TECHNISCHE UNIVERSITÄT WIEN

Master Thesis

Analysis of the optimal lockdown during the COVID-19 pandemic based on a SIRD model

carried out for purpose of obtaining the degree of Master of Science (MSc), submitted at TU Wien, Faculty of Mathematics and Geoinformation, by

Viktor SOMMER

Mat.Nr.: 01325384

under supervision of

Univ. Prof. Dipl.-Ing. Dr.techn. Alexia Fürnkranz-Prskawetz
Institute of Statistics and Mathematical Methods in Economics, E105

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Abstract

This thesis discusses potential mitigation actions for a pandemic, based on a SIRD model using lockdowns. On the one hand for individuals and on the other hand from the perspective of a social planner. The latter has 4 types of mitigation actions available. Conditions are given under which a lockdown would be reasonable. These conditions vary and depend on the type of people affected by the lockdown. Finally, vaccinations are included in the model and potential changes are compared with previous results. The basic model is based on a paper by Łukasz Rachel from December 2020, which I adapted and modified.

Zusammenfassung

Diese Arbeit behandelt potentielle Eindämmungsmöglichkeiten einer Pandemie, anhand eines SIRD Modells mittels Lockdowns. Einerseits für Individuen und andererseits aus der Sicht einer/s sozialen Planerin/sozialen Planers. Letzterer/em stehen 4 Typen von Milderungsmaßnahmen zur Verfügung. Es werden Voraussetzungen angegeben unter denen ein Lockdown sinnvoll wäre. Diese Bedingungen variieren je nach Art der Betroffenen des Lockdowns. Schließlich werden Impfungen in das Modell aufgenommen und potentielle Änderungen mit den bisherigen Ergebnissen verglichen. Das Basismodell, basiert auf einem paper von Łukasz Rachel aus dem Dezember 2020, welches von mir adaptiert und verändert wurde.

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1 Introduction

We have all been affected by the COVID-19 pandemic over the last year and a half. With over 200,000,000 people now infected and more than 4 million deaths[1], we are still in the midst of the pandemic. Overflowing hospitals and exhausted medical staff around the world have been one of the consequences. Each of us has done our part to contain the Coronavirus, initially individually, later with government mandates. Several measures to keep infection rates low and thus relieve hospitals have been deployed, including mandatory masks in public transportation and retail, as well as social distancing and lockdowns. The first strict lockdown in Austria in March 2020, with the goal to "flatten the curve," was quite successful. We learned that many of us can do our jobs just as productive from home. These restrictions were relaxed over time until summer when the second wave of infections arrived in September. This was followed by two more, not quite as severe lockdowns in November and December the same year.

What does it depend on whether a lockdown should be imposed or not? How strict does an effective lockdown have to be? When do you lift the lockdown?

These topics are the main focus of my master thesis. However, lockdowns and other precautionary measures not only help to stop the spread of a disease, at the same time they also hurt the economy. A global recession was the result of the numerous measures. International tourism could not take place as usual, restaurants and cinemas remained closed for a long period of time or only opened to a limited extent to avoid social gatherings. Even some professions, which could not be done from home, did not make any profit. To find an optimal lockdown strategy, we use a Susceptible, Infected, Recovered and Deceased model, SIRD model in short, to maximize the respective lifetime utilities for individuals and society as a whole. This model was analyzed in a paper by Rachel [2020, Dec] on which this master thesis is based. At first we take a look at the problem for individuals, who start with a strict lockdown which lessens over time until it is no longer necessary. Then in the form of a benevolent social planner, we solve the centralized problem for the various mitigation options.

In our model, we do not distinguish between different risk groups as, for example, Acemoglu et al.[2020] did in a multi-risk (MR) SIR model. In the MR-SIR model there are three groups "young", "middle aged" and "old" with different risk parameters and lockdown policies. The argument to create different groups is that the high mortality of the elderly, together with a high value of life, leads to stricter and longer lockdowns.

[1] source: https://covid19.who.int/, from 12.08.2021

Therefore, targeted lockdowns on the different groups are more effective than a lockdown for all. In the model under assessment in this thesis, the cost of death is simply calculated by using the forgeone lifetime consumption utility.

Another issue that has preoccupied us in recent months is the availability of a vaccine, which has been a target since the beginning of the pandemic. By now, more than 62% of the population in Austria has been vaccinated at least once[2]. In the original paper by Rachel [2020, Dec] vaccinations are briefly mentioned with the scenario that at a certain point in time a vaccine will be available. This results in two options for a social planner, either to suppress the infections until a vaccine is availalbe or follow the same strategy as before the introduction of vaccines. We will take a closer look at a model that also includes vaccinations from the beginning in chapter 3.

The thesis is strucured as follows: the next section focuses on the description of the model as well as on the different mitigation actions by individuals and a social planner. In section 3 we take a closer look on how to expand the base model by Rachel [2020, Dec], followed by a final conclusion.

[2] https://info.gesundheitsministerium.at/, as of 13.09.2021

2 Model description

The model is based on the SIR-model (susceptible-infected-removed) of Kermack and McKendrick [1927]. This epidemiological model describes the spread of contagious diseases with immunization, such as Covid-19. In the baseline model it is assumed that an individual can only get the disease once and (if recovered) develops an immunity for life. Rachel used a variant of the baseline model that includes the number of deaths caused by the disease (D). The goal is to find the optimal intensity of a lockdown as an individual as well as for a benevolent social planner. For this aspect it is assumed that a fraction of new infections can be prevented by social distancing and a lockdown of parts of the population. This deliberately simple model has a few drawbacks for modeling Covid-19: the immunity after recovery from a Covid-infection does not last a lifetime, the initial population has no way of increasing (no births), limitations in health care for intensive care patients are not considered and vaccinations are not incorporated. The last point will be the focus in section 3 where I extend the model in this regard. The following model and representations are based on the paper "An Analytical Model of Covid-19 Lockdowns" by Rachel [2020, Dec] and follows his notation.

2.1 Definition

The model is characterized by the following system of equations:

$$\dot{S} = -\beta S I \tag{1}$$

$$\dot{I} = \beta S I - \gamma I \tag{2}$$

$$\dot{R} = \gamma_r I \tag{3}$$

$$\dot{D} = \gamma_d I \tag{4}$$

with $\gamma = \gamma_r + \gamma_d$ and $S(0) = 1 - \epsilon$, $I(0) = \epsilon$ where ϵ is small but positive, $R(0) = D(0) = 0$. Where γ_r and γ_d respectively denote the share of infected that recover or die from the disease.

It is assumed that part of the infections can be mitigated β_n and the rest β_0 cannot be avoided. Therefore the infection rate β is constructed as follows:

$$\beta = \beta_n \lambda_S \lambda_I + \beta_0 \tag{5}$$

where λ_S and λ_I denote the share of susceptibles and infected that are part of the active labour market. No lockdown would mean that $\lambda_S = \lambda_I = 1$ whereas a total lockdown would mean that either λ_S and/or λ_I would be 0. The idea behind this is that only the infections in the workplace can be eliminated and the share β_0 would represent the essential part of the economy that cannot be shut down or cannot work from home e.g. medical staff, food industry, etc. Another way to interpret the share β_0 would be as the part of the infections that cannot be reduced by behaviour. As units of time "weeks" are chosen.

Another important figure is the herd immunity threshold, which describes the level of susceptibles that have to be reached to no longer have rising infections I (assuming $I > 0$ before reaching that threshold). Let $\mathcal{R}_0$ be the basic reproduction number, then this threshold is defined as the inverse of the basic reproduction number:

$$\bar{S} := \frac{\gamma}{\beta} = \frac{1}{\mathcal{R}_0} \tag{6}$$

As soon as the number of susceptibles falls below this threshold, a sufficient amount of people have developed immunity to preventing further spread of the disease. In our model we have no influence on the recovery rate and aim to shift $\bar{S}$ to the right by reducing the infection rate β.

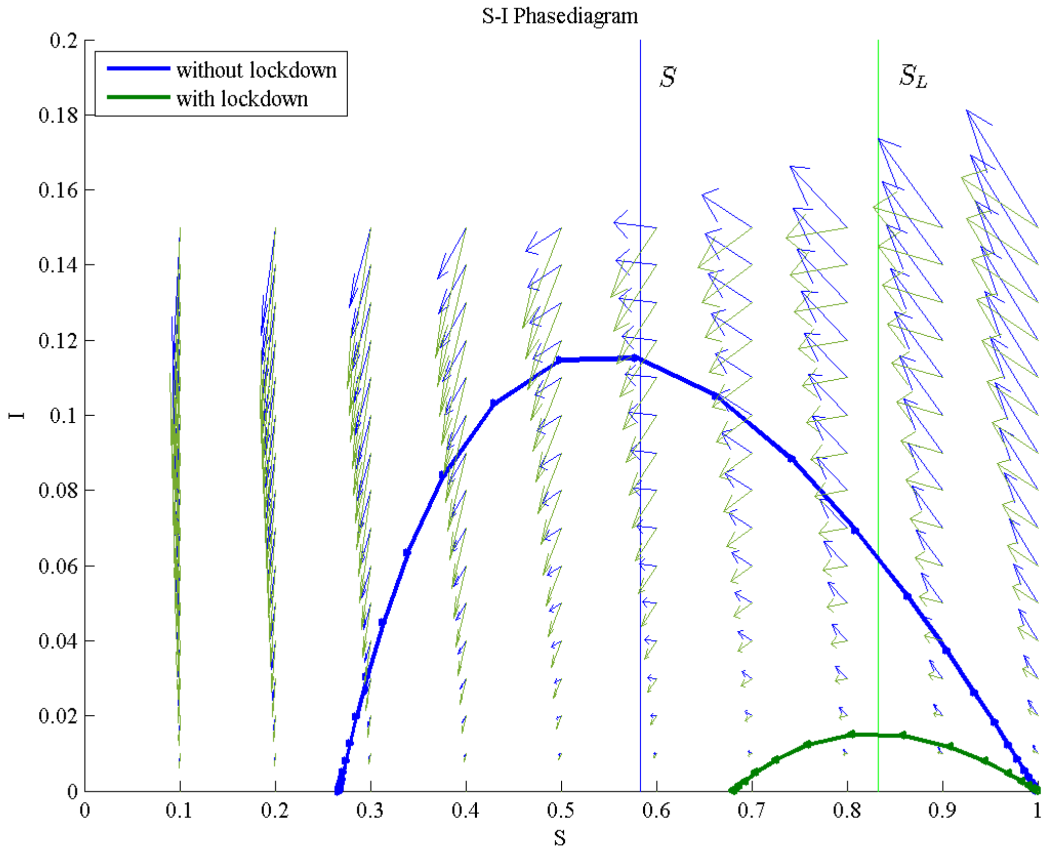


Figure 1: Phase Diagram with and without lockdown measures.
Paramethers: $\gamma_r = 0.99\frac{7}{10}$, $\gamma_d = 0.01\frac{7}{10}$, $\beta_n = 0.8$, $\beta_0 = 0.4$, $\lambda = 0.55$, $\epsilon = 0.0003$

Analyzing the model in a phase diagram in the $S - I$ space, see Figure 1, we start with a large proportion of susceptibles (close to 1) and a small amount of infected (positive but close to 0), see bottom right corner. The blue curve represents the case where no social distancing and remote work is practiced. The corresponding herd immunity threshold $\bar{S}$ is represented by the blue vertical line and marks the point where infections are no longer increasing. A lockdown would increase this threshold and shift the vertical line to the right marked by the straight green line in Figure 1. This would lead to a decrease of total infections (and therefore less deaths) and only a small portion of susceptibles would get the disease, see green curve. The blue and green arrows show the dynamics of the respective versions for any initial point. It is obvious that for a large initial number of infected the spread of the disease is faster and slower for a low number of infected at the initial point.

To ensure that a lockdown is effective enough to reduce the reproduction number $\mathcal{R}_0$

below unity, it is assumed that:

$$\textbf{Assumption 1:} \quad \gamma > \beta_0$$

That means that if total lockdown is imposed, which in our model means $\lambda_S = \lambda_I = 0$, the rate of new infections will be lower than the rate by which individuals recover or pass away from the disease.

2.2 Decentralized equilibrium

For the decentralized problem all individuals are faced with the following maximization problem before the pandemic:

$$\max_{\lambda \in [0,1]} \int_0^\infty e^{-\rho t}(\lambda u^W + (1-\lambda)u^L)dt \tag{7}$$

where ρ is a discount factor and $u^W := u(w, 1)$ and $u^L := u(h, 0)$ denote utility of going to work or staying at home respecively that depends on consumption and labour supply. We assume that utility $u(c, n)$ depends on consumption c and labour supply n. Here λ is equal to the probability of working, i.e. $\lambda = 1$ going to work as usual and $\lambda = 0$ working from home. If an individual decides to work from home she/he would receive an income h otherwise a wage w.[3] It is assumed that individuals prefer going to work over staying at home $u^W > u^L$, which would result in $\lambda = 1$ before the pandemic. We do not distinguish between skilled and unskilled workers and just assume that everyone has the same utility functions. This basic model has a linear production function (Y) in labour and the government taxes on labour income with a tax rate τ_n. Here $N = S + I + R$ is the total number of workers. Wages are then given by the marginal product of labour minus the taxes:

$$\begin{aligned} Y &= AN \\ G &= \tau_n AN \\ w &= A(1-\tau_n) \end{aligned}$$

where A is the production technology factor and G is goverment income. For the market to clear, aggregated income has to equal aggregated expenditure (8), household expenditure (C) must equal houshold income (9) and the effective amount of labour (N) must equal

[3] Income h is made up of three parts: working from home, home production and government transfer, where the respective shares can be summed to one $\psi_{WFH} + \psi_{HPR} + \psi_{GOV} = 1$

the labour input of people going to work and working from home (10).

$$Y + (1-\lambda)h = C + G \tag{8}$$
$$C = \lambda w + (1-\lambda)h \tag{9}$$
$$N = \lambda + (1-\lambda)\psi_{WFH}\frac{h}{A(1-\tau_n)} \tag{10}$$

To get the labour input coming from people working from home in (10), we simply divide income from working from home, $\psi_{WFH}h$, by post tax income, $A(1-\tau_n)$.

In the following chapters we set our initial population at time $t = 0$ to 1. This allows us to interpret all quantities of individuals, i.e. S, I, R, D as shares of the initial population.

We assume that households maximize their expected lifetime utility at time $t = 0$. Additionally we assume that individual preferences do not change after the pandemic outbreak. Since this model has no altruism, infected and recovered individuals would maximize their utility by always going to work since they would not face the risk of infection anymore $\lambda_I = \lambda_r = 1$. Otherwise, assuming that $\lambda_I = 0$ would result in an automatic supression of infections without the need of susceptibles to work from home and participate in social distancing.

Susceptibles are now faced with the following problem:

$$\max_{\lambda(t)\in[0,1]} \int_0^\infty e^{-\rho t}\Big(p_s(t)(\lambda(t)u^W + (1-\lambda(t))u^L) + (p_i(t) + p_r(t))u^W\Big)dt \tag{11}$$

subject to:

$$\dot{p_s} = -p_s(t)(\beta_n\lambda(t) + \beta_0)I(t) \tag{12}$$
$$\dot{p_i} = p_s(t)(\beta_n\lambda(t) + \beta_0)I(t) - \gamma p_i(t) \tag{13}$$
$$\dot{p_r} = \gamma_r p_i(t) \tag{14}$$
$$\lambda(t) \in [0,1] \tag{15}$$

where p_s, p_i, p_r denote the probabilities of being susceptible, infected or recovered at time t, respectively. This problem can be solved using Pontryagins Maximum Principle, derivation of all results below can be found in the appendix B, subsection 7.2.1.

