Scale measures with fuzzy data: analyzing their robustness and application to fuzzy rating scale-based questionnaires

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Doctoral Programme
in Engineering Sciences
Diploma Programme:
Technical Mathematics

PhD Dissertation
2017

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A mis padres y a mi hermano

Acknowledgements

I would like to express my sincere thanks to all the people who have supported me during the course of my PhD project.

First, I would like to especially thank my three PhD supervisors, Ángeles, Asun and Peter for their enormous support. Thanks to Ángeles for introducing me to this exiting world of scientific research, for the invaluable time you devoted to me, always with a smile on your face and good mood, and for being a perfect example of devotion and dedication. Thanks to Asun for always being there to help, for the infinite hours you devoted to this work and for your good and productive ideas. And thanks to Peter, for introducing me to the Robust Statistics field, for your useful suggestions and for making me feel at home in each of every short stay in Vienna.

Thanks to the Departamento de Estadística e I.O. y D.M. of the University of Oviedo, for providing me with all the necessary tools to carry out my thesis work, as well as for the friendly atmosphere which has allowed me to develop this work in a very pleasant way.

I must also say thank you to Bea and Sonia. You have helped every time I needed it and I shall cherish unforgettable memories of our stays in Ghent and Vienna. I am also very grateful to my colleagues Ahmed, Ana, Ángela, Ekra, Itziar, Liudmila and Marta, with whom I have shared much of my time in the office and at conferences.

During the course of my PhD I had the chance to undertake six internships in the Institut für Stochastik und Wirtschaftsmathematik of the Vienna University of Technology, whose members have always warmly welcomed me. I have unforgettable memories from these experiences, thanks in part to my colleagues Maria Anna, Barbara, Gianna, Honza, Petra and Sevinç, and also to Ángel, David, Félix, Irina, Isma, John and Juan, with whom I greatly enjoyed the time spent out of work. All of you made me feel at home despite being so far away. I cannot talk about my stay in Vienna without mentioning Valentin, to whom I am very grateful to for his willingness to help me and his close friendship.

Also thanks to my special friends Belén, Irene, Sara and Yulia. Thank you for your support and encouragement, especially now on the home straight of this work, and for being always there, in good times and in bad.

Finally, thanks to my parents for their advice and support in all my decisions, always trying to help me get the best possible outcomes. I am also grateful for the education they gave me, and for showing me how to face difficulties, always with courage and humility. Thanks to my mum because with her huge help I have been able to make the most of my time when working on this project, and to my dad for the time he devoted to help me to overcome the many mathematical problems I encountered throughout this time, especially in this last step during my thesis preparation. And of course, thanks to my brother, Jorge, for his affection and for being a constant example to me.

I would like to express my gratitude to all the institutions that have economically supported this research along these years. In particular, I would like to thank Gobierno del Principado de Asturias for financing the PhD through the launching of the "Severo Ochoa" plan for the formation of research and teaching staff in Principado de Asturias, which was awarded to me at the beginning of this PhD as a two-years fellowship and a two-years contract, BP12-012, and also for financing the short PhD internships and the Grants GRUPIN14-101 (co-financed by FEDER), CP-PA-11-GI/SPIN-OFF and SV-PA-13-ECOEMP-66, all of which I was highly involved in. For the same reason I wish to thank the Ministerio de Educación y Ciencias for the Grants MTM2009-09440-C02-01 and the Ministerio de Economía y Competitividad for the Grant MTM2013-44212-P. Thanks also to the Fundación Universidad de Oviedo, Banco Santander and the European Science Foundation (through the COST Action IC0702) for economically supporting three of the stays abroad.

My immense gratitude to you all.

Agradecimientos

Quisiera expresar mi más sincero agradecimiento a todas las personas que me han acompañado en estos últimos años de trabajo que han dado como fruto esta tesis doctoral.

En primer lugar, me gustaría agradecer profundamente el apoyo que me han brindado mis tres directores, Ángeles, Asun y Peter. Gracias Ángeles, por introducirme en este mundo apasionante de la investigación, por el incalculable tiempo que me has dedicado, siempre con tu alegría y buen humor, y por ser para mí un ejemplo de dedicación y entrega. Gracias Asun, por estar siempre dispuesta a ayudarme, por tus infinitas horas de trabajo en mi tesis a lo largo de estos años y por tus buenas y productivas ideas. Y gracias Peter, por descubrirme el campo de la Estadística Robusta, por tus valiosas propuestas y por acogerme tan bien en cada una de las estancias que he realizado en Viena.

Gracias al Departamento de Estadística e I.O. y D.M. de la Universidad de Oviedo por poner a mi disposición todos los medios necesarios para poder desarrollar mi trabajo investigador, así como por el buen ambiente y compañerismo que ha hecho mi trabajo mucho más agradable.

No me puedo olvidar de Bea y Sonia. Me habéis ayudado en mi investigación siempre que lo he necesitado y guardo recuerdos inolvidables de nuestras estancias en Gante y Viena. Tampoco me olvido de mis compañeros Ahmed, Ana, Ángela, Ekra, Itziar, Liudmila y Marta, con los que he compartido numerosas horas de trabajo y convivencia en el despacho y congresos.

Durante mis años de doctorado he tenido la enorme suerte de realizar seis estancias breves en el Institut für Stochastik und Wirtschaftsmathematik de la Universidad Técnica de Viena, cuyos miembros me han acogido siempre como una más. De allí me llevo imborrables recuerdos, en parte gracias a mis compañeros de despacho Maria Anna, Barbara, Gianna, Honza, Petra y Sevinç, y en parte a Ángel, David, Félix, Irina, Isma, John y Juan, con los que pasé grandes momentos fuera del trabajo. Todos me habéis hecho sentir como en casa aun estando tan lejos. Y no puedo dejar Viena sin mencionar a Valentin, al que agradezco enormemente su predisposición a ayudarme siempre y su inmejorable trato.

A mis amigas con mayúsculas Belén, Irene, Sara y Yulia, gracias de corazón por apoyarme y darme ánimos en la recta final de este trabajo, y por estar siempre ahí, en los buenos y en los malos momentos.

Finalmente, gracias a mis padres por aconsejarme y apoyarme en todas mis decisiones, siempre buscando lo mejor para mí. Por la educación que me habéis dado, enseñándome a afrontar la vida con valentía y humildad. Gracias a mi madre por regalarme todo su tiempo para que el mío rinda más, y a mi padre, por las horas dedicadas a sacarme de mis "atascos" con las matemáticas a lo largo de todos mis estudios, incluida esta última etapa de preparación de la tesis. Y por supuesto, gracias también a mi hermano Jorge, por su cariño y por ser un ejemplo constante para mí.

Quisiera expresar también mi gratitud a las instituciones que han apoyado la investigación en la que he estado involucrada en estos años de trabajo. Concretamente, al Gobierno del Principado de Asturias, por financiar la ayuda predoctoral BP12-012 del programa "Severo Ochoa" para la formación en investigación y docencia del Principado de Asturias, que me fue concedida al empezar mi doctorado en forma de beca durante dos años y de contrato durante otros dos, y también por sus ayudas dentro de este programa para estancias breves y por los Proyectos GRUPIN14-101 (cofinanciado con Fondos FEDER), CP-PA-11-GI/SPIN-OFF y SV-PA-13-ECOEMP-66, de cuyos equipos de investigación he sido miembro. También, por esta última razón, al Ministerio de Educación y Ciencias por el Proyecto MTM2009-09440-C02-01 y al Ministerio de Economía y Competitividad por el Proyecto MTM2013-44212-P. Y también a la Fundación Universidad de Oviedo, al Banco Santander y a la European Science Foundation (a través de la Acción COST IC0702), por financiar tres de mis estancias de investigación.

Gracias, infinitas gracias a todos.

Prologue

Imprecise data can be found in many real-life situations. Fields like Engineering, Biomedical Sciences or Social Sciences often deal with this kind of data. For instance, the optimization of industrial control systems, the diagnosis determined by a doctor about a patient or the customer valuation about a product are matters that may involve imprecise data. Fuzzy numbers can properly express and model this type of data.

The Likert-type scales are frequently used in designing questionnaires to rate characteristics or attributes that cannot be numerically measured (like satisfaction, perceived quality, perception...). Although they are easy to answer and they do not require a special training to use them, the available statistical methodology to analyze the data from these questionnaires is rather limited. This is mainly due to the fact that Likert scales are discrete with a very small number of responses to choose for each item (often 4 or 5). To overcome this concern, some alternatives have been suggested in the literature:

- On one hand, the visual analogue scales, which allow the respondent to choose the exact value or point along a line between two end-points (i.e., along a compact interval) that better represents his/her level of agreement to a statement or property. Even though it is a continuous scale, so the diversity of responses is ensured and the statistical conclusions are reliable, it does not seem realistic to demand as much accuracy in connection with such an intrinsically imprecise context.

- On the other hand, the posterior encoding (usually made by trained experts) of the Likert responses by means of fuzzy numbers from a linguistic scale. With this scale, the intrinsic imprecision associated with the responses is better captured, but it is a discrete scale with small cardinal, like the Likert scale, and so, the subjectivity and the diversity of ratings are to some extent lost.

The so-called fuzzy rating scales integrate the skills of the two previous alternatives (to be a 'continuous' and fuzzy-valued scale). The questionnaires based on this scale have a free-response format, allowing the rater to draw the fuzzy number that better expresses his/her response to the given item, within a reference bounded interval. In this way, the values can cope (to a full extent) with the intrinsic imprecision associated with the ratings. On the other hand, the diversity and subjectivity of the responses are not lost because it is a continuous scale, and there is a substantial gain of information and accuracy in the conclusions. Moreover, along the last years a statistical methodology is being developed to analyze fuzzy-based data, allowing us to treat them in a similar way to the numerical data.

A substantial part of the thesis work is devoted to summarize the statistical information involved in datasets that are based on a fuzzy rating by analyzing their scale estimate, where it is intended as a representative measure of the dispersion of the imprecise-valued attributes supplying the datasets. More concretely, the objective is summarizing the scale by extending different, mostly robust, scale estimates from the real-valued case. Thus, the Fréchet-type variance (or the corresponding standard deviation) is the best known and used dispersion measure in the fuzzy context. It preserves the main valuable properties of the variance (standard deviation) for real-valued data, but it also inherits its high sensitivity to the presence of outliers or atypical/extreme observations in the data. Therefore, it is desirable to introduce and study other scale measures and to analyze their robust behaviour.

The other main contribution of the work is to compare the mentioned different rating scales to model/deal with imprecise-valued data through some studies based on simulations and also on real-life questionnaires. In particular, the Likert-type scale, through both its numerical and fuzzy linguistic encodings, and the fuzzy rating scale will be compared, mainly on the basis of the dispersion/scale measures studied in the first main part of the work.

Aiming to achieve these goals, the work in this dissertation has been structured as follows:

Chapter 1 introduces the different rating scales currently most employed to measure aspects which are intrinsically imprecise. All of them are illustrated by means of some examples. The main positive and negative features in connection with the use of each scale are analyzed, and the design of questionnaires based on the fuzzy rating scale is explained in detail.

The principal preliminary and supporting tools about fuzzy numbers are gathered, including some of the representations that characterize them, their arithmetic and the metrics which will be used along this work. The random mechanism that models the generation of fuzzy values within a probabilistic setting is presented, and the main summary measures/parameters of its distribution are recalled, both central tendency and dispersion measures. The latter will be deeply analyzed in the next chapter.

Three real-life examples concerning the application of the fuzzy rating and Likert scales in questionnaires are included, and the complete datasets associated to such questionnaires can be found in the appendices section at the end of this work. Besides, the two simulation procedures of fuzzy numbers conducted in different instances of this work are also explained. After a small study consisting of analyzing the sensitivity of Fréchet's variance with respect to the shape of the fuzzy data, the chapter ends motivating the need for robust measures alternative to the variance, as well as the interest of comparing the rating scales from a dispersion perspective.

In **Chapter 2** several measures of scale to deal with real-valued data are extended to deal with fuzzy-valued data. After proving some of their most relevant properties, such as the invariance by translation, the (absolute) equivariance by the product by scalars or the strong consistency, their robust behaviour is analyzed deeply from a theoretical point of view and also by means of simulations. The value of the finite sample breakdown point is calculated for each estimator assuming that there are not two identical observations in the sample, and the results obtained will show that these values are the same as those in the real-valued case.

The three types of outliers considered along the chapter have had to be specifically conceived for this fuzzy setting and they are explained in detail. The empirical breakdown point is obtained for each estimator, confirming the theoretical values. The notion of sensitivity curves is extended from the real- to the fuzzy-valued case and they are graphically displayed.

A brief section is devoted to introduce the M-estimators of scale. It gathers an algorithm to compute them and also an empirical analysis of the robustness for two well-known loss functions: the Huber loss function and the Tukey bisquare loss function.

Finally, the computation of all the scale estimators introduced along this chapter is illustrated for the fuzzy responses from the three questionnaires in Chapter 1.

In **Chapter 3** a comparative study among the rating scales presented in the first chapter is carried out. The first comparative tool to be analyzed will be the diversity, and as we will see it is always higher, under quite general reasonable conditions, for the questionnaires based on the fuzzy rating scale. This supports the idea that this scale allows us to capture better the subjectivity and diversity of responses, as it was mentioned at the beginning of this work.

Then, an inferential approach based on a bootstrapped test about the equality of variances/standard deviations for independent populations is presented and applied to two questionnaires-based case studies. By comparing the p-values obtained for each rating scale, it will be determined if the chosen scale has influence on the test result.

Finally, a descriptive analysis of the three scales is performed in the chapter. On one hand, all the scale measures introduced in Chapter 2, but M-estimators which should be studied in more depth, are calculated for each rating scale, making use of the double response questionnaires-based case studies and of the usual numerical and fuzzy linguistic encodings of Likert data. On the other hand, fuzzy rating responses are simulated and associated with Likert responses by means of a 'Likertization' criterion which will be explained and validated with the questionnaires. Then, each 'Likertized' datum is encoded by means of a fuzzy linguistic scale. In this way, with the responses available in the three scales, the value of the different dispersion estimators is calculated and compared among the scales. We will see that the results obtained with this descriptive approach based on simulations are the same as those obtained for the case studies and they are also coherent with the conclusions for the standard deviation in the inferential analysis.

Each of the chapters ends up with a brief section devoted to summarize its main conclusions and contributions, followed by a list of publications I coauthored where these findings are described in full.

The work for this dissertation concludes with some final comments and suggestions concerning open problems in connection with the topic covered in it.

Prólogo

Los datos imprecisos pueden encontrarse en muchas situaciones de la vida cotidiana. Campos como la Ingeniería, las Ciencias Biomédicas o las Ciencias Sociales tratan habitualmente con este tipo de datos. Por ejemplo, la optimización de sistemas de control industriales, el diagnóstico que determina un médico sobre un paciente o la valoración que hace un cliente de un producto son cuestiones que pueden involucrar datos imprecisos. Los números difusos son los encargados de expresar y modelar este tipo de datos.

En el diseño de cuestionarios que tratan de recabar valoraciones sobre aspectos que no son medibles de forma exacta (como la satisfacción, la calidad percibida, la percepción...) es habitual recurrir al empleo de escalas tipo Likert. A pesar de que su aplicación es sencilla y no requieren preparación previa de los encuestados, cuando se quiere explotar la información contenida en los datos provenientes de estos cuestionarios, la metodología estadística disponible es bastante limitada. Esto se debe en gran parte a que se trata de escalas discretas con un número muy reducido de respuestas a elegir para cada pregunta (normalmente 4 o 5). Para soslayar este inconveniente, se han propuesto en la literatura algunas alternativas al empleo de las escalas Likert:

- Por un lado, las escalas visuales analógicas, que piden al encuestado especificar el punto o valor exacto en una barra o intervalo acotado que mejor exprese su grado de acuerdo con la declaración o propiedad enunciada. Aunque se trata de una alternativa de escala continua, que recoge la diversidad de respuestas y se adapta muy bien al tratamiento estadístico, no parece natural exigir tanta exactitud en un contexto de por sí impreciso.
- Por otro lado, la codificación a posteriori (normalmente realizada por expertos) de las respuestas Likert mediante números difusos procedentes de una escala lingüística. A pesar de que con este tipo de escala captamos mejor la imprecisión inherente a las respuestas, se trata, al igual que la Likert, de una escala discreta con cardinal pequeño, lo que conlleva una clara pérdida en subjetividad y diversidad de valoraciones.

Combinando las ideas de las dos alternativas anteriores (escala continua y difusa) se encuentra la conocida como escala de valoración difusa. Los cuestionarios que utilizan esta escala tienen un formato de respuesta libre, de tal forma que el encuestado representa gráficamente para cada pregunta el numero difuso que mejor expresa su respuesta, dentro un intervalo cuyos extremos son fijados de antemano. Así se garantiza que, por un lado, las respuestas reflejen la imprecisión intrínseca a las valoraciones a las que se refieren, y por otro, que haya una mayor diversidad de las mismas al tratarse de una escala continua, lo que conlleva una ganancia sustancial de información y fiabilidad en las conclusiones. Además, a lo largo de los últimos años se está desarrollando una metodología estadística para el análisis de datos difusos, que permite un tratamiento muy similar al del análisis de datos numéricos.

Buena parte de esta tesis se dedica a resumir la información estadística contenida en varios conjuntos de datos basados en valoraciones difusas. Esto se llevará a cabo mediante el análisis de su estimación de escala, entendiéndola como una medida representativa de la dispersión de los datos. Concretamente, el objetivo es resumir la escala extendiendo del caso real diferentes estimadores, muchos de ellos robustos. La medida de dispersión más conocida y habitual en el contexto difuso es la varianza de Fréchet (o la correspondiente desviación típica). Esta medida conserva las principales propiedades de la varianza (desviación típica) del caso real, entre las que se encuentra su elevada sensibilidad a la presencia de outliers u observaciones atípicas o extremas en los datos. Es deseable, por tanto, tratar de introducir y estudiar otras medidas de escala y analizar su comportamiento robusto.

La otra principal contribución de este trabajo es comparar las escalas de valoración mencionadas mediante estudios basados en simulaciones y también en ejemplos reales de cuestionarios. Concretamente, las escalas que van a compararse son la tipo Likert, codificada numéricamente y a través de escalas lingüísticas difusas, y la escala de valoración difusa. Dicha comparación se hará principalmente sobre la base de las medidas de dispersión/escala estudiadas en la primera parte de la memoria.

Con estos objetivos generales, el trabajo de esta tesis se ha estructurado como detallamos a continuación.

El **Capítulo 1** presenta las escalas de valoración más empleadas en la actualidad para medir magnitudes intrínsicamente imprecisas, ilustrando todas ellas con ejemplos. Se analizan las ventajas e inconvenientes más destacables de cada una, y se explica detalladamente el diseño de cuestionarios basados en la escala de valoración difusa.

Se recogen además los preliminares sobre números difusos, incluyendo algunas de sus representaciones, su aritmética y las distancias que serán usadas a lo largo de este trabajo. Se formaliza el mecanismo que genera aleatoriamente tales datos dentro de un contexto probabilístico y se recuerdan las principales medidas que resumen su distribución, tanto de tendencia central como de dispersión. Estas últimas serán analizadas en profundidad en el siguiente capítulo.

Se incluyen tres ejemplos reales de cuestionarios con respuestas Likert y de valoración difusa, cuyos conjuntos de datos asociados pueden encontrarse en la sección de apéndices al final de esta memoria. Se detallan también los dos procedimientos de simulación de datos difusos que serán empleados en varias ocasiones en este trabajo. Después de un pequeño estudio que analiza la sensibilidad de la varianza de Fréchet respecto a la forma de los datos, se concluye el capítulo motivando la necesidad de buscar alternativas más robustas a ella, así como el interés de comparar, desde un punto de vista de la dispersión, las diferentes escalas de valoración.

En el **Capítulo 2** se extienden del caso real varias medidas de escala para datos difusos. Después de probar algunas de sus propiedades más relevantes, como la invarianza por traslación, la equivarianza (en valor absoluto) por el producto por escalares o la consistencia fuerte, se analiza profundamente el comportamiento robusto de estas medidas, tanto desde un punto de vista teórico como a través de simulaciones. Así, se determina su punto de ruptura muestral finito cuando la muestra no contiene observaciones iguales, siendo estos valores coincidentes con los del caso real para todas las medidas.

Se explican en detalle los tres tipos de outliers que se van a considerar a lo largo del capítulo, los cuales han sido especialmente concebidos para este contexto difuso, y se corrobora de forma empírica e ilustrativa el valor de los puntos de ruptura para cada estimador. También se extiende la noción de curvas de sensibilidad del caso real al caso difuso, y se representan gráficamente.

Se dedica una breve sección a introducir los M-estimadores de escala. En ella se recoge un algoritmo para su computación, así como un análisis empírico de la robustez para dos conocidas funciones de pérdida: la función de pérdida de Huber y la función de pérdida de Tukey.

Por último, se ilustra el cálculo de todos los estimadores de escala presentados a lo largo del capítulo, utilizando para ello las respuestas difusas procedentes de los tres cuestionarios expuestos en el Capítulo 1.

En el **Capítulo 3** se desarrolla el estudio comparativo entre las escalas de valoración que fueron presentadas en el primer capítulo. La primera herramienta comparativa analizada será la diversidad, que como veremos es siempre mayor, bajo condiciones bastante generales, para los cuestionarios basados en la escala de valoración difusa. Esto refrenda la idea de que el uso de esta escala permite captar mejor la subjetividad y la diversidad de respuestas, según se planteó en el inicio de esta memoria.

A continuación, se presenta un enfoque inferencial basado en un test bootstrap de igualdad de varianzas/desviaciones típicas para poblaciones independientes, que es aplicado a dos cuestionarios reales. Comparando los p-valores obtenidos para cada escala de valoración, se determinará si la escala elegida influye en el resultado del test.

Por último y para finalizar este capítulo, se realiza un análisis descriptivo de las tres escalas. Por un lado, haciendo uso de los cuestionarios de respuesta doble y de las codificaciones numéricas y linguísticas difusas más usuales, se calculan todas las medidas de dispersión definidas en el Capítulo 2 para cada escala (no se incluyen los M-estimadores, que deberían ser estudiados con más profundidad). Por otro lado, se simulan respuestas difusas y se asocian a respuestas Likert mediante un criterio de 'Likertización' que será explicado y validado con los cuestionarios. Después, cada dato 'Likertizado' se codifica mediante una escala lingüística difusa. Así, con las respuestas disponibles en las tres escalas, se procede a calcular el valor de los distintos estimadores de dispersión y a compararlos entre las escalas. Veremos que los resultados de este enfoque descriptivo basado en simulaciones coinciden con los obtenidos para los cuestionarios, y que son coherentes con las conclusiones del análisis inferencial para la desviación típica.

Todos los capítulos finalizan con una breve sección dedicada a resumir las principales conclusiones y contribuciones del mismo, además de indicarse las publicaciones, de las que soy coautora, donde han sido recogidas estas aportaciones.

La memoria concluye con algunos comentarios finales y sugerencias de problemas abiertos estrechamente ligados con el tema de esta tesis.

Einleitung

Unscharfe Daten findet man in vielen Lebenssituationen. Themen wie Ingenieurwesen, Biomedizinische Wissenschaften oder Sozialwissenschaften befassen sich oft mit dieser Art von Daten. Beispiele für das Auftreten von unscharfen Daten sind die Optimierung von industriellen Steuerungen, die Bestimmung der Diagnose eines Patienten von einem Arzt, oder die Kundenbewertung eines Produkts. „Fuzzy" Zahlen können dabei helfen, diese Art von Daten korrekt darzustellen.

Die Likert-Skala wird häufig bei der Gestaltung von Fragebögen verwendet, um Eigenschaften oder Merkmale zu bewerten, welche nicht numerisch gemessen werden können (zum Beispiel: Zufriedenheit, wahrgenommene Qualität, Wahrnehmung). Obwohl sie leicht anzuwenden ist und man keine spezielle Ausbildung benötigt um sie zu verwenden, ist die verfügbare statistische Methodik zur Analyse der Daten aus diesen Fragebögen begrenzt. Dies ist vor allem auf die Tatsache zurückzuführen, dass Likert-Skalen sehr diskret mit einer sehr kleinen Anzahl von Antworten für jedes Element zu wählen sind (meist 4 oder 5). Um dies zu überwinden, wurden in der Literatur einige Alternativen vorgeschlagen:

- Auf der einen Seite, visuelle Analogskalen, welche es dem Befragten ermöglichen, den exakten Wert oder Punkt einer Linie zwischen zwei Endpunkten zu wählen (d.h.: entlang eines kompakten Intervalls), der seinen/ihren Stand der Vereinbarung einer Stellungnahme oder Fähigkeit darstellt. Obwohl man damit eine kontinuierliche Skala erhält, die Vielfalt der Antworten gewährleistet ist und auch die statistischen Schlussfolgerungen zulässig sind, ist es nicht realistisch, so viel Genauigkeit in Verbindung mit einem ungenauen Kontext zu verlangen.

- Auf der anderen Seite, die a-posteriori Kodierung (von ausgebildeten Experten) der Likert-Antworten mittels unscharfer Zahlen einer sprachlichen Skala. Mit dieser Skala wird die intrinsische Ungenauigkeit, die mit der Antwort verbunden ist besser erfasst, aber es ist eine diskrete Skala mit einer kleinen Grundzahl, wie die Likert-Skala und daher gehen die Subjektivität und Vielfalt der Bewertungen bis zu einem gewissen Grad verloren.

Die sogenannten „fuzzy" Bewertungsskalen integrieren die Fähigkeiten der beiden vorherigen Alternativen (eine kontinuierliche und unscharf bewertete Skala). Basierend auf diese Skala haben die Fragebögen ein „Freies Antwort"-Format, welches es dem Beurteiler erlaubt, die unscharfe Zahl zu zeichnen, welche seine/ihre Antwort innerhalb eines referenzinternen Intervalls besser ausdrückt. Auf diese Weise können die Werte mit der intrinsischen Ungenauigkeit, die mit den Bewertungen verbunden ist, bewältigt werden. Andererseits gehen Vielfalt und Subjektivität der Antworten nicht verloren, da es eine kontinuierliche Skala ist und es einen erheblichen Gewinn an Informationen und Genauigkeit in den Schlussfolgerungen gibt. Darüber hinaus wurde in den letzten Jahren eine statistische Methode entwickelt, um fuzzy-basierte Daten zu analysieren, sodass wir diese in ähnlicher Weise wie die numerischen Daten behandeln können.

Ein wesentlicher Teil der Arbeit widmet sich der Zusammenfassung der statistischen Information in Datensätzen, basierend auf einer Fuzzy-Bewertung durch die Analyse ihrer Schätzung, wobei sie als repräsentative Maßnahme der Streuung der unscharfen Werte, die die Datensätze liefern, gedacht ist. Konkret ist damit folgendes gemeint: Das Ziel ist die Zusammenfassung der Skala durch die Ausweitung der verschiedenen vor allem robusten Schätzungen aus dem reellwertigen Fall. Somit ist die Fréchet-Varianz (oder die entsprechende Standardabweichung) das bekannteste und auch oft angewendete Streuungsmaß im Fuzzy-Kontext. Es bewahrt die wichtigsten Eigenschaften der Varianz (Standardabweichung) für reellwertige Daten, aber es bewahrt auch die hohe Empfindlichkeit gegenüber dem Vorhandensein von Ausreißern oder atypische/extreme Beobachtungen von Daten. Daher ist es wünschenswert, andere Skalenmaße einzuführen und zu studieren und ihr robustes Verhalten zu analysieren.

Der andere wichtige Teil der Arbeit ist, die beschriebenen unterschiedlichen Bewertungsskalen mit unscharfen Daten durch einige auf Simulation basierenden Studien und auch auf realen Fragebögen zu vergleichen. Insbesondere werden beim Likert-Typ die numerischen und fuzzy-linguistisch codierten Skalen mit den Fuzzy-Rating-Skalen verglichen, hauptsächlich basierend auf den Streuungsmaßen vom ersten Teil der Arbeit.

Um diese Ziele zu erreichen, wurde der Inhalt dieser Dissertation folgend strukturiert:

Kapitel 1 stellt die verschiedenen Rating-Skalen vor, die derzeit am meisten eingesetzt werden, um Aspekte zu messen, die intrinsisch ungenau sind. Diese werden anhand einiger Beispiele dargestellt. Die wichtigsten positiven und negativen Eigenschaften werden im Zusammenhang mit der Verwendung jeder Skala analysiert, und die Gestaltung der Fragebögen, die auf der Rating-Skala basiert, wird im Detail erläutert.

Die vorläufig wichtigsten und unterstützenden Werkzeuge von unscharfen Zahlen werden hier aufgelistet, einschließlich einiger Darstellungen, die sie charakterisieren. Diese beinhalten ebenso die Arithmetik und die Metriken, die für diese Arbeit verwendet werden. Der zufällige Mechanismus, der die Erzeugung von Fuzzy-Werten innerhalb einer probabilistischen Einstellung modelliert wird dargestellt und die wichtigsten zusammenfassenden Maßnahmen/Parameter werden in Erinnerung gerufen, sowohl die Lokation als auch die Streuungsmaße. Die Letzteren werden im nächsten Kapitel genauer analysiert.

Drei Beispiele aus der Praxis für die Anwendung der Fuzzy-Bewertung und Likert-Skalen sind in den Fragebögen enthalten, ebenso findet man die kompletten Datensätze, die mit solchen Fragebögen verknüpft sind, im Anhang dieser Arbeit. Außerdem sind die beiden Simulationsverfahren der Fuzzy-Zahlen, die in verschiedenen Instanzen dieser Arbeit durchgeführt wurden, erläutert. Nach einer kleinen Studie, bestehend aus der Analyse der Empfindlichkeit der Fréchet-Varianz in Bezug auf die Form der Fuzzy-Daten, endet das Kapitel mit der Motivation für die Notwendigkeit für robuste Maße als Alternative zur Varianz, sowie das Interesse des Vergleichs der Rating-Skalen aus der Streuungs-Perspektive.

In **Kapitel 2** werden verschiedene Streuungsmaße erweitert, um mit realwertigen Daten und fuzzy-wertigen Daten umzugehen. Nach der Prüfung einiger der wichtigsten Eigenschaften, wie die Invarianz durch Translation, die absolute Äquivarianz durch das Produkt der Skalare oder die starke Konsistenz, wird das robuste Verhalten aus theoretischer Sicht und auch durch Simulation eingehend analysiert. Der Wert des endlichen Stichproben-Bruchpunktes wird für jede Schätzfunktion berechnet, wobei angenommen wird, dass es keine zwei identischen Beobachtungen in den Proben gibt. Die erzielten Ergebnisse werden zeigen, dass diese Werte der Bruchpunkte die gleichen sind wie im reellwertigen Fall.

Die drei Arten von Ausreißern, die in diesem Kapitel beschrieben werden, mussten speziell für diese Fuzzy-Szenario konzipiert werden, und sie werden ausführlich erklärt. Der empirische Bruchpunkt wird für jeden Schätzer berechnet, wobei die theoretischen Werte bestätigt werden. Der Begriff der Sensitivitätskurven wird vom reellen auf den fuzzy-wertigen Fall erweitert, und diese werden grafisch dargestellt.

Zum Abschluss befasst sich ein kurzer Abschnitt damit, die M-Schätzer für Streuung zu präsentieren. Ein Algorithmus zur Berechnung wird erläutert, sowie eine empirische Analyse der Robustheit für zwei bekannte Verlustfunktionen: die Huber Verlustfunktion und die Tukey bisquare Verlustfunktion.

Schließlich werden die Berechnungen für alle Streuungsschätzer, die in diesem Kapitel eingeführt wurden, für fuzzy-Antworten von den drei Fragebögen aus Kapitel 1 illustriert.

In **Kapitel 3** wird eine Vergleichsstudie der in Kapitel 1 vorgestellten Bewertungsskalen durchgeführt. Das erste Vergleichsinstrument das analysiert werden soll, ist die Diversität, und wie man sehen wird, ist sie immer höher, unter ganz allgemein gültigen Bedingungen, für die Fragebögen basierend auf der Fuzzy-Rating Skala. Wie am Anfang der Arbeit beschrieben, unterstützt dies die Idee, dass diese Skala es uns ermöglicht, die Subjektivität und Vielfalt der Antworten besser zu erfassen.

Es wird ein inferentieller Ansatz, der auf einem Bootstrap-Test über die Gleichheit von Varianzen/Standardabweichungen für unabhängige Populationen basiert, auf zwei Fragebogen-basierte Fallstudien vorgestellt und angewendet. Durch Vergleich der für jede Bewertungsskala erhaltenen p-Werte wird bestimmt, ob die gewählte Skala Einfluss auf das Testergebnis hat.

Zuletzt wird in diesem Kapitel eine beschreibende Analyse der drei Skalen durchgeführt. Einerseits werden alle Maße, die in Kapitel 2 vorgestellt wurden (M-Schätzer sollten näher untersucht werden), für jede Bewertungsskala und unter Verwendung der doppelten Antwortfragebogen-basierten Fallstudie und der üblichen numerischen und unscharfen linguistischen Kodierungen von Likert-Daten berechnet, andererseits werden Fuzzy-Rating-Antworten simuliert und mit Likert Antworten, sogenannte „Likertisierung” verknüpft, welche mit Fragebögen erklärt und validiert werden. Danach wird jedes „Likertisierte”-Datum durch eine fuzzy-linguistische Skala kodiert. Auf diese Weise wird der Wert der verschiedenen Streuungsschätzer verglichen, unter Berücksichtigung der Antworten aus den drei Skalen. Man sieht, dass die Ergebnisse, die man mit dem beschreibenden Ansatz auf Grundlage von Simulationen erhält, die gleichen wie die für die Fallstudien erhaltenen sind und dass sie auch mit den Schlussfolgerungen für die Standardabweichung in der Inferenzanalyse kohärent sind.

Jedes Kapitel endet mit einem kurzen Abschnitt, um die wichtigsten Schlussfolgerungen und Beiträge zusammenzufassen, gefolgt von einer Publikationsliste in der ich Koautor bin, wo die Ergebnisse vollständig beschrieben werden.

Die Arbeit für diese Dissertation endet mit einigen abschließenden Kommentaren und Anregungen zu offenen Problemen, im Zusammenhang mit dem darin besprochenen Thema.

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Chapter 1

On statistics with fuzzy data and measuring scale of fuzzy datasets

When one looks for the definition of Statistics in traditional reputed dictionaries, one usually finds it to be conceived as

- *"A branch of mathematics dealing with the collection, analysis, interpretation, and presentation of masses of numerical data; a collection of quantitative data"* (*Merriam-Webster Dictionary* [119]).
- *"The practice or science of collecting and analysing numerical data in large quantities, ..."* (*Oxford English Dictionary* [120]).

In this way, it has been commonly assumed that available data from the performance of random experiments can be expressed in a numerical scale. However, in practice this assumption sometimes fails. Thus, many human ratings (associated with perceptions/classifications/valuations/judgments/...) in random frameworks lead to data that cannot be expressed in a numerical scale, because they concern intrinsically imprecise-valued attributes.

1.1 Standard scales to rate intrinsically imprecise magnitudes

Aiming to model and handle these imprecise-valued data, some scales have been considered. Among the standard scales to rate intrinsically imprecise magnitudes, the best known are probably the Likert-type (or other categorical) and the visual analogue scales.

Likert scale-based ratings (Likert [73]) allow a rater to choose among a small number of pre-specified 'linguistic values', labeling different degrees of agreement/satisfaction/accomplishment/etc., the one that best represents rater's score.

As an example of a 5-point Likert scale-based items, see those in Figure 1.1. The five possible responses label in this example different degrees of agreement.

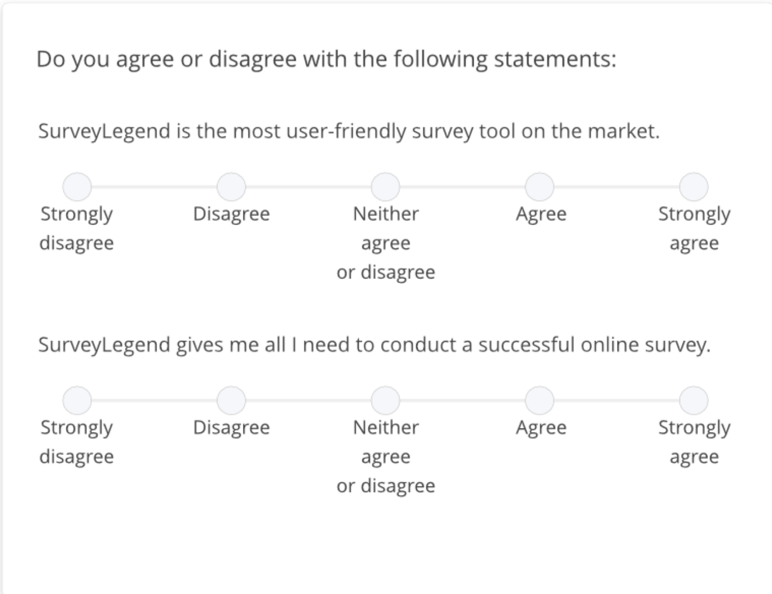


Figure 1.1: Example of two items designed so that possible responses are based on a 5-point Likert scale, in a survey made by *SurveyLegend* (https://www.surveylegend.com/user-guide/likert-scale/)

Among the *pros* of using Likert scales one can highlight the following:

– the ease of rating, irrespectively of the framework the rating is carried out;
– there is no need for a special training to use them, since common sense is generally enough; as a consequence, Likert scale-based ratings can be usually conducted irrespectively of the age, background, knowledge... of raters;
– the linguistic labels are coherent with the intrinsic imprecision associated with the rating based on these scales.

Among the *cons* that have been pointed out in the literature, one can mention the following:

– the number of possible 'values' to choose among is small (i.e., Likert scales are discrete with a small cardinal); consequently, the variability, adjustment, diversity, subjectivity of these ratings cannot be well captured with these scales;

– the choice of the 'value' that best represents rater's score is often a complex task because none of them accurately fit such a score;

– to analyze Likert-type data a posterior numerical-encoding of the involved Likert scale 'values' is usually considered; this makes all differences between consecutive 'values' to coincide, which is often unappropriate;

– the transition from a value to another one within the scale is rather abrupt;

– the number of applicable statistical techniques is quite limited and they are mainly based on the frequencies of different 'values' and, maybe, on their position in accordance with a certain ranking; therefore, relevant statistical information is often lost.

Visual analogue scale-based ratings (introduced by Freyd [38]) allow a rater to draw/choose within a given bounded interval (with labeled extremes) the point that best represents rater's score. The interval is often considered to be 10cm (or units, in general) long, and to get responses the distance of the respondent mark from the left endpoint is the visual analogue scale-based response.

As a usual example of a visual analogue scale, see the one in Figure 1.2 which is related to the rating of pain.

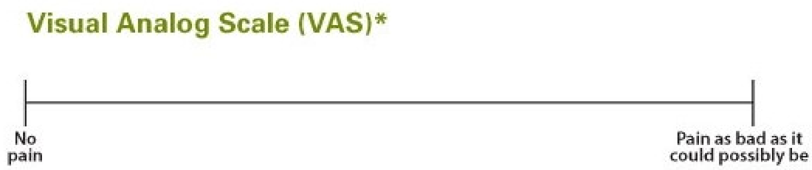


Figure 1.2: Example of a visual analogue scale-based item (http://www.rcemlearning.co.uk/references/pain-management-in-adults/)

Visual analogue scales sometimes appear combined with Likert scales, Likert labels being chosen as 'anchors' and leading to the so-called graphic rating scale. This combination is mainly considered to serve as a reference instead of getting a double (two scales) rating. Alternatively, or simultaneously, visual analogue scales sometimes appear combined with a (discrete) *numerical rating scale*, where numbers in the scale are usually integers, these numbers being chosen as 'anchors' to serve as a frame of references for respondents.

As an example of such a (multiple in this case) combined scale, see the one in Figure 1.3 which is also related to the rating of pain in connection with Crohn's disease.

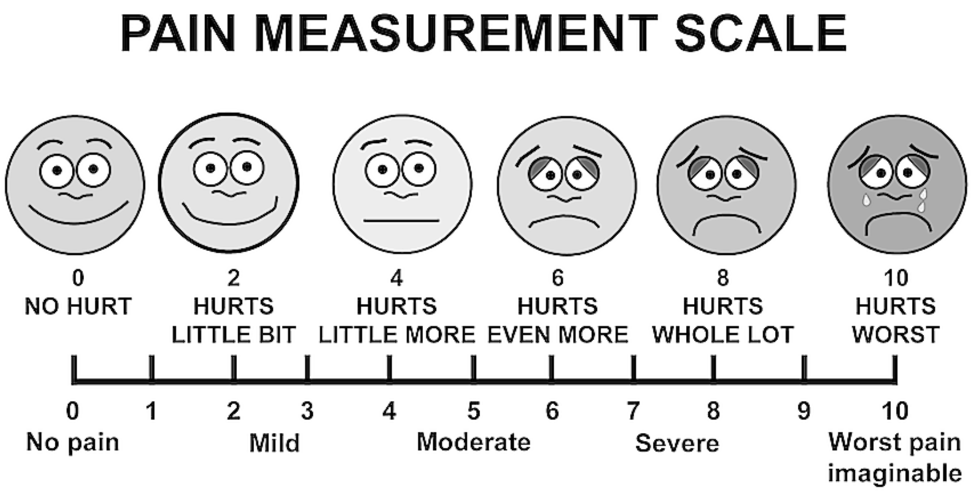


Figure 1.3: Example of a graphic/numerical rating scale-based item (http://cceffect.org/pain-scales-for-crohns-disease/). On the top and the bottom two graphic rating scales, on the middle a numerical one

Among the *pros* of using visual analogue scales one can highlight the following:

– the choice is made within a continuum; so, the ability to capture the variability, subjectivity and diversity which are inherent to these ratings is ensured;

– data drawn from the use of visual analogue scales can be statistically analyzed through traditional techniques and no relevant information is generally lost.

Among the *cons* that have been pointed out in the literature, one can mention the following:

– the choice of the point that best represents rater's score is usually neither easy nor natural;

– to require a full accuracy (i.e., to draw/choose a single real number) seems rather unrealistic in such an intrinsically imprecise context;

– surveys/questionnaries/... involving a visual analogue scale cannot be conducted in any framework, since they require either a paper-and-pencil or a computerized form to be filled by the rater;

– occasionally, problems with subject's ability to conceptually understand the rating method itself have been reported in the literature, so a certain training could be also required.

1.2 Fuzzy scales to rate intrinsically imprecise magnitudes

A rather natural question to be posed in this context is: why not fuzzy scales to rate intrinsically imprecise magnitudes? In the literature one can find several motivating and supporting quotations in this respect. Among some recent ones, one can select the following:

"The fuzzy scales establish a link between strongly defined measurements... and weakly defined measurements" (see Benoit [6]).

"One should consider a rich and expressive scale in which... something can be meaningful although we cannot name it" (see Ghneim [42]).

"Paradoxically, one of the principal contributions of fuzzy logic... is its high power of 'precisiation' of what is imprecise" (see Zadeh [140]).

The first quotation supports, in general, the use of fuzzy scales to rate intrinsically imprecise magnitudes. The second and third quotations are mainly addressed to what has been coined as fuzzy rating scales, that will be soon explained.

Fuzzy scales can be applied to overcome the limitations of standard scales to rate intrinsically imprecise magnitudes associated with random experiments, by modeling such an imprecision in terms of fuzzy numbers so that

– values capture 'differences in location',

– values capture 'differences in imprecision',

– and they can be mathematically treated.

1.2.1 Fuzzy numbers

Fuzzy numbers (also referred to by some authors as fuzzy intervals) are formalized as follows:

Definition 1.2.1. *A (bounded)* ***fuzzy number*** *is a function $\widetilde{U} : \mathbb{R} \to [0,1]$ such that it is upper semi-continuous, quasi-concave, normal (i.e., it takes on the value 1 for at least a real number), and its support (i.e., the set of all real numbers with nonzero image) is bounded. In this view (often referred to as the vertical definition), for each $x \in \mathbb{R}$, the value $\widetilde{U}(x)$ can be interpreted as the 'degree of compatibility of x with the property defined by $\widetilde{U}$'.*

Equivalently, a (bounded) ***fuzzy number*** *is a mapping* $\widetilde{U} : \mathbb{R} \to [0,1]$ *such that for all* $\alpha \in [0,1]$*, the* α*-level set defined as*

$$\widetilde{U}_\alpha = \begin{cases} \{x \in \mathbb{R} : \widetilde{U}(x) \geq \alpha\} & \text{if } \alpha \in (0,1] \\ \\ \mathrm{cl}\{x \in \mathbb{R} : \widetilde{U}(x) > 0\} & \text{if } \alpha = 0 \end{cases}$$

*with '*cl*' denoting the closure of the set, is a nonempty compact interval. This equivalent view is often known as the horizontal definition.*

The space of (bounded) fuzzy numbers will be denoted by $\mathscr{F}_c^*(\mathbb{R})$*.*

Real numbers can be viewed as special fuzzy numbers, since each real number x can be identified with the indicator function of the corresponding singleton $\mathbb{1}_{\{x\}}$.

In dealing with fuzzy number-valued data, some representations have been considered in the literature.

Definition 1.2.2. *The* ***inf/sup representation*** *of the fuzzy number* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ *is the vector-valued function* $\boldsymbol{\iota}_{\widetilde{U}} = (\iota^l_{\widetilde{U}}, \iota^r_{\widetilde{U}}) : [0,1] \to \{(x,y) \in \mathbb{R}^2 : x \leq y\}$ *where* $\iota^l_{\widetilde{U}}(\alpha) = \inf \widetilde{U}_\alpha$*,* $\iota^r_{\widetilde{U}}(\alpha) = \sup \widetilde{U}_\alpha$*.*

It should be emphasized that a fuzzy number in $\mathscr{F}_c^*(\mathbb{R})$ is uniquely determined by its inf/sup representation, i.e., by giving its 'boundaries'. Furthermore, a set of conditions can be established for the inf/sup representation to characterize a fuzzy number. The following result states this set of conditions (see, for instance, Goetschel and Voxman [47] and Ming [83] -Theorem 3.1, pp. 187-188-):

Proposition 1.2.1. *Given a fuzzy number* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$*, there exist two functions* $l : [0,1] \to \mathbb{R}$ *and* $r : [0,1] \to \mathbb{R}$ *satisfying that*

i) l *and* r *are*
  - *left-continuous on* $(0,1]$*,*
  - *right-continuous at* 0*,*
  - *non-increasing on* $[0,1]$*,*

ii) $-l(1) \leq r(1)$*,*

such that

$$\boldsymbol{\iota}_{\widetilde{U}}(\alpha) = (-l(\alpha), r(\alpha)) \text{ for all } \alpha \in [0,1].$$

Conversely, let $l : [0,1] \to \mathbb{R}$ *and* $r : [0,1] \to \mathbb{R}$ *be two functions satisfying Conditions i) and ii). Then, there exists a unique* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ *such that the vector-valued function* $(-l, r)$ *is its inf/sup representation.*

Another representation that has been used in providing an alternative interpretation and expression for the metric by Bertoluzza *et al.* [7] (see Gil *et al.* [45] for the interval-valued case, Trutschnig *et al.* [129] for a fuzzy general one, and recently Casals *et al.* [15] for fuzzy numbers) and in formalizing some statistical developments with fuzzy number-valued data is the following one:

Definition 1.2.3. *The* ***mid/spr representation*** *of the fuzzy number* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ *is the vector-valued function* $\boldsymbol{\eta}_{\widetilde{U}} = (\eta^m_{\widetilde{U}}, \eta^s_{\widetilde{U}}) : [0,1] \to \mathbb{R} \times [0,\infty)$ *such that* $\eta^m_{\widetilde{U}}(\alpha) = \operatorname{mid}\widetilde{U}_\alpha$, $\eta^s_{\widetilde{U}}(\alpha) = \operatorname{spr}\widetilde{U}_\alpha$, *where* $\operatorname{mid}\widetilde{U}_\alpha = (\inf\widetilde{U}_\alpha + \sup\widetilde{U}_\alpha)/2 =$ *centre of* $\widetilde{U}_\alpha$, $\operatorname{spr}\widetilde{U}_\alpha = (\sup\widetilde{U}_\alpha - \inf\widetilde{U}_\alpha)/2 =$ *radius of* $\widetilde{U}_\alpha$.

A fuzzy number in $\mathscr{F}_c^*(\mathbb{R})$ is uniquely determined by its mid/spr representation, i.e., by giving indicators of its 'location' and 'shape/imprecision'. One can verify that $\eta^s_{\widetilde{U}}$ is a left-continuous, non-increasing and non-negative function on $(0,1]$ and right-continuous at 0, whereas $\eta^m_{\widetilde{U}}$ is also left-continuous on $(0,1]$ and right-continuous at 0, but nothing can be ensured in general in connection with its monotonicity. In fact, one cannot establish a set of conditions the mid/spr representation should fulfill to characterize a fuzzy number.

Alternatively, aiming to extend the mid/spr representation of the interval-valued case in a way allowing us to establish a characterizing set of conditions, another representation has been introduced. This representation is based on considering a different indicator of the 'center' (instead of considering the mid function) along with a different indicator of the 'shape' (instead of considering the spr function), and it is possible to establish necessary conditions to determine a fuzzy number.

As indicator of the 'center' of a fuzzy number, the considered one has been that given by Yager [135] and later extended by De Campos and González [24] (as the 0.5-average index) and by Nasibov [86] (as the weighted averaging based on levels -see also Nasibov *et al.* [87]). For any $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$, the *$\varphi$-weighted averaging based on levels* of $\widetilde{U}$ ($\operatorname{wabl}^\varphi(\widetilde{U})$) is defined as the real number in the interior set $\operatorname{int}(\widetilde{U}_0)$ such that $\operatorname{wabl}^\varphi(\widetilde{U}) = \int_{[0,1]} \operatorname{mid}\widetilde{U}_\alpha \, d\varphi(\alpha)$, where φ is a weighting measure on the measurable space $([0,1], \mathcal{B}_{[0,1]})$ that can be formalized by means of an absolutely continuous probability measure with positive mass function on $(0,1)$. No stochastic meaning is actually associated with φ, but it allows us to weight the 'degrees of compatibility' given by the α-levels.

The $\operatorname{wabl}^\varphi$ is one of the three components of the new representation of fuzzy numbers. As indicators of the 'shape' of a fuzzy number, the level-wise 'deviations' with respect to the 'center' (more concretely, the following functions:

$$\mathrm{ldev}^{\varphi}_{\widetilde{U}} : [0,1] \to \mathbb{R}, \quad \alpha \mapsto \mathrm{ldev}^{\varphi}_{\widetilde{U}}(\alpha) = \mathrm{wabl}^{\varphi}(\widetilde{U}) - \inf \widetilde{U}_{\alpha},$$
$$\mathrm{rdev}^{\varphi}_{\widetilde{U}} : [0,1] \to \mathbb{R}, \quad \alpha \mapsto \mathrm{rdev}^{\varphi}_{\widetilde{U}}(\alpha) = \sup \widetilde{U}_{\alpha} - \mathrm{wabl}^{\varphi}(\widetilde{U}))$$

have been considered.

On the basis of these three components, we obtain the φ-wabl/ldev/rdev representation of fuzzy numbers as follows:

Definition 1.2.4. *Let φ be an absolutely continuous probability measure associated with the measurable space $([0,1], \mathcal{B}_{[0,1]})$ and having positive mass function on $(0,1)$. The **φ-wabl/ldev/rdev representation** of the fuzzy number $\widetilde{U} \in \mathscr{F}^*_c(\mathbb{R})$ is the vector-valued function $\boldsymbol{\upsilon}^{\varphi}_{\widetilde{U}} = (\upsilon^{w}_{\widetilde{U}}, \upsilon^{l}_{\widetilde{U}}, \upsilon^{r}_{\widetilde{U}}) : [0,1] \to \mathbb{R}^3$ such that $\upsilon^{w}_{\widetilde{U}}$ is constantly equal to $\mathrm{wabl}^{\varphi}(\widetilde{U})$, $\upsilon^{l}_{\widetilde{U}}(\alpha) = \mathrm{ldev}^{\varphi}_{\widetilde{U}}(\alpha)$ and $\upsilon^{r}_{\widetilde{U}}(\alpha) = \mathrm{rdev}^{\varphi}_{\widetilde{U}}(\alpha)$.*

For symmetric fuzzy number-valued data, the φ-wabl/ldev/rdev representation coincides with the mid/spr one, irrespective of φ. As for the inf/sup representation, one can state a necessary and sufficient set of conditions characterizing fuzzy numbers by their φ-wabl/ldev/rdev representation (see Sinova *et al.* [112]). Thus,

Proposition 1.2.2. *Given a fuzzy number $\widetilde{U} \in \mathscr{F}^*_c(\mathbb{R})$, there exist a value $m \in \mathbb{R}$ and two functions $l^* : [0,1] \to \mathbb{R}$ and $r^* : [0,1] \to \mathbb{R}$ satisfying that*

i) l^* and r^* are
- *left-continuous on $(0,1]$,*
- *right-continuous at 0,*
- *and non-increasing on $[0,1]$,*

ii) $-l^*(1) \leq r^*(1)$,

and such that for all $\alpha \in [0,1]$,

$$\widetilde{U}_{\alpha} = [m - l^*(\alpha), m + r^*(\alpha)] .$$

Conversely, let $m \in \mathbb{R}$ and let $l^* : [0,1] \to \mathbb{R}$ and $r^* : [0,1] \to \mathbb{R}$ be functions satisfying Conditions i) and ii). Then, there exists a unique $\widetilde{U} \in \mathscr{F}^*_c(\mathbb{R})$ such that for all $\alpha \in [0,1]$

$$\widetilde{U}_{\alpha} = [m - l^*(\alpha), m + r^*(\alpha)] .$$

Furthermore, if there is an absolutely continuous probability measure φ on $([0,1], \mathcal{B}_{[0,1]})$ with positive mass function on $(0,1)$ and such that

iii) $\displaystyle\int_{[0,1]} l^*(\alpha)\, d\varphi(\alpha) = \int_{[0,1]} r^*(\alpha)\, d\varphi(\alpha)$,

then, (m, l^*, r^*) is the φ-wabl/ldev/rdev representation of $\widetilde{U}$.

To illustrate the ideas in this subsection, one can consider a well-known and frequently used family of fuzzy numbers: the *trapezoidal fuzzy numbers*. If $a, b, c, d \in \mathbb{R}$ with $a \leq b \leq c \leq d$, the trapezoidal fuzzy number $\mathrm{Tra}(a, b, c, d)$ is given, in accordance with the vertical view, by

$$\mathrm{Tra}(a,b,c,d)(x) = \begin{cases} (x-a)/(b-a) & \text{if } x \in [a,b) \\ 1 & \text{if } x \in [b,c] \\ (d-x)/(d-c) & \text{if } x \in (c,d] \\ 0 & \text{otherwise} \end{cases}$$

and, in accordance with the horizontal view, and for each $\alpha \in [0,1]$ by

$$\left(\mathrm{Tra}(a,b,c,d)\right)_\alpha = [a + \alpha(b-a), d + \alpha(c-d)].$$

And the corresponding representations are given for each $\alpha \in [0,1]$ by

$$\boldsymbol{\iota}_{\mathrm{Tra}(a,b,c,d)}(\alpha) = \Big(\inf\left(\mathrm{Tra}(a,b,c,d)\right)_\alpha, \sup\left(\mathrm{Tra}(a,b,c,d)\right)_\alpha\Big)$$

$$= \Big((1-\alpha)\cdot a + \alpha \cdot b, (1-\alpha)\cdot d + \alpha\cdot c\Big),$$

$$\boldsymbol{\eta}_{\mathrm{Tra}(a,b,c,d)}(\alpha) = \Big(\mathrm{mid}\ \left(\mathrm{Tra}(a,b,c,d)\right)_\alpha, \mathrm{spr}\ \left(\mathrm{Tra}(a,b,c,d)\right)_\alpha\Big)$$

$$= \Big((1-\alpha)\cdot \underline{m} + \alpha\cdot\overline{m}, (1-\alpha)\cdot\underline{s} + \alpha\cdot\overline{s}\Big),$$

$$\boldsymbol{\upsilon}^{\varphi}_{\mathrm{Tra}(a,b,c,d)}(\alpha) = \Big(\mathrm{wabl}^{\varphi}(\mathrm{Tra}(a,b,c,d)), \mathrm{ldev}^{\varphi}_{\mathrm{Tra}(a,b,c,d)}(\alpha), \mathrm{rdev}^{\varphi}_{\mathrm{Tra}(a,b,c,d)}(\alpha)\Big)$$

$$= ((1-\vartheta_\alpha)\cdot\underline{m}+\vartheta_\alpha\cdot\overline{m}, (1-\alpha)\cdot\underline{s}+\alpha\cdot\overline{s}+(\vartheta_\alpha-\alpha)\cdot(\overline{m}-\underline{m}), (1-\alpha)\cdot\underline{s}+\alpha\cdot\overline{s}+(\vartheta_\alpha-\alpha)\cdot(\underline{m}-\overline{m})),$$

where $\underline{m} = \mathrm{mid}\ \left(\mathrm{Tra}(a,b,c,d)\right)_0 = (a+d)/2$, $\overline{m} = \mathrm{mid}\ \left(\mathrm{Tra}(a,b,c,d)\right)_1 = (b+c)/2$, $\underline{s} = \mathrm{spr}\ \left(\mathrm{Tra}(a,b,c,d)\right)_0 = (d-a)/2$, $\overline{s} = \mathrm{spr}\ \left(\mathrm{Tra}(a,b,c,d)\right)_1 = (c-b)/2$, and $\vartheta_\alpha = \int_{[0,1]} \alpha\, d\varphi(\alpha) \in (0,1)$.

A wider interesting family of fuzzy numbers, including the one of trapezoidal fuzzy numbers, is that of the *LR-fuzzy numbers* (see Dubois and Prade [36]). A fuzzy number $\widetilde{U}$ is said to be an LR-fuzzy number $\widetilde{U} = LR(a,b,c,d) = (b, c, b-a, d-c)_{LR}$ if it is given by

$$\widetilde{U}(x) = \begin{cases} L\left(\dfrac{b-x}{b-a}\right) & \text{if } x \in [a,b) \\ 1 & \text{if } x \in [b,c] \\ R\left(\dfrac{x-c}{d-c}\right) & \text{if } x \in (c,d] \\ 0 & \text{otherwise} \end{cases}$$

where $a, b, c, d \in \mathbb{R}$, $a \leq b \leq c \leq d$, and $L, R : [0,1] \to [0,1]$ are continuous non-increasing functions such that $L(x) = R(y) = 1$ iff $x = y = 0$ and $L(x) = R(y) = 0$ iff $x = y = 1$.

In case L and R are invertible functions one can easily check that

$$\inf \Big(LR(a,b,c,d)\Big)_\alpha = a + (b-a)\, L^{-1}(\alpha),\ \sup \Big(LR(a,b,c,d)\Big)_\alpha = d - (d-c)\, R^{-1}(\alpha)$$

for all $\alpha \in [0,1]$.

Along this work, and apart from trapezoidal, some LR-fuzzy numbers for which the so-called LU-parameterized representation (see Stefanini *et al.* [122], Sorini and Stefanini [118], Stefanini and Bede [121]) can be characterized by means of four real numbers (namely the extremes of their 0- and 1-level), are to be considered.

In particular, those involving quadratic functions (Π-curves) and functions with parametric monotonic Hermite-type interpolation, either using (2,2)-rational splines (LU_{1A} and LU_{1B}) or mixed exponential splines (LU_{2A} and LU_{2B}) (see Figure 1.4).

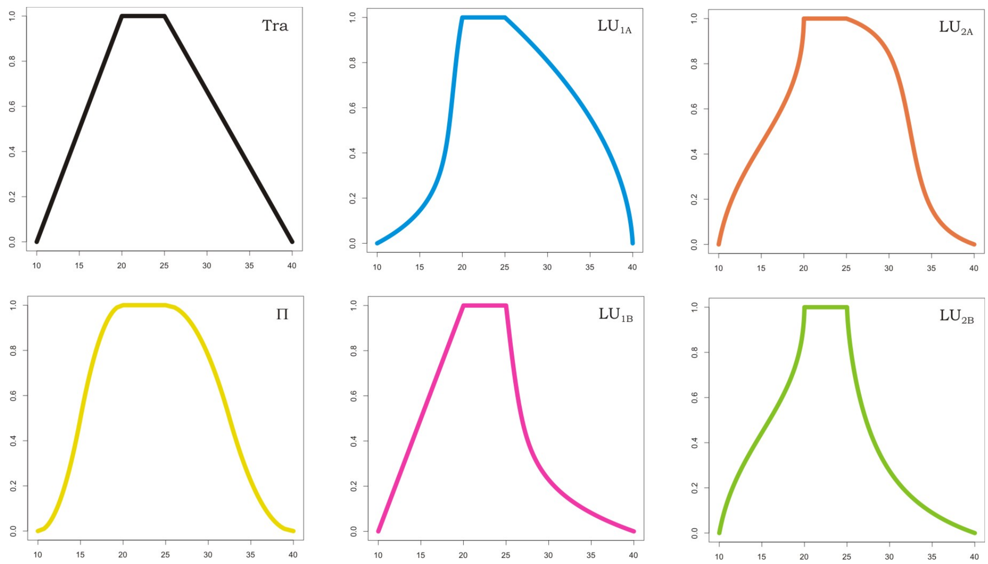


Figure 1.4: Six types of fuzzy numbers sharing core $[20, 25]$ and support $(10, 40)$ and differing in shape. On the left, trapezoidal (top) and Π-curve (bottom), along with four different LU-fuzzy numbers on the middle and the right

More specifically, if $\widetilde{U} \equiv LU(a,b,c,d)$ with $a = \inf \widetilde{U}_0$, $b = \inf \widetilde{U}_1$, $c = \sup \widetilde{U}_1$, $d = \sup \widetilde{U}_0$, and $LU \in \{\text{Tra}, \Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}\}$, then for each $\alpha \in [0,1]$

$$\widetilde{U}_\alpha = [a + l_{LU}(\alpha)(b-a), c + r_{LU}(\alpha)(d-c)]\,,$$

where the functions involved in the left and right arms can be seen in detail in Table 1.1.

Table 1.1: Expressions for functions l_{LU} and r_{LU} in the horizontal view of LU-fuzzy numbers with LU ranging on $\{\text{Tra}, \Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}\}$

LU	$l_{LU}(\alpha)$	$r_{LU}(\alpha)$
Tra	α	$1-\alpha$
Π	$\begin{cases} \sqrt{\alpha/2} & \text{if } \alpha < 1/2 \\ 1-\sqrt{(1-\alpha)/2} & \text{otherwise} \end{cases}$	$\begin{cases} 1-\sqrt{\alpha/2} & \text{if } \alpha < 1/2 \\ \sqrt{(1-\alpha)/2} & \text{otherwise} \end{cases}$
LU_{1A}	$\dfrac{\alpha^2+5\alpha(1-\alpha)}{1+3.5\alpha(1-\alpha)}$	$(1-\alpha)(1+0.9\alpha)$
LU_{1B}	α	$1-\dfrac{\alpha^2+5\alpha(1-\alpha)}{1+3.2\alpha(1-\alpha)}$
LU_{2A}	$\dfrac{\alpha^2(3-2\alpha)-0.5(1-\alpha)^{1.55}+0.5+0.05\alpha^{1.55}}{1.55}$	$1-\dfrac{\alpha^2(3-2\alpha)-5(1-\alpha)^{11}+5+5\alpha^{11}}{11}$
LU_{2B}	$\dfrac{\alpha^2(3-2\alpha)-0.5(1-\alpha)^{1.55}+0.5+0.05\alpha^{1.55}}{1.55}$	$1-\dfrac{\alpha^2(3-2\alpha)-5(1-\alpha)^{6.05}+5+0.05\alpha^{6.05}}{6.05}$

1.2.2 Fuzzy scales

A fuzzy linguistic variable (Zadeh [139]), or its associated FUZZY LINGUISTIC SCALE (FLS), is characterized by a 4-tuple $(\mathbf{X}, \mathsf{T}, \mathcal{S}, \mathbb{R})$, where

- $\mathbf{X}$ is the intrinsically imprecise variable/attribute to be either measured or observed,
- T is the set of imprecise 'values' of $\mathbf{X}$ (usually referred to as terms),
- $\mathcal{S}$ is the (fuzzy) semantic rule, i.e., a mapping

  $$\mathcal{S} : \mathsf{T} \to \mathscr{F}_c^*(\mathbb{R})$$

  where $\mathcal{S}(\mathsf{t})$ is the fuzzy number which has been considered to model the imprecise value $\mathsf{t} \in \mathsf{T}$.

As an example of an FLS, see the one in Figure 1.5. Actually, $\mathcal{S}$ is generally conceived as a posterior fuzzy number-valued encoding of a Likert scale.

Among the *pros* of using fuzzy linguistic scales one can highlight the following:

- the ease of the initial rating and no need for a special training, because of often corresponding to a posterior encoding of Likert-type values, and the encoding is usually made by trained experts;
- the values in the scale can cope (to some extent) with the intrinsic imprecision associated with this rating.

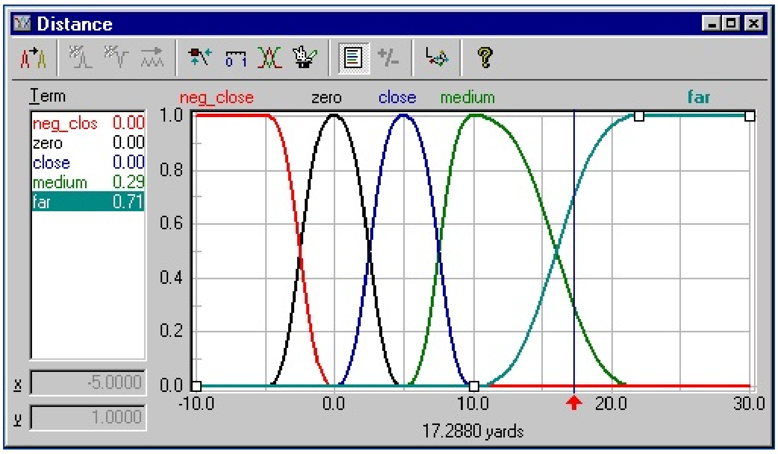


Figure 1.5: Example of a Fuzzy Linguistic Scale taken from fuzzyTECH's Linguistic Variable Editors webpage (http://www.fuzzytech.com/)

Among the *cons* to be pointed out, one could mention the following:

– the number of possible fuzzy values to choose among is small (it is a discrete scale with small cardinal), and the transition from a value to another within the scale is somewhat abrupt; so, the variability, adjustment, diversity, subjectivity of these ratings are to some extent lost;

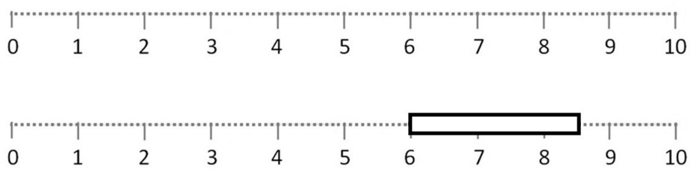
– the choice of the Likert-type 'value' that best represents rater's score is often a complex task because none of them accurately fit such a score, and the same happens with the fuzzy modeling of the chosen value;

– statistical techniques should be developed to analyze fuzzy number-valued data; in fact, this is at present a rather minor concern, since it is being overcome along the last years, as will be commented later.

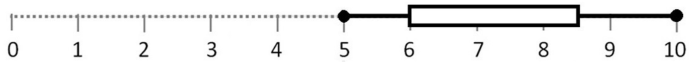
A FUZZY RATING SCALE (FRS), as introduced by Hesketh *et al.* [58], allows a rater to draw the fuzzy number that best represents rater's score. The guideline for the mechanism to draw such a fuzzy number (trapezoidal fuzzy number) is as follows:

Step 1. A reference bounded interval/segment is first considered. This is often chosen to be $[0, 10]$ or $[0, 100]$, but the choice of the intervals is not at all a constraint. The end-points are often labeled in accordance with their meaning referring to the degree of agreement, satisfaction, quality, and so on.

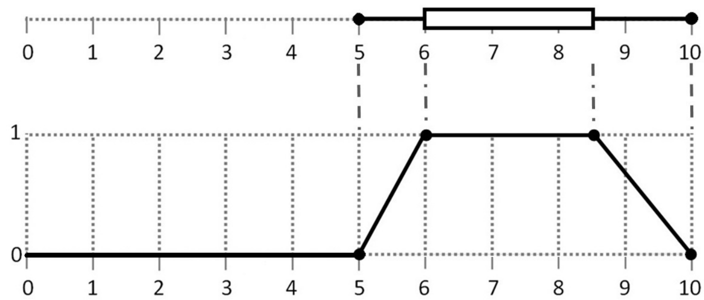
Step 2. The *core*, or 1-level set, associated with the response is determined. It corresponds to the interval consisting of the real values within the reference one which are considered to be as 'fully compatible' with the response.



Step 3. The *support*, or its closure or 0-level set, associated with the response is determined. It corresponds to the interval consisting of the real values within the referential that are considered to be as 'compatible to some extent' with the response, and it should be always included in the reference interval.



Step 4. The two intervals are 'linearly interpolated' to get a trapezoidal fuzzy number.



Among the *pros* of using fuzzy rating scales one can highlight the following:

– values in the FRS's can cope (to a full extent) with the intrinsic imprecision associated with this rating;

– any FRS means a continuum, and the transition from a value to another within the scale is fully gradual (both in location and precision);

– these scales are much richer and more expressive than any one based on a (unavoidably finite) natural language or its real/fuzzy-valued encoding (*"... something can be meaningful although we cannot name it"*);

– the flexibility of the FRS's allows raters to properly capture individual differences, whence the intrinsic variability, diversity and subjectivity are not lost (*"... precisiation of what is imprecise"*);

– values in the FRS's can be mathematically and computationally handled in a suitable way, since one can state arithmetic and distances

- preserving the meaning of fuzzy numbers,
- and allowing us to extend/adapt many concepts and developments from Statistics with real-valued data.

Among the *cons* to be pointed out, one could mention the following:

– surveys/questionnaires for which responses are based on an FRS cannot be conducted in all the frameworks, since they require either a paper-and-pencil or a computerized form to be filled by the rater;

– raters need either to have an adequate background or to be properly trained; it should be remarked that, although this is a clear concern, the training does not need to be highly time-consuming in most of cases, as will be shown in one of the case studies to be considered later;

– statistical techniques should be developed to analyze fuzzy number-valued data; in this respect, Hesketh *et al.* [62] have stated that *"... We are yet to see easily adapted packages that allow for researchers to use the fuzzy concept and then to apply appropriate statistical and other analyses to these in order to both test hypotheses and ensure that meaning is captured"*; as it has already been commented, this is partially a cons.

Regarding the last cons that has been highlighted in connection with the two described fuzzy scales, it should be pointed out that along the last years a methodology is being developed to statistically analyze fuzzy scale-based data (irrespectively of them being or not trapezoidal), and a package in the software R is additionally being stated to support its practical implementation. This work aims to study and discuss new developments related to the analysis of fuzzy data and, more concretely, related to the scale/dispersion of these data.

1.3 Fuzzy arithmetic and metrics

The key tools for the statistical methodology that is being developed around fuzzy data are:

- Arithmetic with fuzzy numbers + Metrics between fuzzy numbers;
- Random fuzzy numbers.

Combining arithmetic + metrics constitutes a key tool in this setting; why? To handle fuzzy data from a mathematical perspective, one can first pose a relevant question:

Can fuzzy data be treated as special functional data?

There is not a single answer to the last question, but the two following answers are compatible:

– Directly, NO. In applying functional arithmetic to handle elements in the space of (functional-valued) fuzzy numbers, one often moves out of the space and the fuzzy meaning is generally lost.

– Indirectly, YES. By using an appropriate arithmetic and suitable metrics, fuzzy numbers can be identified with elements in a convex cone of a Hilbert space of functions, and the arithmetic and metrics with fuzzy numbers with those in the Hilbert space of functions (see, for instance, González-Rodríguez *et al.* [49]).

Implications from the last identification will be commented along the work.

1.3.1 Arithmetic with fuzzy data

When fuzzy numbers are considered to model experimental data, statistics to analyze them are frequently based on two arithmetical operations, namely the sum and the product by scalars.

The common way to extend the sum and the product by a scalar from $\mathbb{R}$ to $\mathscr{F}_c^*(\mathbb{R})$ is to use Zadeh's extension principle [139], which is equivalent to consider level-wise the usual and more natural interval arithmetic. More concretely,

Definition 1.3.1. *Given $\widetilde{U}, \widetilde{V} \in \mathscr{F}_c^*(\mathbb{R})$, the **sum** of $\widetilde{U}$ and $\widetilde{V}$ is the fuzzy number $\widetilde{U} + \widetilde{V} \in \mathscr{F}_c^*(\mathbb{R})$ given by*

$$(\widetilde{U} + \widetilde{V})(t) = \sup_{y,z \in \mathbb{R}\,:\, y+z=t} \min\left\{\widetilde{U}(y), \widetilde{V}(z)\right\}.$$

Equivalently, for each $\alpha \in [0,1]$

$$(\widetilde{U} + \widetilde{V})_\alpha = \textit{Minkowski sum of } \widetilde{U}_\alpha \textit{ and } \widetilde{V}_\alpha = \left[\inf \widetilde{U}_\alpha + \inf \widetilde{V}_\alpha, \sup \widetilde{U}_\alpha + \sup \widetilde{V}_\alpha\right].$$

Definition 1.3.2. *Given $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ and $\gamma \in \mathbb{R}$, the **product** of $\widetilde{U}$ **by the scalar** γ is the fuzzy number $\gamma \cdot \widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ given by*

$$(\gamma \cdot \widetilde{U})(t) = \sup_{y \in \mathbb{R}\,:\, y=\gamma t} \widetilde{U}(y) = \begin{cases} \widetilde{U}(t/\gamma) & \textit{if } \gamma \neq 0 \\ \\ \mathbb{1}_{\{0\}}(t) & \textit{if } \gamma = 0. \end{cases}$$

Equivalently, for each $\alpha \in [0, 1]$

$$(\gamma \cdot \widetilde{U})_\alpha = \gamma \cdot \widetilde{U}_\alpha = \begin{cases} \left[\gamma \cdot \inf \widetilde{U}_\alpha, \gamma \cdot \sup \widetilde{U}_\alpha\right] & \text{if } \gamma \geq 0 \\ \\ \left[\gamma \cdot \sup \widetilde{U}_\alpha, \gamma \cdot \inf \widetilde{U}_\alpha\right] & \text{otherwise.} \end{cases}$$

which corresponds to consider level-wise the natural product of a set by a scalar.

It can be easily proved that for fixed invertible functions L and R, the family of LR-fuzzy numbers is closed under the sum and the product by scalars. More concretely,

$$LR(a, b, c, d) + LR(a', b', c', d') = LR(a + a', b + b', c + c', d + d'),$$

$$\gamma \cdot LR(a, b, c, d) = \begin{cases} LR(\gamma a, \gamma b, \gamma c, \gamma d) & \text{if } \gamma \geq 0 \\ \\ LR(\gamma d, \gamma c, \gamma b, \gamma a) & \text{otherwise.} \end{cases}$$

Remark 1.3.1. It has been above asserted that, in applying the functional arithmetic to handle elements in the space of fuzzy numbers, one often moves out of the space and the fuzzy meaning is generally lost. Actually, one can easily check that the fuzzy arithmetic differs substantially from the functional one (see Figure 1.6 for the sum and Figure 1.7 for the product by a scalar).

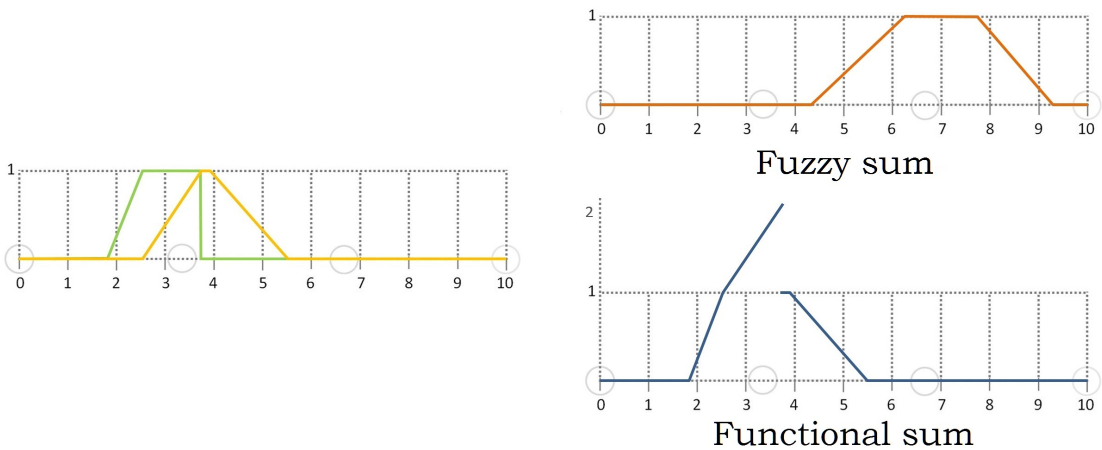


Figure 1.6: The sum of two fuzzy numbers (e.g., those on the left) is generally different depending on the fuzzy arithmetic (on the right top) or the functional arithmetic (on the right bottom) being considered

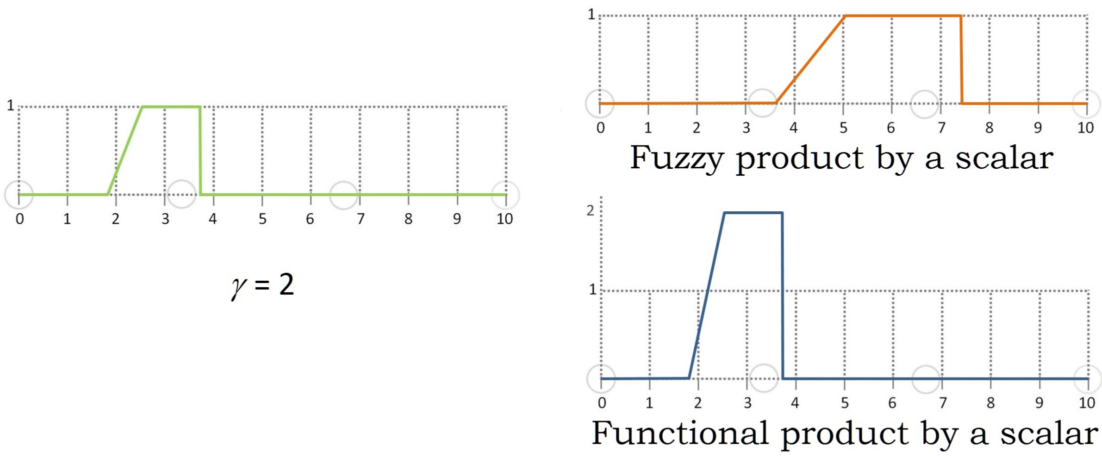


Figure 1.7: The product of a fuzzy number (on the left) by a scalar ($\gamma = 2$) is generally different depending on the fuzzy arithmetic (on the right top) or the functional arithmetic (on the right bottom) being considered

Remark 1.3.2. It should be especially highlighted that the space $(\mathscr{F}_c^*(\mathbb{R}), +, \cdot)$ has not linear but semilinear structure since

$$\widetilde{U} + (-1 \cdot \widetilde{U}) \neq \mathbb{1}_{\{\mathbf{0}\}} \text{ (neutral element of } +).$$

1.3.2 Metrics between fuzzy data

Due to the nonlinearity that has been pointed out in Remark 1.3.2, one cannot state a definition for the difference between fuzzy numbers that is always well-defined and simultaneously preserves the main properties of the difference between real values in connection with the sum. In fact, there exists a difference notion (Hukuhara's one) satisfying the last condition, but it cannot be defined for many interval and fuzzy number values.

This crucial drawback has been substantially overcome in developing statistics with fuzzy data by incorporating suitable distances between them. On one hand, distances will allow to 'translate' the equality of fuzzy numbers into the vanishing of the distance between them, as in the case of real values. On the other hand, appropriate distances also allow us *via* the support function to 'identify' fuzzy data with functional ones and fuzzy arithmetic with functional arithmetic (as will be remarked later). Furthermore, statistical concepts and methods for real-valued datasets involving metrics (e.g., dispersion measures, mean distance approaches, classification problems, etc.) could be extended by considering extended metrics.

Among the L^2 metrics between fuzzy numbers, the one introduced by Bertoluzza *et al.* [7] (see Montenegro *et al.* [84] for the generalized version in this work, Gil *et al.* [45] to justify its mid/spr expression, and Trutschnig *et al.* [129] for fuzzy numbers and higher dimensional fuzzy values), that generalizes the one by Diamond and Kloeden [30] and, hence, extending Vitale's [133] one for interval values (in fact introduced for compact values) is given as follows:

Definition 1.3.3. (Bertoluzza *et al.* [7]) *Let* $\widetilde{U}, \widetilde{V} \in \mathscr{F}_c^*(\mathbb{R})$, *and assume* $\theta \in (0,1]$ *and* $\varphi =$ *probability measure associated with a continuous distribution with support in* $(0,1)$. *The* $\boldsymbol{(\varphi, \theta)}$***-distance*** *between* $\widetilde{U}$ *and* $\widetilde{V}$ *is defined as*

$$D_\theta^\varphi(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left([\operatorname{mid} \widetilde{U}_\alpha - \operatorname{mid} \widetilde{V}_\alpha]^2 + \theta[\operatorname{spr} \widetilde{U}_\alpha - \operatorname{spr} \widetilde{V}_\alpha]^2\right) \, d\varphi(\alpha)}.$$

In particular, the ***2-norm-distance*** (Diamond and Kloeden [30]) *corresponds to* $\rho_2 = D_1^\ell$, *with* ℓ *being the Lebesgue measure on* $[0,1]$, *that is,*

$$\rho_2(\widetilde{U}, \widetilde{V}) = D_1^\ell(\widetilde{U}, \widetilde{V}) = \sqrt{\frac{1}{2}\int_{[0,1]} \left(\left[\inf \widetilde{U}_\alpha - \inf \widetilde{V}_\alpha\right]^2 + \left[\sup \widetilde{U}_\alpha - \sup \widetilde{V}_\alpha\right]^2\right) d\alpha}.$$

In connection with ρ_2, one can later make use of the fact that

- for $LU \in \{\Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}, \mathrm{Tri}, \mathrm{TriS}\}$ (where $\mathrm{Tri}(a,b,c,d) = \mathrm{Tra}(a, (b+c)/2, (b+c)/2, d)$ and $\mathrm{TriS}(a,b,c,d) = \mathrm{Tra}(a, (a+d)/2, (a+d)/2, d)$), distances between $\mathrm{Tra}(a,b,c,d)$ and $LU(a,b,c,d)$ can be obtained exactly and have been gathered in Table 1.2;

Table 1.2: Exact expressions for the ρ_2 distances between fuzzy numbers $\mathrm{Tra}(a,b,c,d)$ and $LU(a,b,c,d)$ with LU ranging on $\{\Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}, \mathrm{Tri}, \mathrm{TriS}\}$

$\boldsymbol{LU}$	$\left[\boldsymbol{\rho_2(\mathrm{Tra}(a,b,c,d), LU(a,b,c,d))}\right]^2$
Π	$0.00416667\,(a-b)^2 + 0.00416667\,(c-d)^2$
LU_{1A}	$0.03129046\,(a-b)^2 + 0.0135\,(c-d)^2$
LU_{1B}	$0.03724265\,(c-d)^2$
LU_{2A}	$0.00171940\,(a-b)^2 + 0.01325610\,(c-d)^2$
LU_{2B}	$0.00171940\,(a-b)^2 + 0.05307135\,(c-d)^2$
Tri	$(c-b)^2/12$
TriS	$[(b-a+c-d)^2 + (c-b)^2]/12$

Table 1.3: Exact expressions for the ρ_2 distances between fuzzy numbers $LU(a_1, b_1, c_1, d_1)$ and $LU(a_2, b_2, c_2, d_2)$ with LU ranging on $\{\mathrm{Tra}, \Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}\}$

$\boldsymbol{LU}$	$\left[\boldsymbol{\rho_2(LU(a_1,b_1,c_1,d_1), LU(a_2,b_2,c_2,d_2))}\right]^{\mathbf{2}}$
Tra	$[(a_1-a_2)^2+(b_1-b_2)^2+(a_1-a_2)(b_1-b_2)]/6$ $+[(c_1-c_2)^2+(d_1-d_2)^2+(c_1-c_2)(d_1-d_2)]/6$
Π	$[7(a_1-a_2)^2+7(b_1-b_2)^2+10(a_1-a_2)(b_1-b_2)]/48$ $+[7(c_1-c_2)^2+7(d_1-d_2)^2+10(c_1-c_2)(d_1-d_2)]/48$
LU_{1A}	$0.06622634(a_1-a_2)^2+0.2900244(b_1-b_2)^2+0.14374922(a_1-a_2)(b_1-b_2)$ $+0.10516667(c_1-c_2)^2+0.25516667(d_1-d_2)^2+0.13966667(c_1-c_2)(d_1-d_2)$
LU_{1B}	$0.16666666(a_1-a_2)^2+0.16666666(b_1-b_2)^2+0.16666666(a_1-a_2)(b_1-b_2)$ $+0.30881332(c_1-c_2)^2+0.06176266(d_1-d_2)^2+0.12942402(c_1-c_2)(d_1-d_2)$
LU_{2A}	$0.16429256(a_1-a_2)^2+0.19560186(b_1-b_2)^2+0.14010559(a_1-a_2)(b_1-b_2)$ $+0.13773163(c_1-c_2)^2+0.13773163(d_1-d_2)^2+0.22453675(c_1-c_2)(d_1-d_2)$
LU_{2B}	$0.16429256(a_1-a_2)^2+0.19560186(b_1-b_2)^2+0.14010559(a_1-a_2)(b_1-b_2)$ $+0.34394341(c_1-c_2)^2+0.05090666(d_1-d_2)^2+0.10514993(c_1-c_2)(d_1-d_2)$

- for $LU \in \{\mathrm{Tra}, \Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}\}$, distances between $LU(a_1, b_1, c_1, d_1)$ and $LU(a_2, b_2, c_2, d_2)$ can be exactly obtained and have been gathered in Table 1.3.

It should be pointed out that φ has not a stochastic but a simple weighting meaning. Actually, it weighs the importance of distances at different levels, and a suitable model for this purpose is to consider beta density functions.

On the other hand, the choice of θ weighs the importance of the squared distance between the spreads in contrast to that between the midpoints. In particular, the choice $\theta = 1$ is equivalent to weigh only and uniformly the two squared Euclidean distances between the extreme points of the level sets, so that

$$D_1^{\varphi}(\widetilde{U}, \widetilde{V}) = \sqrt{\frac{1}{2}\int_{[0,1]} \left(\left[\inf \widetilde{U}_\alpha - \inf \widetilde{V}_\alpha\right]^2 + \left[\sup \widetilde{U}_\alpha - \sup \widetilde{V}_\alpha\right]^2 \right) d\varphi(\alpha)},$$

and the choice $\theta = 1/3$ is equivalent to weight uniformly all the squared Euclidean distances between the convex linear extreme points of the level sets, so that

$$D_{1/3}^{\varphi}(\widetilde{U}, \widetilde{V}) = \sqrt{\int_{[0,1]} \left(\int_{[0,1]} \left[\widetilde{U}_\alpha^{[t]} - \widetilde{V}_\alpha^{[t]}\right]^2 dt \right) d\varphi(\alpha)}$$

where $\widetilde{U}_\alpha^{[t]} = t \cdot \sup \widetilde{U}_\alpha + (1-t) \cdot \inf \widetilde{U}_\alpha$.

The following result summarizes several valuable properties of distances D_θ^{φ}.

Proposition 1.3.1. (González-Rodríguez *et al.* [49]) *Let $\theta \in (0,1]$ and let φ be an absolutely continuous probability measure on* $([0,1],\mathcal{B}_{[0,1]})$ *with the mass function being positive in* $(0,1)$.

Let $\mathbb{H}_2 = \{L^2$*-type real-valued functions defined on* $[0,1]\times\{-1,1\}$ *w.r.t.* $\ell\otimes\lambda_1\}$ *(with* $\lambda_1(-1)=\lambda_1(1)=0.5$*). Then, the* L^2 mid/spr*-based metric satisfies that*

i) D_θ^φ *is an* L^2*-type metric on* $\mathscr{F}_c^*(\mathbb{R})$.

ii) D_θ^φ *is translational invariant, i.e.,* $D_\theta^\varphi(\widetilde{U}+\widetilde{W},\widetilde{V}+\widetilde{W}) = D_\theta^\varphi(\widetilde{U},\widetilde{V})$*, and 'rotational' invariant, where this is understood as* $D_\theta^\varphi((-1)\cdot\widetilde{U},(-1)\cdot\widetilde{V}) = D_\theta^\varphi(\widetilde{U},\widetilde{V})$.

iii) D_θ^φ *is topologically equivalent to* ρ_2^φ*, where* $\rho_2^\varphi = D_1^\varphi$.

iv) $(\mathscr{F}_c^*(\mathbb{R}), D_\theta^\varphi)$ *is a separable metric space.*

v) The support function $s : \mathscr{F}_c^*(\mathbb{R}) \to \mathbb{H}_2$ *(with* $s(\widetilde{U}) = s_{\widetilde{U}}$ *and* $s_{\widetilde{U}}(\alpha,-1) = -\inf\widetilde{U}_\alpha$, $s_{\widetilde{U}}(\alpha,1)=\sup\widetilde{U}_\alpha$*) states an isometric embedding of* $\mathscr{F}_c^*(\mathbb{R})$ *with the fuzzy arithmetic and* D_θ^φ *onto a convex cone of the Hilbert space* $\mathbb{H}_2$ *with the functional arithmetic and the distance induced by the norm*

$$\left\|h-h'\right\|_\theta^\varphi = \sqrt{\langle h-h', h-h'\rangle_\theta^\varphi},$$

where the inner product is given by

$$\left\langle f,g\right\rangle_\theta^\varphi = \sum_{u=-1,1}\int_{[0,1]} [\operatorname{mid} f(\alpha,u)\cdot\operatorname{mid} g(\alpha,u) + \operatorname{spr} f(\alpha,u)\cdot\operatorname{spr} g(\alpha,u)]\, d\varphi(\alpha)$$

and

$$\operatorname{mid} f(\alpha,u) = \frac{f(\alpha,u)-f(\alpha,-u)}{2}, \quad \operatorname{spr} f(\alpha,u) = \frac{f(\alpha,u)+f(\alpha,-u)}{2}.$$

Remark 1.3.3. An immediate and crucial implication from Proposition 1.3.1.*v*) is that any fuzzy number $\widetilde{U}\in\mathscr{F}_c^*(\mathbb{R})$ can be identified with the corresponding function $s_{\widetilde{U}}$ and this identification is accompanied by the correspondences between the usual arithmetics and L^2 metrics. Consequently, data in the setting of fuzzy number-valued data with the fuzzy arithmetic and the metric D_θ^φ can be systematically translated into data in the setting of functional data with the functional arithmetic and the metric based on the associated norm. In this way, despite the fact that fuzzy data should not be treated directly as functional data, they can be treated as functional data by considering the identification *via* the support function.

Then, we can now assert formally as a relevant implication for statistical purposes that several developments in Functional Data Analysis could be particularized to fuzzy number-valued data by using the adequate identifications and correspondences. However, it should be guaranteed that the resulting elements/outputs remain in the cone $s\left(\mathscr{F}_c^*(\mathbb{R})\right)$. In case either the functional developments become very complex or the resulting elements/outputs are out of $s\left(\mathscr{F}_c^*(\mathbb{R})\right)$, *ad hoc* techniques should be developed, as we will show in the next chapters.

The mid/spr-based L^2 metric has been shown to be very suitable in the development of statistical methodology for experimental fuzzy number-valued data (see, for instance, the recent reviews by Blanco-Fernández *et al.* [8, 9, 10, 11] and Gil *et al.* [43]) summarizing many of these statistical methods.

Among the L^1 metrics between fuzzy numbers, the one introduced by Sinova *et al.* [109] (see Sinova *et al.* [108] for fuzzy numbers and higher dimensional fuzzy values), and the one extending Diamond and Kloeden [30] and, hence, extending Vitale's [133] one for interval values (in fact introduced for compact values), are given as follows:

Definition 1.3.4. (Sinova *et al.* [109]) *Let $\widetilde{U}, \widetilde{V} \in \mathscr{F}_c^*(\mathbb{R})$, and assume $\theta \in (0,1]$ and $\varphi =$ probability measure associated with a continuous distribution with support in $(0,1)$. The **(φ,θ)-wabl/ldev/rdev-based** L^1 **metric** between $\widetilde{U}$ and $\widetilde{V}$ is defined as*

$$\mathscr{D}_\theta^\varphi(\widetilde{U}, \widetilde{V}) = |\mathrm{wabl}^\varphi(\widetilde{U}) - \mathrm{wabl}^\varphi(\widetilde{V})|$$
$$+\frac{\theta}{2}\int_{[0,1]} |\mathrm{ldev}^\varphi_{\widetilde{U}}(\alpha) - \mathrm{ldev}^\varphi_{\widetilde{V}}(\alpha)|\, d\varphi(\alpha) + \frac{\theta}{2}\int_{[0,1]} |\mathrm{rdev}^\varphi_{\widetilde{U}}(\alpha) - \mathrm{rdev}^\varphi_{\widetilde{V}}(\alpha)|\, d\varphi(\alpha).$$

*The **1-norm-distance*** (Diamond and Kloeden [30]) *between $\widetilde{U}$ and $\widetilde{V}$ is defined as*

$$\rho_1(\widetilde{U}, \widetilde{V}) = \frac{1}{2}\int_{[0,1]} \left(\left|\inf \widetilde{U}_\alpha - \inf \widetilde{V}_\alpha\right| + \left|\sup \widetilde{U}_\alpha - \sup \widetilde{V}_\alpha\right|\right) d\alpha.$$

If $d\alpha$ is extended to $d\varphi(\alpha)$, the metric is denoted by ρ_1^φ.

In connection with ρ_1, we will later make use of the fact that the distance between $\mathrm{Tra}(a_1, b_1, c_1, d_1)$ and $\mathrm{Tra}(a_2, b_2, c_2, d_2)$ can be exactly obtained and equals

$$\rho_1\Big(\mathrm{Tra}(a_1, b_1, c_1, d_1), \mathrm{Tra}(a_2, b_2, c_2, d_2)\Big) = G(a_1 - a_2, b_1 - b_2) + G(c_1 - c_2, d_1 - d_2),$$

where

$$G(x,y) = \begin{cases} \dfrac{x|x| - y|y|}{4(x-y)} & \text{if } x \neq y \\[2ex] \dfrac{|y|}{2} & \text{otherwise.} \end{cases}$$

The following result summarizes several valuable properties of distances $\mathscr{D}_\theta^\varphi$ and ρ_1^φ.

Proposition 1.3.2. (Sinova *et al.* [109], Sinova and López [114]) *Let* $\theta \in (0,1]$ *and let* φ *be an arbitrarily fixed absolutely continuous probability measure on* $([0,1], \mathcal{B}_{[0,1]})$ *with positive mass function on* $(0,1)$. *Then,*

i) $\mathscr{D}_\theta^\varphi$ *and* ρ_1^φ *are* L^1 *metrics on* $\mathscr{F}_c^*(\mathbb{R})$*, and they are both translational and rotational invariant.*

ii) $\mathscr{D}_\theta^\varphi$ *is topologically equivalent to* ρ_1^φ.

iii) $(\mathscr{F}_c^*(\mathbb{R}), \mathscr{D}_\theta^\varphi)$ *and* $(\mathscr{F}_c^*(\mathbb{R}), \rho_1^\varphi)$ *are separable metric spaces.*

iv) For a fixed φ*, the function* $\boldsymbol{\upsilon}^\varphi : \mathscr{F}_c^*(\mathbb{R}) \to \mathbb{B}_1^\star = \{L^1$*-type 3-dimensional vector-valued functions defined on* $[0,1]\}$ *satisfies that*

- $\boldsymbol{\upsilon}^\varphi(\widetilde{U} + \widetilde{V}) = \boldsymbol{\upsilon}^\varphi(\widetilde{U}) + \boldsymbol{\upsilon}^\varphi(\widetilde{V})$ *for all* $\widetilde{U}, \widetilde{V} \in \mathscr{F}_c^*(\mathbb{R})$,
- $\boldsymbol{\upsilon}^\varphi(\gamma \cdot \widetilde{U}) = \gamma \cdot \boldsymbol{\upsilon}^\varphi(\widetilde{U})$ *for all* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ *and* $\gamma > 0$,
- *and the* $\boldsymbol{\upsilon}^\varphi$ *function preserves the semilinearity of* $\mathscr{F}_c^*(\mathbb{R})$ *and relates the fuzzy arithmetic to the functional arithmetic in such a way that* $(\mathscr{F}_c^*(\mathbb{R}), \mathscr{D}_\theta^\varphi)$ *can be isometrically embedded into a convex cone of the Banach space* $\left(\mathbb{B}_1^\star, \|\cdot\|_\theta^{\varphi\star}\right)$ *where*

$$\|f-g\|_\theta^{\varphi\star} = \int_{[0,1]} \left(|f_1(\alpha) - g_1(\alpha)| + \frac{\theta}{2} \cdot |f_2(\alpha) - g_2(\alpha)| + \frac{\theta}{2} \cdot |f_3(\alpha) - g_3(\alpha)|\right) d\varphi(\alpha)$$

  for $f = (f_1, f_2, f_3), g = (g_1, g_2, g_3) \in \mathbb{B}_1^\star$.

The function $\boldsymbol{\iota} : \mathscr{F}_c^*(\mathbb{R}) \to \mathbb{B}_1 = \{L^1$*-type 2-dimensional vector-valued functions defined on* $[0,1]\}$ *satisfies that*

- $\boldsymbol{\iota}(\widetilde{U} + \widetilde{V}) = \boldsymbol{\iota}(\widetilde{U}) + \boldsymbol{\iota}(\widetilde{V})$ *for all* $\widetilde{U}, \widetilde{V} \in \mathscr{F}_c^*(\mathbb{R})$,
- $\boldsymbol{\iota}(\gamma \cdot \widetilde{U}) = \gamma \cdot \boldsymbol{\iota}(\widetilde{U})$ *for all* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ *and* $\gamma > 0$,
- *and the* $\boldsymbol{\iota}$ *function preserves the semilinearity of* $\mathscr{F}_c^*(\mathbb{R})$ *and relates the fuzzy arithmetic to the functional arithmetic in such a way that* $(\mathscr{F}_c^*(\mathbb{R}), \rho_1^\varphi)$ *can be isometrically embedded into a convex cone of the Banach space* $\left(\mathbb{B}_1, \|\cdot\|_1^\varphi\right)$ *with*

$$\left\|f-g\right\|_1^\varphi = \int_{[0,1]} \left(\frac{1}{2}|f_1(\alpha) - g_1(\alpha)| + \frac{1}{2}|f_2(\alpha) - g_2(\alpha)|\right) d\varphi(\alpha)$$

  for $f = (f_1, f_2), g = (g_1, g_2) \in \mathbb{B}_1$.

As we will see later, the L^1 metrics have been shown to be valuable in connection with the *ad hoc* development of some robust location measures (see Sinova *et al.* [111, 109]).

1.4 Random fuzzy numbers and relevant summary measures

In developing statistics with fuzzy data coming from intrinsically imprecise-valued attributes, random fuzzy numbers constitute a well-formalized model within the probabilistic setting for the random mechanisms generating such data. Random fuzzy numbers, originally coined as (one-dimensional) fuzzy random variables by Puri and Ralescu [94], integrate *randomness* (associated with the data generation) and *fuzziness* (associated with the data nature).

Definition 1.4.1. (Puri and Ralescu [94]) *Given a probability space* $(\Omega, \mathcal{A}, P)$*, an associated* ***random fuzzy number*** *(for short RFN) is a mapping* $\mathcal{X} : \Omega \to \mathscr{F}_c^*(\mathbb{R})$ *such that for all* $\alpha \in [0,1]$ *the interval-valued mapping* $\mathcal{X}_\alpha$*, such that* $\mathcal{X}_\alpha(\omega) = \big(\mathcal{X}(\omega)\big)_\alpha$ *for all* $\omega \in \Omega$*, is a compact random interval (i.e., a Borel-measurable mapping w.r.t. the topology induced by the Hausdorff metric in the space of the nonempty compact intervals).*

Equivalently, $\mathcal{X}$ *is an RFN if and only if* $s(\mathcal{X})$ *is an* $\mathbb{H}_2/\mathbb{B}_1^\star$*-valued random element, that is, a Borel mesurable function w.r.t. the Borel* σ*-field generated by the topology induced by the metric associated with* $D_\theta^\varphi/\mathscr{D}_\theta^\varphi$ via $s/\boldsymbol{\upsilon}^\varphi$*.*

Remark 1.4.1. The above considered definitions are equivalent to state that

- a mapping $\mathcal{X} : \Omega \to \mathscr{F}_c^*(\mathbb{R})$ is an RFN if and only if for all $\alpha \in [0,1]$ the real-valued mappings $\inf \mathcal{X}_\alpha$ and $\sup \mathcal{X}_\alpha$ are real-valued random variables;
- a mapping $\mathcal{X} : \Omega \to \mathscr{F}_c^*(\mathbb{R})$ is an RFN if and only if for all $\alpha \in [0,1]$ the real-valued mappings $\operatorname{mid} \mathcal{X}_\alpha$ and $\operatorname{spr} \mathcal{X}_\alpha$ are real-valued random variables.

Remark 1.4.2. Also equivalently, a mapping $\mathcal{X} : \Omega \to \mathscr{F}_c^*(\mathbb{R})$ is said to be an RFN if and only if it is a Borel-measurable mapping w.r.t. the Borel σ-field generated on $\mathscr{F}_c^*(\mathbb{R})$ by the topology induced by several different metrics, among them D_θ^φ or $\mathscr{D}_\theta^\varphi$. This Borel-measurability ensures that one can properly and trivially refer to the *distribution induced by an RFN*, the *stochastic independence of RFN's*, and so on, without needing to state expressly these notions.

As it has just been highlighted, one can properly refer to the distribution induced by an RFN so that if $\mathfrak{B}$ is a Borel set belonging to the Borel σ-field in the last remark,

$$P(\mathcal{X} \in \mathfrak{B}) = P\Big(\{\omega \in \Omega \,:\, \mathcal{X}(\omega) \in \mathfrak{B}\}\Big)$$

and, in particular,

$$P(\mathcal{X} = \widetilde{U}) = P\Big(\{\omega \in \Omega \,:\, \mathcal{X}(\omega) = \widetilde{U}\}\Big),$$

whatever $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ may be. Nevertheless, in contrast to what happens for the real- and vectorial-valued cases, one can not universally define an extension of the distribution function to characterize the distribution induced by a random fuzzy number. This is due to the fact that there is no universally accepted ranking between fuzzy numbers.

In analyzing the induced distribution of a random fuzzy number, two main types of summary measures/parameters could be distinguished:

- Central tendency or location measures, which are fuzzy-valued summary indicators. The Aumann-type mean (see Puri and Ralescu [94]) is the best known and used location measure.
- Measures for the dispersion/variability or scale, which are real-valued summary indicators. The best known and used dispersion measure is the Fréchet-type variance, defined in terms of a convenient L^2-type metric (see, for instance, Körner [70], Lubiano *et al.* [76], Ramos-Guajardo and Lubiano [95], Blanco-Fernández *et al.* [10] or Gil *et al.* [43]).

The Aumann-type mean value of a random fuzzy number extends the mean of a random variable as well as the Aumann expectation of a random set, and it is formalized as follows:

Definition 1.4.2. (Puri and Ralescu [94]) *Let $\mathcal{X}$ be a random fuzzy number associated with the probability space $(\Omega, \mathcal{A}, P)$. The* ***(population) Aumann-type mean*** *or* ***expected value*** *of $\mathcal{X}$ is the fuzzy number $\widetilde{E}(\mathcal{X}) \in \mathscr{F}_c^*(\mathbb{R})$, if it exists (a sufficient condition for such an existence would be that of $\mathcal{X}$ being integrably bounded, that is, there exists a real-valued function $h \in L^1(\Omega, \mathcal{A}, P)$ such that $\sup_{x\in\mathcal{X}_0}|x| \leq h$ a.s. $[P]$), such that for each $\alpha \in [0,1]$*

$$\Big(\widetilde{E}(\mathcal{X})\Big)_\alpha = \text{Aumann integral of } \mathcal{X}_\alpha \text{ (see [3])},$$

that is, $\Big(\widetilde{E}(\mathcal{X})\Big)_\alpha = [E(\inf \mathcal{X}_\alpha), E(\sup \mathcal{X}_\alpha)]$ with E denoting the expected value of a real-valued random variable. Equivalently, and whenever $s_{\mathcal{X}} \in L^1(\Omega, \mathcal{A}, P)$, it is the fuzzy number $\widetilde{E}(\mathcal{X}) \in \mathscr{F}_c^*(\mathbb{R})$ such that $s_{\widetilde{E}(\mathcal{X})} = \mathrm{E}(s_{\mathcal{X}})$, with E denoting the Bochner expectation of a Banach space-valued random element.

In particular, if $\widetilde{\mathbf{x}}_n = (\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_n)) = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *is a sample of observations from* $\mathcal{X}$ *when measured on a sample of individuals* $(\omega_1, \ldots, \omega_n)$*, the* **(sample) Aumann-type mean** *is the fuzzy number* $\overline{\widetilde{\mathbf{x}}}_n$ *given for all* $\alpha \in [0,1]$ *by*

$$\left(\overline{\widetilde{\mathbf{x}}}_n\right)_\alpha = \left(\frac{1}{n} \cdot (\widetilde{x}_1 + \ldots + \widetilde{x}_n)\right)_\alpha = \left[\frac{1}{n}\sum_{i=1}^n \inf(\widetilde{x}_i)_\alpha, \frac{1}{n}\sum_{i=1}^n \sup(\widetilde{x}_i)_\alpha\right].$$

Remark 1.4.3. If $\mathcal{X}$ is an LR-valued random fuzzy number for fixed invertible functions L and R, then $\widetilde{E}(\mathcal{X}) = LR\Big(E(\inf \mathcal{X}_0), E(\inf \mathcal{X}_1), E(\sup \mathcal{X}_1), E(\sup \mathcal{X}_0)\Big)$.

Due to the properties of the support function and the Hilbertian random elements, the Aumann-type mean preserves the main valuable properties from the real-valued case. In this way (see, for instance, Puri and Ralescu [94], González-Rodríguez *et al.* [49]),

Proposition 1.4.1. $\widetilde{E}$ *is equivariant under affine transformations on* $\mathscr{F}_c^*(\mathbb{R})$*, that is, if* $\gamma \in \mathbb{R}$*,* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ *and* $\widetilde{E}(\mathcal{X})$ *exists, then*

$$\widetilde{E}(\gamma \cdot \mathcal{X} + \widetilde{U}) = \gamma \cdot \widetilde{E}(\mathcal{X}) + \widetilde{U}.$$

Consequently, if $\mathcal{X}$ *is a random fuzzy number associated with the probability space* $(\Omega, \mathcal{A}, P)$ *and the distribution of* $\mathcal{X}$ *is degenerate at a fuzzy number* $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ *(i.e.,* $\mathcal{X} = \widetilde{U}$ *a.s.* $[P]$*), then* $\widetilde{E}(\mathcal{X}) = \widetilde{U}$*.*

Proposition 1.4.2. $\widetilde{E}$ *is additive, that is, if* $\mathcal{X}$ *and* $\mathcal{Y}$ *are random fuzzy numbers associated with the same probability space, for which* $\widetilde{E}(\mathcal{X})$ *and* $\widetilde{E}(\mathcal{Y})$ *exist, then*

$$\widetilde{E}(\mathcal{X} + \mathcal{Y}) = \widetilde{E}(\mathcal{X}) + \widetilde{E}(\mathcal{Y}).$$

Furthermore, the Aumann-type mean is coherent with the usual above-described fuzzy arithmetic because of the equivalent expression it takes for countable-valued RFN's. More concretely,

Proposition 1.4.3. $\widetilde{E}$ *is coherent with the usual fuzzy arithmetic, in the sense that if* $\mathcal{X}$ *is an RFN associated with the probability space* $(\Omega, \mathcal{A}, P)$ *and such that the set of the RFN values is finite or countable, that is,* $\mathcal{X}(\Omega) = \{\widetilde{x}_1^*, \ldots, \widetilde{x}_m^*, \ldots\} \subset \mathscr{F}_c^*(\mathbb{R})$*, then*

$$\widetilde{E}(\mathcal{X}) = P\left(\{\omega \in \Omega \,:\, \mathcal{X}(\omega) = \widetilde{x}_1^*\}\right) \cdot \widetilde{x}_1^* + \ldots + P\left(\{\omega \in \Omega \,:\, \mathcal{X}(\omega) = \widetilde{x}_m^*\}\right) \cdot \widetilde{x}_m^* + \ldots$$

Remark 1.4.4. Another way to show the coherence of the Aumann-type mean with the usual fuzzy arithmetic is that of the fulfilment of Strong Laws of Large Numbers (see, for instance, Colubi *et al.* [21]), so that the fuzzy arithmetic-based sample mean for a sample of independent and identically distributed RFN's converges to the population Aumann-type mean, as the sample size tends to ∞, in the sense most of the metrics one can consider on $\mathscr{F}_c^*(\mathbb{R})$.

The Aumann-type mean is also supported by Fréchet's approach [37] w.r.t. D_θ^φ, irrespectively of the choice of θ and φ, so that

Proposition 1.4.4. *$\widetilde{E}(\mathcal{X})$ is the 'Fréchet expectation' of $\mathcal{X}$ w.r.t. D_θ^φ, that is,*

$$\widetilde{E}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})} E\left(\left[D_\theta^\varphi(\mathcal{X}, \widetilde{U})\right]^2\right),$$

so that the mean is the fuzzy value leading to the lowest mean squared D_θ^φ-distance (or error) w.r.t. the RFN distribution, and this corroborates the fact that it is a central tendency measure.

In extending the variance of real-valued random variables to RFN's, Fréchet's approach [37] has been considered (see Körner [70], Lubiano *et al.* [76], González-Rodríguez *et al.* [49]). In accordance with this approach, the variance of an RFN can be interpreted as a measure of the 'least squares error/distance' in approximating the values of the RFN by a (non-random) fuzzy number. When Fréchet's approach is applied in D_θ^φ's sense we have

Definition 1.4.3. *The* ***(population) Fréchet-type variance*** *is the real number $\sigma_{\mathcal{X}}^2$, if it exists (a necessary and sufficient condition for this existence being that of $\widetilde{E}(\mathcal{X})$ along with $D_\theta^\varphi(\mathcal{X}, \widetilde{E}(\mathcal{X})) \in L^1(\Omega, \mathcal{A}, P)$), given by*

$$\sigma_{\mathcal{X}}^2 = E\left(\left[D_\theta^\varphi\left(\mathcal{X}, \widetilde{E}(\mathcal{X})\right)\right]^2\right) = \int_{[0,1]} \sigma^2_{\operatorname{mid} \mathcal{X}_\alpha}\, d\varphi(\alpha) + \theta\, \sigma^2_{\operatorname{spr} \mathcal{X}_\alpha}\, d\varphi(\alpha),$$

where σ_X^2 denotes the variance of the real-valued random variable X. Equivalently, if $s_{\mathcal{X}} \in L^2(\Omega, \mathcal{A}, P)$, it is the real number $\sigma_{\mathcal{X}}^2$ such that $\sigma_{\mathcal{X}}^2 = \operatorname{Var}(s_{\mathcal{X}})$, with Var *being intended in terms of the metric on $\mathbb{H}_2$ associated with D_θ^φ.*

The ***(population) Fréchet-type standard deviation*** *is the value $\sigma_{\mathcal{X}} = \sqrt{\sigma_{\mathcal{X}}^2}$.*

In particular, if $\widetilde{\mathbf{x}}_n = (\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_n)) = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is a sample of observations from $\mathcal{X}$ when measured on a sample of individuals $(\omega_1, \ldots, \omega_n)$, the ***(sample) Fréchet-type variance*** *is the real number $S^2_{\widetilde{\mathbf{x}}_n}$ given by*

$$S^2_{\widetilde{\mathbf{x}}_n} = \frac{1}{n} \sum_{i=1}^n \left[D_\theta^\varphi(\widetilde{x}_i, \overline{\widetilde{\mathbf{x}}}_n)\right]^2.$$

The ***(sample) Fréchet-type standard deviation*** *corresponds to $S_{\widetilde{\mathbf{x}}_n} = \sqrt{S^2_{\widetilde{\mathbf{x}}_n}}$.*

Due to the properties of the support function and the Hilbertian random elements, the Fréchet variance of an RFN satisfies the usual properties for this concept. In this way (see Körner [70], Lubiano *et al.* [76], González-Rodríguez *et al.* [49]),

Proposition 1.4.5. *The Fréchet-type variance $\sigma^2_{\mathcal{X}}$ is non-negative, with $\sigma^2_{\mathcal{X}} = 0$ if, and only if, $\mathcal{X}$ is degenerate at a fuzzy number, that is, there exists $\widetilde{U} \in \mathscr{F}^*_c(\mathbb{R})$ such that $\mathcal{X} = \widetilde{U}$* a.s. $[P]$.

Proposition 1.4.6. *If $\gamma \in \mathbb{R}$, $\widetilde{U} \in \mathscr{F}^*_c(\mathbb{R})$ and $\mathcal{X}$ is an RFN associated with the probability space $(\Omega, \mathcal{A}, P)$ for which the variance exists, then*

$$\sigma^2_{\gamma\cdot\mathcal{X}+\widetilde{U}} = \gamma^2 \cdot \sigma^2_{\mathcal{X}},$$

that is, the Fréchet-type variance is invariant under (fuzzy number-valued) translations (also referred to as location invariant) and it is squared equivariant under the product by a scalar (also referred to as squared scale equivariant).

The Fréchet-type variance is not additive in general, but it is under independence of the involved RFN's. Thus,

Proposition 1.4.7. *If $\mathcal{X}$ and $\mathcal{Y}$ are independent RFN's associated with the same probability space $(\Omega, \mathcal{A}, P)$ and for which variances exist, we have that*

$$\sigma^2_{\mathcal{X}+\mathcal{Y}} = \sigma^2_{\mathcal{X}} + \sigma^2_{\mathcal{Y}}.$$

The covariance of two RFN's has been extended in the literature (see, for instance, González-Rodríguez *et al.* [49]) on the basis of the inner product $\langle \cdot, \cdot \rangle^{\varphi}_{\theta}$ in $\mathbb{H}_2$ *via* the corresponding isometric embedding. Notice that the existence of such an inner product could be only guaranteed for the L^2 metrics but not for the L^1 ones. Nevertheless, in the study in this work we are not going to make special use of this summary measure of the joint distribution of two RFN's.

1.5 Introduction to robustness with RFN's: some robust location measures

In the preceding section it has been shown that the most commonly used summary measures in dealing with random fuzzy numbers, namely the Aumann-type mean and the Fréchet-type variance, preserve the main valuable properties of the classical location and scale parameters they extend. However, they also inherit some of their drawbacks, especially those in connection with their high sensitivity to the presence of 'outliers', data changes or asymmetries. This assertion can easily be illustrated by means of real-life and synthetic examples, as we will later see.

This relevant concern motivates the development of robust location and scale measures in the setting of random fuzzy numbers. For this purpose, a first natural attempt is to extend the main developments for robust location and scale measurement in the real-valued case. However, one should be aware of the already highlighted handicaps associated with the induced distribution of an RFN, namely,

– the lack of a general extension of the distribution function for RFN's,
– and the lack of general, realistic and well-supported models for these distributions.

Robust statistics studies with real-valued data often involve either distribution functions or normal (or more generally symmetric) distributions, and sometimes asymmetric models, like the exponential. The use of mathematically convenient parametric models helps to solve the question of what to estimate for the location and scale of the distribution of a real-valued random variable when it is not symmetrically distributed.

Although the notion of symmetry of the induced distribution of an RFN can be properly formalized (see Sinova *et al.* [106]), to date the above-mentioned handicaps make unfeasible both the extension of some location and scale measures from the real-valued case, as well as of some tools to evaluate their robustness. In Remark 1.3.3 it has been emphasized that methods for the statistical analysis of fuzzy data are being developed either as particularization of Functional Data Analysis techniques or *ad hoc*.

Recently, several developments have been carried out in connection with the robust estimation of the location of RFN's (see, for instance, Sinova *et al.* [111, 112, 115, 113], Colubi and González-Rodríguez [19] and, especially, Sinova [105]). In estimating/measuring location in the real-valued case, although many robust estimators/measures have been suggested in the literature, the median is certainly the most widely known and considered, and the first approaches in dealing with fuzzy data have aimed to extend it. Since there is no universally accepted ranking between fuzzy numbers, the extension cannot be based on that of the distribution function. Therefore, a different way to proceed should be taking into account: the median minimizes the average deviation. And such a deviation could be extended in the fuzzy number-valued case by using L^1 metrics. This has been the key idea underlying the *ad hoc* approaches that are now recalled.

Definition 1.5.1. (Sinova *et al.* [111]) *Let $\mathcal{X}$ be a random fuzzy number. The **(population) 1-norm median** of $\mathcal{X}$ is the fuzzy number $\widetilde{\mathrm{Me}}(\mathcal{X})$ such that for each $\alpha \in [0,1]$*

$$\left(\widetilde{\mathrm{Me}}(\mathcal{X})\right)_\alpha = [\mathrm{Me}(\inf \mathcal{X}_\alpha), \mathrm{Me}(\sup \mathcal{X}_\alpha)],$$

where in case $\mathrm{Me}(\inf \mathcal{X}_\alpha)$ or $\mathrm{Me}(\sup \mathcal{X}_\alpha)$ are non-unique, we will follow the most usual convention, that is, we will consider the midpoint of the interval of medians.

*In particular, if $\widetilde{\mathbf{x}}_n = (\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_n)) = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is a sample of observations from $\mathcal{X}$ when measured on a sample of individuals $(\omega_1, \ldots, \omega_n)$, the **(sample) 1-norm median** is the fuzzy number $\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)$ such that for all $\alpha \in [0,1]$*

$$\inf\left(\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)\right)_\alpha = \mathrm{Me}\left\{\inf(\widetilde{x}_1)_\alpha, \ldots, \inf(\widetilde{x}_n)_\alpha\right\},$$

$$\sup\left(\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)\right)_\alpha = \mathrm{Me}\left\{\sup(\widetilde{x}_1)_\alpha, \ldots, \sup(\widetilde{x}_n)_\alpha\right\},$$

where in case any of the two involved medians of real-valued data is non-unique, we will follow the most usual convention, that is, we will consider the midpoint of the interval of medians.

The notion above is in fact the solution (or one of the possible solutions, in case the usual convention is not applied to avoid non-uniqueness) of minimizing the average ρ_1^φ-deviation. That is,

Proposition 1.5.1. (Sinova *et al.* [111]) *Let $\mathcal{X}$ be a random fuzzy number. Then,*

$$\widetilde{\mathrm{Me}}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})} E\left(\rho_1^\varphi(\mathcal{X}, \widetilde{U})\right),$$

whatever the weighting measure φ may be and whenever the corresponding involved expectation exists.

Remark 1.5.1. Another solution of the last minimization problem, based on a different convention and Zadeh's extension principle, has been introduced by Grzegorzewski [50]. As it has been observed (Pérez-Fernández and Sinova [91]), the behaviour of both extended ρ_1^φ-medians is much more robust towards contamination than the Aumann-type mean's one. With respect to the comparison between the two alternatives for the median, the 1-norm median behaves a little better than Grzegorzewski's proposal in the sense that the sample estimate is closer (in ρ_1^φ's sense) to the corresponding population value.

On the other hand,

Definition 1.5.2. (Sinova *et al.* [109]) *Let $\mathcal{X}$ be a random fuzzy number and φ be an absolutely continuous probability measure associated with the measurable space* $([0,1], \mathcal{B}_{[0,1]})$ *and having positive mass function on* $(0,1)$*. The **(population) φ-wabl/ldev/rdev median** of $\mathcal{X}$ is the fuzzy number $\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X})$ such that for each* $\alpha \in [0,1]$

$$\left(\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X})\right)_{\alpha} = \left[\mathrm{Me}(\mathrm{wabl}^{\varphi}(\mathcal{X})) - \mathrm{Me}(\mathrm{ldev}^{\varphi}_{\mathcal{X}}(\alpha)), \mathrm{Me}(\mathrm{wabl}^{\varphi}(\mathcal{X})) + \mathrm{Me}(\mathrm{rdev}^{\varphi}_{\mathcal{X}}(\alpha))\right],$$

where in case $\mathrm{Me}(\mathrm{wabl}^{\varphi}(\mathcal{X}))$, $\mathrm{Me}(\mathrm{ldev}^{\varphi}_{\mathcal{X}}(\alpha))$ *or* $\mathrm{Me}(\mathrm{rdev}^{\varphi}_{\mathcal{X}}(\alpha))$ *are non-unique, the most usual convention is considered.*

*In particular, if $\widetilde{\mathbf{x}}_n = (\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_n)) = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is a sample of observations from $\mathcal{X}$ when measured on a sample of individuals $(\omega_1, \ldots, \omega_n)$, the **(sample) φ-wabl/ldev/rdev median** is the fuzzy number $\widehat{\widetilde{\mathrm{M}^{\varphi}}}(\widetilde{\mathbf{x}}_n)$ such that for all* $\alpha \in [0,1]$

$$\begin{aligned} \inf\left(\widehat{\widetilde{\mathrm{M}^{\varphi}}}(\widetilde{\mathbf{x}}_n)\right)_{\alpha} &= \mathrm{Me}\{\mathrm{wabl}^{\varphi}(\widetilde{x}_1), \ldots, \mathrm{wabl}^{\varphi}(\widetilde{x}_n)\} \\ &\quad -\mathrm{Me}\{\mathrm{ldev}^{\varphi}_{\widetilde{x}_1}(\alpha), \ldots, \mathrm{ldev}^{\varphi}_{\widetilde{x}_n}(\alpha)\}, \\ \sup\left(\widehat{\widetilde{\mathrm{M}^{\varphi}}}(\widetilde{\mathbf{x}}_n)\right)_{\alpha} &= \mathrm{Me}\{\mathrm{wabl}^{\varphi}(\widetilde{x}_1), \ldots, \mathrm{wabl}^{\varphi}(\widetilde{x}_n)\} \\ &\quad +\mathrm{Me}\{\mathrm{rdev}^{\varphi}_{\widetilde{x}_1}(\alpha), \ldots, \mathrm{rdev}^{\varphi}_{\widetilde{x}_n}(\alpha)\}. \end{aligned}$$

The notion above is in fact the solution (or one of the possible solutions) of minimizing the average $\mathscr{D}^{\varphi}_{\theta}$-deviation. That is,

Proposition 1.5.2. (Sinova *et al.* [109]) *Let $\mathcal{X}$ be a random fuzzy number and φ be an absolutely continuous probability measure associated with the measurable space* $([0,1], \mathcal{B}_{[0,1]})$ *and having positive mass function on* $(0,1)$*. Then,*

$$\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X}) = \arg \min_{\widetilde{U} \in \mathscr{F}^*_c(\mathbb{R})} E\Big(\mathscr{D}^{\varphi}_{\theta}(\mathcal{X}, \widetilde{U})\Big),$$

whatever θ may be and whenever the corresponding involved expectation exists.

Remark 1.5.2. Sinova and López [114] have recently shown that neither the position nor the shape of the sample φ-wabl/ldev/rdev median are very influenced by the chosen measure φ, but they are scarcely affected.

The two above-recalled L^1-medians preserve the main valuable properties from the real-valued case. In this way,

Proposition 1.5.3. (Sinova *et al.* [111, 109]) $\widetilde{\mathrm{Me}}$ *and* $\widetilde{\mathrm{M}^{\varphi}}$ *are equivariant under affine transformations on* $\mathscr{F}^*_c(\mathbb{R})$*, that is, if* $\gamma \in \mathbb{R}$, $\widetilde{U} \in \mathscr{F}^*_c(\mathbb{R})$*, then*

$$\widetilde{\mathrm{Me}}(\gamma \cdot \mathcal{X} + \widetilde{U}) = \gamma \cdot \widetilde{\mathrm{Me}}(\mathcal{X}) + \widetilde{U}, \quad \widetilde{\mathrm{M}^{\varphi}}(\gamma \cdot \mathcal{X} + \widetilde{U}) = \gamma \cdot \widetilde{\mathrm{M}^{\varphi}}(\mathcal{X}) + \widetilde{U}.$$

Consequently, if $\mathcal{X}$ is a random fuzzy number associated with the probability space $(\Omega, \mathcal{A}, P)$ and the distribution of $\mathcal{X}$ is degenerate at a fuzzy number $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$ (i.e., $\mathcal{X} = \widetilde{U}$ a.s. $[P]$), then $\widetilde{\mathrm{Me}}(\mathcal{X}) = \widetilde{U}$, $\widetilde{\mathrm{M}^\varphi}(\mathcal{X}) = \widetilde{U}$.

As for the real-valued case, under rather mild conditions the sample extended medians are shown to be a strongly consistent estimator of the corresponding population medians, that is,

Proposition 1.5.4. (Sinova *et al.* [111, 109]) *Let $\mathcal{X}$ be a random fuzzy number associated with a probability space $(\Omega, \mathcal{A}, P)$.*

Assume that for each $\alpha \in [0,1]$ the real-valued population medians $\mathrm{Me}\Big(\inf \mathcal{X}_\alpha\Big)$ and $\mathrm{Me}\Big(\sup \mathcal{X}_\alpha\Big)$ exist and they are actually unique. If $\widehat{\widetilde{\mathrm{Me}}(\mathcal{X})}_n$ denotes the sample median statistic corresponding to a simple random sample $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ from $\mathcal{X}$, and the two sequences of the real-valued sample medians $\Big\{\mathrm{Me}\widehat{(\inf \mathcal{X}_\alpha})_n\Big\}_n$ and $\Big\{\mathrm{Me}\widehat{(\sup \mathcal{X}_\alpha})_n\Big\}_n$ as functions of α over $[0,1]$ are both uniformly integrable, then $\widehat{\widetilde{\mathrm{Me}}(\mathcal{X})}_n$ is a strongly consistent estimator of $\widetilde{\mathrm{Me}}(\mathcal{X})$ in ρ_1-sense (and hence in the sense of all the topologically equivalent metrics), i.e.

$$\lim_{n\to\infty} \rho_1\left(\widehat{\widetilde{\mathrm{Me}}(\mathcal{X})}_n, \widetilde{\mathrm{Me}}(\mathcal{X})\right) = 0 \quad a.s.\,[P].$$

Actually, the ρ_1^φ-convergence also holds.

Assume that φ is an absolutely continuous probability measure on $([0,1], \mathcal{B}_{[0,1]})$ with positive mass function on $(0,1)$, and that for each $\alpha \in [0,1]$ the real-valued population medians $\mathrm{Me}(\mathrm{wabl}^\varphi(\mathcal{X}))$, $\mathrm{Me}(\mathrm{ldev}_{\mathcal{X}}^\varphi(\alpha))$ and $\mathrm{Me}(\mathrm{rdev}_{\mathcal{X}}^\varphi(\alpha))$ exist and they are actually unique. If $\widehat{\widetilde{\mathrm{M}^\varphi}(\mathcal{X})}_n$ denotes the sample median corresponding to a simple random sample $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ from $\mathcal{X}$, and the two sequences of the real-valued sample medians $\Big\{\mathrm{Me}\widehat{(\mathrm{ldev}_{\mathcal{X}}^\varphi(\alpha)})_n\Big\}_n$ and $\Big\{\mathrm{Me}\widehat{(\mathrm{rdev}_{\mathcal{X}}^\varphi(\alpha)})_n\Big\}_n$ as functions of α over $[0,1]$ are both uniformly integrable, then for each $\theta \in (0,1]$ the estimator $\widehat{\widetilde{\mathrm{M}^\varphi}(\mathcal{X})}_n$ is strongly consistent in $\mathscr{D}_\theta^\varphi$-sense (and hence in the sense of all the topologically equivalent metrics), i.e.

$$\lim_{n\to\infty} \mathscr{D}_\theta^\varphi\left(\widehat{\widetilde{\mathrm{M}^\varphi}(\mathcal{X})}_n, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})\right) = 0 \quad a.s.\,[P].$$

Regarding robustness, to check it a suitable tool that makes immediate sense in the fuzzy number-valued setting is that of the *finite sample breakdown point* (fsbp for short), which is the minimum proportion of sample data which should be perturbed to get an arbitrarily large or small location estimator value. It has been proved that whereas the fsbp of the Aumann-type sample mean is $1/n$, the one for the two extended medians equal $\left\lfloor \frac{n+1}{2} \right\rfloor / n$ where $\lfloor\cdot\rfloor$ denotes the floor function (which is the highest possible fsbp for location estimators).

Proposition 1.5.5. (Sinova *et al.* [111, 109]) *The finite sample breakdown point of the sample 1-norm median from a random fuzzy number* $\mathcal{X}$*,* $\mathrm{fsbp}\Big(\widehat{\widetilde{\mathrm{Me}}(\mathcal{X})}_n, \widetilde{\mathbf{x}}_n, \rho_1\Big)$*, and the finite sample breakdown point of the sample* φ*-wabl/ldev/rdev median from a random fuzzy number* $\mathcal{X}$*,* $\mathrm{fsbp}(\widehat{\widetilde{\mathrm{M}}^{\varphi}(\mathcal{X})}_n, \widetilde{\mathbf{x}}_n, \mathscr{D}_\theta^\varphi)$*, equal*

$$\mathrm{fsbp}\Big(\widehat{\widetilde{\mathrm{Me}}(\mathcal{X})}_n, \widetilde{\mathbf{x}}_n, \rho_1\Big) = \mathrm{fsbp}(\widehat{\widetilde{\mathrm{M}}^{\varphi}(\mathcal{X})}_n, \widetilde{\mathbf{x}}_n, \mathscr{D}_\theta^\varphi) = \frac{1}{n} \cdot \left\lfloor \frac{n+1}{2} \right\rfloor .$$

In contrast to what happens for the Aumann-type mean, the two preceding extended medians do not keep the data shape in general. More concretely, if one considers an LR-valued random fuzzy number for fixed invertible functions L and R, the 1-norm and φ-wabl/ldev/rdev medians are not necessarily LR-valued (see Sinova [105]).

