they are also fulfilled by $(\tilde{\chi}, \tilde{\omega})$. Those pairs are called conjugate solutions.

3.2.7. The Papapetrou transformation

In this section we will perform a coordinate transformation that will help to show that a particular choice we will make below does not sacrifice generality. For this, we choose the gauge

$$\mu_2 = \mu_3 =: \mu, \qquad \Delta = 1, \tag{3.137}$$

which leaves us with the line element

$$\mathrm{d}s^2 = e^{\beta}\left[\chi \mathrm{d}t^2 - \frac{1}{\chi}(\mathrm{d}\phi - \omega \mathrm{d}t)^2\right] - e^{2\mu}\left[(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\right]. \tag{3.138}$$

In this gauge, equation (3.129) turns into the Laplace equation for e^{β}:

$$\sum_i \partial_i^2 e^{\beta} = 0. \tag{3.139}$$

According to theorem 1 (page 35), we can transform the non-Killing part of the metric to

$$\mathrm{d}\tilde{s}^2 = e^{2\tilde{\mu}} \left[\mathrm{d}\rho^2 + \mathrm{d}z^2\right], \tag{3.140}$$

where ρ and z are the new coordinates, if one of them satisfies the equation

$$0 = \sum_i \partial_i(\sqrt{g}\partial^i\rho) = \sum_i \partial_i(e^{2\mu}e^{-2\mu}\partial_i\rho) = \sum_i \partial_i\partial_i\rho. \tag{3.141}$$

For this, we have used that

$$\partial^i = \sum_j g^{ij}\partial_j = e^{-2\mu}\partial_i. \tag{3.142}$$

Therefore we only need a function that satisfies the Laplace equation. Fortunately, as we have seen earlier in this chapter, e^{β} is such a function. We then define

$$\rho := e^{\beta}, \tag{3.143}$$

and get the line element

$$\mathrm{d}s^2 = \rho\left[\chi\mathrm{d}t^2 - \frac{1}{\chi}(\mathrm{d}\phi - \omega\mathrm{d}t)^2\right] - e^{2\mu}\left[\mathrm{d}\rho^2 + \mathrm{d}z^2\right], \tag{3.144}$$

where we dropped the tilde on $\tilde{\mu}$ and μ, χ, and ω are now functions of ρ and z.

3.2.8. A choice of gauge

The symmetry choices we made in the beginning of this chapter are consistent with the Schwarzschild metric. The aim of this section is to choose the gauge in such a way, that by appropriate parameter choice we obtain the Schwarzschild metric in standard form. We will therefore assume properties we observe in the Schwarzschild metric and show later that these assumptions do not affect generality.

First we will choose as x^3 the polar angle θ with respect to the symmetry axis and give x^2 the name r, which is just a renaming and keeps generality.

The first assumption will be that we allow for an event horizon which is a null surface spanned by the Killing vectors ∂_t and ∂_ϕ, which is a natural choice, because we observe this in the Schwarzschild case. Also in Schwarzschild, the location of the horizon can be described by one equation

$$N(r,\theta) = 0. \tag{3.145}$$

The condition for the surface to be null, expresses itself through the equation

$$g^{ij}\partial_i N \partial_j N = e^{-2\mu_2}(\partial_r N)^2 + e^{-2\mu_3}(\partial_\theta N)^2 = 0, \tag{3.146}$$

or

$$e^{2(\mu_3-\mu_2)}(\partial_r N)^2 + (\partial_\theta N)^2 = \Delta(\partial_r N)^2 + (\partial_\theta N)^2 = 0. \tag{3.147}$$

Since the two squares must be positive, this results in the condition that at the horizon

$$\Delta = 0. \tag{3.148}$$

We use our gauge freedom to choose

$$\Delta = \Delta(r). \tag{3.149}$$

Because of the assumption that the horizon is a null surface spanned by ∂_ϕ and ∂_t, according to proposition 2 (page 26), the determinant of the Killing part of the metric must vanish on the horizon. This determinant is given by $-e^{2\beta}$, and therefore

$$e^{2\beta} = 0 \quad \text{on} \quad \Delta = 0. \tag{3.150}$$

We will now restrict e^β to be separable and of the form:

$$e^\beta =: \sqrt{\Delta} f(\theta). \tag{3.151}$$

Equation (3.129) now gives

$$\partial_r \left[\sqrt{\Delta}\partial_r\sqrt{\Delta}\right] + \frac{1}{f}\partial_\theta^2 f = 0, \tag{3.152}$$

and can be solved by

$$\Delta = k^2r^2 + br + h, \qquad f = P\sin(k\theta) + Q\cos(k\theta), \tag{3.153}$$

where k, b, h, P, and Q are constants. A choice which is compatible with the Schwarzschild solution is

$$k = 1, \quad b =: -2M, \quad h =: a^2, \quad P = 1, \quad Q = 0, \tag{3.154}$$

yielding

$$\Delta = r^2 - 2Mr + a^2, \qquad f = \sin\theta. \tag{3.155}$$

If we plug this into the horizon equation $\Delta = 0$, we see that the horizons are at

$$r_\pm = M \pm \sqrt{M^2 - a^2}. \tag{3.156}$$

From this we see that the parameter a measures, how much the metric differs from Schwarzschild, i.e. the "rotation" or non-stationarity of space-time. There are also two horizons. In the Schwarzschild case, the second horizon coincides with the singularity $r = 0$.

In Appendix A.2 we show that the previous choices do not restrict the generality of our metric. We will now use a new coordinate

$$\mu =: \cos\theta. \tag{3.157}$$

Equations (3.122) and (3.130) can be written as

$$\sin\theta\partial_r\left(\Delta e^{2(\psi-\nu)}\partial_r\omega\right)+\partial_\theta\left(\sin\theta e^{2(\psi-\nu)}\partial_\theta\omega\right)=0, \tag{3.158}$$

$$\sin\theta\partial_r\left(\Delta\partial_r(\psi-\nu)\right)+\partial_\theta\left(\sin\theta\partial_\theta(\psi-\nu)\right)=-e^{2(\psi-\nu)}\sin\theta\left[\Delta(\partial_r\omega)^2+(\partial_\theta\omega)^2\right]. \tag{3.159}$$

With

$$\partial_\theta=-\sin\theta\partial_\mu, \tag{3.160}$$

and the definition

$$\delta:=1-\mu^2=\sin^2\theta, \tag{3.161}$$

they become

$$\partial_r\left(\Delta e^{2(\psi-\nu)}\partial_r\omega\right)+\partial_\mu\left(\delta e^{2(\psi-\nu)}\partial_\mu\omega\right)=0, \tag{3.162}$$

$$\partial_r\left(\Delta\partial_r(\psi-\nu)\right)+\partial_\mu\left(\delta\partial_\mu(\psi-\nu)\right)=-e^{2(\psi-\nu)}\left[\Delta(\partial_r\omega)^2+\delta(\partial_\mu\omega)^2\right]. \tag{3.163}$$