At the beginning and end of the pandemic there is no social distancing, i.e. $\lambda = 1$. After the initial outbreak the number of infected is rising until it peaks at a point in time $T_0 > 0$. At this level the risk of getting infected is high and individuals rather work from home and accept a utility loss. The optimal proportion of going to work at time t ($\lambda^*(t)$)

mainly depends on how much of the spread of the disease can be prevented by lockdown (β_n) and on the current number of susceptibles ($S(t)$). For a small enough discount rate ρ, an approximation for λ^* is given by (see appendix B, subsection 7.2.1, equation (75)):

$$\lambda^*(t) \approx \frac{\gamma-\beta_0 S(t)}{\beta_n S(t)}$$

As susceptibles decrease over time, due to infections, λ^* will increase over time until working from home is no longer deemed necessary and herd immunity is reached at $T_1 < \infty$. This can clearly be seen in figure 3 e). During the lockdown, in the interval $[T_0, T_1]$, the number of infected can be approximated by (see appendix B, subsection 7.2.1, equation (71)):

$$I(t) = \frac{u^W - u^L}{\beta_n(\eta_s(t) - \eta_i(t))}$$

From this result it follows that the share of infected individuals is determined by utility loss when working from home divided by the effectiveness of the lockdown times the difference of shadow values of beeing susceptible or infected. Since it is easier to work without the costates (η_s and η_i) we approximate $I(t)$ with (see appendix B, subsection 7.2.1, equation (81)):

$$I(t) \approx S(t)\frac{\frac{u^W-u^L}{u^W}\rho}{\beta_n \cdot \bar{S} \cdot IFR}$$

where $IFR := \frac{\gamma_d}{\gamma}$ is the infection fatality rate. Since people no longer go to work as usual, but work almost exclusively from home, the number of infected immediately decreases, as can be seen in figure 3 c). The amount of infected is so small that the number of new infections roughly matches the number of recovered individuals, this can be seen in the constant level of infected during the time of the lockdown, see figure 2. As soon as the herd immunity threshold is reached and enough people have developd an immunity, the disease cannot spread any further and people no longer have to work from home.

With individual response to the pandemic the number of susceptibles decreases much slower over time and finally stays at a considerable higher level than without any behavioural response, which is clearly visible from the less steep green curve compared to the blue one in figure 3 a). Recovered and deceased individuals both remain at a slightly lower level than without any social distancing, see figure 3 c) and d). In figure 3 b) rather than an initial peak in infections, we see that the level of infected individuals remains the same until it decreases much later than without an individual lockdown.

If each individual maximizes her/his own expected lifetime utility by deciding to go to work or not, she/he reduces the total number of infected and consequently reduces deaths. Even though it would take longer to reach herd immunity compared to the scenario where no one decides to work from home and practice social distancing, the goal to "flatten the curve" has been reached and essentially, lives have been saved.

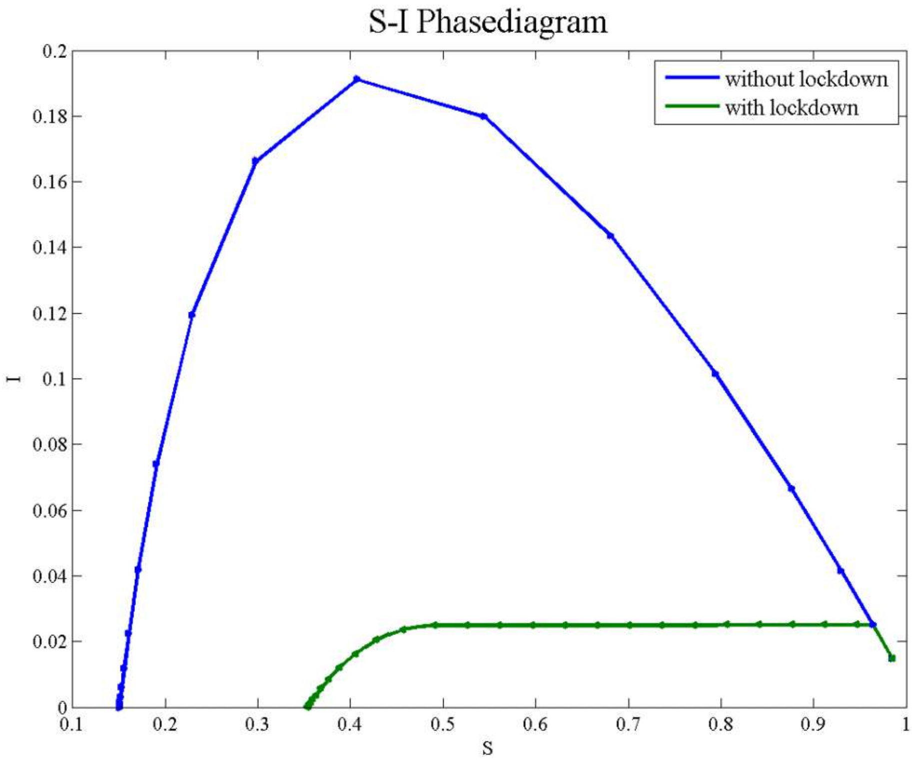
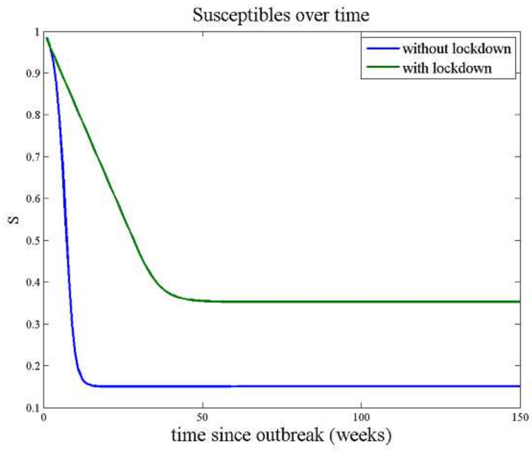
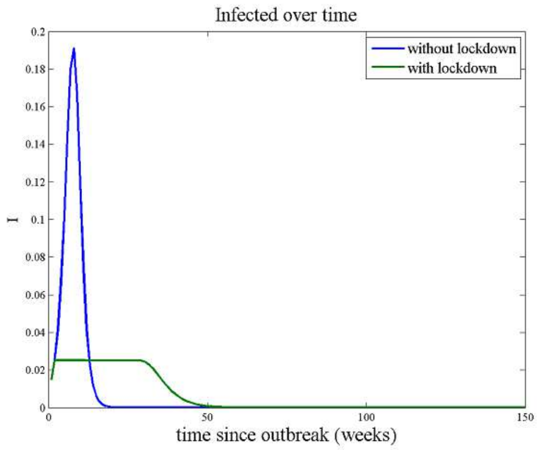


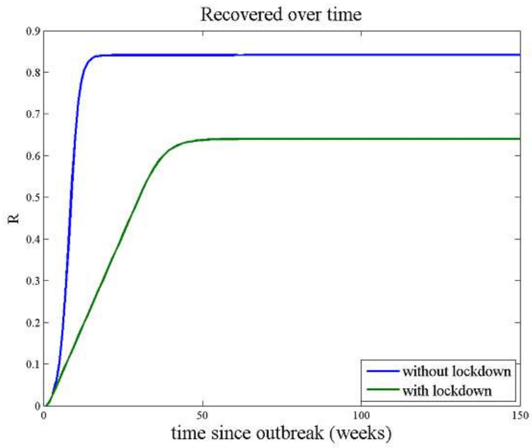
Figure 2: $S - I$ phasediagram, with and without self isolation,
Source: own calculation, see appendix A for parameter values



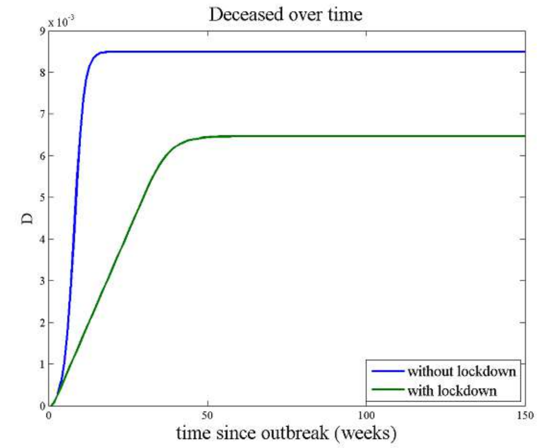
(a) susceptibles



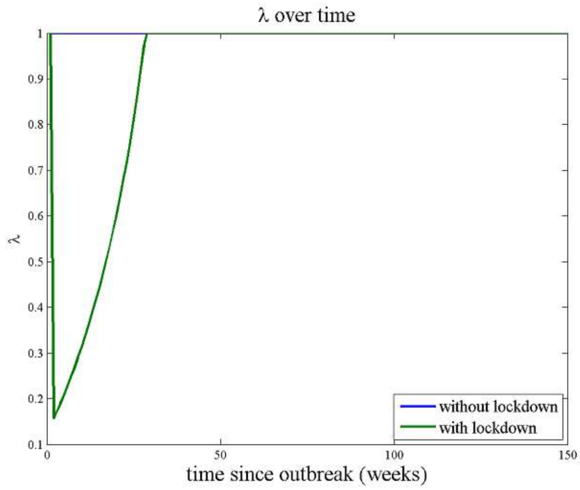
(b) infected



(c) recovered



(d) deceased



(e) Probability of working, λ

Figure 3: Dynamics of the subpopulations S, I, R and D as well as λ over time in the decentralized problem,
Source: own calculation, see appendix A for parameter values

2.3 Centralized problem - social planner view

In this section we put ourself in the position of a social planner and determine an optimal lockdown strategy. Our goal is to maximize the present value lifetime utility of all individuals. We are now taking a closer look at the following mitigation actions and their optimal lockdown strategies:

- **Type 1:** isolation of the infected: planner sets $\lambda_i(t) \in [0,1]$. $\lambda_s(t) = \lambda_r(t) = 1 \forall t$
- **Type 2:** susceptibles-only mitigation: planner sets $\lambda_s(t) \in [0,1]. \lambda_i(t) = \lambda_r(t) = 1 \forall t$
- **Type 3:** immunity passports: planner sets $\lambda_s(t) = \lambda_i(t) \in [0,1]$. $\lambda_r(t) = 1 \forall t$
- **Type 4:** all-in mitigation: planner sets $\lambda_s(t) = \lambda_i(t) = \lambda_r(t) \in [0,1]$

2.3.1 Type 1: isolation of infected

Assuming that tests to detect infected individuals are already available and the results can be evaluated in a reasonable amount of time, a social planner could decide to only isolate infected individuals. For that the following problem has to be solved. Please note that the objective function differs from the one used by Rachel [2020, Dec]. The amount of deceased at time t multiplied by the potential utility they would produce if they were cured or still susceptible, is now subtracted from the maximization problem, i.e. $-u^W D(t)$. This term was added so that death has a negative impact in this optimization problem in the form of utility loss. Since we are considering a lockdown for infected only, we assume that susceptibles and recovered are going to work as usual.

$$\max_{\lambda \in [0,1]} \int_0^{\hat{T}} e^{-\rho t} \Big(S(t)u^W + I(t)(\lambda u^W + (1-\lambda)u^L) + R(t)u^W - D(t)u^W \Big) dt + e^{-\rho \hat{T}} (S(\hat{T}) + R(\hat{T})) \frac{u_\tau}{\rho}$$

subject to (1), (2), (3), (4) and taking S_0, I_0, R_0 as given. Here $u_\tau := u(w - \tau, 1)$ is the post-pandemic instantaneous utility flow and τ is the Corona-tax collected by the government from the survivors of the pandemic. The collection of this tax starts at $\hat{T}$ when $I(t)$ is negligible small and mitigation actions are no longer necessary. At this time the goverment would start to collect taxes from the survivors to fund the debt caused by the pandemic, e.g. goverment aid for businesses that could not operate during a lockdown. Therefore we added the final term and replaced the infinite time horizon with a finite one, that ends at $\hat{T}$.

Furthermore it has to be noted that instead of probabilities we now work with population shares for the centralized problem. Another difference compared to the decentralized problem is that only infected individuals are considered for mitigation actions.

Again with the help of Pontryagins Maximum Principle we can solve this problem, arriving at the following results[4], (derivation of results can be found in appendix B, section 7.2.2):

If the identification of infected is possible, intuitively the immediate supression of infected at the start of the pandemic makes sense, therefore $\lambda = 0\ \forall t$. But this raises the question when a lockdown is necessary. Clearly not every outbreak of a disease would result in a lockdown. We determined that the two main factors to justify an immediate lockdown of infected are the infection fatality rate $IFR = \frac{\gamma_d}{\gamma}$ and the effectiveness of a lockdown β_n. A lockdown would be optimal if the infection fatality rate (IFR) is greater than (see appendix B, subsection 7.2.2, equation (93)):

$$\frac{u^W - u^L}{u^W} \frac{\rho}{2\gamma\beta_n}\Big(\rho + \gamma - \beta_0 - \beta_n\Big) \tag{16}$$

Since the utility cost of a lockdown, $\frac{u^W - u^L}{u^W}$, are smaller than 1 we can argue that IFR is greater than (16) if:

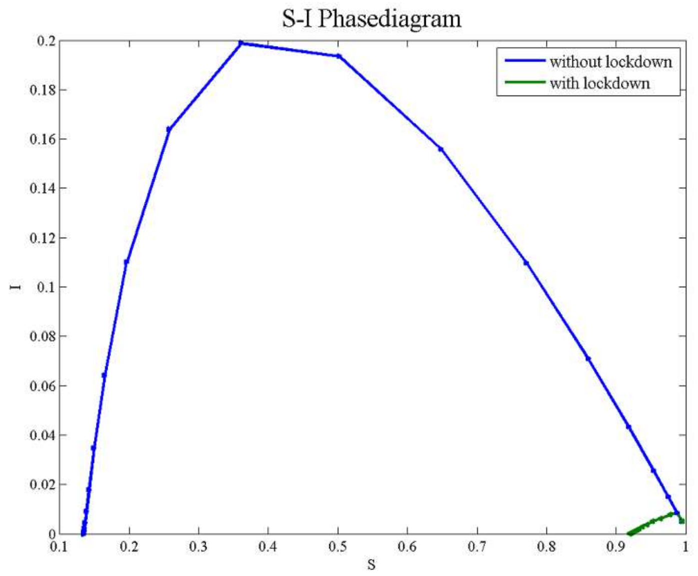
$$IFR > \frac{\rho}{2\gamma\beta_n}\Big(\rho + \gamma - \beta_0 - \beta_n\Big)$$

If β_n is large enough, i.e. $\beta_n \geq (\rho + \gamma - \beta_0)$ this inequality will always be true for a positive IFR, since the right hand side will be smaller or equal to 0. Let's take a closer look at the effectiveness of the lockdown. If the discount rate ρ is small, we see that a lockdown becomes more likely if there are too many new infections that cannot be prevented and infected people do not recover fast enough. Assumption 1, i.e. $\gamma > \beta_0$, again ensures that $\beta_n > 0$.

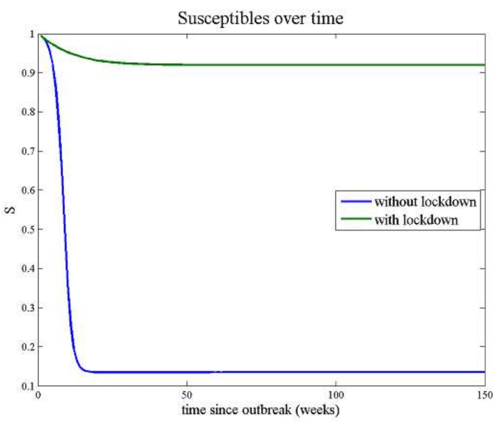
This result makes sense from a social planners point of view, a highly ineffective lockdown for a pandemic or disease with a very low fatality rate would not be reasonable[5]. On the other hand a high fatality rate would almost always lead to a lockdown of infected. This result also shows the intuitive conclusion that if the lockdown is not effective, $\beta_n \approx 0$, a lockdown of infected would not make sense, since the spread of the disease will not be stopped by this measure.

In figure 4, the dynamics for a high enough fatality rate of infected and a sufficiently effective lockdown are pictured. As expected a large outbreak of the disease can be prevented. It is worth mentioning that the lockdown keeps the number of infected and deceased individuals at a minimum but lengthens the duration of the pandemic compared to no lockdown, 24 weeks vs 46 weeks.