At this point, one should notice that Sinova *et al.* [113], by particularizing and adapting some developments from Functional Data Analysis, have stated (robust) *fuzzy M-estimates of location.* These M-estimates include the two preceding medians as especial cases. Under rather general assumptions, which are not fulfilled by the 1-norm and φ-wabl/ldev/rdev medians, M-estimates are given by some weighted averages of the sample fuzzy data (although weights should be obtained in an approximate way; in fact it has been suggested to approximate them by standard iteratively re-weighted least squares algorithms -see Sinova and Terán [116]-). As a consequence, if one considers an LR-valued random fuzzy number for fixed invertible functions L and R, the M-estimates fulfilling the above-mentioned assumptions are LR-valued, what is quite natural and makes several posterior statistical computations easier. Nevertheless, M-estimates cannot be guaranteed to be scale equivariant (not even absolute or squared).

Another recent approach (see Sinova *et al.* [107], Sinova [105], Sinova and Terán [116], Colubi and González-Rodríguez [19]) is that corresponding to the particularization from Functional Data Analysis of the trimmed means approach. *Fuzzy trimmed means* are given by the arithmetic means of some of the available sample data, so they also preserve the shape of the LR-valued fuzzy data for fixed invertible functions L and R. Furthermore, trimmed means are scale equivariant.

It has been proved, by means of their finite sample breakdown point, that the behaviour of these two alternatives is more robust than the one of the Aumann-type mean and coincides with the one for the 1-norm and φ-wabl/ldev/rdev medians. In the comparative analysis in Sinova [105] and Sinova and Terán [116], it has been proved that fuzzy M-estimators of location are usually the most robust approach.

If the trimming proportion is 0.5, fuzzy trimmed mean estimators are as robust as fuzzy M-estimators of location.

Along this work, we will not make use of the Functional Data Analysis-based but only of the *ad hoc* robust location estimates/measures. Of course, the use of the general M-estimates and trimmed means is left as an open direction.

1.6 Case studies to be analyzed in the work

The following two chapters, that will be motivated at the end of this one, and Section 1.8 will make use sometimes of three real-life examples. This section is to be devoted to present these examples in detail. They concern the application of the fuzzy rating and Likert (and, hence and if desired, the fuzzy linguistic) scales.

The three examples have a common guideline. They correspond to standard questionnaires in which most of items are traditionally assumed to be responded in accordance with Likert-type scales (4-point and 5-point ones). The novelty lies in the fact that items have been adapted to allow a double response type, namely the original Likert-type response and a fuzzy rating scale-based one. Although they have been mainly designed aiming to support the convenience of using fuzzy rating scales in this setting, by comparing statistical conclusions with Likert or fuzzy linguistic ones, they will also be considered to illustrate the measures/estimates introduced along the work as well as some results and other conclusions.

Example 1.6.1. (Sinova [105], Gil *et al.* [44], Lubiano *et al.* [75], Sinova *et al.* [113]) The first example is related to the well-known questionnaire TIMSS-PIRLS 2011 which is conducted on the **population** of Grade 4 students (i.e., nine to ten years old) and concerns their opinion and feeling on aspects regarding reading, math, and science. This questionnaire is rather standard and most of the involved questions have to be answered according to a 4-point Likert scale, responses being DISAGREE A LOT, DISAGREE A LITTLE, AGREE A LITTLE, and AGREE A LOT.

To get more expressive responses and informative conclusions, the original **questionnaire form** has been adapted to allow a double-type response: the original Likert and a fuzzy rating scale-based one with reference interval $[0, 10]$ (see Figure 1.8 for one of the items, and Pages 233 to 237 for the full paper-and-pencil form, and http://carleos.epv.uniovi.es:8080/ for the -Spanish- computerized form by Professor Carlos Carleos).

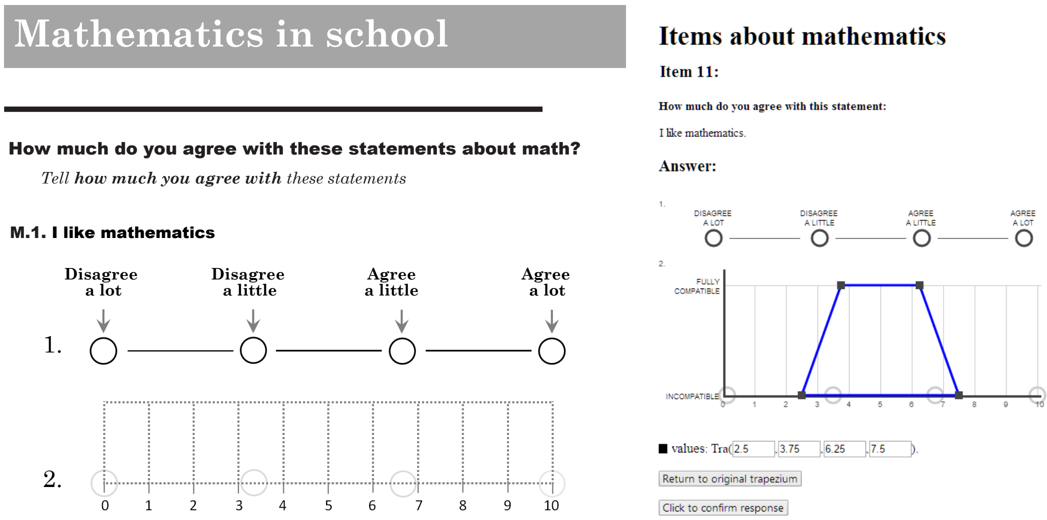


Figure 1.8: Example of the double response paper-and-pencil (on the left) and computerized (on the right) form to an item in Example 1.6.1

The questionnaire involving these double response questions has been conducted in 2014 on a **sample** of 69 fourth grade students from Colegio San Ignacio (Oviedo-Asturias, Spain). These students have been distributed in accordance with (their usual) three groups, so that the teachers have decided that the 24 students in one of the three classrooms have to fill out the paper-and-pencil format and the 45 students from the other two groups have to complete the computerized version. To ‘ease’ the relationship between the two scales for these very young respondents, each numerically encoded Likert response has been superimposed upon the reference interval of the fuzzy rating scale part, as we can see in Figure 1.8.

The **training** of the students to let them know about the meaning and purpose of the case study, as well as the aim of the double response, has been carried out in up to 15 minutes, and three researchers from the Department of Statistics, OR and Math Teaching have been in charge of the explanation and conduction of the survey. At this point, it should be remarked that the students had no idea on the concept of real-valued functions and they have just learned that of a trapezium. With the guideline described in Pages 234 and 235, the students have not had understanding problems, they have catched the philosophy behind and they have been able to provide us with quite coherent responses in most of the cases. Actually, for all the questions, the number of ‘no response”s has been very small and smaller for the fuzzy rating than for the Likert scale. In summary, the training has been surprisingly much easier and more effective than we had expected.

Datasets associated with responses to this questionnaire can be found in Tables A.1 to A.4 in the appendices section.

Example 1.6.2. (De la Rosa de Sáa *et al.* [27], Sinova *et al.* [110]) The second example is related to a typical restaurant customer satisfaction questionnaire which is usually conducted on the **population** of people of different ages and often showing a wide diversity in background, occupation, and so on. This type of questionnaires frequently involves items concerning quality of food and beverage, quality of service, and price; the items are assiduously to be answered according to a 5-point Likert scale where responses are STRONGLY DISAGREE, SOMEWHAT DISAGREE, NEUTRAL, SOMEWHAT AGREE, and STRONGLY AGREE.

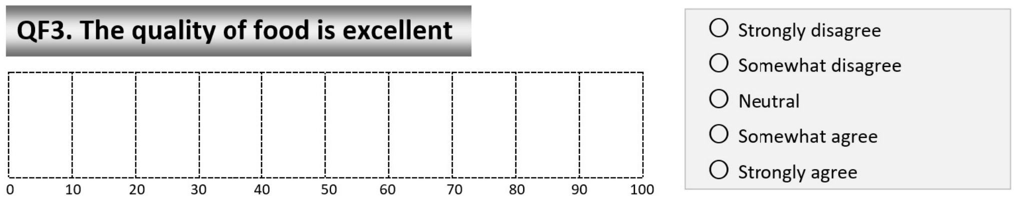


Figure 1.9: Excerpt of the questionnaire about the satisfaction with the quality of restaurants in Example 1.6.2

As for the first example, to get more expressive responses and informative conclusions, a standard paper-and-pencil **questionnaire form** has been adapted to allow the questions associated with intrinsically imprecise matters a double-type response: the above-mentioned 5-point Likert and a fuzzy rating scale-based one with reference interval $[0, 100]$ (see Figure 1.9 for one of the items, and Pages 243 to 245 for the full form).

A **sample** of 70 people has been considered to fill this questionnaire. It has been conducted by students of a Master on Soft Computing and Intelligent Data Analysis delivered in Mieres (Asturias, Spain) in 2011-2012, and individuals in the sample have mainly corresponded to colleagues, friends and relatives of these students. The **training** of respondents has been certainly variable, mainly due to the fact that the background of these respondents is quite diverse. Anyway, this training has never lasted more than 15 minutes.

Datasets associated with responses to this questionnaire can be found in Tables B.1 to B.4 in the appendices section.

Example 1.6.3. (Colubi *et al.* [20], González-Rodríguez *et al.* [49]) By using an **online (computerized) application**, an experiment has been conducted online in which people have been asked for their perception of the relative length of different line segments with respect to a pattern longer one. The **population** has corresponded to people contacted for this purpose. People have participated online by providing with their perception of relative length for each of several line segments.

More concretely, on the center top of the screen the longest (pattern) line segment has been drawn in magenta. This segment is fixed for all the trials, so that there is always a reference for the maximum length. At each trial, a grey shorter line segment is generated and placed below the pattern one, parallel and without considering a concrete location (i.e., indenting or centering). For each respondent, line segments are generated at random, although to avoid the variation in the perception of different respondents can mainly be due to the variation in length of different generated segments, the (27 first) trials for two respondents refer to the same segments but appearing in different position.

Each of the perceptions can be doubly expressed, namely by choosing a label from a 5-point Likert-like scale (VERY SMALL, SMALL, MEDIUM, LARGE, and VERY LARGE), and by using the fuzzy rating scale with reference interval $[0, 100]$ so that they can be thought as imprecise percentages (see Figure 1.10 for a screen capture).

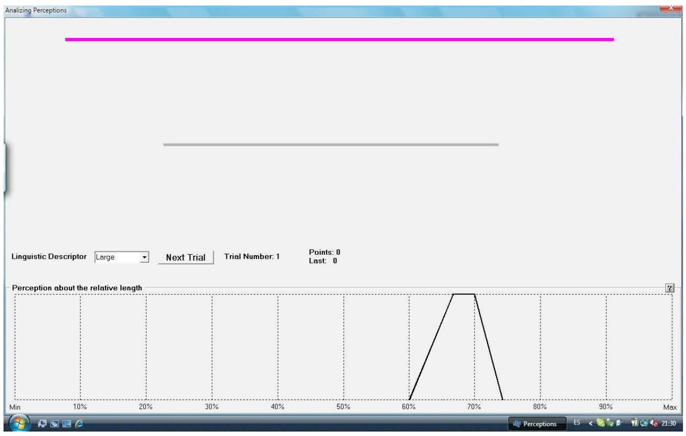


Figure 1.10: Example of a double response from the online application in Example 1.6.3

The online application explains the formalization and meaning of the fuzzy rating values (see Pages 251 and 252).

A **sample** of 25 respondents (all of them with a university scientific background and they have needed a minor **training**, mostly consisting of simply reading the instructions in Pages 251 and 252) has been contacted for this experiment, and they have supplied the responses in Tables C.1 to C.7 in the appendices section.

By looking at datasets in the three examples, gathered in Tables in Appendices A, B and C, it is quite clear that data from the computerized forms and experiments are frequently more detailed (i.e., they have more decimals) than the paper-and-pencil ones.

Remark 1.6.1. It should be emphasized that in Examples 1.6.1 and 1.6.2 the involved random variables to be observed/measured relate to intrinsically imprecise-valued magnitudes, and there is no underlying real-valued attribute. However, in Example 1.6.3 there is an underlying real-valued random variable (the exact relative length), that has been printed in grey in Tables C.1 to C.7. It has been assumed that such an exact measurement has not been accessible to the respondent, but the available information would be only that of respondent's human perception, which is also intrinsically imprecise-valued. In the three examples the statistical analysis is to be referred to either the Likert or the fuzzy data, irrespectively of underlying (albeit unknown) real-valued data existing or not.

1.7 Generating fuzzy data for simulation studies

Simulation studies are to be considered along this work for two different purposes:

- to provide with a complementary view supporting some properties that can furthermore be theoretically supported by means of some formal tools,
- to show how some measures and procedures behave, mainly when it is not possible, feasible or necessary to develop general theoretical properties or conclusions.

A crucial thought at this stage is that, as it has already been commented in Page 28, there are not yet suitable realistic models for the distribution of random fuzzy numbers. This makes the simulation process a novel endeavor, especially in simulating outliers in this setting.

In this work, simulations are to be basically related to two simulation procedures. The first one can be seen as a kind of extension of the normal model, which plays a key role in robust developments with real-valued data. The second one has been inspired by real-life datasets in connection with fuzzy rating scale-based experiments.

To generate fuzzy data from an LR-valued random fuzzy number $\mathcal{X} = LR(\inf \mathcal{X}_0, \inf \mathcal{X}_1, \sup \mathcal{X}_1, \sup \mathcal{X}_0)$, Sinova *et al.* [111] suggest to use an alternative characterization, $\mathcal{X} = LR\langle X_1, X_2, X_3, X_4\rangle$, where

$$X_1 = \mathrm{mid}\mathcal{X}_1 = (\inf \mathcal{X}_1 + \sup \mathcal{X}_1)/2, \quad X_2 = \mathrm{spr}\mathcal{X}_1 = (\sup \mathcal{X}_1 - \inf \mathcal{X}_1)/2,$$
$$X_3 = \mathrm{lspr}\mathcal{X}_0 = \inf \mathcal{X}_1 - \inf \mathcal{X}_0, \quad X_4 = \mathrm{uspr}\mathcal{X}_0 = \sup \mathcal{X}_0 - \sup \mathcal{X}_1,$$

whence

$$\mathcal{X} = LR\langle X_1, X_2, X_3, X_4\rangle = LR(X_1 - X_2 - X_3, X_1 - X_2, X_1 + X_2, X_1 + X_2 + X_4).$$

In fact, irrespective of L and R, fuzzy data will be generated by simulating the four real-valued random variables X_1, X_2, X_3 and X_4 so that the $\mathbb{R} \times [0, \infty) \times [0, \infty) \times [0, \infty)$-valued random vector (X_1, X_2, X_3, X_4) will provide us with the 4-tuples (x_1, x_2, x_3, x_4) with $x_1/x_2 =$ center/radius of the core, and $x_3/x_4 =$ lower/upper spread of the fuzzy number. To each generated 4-tuple (x_1, x_2, x_3, x_4) we associate the fuzzy number $LR\langle x_1, x_2, x_3, x_4\rangle$.

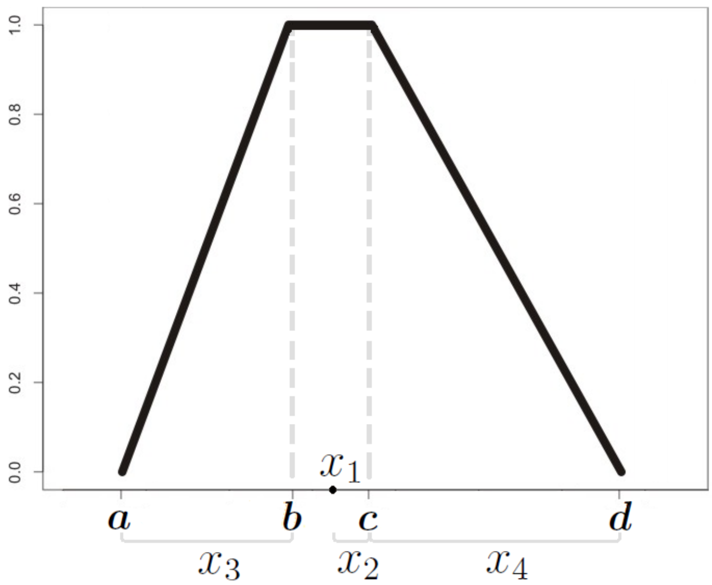


Figure 1.11: 4-Tuples to be generated for the simulation procedures

Most of the simulations to be considered will involve trapezoidal fuzzy numbers, but they could be similarly done with LR-fuzzy numbers with invertible L and R. The choice of the trapezoidal shape is justified by both the simplicity in computations and the conclusions from the discussion presented in the next section.

As we have announced before, the ***simulation developments*** have been carried out in two ways.

According to the FIRST SIMULATION PROCEDURE (1stSP), data have been generated from random fuzzy numbers with an unbounded reference set and showing a symmetrical behavior in what concerns the centers of their cores. Moreover, two different cases have been considered in this 1stSP. More concretely, and once fixed the L and R functions, data will be generated from a random vector (X_1, X_2, X_3, X_4) for which X_1 is normally distributed in the two cases, $X_1 \sim \mathcal{N}(0, 1)$ and $X_2, X_3, X_4 \sim \chi_1^2$ in Case 1, all of these variables being independent, whereas $X_2, X_3, X_4 \sim 1/(X_1^2 + 1)^2 + 0.1 \cdot \chi_1^2$ in Case 2 (see Sinova *et al.* [109], [111]).

However, although this model is appropriate for several discussions in the work, real-life examples scarcely support it. For most of the real-life examples one can consider, random fuzzy numbers make use of a bounded referential and the distribution of X_1 is skewed. This motivates the use of an alternative model.

According to the SECOND SIMULATION PROCEDURE (2ndSP), data have been generated from random fuzzy numbers with a bounded reference set and abstracting and mimicking what we have observed in real-life examples employing the fuzzy rating scale. More concretely, fuzzy data have been generated so that

– 5% (or, more generally, $100 \cdot \omega_1\%$) of the data have been obtained by first considering a simulation from a simple random sample of size 4 from a beta $\beta(p, q)$ distribution, the ordered 4-tuple, and finally computing the values of the x_i. The values of p and q vary in most of cases to cover six quite different distributions (namely symmetrical weighting central values like $p = q = 1$, $p = q = 2$, symmetrical weighting extreme values like $p = q = 0.75$, and three types of asymmetric ones like $p = 4 > 2 = q$, $p = 6 > 1 = q$ and $p = 6 < q = 10$, see Figure 1.12). In most of the comparative studies involving simulations, the values from the beta distribution are re-scaled and translated to an interval $[l_0, u_0]$ different from $[0, 1]$.

– 35% (or, more generally, $100 \cdot \omega_2\%$) of the data have been obtained considering a simulation of four random variables $X_i = (u_0 - l_0) \cdot Y_i + l_0$ as follows:

$$\begin{aligned}
&Y_1 \sim \beta(p, q),\\
&Y_2 \sim \text{Uniform}\Big[0, \min\{1/10, Y_1, 1 - Y_1\}\Big],\\
&Y_3 \sim \text{Uniform}\Big[0, \min\{1/5, Y_1 - Y_2\}\Big],\\
&Y_4 \sim \text{Uniform}\Big[0, \min\{1/5, 1 - Y_1 - Y_2\}\Big].
\end{aligned}$$

– 60% (or, more generally, $100 \cdot \omega_3\%$) of the data have been obtained considering a simulation of four random variables $X_i = (u_0 - l_0) \cdot Y_i + l_0$ as follows:

$$Y_1 \sim \beta(p, q),$$

$$Y_2 \sim \begin{cases} \text{Exp}(200) & \text{if } Y_1 \in [0.25, 0.75] \\ \text{Exp}(100 + 4\,Y_1) & \text{if } Y_1 < 0.25 \\ \text{Exp}(500 - 4\,Y_1) & \text{otherwise} \end{cases}$$

$$Y_3 \sim \begin{cases} \gamma(4, 100) & \text{if } Y_1 - Y_2 \geq 0.25 \\ \gamma(4, 100 + 4\,Y_1) & \text{otherwise} \end{cases}$$

$$Y_4 \sim \begin{cases} \gamma(4, 100) & \text{if } Y_1 + Y_2 \geq 0.25 \\ \gamma(4, 500 - 4\,Y_1) & \text{otherwise.} \end{cases}$$

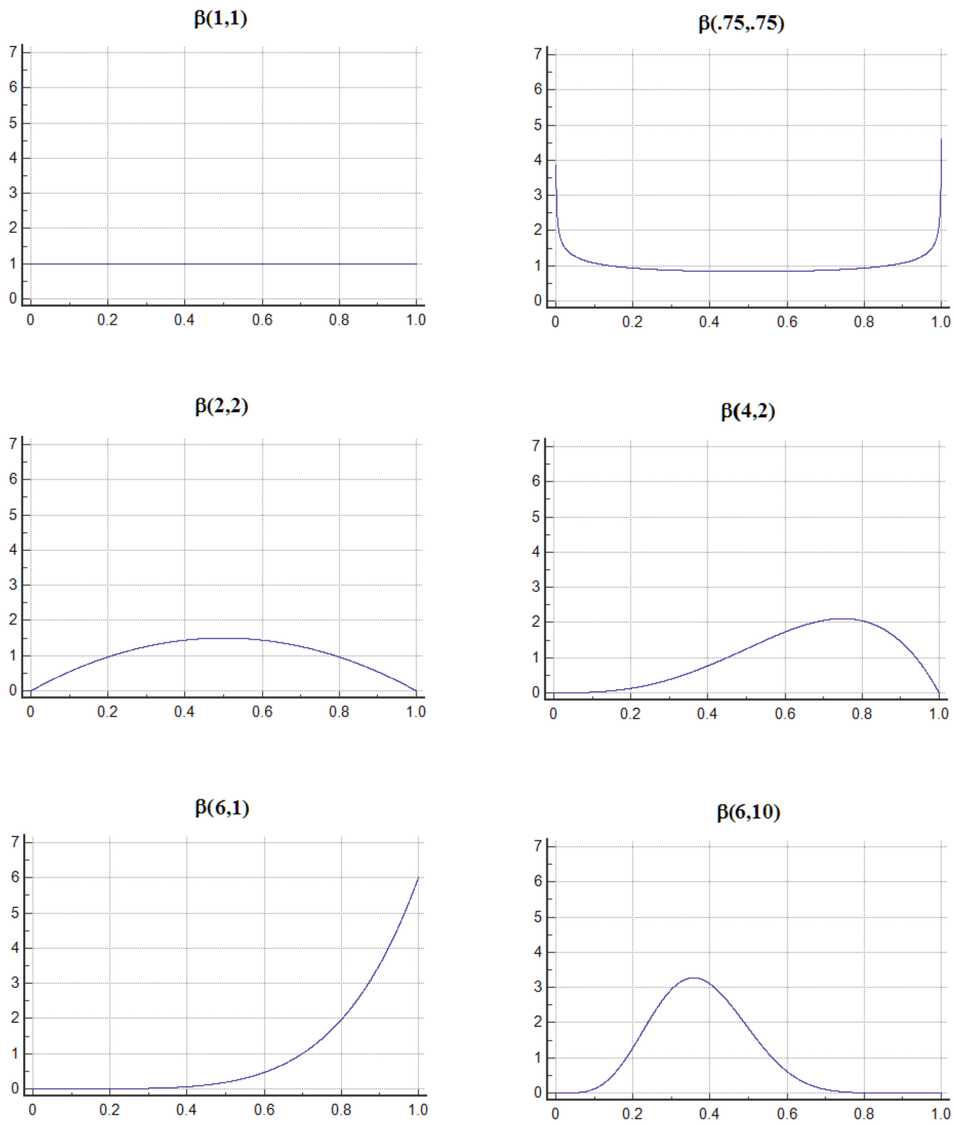


Figure 1.12: Density functions of different Beta(p, q) which have been used in the comparative studies

1.8 Analyzing sensitivity of Fréchet's variance with respect to the shape of fuzzy data

For practical purposes, and to ease both the drawing and the computing processes, the fuzzy rating scale has been introduced assuming responses to be modeled by means of trapezoidal fuzzy numbers. Moreover, Pedrycz [89], Grzegorzewski [51, 52, 53], Grzegorzewski and Pasternak-Winiarska [54], Ban *et al.* [5], and others, have provided with different arguments to employ trapezoidal (in particular, triangular) fuzzy numbers or approximations preserving ambiguity, expected interval, etc.

It should be highlighted that the assumption of the trapezoidal shape is not at all mandatory from a formal viewpoint, although computations could become more cumbersome for some other choices. In this section, we are going to show, by means of simulations and a real-life case study, that when fuzzy numbers are modelling imprecise data coming from random experiments and the dispersion of fuzzy datasets is summarized through their Fréchet-type variances, data shape is usually not relevant.

The discussion is carried out on the basis of the test about the equality of variances with fuzzy data developed by Ramos-Guajardo and Lubiano [95]. Both studies make use of the ***bootstrapped homoscedasticity test of*** k ***independent RFN's*** in Ramos-Guajardo and Lubiano [95], which is now algorithmically summarized. If $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are independent RFN's, consider a sample of independent observations $\widetilde{\boldsymbol{x}}_{\boldsymbol{i}} = (\widetilde{x}_{i1}, \ldots, \widetilde{x}_{in_i})$ from $\mathcal{X}_i$, $i = 1, \ldots, k$, the k samples being also independent, with $n = n_1 + \ldots + n_k$. Denote $\overline{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}} = \frac{1}{n_i} \cdot (\widetilde{x}_{i1} + \ldots + \widetilde{x}_{in_i})$ the sample Aumann-type mean for $\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}$, $S^2_{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}} = \sum_{j=1}^{n_i} \left[D_\theta^\varphi(\widetilde{x}_{ij}, \overline{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}})\right]^2 / n_i$ the sample Fréchet-type variance for $\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}$, and $\overline{S^2_{\widetilde{\boldsymbol{x}}}} = \sum_{i=1}^k n_i \cdot S^2_{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}}/n$.

Then, for fixed arbitrary θ and φ, the bootstrapped algorithm to test the null hypothesis $H_0 : \sigma^2_{\mathcal{X}_1} = \ldots = \sigma^2_{\mathcal{X}_k}$ proceeds as follows:

Step 1. Compute the value of the statistic

$$T_{n_1,\ldots,n_k} = \frac{\displaystyle\sum_{i=1}^k n_i \left(S^2_{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}} - \overline{S^2_{\widetilde{\boldsymbol{x}}}}\right)^2}{\displaystyle\sum_{i=1}^k \frac{1}{n_i} \sum_{j=1}^{n_i} \left(\left[D_\theta^\varphi(\widetilde{x}_{ij}, \overline{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}})\right]^2 - S^2_{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}}\right)^2}$$

Step 2. For each $i \in \{1, \ldots, k\}$, obtain a bootstrap sample from $\left(\widetilde{x}_{i1} \cdot \sqrt{\overline{S^2_{\widetilde{\boldsymbol{x}}}}/S^2_{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}}}, \ldots, \widetilde{x}_{in_i} \cdot \sqrt{\overline{S^2_{\widetilde{\boldsymbol{x}}}}/S^2_{\widetilde{\boldsymbol{x}}_{\boldsymbol{i}}}}\right)$, $\widetilde{\boldsymbol{x}}^*_{\boldsymbol{i}} = (\widetilde{x}^*_{i1}, \ldots, \widetilde{x}^*_{in_i})$, and compute the value of the bootstrap statistic

$$T^*_{n_1,\ldots,n_k} = \frac{\displaystyle\sum_{i=1}^k n_i \left(S^2_{\widetilde{\boldsymbol{x}}^*_{\boldsymbol{i}}} - \overline{S^2_{\widetilde{\boldsymbol{x}}^*}}\right)^2}{\displaystyle\sum_{i=1}^k \frac{1}{n_i} \sum_{j=1}^{n_i} \left(\left[D_\theta^\varphi(\widetilde{x}^*_{ij}, \overline{\widetilde{\boldsymbol{x}}^*_{\boldsymbol{i}}})\right]^2 - S^2_{\widetilde{\boldsymbol{x}}^*_{\boldsymbol{i}}}\right)^2}$$

Step 3. *Step 2* should be repeated a large number B of times to get a set of estimates, denoted by $\{t^*_1, \ldots, t^*_B\}$.

Step 4. Compute the bootstrap p-value as the proportion of values in $\{t^*_1, \ldots, t^*_B\}$ being greater than $T_{n_1,\ldots,n_k}$.

1.8.1 Simulations-based comparative analysis

The preceding k-sample algorithm is to be particularized to $k = 2$ aiming to test whether the population variances of trapezoidal- and LU-valued random fuzzy numbers can be considered or not as significantly different, where $LU \in \{\Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}, \mathrm{Tri}, \mathrm{TriS}\}$. The **scheme of the simulations-based analysis** for an arbitrary choice of $LU \in \{\Pi, LU_{1A}, LU_{1B}, LU_{2A}, LU_{2B}, \mathrm{Tri}, \mathrm{TriS}\}$ is as follows:

S.1. A sample of 4-tuples, $\left\{(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})\right\}_{i=1}^n$, is generated by the selected simulation procedure (either 1stSP or 2ndSP). To each of the 4-tuples we can associate two samples of size n of fuzzy data, $\left\{\mathrm{Tra}\langle x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)}\rangle\right\}_{i=1}^n$ and $\left\{LU\langle x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)}\rangle\right\}_{i=1}^n$. Then, to test the null hypothesis

$$H_0 : \sigma^2_{\mathrm{Tra}\langle X_1,X_2,X_3,X_4\rangle} = \sigma^2_{LU\langle X_1,X_2,X_3,X_4\rangle}$$

on the basis of the two (treated as independent)[1] samples

$$(\mathrm{Tra}\langle x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\rangle, \ldots, \mathrm{Tra}\langle x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, x_4^{(n)}\rangle)$$

$$(LU\langle x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\rangle, \ldots, LU\langle x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, x_4^{(n)}\rangle),$$

the value of the statistic $T_{n,n}$ in the preceding *Step 1* is computed.

S.2. For each of the two samples (in this step viewed as the bootstrap populations), obtain a sample of n independent observations and compute the value of the bootstrap statistic $T^*_{n,n}$ in the preceding *Step 2*.

S.3. Repeat 1000 times ***S.2*** to get a set of statistic values, $\{t_1^*, \ldots, t_{1000}^*\}$.

S.4. Compute the bootstrap p-value as the proportion of values in $\{t_1^*, \ldots, t_{1000}^*\}$ being greater than the value of $T_{n,n}$.

Table 1.4 shows the p-values obtained when the above-described analysis is repeated 30 times, for $\theta = 1/3$ and $\varphi = \ell$, and simulations are performed in accordance with procedure 1stSP-Case 1. For Case 2 similar conclusions would be achieved.

The obtained p-values indicate that there are no significant differences between population Fréchet's variances for

[1]It should be pointed out that to avoid differences in variances coming from differences in the starting 4-tuples, the sample of Tra's and the sample of LU's in these simulations share the same sample of 4-tuples. Nevertheless, they should not be treated as linked since they are actually assumed to come from two independent samples, each sample being associated with one of the two different shapes. Of course, the bootstrap samples in ***S.2*** do not share in general the same sample of 4-tuples.

- most of the significance levels one can consider,
- almost all the generated samples
- and all the seven developed comparisons.

Table 1.4: Simulation (1stSP)-based bootstrapped two-sample test p-values for the equality of population Fréchet's variances ($\theta = 1/3$) of trapezoidal *vs* other *LU*'s RFN's in 30 simulations of samples of size $n = 10 \,|\, n = 100$

$\theta = 1/3$ ($n = 10 \,\|\, n = 100$)	Tra *vs* Π	Tra *vs* LU_{1A}	Tra *vs* LU_{1B}	Tra *vs* LU_{2A}	Tra *vs* LU_{2B}	Tra *vs* Tri	Tra *vs* TriS
SAMPLE 1	0.935 | 0.898	1.000 | 0.680	0.878 | 0.845	0.957 | 0.898	0.880 | 0.800	0.099 | 0.376	0.569 | 0.434
SAMPLE 2	0.960 | 0.832	0.838 | 0.917	0.765 | 0.672	0.954 | 0.872	0.744 | 0.651	0.913 | 0.199	0.715 | 0.347
SAMPLE 3	0.956 | 0.921	0.814 | 0.875	0.836 | 0.768	0.978 | 0.970	0.785 | 0.776	0.682 | 0.204	0.546 | 0.883
SAMPLE 4	0.981 | 0.912	0.901 | 0.933	0.917 | 0.783	0.964 | 0.957	0.898 | 0.759	0.561 | 0.126	0.584 | 0.958
SAMPLE 5	0.908 | 0.823	0.732 | 0.936	0.640 | 0.724	0.866 | 0.886	0.599 | 0.708	0.686 | 0.204	0.489 | 0.580
SAMPLE 6	0.981 | 0.904	0.919 | 0.903	0.857 | 0.867	0.989 | 0.941	0.850 | 0.880	0.510 | 0.125	0.719 | 0.783
SAMPLE 7	0.734 | 0.854	0.585 | 0.968	0.557 | 0.542	0.796 | 0.932	0.605 | 0.493	0.885 | 0.323	0.626 | 0.277
SAMPLE 8	0.901 | 0.893	0.938 | 0.970	0.644 | 0.843	0.904 | 0.913	0.640 | 0.831	0.479 | 0.263	0.363 | 0.926
SAMPLE 9	0.987 | 0.869	0.963 | 1.000	0.949 | 0.608	0.988 | 0.915	0.931 | 0.559	0.965 | 0.246	0.711 | 0.372
SAMPLE 10	0.987 | 0.877	0.900 | 0.855	0.935 | 0.733	0.999 | 0.897	0.942 | 0.726	0.623 | 0.174	0.634 | 0.903
SAMPLE 11	0.713 | 0.900	0.449 | 0.782	0.394 | 0.781	0.677 | 0.935	0.377 | 0.742	0.659 | 0.356	0.188 | 0.535
SAMPLE 12	0.947 | 0.827	0.634 | 0.760	0.995 | 0.740	0.968 | 0.889	0.992 | 0.669	0.656 | 0.342	0.832 | 0.227
SAMPLE 13	0.975 | 0.929	0.710 | 0.963	0.944 | 0.864	0.969 | 0.944	0.949 | 0.849	0.390 | 0.275	0.962 | 0.866
SAMPLE 14	0.962 | 0.860	0.976 | 0.858	0.813 | 0.635	0.963 | 0.913	0.771 | 0.616	0.727 | 0.234	0.580 | 0.470
SAMPLE 15	0.995 | 0.931	0.949 | 0.891	0.931 | 0.827	0.992 | 0.926	0.939 | 0.811	0.507 | 0.236	0.545 | 0.846
SAMPLE 16	0.979 | 0.954	0.946 | 0.913	0.870 | 0.893	0.975 | 0.971	0.820 | 0.893	0.694 | 0.200	0.867 | 0.956
SAMPLE 17	0.954 | 0.894	0.958 | 0.995	0.985 | 0.700	0.977 | 0.930	1.000 | 0.673	0.851 | 0.095	0.902 | 0.697
SAMPLE 18	0.965 | 0.819	0.859 | 0.971	0.992 | 0.580	0.980 | 0.852	0.984 | 0.561	0.630 | 0.348	0.896 | 0.321
SAMPLE 19	0.985 | 0.836	0.798 | 0.710	0.993 | 0.628	0.970 | 0.916	0.973 | 0.600	0.552 | 0.318	0.955 | 0.187
SAMPLE 20	0.955 | 0.912	0.790 | 0.874	0.885 | 0.615	0.974 | 0.891	0.859 | 0.517	0.845 | 0.322	0.672 | 0.418
SAMPLE 21	0.954 | 0.889	0.984 | 0.904	0.885 | 0.659	0.963 | 0.909	0.882 | 0.609	0.719 | 0.246	0.991 | 0.397
SAMPLE 22	0.795 | 0.903	0.538 | 0.907	0.484 | 0.634	0.752 | 0.906	0.509 | 0.591	0.703 | 0.273	0.368 | 0.724
SAMPLE 23	0.968 | 0.916	0.858 | 0.897	0.932 | 0.664	0.945 | 0.896	0.910 | 0.623	0.809 | 0.559	0.705 | 0.259
SAMPLE 24	0.912 | 0.941	0.849 | 0.901	0.904 | 0.872	0.926 | 0.952	0.874 | 0.870	0.853 | 0.214	0.392 | 0.937
SAMPLE 25	0.870 | 0.923	0.697 | 0.936	0.710 | 0.826	0.949 | 0.963	0.730 | 0.789	0.568 | 0.219	0.470 | 0.831
SAMPLE 26	0.935 | 0.921	0.719 | 0.832	0.618 | 0.658	0.887 | 0.943	0.518 | 0.596	0.506 | 0.251	0.608 | 0.841
SAMPLE 27	0.935 | 0.804	0.876 | 0.800	0.850 | 0.693	0.949 | 0.903	0.840 | 0.672	0.712 | 0.409	0.521 | 0.219
SAMPLE 28	0.972 | 0.926	0.922 | 0.917	0.976 | 0.708	0.951 | 0.924	0.963 | 0.603	0.836 | 0.320	0.841 | 0.524
SAMPLE 29	0.938 | 0.871	0.966 | 0.885	0.855 | 0.570	0.992 | 0.898	0.799 | 0.473	0.905 | 0.517	0.562 | 0.374
SAMPLE 30	0.921 | 0.882	0.980 | 0.926	0.761 | 0.696	0.896 | 0.941	0.717 | 0.693	0.896 | 0.327	0.463 | 0.503

When the compared Fréchet's variances correspond to other values of $\theta \in (0,1]$, conclusions are very close. In fact, for all the considered trials we have only found occasionally some significant differences between trapezoidal and triangular shapes (which, actually, share the support but not the core) for $n = 100$ and $\theta = 10$. It should be emphasized that this choice for θ is not usual, although it is possible (i.e., the corresponding D_θ^φ makes sense as a distance, but it is not equivalent to a Bertoluzza *et al.*'s metric, as originally formalized). Because of the weighting role played by θ in the metric, the greater its value the greater the weight for difference in shape, what certainly affects the statistical conclusions.

Table 1.5 illustrates these comments by collecting the average (over the corresponding 30 samples) p-values for $\theta = 1/3, 1$ and 10.

Table 1.5: Simulation (1stSP)-based bootstrapped two-sample test average p-values for the equality of population Fréchet's variances ($\theta = 1/3, \theta = 1, \theta = 10$) of trapezoidal *vs* other LU's RFN's over 30 simulations of samples of size $n = 10 \mid n = 100$

average p-values	Tra *vs* Π	Tra *vs* LU_{1A}	Tra *vs* LU_{1B}	Tra *vs* LU_{2A}	Tra *vs* LU_{2B}	Tra *vs* Tri	Tra *vs* TriS
$\theta = 1/3$ ($n = 10 \mid n = 100$)	0.932 | 0.887	0.835 | 0.889	0.825 | 0.721	0.935 | 0.919	0.809 | 0.688	0.681 | 0.277	0.643 | 0.587
$\theta = 1$ ($n = 10 \mid n = 100$)	0.934 | 0.923	0.857 | 0.928	0.829 | 0.802	0.940 | 0.942	0.813 | 0.777	0.508 | 0.108	0.689 | 0.554
$\theta = 10$ ($n = 10 \mid n = 100$)	0.950 | 0.955	0.860 | 0.958	0.836 | 0.840	0.954 | 0.966	0.826 | 0.869	0.427 | 0.075	0.467 | 0.095

Table 1.6 shows the p-values obtained when the above-described analysis is repeated 30 times, for $\theta = 1/3$ and $\varphi = \ell$, and simulations are performed in accordance with procedure 2ndSP.

Table 1.6: Simulation (2ndSP)-based bootstrapped two-sample test p-values for the equality of population Fréchet's variances ($\theta = 1/3$) of trapezoidal *vs* other LU's RFN's in 30 simulations of samples of size $n = 10 \mid n = 100$

$\theta = 1/3$ ($n = 10 \mid n = 100$)	Tra *vs* Π	Tra *vs* LU_{1A}	Tra *vs* LU_{1B}	Tra *vs* LU_{2A}	Tra *vs* LU_{2B}	Tra *vs* Tri	Tra *vs* TriS
SAMPLE 1	0.999 | 0.996	0.983 | 0.946	0.999 | 0.971	0.999 | 0.996	0.992 | 0.954	0.997 | 0.957	0.936 | 0.727
SAMPLE 2	0.997 | 0.990	0.974 | 0.999	0.954 | 0.934	1.000 | 1.000	0.943 | 0.909	0.992 | 0.974	0.893 | 0.686
SAMPLE 3	1.000 | 0.990	0.965 | 0.928	0.992 | 0.962	0.997 | 0.993	0.989 | 0.948	0.999 | 0.984	0.912 | 0.706
SAMPLE 4	0.996 | 0.988	0.985 | 0.905	0.933 | 0.983	0.996 | 0.986	0.922 | 0.962	0.988 | 0.967	0.824 | 0.707
SAMPLE 5	0.998 | 0.981	0.998 | 0.896	0.983 | 0.979	1.000 | 0.987	0.978 | 0.957	0.995 | 0.970	0.923 | 0.678
SAMPLE 6	0.995 | 0.992	0.957 | 0.996	0.975 | 0.927	0.998 | 0.987	0.983 | 0.903	0.997 | 0.970	0.981 | 0.774
SAMPLE 7	0.997 | 0.990	0.937 | 0.960	0.971 | 0.935	0.994 | 0.992	0.979 | 0.895	0.988 | 0.965	0.957 | 0.635
SAMPLE 8	0.998 | 0.994	0.962 | 0.977	0.990 | 0.914	0.997 | 0.998	0.982 | 0.902	0.991 | 0.971	0.991 | 0.650
SAMPLE 9	0.998 | 0.995	0.986 | 0.994	0.987 | 0.939	0.998 | 0.997	0.969 | 0.918	0.987 | 0.978	0.854 | 0.775
SAMPLE 10	0.999 | 0.992	0.958 | 0.985	0.986 | 0.952	0.997 | 0.995	0.988 | 0.951	1.000 | 0.992	0.964 | 0.872
SAMPLE 11	0.998 | 0.985	0.967 | 0.943	0.946 | 0.951	0.998 | 0.993	0.941 | 0.928	0.990 | 0.976	0.896 | 0.721
SAMPLE 12	0.998 | 0.992	0.980 | 0.988	0.992 | 0.947	0.996 | 0.996	0.985 | 0.951	0.985 | 0.980	0.936 | 0.799
SAMPLE 13	0.995 | 0.985	0.994 | 0.976	1.000 | 0.933	0.999 | 0.999	0.995 | 0.920	0.993 | 0.971	0.993 | 0.715
SAMPLE 14	1.000 | 0.991	0.988 | 0.988	0.977 | 0.947	0.997 | 0.997	0.970 | 0.946	0.992 | 0.982	0.897 | 0.805
SAMPLE 15	0.998 | 0.998	0.953 | 0.960	0.952 | 0.974	0.998 | 0.992	0.935 | 0.954	0.993 | 0.981	0.931 | 0.778
SAMPLE 16	0.996 | 0.997	0.980 | 0.950	0.965 | 0.956	0.996 | 0.992	0.944 | 0.948	0.995 | 0.958	0.809 | 0.719
SAMPLE 17	0.998 | 0.988	0.958 | 0.973	0.941 | 0.945	0.999 | 0.998	0.928 | 0.906	0.994 | 0.975	0.899 | 0.737
SAMPLE 18	0.998 | 0.990	0.992 | 0.995	0.968 | 0.924	0.998 | 0.995	0.963 | 0.901	0.999 | 0.971	0.864 | 0.698
SAMPLE 19	0.998 | 0.990	0.944 | 0.982	0.993 | 0.936	0.991 | 0.997	0.999 | 0.916	0.995 | 0.984	0.908 | 0.735
SAMPLE 20	0.999 | 0.992	0.990 | 0.935	0.992 | 0.868	0.999 | 0.997	0.990 | 0.820	0.990 | 0.975	0.951 | 0.713
SAMPLE 21	1.000 | 0.988	0.974 | 0.958	0.944 | 0.968	0.999 | 0.994	0.918 | 0.953	0.998 | 0.947	0.823 | 0.755
SAMPLE 22	1.000 | 0.993	0.997 | 0.962	0.965 | 0.937	0.998 | 0.999	0.958 | 0.902	0.993 | 0.988	0.904 | 0.806
SAMPLE 23	1.000 | 0.988	0.977 | 0.976	0.970 | 0.944	0.996 | 1.000	0.951 | 0.918	0.988 | 0.973	0.933 | 0.773
SAMPLE 24	0.999 | 0.994	0.958 | 0.976	0.985 | 0.967	0.995 | 0.995	0.977 | 0.954	0.989 | 0.969	0.984 | 0.826
SAMPLE 25	1.000 | 0.991	0.998 | 0.994	0.990 | 0.943	1.000 | 0.999	0.990 | 0.918	0.992 | 0.977	0.955 | 0.730
SAMPLE 26	0.999 | 0.985	1.000 | 0.968	0.991 | 0.974	0.997 | 0.998	0.996 | 0.953	0.997 | 0.952	0.984 | 0.758
SAMPLE 27	0.999 | 0.993	0.993 | 0.991	0.998 | 0.924	0.998 | 0.995	0.997 | 0.901	0.998 | 0.988	0.974 | 0.728
SAMPLE 28	0.997 | 0.989	0.994 | 0.931	0.979 | 0.941	1.000 | 0.985	0.983 | 0.918	0.993 | 0.978	0.958 | 0.654
SAMPLE 29	0.999 | 0.996	0.991 | 0.945	0.952 | 0.959	0.997 | 0.986	0.959 | 0.941	0.983 | 0.986	0.856 | 0.700
SAMPLE 30	0.998 | 0.996	0.992 | 0.980	1.000 | 0.969	0.998 | 0.997	1.000 | 0.951	0.987 | 0.978	0.994 | 0.806

The obtained p-values indicate, even more conclusively than for the 1stSP, that there are no significant differences between population Fréchet's variances for

- almost all the significance levels one can consider,
- all the generated samples
- and all the seven developed comparisons.

When the compared Fréchet's variances correspond to other values of $\theta \in (0, 1]$, conclusions are very close. In fact, for all the considered trials we have only found a significant difference between trapezoidal and triangular symmetric shapes for $n = 100$ and $\theta = 10$.

Table 1.7 illustrates these comments by collecting the average (over the corresponding 30 samples) p-values for $\theta = 1/3, 1$ and 10.

Table 1.7: Simulation (2ndSP)-based bootstrapped two-sample test average p-values for the equality of population Fréchet's variances ($\theta = 1/3, \theta = 1, \theta = 10$) of trapezoidal *vs* other LU's RFN's over 30 simulations of samples of size $n = 10 \mid n = 100$

average p-values	Tra *vs* Π	Tra *vs* LU_{1A}	Tra *vs* LU_{1B}	Tra *vs* LU_{2A}	Tra *vs* LU_{2B}	Tra *vs* Tri	Tra *vs* TriS
$\theta = 1/3$ ($n = 10 \mid n = 100$)	0.998 | 0.991	0.978 | 0.965	0.976 | 0.947	0.998 | 0.995	0.970 | 0.927	0.993 | 0.974	0.923 | 0.739
$\theta = 1$ ($n = 10 \mid n = 100$)	0.997 | 0.985	0.978 | 0.965	0.976 | 0.969	0.997 | 0.994	0.972 | 0.953	0.979 | 0.920	0.911 | 0.684
$\theta = 10$ ($n = 10 \mid n = 100$)	0.982 | 0.923	0.953 | 0.926	0.957 | 0.766	0.986 | 0.933	0.951 | 0.737	0.814 | 0.411	0.747 | 0.280

1.8.2 Case study-based comparative analysis

The analysis in Subsection 1.8.1 is carried out aiming to test the influence of the shape of fuzzy data on the Fréchet variance. By means of some of the data in Example 1.6.1, this subsection follows two different comparative approaches. More concretely, this subsection is first devoted to compare the p-values of two-sample and k-sample test about the equality of variances for different choices of the shape.

Table 1.8 gathers the p-values of the two-sample test about the equality of variances on the basis of the fuzzy rating scale responses to Item $M.2$ in Example 1.6.1 when the two considered populations are 'boys' and 'girls' (see Pages 238 and 240), when the 4-tuples are associated not only with trapezoidal fuzzy numbers (as it has actually been made) but also with other LU's. The p-values have been computed for $\theta = 1/3, 1$ and 10.

Table 1.8: p-Values for the equality of population Fréchet's variances ($\theta = 1/3, \theta = 1, \theta = 10$) of boys' and girls' LU's responses to Item $M.2$ in Example 1.6.1, depending on the considered shape

p-values	Tra	Π	LU_{1A}	LU_{1B}	LU_{2A}	LU_{2B}	Tri	TriS
$\theta = 1/3$	0.416	0.478	0.539	0.466	0.473	0.456	0.466	0.397
$\theta = 1$	0.414	0.443	0.512	0.452	0.467	0.456	0.450	0.376
$\theta = 10$	0.461	0.461	0.406	0.555	0.447	0.544	0.334	0.263

For the usually selected significance levels (those being lower than 0.25), there are no significant differences between boys and girls in responding to $M.2$, irrespectively of the considered shape of fuzzy data and even of the choice of θ.

Table 1.9 gathers the p-values of the two-sample test about the equality of variances on the basis of the fuzzy rating scale responses to Item $M.2$ in Example 1.6.1 when the two considered populations are 'paper-and-pencil' and 'computerized' form (see Pages 238 and 240), when the 4-tuples are associated with several LU-valued fuzzy numbers. The p-values have been computed for $\theta = 1/3, 1$ and 10.

Table 1.9: p-Values for the equality of population Fréchet's variances ($\theta = 1/3, \theta = 1, \theta = 10$) of 'paper-and-pencil' and 'computerized' form's LU's responses to Item $M.2$ in Example 1.6.1, depending on the considered shape

p-values	Tra	Π	LU_{1A}	LU_{1B}	LU_{2A}	LU_{2B}	Tri	TriS
$\theta = 1/3$	0.215	0.220	0.239	0.217	0.233	0.198	0.234	0.190
$\theta = 1$	0.149	0.166	0.186	0.165	0.161	0.165	0.184	0.161
$\theta = 10$	0.014	0.017	0.017	0.017	0.009	0.013	0.033	0.030

In this second situation, the effect of the choice of $\theta \in (0, 1]$ is not very relevant, but it is for $\theta >> 1$ (a rather rare choice, since it does not seem highly recommended for a dispersion measure to weigh 'difference in shape' more than 'difference in location'). Anyway, statistical conclusions scarcely depend on the considered shape of fuzzy data.

Table 1.10 gathers the p-values of the four-sample test about the equality of variances on the basis of the fuzzy rating scale responses to Item $M.2$ in Example 1.6.1 when the four considered populations are four groups of students based on their 'mark taken in the last examination of math' (see $M4$, Page 240) given by $\boldsymbol{G1} = [0, 6]$, $\boldsymbol{G2} = (6, 8]$, $\boldsymbol{G3} = (8, 9]$ and $\boldsymbol{G4} = (9, 10]$, according to the usual range $[0, 10]$ which is considered in Spain. The p-values have been computed for $\theta = 1/3, 1$ and 10, when the 4-tuples are associated with several LU-valued fuzzy numbers.

Table 1.10: p-Values for the equality of population Fréchet's variances ($\theta = 1/3, \theta = 1, \theta = 10$) of the four groups, $\boldsymbol{G1}$ to $\boldsymbol{G4}$, LU's responses to Item $M.2$ in Example 1.6.1, depending on the considered shape

p-values	Tra	Π	LU_{1A}	LU_{1B}	LU_{2A}	LU_{2B}	Tri	TriS
$\theta = 1/3$	0.270	0.255	0.255	0.282	0.263	0.275	0.247	0.218
$\theta = 1$	0.258	0.260	0.274	0.239	0.247	0.268	0.251	0.241
$\theta = 10$	0.157	0.156	0.130	0.177	0.157	0.171	0.201	0.172

Once more, in this third situation statistical conclusions scarcely depend on the considered shape of fuzzy data.

A second way to analyze the influence of the shape of fuzzy data by means of the real-life Example 1.6.1, is to compare by means of the two-sample test about the equality of variances trapezoidal data *vs* other LU data in the responses to Item $M.2$ for different populations involved in the preceding tables in this subsection.

Table 1.11 collects the corresponding p-values for $\theta = 1/3$.

Table 1.11: p-Values for the equality of population Fréchet's variances ($\theta = 1/3$) of trapezoidal *vs* other LU's responses for populations in Tables 1.8, 1.9 and 1.10

p-values	Tra *vs* Π	Tra *vs* LU_{1A}	Tra *vs* LU_{1B}	Tra *vs* LU_{2A}	Tra *vs* LU_{2B}	Tra *vs* Tri	Tra *vs* TriS
Boys	0.998	0.909	0.916	0.994	0.902	0.970	0.897
Girls	1.000	0.974	0.951	0.998	0.940	0.972	0.843
Paper-and-pencil	0.997	0.998	0.964	0.997	0.938	0.994	0.890
Computerized form	0.995	0.902	0.914	0.992	0.901	0.961	0.852
$\boldsymbol{G1}$	0.991	0.866	0.923	0.980	0.904	0.832	0.820
$\boldsymbol{G2}$	0.997	0.922	0.927	0.995	0.920	0.978	0.923
$\boldsymbol{G3}$	1.000	0.979	0.949	0.998	0.949	0.986	0.927
$\boldsymbol{G4}$	0.996	0.999	0.975	0.995	0.968	0.986	0.917

The conclusions from the last table are similar to those in the simulations studies following the procedure 2ndSP: there are no significant differences between population Fréchet's variances for almost all the significance levels one can consider and all the seven developed comparisons.

1.9 Concluding remarks of this chapter

This chapter has presented the ideas motivating the interest of fuzzy scales to deal with intrinsically imprecise magnitudes as well as the preliminary and supporting concepts and results corresponding to fuzzy data. In recalling the basic models and tools to formalize and handle these data in a probabilistic/statistical setting, the need for robust measures/estimates of location of fuzzy datasets has been reviewed. Furthermore, simulations-based and case study-based discussions on the sensitivity of the usual summary dispersion measure (the Fréchet-type variance or the corresponding standard deviation) w.r.t. the shape of fuzzy data have been carried out to conclude that the shape is not relevant in general.

These preliminaries lead to consider two immediate open directions, that will be dealt with in next two chapters and are now briefly motivated.

1.9.1 Motivating the need for robust measures alternative to the Fréchet-type variance

As it has already been commented, the Fréchet-type variance, and hence the standard deviation, of an RFN preserves the main valuable properties of the variance, respectively the standard deviation, of real-valued random variables, but it also inherits its high sensitivity to the presence of outliers, data changes or asymmetries. Actually, the presence of outliers, data changes, etc., can affect the Fréchet-type variance even more than the Aumann-type mean.

To empirically corroborate this assertion, one can consider, for instance, the dataset of 23 FRS-based responses to Item $M.2$ in Example 1.6.1 for students who have filled out the paper-and-pencil form (see Figure 1.13), denoted by $\widetilde{\mathbf{x}}_{23}$.

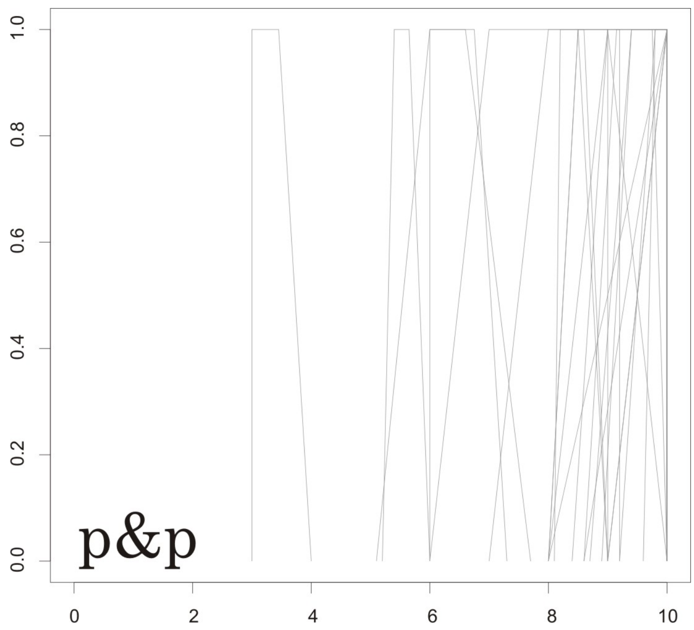


Figure 1.13: FRS responses to Item $M.2$
by students who have filled out the paper-and-pencil form

If we consider the metric ρ_2, then $S_{\widetilde{\mathbf{x}}_{23}} = 1.69$.

If the fuzzy datum on the left of Figure 1.13 (an evident outlier of the sample) is removed and the reduced sample is denoted by $\widetilde{\mathbf{x}}_{22}$, then we get $S_{\widetilde{\mathbf{x}}_{22}} = 1.29$.

This fact motivates the development of a study on the robust measurement/ estimation of scale or dispersion in the setting of random fuzzy numbers. Actually, this will be the objective of Chapter 2 in this work.

1.9.2 Motivating the interest of comparing statistical conclusions for different scales of measurement from a dispersion perspective

The interest of the fuzzy scales to rate intrinsically imprecise magnitudes has been argued in Subsection 1.2.2. It is intuitively obvious that the use of fuzzy rating scales provides with a much richer and diverse information than Likert-type or fuzzy linguistic ones. This results in more accurate statistical conclusions, sometimes differing importantly from the conclusions drawn from Likert or fuzzy linguistic.

In previous studies (see Gil *et al.* [44], Lubiano *et al.* [78, 75]) we have confirmed the preceding comments mainly from the location perspective. Nevertheless, differences can often be clearer from the dispersion perspective. This is mainly due to the fact that many values or responses matching for the Likert-type or fuzzy linguistic scales (and hence showing no dispersion) do not match at all for the fuzzy rating scale (and hence showing a certain dispersion).

To support and illustrate this last assertion, one can consider a combined graphical display of the double response to Item $M.2$ in Example 1.6.1 for which the Likert scale-based response chosen by four students has corresponded to DISAGREE A LITTLE, and the fuzzy rating scale-based responses for the same students have been definitely different (see Figure 1.14).

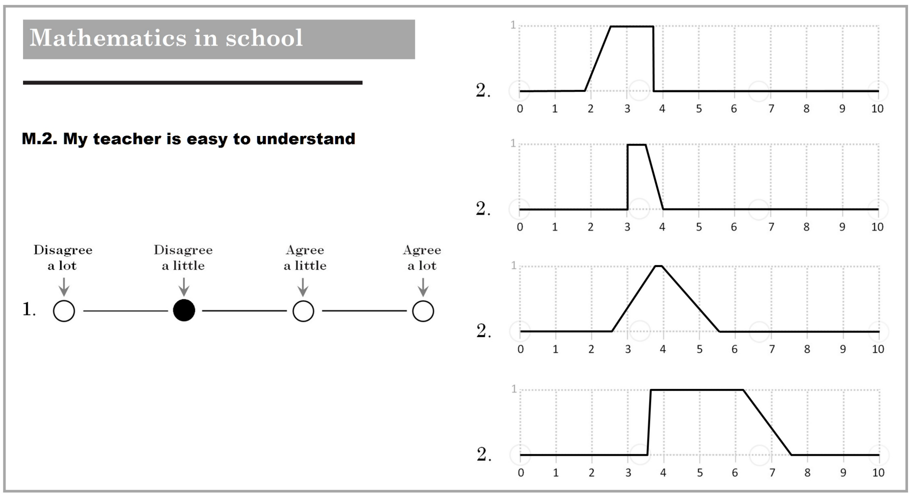


Figure 1.14: Example of 4 double responses to Item $M.2$
for which the Likert-type ones coincide while the fuzzy rating scale-type clearly differ

Of course the sample standard deviation for the four Likert data (or their fuzzy linguistic counterpart) equals 0, whereas the one for the four FRS-based data

$$\mathrm{Tra}(1.75, 2.5, 3.675, 3.675),\ \ \mathrm{Tra}(3, 3, 3.45, 4),$$

$$\mathrm{Tra}(2.5, 3.75, 3.9, 5.45),\ \ \mathrm{Tra}(3.5, 3.55, 6.25, 7.5),$$

and considering ρ_2 as the involved metric, equals 1.

Chapter 3 in this work aims to enter in more detail a comparative statistical analysis between the three scales, Likert or numerically encoded Likert, fuzzy linguistic and fuzzy rating, from a dispersion perspective. More concretely, the purpose is that of corroborating that statistical results often vary with the employed scale of rating.

Chapter 2

Robust scale measures in dealing with fuzzy data

In summarizing the distribution of an RFN, two key aspects are usually taken into account: the measurement of the central tendency or location and the measurement of the dispersion or scale. Whereas the former constitutes fuzzy-valued summary indicators, the latter refers to real-valued summary measurements.

In the previous chapter, it has already been introduced the most commonly used location measure in dealing with fuzzy-valued data: the Aumann-type mean, which inherits from the real case its high sensitivity to the presence of atypical observations in the data. Morever, the 1-norm median and φ-wabl/ldev/rdev median have also been recalled as good alternatives to the Aumann-type mean because of their good robust behaviour.

However, in this work we are focussing on the study of the dispersion or scale measurement for fuzzy-valued data. The estimators of scale, in addition to summarize and inform about the distribution of a random element, can be used for many other purposes, like for instance, as auxiliary estimates for location, as objective functions in regression or multivariate analysis, as a basis to formulate rules to detect outliers and so on. Therefore, although there is less literature on robust estimation of scale than on robust estimation of location, scale estimators are very important in practice.

The Fréchet-type variance (or standard deviation) is the best known and used dispersion measure, despite the fact that it is a measure highly influenced by the outliers, as previously illustrated in Subsection 1.9.1. It is for that reason that the purpose of this chapter consists of looking for other estimators of scale that are resistant to the presence of extreme/atypical observations among the data, that is, estimators with a good robust behaviour.

In Section 2.1, several measures of scale to deal with real-valued data are extended to deal with fuzzy-valued data. Some of them are location-free estimators whereas others are location-based estimators, that is, there is an estimator of location in their expression that needs to be computed first.

Section 2.2 is devoted to the study of some of the main properties of these measures, such as certain in/equi-variances and the strong consistency. The robustness, a highly important and valuable property of an estimator which makes it insensitive to the influence of atypical observations among the data, is proved in Section 2.3 by means of the finite sample breakdown point. A key novel idea at this stage is that of outlier in a fuzzy-valued setting; Section 2.4 explains the three types of outliers in dealing with fuzzy data that will be used along this work. Morover, it shows a simulation study about the empirical finite sample breakdown point and the sensitivity curves of the estimators presented in Section 2.1.

An additional approach to estimating scale consists of what are called M-estimators, which are briefly introduced and their empirical robust behaviour is analyzed in Section 2.5. The three case studies presented in Chapter 1 are used to illustrate the computation of all the scale estimators in Section 2.6. The chapter ends with a Section 2.7 containing the summary of the contribution of the chapter as well as the related publications derived from it.

2.1 Scale measures for fuzzy data

Let $\mathcal{X}$ be a random fuzzy number associated with the probability space $(\Omega, \mathcal{A}, P)$, $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ a sample of observations from $\mathcal{X}$ and $D \in \{\rho_1, \mathscr{D}_\theta^\varphi, D_\theta^\varphi\}$ a metric between fuzzy data (see Subsection 1.3.2). Then, as we have already reminded in Chapter 1,

Definition 2.1.1. *The **(population) Fréchet-type D_θ^φ-Standard Deviation** is the real number D_θ^φ-SD$(\mathcal{X})$ (also shortly denoted by $\sigma_{\mathcal{X}}$), if it exists, given by*

$$D_\theta^\varphi\text{-SD}(\mathcal{X}) = \sqrt{E\left(\left[D_\theta^\varphi\big(\mathcal{X}, \widetilde{E}(\mathcal{X})\big)\right]^2\right)}.$$

*In particular, the **(sample) Fréchet-type D_θ^φ-Standard Deviation** is the real number $\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n)$ (also shortly denoted by $S_{\widetilde{\mathbf{x}}_n}$) given by*

$$\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n) = \sqrt{\frac{1}{n}\sum_{i=1}^n \left[D_\theta^\varphi(\widetilde{x}_i, \overline{\widetilde{\mathbf{x}}}_n)\right]^2}$$

The first suggested alternative location-based scale measures for RFN's are those extending to the fuzzy-valued case the well-known average absolute deviation. Since in the fuzzy setting several distances extending the Euclidean metric on the space of real numbers could be considered, it seems often natural for the location-based scale measures to choose the related metric (i.e., the one for which the average is minimized by the involved location measure).

Definition 2.1.2. *The* ***(population) D-Average Distance Deviation*** *of $\mathcal{X}$ with respect to the location measure $\widetilde{U}$ is the real number $D\text{-ADD}(\mathcal{X},\widetilde{U})$ given by*

$$D\text{-ADD}(\mathcal{X},\widetilde{U}) = E\left[D(\mathcal{X},\widetilde{U})\right].$$

Specifically, the following D-average distance deviations are analyzed in this work:

- *The* ***(population) $\boldsymbol{\rho_2}$-Average Distance Deviation of $\mathcal{X}$ with respect to the Aumann-type mean*** *$\widetilde{E}(\mathcal{X})$ is the real number $\rho_2\text{-ADD}(\mathcal{X},\widetilde{E}(\mathcal{X}))$, if it exists, given by*

  $$\rho_2\text{-ADD}(\mathcal{X},\widetilde{E}(\mathcal{X})) = E\left[\rho_2(\mathcal{X},\widetilde{E}(\mathcal{X}))\right].$$

  In particular, the ***(sample) $\boldsymbol{\rho_2}$-Average Distance Deviation of $\widetilde{\mathbf{x}}_n$ with respect to the Aumann-type mean*** *$\overline{\widetilde{\mathbf{x}}}_n$ is the real number $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n,\overline{\widetilde{\mathbf{x}}}_n)$ given by*

  $$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n,\overline{\widetilde{\mathbf{x}}}_n) = \frac{1}{n}\sum_{i=1}^{n}\rho_2(\widetilde{x}_i,\overline{\widetilde{\mathbf{x}}}_n).$$

- *The* ***(population) $\boldsymbol{\rho_1}$-Average Distance Deviation of $\mathcal{X}$ with respect to the 1-norm median*** *$\widetilde{\mathrm{Me}}(\mathcal{X})$ is the real number $\rho_1\text{-ADD}(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X}))$, if it exists, given by*

  $$\rho_1\text{-ADD}(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X})) = E\left[\rho_1(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X}))\right] = \min_{\widetilde{U}\in\mathcal{F}_c^*(\mathbb{R})} E\left[\rho_1(\mathcal{X},\widetilde{U})\right].$$

  In particular, the ***(sample) $\boldsymbol{\rho_1}$-Average Distance Deviation of $\widetilde{\mathbf{x}}_n$ with respect to the 1-norm median*** *$\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)$ is the real number $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$ given by*

  $$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n}\sum_{i=1}^{n}\rho_1\left(\widetilde{x}_i,\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)\right).$$

- *The* ***(population)*** $\mathscr{D}_\theta^\varphi$***-Average Distance Deviation of*** $\mathcal{X}$ ***with respect to the*** φ***-wabl/ldev/rdev median*** $\widetilde{\mathrm{M}^\varphi}(\mathcal{X})$ *is the real number* $\mathscr{D}_\theta^\varphi\text{-ADD}(\mathcal{X},\widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$*, if it exists, given by*

$$\mathscr{D}_\theta^\varphi\text{-ADD}(\mathcal{X},\widetilde{\mathrm{M}^\varphi}(\mathcal{X})) = E\left[\mathscr{D}_\theta^\varphi(\mathcal{X},\widetilde{\mathrm{M}^\varphi}(\mathcal{X}))\right] = \min_{\widetilde{U}\in\mathscr{F}_c^*(\mathbb{R})} E\left[\mathscr{D}_\theta^\varphi(\mathcal{X},\widetilde{U})\right].$$

In particular, the ***(sample)*** $\mathscr{D}_\theta^\varphi$***-Average Distance Deviation of*** $\widetilde{\mathbf{x}}_n$ ***with respect to the*** φ***-wabl/ldev/rdev median*** $\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)$ *is the real number* $\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))$ *given by*

$$\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n}\sum_{i=1}^n \mathscr{D}_\theta^\varphi\left(\widetilde{x}_i,\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)\right).$$

Remark 2.1.1. As it has been pointed out, the ρ_1-ADD and the $\mathscr{D}_\theta^\varphi$-ADD are the minima of the corresponding average distance. This is not true, in general, for the ρ_2-ADD, since it would not be minimized at the Aumann-type mean value but at the extended spatial median, that in the fuzzy case would be extremely complex to determine (in this respect, see Sinova and Van Aelst [117] for the interval-valued case).

The second suggested alternative location-based scale measures for RFN's are those extending to the fuzzy-valued case the well-known median absolute deviation. In these definitions, the real-valued median is assumed to be intended in accordance with the usual convention and, whenever the involved location fuzzy measure is not unique, the commented conventions in Chapter 1 will be considered.

Definition 2.1.3. *The* ***(population) D-Median Distance Deviation*** *of* $\mathcal{X}$ *with respect to the location measure* $\widetilde{U}$ *is the real number* $D\text{-MDD}(\mathcal{X},\widetilde{U})$ *given by*

$$D\text{-MDD}(\mathcal{X},\widetilde{U}) = \mathrm{Me}\left[D(\mathcal{X},\widetilde{U})\right].$$

Specifically, the following D-median distance deviations are analyzed in this work:

- *The* ***(population)*** $\boldsymbol{\rho_2}$***-Median Distance Deviation of*** $\mathcal{X}$ ***with respect to the Aumann-type mean*** $\widetilde{E}(\mathcal{X})$ *is the real number* $\rho_2\text{-MDD}(\mathcal{X},\widetilde{E}(\mathcal{X}))$*, if it exists, given by*

$$\rho_2\text{-MDD}(\mathcal{X},\widetilde{E}(\mathcal{X})) = \mathrm{Me}\left[\rho_2(\mathcal{X},\widetilde{E}(\mathcal{X}))\right].$$

In particular, the ***(sample)*** $\boldsymbol{\rho_2}$***-Median Distance Deviation of*** $\widetilde{\mathbf{x}}_n$ ***with respect to the Aumann-type mean*** $\overline{\widetilde{\mathbf{x}}}_n$ *is the real number* $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n,\overline{\widetilde{\mathbf{x}}}_n)$ *given by*

$$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n,\overline{\widetilde{\mathbf{x}}}_n) = \mathrm{Me}\left\{\rho_2(\widetilde{x}_i,\overline{\widetilde{\mathbf{x}}}_n)\,;\ i=1,\dots,n\right\}.$$

- *The* ***(population)*** $\boldsymbol{\rho_1}$***-Median Distance Deviation of*** $\mathcal{X}$ ***with respect to the 1-norm median*** $\widetilde{\mathrm{Me}}(\mathcal{X})$ *is the real number* $\rho_1\text{-MDD}(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X}))$*, if it exists, given by*

$$\rho_1\text{-MDD}(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X})) = \mathrm{Me}\left[\rho_1(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X}))\right].$$

  In particular, the ***(sample)*** $\boldsymbol{\rho_1}$***-Median Distance Deviation of*** $\widetilde{\mathbf{x}}_n$ ***with respect to the 1-norm median*** $\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)$ *is the real number* $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$ *given by*

$$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)) = \mathrm{Me}\left\{\rho_1\left(\widetilde{x}_i,\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)\right)\ ;\ i=1,\ldots,n\right\}.$$

- *The* ***(population)*** $\boldsymbol{\mathscr{D}_\theta^\varphi}$***-Median Distance Deviation of*** $\mathcal{X}$ ***with respect to the*** φ***-wabl/ldev/rdev median*** $\widetilde{\mathrm{M}^\varphi}(\mathcal{X})$ *is the real number* $\mathscr{D}_\theta^\varphi\text{-MDD}(\mathcal{X},\widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$*, if it exists, given by*

$$\mathscr{D}_\theta^\varphi\text{-MDD}(\mathcal{X},\widetilde{\mathrm{M}^\varphi}(\mathcal{X})) = \mathrm{Me}\left[\mathscr{D}_\theta^\varphi(\mathcal{X},\widetilde{\mathrm{M}^\varphi}(\mathcal{X}))\right].$$

  In particular, the ***(sample)*** $\boldsymbol{\mathscr{D}_\theta^\varphi}$***-Median Distance Deviation of*** $\widetilde{\mathbf{x}}_n$ ***with respect to the*** φ***-wabl/ldev/rdev median*** $\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)$ *is the real number* $\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))$ *given by*

$$\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)) = \mathrm{Me}\left\{\mathscr{D}_\theta^\varphi\left(\widetilde{x}_i,\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)\right)\ ;\ i=1,\ldots,n\right\}.$$

Next measures are inspired by the scale estimates for data from real-valued random variables introduced by Rousseeuw and Croux [99, 100] as alternatives to the median absolute deviation. They have a simple and explicit formula and they are location-free estimates of scale, that is, in contrast to the measures in Definitions 2.1.1, 2.1.2 and 2.1.3, it is not necessary to compute a location measure because these estimates only consider deviations between data values.

Definition 2.1.4. *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *independent and identically distributed random fuzzy numbers associated with the probability space* $(\Omega,\mathcal{A},P)$*. The* ***(population) scale estimate*** $\boldsymbol{D}$**-S**$(\boldsymbol{\mathcal{X}},\boldsymbol{\mathcal{Y}})$ *is the real number given by*

$$D\text{-S}(\mathcal{X},\mathcal{Y}) = \mathrm{Me}_{\mathcal{X}}\mathrm{Me}_{\mathcal{Y}}D(\mathcal{X},\mathcal{Y}).$$

In particular, the ***(sample) scale estimate*** $\widehat{\boldsymbol{D}\text{-}\mathbf{S}}(\widetilde{\mathbf{x}}_n)$ *is the real number given by*

$$\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n) = \underline{\mathrm{Me}}_i\left\{\overline{\mathrm{Me}}_j\left\{D(\widetilde{x}_i,\widetilde{x}_j)\right\}\right\},$$

where $\underline{\mathrm{Me}}$ *is a low median, that is, the order statistic of rank* $\lfloor (n+1)/2 \rfloor$ *and* $\overline{\mathrm{Me}}$ *is a high median, which is the order statistic of rank* $h := \lfloor n/2 \rfloor + 1$ *(this notation will be used throughout this work). Therefore, for each* i *we compute the high median of* $\{D(\widetilde{x}_i, \widetilde{x}_j) : j = 1, \ldots, n\}$*. This leads to* n *non-negative numbers, the low median of which gives the estimate* $\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n)$*.*

When we are dealing with fuzzy data, next two scale measures only can be defined in their sample version, but not in their population version. This is due to the fact that there is not a distribution function to characterize the distribution induced by a random fuzzy number, as it has already been highlighted in Page 28. It can be seen in Rousseeuw and Croux [99, 100] that the population version of the following estimators involves the class of generalized L-statistics introduced by Serfling [102], in which the distribution function is involved.

Definition 2.1.5. *The* ***(sample) scale estimate*** $\widehat{\boldsymbol{D}\text{-}\mathbf{Q}}(\widetilde{\mathbf{x}}_n)$ *is the real number given by*

$$\widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n) = \{D(\widetilde{x}_i, \widetilde{x}_j) : i < j\}_{(m)},$$

where $m := \binom{h}{2}$ *and* h *is given as in Definition 2.1.4. That is, it takes the order statistic of range* m *of the* $\binom{n}{2}$ *distances between the fuzzy values.*

Definition 2.1.6. *The* ***(sample) scale estimate*** $\widehat{\boldsymbol{D}\text{-}\mathbf{T}}(\widetilde{\mathbf{x}}_n)$ *is the real number given by*

$$\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n) = \frac{1}{h} \sum_{r=1}^{h} \left\{ \overline{\mathrm{Me}}_j \left\{ D(\widetilde{x}_i, \widetilde{x}_j) \right\} \; ; \; i = 1, \ldots, n \right\}_{(r)},$$

where h *is given as in Definition 2.1.4. That is, for each* i *the estimate calculates the high median of* $\{D(\widetilde{x}_i, \widetilde{x}_j) : j = 1, \ldots, n\}$*, the process leading to* n *medians. Then, it computes the average of the* h *first ordered medians.*

Remark 2.1.2. As it has already been mentioned, the calculation of the three last scale estimates does not require a previous computation of a location measure because of they only take into account distances between observations. This entails a computational advantage with respect to the scale estimates introduced in Definitions 2.1.1-2.1.3.

2.2 Formal general properties as scale measures

This section shows that the scale measures introduced in Section 2.1 preserve the main properties from those in the real-valued case. First, we prove that they actually extend well-known measures for real-valued random variables.

Let $\mathcal{X} : \Omega \to \mathscr{F}_c^*(\mathbb{R})$ be an RFN associated with the probability space $(\Omega, \mathcal{A}, P)$, $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample of observations from $\mathcal{X}$, and $D \in \{\rho_1, \mathscr{D}_\theta^\varphi, D_\theta^\varphi\}$ be a metric between fuzzy data.

Proposition 2.2.1. ***(Extension from the real-valued case).*** *If there exists a real-valued random variable X associated with $(\Omega, \mathcal{A}, P)$ such that $\mathcal{X} = \mathbb{1}_{\{X\}}$ a.s. $[P]$, then the scale measures introduced in the Definitions 2.1.1-2.1.6 coincide with the corresponding measures for the real-valued case.*

Proof. This is immediate taking into account that for notions in Definitions 2.1.1 to 2.1.6,

- the Aumann-type mean and the 1-norm median and the φ-wabl/ldev/rdev median reduce in the real-valued case to the mean and median, respectively,
- and the metrics ρ_1, $\mathscr{D}_\theta^\varphi$ and D_θ^φ are generalizations of the Euclidean distance in $\mathbb{R}$. □

In the following propositions it will be shown that the scale measures introduced in Definitions 2.1.1-2.1.6 preserve the main properties of the corresponding scale estimates for real-valued random variables (or even for multidimensional ones -see, for instance, Kołacz and Grzegorzewski [69]-). More concretely,

Proposition 2.2.2. ***(Nonnegativeness).*** *The scale measures in Definitions 2.1.1-2.1.6 are non-negative estimates.*

Proof. The result is obvious, by simply taking into account that D is a metric and, hence, it is non-negative. □

Besides, all the measures are equal to zero if the RFN $\mathcal{X}$ is degenerate at a fuzzy number, that is,

Proposition 2.2.3. *If the RFN $\mathcal{X}$ is degenerate at a fuzzy number (i.e., there exists a fuzzy number $\widetilde{U}$ such that $\mathcal{X} = \widetilde{U}$ a.s. $[P]$), then the scale measures in Definitions 2.1.1-2.1.6 are equal to zero.*

Proof. If $\mathcal{X} = \widetilde{U}$ a.s. $[P]$, then $D_\theta^\varphi(\mathcal{X}, \widetilde{E}(\mathcal{X})) = \rho_1(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})) = \mathscr{D}_\theta^\varphi(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})) = D(\mathcal{X}, \widetilde{U}) = D(\widetilde{U}, \widetilde{U}) = 0$ a.s. $[P]$, whence all measures in Definitions 2.1.1-2.1.3 vanish.

On the other hand, if $\mathcal{X}$ and $\mathcal{Y}$ are independent and identically distributed random fuzzy numbers, then $D\text{-S}(\mathcal{X}, \mathcal{Y}) = \mathrm{Me}_\mathcal{X}\mathrm{Me}_\mathcal{Y} D(\mathcal{X}, \mathcal{Y}) = \mathrm{Me}_\mathcal{X}\mathrm{Me}_\mathcal{Y} D(\widetilde{U}, \widetilde{U}) = 0$ a.s. $[P]$.

Finally, if $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ is a sample of observations from $\mathcal{X} = \widetilde{U}$ a.s. $[P]$, then $\widetilde{\mathbf{x}}_n = (\widetilde{U}, \overset{(n\,\text{times})}{\ldots}, \widetilde{U})$, and so, $\widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n) = \widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n) = 0$. □

As it happens for the real-valued case, the reciprocal implication is only true for some of the measures. Thus,

Proposition 2.2.4. ***(Minimality).*** *D_θ^φ-SD$(\mathcal{X}) = 0$, or ρ_2-ADD$(\mathcal{X}, \widetilde{E}(\mathcal{X})) = 0$, or ρ_1-ADD$(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})) = 0$, or $\mathscr{D}_\theta^\varphi$-ADD$(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})) = 0$, holds if, and only if, the RFN $\mathcal{X}$ is degenerate at the involved location measure.*

Proof. Indeed, because of the non-negativeness of the distances between fuzzy numbers, one can conclude that

$$D_\theta^\varphi\text{-SD}(\mathcal{X}) = 0 \text{ iff } D_\theta^\varphi(\mathcal{X}, \widetilde{E}(\mathcal{X})) = 0 \text{ a.s.} [P]$$
$$\text{iff } \mathcal{X} = \widetilde{E}(\mathcal{X}) \text{ a.s.} [P].$$
$$\rho_2\text{-ADD}(\mathcal{X}, \widetilde{E}(\mathcal{X})) = 0 \text{ iff } \rho_2(\mathcal{X}, \widetilde{E}(\mathcal{X})) = 0 \text{ a.s.} [P]$$
$$\text{iff } \mathcal{X} = \widetilde{E}(\mathcal{X}) \text{ a.s.} [P].$$
$$\rho_1\text{-ADD}(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})) = 0 \text{ iff } \rho_1(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})) = 0 \text{ a.s.} [P]$$
$$\text{iff } \mathcal{X} = \widetilde{\mathrm{Me}}(\mathcal{X}) \text{ a.s.} [P].$$
$$\mathscr{D}_\theta^\varphi\text{-ADD}(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})) = 0 \text{ iff } \mathscr{D}_\theta^\varphi(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})) = 0 \text{ a.s.} [P]$$
$$\text{iff } \mathcal{X} = \widetilde{\mathrm{M}^\varphi}(\mathcal{X}) \text{ a.s.} [P].$$

□

Remark 2.2.1. As for the real-valued case, the vanishing of ρ_2-MDD$(\mathcal{X}, \widetilde{E}(\mathcal{X}))$, ρ_1-MDD$(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X}))$, $\mathscr{D}_\theta^\varphi$-MDD$(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$, D-S$(\mathcal{X}, \mathcal{Y})$, $\widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n)$, or $\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)$, does not generally imply that the RFN $\mathcal{X}$ is degenerate at a fuzzy number. For instance, if $\mathcal{X}$ is a non-degenerate random fuzzy number taking on the triangular fuzzy values Tra$(0, 1, 1, 2)$, Tra$(1, 2, 2, 3)$ and Tra$(2, 3, 3, 4)$ with probabilities $1/5$, $3/5$ and $1/5$, respectively, then $\widetilde{E}(\mathcal{X}) = \widetilde{\mathrm{Me}}(\mathcal{X}) = \widetilde{\mathrm{M}^\varphi}(\mathcal{X}) = \mathrm{Tra}(1, 2, 2, 3)$, whence ρ_2-MDD$(\mathcal{X}, \widetilde{E}(\mathcal{X})) = \rho_1$-MDD$(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})) = \mathscr{D}_\theta^\varphi$-MDD$(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})) = \mathrm{Me}\{1, 0, 0, 0, 1\} = 0$, and can straightforwardly be checked that measures D-S$(\mathcal{X}, \mathcal{Y})$, $\widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)$ equal zero. Consequently, the vanishing of these scale estimators does not necessarily ensure the lack of variability. This is due to the fact that these estimates should be viewed as representative summary measures of the dispersion of sample fuzzy data, instead of measures of variability.

Other valuable properties have also been inherited from their real-valued counterparts, namely,

Proposition 2.2.5. ***(Absolute equivariance by the product by scalar and invariance by translation).*** *The scale measures introduced in Definitions 2.1.1-2.1.6 satisfy the shift (or location or translation) invariance and scale (absolute) equivariance conditions. That is, if $\gamma \in \mathbb{R}$, $\widetilde{U} \in \mathscr{F}_c^*(\mathbb{R})$, $\mathcal{X}$ is an RFN for which the involved scale measures exist, $\mathcal{Y}$ is an RFN which is identically distributed as and independent of $\mathcal{X}$, and $\widetilde{\mathbf{x}}_n$ is a sample of observations from $\mathcal{X}$, then,*

$$D_\theta^\varphi\text{-SD}(\gamma \cdot \mathcal{X} + \widetilde{U}) = |\gamma| \cdot D_\theta^\varphi\text{-SD}(\mathcal{X}),$$

$$\rho_2\text{-ADD}(\gamma \cdot \mathcal{X} + \widetilde{U}, \widetilde{E}(\gamma \cdot \mathcal{X} + \widetilde{U})) = |\gamma| \cdot \rho_2\text{-ADD}(\mathcal{X}, \widetilde{E}(\mathcal{X})),$$

$$\rho_1\text{-ADD}(\gamma \cdot \mathcal{X} + \widetilde{U}, \widetilde{\mathrm{Me}}(\gamma \cdot \mathcal{X} + \widetilde{U})) = |\gamma| \cdot \rho_1\text{-ADD}(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})),$$

$$\mathscr{D}_\theta^\varphi\text{-ADD}(\gamma \cdot \mathcal{X} + \widetilde{U}, \widetilde{\mathrm{M}^\varphi}(\gamma \cdot \mathcal{X} + \widetilde{U})) = |\gamma| \cdot \mathscr{D}_\theta^\varphi\text{-ADD}(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})),$$

$$\rho_2\text{-MDD}(\gamma \cdot \mathcal{X} + \widetilde{U}, \widetilde{E}(\gamma \cdot \mathcal{X} + \widetilde{U})) = |\gamma| \cdot \rho_2\text{-MDD}(\mathcal{X}, \widetilde{E}(\mathcal{X})),$$

$$\rho_1\text{-MDD}(\gamma \cdot \mathcal{X} + \widetilde{U}, \widetilde{\mathrm{Me}}(\gamma \cdot \mathcal{X} + \widetilde{U})) = |\gamma| \cdot \rho_1\text{-MDD}(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})),$$

$$\mathscr{D}_\theta^\varphi\text{-MDD}(\gamma \cdot \mathcal{X} + \widetilde{U}, \widetilde{\mathrm{M}^\varphi}(\gamma \cdot \mathcal{X} + \widetilde{U})) = |\gamma| \cdot \mathscr{D}_\theta^\varphi\text{-MDD}(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X})),$$

$$D\text{-S}(\gamma \cdot \mathcal{X} + \widetilde{U}, \gamma \cdot \mathcal{Y} + \widetilde{U}) = |\gamma| \cdot D\text{-S}(\mathcal{X}, \mathcal{Y}),$$

$$\widehat{D\text{-Q}}(\gamma \cdot \widetilde{\mathbf{x}}_n + \widetilde{U}) = |\gamma| \cdot \widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n),$$

$$\widehat{D\text{-T}}(\gamma \cdot \widetilde{\mathbf{x}}_n + \widetilde{U}) = |\gamma| \cdot \widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n).$$

Proof. Indeed, whatever the metric $D \in \{\rho_1, \mathscr{D}_\theta^\varphi, D_\theta^\varphi\}$ and the location measure $\widetilde{C}(\mathcal{X}) \in \{\widetilde{E}(\mathcal{X}), \widetilde{\mathrm{Me}}(\mathcal{X}), \widetilde{\mathrm{M}^\varphi}(\mathcal{X})\}$ may be, because of the properties of the metrics and the property of affine equivariance fulfilled by the Aumann-type mean (see Proposition 1.4.1) and the 1-norm and φ-wabl/ldev/rdev medians (see Proposition 1.5.3) we have that

$$D\Big(\gamma \cdot \mathcal{X} + \widetilde{U}, \widetilde{C}(\gamma \cdot \mathcal{X} + \widetilde{U})\Big) = D\Big(\gamma \cdot \mathcal{X} + \widetilde{U}, \gamma \cdot \widetilde{C}(\mathcal{X}) + \widetilde{U}\Big) = |\gamma| \cdot D\Big(\mathcal{X}, \widetilde{C}(\mathcal{X})\Big),$$

and the proof is immediately concluded for the location-based measures D_θ^φ-SD$(\mathcal{X})$, ρ_2-ADD$(\mathcal{X}, \widetilde{E}(\mathcal{X}))$, ρ_1-ADD$(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X}))$, $\mathscr{D}_\theta^\varphi$-ADD$(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$, ρ_2-MDD$(\mathcal{X}, \widetilde{E}(\mathcal{X}))$, ρ_1-MDD$(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X}))$ and $\mathscr{D}_\theta^\varphi$-MDD$(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$.

Regarding the location-free estimators, we have that

$$D\text{-S}(\gamma \cdot \mathcal{X} + \widetilde{U}, \gamma \cdot \mathcal{Y} + \widetilde{U}) = \mathrm{Me}_{\gamma \cdot \mathcal{X} + \widetilde{U}} \mathrm{Me}_{\gamma \cdot \mathcal{Y} + \widetilde{U}} D(\gamma \cdot \mathcal{X} + \widetilde{U}, \gamma \cdot \mathcal{Y} + \widetilde{U})$$

$$= \mathrm{Me}_{\mathcal{X}} \mathrm{Me}_{\mathcal{Y}}\, |\gamma| \cdot D(\mathcal{X}, \mathcal{Y}) = |\gamma| \cdot D\text{-S}(\mathcal{X}, \mathcal{Y}),$$

$$\widehat{D\text{-Q}}(\gamma \cdot \widetilde{\mathbf{x}}_n + \widetilde{U}) = \Big\{D\left(\gamma \cdot \widetilde{x}_i + \widetilde{U}, \gamma \cdot \widetilde{x}_j + \widetilde{U}\right) : i < j\Big\}_{(m)}$$

$$= \{|\gamma| \cdot D(\widetilde{x}_i, \widetilde{x}_j) : i < j\}_{(m)} = |\gamma| \cdot \{D(\widetilde{x}_i, \widetilde{x}_j) : i < j\}_{(m)} = |\gamma| \cdot \widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n),$$

$$\widehat{D\text{-T}}(\gamma \cdot \widetilde{\mathbf{x}}_n + \widetilde{U}) = \frac{1}{h}\sum_{r=1}^{h}\left\{\overline{\mathrm{Me}}_j\left\{D\left(\gamma \cdot \widetilde{x}_i + \widetilde{U}, \gamma \cdot \widetilde{x}_j + \widetilde{U}\right)\right\} \; ; \; i = 1, \ldots, n\right\}_{(r)}$$

$$= \frac{1}{h}\sum_{r=1}^{h}\left\{\overline{\mathrm{Me}}_j\left\{|\gamma| \cdot D(\widetilde{x}_i, \widetilde{x}_j)\right\} \; ; \; i = 1, \ldots, n\right\}_{(r)}$$

$$= |\gamma| \cdot \frac{1}{h}\sum_{r=1}^{h}\left\{\overline{\mathrm{Me}}_j\left\{D(\widetilde{x}_i, \widetilde{x}_j)\right\} \; ; \; i = 1, \ldots, n\right\}_{(r)} = |\gamma| \cdot \widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n). \qquad \square$$

Remark 2.2.2. It should be pointed out that while the shift invariance would hold in case of involving as location measures the M-estimates by Sinova *et al.* [113], the scale (absolute) equivariance would not necessarily hold. This is due to the fact that, generally speaking, M-estimates of location are not scale equivariant.

Proposition 2.2.6. ***(Strong consistency).*** *Let $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ be a simple random sample from the RFN $\mathcal{X}$. Let $\overline{\mathcal{X}}_n$ denote the sample mean estimator.*

i) *If $\widetilde{E}(\mathcal{X})$ and D_θ^φ-SD$(\mathcal{X})$ exist, then the statistic $\widehat{D_\theta^\varphi\text{-SD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ is a strongly consistent estimator of D_θ^φ-SD$(\mathcal{X})$, that is,*

$$\lim_{n\to\infty} \widehat{D_\theta^\varphi\text{-SD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n) = D_\theta^\varphi\text{-SD}(\mathcal{X}) \quad a.s.\,[P].$$

ii) *If $\widetilde{E}(\mathcal{X})$ and ρ_2-ADD$(\mathcal{X}, \widetilde{E}(\mathcal{X}))$ exist, then the statistic $\widehat{\rho_2\text{-ADD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \overline{\mathcal{X}}_n)$ is a strongly consistent estimator of ρ_2-ADD$(\mathcal{X}, \widetilde{E}(\mathcal{X}))$, that is,*

$$\lim_{n\to\infty} \widehat{\rho_2\text{-ADD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \overline{\mathcal{X}}_n) = \rho_2\text{-ADD}(\mathcal{X}, \widetilde{E}(\mathcal{X})) \quad a.s.\,[P].$$

iii) *If for each $\alpha \in [0,1]$ the medians of the real-valued random variables $\inf \mathcal{X}_\alpha$ and $\sup \mathcal{X}_\alpha$ are unique (without making use of any convention), and the sequences of the real-valued sample medians $\left\{\mathrm{Me}\{\inf(\mathcal{X}_1)_\alpha, \ldots, \inf(\mathcal{X}_n)_\alpha\}\right\}_n$ and $\left\{\mathrm{Me}\{\sup(\mathcal{X}_1)_\alpha, \ldots, \sup(\mathcal{X}_n)_\alpha\}\right\}_n$ as functions of α over $[0,1]$ are both uniformly integrable, then the statistic $\widehat{\rho_1\text{-ADD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \widehat{\widetilde{\mathrm{Me}}}(\mathcal{X}_1, \ldots, \mathcal{X}_n))$ is a strongly consistent estimator of ρ_1-ADD$(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X}))$, if it exists, that is,*

$$\lim_{n\to\infty} \widehat{\rho_1\text{-ADD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \widehat{\widetilde{\mathrm{Me}}}(\mathcal{X}_1, \ldots, \mathcal{X}_n)) = \rho_1\text{-ADD}(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X})) \quad a.s.\,[P].$$

iv) *If the population median of the real-valued random variable $\mathrm{wabl}^\varphi(\mathcal{X})$ is unique (without making use of any convention), and also for each $\alpha \in [0,1]$ the population medians of the real-valued random variables $\mathrm{ldev}^\varphi_{\mathcal{X}}(\alpha)$ and $\mathrm{rdev}^\varphi_{\mathcal{X}}(\alpha)$ are* *actually unique, and the two sequences of sample medians* $\left\{\mathrm{Me}\{\mathrm{ldev}^{\varphi}_{\mathcal{X}_1}(\alpha),\ldots,\mathrm{ldev}^{\varphi}_{\mathcal{X}_n}(\alpha)\}\right\}_n$ *and* $\left\{\mathrm{Me}\{\mathrm{rdev}^{\varphi}_{\mathcal{X}_1}(\alpha),\ldots,\mathrm{rdev}^{\varphi}_{\mathcal{X}_n}(\alpha)\}\right\}_n$ *as functions of* α *over* $[0,1]$ *are both uniformly integrable, then the statistic* $\widehat{\mathscr{D}^{\varphi}_{\theta}\text{-ADD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\widehat{\widetilde{\mathrm{M}^{\varphi}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n))$ *is a strongly consistent estimator of* $\mathscr{D}^{\varphi}_{\theta}\text{-ADD}(\mathcal{X},\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X}))$*, if it exists, that is,*

$$\lim_{n\to\infty}\widehat{\mathscr{D}^{\varphi}_{\theta}\text{-ADD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\widehat{\widetilde{\mathrm{M}^{\varphi}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n))=\mathscr{D}^{\varphi}_{\theta}\text{-ADD}(\mathcal{X},\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X}))\quad a.s.\,[P].$$

v) If $\widetilde{E}(\mathcal{X})$ *and* $\rho_2\text{-MDD}(\mathcal{X},\widetilde{E}(\mathcal{X}))$ *exist and the second one is unique (without making use of any convention), then the statistic* $\widehat{\rho_2\text{-MDD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\overline{\mathcal{X}}_n)$ *is a strongly consistent estimator of* $\rho_2\text{-MDD}(\mathcal{X},\widetilde{E}(\mathcal{X}))$*, that is,*

$$\lim_{n\to\infty}\widehat{\rho_2\text{-MDD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\overline{\mathcal{X}}_n)=\rho_2\text{-MDD}(\mathcal{X},\widetilde{E}(\mathcal{X}))\quad a.s.\,[P].$$

vi) If for each $\alpha\in[0,1]$ *the medians of the real-valued random variables* $\inf\mathcal{X}_\alpha$ *and* $\sup\mathcal{X}_\alpha$ *are unique (without making use of any convention),* $\rho_1\text{-MDD}(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X}))$ *exists and it is unique (without making use of any convention), and the sequences of the real-valued sample medians* $\left\{\mathrm{Me}\{\inf(\mathcal{X}_1)_\alpha,\ldots,\inf(\mathcal{X}_n)_\alpha\}\right\}_n$ *and* $\left\{\mathrm{Me}\{\sup(\mathcal{X}_1)_\alpha,\ldots,\sup(\mathcal{X}_n)_\alpha\}\right\}_n$ *as functions of* α *over* $[0,1]$ *are both uniformly integrable, then the statistic* $\widehat{\rho_1\text{-MDD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\widehat{\widetilde{\mathrm{Me}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n))$ *is a strongly consistent estimator of* $\rho_1\text{-MDD}(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X}))$*, that is,*

$$\lim_{n\to\infty}\widehat{\rho_1\text{-MDD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\widehat{\widetilde{\mathrm{Me}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n))=\rho_1\text{-MDD}(\mathcal{X},\widetilde{\mathrm{Me}}(\mathcal{X}))\quad a.s.\,[P].$$

vii) If the population median of the real-valued random variable $\mathrm{wabl}^{\varphi}(\mathcal{X})$ *is unique (without using any convention), and also for each* $\alpha\in[0,1]$ *the population medians of the real-valued random variables* $\mathrm{ldev}^{\varphi}_{\mathcal{X}}(\alpha)$ *and* $\mathrm{rdev}^{\varphi}_{\mathcal{X}}(\alpha)$ *are actually unique,* $\mathscr{D}^{\varphi}_{\theta}\text{-MDD}(\mathcal{X},\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X}))$ *exists and it is unique (without making use of any convention), and the two sequences of sample medians* $\left\{\mathrm{Me}\{\mathrm{ldev}^{\varphi}_{\mathcal{X}_1}(\alpha),\ldots,\mathrm{ldev}^{\varphi}_{\mathcal{X}_n}(\alpha)\}\right\}_n$ *and* $\left\{\mathrm{Me}\{\mathrm{rdev}^{\varphi}_{\mathcal{X}_1}(\alpha),\ldots,\mathrm{rdev}^{\varphi}_{\mathcal{X}_n}(\alpha)\}\right\}_n$ *are both uniformly integrable when viewed as functions of* α *over* $[0,1]$*, then the statistic* $\widehat{\mathscr{D}^{\varphi}_{\theta}\text{-MDD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\widehat{\widetilde{\mathrm{M}^{\varphi}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n))$ *is a strongly consistent estimator of* $\mathscr{D}^{\varphi}_{\theta}\text{-MDD}(\mathcal{X},\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X}))$*, that is,*

$$\lim_{n\to\infty}\widehat{\mathscr{D}^{\varphi}_{\theta}\text{-MDD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n,\widehat{\widetilde{\mathrm{M}^{\varphi}}}(\mathcal{X}_1,\ldots,\mathcal{X}_n))=\mathscr{D}^{\varphi}_{\theta}\text{-MDD}(\mathcal{X},\widetilde{\mathrm{M}^{\varphi}}(\mathcal{X}))\quad a.s.\,[P].$$

viii) Let $\mathcal{X}$ *and* $\mathcal{Y}$ *independent and identically distributed random fuzzy numbers. If for any value of* $\mathcal{X}$ *the median* $\mathrm{Me}_{\mathcal{Y}}(D(\mathcal{X},\mathcal{Y}))$ *and the median* $\mathrm{Me}_{\mathcal{X}}$ *of the real-valued random variable* $\mathrm{Me}_{\mathcal{Y}}(D(\mathcal{X},\mathcal{Y}))$ *exist and are actually unique (without* *making use of any convention), then the statistic* $\widehat{D\text{-S}}(\mathcal{X}_1,\ldots,\mathcal{X}_n)$ *is a strongly consistent estimator of* D-S$(\mathcal{X},\mathcal{Y})$*, that is,*

$$\lim_{n\to\infty}\widehat{D\text{-S}}(\mathcal{X}_1,\ldots,\mathcal{X}_n)=D\text{-S}(\mathcal{X},\mathcal{Y})\quad a.s.\,[P].$$

Proof.

i) On the basis of the triangle inequality for D^{φ}_{θ}, we have that for any $i\in\{1,\ldots,n\}$

$$D^{\varphi}_{\theta}(\mathcal{X}_i,\overline{\mathcal{X}}_n)\le D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))+D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n),$$
$$D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))\le D^{\varphi}_{\theta}(\mathcal{X}_i,\overline{\mathcal{X}}_n)+D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)$$

and

$$D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)\le D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))+D^{\varphi}_{\theta}(\mathcal{X}_i,\overline{\mathcal{X}}_n).$$

Therefore,

$$\begin{aligned}\left|D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))-D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)\right| &\le D^{\varphi}_{\theta}(\mathcal{X}_i,\overline{\mathcal{X}}_n)\\ &\le D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))+D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)\end{aligned}$$

and, hence, by taking squares on the three members of the inequality and averaging later over i we have that

$$\begin{aligned}&\frac{1}{n}\sum_{i=1}^{n}\left[D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))\right]^2+\left[D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)\right]^2-D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)\frac{2}{n}\sum_{i=1}^{n}D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))\\ &\le\left[\widehat{D^{\varphi}_{\theta}\text{-SD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n)\right]^2\le\frac{1}{n}\sum_{i=1}^{n}\left[D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))\right]^2+\left[D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)\right]^2\\ &\quad+D^{\varphi}_{\theta}(\widetilde{E}(\mathcal{X}),\overline{\mathcal{X}}_n)\frac{2}{n}\sum_{i=1}^{n}D^{\varphi}_{\theta}(\mathcal{X}_i,\widetilde{E}(\mathcal{X}))\end{aligned}$$

whence, by applying the Strong Law of Large Numbers for the real-valued random variables $D^{\varphi}_{\theta}(\mathcal{X},\widetilde{E}(\mathcal{X}))$ and $\left[D^{\varphi}_{\theta}(\mathcal{X},\widetilde{E}(\mathcal{X}))\right]^2$ and the strong consistency of $\overline{\mathcal{X}}_n$ in D^{φ}_{θ}-sense (as a consequence from Colubi *et al.* [21], in which an SLLN for random fuzzy sets has been obtained for the stronger metric of the supremum between fuzzy sets), we have that

$$\lim_{n\to\infty}\left[\widehat{D^{\varphi}_{\theta}\text{-SD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n)\right]^2=[D^{\varphi}_{\theta}\text{-SD}(\mathcal{X})]^2\quad a.s.\,[P],$$

and therefore

$$\lim_{n\to\infty}\widehat{D^{\varphi}_{\theta}\text{-SD}}(\mathcal{X}_1,\ldots,\mathcal{X}_n)=D^{\varphi}_{\theta}\text{-SD}(\mathcal{X})\quad a.s.\,[P].$$

ii) On the basis of the triangle inequality for ρ_2, we have that for any $i \in \{1, \ldots, n\}$

$$\rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) \leq \rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) + \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n)$$

and

$$\rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) \leq \rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) + \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n).$$

Therefore,

$$\rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) - \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n) \leq \rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) \leq \rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) + \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n),$$

$$-\rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n) \leq \rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) - \rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) \leq \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n)$$

and

$$\left|\rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) - \rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X}))\right| \leq \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n)$$

and, hence, by averaging over i we have that

$$\left|\widehat{\rho_2\text{-ADD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \overline{\mathcal{X}}_n) - \frac{1}{n}\sum_{i=1}^{n} \rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X}))\right| \leq \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n),$$

whence by applying the Strong Law of Large Numbers for the real-valued random variable $\rho_2(\mathcal{X}, \widetilde{E}(\mathcal{X}))$ (that can trivially be drawn from Klement *et al.* [68] and Diamond and Kloeden [30]) and the strong consistency of $\overline{\mathcal{X}}_n$ in ρ_2-sense we have that

$$\left|\lim_{n\to\infty} \widehat{\rho_2\text{-ADD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \overline{\mathcal{X}}_n) - \rho_2\text{-ADD}(\mathcal{X}, \widetilde{E}(\mathcal{X}))\right| = 0 \quad a.s.\,[P],$$

what implies that

$$\lim_{n\to\infty} \widehat{\rho_2\text{-ADD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \overline{\mathcal{X}}_n) = \rho_2\text{-ADD}(\mathcal{X}, \widetilde{E}(\mathcal{X})) \quad a.s.\,[P].$$

iii) The proof is analogous to that for *ii*) by applying the triangle inequality for the metric ρ_1, the Strong Law of Large Numbers for the real-valued random variable $\rho_1(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X}))$ and the strong consistency of the fuzzy-valued sample 1-norm median in ρ_1-sense (see Sinova *et al.* [111]).

iv) The proof is analogous to that for *ii*) by applying the triangle inequality for the metric $\mathscr{D}_\theta^\varphi$, the Strong Law of Large Numbers for the real-valued random variable $\mathscr{D}_\theta^\varphi(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$ and the strong consistency of the fuzzy-valued sample φ-wabl/ldev/rdev median in $\mathscr{D}_\theta^\varphi$-sense (see Sinova *et al.* [109]).

v) On the basis of the triangle inequality for ρ_2, we have that for any $i \in \{1, \ldots, n\}$

$$\rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) \leq \rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) + \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n)$$

and

$$\rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) \leq \rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) + \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n).$$

Therefore,

$$\rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) - \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n) \leq \rho_2(\mathcal{X}_i, \overline{\mathcal{X}}_n) \leq \rho_2(\mathcal{X}_i, \widetilde{E}(\mathcal{X})) + \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n)$$

and hence,

$$\begin{aligned} &\mathrm{Me}\Big\{\rho_2(\mathcal{X}_1, \widetilde{E}(\mathcal{X})), \ldots, \rho_2(\mathcal{X}_n, \widetilde{E}(\mathcal{X}))\Big\} - \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n) \\ &\leq \widehat{\rho_2\text{-MDD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \overline{\mathcal{X}}_n) \\ &\leq \mathrm{Me}\Big\{\rho_2(\mathcal{X}_1, \widetilde{E}(\mathcal{X})), \ldots, \rho_2(\mathcal{X}_n, \widetilde{E}(\mathcal{X}))\Big\} + \rho_2(\widetilde{E}(\mathcal{X}), \overline{\mathcal{X}}_n) \end{aligned}$$

whence, by applying the strong consistency of $\overline{\mathcal{X}}_n$ in ρ_2-sense and the strong consistency of the sample median from the real-valued random variable $\rho_2(\mathcal{X}, \widetilde{E}(\mathcal{X}))$, we have that

$$\lim_{n\to\infty} \widehat{\rho_2\text{-MDD}}(\mathcal{X}_1, \ldots, \mathcal{X}_n, \overline{\mathcal{X}}_n) = \rho_2\text{-MDD}(\mathcal{X}, \widetilde{E}(\mathcal{X})) \quad a.s.\,[P].$$

vi) The proof is analogous to that for *v*) by applying the triangle inequality for the metric ρ_1, the strong consistency of the fuzzy-valued sample 1-norm median in ρ_1-sense (see Sinova *et al.* [111]) and the strong consistency under the assumed conditions of the sample median from the real-valued random variable $\rho_1(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X}))$.

vii) The proof is analogous to that for *v*) by applying the triangle inequality for the metric $\mathscr{D}_\theta^\varphi$, the strong consistency of the fuzzy-valued sample φ-wabl/ldev/rdev median in $\mathscr{D}_\theta^\varphi$-sense (see Sinova *et al.* [109]) and the strong consistency under the assumed conditions of the sample median from the real-valued random variable $\mathscr{D}_\theta^\varphi(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$.

viii) For fixed $i \in \{1, \ldots, n\}$ and by applying the strong consistency of the sample median of the real-valued random variable $D(\mathcal{X}_i, \mathcal{X}_j)$ w.r.t $\mathcal{X}_j$, we have that

$$\lim_{n\to\infty} \overline{\mathrm{Me}}_j \left\{D(\mathcal{X}_i, \mathcal{X}_j)\right\} = \mathrm{Me}(D(\mathcal{X}_i, \mathcal{Y})) \quad a.s.\,[P].$$

Now, by applying the strong consistency of the sample median w.r.t. $\mathcal{X}_i$ for the real-valued random variable $\mathrm{Me}(D(\mathcal{X}_i, \mathcal{Y}))$, we have that

$$\lim_{n\to\infty} \underline{\mathrm{Me}}_i \left\{\mathrm{Me}(D(\mathcal{X}_i, \mathcal{Y}))\right\} = \mathrm{Me}_{\mathcal{X}}\mathrm{Me}_{\mathcal{Y}}(D(\mathcal{X}, \mathcal{Y})) \quad a.s.\,[P]. \qquad \square$$

Remark 2.2.3. It should be pointed out that the strong consistency in Proposition 2.2.6 has been understood in the sense of the convergence of the sample scale measure to the population one, instead of the convergence (after maybe a correction) of the sample measure to the population standard deviation as often made in case of dealing with real-valued random variables. Actually, the last convergence is usually discussed in the real-valued case by considering some outstanding models for the distribution of the involved random variables; this type of models has not been yet realistically stated for RFN's.

2.3 Formal analysis of the robustness of the scale measures: the finite sample breakdown point

Consider the sample of trapezoidal fuzzy numbers collected in Table 2.1 and displayed in Figure 2.1. The last datum, the trapezoidal $\mathrm{Tra}(39.70, 39.73, 60.13, 73.05)$ (in red) clearly stands out from the rest of the values and it can be considered as an outlier. Let's see now Table 2.2, where we can find the values of some of the scales estimates defined in Section 2.1 for the whole dataset (with the outlier) and also for the dataset in which the suspicious outlier has been removed. We can observe the lack of robustness of the estimates $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$. The sample standard deviation is the most influenced estimator by the outlier. The $\widehat{\rho_2\text{-ADD}}$ with respect to the mean is also very sensitive to the presence of this atypical value, although to a lesser extent than the standard deviation. In contrast, good robust alternatives are the estimates $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$, since their values slightly change when the outlier is included in the dataset.

A popular and powerful tool allowing us to describe the robustness of an estimate is its breakdown point. Donoho and Huber [34] pointed out that "the notion of breakdown point was coined, formally defined, and very briefly discussed by Frank Hampel, at that time a student of Erich Lehman, in his PhD in 1968" [55]. Although it was originally presented for location estimates, the concept has also been generalized to scale estimates.

Table 2.1: Sample of 22 trapezoidal fuzzy numbers with one outlier (in red), and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$

a_i	b_i	c_i	d_i	a_i	b_i	c_i	d_i
-2.51	-1.57	-1.35	-1.12	-0.41	0.18	3.92	4.41
-4.26	-4.00	-0.49	-0.34	-1.24	-0.59	-0.46	-0.32
-3.80	-0.58	2.59	2.60	0.16	0.21	0.21	0.48
-4.28	-0.82	-0.77	0.24	-1.31	-1.02	0.35	2.17
-0.56	-0.56	3.66	4.57	-5.18	-1.56	3.27	3.64
-1.44	0.42	0.45	0.77	-2.12	-1.54	1.19	1.26
-0.46	0.48	0.83	1.16	1.28	1.43	1.96	2.79
0.99	1.15	1.39	1.78	-2.08	0.18	3.69	5.01
-0.17	-0.16	-0.16	-0.14	-1.44	-0.86	-0.28	-0.28
-1.90	-1.06	-0.72	-0.65	-3.78	-0.25	0.81	0.85
-1.33	-0.18	0.07	3.37	39.70	39.73	60.13	73.05

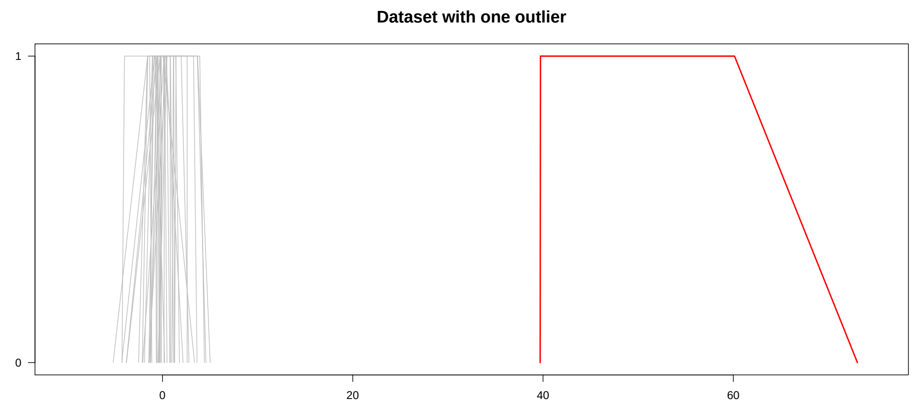


Figure 2.1: Sample of 22 trapezoidal fuzzy numbers with one outlier (in red)

Table 2.2: The effect of one outlier in some dispersion estimates

Scale measure	For dataset WITHOUT outlier	For dataset WITH outlier
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.53	11.46
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	1.40	4.95
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.21	3.57
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.74	4.63
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	1.30	2.79
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.05	1.20
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.56	1.64
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.41	1.48
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.13	1.21
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.24	1.35

A simple and intuitive notion of the breakdown point of a scale estimate constrained to finite samples, the so-called **finite sample breakdown point** (fsbp for short), was introduced by Donoho [33] and Donoho and Huber [34]. For scale estimates, it is defined as the minimum proportion of sample data which should be perturbed in order to let the estimate acquire either an arbitrary large value or the value zero. The higher the breakdown point of an estimate, the more robust it is.

Therefore, two situations need to be taken into account in scale estimation studies: the one consisting of contaminating the sample by means of outliers, which can make the estimate overestimate the true scale up to infinity (*explosion*), and the one consisting of contaminating the sample by means of inliers, which may result in underestimation of the true scale to zero (*implosion*). Notice that when dealing with location estimates, only the explosion case makes sense, this being defined as the minimum proportion of sample data that should be perturbed to get an arbitrarily large or small estimator value.

Next, the replacement version of the finite sample breakdown point for scale estimates (see Donoho and Huber [34]) is recalled, but applied in this case to fuzzy data.

Definition 2.3.1. *For any sample of observations* $\widetilde{\mathbf{x}}_n$ *from an RFN* $\mathcal{X}$*, the* ***finite sample breakdown point of a scale estimate*** $\widehat{\mathrm{D}}$ *is defined by*

$$\mathrm{fsbp}^*(\widehat{\mathrm{D}}(\widetilde{\mathbf{x}}_n)) = \min\left\{\mathrm{fsbp}^+(\widehat{\mathrm{D}}(\widetilde{\mathbf{x}}_n)),\, \mathrm{fsbp}^-(\widehat{\mathrm{D}}(\widetilde{\mathbf{x}}_n))\right\}$$

where

$$\mathrm{fsbp}^+(\widehat{\mathrm{D}}(\widetilde{\mathbf{x}}_n)) = \min\left\{\frac{k}{n}; \sup_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\mathrm{D}}(\widetilde{\mathbf{y}}_{n,k}) = \infty\right\}$$

and

$$\mathrm{fsbp}^-(\widehat{\mathrm{D}}(\widetilde{\mathbf{x}}_n)) = \min\left\{\frac{k}{n}; \inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\mathrm{D}}(\widetilde{\mathbf{y}}_{n,k}) = 0\right\}$$

with $\widetilde{\mathbf{y}}_{n,k}$ *obtained by replacing any* k *observations of* $\widetilde{\mathbf{x}}_n$ *by arbitrary values. The quantities* fsbp^+ *and* fsbp^- *are called the* ***explosion breakdown point*** *and the* ***implosion breakdown point****.*

The following theorems formalize the comparison of the robustness of different scale estimators through the fsbp. They prove that if the considered sample of fuzzy observations $\widetilde{\mathbf{x}}_n$ does not contain two coinciding observations, then their fsbp equals

- $\frac{1}{n}$ for the estimates $\widehat{D^{\varphi}_{\theta}\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}^{\varphi}_{\theta}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\varphi}}}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, which is the lowest possible fsbp, and
- $\frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor$ for the estimates $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}^{\varphi}_{\theta}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\varphi}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)$, which is the highest possible fsbp for a scale estimate.

Therefore, these estimators inherit the value of the fsbp from the real-valued case (see, for instance, Rousseeuw and Croux [100]).

Theorem 2.3.1. *For any sample of observations* $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ *from an RFN* $\mathcal{X}$ *in which there are not two identical observations, we have that*

$$\mathrm{fsbp}^+(\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n}, \quad \mathrm{fsbp}^-(\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n)) = \frac{n-1}{n}.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n)$ *is given by*

$$\mathrm{fsbp}^*(\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n},$$

which is the lowest possible fsbp^* *of an estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample in which there are not two identical observations, and denote $\mathrm{fsbp}^+ = \mathrm{fsbp}^+(\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n))$ and $\mathrm{fsbp}^- = \mathrm{fsbp}^-(\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{x}}_n))$. The proof of the two first conclusions is to be split in three steps.

Step 1: We begin showing that $\mathrm{fsbp}^- \leq (n-1)/n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,n-1} = \{\widetilde{y}_1, \ldots, \widetilde{y}_n\}$ with $n-1$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$ such that $\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{y}}_{n,n-1}) = 0$.

The sample $\widetilde{\mathbf{y}}_{n,n-1}$ is constructed by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_n$ by $\widetilde{x}_1$. The considered sample $\widetilde{\mathbf{y}}_{n,n-1}$ has $k = n-1$ replaced observations. Since $\overline{\widetilde{\mathbf{y}}}_{n,n-1} = \widetilde{x}_1$, we have that for all $i \in \{1, \ldots, n\}$, $D_\theta^\varphi\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,n-1}\right) = D_\theta^\varphi(\widetilde{x}_1, \widetilde{x}_1) = 0$, whence $\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{y}}_{n,n-1}) = \sqrt{\sum_{i=1}^n \left[D_\theta^\varphi\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,n-1}\right)\right]^2 / n} = 0$.

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,n-1}} \widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{z}}_{n,n-1}) = 0$ with $\widetilde{\mathbf{z}}_{n,n-1}$ any sample with $n-1$ replaced observations of $\widetilde{\mathbf{x}}_n$, whence $\mathrm{fsbp}^- \leq (n-1)/n$.

Step 2: Now we show that $\mathrm{fsbp}^- \geq (n-1)/n$.

Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < n-1$ replaced observations of $\widetilde{\mathbf{x}}_n$. There exist at least two observations $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \widetilde{\mathbf{y}}_{n,k}$. Let $\delta := \sqrt{\min_{i,j \in \{1,\ldots,n\}} [D_\theta^\varphi(\widetilde{x}_i, \widetilde{x}_j)]^2 / n^2} > 0$. We have that

$$\delta = \sqrt{\frac{\min_{i,j\in\{1,\ldots,n\}} D_\theta^\varphi[(\widetilde{x}_i, \widetilde{x}_j)]^2}{n^2}} \leq \sqrt{\left[\frac{1}{n} D_\theta^\varphi(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}})\right]^2}$$

$$\leq \sqrt{\left[\frac{D_\theta^\varphi(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \overline{\widetilde{\mathbf{y}}}_{n,k}) + D_\theta^\varphi(\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}}, \overline{\widetilde{\mathbf{y}}}_{n,k})}{n}\right]^2} \leq \sqrt{\left[\frac{1}{n}\sum_{i=1}^n D_\theta^\varphi(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,k})\right]^2}$$

$$\leq \sqrt{\frac{1}{n}\sum_{i=1}^n \left[D_\theta^\varphi(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,k})\right]^2} = \widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{y}}_{n,k}).$$

Therefore, $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{y}}_{n,k}) \geq \delta > 0$ with $k < n-1$, and hence $\mathrm{fsbp}^- \geq (n-1)/n$.

Step 3: Finally, we will prove that $\text{fsbp}^+ = 1/n$. For this purpose, we construct the sample $\widetilde{\mathbf{y}}_{n,1}$ by replacing the observation $\widetilde{x}_1$ by $\widetilde{x}_{(n)} + L$, with $L \in \mathbb{R}$, $L > 0$ and $\widetilde{x}_{(n)}$ the extended maximum $\widetilde{\max}_{i\in\{1,\ldots,n\}}\widetilde{x}_i$ (see, for instance, [35]) defined so that $\left(\widetilde{\max}_{i\in\{1,\ldots,n\}}\widetilde{x}_i\right)_\alpha = [\max_{i\in\{1,\ldots,n\}} \inf(\widetilde{x}_i)_\alpha, \max_{i\in\{1,\ldots,n\}} \sup(\widetilde{x}_i)_\alpha]$ for each $\alpha \in [0,1]$.

The considered sample $\widetilde{\mathbf{y}}_{n,1}$ has $k = 1$ replaced observation. If $\widetilde{\mathbf{x}'}_n = \{\widetilde{x}_{(n)}, \widetilde{x}_2, \ldots, \widetilde{x}_n\}$, by taking into account that $\text{mid}(\widetilde{x}_{(n)})_\alpha \geqslant \text{mid}\left(\overline{\widetilde{\mathbf{x}'}}_n\right)_\alpha$ for all $\alpha \in [0,1]$, and $L \geqslant L/n$, it is satisfied that

$$\left[D_\theta^\varphi\left(\widetilde{y}_1, \overline{\widetilde{\mathbf{y}}}_{n,1}\right)\right]^2 = \left[D_\theta^\varphi\left(\widetilde{x}_{(n)} + L, \overline{\widetilde{\mathbf{y}}}_{n,1}\right)\right]^2 = \left[D_\theta^\varphi\left(\widetilde{x}_{(n)} + L, \overline{\widetilde{\mathbf{x}'}}_n + \frac{L}{n}\right)\right]^2$$

$$\geq \left[D_\theta^\varphi\left(\widetilde{x}_{(n)}, \overline{\widetilde{\mathbf{x}'}}_n\right)\right]^2 + \left[\frac{(n-1)L}{n}\right]^2,$$

whence

$$D_\theta^\varphi\left(\widetilde{y}_1, \overline{\widetilde{\mathbf{y}}}_{n,1}\right) \geq \frac{(n-1)L}{n}.$$

Thus, $\widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{y}}_{n,1}) = \sqrt{\sum_{i=1}^n \left[D_\theta^\varphi(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,1})\right]^2/n} \geq \sqrt{\left[D_\theta^\varphi(\widetilde{y}_1, \overline{\widetilde{\mathbf{y}}}_{n,1})\right]^2/n} = D_\theta^\varphi(\widetilde{y}_1, \overline{\widetilde{\mathbf{y}}}_{n,1})/\sqrt{n} \geq (n-1)L/(n\sqrt{n})$.

Letting $L \to \infty$, $\sup_{\widetilde{\mathbf{z}}_{n,1}} \widehat{D_\theta^\varphi\text{-SD}}(\widetilde{\mathbf{z}}_{n,1}) = \infty$ with $\widetilde{\mathbf{z}}_{n,1}$ any sample with 1 replaced observation of $\widetilde{\mathbf{x}}_n$. Therefore, $\text{fsbp}^+ = 1/n$. □

Theorem 2.3.2. *For any sample of observations $\widetilde{\mathbf{x}}_n$ from an RFN $\mathcal{X}$ in which there are not two identical observations, we have that*

$$\text{fsbp}^+(\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)) = \frac{1}{n}, \quad \text{fsbp}^-(\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)) = \frac{n-1}{n}.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$ is given by

$$\text{fsbp}^*(\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)) = \frac{1}{n},$$

which is the lowest possible fsbp* *of an estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample in which there are not two identical observations, and denote $\text{fsbp}^+ = \text{fsbp}^+(\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n))$ and $\text{fsbp}^- = \text{fsbp}^-(\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n))$. The proof of the two first conclusions is to be split in three steps.

Step 1: We begin showing that $\text{fsbp}^- \leq (n-1)/n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,n-1} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = n-1$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$ such that $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{y}}_{n,n-1}, \overline{\widetilde{\mathbf{y}}}_{n,n-1}) = 0$.

We construct the sample $\widetilde{\mathbf{y}}_{n,n-1}$ by replacing the observations $\widetilde{x}_2,\dots,\widetilde{x}_n$ by $\widetilde{x}_1$. The considered sample $\widetilde{\mathbf{y}}_{n,n-1}$ has $k=n-1$ replaced observations. Since $\overline{\widetilde{\mathbf{y}}}_{n,n-1}=\widetilde{x}_1$, we have that for all $i\in\{1,\dots,n\}$, $\rho_2\Big(\widetilde{y}_i,\overline{\widetilde{\mathbf{y}}}_{n,n-1}\Big)=\ \rho_2(\widetilde{x}_1,\widetilde{x}_1)=0$, whence

$$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{y}}_{n,n-1},\overline{\widetilde{\mathbf{y}}}_{n,n-1})=\sum_{i=1}^{n}\rho_2\Big(\widetilde{y}_i,\overline{\widetilde{\mathbf{y}}}_{n,n-1}\Big)/n=0.$$

Consequently, $\inf_{\widetilde{\mathbf{z}}_{n,n-1}}\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{z}}_{n,n-1},\overline{\widetilde{\mathbf{z}}}_{n,n-1})=0$ with $\widetilde{\mathbf{z}}_{n,n-1}$ any sample with $n-1$ replaced observations of $\widetilde{\mathbf{x}}_n$ and, for this reason, $\text{fsbp}^-\le(n-1)/n$.

Step 2: Now we show that $\text{fsbp}^-\ge(n-1)/n$.

Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k<n-1$ replaced observations of $\widetilde{\mathbf{x}}_n$. There exist at least two observations $\widetilde{x1}_{\widetilde{\mathbf{y}}_{n,k}},\widetilde{x2}_{\widetilde{\mathbf{y}}_{n,k}}\in\{\widetilde{x}_1,\dots,\widetilde{x}_n\}$ such that $\widetilde{x1}_{\widetilde{\mathbf{y}}_{n,k}},\widetilde{x2}_{\widetilde{\mathbf{y}}_{n,k}}\in\ \widetilde{\mathbf{y}}_{n,k}$. Let $\delta:=\min_{i,j\in\{1,\dots,n\}}\rho_2(\widetilde{x}_i,\widetilde{x}_j)/n>0$. We have that

$$\delta=\frac{\min_{i,j\in\{1,\dots,n\}}\rho_2(\widetilde{x}_i,\widetilde{x}_j)}{n}\le\frac{\rho_2(\widetilde{x1}_{\widetilde{\mathbf{y}}_{n,k}},\widetilde{x2}_{\widetilde{\mathbf{y}}_{n,k}})}{n}$$

$$\le\frac{\rho_2(\widetilde{x1}_{\widetilde{\mathbf{y}}_{n,k}},\overline{\widetilde{\mathbf{y}}}_{n,k})+\rho_2(\widetilde{x2}_{\widetilde{\mathbf{y}}_{n,k}},\overline{\widetilde{\mathbf{y}}}_{n,k})}{n}\le\frac{\sum_{i=1}^{n}\rho_2(\widetilde{y}_i,\overline{\widetilde{\mathbf{y}}}_{n,k})}{n}=\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{y}}_{n,k},\overline{\widetilde{\mathbf{y}}}_{n,k}).$$

Therefore, $\inf_{\widetilde{\mathbf{y}}_{n,k}}\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{y}}_{n,k},\overline{\widetilde{\mathbf{y}}}_{n,k})\ge\delta>0$ with $k<n-1$, and hence $\text{fsbp}^-\ \ge\ (n-1)/n$.

Step 3: Finally, we will prove that $\text{fsbp}^+=1/n$. For this purpose, we construct the sample $\widetilde{\mathbf{y}}_{n,1}$ in the same way as was constructed in *Step 3* of the proof of Theorem 2.3.1. Since $\rho_2=D_1^{\ell}$, then by looking at *Step 3* of the proof of Theorem 2.3.1 we can conclude that

$$\rho_2\left(\widetilde{y}_1,\overline{\widetilde{\mathbf{y}}}_{n,1}\right)\ge\frac{(n-1)L}{n}.$$

Thus, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{y}}_{n,1},\overline{\widetilde{\mathbf{y}}}_{n,1})=\sum_{i=1}^{n}\rho_2\Big(\widetilde{y}_i,\overline{\widetilde{\mathbf{y}}}_{n,1}\Big)/n\ge\rho_2\Big(\widetilde{y}_1,\overline{\widetilde{\mathbf{y}}}_{n,1}\Big)/n\ge(n-1)L/n^2$.

Letting $L\to\infty$, $\sup_{\widetilde{\mathbf{z}}_{n,1}}\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{z}}_{n,1},\overline{\widetilde{\mathbf{z}}}_{n,1})=\infty$ with $\widetilde{\mathbf{z}}_{n,1}$ any sample with 1 replaced observation of $\widetilde{\mathbf{x}}_n$. Consequently, $\text{fsbp}^+=1/n$. $\square$

Theorem 2.3.3. *For any sample of observations $\widetilde{\mathbf{x}}_n$ from an RFN $\mathcal{X}$ in which there are not two identical observations, we have that*

$$\text{fsbp}^+(\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n)))=\frac{1}{n},\quad\text{fsbp}^-(\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n)))=\frac{n-1}{n}.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$ is given by

$$\text{fsbp}^*(\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n,\widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n)))=\frac{1}{n},$$

which is the lowest possible fsbp* *of an estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample which does not contain two identical observations, and denote now $\mathrm{fsbp}^+ = \mathrm{fsbp}^+(\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)))$ and $\mathrm{fsbp}^- = \mathrm{fsbp}^-(\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)))$. The proof of the two first conclusions is to be split in three steps.