With the former definition for $\chi:=e^{\nu-\psi}$ we get

$$\partial_r\left(\frac{\Delta}{\chi^2}\partial_r\omega\right)+\partial_\mu\left(\frac{\delta}{\chi^2}\partial_\mu\omega\right)=0 \tag{3.164}$$

$$\partial_r\left(\frac{\Delta}{\chi}\partial_r\chi\right)+\partial_\mu\left(\frac{\delta}{\chi}\partial_\mu\chi\right)=\frac{1}{\chi^2}\left[\Delta(\partial_r\omega)^2+\delta(\partial_\mu\omega)^2\right], \tag{3.165}$$

which can, by multiplying them with χ^3 and χ^2, respectively, also be written as

$$\chi\left[\partial_r(\Delta\partial_r\omega)+\partial_\mu(\delta\partial_\mu\omega)\right]=2\Delta\partial_r\chi\partial_r\omega+2\delta\partial_\mu\chi\partial_\mu\omega, \tag{3.166}$$

$$\chi\left[\partial_r(\Delta\partial_r\chi)+\partial_\mu(\delta\partial_\mu\chi)\right]=\Delta\left[(\partial_r\chi)^2+(\partial_r\omega)^2\right]+\delta\left[(\partial_\mu\chi)^2+(\partial_\mu\omega)^2\right]. \tag{3.167}$$

With the new functions

$$X:=\chi+\omega,\quad Y:=\chi-\omega, \tag{3.168}$$

these equations take the form

$$\begin{aligned}\frac{X+Y}{2}&\left[\partial_r(\Delta\partial_r X)+\partial_\mu(\delta\partial_\mu X)-\partial_r(\Delta\partial_r Y)-\partial_\mu(\delta\partial_\mu Y)\right]\\&=\Delta\left[(\partial_r X)^2-(\partial_r Y)^2\right]+\delta\left[(\partial_\mu X)^2-(\partial_\mu Y)^2\right]\end{aligned} \tag{3.169}$$

$$\begin{aligned}\frac{X+Y}{2}&\left[\partial_r(\Delta\partial_r X)+\partial_\mu(\delta\partial_\mu X)+\partial_r(\Delta\partial_r Y)+\partial_\mu(\delta\partial_\mu Y)\right]\\&=\Delta\left[(\partial_r X)^2+(\partial_r Y)^2\right]+\delta\left[(\partial_\mu X)^2+(\partial_\mu Y)^2\right]\end{aligned} \tag{3.170}$$

Adding and subtracting these equations gives

$$\frac{1}{2}(X+Y)\left[\partial_r\left(\Delta\partial_r X\right)+\partial_\mu\left(\delta\partial_\mu X\right)\right]=\Delta\left(\partial_r X\right)^2+\delta\left(\partial_\mu X\right)^2, \tag{3.171}$$

$$\frac{1}{2}(X+Y)\left[\partial_r\left(\Delta\partial_r Y\right)+\partial_\mu\left(\delta\partial_\mu Y\right)\right]=\Delta\left(\partial_r Y\right)^2+\delta\left(\partial_\mu Y\right)^2. \tag{3.172}$$

Now, equations (3.123) and (3.131) can be written as equations for $(\mu_2 + \mu_3)$ (details in appendix A.3).

$$-\frac{\mu}{\delta}\partial_r(\mu_2+\mu_3) + \frac{r-M}{\Delta}\partial_\mu(\mu_2+\mu_3) = \frac{2}{(X+Y)^2}\left(\partial_r X\partial_\mu Y + \partial_\mu X\partial_r Y\right) \tag{3.173}$$

$$\begin{aligned} &2(r-M)\partial_r(\mu_2+\mu_3) + 2\mu\partial_\mu(\mu_2+\mu_3) \\ &\qquad = \frac{4}{(X+Y)^2}\left(\Delta\partial_r X\partial_r Y - \delta\partial_\mu X\partial_\mu Y\right) - 3\frac{M^2-a^2}{\Delta} + \frac{1}{\delta} \end{aligned} \tag{3.174}$$

Once we know X and Y, we can simply integrate those equations to obtain $\mu_2 + \mu_3$.

3.2.9. The Ernst equation

Equation (3.164) allows for the introduction of a potential Φ:

$$\partial_r\Phi := \frac{\delta}{\chi^2}\partial_\mu\omega, \quad \partial_\mu\Phi := -\frac{\Delta}{\chi^2}\partial_r\omega. \tag{3.175}$$

The commutation of the partial derivatives demands

$$\partial_r\left(\frac{\chi^2}{\delta}\partial_r\Phi\right) + \partial_\mu\left(\frac{\chi^2}{\Delta}\partial_\mu\Phi\right) = 0. \tag{3.176}$$

We now introduce a new function

$$\Psi := \frac{\sqrt{\delta\Delta}}{\chi}. \tag{3.177}$$

First, remember that

$$\Delta = r^2 - 2Mr + a^2, \quad \delta = 1 - \mu^2, \tag{3.178}$$

and therefore

$$\partial_r^2\Delta = 2, \quad \partial_\mu^2\delta = -2. \tag{3.179}$$

We then get

$$\partial_r\left(\Delta\frac{\partial_r\chi}{\chi}\right) = \underbrace{\frac{1}{2}\partial_r^2\Delta}_{=1} \underbrace{-\partial_r\Delta\frac{\partial_r\Psi}{\Psi} - \Delta\frac{\partial_r^2\Psi}{\Psi}}_{-\frac{1}{\Psi}\partial_r(\Delta\partial_r\Psi)} + \Delta\frac{(\partial_r\Psi)^2}{\Psi^2}, \tag{3.180}$$

or

$$\Psi^2\partial_r\left(\Delta\frac{\partial_r\chi}{\chi}\right) = \Psi^2 - \Psi\partial_r\left(\Delta\partial_r\Psi\right) + \Delta\left(\partial_r\Psi\right)^2, \tag{3.181}$$

and similarly

$$\Psi^2\partial_\mu\left(\delta\frac{\partial_\mu\chi}{\chi}\right) = -\Psi^2 - \Psi\partial_\mu\left(\delta\partial_\mu\Psi\right) + \delta\left(\partial_\mu\Psi\right)^2. \tag{3.182}$$

With that, equation (3.165), multiplied by Ψ^2, can be written as

$$\Psi \left[\partial_r \left(\Delta \partial_r \Psi\right) + \partial_\mu \left(\delta \partial_\mu \Psi\right)\right] = \Delta \left[\left(\partial_r \Psi\right)^2 - \left(\partial_r \Phi\right)^2\right] + \delta \left[\left(\partial_\mu \Psi\right)^2 - \left(\partial_\mu \Phi\right)^2\right]. \tag{3.183}$$