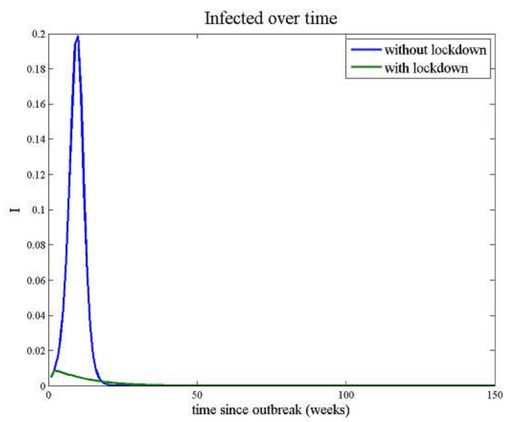
[4]Our derivation of results slightly deviates from the one by Rachel [2020, Dec]. Nevertheless similar qualitative results were obtained. An immediate lockdown is optimal if the IFR is sufficiently large.
[5]For example, the common cold is not fatal enough to warrant a lockdown



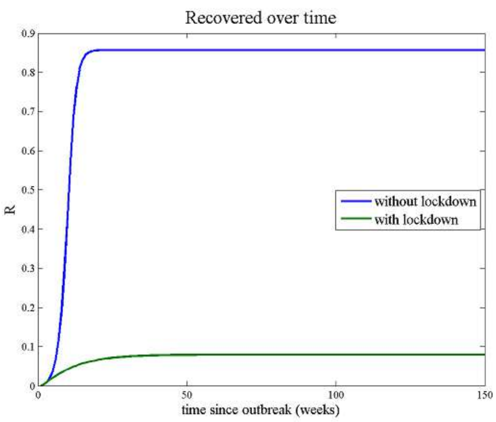
(a) S-I phasediagram



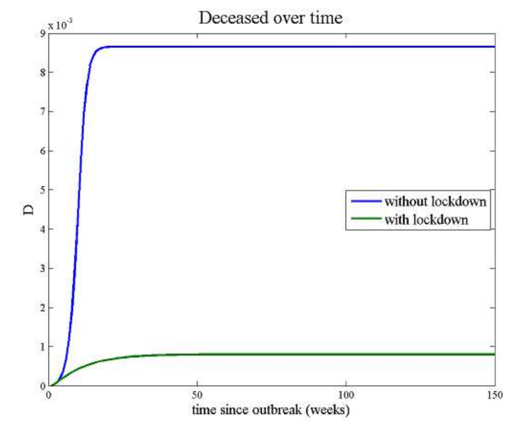
(b) susceptibles



(c) infected



(d) recovered



(e) deceased

Figure 4: Dynamics of the subpopulations S, I, R and D over time for the social planner problem with immediate lockdown of infected.
Source: own calculation, see appendix A for parameter values

As mentioned above this would be the ideal scenario, where tests to detect the disease are available, fast to evaluate and easily produced. In figure 5 an example is shown where we assume that mass testing is only available seven weeks[6] after the outbreak and this type of mitigation action would be feasible. For this we simply set $\lambda = 0$ after seven weeks, when we finally can distinguish between susceptibles, infected and recovered. As expected infections would increase the same way as without any lockdown but as soon as testing is possible and infected individuals are immediatley placed in quarantine, a supression of the disease follows. Compared to an immediate lockdown a delay of mass testing would significantly increase the number of infected. In the next sections we will see that a similar strategy can be optimal to prevent a second wave of infections.

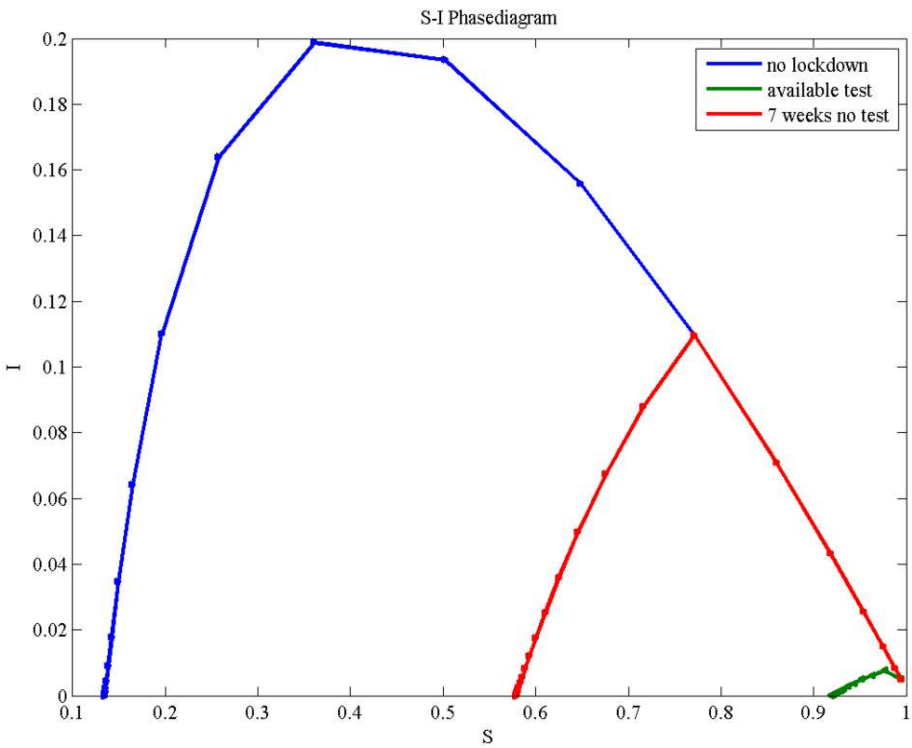


Figure 5: $S-I$ Phase-diagram with no lockdown, immediate lockdown and lockdown after 7 weeks. Source: own calculation, see appendix A for parameter values

[6]This length of the delay is randomly chosen for demonstration purposes.

2.3.2 Type 2: susceptibles-only mitigation

This type of mitigation is very similar to the decentralized problem, since both times we focus on the susceptible population. Additionally to the decentralized problem, we integrate deceased individuals in the objective function. For deceased individuals we again consider the utility loss that this group would have gotten if they survived the infection. Hence we are faced with the following problem:

$$\max_{\lambda \in [0,1]} \int_0^{\hat{T}} e^{-\rho t}\Big(S(t)(\lambda(t)u^W + (1-\lambda(t))u^L)) + \Big(I(t)+R(t)-D(t)\Big)u^W\Big)dt + e^{-\rho\hat{T}}(S(\hat{T})+R(\hat{T}))\frac{u_\tau}{\rho}$$

subject to (1), (2), (3), (4) and taking S_0, I_0, R_0 as given. Similar to type-1-mitigation, we again have a finite time horizon and the Corona-tax at the end of the pandemic. For this type of mitigation action we assume that infected as well as recovered individuals are free to go to work as usual and only susceptibles are subject to a possible lockdown.

The idea behind the solution is to reach the herd immunity threshold at the exact time when there are no longer any infected left, to ensure another outbreak cannot happen and an additional lockdown is not needed. This is achieved by initially not imposing any restrictions and, as soon as a certain level of susceptibles has been infected, take strict measures to contain the disease. This level of susceptibles is given by, (see appendix B, subsection 7.2.3, equation (109)):

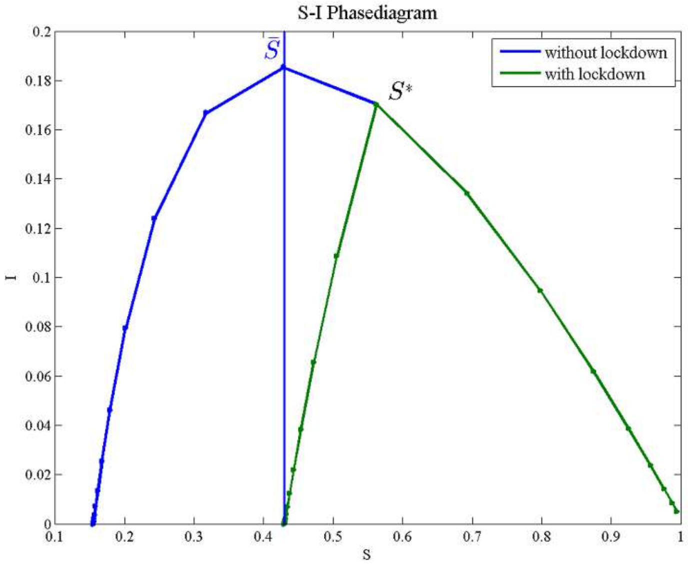
$$S^* = exp\Big(\frac{1 - \bar{S} + \bar{S}_L log(\bar{S})}{\bar{S}_L - \bar{S}}\Big) \tag{17}$$

In figure 6 (a) it can be clearly seen that up to the point S^* the paths with and without lockdown are identical. As soon as the number of susceptibles fall below S^*, a strict lockdown starts, i.e. $\lambda = 0$. The point S^* is chosen in such a way that a strict lockdown will reduce the number of infected until no infected are left ($I = 0$) at exactly the level of susceptibles where the "natural" herd immunity threshold ($\bar{S}$) is reached. This is to ensure that there is no potential second wave in case of a new outbreak of the disease.

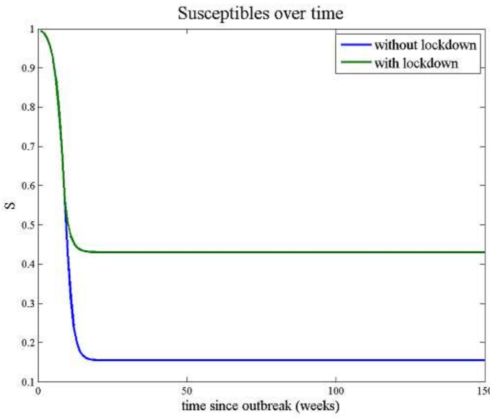
The point (S^*) is optimal because a lockdown at a higher level of susceptibles would increase the duration of the lockdown and would not ensure that the herd immunity threshold is reached. Therefore risking a second wave, which would increase the number of deceased individuals even further. A lockdown at a later point in time would only increase the number infected unnecessarily and risk additional deaths.

Taking a look at the number of susceptibles, infected, recovered and deceased in figure 6, it is nice to see that the peak of infections is reached two weeks earlier than without any mitigation actions, and the level of deceased and recovered is lower while susceptibles remain at a higher level than without a lockdown. Further, the final level of susceptibles is now exactly when herd immunity is reached, i.e. $S = \bar{S}$. In figure 6 a) and b) we can explicitly see that fewer susceptibles have been infected compared to a no lockdown strategy. The level of infected and therefore recovered and deceased individuals as well, could be reduced when imposing a lockdown at S^*, see figure 6 c), d) and e)

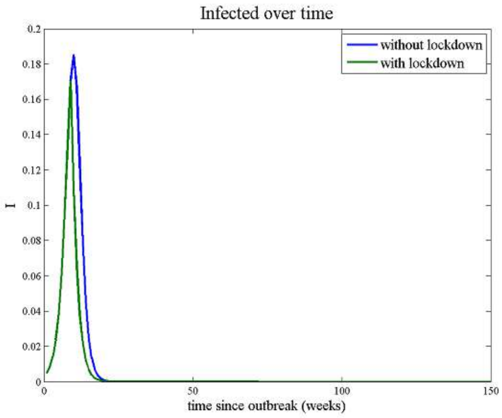
Since we make the same assumptions for type 1 and type 2 mitigation, that we can immediately distinguish between infected and susceptibles, we can easily compare the two methods. The number of individuals that would be affected by a lockdown is obviously higher for a susceptible only lockdown (type 2) compared to an infected only lockdown (type 1). Hence, the costs of this lockdown would be greater as well. A benefit of this susceptible only lockdown would be that herd immunity is reached and a further outbreak could be prevented without any additional costs. In sections 2.3.3 and 2.3.4 we no longer have to differentiate between susceptibles and infected giving us a more realistic approach for our model.



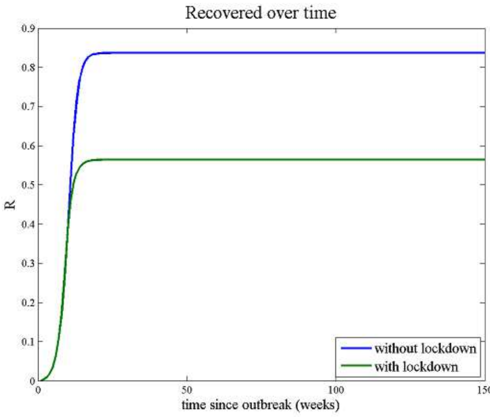
(a) S-I phasediagram



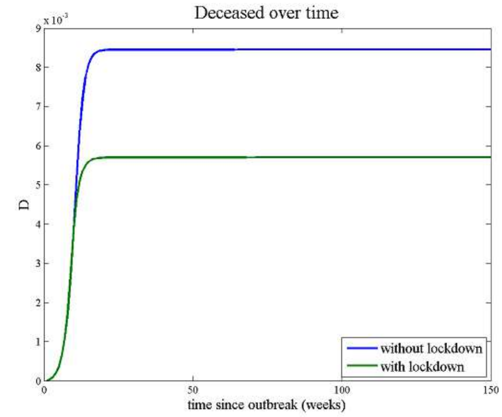
(b) susceptibles



(c) infected



(d) recovered



(e) deceased

Figure 6: Dynamics of the subpopulations S, I, R and D over time for the social planner problem with susceptible only lockdown.
Source: own calculation, see appendix A for parameter values

2.3.3 Type 3: immunity passports

Type 3 and type 4 mitigation actions will always affect susceptible and infected individuals together. Therefore, we combine λ_S and λ_I in equation (5) to λ and are again faced with a linear problem in λ.

The solutions to these two types are quite similar in nature as well. First let's take a look at the possibility of immunity passports. Assuming that only recovered individuals are free to work ($\lambda_r = 1$), provided that it is easy to prove an individual has recovered. Whereas, susceptibles and infected can be placed under lockdown. To find the optimal severity of this lockdown we have to solve the following problem:

$$\max_{\lambda\in[0,1]} \int_0^{\hat{T}} e^{-\rho t}\Big((S(t)+I(t))(\lambda u^W+(1-\lambda)u^L)+R(t)u^W-D(t)u^W\Big)dt+e^{-\rho\hat{T}}(S(\hat{T})+R(\hat{T}))\frac{u_\tau}{\rho}$$

subject to (1), (2), (3), (4) and taking $S(0), I(0), R(0)$ and $D(0)$ as given.

Since the derivation of type 3 and type 4 are not directly part of the original paper by Rachel [2020, Dec], I decided to include the derivation of the results in the main text. The Hamiltonian for this problem is given by:

$$H = \Big((S+I)(\lambda u^W+(1-\lambda)u^L)+R(t)u^W-D(t)u^W\Big)-\eta_s(\beta SI)+\eta_i(\beta SI-\gamma I)+\eta_r\gamma_r I+\eta_d\gamma_d I$$

where $\beta = \beta_n\lambda + \beta_0$. With the following FOC:

$$\begin{aligned}\frac{dH}{d\lambda} &= (S+I)(u^W-u^L)-(\eta_s-\eta_i)SI\beta_n = 0\\ \psi &= (u^W-u^L)(S+I)-(\eta_s-\eta_i)SI\beta_n\end{aligned} \tag{18}$$

$$\frac{dH}{dS} = \lambda u^W + (1-\lambda)u^L - (\eta_s-\eta_i)(\beta_n\lambda+\beta_0)I = \eta_s\rho - \dot{\eta}_s \tag{19}$$

$$\frac{dH}{dI} = \lambda u^W + (1-\lambda)u^L - (\eta_s-\eta_i)(\beta_n\lambda+\beta_0)S - \gamma\eta_i + \eta_r\gamma_r + \eta_d\gamma_d = \eta_i\rho - \dot{\eta}_i \tag{20}$$

$$\frac{dH}{dR} = u^W = \eta_r\rho - \dot{\eta}_r \tag{21}$$

$$\frac{dH}{dD} = -u^W = \eta_d\rho - \dot{\eta}_d \tag{22}$$

and transversality conditions:

$$\eta_s(\hat{T}) = \eta_r(\hat{T}) = \frac{u_\tau}{\rho} \tag{23}$$

In the final steady-state equilibrium, which implies that the time derivative of the costates are zero, we gather from equation (21) and (22) that:

$$\eta_d = -\frac{u^W}{\rho} = -\eta_r \tag{24}$$

Further, comparing (23) with (24) for consistency we conclude that $u_\tau \equiv u^W$.

First assume that $\lambda \in (0,1)$ and $\psi = 0$ over a longer period of time, which gives us:

$$(\eta_s - \eta_i)SI\beta_n = (S+I)(u^W - u^L)$$

Taking the derivative of both sides with respect to time t results in:

$$(\dot{\eta}_s - \dot{\eta}_i)SI\beta_n + (\eta_s - \eta_i)\beta_n(\dot{S}I + S\dot{I}) = (\dot{S} + \dot{I})(u^W - u^L)$$

with equation (19) and (20) we obtain the following

$$\Rightarrow SI\beta_n\Big((\eta_s - \eta_i)\rho - \gamma\eta_i + \frac{u^W}{\rho}(\gamma_r - \gamma_d) - (\eta_s - \eta_i)\gamma)\Big) = -\gamma I(u^W - u^L) \tag{25}$$

detailed derivation of (25) can be found in the appendix B, section 7.2.4.