Step 1: We begin showing that $\mathrm{fsbp}^- \leq (n-1)/n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,n-1} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = n-1$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$ such that $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{y}}_{n,n-1}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,n-1})) = 0$.

We construct the sample $\widetilde{\mathbf{y}}_{n,n-1}$ by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_n$ by $\widetilde{x}_1$. The considered sample $\widetilde{\mathbf{y}}_{n,n-1}$ has $k = n-1$ replaced observations. Since $\widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,n-1}) = \widetilde{x}_1$, we have that $\rho_1\Big(\widetilde{y}_i, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,n-1})\Big) = \rho_1(\widetilde{x}_1, \widetilde{x}_1) = 0$ for all $i \in \{1, \ldots, n\}$, whence $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{y}}_{n,n-1}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,n-1})) = \sum_{i=1}^n \rho_1\Big(\widetilde{y}_i, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,n-1})\Big)/n = 0$.

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,n-1}} \widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{z}}_{n,n-1}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{z}}_{n,n-1})) = 0$ with $\widetilde{\mathbf{z}}_{n,n-1}$ any sample with $n-1$ replaced observations of $\widetilde{\mathbf{x}}_n$ and, for this reason, $\mathrm{fsbp}^- \leq (n-1)/n$.

Step 2: Now we show that $\mathrm{fsbp}^- \geq (n-1)/n$.

Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < n-1$ replaced observations from the original. There exist at least two observations $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \widetilde{\mathbf{y}}_{n,k}$. Let $\delta := \min_{i,j \in \{1,\ldots,n\}} \rho_1(\widetilde{x}_i, \widetilde{x}_j)/n > 0$. We have that

$$\begin{aligned} \delta = \frac{\min_{i,j\in\{1,\ldots,n\}} \rho_1(\widetilde{x}_i, \widetilde{x}_j)}{n} &\leq \frac{\rho_1(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}})}{n} \\ \leq \frac{\rho_1(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})) + \rho_1(\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k}))}{n} &\leq \frac{\sum_{i=1}^n \rho_1(\widetilde{y}_i, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k}))}{n} \\ &= \widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{y}}_{n,k}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})). \end{aligned}$$

Therefore, $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{y}}_{n,k}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})) \geq \delta > 0$ with $k < n-1$, whence $\mathrm{fsbp}^- \geq (n-1)/n$.

Step 3: Finally, we will prove that $\mathrm{fsbp}^+ = 1/n$. For this purpose, we construct the sample $\widetilde{\mathbf{y}}_{n,1}$ in the same way as it was constructed in *Step 3* of the proof of Theorem 2.3.1. The considered sample $\widetilde{\mathbf{y}}_{n,1}$ has $k = 1$ replaced observation. By taking into account that $\inf(\widetilde{x}_{(n)})_\alpha \geqslant \inf\Big(\widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})\Big)_\alpha$ and $\sup(\widetilde{x}_{(n)})_\alpha \geqslant \sup\Big(\widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})\Big)_\alpha$ for all $\alpha \in [0,1]$, it is satisfied that

$$\rho_1\left(\widetilde{y}_1, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})\right) = \rho_1\left(\widetilde{x}_{(n)} + L, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})\right) = \rho_1\left(\widetilde{x}_{(n)}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})\right) + L \geq L.$$

Thus, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{y}}_{n,1}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})) = \sum_{i=1}^n \rho_1\Big(\widetilde{y}_i, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})\Big)/n \geq \rho_1\Big(\widetilde{y}_1, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,1})\Big)/n \geq L/n$.

Letting $L \to \infty$, $\sup_{\widetilde{\mathbf{z}}_{n,1}} \widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{z}}_{n,1}, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{z}}_{n,1})) = \infty$ with $\widetilde{\mathbf{z}}_{n,1}$ any sample with 1 replaced observation of $\widetilde{\mathbf{x}}_n$. Consequently, $\mathrm{fsbp}^+ = 1/n$. $\square$

Theorem 2.3.4. *For any sample of observations* $\widetilde{\mathbf{x}}_n$ *from an RFN* $\mathcal{X}$ *in which there are not two identical observations, we have that*

$$\mathrm{fsbp}^+(\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n}, \quad \mathrm{fsbp}^-(\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))) = \frac{n-1}{n}.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))$ *is given by*

$$\mathrm{fsbp}^*(\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n},$$

which is the lowest possible fsbp* *of an estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample which does not contain two identical observations, and denote now $\mathrm{fsbp}^+ = \mathrm{fsbp}^+(\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)))$ and $\mathrm{fsbp}^- = \mathrm{fsbp}^-(\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)))$. The proof of the two first conclusions is to be split in three steps.

Step 1: We begin showing that $\mathrm{fsbp}^- \leq (n-1)/n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,n-1} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = n-1$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$ such that $\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{y}}_{n,n-1}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,n-1})) = 0$.

We construct the sample $\widetilde{\mathbf{y}}_{n,n-1}$ by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_n$ by $\widetilde{x}_1$. The considered sample $\widetilde{\mathbf{y}}_{n,n-1}$ has $k = n-1$ replaced observations. Since $\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,n-1}) = \widetilde{x}_1$, we have that $\mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,n-1})\right) = \mathscr{D}_\theta^\varphi(\widetilde{x}_1, \widetilde{x}_1) = 0$ for all $i \in \{1, \ldots, n\}$, whence $\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{y}}_{n,n-1}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,n-1})) = \sum_{i=1}^n \mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,n-1})\right)/n = 0$.

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,n-1}} \widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{z}}_{n,n-1}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{z}}_{n,n-1})) = 0$ with $\widetilde{\mathbf{z}}_{n,n-1}$ any sample with $n-1$ replaced observations of $\widetilde{\mathbf{x}}_n$ and, for this reason, $\mathrm{fsbp}^- \leq (n-1)/n$.

Step 2: Now we show that $\mathrm{fsbp}^- \geq (n-1)/n$.

Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < n-1$ replaced observations from the original. There exist at least two observations $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \widetilde{\mathbf{y}}_{n,k}$. Let $\delta := \min_{i,j\in\{1,\ldots,n\}} \mathscr{D}_\theta^\varphi(\widetilde{x}_i, \widetilde{x}_j)/n > 0$. We have that

$$\delta = \frac{\min_{i,j\in\{1,\ldots,n\}} \mathscr{D}_\theta^\varphi(\widetilde{x}_i, \widetilde{x}_j)}{n} \leq \frac{\mathscr{D}_\theta^\varphi(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}})}{n}$$

$$\leq \frac{\mathscr{D}_\theta^\varphi(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})) + \mathscr{D}_\theta^\varphi(\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k}))}{n} \leq \frac{\sum_{i=1}^n \mathscr{D}_\theta^\varphi(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k}))}{n}$$

$$= \widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})).$$

Therefore, $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})) \geq \delta > 0$ with $k < n-1$, and so, $\mathrm{fsbp}^- \geq (n-1)/n$.

Step 3: Finally, we will prove that $\mathrm{fsbp}^+ = 1/n$. For this purpose, we construct the sample $\widetilde{\mathbf{y}}_{n,1}$ in the same way as it was constructed in *Step 3* of the proof of Theorem 2.3.1. The considered sample $\widetilde{\mathbf{y}}_{n,1}$ has $k = 1$ replaced observation. Taking into account that $\mathrm{wabl}^\varphi(\widetilde{x}_{(n)}) \geqslant \mathrm{wabl}^\varphi(\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,1}))$, it is satisfied that

$$\mathscr{D}_\theta^\varphi\left(\widetilde{y}_1, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,1})\right) = \mathscr{D}_\theta^\varphi\left(\widetilde{x}_{(n)} + L, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,1})\right) = \mathscr{D}_\theta^\varphi\left(\widetilde{x}_{(n)}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,1})\right) + L \geq L.$$

Thus, $\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{y}}_{n,1}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,1})) = \sum_{i=1}^n \mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,1})\right)/n \geq \mathscr{D}_\theta^\varphi\left(\widetilde{y}_1, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,1})\right)/n \geq L/n$.

Letting $L \to \infty$, $\sup_{\widetilde{\mathbf{z}}_{n,1}} \widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{z}}_{n,1}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{z}}_{n,1})) = \infty$ with $\widetilde{\mathbf{z}}_{n,1}$ any sample with 1 replaced observation of $\widetilde{\mathbf{x}}_n$. Consequently, $\mathrm{fsbp}^+ = 1/n$. □

Theorem 2.3.5. *For any sample of observations $\widetilde{\mathbf{x}}_n$ from an RFN $\mathcal{X}$ in which there are not two identical observations, we have that*

$$\mathrm{fsbp}^+(\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)) = \frac{1}{n},$$

$$\mathrm{fsbp}^-(\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)) \in \left\{\frac{1}{n}\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right), \frac{1}{n}\left\lfloor\frac{n}{2}\right\rfloor\right\}.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$ is given by

$$\mathrm{fsbp}^*(\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)) = \frac{1}{n},$$

which is the lowest possible fsbp^* *of an estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample which does not contain two identical observations, and denote now $\mathrm{fsbp}^+ = \mathrm{fsbp}^+(\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n))$ and $\mathrm{fsbp}^- = \mathrm{fsbp}^-(\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n))$. The proof of the two first conclusions is to be split in three steps.

Step 1: We begin showing that $\mathrm{fsbp}^- \leq \left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)/n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor+1} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = \left\lfloor\frac{n}{2}\right\rfloor + 1$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$ such that $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor+1}, \overline{\widetilde{\mathbf{y}}}_{n,\lfloor\frac{n}{2}\rfloor+1}) = 0$.

We construct the sample $\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor+1}$ by replacing the observations $\widetilde{x}_1, \ldots, \widetilde{x}_{\lfloor\frac{n}{2}\rfloor+1}$ by $\overline{\widetilde{\mathbf{x}'}}$, being $\overline{\widetilde{\mathbf{x}'}}$ the mean of the sample $\widetilde{\mathbf{x}'} = (\widetilde{x}_{\lfloor\frac{n}{2}\rfloor+2}, \ldots, \widetilde{x}_n)$. The considered sample $\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor+1}$ has $k = \left\lfloor\frac{n}{2}\right\rfloor+1$ replaced observations. See that $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor+1}, \overline{\widetilde{\mathbf{y}}}_{n,\lfloor\frac{n}{2}\rfloor+1}) = 0$.

Since $\overline{\widetilde{\mathbf{y}}}_{n,\lfloor\frac{n}{2}\rfloor+1} = \overline{\widetilde{\mathbf{x}'}}$, then for all $i \in \{1, \ldots, \left\lfloor\frac{n}{2}\right\rfloor + 1\}$,

$$\rho_2\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,\lfloor\frac{n}{2}\rfloor+1}\right) = \rho_2(\overline{\widetilde{\mathbf{x}'}}, \overline{\widetilde{\mathbf{x}'}}) = 0.$$

Thus, $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor+1}, \overline{\widetilde{\mathbf{y}}}_{n,\lfloor\frac{n}{2}\rfloor+1}) = \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_2\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,\lfloor\frac{n}{2}\rfloor+1}\right)\right) = 0.$

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,\lfloor\frac{n}{2}\rfloor+1}} \widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{z}}_{n,\lfloor\frac{n}{2}\rfloor+1}, \overline{\widetilde{\mathbf{z}}}_{n,\lfloor\frac{n}{2}\rfloor+1}) = 0$ with $\widetilde{\mathbf{z}}_{n,\lfloor\frac{n}{2}\rfloor+1}$ any sample with $\left\lfloor\frac{n}{2}\right\rfloor+1$ replaced observations of $\widetilde{\mathbf{x}}_n$ and, for this reason, $\text{fsbp}^- \leq \left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)/n$.

It should be remarked that if there is an $\widetilde{x}_i$ equalling the sample mean of a subsample of $n-\left\lfloor\frac{n}{2}\right\rfloor$ observations including it, the last inequality can be constrained to be $\text{fsbp}^- \leq \left\lfloor\frac{n}{2}\right\rfloor/n$. Thus, assume without loss of generality that the observation $\widetilde{x}_{\lfloor\frac{n}{2}\rfloor+1}$ equals the mean $\overline{\widetilde{\mathbf{x}''}}$ of the sample $\widetilde{\mathbf{x}''} = (\widetilde{x}_{\lfloor\frac{n}{2}\rfloor+1}, \ldots, \widetilde{x}_n)$. By taking the sample $\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor}$ replacing the observations $\widetilde{x}_1, \ldots, \widetilde{x}_{\lfloor\frac{n}{2}\rfloor}$ by $\overline{\widetilde{\mathbf{x}''}}$, the proof is analogous to the previous one.

Step 2: To show that $\text{fsbp}^- \geq \left\lfloor\frac{n}{2}\right\rfloor/n$, let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor\frac{n}{2}\right\rfloor$ replaced observations from the original. Because of the definition of median for real numbers, there exist at least $\left\lfloor\frac{n}{2}\right\rfloor + 1$ observations $\widetilde{y^j}$ (with $j \in \{1, \ldots, \left\lfloor\frac{n}{2}\right\rfloor + 1\}$) in the sample $\widetilde{\mathbf{y}}_{n,k}$ such that $\rho_2\left(\widetilde{y^j}, \overline{\widetilde{\mathbf{y}}}_{n,k}\right)/2 \leq \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_2\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,k}\right)\right)$.

Moreover, because of $\widetilde{\mathbf{y}}_{n,k}$ having $k < \left\lfloor\frac{n}{2}\right\rfloor$ replaced observations from the sample $\widetilde{\mathbf{x}}_n$, there exist at least two observations $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_1}}$ and $\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_2}}$ with $\widetilde{y^{j_1}}, \widetilde{y^{j_2}} \in \{\widetilde{y^1}, \ldots, \widetilde{y^{\lfloor\frac{n}{2}\rfloor+1}}\}$.

Let $\delta := \min_{i,j\in\{1,\ldots,n\}} \rho_2(\widetilde{x}_i, \widetilde{x}_j)/4 > 0$. We have that

$$\begin{aligned}
\delta = \frac{\min_{i,j\in\{1,\ldots,n\}} \rho_2(\widetilde{x}_i, \widetilde{x}_j)}{4} &\leq \frac{\rho_2(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}})}{4} \\
&\leq \frac{1}{2} \cdot \frac{\rho_2(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \overline{\widetilde{\mathbf{y}}}_{n,k}) + \rho_2(\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}}, \overline{\widetilde{\mathbf{y}}}_{n,k})}{2} \\
\leq \frac{1}{2} \cdot \left(\mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_2\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,k}\right)\right) \right. &+ \left. \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_2\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,k}\right)\right)\right) \\
= \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_2\left(\widetilde{y}_i, \overline{\widetilde{\mathbf{y}}}_{n,k}\right)\right) &= \widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \overline{\widetilde{\mathbf{y}}}_{n,k}).
\end{aligned}$$

Therefore, $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \overline{\widetilde{\mathbf{y}}}_{n,k}) \geq \delta > 0$ with $k < \left\lfloor\frac{n}{2}\right\rfloor$, and hence $\text{fsbp}^- \geq \left\lfloor\frac{n}{2}\right\rfloor/n$.

Step 3: Finally, we will prove that $\text{fsbp}^+ = 1/n$. For this purpose, we construct the sample $\widetilde{\mathbf{y}}_{n,1}$ by replacing the observation $\widetilde{x}_1$ by the real number $n \cdot 2 \cdot s + L$, with $L \in \mathbb{R}$, $L > 0$ and $s := \max_{i\in\{1,\ldots,n\},\, \alpha\in[0,1]}\left\{|\inf(\widetilde{x}_i)_\alpha|, |\sup(\widetilde{x}_i)_\alpha|\right\} = \max_{i\in\{1,\ldots,n\}}\left\{|\inf(\widetilde{x}_i)_0|, |\sup(\widetilde{x}_i)_0|\right\}$.

The considered sample $\widetilde{\mathbf{y}}_{n,1}$ has $k = 1$ replaced observation. If $\widetilde{\mathbf{x}'}_n = \{n \cdot 2 \cdot s, \widetilde{x}_2, \ldots, \widetilde{x}_n\} = \{n \cdot 2 \cdot s, \widetilde{y}_2, \ldots, \widetilde{y}_n\}$, since

$$\inf\left(\overline{\widetilde{\mathbf{x}'}_n}\right)_\alpha = \frac{n \cdot 2 \cdot s + \inf(\widetilde{y}_2)_\alpha + \ldots + \inf(\widetilde{y}_n)_\alpha}{n}$$

$$= \frac{2 \cdot s + (\inf(\widetilde{y}_2)_\alpha + 2 \cdot s) + \ldots + (\inf(\widetilde{y}_n)_\alpha + 2 \cdot s)}{n}$$
$$\geq \frac{n \cdot \inf(\widetilde{y}_i)_\alpha}{n} = \inf(\widetilde{y}_i)_\alpha,$$
$$\sup\left(\overline{\overline{\widetilde{\mathbf{x}}'}}_n\right)_\alpha = \frac{n \cdot 2 \cdot s + \sup(\widetilde{y}_2)_\alpha + \ldots + \sup(\widetilde{y}_n)_\alpha}{n}$$
$$= \frac{2 \cdot s + (\sup(\widetilde{y}_2)_\alpha + 2 \cdot s) + \ldots + (\sup(\widetilde{y}_n)_\alpha + 2 \cdot s)}{n}$$
$$\geq \frac{n \cdot \sup(\widetilde{y}_i)_\alpha}{n} = \sup(\widetilde{y}_i)_\alpha$$

for all $\alpha \in [0,1]$ and for all $i \in \{2, \ldots, n\}$, then it is satisfied that

$$\left[\rho_2\left(\widetilde{y}_i, \overline{\overline{\widetilde{\mathbf{y}}}}_{n,1}\right)\right]^2 = \left[\rho_2\left(\widetilde{y}_i, \overline{\overline{\widetilde{\mathbf{x}}'}}_n + \frac{L}{n}\right)\right]^2 \geq \left[\rho_2\left(\widetilde{y}_i, \overline{\overline{\widetilde{\mathbf{x}}'}}_n\right)\right]^2 + \left[\frac{L}{n}\right]^2,$$

for all $i \in \{2, \ldots, n\}$, whence

$$\rho_2\left(\widetilde{y}_i, \overline{\overline{\widetilde{\mathbf{y}}}}_{n,1}\right) \geq \frac{L}{n}.$$

Thus, $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{y}}_{n,1}, \overline{\overline{\widetilde{\mathbf{y}}}}_{n,1}) = \mathrm{Me}_{i \in \{1,\ldots,n\}}\left(\rho_2\left(\widetilde{y}_i, \overline{\overline{\widetilde{\mathbf{y}}}}_{n,1}\right)\right) \geq L/n$.

Letting $L \to \infty$, $\sup_{\widetilde{\mathbf{z}}_{n,1}} \widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{z}}_{n,1}, \overline{\overline{\widetilde{\mathbf{z}}}}_{n,1}) = \infty$ with $\widetilde{\mathbf{z}}_{n,1}$ any sample with 1 replaced observation of $\widetilde{\mathbf{x}}_n$. Therefore, $\text{fsbp}^+ = 1/n$. □

Theorem 2.3.6. *For any sample of observations* $\widetilde{\mathbf{x}}_n$ *from an RFN* $\mathcal{X}$ *in which there are not two identical observations, we have that*

$$\text{fsbp}^+(\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n}\left\lfloor \frac{n+1}{2} \right\rfloor, \quad \text{fsbp}^-(\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$ *is given by*

$$\text{fsbp}^*(\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor,$$

which is the highest possible fsbp* *of a scale estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample which does not contain two identical observations, and denote now $\text{fsbp}^+ = \text{fsbp}^+(\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)))$ and $\text{fsbp}^- = \text{fsbp}^-(\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n)))$. The proof of the two first conclusions is next presented in four steps.

Step 1: We begin showing that $\text{fsbp}^- \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$ such that $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor})) = 0$.

We construct the sample $\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor}$ by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_{\lfloor \frac{n}{2} \rfloor + 1}$ by $\widetilde{x}_1$. Since $\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor}) = \widetilde{x}_1$, then for all $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor + 1\}$

$$\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor})\Big) = \rho_1(\widetilde{x}_1, \widetilde{x}_1) = 0.$$

Thus, $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor})) = \mathrm{Me}_{i\in\{1,\ldots,n\}} \left(\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor})\Big)\right) = 0.$

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,\lfloor \frac{n}{2} \rfloor}} \widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{z}}_{n,\lfloor \frac{n}{2} \rfloor}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{z}}_{n,\lfloor \frac{n}{2} \rfloor})) = 0$ for any sample $\widetilde{\mathbf{z}}_{n,\lfloor \frac{n}{2} \rfloor}$ with $\lfloor \frac{n}{2} \rfloor$ replaced observations of $\widetilde{\mathbf{x}}_n$, and $\mathrm{fsbp}^- \leq \lfloor \frac{n}{2} \rfloor / n$.

Step 2: Now we show that $\mathrm{fsbp}^- \geq \lfloor \frac{n}{2} \rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \lfloor \frac{n}{2} \rfloor$ replaced observations from the original. Because of the definition of median for real numbers, there exist at least $\lfloor \frac{n}{2} \rfloor + 1$ observations $\widetilde{y^j}$ (with $j \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor + 1\}$) in the sample $\widetilde{\mathbf{y}}_{n,k}$ such that $\rho_1\Big(\widetilde{y^j}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})\Big)/2 \leq \mathrm{Me}_{i\in\{1,\ldots,n\}} \left(\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})\Big)\right)$.

Moreover, because of $\widetilde{\mathbf{y}}_{n,k}$ having $k < \lfloor \frac{n}{2} \rfloor$ replaced observations from the sample $\widetilde{\mathbf{x}}_n$, there exist at least two observations $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_1}}$ and $\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_2}}$ with $\widetilde{y^{j_1}}, \widetilde{y^{j_2}} \in \{\widetilde{y^1}, \ldots, \widetilde{y^{\lfloor \frac{n}{2} \rfloor + 1}}\}$.

Let $\delta := \min_{i,j\in\{1,\ldots,n\}} \rho_1(\widetilde{x}_i, \widetilde{x}_j)/4 > 0$. We have that

$$\begin{aligned}
\delta = \frac{\min_{i,j\in\{1,\ldots,n\}} \rho_1(\widetilde{x}_i, \widetilde{x}_j)}{4} &\leq \frac{\rho_1(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}})}{4} \\
&\leq \frac{1}{2} \cdot \frac{\rho_1(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})) + \rho_1(\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k}))}{2} \\
\leq \frac{1}{2} \cdot \left(\mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})\Big)\right)\right. &+ \left.\mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})\Big)\right)\right) \\
= \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})\Big)\right) &= \widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})).
\end{aligned}$$

Therefore, $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})) \geq \delta > 0$ with $k < \lfloor \frac{n}{2} \rfloor$, and hence $\mathrm{fsbp}^- \geq \lfloor \frac{n}{2} \rfloor / n$.

Step 3: Now we will prove that $\mathrm{fsbp}^+ \leq \lfloor \frac{n+1}{2} \rfloor / n$.

We construct the sample $\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor}$ by replacing the observation $\widetilde{x}_1$ by $\widetilde{x}_{(n)} + L$, $\widetilde{x}_2$ by $\widetilde{x}_{(n)} + 2L$, ..., $\widetilde{x}_{\lfloor \frac{n+1}{2} \rfloor}$ by $\widetilde{x}_{(n)} + \lfloor \frac{n+1}{2} \rfloor L$, with $L \in \mathbb{R}$, $L > 0$ and $\widetilde{x}_{(n)}$ the extended maximum $\widetilde{\max}_{i\in\{1,\ldots,n\}} \widetilde{x}_i$.

The considered sample $\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor}$ has $k = \lfloor \frac{n+1}{2} \rfloor$ replaced observations. It satisfies that for any $i \in \{2, \ldots, n\}$, $\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor})\Big) \geq L/2$ because:

- If n is odd, $\widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor}) = \widetilde{x}_{(n)} + L$ and, hence,

- for all $i \in \{2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$, $\rho_1\Big(\widetilde{y}_i, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\Big) = \rho_1\Big(\widetilde{x}_{(n)} + iL, \widetilde{x}_{(n)} + L\Big)$ $= (i-1)L \geq L$ and
- for all $i \in \{\left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n\}$, $\rho_1\Big(\widetilde{y}_i, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\Big) = \rho_1\Big(\widetilde{x}_i, \widetilde{x}_{(n)} + L\Big)$ $= \rho_1\Big(\widetilde{x}_i, \widetilde{x}_{(n)}\Big) + L \geq L$.

- If n is even, $\widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}) = \widetilde{x'} + L/2$, where $(\widetilde{x'})_\alpha = \Big[\Big(\max_{i\in\{\left\lfloor \frac{n+1}{2} \right\rfloor+1,\ldots,n\}} \inf(\widetilde{x}_i)_\alpha + \inf(\widetilde{x}_{(n)})_\alpha\Big)/2, \Big(\max_{i\in\{\left\lfloor \frac{n+1}{2} \right\rfloor+1,\ldots,n\}} \sup(\widetilde{x}_i)_\alpha + \sup(\widetilde{x}_{(n)})_\alpha\Big)/2\Big]$ and, hence,

  - for all $i \in \{2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$, $\rho_1\Big(\widetilde{y}_i, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\Big) = \rho_1\Big(\widetilde{x}_{(n)} + iL, \widetilde{x'} + L/2\Big)$ $= \rho_1\Big(\widetilde{x}_{(n)}, \widetilde{x'}\Big) + (2i-1)L/2 \geq L$ and
  - for all $i \in \{\left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n\}$, $\rho_1\Big(\widetilde{y}_i, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\Big) = \rho_1\Big(\widetilde{x}_i, \widetilde{x'} + L/2\Big)$ $= \rho_1\Big(\widetilde{x}_i, \widetilde{x'}\Big) + L/2 \geq L/2$.

Hence, $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})) = \mathrm{Me}_{i\in\{1,\ldots,n\}}\Big(\rho_1\Big(\widetilde{y}_i, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\Big)\Big)$ $\geq L/2$. Consequently, letting $L \to \infty$, we have that $\sup_{\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}} \widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})) = \infty$ for any sample $\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ with $\left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations of $\widetilde{\mathbf{x}}_n$, and therefore, $\mathrm{fsbp}^+ \leq \left\lfloor \frac{n+1}{2} \right\rfloor / n$.

Step 4: Finally, we will prove that $\mathrm{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations from the original. Because of the definition of median for real numbers, there exist at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ observations $\widetilde{y^j}$ (with $j \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$) in the sample $\widetilde{\mathbf{y}}_{n,k}$ such that $\rho_1\Big(\widetilde{y^j}, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,k})\Big)$ $\geq \mathrm{Me}_{i\in\{1,\ldots,n\}}\Big(\rho_1\Big(\widetilde{y}_i, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,k})\Big)\Big)$.

Moreover, because of $\widetilde{\mathbf{y}}_{n,k}$ having $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations from the sample $\widetilde{\mathbf{x}}_n$, notice that

- there exists at least one observation $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_1}}$ with $\widetilde{y^{j_1}} \in \{\widetilde{y^1}, \ldots, \widetilde{y^{\left\lfloor \frac{n+1}{2} \right\rfloor}}\}$,
- and for each $\alpha \in [0,1]$, $\inf\Big(\widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,k})\Big)_\alpha$ should be in between the minimum and the maximum of the infima of the α-levels of the unreplaced $\widetilde{x}_i$, and the same happens with the suprema.

Let $M := \max_{i,j\in\{1,\ldots,n\}} \rho_1(\widetilde{x}_i, \widetilde{x}_j) < \infty$. We have that

$$M = \max_{i,j\in\{1,\ldots,n\}} \rho_1(\widetilde{x}_i, \widetilde{x}_j) \geq \rho_1(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{\mathrm{Me}}(\widetilde{\mathbf{y}}_{n,k}))$$

$$\geq \mathrm{Me}_{i\in\{1,\ldots,n\}}\Big(\rho_1\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})\Big)\Big) = \widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k})).$$

Therefore, $\sup_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{y}}_{n,k}))) \leq M < \infty$ with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$, and consequently, $\text{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor / n$. $\square$

Theorem 2.3.7. *For any sample of observations* $\widetilde{\mathbf{x}}_n$ *from an RFN* $\mathcal{X}$ *in which there are not two identical fuzzy numbers, we have that*

$$\text{fsbp}^+(\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n}\left\lfloor \frac{n+1}{2} \right\rfloor, \ \text{fsbp}^-(\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))$ *is given by*

$$\text{fsbp}^*(\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n))) = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor,$$

which is the highest possible fsbp* *of a scale estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample in which there are not two identical fuzzy numbers, and denote $\text{fsbp}^+ = \text{fsbp}^+(\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)))$ and $\text{fsbp}^- = \text{fsbp}^-(\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{x}}_n)))$. The proof is presented in four steps.

Step 1: We begin showing that $\text{fsbp}^- \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$ such that $\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor})) = 0$.

We construct the sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}$ by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_{\left\lfloor \frac{n}{2} \right\rfloor + 1}$ by $\widetilde{x}_1$. Since $\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}) = \widetilde{x}_1$, then for all $i \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\}$,

$$\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor})\Big) = \mathscr{D}_\theta^\varphi(\widetilde{x}_1, \widetilde{x}_1) = 0.$$

Thus, $\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor})) = \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor})\Big)\right) = 0.$

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}} \widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor})) = 0$ for any sample $\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}$ with $\left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of $\widetilde{\mathbf{x}}_n$, and $\text{fsbp}^- \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

Step 2: Now we show that $\text{fsbp}^- \geq \left\lfloor \frac{n}{2} \right\rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations from the original. Because of the definition of median for real numbers, there exist at least $\left\lfloor \frac{n}{2} \right\rfloor + 1$ observations $\widetilde{y^j}$ (with $j \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\}$) in the sample $\widetilde{\mathbf{y}}_{n,k}$ such that $\mathscr{D}_\theta^\varphi\Big(\widetilde{y^j}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})\Big)/2 \leq \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})\Big)\right)$.

Moreover, because of $\widetilde{\mathbf{y}}_{n,k}$ having $k < \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations from the sample $\widetilde{\mathbf{x}}_n$, there exist at least two observations $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that

$\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_1}}$ and $\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_2}}$ with $\widetilde{y^{j_1}}, \widetilde{y^{j_2}} \in \{\widetilde{y^1}, \ldots, \widetilde{y^{\lfloor \frac{n}{2} \rfloor + 1}}\}$. Let $\delta := \min_{i,j\in\{1,\ldots,n\}} \mathscr{D}_\theta^\varphi(\widetilde{x}_i, \widetilde{x}_j)/4 > 0$. We have that

$$\delta = \frac{\min_{i,j\in\{1,\ldots,n\}} \mathscr{D}_\theta^\varphi(\widetilde{x}_i, \widetilde{x}_j)}{4} \leq \frac{\mathscr{D}_\theta^\varphi(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}})}{4}$$
$$\leq \frac{1}{2} \cdot \frac{\mathscr{D}_\theta^\varphi(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})) + \mathscr{D}_\theta^\varphi(\widetilde{x^2}_{\widetilde{\mathbf{y}}_{n,k}}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k}))}{2}$$
$$\leq \frac{1}{2} \cdot \Big(\mathrm{Me}_{i\in\{1,\ldots,n\}}\Big(\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})\Big)\Big) + \mathrm{Me}_{i\in\{1,\ldots,n\}}\Big(\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})\Big)\Big)\Big)$$
$$= \mathrm{Me}_{i\in\{1,\ldots,n\}}\Big(\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})\Big)\Big) = \widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})).$$

Therefore, $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,k})) \geq \delta > 0$ with $k < \left\lfloor \frac{n}{2} \right\rfloor$, and hence $\mathrm{fsbp}^- \geq \left\lfloor \frac{n}{2} \right\rfloor / n$.

Step 3: Now we will prove that $\mathrm{fsbp}^+ \leq \left\lfloor \frac{n+1}{2} \right\rfloor / n$.

We construct the sample $\widetilde{\mathbf{y}}_{n,k}$ by replacing the observation $\widetilde{x}_1$ by $\widetilde{x}_{(n)} + L$, $\widetilde{x}_2$ by $\widetilde{x}_{(n)} + 2L$, $\ldots$, $\widetilde{x}_{\lfloor \frac{n+1}{2} \rfloor}$ by $\widetilde{x}_{(n)} + \left\lfloor \frac{n+1}{2} \right\rfloor L$, with $L \in \mathbb{R}$, $L > 0$ and $\widetilde{x}_{(n)}$ the interval value given by

$$\widetilde{x}_{(n)} = \Big[\max_{i\in\{1,\ldots,n\}} \mathrm{wabl}^\varphi(\widetilde{x}_i), \max_{i\in\{1,\ldots,n\}} \mathrm{wabl}^\varphi(\widetilde{x}_i) + 2s\Big],$$

with $s := \max_{i\in\{1,\ldots,n\},\alpha\in[0,1]}\Big\{\mathrm{ldev}^\varphi_{\widetilde{x}_i}(\alpha), \mathrm{rdev}^\varphi_{\widetilde{x}_i}(\alpha)\Big\} = \max_{i\in\{1,\ldots,n\}}\Big\{\mathrm{ldev}^\varphi_{\widetilde{x}_i}(0), \mathrm{rdev}^\varphi_{\widetilde{x}_i}(0)\Big\}$.

The considered sample $\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor}$ has $k = \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations. It satisfies that for any $i \in \{2, \ldots, n\}$, $\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor})\Big) \geq L/2$ because:

- If n is odd, $\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor}) = \widetilde{x}_{(n)} + L$ and, hence,
  - for all $i \in \{2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$, $\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor})\Big) = \mathscr{D}_\theta^\varphi\Big(\widetilde{x}_{(n)} + iL, \widetilde{x}_{(n)} + L\Big) = (i-1)L \geq L$ and
  - for all $i \in \{\left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n\}$, $\mathscr{D}_\theta^\varphi\Big(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor})\Big) = \mathscr{D}_\theta^\varphi\Big(\widetilde{x}_i, \widetilde{x}_{(n)} + L\Big) = \mathscr{D}_\theta^\varphi\Big(\widetilde{x}_i, \widetilde{x}_{(n)}\Big) + L \geq L$.

- If n is even, $\widehat{\widetilde{\mathrm{M}^\varphi}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n+1}{2} \rfloor}) = \widetilde{x}' + L/2$, where $\widetilde{x}'$ is the fuzzy number such that for each $\alpha \in [0,1]$

$$\inf(\widetilde{x}')_\alpha = \frac{\max\limits_{i\in\{\lfloor \frac{n+1}{2} \rfloor + 1,\ldots,n\}} \mathrm{wabl}^\varphi(\widetilde{x}_i) - \max\limits_{i\in\{\lfloor \frac{n+1}{2} \rfloor + 1,\ldots,n\}} \mathrm{ldev}^\varphi_{\widetilde{x}_i}(\alpha) + \inf(\widetilde{x}_{(n)})_\alpha}{2},$$

$$\sup(\widetilde{x}')_\alpha = \frac{\max\limits_{i\in\{\lfloor \frac{n+1}{2} \rfloor + 1,\ldots,n\}} \mathrm{wabl}^\varphi(\widetilde{x}_i) + \max\limits_{i\in\{\lfloor \frac{n+1}{2} \rfloor + 1,\ldots,n\}} \mathrm{rdev}^\varphi_{\widetilde{x}_i}(\alpha) + \sup(\widetilde{x}_{(n)})_\alpha}{2}$$

and, hence,

– for all $i \in \{2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$, $\mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\right) = \mathscr{D}_\theta^\varphi\left(\widetilde{x}_{(n)} + iL, \widetilde{x}' + L/2\right)$ $= \mathscr{D}_\theta^\varphi\left(\widetilde{x}_{(n)}, \widetilde{x}'\right) + (2i-1)L/2 \geq L$ and

– for all $i \in \{\left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n\}$, $\mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\right) = \mathscr{D}_\theta^\varphi\left(\widetilde{x}_i, \widetilde{x}' + L/2\right)$ $= \mathscr{D}_\theta^\varphi\left(\widetilde{x}_i, \widetilde{x}'\right) + L/2 \geq L/2.$

Therefore,

$$\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})) = \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor})\right)\right) \geq L/2.$$

Consequently, if $L \to \infty$, we have that $\sup_{\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}} \widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}))$ $= \infty$ for any sample $\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ with $\left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations of $\widetilde{\mathbf{x}}_n$, and therefore, $\mathrm{fsbp}^+ \leq \left\lfloor \frac{n+1}{2} \right\rfloor / n$.

Step 4: Finally, we will prove that $\mathrm{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations from the original. Because of the definition of median for real numbers, there exist at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ observations $\widetilde{y^j}$ of the sample $\widetilde{\mathbf{y}}_{n,k}$ such that $\mathscr{D}_\theta^\varphi\left(\widetilde{y^j}, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,k})\right) \geq \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,k})\right)\right)$, with $j \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$.

Moreover, because of $\widetilde{\mathbf{y}}_{n,k}$ having $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations from the sample $\widetilde{\mathbf{x}}_n$, there exist at least one observation $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}} \in \{\widetilde{x}_1, \ldots, \widetilde{x}_n\}$ such that $\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}} = \widetilde{y^{j_1}}$ with $\widetilde{y^{j_1}} \in \{\widetilde{y^1}, \ldots, \widetilde{y^{\left\lfloor \frac{n+1}{2} \right\rfloor}}\}$. Let $M := \max_{i,j\in\{1,\ldots,n\}} \mathscr{D}_\theta^\varphi(\widetilde{x}_i, \widetilde{x}_j) < \infty$. Reasoning as for *Step 4* in the proof of Theorem 2.3.6, we have that

$$\begin{aligned} M = \max_{i,j\in\{1,\ldots,n\}} \mathscr{D}_\theta^\varphi(\widetilde{x}_i, \widetilde{x}_j) &\geq \mathscr{D}_\theta^\varphi(\widetilde{x^1}_{\widetilde{\mathbf{y}}_{n,k}}, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,k})) \\ \geq \mathrm{Me}_{i\in\{1,\ldots,n\}}\left(\mathscr{D}_\theta^\varphi\left(\widetilde{y}_i, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,k})\right)\right) &= \widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,k})). \end{aligned}$$

Therefore, $\sup_{\widetilde{\mathbf{y}}_{n,k}} \widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{y}}_{n,k}, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{y}}_{n,k}))) \leq M < \infty$ with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$, and consequently, $\mathrm{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor / n$. □

Theorem 2.3.8. *For any sample of observations $\widetilde{\mathbf{x}}_n$ from an RFN $\mathcal{X}$ in which there are not two identical observations, we have that*

$$\mathrm{fsbp}^+(\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n}\left\lfloor \frac{n+1}{2} \right\rfloor, \quad \mathrm{fsbp}^-(\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n)$ is given by

$$\mathrm{fsbp}^*(\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n}\left\lfloor \frac{n}{2} \right\rfloor,$$

which is the highest possible fsbp* *of a scale estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample in which there are not two identical observations, and denote $\text{fsbp}^+ = \text{fsbp}^+(\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n))$ and $\text{fsbp}^- = \text{fsbp}^-(\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n))$. The proof for the two first conclusions is presented in four steps.

Step 1: We begin showing that $\text{fsbp}^- \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$, such that $\widehat{D\text{-S}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}) = 0$.

We construct the contaminated sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}$ from $\widetilde{\mathbf{x}}_n$ by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_{\left\lfloor \frac{n}{2} \right\rfloor + 1}$ by $\widetilde{x}_1$. Since for all $i \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\}$, $\overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} = 0$, then $\widehat{D\text{-S}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}) = \underline{\text{Me}}_i \left\{ \overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} \right\} = 0$.

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}} \widehat{D\text{-S}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}) = 0$ being $\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}$ any sample with $\left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$, and as a consequence, $\text{fsbp}^- \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

Step 2: Now we show that $\text{fsbp}^- \geq \left\lfloor \frac{n}{2} \right\rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations from the original sample $\widetilde{\mathbf{x}}_n$.

Fix an arbitrary $\widetilde{y}_i \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$. Since $k < \left\lfloor \frac{n}{2} \right\rfloor$ observations are replaced of the original sample $\widetilde{\mathbf{x}}_n$, there exist at least $n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$ observations $\widetilde{y}_l$ in the sample $\widetilde{\mathbf{y}}_{n,k}$ with $\widetilde{y}_l \neq \widetilde{y}_i$ such that $\widetilde{y}_l \in \widetilde{\mathbf{x}}_n$. So, for at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ observations $\widetilde{y}_l$: $D(\widetilde{y}_i, \widetilde{y}_l) \geq \min_{i,j \in \{1,\ldots,n\}} D(\widetilde{x}_i, \widetilde{x}_j) := \delta > 0$ and hence, $\overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} \geq \delta > 0$. Since at least there exist $n - (\left\lfloor \frac{n}{2} \right\rfloor - 1) = \left\lfloor \frac{n+1}{2} \right\rfloor + 1$ observations $\widetilde{y}_i \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$, whence $\widehat{D\text{-S}}(\widetilde{\mathbf{y}}_{n,k}) = \underline{\text{Me}}_i \left\{ \overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} \right\} \geq \delta$.

Thus, $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{D\text{-S}}(\widetilde{\mathbf{y}}_{n,k}) \geq \delta > 0$ with $k < \left\lfloor \frac{n}{2} \right\rfloor$, and hence $\text{fsbp}^- \geq \left\lfloor \frac{n}{2} \right\rfloor / n$.

Step 3: Now we will prove that $\text{fsbp}^+ \leq \left\lfloor \frac{n+1}{2} \right\rfloor / n$.

We construct the sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ from $\widetilde{\mathbf{x}}_n$ by replacing the observation $\widetilde{x}_1$ by $\widetilde{x}_{(n)} + L$, $\widetilde{x}_2$ by $\widetilde{x}_{(n)} + 2L$, ..., $\widetilde{x}_{\left\lfloor \frac{n+1}{2} \right\rfloor}$ by $\widetilde{x}_{(n)} + \left\lfloor \frac{n+1}{2} \right\rfloor L$, with $0 < L \in \mathbb{R}$ and $\widetilde{x}_{(n)}$ the extended maximum $\widetilde{\max}_{i \in \{1,\ldots,n\}} \widetilde{x}_i$.

The considered sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ has $k = \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations. It satisfies that for any $i \in \{1, \ldots, n\}$, $\overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} \geq L$ since

- if $i \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$, $D(\widetilde{y}_i, \widetilde{y}_j) \geq L$ for all $j \neq i$, whereas
- if $i \in \{\left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n\}$, $D(\widetilde{y}_i, \widetilde{y}_j) \geq L$ for all $j \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$.

Thus, $\widehat{D\text{-S}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}) = \underline{\text{Me}}_i \left\{ \overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} \right\} \geq L$.

Letting $L \to \infty$, $\sup_{\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}} \widehat{D\text{-S}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}) = \infty$ being $\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ any sample with $\left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations of $\widetilde{\mathbf{x}}_n$, and $\text{fsbp}^+ \leq \left\lfloor \frac{n+1}{2} \right\rfloor / n$.

Step 4: Finally, we will prove that $\mathrm{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations from the original sample $\widetilde{\mathbf{x}}_n$.

Fix an arbitrary $\widetilde{y}_i \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$. Since $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ observations are replaced of the original sample $\widetilde{\mathbf{x}}_n$, there exist at least $n - (\left\lfloor \frac{n+1}{2} \right\rfloor - 1) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ observations $\widetilde{y}_l$ in the sample $\widetilde{\mathbf{y}}_{n,k}$ such that $\widetilde{y}_l \in \widetilde{\mathbf{x}}_n$. So $D(\widetilde{y}_i, \widetilde{y}_l) \leq \max_{i,j \in \{1,\ldots,n\}} D(\widetilde{x}_i, \widetilde{x}_j) := M < \infty$ and $\overline{\mathrm{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} \leq M < \infty$. And, there exist at least $n - (\left\lfloor \frac{n+1}{2} \right\rfloor - 1) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ observations $\widetilde{y}_i \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$, whence

$$\widehat{D\text{-}\mathrm{S}}(\widetilde{\mathbf{y}}_{n,k}) = \underline{\mathrm{Me}}_i \left\{ \overline{\mathrm{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} \right\} \leq M.$$

Hence, $\sup_{\widetilde{\mathbf{y}}_{n,k}} \widehat{D\text{-}\mathrm{S}}(\widetilde{\mathbf{y}}_{n,k}) \leq M < \infty$ with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ and, consequently, $\mathrm{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor / n$. $\square$

Theorem 2.3.9. *For any sample of observations $\widetilde{\mathbf{x}}_n$ from an RFN $\mathcal{X}$ in which there are not two identical observations, we have that*

$$\mathrm{fsbp}^+(\widehat{D\text{-}\mathrm{Q}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n} \left\lfloor \frac{n+1}{2} \right\rfloor, \quad \mathrm{fsbp}^-(\widehat{D\text{-}\mathrm{Q}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor.$$

Therefore, the finite sample breakdown point of the scale estimate $\widehat{D\text{-}\mathrm{Q}}(\widetilde{\mathbf{x}}_n)$ is given by

$$\mathrm{fsbp}^*(\widehat{D\text{-}\mathrm{Q}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor,$$

which is the highest possible fsbp* *of a scale estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample in which there are not two identical observations, and denote $\mathrm{fsbp}^+ = \mathrm{fsbp}^+(\widehat{D\text{-}\mathrm{Q}}(\widetilde{\mathbf{x}}_n))$ and $\mathrm{fsbp}^- = \mathrm{fsbp}^-(\widehat{D\text{-}\mathrm{Q}}(\widetilde{\mathbf{x}}_n))$. The proof of the two first conclusions is presented in four steps.

Step 1: We begin showing that $\mathrm{fsbp}^- \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$, such that $\widehat{D\text{-}\mathrm{Q}}(\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor}) = 0$.

We construct the contaminated sample $\widetilde{\mathbf{y}}_{n,\lfloor \frac{n}{2} \rfloor}$ from $\widetilde{\mathbf{x}}_n$ by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_{\lfloor \frac{n}{2} \rfloor + 1}$ by $\widetilde{x}_1$. Since $\widetilde{y}_i = \widetilde{x}_1$ for all $i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1$, then we have that

$$D(\widetilde{y}_i, \widetilde{y}_j) = 0 \text{ for all } i \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\} \text{ and for all } j \in \{2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\}.$$

Therefore, by denoting $h = \lfloor n/2 \rfloor + 1$, and $m = \binom{h}{2}$, we have that

$$\mathrm{card}\{D(\widetilde{y}_i, \widetilde{y}_j) = 0 : i < j\} = \binom{\left\lfloor \frac{n}{2} \right\rfloor + 1}{2} = \binom{h}{2} = m.$$

Then, $\widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,\lfloor\frac{n}{2}\rfloor}) = \{D(\widetilde{y}_i, \widetilde{y}_j) : i < j\}_{(m)} = 0$, whence, $\inf_{\widetilde{\mathbf{z}}_{n,\lfloor\frac{n}{2}\rfloor}} \widehat{D\text{-Q}}(\widetilde{\mathbf{z}}_{n,\lfloor\frac{n}{2}\rfloor}) = 0$, being $\widetilde{\mathbf{z}}_{n,\lfloor\frac{n}{2}\rfloor}$ any sample with $\lfloor\frac{n}{2}\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$, then $\text{fsbp}^- \leq \lfloor\frac{n}{2}\rfloor/n$.

Step 2: Now we show that $\text{fsbp}^- \geq \lfloor\frac{n}{2}\rfloor/n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \lfloor\frac{n}{2}\rfloor$ replaced observations from the original sample $\widetilde{\mathbf{x}}_n$, and let $\delta := \min_{i,j\in\{1,\dots,n\}} D(\widetilde{x}_i, \widetilde{x}_j) > 0$.

The number of distances $D(\widetilde{y}_i, \widetilde{y}_j)$ which could be lower than δ for the sample $\widetilde{\mathbf{y}}_{n,k}$ is at most $\binom{k+2}{2} - 1$, which is necessarily lower than or equal to

$$\binom{\lfloor\frac{n}{2}\rfloor - 1 + 2}{2} - 1 = \binom{\lfloor\frac{n}{2}\rfloor + 1}{2} - 1 = m - 1.$$

Then, $\widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,k}) = \{D(\widetilde{y}_i, \widetilde{y}_j) : i < j\}_{(m)} \geq \delta$, so $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,k}) \geq \delta > 0$ with $k < \lfloor\frac{n}{2}\rfloor$, and therefore $\text{fsbp}^- \geq \lfloor\frac{n}{2}\rfloor/n$.

Step 3: Now we will prove that $\text{fsbp}^+ \leq \lfloor\frac{n+1}{2}\rfloor/n$.

We construct the sample $\widetilde{\mathbf{y}}_{n,\lfloor\frac{n+1}{2}\rfloor}$ from $\widetilde{\mathbf{x}}_n$ by replacing the observation $\widetilde{x}_1$ by $\widetilde{x}_{(n)} + L$, $\widetilde{x}_2$ by $\widetilde{x}_{(n)} + 2L$, ..., $\widetilde{x}_{\lfloor\frac{n+1}{2}\rfloor}$ by $\widetilde{x}_{(n)} + \lfloor\frac{n+1}{2}\rfloor L$, with $0 < L \in \mathbb{R}$ and $\widetilde{x}_{(n)}$ the extended maximum $\widetilde{\max}_{i\in\{1,\dots,n\}}\widetilde{x}_i$.

The considered sample $\widetilde{\mathbf{y}}_{n,\lfloor\frac{n+1}{2}\rfloor}$ has $k = \lfloor\frac{n+1}{2}\rfloor$ replaced observations, and it verifies that

$$\begin{aligned} &\text{card}\{D(\widetilde{y}_i, \widetilde{y}_j) \geqslant L : i < j\} \\ &= \text{card}\{D(\widetilde{y}_1, \widetilde{y}_j) \geqslant L : 1 < j\} + \text{card}\{D(\widetilde{y}_2, \widetilde{y}_j) \geqslant L : 2 < j\} + \dots \\ &+ \text{card}\left\{D(\widetilde{y}_{\lfloor\frac{n+1}{2}\rfloor}, \widetilde{y}_j) \geqslant L : \left\lfloor\frac{n+1}{2}\right\rfloor < j\right\} + \dots + \text{card}\{D(\widetilde{y}_{n-1}, \widetilde{y}_n) \geqslant L\} \\ &\geq (n-1) + (n-2) + \dots + \left(n - \left\lfloor\frac{n+1}{2}\right\rfloor\right) = \left\lfloor\frac{n+1}{2}\right\rfloor \left(\frac{2n - \lfloor\frac{n+1}{2}\rfloor - 1}{2}\right). \end{aligned}$$

- If n is even, the number of distances which are greater than or equal to L is at least equal to

  $$\left\lfloor\frac{n+1}{2}\right\rfloor \left(\frac{2n - \lfloor\frac{n+1}{2}\rfloor - 1}{2}\right) = \frac{n(3n-2)}{8},$$

  and therefore, the number of distances which are lower than L is lower than or equal to

  $$\binom{n}{2} - \frac{n(3n-2)}{8} = \frac{n(n-2)}{8}.$$

Since $\widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,\lfloor\frac{n+1}{2}\rfloor}) = \{D(\widetilde{y}_i, \widetilde{y}_j) : i < j\}_{(m)}$, and in the case that n is even,

$$m = \binom{\lfloor\frac{n}{2}\rfloor + 1}{2} = \frac{n(n+2)}{8} > \frac{n(n-2)}{8},$$

then $\widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,\lfloor\frac{n+1}{2}\rfloor}) = \{D(\widetilde{y}_i, \widetilde{y}_j) : i < j\}_{(m)} \geq L$.

- In case that n is odd, the number of distances which are greater than or equal to L is at least equal to

$$\left\lfloor\frac{n+1}{2}\right\rfloor \left(\frac{2n - \lfloor\frac{n+1}{2}\rfloor - 1}{2}\right) = \frac{(n+1)(3n-3)}{8},$$

  and therefore, the number of distances which are lower than L is at most equal to

$$\binom{n}{2} - \frac{(n+1)(3n-3)}{8} = \frac{n^2 - 4n + 3}{8}.$$

  In this case, since n is odd,

$$m = \binom{\lfloor\frac{n}{2}\rfloor + 1}{2} = \binom{\frac{n-1}{2} + 1}{2} = \frac{(n+1)(n-1)}{8} > \frac{n^2 - 4n + 3}{8}$$

  since $n^2 - 1 > n^2 - 4n + 3$ if $n > 1$. Therefore, for n odd it is also verified that $\widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,\lfloor\frac{n+1}{2}\rfloor}) = \{D(\widetilde{y}_i, \widetilde{y}_j) : i < j\}_{(m)} \geq L$.

Letting $L \to \infty$, $\sup_{\widetilde{\mathbf{z}}_{n,\lfloor\frac{n+1}{2}\rfloor}} \widehat{D\text{-Q}}(\widetilde{\mathbf{z}}_{n,\lfloor\frac{n+1}{2}\rfloor}) = \infty$ being $\widetilde{\mathbf{z}}_{n,\lfloor\frac{n+1}{2}\rfloor}$ any sample with $\lfloor\frac{n+1}{2}\rfloor$ replaced observations of $\widetilde{\mathbf{x}}_n$, and $\text{fsbp}^+ \leq \lfloor\frac{n+1}{2}\rfloor / n$.

Step 4: Finally, we will prove that $\text{fsbp}^+ \geq \lfloor\frac{n+1}{2}\rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \lfloor\frac{n+1}{2}\rfloor$ replaced observations from the original sample $\widetilde{\mathbf{x}}_n$.

At least $n - (\lfloor\frac{n+1}{2}\rfloor - 1)$ observations belong to the original sample of observations $\widetilde{\mathbf{x}}_n$, therefore at least $\binom{n - \lfloor\frac{n+1}{2}\rfloor + 1}{2}$ distances belong to the original sample of distances.

- If n is even, then

$$\binom{n - \lfloor\frac{n+1}{2}\rfloor + 1}{2} = \frac{n(n+2)}{8}.$$

  We have seen in *Step 3* that when n is even, $m = n(n+2)/8$, whence $\widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,k}) = \{D(\widetilde{y}_i, \widetilde{y}_j) : i < j\}_{(m)} \leq \max_{i,j\in\{1,\ldots,n\}} D(\widetilde{x}_i, \widetilde{x}_j) := M < \infty$.

- If n is odd, then
$$\binom{n - \left\lfloor \frac{n+1}{2} \right\rfloor + 1}{2} = \frac{(n+1)(n-1)}{8},$$
which is equal to m when n is odd, whence $\widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,k}) \leq M$.

Thus, for any sample size n, $\sup_{\widetilde{\mathbf{y}}_{n,k}} \widehat{D\text{-Q}}(\widetilde{\mathbf{y}}_{n,k}) \leq M < \infty$ with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$, and consequently, $\text{fsbp}^{+} \geq \left\lfloor \frac{n+1}{2} \right\rfloor / n$. $\square$

Theorem 2.3.10. *For any sample of observations $\widetilde{\mathbf{x}}_n$ from an RFN $\mathcal{X}$ in which there are not two identical observations, we have that*
$$\text{fsbp}^{+}(\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n} \left\lfloor \frac{n+1}{2} \right\rfloor, \quad \text{fsbp}^{-}(\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor.$$
Therefore, the finite sample breakdown point of the scale estimate $\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)$ is given by
$$\text{fsbp}^{*}(\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)) = \frac{1}{n} \left\lfloor \frac{n}{2} \right\rfloor,$$
which is the highest possible fsbp* *of a scale estimate.*

Proof. Let $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample in which there are not two identical observations, and denote $\text{fsbp}^{+} = \text{fsbp}^{+}(\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n))$ and $\text{fsbp}^{-} = \text{fsbp}^{-}(\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n))$. The proof of the two first conclusions is presented in four steps.

Step 1: We begin showing that $\text{fsbp}^{-} \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

We are going to find a sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor} = (\widetilde{y}_1, \ldots, \widetilde{y}_n)$ with $k = \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$, such that $\widehat{D\text{-T}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}) = 0$.

We construct the contaminated sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}$ from $\widetilde{\mathbf{x}}_n$ by replacing the observations $\widetilde{x}_2, \ldots, \widetilde{x}_{\left\lfloor \frac{n}{2} \right\rfloor + 1}$ by $\widetilde{x}_1$. Since for all $i \in \{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor + 1\}$, $\overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} = 0$, then, by denoting $h = \lfloor n/2 \rfloor + 1$
$$\widehat{D\text{-T}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}) = \frac{1}{h} \sum_{r=1}^{h} \left\{ \overline{\text{Me}}_j \left\{ D(\widetilde{y}_i, \widetilde{y}_j) \right\} ; \; i = 1, \ldots, n \right\}_{(r)} = 0.$$

Therefore, $\inf_{\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}} \widehat{D\text{-T}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}) = 0$ being $\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n}{2} \right\rfloor}$ any sample with $\left\lfloor \frac{n}{2} \right\rfloor$ replaced observations of the original sample $\widetilde{\mathbf{x}}_n$, and as a consequence, $\text{fsbp}^{-} \leq \left\lfloor \frac{n}{2} \right\rfloor / n$.

Step 2: Now we show that $\text{fsbp}^{-} \geq \left\lfloor \frac{n}{2} \right\rfloor / n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor \frac{n}{2} \right\rfloor$ replaced observations from the original sample $\widetilde{\mathbf{x}}_n$.

Fix $\widetilde{y}_i \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$. Since $k < \left\lfloor \frac{n}{2} \right\rfloor$ observations are replaced of the original sample $\widetilde{\mathbf{x}}_n$, there exist at least $n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$ observations $\widetilde{y}_l$ in the sample $\widetilde{\mathbf{y}}_{n,k}$

with $\widetilde{y}_l \neq \widetilde{y}_i$ such that $\widetilde{y}_l \in \widetilde{\mathbf{x}}_n$. So, for at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ observations $\widetilde{y}_l$: $D(\widetilde{y}_i, \widetilde{y}_l)$ $\geq \min_{i,j\in\{1,\ldots,n\}} D(\widetilde{x}_i, \widetilde{x}_j) := \delta > 0$ and hence, $\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_i, \widetilde{y}_j)\right\} \geq \delta > 0$.

Moreover, at most $\left\lfloor \frac{n}{2} \right\rfloor - 1 = h - 2$ observations are replaced from the original sample $\widetilde{\mathbf{x}}_n$, then there exist at least two observations $\widetilde{y}_{i_1}, \widetilde{y}_{i_2} \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$ such that $\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_{i_1}, \widetilde{y}_j)\right\}$ and $\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_{i_2}, \widetilde{y}_j)\right\}$ belong to the first h elements of the set $\left\{\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_i, \widetilde{y}_j)\right\}\ ;\ i = 1, \ldots, n\right\}$. Therefore,

$$\widehat{D\text{-T}}(\widetilde{\mathbf{y}}_{n,k}) = \frac{1}{h} \sum_{r=1}^{h} \left\{\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_i, \widetilde{y}_j)\right\}\ ;\ i = 1, \ldots, n\right\}_{(r)} \geq \frac{2\delta}{h},$$

whence $\inf_{\widetilde{\mathbf{y}}_{n,k}} \widehat{D\text{-T}}(\widetilde{\mathbf{y}}_{n,k}) \geq 2\delta/h > 0$ with $k < \left\lfloor \frac{n}{2} \right\rfloor$, and hence $\mathrm{fsbp}^- \geq \left\lfloor \frac{n}{2} \right\rfloor /n$.

Step 3: Now we will prove that $\mathrm{fsbp}^+ \leq \left\lfloor \frac{n+1}{2} \right\rfloor /n$.

We construct the sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ from $\widetilde{\mathbf{x}}_n$ by replacing the observation $\widetilde{x}_1$ by $\widetilde{x}_{(n)} + L$, $\widetilde{x}_2$ by $\widetilde{x}_{(n)} + 2L$, $\ldots$, $\widetilde{x}_{\left\lfloor \frac{n+1}{2} \right\rfloor}$ by $\widetilde{x}_{(n)} + \left\lfloor \frac{n+1}{2} \right\rfloor L$, with $0 < L \in \mathbb{R}$ and $\widetilde{x}_{(n)}$ the extended maximum $\widetilde{\max}_{i\in\{1,\ldots,n\}} \widetilde{x}_i$.

The considered sample $\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ has $k = \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations. It satisfies that for any $i \in \{1, \ldots, n\}$, $\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_i, \widetilde{y}_j)\right\} \geq L$, since

- if $i \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$, then $D(\widetilde{y}_i, \widetilde{y}_j) \geq L$ for all $j \neq i$, and
- if $i \in \{\left\lfloor \frac{n+1}{2} \right\rfloor + 1, \ldots, n\}$, then $D(\widetilde{y}_i, \widetilde{y}_j) \geq L$ for all $j \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$.

Consequently, $\widehat{D\text{-T}}(\widetilde{\mathbf{y}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}) = \frac{1}{h} \sum_{r=1}^{h} \left\{\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_i, \widetilde{y}_j)\right\}\ ;\ i = 1, \ldots, n\right\}_{(r)} \geq L$.

Letting $L \to \infty$, $\sup_{\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}} \widehat{D\text{-T}}(\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}) = \infty$ being $\widetilde{\mathbf{z}}_{n,\left\lfloor \frac{n+1}{2} \right\rfloor}$ any sample with $\left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations of $\widetilde{\mathbf{x}}_n$, and $\mathrm{fsbp}^+ \leq \left\lfloor \frac{n+1}{2} \right\rfloor /n$.

Step 4: Finally, we will prove that $\mathrm{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor /n$. Let $\widetilde{\mathbf{y}}_{n,k}$ be an arbitrary sample with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ replaced observations from the original sample $\widetilde{\mathbf{x}}_n$.

Fix $\widetilde{y}_i \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$. Since $k < \left\lfloor \frac{n+1}{2} \right\rfloor$ observations are replaced of the original sample $\widetilde{\mathbf{x}}_n$, there exist at least $n - (\left\lfloor \frac{n+1}{2} \right\rfloor - 1) = \left\lfloor \frac{n}{2} \right\rfloor + 1 = h$ observations $\widetilde{y}_l$ in the sample $\widetilde{\mathbf{y}}_{n,k}$ such that $\widetilde{y}_l \in \widetilde{\mathbf{x}}_n$, whence $D(\widetilde{y}_i, \widetilde{y}_l) \leq \max_{i,j\in\{1,\ldots,n\}} D(\widetilde{x}_i, \widetilde{x}_j) := M < \infty$ and $\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_i, \widetilde{y}_j)\right\} \leq M < \infty$.

Moreover, at least there exist $n - (\left\lfloor \frac{n+1}{2} \right\rfloor - 1) = h$ observations $\widetilde{y}_i \in \widetilde{\mathbf{y}}_{n,k} \bigcap \widetilde{\mathbf{x}}_n$ and, hence,

$$\widehat{D\text{-T}}(\widetilde{\mathbf{y}}_{n,k}) = \frac{1}{h} \sum_{r=1}^{h} \left\{\overline{\mathrm{Me}}_j \left\{D(\widetilde{y}_i, \widetilde{y}_j)\right\}\ ;\ i = 1, \ldots, n\right\}_{(r)} \leq M.$$

Consequently, $\sup_{\widetilde{\mathbf{y}}_{n,k}} \widehat{D\text{-T}}(\widetilde{\mathbf{y}}_{n,k}) \leq M < \infty$ with $k < \left\lfloor \frac{n+1}{2} \right\rfloor$, whence $\mathrm{fsbp}^+ \geq \left\lfloor \frac{n+1}{2} \right\rfloor /n$. $\square$

2.4 Analysis of the robustness of the different scale measures by means of simulation studies

In this section, several simulations are set up to analyze and illustrate the robust behaviour of the different scale estimates from an empirical point of view.

For this purpose, a key question should be posed, namely, what outliers mean when we are dealing with fuzzy-valued data instead of real-valued data?

Firstly, a response for this question is presented. Secondly, a simulation study to obtain the empirical value of the finite sample breakdown point of the different scale estimates will be performed; this study will confirm the theoretical results obtained in Section 2.3. Finally, the sensitivity curves, which inform us of the effect that a single outlier produces in the value of the estimators, will be graphically plotted for each estimator.

All simulations in this section are developed according to the two simulation procedures explained in Section 1.7. Regarding the 2ndSP, data have been generated by assuming the reference interval to be $[0, 100]$ and two different beta $\beta(p, q)$ distributions have been considered: a symmetrical and an asymmetric one.

It should also be commented that in the conducted simulations in this section, the standard deviation has been calculated by using the 2-norm distance $\rho_2 = D_1^\ell$, and for $\widehat{\mathscr{D}_\theta^\varphi\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\mathscr{D}_\theta^\varphi\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\varphi}(\widetilde{\mathbf{x}}_n))$ the choice has corresponded to $\theta = 1$ and $\varphi = \ell$, whereas for $\widehat{D\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{D\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{D\text{-T}}(\widetilde{\mathbf{x}}_n)$ the 1-norm distance ρ_1 has been employed.

2.4.1 Outliers in dealing with fuzzy number-valued data

The meaning of the outliers when we are dealing with real-valued data is well-known. They are atypical observations, that is, as said by Maronna *et al.* [81], they are “observations that are well separated from the majority or ‘bulk’ of the data, or in some way deviate from the general pattern of the data”.

In the setting of the fuzzy data, the notion of outlier for real-valued case still makes sense: they are observations that are ‘separated’ from the majority of the data because of they having either a different ‘location’ or a different ‘imprecision’ (more concretely, observations with a similar location to the majority of the data but with a different scale on the core and the support, that is, observations that are ‘wider’ or ‘narrower’ than the rest of the data).

Taking this into account, three different types of outliers will be considered for all the simulations conducted in this work. We explain now how to generate these outliers.

First, for each type of outlier, the four-tuple (a, b, c, d) is generated from the distribution of the random vector (X_1, X_2, X_3, X_4) in the non-contaminated sample. Then, we construct the outlier $\widetilde{y}_i = \mathrm{Tra}\langle y_i^1, y_i^2, y_i^3, y_i^4\rangle$ in the following way. Let $r_i^1, r_i^2 \in \mathbb{R}$:

- **Outlier of translation:** $y_i^1 = a + r_i^1$, $y_i^2 = b$, $y_i^3 = c$, $y_i^4 = d$.
- **Outlier of scale on the core and support:** $y_i^1 = a$, $y_i^2 = |r_i^2| \cdot b$, $y_i^3 = |r_i^2| \cdot c$, $y_i^4 = |r_i^2| \cdot d$.
- **Outlier of both translation and scale:** $y_i^1 = a + r_i^1$, $y_i^2 = |r_i^2| \cdot b$, $y_i^3 = |r_i^2| \cdot c$, $y_i^4 = |r_i^2| \cdot d$.

The three types of outliers are shown in Figures 2.2 and 2.3 for the first simulation procedure (1stSP) and Figures 2.4 and 2.5 for the second simulation procedure (2ndSP). In all cases, the size of the original sample is 10 and two outliers have been added to the sample.

Regarding the 1stSP, Figure 2.2 illustrates Case 1 and Figure 2.3 illustrates Case 2. For the outliers of translation, we have chosen $r_1^1 = 10$ and $r_2^1 = -10$, for the outliers of scale $r_1^2 = 5$ and $r_2^2 = 10$ and for the outliers of both translation and scale $r_1^1 = 10$, $r_2^1 = -10$, $r_1^2 = 5$ and $r_2^2 = 10$.

Regarding the 2ndSP, Figure 2.4 illustrates the symmetric case, generating the non-contaminated sample from a $\beta(100, 100)$, and Figure 2.5 illustrates the asymmetric case, generating the non-contaminated sample from a $\beta(1, 100)$. The weights in both cases have been $\omega_1 = 0.8$, $\omega_2 = 0.1$ and $\omega_3 = 0.1$.

In the symmetric case, for the outliers of translation we have chosen $r_1^1 = 30$ and $r_2^1 = -30$, for the outliers of scale $r_1^2 = 5$ and $r_2^2 = 10$ and for the outliers of both translation and scale $r_1^1 = 30$, $r_2^1 = -30$, $r_1^2 = 5$ and $r_2^2 = 10$.

In the asymmetric case, for the outliers of translation we have chosen $r_1^1 = 30$ and $r_2^1 = 60$, for the outliers of scale $r_1^2 = 40$ and $r_2^2 = 80$ and for the outliers of both translation and scale $r_1^1 = 30$, $r_2^1 = 60$, $r_1^2 = 40$ and $r_2^2 = 80$.

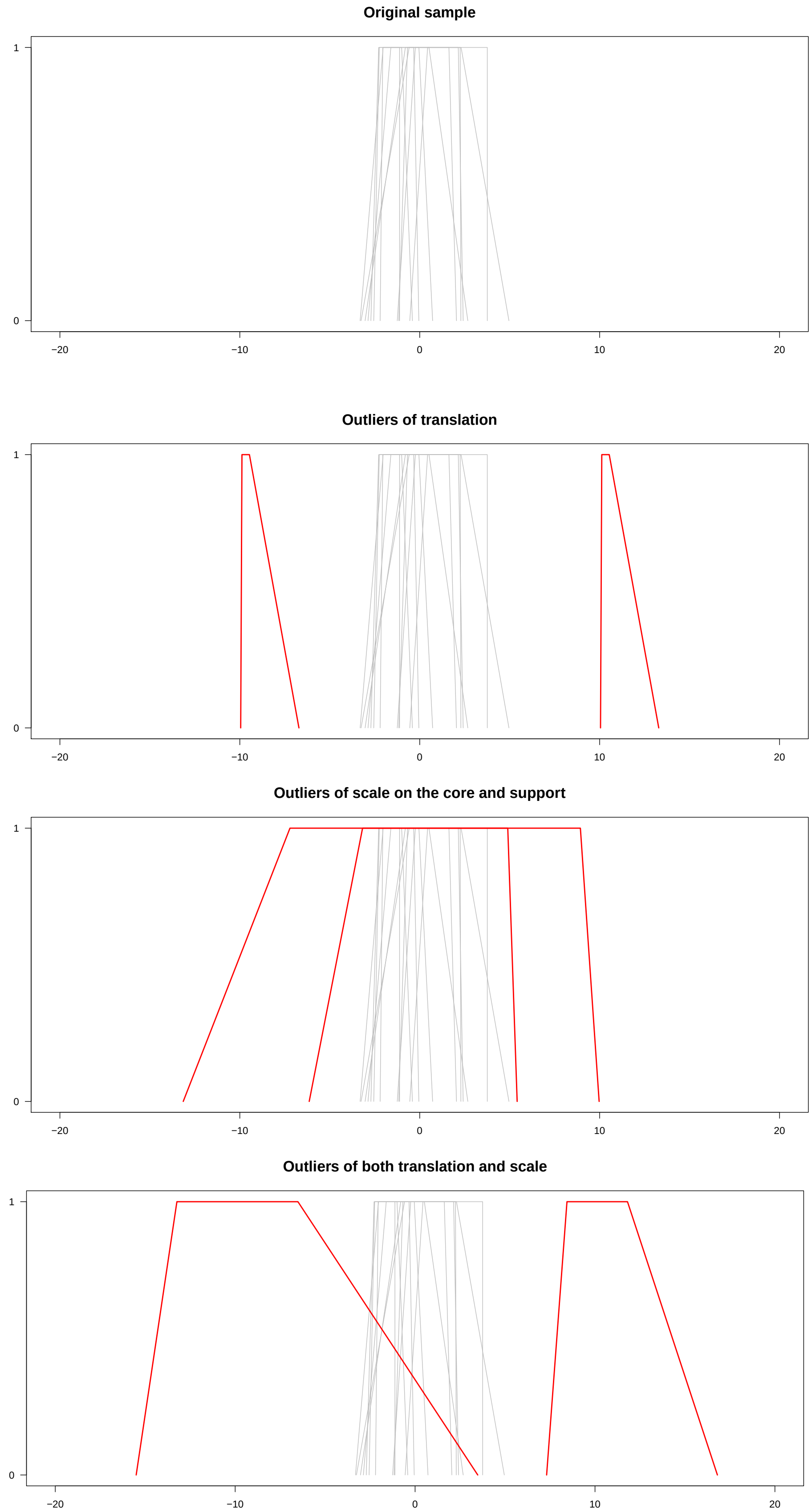


Figure 2.2: From top to bottom, the non-contaminated sample, the contaminated sample by two outliers of translation (in red), the contaminated sample by two outliers of scale on core and support and the contaminated sample by two outliers of both - 1stSP Case 1

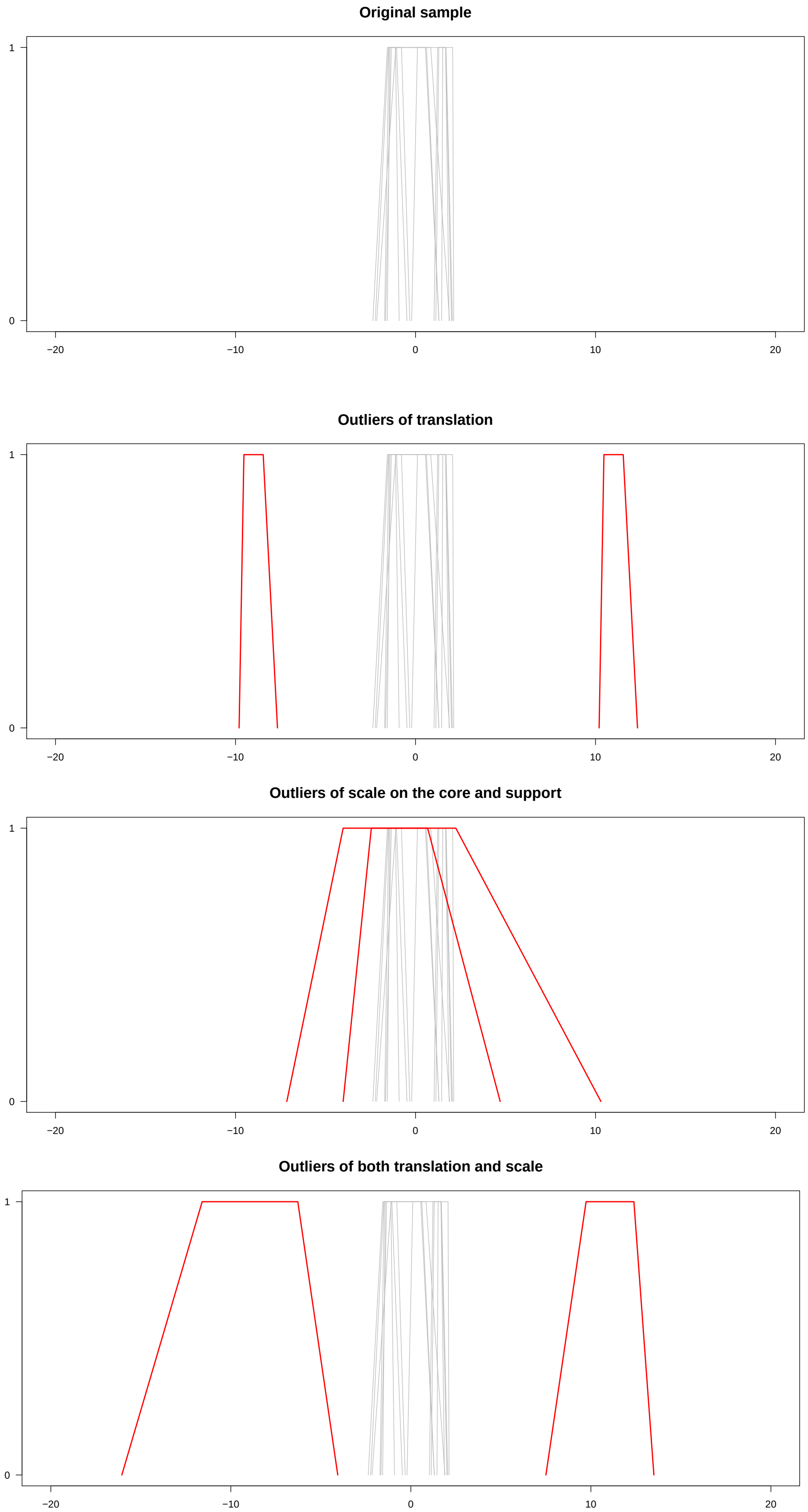


Figure 2.3: From top to bottom, the non-contaminated sample, the contaminated sample by two outliers of translation (in red), the contaminated sample by two outliers of scale on core and support and the contaminated sample by two outliers of both - 1stSP Case 2

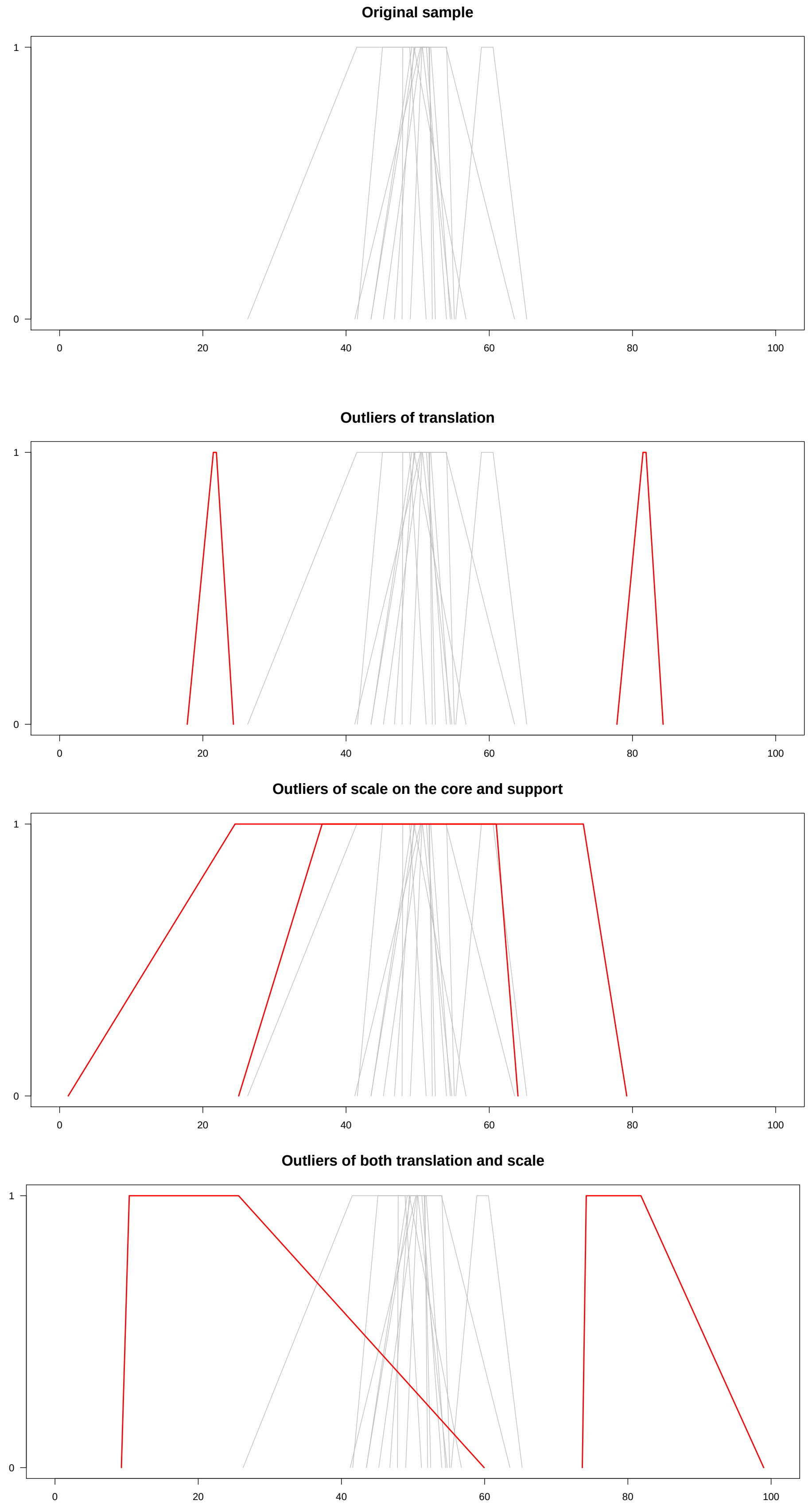


Figure 2.4: From top to bottom, the non-contaminated sample, the contaminated sample by two outliers of translation (in red), the contaminated sample by two outliers of scale on core and support and the contaminated sample by two outliers of both - 2ndSP Symmetric distribution

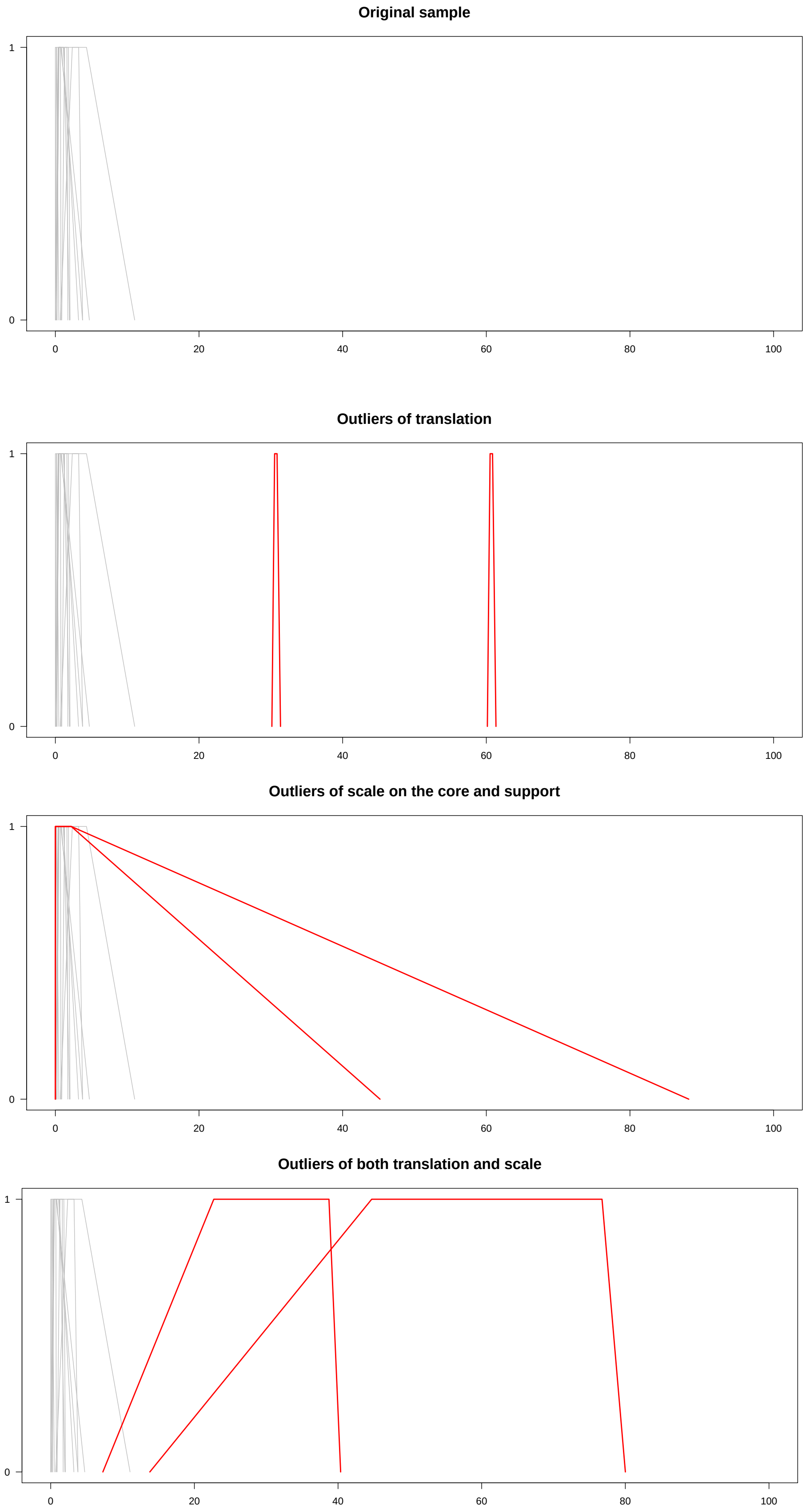


Figure 2.5: From top to bottom, the non-contaminated sample, the contaminated sample by two outliers of translation (in red), the contaminated sample by two outliers of scale on core and support and the contaminated sample by two outliers of both - 2ndSP Asymmetric distribution

2.4.2 Simulations-based analysis of the finite sample breakdown point: explosion and implosion breakdown point

In Section 2.3 the value of the finite sample breakdown point has been obtained for the different scale measures when fuzzy-valued data are considered. This value is now to be empirically corroborated and illustrated.

Recall that when we deal with estimates of scale, two different types of breakdown point need to be studied: the one caused by the presence of outliers in the sample, which can make the estimate explode to infinite (explosion breakdown point), and the one caused by the presence of inliers in the sample, which can make the estimate implode to zero (implosion breakdown point).

By means of simulations according to the FIRST SIMULATION PROCEDURE (1stSP), we are first going to see what the empirical value is for all estimates. The non-contaminated sample has been simulated from the 1stSP considering the two cases (see Page 38). Moreover, two sample sizes have been considered in the simulation study, namely, an even sample size ($n = 100$) and an odd sample size ($n = 101$).

Explosion breakdown point:

To study the breakdown point for explosion, we have considered the three types of outliers explained in Subsection 2.4.1. Namely, if $\widetilde{y}_i$ is the i-th outlier, then we have chosen

$$r_i^1 = \begin{cases} \frac{i+1}{2} \cdot 10^{10} & \text{if } i \text{ is odd} \\ \\ -\frac{i}{2} \cdot 10^{10} & \text{if } i \text{ is even} \end{cases}$$

that is, we have considered observations that are increasingly distant from the data at both right and left sides, and

$$r_i^2 = i \cdot 10^{10},$$

that is, observations that are getting wider.

For the estimator $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$ we have chosen

$$r_i^1 = i \cdot 10^{10},$$

that is, we have considered observations that are increasingly distant from the data only at the right side. In this way, we can see better what the explosion breakdown point is for this estimator.

For each type of outlier, the general scheme of the simulation has been structured as follows:

Step 1. A sample $\widetilde{\mathbf{x}}_n$ of n trapezoidal fuzzy numbers has been simulated from the 1stSP, with $n \in \{100, 101\}$ and considering the two cases of simulation involved in this procedure.

Step 2. Contaminated samples $\widetilde{\mathbf{y}}_{n,k}$ have been obtained by replacing k observations of the original sample $\widetilde{\mathbf{x}}_n$ by k outliers $\widetilde{y}_i$, with $k \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$ and $i \in \{1, \ldots, k\}$. Overall, k contaminated samples, one for each k value, have been considered.

Step 3. The values of the different scale measures have been calculated for the original sample without contamination $\widetilde{\mathbf{x}}_n$, and for each of k contaminated samples $\widetilde{\mathbf{y}}_{n,k}$.

The simulation-based conclusions in this study are presented through Tables 2.3 to 2.6 and Figures 2.6 to 2.17. More concretely, tables gather the values of the different estimators when outliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Implosion breakdown point:

To study the breakdown point for implosion, we have considered the inliers being all of them equal to one observation chosen randomly from the non-contaminated sample, but for the estimator $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, for which to make it implode to zero we have considered the inliers being all of them equal to the mean of the rest of observations.

The general scheme of the simulation has been structured as follows:

Step 1. A sample $\widetilde{\mathbf{x}}_n$ of n trapezoidal fuzzy numbers has been simulated from the 1stSP, with $n \in \{100, 101\}$ and considering the two cases of simulation involved in this procedure.

Step 2. Contaminated samples $\widetilde{\mathbf{y}}_{n,k}$ have been obtained by replacing k observations of the original sample $\widetilde{\mathbf{x}}_n$ by k inliers $\widetilde{y}$, with $k \in \{1, \ldots, n-1\}$. In total, k contaminated samples, one for each k value.

Step 3. The values of the different scale measures have been calculated for the original sample without contamination $\widetilde{\mathbf{x}}_n$, and for each of k contaminated samples $\widetilde{\mathbf{y}}_{n,k}$.

The simulation-based conclusions in this study are presented through Tables 2.7 to 2.10 and Figures 2.18 to 2.21. More concretely, tables gather the values of the different estimators when inliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Table 2.3: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 2, 10, 20, 30, 40, 48, 49, 50\}$ - 1stSP Case 1

# outliers (translation)	0	1	2	10	20	30	40	48	49	50
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.69	9.95E+08	1.41E+09	1.05E+10	2.77E+10	4.98E+10	7.58E+10	9.90E+10	1.02E+11	1.05E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.48	1.98E+08	2.00E+08	3.00E+09	1.10E+10	2.40E+10	4.20E+10	6.00E+10	6.38E+10	6.50E+10
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.31	1.00E+08	2.00E+08	3.00E+09	1.10E+10	2.40E+10	4.20E+10	6.00E+10	6.25E+10	6.50E+10
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.81	1.00E+08	2.00E+08	3.00E+09	1.10E+10	2.40E+10	4.20E+10	6.00E+10	6.25E+10	6.50E+10
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.32	1.00E+08	3.00E+08	5.50E+09	2.10E+10	4.65E+10	8.20E+10	1.18E+11	1.23E+11	1.28E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.09	1.10	1.11	1.17	1.25	1.37	1.76	2.92	3.79	5.00E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.59	1.57	1.57	1.71	1.96	2.33	2.94	3.97	4.85	5.00E+09
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.57	1.57	1.60	1.76	1.91	2.30	3.13	4.80	5.88	1.00E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.09	1.09	1.11	1.24	1.44	1.71	2.29	4.09	5.88	1.00E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.35	1.35	1.37	1.49	1.67	1.91	2.40	3.72	4.68	1.00E+10
# outliers (scale)	0	1	2	10	20	30	40	48	49	50
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.53	1.09E+09	2.44E+09	2.07E+10	5.42E+10	9.40E+10	1.37E+11	1.71E+11	1.75E+11	1.80E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.35	2.18E+08	6.47E+08	1.09E+10	3.72E+10	7.35E+10	1.15E+11	1.49E+11	1.53E+11	1.57E+11
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.18	1.10E+08	3.29E+08	6.02E+09	2.30E+10	5.09E+10	8.98E+10	1.29E+11	1.34E+11	1.40E+11
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.63	1.18E+08	3.54E+08	6.48E+09	2.47E+10	5.48E+10	9.66E+10	1.39E+11	1.44E+11	1.50E+11
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.18	1.10E+08	3.30E+08	6.05E+09	2.31E+10	5.12E+10	9.02E+10	1.29E+11	1.35E+11	1.40E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.98	1.01	1.02	1.08	1.33	1.65	2.23	3.42	4.26	5.48E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.46	1.49	1.51	1.62	1.93	2.25	3.44	6.29	7.97	5.89E+09
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.45	1.47	1.53	1.67	1.93	2.22	2.69	3.71	5.50	1.10E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.99	1.01	1.02	1.17	1.39	1.66	2.04	3.19	5.50	1.10E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.21	1.23	1.24	1.41	1.63	1.85	2.16	2.88	3.80	1.10E+10
# outliers (both)	0	1	2	10	20	30	40	48	49	50
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.07	1.68E+09	3.42E+09	2.84E+10	7.44E+10	1.29E+11	1.90E+11	2.39E+11	2.45E+11	2.52E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.55	3.34E+08	8.74E+08	1.45E+10	4.99E+10	9.96E+10	1.58E+11	2.07E+11	2.13E+11	2.20E+11
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.33	1.39E+08	4.18E+08	7.67E+09	2.93E+10	6.48E+10	1.14E+11	1.64E+11	1.71E+11	1.78E+11
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.85	2.34E+08	6.24E+08	1.07E+10	4.03E+10	8.89E+10	1.56E+11	2.24E+11	2.33E+11	2.43E+11
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.31	1.69E+08	5.07E+08	9.29E+09	3.55E+10	7.85E+10	1.39E+11	1.99E+11	2.07E+11	2.15E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.05	1.06	1.06	1.20	1.38	1.74	2.52	4.72	5.62	6.97E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.53	1.53	1.54	1.68	1.80	2.31	3.04	5.74	6.43	1.17E+10
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.46	1.52	1.52	1.72	1.89	2.28	3.00	4.64	5.88	1.39E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.05	1.06	1.07	1.22	1.37	1.70	2.33	3.85	5.88	1.39E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.26	1.28	1.28	1.47	1.61	1.89	2.43	3.64	4.27	1.39E+10

Table 2.4: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 2, 10, 20, 30, 40, 49, 50, 51\}$ - 1stSP Case 1

# outliers (translation)	0	1	2	10	20	30	40	49	50	51
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.01	9.90E+08	1.41E+09	1.04E+10	2.76E+10	4.96E+10	7.54E+10	1.02E+11	1.05E+11	1.08E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.66	1.96E+08	1.98E+08	2.97E+09	1.09E+10	2.38E+10	4.16E+10	6.31E+10	6.44E+10	6.82E+10
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.45	9.90E+07	1.98E+08	2.97E+09	1.09E+10	2.38E+10	4.16E+10	6.19E+10	6.44E+10	6.69E+10
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	2.05	9.90E+07	1.98E+08	2.97E+09	1.09E+10	2.38E+10	4.16E+10	6.19E+10	6.44E+10	6.69E+10
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.43	9.90E+07	2.97E+08	5.45E+09	2.08E+10	4.60E+10	8.12E+10	1.21E+11	1.26E+11	1.31E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.22	1.23	1.23	1.26	1.34	1.65	2.04	5.66	5.70	1.00E+10
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	1.77	1.75	1.80	1.90	2.12	2.57	3.49	5.48	5.79	1.00E+10
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.70	1.73	1.76	1.88	2.09	2.65	3.36	6.98	7.02	1.00E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.17	1.19	1.21	1.31	1.50	1.91	2.46	5.87	7.02	1.00E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.47	1.48	1.51	1.60	1.77	2.12	2.67	5.74	5.88	1.00E+10

# outliers (scale)	0	1	2	10	20	30	40	49	50	51
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.51	1.88E+09	4.18E+09	3.55E+10	9.31E+10	1.61E+11	2.35E+11	3.02E+11	3.09E+11	3.16E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.82	3.72E+08	1.10E+09	1.86E+10	6.37E+10	1.26E+11	1.97E+11	2.62E+11	2.69E+11	2.76E+11
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.58	1.87E+08	5.62E+08	1.03E+10	3.93E+10	8.71E+10	1.54E+11	2.29E+11	2.39E+11	2.48E+11
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	2.08	1.97E+08	5.92E+08	1.08E+10	4.14E+10	9.17E+10	1.62E+11	2.42E+11	2.51E+11	2.61E+11
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.52	1.88E+08	5.63E+08	1.03E+10	3.94E+10	8.73E+10	1.54E+11	2.30E+11	2.39E+11	2.49E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.04	1.05	1.09	1.33	1.62	1.99	3.14	6.70	7.23	1.89E+10
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	1.58	1.61	1.67	1.83	2.05	2.60	4.21	9.86	11.35	1.99E+10
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.48	1.49	1.51	1.81	2.13	2.60	3.61	6.34	7.12	1.89E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.05	1.08	1.11	1.29	1.46	1.78	2.73	6.08	7.12	1.89E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.24	1.27	1.30	1.49	1.67	2.02	2.88	5.37	6.08	1.89E+10

# outliers (both)	0	1	2	10	20	30	40	49	50	51
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.20	4.44E+09	9.64E+09	8.18E+10	2.14E+11	3.72E+11	5.41E+11	6.97E+11	7.14E+11	7.31E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.79	8.79E+08	2.54E+09	4.27E+10	1.46E+11	2.90E+11	4.54E+11	6.07E+11	6.23E+11	6.40E+11
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.57	4.28E+08	1.28E+09	2.35E+10	8.99E+10	1.99E+11	3.51E+11	5.24E+11	5.46E+11	5.67E+11
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	2.11	5.44E+08	1.47E+09	2.64E+10	1.01E+11	2.23E+11	3.92E+11	5.87E+11	6.10E+11	6.34E+11
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.52	4.44E+08	1.33E+09	2.44E+10	9.32E+10	2.06E+11	3.64E+11	5.44E+11	5.66E+11	5.89E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.22	1.23	1.21	1.41	1.74	2.11	3.28	8.95	11.50	4.32E+10
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	1.60	1.63	1.66	1.79	1.95	2.59	3.59	10.30	13.73	4.32E+10
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.72	1.75	1.76	1.90	2.13	2.56	3.71	8.76	11.50	4.32E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.17	1.19	1.21	1.34	1.58	1.87	2.62	8.51	11.50	4.32E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.45	1.48	1.49	1.64	1.81	2.06	2.78	7.42	9.95	4.32E+10

Table 2.5: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 2, 10, 20, 30, 40, 48, 49, 50\}$ - 1stSP Case 2

# outliers (translation)	0	1	2	10	20	30	40	48	49	50
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.01	9.95E+08	1.41E+09	1.05E+10	2.77E+10	4.98E+10	7.58E+10	9.90E+10	1.02E+11	1.05E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	0.91	1.98E+08	2.00E+08	3.00E+09	1.10E+10	2.40E+10	4.20E+10	6.00E+10	6.38E+10	6.50E+10
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.75	1.00E+08	2.00E+08	3.00E+09	1.10E+10	2.40E+10	4.20E+10	6.00E+10	6.25E+10	6.50E+10
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.17	1.00E+08	2.00E+08	3.00E+09	1.10E+10	2.40E+10	4.20E+10	6.00E+10	6.25E+10	6.50E+10
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	0.79	1.00E+08	3.00E+08	5.50E+09	2.10E+10	4.65E+10	8.20E+10	1.18E+11	1.23E+11	1.28E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.69	0.69	0.70	0.75	0.90	0.92	1.03	1.67	1.88	5.00E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	0.99	1.00	1.04	1.16	1.29	1.46	1.57	1.81	2.09	5.00E+09
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.91	0.96	0.97	1.09	1.30	1.45	1.80	3.02	3.76	1.00E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.50	0.51	0.52	0.63	0.78	1.02	1.36	2.28	3.76	1.00E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.80	0.82	0.83	0.95	1.08	1.22	1.41	2.16	2.58	1.00E+10
# outliers (scale)	0	1	2	10	20	30	40	48	49	50
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.07	7.89E+08	1.76E+09	1.49E+10	3.91E+10	6.77E+10	9.84E+10	1.23E+11	1.26E+11	1.29E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	0.94	1.57E+08	4.66E+08	7.85E+09	2.68E+10	5.30E+10	8.27E+10	1.07E+11	1.10E+11	1.13E+11
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.79	7.77E+07	2.33E+08	4.27E+09	1.63E+10	3.61E+10	6.37E+10	9.14E+10	9.52E+10	9.91E+10
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.19	7.93E+07	2.38E+08	4.36E+09	1.67E+10	3.69E+10	6.50E+10	9.33E+10	9.72E+10	1.01E+11
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	0.75	7.93E+07	2.38E+08	4.36E+09	1.66E+10	3.69E+10	6.50E+10	9.32E+10	9.71E+10	1.01E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.61	0.62	0.62	0.71	0.82	1.13	1.47	2.01	2.33	3.89E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	0.93	0.92	0.97	1.14	1.23	1.99	2.40	4.98	5.72	3.97E+09
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.87	0.87	0.87	0.97	1.13	1.41	2.16	3.64	4.61	7.77E+09
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.54	0.54	0.55	0.65	0.78	1.01	1.42	2.74	4.61	7.77E+09
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.77	0.77	0.77	0.85	0.94	1.15	1.53	2.52	3.01	7.77E+09
# outliers (both)	0	1	2	10	20	30	40	48	49	50
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.30	9.99E+08	1.52E+09	1.14E+10	3.00E+10	5.36E+10	8.11E+10	1.05E+11	1.08E+11	1.12E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	1.13	1.99E+08	2.82E+08	4.37E+09	1.56E+10	3.29E+10	5.55E+10	7.70E+10	7.97E+10	8.28E+10
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.96	9.78E+07	2.02E+08	3.01E+09	1.10E+10	2.40E+10	4.20E+10	6.01E+10	6.24E+10	6.51E+10
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.38	1.21E+08	2.71E+08	4.27E+09	1.58E+10	3.47E+10	6.08E+10	8.70E+10	9.05E+10	9.42E+10
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\overline{\mathbf{x}}}_n)$	0.94	1.00E+08	3.01E+08	5.52E+09	2.11E+10	4.67E+10	8.23E+10	1.18E+11	1.23E+11	1.28E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.86	0.86	0.89	0.93	1.08	1.31	1.53	2.89	2.91	4.89E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.20	1.21	1.21	1.23	1.58	2.21	2.81	4.32	4.52	6.03E+09
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.10	1.10	1.15	1.29	1.53	2.03	2.71	5.73	5.82	9.78E+09
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.65	0.67	0.69	0.80	1.02	1.32	1.87	3.71	5.82	9.78E+09
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.95	0.95	0.97	1.05	1.29	1.52	2.04	3.80	3.88	9.78E+09

Table 2.6: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 2, 10, 20, 30, 40, 49, 50, 51\}$ - 1stSP Case 2

# outliers (translation)	0	1	2	10	20	30	40	49	50	51
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.14	9.90E+08	1.41E+09	1.04E+10	2.76E+10	4.96E+10	7.54E+10	1.02E+11	1.05E+11	1.08E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.98	1.96E+08	1.98E+08	2.97E+09	1.09E+10	2.38E+10	4.16E+10	6.31E+10	6.44E+10	6.82E+10
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.78	9.90E+07	1.98E+08	2.97E+09	1.09E+10	2.38E+10	4.16E+10	6.19E+10	6.44E+10	6.69E+10
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.22	9.90E+07	1.98E+08	2.97E+09	1.09E+10	2.38E+10	4.16E+10	6.19E+10	6.44E+10	6.69E+10
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.73	9.90E+07	2.97E+08	5.45E+09	2.08E+10	4.60E+10	8.12E+10	1.21E+11	1.26E+11	1.31E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.57	0.57	0.57	0.67	0.77	1.05	1.27	2.21	2.63	1.00E+10
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	0.81	0.81	0.81	0.89	1.12	1.24	1.62	2.19	2.79	1.00E+10
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.80	0.80	0.80	0.88	1.10	1.45	2.22	4.17	4.84	1.00E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.51	0.51	0.52	0.58	0.75	1.02	1.44	2.89	4.84	1.00E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.69	0.69	0.69	0.74	0.92	1.19	1.63	2.78	3.21	1.00E+10
# outliers (scale)	**0**	**1**	**2**	**10**	**20**	**30**	**40**	**49**	**50**	**51**
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.08	2.62E+08	5.83E+08	4.96E+09	1.30E+10	2.25E+10	3.27E+10	4.21E+10	4.31E+10	4.41E+10
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.96	5.18E+07	1.54E+08	2.59E+09	8.88E+09	1.76E+10	2.74E+10	3.66E+10	3.76E+10	3.85E+10
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.78	2.55E+07	7.66E+07	1.40E+09	5.36E+09	1.19E+10	2.09E+10	3.13E+10	3.25E+10	3.38E+10
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.20	2.73E+07	8.20E+07	1.50E+09	5.74E+09	1.27E+10	2.24E+10	3.35E+10	3.49E+10	3.62E+10
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.79	2.62E+07	7.85E+07	1.44E+09	5.50E+09	1.22E+10	2.15E+10	3.21E+10	3.34E+10	3.47E+10
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.65	0.67	0.67	0.73	0.99	1.24	1.37	2.21	2.73	2.58E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	0.97	0.97	0.95	1.00	1.28	1.81	3.17	5.13	6.55	2.76E+09
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.88	0.90	0.90	1.07	1.29	1.70	2.29	3.87	4.92	2.58E+09
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.55	0.56	0.57	0.66	0.81	1.12	1.50	2.98	4.92	2.58E+09
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.77	0.78	0.79	0.86	1.03	1.32	1.74	2.73	3.26	2.58E+09
# outliers (both)	**0**	**1**	**2**	**10**	**20**	**30**	**40**	**49**	**50**	**51**
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.03	1.39E+09	2.60E+09	2.13E+10	5.58E+10	9.76E+10	1.44E+11	1.87E+11	1.92E+11	1.97E+11
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.94	2.75E+08	6.42E+08	1.06E+10	3.65E+10	7.35E+10	1.18E+11	1.61E+11	1.66E+11	1.71E+11
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.78	1.04E+08	2.98E+08	5.32E+09	2.03E+10	4.49E+10	7.92E+10	1.18E+11	1.23E+11	1.28E+11
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.17	1.95E+08	4.88E+08	8.29E+09	3.12E+10	6.87E+10	1.21E+11	1.80E+11	1.88E+11	1.95E+11
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.83	1.39E+08	4.16E+08	7.63E+09	2.91E+10	6.45E+10	1.14E+11	1.70E+11	1.77E+11	1.84E+11
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.71	0.74	0.74	0.82	0.95	1.12	1.32	1.76	1.81	5.55E+09
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	0.99	1.02	1.01	1.09	1.48	1.76	1.99	3.03	3.41	9.93E+09
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.94	0.97	0.97	1.10	1.23	1.50	2.11	3.15	3.40	1.05E+10
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.52	0.53	0.54	0.64	0.79	1.01	1.39	2.31	3.40	1.05E+10
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.83	0.85	0.85	0.95	1.05	1.24	1.56	2.18	2.44	1.05E+10

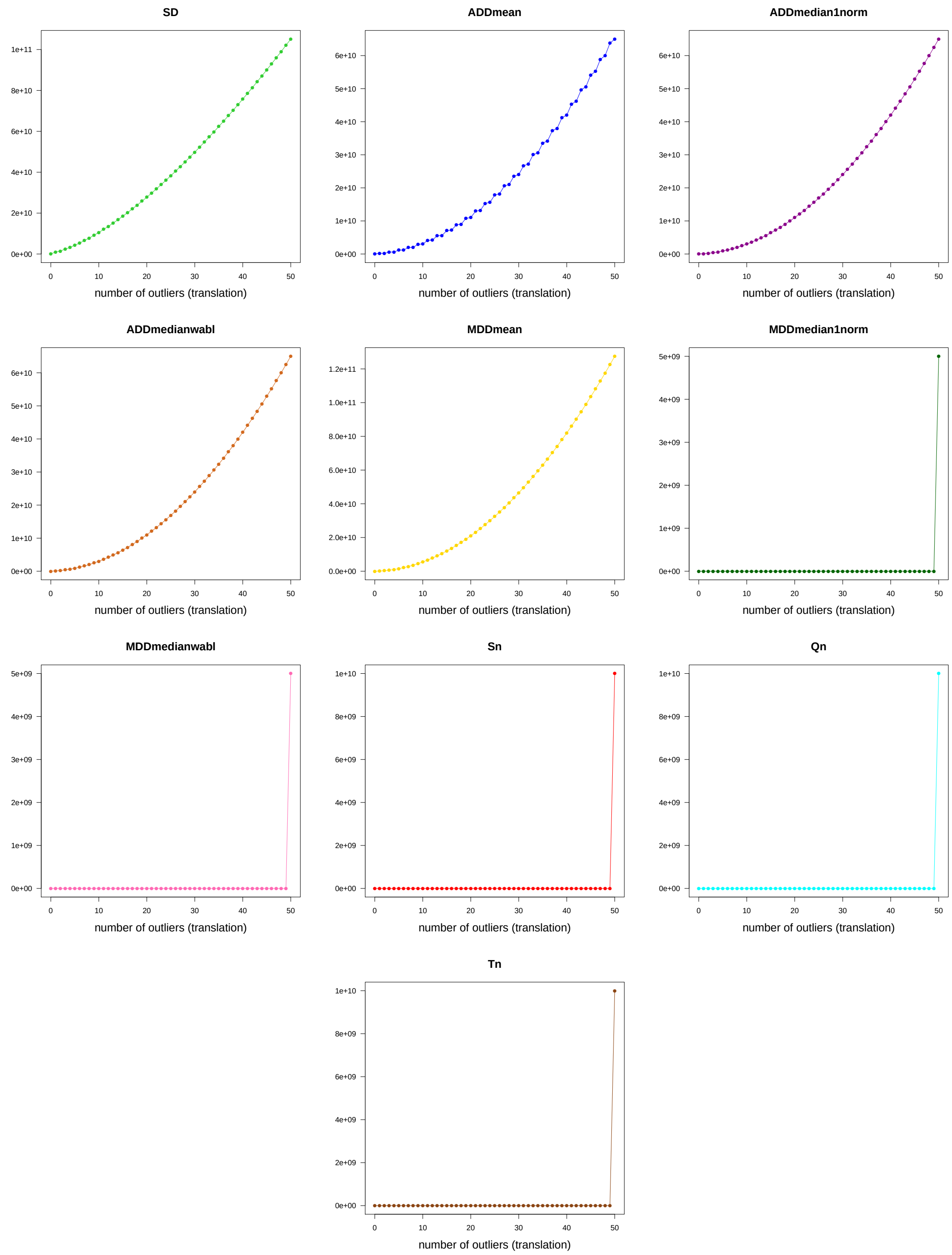


Figure 2.6: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of translation, k varying from 0 to 50 - 1stSP Case 1

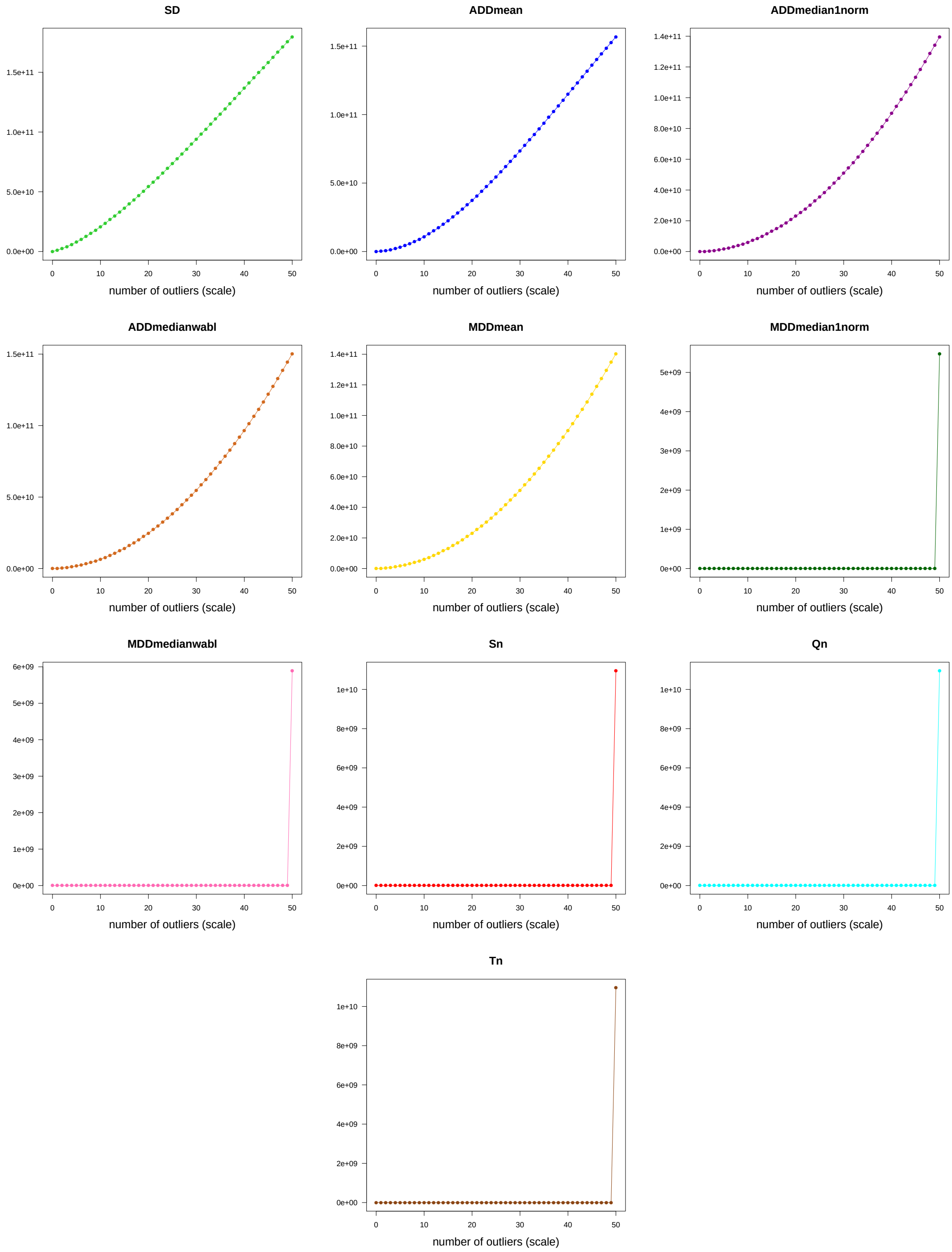


Figure 2.7: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of scale on core and support, k varying from 0 to 50 - 1stSP Case 1

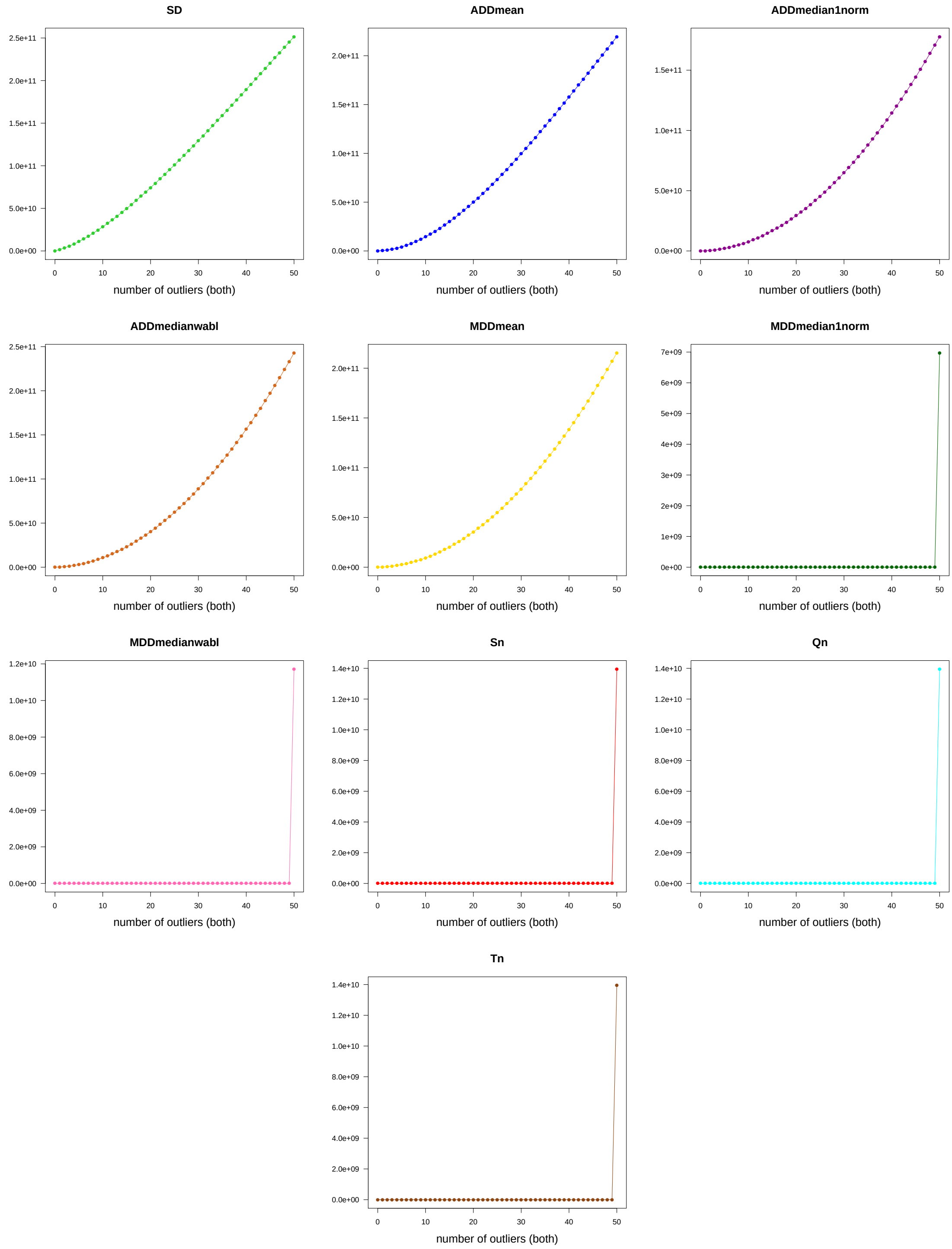


Figure 2.8: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 50 - 1stSP Case 1

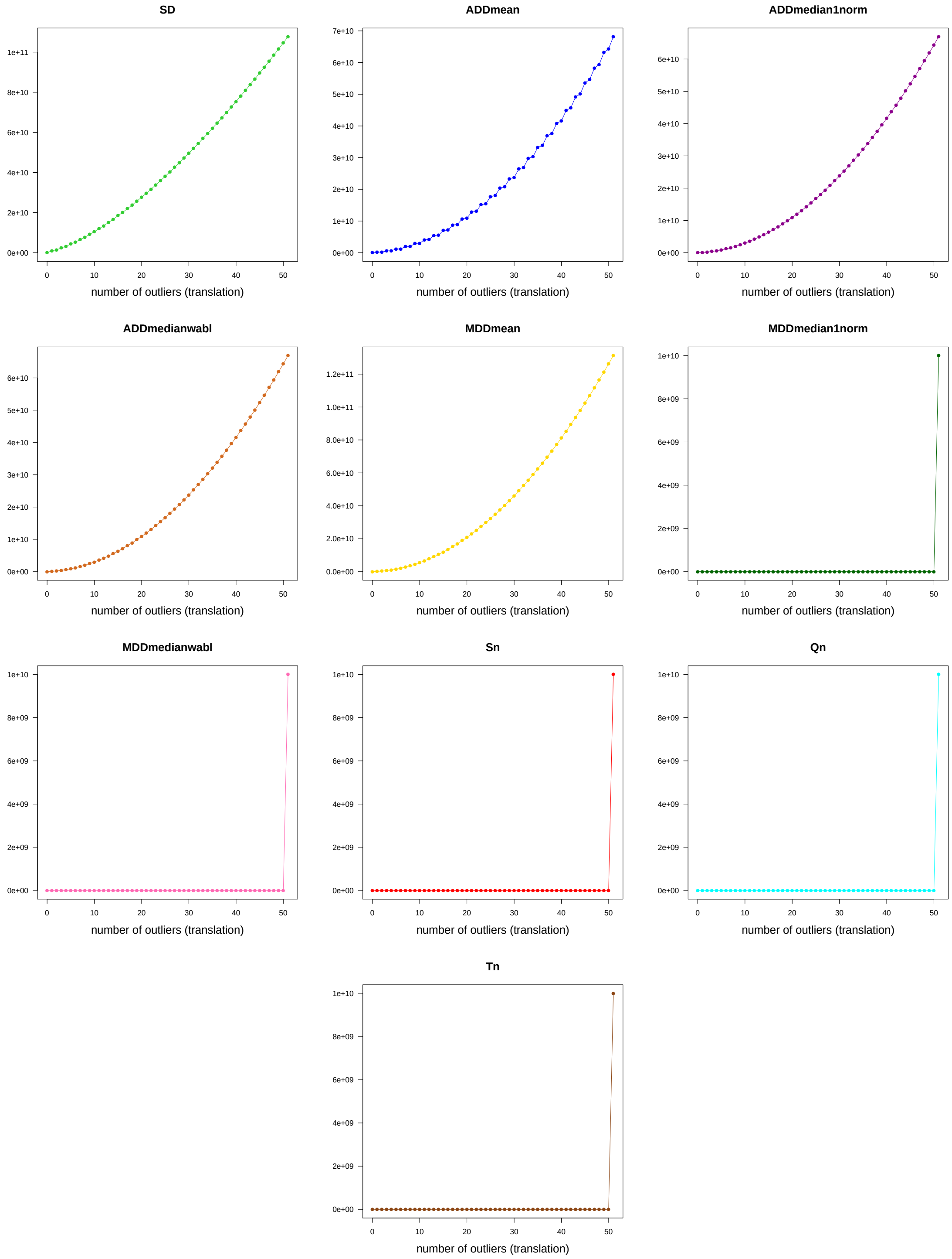


Figure 2.9: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of translation, k varying from 0 to 51 - 1stSP Case 1

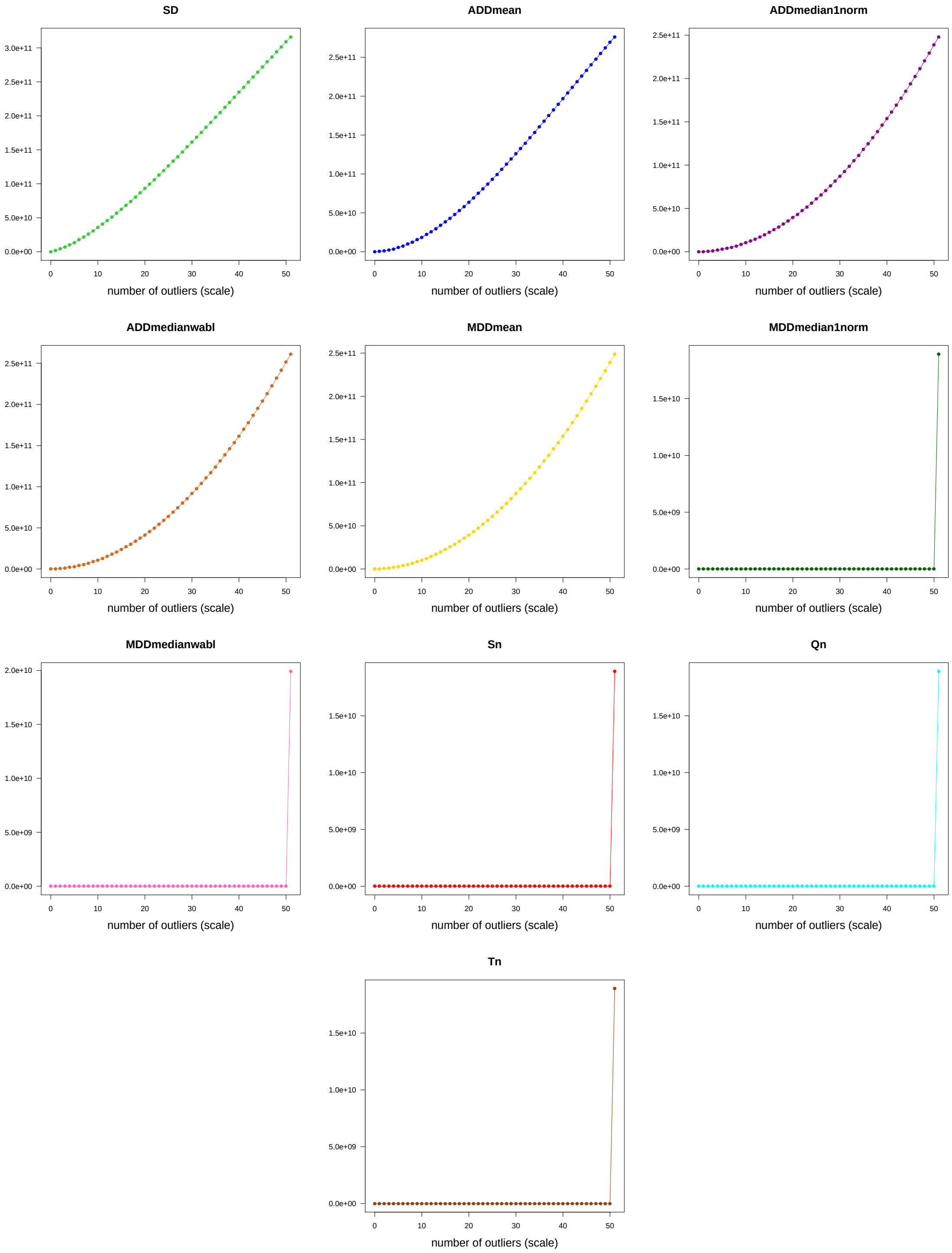


Figure 2.10: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of scale on core and support, k varying from 0 to 51 - 1stSP Case 1

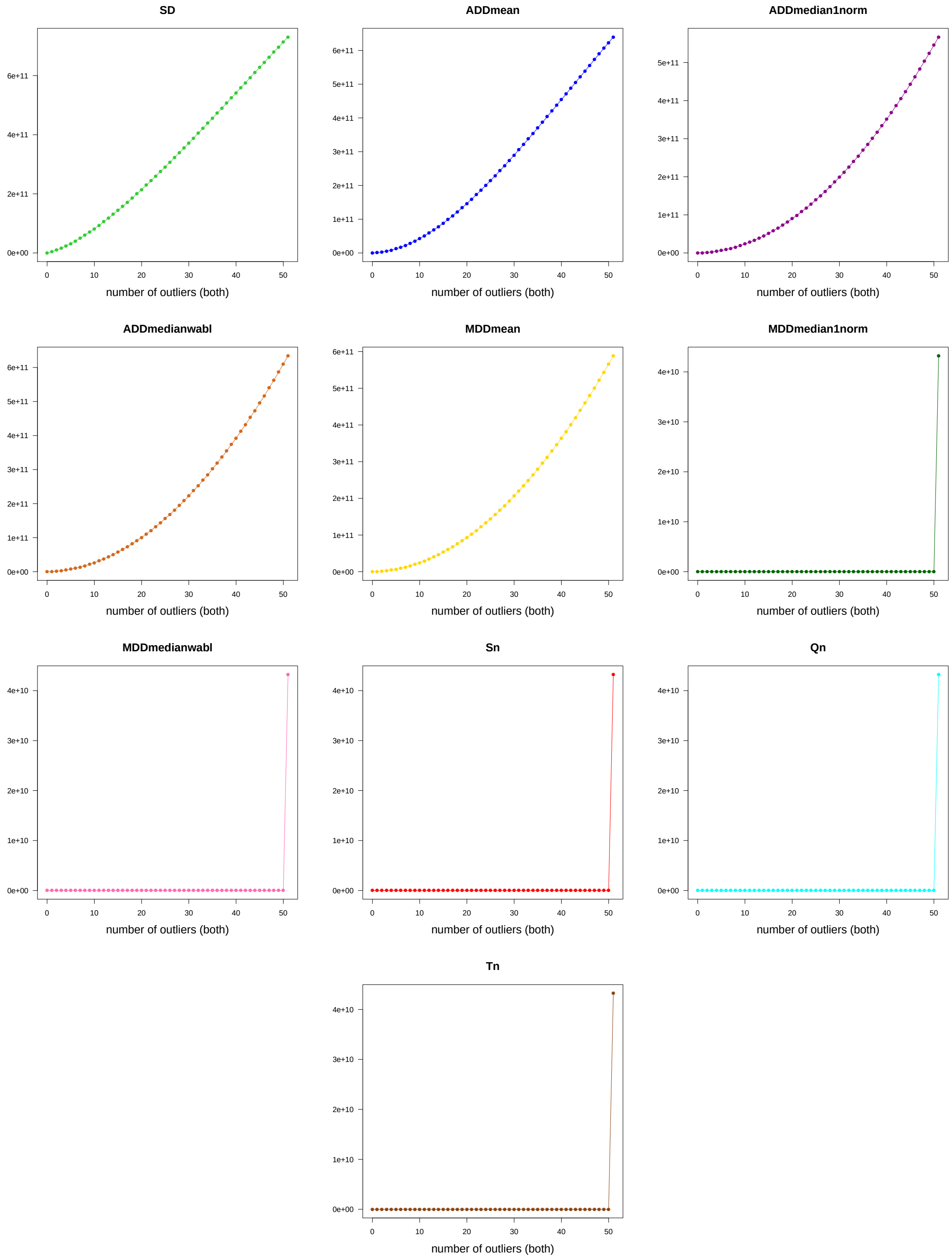


Figure 2.11: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 51 - 1stSP CASE1

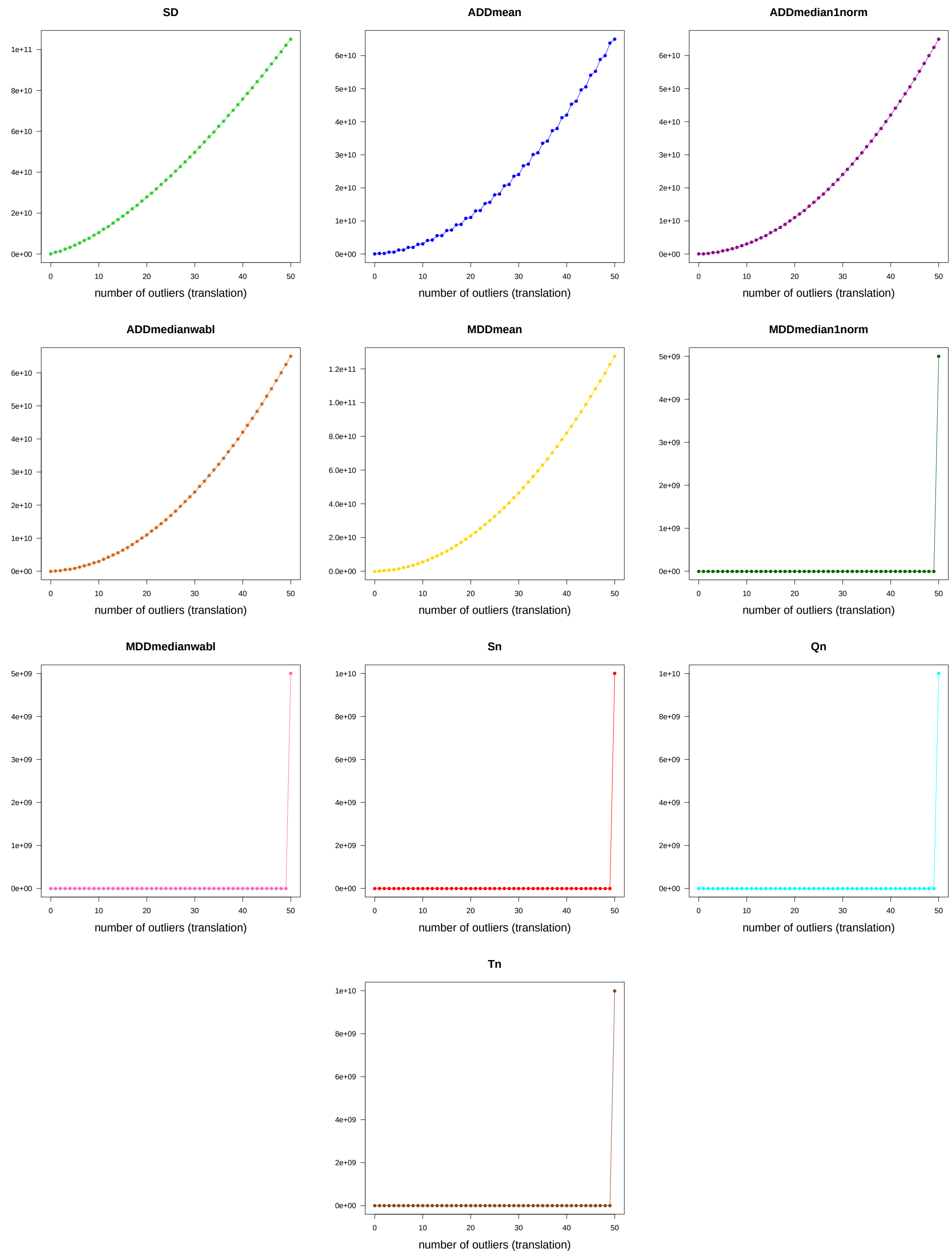


Figure 2.12: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of translation, k varying from 0 to 50 - 1stSP Case 2