Plugging Ψ into equation (3.176) and multiplying by Ψ^3 gives

$$\Psi \left[\partial_r \left(\Delta \partial_r \Phi\right) + \partial_\mu \left(\delta \partial_\mu \Phi\right)\right] = 2\Delta \partial_r \Psi \partial_r \Phi + 2\delta \partial_\mu \Psi \partial_\mu \Phi. \tag{3.184}$$

These two equations can be summarized by expressing Ψ and Φ as the real and imaginary part of a complex function

$$Z := \Psi + \mathrm{i}\Phi, \tag{3.185}$$

to the complex equation

$$\Re(Z) \left[\partial_r \left(\Delta \partial_r Z\right) + \partial_\mu \left(\delta \partial_\mu Z\right)\right] = \Delta \left(\partial_r Z\right)^2 + \delta \left(\partial_\mu Z\right)^2. \tag{3.186}$$

Introducing as a new function the idempotent transformation

$$Z =: \frac{E+1}{E-1}, \tag{3.187}$$

and using

$$\partial_i Z = -2\frac{\partial_i E}{(E-1)^2}, \qquad i \in \{r, \mu\}, \tag{3.188}$$

$$\partial_i \left(C \partial_i Z\right) = \frac{2}{(E-1)^2}\left[2C\frac{\left(\partial_i E\right)^2}{E-1} - \partial_i \left(C \partial_i E\right)\right], \qquad \text{where } C = C(i), \tag{3.189}$$

$$\Re(Z) = \frac{1}{2}\left(Z + Z^*\right) = \frac{EE^* - 1}{(E-1)(E^*-1)}, \tag{3.190}$$

leads us to the Ernst equation

$$\left(1 - EE^*\right)\left[\partial_r \left(\Delta \partial_r E\right) + \partial_\mu \left(\delta \partial_\mu E\right)\right] = -2E^* \left[\Delta \left(\partial_r E\right)^2 + \delta \left(\partial_\mu E\right)^2\right]. \tag{3.191}$$

To get the equation into a more symmetrical form (regarding $\delta = 1 - \mu^2$), we introduce the new coordinate

$$\eta = \frac{r - M}{\sqrt{M^2 - a^2}}, \qquad \partial_r = \frac{1}{\sqrt{M^2 - a^2}}\partial_\eta, \qquad \Delta = (M^2 - a^2)(\eta^2 - 1). \tag{3.192}$$

We then have

$$\begin{aligned} \left(1 - EE^*\right) &\left[\partial_\eta \left((\eta^2 - 1)\partial_\eta E\right) - \partial_\mu \left((\mu^2 - 1)\partial_\mu E\right)\right] \\ &= -2E^* \left[(\eta^2 - 1)\left(\partial_\eta E\right)^2 - (\mu^2 - 1)\left(\partial_\mu E\right)^2\right]. \end{aligned} \tag{3.193}$$

We have seen in section (3.2.6) that with solutions χ and ω there come functions $\tilde{\chi}$ and $\tilde{\omega}$ that are also solutions and are of the form

$$\tilde{\chi} = \frac{\chi}{\chi^2 - \omega^2}, \qquad \tilde{\omega} = \frac{\omega}{\chi^2 - \omega^2}. \tag{3.194}$$

Thus there are also conjugate functions $\tilde{Z}$, $\tilde{\Phi}$, $\tilde{\Psi}$, and $\tilde{E}$ arising in the same way from $\tilde{\chi}$ and $\tilde{\omega}$ as Z, Φ, Ψ, and E do from χ and ω, that also solve the Ernst equation.

$$\tilde{\Psi} := \frac{\sqrt{\Delta\delta}}{\tilde{\chi}} = e^{\nu+\psi}\frac{\chi^2-\omega^2}{\chi} = e^{2\nu} - \omega^2 e^{2\psi} \tag{3.195}$$

$$\partial_r\tilde{\Phi} := \frac{\delta}{\tilde{\chi}^2}\partial_\mu\tilde{\omega} = \frac{\tilde{\Psi}^2}{\Delta}\partial_\mu\tilde{\omega}, \quad \partial_\mu\tilde{\Phi} := -\frac{\Delta}{\tilde{\chi}^2}\partial_r\tilde{\omega} = -\frac{\tilde{\Psi}^2}{\delta}\partial_r\tilde{\omega} \tag{3.196}$$

$$\tilde{Z} := \tilde{\Psi} + \mathrm{i}\tilde{\Phi} =: \frac{\tilde{E}+1}{\tilde{E}-1} \tag{3.197}$$

In reverse, $\tilde{\Psi}$ and $\tilde{\Phi}$ can be obtained by

$$\tilde{\Psi} = \Re\tilde{Z} = \frac{\tilde{E}\tilde{E}^*-1}{|\tilde{E}-1|^2}, \qquad \tilde{\Phi} = \Im\tilde{Z} = \frac{\mathrm{i}(\tilde{E}-\tilde{E}^*)}{|\tilde{E}-1|^2}. \tag{3.198}$$

3.2.10. The Kerr metric

Solving the Ernst equation is everything we need to get an axisymmetric stationary metric, because we get χ and ω from the Ernst equation and then obtain $\mu_2+\mu_3$ from (3.123) and (3.131).

One solution of the Ernst equation is

$$\tilde{E} = -p\eta - \mathrm{i}q\mu \quad \text{with} \quad p^2+q^2=1, \tag{3.199}$$

p and q being real constants. We then get

$$\tilde{Z} = \tilde{\Psi} + \mathrm{i}\tilde{\Phi} = \frac{p\eta+\mathrm{i}q\mu-1}{p\eta+\mathrm{i}q\mu+1} \tag{3.200}$$

and

$$\tilde{\Psi} = \frac{p^2(\eta^2-1)+q^2(\mu^2-1)}{(p\eta+1)^2+q^2\mu^2} = \frac{\Delta - \frac{q^2}{p^2}(M^2-a^2)(1-\mu^2)}{\left(r-M+\frac{\sqrt{M^2-a^2}}{p}\right)^2 + \frac{q^2}{p^2}(M^2-a^2)\mu^2} \tag{3.201}$$

$$\tilde{\Phi} = \frac{2q\mu}{(p\eta+1)^2+q^2\mu^2} = \frac{2\frac{q}{p^2}(M^2-a^2)\mu}{\left(r-M+\frac{\sqrt{M^2-a^2}}{p}\right)^2 + \frac{q^2}{p^2}(M^2-a^2)\mu^2} \tag{3.202}$$

The coices

$$p = \frac{\sqrt{M^2-a^2}}{M}, \quad q = \frac{a}{M} \tag{3.203}$$

are consistent with the condition in (3.199), and with the definition

$$\rho^2 := r^2 + a^2\mu^2, \tag{3.204}$$

the potentials read

$$\tilde{\Psi} = \frac{\Delta - a^2\delta}{\rho^2}, \quad \tilde{\Phi} = \frac{2aM\mu}{\rho^2}. \tag{3.205}$$