From (25) we see that λ is no longer part of this optimization and we conclude that the solution is not singular and has to be bang-bang.

$$\lambda(t) = \begin{cases} 1 & \text{if } \psi > 0 \Leftrightarrow (S+I)(u^W - u^L) > (\eta_s - \eta_i)\beta_n SI \\ 0 & \text{if } \psi < 0 \Leftrightarrow (S+I)(u^W - u^L) < (\eta_s - \eta_i)\beta_n SI \end{cases} \tag{26}$$

From (26) we obtain that the condition whether λ is 0 or 1 is determined by the dynamics of $S+I$ on the left side and SI on the right side of the inequality. At the beginning of the pandemic SI is close to 0 whereas $S+I$ is close to 1. Since $\dot{S}+\dot{I} = -\gamma I$, $S+I$ is declining over time, whereas SI initially increases, peaks roughly when $S = \bar{S}$ and decreases again, as can be seen in figure 7.

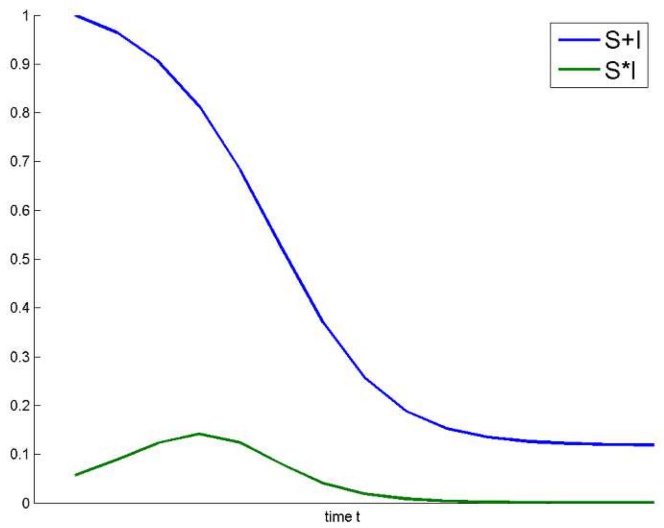


Figure 7: progression of $S + I$ and SI over time, parameters: $\beta = 1.45$, $\gamma = 0.7$, $\epsilon = 0.05$

Since $(u^W - u^L)$ is constant, the plot for the left hand side of (26) will qualitatively look the same as the blue line in figure 7 but shifted up or down depending if $(u^W - u^L)$ is greater or smaller than 1. On the other hand $(\eta_s - \eta_i)$ changes over time and depending on its value we can determine conditions for a lockdown to be optimal.

Assuming $\psi < 0$ for $\lambda = 0$, we can derive a condition when an immediate lockdown is plausible. For easier calculation we use an approximation for the initial values of $S(0) \approx 1$ and $I(0) \approx 0$ to derive an approximation of the initial η_s and η_i from (19) and (20), (note that, we assume that $\dot{\eta_s} \approx 0$ and $\dot{\eta_i} \approx 0$ at the start of the pandemic).

$$\begin{aligned} \eta_s &\approx \frac{u^L}{\rho} \\ \eta_i &\approx \frac{u^L(1 - \frac{\beta_0}{\rho}) + \frac{u^W}{\rho}(\gamma_r - \gamma_d)}{\rho + \gamma - \beta_0} \\ \Rightarrow \eta_s - \eta_i &\approx \frac{u^L(\gamma_r + \gamma_d) - u^W(\gamma_r - \gamma_d)}{\rho(\rho + \gamma - \beta_0)} \end{aligned} \tag{27}$$

Now substituting (27) in (26) we get:

$$\begin{aligned}
\Big(\frac{u^L(\gamma_r+\gamma_d)-u^W(\gamma_r-\gamma_d)}{\rho(\rho+\gamma-\beta_0)}\Big)SI\beta_n &> (u^W-u^L)(S+I)\\
u^L(\gamma_r+\gamma_d)-u^W(\gamma_r-\gamma_d) &> \frac{(u^W-u^L)(S+I)}{SI\beta_n}\rho(\rho+\gamma-\beta_0)\\
\gamma_d(u^W+u^L)-\gamma_r(u^W-u^L) &> \frac{(u^W-u^L)(S+I)}{SI\beta_n}\rho(\rho+\gamma-\beta_0)\\
\gamma_d(u^W+u^L) &> (u^W-u^L)\Big(\frac{(S+I)}{SI\beta_n}\rho(\rho+\gamma-\beta_0)+\gamma_r\Big)
\end{aligned}$$

For a small enough ρ the right side simplifies to $(u^W-u^L)\gamma_r$ giving us our final condition for a lockdown to be optimal at the beginning of the pandemic.

$$\frac{\gamma_d}{\gamma_r} > \frac{(u^W-u^L)}{(u^W+u^L)} \tag{28}$$

From this we gather that if the immediate costs of a lockdown (u^W-u^L) are too big or if the disease is not deadly enough a lockdown at the beginning of the pandemic would not make sense in this case. This does not mean that there will be no lockdown at all. From figure 7 and equation (26) we conclude that even if initially the costs of a lockdown outweigh its benefits, with rising infections this might change and at a point in time a lockdown may be beneficial. Since the number of infected I reaches its maximum at $\bar{S}$ we get a good approximation for the maximum of SI if we multiply I_{max} with $\bar{S}$. After this point we definitely know that SI is decreasing. For a lockdown to be plausible we have the condition that $\psi < 0$, if we can show that

$$(\eta_s-\eta_i)\beta_n\bar{S}I_{max} > (u^W-u^L)(\bar{S}+I_{max}) \tag{29}$$

then we know that at least at this point the benefits of a lockdown are greater than its costs. Otherwise, if the left hand side, at its approximate maximum, is not greater than the right hand side, a lockdown would be unfavorable. Starting with the assumption that until then $\lambda = 1$ we can approximate η_s from (19), at the time when $S = \bar{S}$, as follows. (Note that we assume that η_s will not change much at this point in time and therefore we assume that $\dot{\eta_s} \approx 0$):

$$\begin{aligned}
\eta_s\rho - \dot{\eta_s} &= u^W - \eta_s\beta I_{max} + \eta_i\beta I_{max} \\
\eta_s &= \frac{u^W + \eta_i\beta I_{max} + \dot{\eta_s}}{\rho + \beta I_{max}} \\
\eta_s &\approx \frac{u^W}{\beta I_{max}} + \eta_i \qquad \text{for a small enough } \rho
\end{aligned} \tag{30}$$

This gives us the very simple approximation for $(\eta_s - \eta_i) = \frac{u^W}{\beta I_{max}}$ which we can plug into (29) to get the following condition to impose a lockdown.

$$\begin{aligned}
(\eta_s - \eta_i)\beta_n \bar{S} I_{max} &> (u^W - u^L)(\bar{S} + I_{max}) \\
\frac{\beta_n u^W}{\beta}\bar{S} &> (u^W - u^L)(\bar{S} + I_{max}) \\
\frac{\beta_n}{\beta} &> \frac{(u^W - u^L)}{u^W}(1 + \frac{I_{max}}{\bar{S}}) \\
\frac{\beta_n}{\beta} &> \frac{(u^W - u^L)}{u^W}(\frac{1}{\bar{S}} + \log(\bar{S}))
\end{aligned} \tag{31}$$

Here we see again that whether a lockdown is imposed or not, depends on its effectiveness and its costs. As mentioned above $\bar{S}I_{max}$ is only an approximation of the maximum of SI. If $\psi < 0$ at this point then this indicates that a lockdown at an earlier point in time may be beneficial. Further, every infected individual at this point is basically avoidable and the costs of a lockdown are less then its benefits. In this case a lockdown at $S = S^*$, as in susceptible only mitigation may be the optimal solution to avoid unecessary infected after the herd immunity threshold is reached.

In the case that the costs always outweigh the benefits of a lockdown, one would not impose a lockdown.

2.3.4 Type 4: all-in mitigation

For type 4 we have a similar setup as for type 3 but this time susceptibles, infected and recovered individuals are affected by a lockdown. Again we assume that deceased individuals will lose the same utility they would have produced if they survived the infection, in our case the same utility as the rest of the individuals, which depends on the intensity of the lockdown. The severity of the lockdown for everyone is again given by λ. Resulting in the following optimization problem:

$$\max_{\lambda\in[0,1]} \int_0^{\hat{T}} e^{-\rho t}\Big((S(t)+I(t)+R(t)-D(t))(\lambda u^W+(1-\lambda)u^L)\Big)dt + e^{-\rho\hat{T}}(S(\hat{T})+R(\hat{T}))\frac{u_\tau}{\rho}$$

subject to (1), (2), (3), (4) and taking $S(0), I(0), R(0)$ and $D(0)$ as given.

Again we solve this problem with Pontryagins Maximum Principle, the Hamiltonian is given by:

$$H = (S+I+R-D)(\lambda u^W+(1-\lambda)u^L) - \eta_s(\beta SI) + \eta_i(\beta SI - \gamma I) + \eta_r\gamma_r I + \eta_d\gamma_d I$$

where $\beta = \beta_n\lambda + \beta_0$. With the following FOC:

$$\begin{aligned}\frac{dH}{d\lambda} &= (S+I+R-D)(u^W-u^L) - (\eta_s-\eta_i)SI\beta_n = 0\\ &\Rightarrow \psi = (u^W-u^L)(S+I+R-D) - (\eta_s-\eta_i)SI\beta_n\end{aligned} \tag{32}$$

$$\frac{dH}{dS} = \lambda u^W + (1-\lambda)u^L - (\eta_s-\eta_i)(\beta_n\lambda+\beta_0)I = \eta_s\rho - \dot{\eta}_s \tag{33}$$

$$\frac{dH}{dI} = \lambda u^W + (1-\lambda)u^L - (\eta_s-\eta_i)(\beta_n\lambda+\beta_0)S - \gamma\eta_i + \eta_r\gamma_r + \eta_d\gamma_d = \eta_i\rho - \dot{\eta}_i \tag{34}$$

$$\frac{dH}{dR} = \lambda u^W + (1-\lambda)u^L = \eta_r\rho - \dot{\eta}_r \tag{35}$$

$$\frac{dH}{dD} = -(\lambda u^W + (1-\lambda)u^L) = \eta_d\rho - \dot{\eta}_d \tag{36}$$

and transversality conditions:

$$\eta_s(\hat{T}) = \eta_r(\hat{T}) = \frac{u_\tau}{\rho} \tag{37}$$

In the final steady-state equilibrium, which implies that the time derivative of the costates are zero, we gather from equation (35) and (36) that:

$$\eta_d = -\frac{\lambda u^W + (1-\lambda)u^L}{\rho} = -\eta_r \tag{38}$$

First assume that $\lambda \in (0,1)$ and $\psi = 0$ over a longer period of time, which gives us:

$$(\eta_s - \eta_i)SI\beta_n = (S + I + R - D)(u^W - u^L)$$

Taking the derivative of both sides with respect to time t results in:

$$\begin{aligned}
&(\dot{\eta}_s - \dot{\eta}_i)SI\beta_n + (\eta_s - \eta_i)\beta_n(\dot{S}I + S\dot{I}) = (\dot{S} + \dot{I} + \dot{R} - \dot{D})(u^W - u^L) \\
&SI\beta_n\Big((\eta_s - \eta_i)\rho - \gamma\eta_i + \frac{\lambda u^W + (1-\lambda)u^L}{\rho}(\gamma_r - \gamma_d) - (\eta_s - \eta_i)\gamma\Big) = -2\gamma_d I(u^W - u^L) \\
&\frac{SI\beta_n(\gamma_r - \gamma_d)\lambda(u^W - u^L)}{\rho} = -2\gamma_d I(u^W - u^L) - SI\beta_n\Big(\eta_s(\rho - \gamma) - \eta_i\rho + \frac{u^L}{\rho}(\gamma_r - \gamma_d)\Big) \\
&\Rightarrow \lambda = -\frac{2\gamma_d I(u^W - u^L)\rho + SI\beta_n\Big(\eta_s\rho(\rho - \gamma) - \eta_i\rho^2 + u^L(\gamma_r - \gamma_d)\Big)}{SI\beta_n(\gamma_r - \gamma_d)(u^W - u^L)} \\
&\lambda \approx -\frac{u^L}{u^W - u^L} < 0 \qquad \text{, for } \rho \text{ small enough}
\end{aligned} \tag{39}$$

This contradicts our assumption that $\lambda \in (0,1)$. A negative λ suggests that the optimal solution is as small as possible, in our case that would mean an immediate lockdown might be optimal. Again we are faced with a bang-bang solution and with the following conditions for λ to be either 1 or 0:

$$\lambda(t) = \begin{cases} 1 & \text{if } \psi > 0 \Leftrightarrow (S + I + R - D)(u^W - u^L) > (\eta_s - \eta_i)\beta_n SI \\ 0 & \text{if } \psi < 0 \Leftrightarrow (S + I + R - D)(u^W - u^L) < (\eta_s - \eta_i)\beta_n SI \end{cases} \tag{40}$$

We know that $S + I + R + D = 1$ and assuming that a majority recoveres from the disease and only a small part of infected actually dies, i.e. $D \approx 0$ we get a conservative approximation for the left hand side of (40) with $(u^W - u^L) \gtrless (\eta_s - \eta_i)\beta_n SI$. An example can be seen in figure 8 where we have plotted SI and $S + I + R - D$ over time.

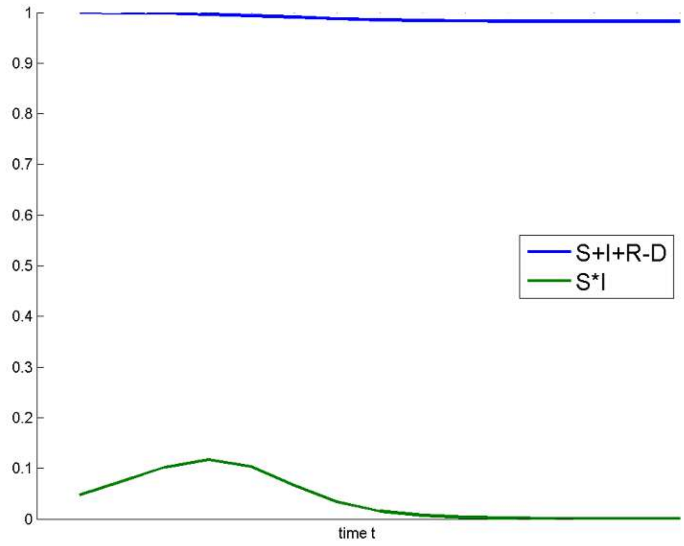


Figure 8: progression of $S + I + R - D$ and SI over time, parameters: $\beta = 1.45$, $\gamma = 0.7$, $\epsilon = 0.05$

First, let us look at the option that an immediate lockdown is optimal. Starting again with the assumption that $S(0) \approx 1$ and $I(0) \approx 0$ to derive an approximation of the initial η_s and η_i from (33) and (34), we assume that $\dot{\eta_s} \approx 0$ and $\dot{\eta_i} \approx 0$ at the beginning of the pandemic:

$$\eta_s \approx \frac{u^L}{\rho}$$

$$\eta_i \approx \frac{u^L(\rho - \beta_0 + \gamma_r - \gamma_d)}{\rho(\rho + \gamma - \beta_0)}$$

$$\Rightarrow \eta_s - \eta_i \approx \frac{u^L}{\rho}\frac{2\gamma_d}{\rho + \gamma - \beta_0} \tag{41}$$

Note that, under assumption 1: $\gamma > \beta_0$, $(\eta_s - \eta_i)$ is greater than 0. We now substitute (41) into our approximation of (40) to determine a condition that guarantees that an immediate lockdown is optimal.

$$\frac{u^L}{\rho}\frac{2\gamma_d}{\rho + \gamma - \beta_0} SI\beta_n > u^W - u^L$$

$$IFR = \frac{\gamma_d}{\gamma} > \frac{u^W - u^L}{u^L}\frac{\rho + \gamma - \beta_0}{2\gamma SI\beta_n}\rho \tag{42}$$

Again for a disease that is deadly enough and/or a lockdown with low costs, given by $u^W - u^L$, a strict lockdown at the beginning of the pandemic would be optimal.