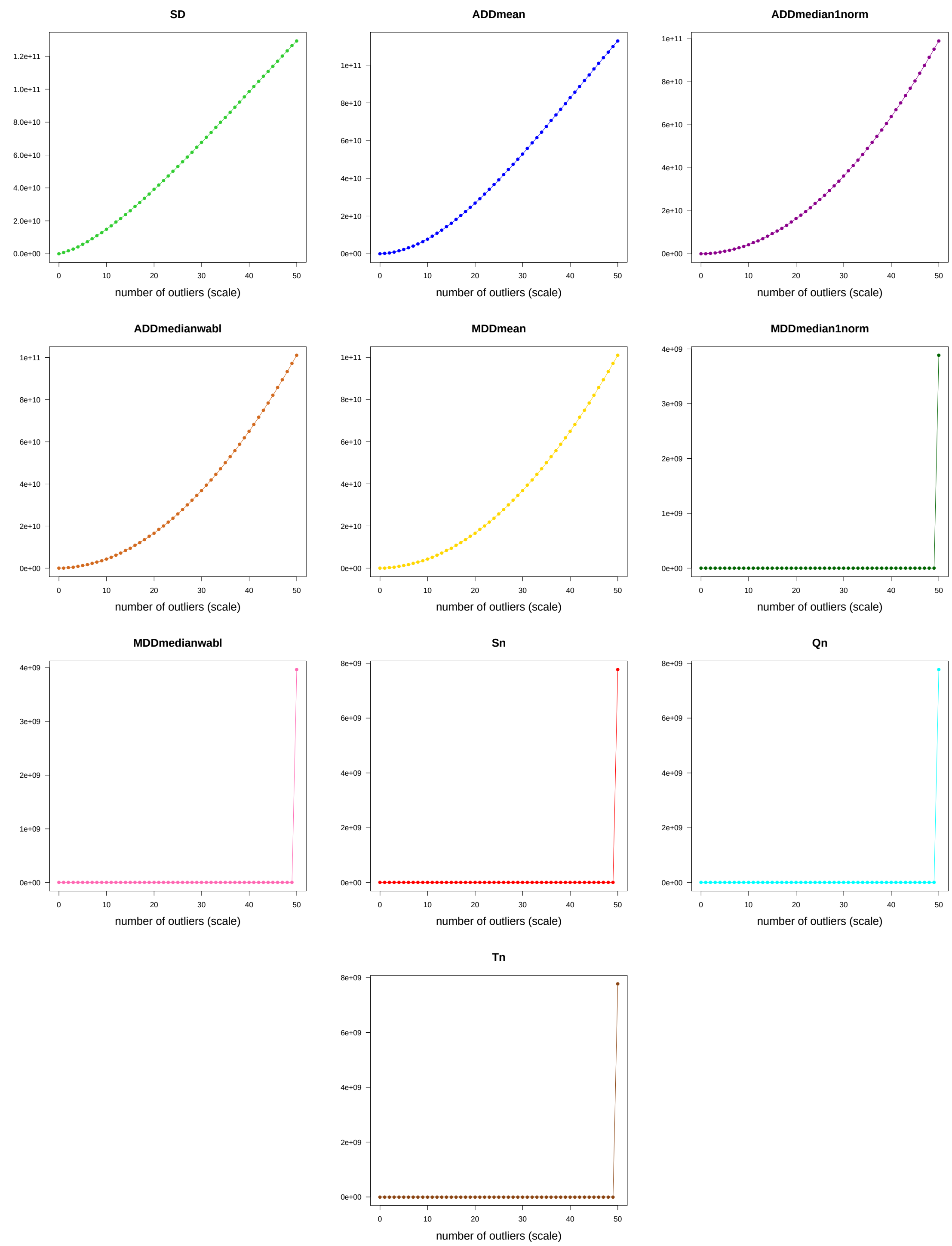


Figure 2.13: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of scale on core and support, k varying from 0 to 50 - 1stSP Case 2

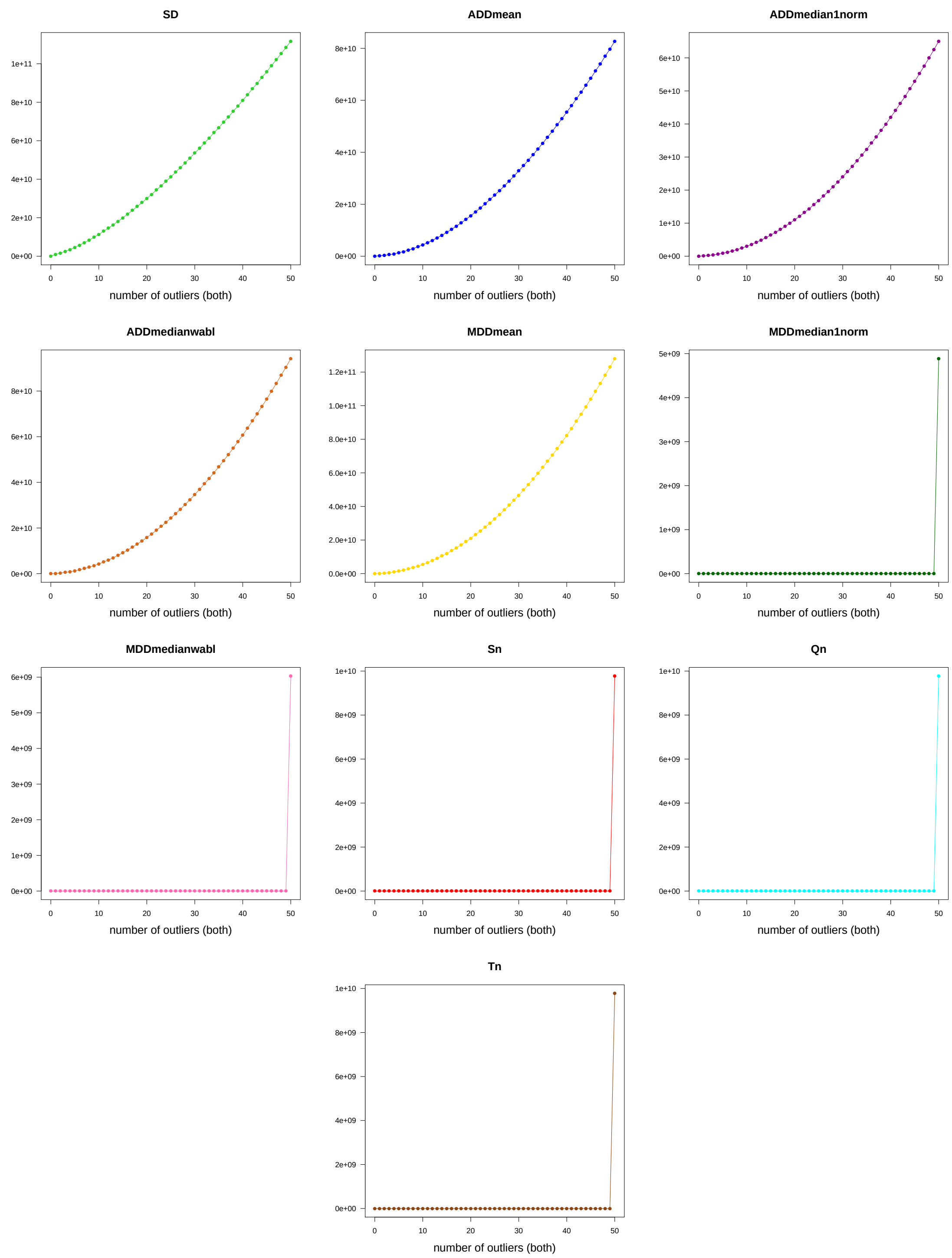


Figure 2.14: Explosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 50 - 1stSP Case 2

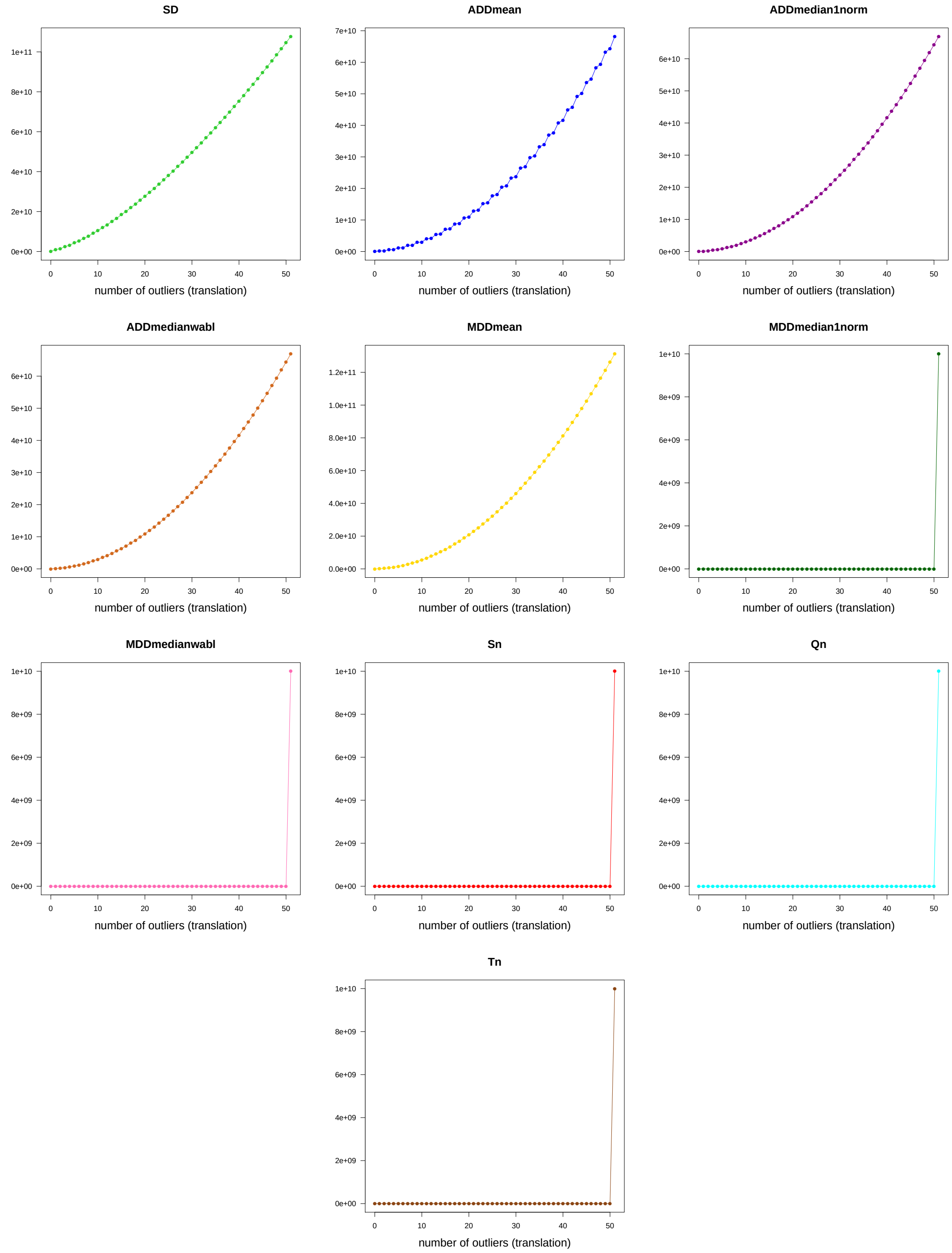


Figure 2.15: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of translation, k varying from 0 to 51 - 1stSP Case 2

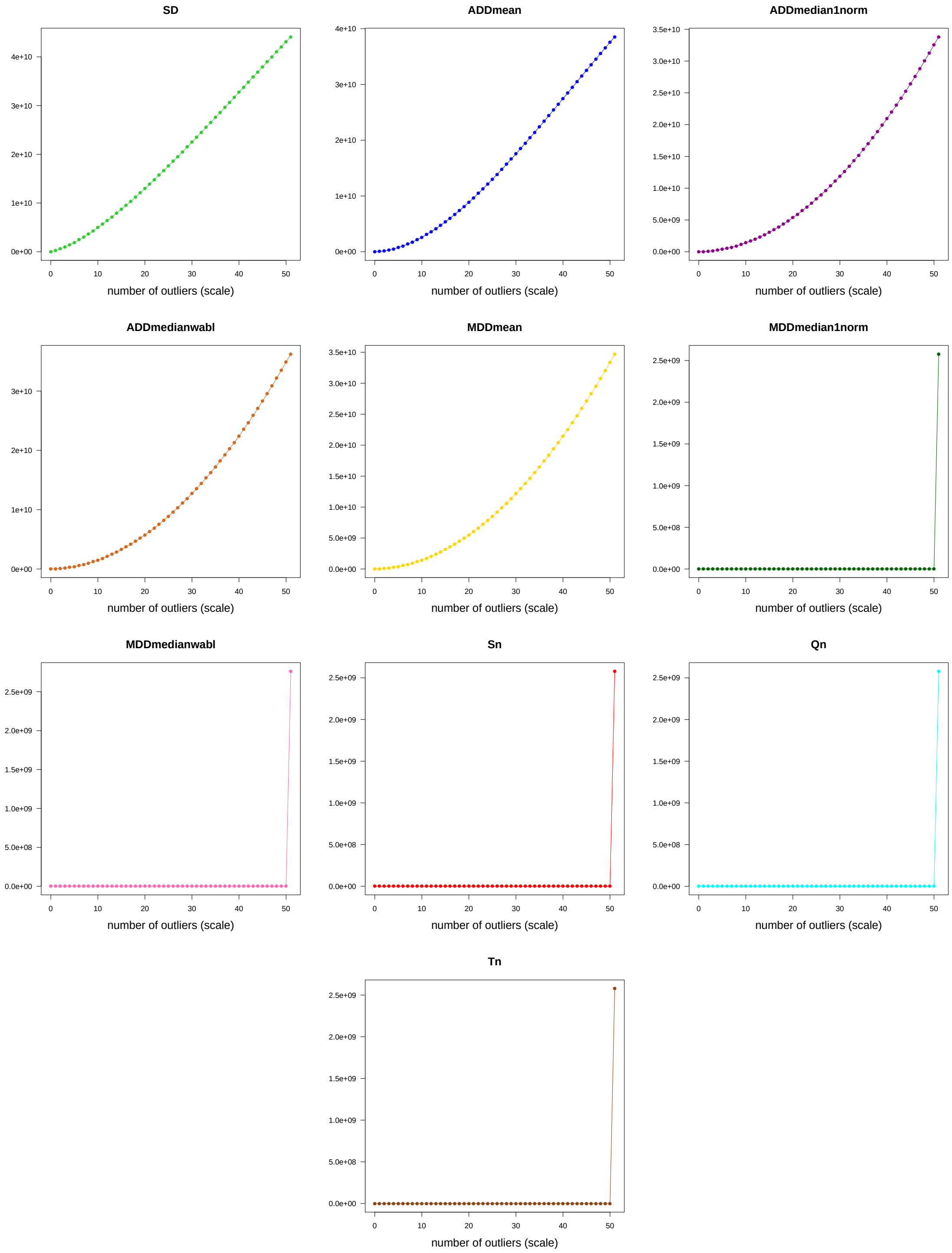


Figure 2.16: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of scale on core and support, k varying from 0 to 51 - 1stSP Case 2

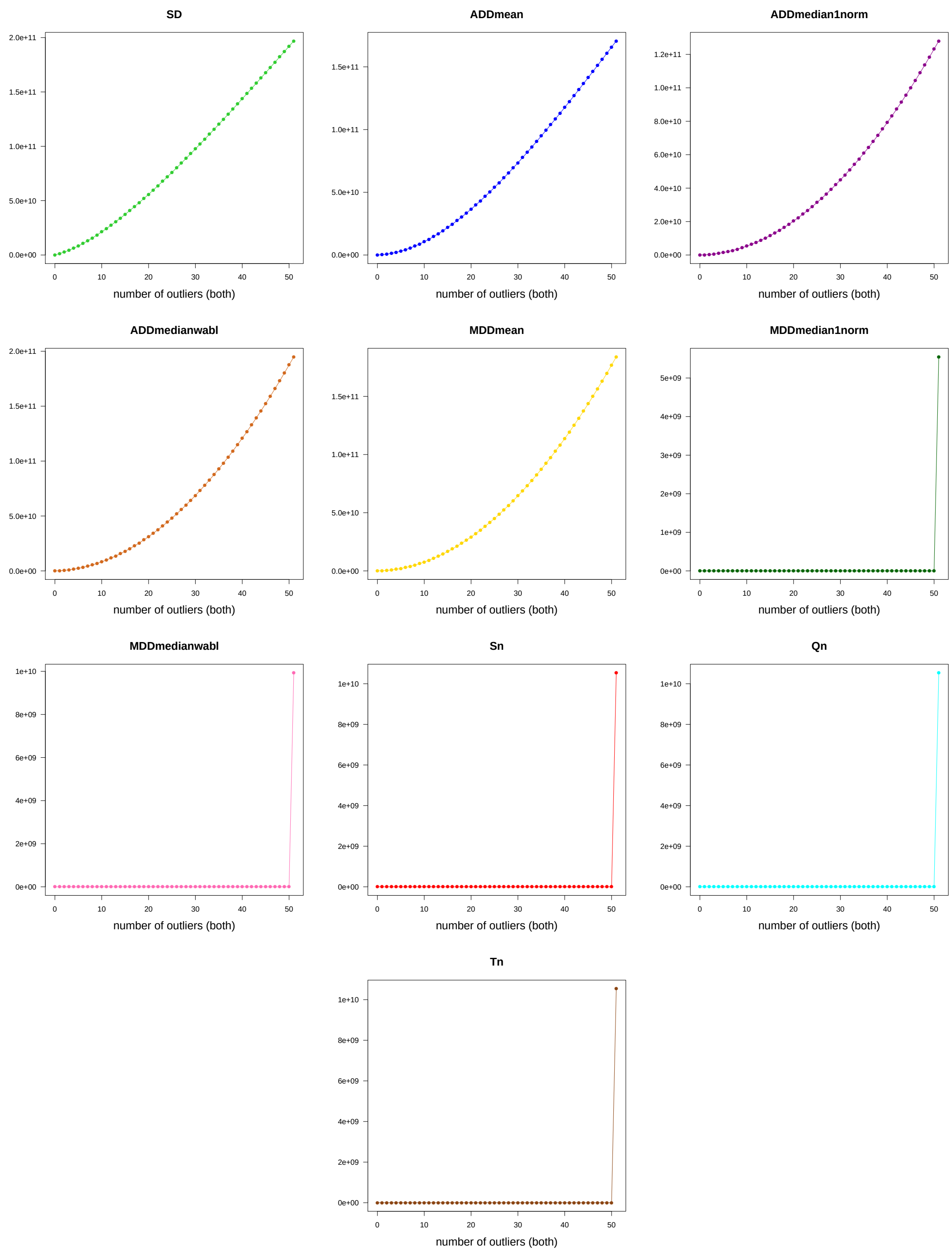


Figure 2.17: Explosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 51 - 1stSP Case 2

Table 2.7: Implosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by inliers, with $k \in \{0, 1, 2, 10, 49, 50, 51, 98, 99\}$ - 1stSP Case 1

# inliers	0	1	2	10	49	50	51	98	99
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.93	1.93	1.93	1.92	1.56	1.56	1.55	0.31	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.62	1.62	1.62	1.61	1.15	1.15	1.12	0.06	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	1.40	1.40	1.40	1.38	0.86	0.85	0.83	0.03	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.96	1.96	1.95	1.91	1.13	1.12	1.09	0.04	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.34	1.35	1.36	1.37	0.43	0.18	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	1.08	1.09	1.08	0.95	0.11	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.50	1.50	1.50	1.33	0.16	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.44	1.44	1.44	1.34	0.22	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.02	1.02	1.01	0.97	0.22	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.22	1.22	1.20	1.12	0.22	0	0	0	0

Table 2.8: Implosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by inliers, with $k \in \{0, 1, 2, 10, 49, 50, 51, 99, 100\}$ - 1stSP Case 1

# inliers	0	1	2	10	49	50	51	99	100
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.80	1.80	1.80	1.89	1.96	1.95	1.95	0.29	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.45	1.45	1.45	1.58	1.70	1.69	1.69	0.06	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	1.27	1.27	1.27	1.37	1.34	1.32	1.30	0.02	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.75	1.75	1.76	1.92	1.98	1.91	1.90	0.04	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.25	1.23	1.21	1.14	0.50	0.35	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	1.02	1.02	1.01	1.10	0.29	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.38	1.40	1.42	1.66	0.41	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.40	1.40	1.40	1.60	0.29	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.02	1.02	1.02	1.12	0.29	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.21	1.21	1.21	1.35	0.29	0	0	0	0

Table 2.9: Implosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by inliers, with $k \in \{0, 1, 2, 10, 49, 50, 51, 98, 99\}$ - 1stSP Case 2

# inliers	0	1	2	10	49	50	51	98	99
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.20	1.20	1.20	1.20	1.15	1.15	1.15	0.23	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.05	1.04	1.04	1.06	1.02	1.02	1.02	0.04	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.87	0.87	0.87	0.89	0.79	0.78	0.78	0.02	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.28	1.28	1.28	1.29	1.11	1.09	1.08	0.03	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.81	0.79	0.79	0.78	0.36	0.15	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.72	0.71	0.71	0.75	0.08	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	0.98	0.98	0.98	1.14	0.13	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.94	0.94	0.94	1.05	0.16	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.59	0.59	0.59	0.59	0.13	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.83	0.83	0.83	0.89	0.16	0	0	0	0

Table 2.10: Implosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by inliers, with $k \in \{0, 1, 2, 10, 49, 50, 51, 99, 100\}$ - 1stSP Case 2

# inliers	0	1	2	10	49	50	51	99	100
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.16	1.16	1.16	1.13	0.92	0.91	0.91	0.16	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.04	1.04	1.03	1.01	0.71	0.69	0.69	0.03	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.86	0.86	0.86	0.85	0.54	0.53	0.53	0.02	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.30	1.29	1.28	1.23	0.76	0.74	0.74	0.02	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.82	0.83	0.83	0.80	0.47	0.43	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.73	0.73	0.72	0.69	0.09	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.07	1.08	1.09	1.05	0.14	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.99	0.99	1.00	1.01	0.09	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.61	0.61	0.61	0.61	0.09	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.88	0.89	0.88	0.86	0.09	0	0	0	0

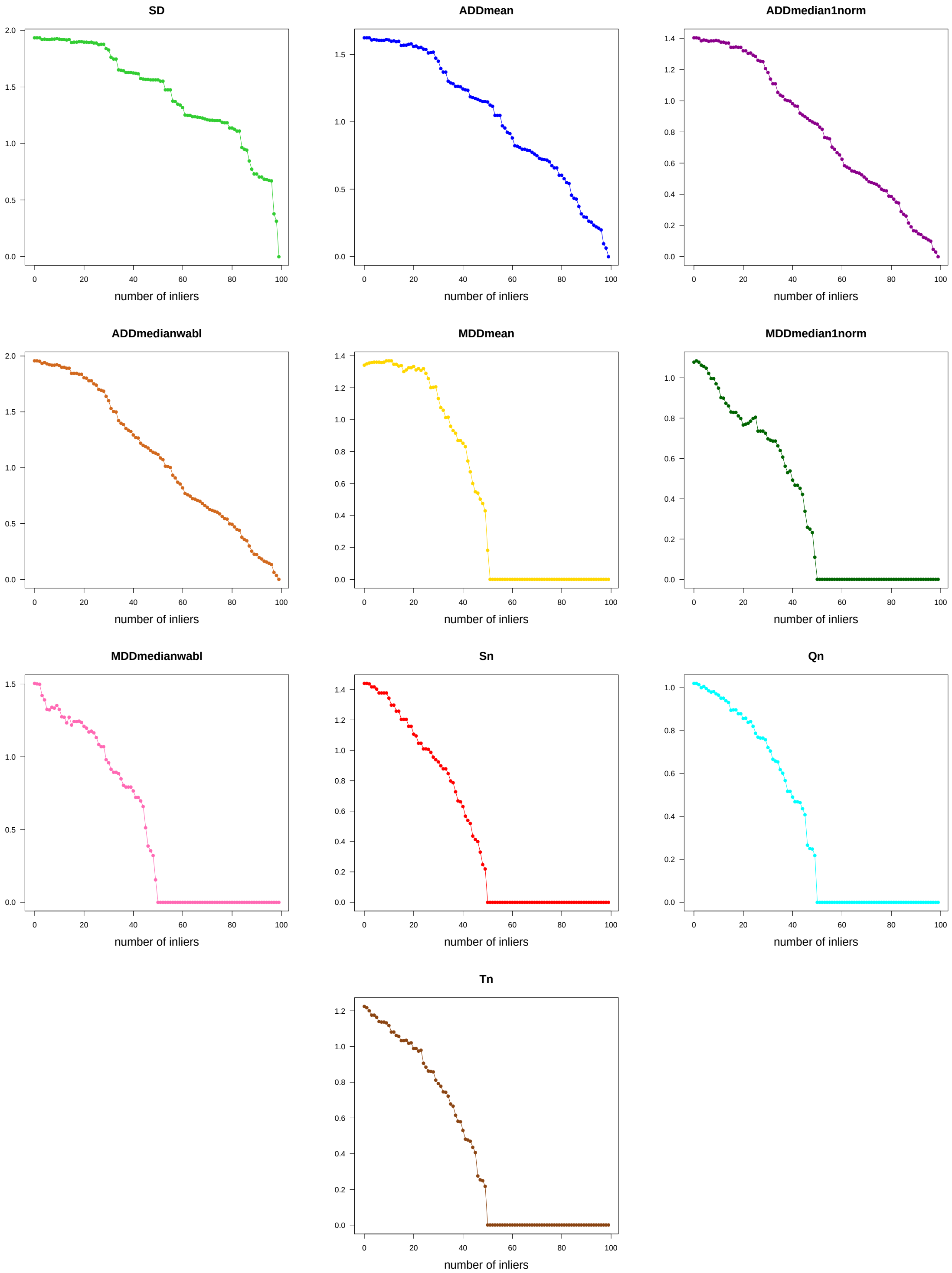


Figure 2.18: Implosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by inliers, k varying from 0 to 99 - 1stSP Case 1

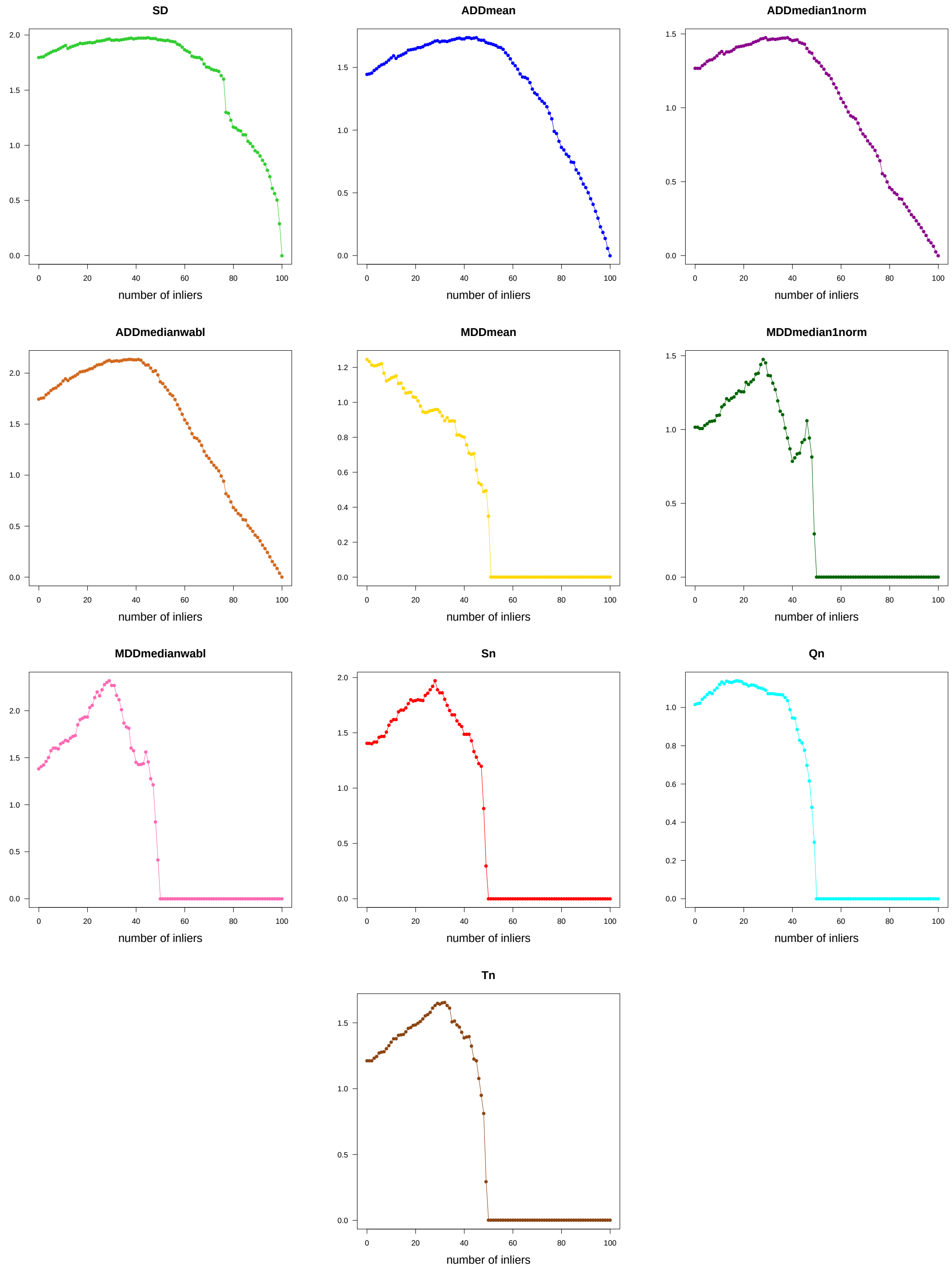


Figure 2.19: Implosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by inliers, k varying from 0 to 100 - 1stSP Case 1

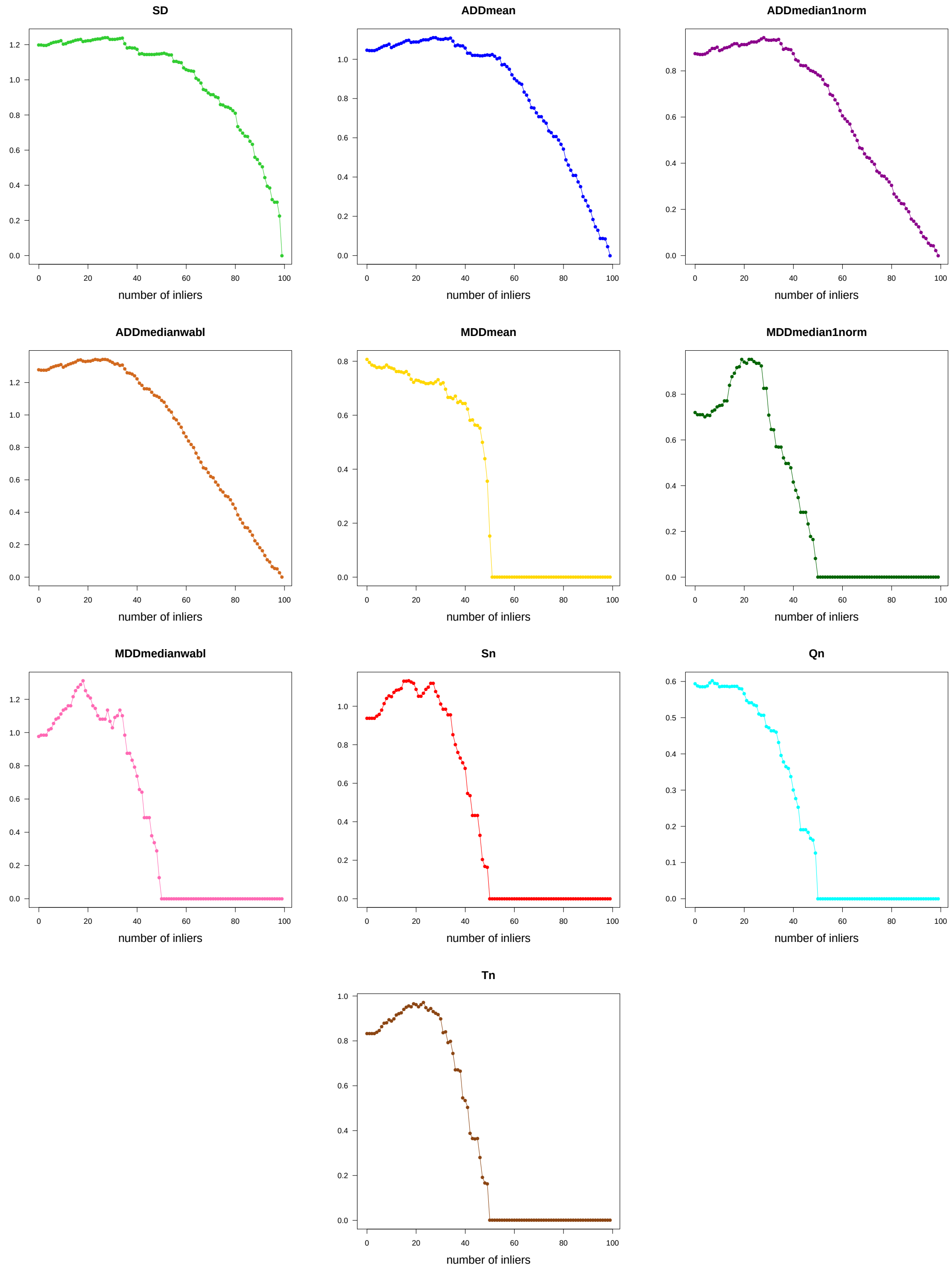


Figure 2.20: Implosion breakdown point: values of the scale estimators for a sample of size 100 with k observations replaced by inliers, k varying from 0 to 99 - 1stSP Case 2

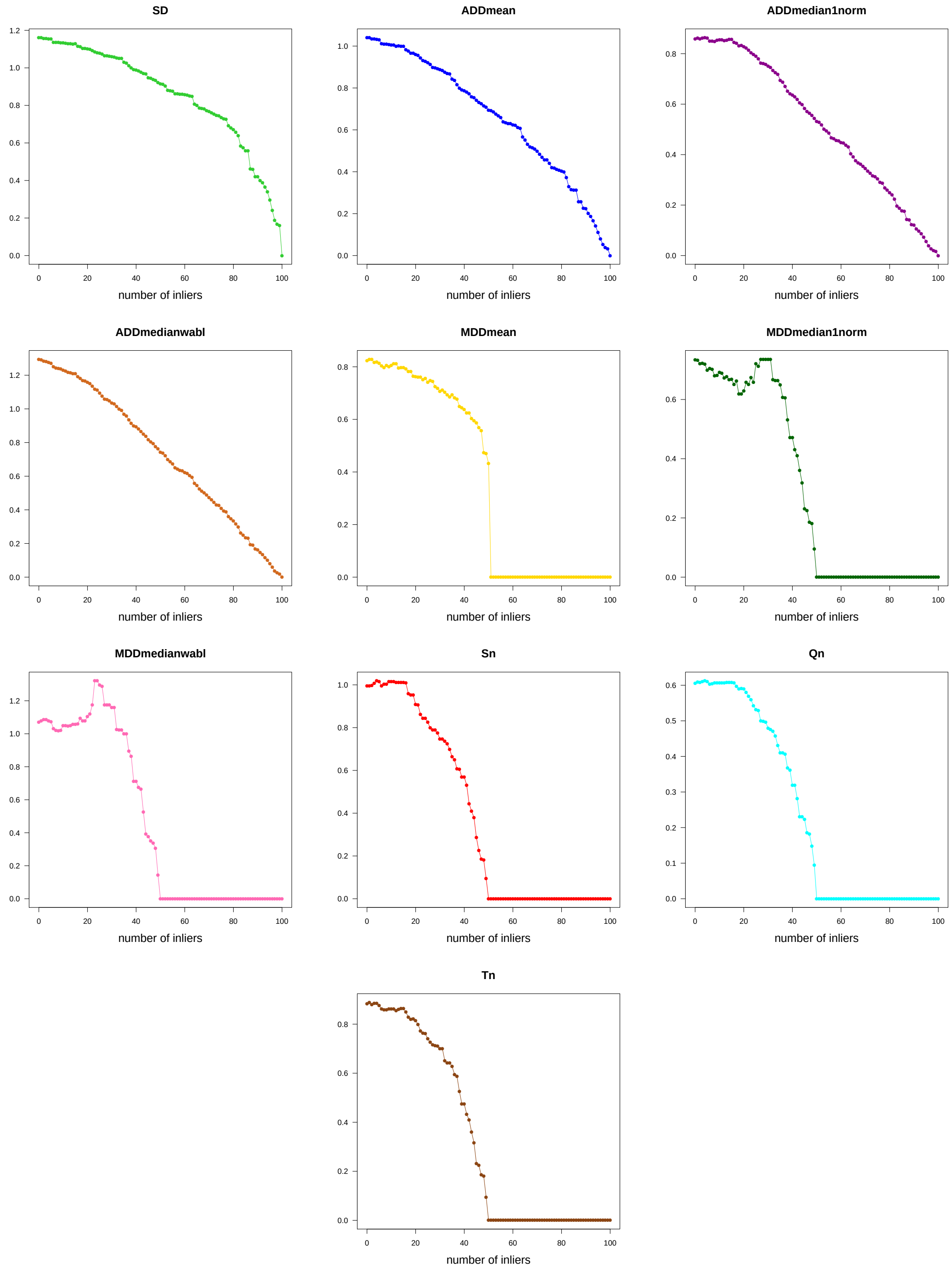


Figure 2.21: Implosion breakdown point: values of the scale estimators for a sample of size 101 with k observations replaced by inliers, k varying from 0 to 100 - 1stSP Case 2

Regarding explosion we can conclude that for the sample size $n = 100$ (101), by looking at Table 2.3 (2.4) and Figures 2.6 to 2.8 (2.9 to 2.11) for Case 1 and Table 2.5 (2.6) and Figures 2.12 to 2.14 (2.15 to 2.17) for Case 2, we can see that the minimum number of perturbed observations by outliers that makes the estimator to increase arbitrarily, independently of the simulation case and the type of outlier considered, has been

- 1 for the estimators $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$;
- 50 (51) for the estimators $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$.

Therefore, these results show that the empirical value for the explosion breakdown point matches the theoretical value obtained in Section 2.3.

On the other hand, and concerning implosion, for the sample size $n = 100$ (101), by looking at Table 2.7 (2.8) and Figure 2.18 (2.19) for Case 1 and Table 2.9 (2.10) and Figure 2.20 (2.21) for Case 2, we can see that the minimum number of perturbed observations by inliers that makes the estimator to implode to zero, independently of the simulation case considered, has been

- 99 (100) for the estimators $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$;
- 51 for the estimator $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$;
- 50 for the estimators $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$.

Therefore, these results show that the empirical value for the implosion breakdown point matches the theoretical value obtained in the previous section.

Similar developments are now to be carried out in accordance with the SECOND SIMULATION PROCEDURE (2ndSP) (see Page 39). Notice that the notion of breakdown point does not make sense with this procedure, because all generated fuzzy numbers must be within a bounded interval ($[0, 100]$ for this study) and, for this reason, any scale estimator never can explode to infinity. Therefore, instead of breakdown point or finite sample breakdown point, we will refer to pseudo-breakdown point or *pseudo-finite sample breakdown point* when we are dealing with data generated from random fuzzy numbers with a bounded reference set.

Two sizes of sample have been considered in the simulation study, namely, an even sample size ($n = 20$) and an odd sample size ($n = 21$). For the sample size $n = 20$ we have chosen the weights $\omega_1 = 0.8$, $\omega_2 = 0.1$ and $\omega_3 = 0.1$, and for the sample size $n = 21$ the weights have been $\omega_1 = 16/21$, $\omega_2 = 3/21$ and $\omega_3 = 2/21$.

Explosion pseudo-breakdown point:

The outlier $\widetilde{y}_i$ has been constructed as follows, depending of the type of distribution generating the data of the non-contaminated sample.

- *Symmetric distribution:*
  - Outlier of translation: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1000, 1000)$ and the fuzzy numbers have been constrained to be in the interval $[47.5, 52.5]$. Then, we have chosen
  $$r_i^1 = \begin{cases} 8 + \frac{i+1}{2} \cdot 4 & \text{if } i \text{ is odd} \\ -(8 + \frac{i}{2} \cdot 4) & \text{if } i \text{ is even.} \end{cases}$$
  For the estimator $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$ we have chosen $r_i^1 = 8 + i \cdot 4$.
  - Outlier of scale on the core and support: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(5000, 5000)$ and the fuzzy numbers have been constrained to be in the interval $[49.5, 50.5]$. Then, we have chosen $r_i^2 = i \cdot 10$.
  - Outlier of both translation and scale: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1000, 1000)$ and the fuzzy numbers have been constrained to be in the interval $[47.5, 52.5]$. Then, we have chosen
  $$r_i^1 = \begin{cases} 8 + \frac{i+1}{2} \cdot 4 & \text{if } i \text{ is odd} \\ -(8 + \frac{i}{2} \cdot 4) & \text{if } i \text{ is even} \end{cases}$$
  $$r_i^2 = 2.$$
  For the estimator $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$ we have chosen $r_i^1 = 8 + i \cdot 4$.
- *Asymmetric distribution:*
  - Outlier of translation: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1, 100)$ and the fuzzy numbers have been constrained to be in the interval $[0, 5]$. Then, we have chosen $r_i^1 = 40 + i \cdot 5$.

- Outlier of scale on the core and support: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1, 1000)$ and the fuzzy numbers have been constrained to be in the interval $[0, 1]$. Then, we have chosen $r_i^2 = i \cdot 100$.
- Outlier of both translation and scale: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1, 100)$ and the fuzzy numbers have been constrained to be in the interval $[0, 5]$. Then, we have chosen
$$r_i^1 = 40 + i \cdot 5, \quad r_i^2 = 2.$$

For each type of outlier, the general scheme of the simulation is the same that the one for the 1stSP (see Page 93).

Remark 2.4.1. It should be pointed out that, in case any of the four numbers characterizing the outlier $\widetilde{y}_i = \mathrm{Tra}(y_i^{[1]}, y_i^{[2]}, y_i^{[3]}, y_i^{[4]})$ falls outside the reference interval $[0, 100]$, then it is automatically replaced by 0 if it is negative, or by 100 if it is over 100.

The simulation-based conclusions in this study are presented through Tables 2.11 to 2.14 and Figures 2.22 to 2.33. More concretely, tables gather the values of the different estimators when outliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Implosion pseudo-breakdown point:

The non-contaminated sample has been generated on the basis of a beta distribution $\beta(100, 100)$ regarding the symmetric distribution, and on the basis of a beta distribution $\beta(1, 100)$ regarding the asymmetric distribution.

The general scheme of the simulation is the same that the one for the 1stSP (see Page 94).

The simulation-based conclusions in this study are presented through Tables 2.15 to 2.18 and Figures 2.34 to 2.37. More concretely, tables gather the values of the different estimators when inliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Table 2.11: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, \ldots, 10\}$ - 2ndSP Symmetric distribution

# outliers (translation)	0	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.84	2.84	3.89	5.29	6.38	7.77	8.98	10.42	11.75	13.26	14.71
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.78	1.51	1.92	2.95	3.44	4.75	5.36	6.94	7.65	9.47	10.38
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.69	1.29	1.85	2.62	3.36	4.31	5.28	6.45	7.58	8.95	10.29
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^\ell}(\widetilde{\mathbf{x}}_n))$	0.93	1.55	2.11	2.88	3.64	4.58	5.56	6.74	7.85	9.21	10.55
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.74	1.00	1.90	2.74	3.91	5.42	7.03	8.76	10.70	12.90	15.27
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.68	0.75	0.71	0.71	0.76	0.76	0.84	0.84	0.81	1.19	6.55
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^\ell}(\widetilde{\mathbf{x}}_n))$	0.89	0.97	1.04	1.00	1.16	1.14	1.29	1.41	1.54	1.90	6.81
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.88	0.99	1.09	1.10	1.24	1.24	1.29	1.41	1.41	2.16	11.98
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.64	0.69	0.74	0.78	0.91	0.93	1.10	1.22	1.30	2.16	8.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.76	0.86	0.94	0.97	1.05	1.06	1.18	1.23	1.26	1.72	11.56

# outliers (scale)	0	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.18	1.15	2.55	4.21	6.06	8.04	9.97	11.69	13.26	14.56	15.63
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.17	0.53	1.45	2.74	4.31	6.09	7.95	9.80	11.50	12.98	14.29
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.14	0.38	0.88	1.65	2.67	3.97	5.49	7.15	8.97	10.86	12.85
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^\ell}(\widetilde{\mathbf{x}}_n))$	0.21	0.48	1.06	1.94	3.10	4.57	6.25	8.01	9.82	11.70	13.68
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.19	0.32	0.84	1.67	2.75	4.11	5.69	7.43	9.26	11.17	13.20
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.13	0.11	0.13	0.18	0.18	0.20	0.23	0.24	0.24	0.26	2.66
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^\ell}(\widetilde{\mathbf{x}}_n))$	0.19	0.17	0.16	0.21	0.23	0.26	0.30	0.42	0.54	0.57	3.06
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.19	0.19	0.22	0.22	0.22	0.22	0.27	0.28	0.32	0.42	5.07
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.14	0.15	0.15	0.16	0.17	0.18	0.22	0.23	0.27	0.42	4.89
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.17	0.17	0.17	0.19	0.19	0.20	0.22	0.25	0.27	0.31	4.96

# outliers (both)	0	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.71	2.88	3.91	5.34	6.41	7.82	9.01	10.47	11.79	13.31	14.74
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.65	1.47	1.84	2.96	3.42	4.84	5.40	7.08	7.82	9.68	10.62
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.57	1.19	1.74	2.50	3.25	4.25	5.18	6.38	7.54	8.96	10.29
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^\ell}(\widetilde{\mathbf{x}}_n))$	0.72	1.44	2.08	2.92	3.74	4.81	5.78	7.05	8.23	9.67	11.04
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.53	0.87	1.61	2.62	3.86	5.33	6.87	8.69	10.69	12.90	15.32
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.47	0.50	0.57	0.55	0.60	0.62	0.62	0.75	0.93	1.26	6.43
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^\ell}(\widetilde{\mathbf{x}}_n))$	0.62	0.66	0.75	0.74	0.79	0.89	1.03	1.16	1.44	2.07	7.18
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.69	0.78	0.85	0.87	0.89	1.01	1.10	1.16	1.47	1.91	11.55
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.46	0.56	0.67	0.69	0.72	0.79	0.86	1.04	1.21	1.91	8.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.59	0.69	0.76	0.77	0.80	0.85	0.91	1.02	1.20	1.56	11.28

Table 2.12: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, \ldots, 11\}$ - 2ndSP Symmetric distribution

# outliers (translation)	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.82	2.59	3.79	5.04	6.22	7.50	8.76	10.11	11.46	12.90	14.36	15.87
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	0.75	1.38	1.81	2.70	3.26	4.41	5.06	6.42	7.25	8.85	9.88	11.61
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.66	1.15	1.73	2.45	3.19	4.11	5.02	6.06	7.21	8.49	9.84	11.26
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.83	1.32	1.90	2.60	3.34	4.24	5.12	6.15	7.30	8.57	9.92	11.33
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	0.68	1.01	1.55	2.47	3.62	4.85	6.30	7.98	9.86	11.88	14.14	16.59
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.49	0.55	0.51	0.65	0.55	0.71	0.79	0.71	0.98	1.10	1.81	11.57
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.75	0.79	0.83	0.88	0.88	0.95	1.00	1.00	1.01	1.16	1.97	11.59
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.69	0.69	0.74	0.79	0.79	1.14	1.16	1.16	1.32	1.39	2.12	11.96
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.49	0.50	0.57	0.64	0.69	0.79	0.81	0.83	1.14	1.37	2.12	8.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.55	0.55	0.58	0.67	0.69	0.89	0.90	0.90	1.07	1.27	1.84	11.58
# outliers (scale)	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.22	0.93	2.05	3.39	4.88	6.50	8.19	9.67	11.00	12.15	13.06	13.77
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	0.19	0.44	1.15	2.17	3.42	4.84	6.40	7.96	9.41	10.69	11.82	12.71
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.17	0.34	0.72	1.29	2.05	3.02	4.19	5.48	6.88	8.35	9.87	11.24
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.23	0.43	0.86	1.52	2.37	3.48	4.81	6.20	7.65	9.12	10.64	12.01
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	0.20	0.25	0.66	1.30	2.17	3.24	4.46	5.86	7.34	8.92	10.51	12.13
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.15	0.14	0.17	0.20	0.21	0.24	0.26	0.26	0.26	0.26	0.39	4.11
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.21	0.20	0.20	0.19	0.23	0.27	0.35	0.34	0.40	0.55	0.57	4.29
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.23	0.25	0.26	0.26	0.26	0.26	0.26	0.26	0.31	0.39	0.46	4.11
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.14	0.15	0.17	0.18	0.18	0.18	0.20	0.22	0.25	0.26	0.46	3.72
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.18	0.19	0.22	0.22	0.22	0.23	0.23	0.23	0.25	0.26	0.35	3.89
# outliers (both)	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.00	2.80	3.84	5.17	6.25	7.60	8.78	10.19	11.48	12.96	14.36	15.92
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	0.91	1.57	2.00	2.91	3.44	4.64	5.29	6.70	7.43	9.13	9.99	11.92
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.81	1.38	1.91	2.65	3.36	4.30	5.21	6.30	7.36	8.68	9.92	11.44
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	1.03	1.62	2.13	2.85	3.57	4.51	5.44	6.52	7.56	8.87	10.08	11.60
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	0.85	0.99	1.69	2.60	3.73	5.12	6.62	8.33	10.25	12.31	14.57	17.05
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.68	0.73	0.82	0.82	0.91	1.31	1.29	1.35	1.41	1.53	1.64	11.57
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.97	1.02	1.14	1.15	1.15	1.74	1.78	1.80	1.82	1.91	1.91	11.51
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.99	1.08	1.11	1.37	1.51	1.80	1.92	1.98	2.16	2.18	2.29	11.72
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.62	0.73	0.73	0.78	1.03	1.25	1.49	1.55	1.67	1.92	2.29	8.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.77	0.84	0.93	1.05	1.21	1.47	1.69	1.71	1.78	1.89	2.00	11.34

Table 2.13: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, \ldots, 10\}$ - 2ndSP Asymmetric distribution

# outliers (translation)	0	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.88	9.73	14.12	17.75	20.96	23.86	26.58	29.05	31.32	33.34	35.09
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.76	4.34	8.49	12.64	16.65	20.43	23.98	27.13	29.85	32.04	33.59
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.58	2.77	5.21	7.87	10.81	14.00	17.39	21.09	25.04	29.22	33.59
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^{\ell}}(\widetilde{\mathbf{x}}_n))$	0.98	3.16	5.61	8.20	11.11	14.29	17.59	21.26	25.18	29.38	33.78
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.75	2.36	4.86	7.66	10.72	13.96	17.47	21.26	25.12	29.33	33.62
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.53	0.52	0.56	0.56	0.57	0.69	0.69	0.78	0.84	1.14	23.20
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^{\ell}}(\widetilde{\mathbf{x}}_n))$	0.89	0.84	0.84	0.86	0.82	0.89	0.91	0.92	0.94	1.06	23.16
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.68	0.68	0.79	0.79	0.82	0.96	0.96	0.96	0.98	1.11	45.05
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.51	0.53	0.57	0.57	0.58	0.66	0.66	0.73	0.87	1.11	10.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.60	0.60	0.69	0.69	0.70	0.80	0.80	0.81	0.84	0.94	44.63
# outliers (scale)	0	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.17	0.71	1.52	2.49	3.55	4.70	5.90	7.13	8.38	9.63	10.86
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.14	0.36	0.87	1.62	2.54	3.57	4.69	5.90	7.11	8.33	9.52
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.09	0.19	0.39	0.68	1.07	1.55	2.13	2.79	3.55	4.40	5.35
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^{\ell}}(\widetilde{\mathbf{x}}_n))$	0.17	0.38	0.79	1.41	2.23	3.25	4.48	5.89	7.51	9.33	11.34
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.10	0.25	0.57	1.05	1.69	2.49	3.45	4.57	5.83	7.20	8.72
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.05	0.06	0.07	0.09	0.11	0.14	0.19	0.22	0.27	0.35	1.22
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^{\ell}}(\widetilde{\mathbf{x}}_n))$	0.10	0.12	0.13	0.16	0.20	0.25	0.35	0.42	0.50	0.61	2.46
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.08	0.09	0.10	0.13	0.16	0.20	0.22	0.23	0.32	0.40	2.03
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.06	0.06	0.08	0.09	0.12	0.15	0.19	0.21	0.28	0.40	1.95
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.07	0.07	0.08	0.10	0.13	0.16	0.20	0.21	0.28	0.33	1.92
# outliers (both)	0	1	2	3	4	5	6	7	8	9	10
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.75	9.81	14.26	17.90	21.12	24.05	26.70	29.17	31.40	33.42	35.18
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.67	4.33	8.56	12.74	16.78	20.60	24.09	27.25	29.93	32.12	33.68
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.53	2.75	5.22	7.95	10.94	14.17	17.57	21.28	25.18	29.34	33.68
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^{\ell}}(\widetilde{\mathbf{x}}_n))$	0.84	3.09	5.58	8.34	11.32	14.55	17.93	21.66	25.56	29.75	34.09
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.74	2.30	4.85	7.64	10.77	14.11	17.65	21.48	25.33	29.55	33.63
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{Me}}(\widetilde{\mathbf{x}}_n))$	0.55	0.59	0.56	0.57	0.68	0.85	0.95	1.12	1.54	2.03	23.93
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\mathrm{M}^{\ell}}(\widetilde{\mathbf{x}}_n))$	0.90	0.98	1.05	0.98	1.01	1.25	1.37	1.67	2.29	2.72	24.84
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.73	0.73	0.74	0.95	1.07	1.16	1.16	1.29	1.68	1.80	45.57
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.52	0.55	0.60	0.66	0.74	0.84	0.95	1.08	1.34	1.80	10.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.61	0.62	0.64	0.78	0.88	0.98	0.98	1.11	1.35	1.55	44.80

Table 2.14: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, \ldots, 11\}$ - 2ndSP Asymmetric distribution

# outliers (translation)	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.66	9.54	13.89	17.48	20.65	23.57	26.25	28.72	30.99	33.06	34.98	36.63
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.62	4.10	8.15	12.19	16.11	19.86	23.35	26.52	29.30	31.63	33.50	34.76
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.50	2.61	4.97	7.54	10.37	13.41	16.68	20.18	23.89	27.82	32.03	34.36
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.75	2.88	5.25	7.81	10.65	13.72	17.00	20.49	24.21	28.14	32.32	34.62
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.63	2.34	4.71	7.40	10.26	13.42	16.79	20.38	24.07	28.00	32.22	36.50
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.43	0.49	0.57	0.60	0.60	0.68	0.80	0.92	1.14	1.54	1.68	44.03
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.67	0.75	0.79	0.81	0.92	0.93	1.13	1.23	1.31	1.53	1.61	44.56
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.54	0.65	0.78	0.78	0.84	0.85	1.00	1.12	1.22	1.36	1.36	44.23
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.42	0.45	0.48	0.50	0.63	0.64	0.71	0.83	0.92	1.08	1.36	5.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.47	0.54	0.63	0.63	0.72	0.74	0.84	0.92	1.01	1.11	1.12	37.38
# outliers (scale)	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.19	0.98	2.14	3.50	5.02	6.65	8.36	10.13	11.92	13.67	14.97	15.92
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.15	0.46	1.19	2.24	3.52	4.98	6.54	8.28	10.03	11.75	13.21	14.31
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.09	0.22	0.49	0.89	1.42	2.09	2.89	3.82	4.87	6.04	7.20	8.24
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.17	0.44	1.00	1.86	2.99	4.41	6.11	8.08	10.32	12.82	15.29	17.48
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.13	0.30	0.73	1.38	2.26	3.35	4.66	6.22	7.93	9.86	11.79	13.78
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.04	0.04	0.05	0.06	0.07	0.08	0.10	0.24	0.25	0.30	0.48	2.89
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.07	0.07	0.09	0.09	0.12	0.19	0.21	0.46	0.49	0.59	0.85	6.10
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.06	0.06	0.06	0.07	0.10	0.13	0.14	0.24	0.25	0.31	0.49	2.89
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.05	0.05	0.05	0.06	0.07	0.08	0.13	0.20	0.24	0.30	0.48	2.81
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.05	0.05	0.06	0.06	0.07	0.10	0.11	0.22	0.22	0.28	0.43	2.78
# outliers (both)	0	1	2	3	4	5	6	7	8	9	10	11
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	0.70	9.60	13.96	17.55	20.74	23.66	26.34	28.84	31.14	33.26	35.13	36.80
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.67	4.15	8.21	12.25	16.18	19.94	23.43	26.64	29.46	31.83	33.66	34.93
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.51	2.63	4.99	7.57	10.40	13.46	16.73	20.26	24.02	28.00	32.18	34.53
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.82	2.94	5.30	7.88	10.71	13.78	17.03	20.57	24.35	28.34	32.53	34.87
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	0.70	2.18	4.56	7.18	10.18	13.34	16.70	20.37	24.16	28.07	32.27	36.62
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.51	0.51	0.53	0.59	0.62	0.65	0.69	0.78	1.16	1.29	1.73	44.54
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	0.74	0.82	0.85	0.86	0.93	1.07	1.20	1.25	1.61	1.73	1.98	45.00
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.73	0.77	0.83	0.87	0.93	1.00	1.03	1.03	1.12	1.17	1.39	44.58
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.41	0.49	0.55	0.58	0.63	0.69	0.73	0.83	1.00	1.07	1.39	5.00
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.61	0.69	0.71	0.73	0.79	0.83	0.85	0.87	1.01	1.04	1.19	36.83

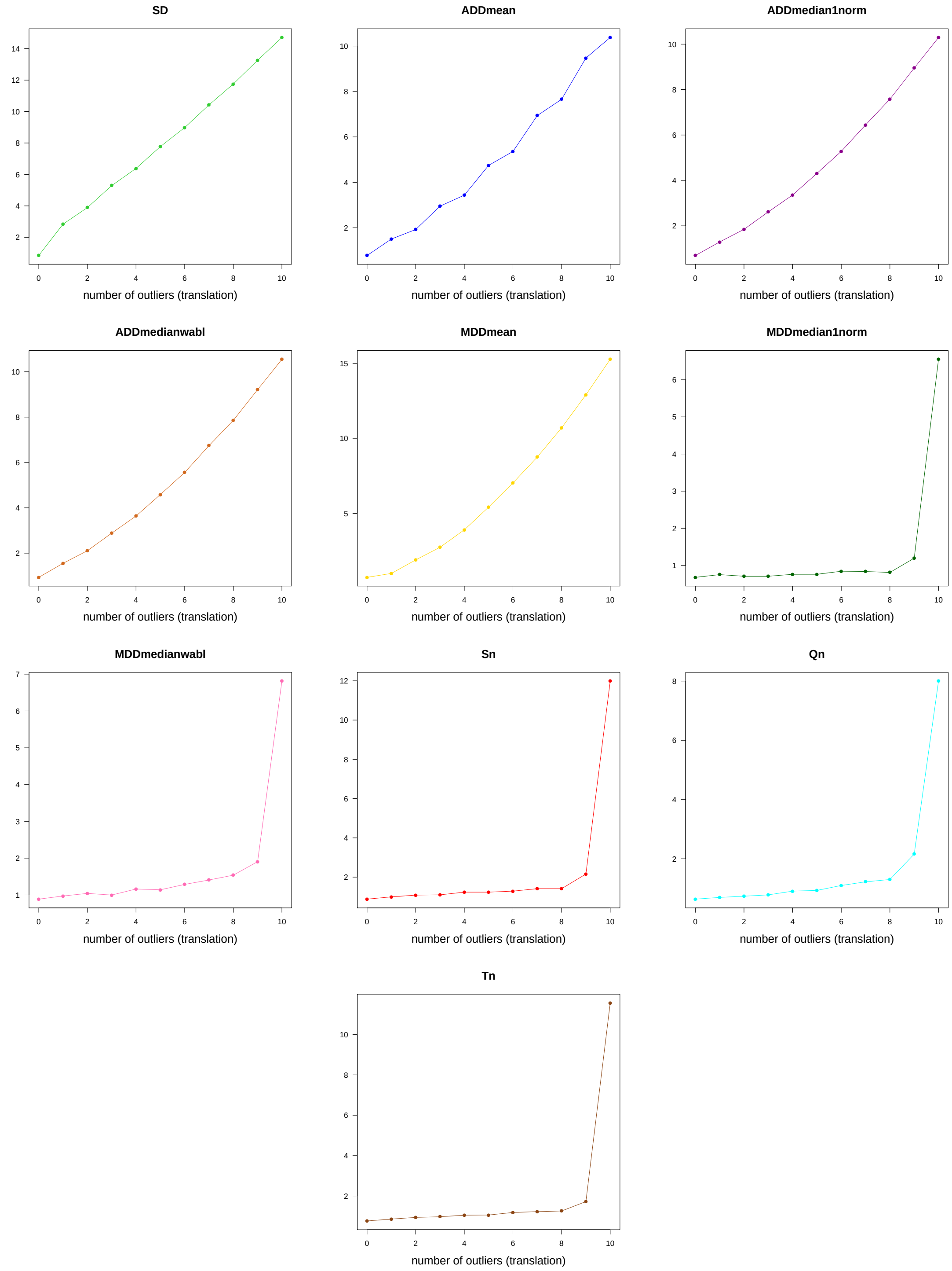


Figure 2.22: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of translation, k varying from 0 to 10 - 2ndSP
Symmetric distribution

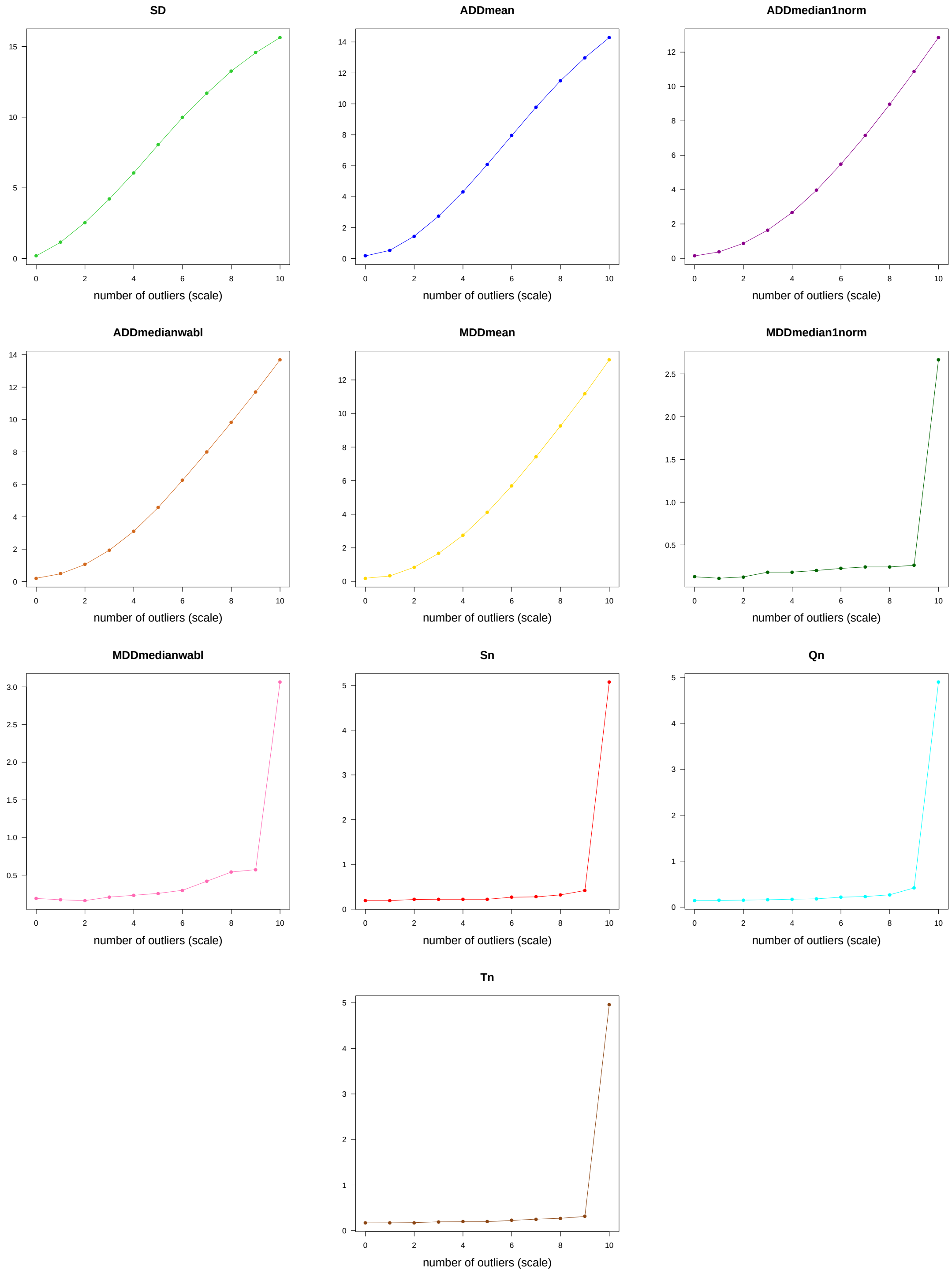


Figure 2.23: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of scale on core and support, k varying from 0 to 10 - 2ndSP Symmetric distribution

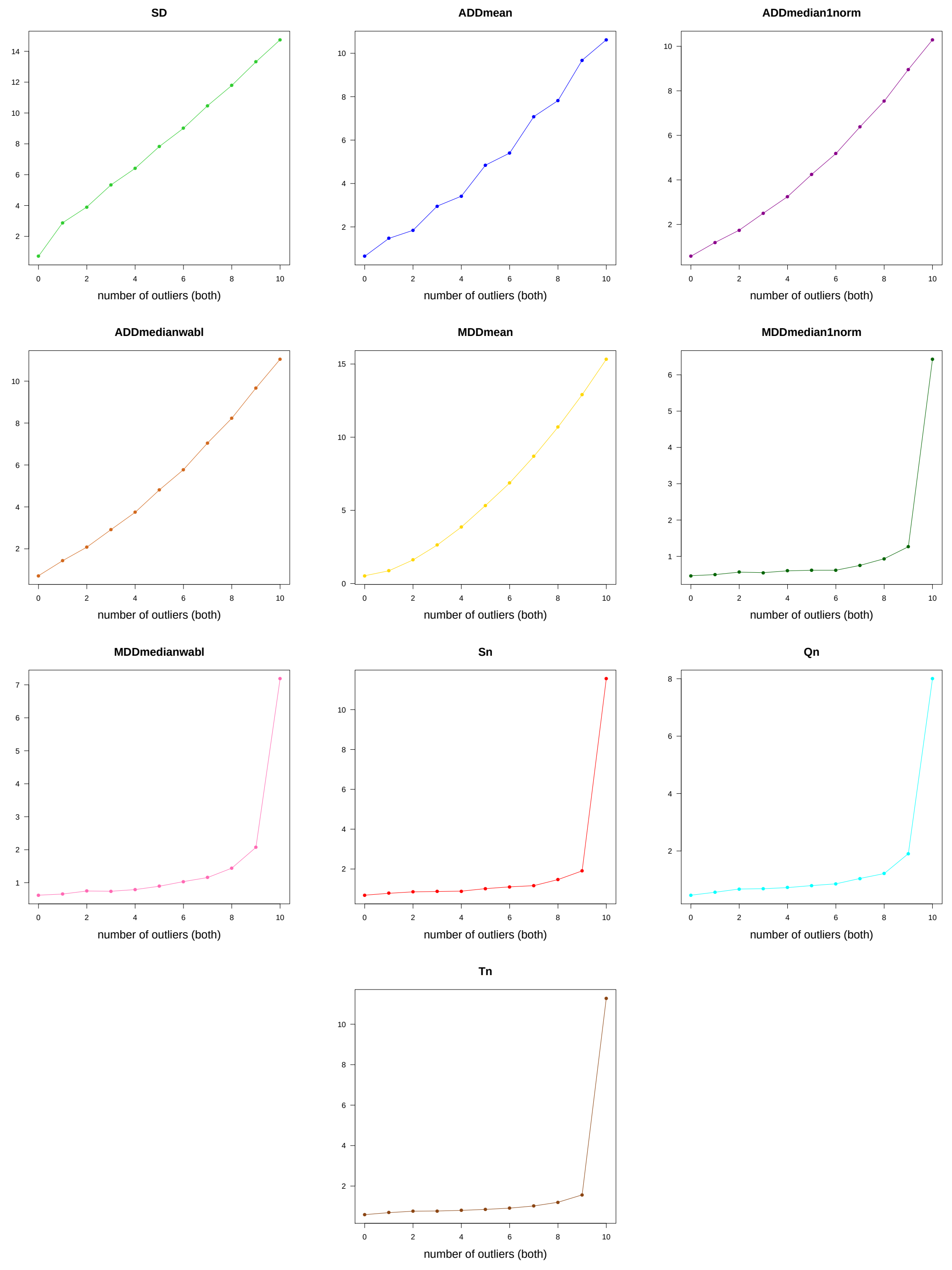


Figure 2.24: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 10 - 2ndSP
Symmetric distribution

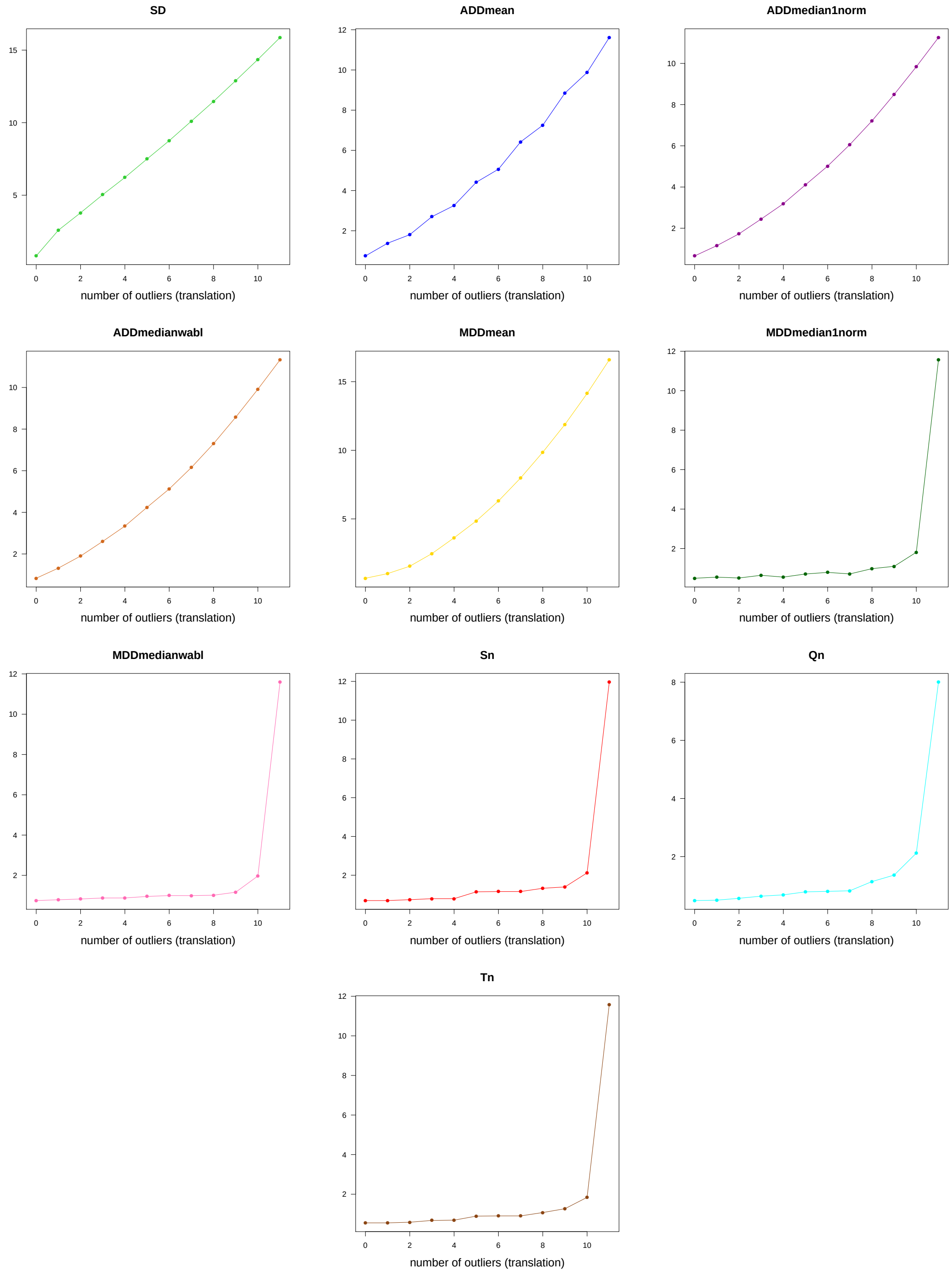


Figure 2.25: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of translation, k varying from 0 to 11 - 2ndSP
Symmetric distribution

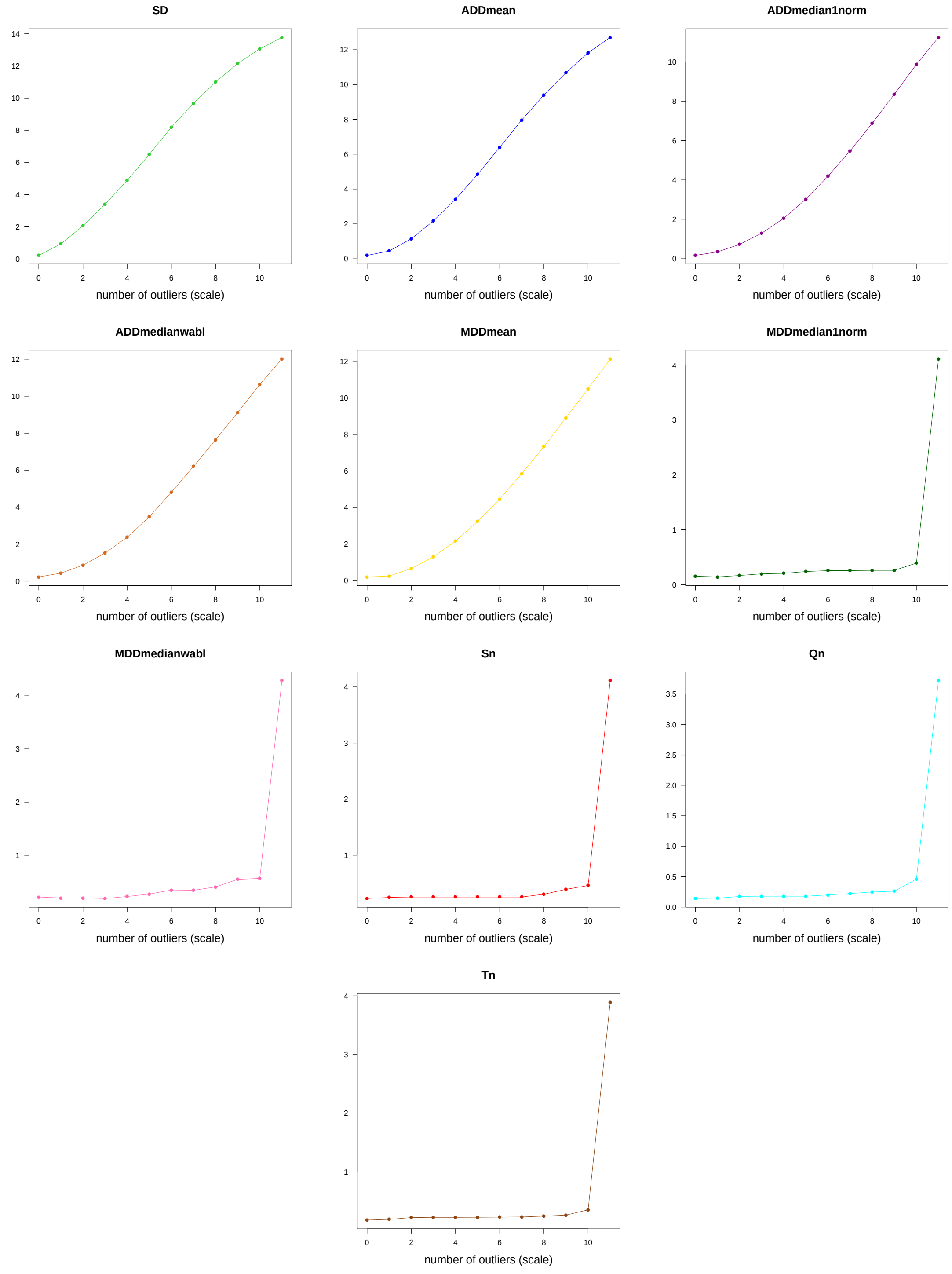


Figure 2.26: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of scale on core and support, k varying from 0 to 11 - 2ndSP Symmetric distribution

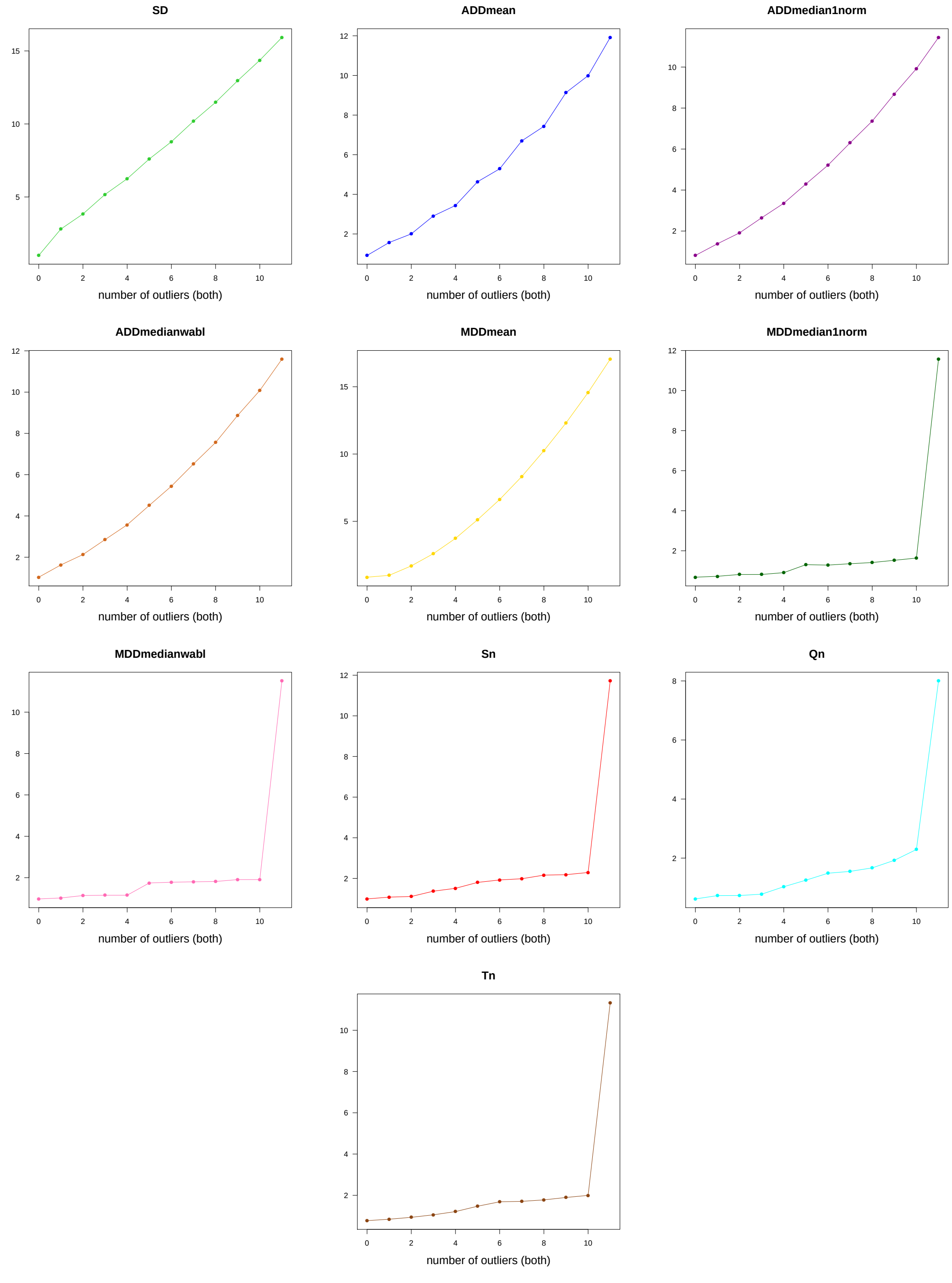


Figure 2.27: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 11 - 2ndSP
Symmetric distribution

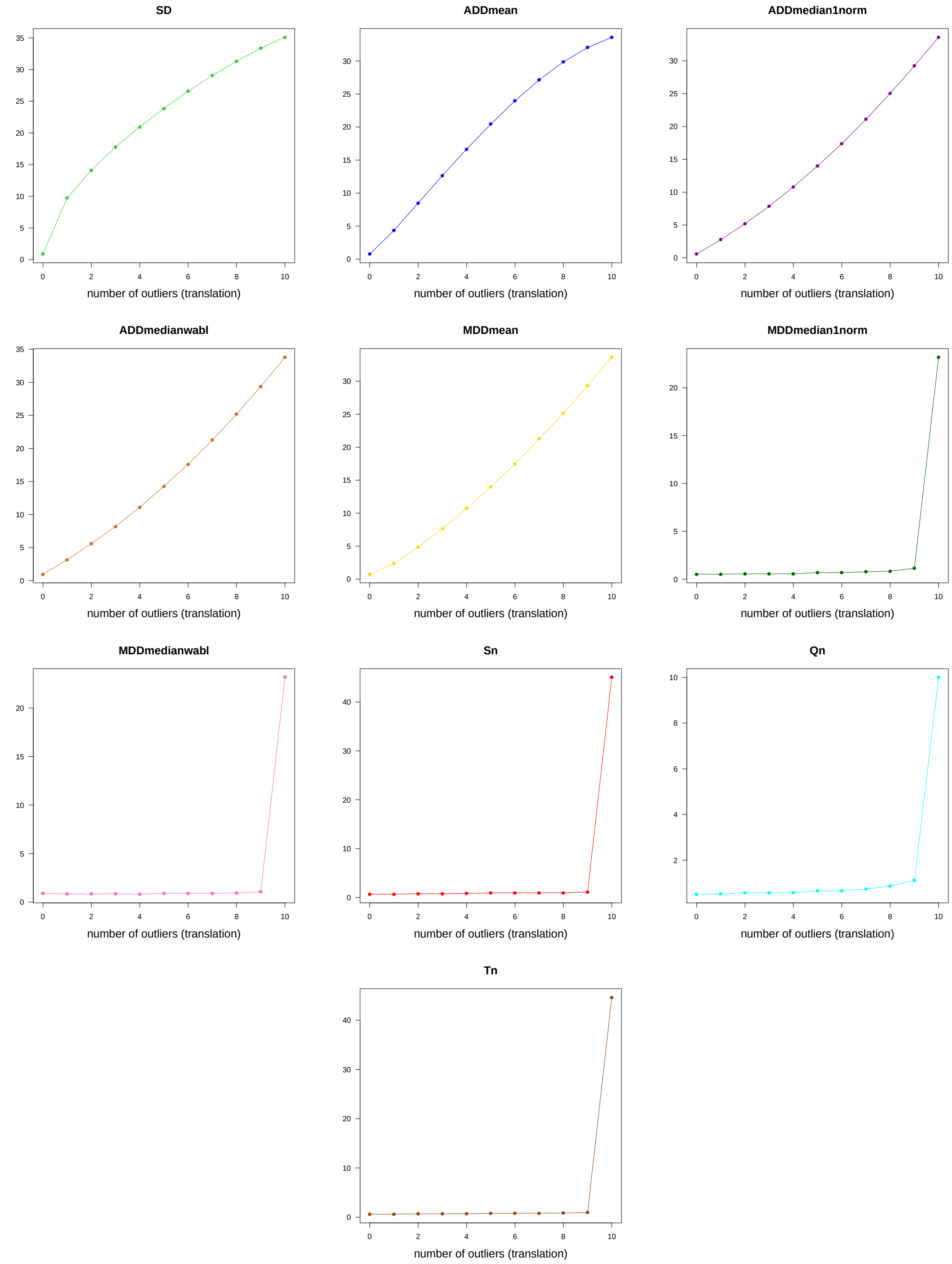


Figure 2.28: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of translation, k varying from 0 to 10 - 2ndSP Asymmetric distribution

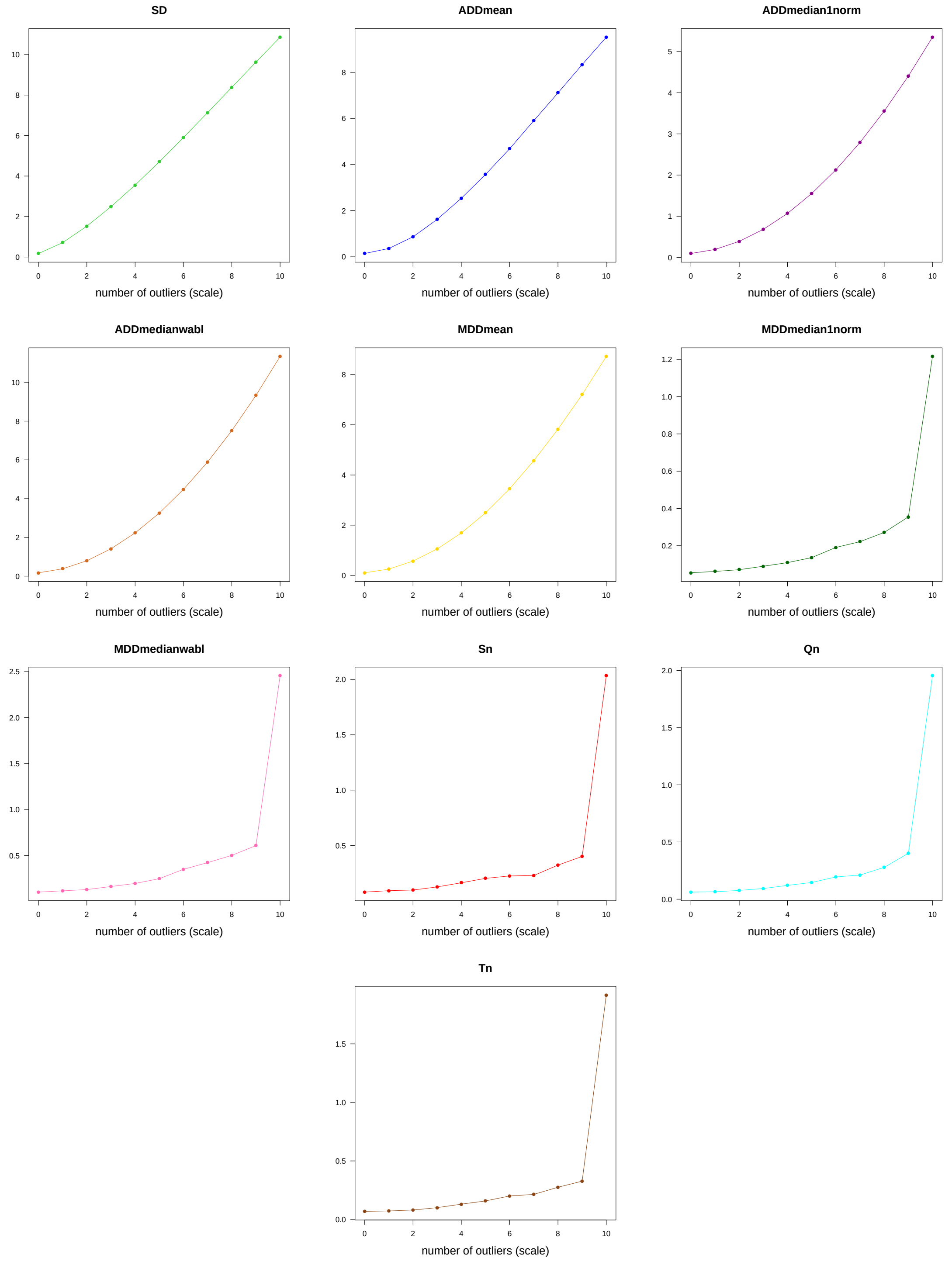


Figure 2.29: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of scale on core and support, k varying from 0 to 10 - 2ndSP Asymmetric distribution

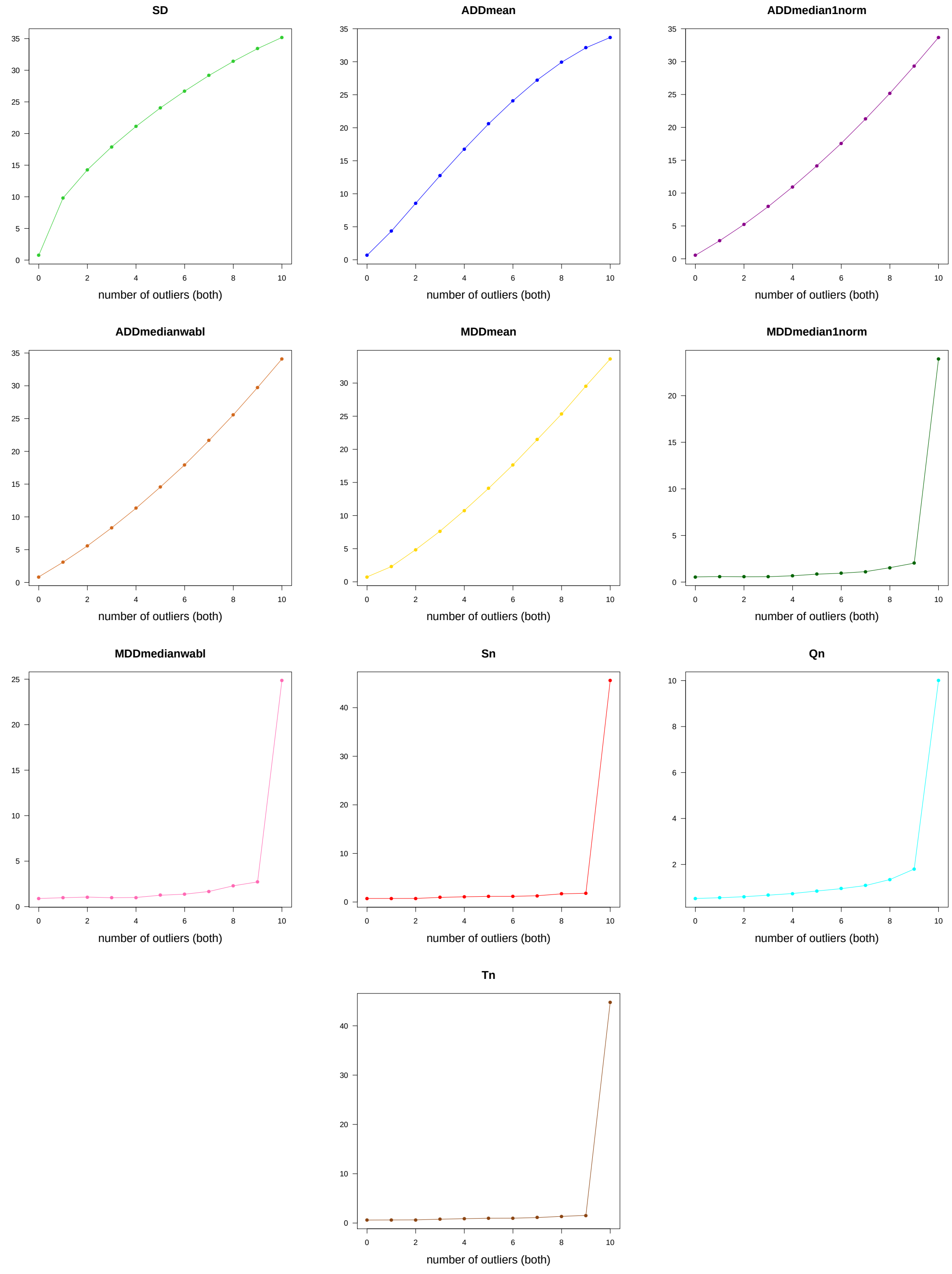


Figure 2.30: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 10 - 2ndSP
Asymmetric distribution

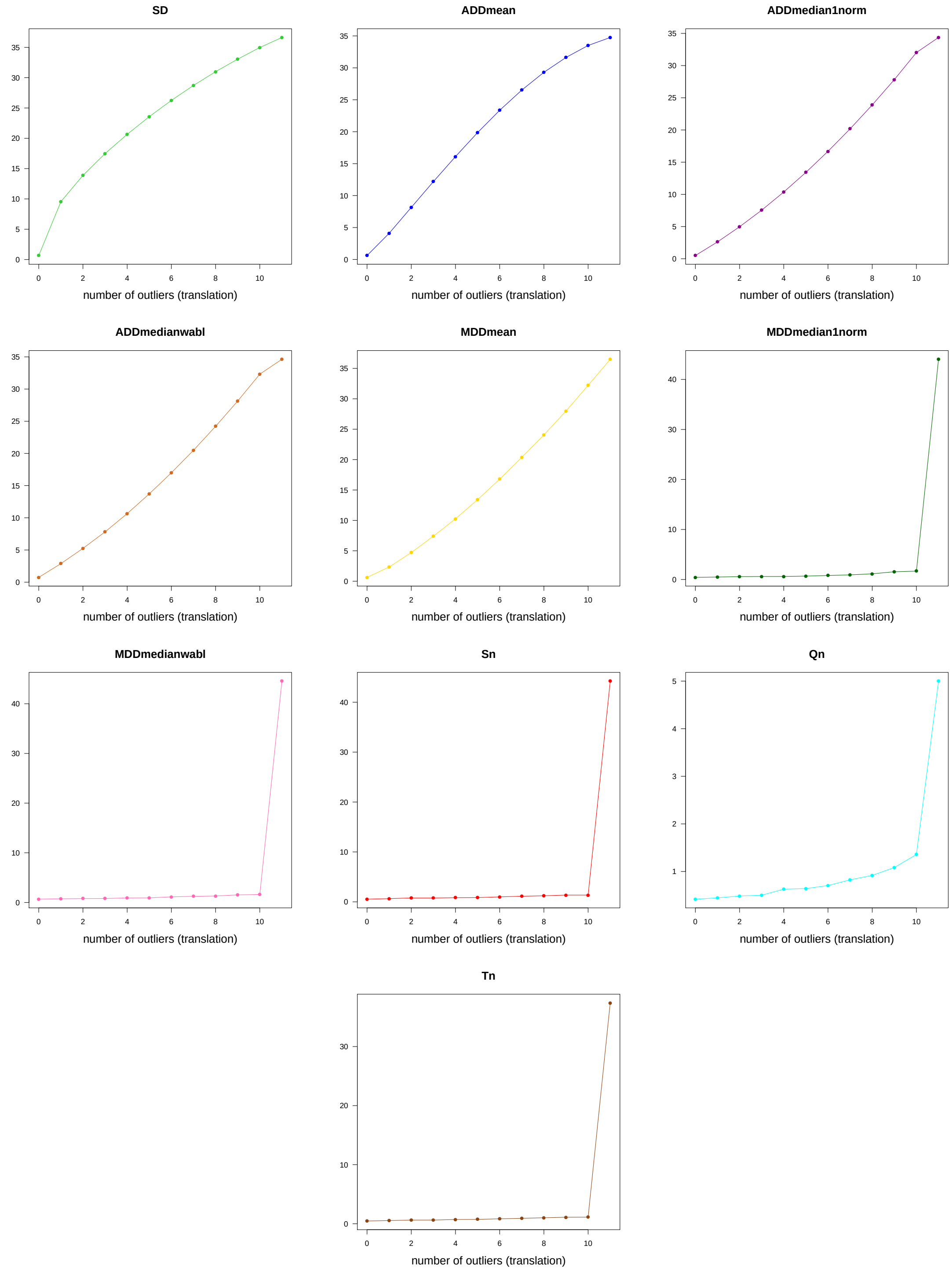


Figure 2.31: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of translation, k varying from 0 to 11 - 2ndSP
Asymmetric distribution

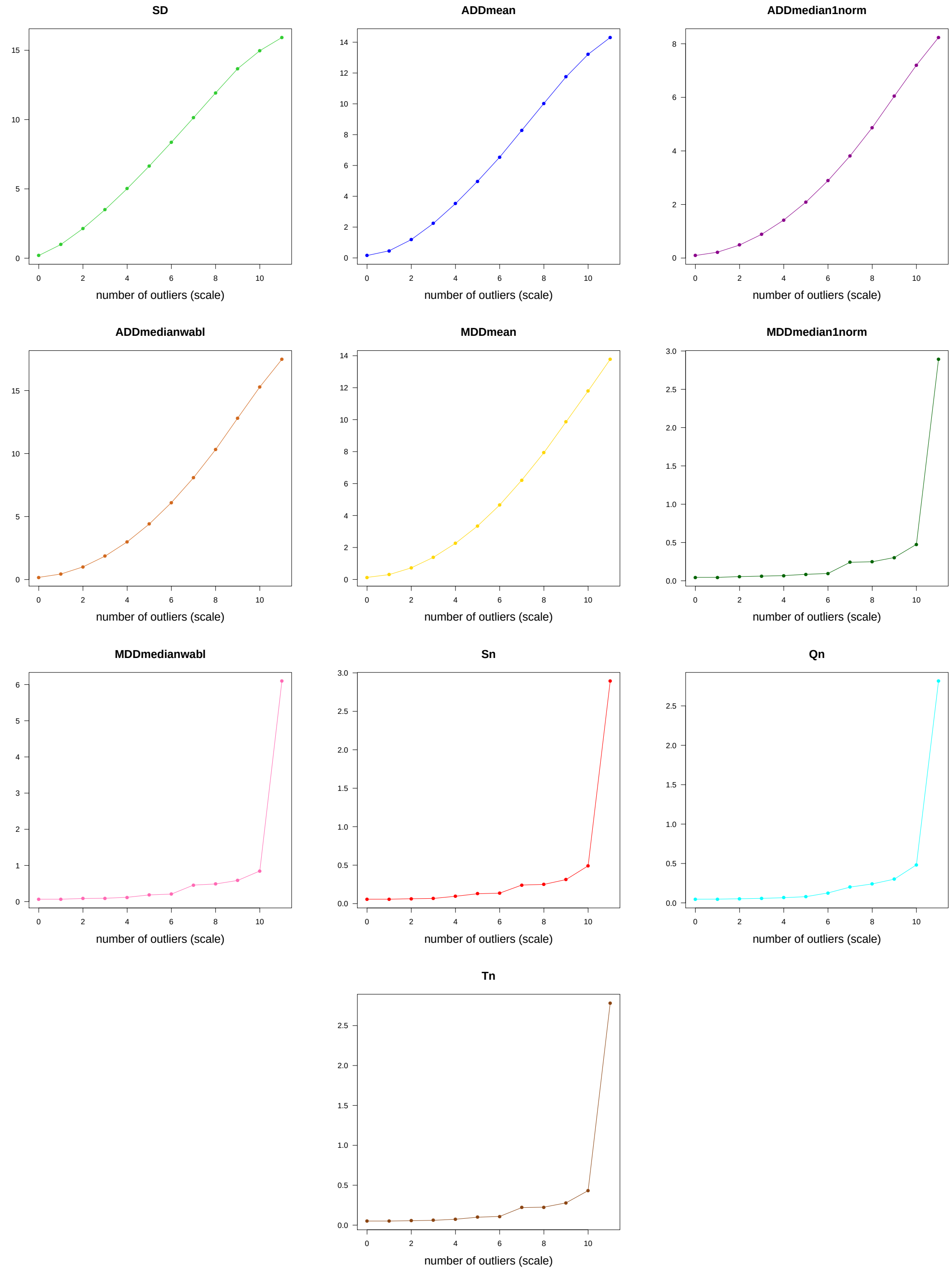


Figure 2.32: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of scale on core and support, k varying from 0 to 11 - 2ndSP Asymmetric distribution

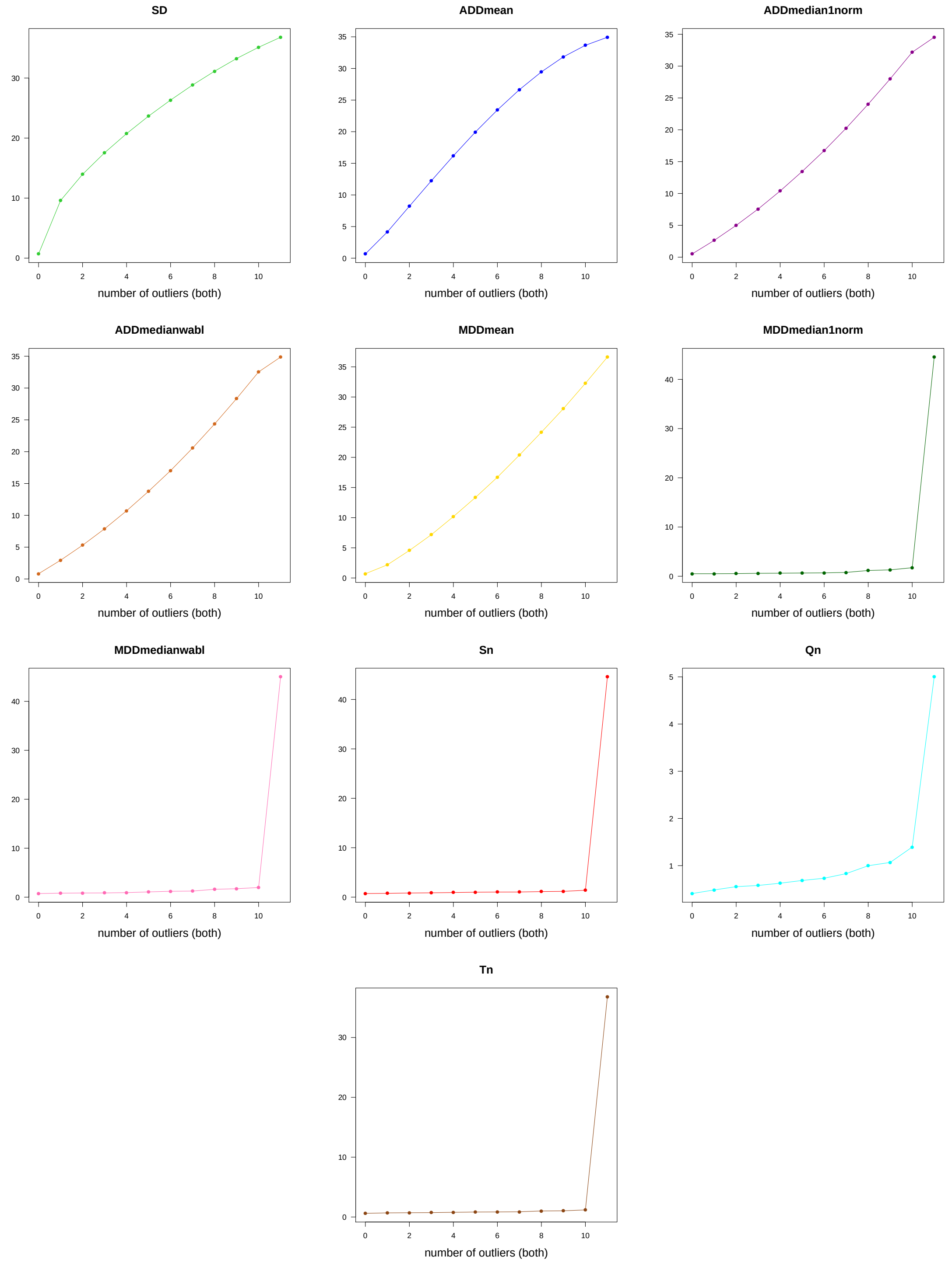


Figure 2.33: Explosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 11 - 2ndSP Asymmetric distribution

Table 2.15: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by inliers, with $k \in \{0, 1, 2, 3, 9, 10, 11, 18, 19\}$ - 2ndSP Symmetric distribution

# inliers	0	1	2	3	9	10	11	18	19
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	3.69	3.69	3.69	3.55	3.19	3.16	3.12	0.28	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	2.83	2.86	2.83	2.66	2.12	2.05	1.92	0.12	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	2.36	2.42	2.39	2.24	1.50	1.41	1.29	0.06	0
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	3.38	3.44	3.40	3.15	2.33	2.24	2.07	0.08	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	1.91	1.91	1.93	1.89	1.92	0.99	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.61	1.71	1.54	1.57	0.54	0	0	0	0
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	2.33	2.33	2.29	2.50	0.82	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	2.55	2.55	2.47	2.26	1.09	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	2.09	2.16	2.10	2.00	1.09	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	2.23	2.32	2.29	2.18	1.09	0	0	0	0

Table 2.16: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by inliers, with $k \in \{0, 1, 2, 3, 9, 10, 11, 19, 20\}$ - 2ndSP Symmetric distribution

# inliers	0	1	2	3	9	10	11	19	20
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	3.88	3.85	3.82	3.77	3.56	3.50	3.43	0.95	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	3.04	2.99	2.93	2.85	2.53	2.45	2.38	0.41	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	2.50	2.46	2.40	2.31	1.84	1.74	1.64	0.17	0
$\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	3.52	3.48	3.39	3.30	2.63	2.44	2.27	0.29	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	2.36	2.38	2.44	2.44	1.83	1.39	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.69	1.66	1.44	1.61	1.48	0	0	0	0
$\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	2.48	2.45	2.24	2.16	2.19	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	2.44	2.28	2.24	2.18	1.48	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.89	1.84	1.83	1.76	1.48	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	2.15	2.09	1.99	1.86	1.48	0	0	0	0

Table 2.17: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by inliers, with $k \in \{0, 1, 2, 3, 9, 10, 11, 18, 19\}$ - 2ndSP Asymmetric distribution

# inliers	0	1	2	3	9	10	11	18	19
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	1.16	1.16	1.13	1.09	0.95	0.90	0.89	0.23	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\mathbf{x}}}_n)$	0.88	0.90	0.88	0.85	0.69	0.63	0.61	0.10	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	0.65	0.66	0.64	0.61	0.43	0.38	0.34	0.04	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	1.10	1.13	1.11	1.08	0.80	0.65	0.61	0.07	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\mathbf{x}}}_n)$	0.69	0.65	0.62	0.52	0.33	0.12	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	0.54	0.59	0.51	0.42	0.17	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	0.77	0.88	0.94	0.82	0.23	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.68	0.72	0.77	0.74	0.33	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.52	0.56	0.57	0.56	0.33	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.61	0.66	0.69	0.68	0.33	0	0	0	0

Table 2.18: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by inliers, with $k \in \{0, 1, 2, 3, 9, 10, 11, 19, 20\}$ - 2ndSP Asymmetric distribution

# inliers	0	1	2	3	9	10	11	19	20
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.23	2.25	2.26	2.28	2.36	2.38	2.39	0.34	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\mathbf{x}}}_n)$	1.34	1.37	1.38	1.39	1.51	1.53	1.55	0.15	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	0.86	0.89	0.90	0.91	0.96	0.90	0.87	0.05	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	1.48	1.54	1.56	1.61	1.74	1.65	1.59	0.10	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\mathbf{x}}}_n)$	0.89	0.80	0.82	0.87	0.67	0.66	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	0.54	0.53	0.60	0.65	0.05	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	0.52	0.71	0.71	0.77	0.06	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.61	0.70	0.72	0.83	0.07	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.49	0.53	0.55	0.55	0.07	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.53	0.57	0.62	0.66	0.07	0	0	0	0

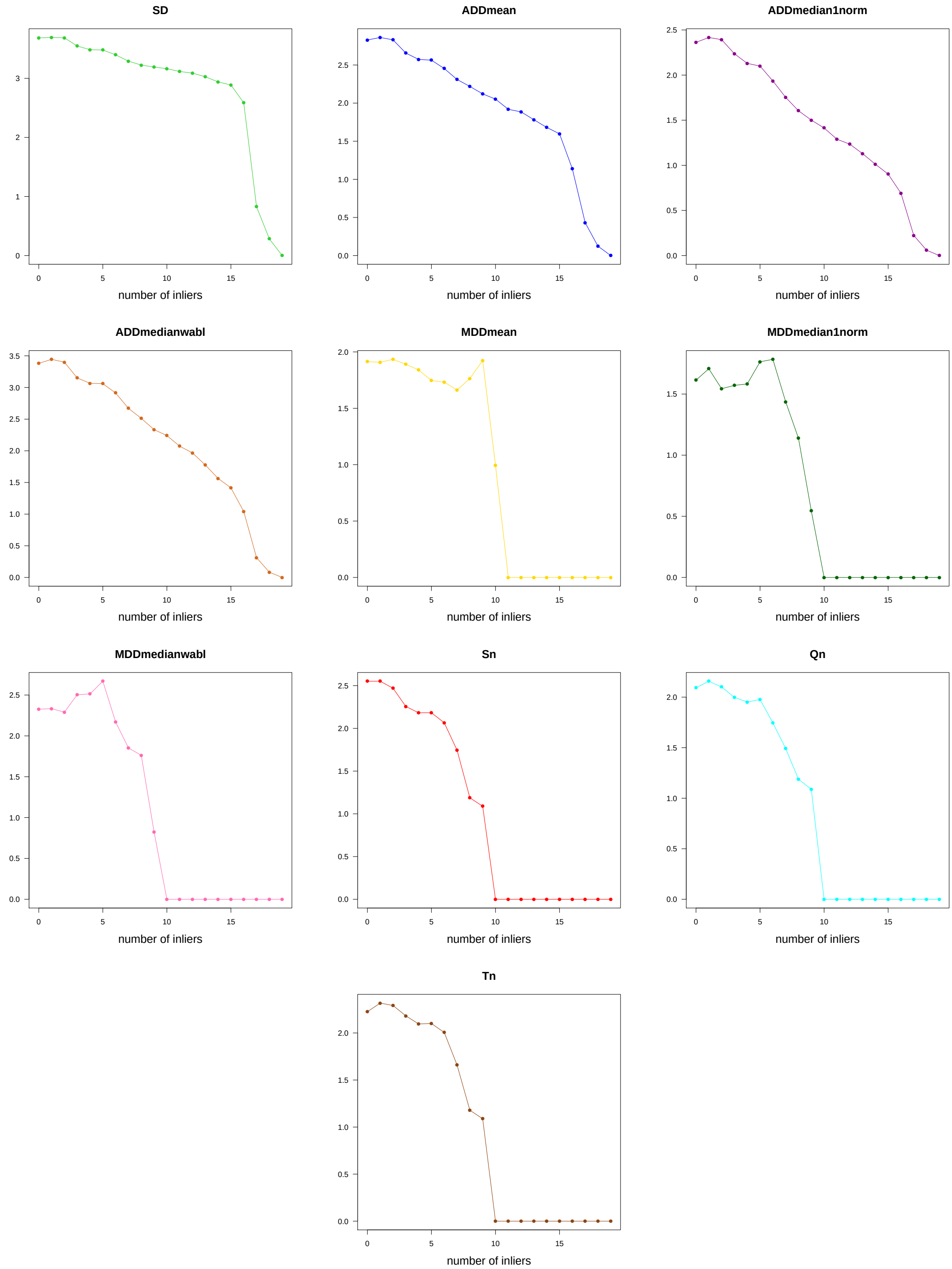


Figure 2.34: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by inliers, k varying from 0 to 19 - 2ndSP Symmetric distribution

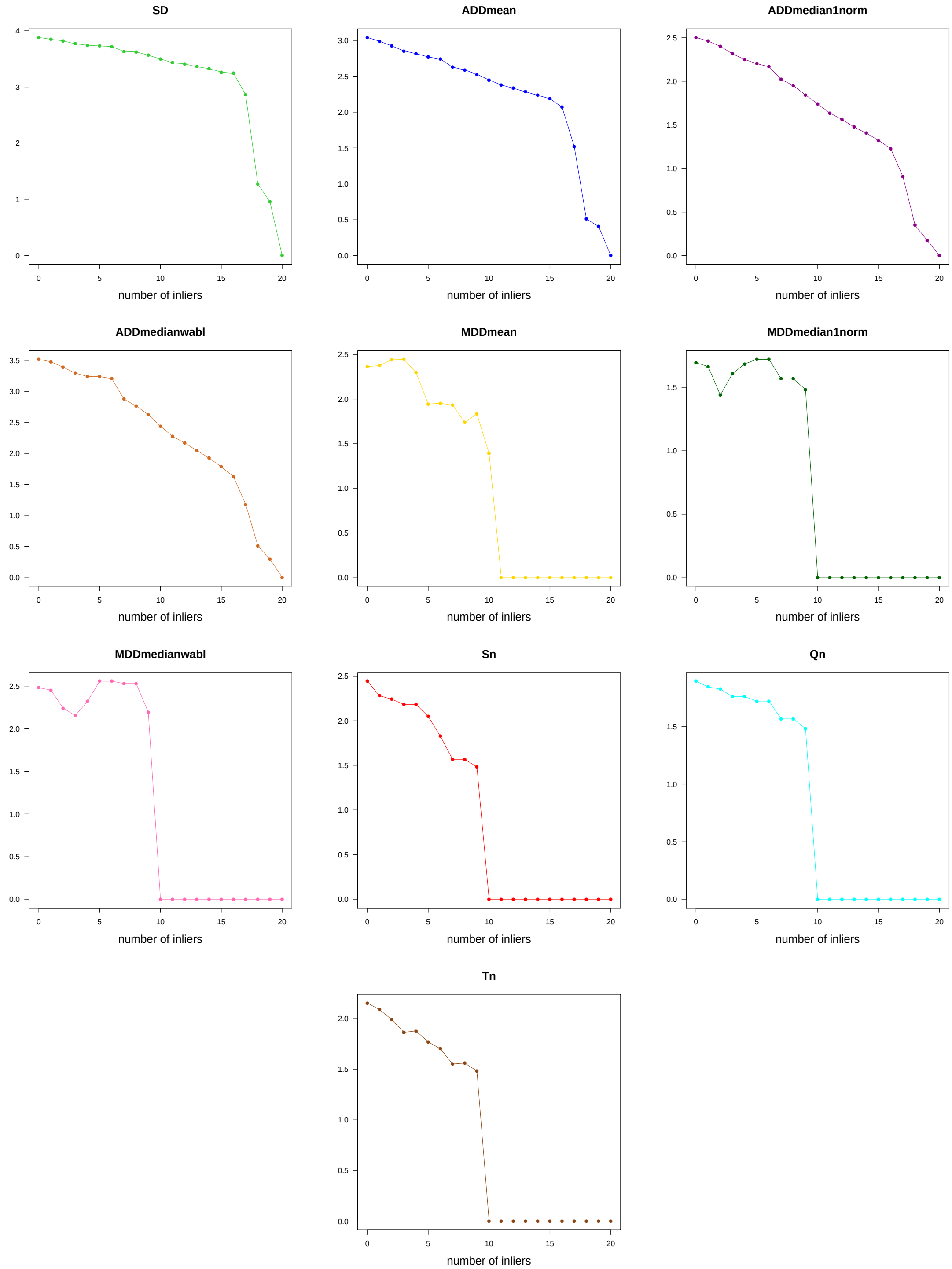


Figure 2.35: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by inliers, k varying from 0 to 20 - 2ndSP Symmetric distribution

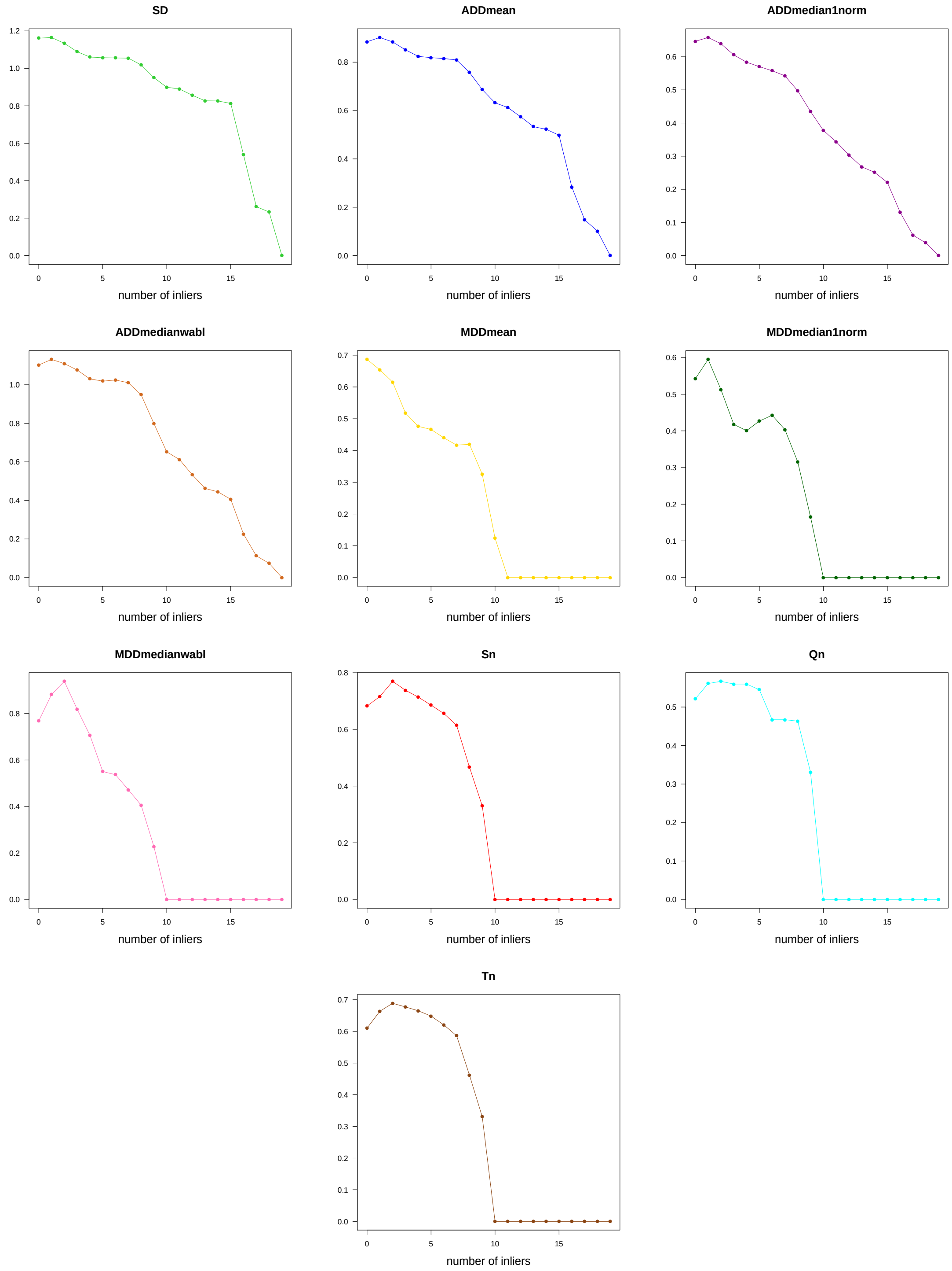


Figure 2.36: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 20 with k observations replaced by inliers, k varying from 0 to 19 - 2ndSP Asymmetric distribution

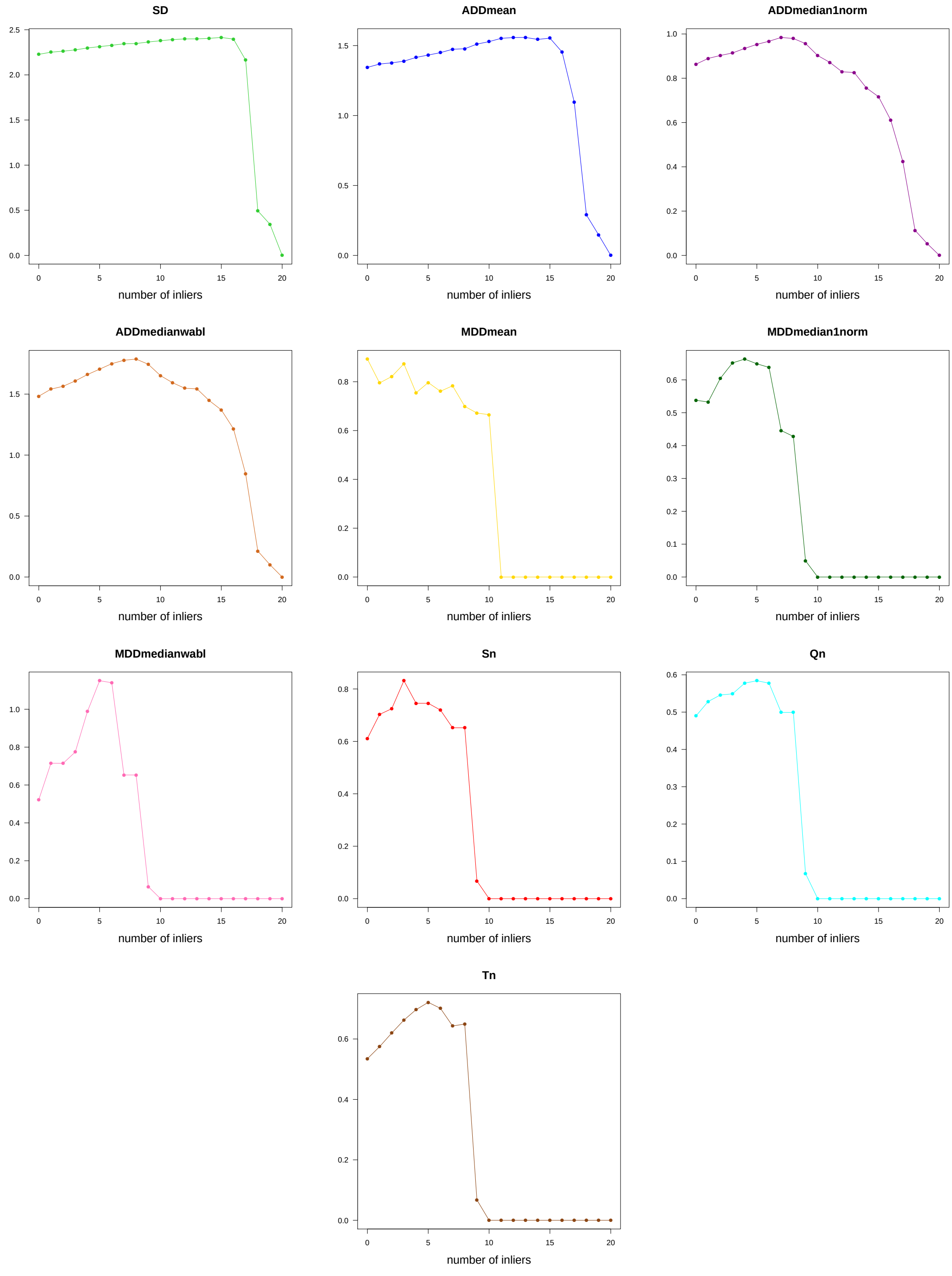


Figure 2.37: Implosion pseudo-breakdown point: values of the scale estimators for a sample of size 21 with k observations replaced by inliers, k varying from 0 to 20 - 2ndSP Asymmetric distribution

Regarding explosion we can conclude that for the sample size $n = 20$ (21), by looking at Table 2.11 (2.12) and Figures 2.22 to 2.24 (2.25 to 2.27) for the symmetric distribution and Table 2.13 (2.14) and Figures 2.28 to 2.30 (2.31 to 2.33) for the asymmetric distribution, we can see that the minimum number of perturbed observations by outliers that makes the estimator to increase noticeably, independently of the distribution and the type of outlier considered, has been

- 1 for the estimators $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$;
- 10(11) for the estimators $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$.

Therefore, the empirical value for the explosion pseudo-breakdown point coincides with the theoretical explosion breakdown point in Section 2.3.

On the other hand, and concerning implosion, for the sample size $n = 20$ (21), by looking at Table 2.15 (2.16) and Figure 2.34 (2.35) for the symmetric case and Table 2.17 (2.18) and Figure 2.36 (2.37) for the asymmetric case, we can see that the minimum number of perturbed observations by inliers that makes the estimator to implode to zero, independently of the distribution case considered, has been

- 19 (20) for the estimators $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$;
- 11 for the estimator $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$;
- 10 for the estimators $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$.

Therefore, the empirical value for the implosion pseudo-breakdown point coincides with the theoretical implosion breakdown point in the previous section.

2.4.3 Sensitivity curves

Another important and useful tool to measure the robustness of an estimator from an empirical point of view is now to be analyzed: the sensitivity curves, which represent the sample version of the influence functions (see, for instance, Maronna *et al.* [81] and Rossello [98]).

Previously, the finite sample breakdown point, which tell us how much the estimate changes when a percentage of the data is contaminated by outliers or inliers, has been studied for the different estimators. In contrast, the sensitivity curves describe how the estimator reacts to a single outlier in the data.

Recall that we are dealing with three different types of outliers in the fuzzy setting (see Subsection 2.4.1): outliers of translation, outliers of scale on the core and support, and outliers of both translation and scale. Therefore, some of the sensitivity curves will allow us to measure the effect of different 'locations' of a single outlier in the sample, other curves will measure the effect of considering different 'widths' of a single outlier, and other ones will measure the effect of different 'locations' and 'widths' of a single outlier in the sample.

The non-contaminated sample has first been simulated from the FIRST SIMULATION PROCEDURE (1stSP) considering the two cases (see Page 38) and an only sample size $n = 100$.

The outliers have been constructed so that if $\widetilde{y}_s$ is the outlier, then,

- Outlier of translation: $r_s^1 = s$, with s varying from -20 to 20 with a step equals 0.1.
- Outlier of scale on the core and support: $r_s^2 = s$, with s varying from 0 to 20 with a step equals 0.1.
- Outlier of both translation and scale: $r_s^1 = r_s^2 = s$, with s varying from -20 to 20 with a step equals 0.1.

For each type of outlier, the general scheme of the construction of the sensitivity curves has been as follows:

Step 1. A sample $\widetilde{\mathbf{x}}_{100}$ of 100 trapezoidal fuzzy numbers has been simulated from the 1stSP, considering the two cases of simulation involved in this procedure.

Step 2. One observation from the original sample $\widetilde{\mathbf{x}}_{100}$ has been chosen randomly and replaced by the outlier $\widetilde{y}_s$.

Step 3. For each s, the value of the sensitivity curve has been calculated for each estimator of scale as follows:

Definition 2.4.1. *Let $\mathcal{X}$ be an RFN, $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ be a sample of observations from $\mathcal{X}$ and $\widehat{\mathrm{D}}$ a dispersion estimate. The **sensitivity curve** of the estimate $\widehat{\mathrm{D}}$ for the sample $\widetilde{\mathbf{x}}_n$ is the function associating with each $s \in \mathbb{R}$ the difference*

$$\mathrm{SC}(s) = \widehat{\mathrm{D}}(\widetilde{\mathbf{x}}_n^{[s]}) - \widehat{\mathrm{D}}(\widetilde{\mathbf{x}}_n)$$

where the sample $\widetilde{\mathbf{x}}_n^{[s]}$ is obtained by replacing an arbitrary observation of $\widetilde{\mathbf{x}}_n$ by the outlier $\widetilde{y}_s$.

The sensitivity curves have been graphically displayed for each estimator in Figures 2.38 to 2.40 for Case 1 and in Figures 2.41 to 2.43 for Case 2.

