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Isoperimetric Problems for Minkowski Valuations

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von

Dipl.-Ing. Georg Hofstätter, BSc BSc
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Georg Hofstätter

Kurzfassung

Diese Arbeit enthält Beiträge zur Lösung von isoperimetrischen Problemen für Minkowski-Bewertungen und zu funktionalen Ungleichungen.

Zuerst wird gezeigt, dass jeder monotone Minkowski-Endomorphismus auf der Menge der konvexen Körper eine isoperimetrische Ungleichung erfüllt, die die klassische Urysohn-Ungleichung impliziert. Dabei ist die Blaschke–Santaló-Ungleichung für ursprungssymmetrische Körper – die einzige Ungleichung in dieser neuen Familie, die invariant unter affinen Transformationen ist – die stärkste Ungleichung. Weiters wird gezeigt, dass sich diese Familie von Ungleichungen nicht auf die Menge der schwach-monotonen Minkowski-Endomorphismen ausdehnen lässt.

Im zweiten Teil der Arbeit werden sogenannte Asplund-Endomorphismen eingeführt, die das Konzept der Minkowski-Endomorphismen auf (koerzive) log-konkave Funktionen verallgemeinern. Es wird eine umfassende Familie von monotonen Asplund-Endomorphismen konstruiert, von denen jeder durch Einschränkung auf Indikatorfunktionen konvexer Körper auf einen monotonen Minkowski-Endomorphismus zurückgeführt werden kann. Für die konstruierte Familie wird anschließend eine Familie analytischer Ungleichungen gezeigt, von denen jede stärker als die funktionale Urysohn-Ungleichung ist. Die stärkste Ungleichung der neuen Familie ist die funktionale Blaschke–Santaló-Ungleichung für gerade Funktionen. Durch Einschränken der funktionalen Ungleichung auf Indikatorfunktionen erhält man die geometrischen Ungleichungen aus dem ersten Teil der Arbeit in einer asymptotisch optimalen Form zurück.

Im dritten Teil der Arbeit wird gezeigt, dass jede stetige gerade Minkowski-Bewertung vom Grad $1 \leq i \leq n-1$ auf der Menge der konvexen Körper, die mit Drehungen vertauscht, durch Faltung der i-ten Projektionsfunktion mit einer eindeutig bestimmten sphärischen Crofton-Distribution dargestellt werden kann. Ist diese Distribution nicht-negativ, dann existieren isoperimetrische Ungleichungen für das polare Volumen der assoziierten Minkowski-Bewertung, die die klassische Ungleichung zwischen i-tem Quermaßintegral und Volumen verschärfen. Diese große Familie an Ungleichungen vereinheitlicht frühere Ergebnisse für $i = 1$ (aus dem ersten Teil der Arbeit) und $i = n-1$. In diesen beiden Fällen wurde gezeigt, dass die isoperimetrischen Ungleichungen für die affinen Quermaßintegrale, genauer die Blaschke–Santaló-Ungleichung für $i = 1$ und die polare Petty-Projektionenungleichung für $i = n-1$, die stärksten Ungleichungen sind. Hier wird ein analoges Resultat für die dazwischen liegenden Homogeneitätsgrade bewiesen.

Schließlich wird eine neue hinreichende Bedingung für die Existenz von minimierenden beziehungsweise maximierenden Körpern für das Volumen oder das polare Volumen von Minkowski-Bewertungen, die mit Drehungen vertauschen, gezeigt. Dieses Resultat führt zu unerwarteten Beispielen von isoperimetrischen Problemen mit maximierenden Körpern, die von Kugeln verschieden sind (und diese nicht inkludieren).

Abstract

This thesis contains contributions to the solution of isoperimetric problems for Minkowski valuations, as well as to functional inequalities.

First, it is shown that each monotone Minkowski endomorphism of convex bodies, gives rise to an isoperimetric inequality which directly implies the classical Urysohn inequality. Among this large family of new inequalities, the only affine invariant one – the Blaschke–Santaló inequality (for origin-symmetric convex bodies) – turns out to be the strongest one. A further extension of these inequalities to merely weakly monotone Minkowski endomorphisms is proven to be impossible.

Secondly, the new notion of Asplund endomorphisms that generalizes Minkowski endomorphisms to the setting of (coercive) log-concave functions is introduced. A large family of monotone Asplund endomorphisms is constructed, each restricting to a monotone Minkowski endomorphism on indicators of convex bodies. Moreover, a family of analytic inequalities is proven for the constructed Asplund endomorphisms, where every inequality is stronger than the functional Urysohn inequality. The strongest one among the new family of inequalities is the functional Blaschke–Santaló inequality for even functions. By restricting the inequalities to indicators, the geometric inequalities of the first part of this thesis are recovered in an asymptotically optimal form.

Thirdly, it is shown that each continuous even Minkowski valuation on convex bodies of degree $1 \leq i \leq n-1$ intertwining rigid motions is obtained from convolution of the ith projection function with a unique spherical Crofton distribution. In case of a non-negative distribution, the polar volume of the associated Minkowski valuation gives rise to an isoperimetric inequality which strengthens the classical relation between the ith quermassintegral and the volume. This large family of inequalities unifies earlier results obtained for $i = 1$ (in the first part of the thesis) and $i = n-1$. In these cases, isoperimetric inequalities for affine quermassintegrals, specifically the Blaschke–Santaló inequality for $i = 1$ and the polar Petty projection inequality for $i = n-1$, were proven to be the strongest inequalities. An analogous result for the intermediate degrees is established here.

Finally, a new sufficient condition for the existence of extremals for the volume and the polar volume of Minkowski valuations intertwining rigid motions reveals unexpected examples of isoperimetric inequalities having extremals which are not (and do not include) Euclidean balls.

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Contents

1 Introduction

The classical isoperimetric problem is the question which convex bodies, that is, compact and convex subsets of the Euclidean space, of a given volume have minimal surface area. Dating back to ancient times, variants of the isoperimetric problem appear, e.g., in tales about Dido, the first queen of Carthage, who faced the task to enclose the maximal area with a string made from a bull's hide. Albeit the solutions of the isoperimetric problem were known since then to be Euclidean balls, this was rigorously proved only in the nineteenth century using, e.g., a symmetrization technique developed by J. Steiner.

Following this proof, a series of similar isoperimetric problems were posed and (partly) solved for other geometric functionals that arose in the development of the theory of convex bodies, the most classical ones being surface area, mean width and all other quermassintegrals. Typically, the functionals for isoperimetric problems are invariant under rigid motions. While these functionals are often related to variations of the volume, during the last century, functionals that can be written as the (polar) volume composed with an operator on the space of convex bodies, based on natural constructions involving, e.g., projections and sections, gained a lot of interest. Indeed, some of the most fundamental inequalities like the Blaschke–Santaló inequality (for the difference body map for origin-symmetric convex bodies) and Petty's polar projection inequality (for the projection body map) are of this type. Moreover, two of the major open problems of convex geometry, Mahler's conjecture and the conjectured Petty projection inequality, can be stated as isoperimetric problems for the difference body and the projection body.

This thesis contributes to the solution of isoperimetric problems for classes of operators on the space of convex bodies which are finitely additive (in a set-theoretic sense), so-called Minkowski valuations, and compatible with rigid motions. Examples include the difference and projection body maps. We consider mainly polar isoperimetric inequalities similar to Petty's polar projection inequality and the Blaschke–Santaló inequality for large subclasses of Minkowski valuations, extending previous results by Haberl and Schuster [61] and Berg and Schuster [25].

In the first part of this thesis, which is joint work with F.E. Schuster, we consider inequalities for Minkowski endomorphisms, that is, Minkowski additive maps which are compatible with rigid motions, – these are exactly rigid motion compatible, one-homogeneous Minkowski valuations. Based on a representation result by Kiderlen [67], we prove that every monotone (with respect to set-inclusion) Minkowski endomorphism satisfies a sharp polar isoperimetric inequality, whereas there exists a non-monotone Minkowski endomorphism for which the functional in the isoperimetric problem is unbounded.

Moreover, we show that the ideas for Minkowski endomorphisms can be used to solve an analogous functional isoperimetric problem. For this reason, we introduce the new notion of Asplund endomorphisms that generalizes Minkowski endomorphisms to the setting of (coercive) log-concave functions, extending Minkowski additivity to additivity with respect to the Asplund sum. After constructing a large family of monotone Asplund endomorphisms, we prove sharp polar isoperimetric inequalities for this family. By restricting the inequalities to indicators, the corresponding geometric inequalities for monotone Minkowski endomorphisms are recovered in an asymptotically optimal form.

The second part of this thesis, which is joint work with P. Kniefacz and F.E. Schuster, is concerned with isoperimetric problems for Minkowski valuations of arbitrary degree of homogeneity. Here, even Minkowski valuations, which are compatible with rigid motions, play an important role as they admit a representation that generalizes the one for even and monotone Minkowski endomorphisms by Kiderlen. Indeed, extending a result by Schuster [113] and Schuster and Wannerer [115] for smooth and even Minkowski valuations and using results by Alesker and Faifman [16] for real-valued valuations, we prove that every even and continuous Minkowski valuation, compatible with rigid motions, is obtained from a convolution of a projection function with a unique spherical Crofton distribution. Based on our representation result, we prove that every even Minkowski valuation with non-negative spherical Crofton distribution satisfies a sharp polar isoperimetric inequality.

Finally, we take one step back and ask the question when isoperimetric problems for Minkowski valuations do possess extremals. In contrast to isoperimetric problems which are invariant under volume-preserving linear transformations and therefore possess both maximizers and minimizers, like the problems for the difference body and projection body maps, it is not at all clear why this should be true for general Minkowski valuations intertwining only rotations. Indeed, our example of a non-monotone Minkowski endomorphism does not possess maximizers. However, with our next result we give a new sufficient condition for the existence of extremals of (polar and non-polar) isoperimetric problems for Minkowski valuations compatible with rigid motions. Moreover, our sufficient condition in connection with the example yields Minkowski valuations that possess maximizers which are not (and do not include) Euclidean balls.

This thesis is structured as follows: Chapter 2 contains necessary background material. This chapter contains no new results. However, we sometimes give proofs for the reader's convenience. In Chapters 3 and 4, we prove the inequalities for monotone Minkowski endomorphisms and the constructed family of Asplund endomorphisms. The results of these chapters are joint work with F.E. Schuster and already pre-published in [65]. Chapter 5 contains representations and sharp inequalities for (subclasses of) even Minkowski valuations and sufficient conditions for the existence of extremals for Minkowski valuations of arbitrary degree. The results of this chapter are based on joint work with P. Kniefacz and F.E. Schuster and will be published in [64]. Partial results of Section 5.4 were already published in [72].

2 Background

Before stating and proving the main results, we have to recall the required notions and facts from the underlying theories and to fix notation. In this chapter, we first revisit functions and distributions on Grassmanians and on the sphere, their convolution and give the definitions of well-known transforms such as the Cosine and the Radon transforms. We then turn to the Brunn–Minkowski theory of convex bodies including the definition of mixed volumes and the statement of important isoperimetric inequalities. This is complemented by a section about convex and log-concave functions, the Legendre transform and functional inequalities. In the remaining section, we recall definitions and principles from the theory of valuations.

2.1 General Notions

In this section, we fix the notation for general notions regarding the geometry of the Euclidean space as well as functions and distributions. As general references we refer to the books by Rudin [102, 103].

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 3$, with the usual inner product and norm, denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ or $\|\cdot\|_2$, respectively, the unit ball $B^n = \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$ and the unit sphere $\mathbb{S}^{n-1} = \operatorname{bd} B^n$. We denote by $e_1, \ldots, e_n$ the standard orthonormal basis of $\mathbb{R}^n$. The Grassmanian manifold of i-dimensional subspaces of $\mathbb{R}^n$ is denoted by $\operatorname{Gr}(n, i)$ and we use the unique probability measure ν_i on $\operatorname{Gr}(n, i)$ that is invariant under the action of the special orthogonal group $\mathrm{SO}(n)$. The uniform probability measure on $\mathbb{S}^{n-1}$ is denoted by σ and the uniform probability measure on any i-dimensional subsphere $\mathbb{S}^{n-1} \cap E$, with $E \in \operatorname{Gr}(n, i+1)$ is denoted by σ_i or σ_E. Integration with respect to the spherical Lebesgue measure of total mass $n\kappa_n$ is denoted by du, that is, $du = n\kappa_n d\sigma$.

For every $i \in \{1, \ldots, n\}$, we define the subspace $\bar{E}_i = \operatorname{span}\{e_1, \ldots, e_i\} \in \operatorname{Gr}(n, i)$. The spanning vector of $\bar{E}_1$ is also denoted by $\bar{e} = e_1$. All the constructions we are using are independent of the choice of orthonormal basis or pole $\bar{e}$. We fix this basis just for convenience, although any other choice is equally fine. For every subspace $E \in \operatorname{Gr}(n, i)$ we fix an orientation preserving transformation $\vartheta_E \in \mathrm{SO}(n)$ which satisfies $\vartheta_E \bar{E}_i = E$. This choice is unique up to a right-multiplication of $\eta \in \mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$, where $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ denotes the stabilizer of $\bar{E}_i$ in $\mathrm{SO}(n)$. In the special case $i = 1$, we use the notation $\vartheta_u \in \mathrm{SO}(n)$ with $\vartheta_u \bar{e} = u$ for $u \in \mathbb{S}^{n-1}$ and $\mathrm{SO}(n-1)$ for the stabilizer of $\bar{e}$.

Taking the orthogonal complement, $E \mapsto E^\perp$, provides a continuous bijection between $\operatorname{Gr}(n, i)$ and $\operatorname{Gr}(n, n-i)$. This can be lifted to an operator on continuous functions, $\perp^*\colon C(\mathrm{Gr}(n,i)) \to C(\mathrm{Gr}(n,n-i))$, by setting $\perp^*(f)(E) = f(E^\perp)$, for $E \in \mathrm{Gr}(n,n-i)$ and $f \in C(\mathrm{Gr}(n,i))$. This is sometimes also denoted by $f^\perp$.

When speaking of measures (on $\mathbb{S}^{n-1}$ or on $\mathrm{Gr}(n,i)$) we implicitly assume that they are finite and signed Borel measures. A measure μ on $\mathbb{S}^{n-1}$ is called zonal, if it is invariant under $\mathrm{SO}(n-1)$, that is, for each Borel set $A \subseteq \mathbb{S}^{n-1}$ we have $\mu(A) = \mu(\tau A)$ for any $\tau \in \mathrm{SO}(n-1)$. If μ is zonal and absolutely continuous with respect to the spherical Lebesgue measure on the sphere, then its density f is zonal, in the sense that $f(u) = f(\tau u)$, for any $\tau \in \mathrm{SO}(n-1)$ and $u \in \mathbb{S}^{n-1}$. A zonal function depends only on $\langle u, \bar{e}\rangle$ instead of $u \in \mathbb{S}^{n-1}$. We therefore can associate to it a function $\tilde{f}$ on the closed interval $[-1,1]$ by $f(u) = \tilde{f}(\langle u, \bar{e}\rangle)$. From this representation it is also clear that a zonal function f satisfies $f(\vartheta_u^{-1} v) = f(\vartheta_v^{-1} u)$ for any $u, v \in \mathbb{S}^{n-1}$, or, more directly, $f(\eta\bar{e}) = f(\eta^{-1}\bar{e})$ for every $\eta \in \mathrm{SO}(n)$.

Any function f on $\mathbb{S}^{n-1}$ or $\mathrm{Gr}(n,i)$ gives rise to a zonal function $\bar{f}$ that is obtained by an $\mathrm{SO}(n-1)$ mean of f, that is,

$$\bar{f}(x) = \int_{\mathrm{SO}(n-1)} f(\tau x)\, d\tau \tag{2.1}$$

for $x \in \mathbb{S}^{n-1}$ or $x \in \mathrm{Gr}(n,i)$, respectively.

For the theory of smooth and generalized valuations we need the notions of smooth maps on manifolds and of distributions. Let M be one of the smooth manifolds $\mathbb{R}^n$, $\mathrm{SO}(n)$, $\mathbb{S}^{n-1}$ or $\mathrm{Gr}(n,i), i \in \{1, \dots, n-1\}$. We denote by $C^\infty(M)$ the set of smooth real-valued functions on M (in the sense of smooth functions on a manifold). $C^\infty(M)$ is a Fréchet space endowed with the family of seminorms given by

$$\|f\|_{C^k(K)} = \sum_{j=0}^{k} \max_{x \in K} \|\nabla^j f(x)\|,$$

where $k \in \mathbb{N}$, $K \subseteq M$ is any compact subset of M and $\nabla^j f$ denotes the jth derivative of f. Consequently, a sequence $(f_n)_{n\in\mathbb{N}}$, $f_n \in C^\infty(M)$, converges to a function $f \in C^\infty(M)$, if every derivative $\nabla^k f_n$ converges uniformly on compact subsets to $\nabla^k f$, $k \in \mathbb{N}$.

The space of distributions $C^{-\infty}(M)$ is defined as the topological dual space of the subspace $C_c^\infty(M)$ of functions in $C^\infty(M)$ with compact support, that is, it consists of all continuous linear functionals on $C_c^\infty(M)$. As the spaces $\mathrm{Gr}(n,i)$ and $\mathbb{S}^{n-1}$ are compact, we will sometimes omit the condition of compact support in our notation. Note that $C^{-\infty}(M)$ usually denotes the space of generalized functions, that is, of continuous linear functionals $C_c^\infty(M, |\Lambda|(M)) \to \mathbb{R}$, where $|\Lambda|(M)$ is the one-dimensional space of smooth densities on M (see, e.g., [60, p. 306]). If M is a Riemannian manifold, then the Riemannian volume form induces an isomorphism $\mathbb{R} \cong |\Lambda|(M)$ and therefore an isomorphism between the spaces of distributions and generalized functions. In the following, we will use this identification and denote distributions by $C^{-\infty}(M)$.

We write $\langle \cdot, \cdot \rangle_{C^{-\infty}(M)}$ for the dual pairing, that is, $\langle \delta, f\rangle_{C^{-\infty}(M)} = \delta(f)$ whenever $\delta \in C^{-\infty}(M)$ and $f \in C_c^\infty(M)$. The space $C^{-\infty}(M)$ is endowed with the weak-star topology, that is, a sequence $(\delta_k)_{k\in\mathbb{N}}$, $\delta_k \in C^{-\infty}(M)$, converges to $\delta \in C^{-\infty}(M)$, if $\langle \delta_k, f\rangle_{C^{-\infty}(M)}$ converges to $\langle \delta, f\rangle_{C^{-\infty}(M)}$ for all $f \in C_c^\infty(M)$. Any locally integrable function f on $\mathbb{S}^{n-1}$ or $\mathrm{Gr}(n,i)$ can be embedded into the space of distributions by the identification $f \mapsto \langle f, \cdot\rangle_{L^2(M)}$, $M = \mathbb{S}^{n-1}$ or $M = \mathrm{Gr}(n,i)$, respectively, where $\langle \cdot,\cdot\rangle_{L^2(M)}$ denotes the standard L^2 inner product with respect to the unique $\mathrm{SO}(n)$ invariant probability measure on $\mathbb{S}^{n-1}$ or $\mathrm{Gr}(n,i)$, respectively. For measures μ this works similarly by integrating the test function against μ, that is, $\langle \mu, \varphi\rangle_{C^{-\infty}(M)} = \int_M \varphi\, d\mu$, $\varphi \in C_c^\infty(M)$. We will sometimes not distinguish between a function or measure and its associated distribution.

We will also work with non-negative distributions, that is, distributions δ that satisfy $\langle \delta, \varphi\rangle_{C^{-\infty}(M)} \geq 0$ for every $\varphi \in C_c^\infty(M)$ with $\varphi \geq 0$. A non-negative distribution is a continuous linear functional with respect to the topology of uniform convergence on compact subsets, that is, a sequence φ_n converges if it converges in the norm $\|f\|_\infty = \sup_{x\in M} |f(x)|$ and the supports of all φ_n are contained in a common compact set. The Hahn–Banach theorem therefore implies that δ can be extended to a continuous functional on the set of continuous functions with compact support. The theorem of Riesz–Markov now implies that δ, in fact, must be a regular non-negative Borel measure.

2.2 Convolutions and Multiplier Transforms

Many representation results on Minkowski valuations use the notion of convolution on the sphere or a Grassmanian. Indeed, any formula for a class of real-valued valuations that works as L^2 inner product on some function or measure associated to a convex body with fixed functions or measures can be translated to a convolution representation for a corresponding class of $\mathrm{SO}(n)$ equivariant Minkowski valuations. We recommend the book by Groemer [59] and the article by Grinberg and Zhang [55] as general reference on convolutions and we will follow the expositions in [113, 115].

Convolutions on the sphere and Grassmanians are tightly linked to representations of the Lie group $\mathrm{SO}(n)$ (or any of its subgroups H) on the spaces of functions, measures or distributions on $\mathrm{SO}(n)$. Letting H be a subgroup of $\mathrm{SO}(n)$, the left-action ℓ_ϑ and the right-action r_ϑ of an element $\vartheta \in H$ on a function $f \in C(\mathrm{SO}(n))$ is defined by $(\ell_\vartheta f)(\tau) = f(\vartheta^{-1}\tau)$ and $(r_\vartheta f)(\tau) = f(\tau\vartheta)$, for any $\tau \in \mathrm{SO}(n)$, respectively. We will sometimes abbreviate $\ell_\vartheta f$ by ϑf. The left- and right-action are extended to measures and distributions by

$$\langle \ell_\vartheta \delta, \varphi\rangle_{C^{-\infty}} = \langle \delta, \ell_{\vartheta^{-1}}\varphi\rangle_{C^{-\infty}} \quad \text{and} \quad \langle r_\vartheta \delta, \varphi\rangle_{C^{-\infty}} = \langle \delta, r_{\vartheta^{-1}}\varphi\rangle_{C^{-\infty}},$$

for any $\varphi \in C_c^\infty(\mathrm{SO}(n))$. This definition is compatible with the embeddings of functions and measures in the space of distributions.

A function, measure or distribution is called left- or right-H-invariant, respectively, if it is invariant with respect to the left- or right-action of any element in H, respectively.

Left- and right-action transfer the multiplication of the Lie group $\mathrm{SO}(n)$ to the spaces of functions, measures and distributions. The definition of inversion is given by $\hat{f}(\eta) = f(\eta^{-1})$, for any $\eta \in \mathrm{SO}(n)$ and $f \in C(\mathrm{SO}(n))$. The inversion of a measure or distribution δ is then just defined as $\langle\hat{\delta}, \varphi\rangle_{C^{-\infty}(\mathrm{SO}(n))} = \langle\delta, \hat{\varphi}\rangle_{C^{-\infty}(\mathrm{SO}(n))}$. The inversion maps left-action to right-action and therefore also interchanges left- and right-invariance.

By the identifications $\mathbb{S}^{n-1} = \mathrm{SO}(n)/\,\mathrm{SO}(n-1)$ and $\mathrm{Gr}(n,i) = \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$, the notion of convolution on the Lie group $\mathrm{SO}(n)$ can be defined on the factor spaces. In the following, we will assume that the subgroup H is either $\mathrm{SO}(n-1)$ or $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$. Now, any function $f \in C(\mathrm{SO}(n)/H)$ can be lifted to a function $\breve{f} \in C(\mathrm{SO}(n))$ by $\breve{f}(\eta) = f(\mathrm{pr}_H(\eta))$, $\eta \in \mathrm{SO}(n)$, where $\mathrm{pr}_H : \mathrm{SO}(n) \to \mathrm{SO}(n)/H$ is the canonical projection. $\breve{f}$ is obviously right-H-invariant.

Conversely, a function $f \in C(\mathrm{SO}(n))$ that is right-H-invariant can be projected to a function $\bar{f} \in C(\mathrm{SO}(n)/H)$, as the value of f does not depend on the member of the equivalence class modulo H. This construction can be extended to measures μ on $\mathrm{SO}(n)/H$ by generalizing the known formula of decomposing the Haar measures, that is, by defining the lifted measure $\breve{\mu}$ on $\mathrm{SO}(n)$ by

$$\int_{\mathrm{SO}(n)} f(\eta)\, d\breve{\mu}(\eta) = \int_{\mathrm{SO}(n)/H} \int_H f(\eta_E \tau)\, d\tau\, d\mu(E),$$

where η_E is a fixed but arbitrary element in the preimage $pr_H^{-1}(\{E\})$. This construction is verified by the Riesz–Markov theorem, which guarantees the existence of such a measure. By the same formula, a right-H-invariant measure μ on $\mathrm{SO}(n)$ can be projected to a measure $\bar{\mu}$ on $\mathrm{SO}(n)/H$, that is,

$$\int_{\mathrm{SO}(n)/H} f(E)\, d\bar{\mu}(E) = \int_{\mathrm{SO}(n)} f(\mathrm{pr}_H(\eta))\, d\mu(\eta),$$

for any $f \in C(\mathrm{SO}(n)/H)$.
The convolution of two functions $\varphi, \psi \in C(\mathrm{SO}(n))$ is defined by

$$(\varphi * \psi)(\eta) = \int_{\mathrm{SO}(n)} \varphi(\eta\tau^{-1})\psi(\tau)\, d\tau = \int_{\mathrm{SO}(n)} \varphi(\tau)\psi(\tau^{-1}\eta)\, d\tau, \qquad \eta \in \mathrm{SO}(n).$$

From this definition it is clear that $l_\vartheta(\varphi * \psi) = (l_\vartheta\varphi) * \psi$ and $r_\vartheta(\varphi * \psi) = \varphi * (r_\vartheta\psi)$, for all $\vartheta \in \mathrm{SO}(n)$, and, hence, that the convolution inherits all the left-invariances of φ and right-invariances of ψ. As a consequence, two functions $f \in C(\mathrm{SO}(n)/H_1)$ and $g \in C(\mathrm{SO}(n)/H_2)$, where H_1 and H_2 are either $\mathrm{SO}(n-1)$ or $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$, can be lifted to $\mathrm{SO}(n)$ and be convoluted there to obtain a function $\breve{f} * \breve{g} \in C(\mathrm{SO}(n))$, which is right-$H_2$-invariant. Projecting $\breve{f} * \breve{g}$ down to $C(\mathrm{SO}(n)/H_2)$ yields a convolution of f and g, denoted by $f * g$.
Moreover, we observe that

$$(\varphi * \psi)(\eta) = \langle r_{\eta^{-1}}\hat{\varphi}, \psi\rangle_{L^2(\mathrm{SO}(n))} = \langle\varphi, \ell_{\eta^{-1}}\hat{\psi}\rangle_{L^2(\mathrm{SO}(n))} \tag{2.2}$$

and, in particular, that $(\varphi * \psi)(\eta) = \langle \ell_\eta \varphi, \hat{\psi} \rangle_{L^2(\mathrm{SO}(n))} = (\hat{\psi} * \hat{\varphi})(\eta)$, $\eta \in \mathrm{SO}(n)$. The convolution is therefore commutative on functions φ satisfying $\hat{\varphi} = \varphi$, including the space of all zonal functions on the sphere (when lifted to $\mathrm{SO}(n)$).

Equation (2.2) also allows to extend convolution to distributions and measures. As $\mathrm{SO}(n)$ is a compact Lie group, the convolution $\tau * \mu$ of signed measures τ, μ on $\mathrm{SO}(n)$ can also be defined by

$$\int_{\mathrm{SO}(n)} f(\vartheta)\, d(\tau * \mu)(\vartheta) = \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} f(\eta\theta)\, d\tau(\eta)\, d\mu(\theta), \qquad f \in C(\mathrm{SO}(n)),$$

which is easily verified to coincide with the above definition.

Note that $\tau * (l_{\vartheta^{-1}} \mu) = (r_\vartheta \tau) * \mu$, for all $\vartheta \in \mathrm{SO}(n)$. Hence, if τ is right-H-invariant, we can assume without loss of generality that μ is left-H-invariant (e.g., by taking means).

Overall, we get that any function f on a factor space $\mathrm{SO}(n)/H$ can be convoluted with a function g on the sphere to obtain a function on the sphere by

$$(f * g)(u) = \int_{\mathrm{SO}(n)} f(\mathrm{pr}_H(\vartheta_u \tau^{-1})) g(\tau \bar{e})\, d\tau, \qquad u \in \mathbb{S}^{n-1},$$

where we used that $\mathrm{pr}_{\mathrm{SO}(n-1)}(\eta) = \eta \bar{e}$.

Moreover, it is possible to define $\hat{f}$ for a left-H_1-invariant function on $\mathrm{SO}(n)/H_2$ by lifting to $\mathrm{SO}(n)$, inverting there, and projecting down to a left-H_2-invariant function on $\mathrm{SO}(n)/H_1$ (accordingly for measures and distributions). As mentioned before, we have that $f = \hat{f}$ for any zonal function (or measure) on $\mathbb{S}^{n-1}$.

Putting this all together, we summarize the formulas for the types of convolutions needed in representation formulas for Minkowski valuations. The convolution of a measure μ and a zonal function f both on $\mathbb{S}^{n-1}$ can be calculated by

$$(\mu * f)(u) = \int_{\mathbb{S}^{n-1}} f(\vartheta_u^{-1} v)\, d\mu(v), \qquad u \in \mathbb{S}^{n-1}, \tag{2.3}$$

the convolution of a function f and a zonal measure μ both on $\mathbb{S}^{n-1}$ by

$$(f * \mu)(u) = \int_{\mathbb{S}^{n-1}} f(\vartheta_u v)\, d\mu(v), \qquad u \in \mathbb{S}^{n-1}, \tag{2.4}$$

and the convolution of a function f on $\mathrm{Gr}(n,i)$ and an $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant measure μ on $\mathbb{S}^{n-1}$ by

$$(f * \mu)(u) = \int_{\mathrm{Gr}(n,i)} f(\vartheta_u E)\, d\hat{\mu}(E), \qquad u \in \mathbb{S}^{n-1}, \tag{2.5}$$

all three giving functions on the sphere. Note that the invariance properties of the right function or measure implies that these integrals are independent from the specific choice of ϑ_u. Moreover, it is not difficult to check that the convolution (2.4) with a measure from the right is selfadjoint (with respect to the L^2 inner product on $\mathbb{S}^{n-1}$) and that the convolution of two zonal functions and measures on the sphere is Abelian.

We now turn to multiplier transforms, in particular, the most important examples of multiplier transforms for this thesis: the Cosine, the Radon transforms and the operator $\Box_n$.

To define the Cosine transform, we use the notation $|\cos(E,F)| = V_i(Q|F)$, where $E, F \in \mathrm{Gr}(n,i)$, V_i denotes the i-dimensional volume, $Q \subseteq E$ with $V_i(Q) = 1$ and $Q|F$ is the orthogonal projection of Q onto F. It can be shown by elementary calculations that $|\cos(E,F)|$ does not depend on the choice of Q and that $|\cos(E,F)| = |\cos(E^\perp, F^\perp)|$. The Cosine transform $C_i : C(\mathrm{Gr}(n,i)) \to C(\mathrm{Gr}(n,i))$, $i \in \{1,\dots,n-1\}$, is then defined by

$$(C_i f)(E) = \int_{\mathrm{Gr}(n,i)} |\cos(E,F)| f(F)\, d\nu_i(F), \qquad E \in \mathrm{Gr}(n,i), \tag{2.6}$$

for all $f \in C(\mathrm{Gr}(n,i))$. Equation (2.6) defines a continuous, linear transformation that intertwines the actions of $\mathrm{SO}(n)$. In particular, it maps smooth functions to smooth functions. As the cosine transform is self-adjoint, it can also be extended to distributions by $\langle C_i\delta, \varphi\rangle_{C^{-\infty}(\mathrm{Gr}(n,i))} = \langle \delta, C_i\varphi\rangle_{C^{-\infty}(\mathrm{Gr}(n,i))}$. The special case $i = 1$ is the classical spherical cosine transform $(Cf)(u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| f(v)\, d\sigma(v)$.

The Radon transform $R_{i,j} : C(\mathrm{Gr}(n,i)) \to C(\mathrm{Gr}(n,j))$, $i \neq j \in \{1,\dots,n-1\}$, is defined as

$$(R_{i,j} f)(E) = \int_{\mathrm{Gr}(n,i)^E} f(F)\, d\nu_i^E(F), \qquad E \in \mathrm{Gr}(n,j), \tag{2.7}$$

for all $f \in C(\mathrm{Gr}(n,i))$, where $\mathrm{Gr}(n,i)^E$ denotes the submanifold of all $F \in \mathrm{Gr}(n,i)$ containing E (for $i > j$) or contained in E (for $i < j$), respectively, with its Haar probability measure ν_i^E. $R_{i,j}$ is continuous and linear. The adjoint transform is given by $R_{j,i}$ (making it possible to extend $R_{i,j}$ to distributions). Moreover, the Radon transform is compatible with taking orthogonal complements in the sense that $R_{i,j} \circ \perp^* = \perp^* \circ R_{n-i,n-j}$.

The differential operator $\Box_n$ on $C^\infty(\mathbb{S}^{n-1})$ is given by

$$\Box_n h = h + \frac{1}{n-1}\Delta_S h, \tag{2.8}$$

where Δ_S is the spherical Laplacian. $\Box_n$ is an $\mathrm{SO}(n)$ equivariant linear operator which intertwines convolution, that is, $\Box_n(f * g) = (\Box_n f) * g = f * (\Box_n g)$, for every $f, g \in C^\infty(\mathbb{S}^{n-1})$. The kernel of $\Box_n$ consists of linear functions, that is, functions of the form $u \mapsto \alpha\langle u, v\rangle$, $\alpha \in \mathbb{R}$ and $v \in \mathbb{S}^{n-1}$. When restricted to $C_0^\infty(\mathbb{S}^{n-1})$, the space of smooth functions with center of mass at the origin, the operator $\Box_n : C_0^\infty(\mathbb{S}^{n-1}) \to C_0^\infty(\mathbb{S}^{n-1})$ is an isomorphism.

C. Berg [26] showed in his solution of the Christoffel problem that for every $n \geq 2$ there exist functions $g_n \in C^\infty(-1,1)$ such that the function $\mathbb{S}^{n-1} \ni u \mapsto g_n(\langle u, \bar{e}\rangle)$ is in $L^1(\mathbb{S}^{n-1})$. Using these functions, the inverse of $\square_n$ can be written as

$$f(u) = \int_{\mathbb{S}^{n-1}} g_n(\langle u, v\rangle)(\square_n f)(v)\, dv, \qquad u \in \mathbb{S}^{n-1}, \tag{2.9}$$

for every $f \in C_0^\infty(\mathbb{S}^{n-1})$. Berg further proved that $g_2 \in C^\infty([-1,1])$ and, for $n > 2$, $g_n \in C^\infty([-1,1))$ (see also [54]). In particular, every g_n is bounded on compact subsets $A \subset [-1,1)$. Moreover, for $n > 2$, there exist numbers $t_n < 1$ such that g_n is decreasing on $(t_n, 1)$, and $\lim_{t\to 1} g_n(t) = -\infty$.

2.3 Convex Bodies and the Brunn–Minkowski Theory

In this section, we recall additional basic facts from the theory of convex bodies and the Brunn–Minkowski theory, as well as about isoperimetric inequalities. As general references, we recommend the monographs by Gardner [52] and Schneider [109].

First, let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$, that is, convex and compact subsets of $\mathbb{R}^n$, endowed with the Hausdorff metric d. The subset of full-dimensional convex bodies (equivalently, with non-empty interior) is denoted by $\mathcal{K}_0^n$. $(\mathcal{K}^n, d)$ is a complete and separable metric space, which is locally compact by the Blaschke selection theorem. In particular, every sequence of convex bodies contained in a common Euclidean ball must have a convergent subsequence.

We denote by $V_i(A)$ the i-dimensional Lebesgue measure of a Borel set $A \subseteq \mathbb{R}^i$ and use the abbreviation $\kappa_i = V_i(B^i)$, $i \in \mathbb{N}$.

Each $K \in \mathcal{K}^n$ is uniquely determined by its support function

$$h(K, x) = \max\{\langle x, y\rangle : y \in K\}, \quad x \in \mathbb{R}^n,$$

which is positively homogeneous of degree one and subadditive. Conversely, every function on $\mathbb{R}^n$ satisfying these two properties is the support function of a unique convex body. For $K, L \in \mathcal{K}^n$, the support function of their Minkowski sum $K + L = \{x + y : x \in K, y \in L\}$ is given by

$$h(K + L, \cdot) = h(K, \cdot) + h(L, \cdot). \tag{2.10}$$

Moreover, for every $\vartheta \in \mathrm{SO}(n)$ and $y \in \mathbb{R}^n$, we have

$$h(\vartheta K, \cdot) = h(K, \vartheta^{-1}\cdot) \qquad \text{and} \qquad h(K + y, \cdot) = h(K, \cdot) + \langle \cdot, y\rangle. \tag{2.11}$$

Also the Hausdorff distance $d(K, L)$ of two convex bodies $K, L \in \mathcal{K}^n$ can be expressed conveniently by $d(K, L) = \|h(K, \cdot\,) - h(L, \cdot\,)\|_\infty$, where $\|\cdot\|_\infty$ denotes the maximum norm on $C(\mathbb{S}^{n-1})$. Finally, we have $K \subseteq L$ if and only if $h(K, \cdot\,) \leq h(L, \cdot\,)$, in particular, $h(K, \cdot\,) > 0$ if and only if $o \in \operatorname{int} K$.

If $K \in \mathcal{K}^n$ has non-empty interior, then

$$K^z = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } y \in K - z\}$$

defines the polar body of K with respect to $z \in \operatorname{int} K$, and, if $o \in \operatorname{int} K$, K° denotes the polar body of K with respect to the origin. The Santaló point can be defined as the unique point $\mathbf{s} = \mathbf{s}(K) \in \operatorname{int} K$, for which $V_n(K^{\mathbf{s}}) = \min\{V_n(K^z) : z \in \operatorname{int} K\}$. It is clear that taking the polar reverses set inclusion, that is, $K \subseteq L$ implies $K^\circ \supseteq L^\circ$, for all $K, L \in \mathcal{K}^n$, $o \in \operatorname{int} K$, and that it is affine contravariant, that is, $(\varphi K)^\circ = \varphi^{-T} K^\circ$, for every $\varphi \in \mathrm{GL}(n)$ and $K \in \mathcal{K}^n$, $o \in \operatorname{int} K$.

We will make frequent use of the following polar volume formula,

$$V_n(K^\circ) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(K, u)^{-n}\, du, \tag{2.12}$$

for $K \in \mathcal{K}^n$ with $o \in \operatorname{int} K$. The mean width of a convex body $K \in \mathcal{K}^n$ is defined by

$$w(K) = \frac{2}{n\kappa_n} \int_{\mathbb{S}^{n-1}} h(K, u)\, du. \tag{2.13}$$

The Steiner point $s(K) \in \mathbb{R}^n$ of K is the unique point in $\operatorname{relint} K$, the relative interior of K, defined by

$$s(K) = \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} h(K, u) u\, du. \tag{2.14}$$

The mean width and the Steiner point are uniquely determined by their Minkowski additivity and compatibility with rigid motions. To be more precise, a continuous map $\varrho : \mathcal{K}^n \to \mathbb{R}$ is Minkowski additive and rigid motion invariant if and only if it is a constant multiple of the mean width w, while the Steiner point is the unique continuous map $s : \mathcal{K}^n \to \mathbb{R}^n$ which is Minkowski additive and rigid motion equivariant (cf. [109, Section 3.3]).