Alternativeley if the condition of (42) is not fulfilled and an immediate lockdown would not be optimal we can again determine if we ever would reach the point at which a lockdown would be beneficial. Similar as for type 3 mitigations we approximate the highest point of SI with $\bar{S}I_{max}$. The derivation of η_s at this point is exaclty the same as for type 3 and is given by euqation (30). As above the problem simplifies to:

$$
\begin{aligned}
(\eta_s - \eta_i)\beta_n \bar{S} I_{max} &> (u^W - u^L) \\
\frac{\beta_n u^W}{\beta} \bar{S} &> (u^W - u^L) \\
\frac{\beta_n}{\beta} &> \frac{(u^W - u^L)}{u^W} \frac{1}{\bar{S}}
\end{aligned}
\tag{43}
$$

As before we see, if (43) is fulfilled, that at least at one point in time it will be ideal to impose a lockdown. The optimal time to reduce the number of infected such that no infected are left when herd immunity is reached is again given when the number of susceptibles S falls to S^*, compare (17).

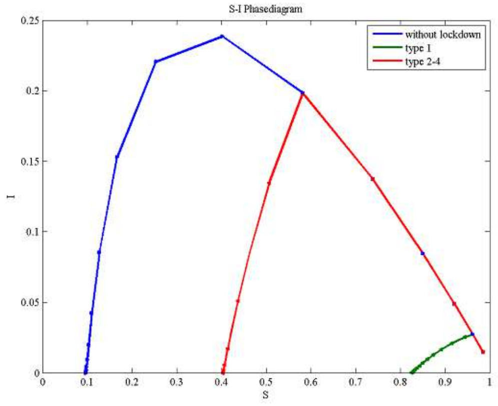
2.4 Summary

Let us conclude this section by comparing the different results of various methods to contain the disease, see figure 9 a). For demonstration purposes we assume that for types 2 to 4 a lockdown will be optimal at a certain point in time and therefore have the same lockdown strategy, i.e. total lockdown as soon as S falls to a level S^*.

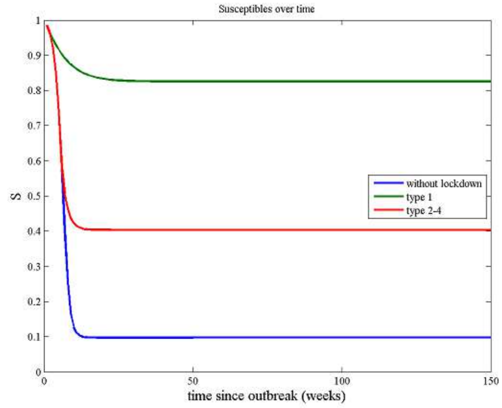
An early, strict lockdown for those who are infected will be a good solution to keep the disease in check at the start of the pandemic, i.e. type 1 mitigation action depicted by the green curves in figure 9. This comes with the assumption that tests, to detect infected individuals, are available at the beginning of the pandemic and mass testing is possible. Type 2 to 4 don't rely on heavy testing at the beginning of the pandmic. To ensure that herd immunity is reached in case of a second wave, a level of susceptibles S^* has to be reached before imposing a strict lockdown, as demonstrated by the red curves in figure 9.

Here you can see that for types 2 to 4 the number of susceptibles, recovered and deceased lies in the middle between an unrestricted pandemic, blue curves, and an immediate containment of infections in green. The number of infections will initially be higher than for type 1, but the duration of the lockdown is shorter than for type 1. This can be seen in figure 9 c) where the red curve falls under the green curve and reaches zero faster. As soon as the infected reach zero, a lockdown is no longer necessary. Furthermore, should type 1 experience one or more renewed outbreaks, additional lockdowns will be necessary to keep the number of losses low, which will not be the case if herd immunity has already been reched, as is the case for type 2 to 4.

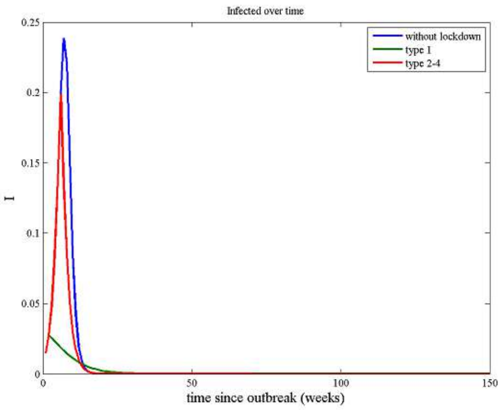
Finally, in figure 9 f), we see the different durations of the lockdowns for the various mitigation actions. In this plot, the dashed black line represents λ from the decentralized solution. It can be seen that the strategies of a social planner always result in a shorter lockdown than an individual lockdown, which ends in week 85. Additionally, an infected only lockdown results in a relatively long lockdown period, roughly 62 weeks, compared to a lockdown of type 2 to 4, which starts 9 weeks after the outbreak and only last for about 20 weeks.



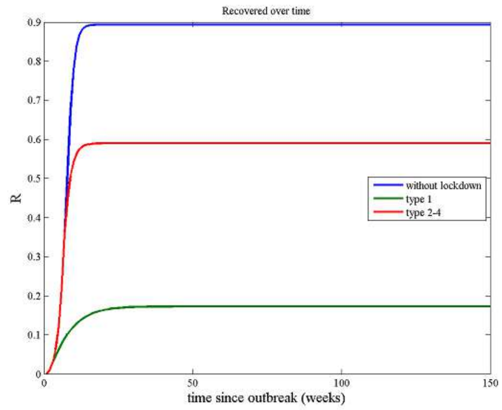
(a) S-I phasediagram



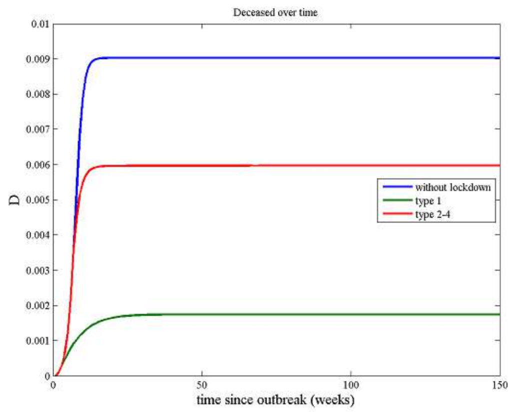
(b) susceptibles



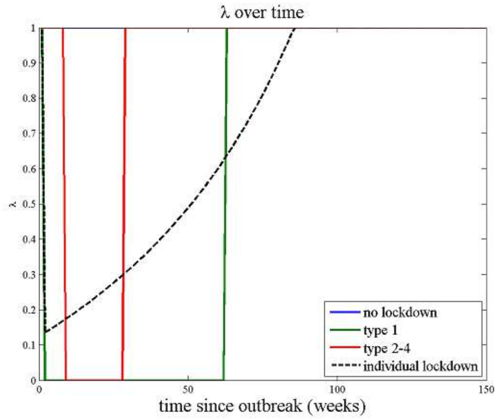
(c) infected



(d) recovered



(e) deceased



(f) λ over time

Figure 9: Comparison of the different mitigation actions available for a social planner. Source: own calculation, see appendix A for parameter values

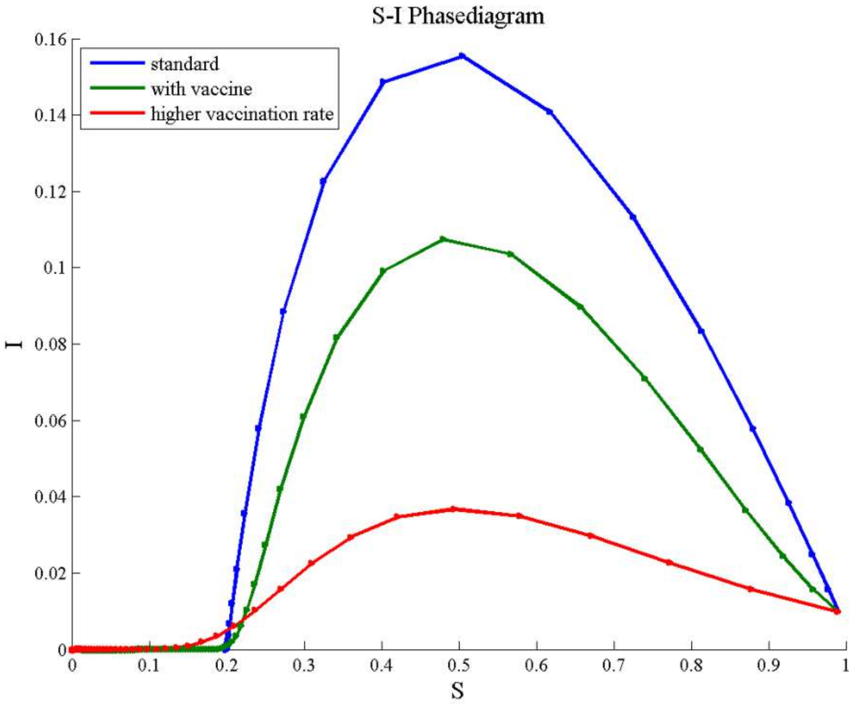
3 Extension of the original model - Vaccinations

An important factor in deciding the optimal lockdown intensity, that was not yet part of the model, is the availability of a vaccine. Depending on the effectiveness of the vaccine and how fast people get vaccinated, the strategy or conditions for a lockdown might change. For this we assume that a fixed proportion δ of susceptibles will be vaccinated each period and will switch directly into the group of recovered individuals. Unlike other papers, like Rao and Brandea [2021], that focus on the allocation and distribution of a vaccine, we assume that the factor, δ, is exogenous. Furthermore, we assume that a vaccine is already available at the beginning of the pandemic. This results in the following system of equations for our SIRD-model:

$$\dot{S} = -\beta SI - \delta S \tag{44}$$
$$\dot{I} = \beta SI - \gamma I \tag{45}$$
$$\dot{R} = \gamma_r I + \delta S \tag{46}$$
$$\dot{D} = \gamma_d I \tag{47}$$

Note that the dynamics for infected individuals has not changed.

In the $S - I$ phasediagram, see figure 10 blue and green curves, we observe that the two curves, with and without vaccines, are very similar. In both cases we start with rising infections until herd immunity is reached, this level has not changed with the introduction of vaccines, followed by a decrease of infections until we reach zero infections at the same level of susceptibles. But unlike the standard model, the number of susceptibles will further decrease as long as people are getting the vaccine. Given enough time, eventually everyone will be vaccinated. The slope for the standard model without any vaccines is much steeper, both at the beginning and end of the pandemic. For demonstration purposes, I added the red curve with a higher rate of vaccination δ_h. As one can see, the number of infected does not come near the levels without vaccination. With these low numbers it might not even be necessary to impose a lockdown, given that enough people are willing to get vaccinated.



(a) S-I phasediagram

Figure 10: S-I Phasediagram with and without the vaccines.
Parameter values: $\beta = 1.3$, $\gamma = 0.7$, $\epsilon = 0.01$, $\delta = 0.02$, $\delta_h = 0.1$

Even though vaccines help to reduce the number of infected we see that there is still room for improvement. Let us start again with the decentralized problem that each individual is faced with. Since utility stays the same, compare equation (11), we only state here the updated system equations for the probabilities of being susceptible, infected, or recovered, compare (12), (13) and (14).

$$\dot{p_s} = -p_s(t)(\beta_n\lambda(t) + \beta_0)I(t) - \delta p_s(t) \tag{48}$$

$$\dot{p_i} = p_s(t)(\beta_n\lambda(t) + \beta_0)I(t) - \gamma p_i(t) \tag{49}$$

$$\dot{p_r} = \gamma_r p_i(t) + \delta p_s(t) \tag{50}$$

The Hamiltonian for this problem is then given by:

$$H = p_s(\lambda u^W + (1-\lambda)u^L) + (p_i + p_r)u^W - \eta_s p_s((\beta_n\lambda + \beta_0)I - \delta) + \eta_i(p_s(\beta_n\lambda + \beta_0)I - \gamma p_i) + \eta_r(\gamma_r p_i + \delta p_s)$$

with the necessary FOC:

$$\frac{dH}{d\lambda} = p_s(u^W - u^L) - \eta_s p_s\beta_n I + \eta_i p_s\beta_n I = 0 \tag{51}$$

Again the switching function is given by $\psi(t) := \Big(u^W - u^L - (\eta_s - \eta_i)I\beta_n\Big)p_s$. The only difference compared to the original problem, as presented in appendix B, section 7.2.1, occurs in (52).

$$\frac{dH}{dp_s} = (\lambda u^W + (1-\lambda)u^L) + \eta_s(\beta_n\lambda + \beta_0)I + \eta_i(\beta_n\lambda + \beta_0)I + \eta_r\delta = \eta_s\rho - \dot{\eta}_s \tag{52}$$

Otherwise nothing changes, for completeness the remaining FOCs are given below.

$$\frac{dH}{dp_i} = u^W - \eta_i\gamma + \eta_r\gamma_r = \eta_i\rho - \dot{\eta}_i \tag{53}$$

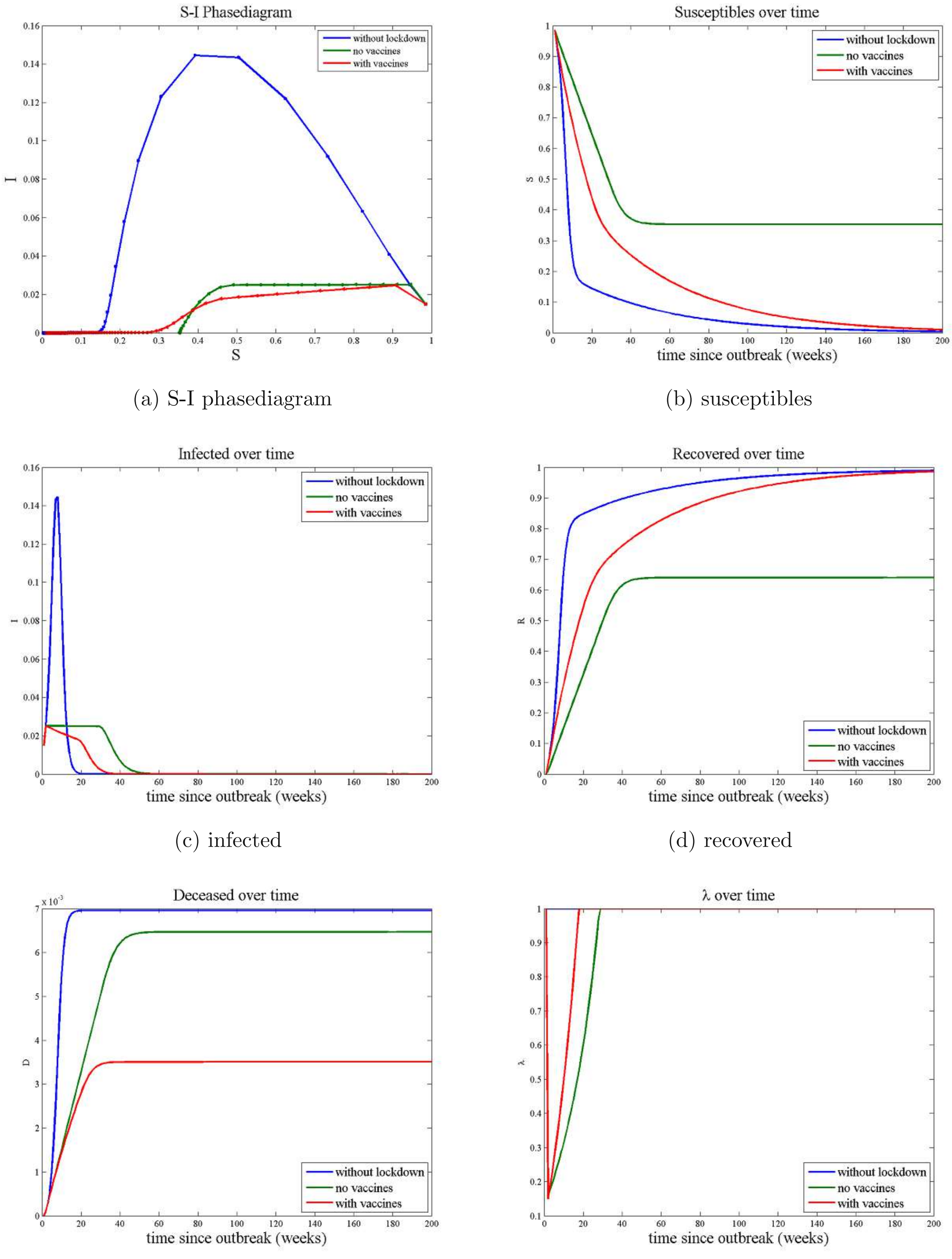
$$\frac{dH}{dp_r} = u^W = \eta_r\rho - \dot{\eta}_r \tag{54}$$

Similar to the problem without vaccination, we can derive the optimal lockdown intensity, λ_V^*, assuming that $\psi = 0$ over a longer period of time, resulting in the following, (see appendix B, subsection 7.2.5, equation (118)):

$$\begin{aligned} \lambda_V^* &= \frac{I}{S} + \frac{1}{\beta_n S}[\beta_0(S+I) - \gamma + \rho] + \frac{I}{S}\frac{\eta_r\gamma_r - \eta_i\gamma + (\eta_s - \eta_r)\delta}{(u^W - u^L)} \\ \lambda_V^* &\approx \frac{\gamma - S\beta_0 - \delta}{\beta_n S} \end{aligned} \tag{55}$$

This approximation is positive iff $\gamma > \beta_0 S + \delta$.