From (3.196) we get

$$\partial_r\tilde{\Phi} = -\frac{4aMr\mu}{\rho^4} = \frac{\tilde{\Psi}^2}{\Delta}\partial_\mu\tilde{\omega} = \frac{\left(\Delta - a^2\delta\right)^2}{\rho^4\Delta}\partial_\mu\tilde{\omega}, \tag{3.206}$$

$$\partial_\mu\tilde{\Phi} = \frac{2aM}{\rho^4}\left(r^2 - a^2\mu^2\right) = -\frac{\tilde{\Psi}^2}{\delta}\partial_r\tilde{\omega} = -\frac{\left(\Delta - a^2\delta\right)^2}{\rho^4\delta}\partial_r\tilde{\omega}, \tag{3.207}$$

or

$$\partial_\mu\tilde{\omega} = -\frac{4aM\Delta\mu r}{\left(\Delta - a^2\delta\right)^2}, \quad \partial_r\tilde{\omega} = -\frac{2aM\delta\left(r^2 - a^2\mu^2\right)}{\left(\Delta - a^2\delta\right)^2}. \tag{3.208}$$

As can be checked by differentiation, a solution for these equations is

$$\tilde{\omega} = \frac{\omega}{\chi^2 - \omega^2} = \frac{2aM\delta r}{\Delta - a^2\delta}. \tag{3.209}$$

The second term comes from the definition of the conjugate solution. Now from (3.195) and (3.205) we get

$$\tilde{\Psi} = e^{2\psi}\left(\chi^2 - \omega^2\right) = e^{2\nu} - \omega^2 e^{2\psi} = \frac{\Delta - a^2\delta}{\rho^2}, \tag{3.210}$$

thus

$$\omega = \frac{2aM\delta r}{\Delta - a^2\delta}\left(\chi^2 - \omega^2\right) = \frac{2aM\delta r}{\rho^2}e^{-2\psi}. \tag{3.211}$$

We now use the definition of $\beta = \nu + \psi$ to obtain

$$\frac{\Delta - a^2\delta}{\rho^2}e^{2\psi} \overset{(3.210)}{=} e^{2\beta} - \omega^2 e^{4\psi} \overset{(3.151)(3.155)}{=} \frac{\delta}{\rho^4}\left[\Delta\rho^4 - 4a^2M^2\delta r^2\right], \tag{3.212}$$

and therefore

$$e^{2\psi} = \frac{\delta}{\rho^2\left(\Delta - a^2\delta\right)}\left[\Delta\rho^4 - 4a^2M^2\delta r^2\right], \tag{3.213}$$

$$\omega = 2aMr\frac{\Delta - a^2\delta}{\Delta\rho^4 - 4a^2M^2\delta r^2}. \tag{3.214}$$

With the new function

$$\Sigma^2 := \left(r^2 + a^2\right)^2 - a^2\Delta\delta, \tag{3.215}$$

we have

$$\Sigma^2\left(\Delta - a^2\delta\right) = \rho^4\Delta - 4a^2M^2\delta r^2. \tag{3.216}$$

Using that, ω and $e^{2\psi}$ take the simple forms

$$e^{2\psi} = \frac{\delta\Sigma^2}{\rho^2}, \quad \omega = \frac{2aMr}{\Sigma^2}. \tag{3.217}$$

Also, the remaining part of the Killing part of the metric can now be calculated:

$$e^{2\nu} = e^{2\beta - 2\psi} = \frac{\rho^2 \Delta}{\Sigma^2}, \tag{3.218}$$

$$\chi = e^{\nu - \psi} = \frac{\rho^2}{\Sigma^2} \sqrt{\frac{\Delta}{\delta}}. \tag{3.219}$$

That leaves only $\mu_2 + \mu_3$ left to determine. We will use the identity

$$\left[\left(r^2 + a^2\right) \mp a\sqrt{\delta\Delta}\right] \left[\sqrt{\Delta} \pm a\sqrt{\delta}\right] = \rho^2 \sqrt{\Delta} \pm 2aMr\sqrt{\delta}. \tag{3.220}$$

Switching back to X and Y, as given in (3.168), where we use $\mathscr{X}$ for both X and Y, X corresponding to the upper sign and Y to the lower, we have

$$\mathscr{X} = \chi \pm \omega = \frac{\rho^2 \sqrt{\Delta} \pm 2aMr\sqrt{\delta}}{\Sigma^2 \sqrt{\delta}} \overset{(3.220)}{=} \frac{\left[(r^2 + a^2) \mp a\sqrt{\Delta\delta}\right] \left[\sqrt{\Delta} \pm a\sqrt{\delta}\right]}{\Sigma^2 \sqrt{\delta}}. \tag{3.221}$$

With

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \delta\Delta = \left[(r^2 + a^2) + a\sqrt{\delta\Delta}\right] \left[(r^2 + a^2) - a\sqrt{\delta\Delta}\right], \tag{3.222}$$

those cancel to

$$\mathscr{X} = \frac{\sqrt{\Delta} \pm a\sqrt{\delta}}{\left[(r^2 + a^2) \pm a\sqrt{\delta\Delta}\right] \sqrt{\delta}}. \tag{3.223}$$

Their derivatives with respect to r and μ are

$$\partial_r \mathscr{X} = \frac{(r - M)\rho^2 - 2r\sqrt{\Delta}\left(\sqrt{\Delta} \pm a\sqrt{\delta}\right)}{\sqrt{\delta\Delta}\left[(r^2 + a^2) \pm a\sqrt{\delta\Delta}\right]^2}, \tag{3.224}$$

$$\partial_\mu \mathscr{X} = \frac{\mu\sqrt{\Delta}\left[r^2 + a^2(1 + \delta) \pm 2a\sqrt{\delta\Delta}\right]}{\left[(r^2 + a^2) \pm a\sqrt{\delta\Delta}\right]^2 \delta^{\frac{3}{2}}}. \tag{3.225}$$

We can now plug those into equations (3.173) and (3.174) for the calculation of $(\mu_2 + \mu_3)$.

$$-\frac{\mu}{\delta}\partial_r(\mu_2 + \mu_3) + \frac{r - M}{\Delta}\partial_\mu(\mu_2 + \mu_3) = \frac{\mu}{\rho^2 \delta\Delta}\left[(r - M)(\rho^2 + 2a^2\delta) - 2r\Delta\right] \tag{3.226}$$

$$2(r - M)\partial_r(\mu_2 + \mu_3) + 2\mu\partial_\mu(\mu_2 + \mu_3) = 4 - \frac{2(r - M)^2}{\Delta} - \frac{4rM}{\rho^2} \tag{3.227}$$

These equations can be solved by elimination and integration and yield

$$e^{\mu_2 + \mu_3} = \frac{\rho^2}{\sqrt{\Delta}}. \tag{3.228}$$

With $e^{\mu_3-\mu_2} = \sqrt{\Delta}$ we have determined all metric components

$$e^{2\nu} = \frac{\rho^2\Delta}{\Sigma^2}, \quad e^{2\psi} = \frac{\delta\Sigma^2}{\rho^2}, \quad \omega = \frac{2aMr}{\Sigma^2}, \quad e^{2\mu_2} = \frac{\rho^2}{\Delta}, \quad e^{2\mu_3} = \rho^2, \tag{3.229}$$

which compose the line element of the Kerr metric

$$\mathrm{d}s^2 = \frac{\rho^2\Delta}{\Sigma^2}\mathrm{d}t^2 - \frac{\Sigma^2\sin^2\theta}{\rho^2}\left(\mathrm{d}\phi - \frac{2aMr}{\Sigma^2}\mathrm{d}t\right)^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2, \tag{3.230}$$

where

$$\Delta = r^2 - 2Mr + a^2, \tag{3.231}$$
$$\rho^2 = r^2 + a^2\cos^2\theta, \tag{3.232}$$
$$\Sigma^2 = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \tag{3.233}$$

and a and M are constant parameters.