A convex body $K \in \mathcal{K}^n$ is said to be of class C^∞_+, if its boundary hypersurface $\operatorname{bd} K$ is a regular submanifold of $\mathbb{R}^n$ that is k-times continuously differentiable, for every $k \in \mathbb{N}$, and all of its principal curvatures are non-zero. We refer to [109, Section 2.5] for further details. An important consequence of this definition is that the ith projection function $\mathrm{Gr}(n, i) \ni E \mapsto V_i(K|E) \in \mathbb{R}$, where $K|E$ denotes the orthogonal projection of K onto E, is a smooth function for every K of class C^∞_+ (see, e.g., the proof of Lemma 5.2.2). Moreover, every convex body $K \in \mathcal{K}^n$ can be approximated arbitrarily well in the Hausdorff metric by convex bodies of class C^∞_+ (cf. [109, Theorem 3.4.1]).

Each even measure μ on $\mathbb{S}^{n-1}$ generates a uniquely determined origin-symmetric convex body $Z^\mu \in \mathcal{K}^n$ by

$$h(Z^\mu, u) = (C\mu)(u) = \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\, d\mu(v), \qquad u \in \mathbb{S}^{n-1}. \tag{2.15}$$

The bodies obtained in this way constitute the class of origin-symmetric zonoids, which naturally arise also in various other contexts (see, e.g., [109, Chapter 3.5]). If the right-hand side of (2.15) defines the support function of a convex body for an arbitrary signed measure μ, this body is called a generalized (origin-symmetric) zonoid. The class of generalized zonoids is dense in the class of origin symmetric convex bodies (see, e.g., [109, Theorem 3.5.4]).

It is a fundamental result in convex geometry, that the volume of a Minkowski combination of convex bodies is a homogeneous polynomial of degree n, that is,

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \dots, K_{i_n}),$$

for $K_1, \dots, K_m \in \mathcal{K}^n$ and $\lambda_1, \dots, \lambda_m \geq 0$. The coefficients $V(K_{i_1}, \dots, K_{i_n})$ are called mixed volumes and play an important role in the Brunn–Minkowski theory of convex bodies. It is a direct consequence of their definition that mixed volumes are non-negative, multilinear and invariant under translations and permutations of the entries. Moreover, mixed volumes are monotone, that is, $K \subseteq L$ implies $V(K, K_2, \dots, K_n) \leq V(L, K_2, \dots, K_n)$, and behave nicely under simultaneous linear transformations of all entries, that is, $V(\varphi K_1, \dots, \varphi K_n) = |\det \varphi| V(K_1, \dots, K_n)$ for any $\varphi \in \mathrm{GL}(n)$.

By taking $K_1, \dots, K_i = K$ and $K_{i+1}, \dots, K_n = B^n$, $i \in \{0, \dots, n\}$, we obtain the quermassintegrals $W_{n-i}(K) = V(K[i], B^n[n-i])$, where the notation $K[i]$ abbreviates K appearing i times. The special cases $i = 0$ and $i = n$ correspond to the constant κ_n and the volume V_n, respectively. Among the other quermassintegrals, $W_1(K) = \frac{1}{n} S(K)$, where $S(K)$ is the surface area of K, and $W_{n-1}(K) = \kappa_n \frac{w(K)}{2}$. For i-dimensional K, $W_{n-i}(K)$ is a constant multiple of the i-dimensional volume of K. W_{n-i} is obviously homogeneous of degree i and $\mathrm{SO}(n)$ invariant.

The mixed volumes can be localized to measures on the unit sphere $\mathbb{S}^{n-1}$. Indeed, for convex bodies $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ there exists a measure $S(K_1, \dots, K_{n-1}, \cdot)$, called the mixed area measure, defined by

$$V(L, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(L, u) \, dS(K_1, \dots, K_{n-1}, u), \qquad L \in \mathcal{K}^n. \tag{2.16}$$

By the properties of the mixed volume, the measure $S(K_1, \dots, K_{n-1}, \cdot)$ is multilinear and symmetric in $K_1, \dots, K_{n-1}$ and non-negative. The diagonal form $S_{n-1}(K, \cdot) = S(K[n-1], \cdot)$ is called surface area measure and allows the interpretation that $S_{n-1}(K, \omega)$ is the surface area of all points on the boundary of K with outer unit normal vectors contained in $\omega \subseteq \mathbb{S}^{n-1}$. The measure $S_{n-1}(B^n, \cdot)$ is the spherical Lebesgue measure with total mass $n\kappa_n$.

By taking $K_1, \dots, K_i = K$ and $K_{i+1}, \dots, K_{n-1} = B^n$, $i \in \{0, \dots, n-1\}$, we obtain the ith area measures $S_i(K, \cdot) = S(K[i], B^n[n-i-1], \cdot)$ of K. Every $S_i(K, \cdot)$ is a finite, non-negative measure on $\mathbb{S}^{n-1}$ which is i-homogeneous in K. As the total mass of $S_i(K, \cdot)$ is given by $nW_{n-i}(K)$, the area measure $S_i(K, \cdot)$ can be seen as a localization of the quermassintegral $W_{n-i}(K)$.

The area measure of order one $S_1(K,\cdot)$ is related to the support function via the operator $\square_n$ (to be understood in a distributional sense) by

$$S_1(K,\cdot) = \square_n h(K,\cdot). \tag{2.17}$$

In particular, $\square_n$ maps the constant function 1 on the sphere to itself. As shown by Weil [120], the set $\{S_1(K,\cdot) : K \in \mathcal{K}^n\}$ is *not* dense in the set of all non-negative measures on $\mathbb{S}^{n-1}$ with barycenter at the origin in the weak topology.

An analogue of quermassintegrals which is invariant under the special linear group $\mathrm{SL}(n)$, in short, equi-affine invariant, are the affine quermassintegrals, introduced in [82]. For a convex body $K \in \mathcal{K}^n$ with non-empty interior, the affine quermassintegral $A_{n-i}(K)$, $i \in \{0,\dots n\}$, is defined by

$$A_{n-i}(K) = \begin{cases} \kappa_n & i=0, \\ \frac{\kappa_n}{\kappa_i}\left(\int_{\mathrm{Gr}(n,i)} V_i(K|E)^{-n}\,d\nu_i(E)\right)^{-\frac{1}{n}} & 1 \le i \le n-1, \\ V_n(K) & i=n. \end{cases} \tag{2.18}$$

The normalization is chosen such that $A_{n-i}(B^n) = W_{n-i}(B^n) = \kappa_n$. It is a simple application of Jensen's inequality and Kubota's formula that $W_{n-i}(K) \ge A_{n-i}(K)$.

Let us now turn to isoperimetric inequalities for mixed volumes and quermassintegrals (see, e.g., [109, Chapter 7]). The famous classical isoperimetric inequality between the surface area and the volume of a convex body, seen as an inequality between the quermassintegrals W_1 and W_0, is a special case of a much bigger class of inequalities. The most basic inequality, which we will need, is due to Minkowski,

$$V(K[n-1],L)^n \ge V_n(K)^{n-1}V_n(L), \tag{2.19}$$

where $K,L \in \mathcal{K}_0^n$. Equality is attained exactly for homothetic $K,L \in \mathcal{K}_0^n$, that is, if one is a translated and dilated copy of the other, $K = \lambda L + x$, or vice versa, for some $\lambda > 0$ and $x \in \mathbb{R}^n$. The isoperimetric inequality corresponds to $L = B^n$.
Minkowski's inequality is an easy consequence of the Brunn–Minkowski inequality,

$$V_n(K+L)^{\frac{1}{n}} \ge V_n(K)^{\frac{1}{n}} + V_n(L)^{\frac{1}{n}}, \tag{2.20}$$

where $K,L \in \mathcal{K}^n$, and is strengthened by the Aleksandrov–Fenchel inequality,

$$V(K_1,K_2,\dots,K_n)^2 \ge V(K_1,K_1,K_3,\dots,K_n)V(K_2,K_2,K_3,\dots,K_n), \tag{2.21}$$

where $K_1,\dots,K_n \in \mathcal{K}_0^n$. Many other inequalities may be deduced by consecutive applications of the Aleksandrov–Fenchel inequality, among them,

$$V(K[j],L[m-j],\mathcal{C})^{k-i} \ge V(K[i],L[m-i],\mathcal{C})^{k-j}V(K[k],L[m-k],\mathcal{C})^{j-i}, \tag{2.22}$$

for $0 \leq i < j < k \leq m \leq n$ and $K, L \in \mathcal{K}_0^n$, abbreviating $\mathcal{C} = (K_{m+1}, \ldots, K_n)$. Choosing $m = n$ and $L = B^n$, we obtain the isoperimetric inequalities between the quermassintegrals, that is,

$$W_{n-j}(K)^{i-k} \geq W_{n-i}(K)^{j-k} W_{n-k}(K)^{i-j}, \tag{2.23}$$

for $0 \leq k < j < i \leq n$ and $K \in \mathcal{K}_0^n$. If we let $k = 0$, we obtain

$$\kappa_n^{-i} W_{n-j}(K)^i \geq \kappa_n^{-j} W_{n-i}(K)^j, \tag{2.24}$$

where $0 < j < i \leq n$, $K \in \mathcal{K}_0^n$, and equality holds if and only if K is a Euclidean ball. Letting $i = n$, we get the inequalities between quermassintegrals and the volume,

$$W_{n-j}(K)^n \geq \kappa_n^{n-j} V_n(K)^j, \tag{2.25}$$

where $0 < j < n$, $K \in \mathcal{K}_0^n$, and equality holds exactly for Euclidean balls. The most well-known members of this family of inequalities are the classical isoperimetric inequality ($j = n-1$) and the Urysohn inequality ($j = 1$). As we can see from the previous sets of inequalities, the inequality for $j = n-1$ is the strongest among this family. Using that $W_1(K) = nS(K)$ and $W_{n-1}(K) = \kappa_n \frac{w(K)}{2}$, the isoperimetric

$$\frac{V_n(K)^{n-1}}{\kappa_n^{n-1}} \leq \frac{S(K)^n}{(n\kappa_n)^n} \tag{2.26}$$

and the Urysohn inequality

$$V_n(K) \leq \left(\frac{w(K)}{2}\right)^n \kappa_n \tag{2.27}$$

take their usual forms.

The classical isoperimetric inequality admits stability results, that is, there exist quantitative bounds on the Hausdorff distance of a specific $K \in \mathcal{K}^n$ to a Euclidean ball, whenever the isoperimetric inequality is close to being an equality for fixed K (see, e.g., [49,51]). We need a result by Gritzmann, Wills and Wrase [57] that points in the same direction (see also [58] or [49, Lemma 4.1]), stating that

$$\operatorname{diam} K \leq c(n) \frac{S(K)^{n-1}}{V_n(K)^{n-2}}, \tag{2.28}$$

for $K \in \mathcal{K}_0^n$ and some constant $c(n) > 0$ depending only on the dimension.

A strengthening of the set of isoperimetric inequalities (2.25) is given by an inequality between affine quermassintegrals and volume of $K \in \mathcal{K}_0^n$, that is,

$$W_{n-j}(K)^n \geq A_{n-j}(K)^n \geq \kappa_n^{n-j} V_n(K)^j. \tag{2.29}$$

This was conjectured by Lutwak [84] and was for a long time only known for the cases $j = 1$, where we have the Blaschke–Santaló inequality for origin-symmetric convex bodies $K \in \mathcal{K}_0^n$,

$$V_n(K^\circ) V_n(K) \leq \kappa_n^2, \tag{2.30}$$

with equality if and only if K is an ellipsoid, that is, an affine image of B^n (reflecting the affine invariance of the affine quermassintegral and therefore also of this inequality), and $j = n-1$, with the polar Petty Projection inequality [96],

$$V_n(\Pi^\circ K)V_n(K)^{n-1} \le \frac{\kappa_n^n}{\kappa_{n-1}^n}, \tag{2.31}$$

where $\Pi^\circ K = (\Pi K)^\circ$ is the polar projection body of $K \in \mathcal{K}_0^n$ (see Chapter 5.1 for the definition) and equality holds exactly for ellipsoids. Lutwak's conjecture was recently proved by E. Milman and Yehudayoff [93] using Steiner symmetrization.

We will also need the following generalization of (2.30) with the polar taken with respect to the Santaló point,

$$V_n(K^{\mathbf{s}})V_n(K) \le \kappa_n^2, \tag{2.32}$$

where $K \in \mathcal{K}^n$ is a convex body with non-empty interior. Equality holds if and only if K is an ellipsoid.

2.4 Convex and Log-concave Functions

We turn now to convex and log-concave functions on $\mathbb{R}^n$. As general reference for this section, we recommend the monographs by Rockafellar [98], and Rockafellar and Wets [99], as well as the survey [38] by Colesanti.

Let $\mathrm{Cvx}(\mathbb{R}^n)$ denote the set of convex and lower semi-continuous functions $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ which are proper, that is, not identically $+\infty$. Two convex sets naturally associated to any $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ are its domain, $\operatorname{dom}\varphi = \{x \in \mathbb{R}^n : \varphi(x) < +\infty\}$, and its epigraph defined by $\operatorname{epi}\varphi = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R} : \varphi(x) \le \xi\}$. Note that for any $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$, $\operatorname{dom}\varphi$ is non-empty and $\operatorname{epi}\varphi$ is closed and non-empty. We call a function $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ coercive if $\lim_{\|x\|\to\infty}\varphi(x) = +\infty$ and we denote by $\mathrm{Cvx}_{\mathrm{c}}(\mathbb{R}^n)$ the set of all coercive $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$. We also note that (see, e.g., [39, Lemma 2.5]), $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ is coercive if and only if there exist $\gamma > 0$ and $\beta \in \mathbb{R}$ such that for every $x \in \mathbb{R}^n$,

$$\varphi(x) \ge \gamma\|x\| + \beta. \tag{2.33}$$

Next we endow the spaces $\mathrm{Cvx}(\mathbb{R}^n)$ and $\mathrm{Cvx}_{\mathrm{c}}(\mathbb{R}^n)$ with the topology induced by epi-convergence. Recall that a sequence of $\varphi_k \in \mathrm{Cvx}(\mathbb{R}^n)$ is called epi-convergent to $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ if for all $x \in \mathbb{R}^n$ the following two conditions hold:

- $\varphi(x) \le \liminf_{k\to\infty}\varphi_k(x_k)$ for every sequence x_k that converges to x.
- There exists a sequence x_k converging to x such that $\varphi(x) = \lim_{k\to\infty}\varphi_k(x_k)$.

In this case, we write $\varphi_k \overset{\mathrm{epi}}{\to} \varphi$. Note that the limiting function φ is again convex and lower semi-continuous. However, in general, φ need not be proper.

Lemma 2.4.1 ([99, Theorem 7.17]). *If $\varphi, \varphi_k \in \mathrm{Cvx}(\mathbb{R}^n)$ and $\operatorname{int}\operatorname{dom}\varphi$ is non-empty, then the following statements are equivalent to φ_k being epi-convergent to φ:*

(i) There exists a dense set $D \subseteq \mathbb{R}^n$ such that $\varphi_k(x) \to \varphi(x)$ for every $x \in D$.

(ii) The sequence φ_k converges uniformly to φ on every compact subset of $\mathbb{R}^n$ that does not intersect the boundary of $\operatorname{dom}\varphi$.

Let us also emphasize that epi-convergence (also known as Γ-convergence) is equivalent to the convergence of the corresponding epigraphs in the so-called Painlevé–Kuratowski sense (cf. [99, Proposition 7.2]).
For $\varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$, their infimal convolution is defined by

$$(\varphi \,\square\, \psi)(x) = \inf_{x_1+x_2=x} \{\varphi(x_1) + \psi(x_2)\}.$$

If $\varphi \,\square\, \psi$ does not attain the value $-\infty$, then it is convex, proper, and

$$\operatorname{epi}(\varphi \,\square\, \psi) = \operatorname{epi}\varphi + \operatorname{epi}\psi.$$

However, $\varphi \,\square\, \psi$ need not be semi-continuous (see, e.g., [109, p. 39]). A quite useful condition to ensure lower semi-continuity of the infimal convolution of $\varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$ can be found in [98, Corollary 9.2.2] and requires that

$$\lim_{\lambda\to\infty} \frac{\varphi(y+\lambda x)}{\lambda} + \lim_{\lambda\to\infty} \frac{\psi(z-\lambda x)}{\lambda} > 0 \tag{2.34}$$

for every non-zero $x \in \mathbb{R}^n$ and arbitrary $y \in \operatorname{dom}\varphi$, $z \in \operatorname{dom}\psi$.
For $t > 0$ and $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$, the Moreau envelope $e_t\varphi$ of φ is defined by

$$e_t\varphi = \varphi \,\square\, \tfrac{1}{2t}\|\cdot\|^2.$$

In the following, we require some of its simple properties.

Lemma 2.4.2 ([99, Theorems 1.25 & 2.26]). *Suppose that $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$. Then the following statements hold:*

(i) $e_t\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ and it is finite for every $t > 0$;

(ii) $e_t\varphi(x)$ converges to $\varphi(x)$ monotonously from below for every $x \in \mathbb{R}^n$ as $t \searrow 0$.

In particular, $e_t\varphi \xrightarrow{\text{epi}} \varphi$ as $t \searrow 0$.

Now, let $\mathrm{LC}(\mathbb{R}^n) = \{f = e^{-\varphi} : \varphi \in \mathrm{Cvx}(\mathbb{R}^n)\}$ denote the set of all proper, that is, not identically 0, log-concave, and upper semi-continuous functions on $\mathbb{R}^n$ and let

$$\mathrm{LC_c}(\mathbb{R}^n) = \left\{ f \in \mathrm{LC}(\mathbb{R}^n) : \lim_{\|x\|\to\infty} f(x) = 0 \right\} = \{f = e^{-\varphi} : \varphi \in \mathrm{Cvx_c}(\mathbb{R}^n)\}$$

be the subspace of all coercive functions in $\mathrm{LC}(\mathbb{R}^n)$. Here, f is called coercive if $\lim_{\|x\|\to\infty} f(x) = 0$. Note that we call $f = e^{-\varphi}$ coercive or proper, respectively, exactly if φ is coercive or proper, respectively.

We call a sequence $f_k = e^{-\varphi_k} \in \mathrm{LC}(\mathbb{R}^n)$ or $\mathrm{LC_c}(\mathbb{R}^n)$, respectively, hypo-convergent to $f = e^{-\varphi}$, if $\varphi_k \in \mathrm{Cvx}(\mathbb{R}^n)$ or $\mathrm{Cvx_c}(\mathbb{R}^n)$, respectively, epi-converges to φ. (Note that for log-concave functions our notion of hypo-convergence coincides with the more general definition used frequently in analysis.)
For log-concave f and g and $\lambda > 0$, let

$$(f \star g)(x) = \sup_{x_1+x_2=x} f(x_1)g(x_2), \qquad (\lambda \cdot f)(x) = f\left(\tfrac{x}{\lambda}\right)^{\lambda}.$$

Then $f \star g$ is called the Asplund sum (or sup-convolution) of f and g (see, e.g., [53]). The above definitions imply that $\mathbb{1}_K \star \mathbb{1}_L = \mathbb{1}_{K+L}$ and $\lambda \cdot \mathbb{1}_K = \mathbb{1}_{\lambda K}$ for all $K, L \in \mathcal{K}^n$ and $\lambda > 0$, where $\mathbb{1}_K$ denotes the indicator function of a convex body $K \in \mathcal{K}^n$ (see Example 2.4.4 (a) for the definition).

The Asplund sum $f \star g$ of $f = e^{-\varphi}, g = e^{-\psi} \in \mathrm{LC}(\mathbb{R}^n)$ is related to the infimal convolution $\varphi \,\Box\, \psi$ of $\varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$ by

$$f \star g = e^{-\varphi \,\Box\, \psi}. \tag{2.35}$$

In particular, since $\mathrm{Cvx}(\mathbb{R}^n)$ is not closed under infimal convolution, the space $\mathrm{LC}(\mathbb{R}^n)$ is not closed under Asplund addition. However, as the following lemma shows, this is no longer the case when considering coercive functions.

Lemma 2.4.3. *Suppose that $f, g \in \mathrm{LC_c}(\mathbb{R}^n)$, $a, b > 0$ and $y \in \mathbb{R}^n$. Then the following statements hold:*

(i) $f \star g \in \mathrm{LC_c}(\mathbb{R}^n)$;

(ii) $(a \cdot f) \star (b \cdot f) = (a + b) \cdot f$;

(iii) $(f \star \mathbb{1}_{\{y\}})(x) = f(x - y)$, $x \in \mathbb{R}^n$.

Proof. In order to prove (i), let $f = e^{-\varphi}$ and $g = e^{-\psi}$ with $\varphi, \psi \in \mathrm{Cvx_c}(\mathbb{R}^n)$. Then, by (2.35), it is sufficient to show that $\varphi \,\Box\, \psi \in \mathrm{Cvx_c}(\mathbb{R}^n)$. First note that since φ and ψ are coercive, they are bounded from below and, consequently, so is $\varphi \,\Box\, \psi$ which is, therefore, convex and proper. Moreover, from the definition of infimal convolution and the triangle inequality, it follows that $\varphi \,\Box\, \psi$ is coercive. It remains to show that $\varphi \,\Box\, \psi$ is lower semi-continuous, for which we check that condition (2.34) is satisfied. To this end, we use (2.33) to conclude that there exist $\gamma_\varphi, \gamma_\psi > 0$ and $\beta_\varphi, \beta_\psi \in \mathbb{R}$ such that $\varphi(w) \geq \gamma_\varphi \|w\| + \beta_\varphi$ and $\psi(w) \geq \gamma_\psi \|w\| + \beta_\psi$ for every $w \in \mathbb{R}^n$. Thus,

$$\lim_{\lambda\to\infty} \frac{\varphi(y + \lambda x)}{\lambda} + \lim_{\lambda\to\infty} \frac{\psi(z - \lambda x)}{\lambda} \geq \gamma_\varphi \|x\| + \gamma_\psi \|x\| > 0$$

for every non-zero $x \in \mathbb{R}^n$ which completes the proof of (i). Statements (ii) and (iii) follow easily from the definition of the Asplund sum and multiplication. $\square$

Example 2.4.4.

(a) If $K \in \mathcal{K}^n$, then the indicator function $\mathbb{1}_K \in \mathrm{LC_c}(\mathbb{R}^n)$ is defined by

$$\mathbb{1}_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Recall that for $K \in \mathcal{K}^n$ containing the origin in its interior, its gauge or Minkowski functional is given by

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}, \qquad x \in \mathbb{R}^n.$$

If K is origin-symmetric, $\|\cdot\|_K$ is the norm with unit ball K. For $K = B^n$, we have $\|\cdot\|_{B^n} = \|\cdot\|$. Another interesting class of log-concave functions consists of those $f = e^{-\varphi} \in \mathrm{LC_c}(\mathbb{R}^n)$, where

$$\varphi = \frac{1}{p}\|\cdot\|_K^p, \qquad p \geq 1. \tag{2.36}$$

In particular, $f \in \mathrm{LC_c}(\mathbb{R}^n)$ is called a Gaussian if there exist $a > 0$, $y \in \mathbb{R}^n$ and an origin-symmetric ellipsoid $E \subseteq \mathbb{R}^n$ such that

$$f(x) = a\, e^{-\frac{1}{2}\|x-y\|_E^2}, \qquad x \in \mathbb{R}^n.$$

For $a = (2\pi)^{-n/2}$, $y = o$ and $E = B^n$, we obtain the standard Gaussian ψ_n.

Let us mention here another useful volume formula for convex bodies, involving the functions defined in (2.36). If $K \in \mathcal{K}^n$ contains the origin in its interior, then

$$V_n(K) = \frac{1}{p^{n/p}\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{p}\|x\|_K^p\right) dx. \tag{2.37}$$

The Legendre transform $\mathcal{L} : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ is defined by

$$(\mathcal{L}\varphi)(x) = \sup_{y \in \mathbb{R}^n} \langle x, y\rangle - \varphi(y), \qquad x \in \mathbb{R}^n.$$

It is a classical notion with many applications in several areas which are extensively covered in the literature (e.g., [98,99]). We collect a number of its well-known properties for quick later reference in the following proposition. Note that the properties from (i) were recently shown to essentially characterize the Legendre transform in a fundamental paper by Artstein-Avidan and Milman [20].

Proposition 2.4.5 (see, e.g., [98,99])**.** *For $\varphi_k, \varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$ and $K \in \mathcal{K}^n$, the following statements hold:*

(i) $\mathcal{L}\mathcal{L}\varphi = \varphi$ and if $\varphi \leq \psi$, then $\mathcal{L}\varphi \geq \mathcal{L}\psi$;

(ii) φ *is coercive if and only if* $\operatorname{dom} \mathcal{L}\varphi$ *contains the origin in its interior;*

(iii) $\varphi_k \overset{\text{epi}}{\to} \varphi$ *if and only if* $\mathcal{L}\varphi_k \overset{\text{epi}}{\to} \mathcal{L}\varphi$*;*

(iv) $\mathcal{L}(-\log \mathbb{1}_K) = h(K, \cdot)$ *and if* $1 < p, q < \infty$ *are such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* K *contains the origin in its interior, then*

$$\mathcal{L}\left(\frac{1}{p}\|\cdot\|_K^p\right) = \frac{1}{q}\|\cdot\|_{K^\circ}^q.$$

The Legendre transform gives rise to several constructions for log-concave functions. For $f \in \mathrm{LC}(\mathbb{R}^n)$, its support function is, following [19], defined by $h(f, \cdot) = \mathcal{L}(-\log f)$ and note that $h(f, \cdot) \in \mathrm{Cvx}(\mathbb{R}^n)$. By Proposition 2.4.5, the map $f \mapsto h(f, \cdot)$ is bijective from $\mathrm{LC}(\mathbb{R}^n)$ to $\mathrm{Cvx}(\mathbb{R}^n)$. For our purposes it is particularly useful to observe that support functions are Asplund additive (cf. [109, p.518]), in the sense that for $f, g \in \mathrm{LC_c}(\mathbb{R}^n)$, we have

$$h(f \star g, \cdot) = h(f, \cdot) + h(g, \cdot) \tag{2.38}$$

which is an extension of (2.10) to $\mathrm{LC_c}(\mathbb{R}^n)$. Moreover, for $\vartheta \in \mathrm{SO}(n)$ and $y \in \mathbb{R}^n$,

$$h(\vartheta f, \cdot) = h(f, \vartheta^{-1} \cdot) \qquad \text{and} \qquad h(f(\,.\, - y), \cdot) = h(f, \cdot) + \langle \cdot, y \rangle, \tag{2.39}$$

extending the properties (2.11) to $\mathrm{LC_c}(\mathbb{R}^n)$.
For $f \in \mathrm{LC}(\mathbb{R}^n)$, its polar function is defined by (following [18])

$$f^\circ = e^{-\mathcal{L}(-\log f)} = e^{-h(f,\cdot)}. \tag{2.40}$$

From the definition of the Legendre transform and Proposition 2.4.5, one obtains:

Lemma 2.4.6. *For* $f, g \in \mathrm{LC_c}(\mathbb{R}^n)$*,* $a > 0$*, and* $K \in \mathcal{K}^n$ *containing the origin in its interior, the following statements hold:*

(i) $(f^\circ)^\circ = f$*;*

(ii) $(Af)^\circ = A^{-\mathrm{T}} f^\circ$ *for every* $A \in \mathrm{GL}(n)$*;*

(iii) $(f \star g)^\circ = f^\circ g^\circ$ *and* $(a \cdot f)^\circ = (f^\circ)^a$*;*

(iv) $(\mathbb{1}_K)^\circ = e^{-\|\cdot\|_{K^\circ}}$ *and if* $1 < p, q < \infty$ *are such that* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$\left(e^{-\frac{1}{p}\|\cdot\|_K^p}\right)^\circ = e^{-\frac{1}{q}\|\cdot\|_{K^\circ}^q}.$$

We next state a general version of the functional Blaschke–Santaló inequality due to Artstein-Avidan, Klartag, and Milman [18] required in Chapter 4. We restrict ourselves to log-concave functions and recall that the centroid of an integrable function f on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} f\, dx > 0$ is defined by

$$\operatorname{cent} f = \frac{\int_{\mathbb{R}^n} x f(x)\, dx}{\int_{\mathbb{R}^n} f(x)\, dx}.$$

Theorem 2.4.7 ([18]). *Suppose that* $f \in \mathrm{LC}(\mathbb{R}^n)$ *is such that* $0 < \int_{\mathbb{R}^n} f(x)\,dx < \infty$ *and let* $\tilde{f}(x) = f(x - \operatorname{cent} f)$. *Then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} \tilde{f}^\circ(x)\,dx \le (2\pi)^n$$

with equality if and only if f *is a Gaussian.*

Theorem 2.4.7 (without equality cases) is due to Ball [22] in the case when f is additionally even. Ball's functional Blaschke–Santaló inequality then reads

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} f^\circ(x)\,dx \le (2\pi)^n \tag{2.41}$$

with equality if and only if f is a Gaussian.

Let us note that for $K \in \mathcal{K}^n$ containing the origin in its interior and $f = e^{-\frac{1}{2}\|\cdot\|_K^2}$ in Theorem 2.4.7, one recovers a version of the geometric Blaschke–Santaló inequality equivalent to (2.32).

2.5 Valuations on Convex Bodies

Underlying many of the results in this thesis, the theory of valuations does not appear directly, but implicitly via representation results for valuations. As general reference on this section, we recommend the books by Klain and Rota [70] and Schneider and Weil [110], as well as the survey by Bernig [28]. Note also that we fixed a Euclidean structure on $\mathbb{R}^n$ in order to simplify the presentation.

A map $\varphi : \mathcal{K}^n \to \mathcal{A}$ with values in an Abelian semigroup $(\mathcal{A}, +)$ is a valuation if

$$\varphi(K) + \varphi(L) = \varphi(K \cup L) + \varphi(K \cap L)$$

whenever $K \cup L$ is convex. We will work mostly with real-valued valuations ($\mathcal{A} = \mathbb{R}$) and convex-body-valued valuations ($\mathcal{A} = \mathcal{K}^n$ with Minkowski addition), so-called Minkowski valuations. In the following, we will therefore restrict ourselves to these two cases. Typical examples of real-valued valuations are the volume V_n and all the quermassintegrals W_{n-i} and mixed volumes.

A valuation is called continuous, if it is continuous with respect to the Hausdorff metric. It is translation invariant, if $\varphi(K + x) = \varphi(K)$ for all $K \in \mathcal{K}^n$ and $x \in \mathbb{R}^n$. The space of continuous and translation invariant, real-valued valuations is denoted by Val and is naturally a vector space with pointwise addition. A valuation $\varphi \in \mathrm{Val}$ is homogeneous of degree i, if $\varphi(\lambda K) = \lambda^i \varphi(K)$, for all $K \in \mathcal{K}^n$ and $\lambda > 0$. The space of i-homogeneous valuations is denoted by Val_i. It is an important result by McMullen [88] that the space Val can be decomposed as direct sum of the spaces Val_i, where $i \in \{0, \dots, n\}$, that is,

$$\mathrm{Val} = \bigoplus_{i=0}^{n} \mathrm{Val}_i \,.$$

In particular, the degrees $i = 0, \dots, n$ are the only possible degrees of homogeneity. The spaces Val_i can be further decomposed into the spaces of even and odd valuations, denoted by Val_i^+ and Val_i^-, respectively, where a valuation φ is called even, if $\varphi(-K) = \varphi(K)$, and it is called odd, if $\varphi(-K) = -\varphi(K)$ for all $K \in \mathcal{K}^n$. Every valuation can be written as a sum of an even and an odd valuation, that is, $\varphi = \varphi^+ + \varphi^-$, where $2\varphi^+(K) = \varphi(K) + \varphi(-K)$ and $2\varphi^-(K) = \varphi(K) - \varphi(-K)$, $K \in \mathcal{K}^n$.

The spaces Val_0 and Val_n are one-dimensional and spanned by the Euler characteristic $\chi(K) = 1$ for every $K \in \mathcal{K}^n$, and the n-dimensional volume (a result by Hadwiger [63]), respectively. All other spaces $\mathrm{Val}_1, \dots, \mathrm{Val}_{n-1}$ are infinite-dimensional.

The space Val is endowed with the topology of uniform convergence on compact sets, also induced by the norm

$$\|\varphi\|_{\mathrm{Val}} = \sup\{|\varphi(K)| : K \subseteq B^n\},$$

that makes Val a Banach space. The supremum in the definition is always finite, since the set $\{K \in \mathcal{K}^n : K \subseteq B^n\}$ is a bounded and, therefore, by Blaschke's selection theorem, compact subset of $\mathcal{K}^n$.

In the last twenty years, a new and very successful algebraic theory of valuations evolved from the seminal works of Alesker [5–9] with many further contributions, e.g., [10, 11, 13, 16, 27, 30–32], focussing on the properties of Val as representation of the general linear group $\mathrm{GL}(n)$.

The (continuous) representation of $\mathrm{GL}(n)$ on Val is defined by

$$(\eta \cdot \varphi)(K) = \varphi(\eta^{-1}K), \qquad K \in \mathcal{K}^n,$$

for all $\varphi \in \mathrm{Val}$ and $\eta \in \mathrm{GL}(n)$. The famous result by Alesker states that

Theorem 2.5.1 ([6]). *The representations of* $\mathrm{GL}(n)$ *on* Val_i^+ *and on* Val_i^- *are irreducible, that is, any closed* $\mathrm{GL}(n)$ *invariant subspace is either zero or the whole space.*

An immediate consequence of this theorem is that every $\mathrm{GL}(n)$-intertwining map (that is, which is compatible with the representation) with co-domain $\mathrm{Val}_i^{\pm}$ must be either zero or have dense image.

Speaking of representations, it is natural to consider smooth vectors of the representation on Val, that is, of valuations $\varphi \in \mathrm{Val}$ such that the map

$$z_\varphi : \begin{cases} \mathrm{GL}(n) \to \mathrm{Val} \\ \vartheta \mapsto \vartheta \cdot \varphi \end{cases}$$

is infinitely differentiable. Such φ is called a smooth valuation (see [7]) and the subspaces of smooth valuations of $\mathrm{Val}_i^{\pm}$ are denoted by adding a superscript ∞, that is, $\mathrm{Val}_i^{\pm,\infty}$. The space Val^∞ also satisfies a homogeneous decomposition theorem, as the homogeneous components of a valuation φ can be calculated smoothly from the coefficients of the polynomial $\varphi(\lambda K)$ and are therefore smooth themself.

It can be shown using a theorem by Bernstein and Krötz [33, Corollary 3.10] that a valuation φ is smooth with respect to the $\mathrm{GL}(n)$ representation if and only if it is smooth with respect to the subrepresentation of $\mathrm{SO}(n) \subseteq \mathrm{GL}(n)$ (see [119, Proposition 1.31]).
Val^∞ can be endowed with the Gårding topology. Namely, consider the map

$$\mathrm{Val}^\infty \ni \varphi \mapsto z_\varphi \in C^\infty(\mathrm{GL}(n), \mathrm{Val}),$$

which yields an identification of Val^∞ with a closed (this has to be shown) subspace of $C^\infty(\mathrm{GL}(n), \mathrm{Val})$, which is a Fréchet space with respect to uniform convergence of every finite derivative on any compact subset. The Gårding topology is the inherited subspace topology from $C^\infty(\mathrm{GL}(n), \mathrm{Val})$ pulled back to Val^∞, that is, a sequence $(\varphi_j)_{j\in\mathbb{N}}$ of smooth valuations converges in the Gårding topology $\varphi_j \xrightarrow{\text{Gårding}} \varphi$, if

$$\begin{aligned} z_{\varphi_j} \xrightarrow{C^\infty(\mathrm{GL}(n),\mathrm{Val})} z_\varphi \quad &\Leftrightarrow \quad \|z_{\varphi_j} - z_\varphi\|_{C^k(C,\mathrm{Val})} \to 0 \quad \forall k \geq 0, C \subseteq \mathrm{GL}(n) \text{ compact} \\ &\Leftrightarrow \quad \|\vartheta \mapsto \varphi_j(\vartheta^{-1}\cdot) - \varphi(\vartheta^{-1}\cdot)\|_{C^k(C,\mathrm{Val})} \to 0 \quad \forall k, C \\ &\Leftrightarrow \quad \max_{\vartheta\in C} \sum_{l=0}^{k} \|\nabla^l(\varphi_j(\vartheta^{-1}\cdot) - \varphi(\vartheta^{-1}\cdot))\|_{\mathrm{Val}} \to 0 \quad \forall k, C. \end{aligned}$$

It was discovered by Alesker [9] that Val^∞ becomes an associative and commutative algebra when endowed with a bilinear and continuous product

$$\cdot : \begin{cases} \mathrm{Val}^\infty \times \mathrm{Val}^\infty \to \mathrm{Val}^\infty \\ (\varphi, \psi) \mapsto \varphi \cdot \psi \end{cases}.$$

This product respects the graduation of Val^∞ by degree of homogeneity, that is, for $\varphi \in \mathrm{Val}_i^\infty$ and $\psi \in \mathrm{Val}_j^\infty$, we have $\varphi \cdot \psi \in \mathrm{Val}_{i+j}^\infty$, where $\mathrm{Val}_k^\infty = \{0\}$ for $k > n$. The Euler characteristic serves as unit element. The Alesker product was originally defined on the subspace of mixed volumes, which is dense according to Alesker's irreducibility theorem, and then extended by continuity to smooth valuations. In [11], Alesker showed that it can be further extended to products of a smooth with a merely continuous valuation.

A special property of the Alesker product arises when we consider only products of two valuations φ_i and ψ_{n-i} of degrees of homogeneity that add up to n. In this case, the product $\varphi_i \cdot \psi_{n-i}$ is n-homogeneous and therefore a constant multiple of the n-dimensional volume V_n. Alesker used this fact to introduce a bilinear form

$$\langle \cdot, \cdot \rangle_{\mathrm{A}} : \begin{cases} \mathrm{Val}_i^\infty \times \mathrm{Val}_{n-i}^\infty \to \mathbb{R} \\ (\varphi_i, \psi_{n-i}) \mapsto \langle \varphi_i \cdot \psi_{n-i}, V_n^* \rangle \end{cases}$$

where V_n^* denotes the dual element of V_n in $\mathrm{Val}_n^* = (\mathrm{Val}_n)^*$, the topological dual space of Val_n, that is, $\langle \varphi, V_n^* \rangle = c$, whenever $\varphi = cV_n$.