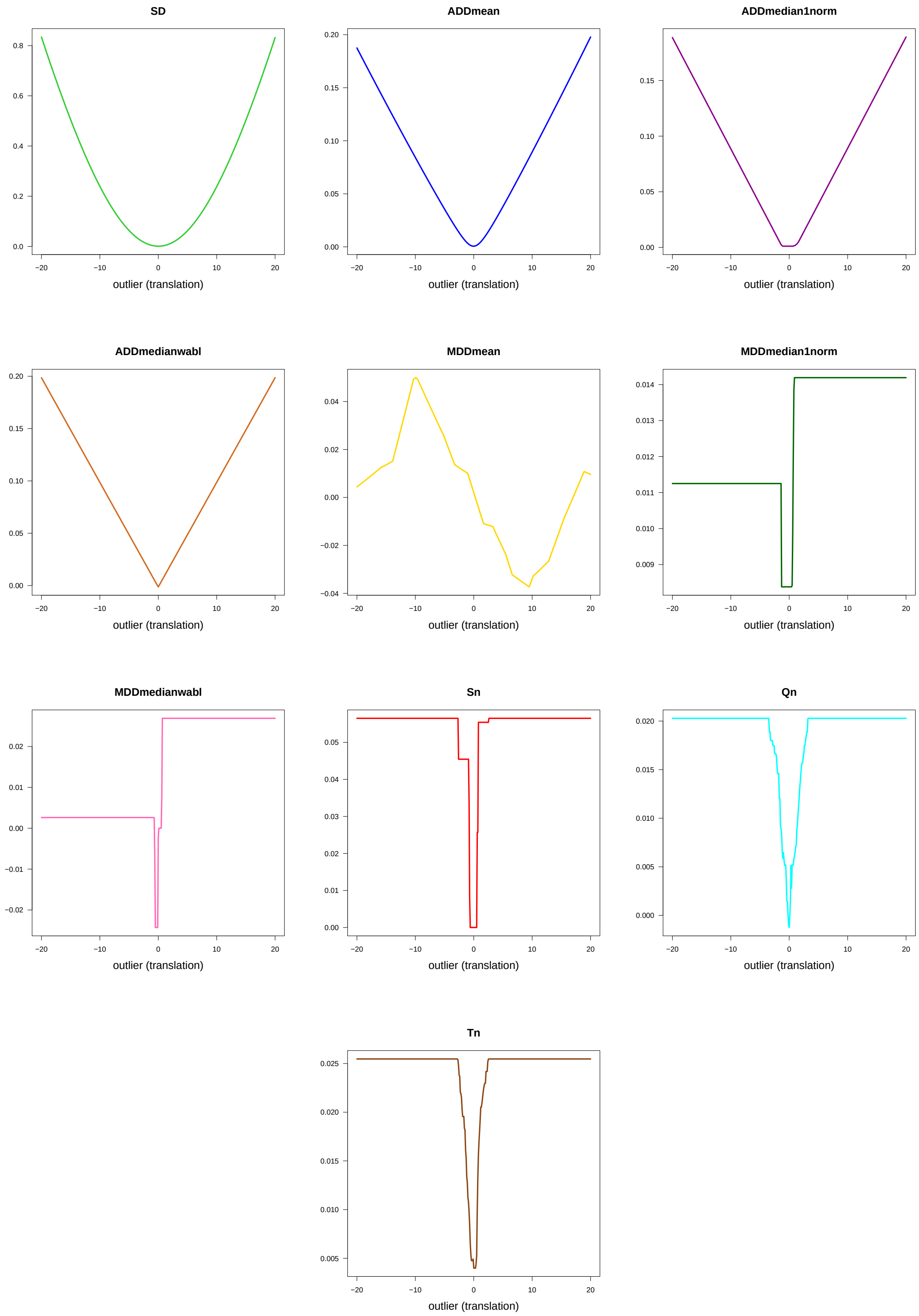


Figure 2.38: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of translation - 1stSP Case 1

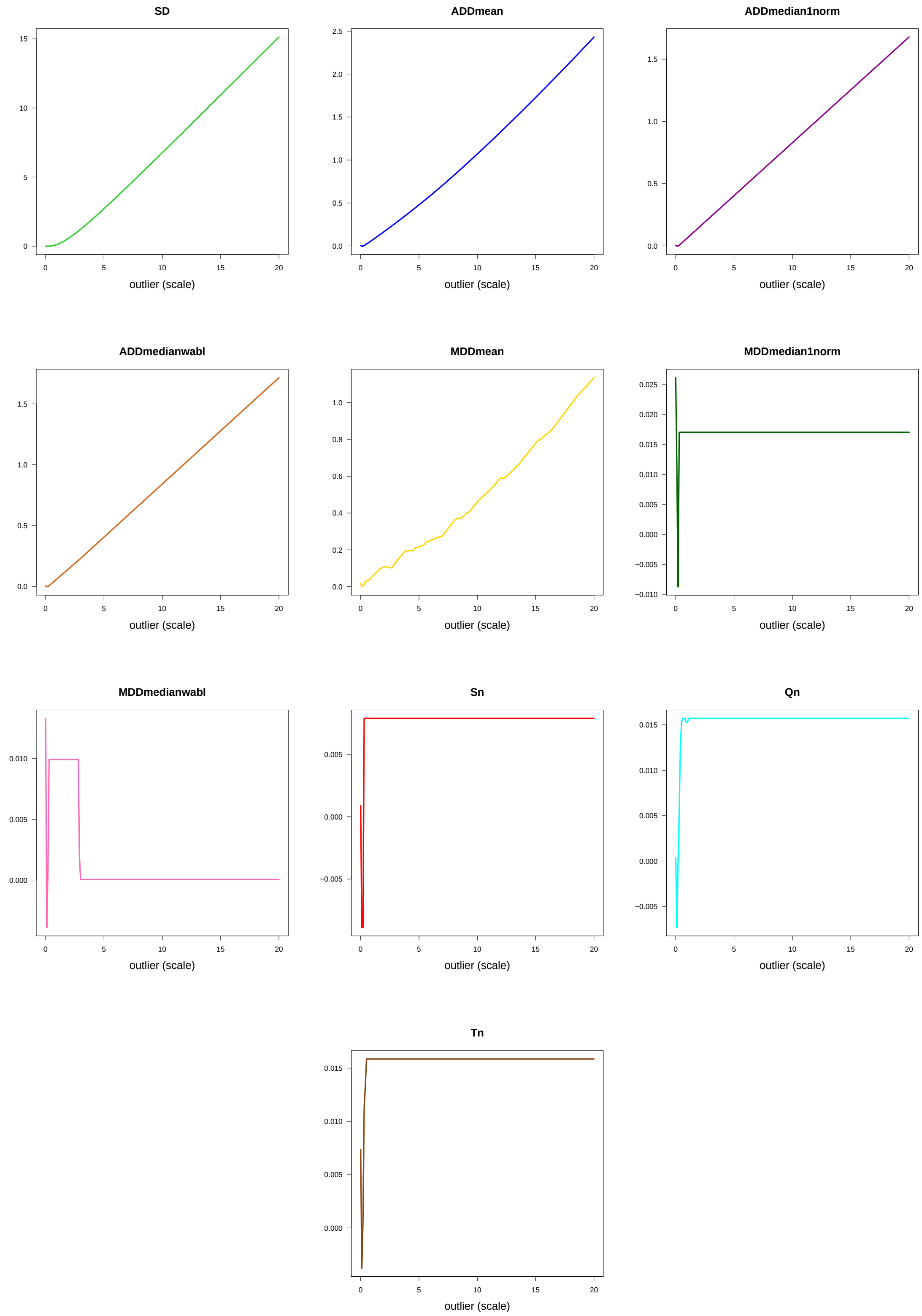


Figure 2.39: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of scale on core and support - 1stSP Case 1

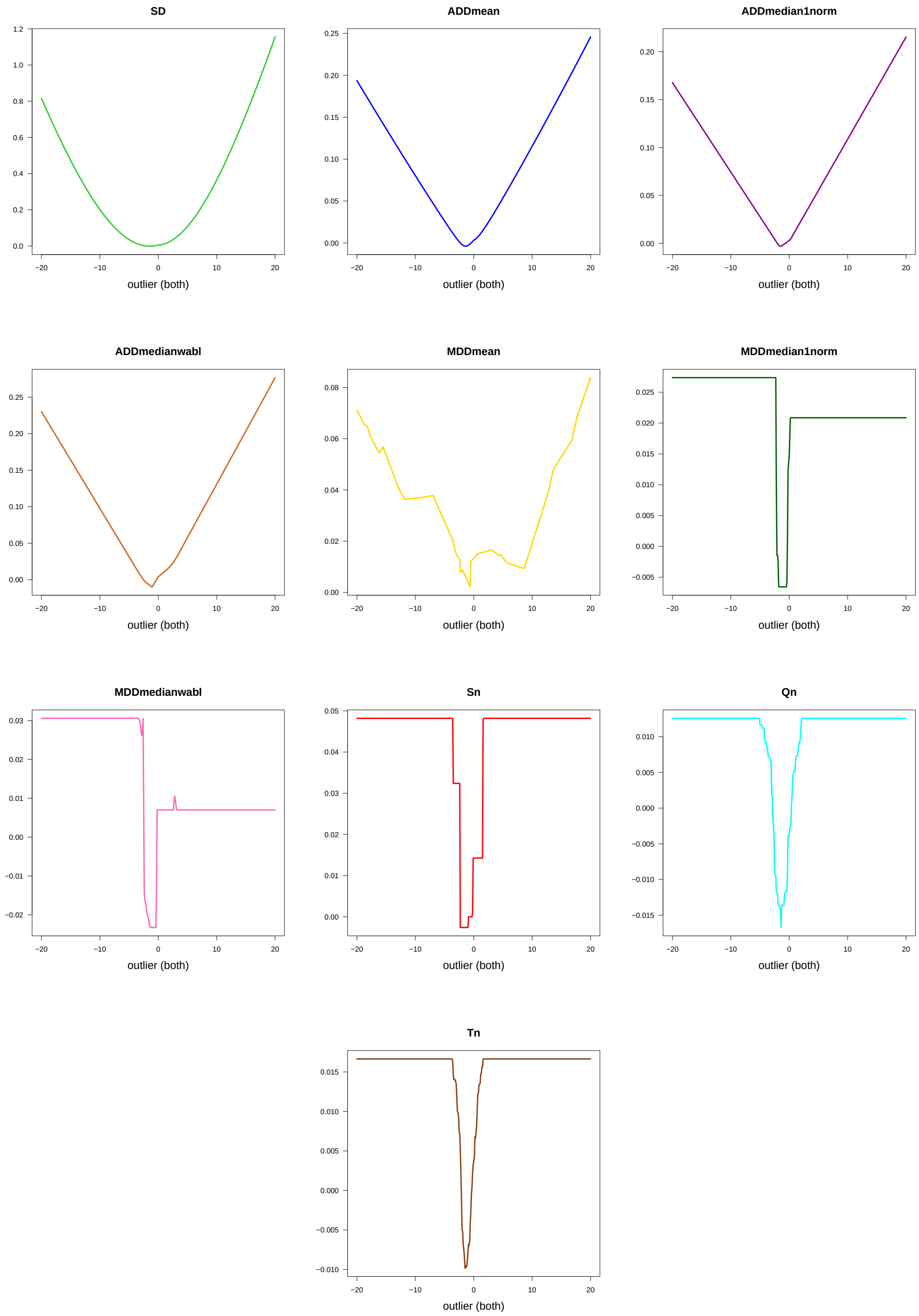


Figure 2.40: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of both translation and scale on core and support - 1stSP Case 1

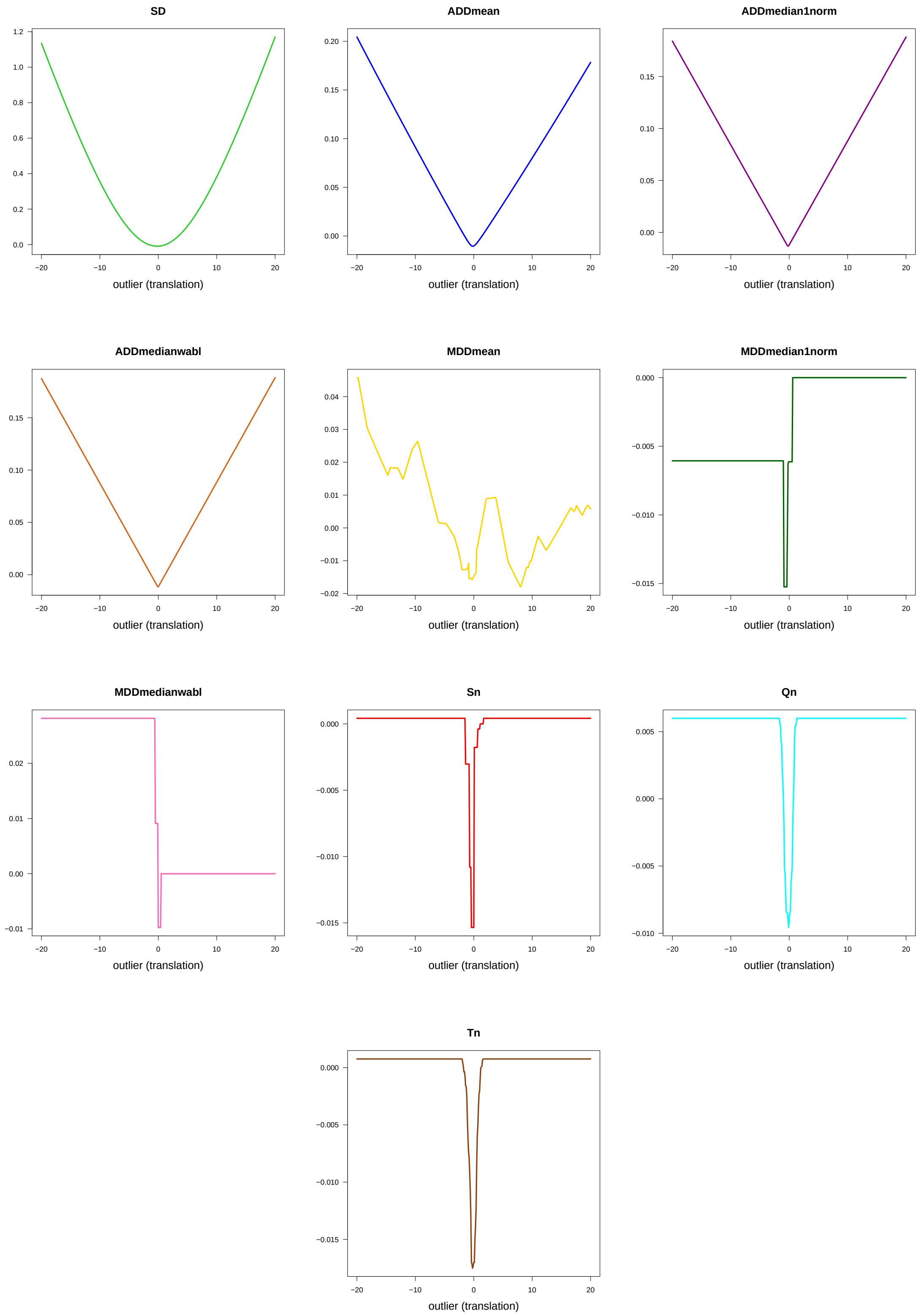


Figure 2.41: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of translation - 1stSP Case 2

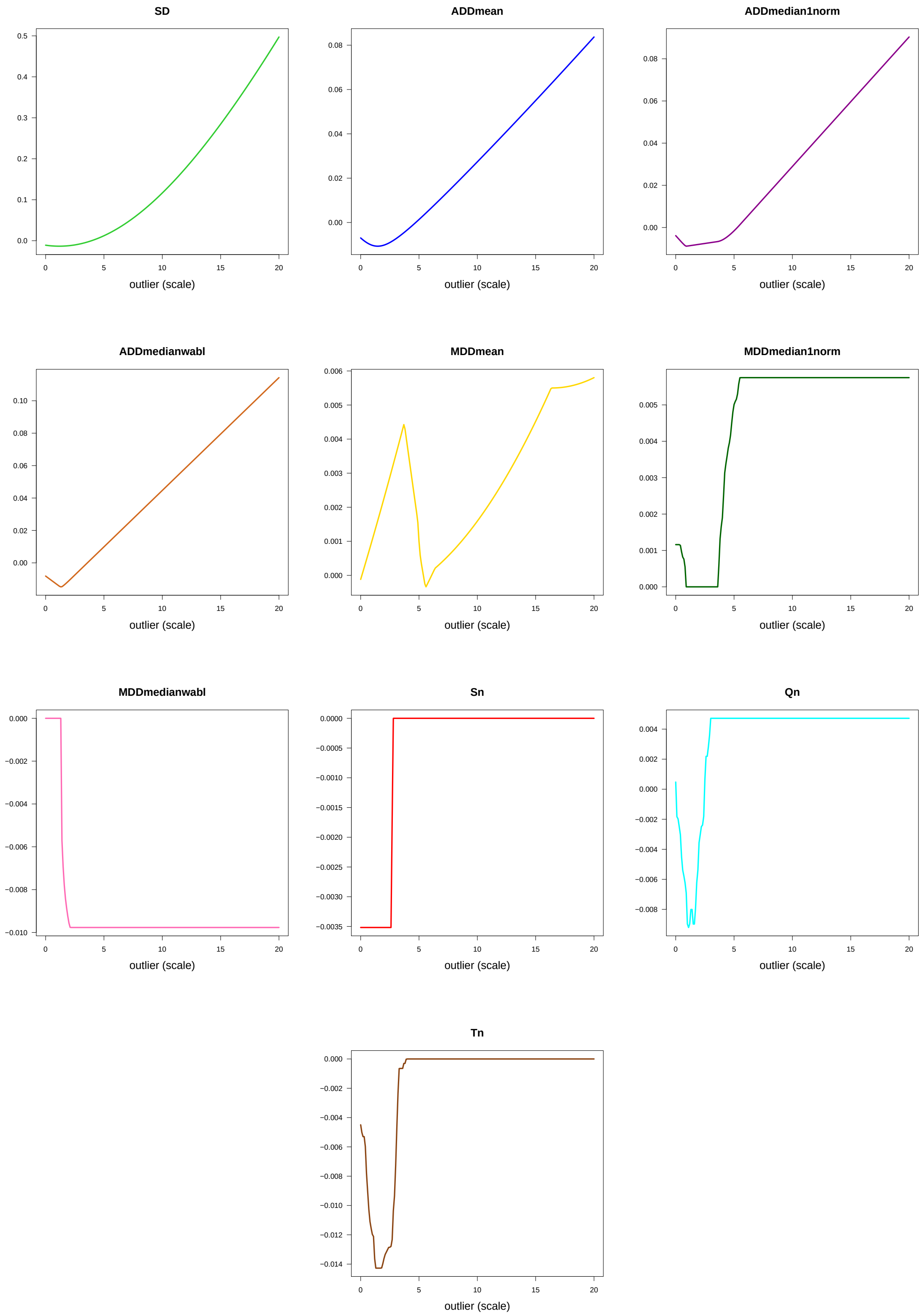


Figure 2.42: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of scale on core and support - 1stSP Case 2

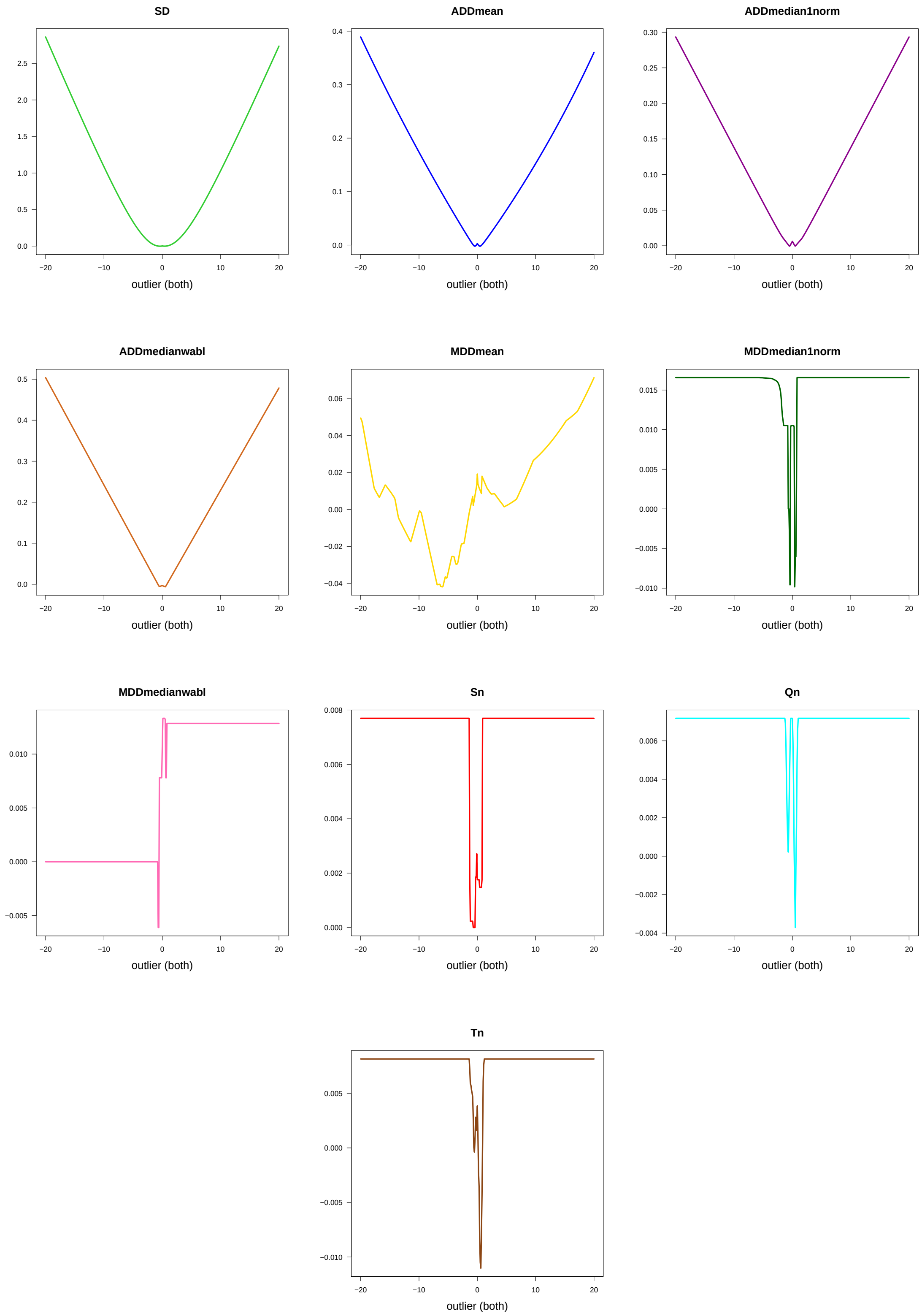


Figure 2.43: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of both translation and scale on core and support - 1stSP Case 2

Irrespective of the type of outlier that we are considering, we can see that the sensitivity curves in the two cases are

- bounded for the robust scale estimates $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$;
- unbounded for the remaining ones $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$.

The preceding developments are now to be carried out by considering the SECOND SIMULATION PROCEDURE (2ndSP) (see Page 39). Notice that, as it happens for the fsbp, when there is a bounded reference interval the notion of sensitivity curve does not make sense, and we will refer to the pseudo-sensitivity curve.

In this study we have only considered the sample size $n = 100$, with the weights $\omega_1 = 0.8$, $\omega_2 = 0.1$ and $\omega_3 = 0.1$.

The outlier $\widetilde{y}_s$ has been constructed as follows:

- *Symmetric distribution:* the non-contaminated sample has been generated from a beta $\beta(100, 100)$.
  - Outlier of translation: $r_s^1 = s$, with s varying from -20 to 20 with a step equals 0.1.
  - Outlier of scale on the core and support: $r_s^2 = s$, with s varying from 0 to 20 with a step equals 0.1.
  - Outlier of both translation and scale: $r_s^1 = r_s^2 = s$, with s varying from -20 to 20 with a step equals 0.1.

- *Asymmetric distribution:* the non-contaminated sample has been generated from a beta $\beta(1, 100)$.
  - Outlier of translation: $r_s^1 = s$, with s varying from 0 to 20 with a step equals 0.1.
  - Outlier of scale on the core and support: $r_s^2 = s$, with s varying from 0 to 20 with a step equals 0.1.
  - Outlier of both translation and scale: $r_s^1 = r_s^2 = s$, with s varying from 0 to 20 with a step equals 0.1.

For each type of outlier, the general scheme of the simulation is the same that the one for the 1stSP (see Page 143), and the criterion in Remark 2.4.1 is also considered.

The pseudo-sensitivity curves have been graphically displayed for each estimator in Figures 2.44 to 2.46 for the symmetric distribution and in Figures 2.47 to 2.49 for the asymmetric one.

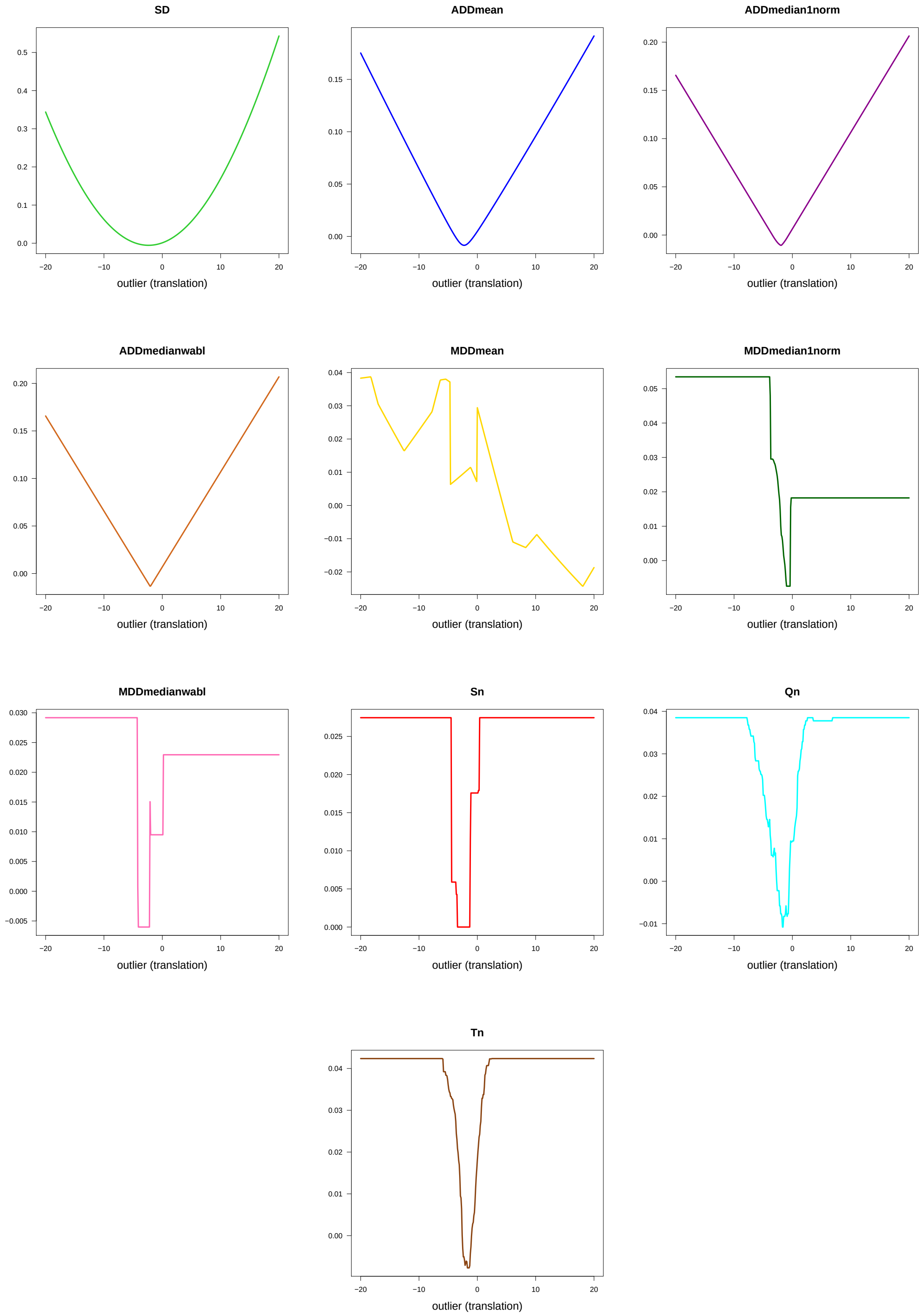


Figure 2.44: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of translation - 2ndSP
Symmetric distribution

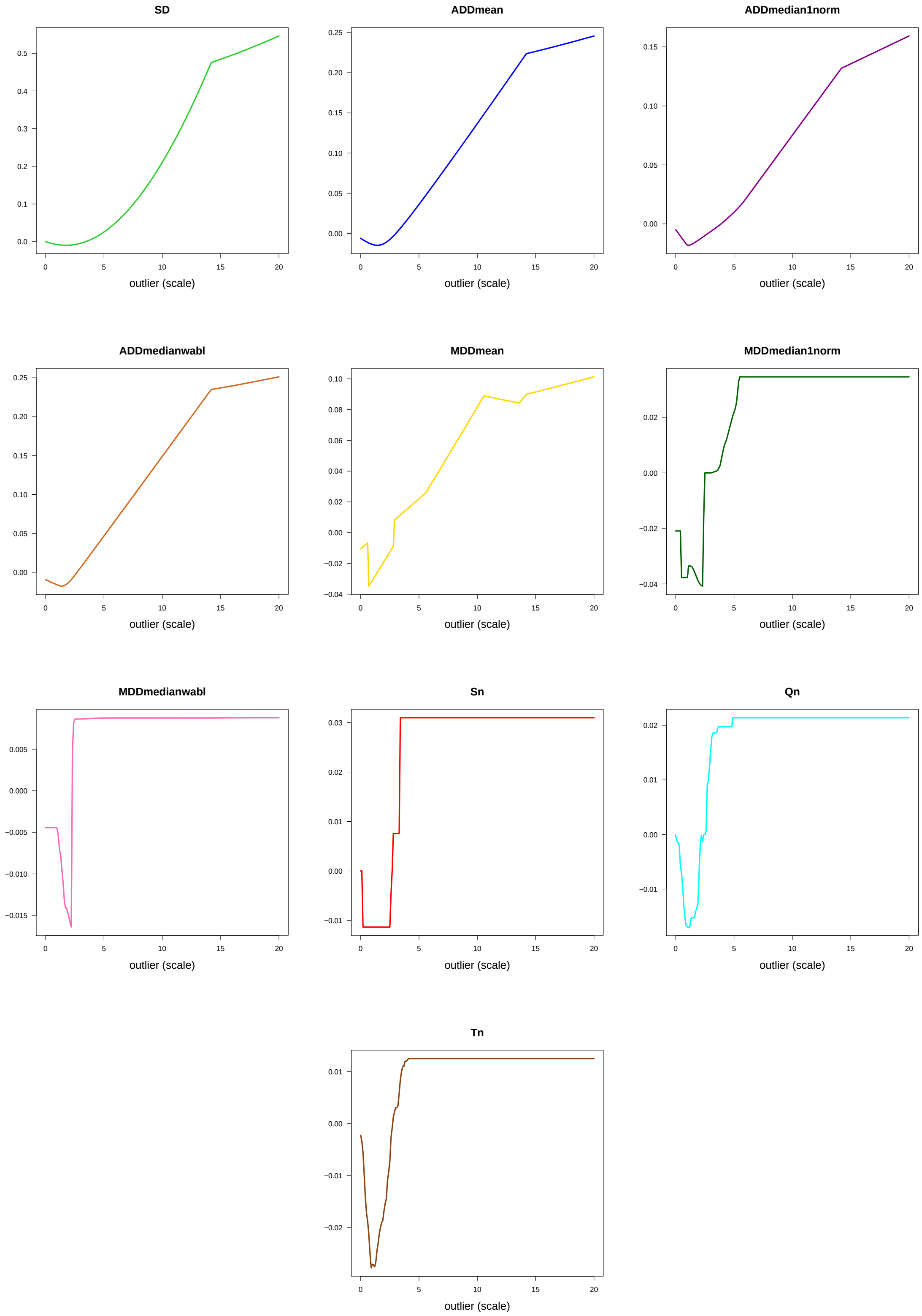


Figure 2.45: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of scale on core and support - 2ndSP Symmetric distribution

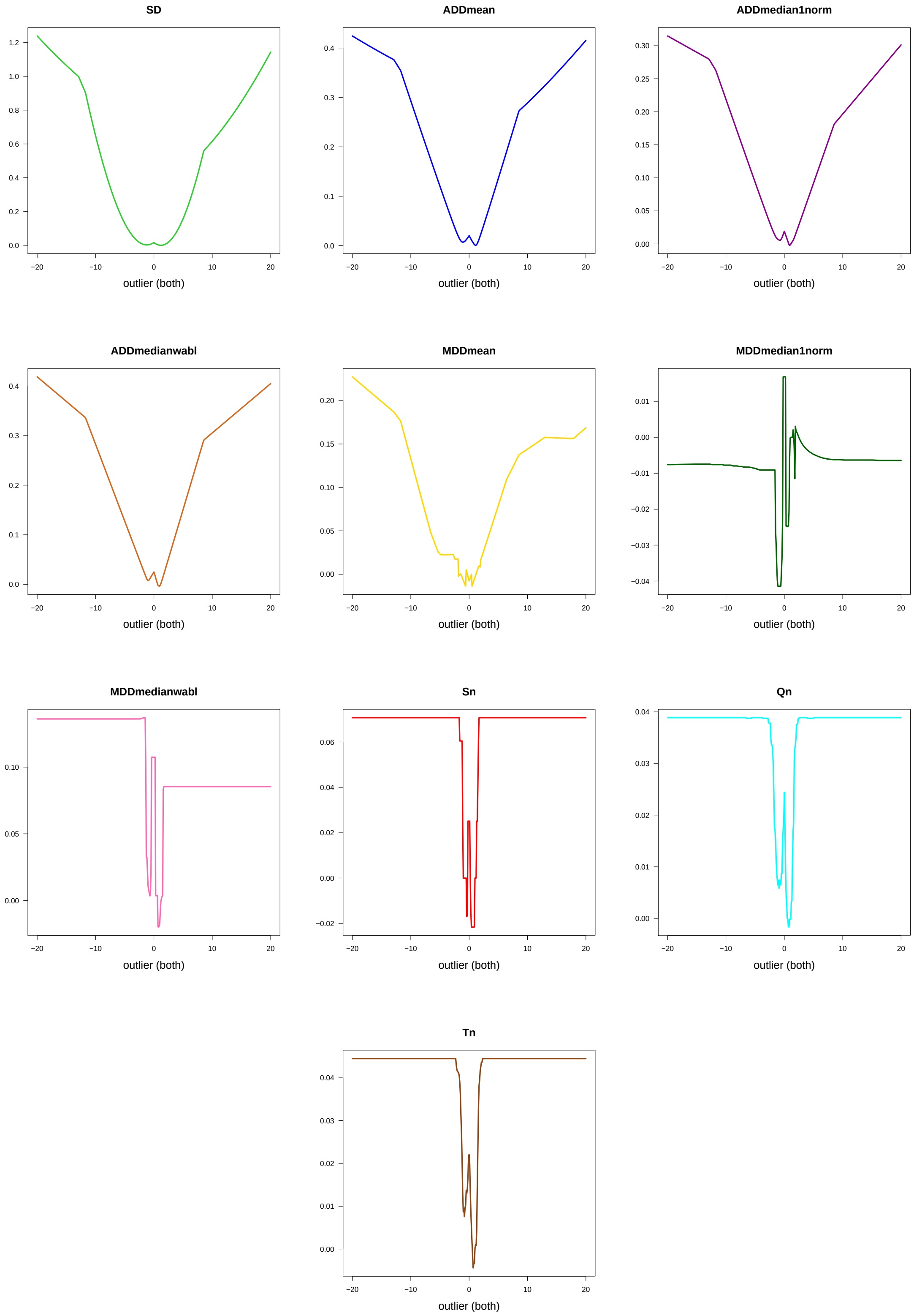


Figure 2.46: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of both translation and scale on core and support - 2ndSP Symmetric distribution

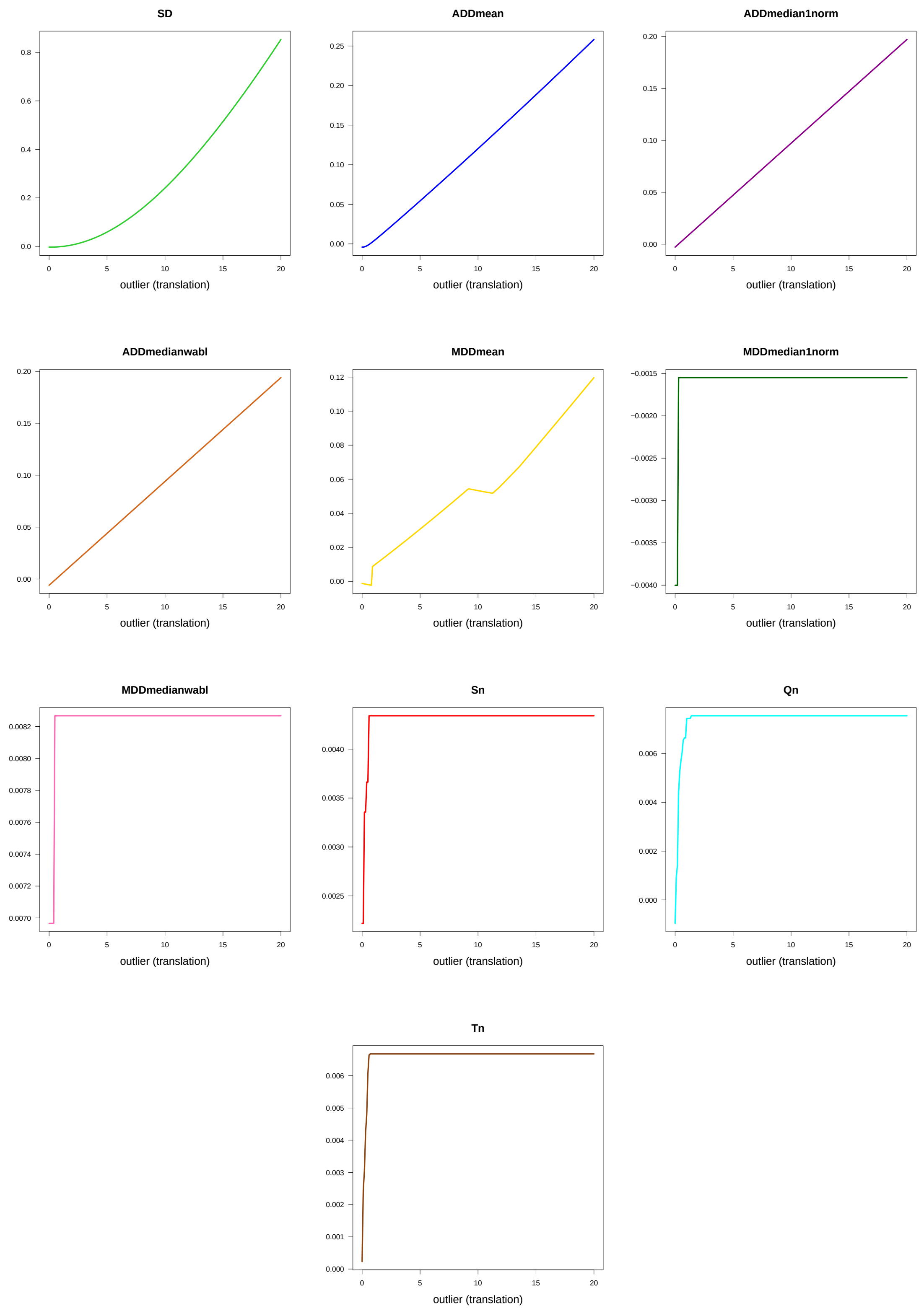


Figure 2.47: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of translation - 2ndSP
Asymmetric distribution

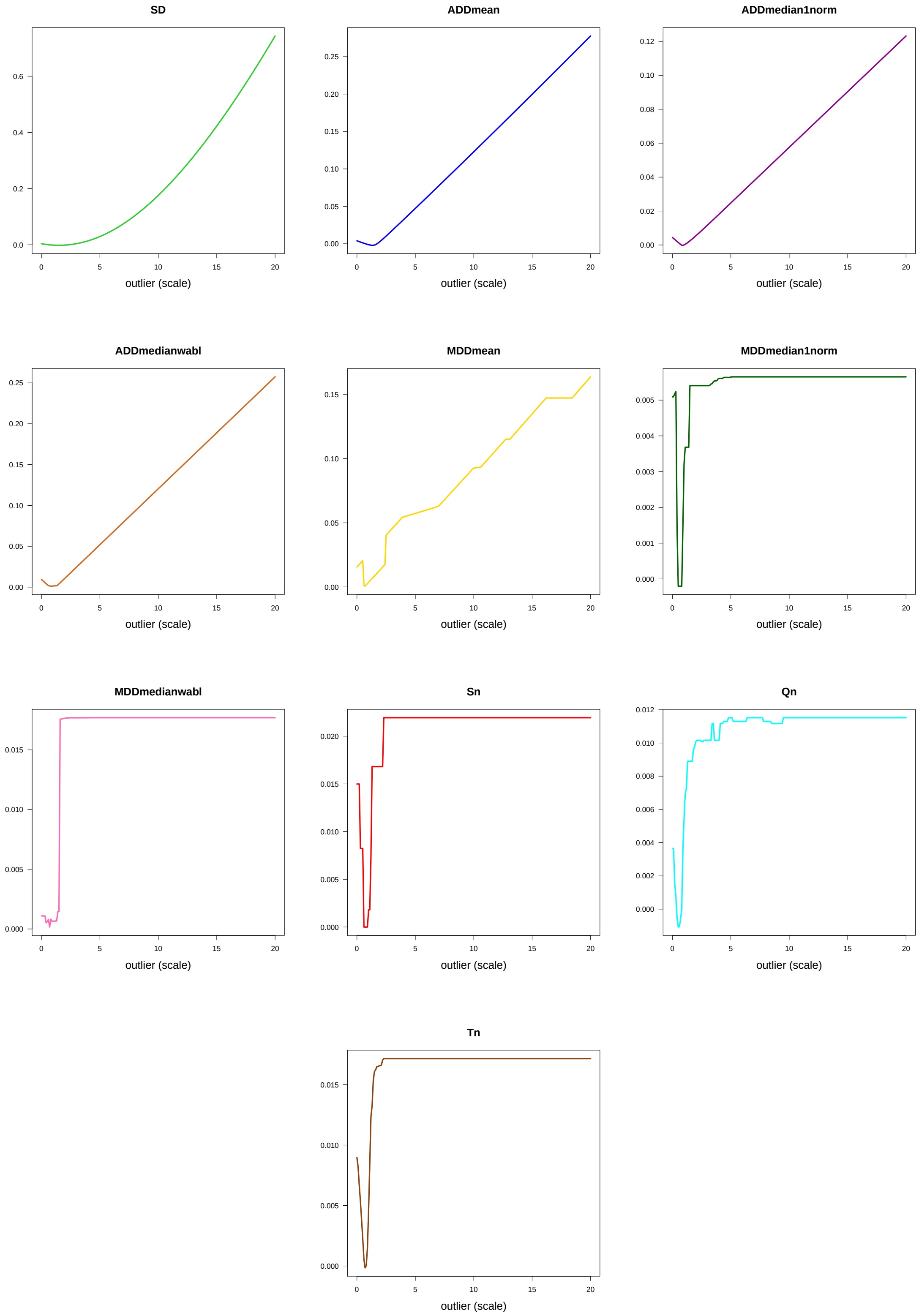


Figure 2.48: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of scale on core and support - 2ndSP Asymmetric distribution

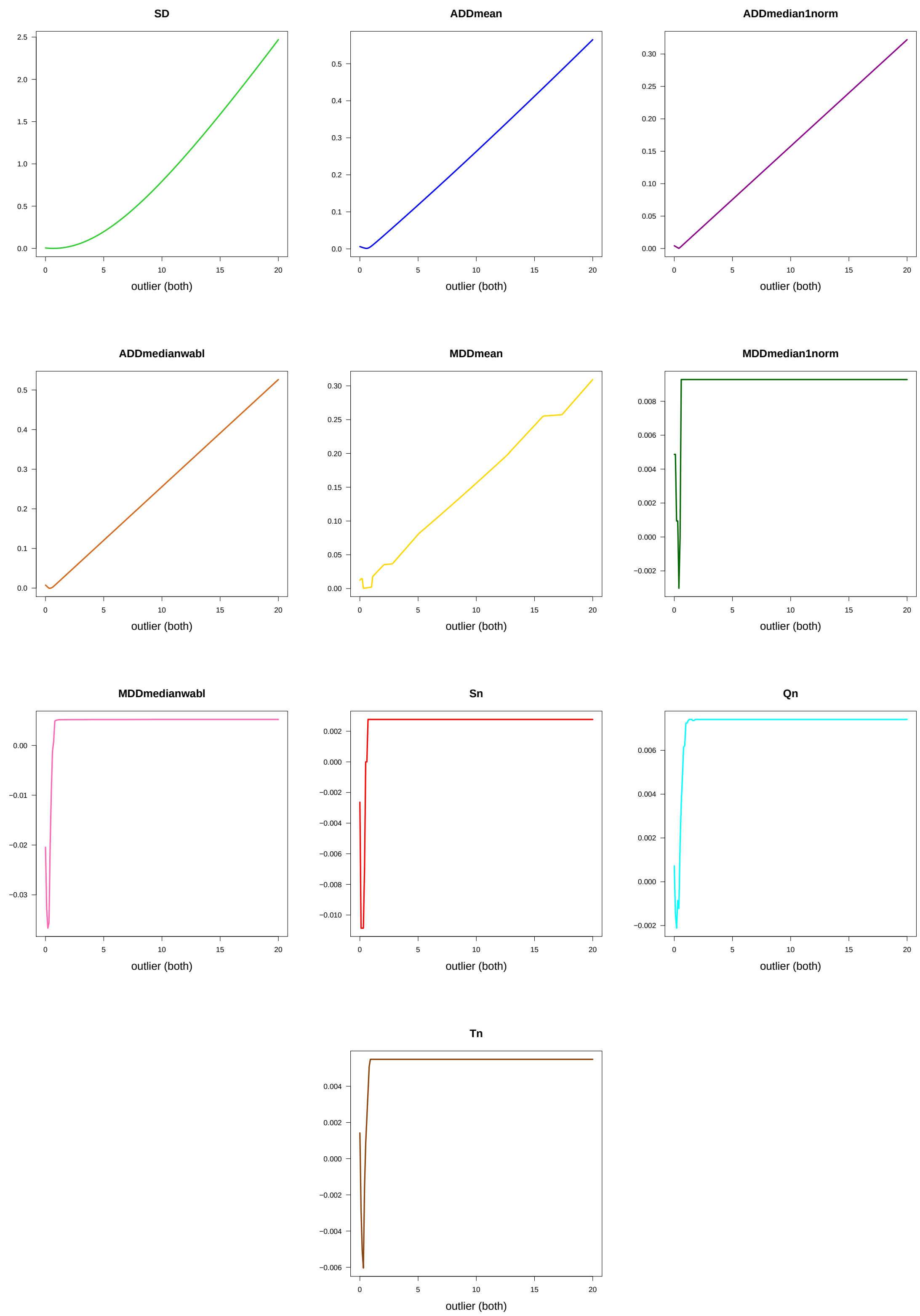


Figure 2.49: Sensitivity curves of the scale estimators for a sample of size 100 and outliers of both translation and scale on core and support - 2ndSP Asymmetric distribution

Irrespective of the type of outlier we are considering, the pseudo-sensitivity curves in the two distributions are

- bounded for the robust scale estimates $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}}^\ell}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$;
- 'unbounded' for the other estimates $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}}^\ell}(\widetilde{\mathbf{x}}_n))$ and $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$.

2.5 Another approach for the scale estimation: the M-estimation of scale

Another approach to estimating scale is now to be tackled. This section is focussed on the extension to the fuzzy setting of a well-known family of estimators of scale which play an important role in the robust estimation because of their suitable properties: the so-called M-estimators of scale. The notion of M-estimator of scale can easily be extended from the real-valued case to the case where the available data are fuzzy number-valued.

Anyway, it should be emphasized that, in spite of their good behaviour in the presence of either outliers or subtle changes in the data, M-estimators have a clear handicap in comparison with the ones in Definitions 2.1.1-2.1.6: they do not have an explicit formula, so iterative algorithms should be designed and performed to compute their value, as will be shown along this section, and they are more computationally expensive. For this reason some authors (see Rousseeuw and Croux [99]) have expressed in the real-valued case a slight preference for robust estimators with an explicit expression, like those in Section 2.1 for the fuzzy framework.

Robust M-estimation is a 'modified' maximum likelihood estimation and, according to Maronna *et al.* [81], it leads to less sensitive estimators to the presence of outlying observations.

Following Maronna *et al.* [81] to formalize the M-scale estimation, observations are assumed to be defined in accordance with a model $\mathcal{X}_i = \sigma \cdot \mathcal{E}_i$ $(i = 1, \ldots, n)$, where $\mathcal{E}_i$ are independent and identically distributed RFN's and σ is the scale parameter to be estimated. Two concepts should first be recalled.

Definition 2.5.1. *A **loss function** $\varrho : \mathbb{R} \to \mathbb{R}$ is a continuous function such that*

- *it is even (actually, it is often expressed as a function of its absolute value),*
- *it is non-decreasing on the positive numbers with $\varrho(0) = 0$, and*
- *if ϱ is bounded it will be assumed, without loss of generality, that $\varrho(\infty) = 1$.*

Loss functions play a key role in M-scale estimation. They should ideally define an estimator close to the classical one (often the maximum likelihood or the least squares estimator) when there is no outlier and, furthermore, provide an estimator close enough to the actual parameter when the data are contaminated.

Robust M-scale estimation makes use also of the so-called weight function associated with a wide class of appropriate loss functions.

Definition 2.5.2. *Given a loss function ϱ with a quadratic behaviour near the origin (more concretely, $\varrho'(0) = 0$ and $\varrho''(0) > 0$), the **weight function** $W : \mathbb{R} \to \mathbb{R}$ associated with ϱ is defined as*

$$W(x) = \begin{cases} \varrho(x)/x^2 & \text{if } x \neq 0 \\ \varrho''(0) & \text{if } x = 0. \end{cases}$$

Adapting Huber's ideas (see Huber [64]) for real-valued data, an M-estimate of scale for fuzzy data is formalized as follows:

Definition 2.5.3. *Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{X} : \Omega \to \mathscr{F}_c^*(\mathbb{R})$ be an associated random fuzzy number. Moreover, let ϱ be a loss function and $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ a simple random sample from $\mathcal{X}$. Then, any estimate $\widehat{\sigma}(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ (for short $\widehat{\sigma}$) satisfying an equation of the form*

$$\frac{1}{n}\sum_{i=1}^{n} \varrho\left(\frac{D(\mathcal{X}_i, \mathbb{1}_{\{0\}})}{\widehat{\sigma}}\right) = \delta,$$

*where δ is a positive constant and D is a metric between fuzzy numbers, is said to be an **M-estimator of scale**.*

In case ϱ has a quadratic behaviour near 0, then M-estimators of scale fulfill the equation

$$\widehat{\sigma}^2 = \frac{1}{n\delta}\sum_{i=1}^{n} W\left(\frac{D(\mathcal{X}_i, \mathbb{1}_{\{0\}})}{\widehat{\sigma}}\right) \cdot D(\mathcal{X}_i, \mathbb{1}_{\{0\}})^2.$$

Remark 2.5.1. Note that in order for the equations in Definition 2.5.3 to have a solution, it should be $0 < \delta < \varrho(\infty)$. Hence, if the loss function ϱ is bounded, then it will be assumed without loss of generality that $\varrho(\infty) = 1$ (as assumed in Definition 2.5.1) and $\delta \in (0, 1)$.

The M-estimators of scale satisfy the scale (absolute) equivariance condition. That is,

Proposition 2.5.1. ***(Scale absolute equivariance).*** *If $\mathcal{X}$ is an RFN, $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$ a simple random sample from $\mathcal{X}$, $\widehat{\sigma}$ a scale M-estimate and $\gamma \in \mathbb{R}$, then*

$$\widehat{\sigma}(\gamma \cdot \mathcal{X}_1, \ldots, \gamma \cdot \mathcal{X}_n) = |\gamma| \cdot \widehat{\sigma}(\mathcal{X}_1, \ldots, \mathcal{X}_n).$$

Proof. If $\widehat{\sigma}$ is the M-estimator of scale for $(\mathcal{X}_1, \ldots, \mathcal{X}_n)$, then,

$$\frac{1}{n}\sum_{i=1}^{n} \varrho\left(\frac{D(\mathcal{X}_i, \mathbb{1}_{\{0\}})}{\widehat{\sigma}}\right) = \delta$$

for some $\delta > 0$. Therefore,

$$\delta = \frac{1}{n}\sum_{i=1}^{n} \varrho\left(\frac{D(\mathcal{X}_i, \mathbb{1}_{\{0\}})}{\widehat{\sigma}}\right) = \frac{1}{n}\sum_{i=1}^{n} \varrho\left(\frac{|\gamma| \cdot D(\mathcal{X}_i, \mathbb{1}_{\{0\}})}{|\gamma| \cdot \widehat{\sigma}}\right) = \frac{1}{n}\sum_{i=1}^{n} \varrho\left(\frac{D(\gamma \cdot \mathcal{X}_i, \mathbb{1}_{\{0\}})}{|\gamma| \cdot \widehat{\sigma}}\right),$$

whence $|\gamma| \cdot \widehat{\sigma}$ is the scale M-estimate for the simple random sample $(\gamma \cdot \mathcal{X}_1, \ldots, \gamma \cdot \mathcal{X}_n)$. □

2.5.1 An algorithm to compute M-estimators of scale for fuzzy data

As we have already mentioned at the beginning of this section, the M-estimators of scale cannot be calculated in a direct way, as they do not have an explicit formula and require the solution of an equation. There are several methods available for computing M-estimates of scale, but we will only focus on an computational method known as iterative reweighting (see Maronna *et al.* [81]), which is explained now in detail.

Let $\mathcal{X}$ be a random fuzzy number associated with the probability space $(\Omega, \mathcal{A}, P)$, $\widetilde{\mathbf{x}}_n = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$ a sample of observations from $\mathcal{X}$ and $D \in \{\rho_1, \mathscr{D}_\theta^\varphi, D_\theta^\varphi\}$ a metric for fuzzy data. Consider a loss function ϱ being quadratic near the origin. The algorithm is the following:

Step 1. Take an initial robust estimate $\widehat{\sigma}_1$. This initial value can be set, for instance, to be the sample $\widehat{\rho_1\text{-MDD}}$ with respect to the 1-norm median, $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$ (see also, for instance, Shahriari *et al.* [103] for a recently corrected suggestion in the real-valued case). Fix a tolerance ϵ.

Step 2. Given $\widehat{\sigma}_k$ $(k = 1, 2...)$, compute the weights

$$\omega_{k,i} = W\left(\frac{D(\widetilde{x}_i, \mathbb{1}_{\{0\}})}{\widehat{\sigma}_k}\right) \quad (i \in \{1, \ldots, n\})$$

where W is the considered weight function, and let

$$\widehat{\sigma}_{k+1} = \sqrt{\frac{1}{n\delta}\sum_{i=1}^{n} \omega_{k,i} \cdot D(\widetilde{x}_i, \mathbb{1}_{\{0\}})^2}.$$

Step 3. Repeat *Step 2* until

$$\left|\frac{\widehat{\sigma}_{k+1}}{\widehat{\sigma}_k} - 1\right| < \epsilon \quad \text{or} \quad \widehat{\sigma}_{k+1} < 10^{-10}.$$

In that case, stop the algorithm.

The simulations conducted in the next subsection involve M-estimators of scale which are determined by means of this algorithm.

2.5.2 Simulations-based analysis of the robustness of some M-estimates of scale

In this subsection the empirical robustness of some M-estimators of scale is examined. By means of simulations dealing with fuzzy data, it is seen what the empirical finite sample breakdown point is and how the sensitivity curves are for two well-known loss functions: the *Huber loss function*, which corresponds to $\varrho(x) = \min\{x^2, 1\}$ and the *Tukey bisquare loss function* (also called biweight loss function), which corresponds to $\varrho(x) = \min\{3 \cdot x^2 - 3 \cdot x^4 + x^6, 1\}$ (see Croux [23]).

The M-estimators are computed using the algorithm explained in Subsection 2.5.1 with a chosen tolerance $\epsilon = 10^{-5}$, three chosen positive constants δ: 0.1, 0.3 and 0.5, and the metric between fuzzy data ρ_1.

All simulations in this section are developed in accordance with the two simulation procedures described in Section 1.7 (Page 37). Regarding the 2ndSP, data have been generated within the reference interval $[0, 100]$ and only the asymmetric distribution has been considered. Symmetric distribution has not been included in the M-estimator analysis as the empirical explosion pseudo-breakdown point cannot be deduced when the outliers fall within the interval $[0, 100]$. If so, we should consider outliers located further than the observations of the non-contaminated sample, which implies they would fall outside this interval.

We start by analyzing the finite sample breakdown point when the simulations are conduced according to the FIRST SIMULATION PROCEDURE (1stSP) and as explained in Subsection 2.4.2. Thus, the non-contaminated sample has been simulated from the 1stSP considering the two cases (see Page 38). Moreover, two sample sizes have been considered in the simulation study, namely, an even sample size ($n = 100$) and an odd sample size ($n = 101$).

Explosion breakdown point:

To study the breakdown point for explosion, we have considered the three types of outliers explained in Subsection 2.4.1. Namely, if $\widetilde{y}_i$ is the i-th outlier, then we have chosen

$$r_i^1 = \begin{cases} \frac{i+1}{2} \cdot 10^{10} & \text{if } i \text{ is odd} \\ -\frac{i}{2} \cdot 10^{10} & \text{if } i \text{ is even} \end{cases}$$

that is, we have considered observations that are increasingly distant from the data at both right and left sides, and $r_i^2 = i \cdot 10^{10}$, that is, observations that are getting wider.

For each type of outlier, the general scheme of the simulation has been as follows:

Step 1. A sample $\widetilde{\mathbf{x}}_n$ of n trapezoidal fuzzy numbers has been simulated from the 1stSP, with $n \in \{100, 101\}$ and considering the two cases of simulation involved in this procedure.

Step 2. Contaminated samples $\widetilde{\mathbf{y}}_{n,k}$ have been obtained by replacing k observations of the original sample $\widetilde{\mathbf{x}}_n$ by k outliers $\widetilde{y}_i$, with $k \in \{1, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor\}$ and $i \in \{1, \ldots, k\}$. Overall, k contaminated samples, one for each k value.

Step 3. The values of the M-estimators of scale have been calculated for the original sample without contamination $\widetilde{\mathbf{x}}_n$, and for each of k contaminated samples $\widetilde{\mathbf{y}}_{n,k}$. We have chosen the $\widehat{\rho_1\text{-MDD}}$ with respect to the 1-norm median, $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, as initial robust scale measure to start the algorithm.

The simulation-based conclusions in this study are presented through Tables 2.19 to 2.22 and Figures 2.50 to 2.61. More concretely, tables gather the values of the M-estimators of scale when outliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Implosion breakdown point:

To study the breakdown point for implosion, we have considered the inliers being all of them equal to $\mathbb{1}_{\{0\}}$. The general scheme of the simulation has been as follows:

Step 1. A sample $\widetilde{\mathbf{x}}_n$ of n trapezoidal fuzzy numbers has been simulated from the 1stSP, with $n \in \{100, 101\}$ and considering the two cases of simulation involved in this procedure.

Step 2. Contaminated samples $\widetilde{\mathbf{y}}_{n,k}$ have been obtained by replacing k observations of the original sample $\widetilde{\mathbf{x}}_n$ by k inliers $\widetilde{y}$, with $k \in \{1, \ldots, n-1\}$. In total, k contaminated samples, one for each k value.

Step 3. The values of the M-estimators of scale have been calculated for the original sample without contamination $\widetilde{\mathbf{x}}_n$, and for each of k contaminated samples $\widetilde{\mathbf{y}}_{n,k}$. We have chosen the ρ_2-standard deviation, $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, as initial robust scale measure to start the algorithm (this estimator is robust in the implosion case).

The simulation-based conclusions in this study are presented through Tables 2.23 to 2.26 and Figures 2.62 to 2.65. More concretely, tables gather the values of the M-estimators of scale when inliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Table 2.19: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 9, 10, 11, 29, 30, 31, 49, 50\}$ - 1stSP Case 1

# outliers (translation)	0	1	9	10	11	29	30	31	49	50
ρ_1-Huber, $\delta = 0.1$	7.06	7.42	22.84	1610.31	1.41E+10	1.50E+11	1.57E+11	1.65E+11	3.23E+11	3.32E+11
ρ_1-Huber, $\delta = 0.3$	3.27	3.28	4.17	4.33	4.50	18.77	760.11	1.41E+10	1.50E+11	1.57E+11
ρ_1-Huber, $\delta = 0.5$	2.08	2.08	2.29	2.31	2.34	3.37	3.45	3.59	17.27	5.00E+09
ρ_1-Tukey, $\delta = 0.1$	11.48	12.10	39.14	2789.10	2.19E+10	2.28E+11	2.40E+11	2.53E+11	5.21E+11	5.38E+11
ρ_1-Tukey, $\delta = 0.3$	5.16	5.21	6.56	6.77	7.01	32.24	1316.54	2.19E+10	2.19E+11	2.30E+11
ρ_1-Tukey, $\delta = 0.5$	3.14	3.14	3.54	3.57	3.62	5.34	5.46	5.67	29.57	5.00E+09
# outliers (scale)	0	1	9	10	11	29	30	31	49	50
ρ_1-Huber, $\delta = 0.1$	7.33	7.72	22.88	1617.07	3.18E+10	4.16E+11	4.38E+11	4.59E+11	9.05E+11	9.32E+11
ρ_1-Huber, $\delta = 0.3$	3.54	3.63	4.54	4.70	4.86	20.15	820.06	3.18E+10	4.16E+11	4.38E+11
ρ_1-Huber, $\delta = 0.5$	2.16	2.20	2.51	2.58	2.63	3.74	3.86	3.97	17.45	7.12E+09
ρ_1-Tukey, $\delta = 0.1$	11.79	12.52	39.36	2800.83	4.67E+10	6.29E+11	6.65E+11	7.02E+11	1.46E+12	1.51E+12
ρ_1-Tukey, $\delta = 0.3$	5.49	5.63	7.10	7.36	7.63	34.59	1420.39	4.67E+10	6.07E+11	6.38E+11
ρ_1-Tukey, $\delta = 0.5$	3.31	3.36	3.88	3.98	4.07	5.78	5.97	6.15	29.87	7.12E+09
# outliers (both)	0	1	9	10	11	29	30	31	49	50
ρ_1-Huber, $\delta = 0.1$	8.35	8.77	25.56	1804.04	1.39E+10	1.55E+11	1.61E+11	1.73E+11	3.50E+11	3.55E+11
ρ_1-Huber, $\delta = 0.3$	3.80	3.88	4.91	5.10	5.27	23.31	949.81	1.39E+10	1.45E+11	1.52E+11
ρ_1-Huber, $\delta = 0.5$	2.34	2.34	2.54	2.58	2.58	4.14	4.33	4.56	20.58	3.87E+09
ρ_1-Tukey, $\delta = 0.1$	13.41	14.18	43.95	3124.67	2.02E+10	2.36E+11	2.46E+11	2.64E+11	5.53E+11	5.64E+11
ρ_1-Tukey, $\delta = 0.3$	6.02	6.14	7.62	7.90	8.14	40.03	1645.10	2.02E+10	2.11E+11	2.22E+11
ρ_1-Tukey, $\delta = 0.5$	3.57	3.58	3.90	3.96	3.98	6.53	6.79	7.12	35.22	3.87E+09

Table 2.20: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 9, 10, 11, 29, 30, 31, 50, 51\}$ - 1stSP Case 1

# outliers (translation)	0	1	9	10	11	29	30	31	50	51
ρ_1-Huber, $\delta = 0.1$	6.00	6.00	18.04	59.45	1.35E+10	1.49E+11	1.57E+11	1.65E+11	3.31E+11	3.41E+11
ρ_1-Huber, $\delta = 0.3$	2.93	2.93	3.51	3.59	3.66	15.18	31.34	1.24E+10	1.55E+11	1.63E+11
ρ_1-Huber, $\delta = 0.5$	1.98	1.98	2.16	2.17	2.17	2.86	2.91	3.02	20.69	1.15E+10
ρ_1-Tukey, $\delta = 0.1$	9.97	9.97	30.86	102.85	2.07E+10	2.26E+11	2.39E+11	2.52E+11	5.35E+11	5.51E+11
ρ_1-Tukey, $\delta = 0.3$	4.61	4.61	5.53	5.65	5.78	25.76	54.02	1.84E+10	2.26E+11	2.37E+11
ρ_1-Tukey, $\delta = 0.5$	2.99	2.99	3.28	3.30	3.31	4.51	4.60	4.78	35.48	1.64E+10
# outliers (scale)	0	1	9	10	11	29	30	31	50	51
ρ_1-Huber, $\delta = 0.1$	6.16	6.56	18.45	60.83	1.57E+10	2.14E+11	2.25E+11	2.36E+11	4.79E+11	4.94E+11
ρ_1-Huber, $\delta = 0.3$	3.09	3.14	3.61	3.69	3.77	14.81	30.49	1.34E+10	2.23E+11	2.34E+11
ρ_1-Huber, $\delta = 0.5$	2.08	2.11	2.24	2.25	2.25	3.10	3.14	3.19	19.14	1.04E+10
ρ_1-Tukey, $\delta = 0.1$	10.07	10.68	31.65	105.26	2.25E+10	3.23E+11	3.41E+11	3.60E+11	7.74E+11	7.98E+11
ρ_1-Tukey, $\delta = 0.3$	4.82	4.92	5.77	5.91	6.05	25.31	52.65	1.93E+10	3.25E+11	3.41E+11
ρ_1-Tukey, $\delta = 0.5$	3.12	3.16	3.39	3.41	3.42	4.83	4.91	5.00	32.98	1.61E+10
# outliers (both)	0	1	9	10	11	29	30	31	50	51
ρ_1-Huber, $\delta = 0.1$	7.22	7.73	22.41	74.18	1.66E+10	1.83E+11	1.97E+11	2.04E+11	4.36E+11	4.42E+11
ρ_1-Huber, $\delta = 0.3$	3.35	3.39	4.02	4.15	4.30	18.88	39.07	1.35E+10	1.83E+11	1.92E+11
ρ_1-Huber, $\delta = 0.5$	2.27	2.27	2.49	2.53	2.58	3.44	3.50	3.50	22.64	1.04E+10
ρ_1-Tukey, $\delta = 0.1$	11.64	12.35	38.36	128.35	2.21E+10	2.78E+11	2.99E+11	3.12E+11	6.90E+11	7.04E+11
ρ_1-Tukey, $\delta = 0.3$	5.24	5.32	6.48	6.70	6.97	32.10	67.38	1.96E+10	2.67E+11	2.80E+11
ρ_1-Tukey, $\delta = 0.5$	3.35	3.36	3.72	3.79	3.88	5.40	5.52	5.53	38.79	1.63E+10

Table 2.21: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 9, 10, 11, 29, 30, 31, 49, 50\}$ - 1stSP Case 2

# outliers (translation)	0	1	9	10	11	29	30	31	49	50
ρ_1-Huber, $\delta = 0.1$	4.44	4.66	13.21	927.34	1.41E+10	1.50E+11	1.57E+11	1.65E+11	3.23E+11	3.32E+11
ρ_1-Huber, $\delta = 0.3$	2.54	2.57	2.87	2.92	2.99	11.57	468.18	1.41E+10	1.50E+11	1.57E+11
ρ_1-Huber, $\delta = 0.5$	1.90	1.91	1.98	1.99	2.01	2.48	2.53	2.58	9.56	5.00E+09
ρ_1-Tukey, $\delta = 0.1$	7.49	7.87	22.82	1606.19	2.19E+10	2.28E+11	2.40E+11	2.53E+11	5.21E+11	5.38E+11
ρ_1-Tukey, $\delta = 0.3$	4.06	4.11	4.66	4.75	4.87	19.97	810.91	2.19E+10	2.19E+11	2.30E+11
ρ_1-Tukey, $\delta = 0.5$	2.89	2.90	3.06	3.08	3.11	4.00	4.07	4.16	16.49	5.00E+09
# outliers (scale)	0	1	9	10	11	29	30	31	49	50
ρ_1-Huber, $\delta = 0.1$	4.35	4.56	13.07	920.84	1.83E+10	2.40E+11	2.52E+11	2.65E+11	5.21E+11	5.37E+11
ρ_1-Huber, $\delta = 0.3$	2.51	2.54	2.85	2.91	2.98	11.66	472.32	1.83E+10	2.40E+11	2.52E+11
ρ_1-Huber, $\delta = 0.5$	1.93	1.94	2.03	2.05	2.07	2.55	2.59	2.64	9.98	4.10E+09
ρ_1-Tukey, $\delta = 0.1$	7.38	7.75	22.59	1594.93	2.69E+10	3.62E+11	3.83E+11	4.04E+11	8.41E+11	8.68E+11
ρ_1-Tukey, $\delta = 0.3$	4.05	4.10	4.68	4.79	4.91	20.14	818.08	2.69E+10	3.50E+11	3.68E+11
ρ_1-Tukey, $\delta = 0.5$	2.93	2.94	3.13	3.17	3.20	4.10	4.18	4.28	17.21	4.10E+09
# outliers (both)	0	1	9	10	11	29	30	31	49	50
ρ_1-Huber, $\delta = 0.1$	4.09	4.30	12.43	874.61	1.91E+10	2.34E+11	2.46E+11	2.58E+11	5.08E+11	5.23E+11
ρ_1-Huber, $\delta = 0.3$	2.36	2.40	2.71	2.77	2.83	11.27	459.28	1.91E+10	2.34E+11	2.46E+11
ρ_1-Huber, $\delta = 0.5$	1.81	1.82	1.94	1.95	1.97	2.46	2.52	2.56	9.44	5.20E+09
ρ_1-Tukey, $\delta = 0.1$	6.92	7.30	21.48	1514.87	2.87E+10	3.53E+11	3.74E+11	3.94E+11	8.20E+11	8.46E+11
ρ_1-Tukey, $\delta = 0.3$	3.79	3.86	4.44	4.54	4.66	19.47	795.50	2.87E+10	3.41E+11	3.58E+11
ρ_1-Tukey, $\delta = 0.5$	2.73	2.76	2.96	2.99	3.03	3.95	4.06	4.14	16.29	5.20E+09

Table 2.22: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, 9, 10, 11, 29, 30, 31, 50, 51\}$ - 1stSP Case 2

# outliers (translation)	0	1	9	10	11	29	30	31	50	51
ρ_1-Huber, $\delta = 0.1$	4.18	4.39	12.23	40.45	1.35E+10	1.49E+11	1.57E+11	1.65E+11	3.31E+11	3.41E+11
ρ_1-Huber, $\delta = 0.3$	2.42	2.45	2.78	2.84	2.89	9.97	20.59	1.24E+10	1.55E+11	1.63E+11
ρ_1-Huber, $\delta = 0.5$	1.82	1.83	1.94	1.97	1.98	2.45	2.49	2.54	13.28	1.15E+10
ρ_1-Tukey, $\delta = 0.1$	7.07	7.44	21.12	70.05	2.07E+10	2.26E+11	2.39E+11	2.52E+11	5.35E+11	5.52E+11
ρ_1-Tukey, $\delta = 0.3$	3.85	3.91	4.52	4.64	4.74	17.19	35.62	1.84E+10	2.26E+11	2.37E+11
ρ_1-Tukey, $\delta = 0.5$	2.75	2.78	2.99	3.03	3.06	3.91	3.98	4.06	22.95	1.64E+10
# outliers (scale)	0	1	9	10	11	29	30	31	50	51
ρ_1-Huber, $\delta = 0.1$	4.41	4.63	12.72	42.09	3.01E+10	4.11E+11	4.32E+11	4.53E+11	9.20E+11	9.48E+11
ρ_1-Huber, $\delta = 0.3$	2.51	2.54	2.87	2.94	3.00	10.53	21.73	2.58E+10	4.28E+11	4.49E+11
ρ_1-Huber, $\delta = 0.5$	1.88	1.89	1.98	2.00	2.01	2.54	2.58	2.64	13.98	1.99E+10
ρ_1-Tukey, $\delta = 0.1$	7.43	7.82	21.97	72.87	4.32E+10	6.20E+11	6.56E+11	6.92E+11	1.49E+12	1.53E+12
ρ_1-Tukey, $\delta = 0.3$	4.02	4.08	4.66	4.78	4.88	18.15	37.60	3.71E+10	6.24E+11	6.55E+11
ρ_1-Tukey, $\delta = 0.5$	2.86	2.88	3.05	3.09	3.11	4.05	4.13	4.23	24.16	3.09E+10
# outliers (both)	0	1	9	10	11	29	30	31	50	51
ρ_1-Huber, $\delta = 0.1$	4.22	4.44	12.19	40.27	1.50E+10	1.79E+11	1.88E+11	1.97E+11	4.00E+11	4.12E+11
ρ_1-Huber, $\delta = 0.3$	2.41	2.45	2.77	2.83	2.89	10.28	21.25	1.38E+10	1.86E+11	1.96E+11
ρ_1-Huber, $\delta = 0.5$	1.84	1.85	1.94	1.96	1.97	2.51	2.56	2.56	13.59	1.29E+10
ρ_1-Tukey, $\delta = 0.1$	7.13	7.51	21.05	69.73	2.28E+10	2.70E+11	2.86E+11	3.02E+11	6.46E+11	6.67E+11
ρ_1-Tukey, $\delta = 0.3$	3.88	3.95	4.51	4.61	4.73	17.73	36.76	2.03E+10	2.72E+11	2.85E+11
ρ_1-Tukey, $\delta = 0.5$	2.78	2.81	2.98	3.02	3.05	4.03	4.11	4.13	23.49	1.80E+10

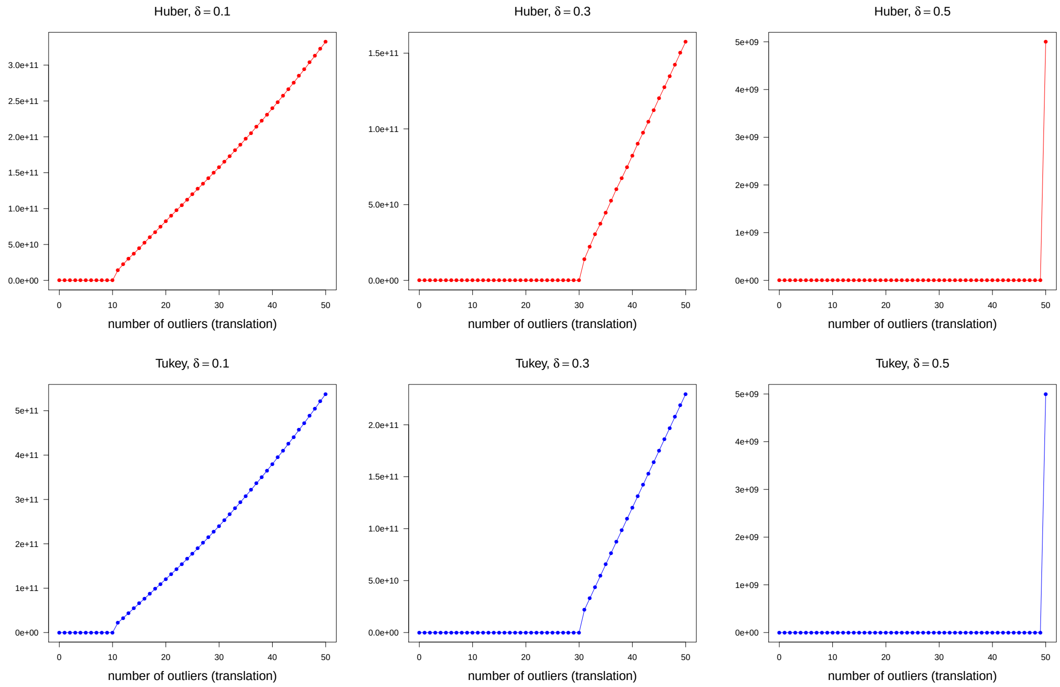


Figure 2.50: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of translation, k varying from 0 to 50 - 1stSP Case 1

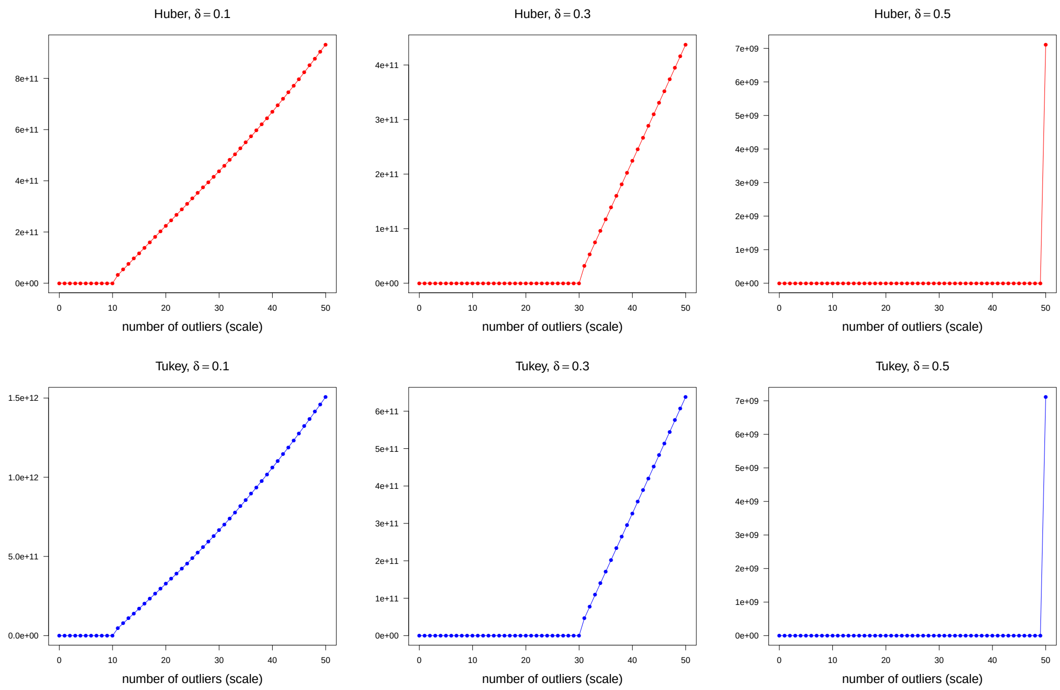


Figure 2.51: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of scale on core and support, k varying from 0 to 50 - 1stSP Case 1

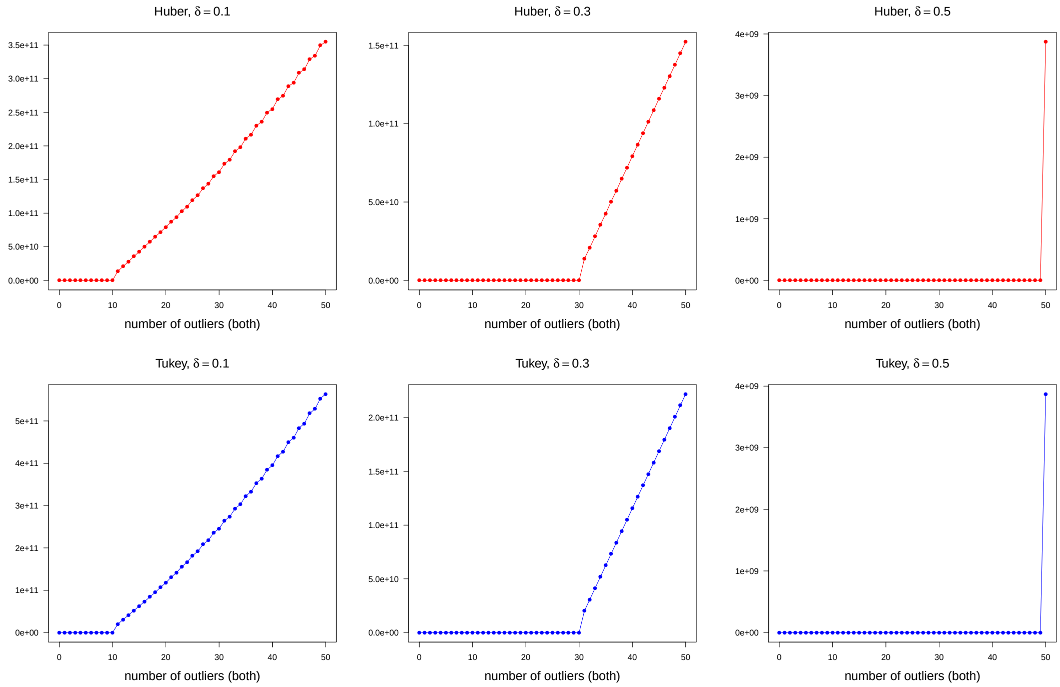


Figure 2.52: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 50 - 1stSP Case 1

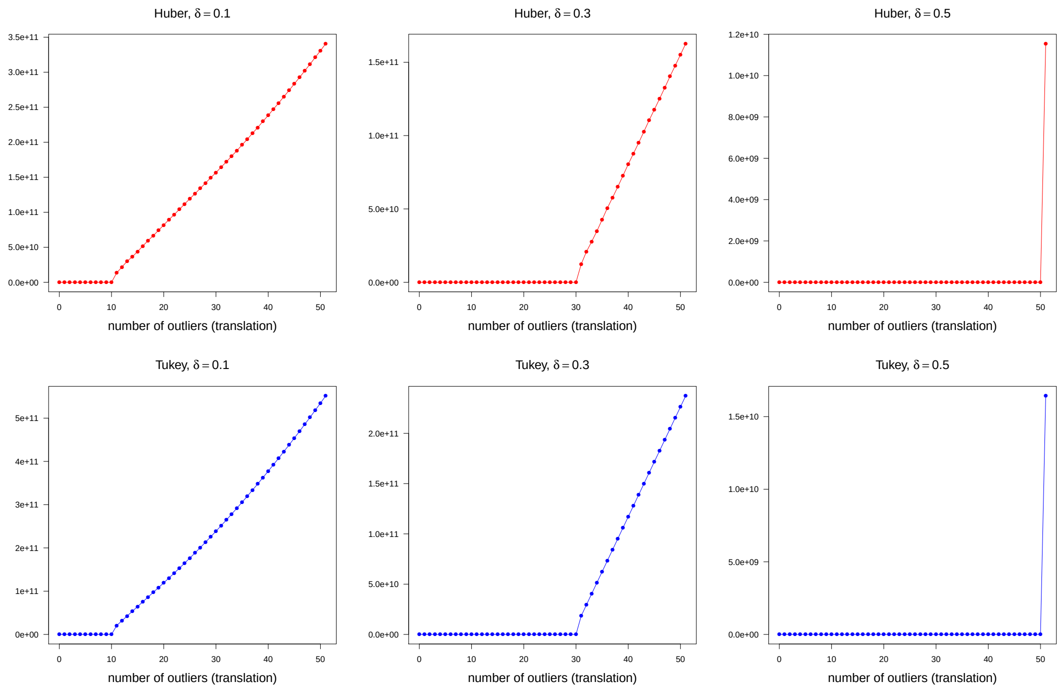


Figure 2.53: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of translation, k varying from 0 to 51 - 1stSP Case 1

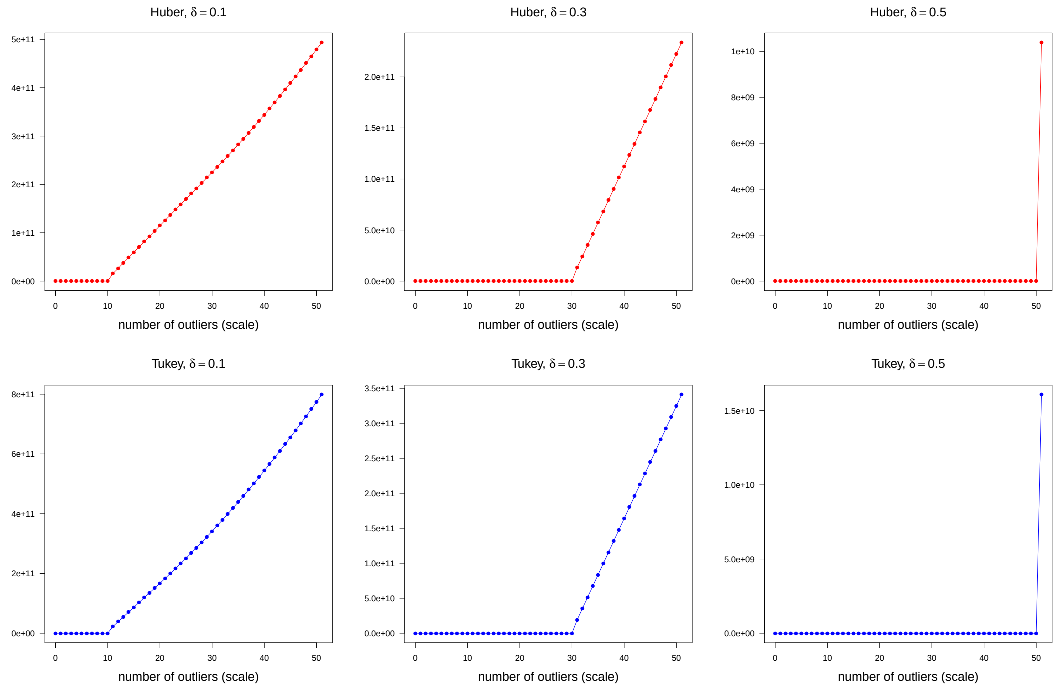


Figure 2.54: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of scale on core and support, k varying from 0 to 51 - 1stSP Case 1

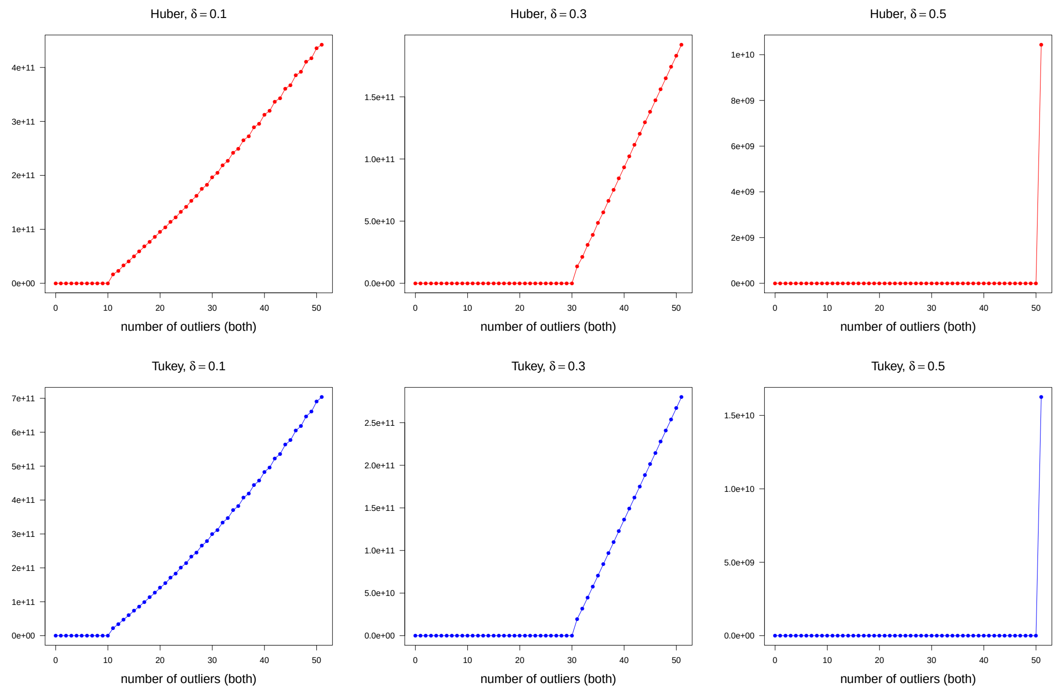


Figure 2.55: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 51 - 1stSP Case 1

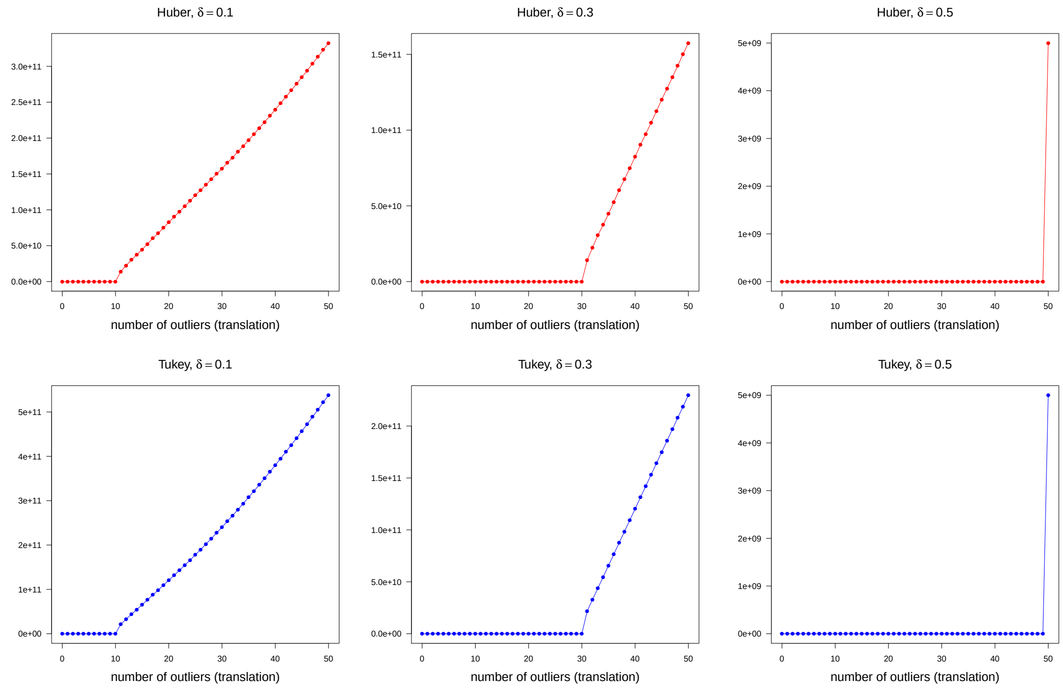


Figure 2.56: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of translation, k varying from 0 to 50 - 1stSP Case 2

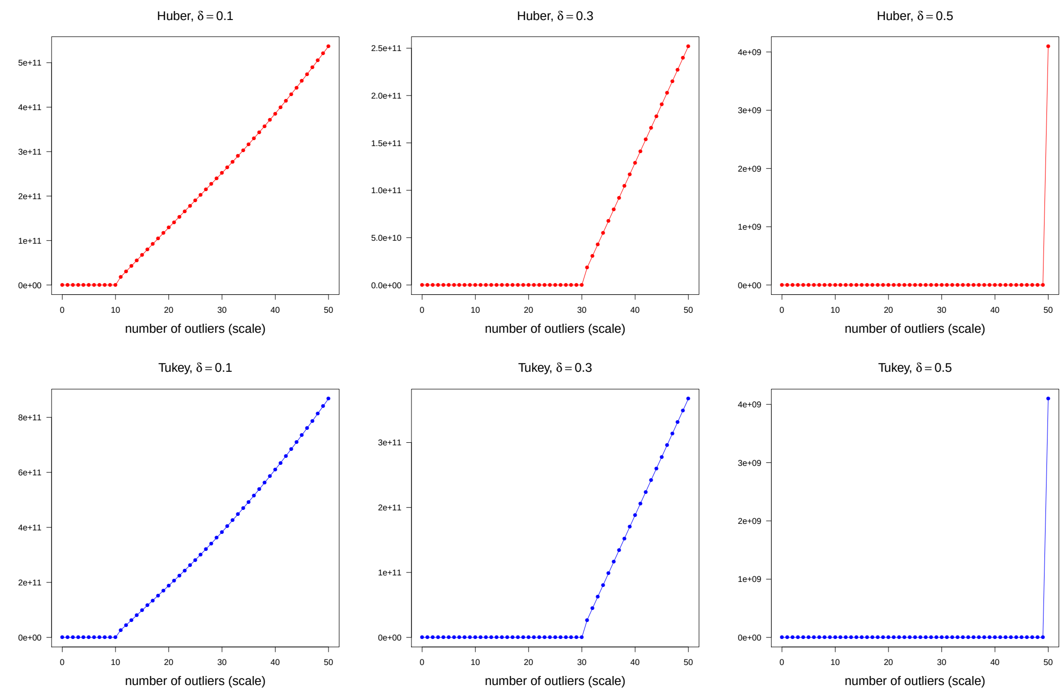


Figure 2.57: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of scale on core and support, k varying from 0 to 50 - 1stSP Case 2

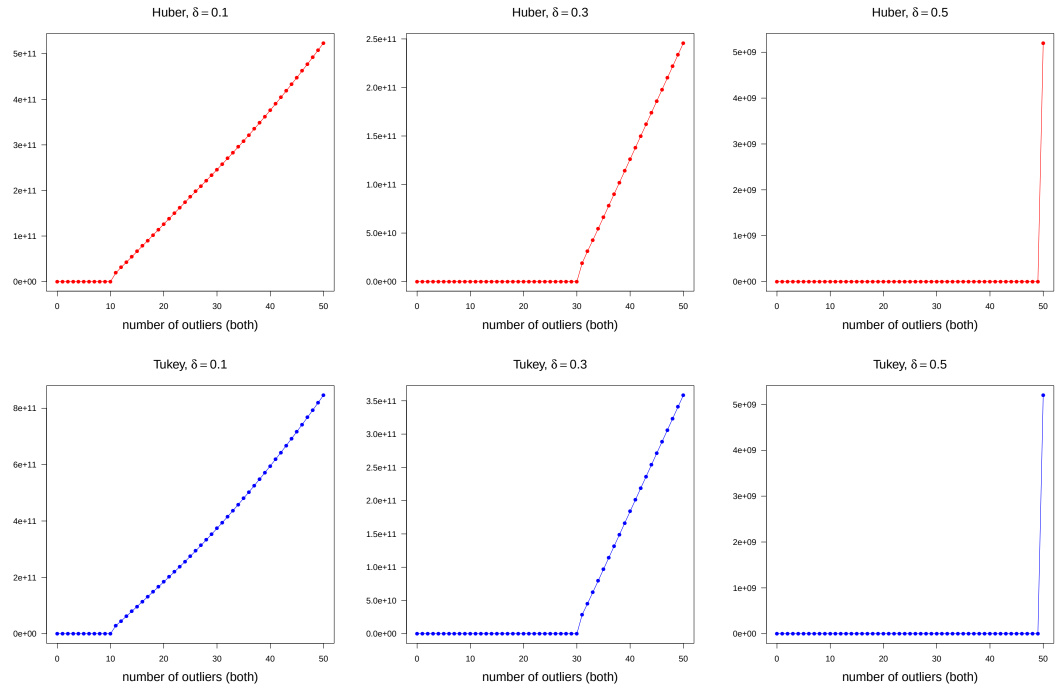


Figure 2.58: Explosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 50 - 1stSP Case 2

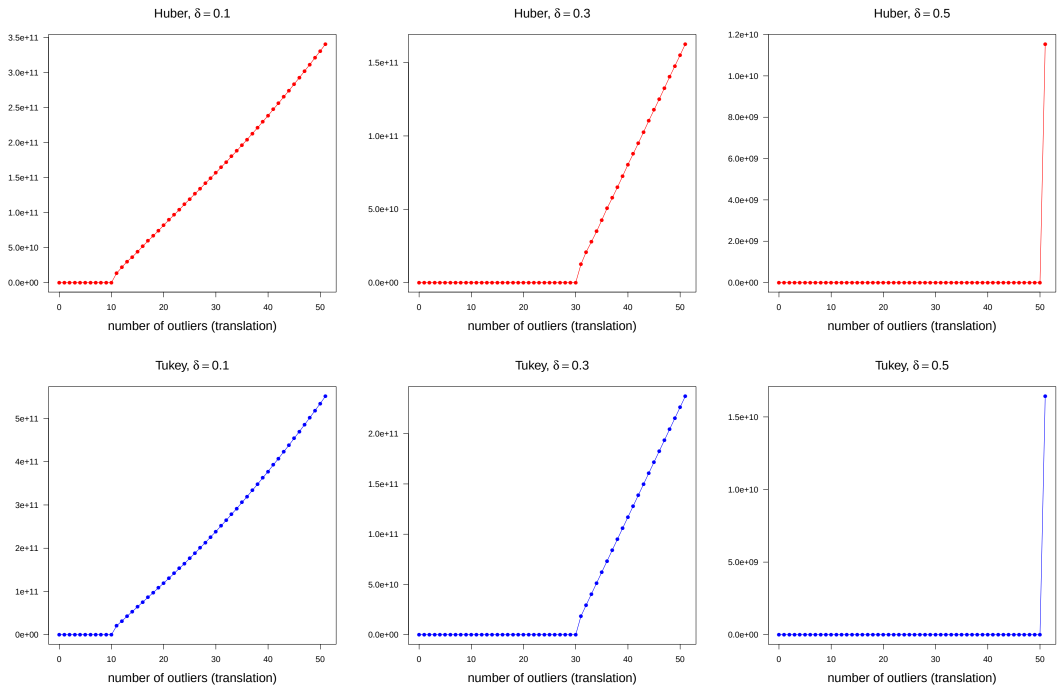


Figure 2.59: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of translation, k varying from 0 to 51 - 1stSP Case 2

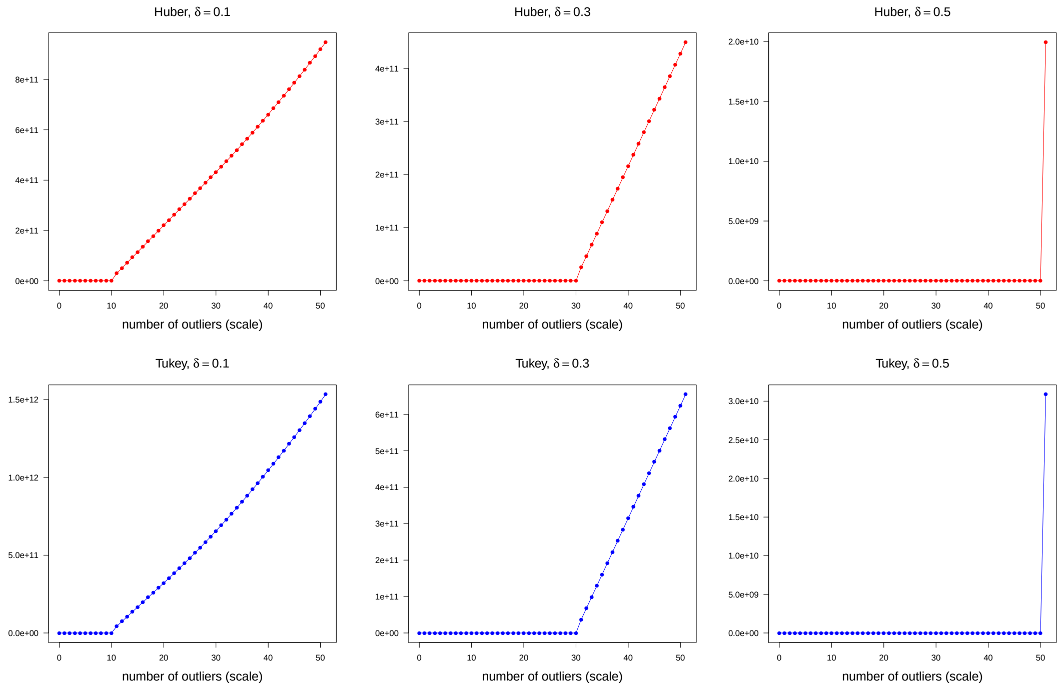


Figure 2.60: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of scale on core and support, k varying from 0 to 51 - 1stSP Case 2

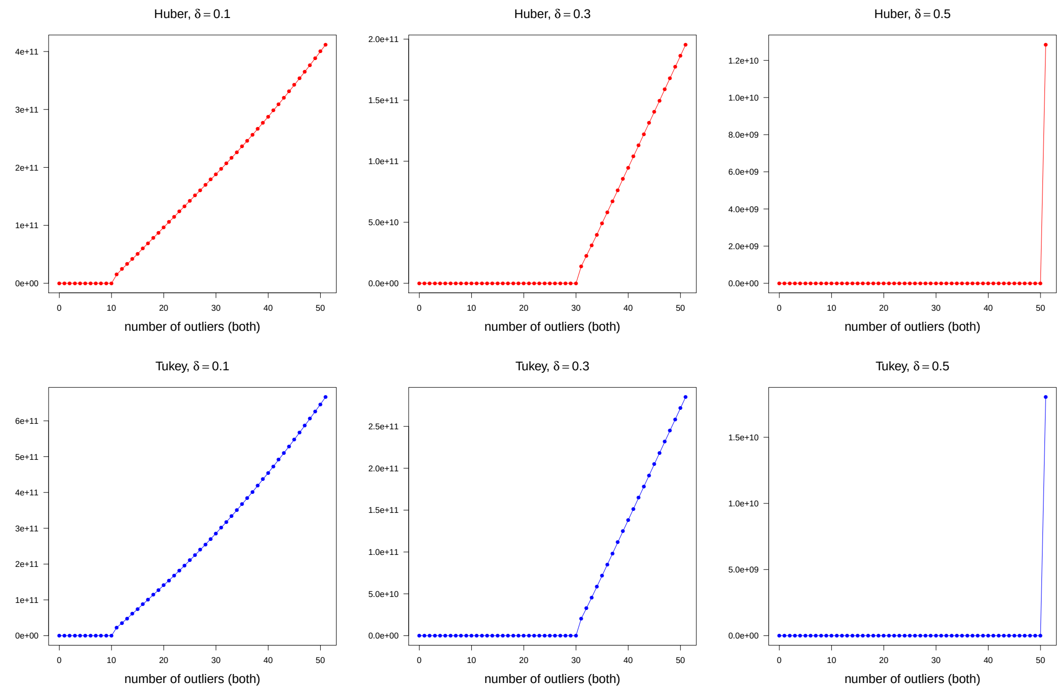


Figure 2.61: Explosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 51 - 1stSP Case 2

Table 2.23: Implosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by inliers, with $k \in \{0, 1, 2, 50, 51, 70, 71, 90, 91\}$ - 1stSP Case 1

# inliers	0	1	2	50	51	70	71	90	91
ρ_1-Huber, $\delta = 0.1$	7.43	7.42	7.38	4.35	4.32	3.34	3.32	0.39	9.58E-11
ρ_1-Huber, $\delta = 0.3$	3.20	3.19	3.14	1.60	1.57	0.39	9.95E-11	5.90E-11	6.98E-11
ρ_1-Huber, $\delta = 0.5$	2.05	2.03	2.00	0.29	9.96E-11	9.91E-11	8.57E-11	4.52E-11	5.31E-11
ρ_1-Tukey, $\delta = 0.1$	11.78	11.76	11.68	6.82	6.78	5.23	5.20	0.40	9.78E-11
ρ_1-Tukey, $\delta = 0.3$	5.09	5.07	4.99	2.36	2.30	0.40	9.85E-11	6.29E-11	7.51E-11
ρ_1-Tukey, $\delta = 0.5$	3.08	3.06	3.00	0.30	9.94E-11	8.87E-11	9.79E-11	4.82E-11	5.71E-11

Table 2.24: Implosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by inliers, with $k \in \{0, 1, 2, 50, 51, 70, 71, 90, 91\}$ - 1stSP Case 1

# inliers	0	1	2	50	51	70	71	90	91
ρ_1-Huber, $\delta = 0.1$	6.96	6.78	6.72	4.33	4.29	3.23	3.20	0.76	9.99E-11
ρ_1-Huber, $\delta = 0.3$	3.64	3.56	3.52	1.91	1.86	0.48	9.98E-11	6.72E-11	6.88E-11
ρ_1-Huber, $\delta = 0.5$	2.44	2.40	2.36	0.37	9.99E-11	8.78E-11	8.90E-11	6.31E-11	7.27E-11
ρ_1-Tukey, $\delta = 0.1$	11.41	11.12	11.02	6.99	6.92	5.11	5.07	1.09	9.99E-11
ρ_1-Tukey, $\delta = 0.3$	5.69	5.56	5.48	2.79	2.71	0.67	9.98E-11	6.72E-11	6.88E-11
ρ_1-Tukey, $\delta = 0.5$	3.69	3.62	3.55	0.54	9.99E-11	7.84E-11	7.77E-11	6.31E-11	7.27E-11

Table 2.25: Implosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by inliers, with $k \in \{0, 1, 2, 50, 51, 70, 71, 90, 91\}$ - 1stSP Case 2

# inliers	0	1	2	50	51	70	71	90	91
ρ_1-Huber, $\delta = 0.1$	4.30	4.29	4.26	2.96	2.93	2.26	2.22	0.46	9.82E-11
ρ_1-Huber, $\delta = 0.3$	2.49	2.48	2.46	1.63	1.60	0.72	9.96E-11	7.68E-11	9.42E-11
ρ_1-Huber, $\delta = 0.5$	1.90	1.89	1.87	0.78	9.99E-11	9.61E-11	8.17E-11	7.60E-11	9.24E-11
ρ_1-Tukey, $\delta = 0.1$	7.29	7.26	7.21	4.88	4.82	3.58	3.51	0.46	9.82E-11
ρ_1-Tukey, $\delta = 0.3$	3.99	3.97	3.94	2.41	2.36	0.72	9.96E-11	7.68E-11	9.42E-11
ρ_1-Tukey, $\delta = 0.5$	2.88	2.86	2.83	0.83	9.96E-11	9.61E-11	8.17E-11	7.60E-11	9.24E-11

Table 2.26: Implosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by inliers, with $k \in \{0, 1, 2, 50, 51, 70, 71, 90, 91\}$ - 1stSP Case 2

# inliers	0	1	2	50	51	70	71	90	91
ρ_1-Huber, $\delta = 0.1$	4.37	4.34	4.33	3.10	3.09	2.44	2.43	1.28	9.99E-11
ρ_1-Huber, $\delta = 0.3$	2.47	2.45	2.44	1.65	1.64	0.88	9.99E-11	7.28E-11	6.58E-11
ρ_1-Huber, $\delta = 0.5$	1.81	1.79	1.78	0.84	9.95E-11	8.79E-11	8.93E-11	6.83E-11	6.95E-11
ρ_1-Tukey, $\delta = 0.1$	7.34	7.29	7.27	5.06	5.04	3.89	3.86	1.71	9.99E-11
ρ_1-Tukey, $\delta = 0.3$	3.93	3.90	3.89	2.46	2.43	1.18	9.96E-11	7.28E-11	6.58E-11
ρ_1-Tukey, $\delta = 0.5$	2.76	2.73	2.72	1.05	9.99E-11	8.81E-11	8.94E-11	6.83E-11	6.95E-11

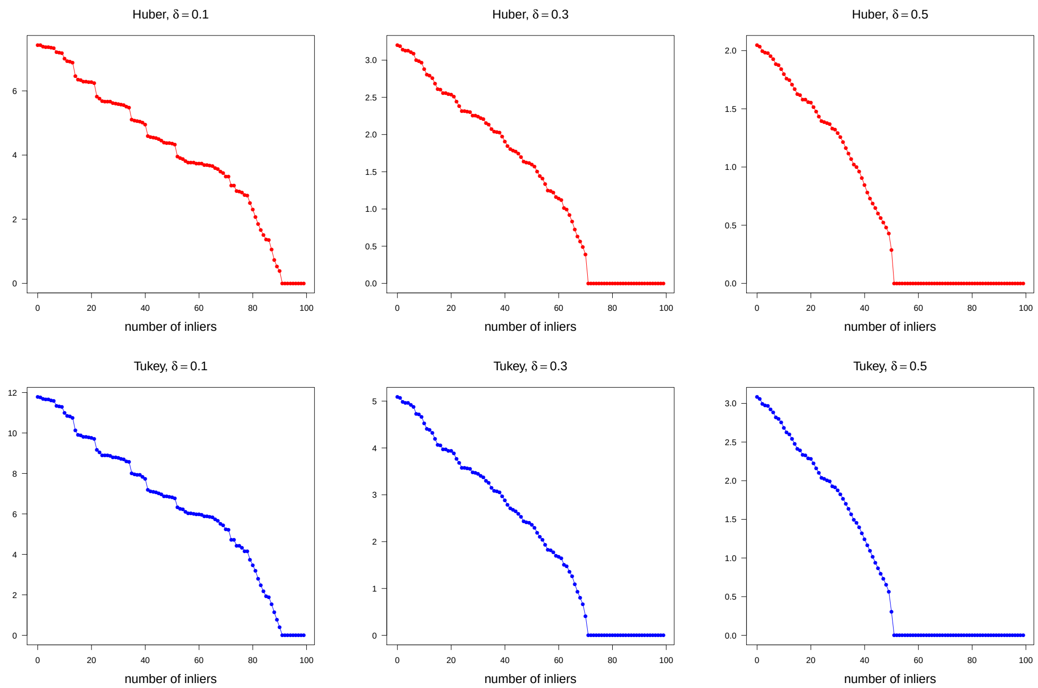


Figure 2.62: Implosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by inliers, k varying from 0 to 99 - 1stSP Case 1

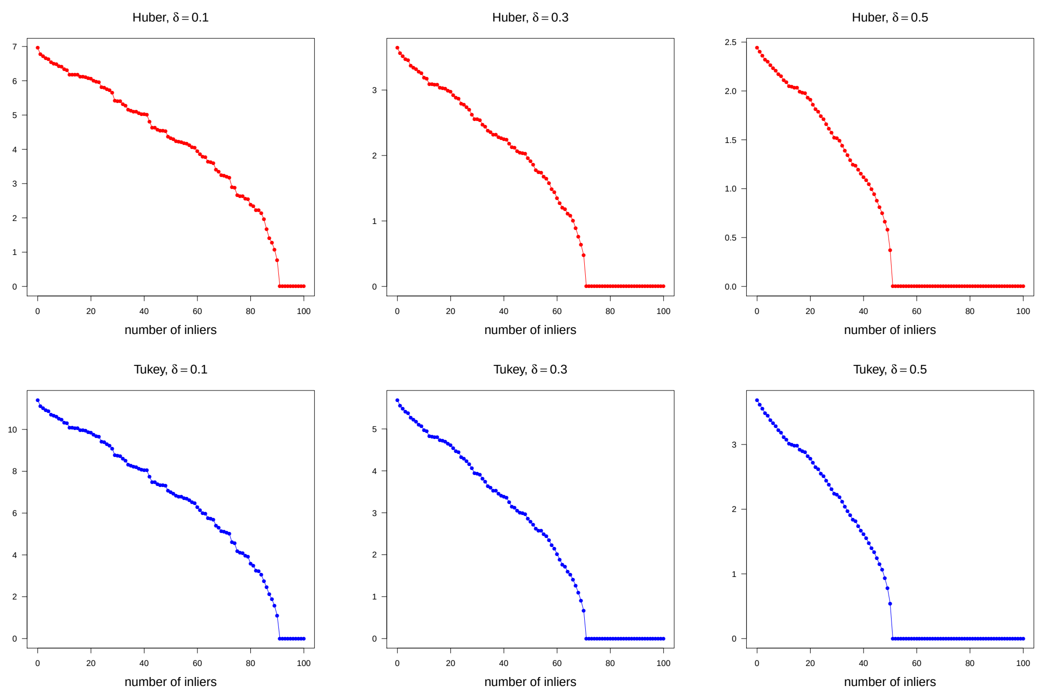


Figure 2.63: Implosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by inliers, k varying from 0 to 100 - 1stSP Case 1

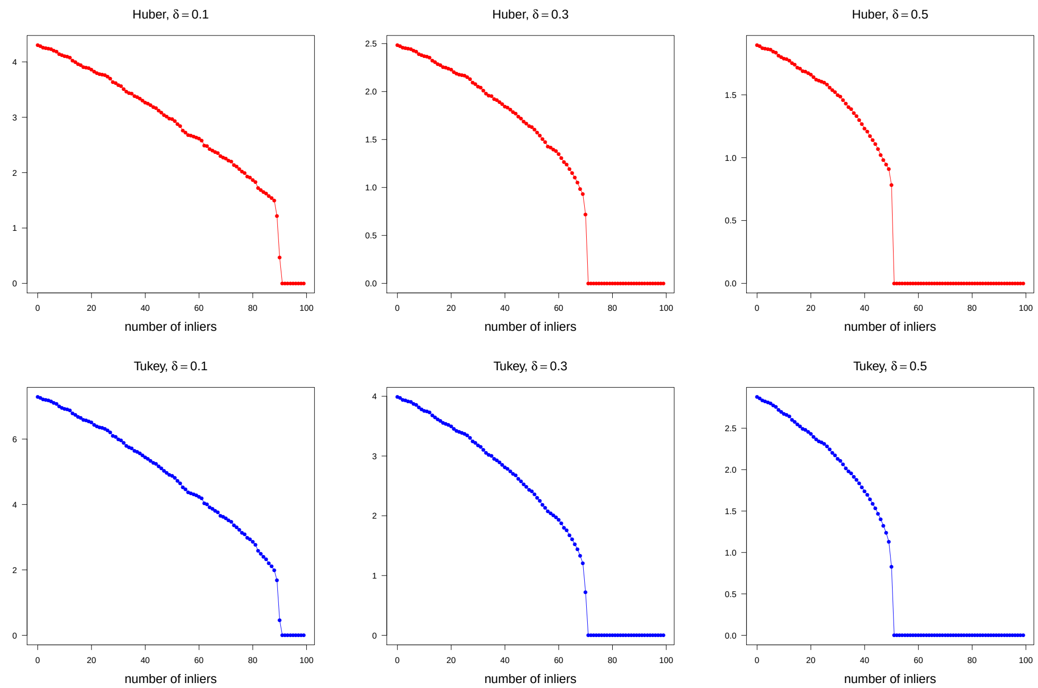


Figure 2.64: Implosion breakdown point: values of the M-estimators of scale for a sample of size 100 with k observations replaced by inliers, k varying from 0 to 99 - 1stSP Case 2

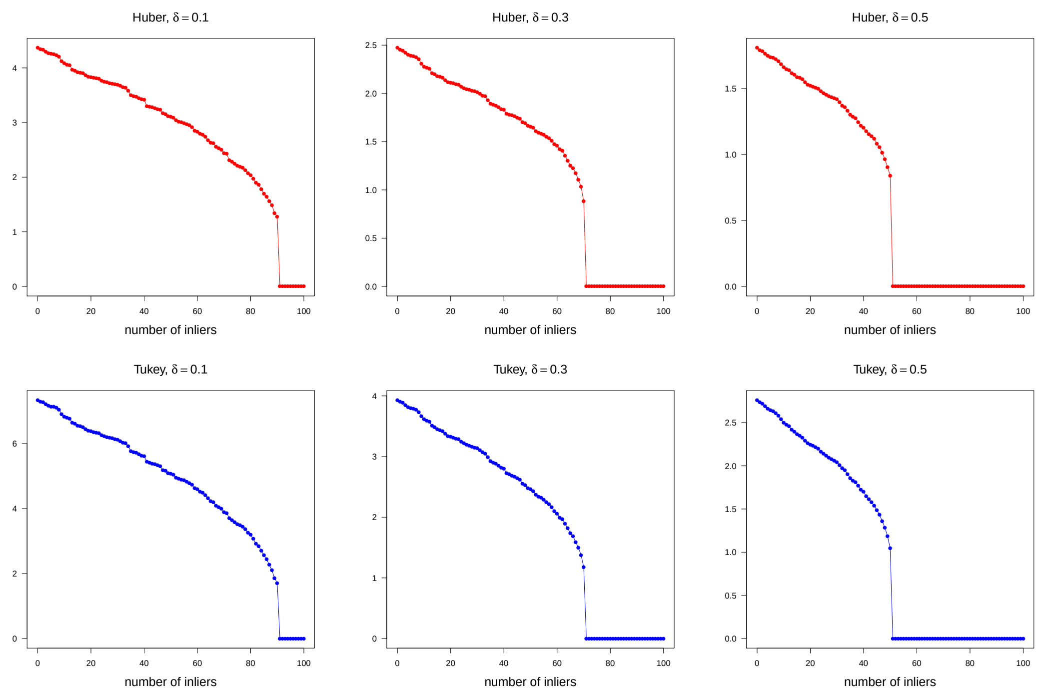


Figure 2.65: Implosion breakdown point: values of the M-estimators of scale for a sample of size 101 with k observations replaced by inliers, k varying from 0 to 100 - 1stSP Case 2

In connection with these first simulations and the explosion analysis, for the sample size $n = 100$ (101), by looking at Table 2.19 (2.20) and Figures 2.50 to 2.52 (2.53 to 2.55) for Case 1 and Table 2.21 (2.22) and Figures 2.56 to 2.58 (2.59 to 2.61) for Case 2, we can see that the minimum number of perturbed observations by outliers that makes the estimator to increase arbitrarily, independently of the simulation case and the type of outlier considered, has been

- 11 for the Huber and Tukey estimators with $\delta = 0.1$;
- 31 for the Huber and Tukey estimators with $\delta = 0.3$;
- 50 (51) for the Huber and Tukey estimators with $\delta = 0.5$.

In connection with the implosion analysis, for the sample size $n = 100$ (101), by looking at Table 2.23 (2.24) and Figure 2.62 (2.63) for Case 1 and Table 2.25 (2.26) and Figure 2.64 (2.65) for Case 2, we can see that the minimum number of perturbed observations by inliers that makes the estimator to implode to zero, independently of the simulation case considered, has been

- 91 for the Huber and Tukey estimators with $\delta = 0.1$;
- 71 for the Huber and Tukey estimators with $\delta = 0.3$;
- 51 for the Huber and Tukey estimators with $\delta = 0.5$.

On the basis of these simulations concerning the explosion and implosion breakdown point, one can empirically conclude that Huber and Tukey M-estimators of scale achieve maximum fsbp when the constant δ equals 0.5.

Next, we will perform the same analyses when the the SECOND SIMULATION PROCEDURE (2ndSP) (see Page 39) is conducted. As in Subsection 2.4.2, two sample sizes have been considered in the simulation study, namely, an even sample size ($n = 20$) and an odd sample size ($n = 21$). For the sample size $n = 20$ we have chosen the weights $\omega_1 = 0.8$, $\omega_2 = 0.1$ and $\omega_3 = 0.1$, and for the sample size $n = 21$ the weights have been $\omega_1 = 16/21$, $\omega_2 = 3/21$ and $\omega_3 = 2/21$.

Explosion pseudo-breakdown point:

The outlier $\widetilde{y}_i$ has been constructed as follows (only for asymmetric distribution):

– Outlier of translation: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1, 100)$ and the fuzzy numbers have been constrained to be in the interval $[0, 5]$. Then, we have chosen $r_i^1 = 40 + i \cdot 5$.

– Outlier of scale on the core and support: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1, 1000)$ and the fuzzy numbers have been constrained to be in the interval $[0, 1]$. Then, we have chosen $r_i^2 = i \cdot 100$.

– Outlier of both translation and scale: the non-contaminated sample has been generated on the basis of a beta distribution $\beta(1, 100)$ and the fuzzy numbers have been constrained to be in the interval $[0, 5]$. Then, we have chosen

$$r_i^1 = 40 + i \cdot 5, \quad r_i^2 = 2.$$

For each type of outlier, the general scheme of the simulation is the same that the one for the 1stSP (see Page 161). The criterion in Remark 2.4.1 is also applied.

The simulation-based conclusions in this study are presented through Tables 2.27 and 2.28 and Figures 2.66 to 2.71. More concretely, tables gather the values of the M-estimators of scale when outliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Implosion pseudo-breakdown point:

The non-contaminated sample has been generated from a beta $\beta(1, 100)$. The general scheme of the simulation is the same that the one for the 1stSP (see Page 161).

The simulation-based conclusions in this study are presented through Tables 2.29 and 2.30 and Figures 2.72 and 2.73. Tables gather the values of the M-estimators of scale when inliers are introduced in the sample by replacement, and figures graphically display these values for each estimator.