4. Outlook

In this section, an outlook is provided on how the content of this thesis could be used in the future. The goal is a general relativistic approximation for slow moving neutrons for experiments on earth's surface. One procedure to achieve this might be the following:

- Calculate the spin coefficients for the Kerr metric.
- Transform the 2-spinor version of the Dirac equation in Weyl representation into the 4-spinor version in Dirac representation.
- Apply the Foldy-Wouthuysen transformation.

Inspired by [6], if we take the 4-spinor of the Dirac equation to be

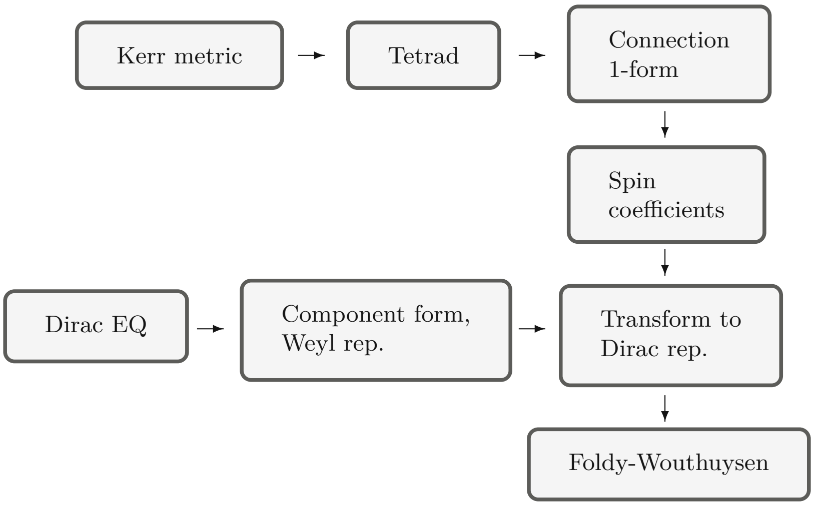
$$\Psi = \begin{pmatrix} \psi_R \\ \chi^{R'} \end{pmatrix}, \tag{4.1}$$

the γ-matrices in Weyl representation take the form

$$\gamma_a = \sqrt{2} \begin{pmatrix} 0 & \varepsilon_{AS}\varepsilon_{A'R'} \\ \varepsilon_{A'}{}^{S'}\varepsilon_{A}{}^{R} & 0 \end{pmatrix}. \tag{4.2}$$

The Dirac equation then reads

$$\begin{aligned} \mathrm{i}\nabla_{AB'}\chi^{B'} &= \frac{m}{\sqrt{2}}\psi_A \\ \mathrm{i}\nabla^{A'B}\psi_B &= \frac{m}{\sqrt{2}}\chi^{A'} \end{aligned} \tag{4.3}$$



To start calculating, (2.4.4) can be used to switch to component notation, which also brings spin coefficients into play via the covariant derivative (2.4.6). The spin coefficients can be related to the connection 1-forms ((3.75-3.78) with (3.229)) via (2.166).

The coordinate form of the 2-spinor version of the Dirac equation can then be rewritten in 4-spinor notation and then be transformed to Dirac representation. This shapes the equation into the right form to perform a Foldy-Wouthuysen transformation (2.2.5). If a Schrödinger equation with the Newton potential term and additional correction terms is desired, the Dirac equation can be expanded not only in $\frac{p}{mc}$, but also, since we assume the distance from the origin r large compared to the traveling distance, in the other dimensionless quantities $\frac{GM}{rc^2}$ and $\frac{\hbar}{mcr}$, with c being the speed of light, m the neutron mass, G the gravitational constant, and $\hbar$ the reduced Planck constant. These quantities can then be compared with $\frac{p}{mc}$ to determine to which order of magnitude the individual terms are relevant.

4.1. Things to consider

4.1.1. Laboratory observer

The calculations in section 3 are from the perspective of an asymptotic inertial observer. Since most experiments are performed from the surface of the rotating earth, it might be advantageous to transform into that perspective first.

4.2. No Birkhoff theorem

While the Schwarzschild metric perfectly describes a spherical resting earth according to (2.8) and further the Birkhoff theorem (section 2.8 in combination with [3]), there is no Birkhoff theorem for general axially symmetric space-times. That means that the metric outside of a rotating earth might not be accurately described by the Kerr metric. In this thesis we use the Kerr metric to describe a rotating earth since $a << M$, but the validity of this is not proved here and further work is required.

A. Supplementation to the main text

A.1. Existence of an inverse on the support of an operator in Hilbert space

Consider operators P with eigenbasis $|p_i>$, and Q on Hilbert space, where

$$P = \sum_i p_i |p_i >< p_i|, \quad Q = \sum_j \frac{1}{p_j} |p_j\rangle\langle p_j|. \tag{A.1}$$

j in this case runs over the indices i, where $p_i \neq 0$. The summation is to be seen as abstract and can also be an integral in the case of a continuous spectrum. Then on the subspace of the support of P,

$$PQ = QP = \sum_{ij} |p_i >< p_i | p_j >< p_j| = \sum_i |p_i >< p_i| = \mathbb{1}. \tag{A.2}$$

A.2. Proof of generality despite the taken choices

In this section we will prove that the choices taken during the derivation of the Ernst equation in section 3.2.8 do not limit generality, following [2]. This will be done by showing that there is a coordinate transformation that brings the metric under the effect of our choices (3.151) into the Papapetrou form (3.144), which we previously obtained via a coordinate transformation of the most general metric for axially symmetric stationary space-times generated by rotating bodies.