The Alesker bilinear form is non-degenerate and yields an injective map with dense image (see [9])

$$\mathrm{pd}: \begin{cases} \mathrm{Val}_i^\infty \to (\mathrm{Val}_{n-i}^\infty)^* \\ \varphi_i \mapsto \langle \varphi_i, \cdot \rangle_{\mathrm{A}} \end{cases},$$

which is called Alesker–Poincaré duality map and can be extended to a map $\widetilde{\mathrm{pd}}$ on continuous valuations. The Alesker–Poincaré duality (and its extension) allows the following identification as (dense) subspaces

$$\mathrm{Val}_i^\infty \quad \subseteq \quad \mathrm{Val}_i \quad \subseteq \quad (\mathrm{Val}_{n-i}^\infty)^*. \tag{2.42}$$

Restricting pd and $\widetilde{\mathrm{pd}}$ to the subspaces of even valuations, one obtains a corresponding chain of inclusion $\mathrm{Val}_i^{+,\infty} \subseteq \mathrm{Val}_i^+ \subseteq (\mathrm{Val}_{n-i}^{+,\infty})^*$.

Relation (2.42) leads to the viewpoint of the dual space $(\mathrm{Val}_{n-i}^\infty)^*$ as an extension of the space of continuous valuations in a similar way as the space of distributions extends the space of continuous functions. Consequently, Alesker and Faifman [16] recently introduced the space of generalized valuations

$$\mathrm{Val}_i^{-\infty} = (\mathrm{Val}_{n-i}^\infty)^*,$$

endowed with the weak topology, that is, a sequence $(\psi_j)_{j\in\mathbb{N}}$ converges to a generalized valuation ψ, if $\psi_j(\varphi_{n-i}) \to \psi(\varphi_{n-i})$ for all $\varphi_{n-i} \in \mathrm{Val}_{n-i}^\infty$. The space $\mathrm{Val}_i^{+,-\infty} = (\mathrm{Val}_{n-i}^{+,\infty})^*$ is defined similarly.

Adopting the usual notation for dual spaces or distributions, we denote the application $\psi(\varphi)$ of a generalized valuation to a smooth valuation by $\langle \psi, \varphi \rangle_{\mathrm{Val}^{-\infty}}$. By definition, we have $\langle \mathrm{pd}\, \varphi_i, \psi_{n-i} \rangle_{\mathrm{Val}^{-\infty}} = \langle \varphi_i, \psi_{n-i} \rangle_{\mathrm{A}}$, for every $\varphi_i \in \mathrm{Val}_i^\infty$ and $\psi_{n-i} \in \mathrm{Val}_{n-i}^\infty$.

We now want to focus on more direct descriptions of the spaces of (smooth) valuations, namely the Klain and the Crofton map. The Klain map associates a continuous function on the Grassmanian $\mathrm{Gr}(n,i)$ to every even and i-homogeneous valuation. It was first introduced by Klain in [69]. Underlying this construction is the next theorem by Klain [68], which can be used to prove the famous Hadwiger theorem on rigid motion invariant valuations. Here, a valuation is called simple, if it vanishes on all lower-dimensional convex bodies.

Theorem 2.5.2 ([68])**.** *Let φ be a simple, continuous and even valuation. If φ is translation invariant, then there exists $c \in \mathbb{R}$ such that $\varphi = cV_n$.*

Together with the homogeneous decomposition theorem, this allows the following construction: Let $\varphi \in \mathrm{Val}_i^+$ and let $F \in \mathrm{Gr}(n, i-1)$ be an $(i-1)$-dimensional subspace, for some $i \in \{1, \dots, n-1\}$. We denote by $\mathcal{K}^n(F)$ all convex bodies contained in F. By McMullen's decomposition theorem, the restriction $\varphi|_{\mathcal{K}^n(F)}$ is a sum of homogeneous valuations of degree up to $\dim F = i-1$. As $\varphi|_{\mathcal{K}^n(F)}$ is

i-homogeneous it must be zero. Hence, we have shown, that $\varphi|_{\mathcal{K}^n(E)}$ is a simple valuation, for every i-dimensional subspace $E \in \mathrm{Gr}(n,i)$, and therefore satisfies the conditions of Klain's theorem.

We conclude that for every $E \in \mathrm{Gr}(n,i)$ there exists $(\mathrm{Kl}_i\,\varphi)(E) \in \mathbb{R}$ such that

$$\varphi(K) = (\mathrm{Kl}_i\,\varphi)(E)V_i(K), \qquad K \in \mathcal{K}^n(E).$$

The function $(\mathrm{Kl}_i\,\varphi) : \mathrm{Gr}(n,i) \to \mathbb{R}$ constructed in this way is called the Klain function of φ. $(\mathrm{Kl}_i\,\varphi)$ is continuous since φ is continuous. It was proved by Klain in [69] that φ is uniquely determined by its Klain function, that is, that the mapping $\varphi \mapsto \mathrm{Kl}_i\,\varphi$ is injective.

The Crofton map associates to any continuous function $f \in C(\mathrm{Gr}(n,i))$ (or any measure in the same way) an even valuation $\mathrm{Cr}_i\,f \in \mathrm{Val}_i^+$ by

$$(\mathrm{Cr}_i\,f)(K) = \int_{\mathrm{Gr}(n,i)} V_i(K|E)f(E)\,d\nu_i(E), \quad K \in \mathcal{K}^n.$$

The function f is called the Crofton function of the valuation $\mathrm{Cr}_i\,f$. It is clear that Cr_i maps continuous functions to continuous, even and translation invariant valuations. This follows from the same properties of the projection function $K \mapsto V_i(K|E)$, for a fixed $E \in \mathrm{Gr}(n,i)$. As $V_i(\eta K|E) = V_i(K|\eta^{-1}E)$ for any $\eta \in \mathrm{SO}(n)$, Cr_i is $\mathrm{SO}(n)$ equivariant. Therefore, Cr_i maps smooth functions on $\mathrm{Gr}(n,i))$ to smooth valuations. The restriction $\mathrm{Cr}_i\,|_{C^\infty(\mathrm{Gr}(n,i))}$ to smooth functions is also continuous with respect to the Fréchet space topology on $C^\infty(\mathrm{Gr}(n,i))$ and the Gårding topology on $\mathrm{Val}_i^{+,\infty}$, as can be seen by unravelling the definitions.

It is also worth noting that both the Klain and the Crofton map are linear functions between vector spaces.

The Alesker product of a smooth valuation $\mathrm{Cr}_i\,f$ with a valuation $\psi \in \mathrm{Val}^\infty$ can be calculated by (see, e.g., [28])

$$(\mathrm{Cr}_i\,f \cdot \psi)(K) = \int_{\mathrm{Gr}(n,n-i)} \int_{E^\perp} \psi(K \cap (E+x))f(E^\perp)\,dx\,d\nu_{n-i}(E). \tag{2.43}$$

The Klain and the Crofton map are connected via the Cosine transform. Indeed, letting f be a smooth function, $F \in \mathrm{Gr}(n,i)$, and $K \in \mathcal{K}^n(F)$, we calculate

$$\begin{aligned}(\mathrm{Cr}_i\,f)(K) &= \int_{\mathrm{Gr}(n,i)} V_i(K|E)f(E)\,d\nu_i(E)\\ &= \int_{\mathrm{Gr}(n,i)} |\cos(E,F)|f(E)\,d\nu_i(E)V_i(K) = (C_i f)(F)V_i(K).\end{aligned}$$

Consequently, the Klain function of $\mathrm{Cr}_i\,f$ is the Cosine transform $C_i f$ of f, that is, we have the following commuting diagram:

$$\begin{tikzcd} & \mathrm{Val}_i^{+,\infty} \arrow[dr, "\mathrm{Kl}_i"] & \\ C^\infty(\mathrm{Gr}(n,i)) \arrow[ur, "\mathrm{Cr}_i"] \arrow[rr, "C_i"] & & C^\infty(\mathrm{Gr}(n,i)) \end{tikzcd}$$

This observation allows to think of the Klain map as "inverse" of the Crofton map, "up to the Cosine transform", and can be used to prove the surjectivity of the Crofton map (for smooth functions and valuations).

Indeed, Alesker and Bernstein [15] proved that the image of the cosine transform and the image of the Klain map coincide. Consequently, for every $\varphi \in \mathrm{Val}_i^{+,\infty}$, there exists a function $f \in C^\infty(\mathrm{Gr}(n,i))$ such that $C_i f = \mathrm{Kl}_i\,\varphi$. By the commuting diagram above and the injectivity of Kl_i, we obtain $\varphi = \mathrm{Cr}_i\, f$.

The relation between the Klain and Crofton maps actually goes deeper. Namely, the maps are adjoint with respect to the Alesker bilinear form, which can be used to extend the Crofton map to generalized valuations (following the arguments of Alesker and Faifman [16]).

Lemma 2.5.3 ([16])**.** *The Crofton map* $\mathrm{Cr}_i : C^\infty(\mathrm{Gr}(n,i)) \to \mathrm{Val}_i^{+,\infty}$ *and the Klain map* $\mathrm{Kl}_{n-i} : \mathrm{Val}_{n-i}^{+,\infty} \to C^\infty(\mathrm{Gr}(n,n-i))$ *of complement degree* $n-i$ *are adjoint, that is,*

$$\langle \mathrm{Cr}_i\, f, \varphi\rangle_{\mathrm{A}} = \langle f, (\mathrm{Kl}_{n-i}\,\varphi)^\perp\rangle_{L^2(\mathrm{Gr}(n,i))} \quad \forall f \in C^\infty(\mathrm{Gr}(n,i)), \varphi \in \mathrm{Val}_{n-i}^{+,\infty}. \tag{2.44}$$

Proof. Let $f \in C^\infty(\mathrm{Gr}(n,i))$ and $\varphi \in \mathrm{Val}_{n-i}^{+,\infty}$. By definition, the Alesker inner product of two valuations is the coefficient of the n-dimensional volume in the homogeneous decomposition of the Alesker product of the valuations. We therefore compute for $K \in \mathcal{K}^n$ the Alesker product using (2.43), the definition of the Klain map and Fubini's theorem

$$\begin{aligned}
(\mathrm{Cr}_i\, f \cdot \varphi)(K) &= \int_{\mathrm{Gr}(n,n-i)} \int_{E^\perp} \underbrace{\varphi(K \cap (E+x))}_{=\varphi((K-x)\cap E) = \mathrm{Kl}_{n-i}\,\varphi(E) V_i((K-x)\cap E)} dx\, f(E^\perp)\, d\nu_{n-i}(E) \\
&= \int_{\mathrm{Gr}(n,n-i)} \mathrm{Kl}_{n-i}\,\varphi(E) f(E^\perp) \left(\int_{E^\perp} V_i((K-x)\cap E)\, dx\right) d\nu_{n-i}(E) \\
&= \int_{\mathrm{Gr}(n,n-i)} \mathrm{Kl}_{n-i}\,\varphi(E) f(E^\perp)\, d\nu_{n-i}(E)\, V_n(K).
\end{aligned}$$

Switching from integration over $E \in \mathrm{Gr}(n,n-i)$ to $E^\perp \in \mathrm{Gr}(n,i)$ yields the claim. □

As was shown before, the Crofton map is surjective and therefore every smooth, even and i-homogeneous valuation can be represented as $\mathrm{Cr}_i\, f$, $f \in C^\infty(\mathrm{Gr}(n,i))$. This leads to the question whether such a representation exists for every, not necessarily smooth valuation in Val_i^+. As can be seen in the original definition, the

Crofton map can be directly extended by replacing $f \in C^\infty(\mathrm{Gr}(n,i))$ with a merely continuous function or even a measure. The image of this map contains the dense subset of smooth, even and i-homogeneous valuations. Moreover, it can be shown that the image is a $\mathrm{GL}(n)$ equivariant subspace (a proof of this statement needs some linear algebra and can be found, e.g., in [48]) and therefore, by Alesker's irreducibility Theorem 2.5.1, is dense in the space of continuous valuations.

Alesker and Faifman [16] showed that the Crofton map can be extended to a surjective map onto the space of generalized valuations using the relation to the Klain function. This fact is the main ingredient for our proof of Proposition 5.0.1 for Minkowski valuations in Chapter 5. In order to be self-contained, we will repeat their proof for valuations in Section 5.2.

3 Blaschke–Santaló Inequalities for Minkowski Endomorphisms

The *Blaschke–Santaló inequality*, roughly stating that the volume product of polar reciprocal convex bodies is maximized by ellipsoids, is one of the most widely known and fundamental affine isoperimetric inequalities. Recalling from Section 2.3, it states more precisely that for $K \in \mathcal{K}_0^n$,

$$V_n(K^{\mathbf{s}})V_n(K) \leq \kappa_n^2 \tag{3.1}$$

with equality if and only if K is an ellipsoid. Here, $K^{\mathbf{s}}$ is the *polar body* of $K \in \mathcal{K}_0^n$ with respect to the *Santaló point* of K, $\mathbf{s} = \mathbf{s}(K) \in \operatorname{int} K$.

Initial proofs of (3.1) were given in the first half of the previous century by Blaschke for $n \leq 3$ and Santaló for all $n \geq 2$, while the equality conditions were completely settled only in 1985 by Petty [97]. In subsequent years, simplified proofs, including the equality cases, were obtained (see, e.g., [50, 89–91]) and it remained an active focus of research due to our evolving understanding of its impact (see [17, 35, 45, 66, 86, 92, 108] and the references therein).

Affine invariant inequalities are often more powerful than related inequalities that are merely invariant under Euclidean rigid motions. This becomes particularly striking for the Blaschke–Santaló inequality which considerably strengthens and directly implies the classical *Urysohn inequality* (2.27) (as observed by Lutwak [81]).

Another affine isoperimetric inequality that plays a special role in this chapter coincides for origin-symmetric bodies with the Blaschke–Santaló inequality but is in general weaker than (3.1). In order to state it, let $\Delta K = \frac{1}{2}(-K + K)$ denote the *central symmetral* of a convex body $K \in \mathcal{K}^n$. If K has non-empty interior, then (3.1), combined with the Brunn–Minkowski inequality (2.20), implies that

$$V_n(\Delta^\circ K)V_n(K) \leq \kappa_n^2 \tag{3.2}$$

with equality if and only if K is an ellipsoid. Here, $\Delta^\circ K$ is the polar body of ΔK with respect to the origin. The central symmetrization Δ has long been a useful tool in the Brunn–Minkowski theory (see, e.g., [52, Chapter 3.2] and [109, Chapter 10.1]). As a continuous operator on the space $\mathcal{K}^n$ endowed with the Hausdorff metric, the importance of Δ stems from its *Minkowski additivity* (that is, $\Delta(K + L) = \Delta K + \Delta L$ for all $K, L \in \mathcal{K}^n$) and compatibility with affine transformations. These are characterizing properties, as the following result by Schneider shows.

Theorem ([107]). *A continuous map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a translation invariant Minkowski additive map such that $\Phi(AK) = A\Phi K$ for every $K \in \mathcal{K}^n$ and $A \in \mathrm{GL}(n)$ if and only if $\Phi = c\,\Delta$ for some $c \geq 0$.*

This theorem was a byproduct of a more general, systematic study of Minkowski additive operators on $\mathcal{K}^n$, initiated about 50 years ago by Schneider [105–107]. Since then, and up to now, the main focus thereby has been on maps that also commute with $\mathrm{SO}(n)$ transforms (see [3, 47, 67, 112, 113, 116]). As such maps are automatically compatible with translations (see, e.g., [67, Section 2.3]), they are often assumed without loss of generality to be translation invariant, leading to the following central definition.

Definition. *A continuous map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a* Minkowski endomorphism *if Φ is Minkowski additive, translation invariant, and commutes with $\mathrm{SO}(n)$ transforms. The* trivial *Minkowski endomorphism maps every convex body to the origin.*

Much of this chapter is motivated by the observation that the Urysohn inequality (2.27) and (3.2) can be cast as volume estimates for polar Minkowski endomorphisms. Another prominent such example was established by Lutwak [85] for polar projection bodies of order one of $K \in \mathcal{K}_0^n$,

$$V_n(\Pi_1^\circ K)V_n(K) \le \kappa_n^2 \tag{3.3}$$

with equality if and only if K is a ball. $\Pi_1 K$ can be defined, using support functions, by $h(\Pi_1 K, u) = c_n w(K|u^\perp)$, where $K|u^\perp$ denotes the orthogonal projection of K onto $u^\perp$ and $c_n \in \mathbb{R}$ is chosen such that $\Pi_1 B^n = B^n$.

The natural question to what degree Urysohn's inequality (2.27) and inequalities (3.2) and (3.3) can be unified was asked by Lutwak. A partial answer was given in [25], deduced from results in [61], where the Urysohn inequality (2.27) and (3.3) were identified as part of a larger family of inequalities for a subcone of Minkowski endomorphisms which are *monotone*, that is, $K \subseteq L$ implies $\Phi K \subseteq \Phi L$ for all $K, L \in \mathcal{K}^n$. For a more precise statement we require the following classification of monotone Minkowski endomorphisms by Kiderlen.

Theorem ([67])**.** *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a monotone Minkowski endomorphism if and only if there exists a non-negative $\mathrm{SO}(n-1)$ invariant measure μ on $\mathbb{S}^{n-1}$ with center of mass at the origin such that*

$$h(\Phi K, \cdot) = h(K, \cdot) * \mu \tag{3.4}$$

for every $K \in \mathcal{K}^n$. Moreover, the measure μ is uniquely determined by Φ.

See Section 2.2 for the definition of the convolution of functions and measures on $\mathbb{S}^{n-1}$ used in (3.4). Note that we assume all measures to be finite Borel measures.

In [25], the Urysohn inequality (2.27) and (3.3) were generalized to monotone Minkowski endomorphisms generated by area measures of order one of zonoids. As a first main result, we generalize these inequalities from [25] to *all* monotone Minkowski endomorphisms Φ. Throughout, we always assume that $n \ge 3$.

Theorem 3.0.1. *Suppose that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a monotone non-trivial Minkowski endomorphism. Among $K \in \mathcal{K}^n$ with non-empty interior the volume product*

$$V_n(\Phi^\circ K)V_n(K)$$

is maximized by Euclidean balls. If $\Phi = c\,\Delta$ for some $c > 0$, then K is a maximizer if and only if it is an ellipsoid. Otherwise, Euclidean balls are the only maximizers.

Let us emphasize that Theorem 3.0.1 not only includes the Urysohn inequality (2.27) and inequalities (3.2) and (3.3) as special cases, but provides an extension of the isoperimetric inequalities from [25] for a non-dense set of Minkowski endomorphisms to all monotone ones. While the proof of Theorem 3.0.1 does not require any results from [61], our approach is very much inspired by techniques from [61] and relies on Kiderlen's classification of monotone Minkowski endomorphisms.

While it was long known that not all Minkowski endomorphisms are monotone, a conjecture that they are all *weakly monotone* (see Section 3.1 for details) was disproved by Dorrek [47] only recently. We will show in Section 3.2 that Theorem 3.0.1 cannot be extended further to merely weakly monotone endomorphisms.

By Schneider's above characterization of the map Δ, inequality (3.2) is the only affine invariant one among the family of isoperimetric inequalities provided by Theorem 3.0.1. With our second main result we show that all these inequalities can be deduced from the Blaschke–Santaló inequality. In particular, among inequalities for even Minkowski endomorphisms, (3.2) is the strongest member of the inequalities from Theorem 3.0.1. This is in contrast to the volume estimates obtained in [25], among which (3.3) was the strongest one, since (3.2) was not included. Finally, we prove that each of the inequalities of Theorem 3.0.1 is stronger than and directly implies the Urysohn inequality (2.27).

Theorem 3.0.2. *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a monotone Minkowski endomorphism such that $\Phi B^n = B^n$ and $K \in \mathcal{K}^n$ has non-empty interior, then*

$$\kappa_n \left(\frac{w(K)}{2}\right)^{-n} \leq V_n(\Phi^\circ K) \leq V_n(K^{\mathrm{s}}). \tag{3.5}$$

There is equality in the left hand inequality if and only if ΦK is a Euclidean ball. Equality in the right hand inequality holds if and only if K is centrally symmetric and $\Phi = \Delta$ or if K is a Euclidean ball.

Note that the right-hand inequality of Theorem 3.0.2 (and its equality case) combined with the Blaschke–Santaló inequality implies Theorem 3.0.1. In Section 3.2, we will therefore first prove Theorem 3.0.2 and then deduce Theorem 3.0.1 as a consequence.

3.1 Preliminary Results

In the following we first review additional background material on Minkowski endomorphisms required in the proofs of Theorems 3.0.1 and 3.0.2 and the discussion thereof in Section 3.2.

We begin by recalling a notion of monotonicity for operators on convex bodies, which is of particular importance for Minkowski endomorphisms.

Definition 3.1.1. *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called* weakly monotone *if $\Phi K \subseteq \Phi L$ for all $K, L \in \mathcal{K}^n$ such that $K \subseteq L$ and $s(K) = s(L) = o$.*

The quest to establish a classification of all Minkowski endomorphisms has its origin in the paper [107] from 1974 by Schneider. The following result – combining theorems by Dorrek and Kiderlen – represents the status quo on this difficult task, which has not yet been completed. Here, we call a measure on $\mathbb{S}^{n-1}$ *linear* if it has a density (with respect to spherical Lebesgue measure) of the form $u \mapsto \langle x, u \rangle$ for some $x \in \mathbb{R}^n$.

Theorem 3.1.2 ([47, 67])**.** *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Minkowski endomorphism, then there exists a signed zonal measure μ on $\mathbb{S}^{n-1}$ with center of mass at the origin such that*

$$h(\Phi K, \cdot) = h(K, \cdot) * \mu \tag{3.6}$$

for every $K \in \mathcal{K}^n$. The measure μ is uniquely determined by Φ.

Moreover, Φ is monotone if and only if μ is non-negative and Φ is weakly monotone if and only if μ is non-negative up to addition of a linear measure.

The measure μ uniquely associated with the Minkowski endomorphism Φ via the relation (3.6) is called the *generating measure* of Φ and we frequently indicate this by writing Φ_μ. Exploiting this one-to-one correspondence, we can endow the cone of Minkowski endomorphisms with the topology induced by the weak convergence of their generating measures. Before we exhibit some prominent examples, let us note that for $n = 2$, Theorem 3.1.2 is due to Schneider [106] who also showed that in this special case *all* Minkowski endomorphisms are weakly monotone. The conjecture that the same is true for $n \geq 3$ was disproved by Dorrek [47].

Example 3.1.3.

(a) Recall that σ denotes the $\mathrm{SO}(n)$ invariant probability measure on $\mathbb{S}^{n-1}$. The Minkowski endomorphism $\Phi_\sigma : \mathcal{K}^n \to \mathcal{K}^n$ generated by σ satisfies

$$\Phi_\sigma K = \frac{w(K)}{2} B^n \tag{3.7}$$

for every $K \in \mathcal{K}^n$ and, thus, the inequality $V_n(\Phi_\sigma^\circ K) V_n(K) \leq \kappa_n^2$ is precisely the Urysohn inequality.

(b) The unique *discrete* zonal probability measure on $\mathbb{S}^{n-1}$ with center of mass at the origin is given by

$$\nu = \frac{1}{2}(\delta_{\bar{e}} + \delta_{-\bar{e}}). \tag{3.8}$$

It is the generating measure of the central symmetrization $\Delta : \mathcal{K}^n \to \mathcal{K}^n$.

(c) The generating measure of the Minkowski endomorphism $\Pi_1 : \mathcal{K}^n \to \mathcal{K}^n$ (recall our normalization $\Pi_1 B^n = B^n$) is the invariant probability measure $\sigma_{\bar{e}^\perp}$ concentrated on the equator $\mathbb{S}^{n-1} \cap \bar{e}^\perp$. Noting that

$$\sigma_{\bar{e}^\perp} = S_1\big(\tfrac{1}{2}[-\bar{e}, \bar{e}], \cdot\big),$$

Berg and Schuster [25] considered, more generally, Minkowski endomorphisms generated by measures of the form $S_1(Z, \cdot)$ for some zonoid $Z \in \mathcal{K}^n$ and established Theorem 3.0.1 for such maps. However, this class is not dense in all monotone Minkowski endomorphisms.

(d) The Minkowski endomorphism $\mathrm{J} : \mathcal{K}^n \to \mathcal{K}^n$, defined by

$$\mathrm{J}K = K - s(K), \tag{3.9}$$

is weakly monotone and its generating measure is given by $\delta_{\bar{e}} - n\langle \bar{e}, \cdot \rangle d\sigma$.

Next, we prove three well-known and simple but useful properties of Minkowski endomorphisms required in the proof of Theorem 3.0.2.

Lemma 3.1.4. *Suppose that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Minkowski endomorphism with generating measure μ on $\mathbb{S}^{n-1}$. Then the following statements hold:*

(i) $w(\Phi K) = \mu(\mathbb{S}^{n-1})\, w(K)$ *for every* $K \in \mathcal{K}^n$*;*

(ii) If Φ is non-trivial and weakly monotone and $K \in \mathcal{K}^n$ has non-empty interior, then ΦK contains the origin in its interior;

(iii) $\Phi\{x\} = \{o\}$ *for every* $x \in \mathbb{R}^n$.

Proof. In order to see (i), we use that the spherical convolution is selfadjoint and Abelian for zonal measures. These facts combined with (2.13) and (3.7) yield,

$$\begin{aligned} w(\Phi K) &= 2\int_{\mathbb{S}^{n-1}} (h(K,\cdot) * \mu)(u)\, d\sigma(u) = 2\int_{\mathbb{S}^{n-1}} (h(K,\cdot) * \sigma)(u)\, d\mu(u) \\ &= w(K) \int_{\mathbb{S}^{n-1}} h(B^n, u)\, d\mu(u) = \mu(\mathbb{S}^{n-1})\, w(K) \end{aligned}$$

for every $K \in \mathcal{K}^n$. For the proof of (ii), first note that every non-trivial Minkowski endomorphism maps Euclidean balls of positive radii to origin-symmetric balls by $\mathrm{SO}(n)$ equivariance and translation invariance. These balls must be of positive radii by (i). Now, using that the Steiner point $s(K) \in \operatorname{int} K$ for every $K \in \mathcal{K}^n$ with non-empty interior (see, e.g., [109, p. 50]), we obtain (ii) from the translation invariance of Φ, which implies $\Phi K = \Phi(K - s(K))$, and its monotonicity on bodies with Steiner points at the origin. Claim (iii) is a direct consequence of the fact that $\{x\} = \{x\} + \{o\}$ and the additivity and translation invariance of Φ. □

3.2 Proof of the main results

In this section we first prove Theorem 3.0.2 and deduce Theorem 3.0.1 from it. We then show that a further extension of Theorem 3.0.1 to merely weakly monotone Minkowski endomorphisms is not possible.

Proof of Theorem 3.0.2. Let $K \in \mathcal{K}^n$ have non-empty interior and note that the normalization $\Phi B^n = B^n$ ensures that there is equality in both inequalities of (3.5) if K is a Euclidean ball. By Theorem 3.1.2, the generating measure μ of Φ is non-negative and, by our normalization, $1 = h(\Phi B^n, \cdot) = \mu(\mathbb{S}^{n-1})$. Moreover, by Lemma 3.1.4 (ii), ΦK contains the origin in its interior.

In order to establish the left hand inequality of (3.5), we use the polar coordinate formula for volume (2.12), Jensen's inequality, and (2.13) to obtain

$$\left(\frac{V_n(\Phi^\circ K)}{\kappa_n}\right)^{-1/n} = \left(\int_{\mathbb{S}^{n-1}} h(\Phi K, u)^{-n} d\sigma(u)\right)^{-1/n} \leq \int_{\mathbb{S}^{n-1}} h(\Phi K, u)\, d\sigma(u) = \frac{w(\Phi K)}{2}.$$

An application of Lemma 3.1.4 (i) now yields the desired inequality. By the equality conditions for Jensen's inequality, equality holds here, and thus in the left hand side of (3.5), if and only if $h(\Phi K, \cdot)$ is constant, that is, if and only if ΦK is a ball.

For the proof of the right hand inequality of (3.5), we may assume, by the translation invariance of both sides, that $\mathbf{s}(K) = o$, which implies $h(K, \cdot) > 0$ and $K^{\mathbf{s}} = K^\circ$. First, we use the polar coordinate formula for volume (2.12), (2.4), and Jensen's inequality to obtain

$$\begin{aligned} V_n(\Phi^\circ K) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(\Phi K, u)^{-n} du = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-1}} h(K, \vartheta_u v)\, d\mu(v)\right)^{-n} du \\ &\leq \frac{1}{n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} h(K, \vartheta_u v)^{-n}\, d\mu(v)\, du. \end{aligned}$$

Since Φ and the polar map commute with $\mathrm{SO}(n)$ transforms, we may replace K here by a rotated copy θK and integrate over $\mathrm{SO}(n)$ with respect to the Haar measure, to arrive at

$$V_n(\Phi^\circ K) = \int_{\mathrm{SO}(n)} V_n(\Phi^\circ(\theta K))\, d\theta \leq \frac{1}{n}\int_{\mathrm{SO}(n)}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} h(K, \theta^{-1}\vartheta_u v)^{-n}\, d\mu(v)\, du\, d\theta, \tag{3.10}$$

where we also used (2.11) in the last step. By Fubini's theorem and the invariance of the Haar measure on $\mathrm{SO}(n)$,

$$V_n(\Phi^\circ K) \leq \frac{1}{n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_{\mathrm{SO}(n)} h(K, \theta^{-1}u)^{-n}\, d\theta\, d\mu(v)\, du.$$

Using the fact that $\mu(\mathbb{S}^{n-1}) = 1$ and again Fubini's theorem, we finally obtain the desired inequality,

$$V_n(\Phi^\circ K) \leq \frac{1}{n} \int_{\mathbb{S}^{n-1}} \int_{\mathrm{SO}(n)} h(K, \theta^{-1}u)^{-n}\, d\theta\, du = \int_{\mathrm{SO}(n)} V_n(\theta K^\circ)\, d\theta = V_n(K^\circ).$$

By the above arguments, equality holds in the right hand inequality of (3.5) if and only if we have equality in (3.10). By the equality condition of Jensen's inequality this is the case if and only if for almost every $u \in \mathbb{S}^{n-1}$ and almost every $\theta \in \mathrm{SO}(n)$ there exist $c_{u,\theta} \in \mathbb{R}^+$ such that $h(K, \theta^{-1}\vartheta_u v) = c_{u,\theta}$ for μ-almost every $v \in \mathbb{S}^{n-1}$. Clearly, by the continuity of $h(K, \cdot)$, this is the case if and only if for every $\eta \in \mathrm{SO}(n)$, there exist $c_\eta \in \mathbb{R}^+$ such that

$$h(K, \eta v) = c_\eta \qquad \text{for } \mu\text{-a.e. } v \in \mathbb{S}^{n-1}. \tag{3.11}$$

Let $A_\eta = \{v \in \mathbb{S}^{n-1} : h(K, \eta v) = c_\eta\} \subseteq \mathbb{S}^{n-1}$ and observe that $\mu(\mathbb{S}^{n-1} \backslash A_\eta) = 0$. Note that A_η is closed, by the continuity of $h(K, \cdot)$, and therefore contains the support of μ. If μ is discrete, by $\mathrm{SO}(n-1)$ invariance, it must coincide with the measure ν given by (3.8) and, hence, $\Phi = \Delta$. Since $\operatorname{supp} \nu = \{-\bar{e}, \bar{e}\}$, it follows immediately from (3.11) that $h(K, \cdot)$ takes the same value on antipodal points, that is, K is origin-symmetric.

It remains to be shown that if μ is *not discrete*, then (3.11) holds if and only if $h(K, \cdot)$ is constant on $\mathbb{S}^{n-1}$, or equivalently, if K is a Euclidean ball. Since μ is non-zero and not discrete, there exists $w \in \operatorname{supp} \mu \backslash \{-\bar{e}, \bar{e}\}$. By the $\mathrm{SO}(n-1)$-invariance of μ, the entire parallel subsphere orthogonal to $\bar{e}$ through w is contained in $\operatorname{supp} \mu$ and therefore in every A_η. Hence, $h(K, \eta v) = c_\eta$ for every $\eta \in \mathrm{SO}(n)$ and all v in this subsphere. Choosing η' such that this subsphere and a copy of it rotated by η' intersect, we see that the value of $h(K, \cdot)$ at the intersection is given by c_{id} and $c_{\eta'}$, thus, these values must be equal. By repeating this argument finitely many times, we can reach every point on $\mathbb{S}^{n-1}$ implying that $h(K, \cdot)$ is constant as desired. □

Theorem 3.0.1 is now an easy consequence of the right-hand inequality of (3.5).

Proof of Theorem 3.0.1. Let $K \in \mathcal{K}^n$ have non-empty interior and assume without loss of generality that $\Phi B^n = B^n$. Then, by the right-hand inequality of (3.5) and the Blaschke–Santaló inequality (3.1), it follows that

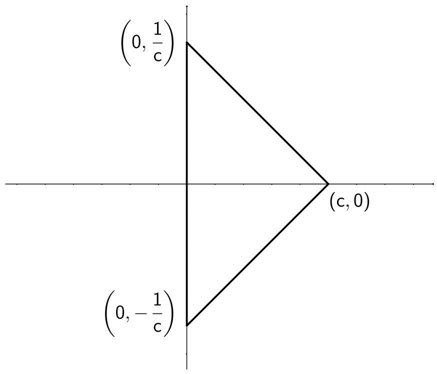
$$V_n(\Phi^\circ K) V_n(K) \leq V_n(K^{\mathrm{s}}) V_n(K) \leq \kappa_n^2 = V_n(\Phi^\circ B^n) V_n(B^n).$$

The equality $V_n(\Phi^\circ K) V_n(K) = \kappa_n^2$ holds if and only if equality holds both in the right-hand inequality of (3.5) and the Blaschke–Santaló inequality (3.1), that is, if and only if $\Phi = \Delta$ and K is an ellipsoid or if K is a Euclidean ball. □

Next, we want to show that an extension of Theorem 3.0.1 to all merely weakly monotone Minkowski endomorphisms is impossible.

Theorem 3.2.1. *For every $n \geq 2$, the volume product $V_n(\mathrm{J}^\circ K)V_n(K)$ is unbounded for the weakly monotone Minkowski endomorphism $\mathrm{J} : \mathcal{K}^n \to \mathcal{K}^n$, $\mathrm{J}K = K - s(K)$.*

Proof. We begin with dimension $n = 2$, where for $c > 0$, we consider the triangle $K_c \in \mathcal{K}^2$ of unit volume defined by $K_c = \operatorname{conv}\left\{(c,0), \left(0,\frac{1}{c}\right), \left(0,-\frac{1}{c}\right)\right\}$.



Then for every $0 < t < c$, the polar body $(K_c - (t,0))^\circ$ is again a triangle given by

$$(K_c - (t,0))^\circ = \operatorname{conv}\left\{\left(\frac{1}{c-t}, \frac{c^2}{c-t}\right), \left(\frac{1}{c-t}, -\frac{c^2}{c-t}\right), \left(-\frac{1}{t}, 0\right)\right\}.$$

Thus, a short calculation yields the volume formula,

$$V_n\left((K_c - (t,0))^\circ\right) = \frac{c^3}{t(c-t)^2}. \tag{3.12}$$

Due to the axial symmetry of K_c, its Steiner point $s(K_c)$ lies on the x-axis and its coordinates are easily calculated to

$$s(K_c) = \left(\frac{c}{\pi}\arctan c^2, 0\right).$$

Plugging this into (3.12), we obtain

$$V_n\left((K_c - s(K_c))^\circ\right)V_n(K_c) = \frac{\pi}{\arctan(c^2)(1 - \frac{1}{\pi}\arctan(c^2))^2},$$

which tends to infinity as c tends to zero.

For arbitrary $n \geq 3$, we consider the body of revolution $L_c \in \mathcal{K}^n$, obtained by rotating the body K_c around the e_1-axis of $\mathbb{R}^n$. The volume of L_c can be easily calculated and is given by

$$V_n(L_c) = \frac{\kappa_{n-1}}{nc^{n-2}}. \tag{3.13}$$

Since for every $K \in \mathcal{K}^n$ containing the origin in its interior and any subspace $H \subseteq \mathbb{R}^n$, we have $K^\circ|H = (K \cap H)^\circ$, where the polar body on the right hand side is taken in the subspace H, it follows, by taking H a 2-dimensional subspace containing e_1, that for every $0 < t < c$, $(L_c - te_1)^\circ$ is a body of revolution obtained by rotating the triangle $(K_c - te_1)^\circ$ around the e_1-axis. Consequently, we obtain the volume formula,

$$V_n((L_c - te_1)^\circ) = \frac{\kappa_{n-1}}{n} \frac{c^{2n-1}}{t(c-t)^n}$$

and, from this and (3.13),

$$V_n((L_c - te_1)^\circ)V_n(L_c) = \left(\frac{\kappa_{n-1}}{n}\right)^2 \frac{c^{n+1}}{t(c-t)^n}.$$

Letting $t = cg(c)$, where $g(c)$ depends only on c and satisfies $0 < g(c) < 1$, this reduces to

$$V_n((L_c - cg(c)e_1)^\circ)V_n(L_c) = \left(\frac{\kappa_{n-1}}{n}\right)^2 \frac{1}{g(c)(1-g(c))^n}$$

which clearly tends to infinity if $g(c)$ tends to zero as c tends to zero. It remains to be shown that the e_1-coordinate of the Steiner point of L_c is of the form $cg(c)$ such that $\lim_{c\to 0} g(c) = 0$ (note that by the rotational symmetry of L_c all other coordinates of $s(L_c)$ are zero). Since $h(L_c, u) = h(K_c, \langle u, e_1\rangle e_1 + \sqrt{1 - \langle u, e_1\rangle^2} e_2)$ for every $u \in \mathbb{S}^{n-1}$, we obtain from (2.14) by integration in cylindrical coordinates (cf. [59, Lemma 1.3.1]),

$$\begin{aligned}\langle s(L_c), e_1\rangle &= \frac{1}{\kappa_n} \int_{\mathbb{S}^{n-1}} h(L_c, u)\langle u, e_1\rangle \, du \\ &= n \int_{-1}^{1} h(K_c, \zeta e_1 + \sqrt{1-\zeta^2} e_2)\zeta(1-\zeta^2)^{\frac{n-3}{2}} \, d\zeta.\end{aligned}$$

Inserting the explicit expression for the support function of K_c, we see that

$$\begin{aligned}\langle s(L_c), e_1\rangle &= n\left(\int_{-1}^{\frac{1}{\sqrt{1+c^4}}} \frac{\zeta}{c}(1-\zeta^2)^{\frac{n-1}{2}} \, d\zeta + \int_{\frac{1}{\sqrt{1+c^4}}}^{1} c\zeta^2(1-\zeta^2)^{\frac{n-3}{2}} \, d\zeta\right) \\ &= c\,n \underbrace{\left(\frac{-c^{2n}}{(n+1)(1+c^4)^{\frac{n+1}{2}}} + \int_{\frac{1}{\sqrt{1+c^4}}}^{1} \zeta^2(1-\zeta^2)^{\frac{n-3}{2}} \, d\zeta\right)}_{=g(c)}\end{aligned}$$

which is of the desired form. □

Theorem 3.2.1 raises the interesting problem whether there exist weakly monotone or even non-monotonic Minkowski endomorphisms Φ (different from multiples of J and −J) such that their volume product $V_n(\Phi^\circ K)V_n(K)$ is unbounded. In view of Theorem 3.0.1, it is also natural to ask whether there exist Minkowski endomorphisms which are maximized by convex bodies of non-empty interior which may be different from Euclidean balls. Partial answers to this question are given in Chapter 5 by different methods.