Given this result we see that it does not differ much from the approximation of the original problem $\lambda^* \approx \frac{\gamma - S\beta_0}{\beta_n S}$. Initially λ_V^* is smaller than λ^* but since the number of susceptibles S is decreasing faster than without vaccines the severity of the optimal lockdown decreases faster over time as well, as can be seen in figure 11 f). Again we see an initial increase of infected before individuals decide to self-isolate, see figure 11 c). Compared to the scenario without vaccines, the number of infections are decreasing slightly faster and the total amount of deceased individuals could again be further reduced, see figure 11 c) and e). Another important difference is that almost all susceptibles are vaccinated or have recovered from the disease and developed immunity. Therefore, in the long run, almost no susceptibles are left while the majority is now recovered and only a small share died from the disease, compare figure 11 b), d) and e).



(a) S-I phasediagram (b) susceptibles

(c) infected (d) recovered

(e) deceased (f) Probability of working, λ

Figure 11: Dynamics of the subpopulations S, I, R and D as well λ over time in the decentralized problem with vaccines, note that in subfigure f) the blue line "without lockdown" is constant and equal to 1.
Source: own calculation, see appendix A for parameter values

In the following we adress the problem with vaccination from a social planner point of view. The focus here lies on an infected only lockdown or type 1 mitigation as it was called before, with the inclusion of vaccines. We have the same objective function as in section 2.3.1 but subject to the system equations (44) - (47).

$$\max_{\lambda\in[0,1]} \int_0^{\hat{T}} e^{-\rho t}\Big(S(t)u^W + I(t)(\lambda u^W + (1-\lambda)u^L) + R(t)u^W - D(t)u^W\Big)dt + e^{-\rho\hat{T}}(S(\hat{T}) + R(\hat{T}))\frac{u_\tau}{\rho}$$

Therefore, the Hamiltonian for this problem looks slightly different from the one derived from section 2.3.1, (see appendix B, section 7.2.2) :

$$\begin{aligned} H = & \\ & \Big(S(t)u^W + I(t)(\lambda u^W + (1-\lambda)u^L) + R(t)u^W - D(t)u^W\Big) \\ & - \eta_s(\beta SI + \delta S) + \eta_i(\beta SI - \gamma I) + \eta_r(\gamma_r I + \delta S) + \eta_d \gamma_d I \end{aligned}$$

where $\beta = \beta_n\lambda + \beta_0$, with the necessary FOC:

$$\begin{gathered} \frac{dH}{d\lambda} = Iu^W - Iu^L - \eta_s SI\beta_n + \eta_i\beta_n SI = 0 \\ \text{with switching function } \psi = (u^W - u^L - (\eta_s - \eta_i)S\beta_n)I \end{gathered} \tag{56}$$

$$\frac{dH}{dS} = u^W + (\eta_i - \eta_s)(\beta_n\lambda + \beta_0)I + (\eta_r - \eta_s)\delta = \eta_s\rho - \dot{\eta}_s \tag{57}$$

Here we have the only change compared to the version without vaccines. The last term, where the vaccination rate δ is multiplied with the difference of shadow values for recovered and susceptibles, is new.

$$\begin{gathered} \frac{dH}{dI} = u^W\lambda + (1-\lambda)u^L - \eta_s\beta S + \eta_i[\beta S - \gamma] + \eta_r\gamma_r + \eta_d\gamma_d = \eta_i\rho - \dot{\eta}_i \\ u^W\lambda + (1-\lambda)u^L + (\eta_i - \eta_s)(\beta_n\lambda + \beta_0)S - \gamma\eta_i + \eta_r\gamma_r + \eta_d\gamma_d = \eta_i\rho - \dot{\eta}_i \end{gathered} \tag{58}$$

$$\frac{dH}{dR} = u^W = \eta_r\rho - \dot{\eta}_r \tag{59}$$

$$\frac{dH}{dD} = -u^W = \eta_d\rho - \dot{\eta}_d \tag{60}$$

The transversality conditions yield:

$$\eta_s(\hat{T}) = \eta_r(\hat{T}) = \frac{u_\tau}{\rho} \tag{61}$$

As can be seen only the FOC for S changes when vaccines are introduced. With this in mind we can proceed the same way as for the problem without vaccines and derive with the fact that in the long run equilibrium, where the time derivative of the costates are zero that $\eta_r = \frac{u^W}{\rho} = -\eta_d$.

Assume that the switching function ψ can be 0 over a longer period of time and $\lambda \in (0,1)$. We can divide the switching function by $I > 0$, otherwise no lockdown would be necessary, and get the following equation from (56).

$$\frac{u^W - u^L}{\beta_n} = (\eta_s - \eta_i)S \tag{62}$$

Taking the derivative of (62) with respect to time t yields:

$$0 = (\dot{\eta_s} - \dot{\eta_i})S + \dot{S}(\eta_s - \eta_i) \tag{63}$$

The term $(\dot{\eta_s} - \dot{\eta_i})$ can be expressed using equations (57) and (58).

$$\begin{aligned}
(\dot{\eta_s} - \dot{\eta_i}) =& \\
&= \eta_s\rho - u^W + (\eta_s - \eta_i)I(\beta_n\lambda + \beta_0) + \lambda u^W + (1-\lambda)u^L - (\eta_s - \eta_i)(\beta_n\lambda + \beta_0)S \\
&+ (\eta_r - \eta_i)\gamma_r + (\eta_d - \eta_i)\gamma_d - \eta_i\rho + (\eta_s - \eta_r)\delta \\
&= (1-\lambda)(u^L - u^W) + (\eta_s - \eta_i)\beta_n\lambda(I - S) + \Theta
\end{aligned}$$

where Θ is defined as:

$$\Theta := (\eta_s - \eta_i)(\rho + \beta_0(I - S)) + (\eta_r - \eta_i)\gamma_r + (\eta_d - \eta_i)\gamma_d + (\eta_s - \eta_r)\delta$$
$$\text{(Note that } \Theta \text{ is independent of } \lambda \text{)}$$

Here the only change compared to the version without vaccines, is again the term $(\eta_s - \eta_r)\delta$, which is included in Θ. We can now substitute the expression for $(\dot{\eta_s} - \dot{\eta_i})$ and $\dot{S}$ into equation (63).

$$\begin{aligned}
0 &= S\overbrace{[(1-\lambda)(u^L - u^W) + (\eta_s - \eta_i)\beta_n\lambda(I - S) + \Theta]}^{(\dot{\eta_s} - \dot{\eta_i})}\overbrace{-((\beta_n\lambda + \beta_0)SI}^{+\dot{S}}(\eta_s - \eta_i) \\
0 &= (1-\lambda)\frac{u^L - u^W}{\eta_s - \eta_i} - \beta_n\lambda S + \frac{\Theta}{\eta_s - \eta_i} - \beta_0 I
\end{aligned}$$

Basically we obtain similar results as in the problem without vaccinations, compare section 7.2.2 in appendix B, where we derive equation (91) and conclude that a singular solution for λ is irrelevant if $\psi = 0$ over a longer period of time.

Faced again with a bang-bang solution for λ under the following conditions:

$$\lambda(t) = \begin{cases} 1 & \text{if } \psi > 0 \\ 0 & \text{if } \psi < 0 \end{cases}$$

With the same argumentation as before an immediate containment of the disease makes sense for a social planner. Hence, we again try to determine the condition for a total lockdown to be optimal at the beginning of the pandemic.

Since infections do not spread so quickly in a lockdown, we can work with $S \approx 1$ and $I \approx 0$ to derive an approximation for η_s and η_i. With equation (57), (58) and (59) we get the following representation for η_s and η_i, (again we assume that $\dot{\eta_s} \approx 0$ and $\dot{\eta_i} \approx 0$ at the start of the pandemic):

$$\begin{aligned} \eta_s \rho &= u^w - \eta_s \delta + \eta_r \delta \\ \eta_s(\rho + \delta) &= \frac{u^w}{\rho}(\rho + \delta) \\ \eta_s &= \frac{u^w}{\rho} \end{aligned}$$

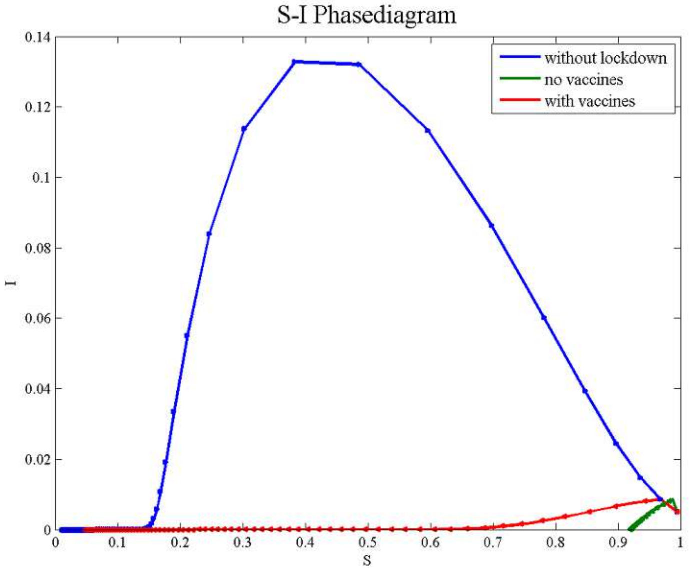
and

$$\begin{aligned} \eta_i \rho &= u^L + (\eta_i - \eta_s)\beta_0 - \eta_i \gamma + \eta_r \gamma_r + \eta_d \gamma_d \\ \eta_i(\rho + \gamma - \beta_0) &= u^L - \eta_s \beta_0 + \gamma_r \eta_r + \eta_d \gamma_d \\ \eta_i &= \frac{u^L + \frac{u^w}{\rho}(\gamma_r - \gamma_d - \beta_0)}{(\rho + \gamma - \beta_0)} \end{aligned}$$

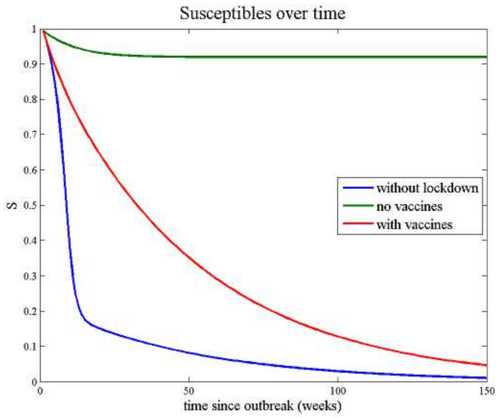
This is exactly the same initial situation as in the original problem which would result in the same condition for the infection fatality rate, compare derivation of equation (93) in appendix B:

$$IFR = \frac{\gamma_d}{\gamma} > \frac{(u^W - u^L)}{u^W} \frac{\rho}{2\beta_n \gamma} \Big[\rho + \gamma - \beta_0 - \beta_n\Big] \tag{64}$$

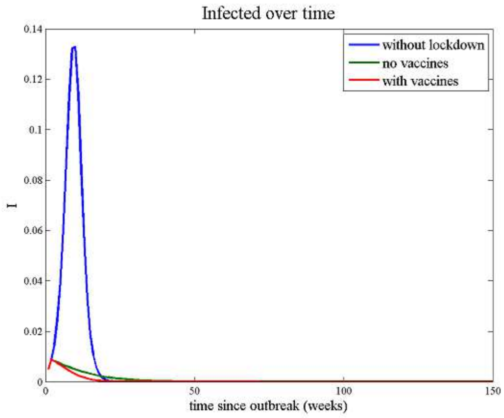
Nevertheless, in figure 12 we can use an example to illustrate the differences between the two variantes, with and without vaccines. An important difference we notice is that the number of susceptibles is steadily decreasing even after infected have reached zero, as expected if individuals keep getting vaccinated. This guarantees that there will be no second outbreak of the disease, as herd immunity is achieved without having to infect a large part of the population. Susceptibles are now decreasing much faster than without vaccines, this can be seen in figure 12 a) where the points on the red line are spread further apart than the points on the green curve. To see the effect of lower infections from vaccines, we take a look at 12 c), where you can see that with vaccines the number of infected individuals approach zero even faster than without vaccines. This effect can also be seen in 12 e), where initially the two paths with and without vaccines are the same but eventually the number of deaths is lower for the variant with vaccines.



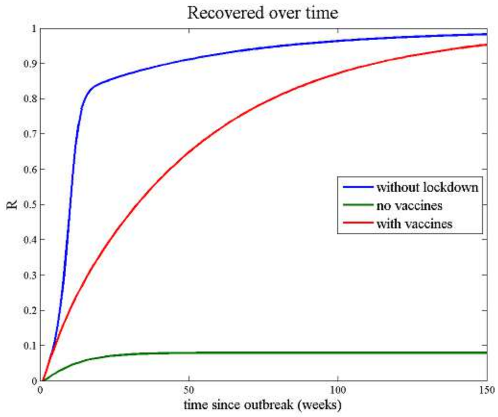
(a) S-I phasediagram



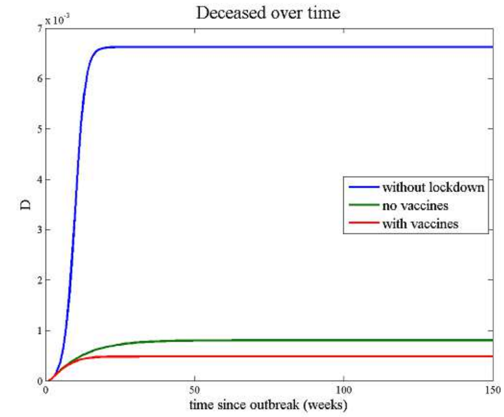
(b) susceptibles



(c) infected



(d) recovered



(e) deceased

Figure 12: Dynamics of the subpopulations S, I, R and D over time for the social planner problem with immediate lockdown of infected and vaccinations.
Source: own calculation, see appendix A for parameter values

4 Conclusion

To conclude my master thesis, I would like to summarize the main results.

Assuming that all individuals behave in the same way, we see that individuals decide to self-isolate as soon as the risk of infection becomes too great. The willingness to work from home slowly decreases over time as infection rates slowly decrease and more and more people develop an immunity. Similar behaviour can be observed, when a vaccine is available and a certain proportion of the population is vaccinated. The time spent in lockdown will be shorter than without vaccines.

For a central planner, the decision of when and for which group to impose a lockdown, depends on several factors. First, the possibility to detect infected and/or recovered individuals. Second, the effectiveness and cost of a lockdown have a great impact on the decision. A short and strict lockdown can stop a pandemic in its early stages, but it would have to be possible to effectively detect infected individuals. If vaccines are already available, together with this early lockdown a second outbreak can be avoided without an additional lockdown. Otherwise, to avoid further outbreaks of the disease, the alternative would be to impose the lockdown only when enough people have developed immunity to provide herd immunity. If there is no way to distinguish between the different groups, this would be a good alternative, as long as the cost of the lockdown is not too high.

Furthermore, it has to be stated that the model used in my thesis takes very basic assumptions about the disease, e.g. that immunity lasts a lifetime. We now know that for Covid-19 the immunization is not permanent. For the introducion of vaccines, we assumed that the vaccines are already available at the start of the pandemic and enough people are willing to get vaccinated. As recent history shows, this is not the case. Further, there is no risk differentiation between the different age groups. Nevertheless, I hope this thesis gave some insight on the dynamics and the potential of lockdowns.