The metric we arrived at in section 3.2.8 is

$$\mathrm{d}s^2 = e^\beta \left[\chi \mathrm{d}t^2 - \frac{1}{\chi}(\mathrm{d}\phi - \omega \mathrm{d}t)^2\right] - \frac{e^{\mu_2+\mu_3}}{\sqrt{\Delta}} \left[\mathrm{d}r^2 + \Delta \mathrm{d}\theta^2\right], \tag{A.3}$$

with

$$e^\beta = \sqrt{\Delta}\sin\theta, \quad \Delta = r^2 - 2Mr + a^2. \tag{A.4}$$

The Papapetrou form is

$$\mathrm{d}s^2 = \rho \left[\chi \mathrm{d}t^2 - \frac{1}{\chi}(\mathrm{d}\phi - \omega \mathrm{d}t)^2\right] - e^{2\mu} \left[\mathrm{d}\rho^2 + \mathrm{d}z^2\right]. \tag{A.5}$$

Keep in mind that the unknown functions $e^{\mu_2+\mu_3}$ and $e^{2\mu}$, despite their similar origin, are not the same function, because we used different gauge functions Δ in both cases. Thus we can absorb a common factor of $\mathrm{d}r^2$ and $\mathrm{d}\theta^2$ into these functions.

Consider the coordinate transformation

$$\rho = e^\beta = \sqrt{\Delta}\sin\theta, \quad z = (r - M)\cos\theta. \tag{A.6}$$

Notice that because we chose $\rho = e^\beta$, the (t, ϕ) parts of the metric are already identical, meaning we can restrict our attention to the (r, θ) part. For convenience we will perform the inverse transformation from (ρ, z) to (r, θ).

It is useful to note that

$$\Delta' = 2(r-M), \tag{A.7}$$

$$(r-M)^2 = r^2 - 2Mr + M^2 =: \Delta + \tilde{M}^2. \tag{A.8}$$

Then the basis forms and their squares are

$$d\rho = \frac{r-M}{\sqrt{\Delta}} \sin\theta dr + \sqrt{\Delta}\cos\theta d\theta, \tag{A.9}$$

$$dz = \cos\theta dr - (r-M)\sin\theta d\theta, \tag{A.10}$$

$$d\rho^2 = \frac{\Delta + \tilde{M}^2}{\Delta} \sin^2\theta dr^2 + \Delta\cos^2\theta d\theta^2 + 2(r-M)\sin\theta\cos\theta dr d\theta, \tag{A.11}$$

$$dz^2 = \cos^2\theta dr^2 + \left(\Delta + \tilde{M}^2\right)\sin^2\theta d\theta^2 - 2(r-M)\sin\theta\cos\theta dr d\theta, \tag{A.12}$$

leading to

$$\begin{aligned} d\rho^2 + dz^2 &= \left[\left(1 + \frac{\tilde{M}^2}{\Delta}\right)\sin^2\theta + \cos^2\theta\right] dr^2 + \Delta\left[\left(1 + \frac{\tilde{M}^2}{\Delta}\right)\sin^2\theta + \cos^2\theta\right] d\theta^2 \\ &= \left[1 + \frac{\tilde{M}^2}{\Delta}\sin^2\theta\right]\left[dr^2 + \Delta d\theta^2\right]. \end{aligned} \tag{A.13}$$

The prefactor can be absorbed into $e^{2\mu}$ and the equivalence is established.

A.3. Calculation of the equations for $(\mu_2 + \mu_3)$

This section is a supplement to section 3.2.8 after the introduction of X and Y in order to get to equations (3.173) and (3.174) from equations (3.123) and (3.131).
We will need the following relations throughout the calculation:

$$\chi = e^{\nu-\psi}, \quad e^{\beta} = e^{\nu+\psi} = \sqrt{\Delta}\sin\theta, \quad \Delta = e^{2(\mu_3-\mu_2)} = r^2 - 2Mr + a^2, \tag{A.14}$$

as well as

$$\mu = \cos\theta, \quad \partial_\theta = -\sin\theta\partial_\mu, \quad \delta = \sin^2\theta = 1 - \mu^2. \tag{A.15}$$

Also we will define

$$\tau := \mu_2 + \mu_3. \tag{A.16}$$

The derivatives of the relevant functions can be written as

$$\partial_r\psi = \frac{1}{2}\partial_r\left[(\nu+\psi)-(\nu-\psi)\right] = \frac{1}{2}\left[\frac{\partial_r e^{\beta}}{e^{\beta}} - \frac{\partial_r\chi}{\chi}\right] = \frac{1}{2}\left[\frac{(r-M)}{\Delta} - \frac{\partial_r\chi}{\chi}\right], \tag{A.17}$$

$$\partial_r\nu = \frac{1}{2}\partial_r\left[(\nu+\psi)+(\nu-\psi)\right] = \frac{1}{2}\left[\frac{\partial_r e^{\beta}}{e^{\beta}} + \frac{\partial_r\chi}{\chi}\right] = \frac{1}{2}\left[\frac{(r-M)}{\Delta} + \frac{\partial_r\chi}{\chi}\right], \tag{A.18}$$

$$\partial_\theta\psi = \frac{1}{2}\partial_\theta\left[(\nu+\psi)-(\nu-\psi)\right] = \frac{1}{2}\left[\frac{\partial_\theta e^{\beta}}{e^{\beta}} - \frac{\partial_\theta\chi}{\chi}\right] = \frac{1}{2}\left[\frac{\cos\theta}{\sin\theta} - \frac{\partial_\theta\chi}{\chi}\right], \tag{A.19}$$

$$\partial_\theta\nu = \frac{1}{2}\partial_\theta\left[(\nu+\psi)+(\nu-\psi)\right] = \frac{1}{2}\left[\frac{\partial_\theta e^{\beta}}{e^{\beta}} + \frac{\partial_\theta\chi}{\chi}\right] = \frac{1}{2}\left[\frac{\cos\theta}{\sin\theta} + \frac{\partial_\theta\chi}{\chi}\right], \tag{A.20}$$

$$\partial_r\mu_3 = \frac{1}{2}\partial_r\left[(\mu_3+\mu_2)+(\mu_3-\mu_2)\right] = \frac{1}{2}\left[\partial_r\tau + \frac{\partial_r\sqrt{\Delta}}{\sqrt{\Delta}}\right] = \frac{1}{2}\left[\partial_r\tau + \frac{(r-M)}{\Delta}\right], \tag{A.21}$$

$$\partial_\theta\mu_2 = \frac{1}{2}\partial_\theta\left[(\mu_3+\mu_2)-(\mu_3-\mu_2)\right] = \frac{1}{2}\left[\partial_\theta\tau - \frac{\partial_\theta\sqrt{\Delta}}{\sqrt{\Delta}}\right] = \frac{1}{2}\partial_\theta\tau. \tag{A.22}$$