Let us also comment on our general assumption that $n \geq 3$. An analogue of Theorem 3.1.2 for $n = 2$ was already obtained by Schneider [106] in 1974 showing also that all Minkowski endomorphisms in $\mathbb{R}^2$ are weakly monotone. The key difference in this case is the commutativity of SO(2), which implies that for every monotone Minkowski endomorphism Φ, ΦK is the limit of combinations $\lambda_1\vartheta_1 K + \cdots + \lambda_m\vartheta_m K$, where $\vartheta_i \in \mathrm{SO}(2)$ and $\lambda_i > 0$ such that $\sum_{i=1}^m \lambda_i$ is fixed. The well-known inequality $V_n((s_1K_1 + s_2K_2)^\circ) \leq s_1V_n(K_1^\circ) + s_2V_n(K_2^\circ)$ for $K_1, K_2 \in \mathcal{K}^2$ with non-empty interior and $s_1, s_2 > 0$ such $s_1 + s_2 = 1$, directly implies an analogue of Theorem 3.0.2 for $n = 2$.

4 Asplund Endomorphisms

A second focus of this thesis concerns the continuing effort to extend notions and results from convex geometry to the class of log-concave functions. The most basic such notions are Minkowski addition and scalar multiplication, which are naturally extended by the Asplund sum (or sup-convolution). As the Asplund sum of two log-concave and semi-continuous functions need not be log-concave and semi-continuous, a frequently used possibility to overcome this issue is to work with the space $\mathrm{LC_c}(\mathbb{R}^n)$ of all proper log-concave functions which are upper semi-continuous and coercive, see Section 2.4 for details.

As a seminal inequality for log-concave functions, we first mention the celebrated Prékopa–Leindler inequality, which is universally recognized as the functional form of the Brunn–Minkowski inequality (see, e.g., [109, Section 7.1]). A functional version of the Blaschke–Santaló inequality, discovered later by Ball [22], is of special importance for us. If f is log-concave, even and $0 < \int_{\mathbb{R}^n} f \, dx < \infty$, then Ball's *functional Blaschke–Santaló inequality* reads

$$\int_{\mathbb{R}^n} f(x)\, dx \int_{\mathbb{R}^n} f^\circ(x)\, dx \leq (2\pi)^n \tag{4.1}$$

with equality if and only if f is a Gaussian. Recall from Section 2.4 that $f^\circ = e^{-\mathcal{L}(-\log f)}$ denotes the polar function of f, using the classical Legendre transform $\mathcal{L}$.

The cases for equality in (4.1) were settled by Artstein, Klartag and Milman [18], who also established a far-reaching extension of (4.1) to not necessarily even functions (cf. Theorem 2.4.7), that has sparked a great deal of research interest in recent years, see [21, 23, 39, 44, 50, 62, 75–77, 101].

As noted by Rotem [100], the functional Blaschke–Santaló inequality implies an analogue of Urysohn's inequality for log-concave functions, a result first obtained by Klartag and Milman [71] by other means. It can be conveniently formulated with the help of the support function of a log-concave $f : \mathbb{R}^n \to [0, \infty)$. If additionally $\int_{\mathbb{R}^n} f\, dx = (2\pi)^{n/2}$ and γ_n is the standard Gaussian measure on $\mathbb{R}^n$, the functional analogue of Urysohn's inequality states that

$$\frac{2}{n} \int_{\mathbb{R}^n} h(f, x)\, d\gamma_n(x) \geq 1 \tag{4.2}$$

with equality if and only if f is a translation of the standard Gaussian. Note that if $f = \mathbb{1}_K$ for some $K \in \mathcal{K}^n$, the left hand side is proportional to $w(K)$. However, the *sharp* geometric Urysohn inequality (2.27) cannot be recovered from (4.2).

Recalling that $\mathrm{SO}(n)$ acts naturally on $\mathrm{LC_c}(\mathbb{R}^n)$, specifically, $(\vartheta f)(x) = f(\vartheta^{-1}x)$ for $\vartheta \in \mathrm{SO}(n)$ and $f \in \mathrm{LC_c}(\mathbb{R}^n)$, we can now introduce 'functional Minkowski endomorphisms' on $\mathrm{LC_c}(\mathbb{R}^n)$ as follows:

Definition. *A continuous map* $\Psi : \mathrm{LC_c}(\mathbb{R}^n) \to \mathrm{LC_c}(\mathbb{R}^n)$ *is an* Asplund endomorphism *if* Ψ *is Asplund additive, translation invariant, and commutes with the* $\mathrm{SO}(n)$ *action. The* trivial *Asplund endomorphism maps every function to the indicator of the origin. (Note that we do not identify functions that coincide almost everywhere.)*

In Section 4.2, we discuss the problem of establishing an analogue of Kiderlen's characterization of monotone Minkowski endomorphisms (Theorem 3.1.2). Here, we use the latter as motivation to define the following rich class of Asplund endomorphisms which are *monotone*, that is, $f \le g$ implies $\Psi f \le \Psi g$ for all $f, g \in \mathrm{LC_c}(\mathbb{R}^n)$.

Theorem 4.0.1. *Each non-negative* $\mathrm{SO}(n-1)$ *invariant measure* μ *on* $\mathbb{S}^{n-1}$ *with center of mass at the origin induces a monotone Asplund endomorphism* Ψ_μ *by*

$$h(\Psi_\mu f, \cdot) = h(f, \cdot) \circledast \mu$$

for $f \in \mathrm{LC_c}(\mathbb{R}^n)$. *Moreover, the measure* μ *is uniquely determined by* Ψ_μ.

For the definition of the convolution $\circledast$ of the convex function $h(f, \cdot)$ on $\mathbb{R}^n$ with the measure μ, we refer to Section 4.1. Let us emphasize that $\Psi_\mu \mathbb{1}_K = \mathbb{1}_{\Phi_\mu K}$ for every $K \in \mathcal{K}^n$, where Φ_μ is the monotone Minkowski endomorphism defined by (3.4). In this sense, the Asplund endomorphisms Ψ_μ extend the class of all monotone Minkowski endomorphisms to $\mathrm{LC_c}(\mathbb{R}^n)$. As our next main result, we prove a functional analogue of Theorem 3.0.1 for the monotone Asplund endomorphisms defined by Theorem 4.0.1.

Theorem 4.0.2. *Let* μ *be an* $\mathrm{SO}(n-1)$ *invariant probability measure on* $\mathbb{S}^{n-1}$ *with center of mass at the origin. If* $f \in \mathrm{LC_c}(\mathbb{R}^n)$ *such that* $\int_{\mathbb{R}^n} f\,dx > 0$, *then*

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\,dx \le (2\pi)^n. \tag{4.3}$$

If μ *is discrete, there is equality if and only if* f *is a Gaussian. Otherwise, equality holds if and only if* f *is proportional to a translation of the standard Gaussian.*

Note that the additional normalization of μ in Theorem 4.0.2 is critical due to the non-homogeneity of the integral as well as the functional polarity with respect to Asplund scalar multiplication.

Example 4.0.3.

(a) In 2006, Colesanti [37] introduced the (Asplund) *difference function* $\Delta_\star f$ of a log-concave function $f \in \mathrm{LC_c}(\mathbb{R}^n)$ by $\Delta_\star f = \frac{1}{2} \cdot f \star \frac{1}{2} \cdot \overline{f}$, where $\overline{f}(x) = f(-x)$. By taking μ to be the even discrete probability measure $\frac{1}{2}\delta_{\bar{e}} + \frac{1}{2}\delta_{-\bar{e}}$, concentrated on the stabilizer $\bar{e} \in \mathbb{S}^{n-1}$ of $\mathrm{SO}(n-1)$ and its antipodal, Theorem 4.0.2 reduces to a functional analogue of (3.2),

$$\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} (\Delta_\star f)^\circ(x)\,dx \le (2\pi)^n \tag{4.4}$$

which can also be deduced from Ball's functional Blaschke–Santaló inequality (4.1) and the Prékopa–Leindler inequality. Clearly, for even f, (4.4) coincides with (4.1) which, thus, is a member of the family of inequalities of Theorem 4.0.2.

(b) If μ is taken to be the uniform spherical probability measure σ, the induced Asplund endomorphism Ψ_σ has the following interesting properties:

- $\Psi_\sigma f$ is radially symmetric for every $f \in \mathrm{LC_c}(\mathbb{R}^n)$;
- $\Psi_\sigma \mathbb{1}_K = \frac{w(K)}{2} \cdot \mathbb{1}_{B^n}$ for every $K \in \mathcal{K}^n$.

Moreover, we will see that inequality (4.3) for Ψ_σ is *strictly* stronger than the functional analogue of Urysohn's inequality (4.2) and, when restricted to indicators of convex bodies, yields a version of the Urysohn inequality (2.27) which is asymptotically sharp (see Section 4.2).

The proof of Theorem 4.0.2 does not make use of Theorem 3.0.1, however, it follows similar arguments as in the geometric setting, replacing the application of the Blaschke–Santaló inequality (3.1) by its functional form. In particular, in Section 4.2 we also prove a functional version of Theorem 3.0.2 showing that each of the inequalities of Theorem 4.0.2 is stronger than the one for the Asplund endomorphism Ψ_σ and, hence, *strictly* stronger than the functional analogue of Urysohn's inequality (4.2). For even μ, inequality (4.4) is proven to be the strongest one among the family of inequalities (4.3). Finally, we will see that Theorem 3.0.1 can be recovered in an asymptotically optimal form by restricting Theorem 4.0.2 to indicators of convex bodies.

4.1 Convolutions of Convex Functions

Motivated by (2.4) and the significance of the spherical convolution $*$ for Minkowski endomorphisms, we now introduce an extension of $*$ to the following important open subset of convex functions in $\mathrm{Cvx}(\mathbb{R}^n)$,

$$\mathrm{Cvx}_{(o)}(\mathbb{R}^n) = \{\varphi \in \mathrm{Cvx}(\mathbb{R}^n) : o \in \operatorname{int}\operatorname{dom}\varphi\}.$$

Note that, by Proposition 2.4.5 (ii), $\varphi \in \mathrm{Cvx}_{(o)}$ if and only if $\mathcal{L}\varphi \in \mathrm{Cvx_c}(\mathbb{R}^n)$ or, equivalently,

$$f \in \mathrm{LC_c}(\mathbb{R}^n) \text{ if and only if } h(f,\cdot) \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n). \tag{4.5}$$

In the following, for $x \in \mathbb{R}^n \backslash \{o\}$, let $\vartheta_x \in \mathrm{SO}(n)$ denote an arbitrary rotation such that $\vartheta_x \bar{e} = \frac{x}{\|x\|}$.

Definition 4.1.1. *Suppose that $\varphi \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$ and let μ be a non-negative zonal measure on $\mathbb{S}^{n-1}$. The convolution $\varphi \circledast \mu$ is defined for $x \in \mathbb{R}^n \backslash \{o\}$ by*

$$(\varphi \circledast \mu)(x) = \int_{\mathbb{S}^{n-1}} \varphi(\|x\|\vartheta_x v)\, d\mu(v) \tag{4.6}$$

and at the origin by $(\varphi \circledast \mu)(o) = \liminf_{\|x\|\to 0}(\varphi \circledast \mu)(x)$.

Note that if φ is homogeneous of some degree $p \in \mathbb{R}$, then, by (2.4), $\varphi \circledast \mu$ coincides with the homogeneous extension of degree p of $\widehat{\varphi} * \mu$ to $\mathbb{R}^n$, where $\widehat{\varphi}$ is the restriction of φ to $\mathbb{S}^{n-1}$. The following result shows that $\varphi \circledast \mu$ is indeed a well defined function in $\mathrm{Cvx}_{(o)}(\mathbb{R}^n)$. This is critical for the proof of Theorem 4.0.1.

Proposition 4.1.2. *Suppose that $\varphi \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$ and μ is a non-negative zonal measure on $\mathbb{S}^{n-1}$. Then $\varphi \circledast \mu$ is a well defined function in $\mathrm{Cvx}_{(o)}(\mathbb{R}^n)$. Moreover, the map $\varphi \mapsto \varphi \circledast \mu$ defines a continuous linear operator from $\mathrm{Cvx}_{(o)}(\mathbb{R}^n)$ to itself which commutes with the action of $\mathrm{SO}(n)$.*

Proof. First note that the right hand side of (4.6) is independent of the choice of ϑ_x by the $\mathrm{SO}(n-1)$ invariance of μ. Since φ is convex on $\mathbb{R}^n$, it is bounded from below by an affine function and, thus, the negative part of φ is bounded on every sphere in $\mathbb{R}^n$. Consequently, the integral in (4.6) is well defined and takes values in $(-\infty, \infty]$. Moreover, since $o \in \operatorname{int} \operatorname{dom} \varphi$, there exists $r > 0$ such that φ takes finite values on rB^n, which implies that also $\varphi \circledast \mu$ takes finite values on rB^n. In particular, $\varphi \circledast \mu$ is proper and $o \in \operatorname{int} \operatorname{dom} \varphi \circledast \mu$.

The proof that $\varphi \circledast \mu$ is convex on $\mathbb{R}^n$ is rather tedious and technical and we therefore postpone it to Section 4.3. In order to see that $\varphi \circledast \mu \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$, it remains to show that it is lower semi-continuous. To this end, let $x_0 \in \mathbb{R}^n \backslash \{o\}$ and note that φ is bounded from below on the compact set $2\|x_0\|B^n$ by semi-continuity. Hence, we may use Fatou's Lemma and the semi-continuity of φ to conclude that

$$\begin{aligned}\liminf_{x \to x_0} (\varphi \circledast \mu)(x) &\geq \int_{\mathbb{S}^{n-1}} \liminf_{x \to x_0} \varphi(\|x\| \vartheta_x u) \, d\mu(u) \\ &\geq \int_{\mathbb{S}^{n-1}} \varphi(\|x_0\| \vartheta_{x_0} u) \, d\mu(u) = (\varphi \circledast \mu)(x_0)\end{aligned}$$

which combined with our definition of $(\varphi \circledast \mu)(o)$ yields the lower semi-continuity of $\varphi \circledast \mu$ on $\mathbb{R}^n$ and completes the proof that $\varphi \circledast \mu \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$.

Since the linearity and commutativity with respect to the action of $\mathrm{SO}(n)$ of the map $\varphi \mapsto \varphi \circledast \mu$ on $\mathrm{Cvx}_{(o)}(\mathbb{R}^n)$ are immediate consequences of the definition of $\circledast$, it only remains to show that this map is continuous with respect to the topology induced by epi-convergence. Therefore, let $\varphi_k \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$ be an epi-convergent sequence with limit $\varphi \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$. In order to prove that

$$\varphi_k \circledast \mu \stackrel{\text{epi}}{\to} \varphi \circledast \mu \tag{4.7}$$

we proceed in three steps. First we claim that the Moreau envelope $e_t\varphi$ of φ satisfies

$$e_t\varphi \circledast \mu \stackrel{\text{epi}}{\to} \varphi \circledast \mu \tag{4.8}$$

as $t \searrow 0$. Indeed, by (4.6), Lemma 2.4.2 (ii) and the monotone convergence theorem,

$$\lim_{t \searrow 0} (e_t\varphi \circledast \mu)(x) = \lim_{t \searrow 0} \int_{\mathbb{S}^{n-1}} e_t\varphi(\|x\| \vartheta_x v) \, d\mu(v) = \int_{\mathbb{S}^{n-1}} \varphi(\|x\| \vartheta_x v) \, d\mu(v) = (\varphi \circledast \mu)(x)$$

for every $x \in \mathbb{R}^n \backslash \{o\}$, which, by Lemma 2.4.1 (i), implies (4.8). In a second step, letting $k \to \infty$, we claim that for every $t > 0$,

$$e_t\varphi_k \circledast \mu \xrightarrow{\text{epi}} e_t\varphi \circledast \mu. \tag{4.9}$$

To see this, we first note that $e_t\varphi_k$ is epi-convergent to $e_t\varphi$ as $k \to \infty$, by Proposition 2.4.5 (iii) and the fact that the Legendre transform maps infimal convolution to pointwise addition. Moreover, by Lemma 2.4.2 (i), $e_t\varphi_k$ and $e_t\varphi$ are both finite and, hence, by Lemma 2.4.1 (ii), their epi-convergence is equivalent to uniform convergence on compact subsets of $\mathbb{R}^n$, which in turn is preserved under the convolution $\circledast$ with the measure μ.
Finally, we show that, letting $k \to \infty$,

$$(\varphi_k \circledast \mu)(x) \to (\varphi \circledast \mu)(x) \text{ for every } \begin{cases} x \in \operatorname{int}\operatorname{dom}(\varphi \circledast \mu)\backslash\{o\}, \\ x \notin \operatorname{cl}\operatorname{dom}(\varphi \circledast \mu), \end{cases} \tag{4.10}$$

which, by Lemma 2.4.1 (i) and the fact that $\mathbb{R}^n$ without the boundary of $\operatorname{dom}(\varphi \circledast \mu)$ and the origin is a dense subset, concludes the proof of (4.7) and the proposition.

To this end, first suppose that $x \notin \operatorname{cl}\operatorname{dom}(\varphi \circledast \mu)$. Then there exists a closed ball B such that $x \in B$ and $B \cap \operatorname{cl}\operatorname{dom}(\varphi \circledast \mu) = \emptyset$. By (4.8) and Lemma 2.4.1 (ii), $e_t\varphi \circledast \mu$ converges to $\varphi \circledast \mu$ uniformly on B as $t \searrow 0$. Thus, for every $c > 0$, we can find $t_0 > 0$ such that for every $t \leq t_0$, we have $e_t\varphi \circledast \mu \geq c$ on B. Similarly, by (4.9) and Lemma 2.4.1 (ii), we can find $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $e_t\varphi_k \circledast \mu > \frac{c}{2}$ on B. Since $e_t\varphi_k \leq \varphi_k$, by Lemma 2.4.2 (ii), and the convolution $\circledast$ is obviously monotone, $\varphi_k \circledast \mu > \frac{c}{2}$ on B for all $k \geq k_0$. Since $c > 0$ was arbitrary, we conclude that $\varphi_k \circledast \mu$ converges to infinity uniformly on B, which proves (4.10) for $x \notin \operatorname{cl}\operatorname{dom}(\varphi \circledast \mu)$.

Suppose now that $x \in \operatorname{int}\operatorname{dom}(\varphi \circledast \mu)$ is non-zero and fix a rotation $\vartheta_x \in \mathrm{SO}(n)$ such that $\vartheta_x\bar{e} = \frac{x}{\|x\|}$. Then there exists an open ε-ball $B_\varepsilon(x) \subseteq \operatorname{int}\operatorname{dom}(\varphi \circledast \mu)$ centered at x . By (4.6), $(\varphi_k \circledast \mu)(x)$ and $(\varphi \circledast \mu)(x)$ are determined by integrating the values of φ_k and φ, respectively, over $\|x\|\mathbb{S}^{n-1}$. Therefore, we consider the compact set

$$C = (\|x\|\mathbb{S}^{n-1}) \cap (\mathbb{R}^n \backslash \operatorname{int}\operatorname{dom}\varphi).$$

If C is empty, then, by Lemma 2.4.1 (ii), φ_k converges uniformly to φ on $\|x\|\mathbb{S}^{n-1}$ and, consequently, by (4.6), we conclude that $(\varphi_k \circledast \mu)(x) \to (\varphi \circledast \mu)(x)$.

Thus, assume that there exists some $y \in C$. Since $o \in \operatorname{int}\operatorname{dom}\varphi$, there exists an open δ-ball $B_\delta(o) \subseteq \operatorname{dom}\varphi$ centered at o. Since $\operatorname{dom}\varphi$ is convex, each ray through $y \in C$ emanating from a point in $B_\delta(o)$ intersects the boundary of $\operatorname{dom}\varphi$ in exactly one point. In particular, the parts of these rays starting at y are completely contained in $\mathbb{R}^n \backslash \operatorname{dom}\varphi$. Hence, for every $y \in C$, there exists an open cone C_y with apex y contained in $\operatorname{int}(\mathbb{R}^n \backslash \operatorname{dom}\varphi)$ and intersecting $R\,\mathbb{S}^{n-1}$, for any $R > \|x\|$, in an open cap whose diameter depends only on R and δ.

Choosing $R = \|x\| + \frac{\varepsilon}{2}$, we have $x_0 = \frac{R}{\|x\|}x \in B_\varepsilon(x)$, that is, $\|x_0\| = R$ and $(\varphi \circledast \mu)(x_0) < \infty$ which imply, by (4.6), that

$$\mu(\{u \in \mathbb{S}^{n-1} : \varphi(R\vartheta_x u) = \infty\}) = 0.$$

Consequently, since φ is infinite on $C_y \cap R\,\mathbb{S}^{n-1}$, we have

$$\mu(\{u \in \mathbb{S}^{n-1} : R\vartheta_x u \in C_y\}) = 0.$$

Using that $R\vartheta_x u \in C_y$ if and only if $\|x\|\vartheta_x u \in \frac{\|x\|}{R}(C_y \cap R\,\mathbb{S}^{n-1}) \subseteq \|x\|\mathbb{S}^{n-1}$, we infer that for every $y \in C$, there exists an open subset U_y of $\mathbb{S}^{n-1}$ of μ-measure zero such that $\|x\|\vartheta_x U_y$ is an open neighborhood of y. The family $(\|x\|\vartheta_x U_y)_{y\in C}$ is an open cover of the compact set C, hence, there exists a finite subcover $(\|x\|\vartheta_x U_{y_i})_{i=1}^m$ and, by the sub-additivity of μ, we have $\mu\,(U) = 0$ for $U = \bigcup_{i=1}^m U_{y_i}$.

Since the compact set $C' = \|x\|\mathbb{S}^{n-1}\backslash\|x\|\vartheta_x U$ is disjoint from $\operatorname{bd}\operatorname{dom}\varphi$, φ_k converges uniformly to φ on C', by Lemma 2.4.1 (ii). For every $\widehat{\varepsilon} > 0$, we thus find k_0 such that for all $k \geq k_0$ we have $|\varphi_k(z) - \varphi(z)| \leq \widehat{\varepsilon}$ for all $z \in C'$. Consequently,

$$\begin{aligned}(\varphi_k \circledast \mu)(x) &= \int_{\mathbb{S}^{n-1}} \varphi_k(\|x\|\vartheta_x v)\, d\mu(v) = \int_{\mathbb{S}^{n-1}\backslash U} \varphi_k(\|x\|\vartheta_x v)\, d\mu(v) \\ &\leq \int_{\mathbb{S}^{n-1}\backslash U} (\varphi(\|x\|\vartheta_x v) + \widehat{\varepsilon})\, d\mu(u) = (\varphi \circledast \mu)(x) + \widehat{\varepsilon}\,\mu(\mathbb{S}^{n-1}),\end{aligned}$$

that is, $(\varphi_k \circledast \mu)(x)$ is finite for all $k \geq k_0$. Hence, we can infer that

$$|(\varphi_k \circledast \mu)(x) - (\varphi \circledast \mu)(x)| \leq \int_{\mathbb{S}^{n-1}\backslash U} |\varphi_k(\|x\|\vartheta_x v) - \varphi(\|x\|\vartheta_x v)|\, d\mu(v) \leq \widehat{\varepsilon}\,\mu(\mathbb{S}^{n-1}),$$

which completes the proof of (4.10). □

4.2 Proof of the main results

In this section, we first complete the proof of Theorem 4.0.1. In order to prove Theorem 4.0.2, we will then establish a counterpart of Theorem 3.0.2 for log-concave functions and continue as in the geometric setting. We conclude this section by showing that each of the inequalities from Theorem 4.0.2 is strictly stronger than the functional analogue of Urysohn's inequality.

Proof of Theorem 4.0.1. It follows from (4.5) and Proposition 4.1.2 that $\Psi_\mu f$ is well defined for every $f \in \mathrm{LC_c}(\mathbb{R}^n)$. By (2.38) and (4.6), $\Psi_\mu : \mathrm{LC_c}(\mathbb{R}^n) \to \mathrm{LC_c}(\mathbb{R}^n)$ is Asplund additive. By Lemma 2.4.3 (iii), $f \star \mathbb{1}_{\{y\}}$ coincides with the translate of $f \in \mathrm{LC_c}(\mathbb{R}^n)$ by $y \in \mathbb{R}^n$. Hence, using $\Psi_\mu \mathbb{1}_{\{y\}} = \mathbb{1}_{\Phi_\mu\{y\}} = \mathbb{1}_{\{o\}}$, by Lemma 3.1.4 (iii), where Φ_μ denotes the Minkowski endomorphism generated by μ, we deduce that

$$\Psi_\mu(f \star \mathbb{1}_{\{y\}}) = \Psi_\mu f \star \mathbb{1}_{\Phi_\mu\{y\}} = \Psi_\mu f \star \mathbb{1}_{\{o\}} = \Psi_\mu f.$$

The commutativity of Ψ_μ with the action of $\mathrm{SO}(n)$ follows from the definitions of $\circledast$ and the support function $h(f,\cdot)$, and the fact that the Legendre transform commutes with the action of $\mathrm{SO}(n)$ on $\mathrm{Cvx}(\mathbb{R}^n)$. Continuity of $\Psi_\mu : \mathrm{LC_c}(\mathbb{R}^n) \to \mathrm{LC_c}(\mathbb{R}^n)$ is a consequence of Proposition 2.4.5 (iii) and Proposition 4.1.2. The monotonicity of Ψ_μ follows from the monotonicity of support functions and that of $\circledast$.

By Proposition 2.4.5 (iv) and the remark following (4.6), $\Psi_\mu \mathbb{1}_K = \mathbb{1}_{\Phi_\mu K}$ for every $K \in \mathcal{K}^n$. Thus, μ is uniquely determined by Ψ_μ by Theorem 3.1.2. □

In view of Theorem 3.1.2, it is a natural question to ask whether Theorem 4.0.1 also holds for measures μ which are non-negative up to addition of a linear measure. In general, this is not possible as the integral of a (convex) function attaining the value $+\infty$ with respect to a signed measure is not well-defined. Restricting to the subspace of finite convex functions would overcome this issue, but it remains to prove that convoluting with μ preserves convexity.

The classification of additive (in a set-theoretic sense) maps on convex and log-concave functions has recently become the focus of intensive investigations (see, e.g., [12, 36, 40–43, 73, 74]). It is certainly an interesting open problem whether there exist monotone Asplund endomorphisms different from the ones provided by Theorem 4.0.1 and, if so, what additional properties characterize the endomorphisms from Theorem 4.0.1.

Using Theorem 4.0.1, we give a functional analogue of Lemma 3.1.4 (i).

Lemma 4.2.1. *If μ is a non-negative zonal measure on $\mathbb{S}^{n-1}$ with center of mass at the origin, then*

$$\int_{\mathbb{R}^n} h(\Psi_\mu f, x)\, d\gamma_n(x) = \mu(\mathbb{S}^{n-1}) \int_{\mathbb{R}^n} h(f, x)\, d\gamma_n(x) \tag{4.11}$$

for every $f \in \mathrm{LC_c}(\mathbb{R}^n)$.

Proof. Using polar coordinates and the density ψ_n of the Gaussian measure, yields

$$\int_{\mathbb{R}^n} h(\Psi_\mu f, x)\, d\gamma_n(x) = n\kappa_n \int_0^\infty \int_{\mathbb{S}^{n-1}} (h(f,\cdot) \circledast \mu)(ru)\, \psi_n(ru)\, r^{n-1}\, d\sigma(u)\, dr.$$

By the $\mathrm{SO}(n)$ invariance of ψ_n, (4.11) follows, if we can show that

$$\int_{\mathbb{S}^{n-1}} (h \circledast \mu)(ru)\, d\sigma(u) = \mu(\mathbb{S}^{n-1}) \int_{\mathbb{S}^{n-1}} h(ru)\, d\sigma(u) \tag{4.12}$$

for every $h \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$ and every $r > 0$. To this end, first assume that $\operatorname{dom} h = \mathbb{R}^n$ and, hence, that h is continuous. Since $(h \circledast \mu)(ru) = (h(r\,\cdot\,) * \mu)(u)$ for every $u \in \mathbb{S}^{n-1}$, we obtain, as in the proof of Lemma 3.1.4 (i), from the fact that spherical convolution is selfadjoint and Abelian for zonal measures, that

$$\int_{\mathbb{S}^{n-1}} (h \circledast \mu)(ru)\, d\sigma(u) = \int_{\mathbb{S}^{n-1}} (h(r\,\cdot\,) * \sigma)(u)\, d\mu(u).$$

By the $\mathrm{SO}(n)$ invariance of σ, $(h(r\,\cdot\,) * \sigma)(u)$ is independent of u and (4.12) follows.

For general $h \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$, we use that the Moreau envelope $e_t h$ of h is convex and finite and converges monotonously to h, by Lemma 2.4.2. By what we have shown above, (4.12) holds for $e_t h$ for every $t > 0$. Hence, by monotone convergence, we conclude that (4.12) holds generally. ∎

The next theorem establishes a counterpart of Theorem 3.0.2 used in the proof of Theorem 4.0.2.

Theorem 4.2.2. *Let μ be an $\mathrm{SO}(n-1)$ invariant probability measure on $\mathbb{S}^{n-1}$ with center of mass at the origin. If $f \in \mathrm{LC_c}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} f\,dx > 0$, then*

$$\int_{\mathbb{R}^n} (\Psi_\sigma f)^\circ(x)\,dx \le \int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\,dx \le \int_{\mathbb{R}^n} f^\circ(x)\,dx. \tag{4.13}$$

There is equality in the left hand inequality if and only if $\Psi_\mu f$ is radially symmetric. Equality in the right hand inequality holds if and only if f is even and $\Psi_\mu = \Delta_\star$ or if f is radially symmetric.

Proof. First note that for $f \in \mathrm{LC_c}(\mathbb{R}^n)$, we always have $\int_{\mathbb{R}^n} f\,dx < \infty$, by (2.33). In order to establish the left hand inequality of (4.13), we use (2.40), polar coordinates and Jensen's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\,dx &= n\kappa_n \int_0^\infty \int_{\mathbb{S}^{n-1}} \exp(-(h(f,\cdot) \circledast \mu)(ru))\, r^{n-1}\,d\sigma(u)\,dr \\ &\ge n\kappa_n \int_0^\infty \exp\left(-\int_{\mathbb{S}^{n-1}} (h(f,\cdot)\circledast\mu)(ru)\,d\sigma(u)\right) r^{n-1}\,dr. \end{aligned} \tag{4.14}$$

To be precise, for the application of Jensen's inequality, we require the function $(h(f,\cdot)\circledast\mu)(r\,\cdot\,)$ to be σ-integrable. However, if this is not the case, then its integral is $+\infty$ and inequality (4.14) still holds.

From an application of (4.12) to the inner integral in (4.14) and the $\mathrm{SO}(n)$ invariance of σ, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\,dx &\ge n\kappa_n \int_0^\infty \exp\left(-\int_{\mathbb{S}^{n-1}} h(f,ru)\,d\sigma(u)\right) r^{n-1}\,dr \\ &= n\kappa_n \int_0^\infty \exp(-(h(f,\cdot)\circledast\sigma)(rv))\, r^{n-1}\,dr\end{aligned}$$

for an arbitrary $v \in \mathbb{S}^{n-1}$. Finally, using that $(h(f,\cdot)\circledast\sigma)(rv)$ does not depend on v, we arrive at the left hand inequality of (4.13),

$$\begin{aligned}\int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\,dx &\ge n\kappa_n \int_0^\infty \int_{\mathbb{S}^{n-1}} \exp(-(h(f,\cdot)\circledast\sigma)(rv))\, r^{n-1}\,d\sigma(v)\,dr \\ &= \int_{\mathbb{R}^n} (\Psi_\sigma f)^\circ(x)\,dx.\end{aligned}$$

If equality holds in the left hand inequality of (4.13), then we must have equality in (4.14) which implies by the equality condition of Jensen's inequality (including the case of non-σ-integrability) that for almost every $r > 0$ there exists $c_r \in (-\infty, \infty]$ such that

$$h(\Psi_\mu f, rv) = (h(f,\cdot)\circledast\mu)(rv) = c_r \quad \text{for } \sigma\text{-a.e. } v \in \mathbb{S}^{n-1}. \tag{4.15}$$

Note that, by continuity, (4.15) yields that $h(\Psi_\mu f, \cdot)$ is constant on every sphere contained in $\operatorname{int}\operatorname{dom} h(\Psi_\mu f, \cdot)$. Next, we want to show that this domain is a ball.

If $c_r < \infty$ for some $r > 0$, then the lower semi-continuity of $h(\Psi_\mu f, \cdot)$ implies that $h(\Psi_\mu f, rv)$ is finite for *every* $v \in \mathbb{S}^{n-1}$. In particular, $r\mathbb{S}^{n-1} \subseteq \operatorname{dom} h(\Psi_\mu f, \cdot)$ which by the convexity of this domain yields $rB^n \subseteq \operatorname{dom} h(\Psi_\mu f, \cdot)$. This implies that (4.15) holds for every $r' \leq r$ and $c_{r'} < \infty$ by continuity. Thus, the set of all $r > 0$ such that (4.15) holds with $c_r < \infty$ is an interval, that is, there exists $R > 0$ such that

$$c_r \begin{cases} < \infty & \text{for all } r < R, \\ = \infty & \text{for all } r > R \text{ for which (4.15) holds.} \end{cases}$$

In order to conclude that $\operatorname{int}\operatorname{dom} h(\Psi_\mu f, \cdot)$ is a ball, it remains to show that for every $r > R$, $h(\Psi_\mu f, \cdot)$ is infinite on $r\mathbb{S}^{n-1}$. To this end, let $x \in r\mathbb{S}^{n-1}$ and assume that $h(\Psi_\mu f, x) < \infty$. Since $\operatorname{dom} h(\Psi_\mu f, \cdot)$ is convex and contains an open ball centered at the origin, the convex hull of x and this ball is contained in $\operatorname{dom} h(\Psi_\mu f, \cdot)$. However, this convex hull must contain an open neighborhood of $r'\,\mathbb{S}^{n-1}$ for some $r' > R$ for which (4.15) holds, which contradicts $c_{r'} = \infty$.

Finally, since $\operatorname{int}\operatorname{dom} h(\Psi_\mu f, \cdot)$ is a ball and, by the comment following (4.15), $h(\Psi_\mu f, \cdot)$ is radially symmetric on this ball, we conclude that $h(\Psi_\mu f, \cdot)$ is radially symmetric on all of $\mathbb{R}^n$, since a convex function depending only on one variable is uniquely determined by its values on the interior of its domain. This concludes the proof of the equality conditions for the left hand inequality of (4.13).

For the proof of the right hand inequality of (4.13), we use (2.40), (4.6), and Jensen's inequality to obtain

$$\int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\, dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \exp\left(-h(f, \|x\|\vartheta_x v)\right)\, d\mu(v)\, dx. \tag{4.16}$$

As in the first part of this proof, for the application of Jensen's inequality, we require $h(f, \|x\|\vartheta_x \cdot)$ to be μ-integrable. However, if this is not the case, then the left hand side of (4.16) is zero and inequality (4.16) still holds.

Since Ψ_μ and the polar map commute with $\mathrm{SO}(n)$ transforms, we may replace f in (4.16) by a rotated copy θf and integrate over $\mathrm{SO}(n)$ with respect to the Haar measure, to arrive at

$$\int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\, dx \leq \int_{\mathrm{SO}(n)} \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \exp\left(-h(f, \|x\|\theta^{-1}\vartheta_x v)\right)\, d\mu(v)\, dx\, d\theta. \tag{4.17}$$

Since we integrate non-negative functions, we may apply Fubini's theorem twice, the invariance of the Haar measure on $\mathrm{SO}(n)$, and the fact that $\mu(\mathbb{S}^{n-1}) = 1$, to obtain the desired inequality

$$\begin{aligned}\int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\,dx &\le \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}}\int_{\mathrm{SO}(n)} \exp\left(-h(f,\|x\|\theta^{-1}\vartheta_x v)\right) d\theta\, d\mu(v)\, dx\\ &= \int_{\mathbb{R}^n}\int_{\mathrm{SO}(n)} \exp\left(-h(f,\theta^{-1}x)\right)\, d\theta\, dx\\ &= \int_{\mathrm{SO}(n)}\int_{\mathbb{R}^n} f^\circ(\theta^{-1}x)\,dx\,d\theta = \int_{\mathbb{R}^n} f^\circ(x)\,dx.\end{aligned}$$

By the above arguments, equality holds in the right hand inequality of (4.13) if and only if we have equality in (4.17) which implies by the equality condition of Jensen's inequality (including the case of non-μ-integrability) that for almost every $\theta \in \mathrm{SO}(n)$ and almost every $x \in \mathbb{R}^n$ there exist constants $c_{\theta,x} \in (-\infty,\infty]$ such that

$$h(f,\|x\|\theta^{-1}\vartheta_x v) = c_{\theta,x} \quad \text{for } \mu\text{-a.e. } v \in \mathbb{S}^{n-1}. \tag{4.18}$$

As in the proof of Theorem 3.0.2, if μ is discrete it must coincide with the measure ν given by (3.8). Thus, $\Psi_\mu = \Delta_\star$ and, since $\operatorname{supp}\nu = \{-\bar{e},\bar{e}\}$, (4.18) reduces to the existence of constants $c_{\theta,x} \in (-\infty,\infty]$ such that for almost every $\theta \in \mathrm{SO}(n)$ and almost every $x \in \mathbb{R}^n$,

$$h(f,\theta^{-1}x) = c_{\theta,x} = h(f,-\theta^{-1}x).$$

Consequently, the interior of the domain of $h(f,\cdot)$ must be origin-symmetric and $h(f,\cdot)$ must be even on it (by continuity, $h(f,\cdot)$ must attain the same value on *all* antipodal points in $\operatorname{int}\operatorname{dom} h(f,\cdot)$). By now considering the restriction of $h(f,\cdot)$ to lines through the origin and using the extendibility of convex, lower semi-continuous functions of one variable, we conclude that $h(f,\cdot)$ must be even on all of $\mathbb{R}^n$.

If μ is *not discrete*, we first want to show that (4.18) implies that $\operatorname{int}\operatorname{dom} h(f,\cdot)$ is an open ball centered at the origin. To this end, note that it follows from (4.18) that for almost every $r > 0$ and almost every $\eta \in \mathrm{SO}(n)$ there exist constants $c_{r,\eta} \in (-\infty,\infty]$ such that

$$h(f,r\eta v) = c_{r,\eta} \quad \text{for } \mu\text{-a.e. } v \in \mathbb{S}^{n-1}. \tag{4.19}$$

If $c_{r,\eta} < \infty$ for some $r > 0$ and $\eta \in \mathrm{SO}(n)$, then the lower semi-continuity of $h(f,\cdot)$ implies that $h(f,r\eta v) \le c_{r,\eta} < \infty$ for all $v \in \operatorname{supp}\mu$. If on the other hand $c_{r,\eta} = \infty$, then the lower semi-continuity of $h(f,\cdot)$ implies that $v \mapsto h(f,r\eta v)$ cannot be bounded on any open subset of $\mathbb{S}^{n-1}$ intersecting $\operatorname{supp}\mu$.