5 Acknowledgments

At this point I would like to thank my supervisor Univ. Prof. Dipl.-Ing. Dr. techn. Alexia Fürnkranz-Prskawetz for always having a quick response and advice when I needed it. Further, I want to thank Miguel Sanchez-Romero for steering me in the right direction and of course Łukasz Rachel for providing further insight on his paper.

I would not be where I am right now without my familiy and friends, who supported me throughout my studies and giving me space and time to recharge my mental batteries. Thank you for always believing in me.

Huge thanks go out to my esteemed friends and colleagues that I met during my time at TU Wien. We shared much frustration and joy, without them I would not have finished my studies. I will always cherish the memories we have, thank you all!

Also I want to thank my dog Kinoko, who forced me to go outside and take breaks these past few months.

Viktor Sommer
Vienna, September 2021

6 References

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7 Appendix

7.1 Appendix A

All quantitative analysis and plots were done in MATLAB R2013b (8.2.0.701), codes can be sent on request.

The parameters[7] for the plots of figure 2, 3 and 11 are provided in table 1 below:

Name	Description	Value
ρ	Discount rate, annualized	4%
γ_d	mortality rate	$0.01\frac{7}{10}$
γ_r	recovery rate	$0.99\frac{7}{10}$
β_0	unavoidable infection rate	0.6
β_n	preventable infection rate	0.8
ϵ	share of infected at time 0	0.015
δ	share of vaccinated individuals per week	2%

Table 1: parameters for decentralized model

The parameters[7] for the plots of figure 4 - 6, 9 and 12 are provided in table 2 below:

Name	Description	Value
ρ	Discount rate, annualized	4%
γ_d	mortality rate	$0.01\frac{7}{10}$
γ_r	recovery rate	$0.99\frac{7}{10}$
β_0	unavoidable infection rate	0.65
β_n	preventable infection rate	0.9
ϵ	share of infected at time 0	0.005

Table 2: parameters for the social planner model

[7]Most of the parameters correspond to the ones used by Rachel [2020, Dec]

7.2 Appendix B

7.2.1 Proof for decentralized equilibrium

The Hamiltonian:

$$H = p_s(\lambda u^W + (1-\lambda)u^L) + (p_i + p_r)u^W - \eta_s p_s(\beta_n\lambda + \beta_0)I + \eta_i(p_s(\beta_n\lambda + \beta_0)I - \gamma p_i) + \eta_r\gamma_r p_i$$

with the necessary FOC:

$$\frac{dH}{d\lambda} = p_s(u^W - u^L) - \eta_s p_s \beta_n I + \eta_i p_s \beta_n I = 0 \tag{65}$$

Since we are faced with a linear optimization problem in λ, the solution will either be 0, 1 or singular in $[0,1]$. From eq. (65) we define the switching function $\psi(t) := \Big(u^W - u^L - (\eta_s - \eta_i)I\beta_n\Big)p_s$ and conclude that $\lambda = 0$ for $\psi(t) < 0$, $\lambda = 1$ for $\psi(t) > 0$ or $\lambda \in (0,1)$ when $\psi(t) = 0$.

$$\begin{aligned}\frac{dH}{dp_s} = (\lambda u^W + (1-\lambda)u^L) - \eta_s(\beta_n\lambda + \beta_0)I + \eta_i(\beta_n\lambda + \beta_0)I &= \eta_s\rho - \dot{\eta}_s \\ (\lambda u^W + (1-\lambda)u^L) + (\eta_i - \eta_s)(\beta_n\lambda + \beta_0)I &= \eta_s\rho - \dot{\eta}_s\end{aligned} \tag{66}$$

$$\frac{dH}{dp_i} = u^W - \eta_i\gamma + \eta_r\gamma_r = \eta_i\rho - \dot{\eta}_i \tag{67}$$

$$\frac{dH}{dp_r} = u^W = \eta_r\rho - \dot{\eta}_r \tag{68}$$

For $t \to \infty$ and the assumption that $\dot{\eta}_s = \dot{\eta}_i = 0$ we can derive from equation (66) and (68) the following:

$$\eta_s = \eta_i = \frac{u^W}{\rho} \tag{69}$$

From equation (67) assuming that $\dot{\eta}_i = 0$ together with (69) we can derive:

$$\eta_i = \frac{u^W + \gamma_r\frac{u^W}{\rho}}{\rho + \gamma} \tag{70}$$

For the cases where $I \approx 0$ i.e. at the beginning and end of the pandemic, $\psi(t)$ is greater than zero. That means individual mitigation starts at $T_0 > 0$ and ends at $T_1 < \infty$. We assume now that $\psi(t) = 0$ can be sustained over an interval of time for $t \in [T_0, T_1]$, which brings us to:

$$\begin{aligned} u^W - u^L &= (\eta_s - \eta_i) I \beta_n \\ (\eta_s - \eta_i) I &= \frac{u^W - u^L}{\beta_n} \end{aligned} \tag{71}$$

Taking the derivative with respect to time of equation (71) we get:

$$(\dot{\eta}_s - \dot{\eta}_i) I + \dot{I}(\eta_s - \eta_i) = 0 \tag{72}$$

Substituting $\dot{\eta}_s$ and $\dot{\eta}_i$ from equation (66) and (67) into equation (72) and together with the model definition equation for $\dot{I}$ we get the following:

$$\begin{aligned} &[-(\lambda u^W + (1-\lambda) u^L) + (\eta_s - \eta_i)(\beta_n \lambda + \beta_0) I + \rho \eta_s + u^W - \eta_i \gamma + \eta_r \gamma_r - \eta_i \rho] I \\ &+ (\eta_s - \eta_i)[(\beta_n \lambda + \beta_0) S I - \gamma I] = 0 \\ \Leftrightarrow &(1-\lambda)(u^W - u^L) + (\eta_s - \eta_i)[(\beta_n \lambda + \beta_0)(S + I) - \gamma + \rho] - \eta_i \gamma + \eta_r \gamma_r = 0 \end{aligned}$$

Substituting now $\eta_s - \eta_i$ from (71) in the equation above yields the following problem, which we solve for λ.

$$(1-\lambda)(u^W - u^L) + \frac{u^W - u^L}{\beta_n I}[(\beta_n \lambda + \beta_0)(S + I) - \gamma + \rho] - \eta_i \gamma + \eta_r \gamma_r = 0 \tag{73}$$

To get to the desired λ^* we start by dividing by $(u^W - u^L)$

$$\begin{aligned} 0 &= 1 - \lambda + \frac{1}{\beta_n I}[(\beta_n \lambda + \beta_0)(S + I) - \gamma + \rho] + \frac{\eta_r \gamma_r - \eta_i \gamma}{(u^W - u^L)} \\ \lambda - \frac{S + I}{\beta_n I} \beta_n \lambda &= 1 + \frac{1}{\beta_n I}[\beta_0 (S + I) - \gamma + \rho] + \frac{\eta_r \gamma_r - \eta_i \gamma}{(u^W - u^L)} \\ -\lambda \frac{S}{I} &= 1 + \frac{1}{\beta_n I}[\beta_0 (S + I) - \gamma + \rho] + \frac{\eta_r \gamma_r - \eta_i \gamma}{(u^W - u^L)} \\ \lambda^* &= \frac{I}{S} + \frac{1}{\beta_n S}[\beta_0 (S + I) - \gamma + \rho] + \frac{I}{S} \frac{\eta_r \gamma_r - \eta_i \gamma}{(u^W - u^L)} \end{aligned} \tag{74}$$

In equilibrium the proportion of infected to susceptible is small and for a small enough discount rate ρ we get a good enough approximation for λ^*

$$\lambda^* \approx \frac{\gamma}{S \beta_n} - \frac{\beta_0}{\beta_n} \tag{75}$$

The stringency of social distancing is decreasing over time:[8]:

$$\frac{d\lambda^*}{dt} = \frac{d\lambda^*}{dS}\frac{dS}{dt} = -\frac{\gamma}{S^2\beta_n}\dot{S} > 0 \tag{76}$$

To get an approximation of the infection rate, let p be the cumulative probability of getting infected in the future. Susceptible individuals will either become infected or stay susceptibles. In the long run we know that $\eta_s = \eta_r$, compare equation (69). Therefore, if we ignore discounting, the shadow value for being susceptible can be approximated by the following:

$$\eta_s \approx p\eta_i + (1-p)\eta_r \tag{77}$$

p is given by, the number of infections that will happen, (which is then given by the current susceptible level minus the long run susceptible level), divided by the amount of current susceptibles, this we approximate as follows, note $\bar{S} := \frac{\gamma}{\beta}$:

$$p = \frac{S - S_\infty}{S} \approx \frac{S - \bar{S}}{S} = 1 - \frac{\bar{S}}{S} \tag{78}$$

where S_∞ is the long-run level of susceptibility in equilibrium. From equation (67) we get an approximation for the steady-state η_i:

$$\begin{aligned} \eta_i &= \frac{u^W + \gamma_r\eta_r}{\rho+\gamma} = \frac{\eta_r\rho + \gamma_r\eta_r}{\rho+\gamma} \\ &= \frac{\eta_r(\rho+\gamma-\gamma_d)}{\rho+\gamma} = \eta_r(1 - \frac{\gamma_d}{\rho+\gamma}) \approx \eta_r(1 - IFR) \end{aligned} \tag{79}$$

Note that $IFR = \frac{\gamma_d}{\gamma}$ is the infection fatality rate. Combining equation (77), (78) and (79) we get:[9]

$$\eta_s - \eta_i \approx \frac{\bar{S}}{S}\eta_r IFR \tag{80}$$

which implies from equation (71) and $\eta_r = \frac{u^W}{\rho}$:

$$I(t) = \frac{u^W - u^L}{\beta_n(\eta_s - \eta_i)} \approx S\frac{\rho\frac{u^W - u^L}{u^W}}{\beta_n \cdot \bar{S} \cdot IFR} \tag{81}$$

□.

[8] In the work-in-progress paper from Rachel [2020, Dec], there is a minor difference when taking the derivative of λ^*. The conclusions, that λ^* increases over time, stays the same though.

[9] In the proof by Rachel [2020, Dec] S and $\bar{S}$ were accidentially switched in the following equation, but were fixed in the subsequent equations.

7.2.2 Proof of infected only isolation

We are now faced with the following Hamiltonian :

$$H = \Big(S(t)u^W + I(t)(\lambda u^W + (1-\lambda)u^L) + R(t)u^W - D(t)u^W\Big) - \eta_s(\beta SI) + \eta_i(\beta SI - \gamma I) + \eta_r\gamma_r I + \eta_d\gamma_d I$$

where $\beta = \beta_n\lambda_i + \beta_0$. Note that since $\lambda_s = \lambda_r = 1$ we drop the subscript for λ_i from here on. The necessary FOC are:

$$\begin{aligned} \frac{dH}{d\lambda} &= Iu^W - Iu^L - \eta_s SI\beta_n + \eta_i\beta_n SI = 0 \\ &\text{with switching function } \psi = (u^W - u^L - (\eta_s - \eta_i)S\beta_n)I \end{aligned} \tag{82}$$

$$\begin{aligned} \frac{dH}{dS} &= u^W - \eta_s(\beta_n\lambda + \beta_0)I + \eta_i((\beta_n\lambda + \beta_0)I = \eta_s\rho - \dot{\eta}_s \\ & u^W + (\eta_i - \eta_s)(\beta_n\lambda + \beta_0)I = \eta_s\rho - \dot{\eta}_s \end{aligned} \tag{83}$$

$$\begin{aligned} \frac{dH}{dI} &= u^W\lambda + (1-\lambda)u^L - \eta_s\beta S + \eta_i[\beta S - \gamma] + \eta_r\gamma_r + \eta_d\gamma_d = \eta_i\rho - \dot{\eta}_i \\ u^W\lambda &+ (1-\lambda)u^L + (\eta_i - \eta_s)(\beta_n\lambda + \beta_0)S - \gamma\eta_i + \eta_r\gamma_r + \eta_d\gamma_d = \eta_i\rho - \dot{\eta}_i \end{aligned} \tag{84}$$

$$\frac{dH}{dR} = u^W = \eta_r\rho - \dot{\eta}_r \tag{85}$$

$$\frac{dH}{dD} = -u^W = \eta_d\rho - \dot{\eta}_d \tag{86}$$

The transversality conditions give us:

$$\eta_s(\hat{T}) = \eta_r(\hat{T}) = \frac{u_\tau}{\rho} \tag{87}$$

In the final steady-state equilibrium, which implies that the time derivative of the costates are zero, we gather from equation (85) and (86) that:

$$\eta_d = -\frac{u^W}{\rho} = -\eta_r \tag{88}$$

Further, comparing (88) with (87) for consistency we conclude that $u_\tau \equiv u^W$. Assume that the switching function ψ can be 0 over a longer period of time and therefore $\lambda \in (0,1)$ we can divide by $I > 0$, otherwise no lockdown would be necessary, and get the following equation from (82).

$$\Rightarrow \frac{u^W - u^L}{\beta_n} = (\eta_s - \eta_i)S \tag{89}$$

Taking the derivative of (89) with respect to time t gives us:

$$0 = (\dot{\eta_s} - \dot{\eta_i})S + \dot{S}(\eta_s - \eta_i) \tag{90}$$

First consider the term $(\dot{\eta_s} - \dot{\eta_i})$ which we can get from (83) and (84).

$$\begin{aligned}
(\dot{\eta_s} - \dot{\eta_i}) = & \\
&= \eta_s\rho - u^W + (\eta_s - \eta_i)I(\beta_n\lambda + \beta_0) + \lambda u^W + (1-\lambda)u^L - (\eta_s - \eta_i)(\beta_n\lambda + \beta_0)S \\
&+ (\eta_r - \eta_i)\gamma_r + (\eta_d - \eta_i)\gamma_d - \eta_i\rho \\
&= (1-\lambda)(u^L - u^W) + (\eta_s - \eta_i)\beta_n\lambda(I - S) + \Theta
\end{aligned}$$

where Θ is defined as:

$$\Theta := (\eta_s - \eta_i)(\rho + \beta_0(I - S)) + (\eta_r - \eta_i)\gamma_r + (\eta_d - \eta_i)\gamma_d$$
(Note that Θ is independent of λ)

We can now substitute the expression for $(\dot{\eta_s} - \dot{\eta_i})$ and $\dot{S}$ into equation (90).

$$0 = S\overbrace{[(1-\lambda)(u^L - u^W) + (\eta_s - \eta_i)\beta_n\lambda(I - S) + \Theta]}^{(\dot{\eta_s} - \dot{\eta_i})}\overbrace{-((\beta_n\lambda + \beta_0)SI}^{+\dot{S}}(\eta_s - \eta_i)$$

In the first step we assume that $S \neq 0$ and divide by S.

$$0 = (1-\lambda)(u^L - u^W) + (\eta_s - \eta_i)\beta_n\lambda(I - S) + \Theta - (\eta_s - \eta_i)\beta_n\lambda I - (\eta_s - \eta_i)\beta_0 I$$

Afterwards the two expressions with $\beta_n\lambda I(\eta_s - \eta_i)$ cancel each other out and to get to the final line we divide by $(\eta_s - \eta_i)$.

$$0 = (1-\lambda)\frac{u^L - u^W}{\eta_s - \eta_i} - \beta_n\lambda S + \frac{\Theta}{\eta_s - \eta_i} - \beta_0 I$$

Now we solve for λ.

$$\lambda(\beta_n S + \frac{u^L - u^W}{\eta_s - \eta_i}) = \frac{u^L - u^W}{\eta_s - \eta_i} + \frac{\Theta}{\eta_s - \eta_i} - \beta_0 I \tag{91}$$

If we substitute now for $(\eta_s - \eta_i)$ with what we know from (89), the left hand side is zero.

$$0 = -\beta_n S + \frac{\Theta\beta_n S}{u^W - u^L} - \beta_0 I$$

We now see that a singular solution for $\lambda \in (0,1)$ is irrelevant in this optimization problem. That means that only a bang-bang solution for λ is eligible. With the switching function

ψ from equation (82) we know that:

$$\lambda(t) = \begin{cases} 1 & \text{if } \psi > 0 \\ 0 & \text{if } \psi < 0 \end{cases}$$

It is obvious that an immediate and total lockdown of individuals at the start of the pandemic, when the number of infected individuals is low, which will keep the lockdown relatively short as well, would be optimal i.e. $\lambda = 0$. Stopping the spread of the disease before the majority of individuals get infected and keeping the number of infected people low will also decrease the costs for a broader lockdown.