Table 2.27: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 20 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, \ldots, 10\}$ - 2ndSP Asymmetric distribution

# outliers (translation)	0	1	2	3	4	5	6	7	8	9	10
ρ_1-Huber, $\delta = 0.1$	3.74	5.27	45.51	61.80	75.04	88.07	101.10	114.23	127.53	141.05	154.79
ρ_1-Huber, $\delta = 0.3$	2.10	2.33	2.61	2.99	3.58	4.88	45.46	61.26	71.48	80.66	89.20
ρ_1-Huber, $\delta = 0.5$	1.40	1.52	1.67	1.84	1.98	2.11	2.39	2.75	3.33	4.69	45.38
ρ_1-Tukey, $\delta = 0.1$	6.24	8.96	53.87	90.77	116.23	139.97	163.18	186.29	209.54	233.05	256.88
ρ_1-Tukey, $\delta = 0.3$	3.25	3.64	4.14	4.85	5.93	8.25	53.40	83.80	100.68	115.94	130.67
ρ_1-Tukey, $\delta = 0.5$	2.14	2.34	2.54	2.78	3.00	3.22	3.68	4.34	5.43	7.89	52.39
# outliers (scale)	0	1	2	3	4	5	6	7	8	9	10
ρ_1-Huber, $\delta = 0.1$	0.61	0.85	5.37	11.83	19.72	26.99	32.28	36.81	40.84	44.51	47.90
ρ_1-Huber, $\delta = 0.3$	0.32	0.36	0.41	0.48	0.58	0.81	5.35	11.82	19.72	25.70	27.66
ρ_1-Huber, $\delta = 0.5$	0.21	0.22	0.24	0.27	0.29	0.33	0.36	0.44	0.46	0.64	5.33
ρ_1-Tukey, $\delta = 0.1$	1.01	1.44	6.64	17.41	28.88	40.58	50.51	58.95	66.38	73.08	79.23
ρ_1-Tukey, $\delta = 0.3$	0.51	0.56	0.64	0.77	0.94	1.37	6.55	17.40	28.35	34.80	39.53
ρ_1-Tukey, $\delta = 0.5$	0.32	0.33	0.36	0.41	0.44	0.51	0.56	0.66	0.72	1.04	6.33
# outliers (both)	0	1	2	3	4	5	6	7	8	9	10
ρ_1-Huber, $\delta = 0.1$	4.00	5.51	45.76	62.08	75.36	88.42	101.48	114.64	127.97	141.51	155.28
ρ_1-Huber, $\delta = 0.3$	2.31	2.46	2.73	3.09	3.65	5.11	45.71	61.52	71.75	80.93	89.50
ρ_1-Huber, $\delta = 0.5$	1.72	1.78	1.89	1.99	2.09	2.28	2.53	2.82	3.19	4.47	45.64
ρ_1-Tukey, $\delta = 0.1$	6.75	9.41	54.37	91.20	116.73	140.55	163.81	186.97	210.26	233.82	257.71
ρ_1-Tukey, $\delta = 0.3$	3.65	3.94	4.44	5.09	6.11	8.69	53.90	84.15	101.05	116.34	131.12
ρ_1-Tukey, $\delta = 0.5$	2.58	2.67	2.87	3.04	3.22	3.56	4.02	4.57	5.27	7.57	53.19

Table 2.28: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 21 with k observations replaced by outliers of translation (at the top), scale on core and support (in the middle) and both (at the bottom), with $k \in \{0, 1, \ldots, 11\}$ - 2ndSP Asymmetric distribution

# outliers (translation)	0	1	2	3	4	5	6	7	8	9	10	11
ρ_1-Huber, $\delta = 0.1$	4.07	5.63	18.59	61.56	74.65	87.52	100.38	113.33	126.43	139.72	153.24	167.01
ρ_1-Huber, $\delta = 0.3$	2.28	2.52	2.83	3.20	3.61	4.55	9.10	58.78	69.81	79.18	87.94	96.42
ρ_1-Huber, $\delta = 0.5$	1.63	1.78	1.90	2.01	2.02	2.08	2.23	2.58	2.90	3.01	4.83	56.34
ρ_1-Tukey, $\delta = 0.1$	6.81	9.57	32.14	89.26	114.79	138.39	161.37	184.21	207.14	230.28	253.76	277.60
ρ_1-Tukey, $\delta = 0.3$	3.58	4.00	4.52	5.19	5.96	7.65	15.65	79.34	97.45	112.95	127.72	142.25
ρ_1-Tukey, $\delta = 0.5$	2.42	2.64	2.86	3.07	3.15	3.28	3.55	4.08	4.60	5.04	8.27	74.58
# outliers (scale)	0	1	2	3	4	5	6	7	8	9	10	11
ρ_1-Huber, $\delta = 0.1$	0.59	0.82	2.71	8.98	14.91	21.27	27.32	32.33	36.66	40.54	44.07	47.34
ρ_1-Huber, $\delta = 0.3$	0.32	0.36	0.41	0.47	0.56	0.73	1.52	7.88	13.71	20.00	25.44	27.33
ρ_1-Huber, $\delta = 0.5$	0.18	0.20	0.23	0.26	0.31	0.34	0.38	0.44	0.49	0.62	1.06	6.09
ρ_1-Tukey, $\delta = 0.1$	0.98	1.38	4.60	12.97	21.95	31.44	41.73	50.89	58.86	65.94	72.36	78.28
ρ_1-Tukey, $\delta = 0.3$	0.48	0.55	0.63	0.75	0.92	1.23	2.60	11.21	20.12	28.75	34.54	39.02
ρ_1-Tukey, $\delta = 0.5$	0.28	0.31	0.35	0.40	0.46	0.50	0.58	0.68	0.77	1.01	1.81	9.44
# outliers (both)	0	1	2	3	4	5	6	7	8	9	10	11
ρ_1-Huber, $\delta = 0.1$	4.79	5.50	17.40	62.08	75.27	88.22	101.14	114.15	127.30	140.65	154.23	168.04
ρ_1-Huber, $\delta = 0.3$	2.47	2.47	2.65	2.93	3.41	4.48	8.77	59.30	70.36	79.76	88.54	97.02
ρ_1-Huber, $\delta = 0.5$	1.58	1.58	1.58	1.63	1.74	1.98	2.08	2.53	2.78	3.49	5.94	56.86
ρ_1-Tukey, $\delta = 0.1$	7.83	9.32	30.08	89.98	115.74	139.50	162.60	185.56	208.59	231.84	255.41	279.34
ρ_1-Tukey, $\delta = 0.3$	3.83	3.83	4.12	4.64	5.53	7.48	15.05	79.99	98.19	113.78	128.61	143.19
ρ_1-Tukey, $\delta = 0.5$	2.41	2.41	2.45	2.54	2.69	3.03	3.20	3.86	4.28	5.64	10.05	75.30

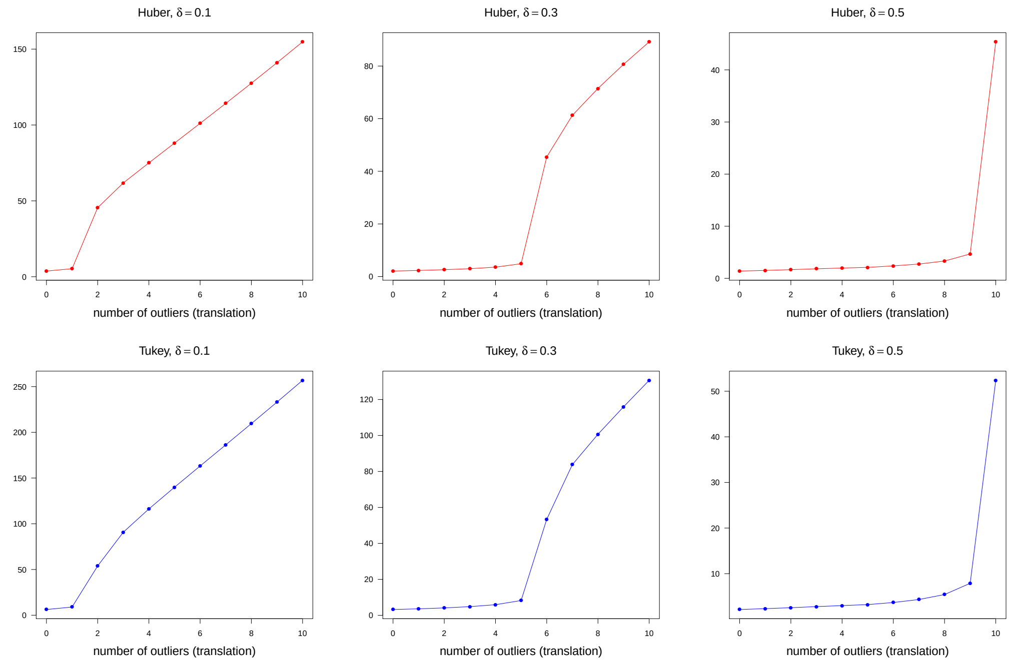


Figure 2.66: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 20 with k observations replaced by outliers of translation, k varying from 0 to 10 - 2ndSP Asymmetric distribution

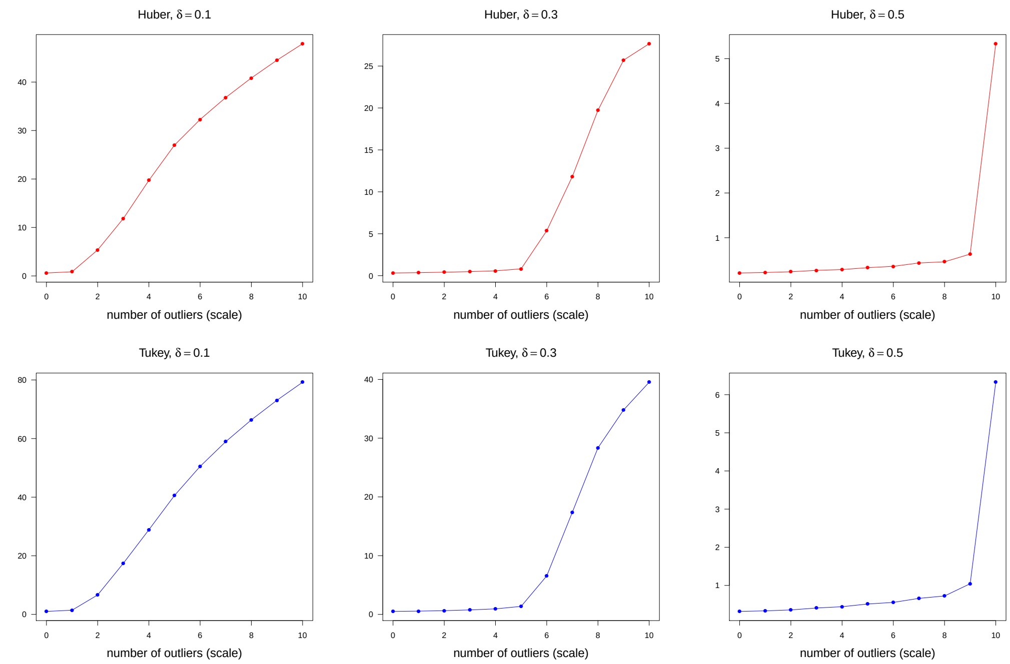


Figure 2.67: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 20 with k observations replaced by outliers of scale on core and support, k varying from 0 to 10 - 2ndSP Asymmetric distribution

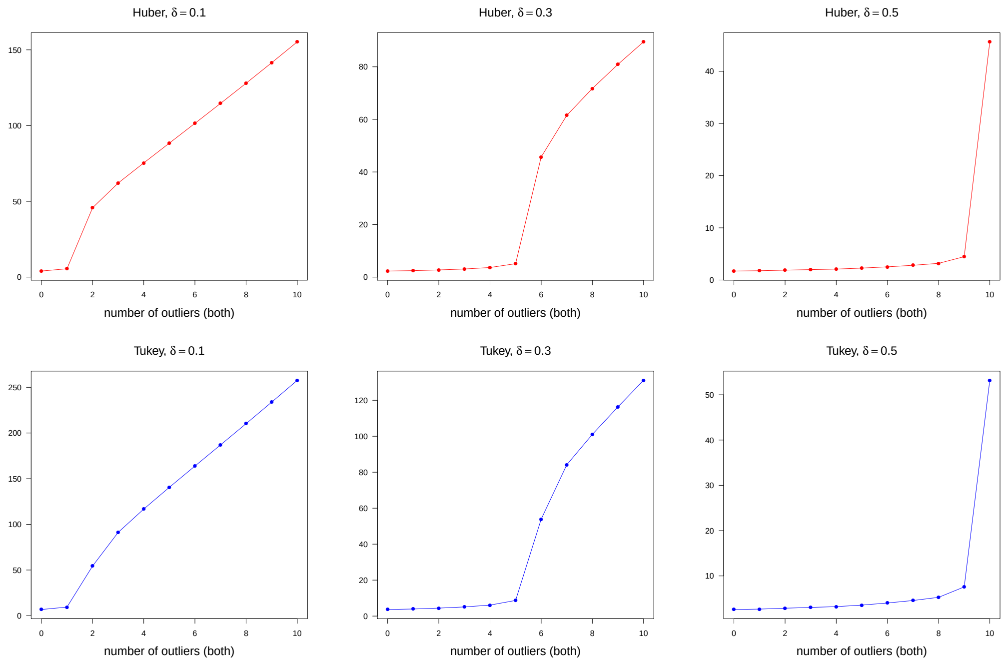


Figure 2.68: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 20 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 10 - 2ndSP Asymmetric distribution

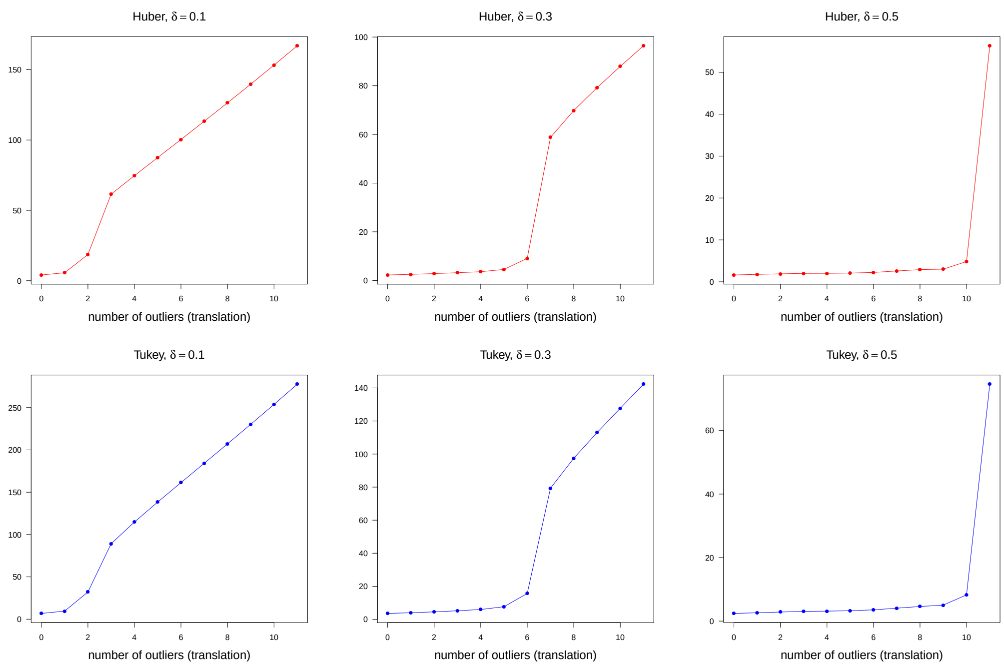


Figure 2.69: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 21 with k observations replaced by outliers of translation, k varying from 0 to 11 - 2ndSP Asymmetric distribution

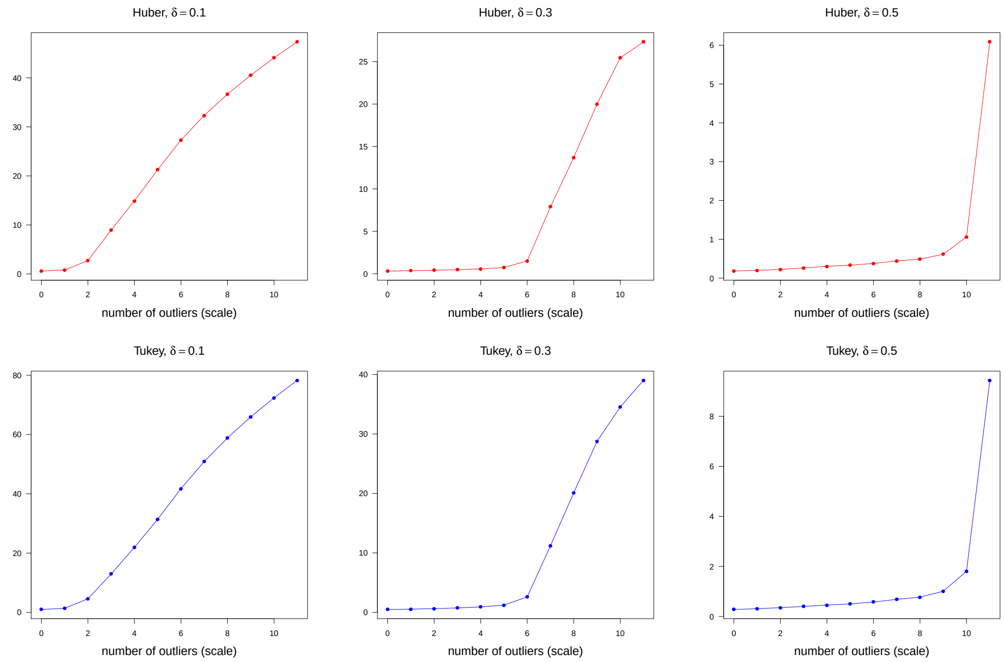


Figure 2.70: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 21 with k observations replaced by outliers of scale on core and support, k varying from 0 to 11 - 2ndSP Asymmetric distribution

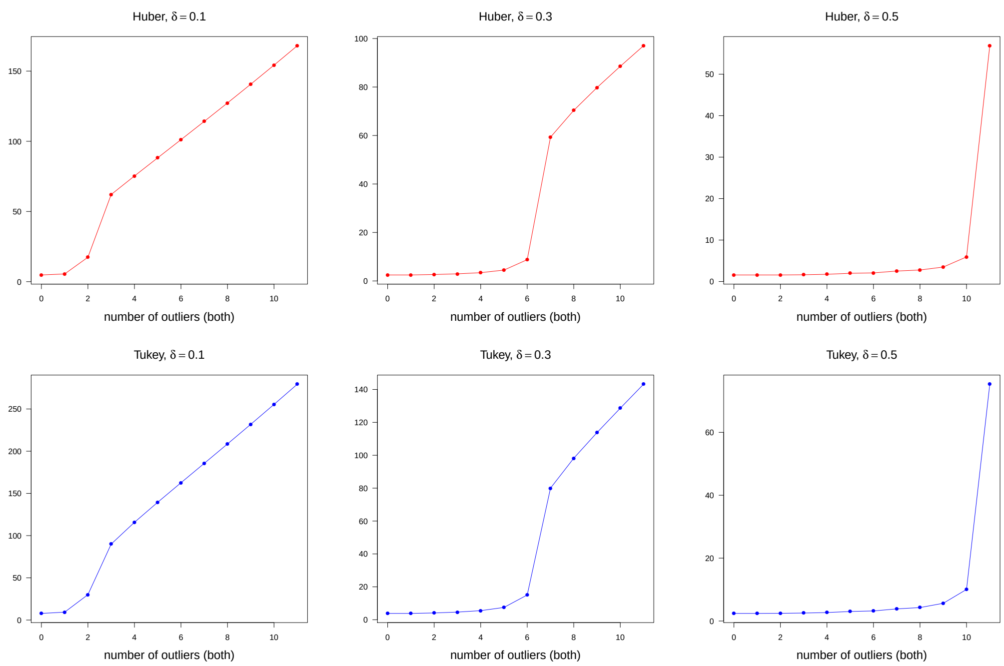


Figure 2.71: Explosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 21 with k observations replaced by outliers of both translation and scale on core and support, k varying from 0 to 11 - 2ndSP Asymmetric distribution

Table 2.29: Implosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 20 with k observations replaced by inliers, with $k \in \{0, 1, 2, 10, 11, 14, 15, 18, 19\}$ - 2ndSP Asymmetric distribution

# inliers	0	1	2	10	11	14	15	18	19
ρ_1-Huber, $\delta = 0.1$	6.01	5.78	5.54	4.22	4.14	3.86	3.69	0.88	8.60E-11
ρ_1-Huber, $\delta = 0.3$	2.69	2.59	2.48	1.66	1.56	1.11	9.92E-11	8.43E-11	6.93E-11
ρ_1-Huber, $\delta = 0.5$	1.86	1.70	1.50	0.50	9.64E-11	7.86E-11	7.29E-11	6.46E-11	3.69E-11
ρ_1-Tukey, $\delta = 0.1$	9.40	9.12	8.82	7.02	6.88	6.35	6.03	0.88	8.60E-11
ρ_1-Tukey, $\delta = 0.3$	4.22	4.00	3.77	2.47	2.33	1.14	9.78E-11	8.43E-11	6.93E-11
ρ_1-Tukey, $\delta = 0.5$	2.76	2.55	2.32	0.51	9.89E-11	9.27E-11	8.22E-11	6.46E-11	3.69E-11

Table 2.30: Implosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 21 with k observations replaced by inliers, with $k \in \{0, 1, 2, 10, 11, 14, 15, 18, 19\}$ - 2ndSP Asymmetric distribution

# inliers	0	1	2	10	11	14	15	18	19
ρ_1-Huber, $\delta = 0.1$	7.51	7.47	7.41	7.28	7.27	7.17	7.06	1.49	9.77E-11
ρ_1-Huber, $\delta = 0.3$	2.50	2.44	2.32	1.99	1.95	1.37	9.97E-11	7.75E-11	8.00E-11
ρ_1-Huber, $\delta = 0.5$	1.53	1.46	1.34	0.85	9.93E-11	9.72E-11	9.66E-11	6.50E-11	7.91E-11
ρ_1-Tukey, $\delta = 0.1$	11.58	11.50	11.35	10.99	10.96	10.69	10.36	2.27	9.77E-11
ρ_1-Tukey, $\delta = 0.3$	4.05	3.92	3.69	3.01	2.93	2.01	9.86E-11	7.75E-11	8.00E-11
ρ_1-Tukey, $\delta = 0.5$	2.37	2.25	2.06	1.14	9.90E-11	9.74E-11	8.67E-11	6.50E-11	7.91E-11

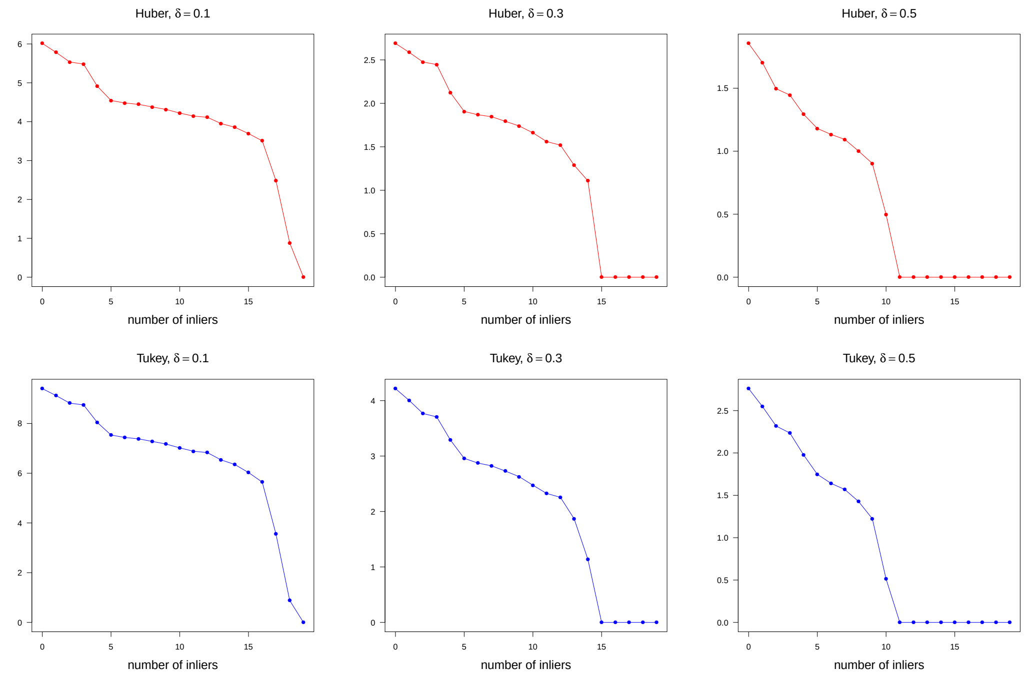


Figure 2.72: Implosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 20 with k observations replaced by inliers, k varying from 0 to 19 - 2ndSP Asymmetric distribution

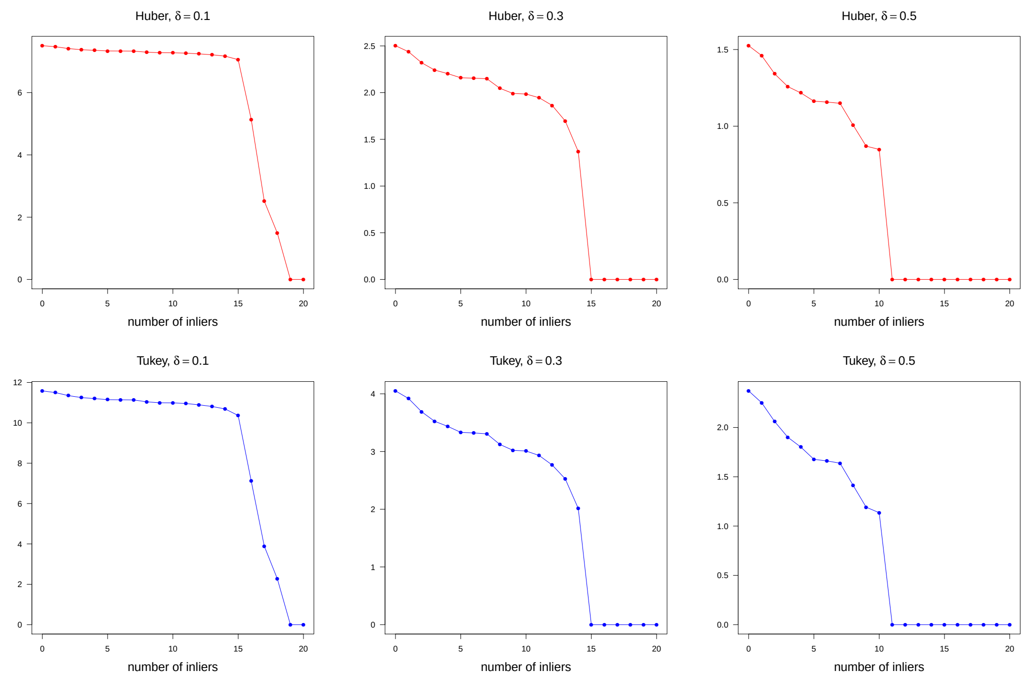


Figure 2.73: Implosion pseudo-breakdown point: values of the M-estimators of scale for a sample of size 21 with k observations replaced by inliers, k varying from 0 to 20 - 2ndSP Asymmetric distribution

In summary, for these second type simulations and in connection with the explosion/implosion, for the sample size $n = 20$ (21), by looking at Table 2.27/2.29 (2.28/2.30) and Figures 2.66 to 2.68/2.72 (2.69 to 2.71/2.73), we can see that 'the minimum number of perturbed observations by outliers that makes the estimator to increase noticeably'/'the minimum number of perturbed observations by inliers that makes the estimator to implode to zero', independently of the considered type of outlier, has been

- 2/19 for the Huber and Tukey estimators with $\delta = 0.1$;
- 6/15 (7/15) for the Huber and Tukey estimators with $\delta = 0.3$;
- 10/11 (11/11) for the Huber and Tukey estimators with $\delta = 0.5$.

We analyze now the sensitivity curves of the M-estimators. The simulations are conduced according to the FIRST SIMULATION PROCEDURE (1stSP) in the same way that was explained in Subsection 2.4.3.

Therefore, the non-contaminated sample has been simulated from the 1stSP considering the two cases (see Page 38) and a unique sample size, $n = 100$.

The outliers have been constructed as follows. Let $\widetilde{y}_s$ be the outlier. Then,

– Outlier of translation: $r_s^1 = s$, with s varying from -20 to 20 with a step equals 0.1.
– Outlier of scale on the core and support: $r_s^2 = s$, with s varying from 0 to 20 with a step equals 0.1.
– Outlier of both translation and scale: $r_s^1 = r_s^2 = s$, with s varying from -20 to 20 with a step equals 0.1.

For each type of outlier, the general scheme of the construction of the sensitivity curves has been as follows:

Step 1. A sample $\widetilde{\mathbf{x}}_{100}$ of 100 trapezoidal fuzzy numbers has been simulated from the 1stSP, considering the two cases of simulation involved in this procedure.

Step 2. One observation from the original sample $\widetilde{\mathbf{x}}_{100}$ has been chosen randomly and replaced by the outlier $\widetilde{y}_s$.

Step 3. For each s, the value of the sensitivity curve has been calculated for each M-estimator of scale. We have chosen the $\widehat{\rho_1\text{-MDD}}$ with respect to the 1-norm median, $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, as initial robust scale measure to start the algorithm.

The sensitivity curves have been graphically displayed for each M-estimator in Figures 2.74 to 2.76 for Case 1 and in Figures 2.77 to 2.79 for Case 2. Irrespective of the type of outlier that we are considering, we can see that the sensitivity curves are bounded for all the M-estimators in both cases.

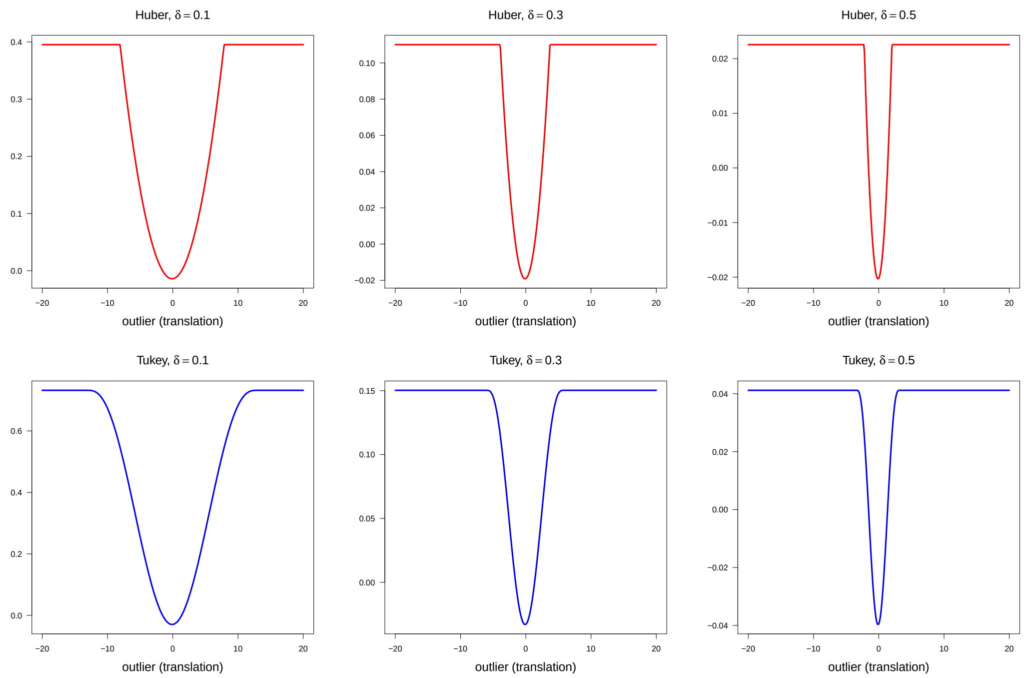


Figure 2.74: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of translation - 1stSP Case 1

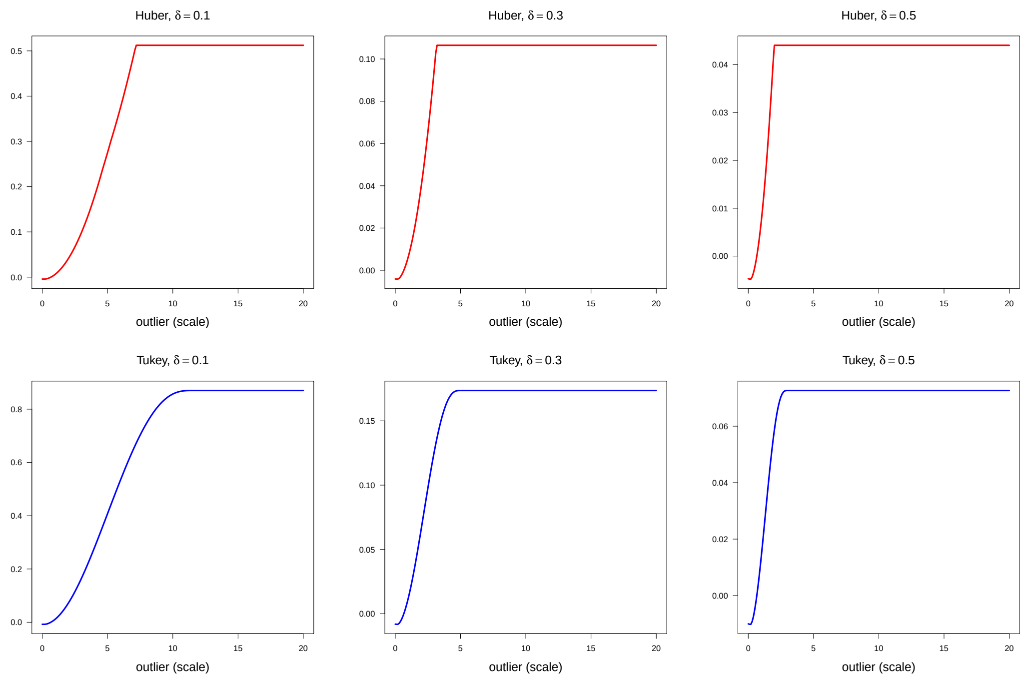


Figure 2.75: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of scale on core and support - 1stSP Case 1

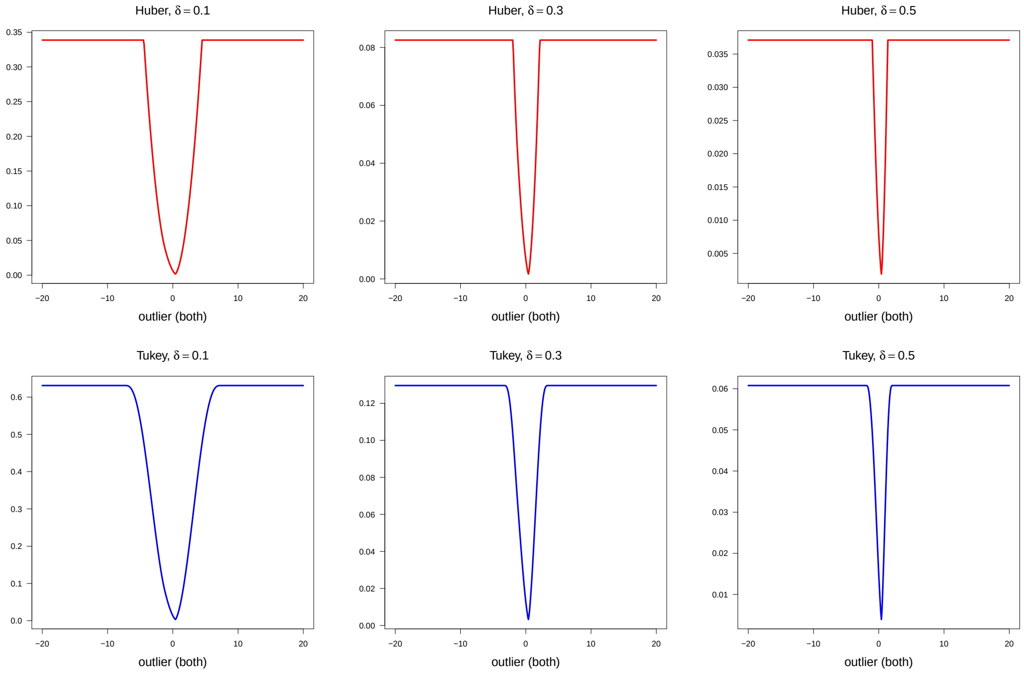


Figure 2.76: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of both translation and scale on core and support - 1stSP Case 1

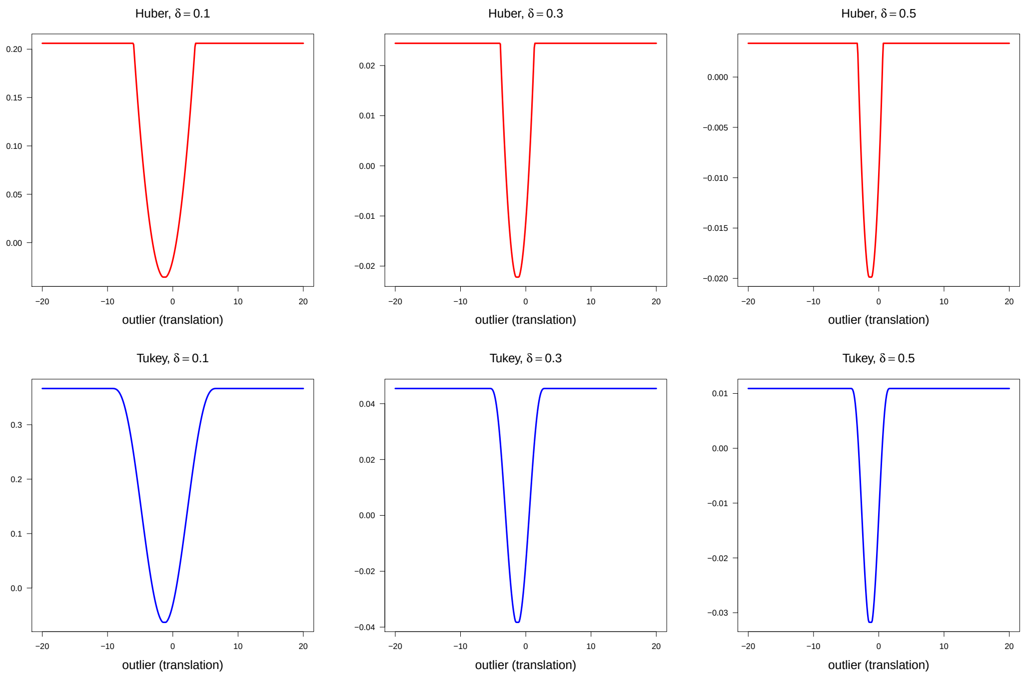


Figure 2.77: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of translation - 1stSP Case 2

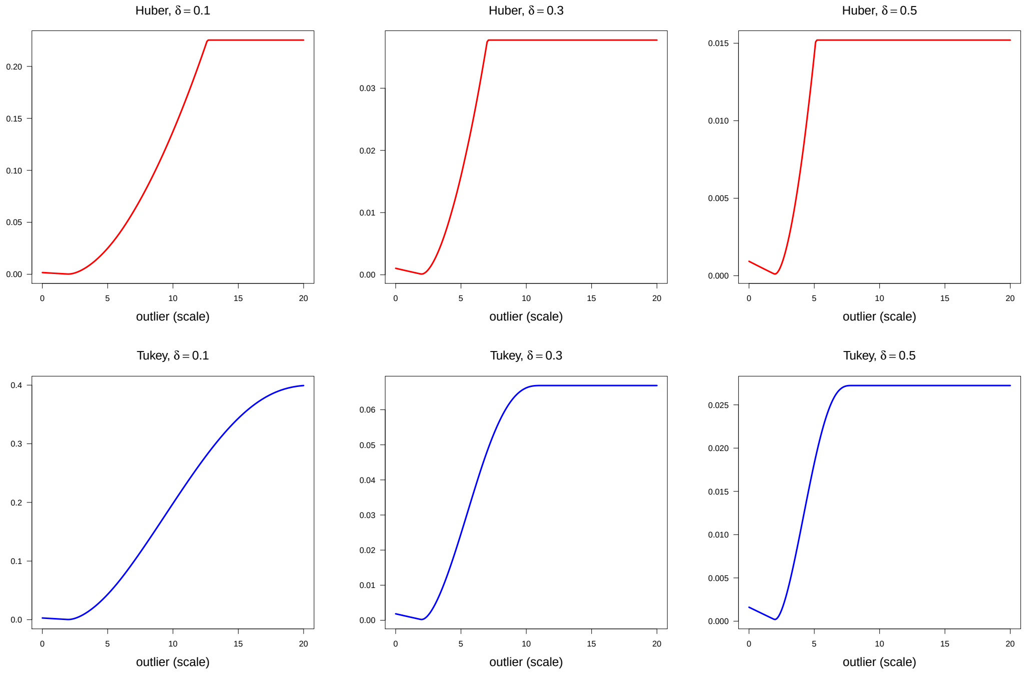


Figure 2.78: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of scale on core and support - 1stSP Case 2

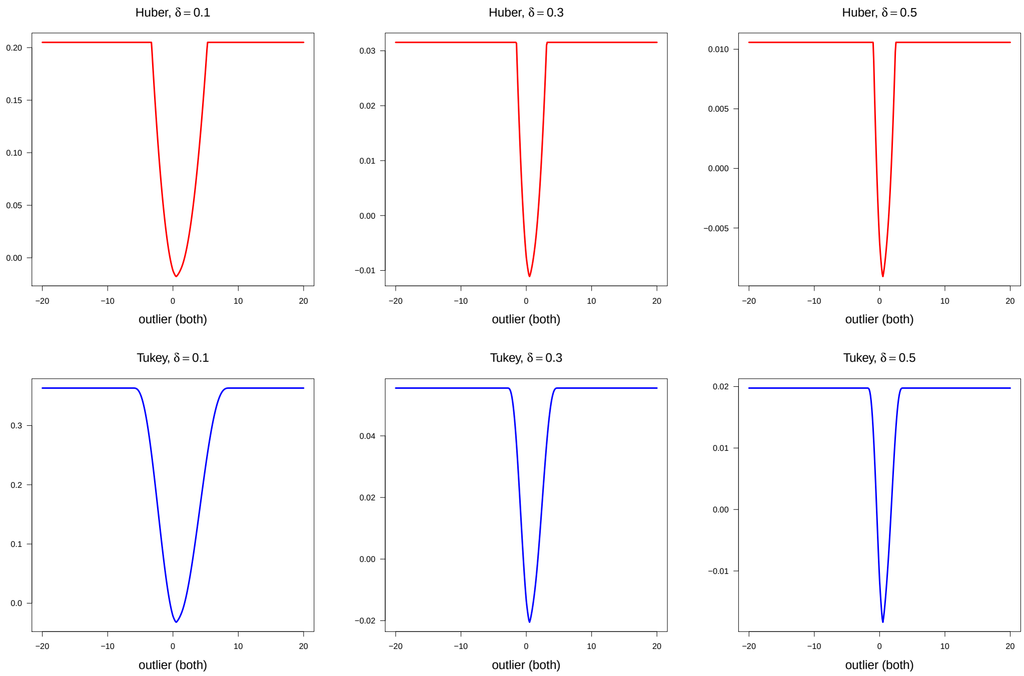


Figure 2.79: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of both translation and scale on core and support - 1stSP Case 2

Now, we follow the SECOND SIMULATION PROCEDURE (2ndSP) (see Page 39). As in Subsection 2.4.3, a unique sample size $n = 100$ has been considered in the study, with weights $\omega_1 = 0.8$, $\omega_2 = 0.1$ and $\omega_3 = 0.1$.

The outlier $\widetilde{y}_s$ for the considered asymmetric distribution has been constructed so that, with the non-contaminated sample being generated from a beta $\beta(1, 100)$,

– Outlier of translation: $r_s^1 = s$, with s varying from 0 to 20 with a step equals 0.1.

– Outlier of scale on the core and support: $r_s^2 = s$, with s varying from 0 to 20 with a step equals 0.1.

– Outlier of both translation and scale: $r_s^1 = r_s^2 = s$, with s varying from 0 to 20 with a step equals 0.1.

For each type of outlier, the general scheme of the simulation is the same that the one for the 1stSP (see Page 181), and the criterion in Remark 2.4.1 has also been applied.

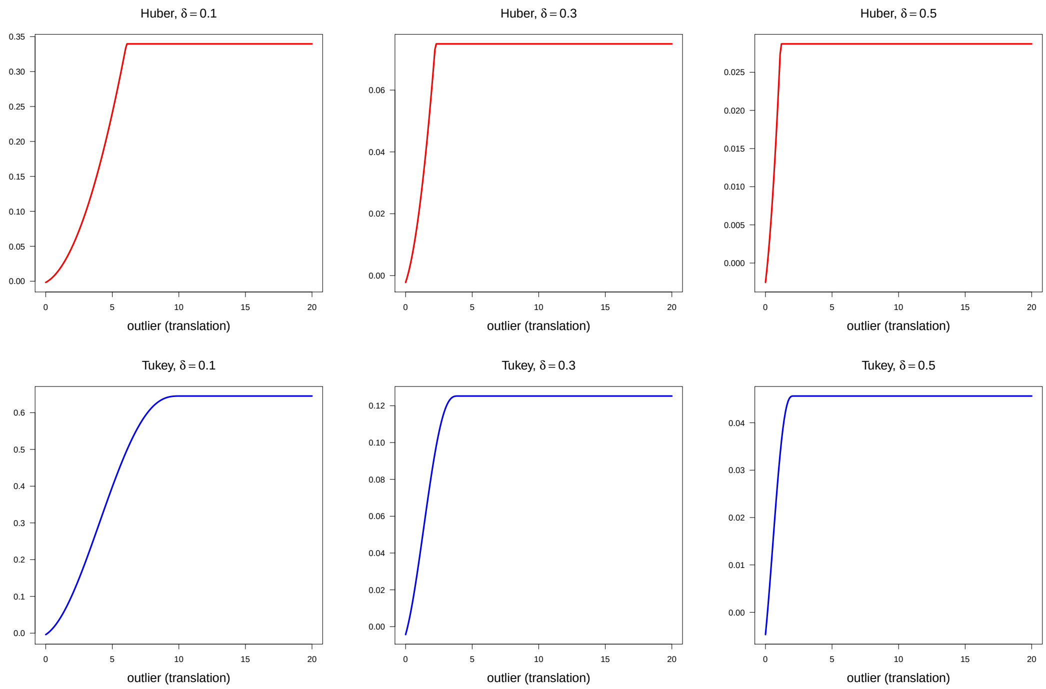


Figure 2.80: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of translation - 2ndSP Asymmetric distribution

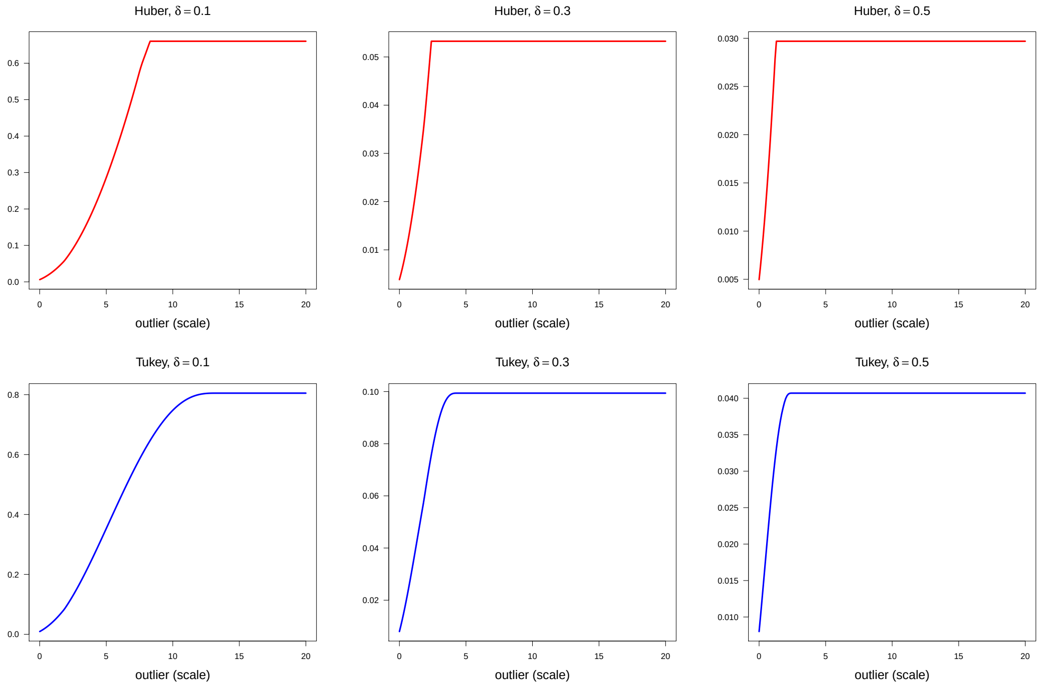


Figure 2.81: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of scale on core and support - 2ndSP Asymmetric distribution

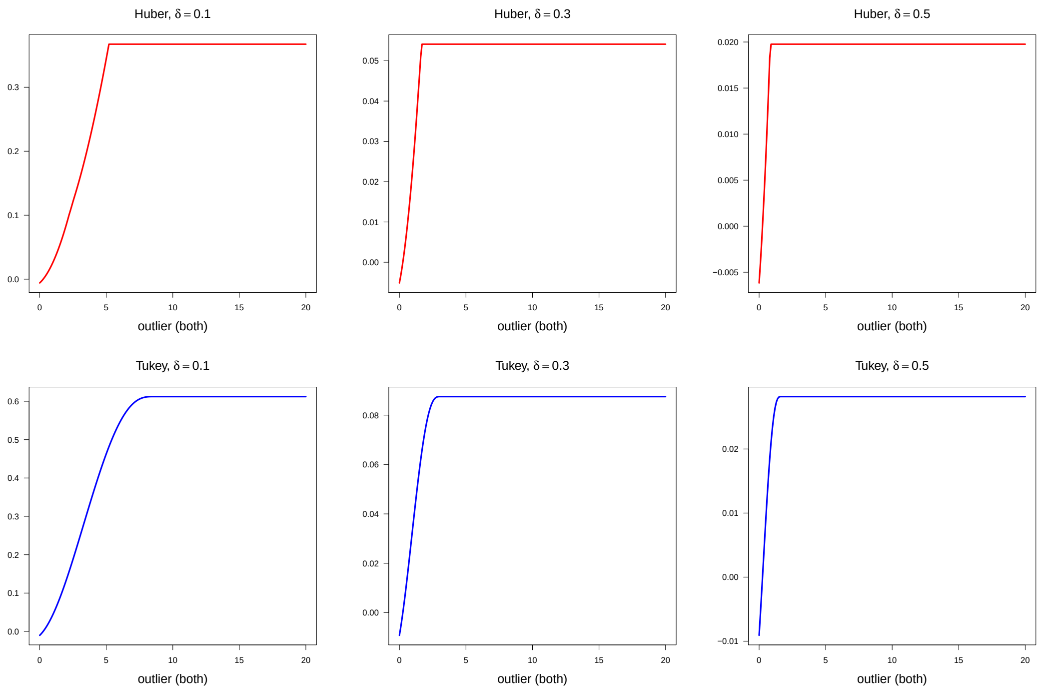


Figure 2.82: Sensitivity curves of the M-estimators of scale for a sample of size 100 and outliers of both translation and scale on core and support - 2ndSP Asymmetric distribution

We can see the pseudo-sensitivity curves graphically represented for each M-estimator in Figures 2.80 to 2.82. Independently of the type of outlier that we are considering, we can see that the pseudo-sensitivity curves are bounded for all the M-estimators.

2.6 Illustrating the use of different scale measures through case studies

This section aims to illustrate the computation of the scale measures that have been introduced in this chapter for FRS-based data in the three case studies in Examples 1.6.1 (Pages 33 and 233), 1.6.2 (Pages 35 and 243) and 1.6.3 (Pages 35 and 251).

In Table 2.32 the values of the scale estimates in Sections 2.1 and 2.5 ($\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, and the robust ones $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widehat{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$, ρ_1-Huber$(\widetilde{\mathbf{x}}_n)$ with $\delta = 0.5$, and ρ_1-Tukey$(\widetilde{\mathbf{x}}_n)$ with $\delta = 0.5$) are gathered for the FRS-based responses to the following nine items from the Example 1.6.1 questionnaire:

Table 2.31: Questions selected from the student questionnaire in Example 1.6.1

	Reading in school
$R.1$	I like to read things that make me think
$R.2$	I learn a lot from reading
$R.3$	Reading is harder for me than any other subject
	Mathematics in school
$M.1$	I like mathematics
$M.2$	My math teacher is easy to understand
$M.3$	Mathematics is harder for me than any other subject
	Science in school
$S.1$	My teacher taught me to discover science in daily life
$S.2$	I read about science in my spare time
$S.3$	Science is harder for me than any other subject

If we analyze Table 2.32 by quantifying the pairwise Pearson's correlation coefficients between scale estimates, where individuals are the nine items, we get a rather strong positive correlation for all pairs of estimates but for M-estimates that behave in a different way. More concretely, the lowest correlation coefficient for pairs of non-robust estimates equals 0.942, and the lowest correlation coefficient for pairs of robust estimates (different from M-estimates) equals 0.872.

Table 2.32: Sample scale estimates for the fuzzy rating responses to Items $R.1$ to $S.3$ in Example 1.6.1

scale measure \ item	$R.1$	$R.2$	$R.3$	$M.1$	$M.2$	$M.3$	$S.1$	$S.2$	$S.3$
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.26	1.88	2.90	2.70	2.34	3.48	2.59	2.34	2.93
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.82	1.63	2.42	2.38	1.98	3.09	2.16	1.93	2.58
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	1.72	1.50	2.24	2.27	1.74	3.03	2.03	1.83	2.47
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	2.08	1.97	2.73	2.79	2.27	3.28	2.54	2.29	2.98
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.44	1.45	2.27	2.31	1.80	3.41	1.71	1.67	2.41
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$	1.37	1.39	1.58	2.17	1.06	3.04	1.69	1.60	2.24
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	1.65	1.86	2.12	2.89	1.79	3.30	2.24	2.03	2.69
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.59	1.56	2.21	2.40	1.50	3.36	2.03	1.95	2.84
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.04	0.94	1.09	1.43	0.81	1.53	1.33	1.14	1.62
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.39	1.32	1.69	2.18	1.18	2.83	1.83	1.68	2.48
ρ_1-Huber, $\delta = 0.5$	9.46	11.10	3.52	9.85	11.57	6.22	8.93	4.07	6.45
ρ_1-Tukey, $\delta = 0.5$	14.09	16.90	5.10	14.37	17.51	9.15	13.26	6.01	9.36

As we have commented, the behaviour of the computed scale M-estimates is quite different from that of the other scale estimates (see Figure 2.83). It would be then interesting to examine whether this behaviour can be more coherent if other loss functions or parameter values are chosen.

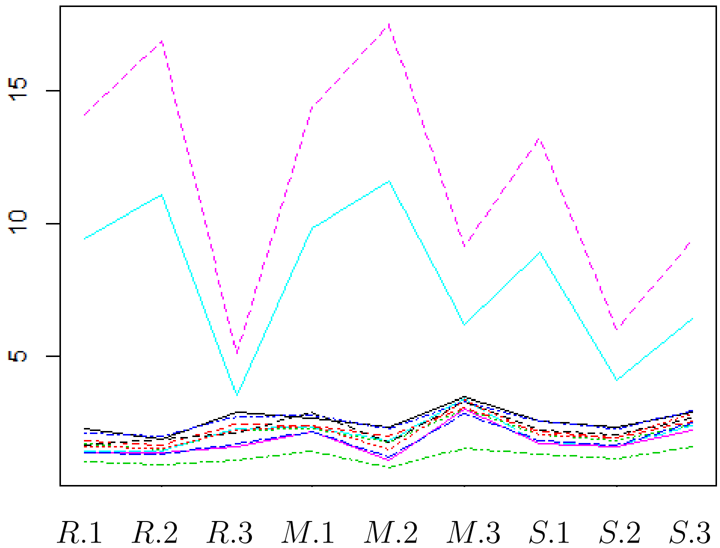


Figure 2.83: Graphical display of the values of different scale estimates for responses to Items $R.1$ to $S.3$ in Example 1.6.1

In Table 2.34 the values of the scale estimates in Sections 2.1 and 2.5 ($\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\text{M}^\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, and the robust ones $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$, ρ_1-Huber$(\widetilde{\mathbf{x}}_n)$ with $\delta = 0.5$, and ρ_1-Tukey$(\widetilde{\mathbf{x}}_n)$ with $\delta = 0.5$) are gathered for the FRS-based responses to the following fourteen items from the Example 1.6.2 questionnaire:

Table 2.33: Questions selected from the restaurant questionnaire in Example 1.6.2

	About your opinion/valuation/rating
$QF1$	The food is served hot and fresh
$QF2$	The menu has a good variety of items
$QF3$	The quality of food is excellent
$QF4$	The food is tasty and flavorful
$QF5$	The quality of beverage is good
	Satisfaction with restaurant service
$QR1$	My food order was correct and complete
$QR2$	Employees are patient when taking my order
$QR3$	I was served promptly
$QR4$	Good availability of sauces, utensils, napkins,...
$QR5$	The menu board was easy to read
$QR6$	Employees are friendly and courteous
$QR7$	The service is excellent
$QR8$	Good cleanness of the restaurant and service
	Opinion about prices
$QP1$	Prices are competitive

Table 2.34: Sample scale estimates for the fuzzy rating responses to Items $QF1$ to $QP1$ in Example 1.6.2

scale measure \ item	$QF1$	$QF2$	$QF3$	$QF4$	$QF5$	$QR1$	$QR2$	$QR3$	$QR4$	$QR5$	$QR6$	$QR7$	$QR8$	$QP1$
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}_n})$	15.76	20.36	19.75	14.05	16.57	16.39	16.79	15.84	22.81	17.59	21.64	16.88	19.06	18.28
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}_n}, \overline{\widetilde{\mathbf{x}_n}})$	12.02	17.41	15.65	11.71	14.21	12.81	13.74	12.81	19.48	13.13	17.11	13.36	15.03	14.68
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}_n}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}_n}))$	11.33	16.71	14.28	10.84	13.38	11.18	12.39	11.99	18.30	11.40	15.17	12.36	14.20	13.04
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}_n}, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}_n}))$	14.31	20.21	17.64	13.59	16.47	14.37	15.26	14.94	21.64	14.76	18.55	15.25	16.82	16.77
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}_n}, \overline{\widetilde{\mathbf{x}_n}})$	8.93	17.32	12.37	10.68	13.69	11.04	12.34	9.59	18.79	9.61	15.07	10.86	13.87	12.91
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}_n}, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}_n}))$	8.48	16.79	10.05	9.18	13.50	6.00	8.96	9.85	17.28	6.88	9.74	9.76	12.63	8.75
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}_n}, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}_n}))$	10.67	18.29	11.96	10.80	16.38	10.78	11.99	13.53	19.58	10.46	12.31	12.88	15.07	11.19
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}_n})$	12.00	17.75	13.25	10.25	13.75	8.75	10.00	12.50	17.50	10.00	12.25	12.50	15.00	12.50
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}_n})$	7.50	10.50	10.00	7.50	10.00	5.00	7.50	7.50	11.25	5.00	7.50	8.25	8.33	7.50
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}_n})$	10.39	15.99	11.45	9.34	12.62	6.82	8.94	10.31	15.47	7.88	9.61	10.62	12.93	9.88
ρ_1-Huber, $\delta = 0.5$	109.97	98.83	102.54	112.67	105.18	121.36	114.12	109.18	104.77	121.47	113.49	106.40	110.48	113.59
ρ_1-Tukey, $\delta = 0.5$	168.85	148.14	155.63	173.30	160.07	186.52	174.77	167.01	156.91	186.49	172.49	162.54	168.22	173.61

If we analyze Table 2.34 by quantifying the pairwise Pearson's correlation coefficients between scale estimates, where individuals are the fourteen items, we get a rather strong positive correlation for all pairs of estimates but for M-estimates that behave in a different way. More concretely, the lowest correlation coefficient for pairs of non-robust estimates equals 0.832, and the lowest correlation coefficient for pairs of robust estimates (different from M-estimates) equals 0.811.

As for the preceding example, the behaviour of the computed scale M-estimates is quite different from that of the other scale estimates (see Figure 2.84).

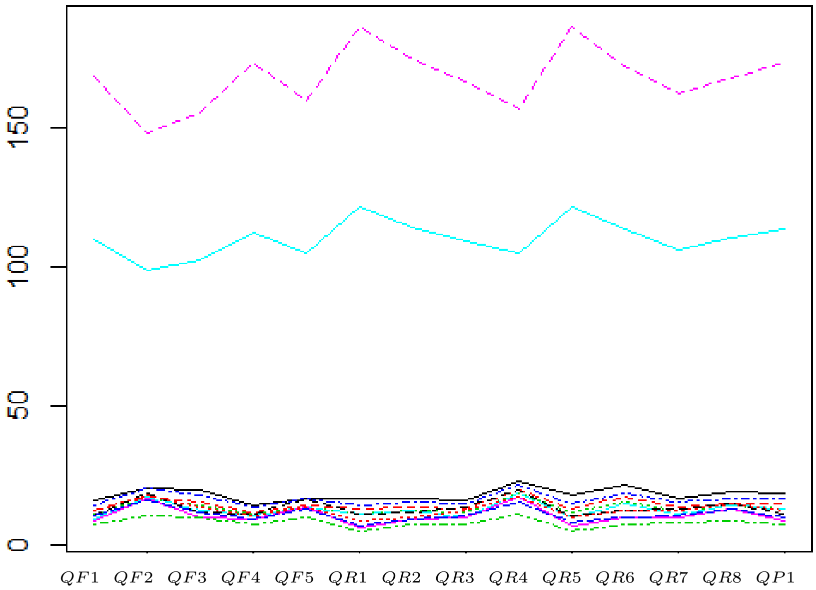


Figure 2.84: Graphical display of the values of different scale estimates for responses to Items $QF1$ to $QP1$ in Example 1.6.2

In Table 2.35 the values of the scale estimates in Sections 2.1 and 2.5 ($\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$, and the robust ones $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$, ρ_1-Huber$(\widetilde{\mathbf{x}}_n)$ with $\delta = 0.5$, and ρ_1-Tukey$(\widetilde{\mathbf{x}}_n)$ with $\delta = 0.5$) are gathered for the unique item in Example 1.6.3, namely, the relative length of different line segments with respect to a pattern longer one.

Table 2.35: Sample scale estimates for the responses to questionnaire in Example 1.6.3

scale measure		scale measure	
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	28.85	$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	24.78
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	24.65	$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	25.98
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	24.81	$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	24.67
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	25.98	$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	25.35
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	13.45	$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	24.58
ρ_1-Huber, $\delta = 0.5$	75.78	ρ_1-Tukey, $\delta = 0.5$	110.81

2.7 Concluding remarks of this chapter

This chapter has been devoted to introduce several scale measures in dealing with fuzzy-valued data. Some of their main properties have been examined, specially the one concerning the robustness. The analysis of the finite sample breakdown point and the sensitivity curves, as powerful tools to verify the robust behaviour of some of these measures, has allowed us to draw similar conclusions to those from the case of real-valued data.

Therefore, the main contributions of this Chapter have been the following:

- The introduction of several scale measures for fuzzy-valued data in Definitions 2.1.1-2.1.6 and the analysis of some of their main properties in Section 2.2.
- The study of the robust behaviour of the different scale estimates by means of the finite sample breakdown point in Section 2.3 from a theoretical point of view; conclusions have been later corroborated by means of simulation studies, including their sensitivity curves, in Section 2.4; this study has confirmed the good robust behaviour of the scale measures ρ_1-MDD$(\mathcal{X}, \widetilde{\mathrm{Me}}(\mathcal{X}))$, $\mathscr{D}_\theta^\varphi$-MDD$(\mathcal{X}, \widetilde{\mathrm{M}^\varphi}(\mathcal{X}))$, D-S$(\mathcal{X}, \mathcal{Y})$, $\widehat{D\text{-Q}}(\tilde{\mathbf{x}}_n)$ and $\widehat{D\text{-T}}(\tilde{\mathbf{x}}_n)$, so their robustness is inherited when they are extended from the real- case to the fuzzy-valued case.
- A brief introduction to the M-estimation of scale for fuzzy-valued data, including an simulation study to check the empirical robustness of the M-estimators of scale based on two of the most popular loss functions, by means of the finite sample breakdown point and the sensitivity curves; the application to illustrate the measurement of scale in the case studies presented in Chapter 1 indicates the behaviour of M-estimators is rather different to the other robust estimators, so a much deeper analysis needs to be performed.

The ideas and results in this chapter have been gathered at this stage in one published paper (De la Rosa de Sáa *et al.* [28]) and one communication to an international conference (De la Rosa de Sáa *et al.* [25]).

Chapter 3

Comparing statistical conclusions for different rating scales. Questionnaires-based case studies and simulations

Different studies can be found in the literature to discuss the influence of the number of categories/points of the Likert-type scales on the reliability of the analysis of these responses. They usually coincide in pointing out that increasing the number of categories results in an increase of the variability, information and reliability (see, for instance, Tomás and Oliver [127] and Lozano *et al.* [74]).

Actually, to some extent, the ideal situation would be increasing the number of choices to a continuum. Several studies (see, for instance, Reips and Funke [96], and Treiblmaier and Filzmoser [128]) have pointed out that the visual analogue scales provide researchers with many advantages in contrast to discrete ones like Likert's. Thus, one can benefit from a metric setting and from the fact that a much wider set of statistical methods can be applied to analyze the data coming from this rating.

However, the choice of a single point within a continuum representing each response is neither easy nor natural in an imprecise context, and it does not seem realistic to demand so much accuracy in rating intrinsically imprecise magnitudes.

If one aims to really exploit the individual differences in responding to questionnaires, there is a need for a rich and expressive scale that can cope with imprecision. Fuzzy scales have been involved in this work aiming to rate intrinsically imprecise magnitudes, like human perceptions, attitudes, opinions, and so on.

In Chapter 1, advantages of using fuzzy rating scales have been commented. Among these advantages, it has been stated that

– these scales are much richer and more expressive than any one based on a (unavoidably finite) natural language or its real/fuzzy-valued encoding, and they allow an uncountable number of modifiers and hedges;
– the flexibility of the FRS's allows raters to properly capture individual differences, whence the intrinsic variability, diversity and subjectivity are not lost.

As a consequence from such a richness and flexibility, one can expect a richer statistical information which will result in more accurate statistical conclusions. This chapter is to be focussed on empirically illustrate some aspects related to the last assertion, by comparing statistical conclusions for three types of scales in this framework, namely

- the LIKERT-TYPE (or its numerical-encoded counterpart),
- the FUZZY LINGUISTIC,
- and the FUZZY RATING scales.

Visual analogue scales (VAS's) and their combined versions are not to be included in the comparative discussion in this chapter. Main reasons supporting such a decision are the following:

– to require a real number-valued full accuracy in such an intrinsically imprecise context seems not to be coherent and realistic;
– although the visual analogue scale allows a high diversity, variability and subjectivity in rating, the fuzzy rating scale includes VAS' values as special elements, so the diversity, variability and subjectivity are even more highly captured and, in contrast to VAS's, FRS's can (fully) cope with the imprecision of the considered magnitudes;
– lastly, as shown in González-Rodríguez *et al.* [48], real-valued data like those based on a VAS could be fuzzified in such a way that the Aumann-type mean of these fuzzy data would contain the whole information on (i.e., would characterize) the distribution of the original data, with the value added of being based on a mean and then showing properties of consistency, and some valuable others.

The comparative studies among the three considered scales have been carried out in this chapter either on questionnaires-based case studies or on simulation developments inspired on such questionnaires (i.e., the 2ndSP). Questionnaires play a major role in many scientific studies, especially in those related to social and biomedical sciences. Evaluation, rating, judgment, perception, etc., are typical in human social lives, and the corresponding data are routinely collected as responses to some questionnaires. Questionnaires were first considered by Galton (see, for instance, [40]) in connection with human communities. They mean a valuable research tool constituted of a series of questions which are posed to gather information from respondents, and they are usually designed for statistical analysis.

The comparative analysis has been empirically performed mainly due to the fact that theoretical results are mostly impossible to achieve in general, since raters not necessarily behave in a completely systematic way, as it is to be commented along the chapter. Actually, for any possible comparison we could prepare a counterexample of any conclusion we could think about. Even for conclusions on aspects/measures/indices for which data values are irrelevant, like happens for instance with the diversity, one should make some assumptions to get theoretical results. And for the case studies- and simulations-based developments we will make use of particular cases of these assumptions.

In this chapter, a first comparison is made through a well-known diversity index to show that, under reasonable conditions, the diversity is more clearly shown for the FRS's than with Likert-type scales or their numerical/fuzzy encodings.

Secondly, on the basis of the bootstrapped algorithm to test the equality of the Fréchet-type variances in Page 41, an additional comparative study is considered. It presents different (two-sample and k-sample) case studies-based tests about the equality of Fréchet's variances (or the corresponding standard deviations), that are to be considered separately for each of the three scales of measurement and for Case studies 1.6.1 (Page 33) and 1.6.3 (Page 35) in which the involved Likert scale is a 4- and 5-point one, respectively. By comparing the p-values associated with different scales we will conclude whether or not they differ depending on the rating scale.

Later, two descriptive comparative analyses are to be carried out in terms of all the, robust and non-robust, scale measures in Chapter 2. The first one compares the estimates of these measures for Case studies 1.6.1 and 1.6.3; in both case studies the considered fuzzy linguistic scales are the most usual (balanced) semantic representations of the linguistic hierarchies of $k = 4$ and $k = 5$ levels. The second descriptive comparison is simulations-based. Since data simulated will be FRS-based in accordance with the 2ndSP in Section 1.7 (Page 39), there are not direct Likert-type linked ones, so they need to be stated in a reasonable way (it will be considered a 'Likertization' process based on the minimum distance and a posterior fuzzy encoding involving a few different fuzzy linguistic scales).

3.1 Comparing the diversity of datasets for different rating scales

The diversity of a dataset from a random element quantifies the variability of data by taking into account neither their magnitude order nor how different they are (that is, irrespectively of how 'distant' they are and even of the nature of these data). It can also be interpreted as the amount of information supplied by the experimental performance.

In quantifying the diversity of a dataset, one of the best known indices is Gini-Simpson one (see Gini [46] and Simpson [104]).

Definition 3.1.1. *Let $\mathcal{X}$ be a random fuzzy number associated with the probability space $(\Omega, \mathcal{A}, P)$, and let $(\omega_1, \ldots, \omega_n)$ a sample of individuals from Ω and providing with the dataset $\widetilde{\mathbf{x}}_n = (\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_n))$. The **(sample) Gini-Simpson diversity index** of $\widetilde{\mathbf{x}}_n$ is the real number given by*

$$\widehat{G}(\widetilde{\mathbf{x}}_n) = 1 - \frac{1}{n^2} \sum_{j=1}^{m} \left[\text{card}\, \mathcal{X}^{-1}\Big(\{\widetilde{x}_j^*\}\Big)\right]^2$$

where $\mathcal{X}(\{\omega_1, \ldots, \omega_n\}) = \{\widetilde{x}_1^*, \ldots, \widetilde{x}_m^*\}$, $\mathcal{X}^{-1}\Big(\{\widetilde{x}_j^*\}\Big) = \{\omega_i : \mathcal{X}(\omega_i) = \widetilde{x}_j^*\}$, $m \leq n$.

If in responding to a given item or in rating an imprecise magnitude, raters can simultaneously make use of both an FRS and a Likert-type scale, then it seems quite reasonable that two coinciding outputs for the FRS are associated with two coinciding Likert outputs. Under such a behaviour, the Gini-Simpson index is definitely higher for the fuzzy rating scale-based questionnaire. Thus, on the basis of the well-known decomposability of the Gini-Simpson index, and because of the Likert assessment establishing a partition on the class of fuzzy responses, one can state that

Proposition 3.1.1. *Let $\mathcal{X}$ be a random fuzzy number associated with the probability space $(\Omega, \mathcal{A}, P)$, and let $(\omega_1, \ldots, \omega_n)$ a sample of individuals from Ω and providing with the dataset $\widetilde{\mathbf{x}}_n = (\mathcal{X}(\omega_1), \ldots, \mathcal{X}(\omega_n))$. Let $\mathfrak{F} : \mathcal{X}(\{\omega_1, \ldots, \omega_n\}) \to \mathscr{S}$ where $\mathscr{S} = \{\mathbf{L}_1, \ldots, \mathbf{L}_k\}$ is either the set of values from a Likert scale or the set of terms of its numerical or fuzzy linguistic encoding. Then,*

$$\widehat{G}(\widetilde{\mathbf{x}}_n) \geq \widehat{G}(\mathfrak{F}(\widetilde{\mathbf{x}}_n)),$$

with equality iff $k = m$.

Proof. Let $\mathcal{X}(\{\omega_1, \ldots, \omega_n\}) = \{\widetilde{x}_1^*, \ldots, \widetilde{x}_m^*\}$. The set $\mathbb{J} = \{1, \ldots, m\}$ can be partitioned as $\mathbb{J} = \mathbb{J}_1 \cup \ldots \cup \mathbb{J}_k$, with $\mathbb{J}_l = \{j \in \mathbb{J} : \mathfrak{F}(\widetilde{x}_j^*) = \mathbf{L}_l\}$. Then, if $n_l = \operatorname{card}(\mathfrak{F} \circ \mathcal{X})^{-1}\Big(\{\mathbf{L}_l\}\Big) = \sum_{j \in \mathbb{J}_l} \operatorname{card} \mathcal{X}^{-1}\Big(\{\widetilde{x}_j^*\}\Big)$, we have that

$$\widehat{G}(\widetilde{\mathbf{x}}_n) = 1 - \frac{1}{n^2} \sum_{j \in \mathbb{J}} \left[\operatorname{card} \mathcal{X}^{-1}\Big(\{\widetilde{x}_j^*\}\Big)\right]^2 = 1 - \sum_{l=1}^{k} \left[\frac{n_l}{n}\right]^2 \sum_{j \in \mathbb{J}_l} \left[\frac{\operatorname{card} \mathcal{X}^{-1}\Big(\{\widetilde{x}_j^*\}\Big)}{n_l}\right]^2$$

$$\geq 1 - \sum_{l=1}^{k} \left[\frac{n_l}{n}\right]^2 \sum_{j \in \mathbb{J}_l} \frac{\operatorname{card} \mathcal{X}^{-1}\Big(\{\widetilde{x}_j^*\}\Big)}{n_l} = 1 - \sum_{l=1}^{k} \left[\frac{n_l}{n}\right]^2 = \widehat{G}(\mathfrak{F}(\widetilde{\mathbf{x}}_n)).$$

□

As a consequence, since the most usual and reasonable double associations would be identified with mappings, and in case of developing simulations we would certainly make use of them, there is no need at this point to show conclusions from a simulations-based approach, which would inevitably ratify the general result above.

3.2 Comparison of rating scales through testing hypotheses about variances. Case studies-based analysis

This section is the first attempt to compare the three considered rating scales for intrinsically imprecise-valued data through scale measures. And this attempt is an inferential one; more concretely, a hypothesis testing one. Since the only available test about the equality of population scale measures for fuzzy data is the bootstrapped one corresponding to variances (or, equivalently, testing the equality of population standard deviations of two or more independent random elements) in Page 41, it is the only procedure we can use at this stage.

As we have stated before, the aim of this chapter is verifying that statistical conclusions can differ depending on the scale that is chosen to respond to items in a questionnaire when such a response is naturally imprecise (or, more generally, to rate intrinsically imprecise-valued magnitudes). To illustrate this assertion we are going to analyze some of the data from case studies in Example 1.6.1 (Page 33) and Example 1.6.3 (Page 35). These two examples relate to questionnaires in which respondents have chosen for each item both a Likert answer and an FRS-based one in [0,10] and [0,100], respectively.

Since an FLS corresponds to a posterior fuzzy number-valued encoding of Likert labels, we have chosen the most usual (see, for instance, Herrera *et al.* [57]) balanced semantic representations of the linguistic hierarchies of $k = 4$ levels (Figure 3.1) and $k = 5$ levels (Figure 3.2).

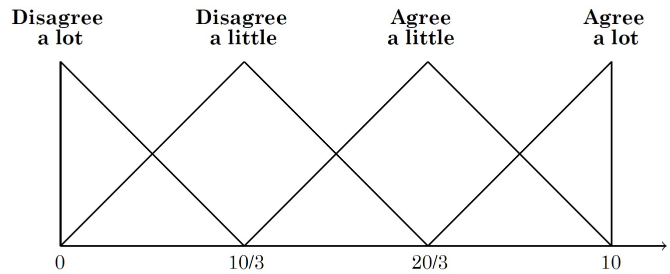


Figure 3.1: Usual balanced semantic representation of the linguistic hierarchies of $k = 4$ levels used in Example 1.6.1

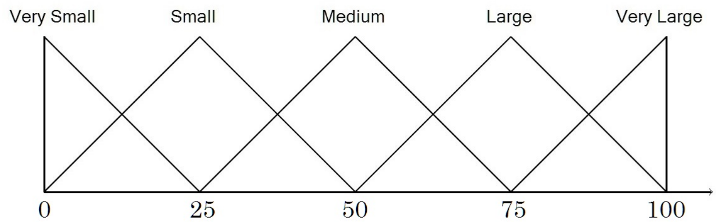


Figure 3.2: Usual balanced semantic representation of the linguistic hierarchies of $k = 5$ levels used in Example 1.6.3

Example 3.2.1. Consider the random experiment in Example 1.6.1 (Page 33, and see Appendix A, Page 233, for more details), and the nine adapted items allowing to receive a double response in Table 3.1. Only data from the students who have provided with both the Likert-type and the FRS responses have been considered in the comparative analysis in this section.

Assume that one wishes to test whether or not responses to Item $R.1$ can be expected to be equally disperse for the two considered forms (computerized and paper-and-pencil). Suppose that, for this purpose, the two-sample version of the bootstrapped homoscedasticity test in Page 41 is applied, separately, for

- the FRS-based data with reference interval [0,10],
- the numerically encoded Likert data (NELikert for short, and identifying DISAGREE A LOT with $\mathbb{1}_{\{0\}}$, DISAGREE A LITTLE with $\mathbb{1}_{\{10/3\}}$, AGREE A LITTLE with $\mathbb{1}_{\{20/3\}}$ and AGREE A LOT with $\mathbb{1}_{\{10\}}$),
- and the fuzzy linguistically encoded Likert data in accordance with the triangular encoding in Figure 3.1 (FLTriLikert for short),

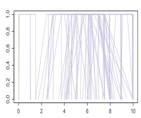
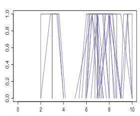
where Likert- and FRS-based data can be found in Table A.2 (Page 239). The L^2 chosen metric to compute the standard deviations in this example has been ρ_2.

Table 3.1: Questions selected from the student questionnaire in Example 1.6.1

	Reading in school
$R.1$	I like to read things that make me think
$R.2$	I learn a lot from reading
$R.3$	Reading is harder for me than any other subject
	Mathematics in school
$M.1$	I like mathematics
$M.2$	My math teacher is easy to understand
$M.3$	Mathematics is harder for me than any other subject
	Science in school
$S.1$	My teacher taught me to discover science in daily life
$S.2$	I read about science in my spare time
$S.3$	Science is harder for me than any other subject

The datasets, distances between the sample standard deviations for both forms and bootstrap p-values are gathered in Table 3.2.

Table 3.2: Two-sample bootstrapped homoscedasticity tests on the effect of the form chosen to respond to Item $R.1$ ($H_0 : \rho_2$-SD(comp) $= \rho_2$-SD(p&p)) in the adapted Questionnaire TIMSS-PIRLS ($\widehat{\mathrm{D}} = \widehat{\rho_2\text{-SD}}$)

rating scale	FRS	NELikert	FLTriLikert
dataset	form Comp.  P&p 	form: 0, 10/3, 20/3, 10 Comp.: 3, 8, 23, 10 P&p: 0, 4, 17, 3	form:     Comp.: 3, 8, 23, 10 P&p: 0, 4, 17, 3
|difference|	$\lvert\widehat{\mathrm{D}}(\text{comp}) - \widehat{\mathrm{D}}(\text{p\&p})\rvert = 0.321$	$\lvert\widehat{\mathrm{D}}(\text{comp}) - \widehat{\mathrm{D}}(\text{p\&p})\rvert = 0.977$	$\lvert\widehat{\mathrm{D}}(\text{comp}) - \widehat{\mathrm{D}}(\text{p\&p})\rvert = 0.808$
p-value	0.47	**0.05**	**0.06**

On the basis of the results in Table 3.2, we can conclude that the ρ_2-SD-dispersion is not significantly influenced by the filled form when the information in FRS-based data is considered. On the contrary, if encoded Likert data are employed, the ρ_2-SD-dispersion is significantly influenced by the filled form, especially for the NELikert data for which the p-value equals 0.05.

If p-values are analogously computed for the nine items, we get the results in Table 3.3.

Table 3.3: Two-sample bootstrapped homoscedasticity tests on the effect of the form chosen to respond to Items $R.1 - S.3$ ($H_0 : \rho_2\text{-SD(comp)} = \rho_2\text{-SD(p\&p)}$) in the adapted Questionnaire TIMSS-PIRLS ($\widehat{\text{D}} = \widehat{\rho_2\text{-SD}}$)

item \ rating scale	FRS	NELikert	FLTriLikert
$R.1$	0.47	**0.05**	**0.06**
$R.2$	0.79	0.49	0.50
$R.3$	0.38	0.57	0.52
$M.1$	0.35	0.24	0.25
$M.2$	0.15	0.20	0.22
$M.3$	0.11	**0.04**	**0.07**
$S.1$	0.27	**0.07**	**0.09**
$S.2$	0.49	0.99	0.95
$S.3$	0.47	**0.09**	**0.07**

The results in Table 3.3 show that statistical conclusions for the FRS approach in this study differ from the encoded Likert approaches substantially for Items $R.1$, $S.1$ and $S.3$ and slightly for Item $M.3$.

In Example 3.2.1 we can see that, for the items for which conclusions are remarkably different and dispersion is significantly different (actually, at significance levels up to 0.1) for the two involved forms, the p-values for the FRS are greater than those for the two encoded Likert. However, this cannot be viewed as a rather general situation, as we will see right now by means of another example.

Example 3.2.2. Consider the random experiment in Example 1.6.3 (Page 35, and see Appendix C, Page 251, for more details). The perception of the relative length of different line segments with respect to a pattern longer one is the unique item to be responded in an online application involving a double response.

An underlying non-explicit factor that could influence the response is Pref = "absolute number of pixels in horizontal sense occupied by the reference line". It is intended as the integer part of the product of the number of pixels in the horizontal direction of the working space (taking on values 1019, 1147, 1275 and 1435) times the relative size of the reference line with respect to the available horizontal working space expressed in percentage (fixed as 80) divided by 100. Levels for factor Pref are then 815, 917, 1020 and 1148.

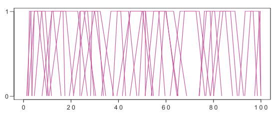
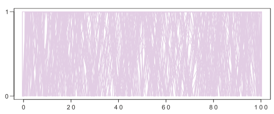
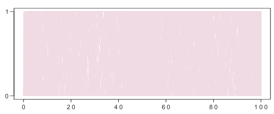
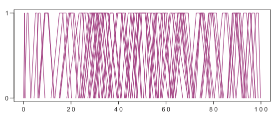
Assume that one wishes to test whether or not responses to the item can be expected to be equally disperse for the four values of Pref. For this purpose, the four-sample version of the bootstrapped homoscedasticity test in Page 41 is applied, separately, for

- the FRS-based data with reference interval [0,100],
- the numerically encoded Likert data (NELikert for short, and identifying VERY SMALL with $\mathbb{1}_{\{0\}}$, SMALL with $\mathbb{1}_{\{25\}}$, MEDIUM with $\mathbb{1}_{\{50\}}$, LARGE with $\mathbb{1}_{\{75\}}$ and VERY LARGE with $\mathbb{1}_{\{100\}}$),
- and the fuzzy linguistically encoded Likert data in accordance with the triangular encoding in Figure 3.2 (FLTriLikert for short).

The L^2 chosen metric to compute the standard deviations in this example has been ρ_2.

The datasets and bootstrap p-values are gathered in Table 3.4.

Table 3.4: Four-sample bootstrapped homoscedasticity tests on the effect of the resolution Pref ($H_0 : \rho_2$-SD(Pref=815) = ρ_2-SD(Pref=917) = ρ_2-SD(Pref=1020) = ρ_2-SD(Pref=1148)) in the online application Perceptions

<table>
<tr><th>rating scale</th><th>FRS</th><th>NELikert</th><th>FLTriLikert</th></tr>
<tr><td>dataset</td>
<td><table>
<tr><th>Pref</th><th>Data</th></tr>
<tr><td>815</td><td></td></tr>
<tr><td>917</td><td></td></tr>
<tr><td>1020</td><td></td></tr>
<tr><td>1148</td><td></td></tr>
</table></td>
<td><table>
<tr><th>Pref</th><th>0</th><th>25</th><th>50</th><th>75</th><th>100</th></tr>
<tr><td>815</td><td>5</td><td>10</td><td>7</td><td>7</td><td>4</td></tr>
<tr><td>917</td><td>73</td><td>92</td><td>90</td><td>92</td><td>65</td></tr>
<tr><td>1020</td><td>119</td><td>173</td><td>244</td><td>194</td><td>130</td></tr>
<tr><td>1148</td><td>10</td><td>18</td><td>24</td><td>17</td><td>13</td></tr>
</table></td>
<td><table>
<tr><th>Pref</th><th></th><th></th><th></th><th></th><th></th></tr>
<tr><td>815</td><td>5</td><td>10</td><td>7</td><td>7</td><td>4</td></tr>
<tr><td>917</td><td>73</td><td>92</td><td>90</td><td>92</td><td>65</td></tr>
<tr><td>1020</td><td>119</td><td>173</td><td>244</td><td>194</td><td>130</td></tr>
<tr><td>1148</td><td>10</td><td>18</td><td>24</td><td>17</td><td>13</td></tr>
</table></td></tr>
<tr><td>p-value</td><td>0.09</td><td>0.15</td><td>0.19</td></tr>
</table>

On the basis of the results in Table 3.4, we can conclude that the ρ_2-SD-dispersion is not very much influenced by the resolution (Pref) of the considered line if the (either numerically or fuzzy linguistically) encoded Likert data are employed. On the contrary, the ρ_2-SD-dispersion is rather significantly influenced by the resolution when the FRS-based data are considered since the p-value equals 0.09.

3.3 Descriptive comparison of the rating scales through different scale estimates. Case studies-based analysis

This section aims to reinforce the conclusions in the preceding section by involving scale measures different from the variance/standard deviation, and especially robust scale measures. However, there are not yet hypothesis testing procedures about other scale measures; in fact, it is a future direction of the research in this work. Consequently, we are limited to consider descriptive developments.

For the descriptive approach, we will first make use of the two case studies inferentially analyzed in the preceding section. Then, we are going to compute the scale measures in Section 2.1 over their datasets. The notations as well as the numerical and fuzzy linguistic encodings of the Likert responses in both case studies are those already considered in Section 3.2.

Example 3.3.1. Consider the random experiment in Example 1.6.1 (Page 33, and see Appendix A, Page 233, for more details), and the nine adapted items allowing to receive a double response in Table 3.1.

A descriptive analysis of the sample Aumann-type mean, the 1-norm median and the ℓ-wabl/ldev/rdev median for the nine items can be found in Table 3.5.

As one can see, to a lesser or greater extent the values of these location measures for the FRS-based responses are different from those for the encoded Likert ones (see also Lubiano *et al.* [75] for a wider analysis about). In this respect, if we look, for instance, at Item $M.3$, the mean value for the NELikert responses equals 5.8935 and the one for the FLTriLikert is $\mathrm{Tra}(3.2850, 5.8937, 5.8937, 7.8261)$, whereas the one for the FRS equals $\mathrm{Tra}(4.0149, 4.2562, 4.8982, 5.1895)$. If the last two fuzzy location estimates are defuzzified by means of their wabl$^{\ell}$, we get the values 5.7246 for the FLTriLikert and 4.5897 for the FRS. The difference is even greater for the medians, since the NELikert median equals 6.6667, the wabl$^{\ell}$ of both the 1-norm and wabl medians of the FLTriLikert equals 6.6667, and the wabl$^{\ell}$ of the 1-norm median of the FRS equals 3.8544 and the one of the wabl median of the FRS equals 3.9365.

Such a difference between different rating scales can also be shown when the sample scale is measured. In Table 3.6 the real values of the scale estimates in Section 2.1 (namely, $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^{\ell}\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^{\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$, and the robust ones $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\mathscr{D}_1^{\ell}\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^{\ell}}(\widetilde{\mathbf{x}}_n))$, $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$) are gathered for the nine items.

Table 3.5: Sample means, 1-norm medians and ℓ-wabl/ldev/rdev medians for the responses to Items $R.1$ to $S.3$ in Example 1.6.1

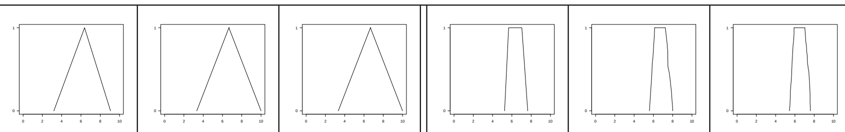
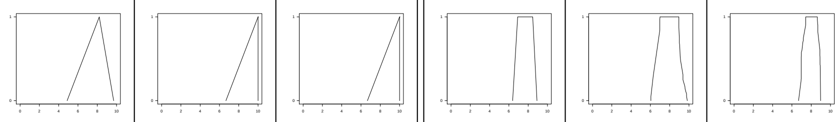
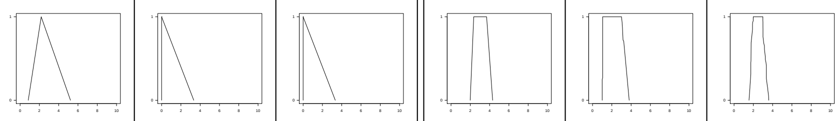
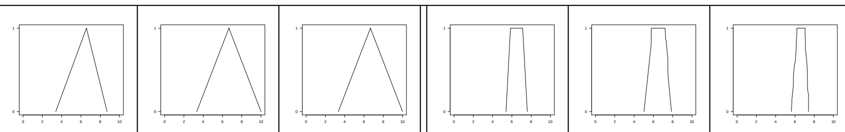
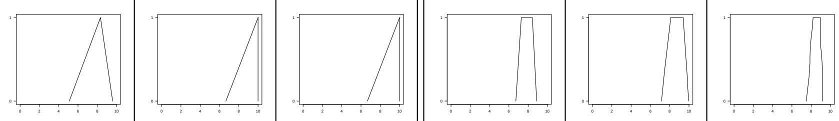
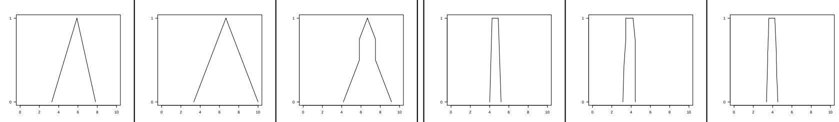
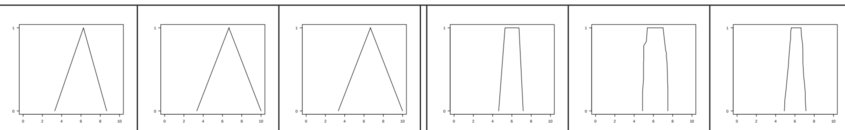
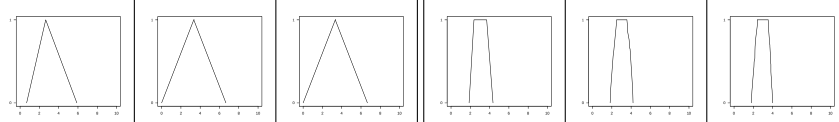
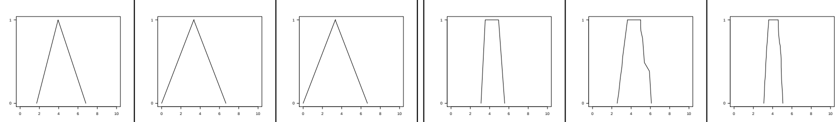
	NELikert mean	NELikert median	FLTriLikert Aumann mean	FLTriLikert 1-norm median	FLTriLikert wabl median	FRS Aumann mean	FRS 1-norm median	FRS wabl median
$R.1$	6.3738	20/3						
$R.2$	8.2099	10						
$R.3$	2.1885	0						
$M.1$	6.5672	20/3						
$M.2$	8.3341	10						
$M.3$	5.8935	20/3						
$S.1$	6.2572	20/3						
$S.2$	2.6553	10/3						
$S.3$	3.9392	10/3						

Table 3.6: Sample scale estimates for the responses to Items $R.1$ to $S.3$ in Example 1.6.1

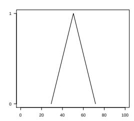
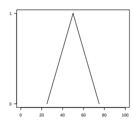
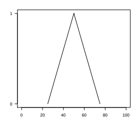
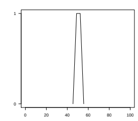
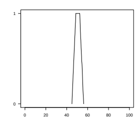
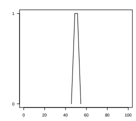
scale measure \ item	$R.1$	$R.2$	$R.3$	$M.1$	$M.2$	$M.3$	$S.1$	$S.2$	$S.3$
NELikert $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.4754	2.1798	3.1834	3.0469	2.4786	4.0192	3.1227	2.6474	3.4780
FLTriLikert $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.2052	1.8873	2.7177	2.6912	2.1186	3.4502	2.7284	2.3084	3.0256
FRS $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	2.2056	1.8435	2.9205	2.7125	2.3641	3.4820	2.6238	2.3081	2.9128
NELikert $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.7262	1.9772	2.6131	2.5207	2.0698	3.6769	2.4285	2.1574	2.9769
FLTriLikert $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.6572	1.6681	2.2253	2.3602	1.7403	3.2370	2.2499	1.8697	2.6990
FRS $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.7674	1.5979	2.4406	2.3890	2.0103	3.0942	2.1883	1.8853	2.5706
NELikert $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.5694	1.7901	2.1885	2.4885	1.6659	3.5757	2.2571	2.0309	2.8292
FLTriLikert $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.3725	1.4179	1.7786	2.1517	1.3258	3.0435	1.9359	1.6667	2.4369
FRS $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.6700	1.4611	2.2770	2.2865	1.7741	3.0276	2.0579	1.7890	2.4635
NELikert $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.5694	1.7901	2.1885	2.4885	1.6659	3.5757	2.2571	2.0309	2.8292
FLTriLikert $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.5931	1.8377	2.1471	2.5295	1.6777	3.4979	2.2965	2.0768	2.8772
FRS $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	2.0320	1.9332	2.7688	2.7927	2.3110	3.2824	2.5621	2.2428	2.9730
NELikert $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.2962	1.7901	2.1885	3.2372	1.6659	4.1065	2.9272	2.6553	2.7308
FLTriLikert $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	0.4795	1.4975	1.8195	2.9364	1.3816	3.4808	2.8077	2.2234	2.6044
FRS $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\widetilde{\mathbf{x}}}_n)$	1.4050	1.3520	2.3321	2.3665	1.8534	3.4066	1.8743	1.5960	2.4658
NELikert $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.0000	0.0000	0.0000	3.3300	0.0000	3.3300	3.3300	3.3300	3.3300
FLTriLikert $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	0.0000	0.0000	0.0000	2.5000	0.0000	2.5000	2.5000	2.5000	2.5000
FRS $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	1.2934	1.3625	1.7149	2.2140	1.2119	3.0357	1.8289	1.4314	2.3375
NELikert $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	0.0000	0.0000	0.0000	3.3300	0.0000	3.3300	3.3300	3.3300	3.3300
FLTriLikert $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	0.0000	0.0000	0.0000	3.3333	0.0000	2.9167	3.3333	3.3333	3.3333
FRS $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	1.6188	1.8240	2.2581	2.9174	1.8625	3.3000	2.2755	1.9329	2.9304
NELikert $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.0000	0.0000	0.0000	3.3300	0.0000	3.3300	3.3300	3.3300	3.3300
FLTriLikert $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	0.0000	0.0000	0.0000	2.8333	0.0000	2.8333	2.8333	2.5000	2.5000
FRS $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	1.5688	1.5625	2.2642	2.4005	1.5250	3.3625	2.0688	1.9250	2.8000
NELikert $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	3.3300
FLTriLikert $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	2.5000
FRS $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	1.0389	0.9313	1.1500	1.3813	0.9192	1.5313	1.3532	1.1625	1.6125
NELikert $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.0000	0.0000	0.0000	3.3300	0.0000	3.3300	3.3300	3.3300	3.3300
FLTriLikert $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	0.0000	0.0000	0.0000	2.8333	0.0000	2.8333	2.8333	2.5000	2.5000
FRS $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	1.3879	1.2980	1.8176	2.1641	1.2321	2.8338	1.8917	1.6460	2.4582

On the basis of Table 3.6 we can empirically conclude that the values of the scale estimates differ to a lesser or greater extent for (encoded) Likert responses and FRS responses. For all the items, the robust scale estimates show a higher sensitivity w.r.t. the rating scale (more concretely, to be either encoded Likert or FRS) than the non-robust ones. In fact for each of the non-robust estimates the highest differences are up to 1.1 whereas for each of the robust ones are over 1.8. In particular, $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ is uniformly quite different for the encoded Likert data (both the NELikert and the FLTriLikert) and the FRS-based ones.

Example 3.3.2. Consider the random experiment in Example 1.6.3 (Page 35, and see Appendix C, Page 251, for more details), the perception of the relative length of different line segments with respect to a pattern longer one being the unique item to be responded in the already referred online application involving a double response.

A descriptive analysis of the sample Aumann-type mean, the 1-norm median and the ℓ-wabl/ldev/rdev median for this item can be found in Table 3.7.

Table 3.7: Sample means, 1-norm medians and ℓ-wabl/ldev/rdev medians for the responses to questionnaire in Example 1.6.3

NELikert mean	NELikert median	FLTriLikert Aumann mean	FLTriLikert 1-norm median	FLTriLikert wabl median	FRS Aumann mean	FRS 1-norm median	FRS wabl median
50.4867	50						

As one can see, these location measures are mainly different for the FRS-based responses and the encoded Likert ones because of their shape, but if they were defuzzified they would be close. This is due to the fact that the distributions of data are close to symmetrical (rather than to the fact that $k = 5$), so that both robust and non-robust are also 'close'.

On the other hand, the analysis of the sample scale measures (those considered in Example 3.3.1) can be found in Table 3.8.

On the basis of Table 3.8 we can empirically conclude that the values of the scale estimates are very close but importantly for the robust $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$ and to a lesser extent for the non-robust $\widehat{\rho_2\text{-SD}}$. Actually, eight of the measures (all but $\widehat{\rho_1\text{-Q}}$ and $\widehat{\rho_2\text{-SD}}$) lead to close estimates irrespective of the measure. This behaviour is also mainly due to the symmetry of the distribution.

As a summary implication of this descriptive analysis for the scale estimation from intrinsically imprecise-valued data, we can conclude that estimates can differ with the rating scale. This implication has an important analogy with what happens when grouping real-valued data by intervals: both, 'identifying' fuzzy rating scale-valued responses with one of a few possible Likert labels (or their numerical/fuzzy linguistic counterpart) and 'identifying' real-valued responses with one of a few possible non-overlapping interval-valued ones, entail a loss of information so that some existing differences can be ignored, whence statistical conclusions are not usually well preserved under such an identification.

Table 3.8: Sample scale estimates for the responses to questionnaire in Example 1.6.3

scale measure		scale measure	
NELikert $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	32.0423	NELikert $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	26.0937
FLTriLikert $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	29.4047	FLTriLikert $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	24.8377
FRS $\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	28.8479	FRS $\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	24.7757
NELikert $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	25.9733	NELikert $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	25.9733
FLTriLikert $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	24.0853	FLTriLikert $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	26.2093
FRS $\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	24.6524	FRS $\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	25.9801
NELikert $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	25.4867	NELikert $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	25.0000
FLTriLikert $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	25.5573	FLTriLikert $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	25.0000
FRS $\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \overline{\overline{\widetilde{\mathbf{x}}}}_n)$	24.8092	FRS $\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	24.6680
NELikert $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	25.0000	NELikert $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	25.0000
FLTriLikert $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	25.0000	FLTriLikert $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	25.0000
FRS $\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}}^\ell(\widetilde{\mathbf{x}}_n))$	25.9756	FRS $\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	25.3450
NELikert $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	25.0000	NELikert $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	25.0000
FLTriLikert $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	18.7500	FLTriLikert $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	25.0000
FRS $\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	13.4500	FRS $\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	24.5803

3.4 Descriptive comparison of the rating scales through different scale estimates. Simulations-based analysis

The descriptive analysis in the last section is to be now completed with a simulation-based one. Along this section, FRS data will first be simulated in accordance with the realistic process in the 2ndSP (Page 39), which to a great extent mimics the response to FRS-based questionnaires. In the case studies we have assumed double response questionnaires for which components of each double response are linked, because of them coming from the same respondent. However, this link cannot be immediately stated in simulation processes. In this section, a reasonable 'Likertization' process is to be considered and later a fuzzy linguistic encoding is to be applied. Next two subsections are devoted to present such processes, as well as to validate the first one by means of the considered case studies.

3.4.1 'Likertization' of fuzzy rating scale-based data: criterion and validation

Fuzzy data based on a fuzzy rating scale can fairly be associated/classified in accordance with labels in a Likert scale (more concretely, with their numerical encoding). This process is to be called "Likertization". Furthermore, the associated Likert values could also be later encoded by means of values from a fuzzy linguistic scale. This subsection is devoted to explain an ease-to-use and ease-to-support association criterion.

For carrying out the Likertization, different association criteria could be employed. The one to be used in next subsections will be the following:

Minimum distance Likertization criterion:

If the considered Likert scale is a k-point one, a reasonable Likertization criterion consists in associating each FRS-based datum with the integer number in $\{1, \ldots, k\}$, if $[1, k]$ is the reference interval in the considered FRS, with the smallest distance to the given datum. That is, given a metric D between fuzzy data and $\widetilde{U}$ the free fuzzy response to be classified, then $\widetilde{U}$ is associated with the integer $\kappa(\widetilde{U})$ such that

$$\kappa(\widetilde{U}) = \arg \min_{j \in \{1,\ldots,k\}} D(\widetilde{U}, \mathbb{1}_{\{j\}}).$$

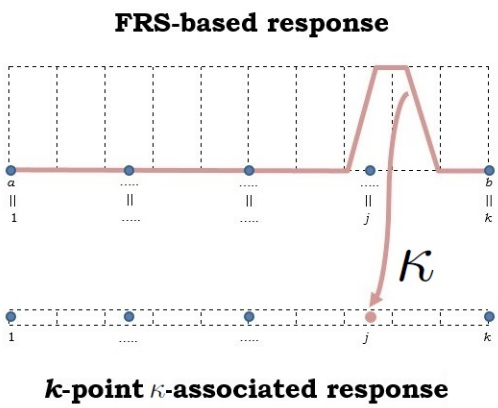


Figure 3.3: Minimum distance criterion scheme when the reference interval equals $[1, k]$

Figure 3.3 graphically illustrates the process when $[a, b] = [1, k]$, but it can immediately be re-scaled to any other $[a, b]$, so that the choice is made in the subset $\{a, a+(b-a)/(k-1), a+2(b-a)/(k-1), \ldots, b\}$ whose elements divide the reference interval $[a, b]$ in $k-1$ equally wide subintervals.

Remark 3.4.1. If in solving the above minimization there exist two involved coincident distances, the associated numerically encoded Likert response is chosen at random among the two corresponding numerical values. It should be pointed out that other reasonable and well-supported Likertization criteria (like, for instance, the supervised classification by Colubi *et al.* [20]) have been considered, but they lead to a similar validation.

Case studies-based validation of the minimum distance criterion:

The three case studies in Examples 1.6.1 (Pages 33 and 233), 1.6.2 (Pages 35 and 243) and 1.6.3 (Pages 35 and 251) are now to be used to make the classification of the FRS-based component of each double response into a Likert one and validate this classification later with the real Likert-based component of the same double response.

Table 3.9 shows the percentages of responses for which the minimum distance-based Likert classification of the FRS responses coincides with the real associated Likert response for different choices of the metric D.

Table 3.9: Validation of the minimum D-distance Likertization of fuzzy rating responses: % of right associations in the Likertization of the FRS-based responses in Examples 1.6.1, 1.6.2 and 1.6.3 ($D \in \{\rho_1, \mathscr{D}_1^\varphi, D_{1/3}^\varphi, D_1^\varphi\}$ with different choices of φ)

% matching				**TIMSS/PIRLS** $n=599$	**Restaurants** $n=980$	**Perceptions** $n=1387$
Minimum distance criterion	ρ_1			81.64	77.96	85.00
	$\mathscr{D}_1^\varphi$	$\varphi \equiv \beta(1,1)$		81.30	77.76	84.93
		$\varphi \equiv \beta(1,2)$		80.63	77.96	85.08
		$\varphi \equiv \beta(2,1)$		80.63	77.96	85.15
		$\varphi \equiv \beta(1,5)$		80.63	76.43	85.15
		$\varphi \equiv \beta(5,1)$		81.64	78.16	85.29
	D_θ^φ	$\varphi \equiv \beta(1,1)$	$\theta = 1/3$	81.30	78.06	84.93
			$\theta = 1$	81.30	78.16	84.93
		$\varphi \equiv \beta(1,2)$	$\theta = 1/3$	80.80	78.06	85.08
			$\theta = 1$	80.97	78.06	85.08
		$\varphi \equiv \beta(2,1)$	$\theta = 1/3$	81.30	78.06	85.15
			$\theta = 1$	80.80	77.96	85.15
		$\varphi \equiv \beta(1,5)$	$\theta = 1/3$	80.47	76.43	85.15
			$\theta = 1$	80.63	76.43	85.15
		$\varphi \equiv \beta(5,1)$	$\theta = 1/3$	81.47	78.06	85.29
			$\theta = 1$	81.47	78.16	85.29

The preceding percentages are rather high. One should not expect to design a Likertization criterion leading to 100% of matching, since humans not necessarily behave in a completely systematized way.

The validation is almost irrespective of the chosen metric, and in the next simulations the minimum ρ_1 Likertization criterion will be considered.

3.4.2 Fuzzy linguistic encoding through the Likertized responses

This subsection is focussed on the fuzzy linguistic encoding of FRS-based responses. Each FRS-based datum will be first Likertized by means of the minimum distance criterion, and it will later be encoded by means of a fuzzy linguistic scale (like, for instance, those in Figures 3.1 and 3.2).

In the simulations studies in this section, we will make use of some fuzzy linguistic scales selected among the most mentioned in the literature to encode 4- and 5- point Likert scales.

Case $k = 4$**:**

In this case we have made use of some of the most frequently fuzzy linguistic scales considered when 4 labels are modelled (see, for instance, Herrera *et al.* [57], Bajpai *et al.* [4], Cai *et al.* [12], Picon *et al.* [92]).

FLS_1^4, FLS_2^4 and FLS_3^4 will denote some of the most usual symmetric representations for a 4-point Likert scale, $\mathrm{FLS}_3^4 = \mathrm{FLTriLikert}$ being the balanced semantic representation of the $k = 4$ linguistic hierarchies, and FLS_4^4 and FLS_5^4 being two asymmetric representations for the same number of labels.

FLS_1^4 to FLS_5^4 have been displayed in Figure 3.4 by considering $[0, 100]$ as the reference interval.

Case $k = 5$**:**

In this case we have made use of some of the most frequently fuzzy linguistic scales considered when 5 labels are modelled (see, for instance, Yeh *et al.* [136], Motawa *et al.* [85], Herrera *et al.* [57]).

FLS_1^5 and FLS_2^5 will denote some of the most usual symmetric representations for a 5-point Likert scale, $\mathrm{FLS}_3^5 = \mathrm{FLTriLikert}$ being the balanced semantic representation of the $k = 5$ linguistic hierarchies, and FLS_4^5 being an asymmetric representation inspired by the unbalanced semantic of the same number of labels (Herrera *et al.* [57]).

FLS_1^5 to FLS_4^5 have been displayed in Figure 3.5 by considering $[0, 100]$ as the reference interval.

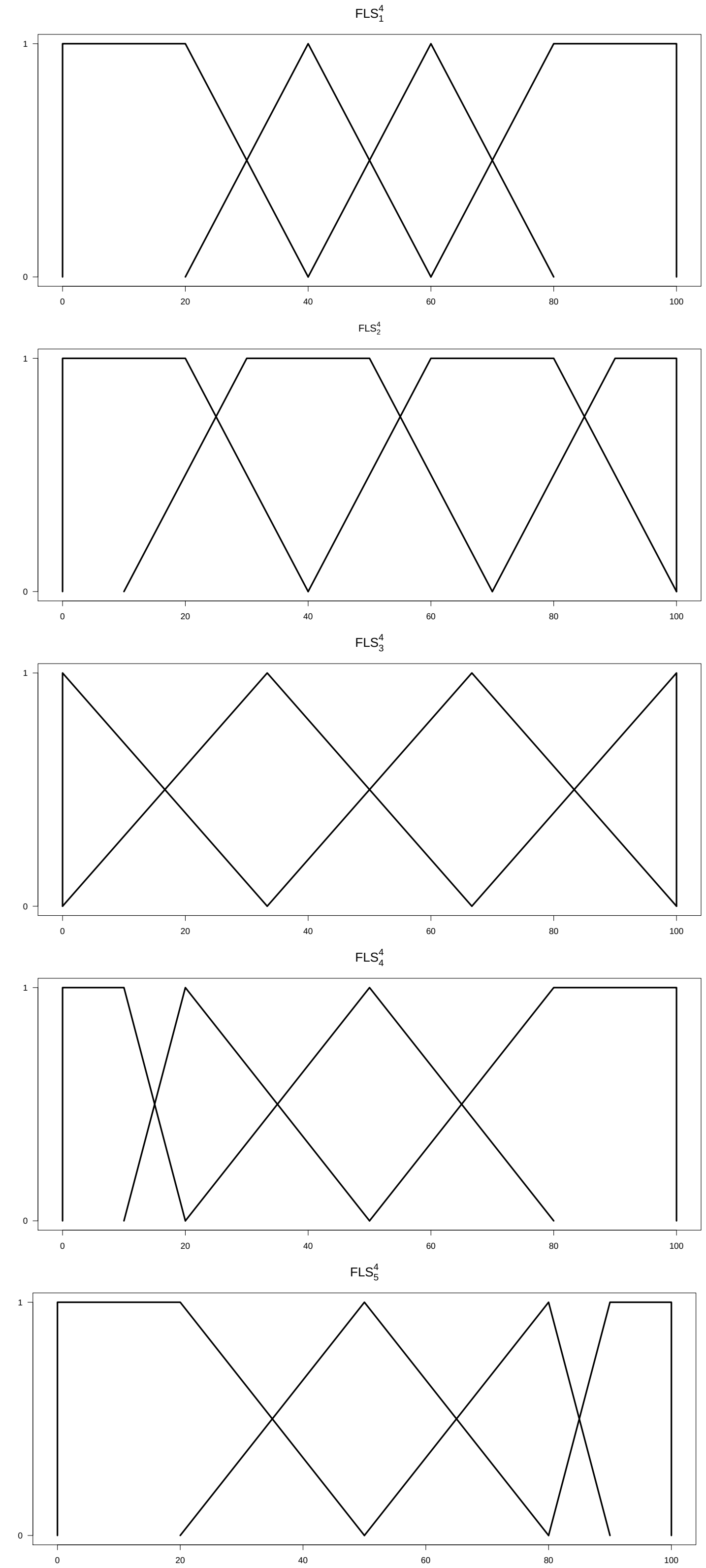


Figure 3.4: Fuzzy linguistic scales with four values and reference interval $[0, 100]$: from top to bottom, the scales FLS_1^4, FLS_2^4, FLS_3^4, FLS_4^4 and FLS_5^4.

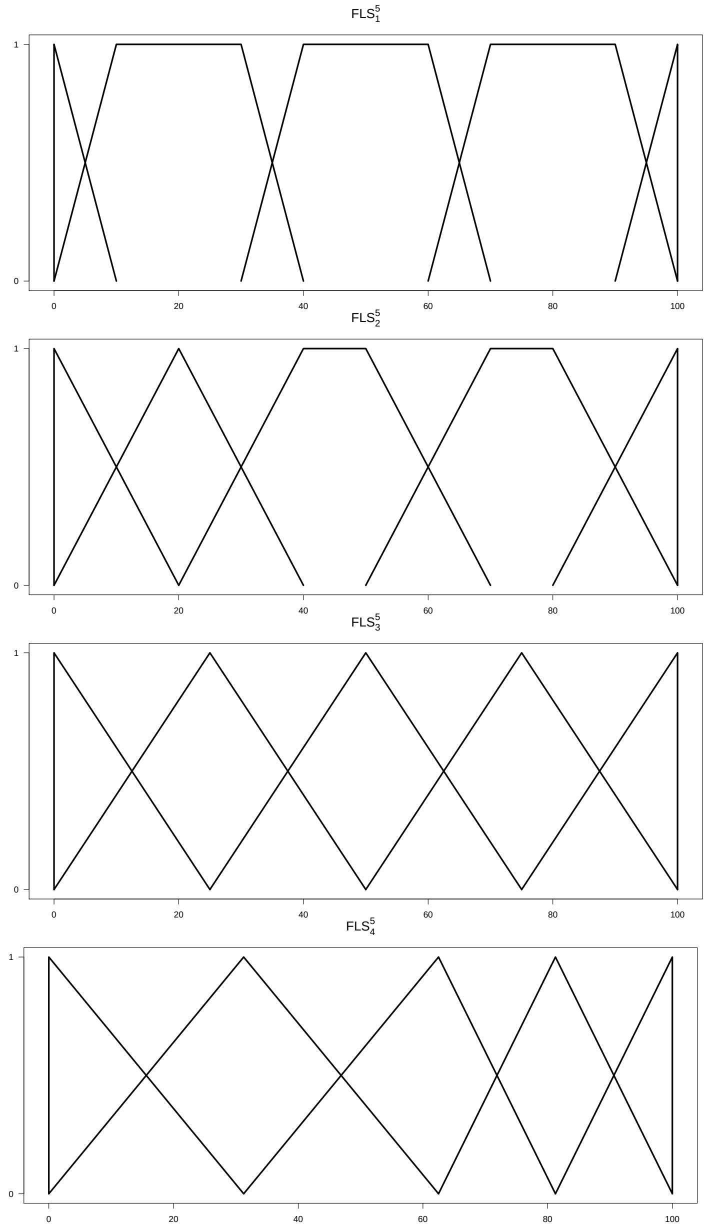


Figure 3.5: Fuzzy linguistic scales with five values and reference interval $[0, 100]$: from top to bottom, the scales FLS_1^5, FLS_2^5, FLS_3^5 and FLS_4^5.

3.4.3 Descriptive simulations-based comparison of different rating scales

Next simulations-based tables (Tables 3.10 to 3.21) collect the percentages of Euclidean distances between the sample scale estimates $\widehat{\mathrm{D}}$ for the FRS-simulated data and for their numerically and fuzzy linguistically encoded ρ_1 Likertization that are over $\varepsilon \in \{1, 5, 10, 15\}$. The percentages have been quantified over 1000 samples of $n \in \{10, 30, 100\}$ FRS simulated (following the 2ndSP with different betas) data with reference interval $[0, 100]$ (this last fact being irrelevant for the study).

Table 3.10: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 4$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p,q) \equiv \beta(1,1)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k=4,\ \beta(p,q) \equiv \beta(1,1))$					
		S = Likert	S = FLS_1^4	S = FLS_2^4	S = FLS_3^4	S = FLS_4^4	S = FLS_5^4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	82.2	99.5	90.1	64.3	90.8	91.6
	30	85.7	100	99	51.9	98.9	98.9
	100	94.9	100	100	44.2	100	100
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	78.2	98.9	82.7	70.1	90.9	92
	30	69	100	90.3	52.2	99.2	98.7
	100	79.1	100	98.8	25.7	100	100
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	79.2	99.3	90.8	76.2	94	94.4
	30	63.5	100	95.6	72.3	99.5	99.2
	100	64.9	100	99.4	58.9	100	100
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	80	99.3	92	79.6	92.1	93.2
	30	79.6	100	97.9	76.4	98.3	97.6
	100	78.5	100	100	71.5	100	99.9
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	91	96.6	88.8	86.3	91.8	93.5
	30	88.7	99.6	92.9	89.1	96.8	95.2
	100	94.9	100	98.8	95.7	99.7	99.8
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	93	93	92.7	91.2	93.8	94.3
	30	97.1	89.1	87.7	86.6	91.3	89.6
	100	100	90	80.8	74.3	87.4	85.7
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	94.3	95.5	94.4	93.5	93.9	94.9
	30	96	96.2	89.1	96.4	89.2	90.3
	100	98.8	100	80.8	98.8	86.6	83.2
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	94.9	93.8	91.4	88.8	90.3	91.7
	30	96.3	96.7	85.4	81.6	87.4	87
	100	100	99.9	78.6	71.5	76.4	76.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	99.9	83.6	89.6	94.1	91.3	89.9
	30	100	100	100	100	88.7	88.7
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	96.4	90.7	89.6	87.6	91.2	90.4
	30	99.9	84.7	84.5	78.1	87.8	87.4
	100	100	92.4	83.4	59.5	87.4	85.3

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k=4,\ \beta(p,q) \equiv \beta(1,1))$					
		S = Likert	S = FLS_1^4	S = FLS_2^4	S = FLS_3^4	S = FLS_4^4	S = FLS_5^4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	16.8	78.7	23.9	4.1	38.7	36.8
	30	4.8	95.2	11.8	0.1	32	31.2
	100	0	100	1.9	0	20.9	19.5
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	11.3	77.6	21.6	6.1	43.1	40
	30	1.6	93.4	7.5	0.1	33.6	34.9
	100	0	99.4	0.3	0	22.5	21.6
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	14.2	80.8	36.3	16.7	54.4	51.3
	30	1.9	95.2	26.1	3.1	57.3	57.8
	100	0	99.4	8.4	0	52.2	51
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	25.5	90	45.3	22.8	47.4	46.7
	30	11.4	99.2	47.4	8.8	47.3	46.5
	100	1	100	31.1	0.5	32.5	31
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	47.9	76.7	50.1	41.1	61.5	59.7
	30	42.8	92.1	58	42.8	75.3	74.2
	100	55.3	99.8	79.4	55.8	91.1	90.9
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	66.2	67.8	61.5	57.5	69.7	70.2
	30	83.5	47	42.1	37.8	54.4	52.2
	100	97.5	31.1	21.6	12.5	42	39.6
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	71.2	78.4	69.3	70.6	70.4	72.1
	30	71.6	77.7	48.2	71.4	52.1	52.4
	100	63	89.7	18.4	63	40.8	38.8
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	72.2	65.4	54.7	52.8	60.1	61.7
	30	68.6	64.8	37.5	27.4	49.7	48.8
	100	88.7	65.3	27.8	8.6	45.5	46.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	99.1	39.3	54.1	63.1	52.3	54.2
	30	100	75	80.2	99.8	58.4	58.4
	100	100	99.9	100	100	97.8	97.8
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	80.7	54.2	50.3	51.8	52.7	56.6
	30	94.9	26.9	26.5	23.9	41	39.8
	100	100	11.9	8.6	8.5	39.4	38.3

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k=4,\ \beta(p,q) \equiv \beta(1,1))$					
		S = Likert	S = FLS_1^4	S = FLS_2^4	S = FLS_3^4	S = FLS_4^4	S = FLS_5^4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0.2	7.5	0.1	0	2	2.9
	30	0	0.6	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.2	11.1	0.3	0	1.6	1.9
	30	0	2.4	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0.2	12.4	1.2	0.1	3.7	4.1
	30	0	3.7	0	0	0.4	0.2
	100	0	0.2	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	1.5	29.7	5.6	1.2	6.1	6.2
	30	0	27.1	0.6	0	0.7	0.6
	100	0	15.7	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	14.3	40.7	15.2	9.4	23.7	24.7
	30	8.7	60.7	15	7.7	30.6	29.2
	100	4.3	85.2	14.4	3.7	34.8	33.8
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	40.7	36.8	29.1	25	35.4	36.6
	30	47.8	20.9	12.7	9	19.8	19.9
	100	44.7	7.9	3.4	0.7	7.3	7.2
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	41.9	54.1	38.4	40.4	40.2	41.9
	30	25.6	39.1	19.7	25.4	22.2	21.4
	100	9.1	22.7	6.9	9.1	7.3	8.3
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	41.9	38.8	29.9	29.1	33.1	34.6
	30	17.1	15.5	8.1	5.8	20.3	20.5
	100	3.5	0.5	0.3	0.1	11.3	11.2
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	88.3	23.9	28.6	29	27.8	28.9
	30	100	58	58.1	62.5	58	58
	100	100	97.8	97.8	98.4	97.8	97.8
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	52	26.8	25.6	25.3	26.5	29.4
	30	45.8	4.6	5.4	7.9	7.6	7.4
	100	35.5	0.1	0.1	1.3	0.1	0.5

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k=4,\ \beta(p,q) \equiv \beta(1,1))$					
		S = Likert	S = FLS_1^4	S = FLS_2^4	S = FLS_3^4	S = FLS_4^4	S = FLS_5^4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	1.1	0	0	0.1	0.2
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	2.3	10.8	1.4	0.7	3.7	4.2
	30	0.4	17.2	1.1	0	3.8	4.5
	100	0	16.5	0.1	0	0.8	0.6
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	15.7	14.2	10.2	8.3	11.8	13.1
	30	11.2	7.8	3.6	3.2	5.6	5.8
	100	1.2	3.2	0.1	0.1	0.4	0.4
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	19.8	29.9	18.9	19.2	19.3	20.8
	30	6.9	17.8	8.1	6.9	7.7	8.1
	100	0.6	6.9	2.5	0.6	1.3	1.3
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	20.5	18.9	17.2	16.5	18.1	20.6
	30	4.6	4.5	4.4	4.4	5.7	5.8
	100	0.1	0.1	0.1	0.1	0.1	0.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	38.7	12.5	13	12.8	12.9	13.2
	30	66.3	26.2	26.2	26.2	26.2	26.2
	100	13.1	10.9	10.9	10.9	10.9	10.9
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	21.4	13.2	13.2	13.2	14	15.6
	30	7.3	4	4	4	4.1	4.1
	100	0.1	0.1	0.1	0.1	0.1	0.1

Table 3.11: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 4$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(0.75, 0.75)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k = 4,\ \beta(p,q) \equiv \beta(0.75, 0.75))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	83	99.7	94.7	67.9	95.6	94.1
	30	90	100	99.9	59.7	99.7	99.7
	100	98.8	100	100	70	100	100
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	76.4	99.4	89.9	71.4	93.4	92.6
	30	70.7	100	96.7	55.4	99.4	99.4
	100	82.1	100	100	40	100	100
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	78	99.4	92.7	78.4	95.6	94.3
	30	65.9	100	98.6	77.8	99.6	99.9
	100	65.3	100	100	77.6	100	100
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	83.1	99.7	90.8	81.2	91.6	89.2
	30	76.9	100	98.7	72.8	97.5	98.9
	100	77.5	100	100	66	100	100
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	87.9	94.8	90.6	87.8	91.3	92.6
	30	91.5	97.9	93.5	92	95.7	97.2
	100	98.5	99.6	99.7	99.2	99.6	99.8
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	94	92.1	90.1	91.1	94.7	92.7
	30	96.5	91.3	87.3	83.9	89.3	91.1
	100	98.8	99.1	88.5	78.1	89.1	88.5
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	93.6	93.8	91.1	93.5	93.1	93.5
	30	90.8	97.3	84.5	90.9	90	91.4
	100	80.1	100	73.7	80.4	90.4	89.1
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	94	92.3	91.6	88.5	92.9	94
	30	87.2	98.6	91.5	91.5	90.1	91.8
	100	93.8	100	94.5	97.4	95.3	95.2
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	99.9	82.8	87.3	89.1	87.5	88.4
	30	100	98.5	98.9	100	68.5	68.5
	100	100	100	100	100	98.8	98.8
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	95.1	90.9	87.9	87.3	90.1	91.4
	30	98.5	93	81	80.9	84.3	83.6
	100	100	100	76.8	76.8	76.3	73.9

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k = 4,\ \beta(p,q) \equiv \beta(0.75, 0.75))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	19.9	83.9	32	3.8	42.5	39.4
	30	6.2	98.6	23.1	0	38.7	41.8
	100	0.1	100	10.8	0	38.4	36.6
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	12.1	80.3	28.9	5.3	48	45.5
	30	2.2	97	13.9	0	43.9	47
	100	0	100	2.5	0	43.1	43.7
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	14.1	81.8	42.1	14	58.6	55.9
	30	1.3	97.9	40.2	2.7	63.4	69.8
	100	0	100	23.4	0	73.3	73.3
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	20.3	86.1	44.4	17.6	49.6	45.4
	30	7.3	99.2	45	5.4	46.6	50.9
	100	0.3	100	35.1	0.1	43.1	39.8
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	53.2	73.3	51.1	41.9	60.9	59.6
	30	59.5	89.5	68	56.1	77.3	76.8
	100	84.8	99.3	95.3	86.8	96.9	96.9
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	68.5	64.5	61.3	57.5	68	69.1
	30	74.7	59.1	47.4	34.6	55.6	57
	100	74.6	77.7	48.8	18.6	57.1	56.3
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	69.1	71.8	62.8	67	66.9	68.9
	30	52.1	83.3	38.6	52.7	56.2	55.7
	100	20.4	99.2	14	21	59	57.4
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	64.2	67.5	55.8	52.5	63.9	64.9
	30	38.7	85.5	55.2	44.3	59.1	60
	100	29.1	98.7	53.4	42.6	56.6	56.2
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	98.8	33.6	45.7	51.6	45	50.4
	30	100	45.2	51.5	97.7	27	27
	100	100	91.9	94.8	100	62.6	62.6
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	72.9	59.4	46.1	44.8	57.3	58.2
	30	75.5	52.9	29.3	20.6	42.3	43.1
	100	89.3	69.3	29.4	6.6	45.8	45.5

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k = 4,\ \beta(p,q) \equiv \beta(0.75, 0.75))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	7.9	0.3	0	2.1	2.4
	30	0	1.1	0	0	0.1	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	12.8	0.2	0	2.4	2
	30	0	4.3	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	12.3	0.9	0	4.6	3.5
	30	0	5.6	0	0	0.2	0.2
	100	0	0.1	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	1.3	27.3	4.4	1.3	5.6	5.5
	30	0	25.9	0.3	0	0.5	0.5
	100	0	16.5	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	17.4	41.7	16.1	9.9	28.7	25.2
	30	20.4	68.9	29.3	15.3	41.7	43.6
	100	31.1	97.2	53.6	28.6	72.1	69.6
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	39.7	33.2	26.1	22.6	35.6	35.4
	30	36.3	21.9	16.7	9.5	26.1	26.4
	100	13.5	17.7	11.6	4.8	25.7	26
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	40.9	45.5	33.3	37.6	38.6	38.9
	30	19.9	46.7	17	19	26.1	28.8
	100	8.1	68.8	8.4	8.2	24.7	26.7
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	35.3	37.4	26.4	24.6	34.9	35
	30	6.2	32.5	18.4	8.9	32.3	33.1
	100	0	24.6	8.4	0.3	39.7	39.6
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	79.8	14.8	19.6	19.5	18.1	20.5
	30	100	26.7	26.7	31.9	26.7	26.7
	100	100	62.6	62.6	66.4	62.6	62.6
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	40.9	25.7	18.9	17.8	27.6	27
	30	20.2	4.4	3.6	2.1	13.2	15
	100	3	0	0.1	0	11.8	11.6

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k = 4,\ \beta(p,q) \equiv \beta(0.75, 0.75))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0.1	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	1.2	0	0	0	0.1
	30	0	0.1	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	1.3	13	1.4	0.2	5.1	4
	30	0.7	31.5	2.1	0.3	7.8	9.4
	100	1.2	60.1	5.1	0.2	11.1	12.4
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	15.5	12.6	7.6	6.1	11.6	12.1
	30	5.6	9.8	3.8	2.1	8.8	8.9
	100	0.3	6.4	1.5	0.3	5	4.4
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	17	24.9	15.7	15.4	17.8	18.7
	30	7.5	17.8	9.3	7.2	10.6	10.5
	100	4.3	12.1	5.8	4.3	5.3	5.4
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	17.2	16.1	12.3	11	17.4	16.8
	30	2.9	3.6	3.1	2.5	9.2	10.3
	100	0	0	0	0	2.5	2.6
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	34.1	7.8	8.2	8	8.4	8.4
	30	78.3	15.5	15.5	15.5	15.5	15.5
	100	71	33.6	33.6	33.6	33.6	33.6
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	17.2	9.7	8.1	8.1	12.3	12.5
	30	3.2	1.4	1.4	1.4	2	1.9
	100	0	0	0	0	0	0

Table 3.12: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 4$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(2, 2)$

		% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 1$ $(k = 4,\ \beta(p,q) \equiv \beta(2,2))$					
$\widehat{\mathrm{D}}$	n	S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	77.9	98	80.8	70.6	85.1	87.9
	30	72.8	100	86.6	44	95.2	95.3
	100	75.2	100	98.2	18.8	99.8	99.9
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	73.9	97.5	76.9	70.7	84.7	86
	30	70.4	99.9	62.6	52.7	92.5	91.8
	100	81.8	100	55.2	47.6	98.7	98.9
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	78.5	97.9	83.8	78.5	90.7	90.5
	30	64.3	100	78.6	62	95.9	95.5
	100	66.8	100	69.2	44.4	99.7	99.8
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	84.6	99.5	91.5	85	90.5	90.9
	30	82	100	96.3	80.6	95.5	94.5
	100	76.3	100	99.6	72.7	98.9	99
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	87.2	94.6	88	86.6	89	90
	30	79.9	97.2	82.4	79	89.3	88
	100	68.7	99.9	68.6	68.3	90.7	90.9
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	90.8	95.1	92.9	92	93.4	94.3
	30	91.5	93	93.7	94.5	92.7	93.6
	100	94.4	94.4	94.6	95.8	84.4	85.9
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	93.2	98.1	94.8	93.2	94.2	95.3
	30	95.6	91.2	98.3	95.6	93.5	94.3
	100	98.8	69.6	100	98.8	87.8	86.8
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	99.1	92.2	94	96.3	93.7	93.5
	30	100	78.3	90.3	95.3	94.5	95.4
	100	100	61.1	86.3	99.6	95.8	95.6
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	96.5	98.2	99.6	94.7	93.8
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	99.8	91.2	95.3	96.5	93.9	93.3
	30	100	84.5	95.2	99.3	91.2	91.8
	100	100	92.8	98	100	79.7	79.6

		% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 5$ $(k = 4,\ \beta(p,q) \equiv \beta(2,2))$					
$\widehat{\mathrm{D}}$	n	S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	14.9	65.7	13.7	4.4	29.7	32.7
	30	1.5	81.6	2.5	0	19.6	18.4
	100	0	96.1	0.1	0	6.7	5.9
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	12.5	64.4	10	7.2	24.5	26.6
	30	1.1	70.7	0.6	0.1	8.9	9.5
	100	0	84.3	0	0	1.2	1.1
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	13.2	68.7	24.8	14.2	42.1	42.3
	30	1.6	78.9	6	0.9	29	29.1
	100	0	83.2	0.3	0	10.1	9.6
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	32.1	89.4	48.1	31.4	45.8	48.6
	30	10.8	98.3	38.2	10.2	34.7	32
	100	1.5	100	16.9	1.2	13.1	12.6
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	39.3	67.8	40.2	38	49.6	51.2
	30	21.5	72.8	23.1	19.1	40.3	39.4
	100	4.4	81.4	8.3	3.4	27.4	24.1
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	65	74.9	66.8	63.6	68.3	70
	30	70.9	63.7	63.7	69.2	64.4	67
	100	85	38	56.8	83.6	56.2	55.2
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	67.5	87.5	71.9	67.5	72.2	73.5
	30	77.5	69	81.7	77.5	66.8	69.1
	100	88	31.4	92.9	88	57.7	57.6
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	93.7	64.9	73.9	77.1	71.4	70.2
	30	99.5	37.8	61.1	61.8	59.8	60.8
	100	100	16.2	54.4	58	53.9	53.2
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	79.7	84.8	93.3	81.5	81.4
	30	100	99.7	99.9	100	99.3	99.3
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	97.7	60.8	77.5	83.7	72.8	72.7
	30	100	39.7	72.8	84.1	67.8	67.5
	100	100	20.6	65.2	93.4	61.7	62.3

		% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 10$ $(k = 4,\ \beta(p,q) \equiv \beta(2,2))$					
$\widehat{\mathrm{D}}$	n	S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	6.7	0.1	0	0.8	1.2
	30	0	0.5	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.1	4.3	0	0	0.5	0.5
	30	0	0.1	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0.2	5.6	0.3	0.1	1.6	2.2
	30	0	0.7	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	2.6	30.3	4.5	2.4	4.6	5.1
	30	0	19.2	0.2	0	0.2	0.2
	100	0	7	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	10	25.7	8.7	6.9	14.2	13.8
	30	1.7	17.9	1.7	0.9	5.8	5.2
	100	0	6.2	0	0	0.4	0.5
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	39.3	41.2	36.3	35.4	39.2	39.7
	30	65.7	36.1	37.2	40.2	38.6	38.7
	100	84.9	16.5	21.1	26.4	21.7	21.2
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	50.4	64.5	50.2	50.3	51.2	51.6
	30	60.2	51.1	47.4	60.2	41.4	40.3
	100	80.5	24.3	39.3	80.5	21.1	21.5
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	77.4	46.6	55	59.1	50.8	50.8
	30	88.9	33.8	38.9	42.5	35.4	35
	100	99.2	16.1	19.8	24.8	16.2	16.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	92.3	65.1	68.7	70.4	67.1	68.6
	30	97.2	96.5	96.5	96.7	96.5	96.5
	100	99.9	99.9	99.9	99.9	99.9	99.9
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	82.4	43.2	56.2	61	49.4	49.7
	30	98.2	33.7	45.1	48	39.1	38.8
	100	100	16.1	27.3	25.6	19.6	19.8

		% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 15$ $(k = 4,\ \beta(p,q) \equiv \beta(2,2))$					
$\widehat{\mathrm{D}}$	n	S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0.1	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	1.6	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.7	3	0.2	0.2	0.9	0.7
	30	0	0.8	0	0	0.1	0.2
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	13.5	12.2	11	10.9	11.5	11.8
	30	37.3	15.9	15.7	15.8	15.9	15.7
	100	68.6	11.2	11.2	11.2	11.2	11.2
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	26.5	31.5	26.2	26.5	26.5	26
	30	35.4	32.2	31.2	35.4	30.7	30.5
	100	21.7	16.3	16.1	21.7	16.1	16.1
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	42	28.9	31.6	34.3	29.1	29.4
	30	39.2	30.7	31	32.3	30.7	30.7
	100	24.9	16.1	16.1	16.5	16.1	16.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	40.2	21.1	21.5	21.4	21.2	21.5
	30	1.6	0.9	0.9	0.9	0.9	0.9
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	43.4	23.7	28	31.6	24.4	24.4
	30	52	23.1	24.7	27.9	23.1	23.1
	100	69.4	14	14	17.6	14	14

Table 3.13: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 4$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(4, 2)$

$\widehat{\mathrm{D}}$	n	$\% \left\vert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\vert > 1$ $(k=4,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	81	87.4	69.5	71.7	73.1	93.6
	30	81	97.7	77.5	50.6	53.4	99.9
	100	93.1	100	93.8	22.3	23.3	100
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	77.6	92.8	81.6	73.1	81.9	97.3
	30	70	98.5	95.1	75.5	83.9	100
	100	85.5	100	99.9	95.9	95.3	100
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	79.2	96.9	93.1	84.2	85.3	98.1
	30	76.5	99.6	99.7	95.7	94	100
	100	92.8	100	100	100	100	100
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	88.4	98.2	94.2	87.5	87.2	98.1
	30	98.3	99.8	99.6	97.2	95.7	100
	100	100	100	100	100	100	100
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	92.5	95.1	94.4	92.8	93.5	94.3
	30	97.1	99.4	99.2	97.4	98.9	99.1
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	96.5	95.9	95.9	96.1	96.4	96.3
	30	98.1	99.4	99.4	99.3	99.3	99.7
	100	100	100	100	100	100	100
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	96.2	97	96.9	96.2	96.6	96
	30	98.6	99.6	99.7	98.6	99.6	99.9
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	100	94.9	94.9	98.5	99	95.8
	30	100	98.6	98.7	99.8	100	97.4
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	99.7	99.9	100	99.9	98
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	100	96.4	97	99.3	99.8	93.7
	30	100	99.7	99.8	100	100	98
	100	100	100	100	100	100	100

$\widehat{\mathrm{D}}$	n	$\% \left\vert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\vert > 5$ $(k=4,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	18.9	27.8	8.6	4.9	8.7	31.9
	30	3.5	20.5	0.8	0	0.4	26.6
	100	0	7.9	0	0	0	13.3
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	12.8	43.6	18.7	10.3	16	45.4
	30	5	62.9	15.7	5.7	12.3	66.5
	100	0.6	81.6	8.9	0.4	4.9	88.7
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	16	54.2	38.5	20.3	27	63
	30	5.6	76.9	51.4	16.4	23	88.6
	100	0.2	96.8	70.3	5.8	13.4	99.9
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	40.9	77.8	54.9	39.7	41	72.1
	30	51.2	94	75.2	46.1	47.5	95.3
	100	66.7	100	96.6	59.2	57.6	100
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	62.4	69.1	61.5	61.9	64.4	70.7
	30	90.7	93.9	92.1	91.4	90	95
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	84.9	80.6	80.3	80.6	81.1	81.8
	30	95.4	96.7	96.7	95.4	95.4	96.4
	100	100	100	100	100	100	100
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	83.9	85.6	84.6	84	84.9	84.8
	30	95.2	97.2	96	95.2	95.7	97.3
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	99.5	80.9	81.9	88.8	93.1	79.8
	30	100	93.2	93.3	97.6	99.3	93.1
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	99.8	96.5	96.9	99.5	99.5	86
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	99.8	83.4	84.5	93.5	96.5	79
	30	100	95.7	96.3	99.6	99.9	93.1
	100	100	100	100	100	100	100

$\widehat{\mathrm{D}}$	n	$\% \left\vert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\vert > 10$ $(k=4,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0.6	1.2	0.6	0.6	0.6	0.8
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.3	1.2	0.3	0.3	0.3	1
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0.1	0.9	0	0	0.1	0.7
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	3.2	14.5	4.9	3.1	3.5	9.3
	30	0.1	9.9	1.1	0.1	0.3	3.9
	100	0	2.3	0	0	0	0.2
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	21.7	22.2	12.9	13	18.1	19.8
	30	51.9	56.5	37.4	37	47	54.1
	100	86.3	85	59.9	61.7	75.1	85
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	42	34.7	34.5	36.5	36.6	34.6
	30	73.7	71.4	71.4	72.4	72.4	71.5
	100	96.1	96.1	96.1	96.1	96.1	96.1
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	65.7	62	62.4	65.6	62	60.3
	30	94.1	92.9	93.7	94.1	93.3	92.3
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	85.5	63.4	63.5	66.2	69.6	62.8
	30	99	91.7	91.8	92.2	92.5	91.7
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	71	52.5	53.3	63.9	68.1	51.1
	30	44.9	44.9	44.9	44.9	44.9	44.9
	100	7.8	7.8	7.8	7.8	7.8	7.8
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	82.2	58.5	58.9	63.3	67.5	57
	30	94.7	87.4	87.5	88.8	91	87.2
	100	99.7	99.7	99.7	99.7	99.7	99.7

$\widehat{\mathrm{D}}$	n	$\% \left\vert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\vert > 15$ $(k=4,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	2.6	0.3	0	0	0.1	0.1
	30	3.2	2.4	0.2	0.2	1.5	1
	100	1.3	0.4	0	0	0.3	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	9.2	3.5	3.5	4	4.1	3.4
	30	7.7	5.6	5.6	5.7	5.7	5.6
	100	0.7	0.7	0.7	0.7	0.7	0.7
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	22.7	19.6	20.5	22.7	20.2	19.6
	30	50.6	49.4	49.6	50.6	49.6	49.4
	100	70.8	70.8	70.8	70.8	70.8	70.8
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	34.3	22.4	22.4	23	24.1	22.1
	30	44.7	41.1	41.1	41.1	41.2	41.1
	100	44.4	44.4	44.4	44.4	44.4	44.4
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	23.5	4.2	4.2	4.8	6.5	4.1
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	31.2	13.6	13.8	14.7	16.1	13.4
	30	20.8	14.3	14.3	14.3	14.6	14.3
	100	3.4	3.4	3.4	3.4	3.4	3.4