For suitable $\delta > 0$, let B_δ denote an open origin-symmetric δ-ball in $\operatorname{dom} h(f,\cdot)$ such that its closure is still contained in $\operatorname{int}\operatorname{dom} h(f,\cdot)$. Next, choose an arbitrary $x \in \operatorname{int}\operatorname{dom} h(f,\cdot)\backslash B_\delta$ and let $r \in [\delta,\|x\|)$. Then the set $\operatorname{conv}\{x,\operatorname{cl} B_\delta\}$ is contained in $\operatorname{int}\operatorname{dom} h(f,\cdot)$, and, thus, $h(f,\cdot)$ is bounded on it. Define the open spherical caps C_x^r as $\operatorname{conv}\{x,B_\delta\}\cap r\mathbb{S}^{n-1}$.

In the following, let d denote the geodesic distance on $\mathbb{S}^{n-1}$ and let d_r denote the geodesic distance on $r\mathbb{S}^{n-1}$ normalized such that $d_r(u,v) = d(\frac{u}{r},\frac{v}{r})$ for any

$u, v \in r\mathbb{S}^{n-1}$. Since μ is not discrete and has center of mass at the origin, there exists $t_0 \in [0, 1)$ such that $H_{\bar{e},t_0} \cap \mathbb{S}^{n-1} \subseteq \operatorname{supp} \mu$, where $H_{\bar{e},t_0} = \{y \in \mathbb{R}^n : \langle \bar{e}, y\rangle = t_0\}$. Let α be the maximal geodesic distance of two points in $H_{\bar{e},t_0} \cap \mathbb{S}^{n-1}$.

Choose now r_0 from the dense subset of all $r \in [\delta, \|x\|)$ such that (4.19) holds for almost all $\eta \in \mathrm{SO}(n)$. Then for $x_0 \in r_0\mathbb{S}^{n-1}$ with $d_{r_0}(x_0, C_x^{r_0}) < \frac{\alpha}{2}$ and $\varepsilon > 0$, we consider the set

$$A_{x_0,\varepsilon} = \{\eta \in \mathrm{SO}(n) : r_0\eta v_1 \in C_x^{r_0}, d_{r_0}(r_0\eta v_2, x_0) < \varepsilon \text{ for some } v_1, v_2 \in H_{\bar{e},t_0} \cap \mathbb{S}^{n-1}\}.$$

Clearly, $A_{x_0,\varepsilon}$ is open and non-empty, hence, there exists $\eta_0 \in A_{x_0,\varepsilon}$ such that (4.19) holds for η_0 and r_0. Since $C_x^{r_0}$ is open and $h(f, \cdot)$ is bounded on $C_x^{r_0}$, we have $c_{r_0,\eta_0} < \infty$ (as we have seen above). In particular, $h(f, r_0\eta_0 v_2) < \infty$, that is, $r_0\eta_0 v_2 \in \operatorname{dom} h(f, \cdot)$ for some $v_2 \in H_{\bar{e},t_0} \cap \mathbb{S}^{n-1}$ such that $d_{r_0}(r_0\eta_0 v_2, x_0) < \varepsilon$ and there exists $v_1 \in H_{\bar{e},t_0} \cap \mathbb{S}^{n-1}$ with $r_0\eta_0 v_1 \in C_x^{r_0}$.

Since $x_0 \in r_0\mathbb{S}^{n-1}$ such that $d_{r_0}(x_0, C_x^{r_0}) < \frac{\alpha}{2}$ and $\varepsilon > 0$ were arbitrary, $h(f, \cdot)$ is finite on a dense subset of $U_{\alpha/2}(C_x^{r_0})$, the set of all points on $r_0\mathbb{S}^{n-1}$ whose distance d_{r_0} to $C_x^{r_0}$ is less than $\frac{\alpha}{2}$. Taking $r' < r_0$, this implies that $U_{\alpha/2}(C_x^{r'}) \subseteq \operatorname{dom} h(f, \cdot)$. Indeed, for every $y \in U_{\alpha/2}(C_x^{r'})$ we may find points $x_1, \ldots, x_n$ in $U_{\alpha/2}(C_x^{r_0}) \cap \operatorname{dom} h(f, \cdot)$ such that $y \in \operatorname{conv}\{0, x_1, \ldots, x_n\} \subseteq \operatorname{dom} h(f, \cdot)$. Thus, we have shown that for every $x \in \operatorname{int} \operatorname{dom} h(f, \cdot)$ and *every* $r \in [\delta, \|x\|)$, the set $U_{\alpha/2}(C_x^r)$ is contained in $\operatorname{dom} h(f, \cdot)$. In particular, $U_{\alpha/2}\left(\frac{r}{\|x\|}x\right) \subseteq \operatorname{dom} h(f, \cdot)$.

Finally, if $\operatorname{int} \operatorname{dom} h(f, \cdot)$ is not a centered open ball, then there exists a sequence $x_k \in \operatorname{int} \operatorname{dom} h(f, \cdot)$ converging to $x \in \operatorname{bd} \operatorname{dom} h(f, \cdot)$ with $\|x_k\| > \|x\| > 0$ for all k (take, e.g., for x any boundary point that is touched non-radially by a closed ball in $\operatorname{int} \operatorname{dom} h(f, \cdot)$).

Since $x_k \to x$, $\frac{\|x\|}{\|x_k\|}x_k$ converges to x as well. Hence, there exists $k_0 \in \mathbb{N}$ such that $d_{\|x\|}\left(\frac{\|x\|}{\|x_{k_0}\|}x_{k_0}, x\right) < \frac{\alpha}{4}$, that is,

$$x \in U_{\alpha/4}\left(\frac{\|x\|}{\|x_{k_0}\|}x_{k_0}\right) \subseteq \bigcup_{r \in (\delta, \|x_{k_0}\|)} U_{\alpha/2}\left(\frac{r}{\|x_{k_0}\|}x_{k_0}\right) \subseteq \operatorname{int} \operatorname{dom} h(f, \cdot)$$

which is a contradiction.

Knowing that $\operatorname{int} \operatorname{dom} h(f, \cdot)$ is a centered open ball, from (4.19) combined with the continuity of $h(f, \cdot)$ on the interior of its domain, it follows, as in the final paragraph of the proof of Theorem 3.0.2, that $h(f, \cdot)$ is radially symmetric on $\operatorname{int} \operatorname{dom} h(f, \cdot)$. Noting again that a convex function depending only on one variable is uniquely determined by its values on the interior of its domain, we infer that $h(f, \cdot)$ is radially symmetric on all of $\mathbb{R}^n$. Since the Legendre transform commutes with the action of $\mathrm{SO}(n)$, f must be radially symmetric itself. As $\Psi_\mu f = f$ for any radially symmetric $f \in \mathrm{LC_c}(\mathbb{R}^n)$, by (4.6), this concludes the proof of the theorem. □

The same way Theorem 3.0.1 was a simple consequence of Theorem 3.0.2 and (3.1), we can now deduce Theorem 4.0.2 easily from Theorems 4.2.2 and 2.4.7.

Proof of Theorem 4.0.2. By the translation-invariance of Ψ_μ, we have $\Psi_\mu f = \Psi_\mu \tilde{f}$, where, as in Theorem 2.4.7, $\tilde{f}(x) = f(x - \operatorname{cent} f)$. Thus, by the right hand inequality of (4.13) and Theorem 2.4.7, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} (\Psi_\mu f)^\circ(x)\,dx &= \int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} (\Psi_\mu \tilde{f})^\circ(x)\,dx \\ &\leq \int_{\mathbb{R}^n} f(x)\,dx \int_{\mathbb{R}^n} \tilde{f}^\circ(x)\,dx \leq (2\pi)^n.\end{aligned}$$

Equality holds in (4.3) if and only if equality holds both in the right hand inequality of (4.13) and in Theorem 2.4.7, that is, if and only if $\Psi_\mu = \Delta_\star$ and f is a Gaussian or if f is proportional to a translation of the standard Gaussian. □

Let us remark at this point again that Theorem 3.0.1 can be recovered from Theorem 4.0.2 in an asymptotically optimal form. More precisely, choosing $f = \mathbb{1}_K$ for $K \in \mathcal{K}^n$ with non-empty interior in Theorem 4.0.2, inequality (4.3) becomes,

$$(2\pi)^n \geq V_n(K) \int_{\mathbb{R}^n} \mathbb{1}^\circ_{\Phi_\mu K}(x)\,dx = V_n(K) \int_{\mathbb{R}^n} \exp\bigl(-\|x\|_{\Phi_\mu^\circ K}\bigr)\,dx = n! V_n(K) V_n(\Phi_\mu^\circ K),$$

where we have used that $\Psi_\mu \mathbb{1}_K = \mathbb{1}_{\Phi_\mu K}$, the definition of the polar map, and (2.37) with $p = 1$. Since the assumption that μ is a probability measure is equivalent to the normalization $\Phi_\mu B^n = B^n$, we obtain

$$V_n(\Phi_\mu^\circ K) V_n(K) \leq \frac{(2\pi)^n}{n!} = c_n^n \kappa_n^2,$$

where $c_n > 1$ and $\lim_{n\to\infty} c_n = 1$.

The reason we do not recover the sharp form of Theorem 3.0.1 is the same reason why Urysohn's inequality (2.27) is not a special case of its functional analogue (4.2), namely, that extremals in the functional inequalities are Gaussians in both cases while the relevant geometric quantities are recovered for indicators of convex bodies. However, let us emphasize that the weakest inequality of Theorem 4.0.2, obtained for $\mu = \sigma$, yields a new functional analogue of Urysohn's inequality from which (2.27) can be deduced in an asymptotically optimal way, in contrast to (4.2). We will make this even more precise with our final result, which shows that all inequalities of Theorem 4.0.2 are strictly stronger than (4.2). The proof uses ideas from [100] and relies on a basic inequality from information theory by Shannon (see [87, Theorem B.1]) which states that if $g, h : \mathbb{R}^n \to \mathbb{R}$ are non-negative measurable functions such that $g > 0$ and $\int_{\mathbb{R}^n} g\,dx = 1$, then

$$\int_{\mathbb{R}^n} g \log \frac{1}{h}\,dx \geq \int_{\mathbb{R}^n} g \log \frac{1}{g}\,dx - \log\left(\int_{\mathbb{R}^n} h\,dx\right) \tag{4.20}$$

with equality if and only if $h = \alpha g$ for some $\alpha \geq 0$ almost everywhere.

Theorem 4.2.3. *Let μ be an $\mathrm{SO}(n-1)$ invariant probability measure on $\mathbb{S}^{n-1}$ with center of mass at the origin. If $f \in \mathrm{LC_c}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} f\,dx > 0$, then*

$$\frac{2}{n}\int_{\mathbb{R}^n} h(f,x)\,d\gamma_n(x) \geq 1 - \frac{2}{n}\log\left(\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}(\Psi_\mu f)^\circ(x)\,dx\right) \tag{4.21}$$

with equality if and only if $\Phi_\mu f$ is a multiple of the standard Gaussian.

Proof. First note that, by Lemma 4.2.1,

$$\frac{2}{n}\int_{\mathbb{R}^n} h(f,x)\,d\gamma_n(x) = \frac{2}{n}\int_{\mathbb{R}^n} h(\Psi_\mu f,x)\,d\gamma_n(x) = \frac{2}{n}\int_{\mathbb{R}^n}\log\left(\frac{1}{e^{-h(\Psi_\mu f,x)}}\right)\psi_n(x)\,dx.$$

Choosing $g = \psi_n$ and $h = e^{-h(\Psi_\mu f,\cdot)}$ in Shannon's inequality (4.20), we thus obtain

$$\frac{2}{n}\int_{\mathbb{R}^n} h(f,x)\,d\gamma_n(x) \geq \frac{2}{n}\int_{\mathbb{R}^n}\psi_n(x)\log\left(\frac{1}{\psi_n(x)}\right)dx - \frac{2}{n}\log\left(\int_{\mathbb{R}^n} e^{-h(\Psi_\mu f,x)}\,dx\right). \tag{4.22}$$

The first integral on the right hand side is the entropy of the standard normal distribution which is well-known to be $\frac{n}{2}(1+\log(2\pi))$. Consequently, we obtain

$$\frac{2}{n}\int_{\mathbb{R}^n} h(f,x)\,d\gamma_n(x) \geq 1 + \log(2\pi) - \frac{2}{n}\log\left(\int_{\mathbb{R}^n}(\Psi_\mu f)^\circ(x)\,dx\right)$$

which is clearly equivalent to (4.21). Equality holds in (4.21) if and only if we have equality in (4.22), that is, by the equality conditions of Shannon's inequality, if and only if $e^{-h(\Psi_\mu f,\cdot)} = \alpha\psi_n$, for some $\alpha > 0$ or, equivalently, if and only if

$$\mathcal{L}(-\log\Psi_\mu f)(x) = h(\Psi_\mu f,x) = \frac{\|x\|^2}{2} + \beta,$$

for some $\beta \in \mathbb{R}$ and every $x \in \mathbb{R}^n$. This shows, by Proposition 2.4.5 (i) and (iv), that equality holds in (4.21) if and only if $\Psi_\mu f = e^{-\beta}(2\pi)^{n/2}\psi_n$. $\square$

4.3 Appendix

The purpose of this appendix is to complete the proof of Proposition 4.1.2 by showing that for $\varphi \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$, the function $\varphi \circledast \mu$ is convex. The proof is based on arguments used in [67] and [107], where variants of this fact were shown under additional assumptions on the function φ. We begin with an auxiliary result.

Lemma 4.3.1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be convex and $H \subseteq \mathbb{R}^n$ a 2-dimensional linear subspace. For every $z \in \mathbb{R}^n$, $a, b \in \mathbb{R}$, the function $g_{z,a,b} : H \to \mathbb{R}$, defined by*

$$g_{z,a,b}(x) = \varphi(ax + b\vartheta_H x + \|x\|z) + \varphi(ax + b\vartheta_H x - \|x\|z), \tag{4.23}$$

where $\vartheta_H \in \mathrm{SO}(n)$ acts as rotation by the angle $\frac{\pi}{2}$ on H and keeps $H^\perp$ fixed, is convex.

Proof. Since φ and, thus, $g_{z,a,b}$ are continuous, it is sufficient to show that

$$g_{z,a,b}\Big(\frac{x+y}{2}\Big) \le \frac{1}{2}\, g_{z,a,b}(x) + \frac{1}{2}\, g_{z,a,b}(y) \tag{4.24}$$

for all distinct $x, y \in H$. As, by definition, $g_{z,a,b}\big(\frac{x+y}{2}\big)$ equals

$$\varphi\Big(a\,\frac{x+y}{2} + b\vartheta_H\frac{x+y}{2} + \frac{\|x+y\|}{2}z\Big) + \varphi\Big(a\,\frac{x+y}{2} + b\vartheta_H\frac{x+y}{2} - \frac{\|x+y\|}{2}z\Big), \tag{4.25}$$

we may only consider the first term for the following computation and then flip the sign of z. In order to see (4.24), first note that for every $\alpha \in [0,1]$,

$$\begin{aligned} &a\,\frac{x+y}{2} + b\vartheta_H\frac{x+y}{2} + \frac{\|x+y\|}{2}z \\ &= \Big(a\,\frac{x}{2} + b\vartheta_H\frac{x}{2} + \alpha\frac{\|x+y\|}{2}z\Big) + \Big(a\,\frac{y}{2} + b\vartheta_H\frac{y}{2} + (1-\alpha)\frac{\|x+y\|}{2}z\Big). \end{aligned}$$

Again, we may consider only the first term and skip the computation for the second (just replace x by y). Choosing $\alpha = \frac{\|x\|}{\|x\|+\|y\|}$, we have $1-\alpha = \frac{\|y\|}{\|x\|+\|y\|}$ and

$$\begin{aligned} &a\,\frac{x}{2} + b\vartheta_H\frac{x}{2} + \alpha\frac{\|x+y\|}{2}z \\ &= \lambda_1(ax + b\vartheta_H x + \|x\|z) + \lambda_2(ax + b\vartheta_H x - \|x\|z), \end{aligned}$$

where

$$\lambda_1 = \frac{1}{4}\Big(1 + \frac{\|x+y\|}{\|x\|+\|y\|}\Big) \qquad \text{and} \qquad \lambda_2 = \frac{1}{4}\Big(1 - \frac{\|x+y\|}{\|x\|+\|y\|}\Big).$$

Hence, we conclude that

$$\begin{aligned} &a\,\frac{x+y}{2} + b\vartheta_H\frac{x+y}{2} + \frac{\|x+y\|}{2}z \\ &= \lambda_1(ax + b\vartheta_H x + \|x\|z) + \lambda_2(ax + b\vartheta_H x - \|x\|z) \\ &\quad + \lambda_1(ay + b\vartheta_H y + \|y\|z) + \lambda_2(ay + b\vartheta_H y - \|y\|z). \end{aligned}$$

Noting that $\lambda_1, \lambda_2 \in \big[0, \frac{1}{2}\big]$ and that $2\lambda_1 + 2\lambda_2 = 1$, the convexity of φ implies that

$$\begin{aligned} &\varphi\Big(a\,\frac{x+y}{2} + b\vartheta_H\frac{x+y}{2} + \frac{\|x+y\|}{2}z\Big) \\ &\le \lambda_1\varphi(ax + b\vartheta_H x + \|x\|z) + \lambda_2\varphi(ax + b\vartheta_H x - \|x\|z) \\ &\quad + \lambda_1\varphi(ay + b\vartheta_H y + \|y\|z) + \lambda_2\varphi(ay + b\vartheta_H y - \|y\|z). \end{aligned}$$

The analogue computation for the second term of (4.25), finally yields the desired inequality (4.24). □

In order to show that for $\varphi \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$, the function $\varphi \circledast \mu$ is convex, we first assume that φ is convex and finite on $\mathbb{R}^n$. Moreover, we may restrict ourselves to convex combinations along lines that lie completely in $\mathbb{R}^n\backslash\{o\}$, the general case then follows by continuity.

Since a zonal function on $\mathbb{S}^{n-1}$ depends only on the value of $\langle u, \bar{e}\rangle$, there is a natural one-to-one correspondence between zonal functions and measures on $\mathbb{S}^{n-1}$ and functions and measures on $[-1,1]$ (see, e.g., [112]). In particular, there exists a unique non-negative measure $\widehat{\mu}$ on $[-1,1]$ such that for every $f \in C(\mathbb{S}^{n-1})$, we have

$$\int_{\mathbb{S}^{n-1}} f(v)\, d\mu(v) = \int_{-1}^{1} \int_{\mathbb{S}^{n-1}\cap \bar{e}^{\perp}} f\Big(\alpha \bar{e} + \sqrt{1-\alpha^2} w\Big)\, d\sigma_{\bar{e}^{\perp}}(w)\, (1-\alpha^2)^{\frac{n-2}{2}}\, d\widehat{\mu}(\alpha),$$

where $\sigma_{\bar{e}^{\perp}}$ is the invariant probability measure on $\mathbb{S}^{n-1} \cap \bar{e}^{\perp}$. Applying this to definition (4.6), we obtain for $x \in \mathbb{R}^n\backslash\{o\}$,

$$\begin{aligned}(\varphi \circledast \mu)(x) &= \int_{-1}^{1} \int_{\mathbb{S}^{n-1}\cap \bar{e}^{\perp}} \varphi\Big(\alpha x + \sqrt{1-\alpha^2}\|x\| \vartheta_x w\Big)\, d\sigma_{\bar{e}^{\perp}}(w)\, (1-\alpha^2)^{\frac{n-2}{2}}\, d\widehat{\mu}(\alpha)\\ &= \int_{-1}^{1} \int_{\mathbb{S}^{n-1}\cap x^{\perp}} \varphi\Big(\alpha x + \sqrt{1-\alpha^2}\|x\| v\Big)\, d\sigma_{x^{\perp}}(v)\, (1-\alpha^2)^{\frac{n-2}{2}}\, d\widehat{\mu}(\alpha).\end{aligned}$$

Since $\widehat{\mu}$ is non-negative, we are done, if we can prove the convexity of the function

$$\varphi_\alpha(x) = \int_{\mathbb{S}^{n-1}\cap x^{\perp}} \varphi\Big(\alpha x + \sqrt{1-\alpha^2}\|x\| v\Big)\, d\sigma_{x^{\perp}}(v),$$

for all $\alpha \in [-1,1]$. To this end, let $x \in \mathbb{R}^n\backslash\{o\}$ and let $H \subseteq \mathbb{R}^n$ be an arbitrary 2-dimensional linear subspace containing x. Then $x^{\perp} = H^{\perp} \oplus \operatorname{span}\{\vartheta_H x\}$.

First, consider the case $n = 3$, where $H = w^{\perp}$ for some non-zero $w \in \mathbb{R}^3$. Using cylindrical coordinates $v = \beta \vartheta_H \frac{x}{\|x\|} \pm \sqrt{1-\beta^2} w$ on $\mathbb{S}^2 \cap x^{\perp}$, we obtain

$$\begin{aligned}\varphi_\alpha(x) &= \frac{1}{4\pi} \int_{-1}^{1} \varphi\Big(\alpha x + \sqrt{1-\alpha^2}\beta \vartheta_H x + \sqrt{(1-\alpha^2)(1-\beta^2)}\|x\| w\Big) \frac{d\beta}{\sqrt{1-\beta^2}}\\ &\quad + \frac{1}{4\pi} \int_{-1}^{1} \varphi\Big(\alpha x + \sqrt{1-\alpha^2}\beta \vartheta_H x - \sqrt{(1-\alpha^2)(1-\beta^2)}\|x\| w\Big) \frac{d\beta}{\sqrt{1-\beta^2}}\\ &= \frac{1}{4\pi} \int_{-1}^{1} g_{z,a,b}(x) \frac{d\beta}{\sqrt{1-\beta^2}},\end{aligned}$$

where $z = \sqrt{(1-\alpha^2)(1-\beta^2)} w$, $a = \alpha$ and $b = \sqrt{1-\alpha^2}\beta$. By Lemma 4.3.1, $g_{z,a,b}$ is convex and, hence, φ_α is convex, as well.

For $n \geq 4$, we again use cylindrical coordinates on $\mathbb{S}^{n-1} \cap x^{\perp}$ in the direction of $\vartheta_H \frac{x}{\|x\|}$ to obtain

$$\varphi_\alpha(x) = c_n \int_{-1}^{1} \int_{\mathbb{S}^{n-1}\cap H^{\perp}} \varphi(\alpha x + \sqrt{1-\alpha^2}\beta \vartheta_H x + \sqrt{(1-\alpha^2)(1-\beta^2)}\|x\| w)\, d\sigma_{H^{\perp}}(v) \frac{d\beta}{(1-\beta^2)^{\frac{5-n}{2}}},$$

where $c_n = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n-1)/2)}$. Taking $\frac{1}{2}$ of this integral twice and replacing v by $-v$ in one copy, we see that again

$$\varphi_\alpha(x) = \frac{c_n}{2}\int_{-1}^{1}\int_{\mathbb{S}^{n-1}\cap H^\perp} g_{z,a,b}(x)\, d\sigma_{H^\perp}(v)\frac{d\beta}{(1-\beta^2)^{\frac{5-n}{2}}},$$

where $z = \sqrt{(1-\alpha^2)(1-\beta^2)}w$, $a = \alpha$ and $b = \sqrt{1-\alpha^2}\beta$ as before. Lemma 4.3.1 implies that $g_{z,a,b}$ is convex and, thus, so is φ_α.

For general $\varphi \in \mathrm{Cvx}_{(o)}(\mathbb{R}^n)$, we use that the Moreau envelope $e_t\varphi$ of φ is convex and finite and converges monotonously to φ, by Lemma 2.4.2. By what we have shown above, each of the functions $e_t\varphi \circledast \mu$, $t > 0$, is convex. Hence, by monotone convergence, we conclude that

$$\varphi \circledast \mu = \lim_{t\searrow 0} e_t\varphi \circledast \mu = \sup_{t>0} e_t\varphi \circledast \mu$$

is convex as well.

5 Isoperimetric Inequalities for Minkowski Valuations

The classical theory of convex bodies, often referred to as the Brunn–Minkowski theory, arises naturally from the interplay between Minkowski addition and volume. The definitions of its fundamental geometric functionals, mixed volumes and *quermassintegrals*, were given in Section 2.3 using variations of the volume. When viewed as coefficients in Weyl's tube formula, the quermassintegrals also appear in differential geometry as integrals of intermediate mean curvatures of convex hypersurfaces. They are central to various integral geometric formulas, such as the principal kinematic, Crofton's, or Kubota's formula. The latter allows to compute quermassintegrals of a convex body $K \in \mathcal{K}^n$ from means of its projection functions,

$$W_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\mathrm{Gr}(n,i)} V_i(K|E)\, d\nu_i(E), \qquad i = 0, \ldots, n.$$

One of the basic classical inequalities for quermassintegrals (2.25) of a convex body $K \in \mathcal{K}_0^n$ relates $W_{n-i}(K)$, $1 \leq i \leq n-1$, to the volume of K in the following way,

$$W_{n-i}(K)^n \geq \kappa_n^{n-i}\, V_n(K)^i \tag{5.1}$$

with equality if and only if K is a Euclidean ball.

More recently, conceptually more involved projection inequalities that directly imply (5.1) were established by Petty [96] for $i = n-1$ and Lutwak [83] for $i = 1, \ldots, n-2$. For $K \in \mathcal{K}_0^n$ and $1 \leq i \leq n-1$, the Lutwak–Petty projection inequalities are given by the right-hand side in the chain of inequalities,

$$W_{n-i}(K)^n \geq \frac{\kappa_n^{n+1}}{\kappa_{n-1}^n} V_n(\Pi_i^\circ K)^{-1} \geq \kappa_n^{n-i}\, V_n(K)^i \tag{5.2}$$

with equality on the right if and only if K is an ellipsoid when $i = n-1$, and a Euclidean ball when $i \leq n-2$. Here, $\Pi_i : \mathcal{K}^n \to \mathcal{K}^n$ is the projection body map of order i, defined by $h(\Pi_i K, u) = c_{n,i} W_{n-i}(K|u^\perp)$, where, in the literature, $c_{n,i} > 0$ is usually chosen such that $\Pi_i B^n = \kappa_{n-1} B^n$. Note that this normalization stems from the original definition of the projection body map of order $n-1$ and differs from the one used in Chapter 3. The left-hand inequalities in (5.2) are due to Lutwak [82, 83] with equality if and only if $\Pi_i K$ is a Euclidean ball.

For $i = n-1$, the quantity $V_n(\Pi_i^\circ K)^{-1/n}$ is (up to a factor) one of the *affine quermassintegrals*, first defined by Lutwak [82] (see (2.18)). Note that while the quermassintegrals W_{n-i}, $1 \leq i \leq n-1$, are merely invariant under rigid motions of $\mathbb{R}^n$, every A_{n-i} is invariant under all volume-preserving affine transformations, as was shown by Grinberg [56]. A major problem in affine convex geometry, first posed by Lutwak [84], was to obtain a sharp lower bound on $A_{n-i}(K)$ for $K \in \mathcal{K}_0^n$, $1 \leq i \leq n-1$, analogous to (5.1),

$$A_{n-i}(K)^n \geq \kappa_n^{n-i}\, V_n(K)^i \tag{5.3}$$

with equality if and only if K is an ellipsoid. Asymptotic confirmations of (5.3) were obtained in [46, 94]. Apart from these, only the rank-one cases $i = 1$ (inequality (3.2), a consequence of the well-known Blaschke–Santaló inequality) and $i = n-1$ (the Petty projection inequality) were known until very recently. However, in a landmark paper, Milman and Yehudayoff [93] established Lutwak's conjectured inequalities (5.3) giving a unified proof for all $i = 1, \dots, n-1$.

Finally, the relation of (5.3) to the chain of inequalities (5.2) was settled recently in [25], where it was shown that for a convex body $K \in \mathcal{K}^n$ with non-empty interior,

$$\frac{\kappa_n^{n+1}}{\kappa_{n-1}^n} V_n(\Pi_i^\circ K)^{-1} \geq A_{n-i}(K)^n. \tag{5.4}$$

Thus, the affine invariant inequality (5.3) fits in seamlessly in (5.2) and implies the other inequalities of the chain.

In the following, we will show that (5.2) and (5.4) can be extended to a much larger family of inequalities, by proving them not only for the projection body maps but for operators (Minkowski valuations) from an infinite dimensional cone, that are compatible with rigid motions.

Valuations (see Section 2.5) have long been tightly linked to quermassintegrals. Indeed, the most widely known classical result on real-valued valuations is the classification of all continuous rigid motion invariant valuations by Hadwiger [63] as precisely the linear combinations of the quermassintegrals, see [70] and [110] for more information on the history of this result and its transformational impact on integral geometry.

Inspired by Hadwiger's theorem, valuations with values in different semigroups became an important focus of interest. In particular, over the last two decades *Minkowski valuations* (with values in $\mathcal{K}^n$) received widespread attention. This line of research has its origins in the seminal work of Ludwig [78, 79] and, first, was mainly concerned with classifying continuous Minkowski valuations compatible with *linear* transformations [1, 2, 4, 80, 114, 118]. In this case, recent results show that these valuations form a cone generated by finitely many maps (such as the projection body map Π_{n-1}). In contrast, results by Kiderlen [67] and Schuster [112] imply that the cone of all translation invariant continuous Minkowski valuations which merely commute with $\mathrm{SO}(n)$ (like the operators Π_i when $i = 1, \dots, n-2$) is infinite-dimensional. The natural problem to also obtain a precise description of this cone has yet to be solved. By McMullen [88], only integer degrees of homogeneity $0 \leq i \leq n$ can occur, with $i = 0$ and $i = n$ being trivial. In [67] and [112], convolution representations were established for the cases $i = 1$ and $i = n - 1$, respectively. For *even* Minkowski valuations these results were subsequently generalized in [113] and [115] to all intermediate degrees under an additional smoothness assumption (see Section 5.1). With our first result of this chapter, we are able to remove this strong regularity condition.

Proposition 5.0.1. *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \ldots, n-1\}$. If Φ_i is even, then there exists an $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant distribution δ on $\mathbb{S}^{n-1}$ uniquely determined by the property that for every $K \in \mathcal{K}^n$ of class C^∞_+,*

$$h(\Phi_i K, \cdot) = V_i(K|\cdot) * \delta. \tag{5.5}$$

The distribution δ is called the spherical Crofton distribution *of Φ_i.*

See Section 2.2 for the definition of the convolution of functions on Grassmanians and measures on $\mathbb{S}^{n-1}$ used in (5.5).

The proof of Proposition 5.0.1 relies on results about generalized valuations by Alesker and Faifman [16] and the techniques to obtain the earlier versions of Proposition 5.0.1 from [113] and [115]. Examples of spherical Crofton distributions and more details on the class of distributions that appear for different degrees will be given in Section 5.1.

In recent years, several isoperimetric-type inequalities involving projection body maps (of arbitrary degree) were shown to hold, in fact, for much larger classes of Minkowski valuations compatible with rigid motions (see [14, 24, 61, 65, 95, 112] and Chapter 3). Based on Proposition 5.0.1, we establish a significant extension of inequalities (5.2) and (5.4) to all even Minkowski valuations admitting a spherical Crofton distribution which is non-negative (and, thus, a spherical Crofton *measure*).

Theorem 5.0.2. *Suppose that the spherical Crofton distribution of an even Minkowski valuation $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$, $1 \leq i \leq n-1$, is non-negative and normalized such that $\Phi_i B^n = \kappa_{n-1} B^n$. If $K \in \mathcal{K}^n$ has non-empty interior, then*

$$W_{n-i}(K)^n \geq \frac{\kappa_n^{n+1}}{\kappa_{n-1}^n} V_n(\Phi_i^\circ K)^{-1} \geq A_{n-i}(K)^n.$$

There is equality in the left-hand inequality if and only if $\Phi_i K$ is a Euclidean ball. Equality holds in the right-hand inequality if and only if K is of constant i-brightness or $i = 1$ and $\Phi_1 = \kappa_{n-1}\Delta$ or $i = n-1$ and $\Phi_{n-1} = \Pi_{n-1}$.

Let us note that (as we shall show) the assumption that the spherical Crofton distribution is non-negative is only necessary for the right-hand inequality. In fact, the left-hand inequality, including its equality conditions, holds for all continuous Minkowski valuations which are translation in- and $\mathrm{SO}(n)$ equivariant mapping bodies with non-empty interior to such bodies. Next, we want to point out that when $i = 1$ or $i = n-1$ *all* non-negative spherical measures which are $\mathrm{SO}(n-1)$ invariant are spherical Crofton measures (see Section 5.1). It is an open problem, whether the same is true also for the degrees $2 \le i \le n-2$. Finally, the special case $i = 1$ of Theorem 5.0.2 was obtained in Chapter 3 and the case $i = n-1$ was obtained in [61]. Partial results for the intermediate degrees, without the equality cases for the right-hand inequality, were obtained in [25]. However, we will see in Section 5.3 that Theorem 5.0.2 provides a significant extension of these earlier results.

An immediate consequence of Theorem 5.0.2 and inequality (5.3) of Milman and Yehudayoff is the solution to the following isoperimetric problem.

Corollary 5.0.3. *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \dots, n-1\}$. If Φ_i is non-trivial, even and its spherical Crofton distribution non-negative, then, among $K \in \mathcal{K}^n$ with non-empty interior,*

$$V_n(\Phi_i^\circ K) V_n(K)^i \tag{5.6}$$

is maximized by Euclidean balls. If $i = 1$ and $\Phi_1 = c\Delta$ or $i = n-1$ and $\Phi_{n-1} = c\Pi_{n-1}$ for some $c > 0$, then K is a maximizer if and only if it is an ellipsoid. Otherwise, Euclidean balls are the only maximizers.

Let us emphasize that the existence of extremals for (5.6) is a-priori not clear. In case of continuous affine invariant functionals, this follows easily from compactness (see, e.g., [109, Chapter 10]). However, as was recently discovered (see Chapter 3, Theorem 3.2.1), there exists a continuous Minkowski valuation Φ_1 of degree 1 compatible with rigid motions such that $V_n(\Phi_1^\circ K) V_n(K)$ is unbounded. As this somewhat surprising example is *not even*, we consider for our next result – a sufficient condition for the existence of maximizers of volume products of the form (5.6) – also Minkowski valuations that are not necessarily even. To this end, we require the following counterpart of Proposition 5.0.1 for such valuations.

Theorem 5.0.4 ([47, 112, 115, 116])**.** *If $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \dots, n-1\}$, then there exists a unique $\mathrm{SO}(n-1)$ invariant $f \in L^1(\mathbb{S}^{n-1})$ with center of mass at the origin such that for every $K \in \mathcal{K}^n$,*

$$h(\Phi_i K, \cdot) = S_i(K, \cdot) * f. \tag{5.7}$$

The function f is called the generating function *of Φ_i.*

The aforementioned sufficient condition can now be stated in terms of generating functions. We also include in our result an immediate application of the condition that uncovers a phenomenon not seen before.

Theorem 5.0.5. *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \dots, n-1\}$. If* *the generating function of Φ_i is a sum of two generating functions one of which is bounded from below by a positive constant, then*

$$V_n(\Phi_i^\circ K)V_n(K)^i$$

attains a maximum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior. Moreover, for $i = 1$, there exist Minkowski valuations $\Psi_1 : \mathcal{K}^n \to \mathcal{K}^n$ such that the maximizers of $V_n(\Psi_1^\circ K)V_n(K)$ are different from Euclidean balls.

Let us point out that even Minkowski valuations with positive generating function need not have a non-negative spherical Crofton distribution (cf. Section 5.1). Let us also note that the positivity condition on generating functions in Theorem 5.0.5 is not very restrictive. Indeed, generating functions of Minkowski valuations of degree $i = n - 1$ are all non-negative (as was shown in [112]). Moreover, strictly positive support functions of convex bodies of revolution generate a large class of examples of Minkowski valuations of arbitrary degree $1 \leq i \leq n - 1$ (see Section 5.1).

It is still open and part of future work, how maximizers of arbitrary Minkowski valuations may look like. Especially for Minkowski valuations generated by support functions of convex bodies of revolution, Euclidean balls seem to be natural candidates. Indeed, it was used by Lutwak [83] for i-projection bodies and can be deduced from results by Schuster [111] that if Euclidean balls are extremals of the volume product/ratio of some Minkowski valuation Φ_i of degree i generated by the support function of a convex body of revolution, then this is also true for the Minkowski valuations of degree $j < i$ generated by the same support function. We will give a different proof of this fact in Section 5.6.

Note that our proof of Theorem 5.0.5 shows that any example of a Minkowski valuation Φ_i with unbounded volume product $V_n(\Phi_i^\circ K)V_n(K)^i$ yields an entire cone with apex at Φ_i (not including Φ_i) of Minkowski valuations Ψ_i such that maximizers of $V_n(\Psi_i^\circ K)V_n(K)^i$ exist and are different from Euclidean balls.

Moreover, we note that the arguments in the proof of Theorem 5.0.5 are not bound to the volume product of Minkowski valuations and can also be applied to non-polar isoperimetric problems for Minkowski valuations. Indeed, with our next result, we obtain an analogous condition for the existence of extremals of the volume ratio.

Theorem 5.0.6. *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \ldots, n-1\}$. If the generating function of Φ_i is a sum of two generating functions one of which is bounded from below by a positive constant, then*

$$\frac{V_n(\Phi_i K)}{V_n(K)^i}$$

attains a minimum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior.

Let us note that, as before, any example of a Minkowski valuation with unbounded volume ratio would yield examples of Minkowski valuations where minimizers exist and are different from Euclidean balls. However, up to now, no such examples are known and it is unclear whether one should expect examples to exist. We will discuss this problem and further related partial results in Sections 5.5 and 5.6.

5.1 Preliminaries and Auxiliary Results

In this section, we recall known constructions and results for Minkowski valuations and give some examples.

Recalling from Section 2.5, a Minkowski valuation $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous map satisfying the following valuation property with respect to Minkowski addition

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L),$$

for every $K, L \in \mathcal{K}^n$, whenever $K \cup L \in \mathcal{K}^n$. We will usually work with Minkowski valuations that are translation invariant and $\mathrm{SO}(n)$ equivariant, that is, $\Phi(\eta K) = \eta\Phi(K)$ for every $\eta \in \mathrm{SO}(n)$ and $K \in \mathcal{K}^n$. Analogously to real-valued valuations, a Minkowski valuation Φ is called homogeneous of degree $1 \leq i \leq n-1$, if $\Phi(\lambda K) = \lambda^i \Phi(K)$, and it is called even, if $\Phi(-K) = \Phi(K)$, for every $K \in \mathcal{K}^n$ and $\lambda > 0$. Note that by a result in [14] every $\mathrm{SO}(n)$ equivariant Minkowski valuation is also $\mathrm{O}(n)$ equivariant, in particular, $\Phi(-K) = -\Phi K$ for every $K \in \mathcal{K}^n$. Consequently, ΦK is origin-symmetric whenever $K \in \mathcal{K}^n$ and Φ is an $\mathrm{SO}(n)$ equivariant even Minkowski valuation.