At the beginning of the pandemic S is close to but below 1 and under Assumption 1 ($\gamma > \beta_0$), we know that $\dot{I} = \beta_0 IS - \gamma I < 0$, meaning that infected won't increase over time. Therefore, we work with $S \approx 1$ and $I \approx 0$ to derive an approximation for η_s and η_i. With equation (83), (84) and (85) we get the following representation for η_s and η_i, (note that, we assume that $\dot{\eta_s} \approx 0$ and $\dot{\eta_i} \approx 0$ at the start of the pandemic):

$$\eta_s \rho = u^w \Leftrightarrow \eta_s = \frac{u^w}{\rho}$$

and

$$\begin{aligned} \eta_i \rho &= u^L + (\eta_i - \eta_s)\beta_0 - \eta_i \gamma + \eta_r \gamma_r + \eta_d \gamma_d \\ \eta_i(\rho + \gamma - \beta_0) &= u^L - \eta_s \beta_0 + \gamma_r \eta_r + \eta_d \gamma_d \\ \eta_i &= \frac{u^L + \frac{u^w}{\rho}(\gamma_r - \gamma_d - \beta_0)}{(\rho + \gamma - \beta_0)} \end{aligned} \tag{92}$$

With equation[10] (92) and $\psi(t) = u^W - u^L - (\eta_s - \eta_i)\beta_n < 0$ we can now derive the conditions for the lockdown to be optimal. Substituting for η_i and η_s in $(\eta_s - \eta_i)$ we have:

$$\eta_s - \eta_i = \frac{u^W}{\rho} - \frac{u^L + \frac{u^W}{\rho}(\gamma_r - \gamma_d - \beta_0)}{\rho + \gamma - \beta_0} = \frac{(u^W - u^L)\rho + 2\gamma_d u^W}{\rho(\rho + \gamma - \beta_0)}$$

[10] The derivation of η_i differs noticeably from the version found in the proof of prop. 2 in the paper by Rachel [2020, Dec].

substituting this in the inequality from above we get:

$$\begin{aligned}(\eta_s - \eta_i)\beta_n &> u^W - u^L \\ (u^W - u^L)\rho + 2\gamma_d u^W &> \frac{(u^W - u^L)\rho(\rho + \gamma - \beta_0)}{\beta_n} \\ 2\gamma_d u^W &> (u^W - u^L)\frac{\rho}{\beta_n}\Big[\rho + \gamma - \beta_0 - \beta_n\Big] \\ IFR = \frac{\gamma_d}{\gamma} &> \frac{(u^W - u^L)}{u^W}\frac{\rho}{2\beta_n\gamma}\Big[\rho + \gamma - \beta_0 - \beta_n\Big] \end{aligned} \tag{93}$$

Therefore, an immediate lockdown of infected would be optimal if the infection fatality rate is greater than $\frac{u^W - u^L}{u^W}\frac{\rho}{2\gamma\beta_n}\Big(\rho + \gamma - \beta_0 - \beta_n\Big)$, which mainly depends on lockdown efficiency β_n, unavoidable infections β_0 and removal rate of infected (either recovered or deceased) γ. Assuming a positive IFR a full lockdown would definitely be optimal for a sufficiently effective lockdown, $\beta_n \geq (\rho + \gamma - \beta_0)$, since the right hand side of (93) will then be smaller or equal to zero. For a less effective lockdown inequality (93) has to be fulfilled to make the lockdown beneficial. $\square$.

7.2.3 Proof of susceptible only mitigation

The Hamiltonian for susceptible only mitigation looks as follows:

$$H = S(\lambda u^W + (1-\lambda)u^L) + (I + R - D)u^W - \eta_s(\beta SI) + \eta_i(\beta SI - \gamma I) + \eta_r\gamma_r I + \eta_d\gamma_d I$$

where $\beta = \beta_n\lambda_s + \beta_0$. Again we drop the subscript for λ_s from here on. The necessary FOC are:

$$\begin{aligned}\frac{dH}{d\lambda} &= S(u^W - u^L) - \eta_s SI\beta_n + \eta_i\beta_n SI = 0 \\ \Rightarrow \psi &= \Big(u^W - u^L - (\eta_s - \eta_i)I\beta_n\Big)S\end{aligned} \tag{94}$$

$$\frac{dH}{dS} = u^W\lambda + (1-\lambda)u^L - (\eta_s - \eta_i)(\beta_n\lambda + \beta_0)I = \eta_s\rho - \dot{\eta}_s \tag{95}$$

$$\frac{dH}{dI} = u^W - (\eta_s - \eta_i)(\beta_n\lambda + \beta_0)S - \eta_i\gamma + \eta_r\gamma_r + \eta_d\gamma_d = \eta_i\rho - \dot{\eta}_i \tag{96}$$

$$\frac{dH}{dR} = u^W = \eta_r\rho - \dot{\eta}_r \tag{97}$$

$$\frac{dH}{dD} = -u^W = \eta_d\rho - \dot{\eta}_d \tag{98}$$

The transversality conditions again give us:

$$\eta_s(\hat{T}) = \eta_r(\hat{T}) = \frac{u_\tau}{\rho} \tag{99}$$

In the final steady-state equilibrium, which implies that the time derivative of the costates are zero, we gather from equation (97) and (98) that:

$$\eta_d = -\frac{u^W}{\rho} = -\eta_r \tag{100}$$

Further, comparing (100) with (99) for consistency we conclude that $u_\tau \equiv u^W$. We assume now that the switching function $\psi = 0$ over a longer period of time, which yields the following:

$$(\eta_s - \eta_i)I = \frac{u^W - u^L}{\beta_n} \tag{101}$$

Taking the derivative of (101) with respect to time and substitute with what we know from (95) and (96):

$$\begin{aligned} 0 &= (\dot{\eta_s} - \dot{\eta_i})I + (\eta_s - \eta_i)\dot{I} = \\ &= [(1-\lambda)(u^w - u^L) + (\eta_s - \eta_i)(\beta_n\lambda + \beta_0)(I - S) - \eta_i\gamma + \eta_r\gamma_r + \eta_d\gamma_d + \rho(\eta_s - \eta_i)]I \\ &\quad + (\eta_s - \eta_i)[(\beta_n\lambda + \beta_0)SI - \gamma I] \end{aligned}$$

A possible solution would be $I = 0$, let's assume that $I > 0$ and divide by I. Which yields the following equation

$$0 = (1-\lambda)(u^W - u^L) + (\eta_s - \eta_i)(\beta_n\lambda + \beta_0)I - \eta_i\gamma + \eta_r\gamma_r + \eta_d\gamma_d + (\rho - \gamma)(\eta_s - \eta_i)$$

Substituting for $(\eta_s - \eta_i)$ from (101) delivers us:

$$(1-\lambda)(u^w - u^L) + \frac{(u^w - u^L)(\beta_n\lambda + \beta_0)}{\beta_n} - \eta_i\gamma + \eta_r\gamma_r + \eta_d\gamma_d + \frac{(u^w - u^L)}{\beta_n I}(\rho - \gamma) = 0$$

if we divide by $(u^W - u^L)$ it is clear that both λ cancel each other out.

$$1 - \lambda + \lambda + \frac{\beta_0}{\beta_n} + \frac{(\rho - \gamma)}{\beta_n I} + \frac{\eta_d\gamma_d - \eta_i\gamma + \eta_r\gamma_r}{(u^w - u^L)} = 0 \tag{102}$$

Similar to the infected only mitigation we conclude that ψ cannot be 0 over a longer period of time and only a bang-bang solution for λ is optimal. Again we get:

$$\lambda(t) = \begin{cases} 1 & \text{if } \psi > 0 \Leftrightarrow S(u^W - u^L) > (\eta_s - \eta_i)\beta_n IS \\ 0 & \text{if } \psi < 0 \Leftrightarrow S(u^W - u^L) < (\eta_s - \eta_i)\beta_n IS \end{cases}$$

If we divide both sides of the condition above by $S > 0$, we see that the left hand side is constant and always positive. Now it all depends on $(\eta_s - \eta_i)\beta_n I$ which starts out very low since $I(0) \approx 0$ which means we begin with $\lambda = 1$. The number of infected will increase over time as long as no lockdown is imposed, which means λ will remain 1 until then. We know that I reaches its maximum (I_{max}) at $\bar{S}$ with that we can derive an approximation of $\eta_s - \eta_i$. Assuming that until now no lockdown was imposed, which means $\lambda = 1$, from (95) we get, (note that for approximation purposes we assume that $\dot{\eta_s} \approx 0$ at this point in time):

$$\eta_s = \frac{u^W + \eta_i \beta I_{max}}{\rho + \beta I_{max}} \approx \frac{u^W}{\beta I_{max}} + \eta_i$$
$$\Rightarrow \eta_s - \eta_i \approx \frac{u^W}{\beta I_{max}} \tag{103}$$

We can now substitute (103) in our condition for $\psi < 0$ and get :

$$u^W - u^L < (\eta_s - \eta_i)\beta_n I_{max} = u^W \frac{\beta_n}{\beta} \tag{104}$$

For a sufficiently effective lockdown, i.e. $\beta_0(\frac{u^W}{u^L} - 1) < \beta_n$, at a certain point in time, the right hand side of (104) will surpass $(u^W - u^L)$ and λ will switch to zero. This means that there is a point, at least once, where a lockdown would make sense. We now want to determine a time at which a lockdown would be optimal, so that it stops exactly when herd immunity is reached and is not too late and does not cause unnecessary deaths. Let S^* and I^* be the levels of susceptibles and infected at this specific point in time. To determine the point (S^*, I^*), similar as Rachel [2020,Jul], we divide the equations for $\dot{I}$ by $\dot{S}$ and get a first order ordinary differential equation:

$$\frac{\dot{I}}{\dot{S}} = \frac{\frac{dI}{dt}}{\frac{dS}{dt}} = \frac{\beta SI - \gamma I}{-\beta SI}$$
$$\frac{dI}{dS} = -1 + \frac{\bar{S}}{S}$$
$$\Leftrightarrow \int_0^t 1dI = \int_0^t \Big(-1 + \frac{\bar{S}}{S}\Big)dS$$
$$\Leftrightarrow I(t) = -S(t) + \bar{S}log(S(t)) + I(0) + S(0) - \bar{S}log(S(0)) \tag{105}$$

The point (S^*, I^*) is now determined by two equations both derived by (105). First we start from the initial conditions $\lambda = 1$ and $S(0) = 1 - \epsilon$ and $I(0) = \epsilon$ or rather with its approximations $S(0) \approx 1$ and $I(0) \approx 0$, which will end at the desired point (S^*, I^*) resulting in:

$$I^* = -S^* + \bar{S}log(S^*) + 1 - \bar{S}log(1) \tag{106}$$

The second equation is given by (S^*, I^*) as starting point and $\lambda = 0$ and the final point is given by $\bar{S}$ with the desired level of infected $I(t) = 0$. This gives us the following equation again derived from eq. (105), here $\bar{S}_L = \frac{\gamma}{\beta_0}$ and denotes the herd immunity threshold level if a lockdown is imposed:

$$0 = -\bar{S} + \bar{S}_L log(\bar{S}) + I^* + S^* - \bar{S}_L log(S^*) \tag{107}$$

We can now substitute equation (106) for I^* in eq. (107):

$$0 = -\bar{S} + \bar{S}_L log(\bar{S}) + \bar{S} log(S^*) + 1 - \bar{S}_L log(S^*) \tag{108}$$

This equation can now be solved for S^* to determine to what level susceptibles have to fall before a total lockdown is imposed.

$$S^* = exp\Big(\frac{1 - \bar{S} + \bar{S}_L log(\bar{S})}{\bar{S}_L - \bar{S}}\Big) \tag{109}$$

□.

7.2.4 Derivation of equation (25)

Starting with:

$$(\dot{\eta}_s - \dot{\eta}_i)SI\beta_n + (\eta_s - \eta_i)\beta_n(\dot{S}I + S\dot{I}) = (\dot{S} + \dot{I})(u^W - u^L) \tag{110}$$

with equation (19) and (20) we obtain the following for $\dot{\eta}_s - \dot{\eta}_i$:

$$\dot{\eta}_s - \dot{\eta}_i = (\eta_s - \eta_i)\beta(I - S) + (\eta_s - \eta_i)\rho - \gamma\eta_i + \frac{u^W}{\rho}(\gamma_r - \gamma_d) \tag{111}$$

For $\dot{S}I + S\dot{I}$ and $\dot{S} + \dot{I}$ we use equations (1) and (2) to get:

$$\begin{aligned} \dot{S}I + S\dot{I} &= SI(\beta(S - I) - \gamma) \\ \dot{S} + \dot{I} &= -\gamma I \end{aligned} \tag{112}$$

Equation (111) and (112) can now be substituted in (110) and we get the following:

$$\begin{aligned} \Big((\eta_s - \eta_i)(\beta(I - S) + \rho) - \gamma\eta_i + \frac{u^W}{\rho}(\gamma_r - \gamma_d)\Big)SI\beta_n + (\eta_s - \eta_i)\beta_n SI(\beta(S - I) - \gamma) \\ = -\gamma I(u^W - u^L) \\ SI\beta_n\Big((\eta_s - \eta_i)\rho - \gamma\eta_i + \frac{u^W}{\rho}(\gamma_r - \gamma_d) - (\eta_s - \eta_i)\gamma\Big) = -\gamma I(u^W - u^L) \end{aligned} \tag{113}$$

Equation (113) is now equivalent to equation (25) □

7.2.5 Derivation of λ_V^*

With the Hamiltonian and the first order conditions as stated in section 3, we assume again that the switching function ψ can be 0 over a longer period of time giving us the following equation:

$$\frac{u^W - u^L}{\beta_n} = (\eta_s - \eta_i)I \tag{114}$$

taking the derivative of both sides with respect to time t results in:

$$(\dot{\eta}_s - \dot{\eta}_i)I + \dot{I}(\eta_s - \eta_i) = 0 \tag{115}$$

Substituting now (52) and (53) into (115) we get:

$$\begin{aligned}
&\Big[-(\lambda u^W + (1-\lambda)u^L) + (\eta_s - \eta_i)(\beta_n\lambda + \beta_0)I + (\eta_s - \eta_r)\delta + \eta_s\rho + u^W - \eta_i\gamma + \eta_r\gamma_r - \eta_i\rho \Big] I \\
&+ (\eta_s - \eta_i)I((\beta_n\lambda + \beta_0)S - \gamma) = 0 \\
&(1-\lambda)(u^W - u^L) + (\eta_s - \eta_i)[(\beta_n\lambda + \beta_0)(S + I) - \gamma + \rho] + (\eta_s - \eta_r)\delta - \eta_i\gamma + \eta_r\gamma_r = 0
\end{aligned} \tag{116}$$

Substituting now (114) for $(\eta_s - \eta_i)I$:

$$\begin{aligned}
0 &= 1 - \lambda + \frac{1}{\beta_n I}[(\beta_n\lambda + \beta_0)(S + I) - \gamma + \rho] + \frac{\eta_r\gamma_r - \eta_i\gamma + (\eta_s - \eta_r)\delta}{(u^W - u^L)} \\
\lambda - \frac{S+I}{\beta_n I}\beta_n\lambda &= 1 + \frac{1}{\beta_n I}[\beta_0(S + I) - \gamma + \rho] + \frac{\eta_r\gamma_r - \eta_i\gamma + (\eta_s - \eta_r)\delta}{(u^W - u^L)} \\
-\lambda\frac{S}{I} &= 1 + \frac{1}{\beta_n I}[\beta_0(S + I) - \gamma + \rho] + \frac{\eta_r\gamma_r - \eta_i\gamma + (\eta_s - \eta_r)\delta}{(u^W - u^L)} \\
\lambda_V^* &= \frac{I}{S} + \frac{1}{\beta_n S}[\beta_0(S + I) - \gamma + \rho] + \frac{I}{S}\frac{\eta_r\gamma_r - \eta_i\gamma + (\eta_s - \eta_r)\delta}{(u^W - u^L)}
\end{aligned} \tag{117}$$

Since $\frac{I}{S}$ is small in equilibrium and for a small enough ρ we can get a good approximation for λ_V^*

$$\lambda_V^* \approx \frac{\gamma - \beta_0 S - \delta}{\beta_n S} \tag{118}$$

□