We first examine equation (3.123),

$$\begin{aligned}\partial_r\partial_\theta(\psi+\nu) - \partial_r(\psi+\nu)\partial_\theta\mu_2 - \partial_\theta(\psi+\nu)\partial_r\mu_3 + \partial_r\psi\partial_\theta\psi + \partial_r\nu\partial_\theta\nu&\\ = \frac{1}{2}e^{2(\psi-\nu)}\partial_r\omega\partial_\theta\omega&,\end{aligned} \tag{A.23}$$

$$\begin{aligned}\partial_r\frac{\partial_\theta e^{\beta}}{e^{\beta}} - \frac{\partial_r e^{\beta}}{2e^{\beta}}\partial_\theta\tau - \frac{\partial_\theta e^{\beta}}{2e^{\beta}}\left[\partial_r\tau + \frac{r-M}{\Delta}\right] + \frac{1}{2}\left[\frac{\cos\theta}{\sin\theta}\frac{r-M}{\Delta} + \frac{\partial_\theta\chi\partial_r\chi}{\chi^2}\right]&\\ = \frac{1}{2\chi^2}\partial_r\omega\partial_\theta\omega&,\end{aligned} \tag{A.24}$$

$$-\frac{r-M}{2\Delta}\partial_\theta\tau - \frac{\cos\theta}{2\sin\theta}\partial_r\tau = \frac{1}{2\chi^2}\left[\partial_r\omega\partial_\theta\omega - \partial_r\chi\partial_\theta\chi\right], \tag{A.25}$$

$$\frac{r-M}{\Delta}\partial_\mu\tau - \frac{\mu}{\delta}\partial_r\tau = -\frac{1}{\chi^2}\left[\partial_r\omega\partial_\mu\omega - \partial_r\chi\partial_\mu\chi\right]. \tag{A.26}$$

Then we bring (3.131) into a similar form,

$$\begin{aligned} &4\sqrt{\Delta}\left(\partial_r\beta\partial_r\mu_3+\partial_r\psi\partial_r\nu\right)-\frac{4}{\sqrt{\Delta}}\left(\partial_\theta\beta\partial_\theta\mu_2+\partial_\theta\psi\partial_\theta\nu\right)\\ &=2e^{-\beta}\left[\partial_r\left(\sqrt{\Delta}\partial_r e^{\beta}\right)-\partial_\theta\left(\frac{1}{\sqrt{\Delta}}\partial_\theta e^{\beta}\right)\right]-\frac{1}{\chi^2}\left[\sqrt{\Delta}\left(\partial_r\omega\right)^2-\frac{1}{\sqrt{\Delta}}\left(\partial_\theta\omega\right)^2\right], \end{aligned} \tag{A.27}$$

$$\begin{aligned} &2\frac{r-M}{\sqrt{\Delta}}\partial_r\tau+3\frac{(r-M)^2}{\Delta\sqrt{\Delta}}-\sqrt{\Delta}\frac{(\partial_r\chi)^2}{\chi^2}-\frac{2}{\sqrt{\Delta}}\frac{\cos\theta}{\sin\theta}\partial_\theta\tau-\frac{1}{\sqrt{\Delta}}\frac{\cos^2\theta}{\sin^2\theta}+\frac{1}{\sqrt{\Delta}}\frac{(\partial_\theta\chi)^2}{\chi^2}\\ &=\frac{4}{\sqrt{\Delta}}-\frac{1}{\chi^2}\left[\sqrt{\Delta}(\partial_r\omega)^2-\frac{1}{\sqrt{\Delta}}(\partial_\theta\omega)^2\right], \end{aligned} \tag{A.28}$$

$$\begin{aligned} &2(r-M)\partial_r\tau-2\frac{\cos\theta}{\sin\theta}\partial_\theta\tau+3\frac{(r-M)^2}{\Delta}-3-\frac{\cos^2\theta}{\sin^2\theta}-1\\ &=-\frac{1}{\chi^2}\left[\Delta\left[(\partial_r\omega)^2-(\partial_r\chi)^2\right]-\left[(\partial_\theta\omega)^2-(\partial_\theta\chi)^2\right]\right], \end{aligned} \tag{A.29}$$

$$\begin{aligned} &2(r-M)\partial_r\tau+2\mu\partial_\mu\tau\\ &=-\frac{1}{\chi^2}\left[\Delta\left[(\partial_r\omega)^2-(\partial_r\chi)^2\right]-\delta\left[(\partial_\mu\omega)^2-(\partial_\mu\chi)^2\right]\right]-3\frac{M^2-a^2}{\Delta}+\frac{1}{\delta}. \end{aligned} \tag{A.30}$$

If we now use

$$X=\chi+\omega, \qquad Y=\chi-\omega, \tag{A.31}$$

in the two equations, we get the desired equations

$$-\frac{\mu}{\delta}\partial_r(\mu_2+\mu_3)+\frac{r-M}{\Delta}\partial_\mu(\mu_2+\mu_3)=\frac{2}{(X+Y)^2}\left(\partial_r X\partial_\mu Y+\partial_\mu X\partial_r Y\right), \tag{A.32}$$

$$\begin{aligned} &2(r-M)\partial_r(\mu_2+\mu_3)+2\mu\partial_\mu(\mu_2+\mu_3)\\ &\qquad=\frac{4}{(X+Y)^2}\left(\Delta\partial_r X\partial_r Y-\delta\partial_\mu X\partial_\mu Y\right)-3\frac{M^2-a^2}{\Delta}+\frac{1}{\delta}. \end{aligned} \tag{A.33}$$

B. Useful proofs

This section supplements the main text with useful and interesting proofs.

B.1. Spinors

B.1.1. Symmetry decomposition of a generic spinor

Proposition 4. *Every spinor can be decomposed into a sum of products of a symmetric spinor and ε-spinors.*

Proof. First it is helpful to introduce some definitions for clumped indices:

$$\mathscr{A} := A_1 ... A_n, \tag{B.1}$$

$$\mathscr{A}^{ij} := A_i ... A_j, \tag{B.2}$$

$$\mathscr{A}_i := A_1 ... A_{i-1} A_{i+1} ... A_n, \tag{B.3}$$

and similar for more than one lower index of clumped indices.

If $\{\}$ is used on indices it is a symmetrisation, only that the indices inside clumped indices are also affected.
We will start with the following two identities.

$$\chi_{\{\mathscr{A}\}\mathscr{D}} = \frac{1}{n} \sum_{j=1}^{n} \chi_{\{\mathscr{A}_j\}A_j\mathscr{D}} \tag{B.4}$$

$$\chi_{\{\mathscr{A}_n\}A_n\mathscr{D}} - \chi_{\{\mathscr{A}_j\}A_j\mathscr{D}} \overset{(2.137)}{=} \varepsilon_{A_j A_n} \chi_{\{\mathscr{A}_{jn}\, X\}}{}^{X}{}_{\mathscr{D}} \tag{B.5}$$