Every Minkowski valuation Φ gives rise to a family of real-valued valuations $(\varphi_u)_{u \in \mathbb{S}^{n-1}}$, defined by

$$\varphi_u(K) = h(\Phi K, u), \qquad u \in \mathbb{S}^{n-1}.$$

Clearly, each φ_u inherits homogeneity and translation invariance from Φ, in particular, $\varphi_u \in \mathrm{Val}$ whenever Φ is translation invariant. Moreover, if Φ is even, so is every φ_u. If Φ is $\mathrm{SO}(n)$ equivariant, then φ_u is invariant under the stabilizer of u in $\mathrm{SO}(n)$ and any φ_u already contains all information about Φ. Indeed, by (2.11), $h(\Phi K, \eta u) = \varphi_u(\eta^{-1}K)$, for every $\eta \in \mathrm{SO}(n)$ and $K \in \mathcal{K}^n$, and therefore $h(\Phi K, \cdot)$ is determined by the values of φ_u. Taking $u = \bar{e}$, we obtain the associated real-valued valuation $\varphi = \varphi_{\bar{e}}$ of Φ.

Unlike real-valued valuations, the set of Minkowski valuations does not form a vector space, but a cone, that is, for Minkowski valuations Φ_1 and Φ_2 and $\lambda_1, \lambda_2 \geq 0$, the combination $\lambda_1\Phi_1 + \lambda_2\Phi_2$, is again a Minkowski valuation. Consequently, there are (many) real-valued valuations that do not appear as associated valuation of a Minkowski valuation.

The connection between Minkowski and associated real-valued valuations provides an important tool which can be used to transfer some results from the rich theory of real-valued valuations to statements for Minkowski valuations. Indeed, it turned out to be highly useful in the task of proving representation results for Minkowski valuations.

First representation results for i-homogeneous Minkowski valuations were obtained by Kiderlen [67] (for $i = 1$) and by Schuster [112] (for $i = n - 1$). These results were successively generalized and unified to all degrees of homogeneity in [47, 113, 115, 116], yielding in principle two types of representation formulas. The first result, given by Theorem 5.0.4, holds for general $\mathrm{SO}(n)$ equivariant, translation invariant, and continuous Minkowski valuations Φ_i of given degree $i \in \{1, \dots, n-1\}$ and makes use of the ith area measure $S_i(K, \cdot)$ and spherical convolution,

$$h(\Phi_i K, \cdot) = S_i(K, \cdot) * f.$$

It is not clear, which functions f generate a Minkowski valuation. However, if f is the support function $h(L, \cdot)$ of a convex body of revolution (that is, $h(L, \cdot)$ is zonal), then $S_i(K, \cdot) * h(L, \cdot)$ is always a support function (see also Example 5.1.2).

While the representation $S_i(K, \cdot) * f$ is in some sense compatible with the notion of quermassintegrals, the second type is more suitable when working with affine quermassintegrals. Based on the Crofton map for valuations, it is a representation for even and smooth Minkowski valuations, where a continuous Minkowski valuation is called smooth, if its associated real-valued valuation is smooth (see Section 2.5).

Theorem 5.1.1 ([113, 115]). *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a smooth, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \dots, n-1\}$. If Φ_i is even, then there exists a unique smooth $\mathrm{S}(\mathrm{O}(i) \times \mathrm{O}(n-i))$ invariant measure μ on $\mathbb{S}^{n-1}$ such that for every $K \in \mathcal{K}^n$,*

$$h(\Phi_i K, \cdot) = V_i(K| \cdot) * \mu. \tag{5.8}$$

The measure μ is called the spherical Crofton measure *of Φ_i.*

Note that in [113], Theorem 5.1.1 was stated for $\mathrm{O}(n)$ equivariant Minkowski valuations (without the uniqueness result) with the convolution in (5.8) also being induced by the convolution on $\mathrm{O}(n)$ instead of $\mathrm{SO}(n)$. However, it is argued in [115, p. 21] that this is equivalent to Theorem 5.1.1, using that every $\mathrm{SO}(n)$ equivariant Minkowski valuation is also $\mathrm{O}(n)$ equivariant, by [14, Lemma 7.1], and that the convolution expressions coincide.

Let us further note that it is not clear which measures μ are the spherical Crofton measures of some Minkowski valuations. As for $i = 1$, (5.8) reduces to

$$h(\Phi_1 K, \cdot) = (h(K, \cdot) + h(-K, \cdot)) * \mu,$$

every non-negative measure μ is a spherical Crofton measure, by Theorem 3.1.2, and there exist 1-homogeneous Minkowski valuations where the spherical Crofton measure is a signed measure, as was shown in [47].

Moreover, for $i = n - 1$, Cauchy's surface area formula (see, e.g., [109, p. 301]) implies that

$$V_{n-1}(K|u^\perp) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| dS_{n-1}(K, v) = \left(S_{n-1}(K, \cdot) * \left(\frac{1}{2} |\langle \bar{e}, \cdot \rangle| \right) \right) (u), \tag{5.9}$$

for $u \in \mathbb{S}^{n-1}$. Consequently, (2.5) and an application of Fubini's theorem (or the associativity of the convolution) yields for a non-negative zonal measure μ that

$$(V_{n-1}(K,\cdot) * \mu)(u) = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} |\langle w, \vartheta_u^{-1} v\rangle| d\hat{\mu}(w) dS_{n-1}(K,v)$$
$$= \frac{1}{2}(S_{n-1}(K,\cdot) * h(Z^\mu,\cdot))(u), \tag{5.10}$$

where Z^μ is the zonoid generated by μ by (2.15). Here, we used that functions and measures on $\mathrm{Gr}(n,n-1)$ can be interpreted as even functions and measures on $\mathbb{S}^{n-1}$ and that zonal measures satisfy $\hat{\mu} = \mu$. As (5.10) always defines the support function of a convex body, every non-negative zonal measure μ on $\mathbb{S}^{n-1}$ is the Crofton measure of an $(n-1)$-homogeneous Minkowski valuation.

For the intermediate cases $2 \le i \le n-2$, however, it is an open question whether non-negative measures are always spherical Crofton measures.

Let us now give some examples of Minkowski valuations including their representations from Theorem 5.0.4 and/or Theorem 5.1.1.

Example 5.1.2.

- For degree of homogeneity $i = 1$, it was already noted by Spiegel [117] (see also [106,109]) that (assuming continuity and translation invariance) the valuation property is equivalent to additivity with respect to Minkowski addition. Hence, the class of continuous 1-homogeneous Minkowski valuations, which are translation invariant and $\mathrm{SO}(n)$ equivariant, coincides with that of Minkowski endomorphisms considered in Chapter 3. The results of this chapter can therefore be seen as generalizations of Chapter 3.

  Using (2.17) and the properties of $\square_n$, the representation of Theorem 3.1.2 (or Theorem 5.1.1) relates to Theorem 5.0.4 by

  $$S_1(K,\cdot) * f = (\square_n h(K,\cdot)) * f = h(K,\cdot) * (\square_n f). \tag{5.11}$$

  Note that, by relation (5.11), the spherical Crofton measure of a Minkowski endomorphism with positive generating function need not be non-negative, in general. Indeed, we will give an example for this in Section 5.5. The converse is also not true, as the example $\Delta K = \frac{1}{2}(-K+K)$ (or any smooth approximation of it) shows.

- The maps $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ generated by the constant function $f(u) = \frac{1}{n}$ are given by

  $$\Phi_i K = W_{n-i}(K) B^n$$

  for $i = 1, \ldots, n-1$. The properties of the quermassintegrals directly imply that the Φ_i are continuous and even Minkowski valuations which are $\mathrm{SO}(n)$ equivariant and translation invariant. By Kubota's formula, the spherical Crofton measure of Φ_i is a multiple of the uniform measure σ on $\mathbb{S}^{n-1}$.

- The projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ (see [34]) is defined by

$$h(\Pi K, u) = V_{n-1}(K|u^\perp) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\, dS_{n-1}(K, v), \quad u \in \mathbb{S}^{n-1}.$$

From (2.5) we directly obtain that the spherical Crofton distribution of Π is given by $\widehat{\delta}_{\bar{e}^\perp}$. The projection body operator is an $(n-1)$-homogeneous Minkowski valuation which is additionally $\mathrm{SL}(n)$ contravariant (or equi-affine contravariant), that is, $\Pi(AK) = A^{-T}\Pi(K)$, for any $A \in \mathrm{SL}(n)$, a property that already characterizes Π up to scalar multiples (see [79]).

The projection body gives rise to the series of i-projection bodies Π_i by taking derivatives of $\Pi = \Pi_{n-1}$, that is,

$$h(\Pi_i K, u) = \frac{i!}{(n-1)!}\, \frac{d^{n-1-i}}{dt^{n-1-i}}\bigg|_{t=0} h(\Pi(K + tB^n), u)$$

for $u \in \mathbb{S}^{n-1}$. Note that the derivative is well-defined by the multilinearity of the mixed area measure $S((K + tB^n)[n-1], \cdot) = S_{n-1}(K + tB^n, \cdot)$ leading to

$$h(\Pi_i K, u) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle u, v\rangle|\, dS_i(K, v) = W_{n-1-i}(K|u^\perp), \quad u \in \mathbb{S}^{n-1}.$$

The derivation operator used in this definition is one of the Hard Lefschetz operators for valuations (the other being an integration operator) introduced and studied in [7, 8, 11, 29] and were applied to Minkowski valuations in [95, 115]. Note that the derivative of any Minkowski valuation is again a Minkowski valuation with the same generating function (up to a scalar multiple). Hence, the set of functions generating i-homogeneous Minkowski valuations is contained in the set generating j-homogeneous ones whenever $i > j$. However, examples show that the other inclusion is not true, that is, there exist Minkowski endomorphisms that are not derivatives of $(n-1)$-homogeneous Minkowski valuations.

The spherical Crofton measure of a derived Minkowski valuation can be calculated using the Radon transform (see [115, Theorem 6.1]).

- The previous example can be generalized by replacing the support function of the interval $[-\bar{e}, \bar{e}]$ by the support function of an arbitrary convex body L of revolution. The maps $\Phi_i^L : \mathcal{K}^n \to \mathcal{K}^n$ are then given by

$$h(\Phi_i^L K, \cdot) = S_i(K, \cdot) * h(L, \cdot).$$

Using the notation $L(u) = \vartheta_u L$, this reduces to

$$h(\Phi_i^L K, u) = \int_{\mathbb{S}^{n-1}} h(L(u), v)\, dS_i(K, v), \quad u \in \mathbb{S}^{n-1}, \tag{5.12}$$

where $h(L(u), v) = h(L(v), u)$ can be exchanged arbitrarily due to the zonality of $h(L, \cdot)$.

Note that the formula $h(\Phi_i^L K, \cdot)$ defines a support function as an integral over the support functions $h(L(v), u)$ with respect to a non-negative measure, and that the Φ_i^L are the derivatives of Φ_{n-1}^L.

Having two representations at hand, it is a natural question how to get one from the other. This was answered completely by [115, Corollary 5.1]. In the following, we will need a special case of their result when the function f in Theorem 5.0.4 is the Cosine transform $C\nu$ of a (signed) measure ν.

Theorem 5.1.3 ([115, Corollary 5.1])**.** *Suppose that Φ_i is an even i-homogeneous Minkowski valuation such that $h(\Phi_i K, \cdot) = S_i(K, \cdot) * (C\nu)$, where ν is a signed measure on $\mathbb{S}^{n-1}$. Then the spherical Crofton measure μ of Φ_i is given by*

$$\hat{\mu} = 2\frac{\kappa_{n-1}}{\kappa_i}(R_{n-1,i}\nu^{\perp}).$$

This theorem implies that if Φ_i is generated by the support function of a zonoid Z^ν (that is, $h(Z^\nu, \cdot) = C\nu$ with $\nu \geq 0$), then the Crofton measure of Φ_i is non-negative. If Z^ν is merely a generalized zonoid (that is, ν is a signed measure), then the Crofton measure of Φ_i is non-negative if and only if $R_{n-1,i}\nu^\perp \geq 0$. The next lemma rewrites this condition for signed measures ν with density g with respect to the uniform probability measure σ on $\mathbb{S}^{n-1}$, using the function $\tilde{g}$ defined by $\tilde{g}(\langle u, \bar{e}\rangle) = g(u)$, $u \in \mathbb{S}^{n-1}$ (see Section 2.1).

Lemma 5.1.4. *Suppose that Φ_i is an even i-homogeneous Minkowski valuation such that $h(\Phi_i K, \cdot) = S_i(K, \cdot) * (Cg)$, where $g \in C(\mathbb{S}^{n-1})$ is even and zonal. Then the spherical Crofton measure of Φ_i is non-negative, if and only if*

$$\int_0^1 \tilde{g}(\alpha t)(1-t^2)^{\frac{n-i-3}{2}}\,dt \geq 0 \quad \forall \alpha \in [0, 1].$$

Proof. By Theorem 5.1.3, the Crofton measure of Φ_i is positively generated, if and only if

$$(R_{n-1,i}g^\perp)(F^\perp) = (R_{1,n-i}g)(F) = \int_{\mathbb{S}^{n-1}\cap F} g(w)\,d\sigma_{E^\perp}(w) \geq 0, \tag{5.13}$$

for every $F \in \mathrm{Gr}(n, n-i)$.

In a next step, we want to get a better representation of the set $\mathbb{S}^{n-1} \cap F$, $F \in \mathrm{Gr}(n, n-i)$. As is proved in Section 5.7, F admits an orthonormal basis $(v_1(F), v_2(F), \ldots, v_{n-i}(F))$ such that $v_2(F), \ldots, v_{n-i}(F) \in F \cap \bar{e}^\perp$ and, for $F \not\subseteq \bar{e}^\perp$, the vector $v_1(F) \in \mathbb{S}^{n-1}$ is uniquely determined by the conditions

$$F = \mathrm{span}(v_1(F)) \oplus (F \cap \bar{e}^\perp) \quad \text{and} \quad \langle v_1(F), \bar{e}\rangle > 0.$$

Any $u \in F \cap \mathbb{S}^{n-1}$ can then be written as $u = \alpha v_1(F) + \sum_{j=2}^{n-i} \alpha_j v_j(F)$, where $\alpha, \alpha_j \in [-1,1]$ and $\alpha^2 + \sum_{j=2}^{n-i} \alpha_j^2 = 1$. The calculation $\langle u, \bar{e}\rangle = \alpha\langle v_1(F), \bar{e}\rangle = \langle u, v_1(F)\rangle\langle v_1(F), \bar{e}\rangle$ shows that any zonal function g depends only on $\langle u, v_1(F)\rangle$, that is,

$$g(\langle u, \bar{e}\rangle) = g(\langle u, v_1(F)\rangle\langle v_1(F), \bar{e}\rangle), \quad u \in F \cap \mathbb{S}^{n-1}.$$

Using [59, Lemma 1.3.1], we can transform the integral from (5.13) to obtain

$$\int_{\mathbb{S}^{n-1}\cap F} g(w)\, d\sigma(w) = (n-i-1)\kappa_{n-i-1} \int_{-1}^{1} \tilde{g}(t\langle v_1(F), \bar{e}\rangle)(1-t^2)^{\frac{n-i-3}{2}}\, dt,$$

where $\tilde{g}$ is determined by $g(w) = \tilde{g}(\langle w, \bar{e}\rangle)$.

As we vary $F \in \mathrm{Gr}(n, n-i)$, $\langle v_1(F), \bar{e}\rangle$ ranges from 0 to 1. Consequently, condition (5.13) is – omitting the positive constants – equivalent to

$$\int_{-1}^{1} \tilde{g}(\alpha t)(1-t^2)^{\frac{n-i-3}{2}}\, dt \geq 0 \quad \forall \alpha \in [0,1].$$

Since g is assumed to be even, the claim follows. $\square$

In the following, we will collect some well-known normalization results on Minkowski valuations generalizing Lemma 3.1.4 that directly follow from Theorem 5.0.4.

Lemma 5.1.5 (see, e.g., [111, Lemma 6.3])**.** *Let Φ_i be an i-homogeneous Minkowski valuation, generated by $f \in L^1(\mathbb{S}^{n-1})$. Then we have*

$$\Phi_i B^n = r_{\Phi_i} B^n \qquad \text{and} \qquad W_{n-1}(\Phi_i K) = r_{\Phi_i} W_{n-i}(K),$$

for every convex body $K \in \mathcal{K}^n$, where $r_{\Phi_i} = \int_{\mathbb{S}^{n-1}} f(u)\, du$.

If $\Phi_i = \Phi_i^L$ for some convex body of revolution $L \in \mathcal{K}^n$, then

- $r_{\Phi_i^L} = nW_{n-1}(L)$,
- $h(\Phi_i^L K, u) = nV(L(u), K[i], B^n[n-i-1])$*, for all* $u \in \mathbb{S}^{n-1}$*, and*
- $\Phi_i^{B^n} K = nW_{n-i}(K)B^n$.

Proof. By the integral representation of mixed volumes, Theorem 5.0.4 and (2.3) we calculate using Fubini's theorem

$$W_{n-1}(\Phi_i K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(\Phi_i K, u)\, du = \int_{\mathbb{S}^{n-1}} \frac{1}{n}\int_{\mathbb{S}^{n-1}} f(\vartheta_v^{-1} u)\, du\, dS_i(K, v).$$

Since the inner integral does not depend on v anymore, we obtain

$$W_{n-1}(\Phi_i K) = \left(\int_{\mathbb{S}^{n-1}} f(u)\, du\right) W_{n-i}(K).$$

The claim now follows from the fact that by $\mathrm{SO}(n)$ equivariance and translation invariance balls must be mapped to balls centered at the origin.

If $f = h(L, \cdot)$, the definition of quermassintegrals shows that $r_{\Phi_i^L} = nW_{n-1}(L)$ and that $h(\Phi_i^L K, u) = nV(L(u), K[i], B^n[n-i-1])$. For $L = B^n$ this simplifies to

$$h(\Phi_i^{B^n} K, u) = nV(B^n, K[i], B^n[n-i-1]) = nW_{n-i}(K).$$

$\square$

The next fact is a direct consequence of the representation of Φ_i^L and Fubini's theorem.

Lemma 5.1.6 (see, e.g., [111, Lemma 6.2]). *Suppose that L is a convex body of revolution. Then*

$$V(K[n-1], \Phi_i^L M) = V(M[i], B^n[n-i-1], \Phi_{n-1}^L K)$$

for every $K, M \in \mathcal{K}^n$.

5.2 Representations of Even Minkowski Valuations

In this section, we first recall results by Alesker and Faifman [16] about generalized even valuations and then apply these results to prove Proposition 5.0.1.

We start with an extension of the Crofton map $\mathrm{Cr}_i : C^\infty(\mathrm{Gr}(n,i)) \to \mathrm{Val}_i^{+,\infty}$, $1 \le i \le n-1$, which was already defined in Section 2.5 for $f \in C^\infty(\mathrm{Gr}(n,i))$ by

$$(\mathrm{Cr}_i\, f)(K) = \int_{\mathrm{Gr}(n,i)} V_i(K|E) f(E)\, d\nu_i(E), \quad K \in \mathcal{K}^n.$$

The map Cr_i is continuous and surjective. Moreover, the Crofton map Cr_i and the Klain map Kl_{n-i} (of complementary degree $n-i$) are adjoint in the sense of Lemma 2.5.3 and satisfy $\mathrm{Kl}_i \circ \mathrm{Cr}_i = C_i$. The adjointness property can be used to give a short proof of the following theorem from [16]. As the notation in [16] differs very much from our notation, we repeat their proof for the reader's convenience.

Theorem 5.2.1 ([16]). *The Crofton map Cr_i and the Klain map Kl_i can be extended uniquely by continuity to*

$$\widetilde{\mathrm{Cr}}_i : C^{-\infty}(\mathrm{Gr}(n,i)) \to \mathrm{Val}_i^{+,-\infty}, \quad \widetilde{\mathrm{Kl}}_i : \mathrm{Val}_i^{+,-\infty} \to C^{-\infty}(\mathrm{Gr}(n,i)),$$

where $\widetilde{\mathrm{Cr}}_i$ is surjective and $\widetilde{\mathrm{Kl}}_i$ is injective. Moreover, $\widetilde{\mathrm{Cr}}_i$ is adjoint to Kl_{n-i},

$$\langle \widetilde{\mathrm{Cr}}_i(\psi), \varphi \rangle_{\mathrm{Val}^{-\infty}} = \langle \psi, (\mathrm{Kl}_{n-i}\, \varphi)^\perp \rangle_{C^{-\infty}}, \tag{5.14}$$

for $\psi \in C^{-\infty}(\mathrm{Gr}(n,i))$ and $\varphi \in \mathrm{Val}_{n-i}^{+,\infty}$, and the extensions satisfy $\widetilde{\mathrm{Kl}}_i \circ \widetilde{\mathrm{Cr}}_i = C_i$.

Let us note, that "extension" in the last theorem means extension with respect to the embeddings $C^\infty(\mathrm{Gr}(n,i)) \hookrightarrow C^{-\infty}(\mathrm{Gr}(n,i))$ and $\mathrm{Val}_i^{+,\infty} \hookrightarrow \mathrm{Val}_i^{+,-\infty}$, that is, for the Crofton map, e.g., the following diagram commutes

$$\begin{tikzcd} C^\infty(\mathrm{Gr}(n,i)) \arrow[r, "\mathrm{Cr}_i"] \arrow[d, hook, "f \mapsto \langle f, \cdot \rangle_{L^2(\mathrm{Gr}(n,i))}"'] & \mathrm{Val}_i^{+,\infty} \arrow[d, hook, "\mathrm{pd}"] \\ C^{-\infty}(\mathrm{Gr}(n,i)) \arrow[r, "\widetilde{\mathrm{Cr}}_i"] & \mathrm{Val}_i^{+,-\infty} \end{tikzcd}$$

Proof. First, note that the uniqueness of the extension by continuity is clear as $C^\infty(\mathrm{Gr}(n,i))$ is dense in $C^{-\infty}(\mathrm{Gr}(n,i))$ and $\mathrm{Val}_i^{+,\infty}$ is dense in $\mathrm{Val}_i^{+,-\infty}$.

In light of Lemma 2.5.3, consider

$$\mathrm{Kl}_{n-i}^* \circ \perp^*\colon C^{-\infty}(\mathrm{Gr}(n,i)) \to \mathrm{Val}_i^{+,-\infty},$$

as a candidate for $\widetilde{\mathrm{Cr}}_i$, that is, the map that acts on $\psi \in C^{-\infty}(\mathrm{Gr}(n,i))$ by

$$\langle (\mathrm{Kl}_{n-i}^* \circ \perp^*)(\psi), \varphi \rangle_{\mathrm{Val}^{-\infty}} = \langle \psi, (\mathrm{Kl}_{n-i}\,\varphi)^\perp \rangle_{C^{-\infty}}, \quad \varphi \in \mathrm{Val}_{n-i}^{+,\infty}.$$

$\mathrm{Kl}_{n-i}^* \circ \perp^*$ is obviously continuous and satisfies the adjointness relation. In order to show that it extends Cr_i, let $f \in C^\infty(\mathrm{Gr}(n,i))$ arbitrary. We have to show that for every $\varphi \in \mathrm{Val}_{n-i}^{+,\infty}$

$$\langle (\mathrm{Kl}_{n-i}^* \circ \perp^*)(\langle f, \cdot \rangle_{L^2(\mathrm{Gr}(n,i))}), \varphi \rangle_{\mathrm{Val}^{-\infty}} = \langle \mathrm{pd}\,\mathrm{Cr}_i\, f, \varphi \rangle_{\mathrm{Val}^{-\infty}},$$

or, after plugging in the definitions,

$$\langle f, (\mathrm{Kl}_{n-i}\,\varphi)^\perp \rangle_{L^2(\mathrm{Gr}(n,i))} = \langle \mathrm{Cr}_i\, f, \varphi \rangle_{\mathrm{A}}.$$

However, this is exactly the statement of Lemma 2.5.3.

As adjoint map (up to the bijective map $\perp^*$) of the injective Klain map, the image of $\widetilde{\mathrm{Cr}}_i = \mathrm{Kl}_{n-i}^* \circ \perp^*$ is dense in $\mathrm{Val}_i^{+,-\infty}$. The image is also closed by an application of Banach's open mapping theorem (see [16, Claim 4.3]), which implies the surjectivity of $\widetilde{\mathrm{Cr}}_i$.

The map $\widetilde{\mathrm{Kl}}_i$ is similarly defined as $\perp^* \circ \mathrm{Cr}_{n-i}^*$. As before, $\widetilde{\mathrm{Kl}}_i$ extends Kl_i, by Lemma 2.5.3, and is continuous and injective, by the surjectivity of Cr_{n-i}.

It remains to show that $\widetilde{\mathrm{Kl}}_i \circ \widetilde{\mathrm{Cr}}_i = C_i$. Letting $\psi \in C^{-\infty}(\mathrm{Gr}(n,i))$ and $\varphi \in C^\infty(\mathrm{Gr}(n,i))$, the definitions of $\widetilde{\mathrm{Kl}}_i$ and $\widetilde{\mathrm{Cr}}_i$ and $\mathrm{Kl}_{n-i} \circ \mathrm{Cr}_{n-i} = C_{n-i}$ imply

$$\langle \widetilde{\mathrm{Kl}}_i(\widetilde{\mathrm{Cr}}_i(\psi)), \varphi \rangle_{\mathrm{Val}^{-\infty}} = \langle \psi, (\mathrm{Kl}_{n-i}(\mathrm{Cr}_{n-i}\,\varphi^\perp))^\perp \rangle_{C^{-\infty}} = \langle \psi, (C_{n-i}\varphi^\perp)^\perp \rangle_{C^{-\infty}}.$$

The fact that $|\cos(E,F)| = |\cos(E^\perp, F^\perp)|$, for $E, F \in \mathrm{Gr}(n,i)$, and, therefore, $(C_{n-i}\varphi^\perp)^\perp = C_i\varphi$, and the self-adjointness of C_i now directly yield the claim. □

By the embedding of continuous valuations into generalized valuations, every continuous and even valuation $\varphi \in \mathrm{Val}_i^+$ admits a Crofton distribution, that is, a distribution δ such that $\widetilde{\mathrm{pd}}\,\varphi = \widetilde{\mathrm{Cr}}_i\delta$. For convex bodies K of class C_+^∞, $\varphi(K)$ can then be directly calculated by δ, as was shown in [16]. The proof given here uses the ideas from [16], but varies in some details.

Lemma 5.2.2 ([16, Lemma 4.7]). *Suppose that* $\varphi \in \mathrm{Val}_i^+$ *and* $\widetilde{\mathrm{pd}}\,\varphi = \widetilde{\mathrm{Cr}}_i\psi$ *for some* $\psi \in C^{-\infty}(\mathrm{Gr}(n,i))$.

If $K \in \mathcal{K}^n$ *is of class* C^∞_+, *then*

$$\varphi(K) = \langle \psi, V_i(K|\cdot)\rangle_{C^{-\infty}}. \tag{5.15}$$

Proof. Let $K \in \mathcal{K}^n$ be of class C^∞_+. Then K naturally defines a generalized valuation $\psi^K \in \mathrm{Val}_{n-i}^{+,-\infty}$ by

$$\langle \psi^K, \varphi\rangle_{\mathrm{Val}^{-\infty}} = \varphi(K), \quad \varphi \in \mathrm{Val}_i^{+,\infty}. \tag{5.16}$$

Letting $f \in C^\infty(\mathrm{Gr}(n,n-i))$ and using the adjointness relation of $\widetilde{\mathrm{Kl}}_{n-i}$, we calculate

$$\langle \widetilde{\mathrm{Kl}}_{n-i}\psi^K, f\rangle_{C^{-\infty}(\mathrm{Gr}(n,n-i))} = \langle \psi^K, \mathrm{Cr}_i(f^\perp)\rangle_{\mathrm{Val}^{-\infty}} = (\mathrm{Cr}_i\, f^\perp)(K) = \langle V_i(K|\cdot)^\perp, f\rangle_{L^2},$$

that is, $\widetilde{\mathrm{Kl}}_{n-i}\psi^K = \langle V_i(K|\cdot)^\perp, \cdot\rangle_{L^2}$.

Consider now the smooth valuation $\xi^K \in \mathrm{Val}_{n-i}^{+,\infty}$ defined by

$$\xi^K(L) = \frac{1}{2}\binom{n}{i}\left(V(L[n-i], K[i]) + V(L[n-i], -K[i])\right).$$

As K is of class C^∞_+, ξ^K is indeed smooth (see, e.g., [31]). Note that ξ^K is (a scalar multiple of) the even part of the mixed volume $V(L[n-i], K[i])$, which was used in [30] for the same purpose. Calculating $\xi^K(L)$ for $L \subseteq E \in \mathrm{Gr}(n, n-i)$ using [109, Theorem 5.3.1],

$$\xi^K(L) = \frac{1}{2}\left(V_{n-i}(L)V_i(K|E^\perp) + V_{n-i}(L)V_i(-K|E^\perp)\right) = V_i(K|E^\perp)V_{n-i}(L), \tag{5.17}$$

we conclude that $\widetilde{\mathrm{Kl}}_{n-i}\psi^K = \langle \mathrm{Kl}_{n-i}\,\xi^K, \cdot\rangle_{L^2}$. Consequently, $\psi^K = \mathrm{pd}\,\xi^K$, by the injectivity of $\widetilde{\mathrm{Kl}}_{n-i}$. Equation (5.16) then reduces to

$$\varphi(K) = \langle \xi^K, \varphi\rangle_A = \langle \mathrm{pd}\,\varphi, \xi^K\rangle_{\mathrm{Val}^{-\infty}}, \quad \varphi \in \mathrm{Val}_i^{+,\infty}, \tag{5.18}$$

which extends to $\varphi \in \mathrm{Val}_i^+$ using $\widetilde{\mathrm{pd}}$.

The claim (5.15) finally follows from (5.18), (5.14) and (5.17),

$$\varphi(K) = \langle \widetilde{\mathrm{Cr}}_i\psi, \xi^K\rangle_{\mathrm{Val}^{-\infty}} = \langle \psi, (\mathrm{Kl}_{n-i}\,\xi^K)^\perp\rangle_{C^{-\infty}} = \langle \psi, V_i(K|\cdot)\rangle_{C^{-\infty}}.$$

Note that the right-hand side is well-defined since $V_i(K|\cdot) \in C^\infty(\mathrm{Gr}(n,i))$ as Klain function of the smooth valuation ξ^K. $\square$

Using Theorem 5.2.1 and Lemma 5.2.2, we are now in position to prove Proposition 5.0.1.

Proof of Proposition 5.0.1. Let Φ_i be a continuous, translation invariant, $\mathrm{SO}(n)$ equivariant, and even Minkowski valuation of a given degree i. Then the associated real-valued valuation φ_i is in Val_i^+ and $\mathrm{SO}(n-1)$ invariant. By the (extended) Poincaré duality map $\widetilde{\mathrm{pd}}$, we embed φ_i into the space of generalized valuations. Hence, by Theorem 5.2.1, there exists $\psi \in C^{-\infty}(\mathrm{Gr}(n,i))$ such that $\widetilde{\mathrm{pd}}\,\varphi_i = \widetilde{\mathrm{Cr}}_i\psi$.

Without loss of generality, we assume that ψ is $\mathrm{SO}(n-1)$ invariant. Indeed, let ψ be in the preimage of φ_i and consider the distribution $\bar{\psi}$, defined by

$$\langle \bar{\psi}, f\rangle_{C^{-\infty}} = \langle \psi, \bar{f}\rangle_{C^{-\infty}} \quad \forall f \in C^{\infty}(\mathrm{Gr}(n,i)),$$

where $\bar{f}$ is defined by (2.1). As $\bar{f}$ can be obtained by a limit of finite means of rotations ϑf of f, where the choice of the $\vartheta \in \mathrm{SO}(n-1)$ is independent of f, $\bar{\psi}$ is a limit of means of rotations of ψ, as well. More precisely, there exists a sequence $(\vartheta_j^{(N)})_{N\in\mathbb{N}, j\leq N}$ of transformations in $\mathrm{SO}(n-1)$ such that

$$\bar{\psi} = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \vartheta_j^{(N)}\psi.$$

By the continuity, linearity and $\mathrm{SO}(n)$ equivariance of $\widetilde{\mathrm{Cr}}_i$, we thus have

$$\widetilde{\mathrm{Cr}}_i\bar{\psi} = \widetilde{\mathrm{Cr}}_i\left(\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \vartheta_j^{(N)}\psi\right) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \vartheta_j^{(N)}\widetilde{\mathrm{Cr}}_i\psi,$$

which is equal to $\widetilde{\mathrm{Cr}}_i\psi = \widetilde{\mathrm{pd}}\varphi_i$ by the $\mathrm{SO}(n-1)$ invariance of φ_i.
Letting now $K \in \mathcal{K}^n$ be of class C^∞_+, Lemma 5.2.2 and (2.2) imply, for $\eta \in \mathrm{SO}(n)$,

$$\begin{aligned} h(\Phi_i K, \eta\bar{e}) &= \varphi_i(\eta^{-1}K) = \langle \psi, \ell_{\eta^{-1}} V_i(K|\cdot)\rangle_{C^{-\infty}} \\ &= \langle \hat{\psi}, r_{\eta^{-1}}\widehat{V_i(K|\cdot)}\rangle_{C^{-\infty}} = (V_i(K|\cdot) * \hat{\psi})(\eta\bar{e}). \end{aligned}$$

As ψ is $\mathrm{SO}(n-1)$ invariant, $\hat{\psi}$ is an $\mathrm{S}(\mathrm{O}(i)\times\mathrm{O}(n-i))$ invariant distribution on the sphere and we obtain (5.5) for $\delta = \hat{\psi}$.

It remains to show the uniqueness of the spherical Crofton distribution $\delta = \hat{\psi}$. For this reason, first note that since ψ is $\mathrm{SO}(n-1)$ invariant and, therefore, $\langle \psi, f\rangle_{C^{-\infty}} = \langle \psi, \bar{f}\rangle_{C^{-\infty}}$, $f \in C^\infty(\mathrm{Gr}(n,i))$, it is completely determined by its value on $\mathrm{SO}(n-1)$ invariant functions in $C^\infty(\mathrm{Gr}(n,i))$.

Next, we apply the (extended) Klain map to $\widetilde{\mathrm{pd}}\,\varphi_i$ and use Theorem 5.2.1 to obtain for $f \in C^\infty(\mathrm{Gr}(n,i))$

$$\langle \widetilde{\mathrm{Kl}}_i(\widetilde{\mathrm{pd}}\,\varphi_i), f\rangle_{C^{-\infty}} = \langle \widetilde{\mathrm{Kl}}_i(\widetilde{\mathrm{Cr}}\,\psi), f\rangle_{C^{-\infty}} = \langle \psi, C_i f\rangle_{C^{-\infty}}. \tag{5.19}$$

By [115, Lemma 3.3], the Cosine transform C_i applied to $\mathrm{SO}(n-1)$ invariant smooth functions $f \in C^\infty(\mathrm{Gr}(n,i))$ is given by

$$C_i f = \frac{n\kappa_i\kappa_{n-i}}{2\kappa_{n-1}}\binom{n}{i}^{-1}\widehat{C\hat{f}},$$

when interpreting $\hat{f}$ as a $\mathrm{S}(\mathrm{O}(i)\times\mathrm{O}(n-i))$ invariant smooth function on the sphere. Moreover, it is proved in [115, Lemma 3.3] that all such $\hat{f}$ are even. Since C is bijective on even functions from $C^\infty(\mathbb{S}^{n-1})$ (see, e.g., [59, Section 3.4]), C_i is surjective onto the space of $\mathrm{SO}(n-1)$ invariant smooth functions on $\mathrm{Gr}(n,i)$. Consequently, (5.19) determines ψ completely. □

5.3 Proof of Theorem 5.0.2

The representation result from Proposition 5.0.1 allows to prove sharp isoperimetric inequalities for Minkowski valuations with non-negative Crofton distributions. In this section, we will first give a proof of the right-hand inequality of Theorem 5.0.2 and deduce from it Corollary 5.0.3 using the inequality of Milman and Yehudayoff (5.3). The proof of the left-hand inequality of Theorem 5.0.2 will be postponed to Section 5.4. In the remainder of this section, we will show by example that Theorem 5.0.2 generalizes the results by Berg and Schuster [25].

The next theorem establishes the right-hand inequality of Theorem 5.0.2 for the intermediate degrees of homogeneity, $2 \le i \le n-2$. As was noted earlier, the case $i=1$ follows from Theorem 3.0.1, whereas the case $i=n-1$ was proved in [61].

Theorem 5.3.1. *Suppose that the spherical Crofton distribution of a non-trivial even Minkowski valuation $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$, $2 \le i \le n-2$, is non-negative.*

If $K \in \mathcal{K}^n$ has non-empty interior, then

$$V_n(\Phi_i^\circ K) A_{n-i}(K)^n \le V_n(\Phi_i^\circ B^n)\kappa_n^n. \tag{5.20}$$

Equality holds if and only if the ith projection function of K is constant, that is, there exists $c \in \mathbb{R}$ such that $V_i(K|F) = c$, for all $F \in \mathrm{Gr}(n,i)$.

Proof. By assumption, the Crofton distribution δ of Φ_i is non-negative and, by Proposition 5.0.1, $\Phi_i K$ can be represented by

$$h(\Phi_i K, \cdot) = V_i(K|\cdot) * \delta, \tag{5.21}$$

for every $K \in \mathcal{K}^n$ of class C^∞_+. As δ is non-negative, δ is, in fact, a non-negative measure on $\mathbb{S}^{n-1}$ and the representation (5.21) also holds for arbitrary $K \in \mathcal{K}^n$. Denoting the measure $\hat{\delta}$ on $\mathrm{Gr}(n,i)$ by μ, we obtain for every $K \in \mathcal{K}^n$

$$h(\Phi_i K, u) = \int_{\mathrm{Gr}(n,i)} V_i(K|\vartheta_u F)\, d\mu(F). \tag{5.22}$$

As (5.20) is invariant under scaling of Φ_i, we may assume that μ is a probability measure or, equivalently, that $\Phi_i B^n = \kappa_i B^n$.