Table 3.14: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 4$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(6, 1)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k=4,\ \beta(p,q)\equiv\beta(6,1))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	95.4	72.9	67	78	94.4	76.7
	30	99.5	51.3	45.1	76.7	98.9	88.7
	100	100	35.3	18.1	90.3	100	98.4
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	92.8	72.3	63.6	80.6	92.5	71.2
	30	99.2	75.7	46.4	91.1	99.1	64.9
	100	100	91.7	35.8	99.4	100	71.3
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	81.1	69.4	76.2	68.7	77.3	84.7
	30	86.7	53.3	70.1	52.9	81.8	91.3
	100	95.2	25.9	83.4	25.9	90.9	99
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	76	75.8	75.1	75.5	78.7	88.7
	30	66.9	61.4	60.5	68.2	77.3	95
	100	48.3	39.7	40.4	57.2	80.1	99.7
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	79.4	70.6	71.6	71.9	78.1	80.7
	30	86.6	57.2	56.5	62.2	85	76.6
	100	98.3	40	30.6	60.9	97.2	88.5
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	100	98.8	95.5	98.7	99.8	95.1
	30	100	99.9	98.4	100	100	97.7
	100	100	100	99.9	100	100	99.9
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	99.5	99.2	99	99.5	99.7	94.4
	30	100	99.9	99.9	100	100	97.6
	100	100	100	100	100	100	99.9
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	100	99.9	99.9	100	100	98.9
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	100	100	100	100	100
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	100	100	99.9	100	100	99.3
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k=4,\ \beta(p,q)\equiv\beta(6,1))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0.3	0	0	0	0.2	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	1.1	0	0	0	0.7	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	1.5	0	0	0	0.8	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0.1	0	0	0.3	1.4	0.1
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	2.6	0.2	0.1	0.3	2.2	0.1
	30	0.1	0	0	0	0.1	0
	100	0.1	0	0	0	0.1	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	7.1	3.5	3.5	3.7	6.5	3.5
	30	2.4	2.2	2.2	2.3	2.4	2
	100	0.1	0.1	0.1	0.1	0.1	0
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	15.9	14.9	15.1	16	16.9	14.8
	30	20.8	20.5	20.8	20.8	20.8	20.5
	100	11.4	11.3	11.4	11.4	11.4	11.4
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	32.1	28.3	22.6	27.9	31.4	17.1
	30	22.2	21.6	19.1	22	22.2	17.8
	100	9.4	9.3	9.2	9.4	9.4	9.2
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	7.9	7.6	7.7	7.8	7.9	7.5
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	24.6	22.5	16.7	22.2	24	10.1
	30	7.7	7.4	6.6	7.7	7.7	3.3
	100	0.2	0.2	0.2	0.2	0.2	0

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k=4,\ \beta(p,q)\equiv\beta(6,1))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	44.2	4.6	2.9	8.6	39.1	14.3
	30	44.9	0.1	0	0.4	31.5	2.7
	100	37.3	0	0	0	17.3	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	46.9	5	1.7	11.1	44.7	5.3
	30	63.8	0.3	0	2.5	57.8	0.1
	100	78.4	0	0	0	69.8	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	23.4	3.6	3.9	3.9	20.4	9.5
	30	20.6	0	0	0	15.2	1
	100	6.4	0	0	0	2.4	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	12.2	9.4	8.4	14.1	22	21.3
	30	2.1	0.1	0.1	3.8	13.5	9.4
	100	0	0	0	0.2	2.2	0.7
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	25.2	5.2	4.6	8.4	23.2	9.6
	30	24.6	0.5	0.1	1.5	21.9	0.8
	100	12.1	0.1	0.1	0.2	9.3	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	66.2	57.7	53.8	57.7	65.7	53
	30	88.6	85.1	84.5	85.2	88.2	84.5
	100	99.1	99	99	99	99.1	99
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	86.2	82.2	81.2	87.1	89.9	77.1
	30	97.7	95.5	95.4	98	99	95.3
	100	100	99.9	99.9	100	100	99.9
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	92.5	91.7	90.2	92.2	92.5	85.5
	30	99.3	99.3	99.3	99.3	99.3	97.7
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	83.2	83.2	83.2	83.2	83.2	83.1
	30	82.7	82.7	82.7	82.7	82.7	82.7
	100	79.3	79.3	79.3	79.3	79.3	79.3
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	88.1	87.3	87	88	88.1	83.4
	30	96.5	96.5	96.5	96.5	96.5	96.3
	100	99.7	99.7	99.7	99.7	99.7	99.7

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k=4,\ \beta(p,q)\equiv\beta(6,1))$ S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.2	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0.2	0	0	0	0	0
	30	0.3	0	0	0.2	0.2	0
	100	0.1	0	0	0.1	0.1	0
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	1.1	0.9	0.9	1.1	0.9	0.9
	30	0.4	0.1	0.4	0.4	0.4	0.1
	100	0.1	0	0.1	0.1	0.1	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	16.2	4.9	1.2	4.2	15.6	1
	30	4.5	0.4	0.1	0.2	4.1	0.1
	100	0.2	0	0	0	0.1	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	0.6	0.1	0.1	0.3	0.4	0.1
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	16.6	6.6	1.2	5.5	15.7	0.7
	30	4.4	1.5	0	0.8	4.1	0
	100	0.2	0.1	0	0	0.2	0

Table 3.15: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 4$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(6, 10)$

$\widehat{\mathrm{D}}$	n	% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 1$ $(k=4,\ \beta(p,q) \equiv \beta(6,10))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	76.1	91.1	71.4	76.2	83.7	80
	30	63.6	99.6	53.7	59.5	90.4	71.3
	100	41.8	100	43.7	30.1	99	80.5
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	84.2	96.1	82	83.8	87.6	84.9
	30	75	100	84.5	74.6	95.2	94.2
	100	80	100	94	80.2	99.9	99.9
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	87	98.8	92.3	87.7	96.6	94.3
	30	95.6	100	99.2	97	100	100
	100	100	100	100	100	100	100
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	95.7	99.9	97.9	95.1	98.1	96.5
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	89.2	98.8	94.1	89	96.5	96.3
	30	97.7	100	99.5	97.5	99.9	99.9
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	99.7	99.7	99.9	99.9	99.5	99.6
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	99.9	100	99.6	99.9	99.7	99.7
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	100	99.8	100	100	99.8	100
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	100	100	100	100	100
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	100	99.9	100	100	99.8	100
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100

$\widehat{\mathrm{D}}$	n	% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 5$ $(k=4,\ \beta(p,q) \equiv \beta(6,10))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	23.1	36.6	17.3	20.4	23.7	19.6
	30	1.2	23.3	1.7	1.1	5.5	3.8
	100	0	8.2	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	23.9	49	25.7	24.6	37.1	29
	30	12.7	59	17.6	12.2	29.8	26
	100	1.7	70	4.2	1.6	15.7	13.4
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	34	64.9	41.3	36.7	55	48.3
	30	27.4	86.5	41.5	30.6	70.4	58.1
	100	16.6	99	43.5	22.9	87.4	74
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	67.6	91	73.2	66.9	75.1	70
	30	85.4	100	92.8	85	93.9	89.5
	100	97.9	100	99.9	97.8	100	99.3
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	46.2	69.6	49.8	45.9	57.6	55.8
	30	47.2	87.3	52.8	43.3	64	67.7
	100	51.4	98.8	62.9	44.2	78.7	85.4
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	89.7	89.3	89.4	89.6	89.6	89.2
	30	99.6	99.6	99.6	99.6	99.6	99.6
	100	100	100	100	100	100	100
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	97.4	97.6	97.2	97.4	97.6	97.2
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	99.4	98.6	99.3	99.1	98.8	99.3
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	98.2	98.2	98.2	98.2	98.1	98.2
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	99	98.3	99	98.9	98.3	99
	30	100	100	100	100	100	100
	100	100	100	100	100	100	100

$\widehat{\mathrm{D}}$	n	% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 10$ $(k=4,\ \beta(p,q) \equiv \beta(6,10))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	3.8	4.1	3.8	3.8	3.8	3.8
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	1.7	1.9	1.7	1.7	1.7	1.7
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0.2	0.6	0.2	0.2	0.2	0.2
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	10.1	21.5	11.7	10.1	12	11
	30	2.5	14.8	4	2.5	4.8	3.4
	100	0.1	6	0.1	0.1	0.3	0.1
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	5.6	7.1	4.8	4.1	4.9	5.1
	30	1.2	1.9	0.8	0.3	0.9	1
	100	0	0.2	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	14.2	14	14	14.1	14	14
	30	8.4	8.4	8.4	8.4	8.4	8.4
	100	0.7	0.7	0.7	0.7	0.7	0.7
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	54.9	55	54.9	54.9	55	54.9
	30	74.3	74.3	74.3	74.3	74.3	74.3
	100	93.3	93.3	93.3	93.3	93.3	93.3
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	61.2	59.6	60.7	60.5	60	60.5
	30	71	71	71	71	71	71
	100	87.1	87.1	87.1	87.1	87.1	87.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	21.8	21.7	21.8	21.8	21.7	21.8
	30	0.2	0.2	0.2	0.2	0.2	0.2
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	44.6	43.1	44.2	44	43.4	44
	30	33.5	33.5	33.5	33.5	33.5	33.5
	100	17.7	17.7	17.7	17.7	17.7	17.7

$\widehat{\mathrm{D}}$	n	% $\left\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\rvert > 15$ $(k=4,\ \beta(p,q) \equiv \beta(6,10))$					
		S = Likert	S = FLS^4_1	S = FLS^4_2	S = FLS^4_3	S = FLS^4_4	S = FLS^4_5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0	0
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0.6	0.6	0.6	0.6	0.6	0.6
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	5.7	5.7	5.7	5.7	5.7	5.7
	30	1.6	1.6	1.6	1.6	1.6	1.6
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	8.1	6.8	7.4	7.5	6.9	7
	30	0.1	0.1	0.1	0.1	0.1	0.1
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	0.2	0.1	0.1	0.1	0.1	0.1
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	4.2	2.5	3.2	3.2	2.7	2.8
	30	0	0	0	0	0	0
	100	0	0	0	0	0	0

Table 3.16: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 5$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(1, 1)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k = 5,\ \beta(p,q) \equiv \beta(1,1))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	71.6	88.3	65.9	58.6	63.1
	30	66.2	97.6	60.8	40.5	41.8
	100	75.2	100	62.5	25.1	15.1
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	65.8	83.7	70	64.9	66
	30	48.6	86.4	53.8	60.2	55.9
	100	25.1	96.5	37.5	72.7	42.1
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	68.3	78.5	67	70.4	71.3
	30	50.2	80.2	51.6	60.8	64.4
	100	20.7	87.6	20.9	75.1	71.1
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	84.4	79.9	71.7	83.4	74.7
	30	90.4	72.4	58.3	87.4	65
	100	99.4	70.6	30.8	98.8	61.5
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	83.9	87.9	83.4	83.2	84.1
	30	79.3	87.3	83.4	79.8	78
	100	71.2	90.5	76.8	70.6	70.2
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	89.7	89.7	88.9	87.3	88.6
	30	85.3	94.5	88.7	85.4	80.8
	100	73.3	98.9	84.9	73.4	73.4
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	91.4	91.1	89.8	91.7	88.4
	30	84.9	88.1	87.4	85	83.7
	100	80.2	80	82.6	80.2	82.4
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	89.3	91	88.5	88.9	86.7
	30	80	91.4	76.1	81.8	83.3
	100	66.3	99	69.2	66.8	82.4
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	95.6	91.8	91.8	89.2	85.2
	30	100	80.2	75.1	95.5	81.3
	100	100	97.7	38	100	17.4
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	87.7	92.9	87.5	85.7	85.4
	30	77.3	97.3	80.5	75.8	75.8
	100	60.5	100	78.4	60.6	66.3

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k = 5,\ \beta(p,q) \equiv \beta(1,1))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	4.4	16.5	2.3	1.1	1.9
	30	0	4	0	0	0
	100	0	0.1	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	2.4	14.7	3.4	1.9	3.1
	30	0.2	3.9	0.1	0.2	0.3
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	3.7	11.5	5.2	5.3	6.2
	30	0.2	2.8	0.1	0.3	0.8
	100	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	20	17.5	8.5	18.3	8.8
	30	11.7	5.3	0.6	10.7	1.5
	100	2.1	0.2	0	1.2	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	34.3	41.5	29.8	28.4	30.7
	30	24.3	39	24.7	21.2	20.3
	100	7.2	38.9	10	6.2	5.8
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	53	51.1	47.1	46.1	49.2
	30	33.1	66.4	38.9	29.7	29.3
	100	7.3	73.6	26.9	7.2	11.2
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	53.1	59.7	52.2	52.7	48.5
	30	30.3	41.9	38.3	30.1	31.8
	100	16.1	14.3	16	16.1	24.1
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	51.7	57.5	47.2	50.6	46.7
	30	20.9	44.1	18.7	25.1	28.2
	100	1.7	39.9	2.7	2.5	13.9
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	67.5	69.5	48.2	47.2	36.1
	30	99.8	10.3	21.2	19.1	44.6
	100	100	3	5.5	20.5	5.5
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	43.2	61.6	42.5	36.8	42.3
	30	15.8	69.2	21.1	12.9	14.7
	100	0.6	84.9	5.8	0.7	0.9

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k = 5,\ \beta(p,q) \equiv \beta(1,1))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0.1	0.4	0	0.1	0.2
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	3.6	6.8	2.6	2.3	1.7
	30	1.9	6.1	1.7	0.9	0.8
	100	0	0.7	0.1	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	16.7	22.4	15.1	13.1	13.1
	30	5.5	19.8	6.3	4.6	4.7
	100	0.5	6.5	0.4	0.4	0.4
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	19	27.1	19	18.9	17.9
	30	7.3	9.8	6.7	7.3	6.1
	100	0.6	0.7	0.3	0.6	0.5
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	21.9	26	17.3	17.4	19.9
	30	3.6	7.9	5.7	4.6	5.7
	100	0.1	0.2	0.6	0.2	0.2
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	22.5	40.3	15.3	14.2	11.2
	30	41.8	7.1	7.1	7.1	7.1
	100	88.9	2.6	2.6	2.6	2.6
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	14.7	25.6	15	11	13.6
	30	1.3	12.3	2.5	1.1	1.6
	100	0	1.5	0	0	0

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k = 5,\ \beta(p,q) \equiv \beta(1,1))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.1	0.2	0	0.1	0
	30	0.1	0.1	0	0.1	0.1
	100	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	2.4	4.7	2.8	1.9	2.2
	30	0.3	1.9	0.8	0.2	0.3
	100	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	5.8	7.1	4.9	5.8	5.1
	30	1.5	1.6	0.7	1.4	0.8
	100	0.2	0	0	0.2	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	8.1	8.7	6.7	5.2	6.7
	30	2.9	2.9	2.8	0.9	1.6
	100	0.1	0.1	0	0	0.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	3.4	8.3	3.1	2.9	2.8
	30	0.7	0.7	0.7	0.7	0.7
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	3.7	6.7	4.3	3	3
	30	0.4	1	0.6	0.4	0.4
	100	0	0	0	0	0

Table 3.17: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k=5$ different values is greater than $\varepsilon \in \{1,5,10,15\}$ for the 2ndSP and $\beta(p,q) \equiv \beta(0.75, 0.75)$

$\widehat{\mathrm{D}}$	n	% $\left\|\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\| > 1$ $(k=5,\ \beta(p,q) \equiv \beta(0.75,0.75))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	72.8	85.8	60.3	64.5	64.2
	30	72.4	97.5	44.3	49.1	45.3
	100	83.8	100	33.1	46.1	27.9
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	68.6	79.7	66.8	69.4	71.2
	30	49.3	89.1	51	61.6	59.4
	100	21	98.5	31.8	78.3	47.8
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	67.3	72.6	67	74.2	75.1
	30	50.8	79.5	48.6	64.6	69
	100	24.4	87.6	21.6	79.7	77.5
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	83.6	83	71.9	82.3	75.4
	30	87.1	83.4	57.4	82.1	62
	100	98.1	91	34.1	96	57.4
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	86.4	85.7	80.6	83.5	84.7
	30	86.3	84.4	80.2	86.8	84
	100	79.4	86.4	69.6	78.9	77.7
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	90	87.5	88.7	85.1	89.9
	30	85.7	87.4	85.3	86.4	84.1
	100	78.3	86.6	78.8	79.2	70
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	90.2	91.1	89.6	90.1	87.7
	30	91.7	84.3	83.6	91.1	83.9
	100	98.1	72.4	74.7	98.1	80
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	89.4	91.1	87.8	90.4	90.3
	30	90.9	84	82.5	93	86.9
	100	96.9	81.6	63.9	97.2	75.6
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	91.6	91.4	87.5	85.8	83.1
	30	100	65.4	79.6	82.6	88.7
	100	100	74.3	22.5	100	56.9
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	87	90.1	85.4	84	86.7
	30	81.2	92.7	75.7	81.6	81.6
	100	71.8	99.7	50.8	72.6	77.8

$\widehat{\mathrm{D}}$	n	% $\left\|\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\| > 5$ $(k=5,\ \beta(p,q) \equiv \beta(0.75,0.75))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	3.8	13.2	0.5	0.8	2.2
	30	0	1.4	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	2.2	14.1	3	2.2	4
	30	0	2.8	0	0	0.2
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	4	9.8	3.1	5.3	7.9
	30	0	1.5	0	0.1	0.5
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	18.9	20.7	7.8	16.8	11.2
	30	7.3	11.8	0.1	5.8	0.7
	100	0.6	1.5	0	0.4	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	36.7	31.5	27.3	29.8	32.2
	30	33.5	32.3	22.6	28.8	28.5
	100	13.7	24.6	7	12.4	11.6
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	52.3	46.2	45.2	43.8	50.9
	30	33.9	43.2	32.8	32.6	29.3
	100	12.1	23.8	13	13.4	6.5
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	54	57.5	49	52.6	50.1
	30	51.4	35.1	29.7	50.7	32.3
	100	68.5	7.6	8.9	68.1	16.1
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	56.1	54.7	47.6	56.6	51.9
	30	49.8	33.4	29.4	57.4	40.8
	100	38.3	6.7	4.2	39.5	18.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	50.8	60.2	37.3	34	31.5
	30	99.2	3.2	9.3	8	30.1
	100	100	0.2	4.2	2.8	9.4
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	41.1	54.1	41.4	36.9	43.8
	30	15.2	44.5	19.9	17.1	21.5
	100	1	31.7	2.9	2.5	2.3

$\widehat{\mathrm{D}}$	n	% $\left\|\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\| > 10$ $(k=5,\ \beta(p,q) \equiv \beta(0.75,0.75))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0.1	0	0.1	0.1
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0.4	0.5	0.1	0.4	0.1
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	4.3	4.6	1.8	2.6	2.3
	30	2.5	5.4	1	1	1
	100	0.2	1.1	0.1	0.1	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	14.1	15.3	11.9	10.8	14.4
	30	6.1	8.3	6.7	4.8	5.7
	100	0.7	0.6	1.4	0.6	1.3
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	21.6	24.2	16.8	21.8	18.8
	30	13.8	10.6	4.9	13.7	6.2
	100	6.2	1.4	1	6.2	1.9
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	24.7	24.2	18.8	21.6	22.3
	30	11.9	14.9	11.2	14.6	12.1
	100	1	3	2.2	2.8	1.7
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	15.2	25.3	9.6	8.7	8.8
	30	20.8	1.4	1.4	1.4	1.6
	100	43.4	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	13.1	17.2	11.5	9.9	12.7
	30	0.8	4.8	2.8	0.3	2.6
	100	0	0.1	0.6	0	0.6

$\widehat{\mathrm{D}}$	n	% $\left\|\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\right\| > 15$ $(k=5,\ \beta(p,q) \equiv \beta(0.75,0.75))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0.1
	30	0	0.2	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	2	2.8	1.7	1.4	1.9
	30	0.2	0.4	0.2	0	0.4
	100	0	0	0	0	0.1
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	4.7	6	4.1	4.5	4.1
	30	1.8	2.2	0.3	1.7	0.5
	100	0.2	0.1	0	0.2	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	7.3	7.7	6.9	4.8	6.7
	30	4.8	4.3	4	0.6	4.4
	100	0.9	0.9	1.4	0	1.1
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	1.9	4.5	1.2	1.3	1.1
	30	0.3	0.3	0.3	0.3	0.3
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	2.5	4.9	2.6	1.7	2.8
	30	0	0.1	0.1	0	0
	100	0	0	0	0	0

Table 3.18: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 5$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(2, 2)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k = 5,\ \beta(p,q) \equiv \beta(2,2))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	69.6	89.2	74.2	63.5	64.4
	30	49.9	98	73.9	40.5	45
	100	36.8	100	88.3	11.7	15.8
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	65.4	82	68.5	64.8	66.3
	30	55.7	77.6	56.3	61.7	60.1
	100	58.6	77.5	31.1	77.8	49.9
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	67	81.4	69.4	68.5	71
	30	57	72	53.6	63.7	65.6
	100	56.1	69.1	21.8	76.8	75.2
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	89.1	72.6	72.6	88	75.4
	30	97	64.6	65.3	97	73
	100	100	46.7	58.6	100	88.7
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	83.2	88.6	84.8	83.9	86.3
	30	86.7	96.8	93.3	87.9	88.5
	100	98.5	99.7	99.4	98.9	98.4
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	88.8	93.1	89.1	88.3	88.9
	30	95.8	97.7	95.6	95.7	94.1
	100	99.5	100	99.5	99.5	98.9
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	90.9	91.5	92	90.9	88.9
	30	92.8	98.2	98.1	92.8	93.9
	100	97.5	100	100	97.5	98.4
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	91.8	95.7	92.3	90.8	92
	30	94.5	99.8	95	94.2	80.5
	100	99.8	100	99.8	99.8	78
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	99.7	98.2	97.9	98.5	95.5
	30	100	99.8	95.8	100	88.8
	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	94.2	97.5	95.1	93.6	91.9
	30	99.4	99.6	99.4	99.2	90.7
	100	100	100	100	100	98.7

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k = 5,\ \beta(p,q) \equiv \beta(2,2))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	2.8	25.2	5.5	1.5	2.4
	30	0.1	12.1	0.5	0	0
	100	0	2.8	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	2.6	13.8	4.4	2.4	4
	30	0.2	3.2	0.3	0.2	0.3
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	4.1	13.4	5.8	4	5.1
	30	0.1	3.6	0.3	0.3	0.3
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	25.3	10	9.8	24.5	12.3
	30	24.6	1	1.1	23.3	3.3
	100	17.4	0	0	16.1	0.1
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	33.9	49	39.2	33.1	33.8
	30	42.2	79.5	63.3	41.8	44.2
	100	64.3	98.8	92.9	66.1	67.8
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	57.4	64.3	58.5	56.5	50.3
	30	81.9	91.3	81.9	81.5	57.7
	100	95.5	99.2	95.5	95.5	55.9
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	57.1	62.6	61.9	57.1	55.9
	30	49.9	87	85.9	49.9	55.1
	100	41	98.1	98.1	41	53.1
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	62.7	81.4	61.5	59.6	52.8
	30	51	96.1	53	50.4	18.8
	100	54.8	100	54.8	54.8	2.9
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	94.7	90.6	87.1	87.9	75.4
	30	100	74.8	77.8	91.1	78.4
	100	100	99.6	99.6	100	99.6
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	70.7	88.1	71.4	66.8	58.1
	30	83.2	99	83.7	82.3	37.7
	100	94.8	100	94.8	94.8	27

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k = 5,\ \beta(p,q) \equiv \beta(2,2))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0.1	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0.1	0.1	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0.2	0	0	0.2	0.2
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	4.4	13.3	5.8	3.8	3.9
	30	5	34.6	14.6	4.8	4.4
	100	1.6	60.7	17.9	1.5	1.9
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	19.6	34.4	19.3	19.5	14.1
	30	28.7	79	30	28.6	9
	100	16.1	95.5	16.1	16.1	0.7
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	24.2	33.7	32.6	24.2	23.2
	30	9.8	46	45.1	9.8	12
	100	0.6	40.6	40.6	0.6	0.6
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	28.1	53.2	30.6	27.2	25.9
	30	9.2	51.1	11.2	9	8.8
	100	0.6	54.8	0.6	0.6	0.6
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	55.2	75.9	49.1	49.1	28.3
	30	94.1	65.6	65.6	65.6	66
	100	99.7	99.3	99.3	99.3	99.3
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	32.5	62.1	34.7	31	25.3
	30	17.1	82.1	19.2	16.7	9.5
	100	3.7	94.8	3.7	3.7	0.6

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k = 5,\ \beta(p,q) \equiv \beta(2,2))$ S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	1	0	0	0.1
	30	0	1.2	0	0	0
	100	0	0.9	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	2.2	10.2	2.4	2.2	1.5
	30	0.5	22.6	0.9	0.5	0.3
	100	0	15.5	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	6.6	8.7	8.4	6.6	6.6
	30	4.1	5.7	5.7	4.1	4.1
	100	0.6	0.6	0.6	0.6	0.6
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	8.4	17.9	10.1	8.2	8.3
	30	4.9	6.2	4.9	4.9	4.9
	100	0.6	0.6	0.6	0.6	0.6
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	4.7	29.9	4.7	4.7	3.1
	30	0.1	0.1	0.1	0.1	0.1
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	5.3	21.4	7.8	5.1	5.8
	30	1	10.6	1.5	1	1.1
	100	0.1	3.2	0.1	0.1	0.1

Table 3.19: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 5$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p, q) \equiv \beta(4, 2)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k=5,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	70.7	83.5	81.9	60.7	70.9
	30	57.5	93.6	90.7	39.1	62.4
	100	53.5	99.5	99.1	12.6	66.2
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	67.9	81	79	64.4	74.2
	30	48.4	91.4	89.2	43.7	62.1
	100	33.1	99.3	98.9	19.5	57.5
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	72.2	74.9	73.9	74.7	84.7
	30	58	67	63.3	66.5	84.3
	100	42.8	50	46.8	61.4	95.6
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	85.6	78.8	78.8	85.9	85.4
	30	91.1	65.7	65.8	90.4	88.8
	100	99	53.8	53.8	98.9	98.2
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	85.1	84.9	84	84.7	86.9
	30	82.3	81.6	83.5	80	86
	100	86.5	80.3	85	83.3	87.3
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	89.7	93.6	91.6	90.4	92.4
	30	92.8	96	96.1	96.8	91.2
	100	95.3	99.6	99.2	100	87
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	93.1	91	91	93.1	93.6
	30	98.2	93.6	93.6	98.2	93.7
	100	99.9	96.6	96.6	99.9	96.8
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	98.5	94.8	96.3	94.6	92.4
	30	99.9	85.7	96.8	94.2	89.6
	100	100	81.7	99.8	97.6	89.5
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	97.5	99.8	99.2	91.7
	30	100	100	100	100	100
	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	99.2	94.7	98.3	96.5	92.9
	30	100	92.3	99.7	98.7	82.1
	100	100	98.1	100	100	72.5

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k=5,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	4.4	17.1	10.9	1.2	5
	30	0	6.9	1.1	0	0.1
	100	0	0.3	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	3.3	15	11.4	2.4	6
	30	0.2	6.7	2.9	0	0.6
	100	0	0.3	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	6.7	10.5	9.1	7.3	14.4
	30	0.2	2.4	1.4	0.5	3.7
	100	0	0	0	0	0.1
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	27	15.7	15.6	26.6	28.4
	30	15.2	1.8	1.8	14.5	18.4
	100	5.3	0	0	4.7	7.8
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	36.8	31.2	33.4	31.1	35.3
	30	32.1	20.7	23.6	26.2	28
	100	29.4	9.2	13.7	19.7	25.7
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	66.8	64.2	64.7	62.9	65.5
	30	79.2	68.1	74.2	70.5	65.3
	100	92.8	73.3	90.3	84.1	66.4
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	71.1	69.2	69	71.2	70.4
	30	82	79.5	79.5	82	71.2
	100	93.6	92.8	92.8	93.6	68.9
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	89.2	76.9	80.8	74	68.7
	30	96.9	62.1	72	64.3	60.9
	100	100	62.2	72.4	63.4	62
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	99.7	79.6	94.6	90.6	72.9
	30	100	99.1	100	100	98.2
	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	94.9	78.6	85.6	79.5	69.3
	30	99.8	62.9	87.1	74.4	59.9
	100	100	62.3	95.7	80.3	62.2

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k=5,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	0.1	0.1	0.1	0.1	0.3
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	4.7	2.3	2.4	2.1	2.9
	30	2.3	1	1.3	0.7	0.6
	100	0.3	0.1	0.1	0.1	0.1
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	23.9	20	19.1	18.8	18.9
	30	53.1	36.4	36.2	35.6	33.5
	100	84.5	56.3	56.4	56.3	56
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	43.6	49.1	48.8	43.6	41.3
	30	55.4	74.7	74.7	55.4	50.9
	100	64.5	92.6	92.6	64.5	61.7
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	57.8	58.9	56.6	49.4	44.7
	30	63.4	59.3	55.9	52.7	49.9
	100	71.9	62.2	61.9	61.7	61.7
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	62.3	42.8	42.7	41	36.8
	30	41.6	39.8	39.9	39.8	39.8
	100	5.7	5.7	5.7	5.7	5.7
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	57.4	56.1	53.5	45.8	39.4
	30	77	56.6	54.6	51.4	46.9
	100	95.3	62.1	62	61.8	61.6

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k=5,\ \beta(p,q)\equiv\beta(4,2))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}^\ell_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.1	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	2.2	2.2	1.4	1.4	1.2
	30	2.2	4.5	1.4	1.4	1
	100	0.2	0.4	0.1	0.1	0.1
$\widehat{\mathscr{D}^\ell_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}}^\ell}(\widetilde{\mathbf{x}}_n))$	10	8.4	11.4	11.3	8.4	8.3
	30	16.6	22.5	22.5	16.6	16.6
	100	34.9	37.7	37.7	34.9	34.9
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	11.2	16.5	14.6	10.5	10
	30	11	14	12.1	10.9	10.9
	100	16.9	16.9	16.9	16.9	16.9
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	4.3	5.5	1.5	1.5	1.2
	30	0.1	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	6.6	13.4	11.1	5.6	4.7
	30	1	6.4	3	0.8	0.8
	100	0	0.2	0	0	0

Table 3.20: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 5$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p,q) \equiv \beta(6,1)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k=5,\ \beta(p,q) \equiv \beta(6,1))$				
		S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	88.5	73.7	79.4	63.3	68.7
	30	96.7	80.1	91.1	48.2	75.6
	100	100	93.4	98.9	39.4	89.8
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	91.1	77.9	83.7	66.3	62.5
	30	99.1	90.5	95.9	69.5	57.3
	100	100	99.3	100	86.7	53.2
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	81.2	60.8	63.6	59.2	73.1
	30	92.6	42.4	64.2	44.3	73.2
	100	99.8	33.1	86	48.1	73.9
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	64.7	82.7	82.7	66.7	75.1
	30	59.8	90.8	90.7	65	66.5
	100	75.1	99.6	99.6	84.9	50.8
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	83.4	73.3	76.3	69.7	71
	30	90.3	80.7	85.7	72.5	55.8
	100	98.7	95.1	97.2	89.5	28.5
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	99.1	91	94.8	93.4	87.8
	30	100	93.9	97.9	96.5	87.4
	100	100	98.1	99.9	99.5	92.4
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	95.7	98.9	98.9	96.4	89.1
	30	99	99.9	99.9	99	86.6
	100	99.9	100	100	99.9	92.9
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	99.9	98.2	99.7	99.8	95.3
	30	100	100	100	100	98.9
	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	99.9	100	100	99.5
	30	100	100	100	100	100
	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	100	99.5	100	99.6	97.7
	30	100	100	100	100	99.9
	100	100	100	100	100	100

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k=5,\ \beta(p,q) \equiv \beta(6,1))$				
		S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	13.6	5.1	9.4	0.5	2.6
	30	2.3	0.4	0.7	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	16.8	3.6	6.7	0.7	1.2
	30	9.5	0.2	1.7	0	0
	100	1.5	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	15.4	0.8	2	0.7	3.4
	30	9.6	0	0.1	0	0
	100	5.9	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	4	21.1	20.8	5.4	6.1
	30	0.2	18.4	18.2	0.6	0.7
	100	0	19.2	18.4	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	22.2	11.5	15.7	9.2	8.6
	30	24.5	7.9	12.6	3.8	2.8
	100	34	3.4	11.4	0.5	0.6
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	57.4	39	42.1	40	37.3
	30	79.2	66.5	67	66.7	64.3
	100	96	89.7	89.7	89.7	89.3
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	62.1	76.3	76.3	64.4	56.2
	30	79	91.6	91.6	79.3	77.7
	100	90.5	98.2	98.2	90.7	90.5
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	89.8	78.4	87.6	84.7	66.2
	30	99.6	95.5	99.3	99	77.2
	100	100	99.9	100	100	79.8
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	81.3	79.6	81.1	80.9	76.7
	30	74.8	74.8	74.8	74.8	74.8
	100	75.8	75.8	75.8	75.8	75.8
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	85.6	78.2	83.9	82.5	66.1
	30	93.9	93.1	93.9	93.9	86.3
	100	99.6	99.6	99.6	99.6	99.1

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k=5,\ \beta(p,q) \equiv \beta(6,1))$				
		S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0.3	0.3	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	1	0	0.3	0	0.2
	30	0.5	0.1	0.2	0	0.1
	100	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	5.4	2.5	3.4	2.9	1.9
	30	11.6	4.8	10.4	8.2	0.7
	100	21	12.8	21	19.9	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	12.6	13.9	13.9	12.6	9.9
	30	20.2	20.2	20.2	20.2	10.5
	100	23.6	23.6	23.6	23.6	4.2
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	44.4	10.9	27.3	20.5	9.8
	30	37.1	5.1	23	12.8	5.1
	100	31.9	1.4	24.6	7.2	1.4
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	12	4.3	8.8	7.1	3.6
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	40.6	7.7	28.4	21.1	5.2
	30	33.5	2.3	30.2	23.3	1.2
	100	30.5	0.3	30.5	28.9	0

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k=5,\ \beta(p,q) \equiv \beta(6,1))$				
		S = Likert	S = FLS_1^5	S = FLS_2^5	S = FLS_3^5	S = FLS_4^5
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	1.5	0	0.1	0	0
	30	9.7	0	0.1	0	0
	100	21	0	0	0	0
$\widehat{\mathscr{D}_1^\ell\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^\ell}}(\widetilde{\mathbf{x}}_n))$	10	1	3.1	3.1	1	0.3
	30	4.8	10.9	10.7	4.9	0
	100	14.8	21	21	14.8	0
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	17.9	0.3	0.5	0.3	0.3
	30	17.9	0	0	0	0
	100	23.2	0	0	0	0
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	5.2	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	23.4	0.1	0.4	0.1	0.1
	30	29	0	0	0	0
	100	30.5	0	0	0	0

Table 3.21: % of simulated samples of size n for which the Euclidean distance between the sample scale estimate $\widehat{\mathrm{D}}$ associated with the FRS and the one associated with either the Likert or the FLS with $k = 5$ different values is greater than $\varepsilon \in \{1, 5, 10, 15\}$ for the 2ndSP and $\beta(p,q) \equiv \beta(6,10)$

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 1$ $(k = 5,\ \beta(p,q) \equiv \beta(6,10))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	69.2	92.4	73.2	68	84.2
	30	69.2	99.4	76.1	67.6	97.1
	100	79.6	100	89.1	77.2	100
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	77.4	93.6	79	77.3	90.2
	30	89.7	99.8	94.1	89.4	98.9
	100	99.7	100	99.9	99.7	100
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	71.9	83.3	71.1	71.1	78.4
	30	72.5	94.5	73.4	72	89.2
	100	92	99.8	93	91.6	99.2
$\widehat{\mathscr{D}^{\ell}_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	76.8	74.3	73.8	76.8	76.6
	30	59.6	67.8	67.1	59.8	76.2
	100	33.8	82.2	81.6	33.7	92.4
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	77.4	82.2	78.3	77.4	80.9
	30	78.1	92.4	80.6	77.9	88
	100	92.6	99.2	93.9	92.5	98.4
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	96.8	99.2	96.6	96.5	98.1
	30	99.5	99.3	98.4	99.1	99
	100	100	97.8	97.4	98.9	100
$\widehat{\mathscr{D}^{\ell}_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	91.8	96.8	97.1	91.8	97.5
	30	90.8	98.9	98.5	90.8	97.5
	100	93.4	99.9	98.4	93.4	99.2
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	100	100	99.7	100	100
	30	100	100	100	100	100
	100	100	100	100	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	100	100	99.9	100	100
	30	100	100	100	100	100
	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	100	100	99.8	100	100
	30	100	100	100	100	100
	100	100	100	100	100	100

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 5$ $(k = 5,\ \beta(p,q) \equiv \beta(6,10))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	3.1	31.2	5.9	2.5	14.2
	30	0	22.4	0.5	0	3
	100	0	8.5	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	4.6	35.6	7.3	4.2	20.5
	30	0.5	47.6	1.7	0.5	17.8
	100	0	60.6	0	0	8.6
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	6.4	24.1	6.7	6.2	14.1
	30	1.2	27.6	1.3	1	10.7
	100	0	43.2	0.1	0	9.1
$\widehat{\mathscr{D}^{\ell}_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	10.4	10.6	10.4	10.4	13.2
	30	0.9	1.3	1.3	0.8	4.7
	100	0	0.1	0	0	1
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	13.2	29.2	14.8	13	21.2
	30	6.4	36	9.2	6.4	23.1
	100	2.8	52.1	5.3	2.9	30.9
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	73.5	83.8	73.2	73.1	78.6
	30	84.6	95.5	83.7	83.7	89
	100	87.3	96.2	86.3	86.4	90.4
$\widehat{\mathscr{D}^{\ell}_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	72.6	79	78.7	72.6	80.7
	30	79	82.9	82.5	79	84.9
	100	83.9	84.8	84.5	83.9	87.9
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	99.4	99.3	98.1	98.9	99.6
	30	100	99.7	98.3	100	100
	100	100	100	95.9	100	100
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	99	98.9	98.3	98.9	99
	30	99.9	99.9	99.9	99.9	99.9
	100	100	100	100	100	100
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	99.3	99.2	98.3	99.1	99.3
	30	100	100	98.9	100	100
	100	100	100	99.3	100	100

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 10$ $(k = 5,\ \beta(p,q) \equiv \beta(6,10))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0.2	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}^{\ell}_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0.4	1.7	0.4	0.4	0.5
	30	0	0.5	0	0	0.1
	100	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	8.6	9.7	8.6	8.6	8.7
	30	8.2	8	7.9	8.1	8.1
	100	8	7	6.6	7.6	8
$\widehat{\mathscr{D}^{\ell}_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	33	33.1	33.1	33	33.1
	30	55.7	55.9	55.7	55.7	55.8
	100	78.4	78.4	77.2	78.4	78.4
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	61.1	63.8	59.9	60	61.5
	30	71.4	69.5	69.1	69.3	68.1
	100	85.2	80.3	80.3	80.4	84.8
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	20.8	20	20	20	20.5
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	51.2	51.5	49.8	49.9	50.2
	30	42.1	40.2	39.8	40.4	41.3
	100	32.9	28.4	28	28.8	32.9

$\widehat{\mathrm{D}}$	n	% $\lvert\widehat{\mathrm{D}}(\mathrm{FRS}) - \widehat{\mathrm{D}}(\mathrm{S})\rvert > 15$ $(k = 5,\ \beta(p,q) \equiv \beta(6,10))$ S = Likert	S = FLS^5_1	S = FLS^5_2	S = FLS^5_3	S = FLS^5_4
$\widehat{\rho_2\text{-SD}}(\widetilde{\mathbf{x}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\mathscr{D}^{\ell}_1\text{-ADD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	0	0	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_2\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widetilde{\widetilde{\mathbf{x}}}_n)$	10	0	0.3	0	0	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{Me}}}(\widetilde{\mathbf{x}}_n))$	10	0.3	0.6	0.3	0.3	0.1
	30	1.7	2.1	1.5	1.5	0.4
	100	7.1	6.1	5.8	5.8	0.9
$\widehat{\mathscr{D}^{\ell}_1\text{-MDD}}(\widetilde{\mathbf{x}}_n, \widehat{\widetilde{\mathrm{M}^{\ell}}}(\widetilde{\mathbf{x}}_n))$	10	2.5	2.8	2.8	2.5	2.5
	30	1.4	3.3	3	1.4	1.7
	100	0.8	7.5	6.1	0.8	2.4
$\widehat{\rho_1\text{-S}}(\widetilde{\mathbf{x}}_n)$	10	9.4	28.4	9.3	9.3	18.4
	30	4.5	21.1	3.9	3.9	12.2
	100	3.1	18.7	2.3	2.3	8.5
$\widehat{\rho_1\text{-Q}}(\widetilde{\mathbf{x}}_n)$	10	0.6	0.9	0.4	0.4	0
	30	0	0	0	0	0
	100	0	0	0	0	0
$\widehat{\rho_1\text{-T}}(\widetilde{\mathbf{x}}_n)$	10	11.2	28.2	10.8	10.8	19.9
	30	12.5	21.1	11.3	11.3	13
	100	18.9	18.7	14.7	14.7	8.8

On the basis of Tables 3.10 to 3.21 we cannot get very general conclusions, but we can definitely assert that scale measures mostly vary from the FRS-based data to the encoded Likert ones.

Furthermore, one can state some approximate behaviour patterns. In this way,

- for almost all situations, robust scale estimates provide us with much higher percentages than non-robust ones; more concretely, robust estimates are almost generally more sensitive to the change in the rating scale type, as it has also been shown for the case studies-based descriptive analyses;
- distances are uniformly lower for $k = 5$ than for $k = 4$ when the midpoint of the 1-level is beta distributed with $(p, q) \in \{(1, 1), (0.75, 0.75), (2, 2), (4, 2)\}$; when $(p, q) = (6, 10)$ such a conclusion is not uniform but close to; when $(p, q) = (6, 1)$ such a conclusion is appropriate for robust estimates and $\varepsilon \in \{1, 5\}$, but there is no clear conclusion for non-robust estimates or greater values of ε.

3.5 Concluding remarks of this chapter

This chapter has been devoted to compare, from a statistical point of view, three types of scale used to rate intrinsically imprecise magnitudes: the Likert-type (or its numerical encoding), the fuzzy linguistic scale (which is generally obtained as a posterior fuzzy number-valued encoding of a Likert scale) and the fuzzy rating scale. The comparison is rather specifically conceived to be carried out in the framework of questionnaires.

The first comparative tool among the three scale has been the diversity, quantified through the Gini-Simpson index. Assuming that two respondents assessing the same fuzzy rating response also assess the same Likert one, it has been proved that the diversity is much better captured through fuzzy rating scale-based questionnaires.

Secondly, an inferential approach has been presented on the basis of datasets from the questionnaires in two of the case studies in Chapter 1. Both questionnaires involve items which have to be answered according to a fuzzy rating scale as well as to a Likert-type scale. Likert data have been numerically encoded and also associated with the terms from the most usual balanced fuzzy linguistic scales of 4 and 5 values. With the available data in the three scales, the bootstrapped test about the equality of population variances of two or more random fuzzy numbers has been applied, separately, for each rating scale. The outputs in this study show that the statistical conclusions for the FRS differ from the ones obtained for the encoded Likert-type data.

Finally, a descriptive analysis has been addressed through two approaches to compare the scale measures in Chapter 2. The first one, making use of the two previous datasets, has allowed us to conclude that the values of these measures often differ for different rating scales.

The second descriptive approach has been carried out through simulations. Fuzzy rating scale responses have been generated according to the simulation procedure inspired by questionnaires. The simulated fuzzy data have been associated with numerically encoded labels in a Likert scale. This association has been performed by means of the minimum distance Likertization criterion, which has been validated with the three cases studies introduced in Chapter 1. The Likertized responses have also been encoded by means of the terms from different fuzzy linguistic scales with 4 and 5 values. Considering different sample sizes, the percentages (over 1000 samples) of Euclidean distances between the scale estimates for the FRS data and for the two encoded Likertized responses have been calculated. On the basis of the outputs of this simulation study, we could assert that the values of the scale measures mostly vary from the FRS-based data to the encoded Likert ones. Moreover, the robust scale measures are almost generally more sensitive to the change of rating scale than the non-robust ones (this fact could also be seen for the case studies-based descriptive analyses).

In summary, the main conclusions/contributions of this Chapter are the following:

- The diversity is higher, under quite general conditions, for the fuzzy rating scale than for the Likert scale or its numerical/fuzzy linguistic encoding.
- An inferential study to test the equality of variances has shown different conclusions depending on the considered scale.
- Two descriptive comparative studies, based on real-life examples and simulations, and involving the scale measures introduced in Chapter 2, have allowed us to conclude that the values of these measures differ to a lesser or greater extent for responses based on the FRS and encoded Likert responses. Moreover, robust scale estimates have shown a higher sensitivity w.r.t. the use of different rating scales than the non-robust ones.

The ideas and results in this chapter have been gathered at this stage in three published papers (De la Rosa de Sáa *et al.* [27], Gil *et al.* [44] and Lubiano *et al.* [75]) and one communication to an international conference (Lubiano *et al.* [79]).

Final conclusions and open problems

In describing some of the most immediate open problems from this work, one should distinguish among the new challenges which could be addressed and those which directly derives from the developments in the work.

Among the new challenges we can mention the following:

- To consider other tools that allow us to measure objectively the robust behaviour of an estimator.
- To analyze the influence of the property of symmetry of a random fuzzy number (see Sinova *et al.* [106]) in the scale measures introduced in Chapter 2, as suggested by Rousseeuw and Croux [100].

Among the future directions in connection with the studies already collected in this work, we can mention the following:

- The study of the computation time and storage space of the estimators of scale.
- Regarding the measures of dispersion that involve measures of location, the Aumann-type mean and the 1-norm and φ-wabl/ldev/rdev medians were considered. It would also be interesting to consider the M-estimates and the trimmed means.
- To analyze in more detail the M-estimation of scale in the fuzzy setting by using other loss functions (like Hampel's one) and different choices for the involved parameters.
- To develop a hypothesis test about the equality of variances for different rating scales (that is, a test for the effect of the rating scale on the variance or standard deviation). Because of the same individual would give his/her response according to each scale, it would be a test for linked samples. Then a homoscedasticity test for dependent samples should first be developed.

- To develop hypothesis testing procedures about scale measures different than the variance (or the standard deviation), especially the robust ones analyzed in this work.

Conclusiones finales y problemas abiertos

Al describir los problemas abiertos más inmediatos en relación con esta memoria, conviene distinguir entre los nuevos retos que interesaría abordar y los que se derivan directamente de los desarrollos recogidos en el trabajo.

Entre los primeros cabe destacar los siguientes:

- La consideración de otras herramientas que nos permitan medir de forma objetiva el comportamiento robusto de un estimador.
- El análisis de la influencia de la propiedad de simetría de un número difuso aleatorio (ver Sinova *et al.* [106]) en las medidas de escala presentadas en el Capítulo 2, sugerido por Rousseeuw y Croux [100].

Entre las futuras líneas de investigación ligadas a los estudios llevados a cabo en este trabajo, deben mencionarse las siguientes:

- El estudio del tiempo computacional y el espacio de almacenamiento de los distintos estimadores de escala.
- En relación con las medidas de dispersión que involucran medidas de localización, además de la media tipo Aumann y las medianas 1-norma y φ-wabl/ldev/rdev, sería interesante considerar los M-estimadores y las medias recortadas.
- Analizar con más detalle la M-estimación de escala en el contexto difuso, empleando otras funciones de pérdida (como la de Hampel) y haciendo diferentes elecciones de sus parámetros.
- El desarrollo de un contraste de hipótesis sobre la igualdad de varianzas para distintas escalas de valoración (es decir, un test para el efecto de la escala de valoración en la varianza o desviación típica). Puesto que un mismo individuo daría su respuesta en cada una de las escalas, se trataría de un contraste para muestras relacionadas. Por lo tanto, se debe desarrollar primero un test de homocedasticidad para muestras dependientes.

- El desarrollo de contrastes de hipótesis sobre la igualdad de medidas de escala poblacionales diferentes a la varianza (o desviación típica), especialmente las medidas robustas analizadas en esta memoria.

Schlussfolgerungen und offene Probleme

Bei der Beschreibung einiger der unmittelbarsten offenen Probleme aus dieser Arbeit, sollte man unter den neuen Herausforderungen unterscheiden, welche angesprochen werden könnten, und jenen, die sich direkt aus den Entwicklungen der Arbeit ergeben.

Unter den neuen Herausforderungen können wir folgende erwähnen:

- Andere Werkzeuge zu betrachten, die es uns erlauben, objektiv das robuste Verhalten eines Schätzers zu messen.
- Den Einfluss der Symmetrieeigenschaft einer zufälligen Fuzzy-Zahl (Sinova *et al.* [106]) in den Skalierungsmaßen von Kapitel 2, die von Rousseeuw und Croux [100] vorgeschlagen wurden, zu analysieren.

Zukünftig können wir, im Zusammenhang mit den bereits in dieser Arbeit gesammelten Studien, folgendes erwähnen:

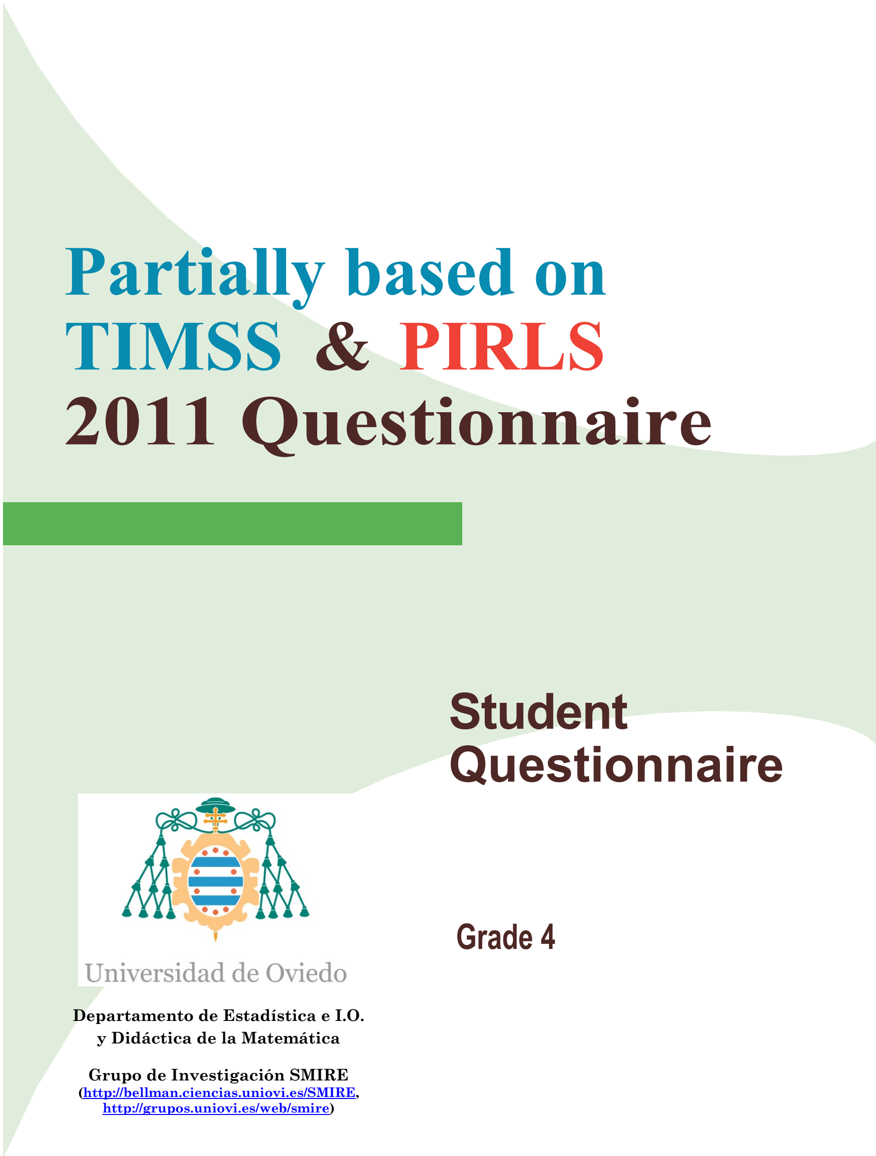
- Untersuchung der Berechnungszeit und des Speicherplatzes der Streuungsschätzer;
- in Bezug auf die Streuungsmaße, die eine Lokationsmaße involvieren, wurden die Aumann-Typ-Mittel, die 1-Norm und die φ-wabl/ldev/rdev Mediane berücksichtigt. Es wäre auch interessant, M-Schätzer und gestutzte Mittel zu betrachten;
- M-Schätzer für Streuung im Fuzzy-Szenario detaillierter zu analysieren, indem andere Verlustfunktionen (wie Hampel's) und verschiedene Optionen für die involvierten Parameter verwendet werden;
- einen Hypothesentest über die Gleichheit der Varianz für verschiedene Bewertungsskalen (ein Test über die Auswirkung der Bewertungsskala auf die Varianz oder Standardabweichung) zu entwickeln. Nachdem die gleiche Person seine/ihre Antwort entsprechend jeder der beiden Skalen geben würde, wäre es

ein Test für verbundene Stichproben. Dann sollte zunächst ein Homoskedastizitäts-Test für abhängige Proben entwickelt werden;

- die Entwicklung von Hypothesentests für Skalierungsmaße, die unterschiedlich von Varianz (oder Standardabweichung) sind, besonders robuste, wie sie in dieser Arbeit analysiert wurden.

Appendix A

Form and datasets for adapted TIMSS/PIRLS student Questionnaire



Contents

Directions

In this booklet you will find questions about you and what you think. For each question, you should either choosing or drawing the answer you think is best.

Let us take a few minutes to practice the kinds of questions you will answer in this booklet.

Example 1 is one kind of question you will find in this booklet.

Example 1

Do you go to school?

*Fill **one c**ircle only*

Yes ----- ○
No ----- ○

Example 2 is another kind of question you will find in this booklet.

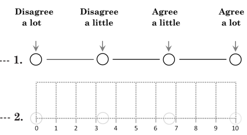
Example 2

What do you think about your school?

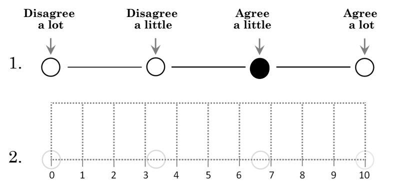
Tell how much you agree with these statement

I Like being in school ------ 1.

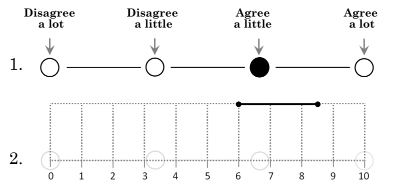
------ 2.



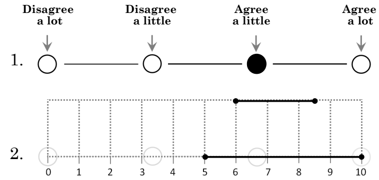
- Read each question carefully, and pick the answer in **1** you think is best to describe your opinion/rating by filling the circle next (if there is only answer **1** to answer the item) or under your answer.



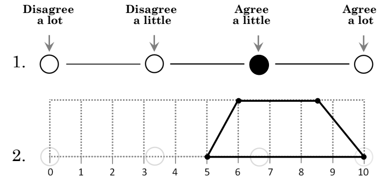
- Read each question carefully, and set the answer in **2** you think is best to describe your opinion/rating by drawing an acute trapezium (or, in particular, a triangle, a parallelogram or even a vertical line) with height equal to 1 and so that
  - the upper base is associated with the values 0 to 10 that best describe numerically your opinion/rating



- the lower base is associated with the values 0 to 10 describing numerically to a (more or less great, but some) certain extent your opinion/rating



- legs of the answer trapezium are obtained by linking the two bases (i.e., by considering a linear interpolation).



In responding in accordance with type **1**, one often considers too demanding having to choose a unique answer, but one would prefer to have the opportunity to choose something ‘in between’. At the same time, it is usually difficult to choose a unique number from 0 to 10 to respond.

Type 2 offers a more flexible and imprecise way to provide with the answer, so that in addition to indicating the “fully preferred” numbers to rate respondent’s valuation, the “somewhat preferred” can be also indicated.

About you

Are you a girl or a boy?

*Fill **one** circle only*

Girl ----- ○ Boy ----- ○

When were you born?

Fill the circles next to the month and year you were born

a) Month	**b) Year**
January -- ○	2000 -- ○
February -- ○	2001 -- ○
March -- ○	2002 -- ○
April -- ○	2003 -- ○
May -- ○	2004 -- ○
June -- ○	2005 -- ○
July -- ○	2006 -- ○
August -- ○	
September -- ○	
October -- ○	
November -- ○	
December -- ○	

Do you have any of these things at your home?

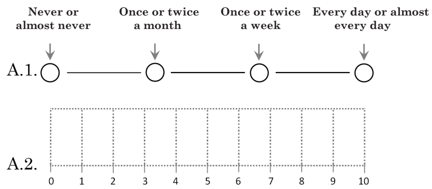
*Fill **one** circle for each line*

	Yes	No
a) Computer	○	○
b) Study desk/table for your use	○	○
c) Books of your very (do not count your school books)	○	○
d) Your own room	○	○
e) Internet connection	○	○

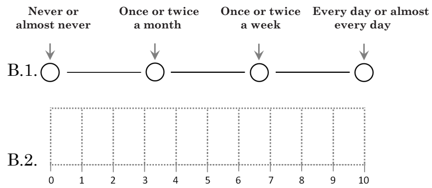
How often do you use a computer in each of these places?

*Tell **how much you agree with** these statements*

A. At home



B. At school



Reading in school

How much do you agree with these statements about reading?

*Tell **how much you agree with** these statements*

R.1. I like to read things that make me think

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

R.2. I learn a lot from reading

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

R.3. Reading is harder for me than any other subject

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

R4. What grade you got in the last reading test? ☐

Mathematics in school

How much do you agree with these statements about math?

*Tell **how much you agree with** these statements*

M.1. I like mathematics

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

M.2. My math teacher is easy to understand

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

M.3. Mathematics is harder for me than any other subject

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

M4. What grade you got in the last math test? ☐

Science in school

How much do you agree with these statements about science?

Tell how much you agree with these statements

S.1. My teacher taught me to discover science in daily life

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

S.2. I read about science in my spare time

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

S.3. Science is harder for me than any other subject

	Disagree a lot	Disagree a little	Agree a little	Agree a lot
1.	○	○	○	○

2. 0 1 2 3 4 5 6 7 8 9 10

S4. What grade you got in the last science test? ☐

Thank you!

Thank you for fillingo ut the questionnaire!

SMIRE Research Group
Statistical Methods with Imprecise Random Elements

Universidad de Oviedo
Departamento de Estadística e I.O.
y Didáctica de la Matemática
Grupo de Investigación SMIRE

Table A.1: Dataset for the "About you" items (Page 235) from the adapted TIMSS/PIRLS student Questionnaire (p&p = paper-and-pencil format, comp = computerized format, D+ = DISAGREE A LOT, D− = DISAGREE A LITTLE, A− = AGREE A LITTLE, A+ = AGREE A LOT, and $(a_i, b_i, c_i, d_i) \equiv \text{Tra}(a_i, b_i, c_i, d_i)$)

ID	Sex	Month born	Comp	Own desk	Own books	Own room	Int	Format	$A.1$ Likert	$A.2$ (a_i, b_i, c_i, d_i)	$B.1$ Likert	$B.2$ (a_i, b_i, c_i, d_i)
ID1	Girl	July	YES	YES	YES	NO	YES	p&p	D-	$(2.2, 3, 4, 5)$	D-	$(1.4, 2, 3, 4.5)$
ID2	Girl	January	NO	YES	YES	YES	NO	comp	A-	$(5.625, 5.625, 7.5, 8.65)$	A-	$(5.4, 5.7, 7.65, 8.275)$
ID3	Boy	June	YES	YES	YES	YES	YES	comp	A+	$(5.95, 5.95, 9.95, 9.975)$	A-	$(5.625, 6.25, 6.25, 7.95)$
ID4	Boy	December	YES	YES	YES	YES	YES	comp	D+	$(0.175, 0.2, 2.025, 4.125)$	A-	$(6.55, 6.55, 8.45, 9.05)$
ID5	Girl	October	YES	YES	YES	YES	YES	comp	D+	$(0.15, 1.975, 5.95, 6.05)$	A-	$(2.5, 4.675, 6.25, 6.9)$
ID6	Boy	March	YES	YES	YES	YES	YES	comp	A-	$(5.975, 6.5, 8.475, 8.975)$	A+	$(6.975, 7.575, 8.925, 9.475)$
ID7	Girl	October	YES	YES	YES	YES	YES	comp	A-	$(4.075, 4.8, 7.425, 8)$	A+	$(5.125, 5.15, 8.55, 10)$
ID8	Boy	October	YES	NO	YES	NO	YES	p&p	D-	$(2, 2.6, 3, 3)$	D-	$(3, 3, 3, 3)$
ID9	Girl	June	YES	NO	YES	YES	NO	p&p	A+	$(6.7, 8, 8, 9)$	D-	$(4.9, 5.9, 6.1, 6.9)$
ID10	Girl	March	YES	YES	YES	NO	YES	comp	A+	$(7, 7.9, 9.475, 10)$	A-	$(3.925, 4.8, 6.8, 6.8)$
ID11	Boy	December	YES	YES	YES	YES	YES	comp	D-	$(2.925, 3.75, 4, 5)$	A-	$(6.075, 6.575, 7.5, 7.5)$
ID12	Girl	October	YES	YES	YES	YES	YES	p&p	A+	$(7, 7.9, 8.5, 9.3)$	D-	$(1.5, 2, 2.5, 3.2)$
ID13	Boy	February	YES	YES	YES	YES	YES	p&p	D-	$(3, 3.45, 4.15, 5)$	D-	$(3, 3, 4, 5)$
ID14	Boy	June	YES	YES	YES	YES	YES	comp	A+	$(7.325, 7.35, 9, 9.975)$	A-	$(4.5, 5.5, 6.65, 8.525)$
ID15	Girl	July	YES	NO	YES	YES	YES	comp	A+	$(7.975, 8, 9.95, 9.95)$	A-	$(4.8, 4.925, 7.625, 7.975)$
ID16	Boy	May	YES	YES	YES		YES	comp	A-	$(5.1, 5.1, 7.975, 7.975)$	A-	$(3.95, 3.95, 7.9, 7.9)$
ID17	Girl	July	YES	YES	YES	YES	YES	comp	A+	$(5.975, 6.95, 9, 10)$	A-	$(4.375, 4.875, 6.25, 7.95)$
ID18	Boy	May	YES	YES	YES		YES	comp	A-	$(5.275, 5.65, 7.575, 7.875)$	A-	$(5.6, 6.175, 8.1, 8.725)$
ID19	Girl	May	YES	YES	YES	NO	NO	comp	A-	$(4.625, 5.525, 7.5, 8.25)$	A-	$(5.55, 5.55, 7.825, 8.55)$
ID20	Girl	June	YES	YES	YES	YES	YES	comp	D+	$(0.175, 0.175, 2.575, 2.625)$	A-	$(3.675, 3.75, 5.775, 6.925)$
ID21	Girl	April	YES	YES	YES	NO	YES	p&p	D-	$(2.35, 3, 3.55, 3.75)$	D-	$(2.9, 3.25, 3.55, 3.9)$
ID22	Boy	April	YES	YES	YES	NO	NO	p&p	D+	$(0, 0, 0, 1)$	D-	$(2, 2.8, 3.55, 4.1)$
ID23	Boy	September	YES	YES	YES	YES	YES	p&p	D-	$(2.8, 3, 3.5, 4)$	D-	$(3, 3.15, 3.55, 4)$
ID24	Boy	March	YES	YES	YES	YES	YES	comp	A-	$(4.075, 4.65, 6.8, 7.5)$	A+	$(6.775, 7.525, 9.4, 10)$
ID25	Boy	January	YES	YES	YES	YES	YES	p&p	A-	$(5, 6, 7, 8)$	D-	$(7, 7.45, 7.45, 8)$
ID26	Girl	August	YES	YES	YES	NO	YES	p&p	D-	$(3, 3, 3.5, 4.2)$	D-	$(2, 2, 2.6, 3)$
ID27	Boy	October	YES	YES	YES	YES	YES	p&p	A-	$(6, 6.2, 6.6, 7.1)$	D-	$(8, 8.3, 8.55, 9)$
ID28	Boy	April	YES	YES	YES	YES	YES	comp	A-	$(4.675, 5.525, 7.5, 7.5)$	A-	$(4.2, 5.275, 7.475, 7.5)$
ID29	Girl	May	YES	YES	YES	YES	YES	comp	D+	$(1.75, 3.075, 3.65, 4.55)$	A+	$(5.9, 6.2, 8.45, 8.975)$
ID30	Boy	May	YES	YES	YES	NO	YES	comp	D+	$(0.1, 0.1, 1.95, 2)$	A-	$(5, 5, 7.5, 7.5)$
ID31	Boy	March	YES	YES	YES	YES	YES	comp	A+	$(6.85, 6.85, 8.975, 10)$		$(2.5, 3.75, 6.25, 7.5)$
ID32	Girl	October	YES	YES	YES	NO	YES	p&p	A-	$(4, 4.5, 5.3, 5.5)$	D-	$(3.3, 3.6, 4.45, 4.5)$
ID33	Girl	May	YES	YES	YES	YES	YES	p&p	A+	$(7.25, 7.55, 8.3, 8.45)$	D-	$(2.35, 2.7, 3.25, 3.65)$
ID34	Boy	April	YES	YES	YES	YES	YES	p&p	A+	$(6, 6.45, 7, 7.4)$	D-	$(2.4, 2.7, 3.3, 3.6)$
ID35	Boy	April	YES	NO	YES	NO	YES	comp		$(3.125, 3.625, 5.125, 5.9)$		$(5.45, 5.9, 7.9, 8.45)$
ID36	Boy	January	YES	YES	NO	NO	YES	p&p	D-	$(2.9, 4, 5, 6.15)$	D-	$(3, 4, 5, 6.1)$
ID37	Boy	November	YES	YES	YES	NO	YES	p&p	D-	$(7, 7, 8, 8)$	A+	$(5, 6, 7, 9)$
ID38	Boy	January	YES	YES	YES	YES	YES	comp	A+	$(8.925, 9, 9.775, 9.775)$	A-	$(5.85, 5.9, 7.425, 7.5)$
ID39	Girl	December	YES	YES	YES	NO	YES	comp	D+	$(0.15, 0.15, 3, 4.05)$	A-	$(4.125, 4.15, 7.95, 7.95)$
ID40	Boy	August	YES	YES	YES	YES	YES	comp	A-	$(5.875, 5.9, 8.9, 8.975)$	A-	$(5, 5.075, 8.075, 8.075)$
ID41	Girl	June	YES	YES	YES	YES	NO	comp	D-	$(2.5, 3.75, 6.25, 6.275)$	A+	$(3.725, 3.75, 6.25, 10)$
ID42	Boy	February	YES	YES	YES	YES	YES	comp		$(2.5, 3.75, 6.25, 7.5)$	A-	$(2.5, 3.75, 7.5, 8.35)$
ID43	Boy	April	YES	YES	YES	YES	YES	comp	D-	$(0.1, 1.975, 3.1, 4.5)$	A-	$(2.825, 3.75, 5.325, 6.1)$
ID44	Boy	July	YES	YES	NO	YES	YES	comp	A+	$(4.975, 7.625, 9.7, 9.7)$	A-	$(7.575, 7.625, 8.1, 8.925)$
ID45	Girl	January	YES	YES	YES	NO	YES	p&p	A-	$(7.65, 8, 9, 9)$	D-	$(0, 0, 1, 1)$
ID46	Boy	June	YES	YES	YES	YES	YES	comp	A+	$(4.125, 5.025, 5.05, 8.075)$	A-	$(5, 5.95, 7.5, 8.95)$
ID47	Boy	June	YES	YES	YES	NO	YES	p&p	D-	$(7, 7.5, 7.75, 7.9)$	D-	$(7.3, 8, 8.5, 8.5)$
ID48	Boy	June	YES	NO	YES	NO	YES	comp	A+	$(6.875, 7.45, 9.525, 10)$	A-	$(3.975, 4.925, 7.5, 8.25)$
ID49	Girl	April	YES	YES	YES	YES	YES	comp	A+	$(5.425, 6.025, 7.975, 10)$	A-	$(3.025, 3.75, 6.25, 6.925)$
ID50	Boy	February	YES	YES	YES	YES	YES	comp	A-	$(4.925, 5.5, 7.5, 8.525)$	A-	$(6, 6, 7.975, 7.975)$
ID51	Girl	April	YES	YES	YES	NO	YES	comp	D-	$(1.125, 1.25, 2.075, 2.15)$	A-	$(1.225, 1.25, 2.025, 2.175)$
ID52	Boy	May	YES	YES	YES	NO	YES	comp	A+	$(9, 9.05, 9.95, 9.95)$	A-	$(6.05, 6.05, 7.5, 7.5)$
ID53	Girl	June	YES	YES	YES	NO	YES	comp	A+	$(7.95, 7.95, 9.075, 9.075)$	A+	$(2.525, 4.975, 7.975, 8.925)$
ID54	Girl	September	YES	NO	YES	NO	NO	comp	A-	$(6.475, 7.625, 9.15, 9.975)$		$(7.05, 7.075, 10, 10)$
ID55	Girl	April	YES	YES	YES	YES	YES	p&p	D-	$(6, 7, 8, 8.8)$	D-	$(6, 6, 7, 8)$
ID56	Girl	May	YES	NO	YES	NO	YES	p&p	A+	$(7.85, 8.35, 8.7, 9.1)$	D-	$(4, 4.2, 4.8, 5)$
ID57	Boy	June	YES	YES	YES	YES	YES	comp	A-	$(5.55, 5.55, 7.375, 7.4)$	A-	$(6.05, 6.125, 7.975, 7.975)$
ID58	Boy	March	YES	YES	YES	NO	YES	p&p	D+	$(0, 0, 0, 2)$	D-	$(2.4, 3, 3.8, 5)$
ID59	Boy	June	YES	YES	YES	NO	YES	p&p	A-	$(6.5, 7, 8, 8.65)$	D-	$(2.45, 3, 4, 4.45)$
ID60	Girl	August	YES	YES	YES	YES	YES	comp	A+	$(9.025, 9.025, 9.975, 10)$	A-	$(4.05, 4.05, 7.025, 7.025)$
ID61	Boy	May	YES	YES	YES	YES	YES	comp	A-	$(4.325, 5.775, 7.675, 8.875)$	A-	$(4.625, 5.475, 7.975, 9)$
ID62	Boy	February	YES	YES	YES	YES	YES	comp	A-	$(6.975, 6.975, 9, 9)$	A+	$(6.925, 8, 8.025, 8.05)$
ID63	Girl	September	YES	YES	YES	YES	YES	comp	A-	$(3.75, 4.475, 7.325, 8.225)$	A+	$(6.225, 6.225, 8.55, 9.9)$
ID64	Girl	June	YES	YES	YES	YES	YES	comp	A+	$(6.825, 9.9, 9.95, 9.975)$	A+	$(2.5, 5.15, 7.975, 9.075)$
ID65	Girl	August	YES	YES	YES	YES	YES	comp	A+	$(7.975, 8.05, 10, 10)$	D-	$(2.025, 2.125, 3.95, 3.95)$
ID66	Girl	November	YES	YES	YES	YES	YES	comp	A+	$(8.95, 9, 9.9, 9.9)$	A+	$(6.85, 6.85, 7.475, 8.125)$
ID67	Boy	July	YES	YES	YES	NO	YES	p&p	D-	$(1.2, 1.55, 2.4, 2.7)$	D-	$(0.2, 0.65, 1.3, 1.55)$
ID68	Girl	November	YES	YES	YES	YES	YES	p&p	D-	$(2.35, 3, 4, 4.5)$	D-	$(3.5, 4.1, 4.9, 5.45)$
ID69	Boy	June	YES	NO	YES	NO	YES	comp	D+	$(0.025, 1.025, 2.025, 3.075)$	A-	$(2.95, 2.95, 2.95, 4.15)$

Table A.2: Dataset for the "Reading in school" items (Page 236) from the adapted TIMSS/PIRLS student Questionnaire (D+ = disagree a lot, D− = disagree a little, A− = agree a little, A+ = agree a lot, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$)

ID	$R.1$ Likert	$R.1$ (a_i, b_i, c_i, d_i)	$R.2$ Likert	$R.2$ (a_i, b_i, c_i, d_i)	$R.3$ Likert	$R.3$ (a_i, b_i, c_i, d_i)	$R4$
ID1	A-	(5.35, 6, 7, 7.75)	A+	(8.1, 9, 10, 10)	D-	(3, 3, 4, 4)	9
ID2	A-	(5.575, 6.15, 7.5, 7.5)	A+	(6.775, 8.45, 9.75, 9.75)	D+	(0.025, 0.15, 0.525, 9.95)	
ID3	A+	(6.35, 6.35, 6.575, 10)	A+	(7.225, 7.275, 9.775, 10)	D+	(2.5, 3.75, 6.25, 9.9)	10
ID4	A-	(6.025, 6.1, 7.525, 9.425)	A-	(3.675, 3.75, 6.25, 6.8)	D+	(0.025, 0.075, 1.075, 3.575)	9
ID5	A+	(6.25, 6.275, 9.95, 9.95)	A-	(5.575, 6.325, 8.475, 9.475)	D+	(0.1, 0.325, 3.05, 3.325)	7
ID6	A+	(8, 8.5, 9.5, 10)	A+	(9, 9, 10, 10)	D+	(0, 0.5, 0.5, 1)	9.5
ID7	A-	(4.65, 4.65, 7.5, 8.4)	A-	(3.975, 3.975, 6.25, 8)	D-	(1.525, 2.525, 4.575, 4.625)	8
ID8	D-	(3, 3, 3.4, 4)	D-	(3, 3, 3.45, 4)	D-	(2, 2.55, 3, 3)	0
ID9	A-	(5, 6.5, 6.5, 7.5)	A+	(8, 9, 9, 10)	D-	(2, 3, 3, 3.8)	8
ID10	A-	(4.025, 4.25, 6.4, 6.775)	A-	(3.575, 4.4, 6.9, 6.9)	D+	(0.175, 0.55, 2.1, 2.45)	9
ID11	A-	(7, 7.025, 7.5, 10)		(8.775, 9.925, 9.925, 9.925)	D+	(1.05, 1.05, 1.675, 2.775)	9
ID12	D-	(2, 2, 3, 3)	A+	(9, 9, 10, 10)	D-	(2, 2, 4, 4)	8
ID13	A-	(6.4, 7, 7.9, 8.5)	A+	(9, 10, 10, 10)	D-	(1.55, 2.45, 3.6, 4)	9
ID14	A-	(5.425, 5.5, 7, 8.275)	A-	(4.275, 5.925, 7.425, 8.275)	D-	(2.95, 2.95, 4.5, 5.95)	9
ID15	A-	(4.375, 4.4, 7.55, 8.05)	A+	(4.775, 4.775, 9.175, 10)	A-	(3.625, 4.15, 7.2, 7.5)	8
ID16	A+	(6.8, 7.025, 9.975, 9.975)	A+	(5.05, 5.05, 10, 10)	A-	(2.975, 3.95, 6.975, 6.975)	8
ID17	D-	(3.75, 3.75, 6.25, 6.25)	A+	(5.9, 5.9, 9.975, 9.975)	D-	(3.9, 3.9, 6.15, 7.075)	8
ID18	D-	(4.175, 4.375, 6.075, 6.275)	A+	(6.75, 7.15, 8.7, 9.525)	D-	(1.55, 2.2, 4.4, 5.6)	8
ID19	A-	(4.975, 5.65, 7.5, 7.5)	D-	(2.5, 2.55, 4.65, 5.5)	D+	(0, 0, 1.7, 1.7)	8
ID20	A+	(7.35, 7.425, 10, 10)	A+	(6.925, 8.5, 10, 10)	D+	(0.05, 0.05, 1.025, 1.025)	10
ID21	A+	(9.2, 9.65, 10, 10)	A+	(10, 10, 10, 10)	D+	(0, 0, 0.15, 0.4)	9
ID22	A-	(6, 6.5, 7.4, 8)	A-	(6, 6.65, 7.3, 8)	D+	(0, 0, 0, 1)	7
ID23	A-	(5.9, 6.25, 6.9, 7)	A-	(6, 6.25, 6.85, 7.1)	D+	(0, 0, 0, 0)	6
ID24	A-	(4.25, 5.05, 7.025, 7.95)	A-	(4.925, 5.9, 7.05, 8.025)	D-	(1.975, 2.525, 4.025, 4.925)	8
ID25	A-	(6, 6.5, 6.5, 7)	A-	(8, 8.5, 8.5, 9)	D+	(0, 0.5, 0.5, 1)	7
ID26	D-	(3, 3, 3.5, 4)	A-	(6, 6, 6.6, 7.2)	D-	(3, 3, 4, 4.6)	7.5
ID27	A+	(9, 9.25, 9.6, 10)	A+	(9, 9.3, 9.55, 10)	D+	(0.1, 0.3, 0.65, 1)	10
ID28	A-	(4, 5.8, 7.425, 9.075)	A+	(6.175, 8.85, 9.85, 9.85)	D+	(0.025, 0.025, 1.025, 1.075)	8
ID29	A+	(7.925, 7.925, 10, 10)	A+	(9, 9.05, 9.925, 9.975)	D+	(9.95, 9.95, 9.95, 9.975)	9.5
ID30	D-	(3.025, 3.025, 5.025, 5.075)	D-	(1.975, 2, 5.025, 5.025)	D+	(1, 1, 3.05, 3.45)	6
ID31	A-	(2.5, 3.75, 6.25, 7.5)	A+	(3.7, 3.75, 9.8, 9.95)	D+	(0.6, 0.675, 6.25, 6.25)	7.75
ID32	A-	(6.75, 6.75, 7.5, 8)	A-	(8, 8, 8.65, 9.2)	D-	(3.5, 4.4, 5.3, 6.2)	8
ID33	A-	(7.3, 7.65, 8.3, 8.6)	A-	(8.4, 8.7, 9.25, 9.65)	D+	(0, 0, 0.2, 0.4)	8.5
ID34	A-	(6, 6.3, 6.9, 7.2)	A+	(8.15, 8.5, 8.9, 9.3)	A+	(8.3, 8.6, 9.25, 9.6)	6
ID35		(0, 1.075, 2.15, 3.525)		(2.875, 3.475, 5.175, 5.75)		(0, 0.475, 1.85, 2.275)	8
ID36	A-	(7.1, 8, 9, 10)	D-	(6, 7, 8, 9)	D+	(1, 2, 3, 4)	9
ID37	A-	(7, 7, 8, 8)	A-	(6, 6, 7, 7)	A-	(8, 8, 9, 9)	6.5
ID38	A-	(5.825, 5.875, 7.95, 8)	A+	(8, 9.975, 9.975, 10)	D+	(0, 0.05, 0.95, 0.95)	
ID39	D-	(2.5, 3, 4.65, 4.65)	A+	(6.975, 6.975, 6.975, 10)	D+	(0.125, 0.125, 0.375, 0.375)	
ID40	A+	(8.95, 8.95, 10, 10)	A+	(9.4, 9.4, 9.95, 9.975)	D+	(0, 0, 1, 1.025)	
ID41	A-	(5.125, 5.125, 6.25, 8.925)	A-	(5.15, 5.175, 7.45, 9.95)	D+	(0.075, 3.75, 6.25, 6.25)	
ID42	D-	(1.725, 3.075, 6.1, 7.5)	A+	(2.5, 5.225, 8.55, 10)	D+	(0.075, 0.075, 3.775, 3.8)	
ID43	A-	(3.5, 4.625, 6.325, 6.775)	A+	(8.725, 9.175, 9.95, 10)	D+	(0, 0, 1.025, 2.025)	
ID44	A+	(8.925, 8.925, 8.925, 8.925)	A-	(5.95, 6.575, 8.375, 9.825)	A+	(2.5, 3.75, 7.45, 7.5)	
ID45	A-	(7, 7, 8, 8.4)	D-	(3, 3, 4, 4)	D-	(1.4, 2, 3, 3)	9
ID46	A-	(4.375, 5.025, 7, 8.025)	A+	(5.925, 7.175, 9.375, 10)	A+	(6.975, 8.025, 9.9, 10)	
ID47	A-	(7, 78.15, 8.65, 8.65)	A+	(8.2, 8.8, 10, 10)	D+	(0, 0.3, 0.65, 1)	8
ID48	D-	(2.425, 3.25, 5.575, 6.675)	A-	(4.425, 5.6, 7.5, 8.25)	D+	(0, 0.5, 2.6, 2.6)	8.5
ID49	A-	(4.95, 5.525, 7.05, 8.05)	A+	(8.95, 8.95, 9.9, 9.95)	D-	(2.825, 2.875, 4.975, 5.05)	
ID50	D+	(0.025, 0.025, 1.5, 1.5)	A-	(5, 6, 7.975, 8)	D-	(3, 3, 5, 6)	
ID51	A-	(4.125, 4.975, 6.9, 7.95)	A-	(4.85, 4.875, 7.375, 7.875)	D+	(1.05, 1.075, 1.775, 2.05)	
ID52	A-	(6.1, 6.15, 7.475, 7.5)	A+	(7.975, 7.975, 9.9, 10)	D+	(8.3, 8.375, 9.825, 9.825)	8
ID53	A-	(7.95, 7.95, 8.95, 9)	A+	(5.125, 7.375, 7.5, 10)	D+	(0, 0, 2.525, 2.525)	
ID54	D-	(2.6, 2.6, 6.25, 6.25)	A-	(3.725, 3.75, 6.875, 6.875)		(0.075, 0.1, 1.725, 1.725)	8
ID55	A-	(7, 7, 8, 9)	A-	(7.05, 8.15, 9, 9)	D+	(0, 0, 0, 0)	7
ID56	A-	(6.9, 7.7, 8.3, 9)	A+	(9, 10, 10, 10)	D+	(0, 0.35, 0.6, 1)	9.5
ID57	D+	(0.025, 0.075, 1.025, 1.05)	A-	(6.05, 6.05, 8.95, 8.95)	D-	(2.15, 3.05, 3.05, 5.025)	
ID58	A-	(6, 7, 8, 9.1)	A+	(9, 10, 10, 10)	A+	(9.4, 9.85, 10, 10)	8
ID59	D-	(2, 2.85, 3.6, 4.15)	A-	(5.6, 6.35, 7.1, 7.5)	A+	(8.65, 9.05, 9.6, 10)	6
ID60	D+	(0.025, 0.025, 1.025, 1.025)	A+	(6.875, 6.95, 9.975, 10)	D+	(0.125, 0.175, 1.075, 1.1)	10
ID61	D-	(1.475, 2.45, 5.025, 6.1)	D-	(1.675, 2.375, 4.825, 5.9)	D+	(0.025, 0.05, 0.05, 0.05)	9
ID62	A+	(8.975, 8.975, 10, 10)	A+	(8, 8, 9.925, 9.975)	A+	(6.9, 7, 9.9, 9.9)	9
ID63	A-	(3.75, 4.7, 6.9, 8.175)	A+	(7.425, 7.475, 9.375, 9.375)	D+	(0.2, 0.2, 0.275, 1.85)	9
ID64	A+	(8.95, 9.075, 9.925, 9.95)	A-	(5.475, 5.475, 8.8, 8.8)	D+	(0.15, 0.175, 7.025, 7.025)	
ID65	A-	(4.2, 4.875, 7.375, 7.5)	A+	(5.875, 5.925, 8.925, 8.95)	D+	(0, 0.1, 0.95, 2.925)	7
ID66	A-	(5.975, 7.025, 7.1, 8.9)	A+	(8.975, 8.975, 9.975, 10)	A-	(5.05, 5.975, 7.775, 9.15)	8
ID67	A-	(5.6, 6.15, 6.9, 7.45)	A+	(8.5, 9.1, 9.65, 10)	D+	(0.2, 0.6, 1.2, 1.6)	8
ID68	A+	(9, 9.6, 9.6, 10)	A+	(8.65, 9.1, 9.9, 10)	A-	(5.55, 6.2, 6.95, 7.3)	8
ID69	D-	(2.1, 2.975, 4, 4.975)	A-	(5.575, 5.575, 7.5, 7.5)	D+	(0, 1.025, 2.05, 3.15)	

Table A.3: Dataset for the “Mathematics in school” items (Page 236) from the adapted TIMSS/PIRLS student Questionnaire (D+ = disagree a lot, D− = disagree a little, A− = agree a little, A+ = agree a lot, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$)

ID	$M.1$ Likert	$M.1$ (a_i, b_i, c_i, d_i)	$M.2$ Likert	$M.2$ (a_i, b_i, c_i, d_i)	$M.3$ Likert	$M.3$ (a_i, b_i, c_i, d_i)	$M4$
ID1	D-	(2.35, 3, 4, 4.8)	A+	(8.4, 9, 10, 10)	A-	(5.2, 6, 7, 7)	8
ID2	A-	(5.65, 6.2, 7.475, 9.8)	A+	(9.2, 9.2, 9.975, 9.975)	D+	(0, 0, 0, 2.95)	8
ID3	A+	(9.85, 9.85, 9.875, 9.975)	A+	(8.925, 9.95, 9.95, 9.95)	A+	(6, 6.425, 7.35, 7.875)	10
ID4	A+	(8.025, 9.05, 9.9, 9.975)	D+	(0.1, 0.1, 0.625, 1.975)	A+	(0, 0, 0.775, 1.225)	8.5
ID5	D-	(3.525, 3.75, 6.25, 6.725)	D-	(3.5, 3.55, 6.25, 7.5)	D+	(0, 0, 2.15, 2.15)	6
ID6	A+	(9, 9.225, 9.775, 10)	A+	(10, 10, 10, 10)	A+	(10, 10, 10, 10)	10
ID7	D-	(2.5, 2.975, 5.3, 5.35)	A-	(4.375, 5.175, 7.5, 7.5)	A+	(7.9, 7.95, 8.7, 8.7)	9
ID8	D-	(2, 2.45, 3, 3)	D-	(3, 3, 3.45, 4)	A+	(9, 9, 9.4, 10)	10
ID9	D-	(3.1, 4, 4, 4.5)	A+	(8.6, 10, 10, 10)	A+	(9, 10, 10, 10)	7
ID10	A-	(4.325, 5.025, 7.925, 8.5)	D-	(1.75, 2.5, 3.675, 3.675)	D+	(4.975, 4.975, 5.325, 5.4)	7
ID11	A+	(7.525, 7.525, 7.55, 9.025)	A-	(6.975, 6.975, 7.5, 7.5)	A+	(7.6, 7.6, 8.35, 8.65)	10
ID12	A+	(9, 9, 10, 10)	A+	(10, 10, 10, 10)	D-	(2, 2, 4, 4)	9
ID13	A+	(9.4, 10, 10, 10)	A+	(8.9, 9.4, 10, 10)	D+	(0, 0, 0, 0.45)	10
ID14	A-	(5.025, 5.95, 7.025, 8.95)	A+	(7.3, 8.05, 9.575, 10)	A+	(9.975, 10, 10, 10)	8
ID15	A+	(5.75, 5.775, 9.55, 9.875)	A+	(5.95, 6, 9.2, 10)	A+	(0, 0, 1.575, 1.575)	9
ID16	A+	(6.85, 8, 10, 10)	A+	(6.75, 7.025, 9.975, 9.975)	A+	(2.225, 2.225, 3.125, 3.125)	10
ID17	A+	(4.025, 5.75, 8.725, 10)	A+	(4.9, 4.9, 8.45, 9.975)	A+	(1.9, 1.95, 3, 3.15)	5
ID18	A-	(4.2, 4.925, 6.975, 7.2)	A-	(4.4, 4.725, 6.25, 7.8)	D-	(4.875, 5.05, 5.45, 5.625)	7
ID19	A-	(3.75, 3.75, 7.5, 7.5)	A-	(6.225, 6.25, 7.5, 7.5)	A+	(6.15, 6.15, 6.75, 6.75)	8
ID20	A+	(5.85, 7.025, 9.05, 9.1)	D+	(0, 0.125, 2.05, 2.55)	D-	(3.45, 3.45, 4.425, 4.425)	8.5
ID21	D-	(3.1, 3.25, 3.85, 4.5)	A+	(10, 10, 10, 10)	D-	(2.5, 3.2, 3.3, 4.45)	8
ID22	A-	(6, 6.7, 7.2, 8)	A+	(8.7, 9.4, 10, 10)	D-	(3, 3.6, 4.2, 5.05)	8
ID23	A-	(6.1, 6.4, 6.75, 7.1)	A+	(9, 10, 10, 10)	D-	(3, 3, 3, 3)	7
ID24	A+	(10, 10, 10, 10)	A+	(9.975, 9.975, 9.975, 9.975)	A+	(0, 0.6, 1.25, 1.65)	10
ID25	A+	(9, 9.5, 9.5, 10)	A-	(8, 8.5, 8.5, 9)	D+	(0, 0.5, 0.5, 1)	9
ID26	D-	(2.4, 3, 3.65, 3.65)	A-	(6, 6, 6.6, 7.7)	D-	(2.5, 3, 3.6, 3.6)	5
ID27	A-	(6, 6.15, 6.55, 7)	A+	(8.1, 8.2, 8.6, 9)	D-	(3, 3.2, 3.6, 4.2)	8
ID28	D-	(2.5, 2.95, 6.25, 7.5)	A+	(3.4, 4.825, 9.95, 9.95)	A+	(10, 10, 10, 10)	5
ID29	A+	(9.975, 10, 10, 10)	A+	(9.975, 9.975, 10, 10)	A+	(10, 10, 10, 10)	9
ID30	A+	(2.975, 3.05, 10, 10)	A-	(3, 3, 7.95, 7.95)	A+	(6.975, 6.975, 7.925, 7.925)	
ID31	D-	(2.5, 3.75, 6.25, 7.5)	A-	(2.5, 3.75, 6.25, 7.5)	D+	(0, 0, 2.575, 2.575)	5
ID32	D-	(3.8, 4.25, 5.5, 6)	A+	(9.6, 9.8, 10, 10)	A-	(6, 6.45, 7.4, 8)	9
ID33	D-	(4.6, 4.75, 5.15, 5.35)	A+	(9.2, 9.8, 10, 10)	D-	(2.35, 2.8, 3.25, 3.5)	7.5
ID34	A-	(6.2, 6.4, 6.85, 7.1)	A-	(5.2, 5.4, 5.65, 6)	D-	(3.15, 3.4, 3.6, 4)	5
ID35		(3.05, 4.05, 7.95, 9.025)		(8.725, 8.95, 9.7, 10)	D+	(0, 0.625, 2.725, 2.75)	7
ID36	A+	(8, 9.15, 10, 10)	A+	(8, 9, 10, 10)	D+	(0, 0, 1, 2)	10
ID37					D-	(0, 1.125, 2.025, 2.625)	
ID38	A+	(9.925, 9.95, 10, 10)	A-	(7, 7.025, 8.9, 8.975)	A+	(4.925, 5.025, 5.95, 6.3)	10
ID39	D+	(0, 0.025, 0.025, 0.025)	A+	(9.975, 9.975, 9.975, 10)	A+	(10, 10, 10, 10)	
ID40	D-	(2.925, 2.975, 5.95, 5.975)	A+	(9.45, 9.45, 9.925, 10)	D+	(0, 0.825, 2.425, 2.425)	
ID41	D+	(0, 1.125, 1.2, 1.275)	D-	(2.5, 3.75, 3.9, 5.45)	D-	(0, 0.325, 1.475, 1.475)	5
ID42	A-	(3.7, 3.75, 7.225, 7.25)	A+	(6.9, 8.175, 9.225, 9.975)	A+	(5.15, 5.35, 6.15, 6.15)	10
ID43	A-	(3.825, 4.9, 6.05, 6.725)	A+	(6.7, 7.775, 8.9, 10)	A+	(8.55, 8.85, 9.625, 10)	8.5
ID44	A-	(8.975, 8.975, 8.975, 10)	A+	(3.175, 5.025, 7.5, 9.95)	A+	(0, 0, 0, 0.725)	7.5
ID45	A+	(10, 10, 10, 10)	A+	(10, 10, 10, 10)	D+	(0, 0, 0, 0)	10
ID46	A+	(10, 10, 10, 10)		(8.05, 8.65, 10, 10)	A+	(10, 10, 10, 10)	9.5
ID47	A-	(6, 6.65, 7.25, 7.25)	A-	(8, 8.5, 9.2, 9.2)	A-	(7, 7.4, 8.2, 8.4)	8
ID48	D-	(2, 2, 5, 5)	A-	(5, 6, 6.125, 8)	A+	(4.05, 4.05, 4.7, 4.775)	9
ID49	A+	(8.975, 8.975, 9.975, 9.975)	A+	(9.025, 9.025, 9.95, 9.95)	A+	(10, 10, 10, 10)	9
ID50	D-	(2.5, 2.975, 5.5, 6.5)	A+	(8, 8.5, 9.85, 9.875)	D+	(0, 0.85, 1.5, 1.825)	9
ID51	A-	(4.85, 5, 7.05, 7.875)	A+	(7.95, 9, 10, 10)	D+	(1.6, 1.825, 2.425, 3.075)	10
ID52	D-	(3.075, 3.1, 4, 7.5)	A+	(9.325, 9.375, 10, 10)	A+	(3.125, 3.275, 3.7, 4.05)	10
ID53	D-	(0.975, 3.875, 4.075, 4.075)	A-	(3.975, 4.925, 6.875, 6.925)	A-	(9.9, 9.9, 10, 10)	9
ID54	A-	(6.675, 6.675, 6.675, 6.7)	A+	(0.225, 3, 6.875, 9.9)	A-	(0, 0, 1.125, 1.125)	
ID55	A-	(7, 7, 8, 9)	A-	(7, 8, 9, 9)	A-	(6, 6, 7, 8)	8
ID56	A-	(8, 8.3, 8.55, 9)	A+	(9, 10, 10, 10)	D-	(1, 1.8, 2.35, 3.1)	10
ID57	A-	(7.925, 7.95, 8, 8)	A-	(6.075, 6.15, 9.05, 9.05)	A-	(0, 0.075, 1, 1.35)	9.5
ID58	A+	(9, 10, 10, 10)	A+	(8, 10, 10, 10)	D+	(0, 0, 0, 0)	10
ID59	A+	(8.3, 9.3, 9.8, 10)	A-	(6, 7, 9, 10)	D+	(0, 0.4, 0.95, 1.75)	9
ID60	D+	(0.05, 0.05, 0.075, 0.075)	A+	(9.025, 9.025, 9.95, 9.95)	A-	(10, 10, 10, 10)	
ID61	D-	(1.45, 1.95, 4.95, 5.725)	A+	(5.6, 6.7, 9.15, 10)	A+	(8.8, 8.8, 9.5, 9.575)	8
ID62	A-	(2.9, 3.75, 6.25, 7.8)	A+	(9.85, 9.85, 9.9, 9.9)	A+	(4.6, 6.15, 6.15, 6.85)	10
ID63	A+	(9.875, 9.95, 9.95, 9.975)	A-	(4.225, 5.7, 7.025, 8.9)	A-	(3.6, 3.925, 4.575, 4.575)	8
ID64	A-	(2.5, 4.075, 7.175, 8.15)	A+	(5.825, 5.85, 9.875, 9.95)	D-	(3.875, 3.875, 5.6, 5.6)	9
ID65	D-	(2.5, 2.55, 4.275, 4.3)	A-	(2.5, 4.625, 4.625, 6.9)	D+	(0, 0.25, 1.025, 1.025)	7
ID66	A+	(8, 8.025, 9.8, 9.975)	A+	(9.8, 9.8, 10, 10)	A+	(10, 10, 10, 10)	9
ID67	A+	(8.55, 9.15, 9.7, 10)	A+	(8.6, 9.15, 9.75, 10)	D-	(0.3, 0.45, 1.15, 1.5)	10
ID68	D-	(3.5, 4.2, 5, 5.45)	A-	(5.1, 6, 6.75, 7.3)	A-	(5.5, 6.1, 6.9, 7.4)	10
ID69	D-	(2.5, 2.5, 5.1, 7.5)	A+	(10, 10, 10, 10)	A+	(6.325, 6.925, 7.175, 7.65)	

Table A.4: Dataset for the “Science in school” items (Page 237) from the adapted TIMSS/PIRLS student Questionnaire (D+ = disagree a lot, D− = disagree a little, A− = agree a little, A+ = agree a lot, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$)

ID	$S.1$ Likert	$S.1$ (a_i, b_i, c_i, d_i)	$S.2$ Likert	$S.2$ (a_i, b_i, c_i, d_i)	$S.3$ Likert	$S.3$ (a_i, b_i, c_i, d_i)	$S4$
ID1	A-	(6.35, 6.95, 8, 8)	D-	(2.4, 2.95, 4.05, 5)	D-	(3.2, 4, 5, 5)	7
ID2	A-	(0.45, 6.225, 7.4, 7.5)	D+	(0.075, 0.075, 0.775, 9.775)	D+	(0.025, 0.025, 0.675, 9.825)	10
ID3		(2.5, 3.75, 6.25, 7.5)	D-	(2.5, 2.5, 3.775, 4.275)	D+	(2.5, 2.5, 2.575, 2.65)	9.5
ID4	A+	(5.975, 7.9, 9.45, 10)	D-	(2.5, 2.5, 4.425, 4.975)	D+	(0, 0, 2, 3.475)	8.5
ID5	A+	(9.175, 9.2, 9.95, 9.95)	D+	(0, 0.125, 2.55, 2.575)	D+	(0.175, 0.225, 6.25, 7.5)	8.5
ID6	A-	(6.5, 7, 7.5, 8)	D-	(2, 2.5, 4.5, 5)	D-	(2.5, 2.5, 3.5, 3.5)	7.75
ID7	A-	(4, 4.8, 7.275, 7.5)	D-	(1.075, 1.5, 3.6, 5.25)	A-	(4.05, 5.075, 7.225, 7.5)	7
ID8	A+	(9, 9, 9.45, 10)	D+	(0, 0, 0.4, 1)	A+	(9, 9, 9.45, 10)	10
ID9	A+	(9, 10, 10, 10)	D-	(2, 3, 3, 3.6)	A-	(6.15, 7, 7, 7.45)	8
ID10	D+	(0.8, 0.8, 1.3, 1.85)	D+	(0.45, 1.175, 2.55, 2.7)	A-	(3.75, 4.675, 6.85, 8.4)	9.5
ID11	A-	(6.925, 6.925, 7.5, 7.5)	D+	(2.5, 2.5, 2.975, 2.975)	D-	(2.5, 2.525, 3.55, 3.575)	8
ID12	A-	(5, 5, 7, 7)	D+	(0, 0, 0, 0)	D+	(0, 0, 1, 1)	10
ID13	D-	(2, 2.8, 4, 4)	D+	(0, 0, 1, 1)	A-	(5.5, 6.1, 6.9, 8.5)	9.5
ID14	A-	(4, 5.65, 6.725, 7.375)	D+	(0.025, 1.175, 2.35, 2.925)	D+	(0.4, 1.475, 2.4, 3)	10
ID15	A-	(4.625, 4.65, 7.475, 7.5)	A-	(3.75, 3.75, 7.2, 7.525)	D-	(3.575, 3.75, 7.15, 7.5)	10
ID16	A-	(3.55, 3.75, 6.95, 6.95)	A-	(3.475, 6.925, 6.975, 6.975)	A-	(4.925, 4.925, 6.8, 6.8)	10
ID17	A-	(3.625, 4.6, 6.95, 7.5)	D+	(0.025, 1, 2.6, 4.175)	D+	(0.075, 0.2, 1.7, 1.75)	7
ID18	A+	(6.175, 6.475, 8.25, 9.525)	D+	(0.45, 1.35, 2.275, 2.675)	A-	(5.225, 6, 7.425, 8.5)	7
ID19	A+	(6.7, 6.7, 9.975, 10)	D+	(0, 0.025, 1.6, 1.6)	A+	(8.025, 8.45, 9.825, 9.85)	7
ID20	A-	(5, 6.05, 6.875, 7.35)	A-	(3.5, 4.95, 6.95, 6.95)	A-	(4.525, 4.95, 7.425, 7.5)	7.5
ID21	A+	(10, 10, 10, 10)	D-	(2.7, 3.2, 3.7, 4)	D+	(0, 0, 0, 0)	8.5
ID22	A-	(5, 5.5, 6.5, 7.1)	D-	(1, 1.65, 2.3, 3.3)	D-	(2, 2.3, 3.1, 3.8)	3
ID23	A+		D+		D-	(3, 3.1, 3.7, 4)	6
ID24	A-	(4.725, 5.375, 7.275, 7.6)	D-	(0, 2.025, 3.225, 3.5)	D+	(0.95, 1.575, 2.425, 3.4)	7
ID25	D-	(3, 3.5, 3.5, 4)	D+	(0, 0.5, 0.5, 1)	D+	(0, 0.5, 0.5, 1)	9
ID26	A-	(5.3, 6, 6.5, 6.5)	0	(0, 0, 0, 0.7)	D-	(3, 3, 3.5, 4.15)	4.5
ID27	A+	(9, 9.15, 9.55, 10)	A+	(8.1, 8.35, 8.8, 10)	D+	(0, 0.2, 0.6, 1)	9
ID28	A+	(1.4, 3.75, 9.9, 9.925)	D-	(2.5, 2.55, 6.25, 7.5)	A-	(3.7, 5.625, 7.5, 7.5)	7
ID29	A+	(9.95, 9.975, 9.975, 9.975)	D-	(3.575, 3.75, 3.75, 6.725)	A-	(3.975, 4.05, 7.025, 7.025)	8
ID30	A-	(2.95, 3.025, 7, 7.025)	D+	(0, 0, 2.075, 2.075)	A+	(3.75, 3.75, 7.95, 8.05)	6
ID31	A-	(3.35, 3.75, 7, 7.5)	D-	(2.5, 2.85, 6.25, 6.3)	D+	(0, 0.025, 6.25, 6.325)	9
ID32	A-	(6.9, 7.6, 8.5, 9.2)	D-	(2.4, 3.2, 3.7, 4.2)	D-	(2.5, 3.5, 4.25, 5.2)	10
ID33	D-	(4.6, 4.7, 5.15, 5.45)	D-	(4.25, 4.55, 5.15, 5.4)	D-	(4.45, 4.8, 5.2, 5.5)	8.5
ID34	A+	(8.25, 8.35, 8.85, 9.1)	D+	(1.2, 1.4, 1.85, 2.15)	A-	(6.15, 6.3, 6.8, 7)	9
ID35		(5.15, 6.05, 7.925, 9)		(0.025, 0.3, 0.5, 0.75)		(8.25, 8.675, 9.425, 10)	8
ID36	A+	(8, 9, 10, 10)	A-	(8, 9, 10, 10)	D+	(0, 0, 1, 1.9)	8
ID37							
ID38	A+	(8.85, 9.975, 9.975, 9.975)	D-	(2.5, 2.5, 4.2, 4.2)	D+	(0.075, 0.075, 1.025, 1.025)	9
ID39	D-	(1.95, 3.75, 6.025, 8.05)	D+	(0.125, 0.15, 0.15, 0.15)	A+	(7.925, 8.925, 10, 10)	3.5
ID40	A-	(4.875, 4.875, 8.025, 8.05)	A-	(3.75, 3.75, 7.5, 7.5)		(2.5, 3.75, 6.25, 7.5)	
ID41	A+	(4.975, 4.975, 6.575, 7.925)	D-	(0.2, 2.5, 3.075, 3.075)	D+	(0.15, 0.6, 2.875, 2.95)	
ID42	D+	(2.5, 2.525, 3.35, 3.775)	D+	(0.35, 0.35, 3.525, 3.575)	D-	(2.5, 3.75, 6.25, 7.5)	6
ID43	D-	(3.55, 3.55, 3.575, 3.6)	D+	(0.025, 0.1, 0.15, 0.15)	D-	(4.975, 4.975, 4.975, 4.975)	10
ID44	A+	(5.075, 7.05, 9.975, 9.975)	A-	(3.75, 3.75, 6.25, 8.8)	A-	(5.625, 5.625, 7.4, 9.9)	
ID45	A-	(7, 7, 8, 8.45)	D-	(1.65, 2, 3, 3)	D+	(0, 0, 1, 1.45)	10
ID46	D-	(0, 0.025, 0.55, 0.55)	A-	(2.5, 3.75, 6.25, 7.5)	A+	(5.1, 5.975, 9.3, 9.9)	
ID47	A-	(7, 7.7, 8.4, 9)	D-	(3, 3.8, 4.6, 5.15)	D-	(2.5, 3, 3.5, 4)	8
ID48	D-	(1.5, 2.5, 4.525, 5.475)	D+	(0.3, 1, 3.075, 4.05)	A+	(8.5, 8.5, 9.925, 9.925)	6
ID49	D+	(0.05, 0.075, 3.025, 3.025)	D+	(1.525, 2.8, 3.875, 4.425)	A-	(4.85, 5.975, 6.9, 8)	6
ID50	D-	(2.5, 3, 5, 6)	D+	(0, 0.025, 0.15, 1)	D+	(0.5, 1, 2, 2.5)	
ID51	A+	(7.975, 8.975, 9.95, 10)	D-	(2.025, 2.025, 3.075, 3.075)	D+	(0.05, 0.05, 0.5, 1.05)	10
ID52	D-	(2.9, 3.125, 6.25, 6.275)	A+	(8.925, 8.925, 10, 10)	A+	(9.675, 9.725, 9.975, 10)	7
ID53	A-	(3.9, 3.9, 6.8, 6.875)	D+	(0.075, 0.125, 1.05, 1.125)	A-	(2.5, 3.75, 6.775, 7)	7
ID54	A+	(0.025, 3.75, 6.25, 9.95)	D-	(3.575, 3.625, 6.75, 6.775)	D-	(0.025, 0.075, 3.625, 3.625)	
ID55	A-	(6, 6, 7, 8)	D+	(0, 0, 0, 0)	A-	(7, 7, 7, 7)	7
ID56	D-	(3, 3, 3, 3.6)	A-	(4.9, 5.5, 5.5, 6)	D+	(0, 0, 0, 1)	10
ID57	D+	(0.125, 0.125, 3, 3.05)	D+	(0, 0.025, 2.025, 2.025)	D+	(0, 0, 0.95, 0.975)	9.5
ID58	D-	(2.9, 4, 5, 6.1)	D-	(2, 3, 4, 4.9)	A-	(4.4, 5.1, 7, 7.8)	6
ID59	D-	(2.2, 3, 3.7, 4.6)	0	(0, 0, 0, 0)	D+	(0.2, 0.45, 1.65, 2.4)	8
ID60	D+	(0.05, 0.05, 1.1, 1.1)	D+	(0, 0.025, 0.05, 0.05)	D+	(0, 0.025, 0.025, 0.025)	7
ID61	D+	(0, 0, 0, 0)	D-	(0.725, 1.6, 4.55, 5.075)	A-	(5.5, 5.9, 7.5, 8.075)	9.5
ID62	A-	(4.925, 6.1, 8.425, 8.425)	D-	(1.375, 2.525, 4.45, 6.325)	D-	(2.5, 4.625, 4.625, 7.5)	8
ID63	A-	(5.4, 5.425, 7.825, 8.425)	D-	(2.425, 3.1, 5, 5.825)	D-	(1.7, 3.075, 4.9, 5.875)	10
ID64	A-	(2.275, 3.75, 6.25, 8.075)	A-	(3.75, 3.75, 8.975, 9.025)	D-	(3.6, 3.75, 7.425, 7.5)	8
ID65	A-	(5.125, 6.775, 7.45, 7.5)	D-	(2.5, 3.125, 3.125, 6)	D-	(2.5, 3.75, 6.25, 7.5)	5
ID66	A-	(6.025, 6.05, 7.5, 9.05)	D-	(2.5, 3.75, 5.45, 6.05)	A+	(8.1, 9.9, 9.975, 10)	8
ID67	A-	(6.1, 6.7, 7.3, 7.65)	D-	(2.25, 2.6, 3.7, 4.3)	D-	(2.7, 3.3, 3.9, 4.3)	9
ID68	A+	(8.5, 9.2, 9.7, 10)	D+	(0.2, 1, 2, 2.4)	D-	(2.6, 3.15, 3.8, 4.3)	8
ID69	D+	(2.5, 2.675, 2.675, 2.725)	D+	(0.025, 0.025, 0.025, 0.075)	A+	(9.925, 9.925, 9.95, 10)	

Appendix B

Form and datasets for adapted restaurant customer satisfaction Questionnaire

QUESTIONNAIRE ON LAUNCH RESTAURANTS

(All information that you provide will be used in the strictest confidence)

PART 1: ABOUT YOURSELF

GENDER
- O Female
- O Male

AGE
- O Under 25 years
- O 25 – 34
- O 35 – 44
- O 45 – 54
- O 55 – 64
- O Over 65 years

CURRENT MAIN RESPONSABILITY
- O Working
- O Training
- O Both
- O Housework
- O None/retired

USUAL RESTAURANT CHOICE
- O Fast food restaurant
- O Self-service restaurant
- O Casual restaurant
- O Fine restaurant

Usual # days/month having lunch at a restaurant ☐

average price/lunch at the usual restaurant ☐

Please indicate an interval of time at which you use to have lunch: –

PART 2: ABOUT YOUR OPINION/VALUATION/RATING ON THE USUAL CHOICE

❖ **Firstly, you will reply to questions concerning the quality of the food and beverage.**
Questions should be replied by using a double type of response:
- **on one hand, the respondent should choose 1 of the 5 possible responses on the right side;**
- **on the other hand, the same respondent should draw a trapezoid as follows:**
  the lower basis for the trapezoid will be the interval of values between 0 (lowest rating) and 100 (highest rating) which are considered by the respondent as being <u>compatible to some extent</u> with their rating;
  the upper basis for the trapezoid will be the interval of values between 0 and 100 which are considered by the respondent as being <u>fully compatible</u> with their rating.
  Then the trapezoid will be immediate to draw.

QF1. The food is served hot and fresh

0 10 20 30 40 50 60 70 80 90 100

- O Strongly disagree
- O Somewhat disagree
- O Neutral
- O Somewhat agree
- O Strongly agree

QF2. The menu has a good variety of items

0 10 20 30 40 50 60 70 80 90 100

- O Strongly disagree
- O Somewhat disagree
- O Neutral
- O Somewhat agree
- O Strongly agree

QF3. The quality of food is excellent

0 10 20 30 40 50 60 70 80 90 100

- Strongly disagree
- Somewhat disagree
- Neutral
- Somewhat agree
- Strongly agree

QF4. The food is tasty and flavorful

0 10 20 30 40 50 60 70 80 90 100

- Strongly disagree
- Somewhat disagree
- Neutral
- Somewhat agree
- Strongly agree

QF5. The quality of beverage is good

0 10 20 30 40 50 60 70 80 90 100

- Strongly disagree
- Somewhat disagree
- Neutral
- Somewhat agree
- Strongly agree

❖ **Secondly, you will reply (in a similar double way) to questions concerning the satisfaction with the restaurant service.**

QR1. My food order was correct and complete

0 10 20 30 40 50 60 70 80 90 100

- Strongly disagree
- Somewhat disagree
- Neutral
- Somewhat agree
- Strongly agree

QR2. Employees are patient when taking my order

0 10 20 30 40 50 60 70 80 90 100

- Strongly disagree
- Somewhat disagree
- Neutral
- Somewhat agree
- Strongly agree

QR3. I was served promptly

0 10 20 30 40 50 60 70 80 90 100

- Strongly disagree
- Somewhat disagree
- Neutral
- Somewhat agree
- Strongly agree

QR4. Good availability of sauces, utensils, napkins,...