If we now plug (B.5) into (B.4) for every term except the $j = n$ one, we obtain

$$\chi_{\{\mathscr{A}\}\mathscr{D}} = \chi_{\{\mathscr{A}_n\}A_n\mathscr{D}} + \frac{1}{n} \sum_{j=1}^{n-1} \varepsilon_{A_n A_j} \chi_{\{\mathscr{A}_{jn}\, X\}}{}^{X}{}_{\mathscr{D}}, \tag{B.6}$$

$$\chi_{\{\mathscr{A}_n\}A_n\mathscr{D}} = \chi_{\{\mathscr{A}\}\mathscr{D}} - \frac{1}{n} \sum_{j=1}^{n-1} \varepsilon_{A_n A_j} \chi_{\{\mathscr{A}_{jn}\, X\}}{}^{X}{}_{\mathscr{D}}. \tag{B.7}$$

We can now start with a general tensor where we assume the first index to be the symmetrised index block of the left hand side of (B.7), which leaves us with a spinor with two symmetrised indices, and spinors with less indices than the original one in a product with an ε-spinor. This procedure can be repeated with the former spinor until it leaves us with a symmetric spinor and a sum of ε-products.

$$
\begin{aligned}
\chi_{\mathscr{A}} = \chi_{\{A_1\}A_2A_3...A_n} &\overset{(B.7)}{=} \chi_{\{A_1A_2\}A_3...A_n} - \frac{1}{2}\sum_{j=1}^{2-1} \varepsilon_{A_2A_j}\chi_{X}{}^{X}{}_{A_3...A_n} \\
\chi_{\{A_1A_2\}A_3A_4...A_n} &\overset{(B.7)}{=} \chi_{\{A_1A_2A_3\}...A_n} - \frac{1}{3}\sum_{j=1}^{3-1} \varepsilon_{A_3A_j}\chi_{\{\mathscr{A}_j^{(1)(2)}X\}}{}^{X}{}_{A_4...A_n} \\
&\vdots
\end{aligned}
\tag{B.8}
$$

Written compactly, this reads

$$
\chi_{\mathscr{A}} = \chi_{\{\mathscr{A}\}} - \sum_{i=2}^{n}\frac{1}{i}\sum_{j=1}^{i-1} \varepsilon_{A_iA_j}\chi_{\{\mathscr{A}^{1(i-1)}{}_j X\}}{}^{X}{}_{\mathscr{A}^{(i+1)n}}. \tag{B.9}
$$

This can be repeated with the spinors lower in index number until there are only symmetric ones left and the original spinor is fully decomposed into symmetric spinors. □

B.1.2. Zero-valued contractions of symmetric spinors

The proofs in this section follow [6]

Proposition 5. *At every point in space-time,*

$$
\chi^{\mathscr{A}}{}_{\mathscr{B}_1...\mathscr{B}_k}\xi^{\mathscr{B}_1}...\xi^{\mathscr{B}_k} = 0 \quad \forall\, \xi^{\mathscr{B}} \in \mathfrak{S}^{\mathscr{B}} \quad \Leftrightarrow \quad \chi^{\mathscr{A}}{}_{(\mathscr{B}_1...\mathscr{B}_k)} = 0. \tag{B.10}
$$

Proof. The backward direction is clear since $\xi^{\mathscr{B}_1}...\xi^{\mathscr{B}_k}$ is symmetric in $\mathscr{B}_1...\mathscr{B}_k$ and therefore

$$
\chi^{\mathscr{A}}{}_{\mathscr{B}_1...\mathscr{B}_k}\xi^{\mathscr{B}_1}...\xi^{\mathscr{B}_k} = \chi^{\mathscr{A}}{}_{(\mathscr{B}_1...\mathscr{B}_k)}\xi^{\mathscr{B}_1}...\xi^{\mathscr{B}_k}. \tag{B.11}
$$

For the forward direction set $\xi^{\mathscr{B}} = \eta^{\mathscr{B}} + \lambda\zeta^{\mathscr{B}}$. Then

$$
\begin{aligned}
\chi^{\mathscr{A}}{}_{\mathscr{B}_1...\mathscr{B}_k}\xi^{\mathscr{B}_1}...\xi^{\mathscr{B}_k} =& \chi^{\mathscr{A}}{}_{(\mathscr{B}_1...\mathscr{B}_k)}\eta^{\mathscr{B}_1}...\eta^{\mathscr{B}_k} + N\lambda\chi^{\mathscr{A}}{}_{(\mathscr{B}_1...\mathscr{B}_k)}\zeta^{\mathscr{B}_1}\eta^{\mathscr{B}_2}...\eta^{\mathscr{B}_k} \\
&+...+\lambda^k\chi^{\mathscr{A}}{}_{(\mathscr{B}_1...\mathscr{B}_k)}\zeta^{\mathscr{B}_1}...\zeta^{\mathscr{B}_k}.
\end{aligned}
\tag{B.12}
$$

Since the left hand side vanishes and λ is arbitrary, every term on the right hand side must vanish independently. But for arbitrary $\xi^{\mathscr{B}}$, for $\chi^{\mathscr{A}}{}_{(\mathscr{B}_1...\mathscr{B}_k)}\zeta^{\mathscr{B}_1}\eta^{\mathscr{B}_2}...\eta^{\mathscr{B}_k}$ to vanish, $\chi^{\mathscr{A}}{}_{(\mathscr{B}_1...\mathscr{B}_k)}\eta^{\mathscr{B}_2}...\eta^{\mathscr{B}_k}$ must vanish. Since $\eta^{\mathscr{B}}$ is arbitrary, this argument can be repeated until the result is obtained. □

Proposition 6. *At every point in space-time,*

$$
\psi_{(\mathscr{A}_1...\mathscr{A}_r}{}^{\mathscr{B}}\phi_{\mathscr{A}_{r+1}...\mathscr{A}_{r+s})}{}^{\mathscr{C}} = 0 \quad \Leftrightarrow \quad \psi_{(\mathscr{A}_1...\mathscr{A}_r)}{}^{\mathscr{B}} = 0 \quad \vee \quad \phi_{(\mathscr{A}_1...\mathscr{A}_s)}{}^{\mathscr{C}} = 0. \tag{B.13}
$$

Proof. According to proposition 5, the left hand side is equivalent to

$$\psi_{\mathscr{A}_1 \dots \mathscr{A}_r}{}^{\mathscr{B}} \xi^{\mathscr{A}_1} \dots \xi^{\mathscr{A}_r} \phi_{\mathscr{A}_{r+1} \dots \mathscr{A}_{r+s}}{}^{\mathscr{C}} \xi^{\mathscr{A}_{r+1}} \dots \xi^{\mathscr{A}_{r+s}}. \tag{B.14}$$

But that means that

$$\psi_{\mathscr{A}_1 \dots \mathscr{A}_r}{}^{\mathscr{B}} \xi^{\mathscr{A}_1} \dots \xi^{\mathscr{A}_r} = 0 \quad \vee \quad \phi_{\mathscr{A}_1 \dots \mathscr{A}_s}{}^{\mathscr{C}} \xi^{\mathscr{A}_1} \dots \xi^{\mathscr{A}_s} = 0. \tag{B.15}$$

Applying proposition 5 to either of those equations proves the claim. □

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