Note that, by (5.22) and since μ is non-negative, Φ_i is monotone with respect to set-inclusion. Consequently, $\Phi_i K$ contains the origin in its interior whenever

$K \in \mathcal{K}_0^n$. By the polar volume formula (2.12) and (5.22),

$$V_n(\Phi_i^\circ K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(\Phi_i K, u)^{-n}\, du = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \left(\int_{\mathrm{Gr}(n,i)} V_i(K|\vartheta_u F)\, d\mu(F)\right)^{-n} du,$$

which can be estimated using Jensen's inequality (as μ is a probability measure) by

$$V_n(\Phi_i^\circ K) \leq \frac{1}{n}\int_{\mathbb{S}^{n-1}}\int_{\mathrm{Gr}(n,i)} V_i(K|\vartheta_u F)^{-n}\, d\mu(F)\, du. \tag{5.23}$$

Since both Φ_i and the polar map are $\mathrm{SO}(n)$ equivariant, the polar volume $V_n(\Phi_i^\circ K)$ is invariant under rotations of K. Consequently, we can replace K in (5.23) by $\vartheta_E^{-1}\tau K$, for $E \in \mathrm{Gr}(n,i)$ and $\tau \in \mathrm{SO}(n)$, and integrate with respect to the Haar measures on $\mathrm{Gr}(n,i)$ and $\mathrm{SO}(n)$ to obtain

$$\begin{aligned} V_n(\Phi_i^\circ K) &= \int_{\mathrm{SO}(n)}\int_{\mathrm{Gr}(n,i)} V_n(\Phi_i^\circ(\vartheta_E^{-1}\tau K))\, d\nu_i(E)\, d\tau \\ &\leq \frac{1}{n}\int_{\mathrm{SO}(n)}\int_{\mathrm{Gr}(n,i)}\int_{\mathbb{S}^{n-1}}\int_{\mathrm{Gr}(n,i)} V_i(\vartheta_E^{-1}\tau K|\vartheta_u F)^{-n}\, d\mu(F)\, du\, d\nu_i(E)\, d\tau. \end{aligned}$$

By Fubini's theorem and the $\mathrm{SO}(n)$ invariance of the Haar measure on $\mathrm{SO}(n)$,

$$\begin{aligned} V_n(\Phi_i^\circ K) &\leq \frac{1}{n}\int_{\mathrm{Gr}(n,i)}\int_{\mathbb{S}^{n-1}}\int_{\mathrm{Gr}(n,i)}\int_{\mathrm{SO}(n)} V_i(K|\tau^{-1}\vartheta_E\vartheta_u\vartheta_F\bar{E}_i)^{-n}\, d\tau\, d\mu(F)\, du\, d\nu_i(E) \\ &= \frac{1}{n}\int_{\mathrm{Gr}(n,i)}\int_{\mathbb{S}^{n-1}}\int_{\mathrm{Gr}(n,i)}\int_{\mathrm{SO}(n)} V_i(K|\eta^{-1}E)^{-n}\, d\eta\, d\mu(F)\, du\, d\nu_i(E), \end{aligned}$$

which, as $\mu(\mathrm{Gr}(n,i)) = 1$ and $S(B^n) = n\kappa_n$, reduces to

$$V_n(\Phi_i^\circ K) \leq \kappa_n \int_{\mathrm{Gr}(n,i)}\int_{\mathrm{SO}(n)} V_i(K|\eta^{-1}E)^{-n}\, d\eta\, d\nu_i(E).$$

Using again Fubini's theorem, (2.18), the $\mathrm{SO}(n)$ invariance of the affine quermassintegrals and the fact that $V_n(\Phi_i^\circ B^n) = \frac{\kappa_n}{\kappa_i^n}$, we finally obtain the desired inequality

$$V_n(\Phi_i^\circ K) \leq \frac{\kappa_n^{n+1}}{\kappa_i^n}\int_{\mathrm{SO}(n)} A_{n-i}(\eta K)^{-n}\, d\eta = \frac{\kappa_n^{n+1}}{\kappa_i^n} A_{n-i}(K)^{-n} = V_n(\Phi_i^\circ B^n)\kappa_n^n A_{n-i}(K)^{-n}.$$

It remains to prove the equality cases. By the above arguments, equality holds in (5.20) if and only if we have equality in Jensen's inequality in (5.23). Using the equality condition of Jensen's inequality, equality holds if and only if for almost every (with respect to the uniform measures on $\mathrm{SO}(n)$, $\mathrm{Gr}(n,i)$ and $\mathbb{S}^{n-1}$, respectively) $\tau \in \mathrm{SO}(n)$, $E \in \mathrm{Gr}(n,i)$ and $u \in \mathbb{S}^{n-1}$ there exist $c_{\tau,E,u} \in \mathbb{R}$ such that

$$V_i(K|\tau^{-1}\vartheta_E\vartheta_u F) = c_{\tau,E,u} \quad \text{for } \mu\text{-a.e. } F \in \mathrm{Gr}(n,i).$$

Since the map $\eta \mapsto V_i(K|\eta F)$ is continuous on $\mathrm{SO}(n)$, this holds for every $\tau \in \mathrm{SO}(n)$, $E \in \mathrm{Gr}(n,i)$ and $u \in \mathbb{S}^{n-1}$. Choosing $E = \bar{E}_i$, $u = \bar{e}$ and $\tau = \eta^{-1}$ (that is, $\vartheta_E = \mathrm{Id} = \vartheta_u$), this can be further reduced to

$$\forall \eta \in \mathrm{SO}(n) \exists c_\eta \in \mathbb{R}: \quad V_i(K|\eta F) = c_\eta \quad \text{for } \mu\text{-a.e. } F \in \mathrm{Gr}(n,i).$$

This is exactly the condition for Theorem 5.7.3 in the appendix which now yields the claimed equality cases. □

By Milman and Yehudayoff's inequality for the affine quermassintegrals (2.29), Theorem 5.3.1 directly implies Corollary 5.0.3.

Proof of Corollary 5.0.3. The cases $i = 1$ and $i = n-1$ follow from Theorem 3.0.1 and [61, Theorem 1], respectively. For $2 \leq i \leq n-2$, we can estimate by Theorem 5.3.1 and (2.29),

$$V_n(\Phi_i^\circ K)V_n(K)^i \leq V_n(\Phi_i^\circ B^n)\kappa_n^n \frac{V_n(K)^i}{A_{n-i}(K)^n} \leq V_n(\Phi_i^\circ B^n)\kappa_n^n \kappa_n^{i-n},$$

for every $K \in \mathcal{K}^n$ with non-empty interior. Equality holds in the left-hand inequality if and only if K has constant ith projection function and in the right-hand inequality exactly if K is an ellipsoid. As the only ellipsoids with constant ith projection function are Euclidean balls, the claimed equality cases follow. □

As mentioned before, Corollary 5.0.3 generalizes earlier results by Berg and Schuster [25]. Indeed, they proved Theorem 5.0.2 for all even Minkowski valuations generated by support functions of zonoids of revolution. By the remark after Theorem 5.1.3, every such Minkowski valuation has a non-negative spherical Crofton measure. We will show in the following that, in fact, there are even Minkowski valuations with non-negative spherical Crofton measure not generated by zonoids. More precisely, we give an example of a class of generalized zonoids L_α (which are not zonoids) such that the Minkowski valuation defined by

$$h(\Phi_i K, \cdot) = S_i(K, \cdot) * h(L_\alpha, \cdot)$$

has non-negative spherical Crofton distribution. This example is a generalization of [52, Rem. 4.1.14] and [104, p. 69] (see also [48, Ex. 5.2(f)]).

Example 5.3.2. Let $P_2^n(t) = \frac{1}{n-1}(nt^2 - 1)$ be the second Legendre polynomial and consider the function $f_\alpha(t) = 1 + \alpha P_2^n(t)$, for $\alpha \in \mathbb{R}$. This function can be lifted to a function on the unit sphere by setting $h_\alpha(u) = f_\alpha(\langle u, \bar{e}\rangle)$, for $u \in \mathbb{S}^{n-1}$.

h_α defines a support function of a convex body of revolution L_α (if extended 1-homogeneously to $\mathbb{R}^n$) exactly for $\alpha \in \left[\frac{n-1}{1-2n}, \frac{n-1}{n+1}\right]$. Indeed, the extension of h_α is subadditive if and only if

$$\alpha \left(\frac{1}{n-1} + \frac{n}{n-1} \left(\frac{(\langle u, \bar{e}\rangle + \langle v, \bar{e}\rangle)^2 - \|u+v\|(\langle u, \bar{e}\rangle^2 + \langle v, \bar{e}\rangle^2)}{\|u+v\|(2 - \|u+v\|)} \right) \right) \leq 1,$$

for all $u, v \in \operatorname{span}\{\bar{e}, e_2\}$, where $e_2 \in \mathbb{S}^{n-1} \cap \bar{e}^{\perp}$ is an arbitrary unit vector (we are dealing with a body of revolution). One can show using e.g. optimization with constraints (Lagrange multipliers) that we have

$$-2 \leq \left(\frac{(\langle u, \bar{e} \rangle + \langle v, \bar{e} \rangle)^2 - \|u+v\|(\langle u, \bar{e} \rangle^2 + \langle v, \bar{e} \rangle^2)}{\|u+v\|(2 - \|u+v\|)} \right) \leq 1,$$

for all $u, v \in \operatorname{span}\{\bar{e}, e_2\}$. Using this fact, we arrive at the claimed interval.

For α in the given interval, we thus get a one-parametric family of convex bodies where the support function is a sum of two (even) spherical harmonics. Hence, h_α lies in the image of the cosine transform, that is,

$$h_\alpha(u) = 1 + \alpha P_2^n(\langle u, \bar{e} \rangle) = C\left(\frac{1}{2\kappa_{n-1}}(1 + (n+1)\alpha P_2^n(\langle \cdot, \bar{e} \rangle)) \right),$$

where we used the multipliers of the cosine transform (see, e.g., [59, Lemma 3.4.5]).

Whenever h_α is a support function, the resulting convex body is therefore (by definition) a generalized zonoid. It is a zonoid, if the preimage of h_α under the cosine transform is non-negative, that is, by a direct calculation, exactly for $\alpha \geq -\frac{1}{n+1}$.

By Lemma 5.1.4, the (generalized) zonoid with support function h_α generates a Minkowski valuation with non-negative Crofton distribution, if and only if

$$\int_0^1 (1 + (n+1)\alpha P_2^n(\tau t))\,(1 - t^2)^{\frac{n-i-3}{2}}\, dt \geq 0 \quad \forall \tau \in [0, 1].$$

Calculating the integral using the Beta function yields that this condition is satisfied exactly for $\frac{n+1}{n-1}\alpha \in \left[-\frac{n-i}{i}, 1\right]$.

Finally, this gives examples for generalized zonoids which positively generate Minkowski valuations and are not zonoids for the range $\alpha \in \left[-\frac{n-i}{i}\frac{n-1}{n+1}, -\frac{1}{n+1}\right) \cap \left[\frac{n-1}{1-2n}, \frac{n-1}{n+1}\right] = \left[\max\left\{-\frac{n-i}{i}\frac{n-1}{n+1}, \frac{n-1}{1-2n}\right\}, -\frac{1}{n+1}\right)$. For a fixed i, we asymptotically get the interval $\left[-\frac{n-1}{2n-1}, -\frac{1}{n+1}\right)$, while for fixed n and $i < n-1$ the interval always has positive length.

5.4 Existence Results

In this section, we prove general criteria for two classes of isoperimetric inequalities to have extremals. Namely, whenever some "sufficiently nice" geometric functional can be estimated by the isoperimetric ratio $\frac{W_{n-i}(K)^n}{V_n(K)^i}$, $1 \leq i \leq n-1$, this is enough to conclude the existence of extremals. We will then apply these results to obtain Theorems 5.0.5 and 5.0.6.

The results in this section are joint work with P. Kniefacz and F.E. Schuster and will be published in [64]. Partial results of this joint work (for $(n-1)$-homogeneous Minkowski valuations) were already published in [72].

The proofs of the main theorems in this section are based on Blaschke's selection theorem and the following property of the isoperimetric ratio. This lemma already appeared in [72]. In order to be self-contained, we repeat the proof.

Lemma 5.4.1 ([64, 72]). *Suppose that $(K_j)_{j\in\mathbb{N}} \subseteq \mathcal{K}_0^n$ is a sequence of convex bodies with non-empty interior. If $(K_j)_{j\in\mathbb{N}}$ converges to a convex body $K_0 \in \mathcal{K}^n$ of dimension $1 \leq \dim K_0 \leq n-1$, then the isoperimetric ratio tends to infinity,*

$$\frac{S(K_j)^n}{V_n(K_j)^{n-1}} \to \infty, \qquad j \to \infty.$$

Proof. We prove the lemma by finding an appropriate minorant to the sequence. For this reason, we use (2.28) to estimate

$$\frac{S(K_j)^n}{V_n(K_j)^{n-1}} = \frac{S(K_j)}{V_n(K_j)} \frac{S(K_j)^{n-1}}{V_n(K_j)^{n-2}} \geq \frac{S(K_j)}{V_n(K_j)} \frac{\operatorname{diam} K_j}{c(n)},$$

where $c(n)$ is a constant depending only on the dimension. Using the isoperimetric inequality (2.26) for the first term, we obtain

$$\frac{S(K_j)}{V_n(K_j)} \geq n\kappa_n^{\frac{1}{n}} \frac{V_n(K_j)^{\frac{n-1}{n}}}{V_n(K_j)} = n\kappa_n^{\frac{1}{n}} \frac{1}{V_n(K_j)^{\frac{1}{n}}}.$$

Overall, the isoperimetric ratio is bounded from below by

$$\frac{S(K_j)^n}{V_n(K_j)^{n-1}} \geq \frac{n\kappa_n^{\frac{1}{n}}}{c(n)} \operatorname{diam} K_j \frac{1}{V_n(K_j)^{\frac{1}{n}}},$$

which tends to infinity as $\operatorname{diam} K_j \to \operatorname{diam} K_0 \neq 0$ and $V_n(K_j) \to V_n(K_0) = 0$ by our assumptions on the dimension of K_0. $\square$

The isoperimetric inequalities for quermassintegrals (2.24) directly imply that the same is true for the surface area replaced by the $(n-i)$th quermassintegral.

Corollary 5.4.2. *Suppose that $(K_j)_{j\in\mathbb{N}} \subseteq \mathcal{K}_0^n$ is a sequence of convex bodies with non-empty interior. If $(K_j)_{j\in\mathbb{N}}$ converges to a convex body $K_0 \in \mathcal{K}^n$ of dimension $1 \leq \dim K_0 \leq n-1$, then the ith isoperimetric ratio tends to infinity,*

$$\frac{W_{n-i}(K_j)^n}{V_n(K_j)^i} \to \infty, \qquad j \to \infty.$$

Proof. By (2.24) applied for $j = i$ and $i = n-1$, that is,

$$\kappa_n^{-n+1} W_{n-i}(K)^{n-1} \geq \kappa_n^{-i} W_1(K)^i,$$

the ith isoperimetric ratio is minorized by the isoperimetric ratio

$$\frac{W_{n-i}(K_j)^n}{V_n(K_j)^i} \geq \frac{\left(\kappa_n^{n-i-1} W_1(K_j)^i\right)^{\frac{n}{n-1}}}{V_n(K_j)^i} = \kappa_n^{\frac{(n-i-1)n}{n-1}} \left(\frac{W_1(K_j)^n}{V_n(K_j)^{n-1}}\right)^{\frac{i}{n-1}}.$$

Lemma 5.4.1 and the fact that $W_1(K) = \frac{1}{n}S(K)$ now imply the claim. $\square$

If the (geometric) functional under consideration can be bounded by an isoperimetric ratio, then no sequence of convex bodies converging to some lower-dimensional convex body can be an extremizing sequence by the asymptotic behavior of this isoperimetric ratio. Formalizing this statement, we obtain the following theorem.

Theorem 5.4.3. *Suppose that $P_1, P_2 : \mathcal{K}_0^n \to (0,\infty)$ are two continuous, translation invariant functionals on the set of convex bodies with non-empty interior, homogeneous of the same degree $i \in \{1,\dots,n-1\}$.*

If the fraction $\frac{P_2(K)^n}{V_n(K)^i}$ is bounded from above and the fraction $\frac{P_1(K)}{W_{n-i}(K)}$ is bounded from below, then there exists a convex body $K_0 \in \mathcal{K}_0^n$ with non-empty interior that minimizes the fraction $\frac{P_1(K)}{P_2(K)}$ for $K \in \mathcal{K}_0^n$.

Proof. We will use the two assumptions in form of the following inequalities

$$P_2(K)^n \le C V_n(K)^i \quad \text{and} \quad P_1(K) \ge c W_{n-i}(K),$$

where $c, C > 0$ are some constants and $K \in \mathcal{K}_0^n$ is arbitrary. Both inequalities together, combined with the isoperimetric inequality for the quermassintegral (2.25), directly imply that $\frac{P_1}{P_2}$ is bounded from below by

$$\frac{P_1(K)}{P_2(K)} \ge \frac{c}{C}\left(\frac{W_{n-i}(K)^n}{V_n(K)^i}\right)^{\frac{1}{n}} \ge \frac{c}{C}\kappa_n^{\frac{n-i}{n}} > 0. \tag{5.24}$$

We therefore can choose a sequence $(K_j)_{j\in\mathbb{N}} \subseteq \mathcal{K}_0^n$, such that

$$\frac{P_1(K_j)}{P_2(K_j)} \to \inf_{K\in\mathcal{K}_0^n}\frac{P_1(K)}{P_2(K)} > 0, \qquad j\to\infty.$$

By the translation and scaling invariance of the fraction $\frac{P_1}{P_2}$ (implied by the same degree of homogeneity), we may assume that every K_j is contained in the unit ball and contains at least a segment of length one. Using Blaschke's selection theorem we obtain that K_j possesses a subsequence converging to a convex body K_0, which, by the previous scaling argument, has a dimension of at least one. To simplify notation, we denote this subsequence again by K_j.

Proceeding by contradiction, we assume that $\dim K_0 < n$. The left-hand inequality of (5.24) and Corollary 5.4.2 show that in this case, the fraction must tend to infinity

$$\frac{P_1(K_j)}{P_2(K_j)} \ge \frac{c}{C}\left(\frac{W_{n-i}(K_j)^n}{V_n(K_j)^i}\right)^{\frac{1}{n}} \to \infty, \qquad j\to\infty,$$

which contradicts the assumption that the sequence $(K_j)_{j\in\mathbb{N}}$ is a minimizing sequence. Hence, K_0 has non-empty interior and by the continuity of P_1 and P_2, K_0 must be a minimizer. $\square$

In the following, the functional P_2 of the theorem will mostly be the n-dimensional volume (to some power) or an affine quermassintegral. For these choices, the assumption on P_2 is trivially fulfilled. Note also that the theorem does not say anything about the explicit form of the minimizer K_0. It is an interesting question to find sufficient conditions for functionals to have a specific class of minimizers (e.g., Euclidean balls).

We will also need the following "reciprocal" version of Theorem 5.4.3, which is a direct consequence of Theorem 5.4.3.

Corollary 5.4.4. *Suppose that $P_1, P_2 : \mathcal{K}_0^n \to (0, \infty)$ are two continuous, translation invariant functionals on the set of convex bodies with non-empty interior, where P_1 is $(-i)$-homogeneous and P_2 is i-homogeneous, for some $i \in \{1, \dots, n-1\}$.*

If the fraction $\frac{P_2(K)^n}{V_n(K)^i}$ and the product $P_1(K)W_{n-i}(K)$ are bounded from above, then there exists a convex body $K_0 \in \mathcal{K}_0^n$ with non-empty interior that maximizes the the product $P_1(K)P_2(K)$ for $K \in \mathcal{K}_0^n$.

Proof. We take $\tilde{P}_1(K) = \frac{1}{P_1(K)}$ and $\tilde{P}_2(K) = P_2(K)$ and apply Theorem 5.4.3. $\square$

Having built the general tools, we are now in position to prove Theorems 5.0.5 and 5.0.6. We first prove the special case of Theorem 5.0.6 for generating functions f which are bounded from below by positive constants. Note that, as $f \in L^1(\mathbb{S}^{n-1})$, this has to be interpreted as $f(u) \geq c > 0$ for σ-almost all $u \in \mathbb{S}^{n-1}$. In particular, this implies that $f^{-n} \in L^1(\mathbb{S}^{n-1})$. Moreover, if $f \in L^1(\mathbb{S}^{n-1})$ is the generating function of a non-trivial Minkowski valuation Φ_i, then

$$h(\Phi_i B^n, u) = n\kappa_n \int_{\mathbb{S}^{n-1}} f(v)d\sigma(v) \neq 0, \qquad u \in \mathbb{S}^{n-1}.$$

Theorem 5.4.5. *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \dots, n-1\}$.*

If the generating function f of Φ_i is bounded from below by $c > 0$, then for $K \in \mathcal{K}_0^n$

$$\kappa_n \int_{\mathbb{S}^{n-1}} f(u)^{-n} d\sigma(u) \geq V_n(\Phi_i^\circ K) n^n W_{n-i}(K)^n \geq \kappa_n \left(\int_{\mathbb{S}^{n-1}} f(u)d\sigma(u) \right)^{-n}, \quad (5.25)$$

and $V_n(\Phi_i^\circ K)V_n(K)^i$ attains a maximum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior. There is equality in the right-hand inequality of (5.25) if and only if $\Phi_i K$ is a ball centered at the origin.

For $\Phi_i = \Phi_i^L$ with $L \in \mathcal{K}^n$ with non-empty interior, (5.25) reads

$$V_n(L^\circ) \geq V_n(\Phi_i^{L,\circ} K) n^n W_{n-i}(K)^n \geq \kappa_n^{n+1} W_{n-1}(L)^{-n}.$$

Proof. By the polar volume formula (2.12), the representation of $h(\Phi_i K, \cdot)$ and Jensen's inequality (with $S_i(K, \mathbb{S}^{n-1}) = nW_{n-i}(K)$),

$$\begin{aligned} V_n(\Phi_i^\circ K) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} (h(\Phi_i K, u)^{-n} \, du \\ &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} (nW_{n-i}(K))^{-n} \left(\int_{\mathbb{S}^{n-1}} f(\vartheta_v^{-1} u) \frac{dS_i(K,v)}{nW_{n-i}(K)} \right)^{-n} du \\ &\leq \frac{(nW_{n-i}(K))^{-n-1}}{n} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} f(\vartheta_v^{-1} u)^{-n} \, dS_i(K, v) \, du. \end{aligned}$$

Interchanging the integrals by Fubini's theorem and using that the inner integral does not depend on v anymore, the left-hand inequality of (5.25) follows

$$\begin{aligned} V_n(\Phi_i^\circ K) &\leq (nW_{n-i}(K))^{-n} \int_{\mathbb{S}^{n-1}} \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(\vartheta_v^{-1} u)^{-n} \, du \, \frac{dS_i(K,v)}{nW_{n-i}(K)} \\ &= (nW_{n-i}(K))^{-n} \left(\frac{1}{n} \int f(u)^{-n} \, du \right). \end{aligned}$$

The right-hand inequality is a consequence of Jensen's inequality and Lemma 5.1.5,

$$\begin{aligned} V_n(\Phi_i^\circ K) = \kappa_n \int_{\mathbb{S}^{n-1}} h(\Phi_i K, u)^{-n} \, d\sigma(u) &\geq \kappa_n \left(\int_{\mathbb{S}^{n-1}} h(\Phi_i K, u) \, d\sigma(u) \right)^{-n} \\ = \kappa_n^{n+1} W_{n-1}(\Phi_i K)^{-n} &= \kappa_n^{n+1} r_{\Phi_i}^{-n} W_{n-i}(K)^{-n}. \end{aligned}$$

Equality holds in the last inequality if and only if $h(\Phi_i K, \cdot)$ is constant, that is, $\Phi_i K$ is a ball centered at the origin, by the equality cases of Jensen's inequality.

For the existence of maximizers, we apply Corollary 5.4.4 with $P_1(K) = V_n(\Phi_i^\circ K)^{\frac{1}{n}}$ and $P_2(K) = V_n(K)^{\frac{i}{n}}$. The conditions of the corollary are asserted by the left-hand inequality of (5.25). □

Let us note that the proof of the right-hand inequality only needs that $h(\Phi_i K, \cdot)$ is positive, that is, that $\Phi_i K$ contains the origin in its interior, whenever $K \in \mathcal{K}_0^n$. Since Minkowski valuations with non-negative spherical Crofton distribution are monotone with respect to set inclusion, and every $\mathrm{SO}(n)$ equivariant and translation invariant Minkowski valuation maps Euclidean balls to Euclidean balls (of positive radii by the scaling assumption in Theorem 5.0.2) centered at the origin, we have therefore proved the left-hand inequality of Theorem 5.0.2. This completes the proof of Theorem 5.0.2.

The proof of Theorem 5.0.5 now follows immediately from Theorem 5.4.5.

Proof of Theorem 5.0.5. Let Φ_i be an i-homogeneous Minkowski valuation with generating function $f = f_1 + f_2$, where f_1 and f_2 generate Minkowski valuations Φ^1 and Φ^2, respectively, and $f_1 \geq c > 0$ for some $c > 0$. Then, by the additivity of the convolution and the fact that $h(\Phi^2 K, \cdot) \geq 0$ as the Steiner point of $\Phi^2 K$ lies at the origin (see, e.g., [14, 24, 95]) and is contained in relint $\Phi^2 K$,

$$h(\Phi_i K, \cdot) = h(\Phi^1 K, \cdot) + h(\Phi^2 K, \cdot) \geq h(\Phi^1 K, \cdot).$$

Consequently, $\Phi_i K \supseteq \Phi^1 K$ and $V_n(\Phi_i^\circ K) \leq V_n(\Phi^{1,\circ} K)$, for every $K \in \mathcal{K}_0^n$. The estimate (5.25) from Theorem 5.4.5 applied to Φ^1 now implies the assumptions of Corollary 5.4.4 for Φ_i, yielding the claim.

The second part of the theorem will be postponed and follows from Theorem 5.5.1. □

In the view of Theorem 5.0.2, we further mention an easy corollary of Theorem 5.4.5.

Corollary 5.4.6. *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \ldots, n-1\}$.*

If the generating function of Φ_i is bounded from below by a positive constant, then

$$V_n(\Phi_i^\circ K) A_{n-i}(K)^n$$

attains a maximum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior.

Proof. We take $P_1(K) = V_n(\Phi_i^{L,\circ} K)^{\frac{1}{n}}$ and $P_2(K) = A_{n-i}(K)$ and apply Corollary 5.4.4. By the last theorem, P_1 satisfies the boundedness assumption. The fraction $\frac{P_2(K)^n}{V_n(K)^i}$ is affine invariant and hence possesses maximizers and minimizers. □

We turn now to the non-polar case. Here, the proof of Theorem 5.0.6 needs some different estimates, but otherwise follows similar arguments as the proof of Theorem 5.0.5. As before, we first prove the special case of Theorem 5.0.6 for generating functions f that are bounded from below by positive constants.

Theorem 5.4.7. *Suppose that $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of a given degree $i \in \{1, \ldots, n-1\}$.*

If the generating function f of Φ_i is bounded from below by $c > 0$, then for $K \in \mathcal{K}_0^n$

$$\kappa_n c^n \leq \frac{V_n(\Phi_i K)}{n^n W_{n-i}(K)^n} \leq \kappa_n \left(\int_{\mathbb{S}^{n-1}} f(u) d\sigma(u) \right)^n, \tag{5.26}$$

and $\frac{V_n(\Phi_i K)}{V_n(K)^i}$ attains a minimum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior. There is equality in the right-hand inequality of (5.26) if and only if $\Phi_i K$ is a ball centered at the origin.

Moreover, if $\Phi_i = \Phi_i^L$ for some convex body $L \in \mathcal{K}^n$ of revolution with non-empty interior, then (5.26) can be strengthened for $K \in \mathcal{K}_0^n$ to

$$V_n(L) \leq \frac{V_n(\Phi_i^L K)}{n^n W_{n-i}(K)^n} \leq \frac{W_{n-1}(L)^n}{\kappa_n^{n-1}}. \tag{5.27}$$

Proof. We first prove the chain of inequalities in (5.27), that is, for $\Phi_i = \Phi_i^L$. Using (2.16), (5.12), Fubini's theorem and again (2.16), we calculate

$$\begin{aligned} V_n(\Phi_i K) &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(\Phi_i K, u)\, dS_{n-1}(\Phi_i K, u) \\ &= \int_{\mathbb{S}^{n-1}} \frac{1}{n}\int_{\mathbb{S}^{n-1}} h(L(v), u)\, dS_{n-1}(\Phi_i K, u)\, dS_i(K, v) \\ &= \int_{\mathbb{S}^{n-1}} V(L(v), \Phi_i K[n-1])\, dS_i(K, v). \end{aligned}$$

The integrand can be estimated using Minkowski's inequality (2.19) to obtain

$$V_n(\Phi_i K) \geq \int_{\mathbb{S}^{n-1}} V_n(L)^{\frac{1}{n}} V_n(\Phi_i K)^{\frac{n-1}{n}}\, dS_i(K, v) = V_n(L)^{\frac{1}{n}} V_n(\Phi_i K)^{\frac{n-1}{n}} n W_{n-i}(K).$$

Rearranging this inequality yields the left-hand inequality of (5.27).

In the general case, that is, for Φ_i generated by $f \in L^1(\mathbb{S}^{n-1})$ bounded from below by $c > 0$, the left-hand inequality of (5.26) is proved in the same way, replacing the application of Minkowski's inequality by the (rougher) estimate

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}} f(\vartheta_v^{-1} u)\, dS_{n-1}(\Phi_i K, u) \geq c W_1(\Phi_i K) \geq c\kappa_n^{\frac{1}{n}} V_n(\Phi_i K[n-1])^{\frac{n-1}{n}}.$$

The right-hand inequalities of (5.26) and (5.27) follow from an application of Urysohn's inequality ((2.25) for $j = 1$) to $\Phi_i K$ and Lemma 5.1.5,

$$V_n(\Phi_i K) \leq \frac{W_{n-1}(\Phi_i K)^n}{\kappa_n^{n-1}} = r_{\Phi_i}^n \frac{W_{n-i}(K)^n}{\kappa_n^{n-1}},$$

where $r_{\Phi_i} = n\kappa_n \int_{\mathbb{S}^{n-1}} f(u) d\sigma(u)$, which is equal to $nW_{n-1}(L)$ for $\Phi_i = \Phi_i^L$.

Equality holds in the last inequality if and only if $\Phi_i K$ is a ball centered at the origin, by the equality cases of Urysohn's inequality.

The existence of minimizers is a direct consequence of Theorem 5.4.3, where we take $P_1(K) = V_n(\Phi_i K)^{\frac{1}{n}}$ and $P_2(K) = V_n(K)^{\frac{i}{n}}$. The left-hand inequality of (5.26) asserts the conditions of the theorem. $\square$

Note that, as before, the assumption on the generating function is not needed for the right-hand inequality.

The proof of Theorem 5.0.6 now follows immediately from Theorem 5.4.7.

Proof of Theorem 5.0.6. As in the proof of Theorem 5.0.5, we can conclude that $\Phi_i K \supseteq \Phi^1 K$ and, consequently, $V_n(\Phi_i K) \geq V_n(\Phi^1 K)$, for some Minkowski valuation Φ^1 with positive generating function. The estimate (5.26) from Theorem 5.4.7 applied to Φ^1 now implies the assumptions of Theorem 5.4.3 for Φ_i, yielding the claim. $\square$

As in the polar case, we mention another easy corollary of Theorem 5.4.7.

Corollary 5.4.8. *Suppose that $\Phi_{n-1} : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of degree $n-1$.*

If the generating function of Φ_i is bounded from below by a positive constant, then

$$\frac{V_n(\Phi_{n-1}K)}{V_n(\Pi K)}$$

attains a minimum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior.

Proof. We take $P_1(K) = V_n(\Phi_{n-1}K)^{\frac{1}{n}}$ and $P_2(K) = V_n(\Pi K)^{\frac{1}{n}}$ and apply Theorem 5.4.3. By Theorem 5.4.7, P_1 satisfies the boundedness assumption. The fraction $\frac{P_2(K)^n}{V_n(K)^{n-1}}$ is affine invariant and hence possesses maximizers and minimizers. $\square$

Although the conditions of Theorem 5.0.5 and Theorem 5.0.6 are not very restrictive, interestingly, they are not satisfied by the i-projection body maps (generated by the support function of the interval). For $i = n-1$, this is not surprising, as the volume product $V_n(\Pi_{n-1}^{\circ}K)V_n(K)^{n-1}$ is invariant under (non-degenerate) affine transformations and therefore bounded from below by a positive constant. For $i \leq n-2$, however, we do not know if Theorems 5.0.5 and 5.0.6 can be further extended to, e.g., some non-negative generating functions.

As an arbitrary generating function f of a Minkowski valuation Φ_i can be approximated by generating functions $f_k = f + \frac{1}{k}$, $k \in \mathbb{N}$, that satisfy the conditions of the theorems, any such f is potentially accessible for limit arguments. However, the example of Theorem 3.2.1 shows that further conditions have to be imposed on f to ensure that the volume product/ratio for Φ_i is bounded and that its extremals for Minkowski valuations generated by the f_k do not converge to a lower-dimensional convex body.

5.5 Minkowski endomorphisms revisited

In this section, we give two examples that demonstrate the variety of behaviours of the volume product $V_n(\Phi_1^{\circ}K)V_n(K)$ in the case of 1-homogeneous Minkowski valuations (Minkowski endomorphisms). In our first example, we show that the Minkowski endomorphism of Theorem 3.2.1 can be used to define a whole family of (weakly monotone) Minkowski endomorphisms that admit maximizers which are different from (and do not include) Euclidean balls. This completes the proof of Theorem 5.0.5. Moreover, although monotonicity is a good property to prove inequalities (see Chapter 3), our second example shows that it is not a necessary condition at all. In the remainder of this section, we finally present a surprising relation between i-homogeneous and 1-homogeneous Minkowski valuations that is used to solve the isoperimetric problem for weakly monotone Minkowski endomorphisms when restricted to convex bodies of revolution.

For the first example recall that the Minkowski endomorphism $\mathrm{J} : \mathcal{K}^n \to \mathcal{K}^n$ is given by $\mathrm{J}K = K - s(K)$, $K \in \mathcal{K}^n$, and $s(K)$ is the Steiner point defined in (2.14).

Theorem 5.5.1. *Suppose that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous, translation invariant, and $\mathrm{SO}(n)$ equivariant Minkowski valuation of degree 1. If the generating function of Φ is a sum of two generating functions one of which is bounded from below by a positive constant and $\Phi_\lambda = \lambda\Phi + (1-\lambda)\mathrm{J}$, for $\lambda \in (0,1)$, then*

$$V_n(\Phi_\lambda^\circ K)V_n(K) \tag{5.28}$$

attains a maximum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior. Moreover, there exists $\varepsilon > 0$ such that Euclidean balls cannot be maximizers whenever $\lambda < \varepsilon$.

Proof. As the generating function of Φ_λ is given by $\lambda f + (1-\lambda)f'$, where f and f' are the generating functions of Φ and J, respectively, Φ_λ satisfies the conditions of Theorem 5.0.5 and therefore admits maximizers for (5.28).

For the second claim, note that, by Lemma 5.1.5,

$$\Phi_\lambda B^n = (\lambda r_\Phi + 1 - \lambda)B^n \supseteq \min\{r_\Phi, 1\}B^n,$$

and, therefore, $V_n(B^n)V_n(\Phi_\lambda^\circ B^n)$ is bounded by $\kappa_n^2\left(\min\{r_\Phi,1\}\right)^{-n}$, for all $\lambda \in (0,1)$. Using that $\Phi_\lambda K \to JK$ for every $K \in \mathcal{K}^n$ as $\lambda \to 0$, we conclude by Theorem 3.2.1 that for $\lambda > 0$ small enough there exists $K \in \mathcal{K}_0^n$ such that

$$V_n(K)V_n(\Phi_\lambda^\circ K) > \frac{\kappa_n^2}{\left(\min\{r_\Phi,1\}\right)^n} \geq V_n(B^n)V_n(\Phi_\lambda^\circ B^n).$$

Consequently, Euclidean balls cannot be maximizers. $\square$

The proof of Theorem 5.5.1 does not use the specific form of J but rather the fact that its volume product is unbounded. Hence, any example with unbounded volume product can be used. Until now, J and $-$J are the only known examples. Note that the set of Minkowski valuations with unbounded volume product is neither open nor convex, as the previous theorem and the example $\Delta = \frac{1}{2}J + \frac{1}{2}(-J)$ show.

Theorem 5.5.2. *There exists an even Minkowski endomorphism Φ, which is not monotone, for which the product*

$$V_n(\Phi^\circ K)V_n(K)$$

attains a maximum on convex bodies $K \in \mathcal{K}^n$ with non-empty interior.

Proof. The proof is based on an example by Dorrek [47] of a non-monotone even Minkowski endomorphism and (5.11). By Theorem 5.0.5 and Theorem 3.1.2, we need to find a zonal, positive and even function f, for which $\square_n f$ takes negative values and $h(\Phi K,\cdot) = S_1(K,\cdot) * f$ defines a Minkowski endomorphism.

In [47], Dorrek proved that for every $h \in C(\mathbb{S}^{n-1})$, which is zonal, non-negative and even, the equation $h(\Phi K,\cdot) = h(K,\cdot) * (1-h)$ defines a Minkowski endomorphism, if h attains its maximum at the pole $\bar{e}$ and h is cumulated around $\pm\bar{e}$, that is, the support of h is contained in $C_\alpha \cup (-C_\alpha)$ with $C_\alpha = \{u \in \mathbb{S}^{n-1} : \langle u, \bar{e}\rangle \geq 1-\alpha\}$

being a spherical cap and $\alpha > 0$ sufficiently small. By assuming that $h(\bar{e}) = C > 1$, the resulting generating function $1 - h$ takes a negative value at $\bar{e}$ and hence Φ is not monotone (and also not weakly monotone since it is even).

We want to show that by taking α small enough (and h smooth), we can ensure that $\square_n^{-1}(1-h) > 0$ and, hence, maximizers exist, by Theorem 5.0.5. As $\square_n 1 = 1$, this is equivalent to $\square_n^{-1} h < 1$. The main tool is the inversion formula (2.9) for $\square_n$,

$$\square_n^{-1} h(u) = \int_{\mathbb{S}^{n-1}} g_n(\langle u, v\rangle) h(v)\, dv, \quad \forall u \in \mathbb{S}^{n-1}.$$

Let $(\alpha_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers converging monotonously to zero and let $(h_j)_{j\in\mathbb{N}}$ be a sequence of zonal, even, non-negative and smooth functions on the sphere, attaining the maximum at the pole $\bar{e}$ (with $h_j(\bar{e}) = C > 1$) and with $\operatorname{supp} h_j \subseteq C_{\alpha_j} \cup (-C_{\alpha_j})$. Obviously, the sequences $(h_j(u))_{j\in\mathbb{N}}$ tend to zero for all $u \neq \pm\bar{e}$.

Let $A_t = \{u \in \mathbb{S}^{n-1} : \langle u, \bar{e}\rangle \le t\}$, for $t < 1$ arbitrary, but fixed, and let j be large enough such that the closure of the cap of size α_j around every $u \in A_t$ does not contain $\bar{e}$. Then $|g_n(\langle \cdot, \bar{e}\rangle)|$ is bounded on the (compact) set of all points $v \in \mathbb{S}^{n-1}$ with $\operatorname{dist}(v, A_t) \le \alpha_j$ by some constant M and we can estimate for $u \in A_t$

$$\begin{aligned} |\square_n^{-1} h_j(u)| &\le \int_{\mathbb{S}^{n-1}} |g_n(\langle u, v\rangle)| h_j(v)\, dv = \int_{C_{\alpha_j} \cup (-C_{\alpha_j})} |g_n(\langle u, v\rangle)| h_j(v)\, dv \\ &\le \int_{C_{\alpha_j} \cup (-C_{\alpha_j})} MC\, dv = 2MC\sigma(C_{\alpha_j}). \end{aligned}$$

As this bound is independent of $u \in A_t$, we can choose $j \in \mathbb{N}$ large enough such that $\square_n^{-1} h_j \le \frac{1}{2}$ on A_t.

This construction works for every $t < 1$. In order to control the behaviour on the complement of A_t, we will now choose t appropriately. Namely, let $t < 1$ such that for every $u \in \mathbb{S}^{n-1} \backslash A_t$ the cap of size α_j (for j large enough for this to make sense) around u does not intersect the set A_{t_n}, where $t_n < 1$ is chosen such that $g_n(t) \le 0$ for all $t > t_n$. Hence, we can estimate $\square_n^{-1} h_j$ on $\mathbb{S}^{n-1} \backslash A_t$ from above by

$$\begin{aligned} \square_n^{-1} h_j(u) &= \int_{\mathbb{S}^{n-1}} g_n(\langle u, v\rangle) h_j(v)\, dv = \int_{C_{\alpha_j} \cup (-C_{\alpha_j})} g_n(\langle u, v\rangle) h_j(v)\, dv \\ &\le 0 + \int_{-C_{\alpha_j}} g_n(\langle u, v\rangle) h_j(v)\, dv \le MC\sigma(C_{\alpha_j}). \end{aligned}$$

By the choice of j above (for $u \in A_t$), we see that $\square_n^{-1} h_j \le \frac{1}{4}$ on $\mathbb{S}^{n-1} \backslash A_t$.

Over all, we see that $1 - \square_n^{-1} h_j(u) \ge 1 - \frac{1}{2} > 0$ for $u \in A_t$ and $1 - \square_n^{-1} h_j(u) \ge 1 - \frac{1}{4} > 0$ for $u \notin A_t$. ☐

Let us note that the example from the last theorem gives rise to many other examples by adding a Minkowski endomorphism $\tilde{\Phi}$ of the form $h(\tilde{\Phi} K, \cdot) = h(K, \cdot) * g$ where g is non-negative and vanishes in caps around $\pm\bar{e}$.

We turn now to the non-polar isoperimetric problem for weakly monotone Minkowski endomorphisms restricted to convex bodies of revolution. Our treatment is based on the observation that the left-hand inequality of (5.27) can be reformulated, using Lemma 5.1.5, to

$$\frac{V_n(\Phi_i^L K)}{V_n(L)} \geq n^n W_{n-i}(K)^n = \frac{V_n(\Phi_i^{B^n} K)}{V_n(B^n)}.$$

Indeed, if both K and L are convex bodies of revolution, then the map $L \mapsto \Phi_i^L(K)$ is a monotone Minkowski endomorphism and we have obtained a sharp isoperimetric inequality for this endomorphism. The same is true if we reformulate (5.25), yielding the inequality of Theorem 3.0.1.

Moreover, the arguments of the proof of Theorem 5.4.5 can be applied to all weakly monotone Minkowski endomorphisms, assuming that all bodies are bodies of revolution. In the following, we will first prove this fact for monotone Minkowski endomorphisms and then extend the argument to weakly monotone Minkowski endomorphisms.

Theorem 5.5.3. *Suppose that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a monotone non-trivial Minkowski endomorphism. Among convex bodies $K \in \mathcal{K}^n$ of revolution with non-empty interior,*

$$\frac{V_n(\Phi K)}{V_n(K)}$$

is minimized by Euclidean balls. If $\Phi = c\Delta$ for some $c > 0$, then K is a minimizer if and only if K is centrally symmetric. Otherwise, Euclidean balls are the only minimizers.

Proof. By Theorem 3.1.2, there exists a zonal measure $\mu \geq 0$ on $\mathbb{S}^{n-1}$ such that

$$h(\Phi K, u) = (h(K, \cdot) * \mu)(u) = \int_{\mathbb{S}^{n-1}} h(K, \vartheta_u v)\, d\mu(v).$$

Let now $K \in \mathcal{K}_0^n$ be a body of revolution. Without loss of generality, we may assume that the axis of revolution is spanned by $\bar{e}$, that is, $h(K, \cdot)$ is a zonal function. By the comment after (2.2), we therefore have $h(K, \cdot) * \mu = \mu * h(K, \cdot)$.

The volume of ΦK can then be calculated using (2.16) twice and Fubini's theorem,

$$\begin{aligned} V_n(\Phi K) &= \frac{1}{n} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} h(K, \vartheta_v^{-1} u)\, dS_{n-1}(\Phi K, u)\, d\mu(v) \\ &= \int_{\mathbb{S}^{n-1}} V(\vartheta_v K, \Phi K[n-1])\, d\mu(v). \end{aligned}$$

Applying Minkowski's inequality (2.19) we obtain

$$V_n(\Phi K) \geq \mu(\mathbb{S}^{n-1}) V_n(K)^{\frac{1}{n}} V_n(\Phi K)^{\frac{n-1}{n}}, \tag{5.29}$$

which can be rewritten by $h(\Phi B^n, u) = \int_{\mathbb{S}^{n-1}} h(B^n, \vartheta_u v)\, d\mu(v) = \mu(\mathbb{S}^{n-1})$ to yield the claimed inequality,

$$\frac{V_n(\Phi K)}{V_n(K)} \geq \mu(\mathbb{S}^{n-1})^n = \frac{V_n(\Phi B^n)}{\kappa_n}. \tag{5.30}$$

It remains to show the equality cases. For this reason, first observe that we may assume that the Steiner points of K (by translation invariance) and of ΦK (this is always true, see, e.g., [14, 24, 67, 95]) lie at the origin and that μ is a probability measure (by scaling invariance). Inequality (5.30) therefore reads $V_n(\Phi K) \geq V_n(K)$.

If there is equality in (5.30), we must have equality in (5.29), that is, equality in Minkowski's inequality. Hence, ΦK and $\vartheta_v K$ must be homothetic for μ-almost all $v \in \mathbb{S}^{n-1}$. As the volumes coincide, there exist vectors $x_v \in \mathbb{R}^n$ such that $\Phi K = \vartheta_v K + x_v$ for μ-almost all $v \in \mathbb{S}^{n-1}$, and, by the assumption on the Steiner points of K and ΦK, we conclude that $x_v = 0$, that is,

$$\Phi K = \vartheta_v K \qquad \text{for } \mu\text{-a.e. } v \in \mathbb{S}^{n-1}. \tag{5.31}$$

As both sides of this equality are continuous in v, we can assume that the statement holds for all v in the support of μ.

If μ is discrete, by zonality, it must be of the form $\mu = \frac{1}{2}\delta_{\bar{e}} + \frac{1}{2}\delta_{-\bar{e}}$. The condition (5.31) therefore reduces to

$$K = \vartheta_{\bar{e}} K = \Phi K = \vartheta_{-\bar{e}} K = -K.$$

As $\Phi = \Delta$ (for this choice of μ) and, hence, $\Phi K = K$ for centrally symmetric K, we conclude that equality holds if and only if K is centrally symmetric. If μ is non-discrete (and non-zero), then (5.31) implies that $\vartheta_v K = \vartheta_w K$ for all $v, w \in \operatorname{supp} \mu$. By the zonality of $h(K, \cdot)$, we deduce

$$h(\vartheta_u K, v) = h(\vartheta_v K, u) = h(\vartheta_w K, u) = h(\vartheta_u K, w),$$

for every $u \in \mathbb{S}^{n-1}$ and $v, w \in \operatorname{supp} \mu$, that is, $h(\vartheta_u K, \cdot)$ is constant on $\operatorname{supp} \mu$, $u \in \mathbb{S}^{n-1}$. Again by the zonality of K and the fact that every $\eta \in \mathrm{SO}(n)$ can be written as $\vartheta_u \tau$ for $u = \eta \bar{e} \in \mathbb{S}^{n-1}$ and $\tau = \vartheta_u^{-1} \eta \in \mathrm{SO}(n-1)$, we obtain that, for every $\eta \in \mathrm{SO}(n)$, $h(\eta K, \cdot)$ is constant μ-almost everywhere, which is exactly the condition (3.11) in the proof of Theorem 3.0.2. Following the same arguments, we conclude that K must be a ball. $\square$

Theorem 5.5.4. *Suppose that $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a weakly monotone non-trivial Minkowski endomorphism. Among convex bodies $K \in \mathcal{K}^n$ of revolution with non-empty interior,*

$$\frac{V_n(\Phi K)}{V_n(K)}$$

is minimized by Euclidean balls. If $\Phi K = aK + b(-K) - \alpha s(K)$ for some $a, b \geq 0$ and $\alpha \in \mathbb{R}$, then K is a minimizer if and only if K is centrally symmetric, for $a \neq 0 \neq b$, or K is any convex body of revolution, for $a = 0$ or $b = 0$. Otherwise, Euclidean balls are the only minimizers.

Proof. By Theorem 3.1.2, there exists a zonal measure μ on $\mathbb{S}^{n-1}$, which is non-negative up to addition of a linear measure, such that $h(\Phi K,\cdot) = h(K,\cdot) * \mu$. Writing $\mu = \tilde{\mu} + \beta\langle \bar{e},\cdot\rangle d\sigma$ with $\tilde{\mu} \geq 0$ and $\beta \in \mathbb{R}$, Φ can be represented as the sum of a map $\tilde{\Phi} : \mathcal{K}^n \to \mathcal{K}^n$ and a constant multiple of the Steiner point, that is, $\Phi K = \tilde{\Phi}K + \alpha s(K)$, $\alpha \in \mathbb{R}$, for every $K \in \mathcal{K}^n$. The map $\tilde{\Phi}$ then has the representation $h(\tilde{\Phi}K, u) = h(K,\cdot) * \tilde{\mu}$.

As the proof of Corollary 5.5.3 only uses the representation as convolution with a non-negative measure (and not the translation invariance of the Minkowski endomorphism), we can conclude in the same way that

$$\frac{V_n(\tilde{\Phi}K)}{V_n(K)} \geq \frac{V_n(\tilde{\Phi}B^n)}{\kappa_n}, \tag{5.32}$$

for every zonal $K \in \mathcal{K}^n_0$, which yields the original claim using that $\tilde{\Phi}K$ is just a translation of ΦK and therefore has the same volume.

For the equality cases, first assume that $s(K) = o$ and that $\Phi B^n = B^n$. Then $\Phi K = \tilde{\Phi}K$ and the equality cases of (5.32), that is, of Minkowski's inequality (5.29), with μ replaced by $\tilde{\mu}$, imply that ΦK and $\vartheta_v K$ must be homothetic for $\tilde{\mu}$-almost all $v \in \mathbb{S}^{n-1}$. As in the proof of Corollary 5.5.3, this condition reduces to

$$\Phi K = \vartheta_v K \qquad \text{for } \tilde{\mu}\text{-a.e. } v \in \mathbb{S}^{n-1}, \tag{5.33}$$

which implies that K must be a ball, whenever $\tilde{\mu}$ is not discrete. If $\tilde{\mu}$ is discrete, then, by zonality, it must be of the form $\tilde{\mu} = a\delta_{\bar{e}} + b\delta_{-\bar{e}}$ with $a + b = 1$ (by the assumption $B^n = \Phi B^n = \tilde{\Phi}B^n$, that is, $\tilde{\mu}$ must be a probability measure). Hence, (5.33) reads for $a \neq 0 \neq b$

$$K = \vartheta_{\bar{e}}K = \Phi K = \vartheta_{-\bar{e}} = -K,$$

and we conclude that equality holds if and only if K is centrally symmetric.

For $a = 0$ or $b = 0$, we have $\Phi K = K$ or $\Phi K = -K$ and (5.33) is a void condition. However, in this case, equality holds trivially for every body $K \in \mathcal{K}^n_0$ of revolution. □

5.6 Trickle-Down Results and Further Estimates

In this section, we prove relations between isoperimetric problems for Minkowski valuations of different degree of homogeneity. More precisely, we show inequalities similar to the inequalities between different quermassintegrals (2.24) with the quermassintegrals $W_{n-i}(K)$ replaced by $V_n(\Phi_i^{L,\circ}K)$ or $V_n(\Phi_i^L K)$, respectively, where $L \in \mathcal{K}^n$ is a body of revolution.

As it turns out, these inequalities may be used to deduce isoperimetric inequalities for Φ_i^L from isoperimetric inequalities for Φ_j^L, $j > i$, motivating the name "trickle-down results". In particular, if Euclidean balls were extremals for some Φ_{n-1}^L, then the equality conditions of the trickle-down inequalities imply that balls are the only extremals for Φ_i^L, $i < n-1$. Consequently, the $(n-1)$-case is the only one where both balls and other convex bodies could be extremals at the same time.

In the remainder of the section, we prove some further estimates for Minkowski valuations Φ_i^L, where $L \in \mathcal{K}^n$ has non-empty interior.

Let us note that Theorems 5.6.1 and 5.6.2 can be deduced from results by Schuster [111], who proved Aleksandrov–Fenchel type inequalities for mixed Minkowski valuations, and generalize results by Lutwak [83] for i-projection bodies. However, we prefer to give a direct proof of Theorem 5.6.1 and Theorem 5.6.2 (not involving dual mixed volumes), using the following consequence of the Aleksandrov–Fenchel inequality (2.22) with $m = n-1$, $i = 0$, $\mathcal{C} = (M)$ and $L = B^n$,

$$V(M, K[j], B^n[n-j-1])^k \geq W_{n-1}(M)^{k-j} V(M, K[k], B^n[n-k-1])^j, \qquad (5.34)$$

for $1 \leq j < k \leq n-1$ and any $K, M \in \mathcal{K}_0^n$.

Theorem 5.6.1. *Suppose that $L \in \mathcal{K}^n$ is a body of revolution, which is not a point, and $1 \leq i < j \leq n-1$. If $K \in \mathcal{K}_0^n$, then*

$$\left(\frac{V_n(\Phi_i^{L,\circ} K)}{c_1(L)} \right)^{\frac{1}{i}} \leq \left(\frac{V_n(\Phi_j^{L,\circ} K)}{c_1(L)} \right)^{\frac{1}{j}} \qquad (5.35)$$

where $c_1(L) = V_n(\Phi_{n-1}^{L,\circ} B^n) = \frac{\kappa_n}{n^n W_{n-1}(L)^n}$ does not depend on K. Equality holds if and only if K is a Euclidean ball.

Proof. By Lemma 5.1.5, the support function of $\Phi_i^L K$, $K \in \mathcal{K}_0^n$, is given by

$$h(\Phi_i^L K, u) = nV(L(u), K[i], B^n[n-i-1]), \quad u \in \mathbb{S}^{n-1}.$$

We estimate the right-hand side using (5.34),

$$V(L(u), K[i], B^n[n-i-1])^j \geq V(L(u), B^n[n-1])^{j-i} V(L(u), K[j], B^n[n-j-1])^i,$$

for $u \in \mathbb{S}^{n-1}$ and $j > i$, and obtain, again by Lemma 5.1.5,

$$\left(\frac{h(\Phi_i^L K, u)}{n} \right)^j \geq W_{n-1}(L)^{j-i} \left(\frac{h(\Phi_j^L K, u)}{n} \right)^i, \quad u \in \mathbb{S}^{n-1}. \qquad (5.36)$$

The claim now follows from the polar volume formula (2.12), (5.36) and Jensen's inequality (for the concave function $x \mapsto x^{\frac{i}{j}}$),

$$\begin{aligned} V_n(\Phi_i^{L,\circ} K)^j &= \left(\frac{1}{n} \int_{\mathbb{S}^{n-1}} h(\Phi_i^L K, u)^{-n} \, du \right)^j \\ &\leq \left(\frac{1}{n} \int_{\mathbb{S}^{n-1}} \left(W_{n-1}(L)^{j-i} n^{j-i} h(\Phi_j^L K, u)^i \right)^{\frac{-n}{j}} du \right)^j \\ &\leq (W_{n-1}(L)^{j-i} n^{j-i})^{-n} \left(\kappa_n \left(\int_{\mathbb{S}^{n-1}} h(\Phi_j^L K, u)^{-n} \frac{du}{n\kappa_n} \right)^{\frac{i}{j}} \right)^j \\ &= (W_{n-1}(L)^{j-i} n^{j-i})^{-n} \kappa_n^{j-i} V_n(\Phi_j^{L,\circ} K)^i. \end{aligned}$$

If there is equality, the equality cases of Jensen's inequality imply that $h(\Phi_j^L K, \cdot)$ must be constant, so $\Phi_j^L K$ a centered ball. Moreover, note that the two applied inequalities may be interchanged, that is, one could first apply Jensen's inequality (for the convex function $x \mapsto x^{\frac{j}{i}}$) and then (5.36) and obtain the same result. For this order of inequalities, the equality cases of Jensen's inequality imply that $\Phi_i^L K$ must be a ball, as well. We write $\Phi_i^L K = r_i B^n$ and $\Phi_j^L K = r_j B^n$ for some $r_i, r_j \geq 0$. As there is equality in (5.35), the two radii are connected by

$$\left(r_i^n \frac{c_1(L)}{\kappa_n}\right)^j = \left(r_j^n \frac{c_1(L)}{\kappa_n}\right)^i.$$

By Lemma 5.1.5, r_i and r_j are multiples of quermassintegrals of K, that is,

$$r_i \kappa_n = W_{n-1}(\Phi_i K) = n W_{n-1}(L) W_{n-i}(K), \tag{5.37}$$

and $r_i^n \frac{c_1(L)}{\kappa_n} = \frac{1}{\kappa_n^n} W_{n-i}(K)^n$, and similarly for j. Equation (5.37) therefore reads

$$\left(\frac{W_{n-i}(K)^j}{\kappa_n^j}\right)^n = \left(\frac{W_{n-j}(K)^i}{\kappa_n^i}\right)^n,$$

which is (up to the power n) the equality case of the (classical) isoperimetric inequality for quermassintegrals (2.24). Equality in this inequality holds exactly for Euclidean balls, hence K must be a ball. □

Theorem 5.6.2. *Suppose that $L \in \mathcal{K}^n$ is a body of revolution, which is not a point, and $1 \leq i < j \leq n-1$. If $K \in \mathcal{K}_0^n$, then*

$$\left(\frac{V_n(\Phi_i^L K)}{c_2(L)}\right)^{\frac{1}{i}} \geq \left(\frac{V_n(\Phi_j^L K)}{c_2(L)}\right)^{\frac{1}{j}}, \tag{5.38}$$

where $c_2(L) = V_n(\Phi_{n-1}^L B^n) = \kappa_n n^n W_{n-1}(L)^n$ does not depend on K. Equality holds if and only if K is a Euclidean ball.

Proof. By Lemma 5.1.6 (twice) and (5.34), we can estimate

$$\begin{aligned} V(\Phi_i^L K, M[n-1])^j &= V(\Phi_{n-1}^L M, K[i], B^n[n-i-1])^j \\ &\geq V(\Phi_{n-1}^L M, B^n[n-1])^{j-i} V(\Phi_{n-1}^L M, K[j], B^n[n-j-1])^i \\ &= V(M[n-1], \Phi_{n-1}^L B^n)^{j-i} V(M[n-1], \Phi_j^L K)^i. \end{aligned}$$

By Minkowski's inequality (2.19), we can further estimate

$$\begin{aligned} V(\Phi_i^L K, M[n-1])^{jn} &\geq \left(V_n(M)^{n-1} V_n(\Phi_{n-1}^L B^n)\right)^{j-i} \left(V_n(M)^{n-1} V_n(\Phi_j^L K)\right)^i \\ &= V_n(\Phi_{n-1}^L B^n)^{j-i} V_n(M)^{j(n-1)} V_n(\Phi_j^L K)^i, \end{aligned}$$

Setting $M = \Phi_i^L K$, we obtain (5.38) (up to rearranging terms),

$$V_n(\Phi_i^L K)^{jn} = V(\Phi_i^L K, \Phi_i^L K[n-1])^{jn} \geq V_n(\Phi_{n-1}^L B^n)^{j-i} V_n(\Phi_i^L K)^{j(n-1)} V_n(\Phi_j^L K)^i.$$

If equality holds in (5.38), the equality cases of Minkowski's inequality imply that $M = \Phi_i^L K$ must be homothetic to $\Phi_{n-1}^L B^n = nW_{n-1}(L)B^n$ and therefore must be a ball, as well. Inequality (5.34) then reduces to the (classical) isoperimetric inequalities (2.24) between the quermassintegrals of K, for which equality holds only for balls. □

Note that the key point in the proof of the previous theorem is the symmetry property of the Minkowski valuations Φ_i^L from Lemma 5.1.6,

$$V(K[n-1], \Phi_i^L M) = V(M[i], B^n[n-i-1], \Phi_{n-1}^L K).$$

In the following, we want to give some estimates similar to the results from Theorem 5.4.5 for the (polar) volume of the Minkowski valuations Φ_{n-1}^L, where $L \in \mathcal{K}^n$ has non-empty interior, in terms of the same quantity for the projection body Π_{n-1}.

Proposition 5.6.3. *Suppose that $L \in \mathcal{K}^n$ is a body of revolution with non-empty interior. If $K \in \mathcal{K}_0^n$, then*

$$\frac{V_n(\Phi_{n-1}^{L,\circ} K)}{V_n(\Pi_{n-1}^\circ K)} \leq \frac{1}{(\operatorname{diam} L)^n}.$$

Proof. Let I be a segment such that $I = [x, y]$ with $x, y \in L$ and $\operatorname{diam} L = \|x - y\|$. We denote by $w \in \mathbb{S}^{n-1}$ a fixed unit vector in direction $x - y$. Then the segment $\vartheta_u(I \cap L) = \vartheta_u I$ is contained in $L(u)$, $u \in \mathbb{S}^{n-1}$, and Lemma 5.1.5 and the monotonicity of the mixed volume imply for $K \in \mathcal{K}_0^n$

$$\begin{aligned} h(\Phi_{n-1}^L K, u) &= nV(L(u), K[n-1]) \geq nV(\vartheta_u I, K[n-1]) \\ &= \frac{n}{2} V_1(I) V([-\vartheta_u w, \vartheta_u w], K[n-1]) = V_1(I) h(\Pi_{n-1} K, \vartheta_u w), \end{aligned} \tag{5.39}$$

where $V_1(I) = \operatorname{diam} L$ is just the length of the segment. By the polar volume formula (2.12) and (5.39), we obtain

$$V_n(\Phi_{n-1}^{L,\circ} K) \leq \frac{(\operatorname{diam} L)^{-n}}{n} \int_{\mathbb{S}^{n-1}} h(\Pi_{n-1} K, \vartheta_u w)^{-n} du. \tag{5.40}$$

Since both Φ_{n-1}^L and the polar map commute with $\mathrm{SO}(n)$ transforms, we may replace K by a rotated copy ηK, $\eta \in \mathrm{SO}(n)$ and integrate over $\mathrm{SO}(n)$ with respect to the Haar measure. By Fubini's theorem (twice), the $\mathrm{SO}(n)$ invariance of the Haar measure and (2.12) again, the claim follows from (5.40)

$$\begin{aligned} V_n(\Phi_{n-1}^{L,\circ} K) &\leq \frac{(\operatorname{diam} L)^{-n}}{n} \int_{\mathbb{S}^{n-1}} \int_{\mathrm{SO}(n)} h(\Pi_{n-1} K, \eta^{-1} \vartheta_u w)^{-n} \, d\eta \, du \\ &= (\operatorname{diam} L)^{-n} \int_{\mathrm{SO}(n)} \frac{1}{n} \int_{\mathbb{S}^{n-1}} h(\Pi_{n-1} K, \tau^{-1} u)^{-n} du \, d\tau \\ &= (\operatorname{diam} L)^{-n} \int_{\mathrm{SO}(n)} V_n((\tau \Pi_{n-1} K)^\circ) \, d\tau = (\operatorname{diam} L)^{-n} V_n(\Pi_{n-1}^\circ K). \end{aligned}$$

□

By the estimate $W_{n-1}(L) \leq \kappa_n \frac{\operatorname{diam} L}{2}$ and Lemma 5.1.5, the inequality of Proposition 5.6.3 can be continued to an inequality in terms of the polar volume of $\Phi_{n-1}^{L,\circ} B^n$. Moreover, Proposition 5.6.3 and the Petty projection inequality (2.31) also give an estimate for the comparison with the volume.

Corollary 5.6.4. *Suppose that $L \in \mathcal{K}^n$ is a body of revolution with non-empty interior. If $K \in \mathcal{K}_0^n$, then*

$$V_n(\Phi_{n-1}^\circ K) V_n(K)^{n-1} \leq \frac{1}{(\operatorname{diam} L)^n} \frac{\kappa_n^n}{\kappa_{n-1}^n}.$$

A similar (yet weaker) estimate yields a non-polar analogue of Proposition 5.6.3.

Proposition 5.6.5. *Suppose that $L \in \mathcal{K}^n$ is a body of revolution with non-empty interior. If $K \in \mathcal{K}_0^n$, then*

$$\frac{V_n(\Phi_{n-1}^L K)}{V_n(\Pi_{n-1} K)} \geq V_1(L \cap \operatorname{span}\{\bar{e}\})^n.$$

Proof. As $\vartheta_u(L \cap \operatorname{span}\{\bar{e}\}) = L(u) \cap \operatorname{span}\{u\} \subseteq L(u)$, $u \in \mathbb{S}^{n-1}$, by Lemma 5.1.5 and the monotonicity of the mixed volume, we can estimate

$$\begin{aligned} h(\Phi_{n-1}^L K, u) &= nV(L(u), K[n-1]) \geq nV(\vartheta_u(L \cap \operatorname{span}\{\bar{e}\}), K[n-1]) \\ &= \frac{n}{2} V_1(L \cap \operatorname{span}\{\bar{e}\}) V([-u,u], K[n-1]) \\ &= V_1(L \cap \operatorname{span}\{\bar{e}\}) h(\Pi_{n-1} K, u). \end{aligned}$$

Hence, $\Phi_{n-1}^L K \supseteq V_1(L \cap \operatorname{span}\{\bar{e}\}) \Pi_{n-1} K$, which directly implies the claim. □

Let us note that when L approaches the interval, the inequalities of Propositions 5.6.3 and 5.6.5 become equality, whereas they are not sharp in general.

5.7 Appendix

In this section, we establish the results needed to conclude the equality cases of Theorem 5.3.1.

We start by introducing a notation for i-dimensional subspaces that fits nicely with the action of $\mathrm{SO}(n-1)$ on the Grassmanian.

For this reason, let $F \in \operatorname{Gr}(n,i)$ and first assume that $F \not\subseteq \bar{e}^\perp$. In this case, the intersection $F \cap \bar{e}^\perp$ has dimension $i-1$. Indeed, we can compute $\dim F \cap \bar{e}^\perp = \dim F + \dim \bar{e}^\perp - \dim(F + \bar{e}^\perp) = i + (n-1) - n = i-1$. As the orthonormal complement of $F \cap \bar{e}^\perp$ inside F is one-dimensional, we find a unique unit vector v_1 spanning it and satisfying $\langle v_1, \bar{e}\rangle > 0$. We thus define $v_1(F) = v_1$ for $F \in \operatorname{Gr}(n,i), F \not\subseteq \bar{e}^\perp$. For the other case, $F \subseteq \bar{e}^\perp$, we choose $v_1(F) \in F$ arbitrarily. Using this representation of $F \in \mathrm{Gr}(n,i)$, we may now directly compute the action of $\mathrm{SO}(n-1)$ on F.

Let again $F \in \mathrm{Gr}(n,i)$ be an i-dimensional subspace and let $\tau \in \mathrm{SO}(n-1)$. Then τ maps $\bar{e}^\perp$ to itself and we have (for $F \not\subseteq \bar{e}^\perp$) that $\tau F = \mathrm{span}\{\tau v_1(F)\} \oplus \tau(F \cap \bar{e}^\perp)$, that is, $v_1(\tau F) = \tau v_1(F)$. In particular, the function $F \mapsto \langle v_1(F), \bar{e}\rangle$ is invariant under the action of $\mathrm{SO}(n-1)$. Indeed, we have more:

Lemma 5.7.1. *The orbit of $F \in \mathrm{Gr}(n,i)$, $1 \le i \le n-1$, under $\mathrm{SO}(n-1)$ satisfies*

$$\mathrm{SO}(n-1) \cdot F = \{\tau F : \tau \in \mathrm{SO}(n-1)\} = \{G \in \mathrm{Gr}(n,i) : \langle v_1(G), \bar{e}\rangle = \langle v_1(F), \bar{e}\rangle\}$$

Proof. The inclusion $\subseteq$ is obvious by the remarks above the lemma. For the other inclusion, we consider the three cases $F \subseteq \bar{e}^\perp$, $v_1(F) = \bar{e}$ and $v_1(F) \notin \bar{e}^\perp \cup \{\bar{e}\}$.

1. For $F \subseteq \bar{e}^\perp$, we have defined that $v_1(F) \in \bar{e}^\perp$, and thus $\langle v_1(F), \bar{e}\rangle = 0$. As $\mathrm{SO}(n-1)$ acts transitively on $\{G \in \mathrm{Gr}(n,i) : G \subseteq \bar{e}^\perp\}$, the claim follows directly.

2. If we have $v_1(F) = \bar{e}$, the action of $\mathrm{SO}(n-1)$ reduces to an action on $\{G \in \mathrm{Gr}(n,i-1) : G \subseteq \bar{e}^\perp\}$. This action is transitive.

3. For the remaining case, $v_1(F) \notin \bar{e}^\perp \cup \{\bar{e}\}$, we may write $v_1(F) = \alpha\bar{e} + v_1(F)|\bar{e}^\perp$, with $\alpha \notin \{0,1\}$ and $v_1(F)|\bar{e}^\perp \neq 0$. Using this, we observe that the orthogonal projection $F|\bar{e}^\perp$ has to be i-dimensional. Indeed, write $F \cap \bar{e}^\perp$ as $\mathrm{span}\{v_2, \dots, v_i\}$, where $v_2, \dots, v_i$ is an orthonormal system, then $F|\bar{e}^\perp = \mathrm{span}\{v_1(F)|\bar{e}^\perp, v_2, \dots, v_i\}$. The right-hand side is a basis as the vectors are non-zero and orthogonal, since $\langle v_1(F)|\bar{e}^\perp, v_j\rangle = \langle v_1(F), v_j\rangle = 0$, for all $j \ge 2$.

   Now let $G \in \mathrm{Gr}(n,i)$ with $\langle v_1(F), \bar{e}\rangle = \langle v_1(G), \bar{e}\rangle$ be an element of the right-hand side. By the same argument, the dimension of $G|\bar{e}^\perp$ is i. We write G as $\mathrm{span}\{v_1(G)\} \oplus \mathrm{span}\{w_2, \dots, w_i\}$ with orthonormal $w_2, \dots, w_i \in G \cap \bar{e}^\perp$. As the dimensions match, there exists a transformation $\tau \in \mathrm{SO}(n-1)$ that maps $F|\bar{e}^\perp$ to $G|\bar{e}^\perp$ and satisfies $\tau(v_1(F)|\bar{e}^\perp) = v_1(G)|\bar{e}^\perp$ and $\tau v_j = w_j$ for $j \ge 2$. This is possible as we map an orthogonal system in $\bar{e}^\perp$ to another and $v_1(F)|\bar{e}^\perp$ and $v_1(G)|\bar{e}^\perp$ have the same length by the assumption that $\langle v_1(F), \bar{e}\rangle = \langle v_1(G), \bar{e}\rangle$.

   This transformation τ already takes F to G. Indeed, it remains to calculate $\tau(v_1(F)) = \alpha\tau\bar{e} + \tau(v_1(F)|\bar{e}^\perp) = \alpha\bar{e} + v_1(G)|\bar{e}^\perp = v_1(G)$, so $G \in \mathrm{SO}(n-1) \cdot F$.

□

For the proof of the equality cases we will need more information about the intersection of an $\mathrm{SO}(n-1)$-orbit with its image under a map $\eta \in \mathrm{SO}(n)$, given by the next lemma. In the lemma, we use the abbreviation $H_{u,s} = \{x \in \mathbb{R}^n : \langle x, u\rangle = s\}$.

Lemma 5.7.2. *Suppose that $\eta \in \mathrm{SO}(n)$, $F \in \mathrm{Gr}(n,i)$, where $i \le n-2$, and write $t = \langle v_1(F), \bar{e}\rangle$.*
If $w_1 \in \eta(H_{\bar{e},t} \cap \mathbb{S}^{n-1})$ with $\langle w_1, \bar{e}\rangle > 0$, then there exists a subspace $G \in \mathrm{Gr}(n,i)$ with $v_1(G) = w_1$ such that $G \in \eta(\mathrm{SO}(n-1) \cdot F)$. If, additionally, $w_1 \in H_{\bar{e},t} \cap \mathbb{S}^{n-1}$, then $\eta(\mathrm{SO}(n-1) \cdot F) \cap (\mathrm{SO}(n-1) \cdot F) \neq \emptyset$.

Proof. Let $w_1 \in \eta(H_{\bar{e},t} \cap \mathbb{S}^{n-1})$ with $\langle w_1, \bar{e} \rangle > 0$. As each of the spaces $(\eta\bar{e})^\perp$, $w_1^\perp$ and $\bar{e}^\perp$ has dimension $n-1$, the intersection of all three spaces has at least dimension $n-3$. By the condition $i \leq n-2$, we have $i-1 \leq n-3$, so we may find an orthonormal system $\{w_2, \ldots, w_i\}$ in $(\eta\bar{e})^\perp \cap w_1^\perp \cap \bar{e}^\perp$. We set $G = \operatorname{span}\{w_1, w_2, \ldots, w_i\}$. This defines an i-dimensional linear subspace with orthonormal basis given by $w_1, w_2, \ldots, w_i$ and we may identify $v_1(G) = w_1$ (or, if $w_1 \in \bar{e}^\perp$, $v_1(G) \in \bar{e}^\perp$). By construction, every $w_j, j \geq 2$, is orthogonal to $\eta\bar{e}$, so η^{-1} maps them into $\bar{e}^\perp$ leading to $\eta^{-1}G = \operatorname{span}\{\eta^{-1}w_1, \eta^{-1}w_2, \ldots \eta^{-1}w_i\}$, where the first vector is contained in $H_{\bar{e},t} \cap \mathbb{S}^{n-1}$ and all the others lie in $\bar{e}^\perp$. If $t = 0$, this implies that $\eta^{-1}G \subseteq \bar{e}^\perp$. If $t > 0$, $v_1(\eta^{-1}G) = \eta^{-1}w_1 \in H_{\bar{e},t} \cap \mathbb{S}^{n-1}$. In both cases we have $\eta^{-1}G \in \mathrm{SO}(n-1) \cdot F$ by Lemma 5.7.1.

On the other hand, if $w_1 \in H_{\bar{e},t} \cap \mathbb{S}^{n-1}$, $G \in \mathrm{SO}(n-1) \cdot F$, as well. □

The following Theorem 5.7.3 combines the previous lemmata to proof the last step for the equality cases.

Theorem 5.7.3. *Let $2 \leq i \leq n-2$ and suppose that $f : \mathrm{Gr}(n,i) \to \mathbb{R}$ is a continuous function and that $\mu \neq 0$ is a non-negative, $\mathrm{SO}(n-1)$ invariant measure on the Grassmanian $\mathrm{Gr}(n,i)$.*
If $\eta^{-1}f$ is constant μ-almost everywhere for every $\eta \in \mathrm{SO}(n)$, that is,

$$\forall \eta \in \mathrm{SO}(n)\, \exists c_\eta \in \mathbb{R} : \quad f(\eta F) = c_\eta \quad \text{for } \mu\text{-a.e. } F \in \mathrm{Gr}(n,i), \tag{5.41}$$

then f is constant.

Proof. As μ is non-zero, there exists a subspace $F \in \operatorname{supp} \mu$ and, by the $\mathrm{SO}(n-1)$ invariance of μ, the whole orbit $\mathrm{SO}(n-1) \cdot F$ is contained in the support. By the continuity of f the assumption (5.41) holds for all F in the support of μ.

We consider the two cases that there exists an $F \in \operatorname{supp} \mu$ with $v_1(F) \neq \bar{e}$ and that there does not exists such an F. This is necessary as we need the images $\{v_1(G) : G \in \mathrm{SO}(n-1) \cdot F\}$ of the $\mathrm{SO}(n-1)$ orbits to be subspheres for our proof. When $v_1(F) = \bar{e}$ this image consists of just the pole $\bar{e}$ and we need different arguments.

1. Let $F \in \operatorname{supp} \mu$ be a subspace such that $v_1(F) \neq \bar{e}$. Taking this F and taking $\eta = \mathrm{Id}$ in (5.41), we obtain that $f(G) = c_{\mathrm{Id}}$ for all $G \in \mathrm{Gr}(n,i)$ in the same orbit, that is, by Lemma 5.7.1, for G satisfying $\langle v_1(G), \bar{e} \rangle = \langle v_1(F), \bar{e} \rangle$. To show the claim, we need to show that f is constant on all subspaces $G \in \mathrm{Gr}(n,i)$ where $v_1(G) \in \mathbb{S}^{n-1}$, that is, every $G \in \mathrm{Gr}(n,i)$.

   Now let $\eta \in \mathrm{SO}(n)$ be a transformation such that $\eta(H_{\bar{e},t} \cap \mathbb{S}^{n-1}) \cap (H_{\bar{e},t} \cap \mathbb{S}^{n-1}) \neq \emptyset$, where $t = \langle v_1(F), \bar{e} \rangle$. By Lemma 5.7.2, there exists a subspace $G \in \eta(\mathrm{SO}(n-1) \cdot F) \cap (\mathrm{SO}(n-1) \cdot F)$, which we may represent as $G = \eta\tilde{F}, \tilde{F} \in \mathrm{SO}(n-1) \cdot F$. As F (and therefore also $\tilde{F}$) is in the support of μ, we may conclude that $c_\eta = f(\eta\tilde{F}) = f(G) = c_{\mathrm{Id}}$. Hence, f is constantly equal to c_{Id} on the subspaces G with $v_1(G) \in \eta(H_{\bar{e},t} \cap \mathbb{S}^{n-1}) \cup (H_{\bar{e},t} \cap \mathbb{S}^{n-1})$.

As we may repeat this argument to reach every vector on the unit surface in finitely many steps, the function f has to be constant on the whole Grassmanian $\mathrm{Gr}(n,i)$.

2. If there is no subspace $F \in \operatorname{supp}\mu$ with $v_1(F) \neq \bar{e}$, then

$$\operatorname{supp}\mu \subseteq \{F \in \mathrm{Gr}(n,i) : v_1(F) = \bar{e}\} = \{F \in \mathrm{Gr}(n,i) : \bar{e} \in F\}.$$

By the $\mathrm{SO}(n-1)$-invariance of μ, all these sets coincide.

We show the claim using that the Grassmanian can be represented as the union of all subsets $\{F \in \mathrm{Gr}(n,i) : u \in F\}$, where $u \in S^{n-1}$ is any unit vector. As it is obvious that $\eta\{F \in \mathrm{Gr}(n,i) : u \in F\} = \{G \in \mathrm{Gr}(n,i) : \eta u \in G\}$, we know by the assumption that f is constant on any of these sets. But as $i \geq 2$, all these sets have non-empty intersection with $\operatorname{supp}\mu = \{F \in \mathrm{Gr}(n,i) : \bar{e} \in F\}$.

$\square$

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