0 10 20 30 40 50 60 70 80 90 100

- ○ Strongly disagree
- ○ Somewhat disagree
- ○ Neutral
- ○ Somewhat agree
- ○ Strongly agree

QR5. The menu board was easy to read

0 10 20 30 40 50 60 70 80 90 100

- ○ Strongly disagree
- ○ Somewhat disagree
- ○ Neutral
- ○ Somewhat agree
- ○ Strongly agree

QR6. Employees are friendly and courteous

0 10 20 30 40 50 60 70 80 90 100

- ○ Strongly disagree
- ○ Somewhat disagree
- ○ Neutral
- ○ Somewhat agree
- ○ Strongly agree

QR7. The service is excellent

0 10 20 30 40 50 60 70 80 90 100

- ○ Strongly disagree
- ○ Somewhat disagree
- ○ Neutral
- ○ Somewhat agree
- ○ Strongly agree

QR8. Good cleanness of the restaurant and service

0 10 20 30 40 50 60 70 80 90 100

- ○ Strongly disagree
- ○ Somewhat disagree
- ○ Neutral
- ○ Somewhat agree
- ○ Strongly agree

❖ **Thirdly, you will reply (in a similar double way) to a single question concerning the price of the restaurant.**

QP1. Prices are competitive

0 10 20 30 40 50 60 70 80 90 100

- ○ Strongly disagree
- ○ Somewhat disagree
- ○ Neutral
- ○ Somewhat agree
- ○ Strongly agree

Table B.1: Dataset for the "About yourself" items (Page 243) from the adapted restaurant customer satisfaction Questionnaire

ID	Gender	Age (years)	Occupation	Restaurant type	# days/month	average price/lunch (€)	lunch interval time
ID1	Female	< 25	Training	Fast food	6	4	$21:00-22:00$
ID2	Male	> 64	None/retired	Casual	1	20	$14:30-16:30$
ID3	Female	45-54	Housework	Casual	1	15	$14:30-16:00$
ID4	Female	< 25	Training	Casual	2	20	$21:00-23:00$
ID5	Female	< 25	Training	Fine	2	45	$21:00-23:00$
ID6	Male	25-34	Working	Casual	6	14	$22:30-00:30$
ID7	Male	< 25	Working	Fine	3	100	$14:15-16:30$
ID8	Female	55-64	Working	Casual	3	15	$14:30-15:30$
ID9	Female	55-64	Housework	Casual	1	15	$14:00-15:15$
ID10	Male	25-34	Training	Casual	3	20	$13:30-14:30$
ID11	Male	< 25	Both	Casual	4	7	$14:00-15:00$
ID12	Male	25-34	Both	Self-service	8	11	$15:00-16:00$
ID13	Female	25-34	Both	Casual	2	17	$14:00-16:00$
ID14	Male	< 25	Training	Casual	2	10	$14:30-15:10$
ID15	Female	25-34	Both	Self-service	4	12	$14:00-15:00$
ID16	Male	55-64	None/retired	Casual	2	15	$14:00-15:15$
ID17	Female	25-34	Training	Casual	30	6	$14:30-15:30$
ID18	Female	< 25	Training	Casual	4	15	$14:00-15:00$
ID19	Female	< 25	Training	Casual	5	15	$13:00-14:00$
ID20	Female	< 25	Training	Casual	1	17	$13:00-13:30$
ID21	Male	55-64	Working	Fine	3	20	$13:30-14:00$
ID22	Male	25-34	Working	Casual	4	20	
ID23	Female	25-34	Working	Fine	4	22	$12:30-14:00$
ID24	Male	25-34	Working	Casual	4	20	$12:30-13:30$
ID25	Female	25-34	Working	Casual	4	20	$13:00-14:00$
ID26	Male	35-44	Working	Casual	2	25	$12:00-13:00$
ID27	Male	35-44	Working	Casual	4	30	
ID28	Male	35-44	Working	Casual	5	25	$12:30-13:30$
ID29	Female	25-34	Working	Casual	2	15	
ID30	Female	25-34	Working	Casual	4	13	$13:00-14:30$
ID31	Female	< 25	Working	Fine	6	30	$12:30-13:30$
ID32	Male	25-34	Training	Casual	5	20	$13:00-14:00$
ID33	Male	< 25	Training	Casual	8	15	$13:00-14:00$
ID34	Male	55-64	None/retired	Casual	1		$14:00-15:30$
ID35	Male	25-34		Fast food	10	5	$14:00-16:00$
ID36	Female	25-34	Working	Casual	2	15	$14:00-15:00$
ID37	Female	25-34	Working	Casual	4	20	$14:00-15:30$
ID38	Female	35-44	Working	Casual	1	15	$14:30-15:00$
ID39	Male	< 25	Both	Casual	3	9	$14:00-15:00$
ID40	Male	< 25	Training	Casual	20	7.5	$14:00-15:00$
ID41	Male	< 25	Both	Fast food	5	5	$13:00-14:00$
ID42	Male	25-34	Working	Self-service	8	8	$13:00-15:00$
ID43	Female	< 25	Training	Fast food	8	6	$14:00-15:00$
ID44	Male	25-34	Working	Casual	5	12	$14:00-15:00$
ID45	Female	25-34	Working	Casual	5	8	$12:00-13:00$
ID46	Female	< 25	Training	Casual	5	8	$12:00-13:00$
ID47	Male	< 25	Training	Fast food	6	4	$14:00-15:00$
ID48	Male	< 25	Both	Casual	12	7.5	$14:00-15:00$
ID49	Male	25-34	Training	Casual	20	7	$14:30-15:30$
ID50	Male	< 25	Both	Casual	4	7.5	$14:00-15:00$
ID51	Male	25-34	Training	Casual	8	10	$13:00-14:00$
ID52	Female	25-34	Training	Casual	20	7	$14:30-15:30$
ID53	Male	25-34	Training	Casual	18	8	$14:30-15:30$
ID54	Male	35-44	Both	Fine	8	12	$20:00-21:00$
ID55	Female	> 64	None/retired	Fine	10	15	$12:30-14:00$
ID56	Female	25-34	Both	Self-service	2	10	$13:00-14:00$
ID57	Female	25-34	Training	Fast food	16	5	$21:00-22:00$
ID58	Male	25-34	Training	Casual	24	3	$13:30-14:30$
ID59	Male	25-34	Training	Casual	20	7	$14:30-15:30$
ID60	Female	25-34	Training	Casual	18	7	$14:30-14:30$
ID61	Male	35-44	Working	Casual	20	8	$13:00-15:00$
ID62	Female	< 25	Training	Fast food	15	4	$13:00-14:00$
ID63	Female	45-54	Working	Fine	20	5	$13:00-15:00$
ID64	Male	45-54	None/retired	Casual	20	7	$15:00-16:00$
ID65	Female	> 64	None/retired	Fine	10	14	$12:30-14:00$
ID66	Female	35-44	None/retired	Casual	24	5	$13:00-14:00$
ID67	Female	35-44	Working	Casual	24	5.5	$13:00-14:00$
ID68	Female	25-34	Working	Casual	4	10	$14:00-15:00$
ID69	Male	55-64	Working	Casual	20	9	$14:00-15:00$
ID70	Male	25-34	Both	Fast food	6	10	$21:00-23:00$

Table B.2: Dataset for the "About your opinion/valuation/rating on the quality of food and beverage" items (Pages 243 and 244) from the adapted restaurant customer satisfaction Questionnaire (SD = strongly disagree, sD = somewhat disagree, N = neutral, sA = somewhat agree, SA = strongly agree, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$)

ID	$QF1$ Likert	$QF1$ (a_i, b_i, c_i, d_i)	$QF2$ Likert	$QF2$ (a_i, b_i, c_i, d_i)	$QF3$ Likert	$QF3$ (a_i, b_i, c_i, d_i)	$QF4$ Likert	$QF4$ (a_i, b_i, c_i, d_i)	$QF5$ Likert	$QF5$ (a_i, b_i, c_i, d_i)
ID1	sA	(70, 80, 90, 100)	SA	(90, 95, 100, 100)	sA	(70, 75, 85, 90)	sA	(75, 80, 85, 90)	sA	(75, 80, 85, 90)
ID2	SA	(80, 90, 90, 100)	SA	(80, 80, 80, 90)	sA	(80, 80, 80, 100)	sA	(80, 90, 90, 100)	SA	(100, 100, 100, 100)
ID3	sA	(70, 80, 80, 90)	sA	(80, 85, 85, 90)	N	(80, 80, 90, 100)	sA	(80, 90, 90, 100)	SA	(100, 100, 100, 100)
ID4	sA	(70, 75, 80, 80)	N	(30, 35, 35, 50)	SA	(80, 80, 90, 90)	SA	(80, 85, 85, 90)	N	(40, 50, 50, 60)
ID5	SA	(80, 95, 95, 100)	sA	(60, 65, 70, 80)	SA	(80, 100, 100, 100)	SA	(90, 100, 100, 100)	sA	(50, 65, 65, 70)
ID6	sA	(60, 65, 75, 80)	SA	(70, 80, 85, 90)	sA	(70, 70, 80, 90)	sA	(60, 70, 80, 80)	sA	(70, 80, 80, 80)
ID7	SA	(90, 98, 100, 100)	sA	(70, 75, 80, 90)	SA	(95, 99, 100, 100)	sA	(65, 70, 85, 90)	SA	(70, 80, 100, 100)
ID8	SA	(80, 90, 100, 100)	SA	(75, 80, 90, 100)	SA	(90, 95, 100, 100)	SA	(80, 90, 100, 100)	SA	(80, 85, 95, 100)
ID9	sA	(60, 70, 80, 90)	N	(30, 40, 60, 70)	sA	(50, 60, 80, 90)	sA	(50, 60, 80, 90)	N	(40, 50, 70, 80)
ID10	sA	(60, 70, 90, 100)	sA	(50, 60, 70, 80)	sA	(60, 70, 90, 100)	N	(40, 50, 60, 70)	N	(50, 60, 70, 80)
ID11	sA	(61, 70, 70, 79)	sD	(25, 30, 40, 48)	sA	(64, 70, 80, 88)	sA	(63, 70, 80, 86)	N	(44, 49, 56, 60)
ID12	sA	(30, 50, 68, 79)	SA	(70, 81, 100, 100)	N	(40, 51, 71, 80)	N	(41, 59, 59, 79)	N	(30, 40, 60, 70)
ID13	sA	(51, 61, 70, 80)	N	(40, 50, 71, 80)	N	(31, 49, 60, 71)	sA	(50, 60, 70, 80)	sA	(40, 59, 70, 80)
ID14	sA	(61, 70, 90, 100)	SA	(72, 80, 89, 100)	N	(50, 60, 90, 100)	sA	(60, 70, 78, 91)	N	(31, 40, 69, 80)
ID15	SA	(60, 70, 81, 90)	sA	(60, 69, 80, 90)	sA	(61, 70, 79, 80)	N	(30, 40, 50, 59)	SA	(91, 100, 100, 100)
ID16	N	(40, 50, 70, 86)	sA	(44, 56, 76, 87)	sA	(50, 63, 74, 86)	N	(45, 55, 76, 83)	sA	(48, 58, 71, 79)
ID17	sA	(60, 68, 70, 80)	N	(44, 50, 60, 80)	N	(60, 68, 75, 80)	SA	(80, 90, 100, 100)	SA	(79, 80, 100, 100)
ID18	SA	(50, 63, 86, 100)	N	(31, 39, 60, 70)	sA	(70, 80, 90, 100)	N	(30, 39, 60, 68)	SA	(70, 80, 100, 100)
ID19	SA	(82, 88, 94, 96)	SA	(95, 100, 100, 100)	SA	(80, 90, 100, 100)	sA	(75, 80, 90, 95)	sA	(70, 75, 85, 90)
ID20	SA	(70, 75, 95, 100)	SA	(70, 74, 94, 100)	SA	(70, 75, 90, 96)	SA	(70, 73, 90, 96)	SA	(65, 70, 90, 95)
ID21	SA	(70, 80, 90, 100)	SA	(68, 80, 90, 100)	SA	(60, 70, 80, 80)	SA	(60, 70, 80, 100)	sA	(50, 70, 90, 100)
ID22	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	SA	(100, 100, 100, 100)	SA	(90, 100, 100, 100)
ID23	SA	(70, 80, 90, 100)	SA	(70, 80, 90, 100)	SA	(70, 80, 90, 100)	SA	(60, 70, 80, 90)	sA	(60, 70, 80, 90)
ID24	sA	(60, 70, 80, 90)	N	(70, 80, 90, 100)	sA	(50, 60, 80, 90)	sA	(50, 60, 80, 90)	sA	(70, 80, 90, 100)
ID25	sA	(60, 70, 80, 90)	sA	(60, 65, 75, 80)	sA	(65, 70, 80, 85)	sA	(70, 80, 90, 95)	N	(45, 50, 60, 65)
ID26	sA	(40, 60, 80, 90)	SA	(80, 90, 100, 100)	SA	(80, 80, 90, 90)	sA	(70, 80, 100, 100)	sA	(60, 70, 90, 100)
ID27	sA	(30, 70, 100, 100)	SA	(60, 90, 100, 100)	sA	(20, 70, 100, 100)	sA	(20, 70, 100, 100)	SA	(40, 90, 100, 100)
ID28	N	(60, 60, 80, 80)	sA	(50, 60, 80, 90)	N	(50, 60, 80, 90)	sA	(60, 60, 80, 80)	sA	(60, 60, 80, 80)
ID29	sA	(40, 50, 80, 100)	sA	(50, 60, 90, 100)	N	(30, 40, 60, 70)	sA	(40, 50, 90, 100)	sD	(20, 30, 60, 80)
ID30	sA	(55, 65, 75, 85)	N	(40, 45, 55, 60)	sD	(20, 25, 35, 40)	sD	(20, 25, 35, 50)	sA	(60, 70, 85, 95)
ID31	SA	(80, 90, 100, 100)	SD	(0, 0, 10, 40)	sD	(50, 60, 70, 100)	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)
ID32	SA	(100, 100, 100, 100)	N	(40, 50, 60, 70)	sA	(70, 75, 85, 90)	sA	(70, 75, 85, 90)	sA	(60, 70, 80, 90)
ID33	sA	(76, 80, 85, 87)	sA	(80, 85, 92, 95)	sA	(65, 69, 81, 83)	sA	(76, 80, 95, 98)	sA	(65, 70, 85, 90)
ID34	sA	(60, 68, 73, 78)	sA	(50, 62, 70, 70)	sA	(70, 76, 83, 86)	sA	(55, 61, 66, 75)	SA	(85, 91, 96, 100)
ID35	sA	(60, 70, 88, 100)	sD	(11, 20, 40, 50)	sD	(10, 20, 39, 52)	sA	(60, 68, 90, 100)	N	(40, 40, 58, 58)
ID36	sA	(70, 80, 85, 100)	N	(50, 60, 75, 75)	N	(40, 50, 70, 75)	sA	(70, 80, 85, 95)	sA	(65, 70, 80, 85)
ID37	sA	(70, 80, 100, 100)	sA	(60, 70, 90, 100)	sA	(70, 80, 90, 100)	SA	(80, 90, 100, 100)	sA	(70, 80, 90, 100)
ID38	sA	(70, 75, 80, 90)	SA	(80, 90, 90, 95)	sA	(70, 75, 80, 85)	SA	(90, 90, 90, 100)	sA	(65, 70, 80, 85)
ID39	N	(45, 50, 50, 65)	N	(40, 50, 60, 70)	sA	(60, 70, 70, 80)	sA	(50, 60, 70, 80)	sA	(42, 50, 70, 80)
ID40	sA	(55, 60, 70, 75)	sD	(35, 40, 40, 45)	sA	(55, 60, 65, 70)	sA	(65, 70, 70, 75)	N	(45, 50, 50, 55)
ID41	N	(40, 50, 50, 60)	sD	(20, 30, 40, 50)	SD	(0, 10, 20, 30)	SA	(70, 80, 90, 100)	N	(30, 40, 50, 60)
ID42	SD	(0, 0, 0, 10)	sA	(65, 70, 80, 85)	N	(35, 40, 50, 60)	sA	(65, 70, 80, 85)	N	(45, 50, 50, 55)
ID43	sA	(55, 60, 70, 75)	N	(45, 48, 52, 55)	sD	(25, 30, 35, 40)	sA	(65, 70, 75, 80)	SA	(85, 90, 100, 100)
ID44	sA	(65, 70, 80, 85)	N	(40, 50, 65, 75)	sA	(70, 75, 85, 90)	sA	(60, 65, 75, 80)	N	(45, 50, 50, 55)
ID45	sA	(50, 60, 70, 80)	SA	(80, 85, 95, 100)	N	(30, 40, 60, 70)	sA	(60, 70, 75, 85)	sA	(50, 60, 70, 80)
ID46	sA	(55, 60, 70, 75)	sA	(60, 70, 80, 90)	sA	(55, 60, 70, 75)	sA	(55, 60, 70, 75)	sA	(65, 70, 75, 80)
ID47	SA	(70, 80, 90, 100)	sD	(25, 30, 35, 40)	SD	(0, 0, 5, 5)	sA	(65, 75, 85, 95)	sD	(20, 30, 30, 40)
ID48	N	(40, 50, 50, 60)	sA	(60, 65, 70, 75)	sA	(50, 60, 70, 80)	SA	(80, 85, 95, 100)	sA	(65, 70, 80, 85)
ID49	SA	(70, 80, 90, 100)	sD	(30, 40, 45, 55)	sA	(60, 65, 75, 80)	SA	(70, 80, 90, 90)	sA	(65, 70, 80, 85)
ID50	sA	(65, 70, 80, 85)	N	(40, 50, 60, 70)	sA	(75, 80, 90, 95)	SA	(80, 85, 95, 100)	sA	(65, 70, 80, 85)
ID51	sA	(70, 75, 80, 85)	SA	(80, 90, 100, 100)	sA	(70, 75, 85, 90)	SA	(80, 90, 100, 100)	N	(45, 50, 60, 65)
ID52	sA	(60, 70, 80, 90)	N	(50, 55, 60, 65)	N	(45, 50, 55, 60)	sA	(75, 80, 85, 90)	SA	(75, 80, 90, 95)
ID53	SA	(80, 90, 100, 100)	N	(40, 45, 55, 60)	sA	(70, 75, 85, 90)	SA	(80, 90, 95, 100)	SA	(80, 90, 95, 100)
ID54	sA	(70, 75, 80, 85)	sA	(50, 60, 70, 80)	sA	(65, 70, 80, 85)	sA	(75, 80, 85, 90)	sA	(70, 75, 80, 85)
ID55	SA	(80, 90, 100, 100)	SA	(70, 80, 90, 100)	sA	(70, 80, 90, 100)	sA	(70, 80, 90, 100)	SA	(80, 90, 100, 100)
ID56	sA	(60, 70, 80, 90)	N	(45, 50, 55, 60)	sA	(60, 70, 80, 90)	sA	(70, 80, 85, 95)	N	(40, 50, 60, 70)
ID57	sA	(60, 70, 80, 90)	sA	(60, 70, 75, 85)	N	(35, 50, 60, 75)	N	(50, 60, 70, 80)	N	(50, 60, 70, 80)
ID58	sA	(65, 75, 80, 100)	SA	(95, 100, 100, 100)	sA	(60, 80, 95, 100)	SA	(70, 95, 100, 100)	sA	(40, 50, 70, 80)
ID59	SA	(70, 80, 90, 100)	sD	(30, 40, 45, 55)	sA	(60, 65, 75, 80)	SA	(70, 80, 90, 90)	sA	(65, 70, 80, 85)
ID60	sA	(65, 80, 85, 95)	N	(20, 40, 50, 80)	sA	(30, 50, 60, 100)	sA	(30, 50, 90, 100)	sA	(40, 70, 80, 100)
ID61	SA	(80, 90, 100, 100)	sA	(40, 50, 70, 90)	N	(10, 20, 50, 80)	sA	(60, 70, 85, 100)	SA	(50, 70, 80, 90)
ID62	sA	(60, 65, 75, 80)	SD	(0, 10, 25, 50)	sA	(20, 70, 90, 95)	SA	(50, 95, 95, 100)	sA	(50, 70, 80, 100)
ID63	N	(30, 40, 90, 100)	SA	(80, 85, 90, 100)	SA	(85, 90, 95, 100)	sA	(80, 85, 100, 100)	N	(80, 90, 100, 100)
ID64	sA	(90, 100, 100, 100)	N	(30, 40, 60, 100)	SA	(40, 85, 100, 100)	SA	(50, 80, 85, 100)	sA	(50, 70, 80, 90)
ID65	SA	(50, 80, 90, 100)	SA	(40, 80, 85, 100)	sA	(40, 60, 70, 100)	SA	(70, 80, 100, 100)	SA	(80, 90, 95, 100)
ID66	sA	(40, 50, 80, 100)	sA	(40, 60, 70, 90)	sA	(30, 60, 75, 95)	SA	(80, 85, 95, 100)	sA	(75, 85, 95, 100)
ID67	sA	(75, 80, 90, 100)	SA	(50, 80, 90, 100)	SA	(40, 70, 90, 100)	SA	(70, 95, 100, 100)	sA	(55, 70, 90, 100)
ID68	SA	(85, 90, 95, 100)	sA	(60, 67, 73, 80)	sA	(70, 80, 80, 85)	SA	(80, 90, 95, 100)	sA	(65, 75, 75, 85)
ID69	sA	(75, 90, 90, 100)	N	(50, 60, 60, 60)	sA	(80, 80, 80, 100)	N	(50, 55, 60, 60)	N	(45, 50, 50, 55)
ID70	N	(37, 47, 56, 62)	sA	(51, 62, 69, 72)	sD	(25, 30, 40, 50)	SA	(90, 100, 100, 100)	sA	(45, 50, 62, 72)

Table B.3: Dataset for the "About your opinion/valuation/rating on the satisfaction with restaurant service" first five items (Pages 244 and 245) from the adapted restaurant customer satisfaction Questionnaire (SD = strongly disagree, sD = somewhat disagree, N = neutral, sA = somewhat agree, SA = strongly agree, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$)

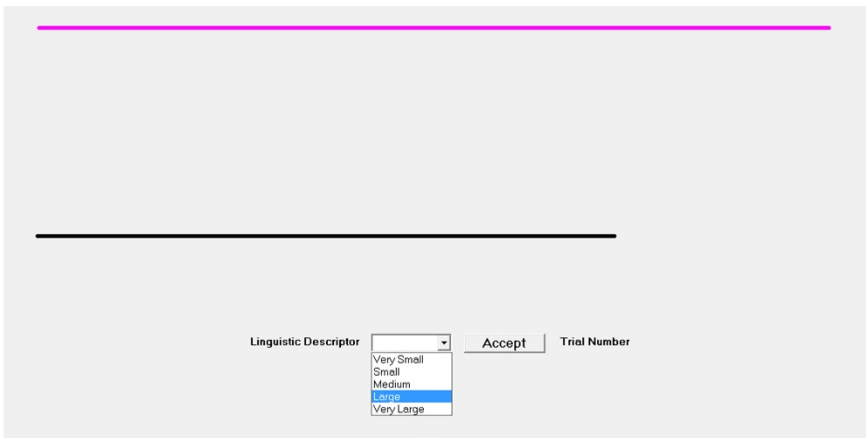
ID	$QR1$ Likert	$QR1$ (a_i, b_i, c_i, d_i)	$QR2$ Likert	$QR2$ (a_i, b_i, c_i, d_i)	$QR3$ Likert	$QR3$ (a_i, b_i, c_i, d_i)	$QR4$ Likert	$QR4$ (a_i, b_i, c_i, d_i)	$QR5$ Likert	$QR5$ (a_i, b_i, c_i, d_i)
ID1	SA	(100, 100, 100, 100)	sA	(80, 85, 90, 100)	N	(40, 50, 60, 70)	sA	(60, 70, 80, 90)	SA	(90, 95, 100, 100)
ID2	SA	(100, 100, 100, 100)	sA	(80, 90, 90, 100)	sA	(60, 70, 80, 80)	SA	(100, 100, 100, 100)	SA	(100, 100, 100, 100)
ID3	SA	(70, 70, 70, 80)	SA	(80, 90, 90, 100)	N	(70, 80, 80, 100)	sA	(60, 65, 75, 80)	SA	(80, 85, 95, 100)
ID4	N	(50, 65, 65, 70)	N	(40, 50, 50, 50)	sA	(60, 80, 80, 80)	sA	(70, 75, 75, 85)	SA	(80, 85, 85, 100)
ID5	SA	(80, 95, 95, 100)	SA	(80, 95, 95, 100)	sA	(60, 80, 80, 85)	SA	(80, 95, 95, 100)	SA	(90, 100, 100, 100)
ID6	sA	(60, 70, 80, 80)	N	(50, 50, 60, 70)	N	(40, 50, 50, 60)	sA	(60, 70, 80, 80)	N	(50, 60, 70, 70)
ID7	SA	(95, 98, 100, 100)	SA	(100, 100, 100, 100)	N	(45, 50, 60, 65)	SA	(90, 95, 100, 100)	sD	(0, 10, 29, 30)
ID8	SA	(75, 80, 90, 95)	SA	(70, 75, 95, 100)	sA	(60, 70, 80, 90)	SA	(70, 70, 90, 90)	sA	(60, 65, 75, 80)
ID9	N	(50, 65, 65, 80)	N	(53, 70, 70, 81)	N	(51, 68, 68, 81)	sA	(62, 77, 77, 90)	N	(52, 69, 69, 82)
ID10	sA	(60, 70, 80, 90)	SA	(70, 80, 100, 100)	sA	(50, 70, 80, 90)	N	(30, 50, 50, 70)	SA	(80, 90, 100, 100)
ID11	SA	(90, 100, 100, 100)	sA	(75, 80, 85, 90)	sA	(75, 80, 90, 96)	sA	(71, 78, 90, 100)	SA	(90, 100, 100, 100)
ID12	SA	(100, 100, 100, 100)	sA	(51, 69, 80, 90)	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	sD	(20, 30, 40, 49)
ID13	N	(40, 50, 60, 70)	sA	(50, 60, 70, 80)	N	(30, 50, 50, 60)	sA	(61, 71, 80, 90)	N	(30, 41, 50, 63)
ID14	sD	(0, 10, 19, 30)	N	(62, 69, 80, 89)	sA	(40, 50, 79, 91)	SD	(0, 11, 20, 38)	SD	(0, 30, 40, 60)
ID15	sA	(71, 80, 80, 90)	SA	(91, 100, 100, 100)	sA	(71, 79, 79, 91)	sA	(71, 79, 79, 91)	N	(41, 50, 60, 71)
ID16	sA	(53, 62, 75, 88)	sA	(52, 62, 79, 90)	sA	(53, 63, 79, 90)	N	(54, 63, 77, 87)	sA	(56, 66, 70, 84)
ID17	SA	(79, 91, 100, 100)	SA	(79, 90, 100, 100)	sA	(62, 70, 80, 86)	sA	(61, 67, 74, 79)	N	(41, 48, 60, 66)
ID18	sA	(80, 90, 90, 100)	sA	(70, 80, 90, 100)	N	(50, 60, 70, 80)	sA	(73, 89, 89, 100)	N	(40, 60, 90, 100)
ID19	sA	(65, 70, 80, 85)	SA	(70, 80, 90, 95)	SA	(95, 100, 100, 100)	SA	(80, 85, 95, 95)	SA	(95, 100, 100, 100)
ID20	sA	(55, 60, 80, 85)	SA	(70, 73, 95, 100)	SA	(60, 65, 84, 90)	sA	(50, 56, 76, 80)	SA	(68, 74, 95, 100)
ID21	sA	(50, 60, 80, 90)	sA	(80, 80, 80, 80)	SA	(70, 80, 90, 100)	SA	(80, 90, 100, 100)	SA	(100, 100, 100, 100)
ID22	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	sA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)
ID23	sA	(50, 60, 70, 80)	sA	(50, 60, 70, 80)	N	(40, 50, 60, 70)	N	(40, 50, 60, 70)	N	(40, 50, 60, 70)
ID24	SA	(70, 80, 90, 100)	N	(70, 80, 90, 100)	sA	(50, 60, 80, 90)	N	(30, 40, 70, 80)	SA	(80, 90, 100, 100)
ID25	SA	(80, 80, 90, 100)	SA	(75, 80, 90, 95)	SA	(75, 80, 90, 95)	N	(45, 50, 60, 65)	sA	(70, 75, 85, 90)
ID26	SA	(70, 80, 100, 100)	SA	(90, 90, 100, 100)	SA	(90, 90, 100, 100)	sA	(50, 60, 80, 80)	sA	(60, 70, 90, 100)
ID27	SA	(80, 90, 100, 100)	sA	(10, 70, 100, 100)	sA	(10, 60, 100, 100)	sA	(10, 70, 100, 100)	SA	(60, 90, 100, 100)
ID28	sA	(60, 60, 80, 80)	N	(60, 60, 80, 80)	N	(60, 60, 80, 80)	sA	(50, 60, 80, 90)	SA	(70, 70, 90, 90)
ID29	N	(40, 50, 80, 90)	N	(30, 40, 70, 80)	sA	(40, 50, 80, 90)	N	(30, 40, 70, 80)	sA	(50, 60, 90, 100)
ID30	SA	(85, 95, 100, 100)	sA	(70, 75, 87, 95)	sA	(64, 73, 86, 95)	sD	(20, 30, 40, 50)	SA	(84, 90, 100, 100)
ID31	sD	(30, 40, 50, 80)	sD	(30, 40, 50, 80)	sD	(40, 50, 60, 70)	sD	(30, 40, 60, 70)	SA	(80, 90, 100, 100)
ID32	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)	sA	(60, 70, 85, 90)	SA	(100, 100, 100, 100)	SA	(100, 100, 100, 100)
ID33	sA	(62, 70, 72, 76)	N	(52, 58, 64, 70)	sA	(65, 70, 80, 90)	SA	(100, 100, 100, 100)	SA	(100, 100, 100, 100)
ID34	SA	(80, 90, 95, 95)	sA	(75, 83, 90, 95)	sA	(84, 84, 94, 97)	SA	(85, 90, 96, 100)	SA	(89, 94, 100, 100)
ID35	SA	(90, 90, 100, 100)	SA	(90, 90, 100, 100)	sA	(62, 70, 88, 97)	N	(32, 40, 60, 70)	sA	(70, 78, 100, 100)
ID36	SA	(90, 95, 100, 100)	sA	(75, 80, 90, 95)	sA	(70, 75, 85, 90)	sA	(75, 80, 90, 100)	SA	(85, 90, 100, 100)
ID37	sA	(70, 80, 100, 100)	SA	(84, 90, 100, 100)	sA	(70, 80, 100, 100)	sA	(70, 80, 100, 100)	SA	(80, 90, 100, 100)
ID38	SA	(90, 95, 100, 100)	sA	(65, 70, 75, 80)	SA	(85, 90, 90, 100)	sA	(75, 80, 80, 85)	SA	(90, 95, 100, 100)
ID39	SA	(60, 70, 80, 90)	SA	(60, 70, 85, 90)	sA	(65, 75, 75, 85)	N	(40, 50, 50, 60)	SA	(75, 80, 90, 95)
ID40	SA	(75, 80, 90, 95)	sA	(65, 70, 80, 85)	N	(45, 50, 50, 55)	N	(45, 50, 60, 65)	SA	(70, 80, 80, 90)
ID41	SA	(80, 90, 90, 100)	sD	(10, 20, 40, 50)	sA	(70, 80, 80, 90)	sD	(10, 20, 30, 40)	sA	(70, 80, 80, 90)
ID42	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	SA	(85, 90, 95, 100)	SA	(85, 90, 95, 100)
ID43	SA	(85, 90, 100, 100)	N	(45, 50, 60, 60)	sA	(70, 75, 85, 90)	sD	(20, 20, 30, 35)	SA	(85, 87, 90, 95)
ID44	SA	(90, 100, 100, 100)	sA	(55, 60, 70, 75)	sA	(65, 70, 80, 85)	N	(35, 40, 60, 60)	sA	(80, 85, 90, 95)
ID45	SA	(100, 100, 100, 100)	sA	(50, 60, 65, 75)	sD	(20, 30, 40, 50)	sD	(25, 30, 40, 45)	SA	(100, 100, 100, 100)
ID46	sD	(20, 30, 40, 50)	N	(35, 40, 50, 55)	N	(50, 55, 60, 65)	sA	(60, 70, 80, 90)	sA	(80, 85, 90, 95)
ID47	sA	(75, 80, 90, 95)	sA	(70, 75, 85, 90)	N	(20, 25, 35, 40)	sD	(10, 15, 25, 30)	sA	(70, 75, 80, 85)
ID48	SA	(80, 100, 100, 100)	SA	(75, 80, 95, 100)	sA	(75, 80, 85, 90)	N	(35, 40, 50, 55)	sA	(80, 85, 85, 90)
ID49	SA	(80, 90, 100, 100)	sA	(65, 70, 80, 85)	sA	(70, 75, 85, 90)	sD	(10, 20, 30, 40)	SA	(80, 90, 100, 100)
ID50	SA	(90, 100, 100, 100)	SA	(90, 95, 100, 100)	sA	(70, 75, 80, 85)	N	(45, 50, 60, 65)	SA	(85, 90, 100, 100)
ID51	sA	(70, 80, 90, 100)	sD	(15, 20, 30, 35)	N	(45, 50, 60, 65)	sA	(70, 75, 85, 90)	SA	(80, 90, 100, 100)
ID52	sA	(70, 75, 80, 85)	SA	(85, 90, 95, 100)	SA	(85, 90, 100, 100)	N	(50, 55, 60, 65)	sA	(70, 80, 90, 100)
ID53	SA	(90, 90, 100, 100)	SA	(90, 90, 100, 100)	sA	(75, 80, 85, 90)	sA	(80, 85, 90, 95)	SA	(85, 90, 100, 100)
ID54	SA	(75, 80, 90, 100)	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)	sA	(70, 80, 90, 100)	sA	(75, 80, 85, 90)
ID55	sA	(70, 80, 90, 100)	sA	(70, 80, 90, 100)	sA	(70, 80, 90, 100)	SA	(80, 90, 100, 100)	sA	(70, 80, 90, 100)
ID56	SA	(90, 95, 100, 100)	sA	(75, 80, 90, 95)	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)	sA	(75, 80, 85, 90)
ID57	SA	(80, 90, 100, 100)	N	(50, 60, 70, 80)	SA	(90, 95, 100, 100)	SA	(90, 95, 100, 100)	SA	(90, 95, 100, 100)
ID58	SA	(80, 90, 100, 100)	sA	(40, 50, 80, 100)	sA	(50, 60, 70, 100)	sA	(40, 80, 85, 100)	SA	(95, 100, 100, 100)
ID59	SA	(80, 90, 100, 100)	sA	(65, 70, 80, 85)	sA	(70, 75, 85, 90)	sD	(10, 20, 30, 40)	SA	(80, 90, 100, 100)
ID60	SA	(70, 90, 95, 100)	sA	(70, 80, 90, 100)	sA	(50, 80, 90, 95)	N	(20, 50, 60, 70)	SA	(75, 90, 95, 100)
ID61	sA	(80, 90, 100, 100)	N	(60, 90, 100, 100)	sA	(40, 50, 70, 80)	N	(30, 60, 80, 90)	SA	(90, 100, 100, 100)
ID62	SA	(80, 90, 100, 100)	sA	(80, 90, 100, 100)	N	(80, 90, 100, 100)	sA	(50, 80, 90, 100)	SA	(90, 100, 100, 100)
ID63	SA	(80, 90, 100, 100)	sA	(85, 95, 100, 100)	sA	(70, 85, 95, 100)	SA	(80, 90, 100, 100)	SA	(70, 80, 90, 100)
ID64	SA	(50, 70, 90, 100)	SA	(70, 90, 100, 100)	SA	(50, 60, 80, 90)	sA	(50, 60, 80, 100)	SA	(95, 100, 100, 100)
ID65	sA	(90, 100, 100, 100)	sA	(70, 85, 90, 100)	sA	(80, 85, 90, 100)	SA	(85, 95, 100, 100)	sA	(70, 80, 90, 100)
ID66	sA	(80, 90, 100, 100)	sA	(80, 90, 100, 100)	sA	(60, 70, 80, 90)	N	(70, 80, 85, 90)	SA	(90, 95, 95, 100)
ID67	SA	(90, 95, 100, 100)	sA	(60, 80, 100, 100)	sA	(50, 65, 70, 80)	sA	(50, 80, 90, 100)	SA	(50, 100, 100, 100)
ID68	sA	(60, 70, 70, 80)	SA	(85, 95, 100, 100)	sA	(65, 75, 80, 90)	SA	(85, 95, 95, 100)	SA	(90, 95, 100, 100)
ID69	SA	(95, 100, 100, 100)	SA	(90, 90, 90, 95)	SA	(100, 100, 100, 100)	sA	(75, 75, 75, 80)	SA	(90, 95, 95, 100)
ID70	SA	(95, 100, 100, 100)	sA	(60, 70, 80, 90)	SA	(85, 91, 97, 99)	SA	(85, 91, 95, 100)	SA	(70, 85, 90, 90)

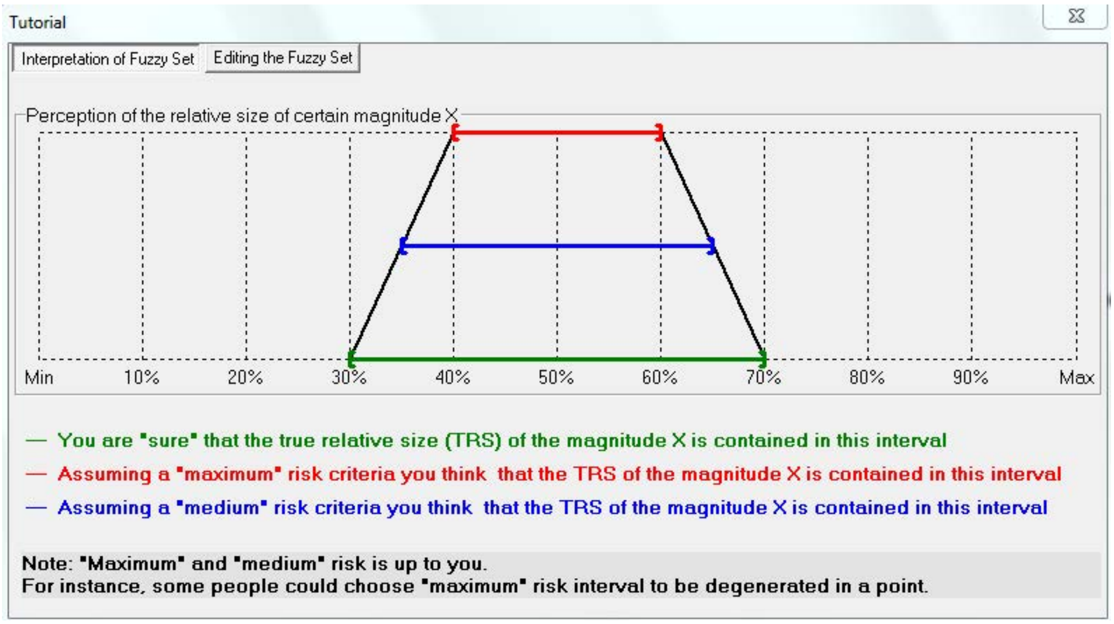
Table B.4: Dataset for the "About your opinion/valuation/rating on the satisfaction with restaurant service" last three and "restaurant price" items (Page 245) from the adapted restaurant customer satisfaction Questionnaire (SD = strongly disagree, sD = somewhat disagree, N = neutral, sA = somewhat agree, SA = strongly agree, and $(a_i, b_i, c_i, d_i) \equiv \text{Tra}(a_i, b_i, c_i, d_i)$)

ID	$QR6$ Likert	$QR6$ (a_i, b_i, c_i, d_i)	$QR7$ Likert	$QR7$ (a_i, b_i, c_i, d_i)	$QR8$ Likert	$QR8$ (a_i, b_i, c_i, d_i)	$QP1$ Likert	$QP1$ (a_i, b_i, c_i, d_i)
ID1	sA	(65, 70, 80, 85)	N	(60, 65, 75, 80)	SA	(80, 90, 95, 100)	SA	(90, 95, 100, 100)
ID2	sA	(90, 90, 90, 90)	sA	(100, 100, 100, 100)	sA	(100, 100, 100, 100)	sA	(80, 80, 80, 80)
ID3	N	(80, 85, 90, 100)	N	(60, 75, 80, 90)	sA	(70, 80, 85, 90)	sA	(80, 85, 95, 100)
ID4	N	(40, 45, 45, 60)	sA	(80, 85, 85, 90)	SA	(90, 90, 90, 100)	SA	(80, 90, 90, 100)
ID5	SA	(90, 100, 100, 100)	sA	(80, 90, 90, 100)	SA	(90, 100, 100, 100)	SA	(80, 90, 90, 100)
ID6	sD	(40, 40, 50, 60)	N	(50, 60, 70, 70)	N	(50, 60, 60, 70)	N	(50, 60, 70, 70)
ID7	SA	(100, 100, 100, 100)	SA	(90, 95, 100, 100)	SA	(100, 100, 100, 100)	N	(50, 50, 70, 70)
ID8	SA	(70, 80, 90, 100)	SA	(80, 85, 95, 100)	SA	(70, 80, 90, 100)	SA	(70, 80, 90, 100)
ID9	sA	(60, 76, 76, 90)	sA	(60, 76, 76, 93)	N	(52, 69, 69, 79)	SA	(71, 85, 100, 100)
ID10	SA	(70, 90, 100, 100)	N	(40, 50, 60, 80)	N	(30, 48, 60, 80)	SA	(69, 90, 100, 100)
ID11	SA	(83, 90, 100, 100)	SA	(85, 95, 100, 100)	N	(53, 60, 65, 71)	sA	(69, 75, 75, 80)
ID12	SA	(71, 90, 100, 100)	N	(40, 50, 50, 70)	N	(40, 50, 50, 61)	SA	(61, 79, 100, 100)
ID13	N	(30, 41, 50, 60)	N	(40, 51, 60, 70)	N	(30, 40, 49, 61)	N	(30, 42, 50, 61)
ID14	SA	(60, 69, 79, 100)	SA	(50, 59, 68, 100)	sA	(51, 59, 68, 80)	SA	(70, 80, 91, 100)
ID15	SA	(81, 90, 100, 100)	sA	(60, 71, 80, 90)	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)
ID16	sA	(45, 54, 75, 85)	N	(46, 57, 69, 82)	N	(56, 67, 71, 84)	sA	(40, 50, 69, 81)
ID17	sA	(60, 68, 80, 86)	sA	(60, 67, 70, 86)	SA	(81, 91, 100, 100)	SA	(83, 90, 100, 100)
ID18	SA	(81, 90, 100, 100)	sA	(62, 70, 90, 100)	N	(28, 50, 69, 80)	sA	(57, 70, 90, 100)
ID19	SA	(95, 100, 100, 100)	SA	(85, 90, 100, 100)	SA	(70, 80, 90, 95)	SA	(80, 85, 95, 100)
ID20	sA	(60, 65, 85, 90)	sA	(60, 65, 85, 90)	sA	(55, 60, 80, 85)	SA	(65, 70, 90, 95)
ID21	SA	(70, 80, 90, 100)	sA	(60, 70, 80, 90)	sA	(60, 70, 80, 90)	SA	(80, 90, 100, 100)
ID22	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)	SA	(90, 100, 100, 100)
ID23	N	(40, 50, 60, 70)	N	(50, 60, 70, 80)	sA	(60, 70, 80, 90)	sA	(60, 70, 80, 90)
ID24	N	(40, 60, 80, 100)	N	(60, 70, 80, 90)	sA	(80, 90, 100, 100)	sA	(50, 60, 90, 100)
ID25	sA	(75, 80, 90, 95)	sA	(70, 75, 85, 90)	sA	(70, 75, 85, 90)	sA	(60, 70, 70, 80)
ID26	SA	(90, 90, 100, 100)	sA	(60, 70, 90, 100)	sA	(40, 60, 80, 80)	N	(10, 40, 50, 80)
ID27	N	(10, 60, 100, 100)	sA	(20, 80, 100, 100)	sA	(20, 80, 100, 100)	sD	(10, 40, 100, 100)
ID28	N	(80, 80, 100, 100)	N	(70, 70, 90, 90)	N	(70, 70, 90, 90)	sA	(70, 70, 90, 90)
ID29	sD	(20, 30, 60, 90)	N	(20, 40, 60, 90)	sD	(10, 20, 60, 80)	sA	(40, 50, 80, 90)
ID30	sA	(60, 75, 90, 97)	N	(34, 44, 55, 67)	sD	(10, 15, 35, 40)	sD	(10, 20, 30, 40)
ID31	sD	(40, 50, 60, 100)	sD	(50, 60, 70, 100)	SA	(80, 90, 100, 100)	sD	(10, 20, 40, 70)
ID32	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)	SA	(70, 80, 90, 100)
ID33	sA	(74, 80, 92, 96)	sA	(70, 75, 80, 86)	SA	(85, 90, 100, 100)	sA	(65, 70, 80, 87)
ID34	N	(60, 64, 70, 74)	N	(63, 70, 77, 80)	sA	(82, 87, 94, 94)	sA	(71, 74, 80, 87)
ID35	SA	(80, 88, 100, 100)	sA	(60, 70, 78, 90)	sA	(62, 70, 84, 90)	SA	(70, 80, 100, 100)
ID36	sA	(80, 85, 95, 95)	N	(50, 55, 65, 70)	sA	(65, 70, 80, 85)	N	(55, 60, 70, 70)
ID37	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)	SA	(80, 90, 100, 100)	sA	(70, 80, 90, 100)
ID38	SA	(85, 90, 100, 100)	SA	(80, 90, 95, 100)	sA	(70, 75, 80, 85)	SA	(90, 90, 95, 100)
ID39	SA	(75, 80, 90, 100)	sA	(62, 75, 85, 91)	sA	(55, 60, 75, 80)	SA	(75, 85, 85, 90)
ID40	SA	(70, 80, 80, 90)	sA	(55, 60, 60, 65)	sA	(65, 70, 80, 85)	sA	(55, 60, 70, 75)
ID41	SD	(0, 10, 20, 30)	SD	(0, 10, 10, 20)	N	(40, 50, 60, 70)	SA	(90, 100, 100, 100)
ID42	N	(45, 50, 60, 65)	N	(45, 50, 60, 65)	sD	(15, 20, 25, 30)	sA	(65, 70, 80, 85)
ID43	N	(45, 48, 52, 55)	N	(45, 48, 52, 55)	sD	(25, 30, 40, 45)	SA	(85, 90, 100, 100)
ID44	sA	(65, 70, 80, 85)	sA	(70, 75, 85, 90)	sA	(75, 80, 80, 85)	N	(45, 50, 65, 70)
ID45	sA	(60, 65, 75, 80)	sA	(60, 65, 75, 80)	sA	(60, 70, 80, 90)	sA	(65, 70, 80, 85)
ID46	sD	(10, 20, 30, 35)	N	(45, 50, 50, 55)	SA	(80, 90, 100, 100)	sA	(60, 70, 80, 90)
ID47	SD	(0, 0, 0, 20)	sD	(10, 20, 30, 40)	N	(10, 20, 20, 30)	SA	(90, 100, 100, 100)
ID48	sA	(60, 65, 70, 75)	sA	(65, 70, 75, 80)	sA	(70, 80, 80, 80)	sA	(70, 80, 90, 100)
ID49	SA	(90, 95, 100, 100)	sA	(60, 70, 80, 90)	SA	(75, 80, 90, 95)	SA	(85, 90, 100, 100)
ID50	SA	(75, 80, 90, 95)	SA	(75, 80, 90, 95)	N	(45, 50, 60, 65)	SA	(85, 90, 100, 100)
ID51	sD	(15, 20, 30, 35)	N	(45, 50, 60, 65)	sA	(65, 70, 80, 85)	sD	(10, 20, 30, 40)
ID52	SA	(85, 90, 100, 100)	sA	(75, 80, 85, 90)	sA	(70, 75, 80, 85)	SA	(80, 90, 100, 100)
ID53	SA	(90, 95, 100, 100)	SA	(80, 85, 95, 100)	SA	(75, 80, 90, 95)	SA	(85, 90, 95, 100)
ID54	sA	(70, 75, 80, 85)	sA	(75, 80, 85, 90)	SA	(80, 90, 100, 100)	sA	(70, 80, 90, 100)
ID55	sA	(70, 80, 90, 100)	sA	(70, 80, 90, 100)	SA	(80, 90, 100, 100)	N	(50, 60, 70, 80)
ID56	sA	(75, 80, 85, 90)	sA	(75, 80, 85, 90)	sA	(75, 80, 85, 90)	SA	(80, 90, 100, 100)
ID57	sA	(60, 70, 80, 90)	N	(40, 50, 60, 70)	sA	(60, 70, 80, 90)	sA	(70, 80, 90, 100)
ID58	N	(40, 70, 85, 100)	sA	(50, 70, 80, 100)	sA	(50, 70, 80, 100)	N	(30, 50, 60, 100)
ID59	SA	(90, 95, 100, 100)	sA	(60, 70, 80, 90)	SA	(75, 80, 90, 95)	SA	(85, 90, 100, 100)
ID60	SA	(85, 90, 95, 100)	sA	(60, 70, 90, 100)	SA	(80, 90, 95, 100)	SA	(90, 95, 100, 100)
ID61	sA	(70, 80, 90, 100)	sA	(50, 80, 95, 100)	SA	(50, 70, 80, 100)	SA	(50, 80, 90, 100)
ID62	SA	(80, 90, 100, 100)	sA	(50, 90, 100, 100)	SA	(40, 80, 90, 100)	sA	(80, 90, 100, 100)
ID63	SA	(90, 95, 100, 100)	SA	(80, 90, 100, 100)	SA	(90, 100, 100, 100)	SA	(50, 70, 80, 90)
ID64	SA	(70, 80, 95, 100)	SA	(50, 70, 80, 100)	SA	(80, 90, 100, 100)	SA	(80, 90, 90, 100)
ID65	sA	(60, 80, 85, 100)	sA	(70, 80, 95, 100)	SA	(85, 95, 100, 100)	SA	(85, 90, 100, 100)
ID66	SA	(50, 90, 100, 100)	sA	(50, 80, 90, 100)	sA	(80, 85, 95, 100)	sA	(50, 80, 85, 100)
ID67	SA	(80, 95, 100, 100)	sA	(50, 60, 70, 90)	SA	(65, 80, 90, 100)	sA	(60, 70, 80, 100)
ID68	SA	(95, 100, 100, 100)	sA	(70, 80, 80, 85)	sA	(65, 75, 80, 84)	sA	(70, 77, 85, 90)
ID69	SA	(100, 100, 100, 100)	sA	(75, 80, 80, 80)	sA	(80, 80, 80, 85)	SA	(100, 100, 100, 100)
ID70	N	(40, 50, 50, 60)	N	(30, 50, 55, 70)	sA	(50, 60, 70, 70)	sD	(30, 34, 38, 45)

Appendix C

Computerized application and datasets for the online double response perceptions question




Tutorial
Interpretation of Fuzzy Set
Editing the Fuzzy Set
Perception of the relative size of certain magnitude X
Min
10%
20%
30%
40%
50%
60%
70%
80%
90%
Max
— You are "sure" that the true relative size (TRS) of the magnitude X is contained in this interval
— Assuming a "maximum" risk criteria you think that the TRS of the magnitude X is contained in this interval
— Assuming a "medium" risk criteria you think that the TRS of the magnitude X is contained in this interval
Note: "Maximum" and "medium" risk is up to you.
For instance, some people could choose "maximum" risk interval to be degenerated in a point.

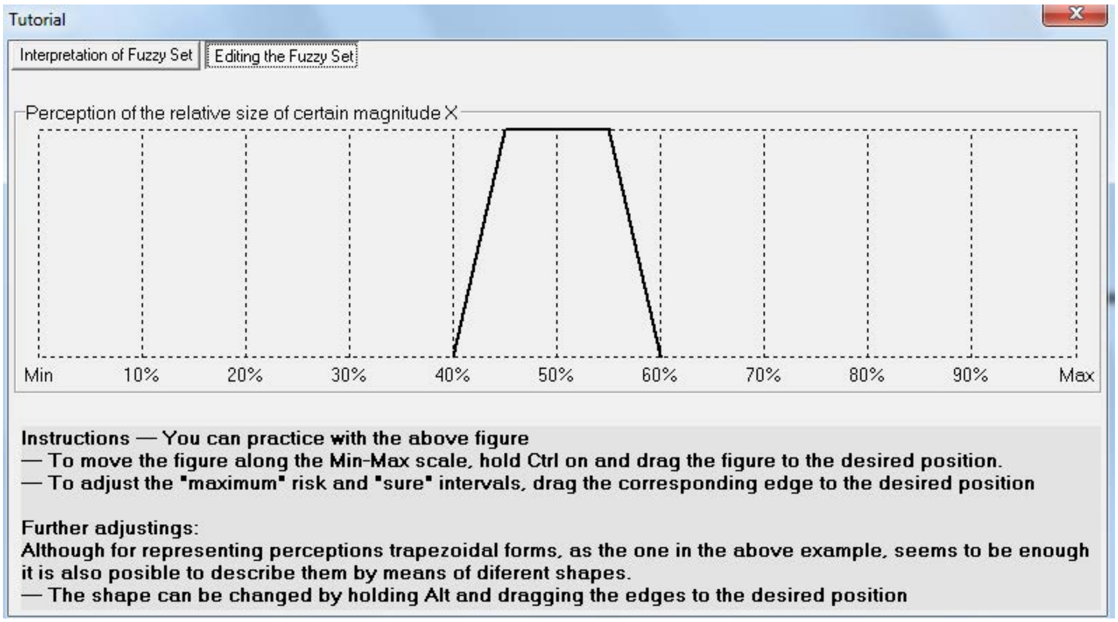

Tutorial
Interpretation of Fuzzy Set
Editing the Fuzzy Set
Perception of the relative size of certain magnitude X
Min
10%
20%
30%
40%
50%
60%
70%
80%
90%
Max
Instructions — You can practice with the above figure
— To move the figure along the Min-Max scale, hold Ctrl on and drag the figure to the desired position.
— To adjust the "maximum" risk and "sure" intervals, drag the corresponding edge to the desired position
Further adjustings:
Although for representing perceptions trapezoidal forms, as the one in the above example, seems to be enough
it is also posible to describe them by means of diferent shapes.
— The shape can be changed by holding Alt and dragging the edges to the desired position

Table C.1: Dataset for the online application Perceptions (Page 251) on the relative length of a given line segment with respect to a fixed longer one (VS = very small, S = small, M = medium, L = large, VL = very large, and $(a_i, b_i, c_i, d_i) \equiv \text{Tra}(a_i, b_i, c_i, d_i)$); data ID1-ID6

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID1	Male	38.39	S	(28.29, 28.92, 31.26, 33.69)
		3.6	VS	(3.6, 4.59, 5.95, 9.91)
		84.62	L	(80.09, 84.05, 87.21, 90.09)
		15.16	S	(9.91, 11.98, 16.67, 17.48)
		49.95	M	(41.71, 44.95, 51.35, 51.35)
		26.72	S	(23.78, 23.96, 27.03, 29.73)
		73.06	L	(74.14, 76.22, 78.65, 80.18)
		61.4	M	(57.03, 59.46, 59.91, 65.14)
		96.29	VL	(97.39, 99.64, 99.64, 100)
		15.16	S	(7.66, 9.82, 11.53, 12.07)
		3.6	VS	(1.89, 2.52, 3.15, 3.6)
		49.95	M	(37.57, 39.55, 40.99, 43.06)
ID2	Male	61.47	M	(56.72, 63.03, 71.59, 74.13)
		26.75	VS	(16.6, 18.74, 26.88, 31.67)
		15.09	VS	(4.48, 9.47, 14.66, 19.55)
		49.94	M	(43.99, 47.05, 52.34, 55.7)
		73.01	L	(77.49, 83.4, 88.49, 93.48)
		96.2	VL	(89.92, 95.01, 98.27, 100)
		3.68	VS	(0, 2.65, 6.01, 9.98)
		38.28	S	(25.56, 30.55, 40.63, 45.62)
		84.66	VL	(83.4, 87.37, 91.65, 95.82)
		49.94	M	(39.92, 47.56, 52.65, 59.98)
		38.28	S	(27.6, 31.16, 37.98, 40.02)
		84.66	VL	(82.99, 85.74, 89.92, 93.38)
		73.01	L	(75.36, 80.24, 83.71, 87.68)
		61.47	M	(45.01, 50, 58.96, 63.03)
		96.2	VL	(89.92, 93.38, 95.01, 99.29)
		26.75	VS	(12.22, 15.27, 19.86, 22.2)
		3.68	VS	(0, 2.65, 6.72, 8.25)
		15.09	VS	(5.5, 7.74, 13.14, 14.26)
		15.09	VS	(4.89, 9.88, 14.05, 17.41)
		3.68	VS	(0, 1.93, 4.89, 6.62)
		26.75	S	(18.64, 23.01, 25.05, 27.6)
		84.66	VL	(80.45, 85.03, 89, 95.32)
		73.01	L	(71.89, 76.27, 83.71, 87.78)
		96.2	VL	(91.96, 94.5, 98.27, 100)
		49.94	M	(41.34, 45.01, 54.18, 56.92)
		61.47	M	(57.43, 59.57, 67.11, 69.35)
		38.28	S	(30.04, 34.01, 41.14, 44.91)
		10.18	VS	(3.16, 8.25, 13.24, 14.97)
		28.1	S	(12.73, 16.29, 23.01, 25.05)
		85.28	VL	(78.21, 83.3, 90.73, 95.82)
		71.9	L	(73.73, 78.31, 86.15, 88.7)
		73.37	L	(69.35, 73.73, 78.92, 81.36)
		13.99	VS	(5.8, 9.06, 14.36, 16.7)
		42.94	S	(27.09, 29.33, 37.07, 40.02)
		43.56	S	(31.67, 34.42, 40.63, 42.36)
		48.83	M	(40.63, 43.48, 49.49, 53.05)
		52.52	M	(36.25, 41.24, 47.76, 50.2)
		87.12	VL	(87.47, 89.51, 92.77, 96.44)
		22.82	S	(16.8, 20.47, 26.17, 29.23)
		94.85	VL	(90.22, 93.18, 96.95, 100)
		37.79	S	(34.32, 38.19, 43.08, 46.13)
		26.63	S	(23.83, 27.09, 33.6, 34.83)
		78.9	VL	(79.23, 81.57, 86.86, 89.61)
		97.06	VL	(87.47, 90.73, 94.4, 98.37)
		64.05	M	(44.3, 46.33, 56.01, 59.47)
		53.5	M	(43.79, 49.59, 56.11, 59.16)
		24.05	S	(16.7, 18.43, 23.32, 25.66)
		22.58	S	(13.85, 18.53, 22.3, 24.95)
		13.01	VS	(3.67, 7.03, 11.3, 15.07)
		21.47	S	(10.39, 12.83, 16.6, 21.38)
ID3	Female	26.76	S	(20.19, 23.99, 27.3, 29.81)
		15.2	S	(13.57, 18.34, 21.57, 24.72)
		61.47	L	(50.24, 56.54, 63.41, 69.71)
		96.27	VL	(89.9, 92.16, 96.61, 100)
		84.61	VL	(80.13, 84.25, 92.57, 95.8)
		73.04	VL	(59.94, 64.54, 75.12, 79.81)
		50	M	(41.36, 46.77, 53.39, 58.08)
		38.33	S	(35.14, 39.98, 45.72, 49.68)
		3.63	VS	(0, 0.73, 8, 9.85)
		26.76	S	(20.11, 21.81, 27.95, 30.21)
		61.47	L	(53.47, 56.14, 64.86, 69.87)
		3.63	VS	(0, 0.97, 7.19, 9.85)
		15.2	S	(4.6, 8, 15.75, 19.87)
		50	M	(44.43, 46.37, 53.39, 55.09)
		38.33	S	(34.65, 38.05, 42.73, 45.23)
		73.04	L	(70.19, 74.15, 82.07, 84.25)
		96.27	VL	(89.98, 91.52, 97.98, 100)
		84.61	VL	(73.59, 79.97, 86.83, 95.32)
		61.47	L	(56.22, 61.07, 71.16, 75.85)
		15.2	VS	(10.18, 15.99, 24.47, 30.05)
		84.61	VL	(74.64, 80.13, 89.9, 94.67)
		73.04	L	(64.3, 70.03, 78.51, 84.49)
		96.27	VL	(85.06, 86.83, 95.32, 100)
		26.76	S	(20.19, 23.67, 31.42, 34.81)
		38.33	L	(30.21, 35.14, 45.8, 49.92)
		50	M	(40.15, 44.99, 54.44, 59.94)
		3.63	VS	(0.48, 2.91, 10.18, 14.94)
		43.63	S	(30.13, 34.17, 44.99, 49.92)
		11.57	VS	(4.68, 10.02, 19.95, 25.93)
		73.33	VL	(69.95, 71.49, 78.84, 79.89)
		18.24	S	(9.85, 15.19, 24.23, 29.89)
		85.78	VL	(79.89, 81.99, 89.9, 95.48)
ID4	Female	38.33	S	(24.72, 29.16, 39.9, 44.83)
		96.27	VL	(90.15, 92.89, 98.14, 100)
		3.63	VS	(0, 2.18, 4.85, 7.27)
		15.2	S	(10.02, 12.76, 18.9, 20.36)
		84.61	VL	(80.21, 86.75, 94.91, 97.74)
		61.47	M	(49.92, 54.77, 63.09, 68.01)
		26.76	S	(19.95, 23.83, 32.15, 34.89)
		73.04	L	(70.19, 71.57, 78.27, 80.53)
		50	S	(44.99, 48.55, 54.28, 59.85)
		15.2	VS	(12.12, 15.75, 20.52, 22.21)
		3.63	VS	(0.4, 0.4, 5.74, 8.08)
		73.04	L	(67.04, 70.6, 77.54, 83.28)
		96.27	VL	(95.8, 100, 100, 100)
		61.47	L	(53.39, 58.64, 62.68, 68.98)
		26.76	S	(20.27, 21.65, 27.63, 30.05)
		84.61	VL	(75.28, 77.79, 85.95, 89.98)
		50	M	(44.18, 46.61, 53.55, 57.67)
		50	M	(44.75, 47.09, 52.83, 55.49)
		3.63	VS	(1.29, 2.5, 4.93, 8)
		15.2	S	(12.6, 15.19, 19.79, 21.16)
		38.33	M	(31.74, 34.57, 44.99, 47.42)
		96.27	VL	(97.9, 99.43, 100, 100)
		73.04	L	(64.86, 67.53, 77.22, 80.13)
		84.61	VL	(75.28, 79, 86.43, 89.82)
		26.76	S	(18.98, 21.65, 28.68, 29.97)
		61.47	M	(53.39, 56.62, 65.11, 69.47)
		35.39	S	(26.49, 28.84, 37.88, 42)
		33.53	S	(28.51, 30.45, 36.35, 38.53)
		71.47	L	(63.81, 68.82, 75.04, 79.81)
ID5	Male	73.01	L	(59.88, 70.16, 78.62, 90.22)
		96.2	VL	(79.84, 89.61, 97.56, 100)
		49.94	M	(39.92, 44.91, 54.99, 59.98)
		38.28	S	(29.94, 37.17, 41.04, 44.5)
		15.09	VS	(7.54, 11.41, 15.89, 20.77)
		3.68	VS	(0.61, 2.24, 3.87, 6.42)
		84.66	L	(72.1, 77.09, 82.69, 86.86)
		26.75	S	(15.48, 19.35, 20.98, 24.44)
		61.47	M	(50.61, 57.64, 62.02, 68.74)
		15.09	VS	(9.98, 13.34, 15.99, 19.04)
		49.94	M	(52.34, 56.82, 58.86, 62.22)
		73.01	L	(70.16, 74.24, 78.11, 82.59)
		3.68	VS	(1.32, 3.26, 5.4, 7.13)
		61.47	L	(58.45, 62.42, 66.7, 69.86)
		38.28	M	(34.01, 38.29, 41.34, 45.32)
		96.2	VL	(91.85, 93.99, 98.27, 99.59)
		84.66	L	(79.33, 84.11, 87.17, 91.85)
		26.75	S	(18.84, 23.52, 26.37, 30.14)
		96.2	VL	(89.21, 92.67, 94.91, 99.19)
		61.47	L	(55.5, 59.27, 62.02, 65.89)
		49.94	M	(49.59, 54.18, 56.62, 59.98)
		84.66	L	(72.1, 78.31, 82.89, 88.39)
		26.75	S	(13.44, 18.43, 21.89, 26.58)
		15.09	VS	(9.06, 14.05, 16.4, 19.76)
		3.68	VS	(0.31, 2.24, 4.68, 7.54)
		73.01	L	(64.56, 69.55, 71.28, 75.66)
		38.28	M	(32.69, 37.68, 42.36, 46.33)
		7.85	VS	(2.14, 5.09, 7.03, 9.47)
		39.26	M	(28.92, 33.91, 37.78, 41.34)
		10.06	VS	(3.56, 7.64, 10.18, 13.24)
		14.23	S	(11.3, 16.29, 19.55, 24.03)
		76.32	L	(69.55, 75.15, 78.21, 84.22)
ID6	Male	96.2	VL	(87.17, 90.12, 94.2, 97.05)
		49.94	M	(44.3, 49.9, 49.9, 55.09)
		61.47	M	(42.36, 46.33, 54.07, 56.62)
		38.28	S	(27.29, 32.69, 38.9, 43.18)
		15.09	VS	(6.62, 8.76, 10.69, 17.82)
		3.68	VS	(2.65, 4.99, 5.09, 10.08)
		84.66	L	(70.16, 73.93, 79.43, 81.67)
		73.01	L	(65.07, 70.06, 80.14, 85.13)
		26.75	S	(19.04, 23.22, 28.31, 30.96)
		49.94	S	(30.24, 37.17, 41.85, 44.6)
		15.09	S	(11.1, 18.13, 22.91, 25.05)
		84.66	L	(70.06, 75.05, 81.47, 85.03)
		61.47	L	(49.59, 54.58, 62.02, 64.36)
		38.28	S	(29.74, 33.4, 43.38, 47.25)
		3.68	VS	(2.14, 3.16, 6.82, 10.08)
		26.75	S	(24.44, 26.48, 36.56, 39.51)
		96.2	VL	(89.31, 93.38, 95.42, 98.07)
		73.01	L	(60.39, 65.17, 72, 75.56)
		3.68	VS	(1.83, 4.58, 7.03, 9.57)
		61.47	L	(53.67, 58.66, 60.79, 65.27)
		49.94	M	(39.92, 43.48, 46.74, 50.2)
		84.66	L	(76.17, 80.96, 86.15, 89)
		26.75	S	(21.79, 25.97, 29.74, 34.73)
		73.01	L	(68.23, 72.3, 75.97, 78.62)
		96.2	VL	(85.03, 87.88, 92.77, 94.81)
		15.09	S	(14.15, 15.89, 20.57, 25.87)
		38.28	S	(28.31, 30.55, 36.15, 39.1)
		35.21	S	(27.9, 29.43, 35.95, 40.84)
		32.15	S	(20.98, 24.64, 27.8, 35.13)
		82.45	L	(64.15, 66.8, 72.71, 79.12)

Table C.2: Dataset for the online application Perceptions (Page 251) on the relative length of a given line segment with respect to a fixed longer one (VS = VERY SMALL, S = SMALL, M = MEDIUM, L = LARGE, VL = VERY LARGE, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$); data ID7-ID13

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID7	Female	61.47	L	(56.18, 59.28, 68.84, 71.92)
		26.76	S	(39.98, 47.5, 52.42, 59.94)
		84.61	L	(75.2, 79.89, 90.15, 95.15)
		3.63	VS	(0.57, 1.21, 8.72, 10.02)
		15.2	S	(6.38, 11.39, 21.65, 26.33)
		38.33	M	(30.05, 35.22, 45.56, 50)
		73.04	L	(68.01, 72.78, 85.3, 87.96)
		50	M	(40.15, 45.8, 54.77, 60.1)
		96.27	VL	(92.16, 94.1, 99.03, 100)
		73.04	L	(70.11, 74.23, 85.7, 90.06)
		38.33	S	(27.46, 30.13, 39.82, 43.62)
		96.27	VL	(92.08, 93.94, 99.84, 100)
		84.61	L	(81.58, 85.3, 95.72, 99.03)
		15.2	S	(5.9, 10.5, 19.47, 25.85)
		26.76	S	(17.04, 20.19, 29.81, 31.99)
		61.47	M	(50.08, 54.2, 64.78, 70.03)
		3.63	VS	(0.73, 1.62, 6.46, 10.99)
		50	M	(48.63, 50.48, 59.61, 63.73)
		38.33	S	(32.39, 36.83, 47.42, 49.68)
		73.04	L	(65.51, 70.6, 80.05, 85.46)
		96.27	VL	(89.42, 93.54, 98.87, 100)
		50	M	(41.84, 45.4, 55.57, 58.64)
		26.76	S	(22.78, 25.77, 35.54, 39.5)
		15.2	S	(10.1, 15.02, 25.04, 30.05)
		3.63	VS	(0, 0.89, 5.25, 10.58)
		61.47	L	(61.63, 65.75, 75.28, 77.54)
		84.61	L	(78.68, 80.94, 89.9, 94.02)
		47.35	M	(39.98, 45.88, 54.77, 59.94)
		26.86	S	(24.96, 28.84, 39.26, 44.91)
		87.94	VL	(82.88, 84.81, 96.45, 98.3)
		2.75	VS	(0.97, 1.37, 6.54, 8.08)
ID8	Male	49.94	M	(40.02, 50.31, 50.31, 59.98)
		3.68	VS	(0, 2.34, 3.77, 4.99)
		73.01	L	(79.94, 84.93, 87.37, 91.75)
		61.47	M	(52.24, 57.23, 62.93, 68.02)
		84.66	L	(79.74, 83.3, 86.56, 89.61)
		15.09	S	(11.71, 13.95, 17.41, 19.65)
		96.2	VL	(93.18, 95.93, 98.07, 100)
		38.28	M	(33.6, 36.86, 39.51, 43.69)
		26.75	S	(13.95, 18.84, 21.28, 24.54)
		73.01	L	(72.81, 77.49, 78.51, 82.89)
ID9	Male	61.47	M	(53.97, 59.67, 62.63, 69.96)
		96.2	VL	(90.02, 93.38, 94.91, 100)
		3.68	VS	(0.51, 1.53, 2.55, 5.19)
		84.66	L	(73.83, 77.7, 80.14, 84.83)
		38.28	S	(29.74, 33.6, 36.15, 39.71)
		26.75	S	(19.96, 23.32, 25.25, 29.84)
		49.94	M	(44.91, 48.78, 51.22, 54.99)
		15.09	VS	(12.53, 15.58, 17.21, 18.84)
		73.01	L	(67.31, 73.22, 75.56, 77.29)
		38.28	S	(35.95, 38.7, 40.63, 44.2)
		3.68	VS	(1.02, 2.65, 3.67, 4.68)
		73.01	L	(72.1, 75.25, 77.19, 79.74)
		61.47	L	(55.3, 58.04, 59.16, 64.15)
		15.09	S	(9.98, 14.05, 15.17, 17.11)
		49.94	M	(44.91, 48.57, 51.93, 54.99)
		26.75	S	(21.28, 24.44, 25.46, 28)
		84.66	L	(82.79, 84.83, 86.15, 88.39)
		96.2	VL	(92.67, 93.58, 97.05, 98.78)
		38.28	S	(30.45, 36.46, 37.17, 38.59)
		3.68	VS	(0.71, 2.04, 3.26, 4.48)
		61.47	L	(60.08, 63.54, 66.29, 68.43)
		26.75	S	(27.8, 29.94, 30.86, 32.08)
		84.66	L	(87.37, 88.9, 90.73, 92.57)
		15.09	S	(12.53, 15.17, 15.68, 17.82)
		49.94	M	(44.91, 48.27, 48.57, 49.49)
		73.01	L	(76.48, 78.82, 80.24, 84.01)
		96.2	VL	(92.26, 95.01, 95.93, 98.07)
		67.36	L	(60.9, 63.75, 67.31, 69.86)
		5.28	VS	(3.97, 5.4, 6.31, 7.23)
		72.88	L	(75.05, 77.7, 79.94, 83.2)
ID10	Female	50	M	(43.21, 47.5, 52.26, 56.46)
		96.27	VL	(91.03, 95.88, 100, 100)
		61.47	L	(50.16, 51.53, 58.64, 65.27)
		73.04	L	(70.03, 72.21, 77.79, 79.97)
		3.63	VS	(1.21, 3.31, 6.22, 8.4)
		84.61	L	(90.06, 90.95, 94.35, 98.14)
		38.33	M	(32.96, 36.19, 42.25, 46.28)
		26.76	S	(24.72, 27.87, 32.79, 35.3)
		15.2	S	(11.15, 13.49, 16.8, 18.9)
		26.76	S	(24.07, 26.98, 32.88, 35.38)
		50	M	(44.99, 48.87, 51.86, 55.57)
		73.04	L	(65.43, 69.87, 74.07, 79)
		15.2	S	(10.02, 12.52, 17.77, 20.03)
		84.61	L	(82.71, 85.46, 92.25, 95.8)
		61.47	M	(48.55, 50.08, 58.48, 59.94)
		3.63	VS	(0, 3.15, 7.03, 10.1)
		38.33	M	(34.57, 38.21, 44.26, 48.14)
		96.27	VL	(95.4, 98.38, 100, 100)
		3.63	VS	(1.21, 3.39, 6.95, 9.53)
		61.47	M	(45.64, 49.11, 51.53, 55.01)
		84.61	L	(80.13, 81.74, 88.61, 89.9)
		50	M	(44.99, 48.3, 52.5, 55.57)
		38.33	S	(24.64, 27.06, 32.88, 34.98)
		96.27	L	(89.98, 92.89, 98.38, 99.35)
		15.2	S	(10.1, 12.68, 17.77, 19.87)
		73.04	L	(68.42, 71.41, 78.84, 81.83)
		26.76	S	(20.03, 21.57, 28.51, 29.97)
		65.1	M	(58.32, 61.79, 67.69, 72.05)
		53.24	M	(44.99, 47.74, 52.99, 55.98)
		51.96	M	(44.1, 47.09, 53.72, 57.67)

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID11	Male	61.47	M	(50.48, 56.54, 63.25, 70.52)
		3.63	VS	(0, 2.42, 7.19, 10.02)
		84.61	VL	(80.13, 87.72, 90.95, 98.22)
		73.04	L	(68.9, 76.01, 80.05, 88.05)
		26.76	S	(30.21, 36.83, 43.05, 50.08)
		96.27	VL	(88.69, 94.83, 99.27, 100)
		38.33	M	(32.63, 37.16, 42.81, 46.77)
		50	M	(45.32, 50, 56.54, 65.11)
		15.2	S	(9.77, 15.91, 24.39, 29.97)
		3.63	VS	(0.4, 1.78, 7.03, 10.18)
		73.04	L	(59.94, 68.34, 76.49, 82.55)
		38.33	M	(28.51, 36.35, 42, 50.08)
		15.2	S	(10.34, 18.98, 24.64, 31.18)
		50	M	(41.84, 48.55, 51.45, 58.4)
		26.76	S	(19.79, 24.39, 29.89, 38.45)
		61.47	L	(52.42, 61.55, 65.02, 71.16)
		96.27	VL	(87.8, 93.7, 97.42, 100)
		84.61	L	(72.62, 82.55, 88.61, 97.42)
		84.61	L	(70.11, 79.97, 85.22, 95.23)
		38.33	M	(36.67, 42.49, 46.45, 55.49)
		3.63	VS	(0, 3.47, 7.75, 11.39)
		26.76	S	(20.11, 28.11, 32.63, 39.9)
		61.47	M	(50.16, 57.35, 62.2, 73.26)
		50	M	(42.81, 49.6, 52.91, 60.26)
		73.04	L	(69.22, 75.12, 80.05, 85.86)
		15.2	S	(9.94, 14.54, 19.22, 26.01)
		96.27	VL	(89.98, 95.88, 100, 100)
		45.1	M	(39.26, 44.99, 48.55, 56.14)
		3.33	VS	(0, 1.45, 5.17, 9.85)
		49.51	M	(41.52, 48.47, 52.1, 59.05)
ID12	Female	50	M	(39.98, 45.07, 54.93, 60.1)
		38.33	S	(20.11, 25.53, 40.31, 46.77)
		15.2	S	(0.24, 4.2, 15.27, 19.95)
		73.04	L	(46.61, 53.63, 72.46, 79.81)
		3.63	VS	(0, 0, 5.17, 9.77)
		26.76	S	(2.26, 4.36, 14.38, 20.19)
		61.47	M	(39.98, 44.99, 60.02, 70.36)
		84.61	L	(59.85, 71.89, 84.57, 95.8)
		96.27	VL	(94.59, 99.68, 99.68, 99.68)
		50	M	(29.64, 34.89, 45.07, 49.92)
		61.47	M	(29.64, 34.81, 46.12, 53.72)
		38.33	S	(15.11, 19.63, 35.54, 39.98)
		73.04	L	(50, 57.27, 71.89, 79.89)
		96.27	VL	(89.98, 99.68, 100, 100)
		84.61	L	(76.01, 79.97, 90.23, 96.69)
		3.63	VS	(0, 0.89, 4.04, 7.27)
		26.76	S	(6.7, 10.02, 24.39, 34.01)
		15.2	S	(0.24, 4.6, 12.76, 25.36)
		15.2	S	(2.99, 7.59, 16.64, 22.21)
		73.04	M	(34.81, 48.14, 60.02, 64.94)
		61.47	M	(44.99, 47.82, 63.41, 63.89)
		50	M	(29.89, 37.08, 52.99, 54.93)
		96.27	VL	(89.98, 95.56, 100, 100)
		38.33	S	(23.02, 29.56, 40.47, 44.18)
		3.63	VS	(0, 5.17, 16.88, 21)
		84.61	L	(67.53, 73.18, 84.25, 89.82)
		26.76	S	(15.51, 21.97, 31.02, 36.19)
		78.73	L	(57.59, 62.68, 77.71, 82.15)
		5	S	(0, 2.83, 14.62, 21.24)
		95.29	VL	(83.36, 89.82, 99.19, 100)
ID13	Female	3.68	VS	(0.1, 1.73, 2.85, 4.07)
		73.01	L	(70.37, 73.52, 76.37, 79.74)
		84.66	VL	(76.58, 81.57, 86.05, 89.51)
		15.09	S	(13.75, 19.65, 23.42, 26.88)
		96.2	VL	(90.02, 93.69, 96.84, 99.9)
		26.75	S	(25.87, 32.59, 36.86, 42.16)
		61.47	L	(50.2, 54.99, 65.07, 70.77)
		38.28	M	(42.36, 46.64, 53.36, 57.94)
		49.94	M	(39.61, 44.6, 51.73, 56.92)
		49.94	M	(44.4, 49.08, 55.8, 61.1)
		96.2	VL	(94.2, 95.42, 97.66, 99.49)
		3.68	VS	(0.31, 1.43, 2.44, 3.26)
		38.28	M	(34.32, 36.97, 45.52, 48.47)
		26.75	S	(20.06, 22.81, 25.66, 27.39)
		61.47	L	(67.52, 70.06, 73.73, 76.37)
		73.01	L	(76.68, 78.62, 81.36, 83.6)
		84.66	VL	(87.78, 89.61, 92.46, 94.7)
		15.09	S	(10.08, 13.34, 16.4, 19.86)
		15.09	S	(10.39, 13.34, 16.29, 20.67)
		84.66	L	(79.12, 83.3, 86.35, 89.41)
		3.68	VS	(0, 2.24, 3.16, 4.38)
		73.01	L	(73.73, 77.39, 82.18, 85.95)
		96.2	VL	(92.67, 94.4, 96.33, 98.27)
		26.75	S	(16.5, 20.26, 25.05, 28.92)
		61.47	L	(57.74, 61.81, 66.5, 70.16)
		49.94	M	(50, 53.26, 56.62, 61.51)
		38.28	S	(28.62, 34.42, 37.68, 41.14)
		83.8	L	(79.33, 83.3, 86.46, 90.63)
		39.51	S	(20.57, 25.25, 29.02, 32.18)
		39.51	M	(44.81, 47.76, 50.81, 53.97)
		42.33	M	(40.43, 44.09, 48.07, 51.53)
		80.98	L	(88.39, 91.34, 95.01, 97.25)
		97.06	VL	(92.97, 95.21, 98.27, 99.69)
		82.7	L	(80.24, 82.08, 86.46, 88.09)
		52.27	M	(46.74, 49.08, 52.55, 54.99)
		71.66	L	(70.47, 73.12, 75.36, 77.8)
		92.15	VL	(86.86, 88.7, 91.24, 93.18)
		8.34	S	(9.47, 11.61, 13.24, 14.66)
		83.68	L	(76.58, 78.72, 81.57, 83.71)
		56.32	L	(65.99, 69.04, 74.03, 77.09)
		8.83	VS	(0.71, 3.05, 6.01, 8.04)

Table C.3: Dataset for the online application Perceptions (Page 251) on the relative length of a given line segment with respect to a fixed longer one (VS = very small, S = small, M = medium, L = large, VL = very large, and $(a_i, b_i, c_i, d_i) \equiv \text{Tra}(a_i, b_i, c_i, d_i)$); data ID14-ID21

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID14	Male	84.66	VL	(74.24, 83.3, 88.39, 94.3)
		26.75	VS	(10.18, 19.45, 24.95, 34.52)
		73.01	L	(60.39, 69.25, 75.05, 85.03)
		3.68	VS	(0.41, 3.26, 6.82, 12.22)
		96.2	VL	(89.82, 91.85, 96.33, 99.49)
		15.09	VS	(9.98, 13.85, 17.01, 21.28)
		49.94	M	(34.83, 39.92, 50.1, 55.4)
		61.47	M	(54.89, 58.66, 61.51, 69.96)
		38.28	M	(29.84, 37.98, 41.85, 45.21)
		38.28	M	(34.83, 39.92, 44.2, 48.88)
		73.01	L	(61.2, 68.94, 73.52, 79.02)
		61.47	M	(60.59, 64.77, 69.35, 74.54)
		26.75	S	(13.03, 20.06, 24.34, 30.14)
		49.94	M	(42.67, 48.17, 53.16, 60.29)
		15.09	VS	(8.66, 11.61, 15.27, 20.16)
		3.68	VS	(0.61, 3.36, 6.62, 9.37)
		84.66	VL	(74.13, 79.33, 83.4, 89.92)
		96.2	VL	(90.02, 94.91, 99.39, 100)
		26.75	S	(20.16, 24.95, 30.04, 39.82)
		49.94	M	(44.7, 49.08, 51.12, 55.91)
		15.09	VS	(9.88, 13.44, 16.7, 26.17)
		3.68	VS	(0.51, 1.43, 5.5, 10.08)
		84.66	L	(74.85, 75.97, 80.65, 85.64)
		38.28	S	(33.4, 37.07, 41.24, 44.91)
		96.2	VL	(89.31, 92.46, 96.74, 98.07)
		73.01	L	(65.58, 69.35, 74.95, 80.04)
		61.47	M	(41.65, 48.27, 52.85, 59.67)
		62.45	M	(52.55, 57.64, 64.05, 65.89)
		57.91	M	(50.2, 55.3, 59.47, 63.03)
		86.01	VL	(76.99, 83.1, 86.05, 90.84)
		12.76	VS	(9.88, 12.32, 18.33, 23.12)
ID15	Male	49.95	M	(44.86, 49.91, 50.18, 54.23)
		38.39	S	(24.41, 29.91, 30, 34.68)
		26.72	S	(19.37, 23.78, 25.5, 30.45)
		61.4	M	(53.87, 57.3, 63.24, 68.74)
		96.29	VL	(90.72, 93.42, 96.76, 98.56)
		84.62	L	(78.83, 82.79, 89.1, 92.7)
		73.06	L	(64.86, 67.84, 74.05, 78.65)
		3.6	VS	(4.59, 7.66, 12.52, 15.86)
		15.16	S	(7.84, 11.98, 17.84, 21.17)
		3.6	VS	(2.16, 4.59, 9.19, 11.8)
		73.06	L	(51.26, 54.86, 60.09, 64.68)
		61.4	M	(48.56, 50.63, 56.94, 59.64)
		38.39	S	(32.88, 36.85, 44.23, 47.75)
		49.95	M	(39.37, 45.05, 51.44, 54.77)
		84.62	VL	(73.96, 77.12, 85.05, 88.11)
		26.72	S	(21.08, 26.04, 32.34, 38.02)
		15.16	S	(10.81, 15.77, 22.88, 26.94)
		96.29	VL	(93.24, 95.77, 99.1, 100)
		3.6	VS	(1.35, 3.15, 7.75, 10.54)
		73.06	L	(61.8, 65.95, 72.7, 76.4)
ID16	Female	38.28	M	(38.59, 45.52, 51.32, 59.98)
		26.75	S	(14.46, 28.31, 33.1, 41.55)
		61.47	L	(49.69, 58.86, 70.98, 80.45)
		73.01	VL	(59.16, 76.07, 85.03, 90.63)
		84.66	VL	(70.26, 81.47, 90.12, 99.9)
		15.09	VS	(0, 9.57, 20.37, 30.14)
		3.68	VS	(0.51, 6.21, 11.61, 19.35)
		49.94	L	(34.22, 44.91, 54.99, 67.72)
		96.2	VL	(84.01, 99.8, 100, 100)
		61.47	L	(38.19, 55.91, 65.27, 80.86)
		49.94	M	(39.21, 46.13, 54.68, 60.08)
		26.75	S	(9.78, 19.45, 31.06, 40.12)
		15.09	VS	(10.08, 15.99, 24.64, 31.06)
		38.28	M	(28.62, 34.73, 44.5, 59.98)
		73.01	VL	(67.82, 84.73, 95.01, 100)
		3.68	VS	(0, 4.18, 10.69, 21.49)
		84.66	VL	(86.05, 91.96, 99.29, 100)
		96.2	VL	(89.71, 98.98, 100, 100)
		15.09	S	(9.88, 15.89, 24.54, 34.62)
		26.75	M	(20.57, 29.12, 32.89, 41.96)
		61.47	L	(49.8, 53.77, 64.26, 80.45)
		38.28	S	(29.02, 40.33, 50.1, 58.15)
		3.68	VS	(0.61, 2.24, 10.18, 20.16)
		49.94	L	(39.21, 46.13, 55.6, 69.65)
		84.66	VL	(79.43, 90.12, 97.05, 100)
		73.01	VL	(67.01, 80.14, 92.26, 98.17)
		96.2	VL	(88.8, 97.25, 100, 100)
		54.85	M	(38.49, 49.49, 60.18, 69.96)
		56.81	L	(39.92, 58.55, 67.41, 79.84)
		79.02	VL	(75.66, 79.23, 89.61, 100)
		0.12	VS	(0, 0, 1.12, 10.29)
ID17	Male	96.2	L	(90.02, 92.77, 95.11, 98.57)
		49.94	M	(44.91, 49.9, 50.31, 54.99)
		73.01	L	(59.88, 60.69, 69.96, 75.05)
		61.47	M	(49.9, 52.55, 61.91, 68.53)
		26.75	S	(13.85, 20.77, 25.56, 29.12)
		3.68	VS	(1.93, 2.85, 6.21, 9.06)
		38.28	M	(29.84, 33.81, 42.46, 46.44)
		84.66	VL	(79.94, 81.57, 85.34, 90.63)
		15.09	VS	(11.51, 15.38, 17.11, 18.74)
		15.09	VS	(12.42, 15.07, 17.62, 19.45)
		96.2	VL	(89.92, 93.18, 97.56, 97.66)
		61.47	M	(55.4, 59.06, 61.71, 71.28)
		73.01	L	(67.01, 74.44, 78.11, 80.14)
		84.66	VL	(69.96, 73.63, 83.1, 87.07)
		26.75	S	(19.76, 22.91, 37.27, 39.92)
		49.94	M	(50.1, 50.1, 50.1, 50.1)
		3.68	VS	(1.53, 1.53, 3.56, 7.13)
		38.28	M	(32.69, 37.07, 42.97, 48.27)
		61.47	L	(52.04, 57.33, 65.38, 69.86)
		38.28	M	(34.22, 39.1, 43.08, 47.66)
		15.09	VS	(14.77, 19.14, 22.2, 29.02)
		49.94	M	(45.11, 49.29, 50.71, 55.6)
		26.75	S	(21.59, 30.14, 33.91, 39.92)
		96.2	VL	(90.73, 96.44, 98.17, 98.17)
		84.66	L	(72.71, 77.8, 82.79, 86.15)
		73.01	M	(68.64, 72.81, 80.04, 82.59)
		3.68	VS	(2.75, 4.07, 5.8, 9.88)
		46.99	M	(44.91, 49.39, 51.02, 54.99)
		92.39	VL	(82.99, 88.39, 90.12, 92.67)
		12.27	VS	(10.9, 16.5, 21.18, 24.03)
		6.26	VS	(5.09, 8.04, 12.93, 13.65)
		69.94	L	(58.96, 61.91, 65.17, 70.26)
		14.48	VS	(11.2, 13.54, 19.76, 21.89)
		47.36	M	(47.25, 49.9, 50.51, 53.05)
		28.1	S	(29.43, 34.01, 37.98, 40.02)
ID18	Female	3.68	VS	(0.51, 1.53, 7.43, 10.08)
		26.75	S	(18.94, 21.28, 29.02, 31.26)
		61.47	M	(55.19, 57.23, 66.8, 69.86)
		73.01	L	(65.48, 69.14, 78.62, 80.65)
		96.2	VL	(78.31, 80.45, 89.31, 92.87)
		38.28	M	(32.18, 34.11, 43.89, 46.44)
		15.09	S	(7.94, 11.51, 21.59, 23.42)
		84.66	VL	(80.24, 83.91, 92.06, 95.21)
		49.94	M	(41.85, 45.32, 55.09, 58.25)
		26.75	S	(18.33, 21.38, 29.02, 31.87)
		15.09	S	(9.88, 13.75, 20.26, 24.64)
		38.28	M	(32.48, 37.37, 44.5, 47.96)
		73.01	L	(61, 65.58, 73.83, 78)
		49.94	M	(42.57, 46.84, 53.56, 57.94)
		96.2	VL	(88.19, 92.57, 98.68, 99.59)
		61.47	M	(52.75, 57.54, 64.66, 69.55)
		3.68	VS	(0.61, 1.53, 7.03, 10.9)
		84.66	L	(78, 83.1, 90.84, 93.38)
		38.28	S	(29.12, 33.3, 42.77, 46.84)
		3.68	VS	(0.41, 1.83, 7.43, 10.49)
		73.01	M	(67.41, 70.16, 77.8, 81.57)
		26.75	S	(18.33, 21.28, 29.02, 32.48)
		96.2	VL	(87.47, 93.18, 100, 100)
		84.66	VL	(80.96, 85.23, 91.55, 96.44)
		61.47	M	(53.36, 57.03, 65.58, 70.06)
		15.09	S	(7.43, 11.61, 18.84, 23.32)
		49.94	M	(41.96, 45.82, 53.97, 58.45)
		66.99	L	(60.9, 64.77, 73.42, 78.41)
		28.96	S	(19.76, 24.03, 31.67, 36.97)
		33.13	S	(32.28, 36.05, 44.91, 49.08)
		54.11	M	(39.61, 43.58, 51.12, 56.31)
		19.51	S	(11.81, 15.58, 23.22, 28.72)
ID19	Male	73.01	L	(62.22, 66.8, 73.93, 80.45)
		26.75	S	(27.6, 32.59, 36.05, 38.49)
		61.47	L	(56.82, 61.91, 66.7, 71.38)
		84.66	VL	(84.93, 87.68, 91.96, 95.01)
		96.2	VL	(92.57, 94.91, 99.39, 100)
		49.94	M	(44.81, 44.91, 54.99, 54.99)
		38.28	M	(40.02, 42.87, 47.45, 50)
		15.09	VS	(9.37, 10.69, 14.15, 15.58)
		3.68	VS	(0, 0, 3.05, 5.4)
		61.47	M	(55.09, 58.45, 63.54, 66.09)
		15.09	S	(12.63, 15.58, 20.88, 23.63)
		49.94	M	(46.54, 49.08, 52.65, 56.31)
		3.68	VS	(1.83, 2.55, 5.19, 6.92)
		73.01	L	(64.26, 69.25, 71.08, 75.15)
		84.66	L	(75.36, 78.72, 82.59, 86.35)
		38.28	S	(30.55, 32.59, 37.88, 39.31)
		26.75	S	(20.57, 24.34, 28.62, 30.86)
		96.2	VL	(89.92, 92.16, 97.15, 98.57)
		26.75	S	(20.67, 24.13, 29.12, 29.94)
		38.28	M	(37.68, 38.49, 42.06, 44.5)
		96.2	VL	(92.16, 93.89, 97.45, 100)
		61.47	M	(54.58, 57.84, 62.12, 64.97)
		73.01	L	(67.92, 70.16, 76.07, 79.23)
		3.68	VS	(0, 1.53, 3.77, 6.82)
		84.66	L	(74.64, 77.7, 81.98, 85.34)
		15.09	S	(11.2, 12.42, 17.52, 20.77)
		49.94	M	(44.7, 47.66, 51.93, 56.72)
		97.91	VL	(94.4, 96.54, 99.08, 100)
		9.94	VS	(7.54, 11.41, 16.19, 17.92)
		20.49	S	(16.19, 17.92, 23.73, 24.44)
ID20	Male	38.28	S	(24.24, 33.4, 38.09, 42.16)
		15.09	VS	(12.93, 17.72, 22.81, 23.83)
		61.47	M	(73.12, 77.19, 82.69, 94.2)
		73.01	L	(77.49, 82.38, 90.73, 97.15)
		3.68	VS	(0.81, 1.12, 4.68, 15.07)
		49.94	M	(26.99, 44.91, 53.67, 59.16)
		26.75	S	(8.86, 15.48, 23.22, 29.02)
		84.66	L	(81.06, 85.34, 89.51, 95.52)
		96.2	VL	(92.16, 94.4, 97.15, 99.29)
		15.09	S	(5.5, 8.35, 12.42, 16.19)
		61.47	M	(44.5, 51.83, 57.03, 65.48)
		96.2	VL	(89.82, 96.54, 99.19, 99.19)
		49.94	M	(36.35, 40.63, 43.18, 44.5)
		3.68	VS	(1.22, 6.82, 8.66, 9.16)
		38.28	M	(22.91, 27.6, 31.77, 36.76)
		26.75	S	(12.93, 18.23, 23.01, 29.63)
		84.66	L	(83.91, 87.68, 91.04, 95.82)
		73.01	L	(72.91, 75.97, 80.24, 84.52)
		61.47	M	(65.68, 69.25, 72.91, 79.02)
		3.68	VS	(1.93, 4.07, 6.62, 9.16)
		15.09	S	(10.39, 13.44, 16.8, 19.45)
		73.01	L	(65.68, 67.31, 74.13, 80.04)
		26.75	S	(14.66, 22.51, 26.17, 29.43)
		84.66	L	(79.84, 81.98, 86.56, 96.64)
		49.94	M	(28.82, 35.34, 40.63, 44.5)
		96.2	VL	(94.2, 96.23, 97.76, 98.68)
		38.28	M	(29.63, 35.13, 38.29, 42.87)
		14.85	S	(6.52, 12.93, 15.48, 16.9)
		20.49	S	(13.44, 17.72, 22, 25.25)
		36.81	M	(30.04, 33.2, 35.64, 44.3)
ID21	Female	3.68	VS	(3.36, 8.04, 8.96, 8.96)
		84.66	VL	(80.75, 81.77, 83.6, 95.52)
		49.94	M	(45.01, 46.95, 51.02, 55.5)
		96.2	VL	(89.92, 94.81, 99.9, 99.9)
		61.47	L	(50.31, 52.44, 56.01, 65.78)
		73.01	L	(56.82, 63.14, 65.27, 71.18)
		15.09	S	(10.08, 14.46, 17.72, 20.16)
		26.75	S	(17.72, 20.37, 23.32, 26.58)
		49.94	M	(39.92, 47.45, 52.44, 59.98)
		38.28	S	(39.92, 42.36, 47.66, 49.8)
		96.2	L	(93.48, 98.17, 100, 100)
		61.47	L	(52.65, 56.82, 63.03, 66.6)
		26.75	S	(19.96, 22.1, 24.95, 29.84)
		84.66	L	(75.87, 77.6, 82.28, 86.86)
		3.68	VS	(2.75, 5.8, 7.33, 9.78)
		73.01	L	(62.63, 64.46, 73.93, 77.49)
		15.09	S	(10.08, 13.03, 16.6, 19.76)
		3.68	VS	(1.73, 3.05, 7.43, 9.47)
		38.28	S	(35.64, 42.26, 45.01, 49.59)
		26.75	S	(30.04, 34.11, 35.95, 39.71)
		96.2	VL	(90.22, 92.26, 97.96, 100)
		15.09	S	(9.67, 11.1, 14.97, 16.29)
		61.47	L	(53.05, 56.62, 59.98, 65.38)
		73.01	L	(59.67, 61.81, 69.96, 77.49)
		84.66	L	(69.86, 72.4, 76.27, 80.24)
		49.94	M	(39.92, 46.54, 49.59, 54.28)
		8.71	VS	(8.66, 9.37, 11.1, 12.73)
		22.21	VS	(19.35, 21.18, 24.34, 29.23)
		81.23	L	(71.69, 75.97, 81.77, 83.71)
		70.43	L	(61.51, 65.48, 71.89, 78)

Table C.4: Dataset for the online application Perceptions (Page 251) on the relative length of a given line segment with respect to a fixed longer one (VS = very small, S = small, M = medium, L = large, VL = very large, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$); data ID22

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)	ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID22	Male	84.61	L	(78.27, 80.94, 84.41, 87.4)	ID22 (cont.)		19.8	S	(17.53, 20.03, 21.41, 23.67)
		73.04	L	(54.93, 58, 62.2, 65.67)			37.25	S	(33.52, 36.75, 37.96, 40.47)
		50	M	(47.25, 49.43, 50.89, 53.31)			66.86	L	(61.07, 64.7, 65.75, 69.79)
		96.27	VL	(92.65, 95.72, 97.58, 99.11)			57.75	M	(55.57, 59.61, 61.39, 65.35)
		15.2	VS	(12.92, 15.51, 17.77, 20.03)			97.16	VL	(96.85, 97.98, 98.63, 99.43)
		38.33	S	(32.55, 36.03, 39.9, 42.89)			55.2	M	(52.02, 53.63, 55.01, 57.27)
		3.63	VS	(2.5, 4.44, 6.22, 9.21)			41.47	M	(39.98, 42, 43.21, 45.4)
		26.76	S	(24.8, 28.19, 30.45, 33.28)			40.78	M	(38.29, 39.98, 41.6, 44.02)
		61.47	L	(55.17, 58.4, 61.79, 65.75)			71.27	L	(70.03, 71.89, 72.7, 75.12)
		3.63	VS	(2.26, 3.63, 5.57, 8.08)			27.06	S	(24.88, 27.14, 28.43, 30.86)
		61.47	M	(53.23, 56.62, 59.53, 63)			77.84	L	(75.28, 77.87, 79, 81.99)
		38.33	S	(36.75, 38.85, 41.68, 44.91)			15.39	S	(12.6, 14.38, 15.35, 18.34)
		26.76	S	(20.6, 23.18, 25.85, 29.56)			59.51	M	(55.9, 58.56, 60.26, 63)
		84.61	L	(74.23, 77.71, 80.45, 85.3)			18.53	S	(16.48, 19.14, 20.52, 23.26)
		73.04	L	(69.39, 72.54, 75.53, 79.16)			69.12	L	(66.48, 69.31, 71, 74.15)
		15.2	S	(14.14, 17.53, 20.19, 23.59)			40.59	M	(38.13, 41.11, 42.33, 45.07)
		50	M	(47.17, 49.11, 51.13, 53.39)			90.98	VL	(87.24, 89.66, 92, 96.28)
		96.27	VL	(94.02, 96.45, 97.74, 98.95)			59.51	L	(57.35, 59.53, 61.39, 63.65)
		50	M	(46.85, 49.27, 50.73, 52.99)			53.14	M	(51.13, 53.47, 54.6, 57.19)
		84.61	L	(74.88, 79, 81.5, 85.38)			56.08	M	(53.55, 55.9, 57.27, 60.02)
		26.76	S	(21, 23.99, 25.69, 29.97)			1.37	VS	(0.97, 1.29, 1.78, 1.94)
		96.27	VL	(94.43, 96.12, 98.06, 99.43)			46.76	M	(43.94, 46.28, 47.5, 48.95)
		38.33	S	(35.54, 38.61, 41.6, 45.8)			72.75	L	(73.67, 75.93, 77.63, 80.05)
		3.63	VS	(0.89, 2.5, 3.63, 5.25)			19.9	S	(18.98, 20.44, 21.81, 24.39)
		61.47	M	(53.8, 56.62, 59.37, 63.81)			27.84	S	(22.46, 25.2, 26.74, 28.84)
		73.04	L	(70.27, 73.51, 76.58, 79.56)			96.76	VL	(95.48, 97.01, 97.98, 99.11)
		15.2	S	(12.44, 15.02, 17.12, 19.95)			36.76	M	(34.25, 36.51, 38.29, 40.06)
		7.65	VS	(4.85, 8.24, 10.18, 13)			22.84	S	(21.32, 23.51, 24.72, 27.06)
		1.76	VS	(0.4, 1.13, 2.42, 4.36)			80.39	L	(75.93, 78.59, 80.29, 85.22)
		57.94	M	(50.24, 52.91, 55.65, 59.29)			95.49	VL	(91.36, 94.35, 95.32, 97.42)
		24.31	S	(21.08, 24.47, 26.09, 29.56)			32.35	S	(30.86, 34.73, 36.43, 39.01)
		40	M	(41.2, 43.78, 46.2, 49.76)			25.1	S	(20.92, 24.64, 25.69, 30.05)
		13.14	S	(12.12, 15.43, 17.29, 20.11)			67.25	L	(64.46, 66.32, 67.85, 70.44)
		13.04	S	(10.34, 12.84, 14.7, 18.42)			50.1	M	(44.83, 48.3, 49.52, 50.48)
		64.61	L	(55.74, 59.61, 63, 68.34)			98.92	VL	(98.38, 99.27, 99.76, 100)
		17.16	S	(13.25, 17.21, 18.42, 21.16)			11.76	VS	(8.48, 9.29, 10.58, 12.68)
		43.82	M	(39.26, 41.84, 44.18, 47.9)			44.31	M	(43.21, 44.99, 46.37, 48.71)
		62.06	L	(60.1, 62.92, 65.11, 69.95)			8.92	VS	(5.82, 8, 8.89, 9.77)
		95	VL	(90.63, 94.26, 96.12, 99.68)			76.27	L	(73.51, 76.98, 79.16, 83.04)
		52.45	M	(47.66, 51.7, 55.01, 59.29)			20	S	(20.76, 23.42, 24.8, 28.11)
		26.08	S	(20.76, 25.44, 26.9, 30.94)			46.37	M	(42.25, 44.59, 46.04, 48.79)
		31.08	S	(30.13, 34.09, 36.19, 39.98)			12.16	VS	(8.32, 10.26, 11.47, 13.89)
		65.39	L	(61.07, 65.43, 66.8, 69.95)			96.86	VL	(94.26, 96.77, 98.14, 99.68)
		66.96	L	(61.15, 65.35, 66.07, 69.31)			66.18	L	(64.46, 67.45, 69.06, 72.78)
		19.31	S	(18.34, 21.08, 22.7, 26.01)			74.61	L	(73.83, 77.87, 79.4, 82.55)
		42.65	M	(39.98, 43.38, 45.88, 50.08)			53.43	M	(52.26, 54.28, 56.14, 58.48)
		91.76	VL	(91.84, 93.94, 94.83, 98.22)			86.67	VL	(83.76, 86.27, 87.8, 90.95)
		28.43	S	(28.76, 32.71, 34.81, 40.06)			14.9	VS	(11.79, 14.22, 15.19, 17.69)
		68.63	L	(63.57, 67.21, 68.42, 71.16)			28.82	S	(23.83, 25.93, 26.82, 29.16)
		71.08	L	(64.7, 67.21, 68.66, 70.19)			35.49	S	(34.25, 37.16, 38.05, 40.95)
		85.98	VL	(75.93, 78.84, 81.5, 85.06)			23.33	S	(20.27, 23.1, 24.15, 27.06)
		81.57	L	(76.17, 79.73, 81.83, 84.89)			95.39	VL	(94.43, 96.28, 97.09, 98.14)
		54.41	M	(50.32, 54.04, 55.57, 60.02)			61.96	L	(60.99, 63.57, 64.62, 68.01)
		16.57	VS	(14.7, 17.37, 19.39, 22.78)			81.76	L	(80.05, 83.44, 86.51, 89.58)
		63.82	L	(58.08, 61.31, 63.81, 68.09)			60.49	M	(57.67, 59.85, 61.39, 63.57)
		95.88	VL	(94.91, 96.45, 97.25, 99.35)			17.84	S	(15.75, 18.34, 19.95, 22.86)
		96.18	VL	(93.78, 95.72, 96.77, 99.03)			11.47	VS	(7.43, 9.05, 10.02, 11.87)
		11.18	VS	(5.98, 9.05, 10.26, 13.65)			33.33	S	(28.03, 30.78, 31.83, 35.95)
		81.47	L	(75.04, 79.32, 80.86, 84.98)			75.88	L	(70.84, 73.83, 74.88, 78.51)
		76.08	L	(71.41, 75.04, 76.25, 80.21)			41.27	M	(43.54, 46.2, 47.01, 49.92)
		56.18	M	(52.26, 55.09, 56.38, 59.85)			83.63	L	(81.66, 84.25, 85.14, 88.37)
		12.16	VS	(10.1, 12.04, 13.41, 16.48)			74.9	L	(73.51, 76.01, 77.22, 80.61)
		4.12	VS	(2.42, 3.55, 4.12, 5.25)			99.51	VL	(97.42, 98.55, 99.35, 99.92)
		34.51	S	(30.13, 32.31, 33.28, 36.27)			96.27	VL	(94.91, 95.88, 96.85, 97.98)
		39.61	M	(33.52, 37.08, 37.96, 41.36)			58.82	L	(58, 60.42, 61.55, 63.97)
		60.69	L	(56.95, 60.02, 62.04, 66.24)			70.49	L	(72.46, 75.36, 76.33, 79.89)
		90.49	VL	(85.06, 88.21, 89.9, 93.86)			17.65	VS	(16.4, 18.58, 20.27, 22.46)
		43.92	M	(45.32, 47.74, 48.87, 50.08)			57.94	L	(56.7, 59.37, 60.99, 63.73)
		83.92	L	(79.16, 82.55, 83.76, 86.83)			98.73	VL	(97.9, 99.03, 99.68, 100)
		95.78	VL	(95.23, 96.53, 97.74, 99.68)			55.78	M	(53.15, 55.09, 55.65, 57.35)
		63.82	L	(60.26, 62.76, 63.89, 66.96)			70.59	L	(67.29, 70.19, 72.13, 75.69)
		43.04	M	(38.37, 41.6, 42.89, 46.69)			17.45	VS	(15.59, 18.34, 19.39, 22.21)
		42.55	M	(40.23, 43.46, 44.51, 47.82)			4.22	VS	(3.07, 4.44, 5.17, 6.7)
		40.1	M	(39.98, 42.57, 44.43, 47.25)			54.61	M	(53.15, 54.85, 55.57, 57.19)
		20.59	S	(18.42, 21.08, 22.62, 25.93)			37.45	S	(34.89, 38.21, 39.9, 42.65)
		26.37	S	(26.98, 30.05, 31.66, 35.06)			35.1	S	(30.13, 31.91, 32.79, 34.57)
		99.31	VL	(96.28, 99.19, 99.92, 100)			41.37	M	(36.83, 39.1, 40.95, 43.62)
		13.04	VS	(8.72, 10.66, 12.2, 14.54)			14.71	VS	(10.02, 11.87, 12.84, 14.7)
		48.82	M	(46.53, 48.87, 51.29, 53.63)			54.12	M	(53.23, 54.44, 55.25, 56.79)
		59.41	M	(57.43, 59.21, 61.55, 65.59)			20.59	S	(16.48, 19.47, 20.68, 23.26)
		50.29	M	(49.6, 51.21, 52.67, 55.49)			21.08	S	(15.67, 18.58, 20.11, 22.29)
		58.92	L	(58.16, 60.99, 62.84, 66.56)			55.69	M	(53.07, 54.68, 55.57, 58.08)
		49.12	M	(45.48, 48.95, 51.29, 54.52)			14.8	VS	(11.15, 13.17, 14.38, 16.96)
		68.43	L	(65.99, 68.66, 71.24, 74.07)			78.14	L	(72.62, 75.04, 76.41, 79.16)
		97.45	VL	(92, 94.18, 95.4, 98.63)			17.45	S	(14.46, 16.48, 17.69, 19.55)
		38.33	M	(35.22, 38.21, 40.79, 44.99)			2.84	VS	(1.29, 2.5, 2.83, 3.8)
		88.63	VL	(84.98, 87.32, 89.01, 92.41)			99.51	VL	(98.38, 99.27, 99.68, 100)
		46.67	M	(43.94, 45.8, 47.66, 49.6)			44.31	M	(42.16, 44.67, 45.64, 48.38)
		16.67	S	(12.52, 14.94, 16.32, 18.66)			75.88	L	(71.57, 74.31, 75.61, 78.59)
		52.65	M	(50.97, 53.47, 55.09, 58.4)			56.86	M	(53.63, 55.25, 56.87, 59.29)
		59.71	M	(57.43, 59.53, 61.63, 64.7)			8.33	VS	(4.2, 5.82, 6.46, 8.48)
		65	L	(62.2, 65.19, 66.8, 69.95)			50.49	M	(50.4, 51.62, 52.34, 54.04)
		83.92	L	(83.2, 85.78, 87.24, 90.06)			15	VS	(10.42, 11.47, 12.44, 14.3)
		71.18	L	(66.4, 69.22, 70.84, 74.8)			38.43	M	(34.89, 36.83, 37.64, 40.06)
		46.86	M	(46.12, 48.14, 49.19, 50)					

Table C.5: Dataset for the online application Perceptions (Page 251) on the relative length of a given line segment with respect to a fixed longer one (VS = very small, S = small, M = medium, L = large, VL = very large, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$); data ID23 (182 trials)

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)	ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID23	Male	26.76	S	(27.14, 31.91, 35.14, 39.26)	ID23 (cont.1)		26.67	S	(28.92, 31.66, 33.2, 36.67)
		96.27	VL	(95.64, 97.66, 98.87, 100)			73.82	L	(68.01, 72.78, 74.07, 76.9)
		84.61	VL	(81.5, 84.65, 87.72, 90.06)			84.71	VL	(84.17, 88.77, 90.06, 92.41)
		50	M	(45.15, 48.79, 51.78, 55.65)			57.75	M	(52.42, 57.35, 59.05, 60.66)
		15.2	VS	(13.33, 16.07, 18.58, 21.32)			59.61	M	(53.31, 57.43, 59.05, 59.94)
		38.33	M	(34.89, 37.96, 40.39, 43.13)			4.9	VS	(2.83, 4.12, 4.85, 5.82)
		73.04	L	(71.97, 75.12, 78.84, 82.39)			81.47	VL	(81.42, 85.22, 86.59, 88.93)
		61.47	L	(58, 61.71, 63.57, 67.69)			38.43	M	(35.86, 40.15, 41.44, 44.91)
		3.63	VS	(1.94, 3.15, 3.8, 4.44)			33.63	S	(31.02, 35.78, 37.32, 40.71)
		73.04	L	(68.09, 70.92, 72.29, 76.09)			63.73	M	(55.33, 60.58, 62.12, 65.02)
		96.27	VL	(96.61, 97.9, 98.63, 99.84)			2.16	VS	(1.13, 2.34, 2.99, 3.15)
		3.63	VS	(2.42, 3.47, 3.8, 4.77)			42.06	M	(37.32, 41.6, 42.57, 45.07)
		38.33	S	(34.25, 36.91, 38.45, 40.95)			70	L	(57.59, 62.52, 64.3, 67.53)
		50	M	(47.33, 49.76, 51.21, 54.28)			68.43	L	(59.37, 64.86, 66.32, 68.66)
		61.47	L	(58.97, 61.31, 62.84, 66.32)			53.14	M	(46.28, 49.35, 51.05, 54.28)
		84.61	L	(81.1, 83.84, 86.67, 89.98)			56.47	M	(48.63, 52.02, 53.15, 59.37)
		15.2	S	(13, 15.91, 17.04, 20.11)			1.08	VS	(0.73, 1.29, 1.62, 2.02)
		26.76	S	(24.96, 28.68, 31.5, 35.7)			30.29	S	(28.76, 31.5, 32.55, 35.86)
		38.33	M	(37.56, 40.87, 42.08, 45.8)			47.84	M	(44.1, 47.42, 48.38, 50.65)
		26.76	S	(22.86, 26.17, 28.19, 30.94)			34.31	S	(34.65, 39.42, 40.63, 43.62)
		50	M	(46.37, 49.11, 50.97, 53.88)			48.63	M	(45.23, 47.9, 49.35, 51.29)
		3.63	VS	(2.34, 3.47, 4.2, 4.93)			1.67	VS	(1.05, 1.86, 2.34, 2.58)
		15.2	VS	(15.51, 18.66, 21, 23.1)			10.1	VS	(6.62, 9.21, 10.34, 12.04)
		61.47	L	(58.8, 61.63, 62.84, 66.07)			67.25	L	(60.99, 66.07, 67.37, 69.63)
		96.27	VL	(95.4, 96.77, 97.5, 98.63)			67.35	M	(59.05, 62.04, 62.68, 67.85)
		73.04	L	(70.44, 73.67, 75.04, 78.76)			56.08	M	(49.6, 52.58, 53.55, 57.67)
		84.61	VL	(83.52, 85.54, 86.59, 90.06)			76.57	L	(70.76, 74.56, 75.85, 78.76)
		48.43	M	(44.99, 49.27, 50.4, 52.02)			54.8	M	(51.29, 55.82, 56.7, 59.29)
		71.57	L	(69.95, 73.75, 75.36, 78.03)			12.65	VS	(9.37, 11.79, 12.44, 14.62)
		63.04	M	(44.91, 48.71, 51.37, 54.85)			46.76	M	(45.48, 48.47, 49.6, 50.65)
		48.53	M	(44.67, 47.42, 49.52, 52.67)			68.53	L	(68.74, 70.03, 72.29, 76.25)
		13.33	VS	(9.37, 11.79, 12.76, 15.35)			36.86	M	(34.98, 40.79, 42, 44.91)
		25.98	S	(22.78, 25.2, 27.38, 30.13)			83.33	L	(74.88, 81.02, 82.15, 82.96)
		3.82	VS	(2.42, 3.88, 4.52, 6.38)			94.71	VL	(92.41, 94.18, 95.56, 97.9)
		73.92	VL	(70.68, 76.17, 80.21, 83.93)			6.57	VS	(2.99, 4.52, 5.17, 5.9)
		37.84	M	(37.32, 41.28, 43.21, 46.85)			37.55	M	(39.58, 44.26, 45.48, 47.66)
		89.9	VL	(91.68, 94.43, 96.04, 97.98)			51.37	M	(50.08, 52.67, 53.55, 56.22)
		21.18	S	(20.03, 23.75, 25.85, 29.89)			74.02	L	(66.4, 70.11, 72.46, 75.61)
		27.84	S	(25.04, 29.89, 32.15, 36.03)			20.78	S	(16.56, 19.95, 21.16, 25.04)
		39.22	M	(38.37, 40.55, 41.84, 46.85)			55.29	M	(51.45, 56.14, 57.35, 60.74)
		40.78	M	(44.02, 46.85, 48.95, 52.26)			24.12	S	(23.99, 26.82, 29.64, 30.61)
		1.47	VS	(0.89, 1.78, 2.26, 3.55)			69.12	L	(68.58, 71.57, 72.7, 77.46)
		6.67	VS	(3.96, 5.49, 6.7, 8)			71.37	L	(66.64, 70.36, 71.57, 75.44)
		96.76	VL	(97.66, 98.71, 99.35, 100)			2.45	VS	(1.94, 2.67, 3.15, 3.63)
		25.88	S	(26.9, 31.91, 34.33, 36.75)			85.78	VL	(81.66, 85.06, 86.03, 90.39)
		30.39	S	(30.86, 35.06, 36.11, 40.15)			97.16	VL	(95.4, 97.5, 98.22, 99.68)
		18.33	VS	(15.51, 18.58, 20.44, 22.78)			13.04	VS	(10.02, 13.57, 14.3, 16.64)
		5.88	VS	(3.31, 5.17, 6.14, 7.84)			35.69	M	(40.15, 41.28, 42.49, 46.45)
		70.39	L	(70.03, 71.97, 73.18, 79.08)			87.65	VL	(88.53, 90.15, 91.11, 94.83)
		15.88	S	(15.43, 17.93, 19.55, 24.23)			47.84	M	(50.16, 52.91, 53.88, 56.3)
		89.8	VL	(87.96, 91.44, 92.65, 94.83)			63.33	M	(56.62, 60.66, 61.31, 63.49)
		96.86	VL	(94.35, 96.2, 96.93, 98.63)			87.45	VL	(89.5, 90.63, 91.76, 94.67)
		95.59	VL	(92.08, 95.8, 96.45, 97.74)			74.31	L	(69.47, 74.64, 75.61, 78.03)
		40.78	M	(41.36, 46.04, 46.93, 48.22)			7.84	VS	(5.17, 8, 9.13, 9.94)
		12.75	VS	(9.94, 11.55, 12.44, 16.4)			73.73	L	(71.16, 73.42, 74.39, 76.82)
		51.47	M	(49.6, 54.44, 55.41, 58.48)			65.98	L	(60.74, 63.97, 64.78, 69.39)
		69.22	L	(68.26, 72.37, 73.42, 75.28)			59.61	M	(54.12, 57.75, 58.8, 61.39)
		36.27	M	(40.31, 42.08, 43.3, 47.58)			53.24	M	(51.78, 55.57, 56.46, 59.05)
		44.41	M	(43.62, 46.69, 48.38, 50.08)			95.69	VL	(92.65, 96.04, 96.85, 99.11)
		97.16	VL	(96.45, 98.14, 98.79, 100)			2.84	VS	(1.37, 2.18, 2.83, 3.8)
		35.98	M	(39.34, 42.08, 42.89, 47.66)			65.39	L	(60.99, 65.59, 66.72, 70.11)
		90.1	VL	(86.83, 89.66, 91.03, 92.65)			51.76	M	(50.48, 53.23, 53.88, 57.84)
		1.37	VS	(0.57, 1.62, 2.02, 2.34)			12.35	VS	(9.61, 10.9, 11.71, 14.38)
		0.1	VS	(0, 0.32, 0.57, 1.21)			51.27	M	(50.4, 52.99, 53.63, 56.62)
		32.35	M	(35.06, 39.01, 39.9, 42.25)			4.02	VS	(2.26, 2.67, 3.15, 3.96)
		26.47	S	(27.06, 30.21, 31.26, 35.54)			76.27	L	(70.76, 74.47, 75.69, 79.97)
		91.86	VL	(86.83, 90.47, 91.84, 94.51)			86.67	VL	(85.06, 88.21, 89.26, 90.87)
		72.94	L	(69.31, 72.78, 75.04, 76.9)			14.8	S	(10.42, 14.22, 15.19, 15.99)
		73.63	L	(60.66, 65.27, 67.21, 73.99)			47.94	M	(41.68, 44.35, 45.32, 49.92)
		20.98	S	(20.36, 25.04, 26.41, 29.64)			9.02	VS	(5.9, 8.08, 8.72, 10.26)
		48.92	M	(44.99, 47.74, 49.27, 50.81)			59.51	M	(55.01, 58.56, 59.77, 62.36)
		44.8	M	(44.99, 47.66, 48.63, 49.68)			31.86	S	(29.24, 31.83, 32.79, 36.19)
		71.67	L	(69.14, 72.7, 74.47, 76.82)			83.53	VL	(82.15, 86.75, 88.93, 90.15)
		50.49	M	(46.69, 49.11, 50.48, 53.31)			52.06	M	(47.9, 51.29, 52.75, 53.88)
		40.49	M	(42.33, 44.26, 45.07, 49.11)			80.88	L	(79.08, 81.74, 83.28, 87.8)
		38.73	M	(40.31, 43.05, 44.18, 49.03)			48.63	M	(45.96, 48.71, 49.84, 52.26)
		70.98	L	(59.69, 65.02, 65.99, 69.14)			64.71	L	(58.56, 61.55, 62.52, 64.94)
		25.39	S	(22.05, 26.82, 28.11, 29.97)			49.02	M	(46.2, 47.82, 48.55, 50.08)
		60.59	M	(52.42, 56.87, 60.58, 65.91)			9.9	VS	(5.57, 7.03, 8.24, 10.02)
		37.35	M	(33.36, 39.26, 41.28, 43.21)			69.12	L	(66.24, 67.85, 69.71, 74.23)
		83.73	L	(82.96, 85.62, 87.16, 89.82)			63.82	L	(61.15, 65.75, 67.21, 69.79)
		74.02	L	(68.98, 75.04, 76.17, 78.35)			24.9	S	(24.88, 28.92, 30.21, 31.74)
		70.1	L	(59.77, 61.95, 63.25, 68.66)			71.08	L	(66.07, 68.5, 69.55, 72.37)
		6.86	VS	(3.39, 5.09, 5.98, 6.87)			71.86	L	(68.01, 71.89, 73.18, 75.93)
		60.2	M	(55.74, 60.1, 61.55, 66.24)			11.57	VS	(8.4, 9.05, 10.18, 13.17)
		85.2	VL	(80.29, 87.08, 88.77, 90.47)			67.16	L	(67.04, 70.11, 71.41, 75.44)
		41.47	M	(40.47, 45.07, 46.37, 49.03)			7.45	VS	(4.52, 6.38, 7.43, 8.16)
		68.24	L	(66.64, 69.63, 71.49, 76.41)			88.92	VL	(84.09, 88.21, 89.5, 93.21)
		54.12	M	(51.37, 53.55, 54.44, 58)			32.06	M	(37.72, 41.52, 43.13, 45.4)
		84.61	L	(72.62, 76.17, 78.35, 80.94)			53.63	M	(51.86, 54.2, 55.49, 58.56)
		69.9	L	(63.81, 68.5, 69.95, 73.59)			42.55	M	(42.25, 44.18, 46.2, 49.35)

Table C.6: Dataset for the online application Perceptions (Page 251) on the relative length of a given line segment with respect to a fixed longer one (VS = very small, S = small, M = medium, L = large, VL = very large, and $(a_i, b_i, c_i, d_i) \equiv \mathrm{Tra}(a_i, b_i, c_i, d_i)$); data ID23 (180 trials)

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)	ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID23 (cont.2)		79.8	L	(73.51, 75.44, 76.41, 80.45)	ID23 (cont.3)		17.84	VS	(13.81, 17.69, 19.31, 21.73)
		38.04	M	(37.8, 42.89, 44.59, 46.28)			63.43	M	(59.37, 62.12, 63.49, 68.42)
		64.22	L	(56.14, 60.74, 62.76, 65.02)			7.35	VS	(5.49, 7.11, 8.56, 9.94)
		19.51	S	(15.35, 19.71, 21.24, 24.15)			33.63	S	(32.79, 35.62, 37.48, 39.42)
		20.59	S	(14.38, 18.17, 20.84, 23.91)			80.98	L	(78.19, 81.34, 83.44, 87.08)
		48.14	M	(46.04, 48.95, 50.48, 53.55)			43.14	M	(39.66, 42.97, 45.56, 48.47)
		35.59	M	(42.08, 44.67, 46.28, 49.35)			81.76	L	(73.75, 78.11, 80.86, 84.81)
		89.51	VL	(87.72, 89.74, 91.76, 93.46)			94.41	VL	(90.31, 93.05, 94.83, 97.01)
		26.86	S	(25.28, 27.46, 29.24, 32.71)			58.1	M	(50.72, 52.43, 54.22, 56.65)
		2.75	VS	(1.86, 2.26, 2.99, 3.55)			52.96	M	(45.21, 48.78, 52.5, 56.44)
		37.84	M	(39.01, 41.76, 42.81, 44.83)			63.5	L	(61.16, 63.73, 66.24, 69.31)
		49.61	M	(47.5, 51.45, 52.1, 54.36)			29.88	S	(25.89, 30.11, 31.62, 36.62)
		29.9	S	(27.3, 30.78, 32.79, 35.14)			45.38	M	(43.13, 46.49, 47.85, 49.71)
		59.61	M	(54.52, 57.67, 59.21, 63.25)			4.7	VS	(3.29, 5.22, 5.94, 6.29)
		48.14	M	(47.5, 49.35, 50.4, 52.58)			52.87	M	(51.36, 53.79, 55.01, 58.51)
		63.73	L	(63.89, 66.96, 68.09, 71.65)			38.59	M	(36.12, 39.41, 41.63, 45.28)
		23.53	S	(21.16, 24.07, 25.69, 28.59)			32.58	S	(27.83, 31.62, 33.4, 37.2)
		93.53	VL	(91.52, 94.43, 95.88, 97.98)			53.75	M	(45.99, 49.14, 51.72, 56.08)
		63.14	M	(58.32, 60.9, 62.12, 64.78)			34.58	S	(33.26, 36.62, 37.91, 41.13)
		32.06	S	(31.5, 35.22, 37, 39.98)			84.84	VL	(80.54, 83.98, 85.62, 88.56)
		94.02	VL	(90.06, 91.52, 92.49, 96.04)			64.63	L	(58.66, 62.37, 63.45, 69.46)
		21.96	S	(20.36, 23.99, 24.8, 27.14)			36.41	S	(29.9, 34.33, 35.62, 40.27)
		55.69	M	(52.26, 55.65, 56.38, 58.97)			71.78	L	(69.89, 72.96, 74.39, 78.4)
		50.98	M	(45.56, 48.71, 50.57, 52.99)			17.68	VS	(15.24, 19.1, 20.03, 23.32)
		94.51	VL	(90.63, 92.33, 94.1, 95.8)			0.44	VS	(0.29, 0.64, 0.64, 0.93)
		90.98	VL	(87.32, 89.5, 90.31, 93.3)			93.64	VL	(92.85, 95.06, 95.64, 98)
		94.12	VL	(90.06, 90.87, 92.16, 95.07)			35.1	M	(35.91, 39.91, 41.34, 44.56)
		32.75	M	(35.14, 38.37, 39.82, 42.25)			50.7	M	(48.07, 50.07, 51.36, 54.86)
		74.51	L	(69.87, 73.59, 74.72, 76.58)			30.92	S	(29.69, 32.9, 34.69, 37.48)
		28.53	S	(23.34, 26.82, 27.95, 32.15)			42.16	M	(39.84, 41.49, 42.56, 45.21)
		87.75	VL	(86.59, 89.18, 91.2, 92.73)			16.11	VS	(13.45, 16.88, 18.17, 20.46)
		21.57	S	(15.59, 19.79, 21.24, 24.23)			44.51	M	(41.13, 44.28, 45.49, 47.78)
		4.61	VS	(3.07, 3.8, 4.36, 5.25)			1.22	VS	(0.64, 1.72, 2.15, 3.22)
		79.02	L	(69.06, 71, 72.62, 75.85)			63.59	L	(61.87, 65.16, 66.17, 70.03)
		35.29	M	(35.22, 39.34, 41.03, 43.86)			65.68	L	(64.31, 67.88, 69.6, 72.03)
		82.75	L	(74.15, 77.14, 78.59, 82.47)			36.67	S	(30.62, 34.12, 35.98, 39.56)
		0.49	VS	(0.4, 0.89, 1.21, 1.45)			96.17	VL	(93.13, 94.99, 95.99, 97.78)
		79.61	L	(77.79, 80.69, 82.39, 85.14)			20.64	S	(18.45, 23.61, 25.04, 27.97)
		2.35	VS	(1.62, 2.26, 2.5, 3.31)			56.27	M	(55.65, 58.3, 59.51, 62.23)
		85	VL	(79.64, 82.55, 83.68, 86.67)			47.39	M	(45.42, 48, 49.28, 51.86)
		76.08	L	(74.64, 77.38, 79.08, 83.04)			55.75	M	(53.58, 56.22, 58.01, 62.95)
		53.82	M	(52.1, 56.06, 57.43, 60.02)			28.48	S	(24.54, 28.33, 29.69, 32.26)
		2.25	VS	(1.53, 1.86, 2.34, 3.07)			79.01	L	(74.96, 78.18, 79.69, 83.76)
		16.57	VS	(10.02, 13.17, 14.05, 15.75)			29.01	S	(25.32, 28.83, 30.11, 33.26)
		57.45	M	(55.17, 57.27, 58.4, 62.6)			88.85	VL	(86.27, 88.34, 89.91, 92.27)
		55.39	M	(54.68, 56.62, 58.56, 62.28)			28.05	S	(25.18, 29.33, 31.4, 34.76)
		71.67	L	(68.66, 71.32, 73.34, 75.44)			6.36	VS	(4.65, 6.58, 7.22, 9.23)
		24.31	S	(19.87, 22.78, 24.23, 27.46)			99.74	VL	(95.64, 98.21, 99.14, 100)
		81.67	L	(80.86, 84.33, 86.11, 88.69)			41.72	M	(43.49, 47.21, 48.43, 50)
		11.27	VS	(6.79, 8.24, 9.29, 10.5)			92.16	VL	(88.56, 90.27, 91.63, 94.92)
		86.18	VL	(84.01, 86.59, 88.13, 91.03)			81.97	L	(75.25, 78.97, 80.33, 84.55)
		93.43	VL	(94.35, 95.72, 96.69, 98.87)			97.82	VL	(89.7, 92.85, 94.13, 97.21)
		36.57	M	(34.17, 37.64, 40.15, 42.25)			23.95	S	(22.25, 24.46, 26.11, 30.11)
		14.31	VS	(10.02, 13.49, 15.02, 17.93)			90.07	VL	(86.19, 88.84, 90.13, 92.7)
		63.14	L	(61.47, 66.4, 67.61, 70.92)			59.84	M	(57.8, 59.8, 60.94, 64.31)
		35.2	S	(33.28, 37.64, 40.87, 44.18)			72.82	L	(71.96, 74.61, 75.46, 78.68)
		70.98	L	(67.37, 70.44, 71.32, 74.31)			32.32	S	(28.25, 32.12, 33.4, 37.41)
		43.92	M	(44.02, 47.9, 48.79, 50.48)			63.76	L	(62.23, 66.31, 67.53, 71.39)
		6.37	VS	(3.31, 4.52, 4.93, 5.9)			13.94	VS	(11.8, 15.31, 16.74, 19.17)
		62.06	M	(63, 65.67, 67.29, 70.36)			17.68	S	(15.59, 19.6, 21.1, 23.25)
		98.33	VL	(95.15, 98.22, 98.79, 99.84)			64.11	L	(63.3, 67.1, 68.17, 71.96)
		71.27	L	(65.19, 68.34, 69.71, 72.94)			10.45	VS	(6.8, 9.3, 10.52, 13.02)
		57.06	M	(52.42, 54.6, 55.49, 59.05)			75.35	L	(71.75, 75.11, 76.18, 80.62)
		80.2	L	(73.91, 77.95, 79.32, 82.15)			19.25	S	(15.45, 19.67, 21.6, 25.82)
		76.67	L	(72.29, 75.69, 76.98, 79.97)			10.98	VS	(6.37, 9.08, 10.66, 13.09)
		92.06	VL	(85.95, 89.74, 91.11, 92.49)			59.84	M	(56.8, 58.87, 60.23, 62.3)
		44.51	M	(38.69, 42.49, 45.32, 48.79)			32.32	S	(27.32, 31.26, 33.19, 35.41)
		34.8	S	(30.13, 34.41, 36.03, 39.01)			44.08	M	(42.7, 45.42, 46.42, 48.07)
		69.02	L	(64.14, 68.42, 69.79, 72.78)			27.79	S	(24.25, 27.75, 29.33, 31.9)
		92.94	VL	(92.97, 94.67, 95.8, 97.17)			78.31	L	(74.54, 78.4, 79.9, 82.62)
		56.37	M	(52.1, 56.22, 58.48, 60.26)			65.24	M	(63.09, 66.45, 67.53, 69.96)
		20.98	S	(18.17, 21.97, 23.99, 26.9)			34.49	M	(30.11, 33.48, 34.69, 38.2)
		81.76	L	(75.2, 78.59, 80.53, 82.47)			55.14	M	(53.29, 55.29, 56.22, 57.08)
		99.41	VL	(97.98, 98.87, 99.6, 100)			66.99	L	(62.8, 65.38, 66.74, 69.96)
		13.53	VS	(8.48, 10.66, 11.63, 13.97)			39.37	M	(39.84, 42.06, 43.2, 45.28)
		53.63	M	(50, 51.94, 53.07, 57.19)			28.75	S	(25.11, 29.18, 30.9, 34.26)
		13.04	VS	(8.32, 9.61, 10.82, 12.92)			89.02	VL	(86.05, 88.13, 89.34, 92.7)
		33.82	M	(37.96, 39.98, 41.68, 45.64)			35.02	M	(33.48, 36.7, 37.98, 41.63)
		43.33	M	(44.1, 47.33, 49.19, 50.32)			48.17	M	(46.92, 48.78, 51.72, 53.51)
		35.1	M	(30.78, 35.54, 36.67, 39.66)			25.61	S	(23.75, 25.68, 27.25, 30.76)
		7.06	VS	(3.63, 4.44, 5.33, 6.3)			86.32	VL	(83.12, 85.19, 86.27, 89.7)
		85	VL	(79.48, 82.63, 84.01, 87.08)			99.3	VL	(93.78, 96.42, 97.57, 99)
		42.84	M	(42.97, 46.28, 48.38, 49.43)			87.89	VL	(87.12, 89.77, 90.99, 93.35)
		52.45	M	(48.63, 50.81, 52.75, 56.7)			74.3	L	(71.17, 74.75, 75.75, 79.11)
		6.67	VS	(3.96, 5.09, 5.74, 6.79)			89.29	VL	(87.05, 89.77, 91.06, 93.99)
		21.67	S	(19.14, 21.65, 23.34, 26.33)			9.41	VS	(6.72, 9.08, 10.44, 13.16)
		70.49	L	(65.11, 68.09, 70.27, 75.04)			82.14	L	(76.97, 79.54, 81.47, 84.05)
		41.47	M	(43.54, 47.42, 49.19, 50.81)			77.53	L	(72.68, 74.82, 76.25, 79.97)
		84.41	VL	(83.52, 86.03, 88.21, 90.23)			60.1	L	(58.08, 61.87, 62.8, 67.24)
		68.63	L	(63.57, 67.77, 69.55, 73.26)			51.22	M	(53.22, 55.94, 56.72, 61.09)

Table C.7: Dataset for the online application Perceptions (Page 251) on the relative length of a given line segment with respect to a fixed longer one (VS = very small, S = small, M = medium, L = large, VL = very large, and $(a_i, b_i, c_i, d_i) \equiv \text{Tra}(a_i, b_i, c_i, d_i)$); data ID24-ID25

ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)	ID	Gender	Underlying exact relative length (%)	Likert	(a_i, b_i, c_i, d_i)
ID24	Female	15.2	S	(7.27, 14.05, 18.34, 23.26)	ID25	Female	61.47	M	(51.13, 56.7, 62.04, 66.88)
		38.33	M	(39.82, 44.1, 47.25, 50)			26.76	S	(21.16, 26.58, 32.55, 37.4)
		26.76	S	(21.08, 26.58, 30.45, 34.89)			84.61	VL	(88.93, 93.86, 97.66, 100)
		61.47	L	(65.67, 67.93, 74.15, 79.73)			3.63	VS	(0, 3.31, 5.17, 8.16)
		50	M	(50.08, 52.58, 54.93, 59.94)			73.04	L	(82.07, 86.67, 93.13, 95.15)
		3.63	VS	(0, 1.86, 4.6, 8)			38.33	M	(30.78, 35.7, 40.23, 42.73)
		84.61	VL	(85.54, 89.1, 93.13, 96.53)			96.27	VL	(96.53, 97.42, 99.11, 100)
		96.27	VL	(94.75, 95.56, 98.55, 100)			15.2	S	(11.95, 13.33, 18.17, 21.73)
		73.04	L	(70.19, 76.25, 81.1, 85.78)			50	M	(45.72, 47.58, 52.34, 54.2)
		15.2	S	(10.02, 14.46, 16.48, 19.79)			3.63	VS	(0, 2.75, 6.54, 11.63)
		61.47	L	(59.94, 63.73, 69.39, 74.23)			84.61	L	(87.32, 88.21, 94.91, 98.47)
		38.33	M	(34.49, 38.93, 43.7, 46.61)			15.2	S	(14.14, 17.29, 22.78, 24.31)
		3.63	VS	(0.32, 2.58, 4.36, 6.87)			38.33	M	(39.9, 42.49, 44.26, 47.74)
		73.04	L	(75.61, 77.87, 83.04, 85.95)			61.47	L	(69.14, 70.68, 74.56, 78.03)
		84.61	VL	(86.75, 89.26, 91.03, 95.32)			73.04	L	(76.17, 78.43, 82.15, 85.22)
		26.76	S	(23.18, 28.59, 32.23, 37.32)			50	M	(49.43, 51.37, 53.88, 57.84)
		96.27	VL	(94.99, 96.2, 98.22, 99.68)			96.27	VL	(90.95, 93.54, 97.82, 100)
		50	M	(53.23, 56.38, 59.13, 63.41)			26.76	S	(17.21, 19.14, 23.99, 26.58)
		50	M	(49.52, 52.34, 55.09, 57.35)			15.2	VS	(13.17, 15.02, 17.69, 19.22)
		38.33	M	(32.63, 37.64, 41.11, 44.02)			96.27	VL	(93.46, 95.48, 98.71, 100)
		84.61	VL	(83.04, 85.14, 87.72, 89.98)			50	M	(50.65, 52.34, 55.82, 60.74)
		26.76	S	(23.67, 27.06, 29.4, 31.34)			26.76	S	(27.63, 30.29, 34.49, 35.54)
		73.04	L	(65.91, 67.12, 70.68, 72.78)			73.04	L	(75.2, 76.41, 81.74, 85.7)
		96.27	VL	(95.07, 96.37, 97.98, 99.52)			61.47	L	(65.02, 66.4, 72.13, 75.44)
		61.47	M	(54.85, 57.92, 61.55, 64.46)			84.61	VL	(81.5, 85.22, 87.96, 92.41)
		15.2	S	(13.81, 15.51, 18.17, 20.11)			3.63	VS	(0.97, 2.18, 4.77, 7.75)
		3.63	VS	(0.97, 3.07, 4.28, 5.82)			38.33	S	(26.82, 31.83, 34.98, 36.43)
		64.02	L	(68.66, 72.62, 74.96, 77.95)			8.53	VS	(4.12, 8.48, 12.04, 14.05)
		85.59	VL	(82.39, 84.57, 85.95, 87.8)			21.57	S	(13.41, 17.45, 19.55, 21.73)
		54.9	M	(44.59, 48.06, 49.35, 51.94)			51.37	M	(50.97, 52.75, 54.93, 57.84)
		58.92	M	(53.96, 57.27, 58.72, 61.79)			6.67	VS	(7.03, 10.58, 14.3, 16.24)
		74.61	L	(71.49, 75.44, 77.63, 79.73)			74.02	L	(68.42, 72.54, 75.36, 77.71)
		83.14	L	(79.97, 81.74, 84.01, 86.59)			42.94	M	(38.45, 42.81, 46.04, 48.63)
		15.39	S	(11.15, 14.3, 16.48, 17.69)			70	L	(67.29, 68.26, 72.37, 75.69)
		87.75	VL	(93.38, 94.75, 96.61, 98.14)			73.73	L	(65.35, 68.09, 74.31, 78.11)
		12.65	VS	(6.7, 9.21, 12.44, 16.32)			5.69	VS	(2.67, 5.9, 8.72, 13)
		15.39	S	(11.79, 14.46, 16.32, 18.98)			74.9	L	(74.64, 76.9, 79.32, 81.1)
		92.75	VL	(90.71, 92.57, 94.43, 96.69)			74.41	L	(67.04, 68.9, 73.67, 77.54)
		27.94	S	(28.03, 30.37, 32.79, 36.35)			67.55	L	(62.28, 63.33, 66.24, 70.11)
		8.63	VS	(7.11, 9.29, 11.63, 12.44)			71.18	L	(65.27, 67.21, 70.84, 74.07)
		25.49	S	(27.14, 29.97, 32.23, 36.59)			82.84	L	(76.33, 81.83, 85.38, 88.05)
		57.65	M	(48.87, 52.02, 54.04, 59.45)			60.78	M	(50.81, 54.04, 57.92, 63.49)
		37.94	M	(29.81, 34.89, 36.91, 39.5)			14.61	VS	(4.36, 10.18, 12.6, 15.43)
		2.84	VS	(0.24, 1.53, 2.83, 4.68)			48.73	M	(43.7, 46.61, 50.57, 54.93)
		91.67	VL	(89.5, 92.08, 93.46, 95.72)			29.12	S	(21.97, 24.23, 31.02, 34.33)
		16.76	S	(17.69, 20.36, 24.31, 27.22)			57.75	M	(55.25, 57.35, 63.09, 65.59)
		46.47	M	(38.85, 42.97, 45.56, 49.11)			45.2	M	(36.43, 38.93, 44.18, 47.9)
		10.1	VS	(8.4, 9.85, 11.15, 14.86)			33.82	S	(28.68, 31.58, 39.01, 42.25)
		26.67	S	(20.36, 23.75, 26.33, 29.73)			80.1	L	(76.33, 77.79, 81.5, 84.89)
		62.55	M	(56.62, 60.42, 63.17, 68.01)			92.25	VL	(85.3, 88.69, 93.78, 95.23)
		71.27	L	(68.17, 70.6, 73.02, 77.22)			53.92	M	(44.99, 48.14, 52.67, 54.6)
		73.14	L	(65.19, 67.61, 69.14, 72.7)			7.16	VS	(2.83, 8.08, 10.18, 11.47)
		43.73	M	(34.65, 38.37, 41.6, 43.7)			93.82	VL	(91.28, 92.16, 95.64, 98.47)
		61.27	M	(58.4, 60.9, 63.33, 65.75)			96.18	VL	(92, 93.86, 96.77, 98.63)
		14.22	S	(11.95, 13.65, 16.16, 18.09)			38.73	S	(29.64, 33.12, 38.53, 42.81)
		26.37	S	(23.02, 26.17, 29.16, 31.02)			13.43	VS	(10.42, 12.68, 16.8, 19.55)
		15.2	S	(12.36, 15.75, 17.29, 19.06)			62.16	M	(55.09, 58.56, 63.41, 68.26)
		13.33	S	(10.82, 12.76, 14.14, 16.64)			94.71	VL	(85.14, 86.43, 88.93, 91.84)
		95.29	VL	(92.16, 93.62, 95.07, 97.25)			23.73	S	(22.13, 26.17, 31.1, 32.23)
		20.39	S	(20.27, 22.62, 24.56, 27.46)			45.69	M	(34.01, 38.77, 42.65, 46.69)
		44.22	M	(33.04, 36.51, 40.87, 44.26)			27.25	S	(20.84, 23.59, 27.14, 31.42)
		52.94	M	(46.61, 49.03, 51.13, 53.23)			34.02	S	(30.78, 37.88, 40.47, 43.62)
		57.25	M	(53.47, 55.57, 57.27, 60.9)			98.14	VL	(93.62, 95.23, 97.33, 100)
		25.88	S	(20.52, 22.7, 23.91, 27.14)			9.41	VS	(7.35, 10.74, 14.54, 16.72)
		54.12	M	(50.48, 51.62, 52.91, 56.62)			73.92	L	(65.67, 69.31, 72.05, 76.09)
		65.1	L	(66.88, 68.98, 71.32, 74.56)			88.92	VL	(84.41, 85.7, 88.13, 92.41)
		29.31	S	(25.61, 27.87, 30.53, 34.98)			37.84	S	(28.43, 31.74, 35.7, 37.4)
		70.49	L	(73.91, 74.47, 77.3, 80.05)			84.31	L	(73.91, 75.85, 80.37, 84.65)
		74.41	L	(69.39, 70.84, 73.59, 77.87)			19.31	S	(15.19, 19.71, 21.65, 25.93)
		8.53	VS	(3.31, 6.06, 8.64, 9.94)			25.59	S	(20.68, 24.47, 27.54, 29.56)
		61.08	M	(61.31, 65.27, 68.34, 70.36)			42.16	M	(34.65, 38.37, 41.11, 45.07)
		62.55	M	(59.45, 62.68, 66.72, 70.52)			68.14	L	(56.95, 58.56, 62.2, 67.77)
		89.02	VL	(85.46, 88.37, 90.47, 93.13)			62.35	M	(54.93, 56.95, 61.31, 63.73)
		30.49	S	(24.56, 29.08, 32.47, 34.81)			57.06	M	(52.1, 57.03, 60.74, 62.52)
		57.16	M	(56.3, 61.23, 64.3, 68.66)			48.04	M	(46.04, 48.47, 51.21, 53.72)
		68.92	M	(58.24, 60.02, 64.94, 68.9)			81.27	L	(71.97, 77.22, 80.05, 83.2)
		36.86	M	(35.86, 40.06, 42.97, 46.45)			7.94	VS	(5.98, 9.45, 14.3, 15.75)
		10.1	VS	(6.87, 8.89, 10.74, 14.7)			43.14	M	(37.8, 39.74, 43.78, 46.85)
		39.02	M	(40.95, 45.88, 48.22, 51.21)			48.53	M	(46.93, 48.71, 51.78, 54.93)
		35.69	M	(34.41, 38.29, 41.36, 44.75)			45	M	(37.08, 43.86, 47.17, 49.52)
		86.18	L	(84.49, 86.19, 89.01, 90.79)			21.67	S	(19.47, 21.97, 25.85, 28.76)
		23.92	S	(20.52, 22.37, 24.39, 25.93)			24.8	S	(23.34, 25.77, 28.76, 31.1)
		2.75	VS	(0.48, 1.53, 2.67, 3.55)			83.33	L	(81.42, 81.42, 85.14, 88.29)
		47.55	M	(45.15, 48.87, 50.48, 52.91)			29.02	S	(24.88, 27.71, 29.81, 33.76)
		44.41	M	(44.83, 47.98, 50.57, 52.99)			38.82	M	(34.73, 37.96, 41.52, 44.1)
		35	M	(32.47, 34.57, 37.8, 41.03)			84.71	VL	(78.11, 82.23, 84.73, 87.08)
		80.1	L	(79.48, 82.88, 84.65, 87.48)			84.02	L	(77.06, 77.87, 82.23, 84.65)
		92.35	VL	(87.08, 89.34, 91.6, 94.99)			50.78	M	(47.42, 48.87, 52.83, 56.22)
		79.41	L	(74.88, 78.51, 80.94, 82.15)			21.37	S	(15.75, 18.26, 22.29, 25.12)
		63.14	M	(59.61, 61.79, 63.73, 67.21)			78.43	L	(75.85, 80.86, 82.55, 84.73)
		62.25	M	(53.96, 57.67, 60.02, 62.2)			45.49	M	(37.64, 41.36, 44.43, 48.06)
		94.22	VL	(92.33, 94.91, 96.37, 98.71)			57.45	M	(53.07, 55.49, 58, 61.63)
		93.43	VL	(91.36, 93.05, 94.26, 97.42)			60.29	M	(58.24, 60.82, 64.05, 68.5)
		74.02	L	(74.64, 77.06, 78.92, 80.45)			88.73	VL	(76.33, 80.53, 83.04, 84.89)
		43.82	M	(37.32, 40.79, 44.02, 46.04)			71.08	L	(65.51, 70.03, 73.91, 75.93)
		89.22	VL	(85.62, 88.37, 90.23, 92.16)			74.8	L	(64.54, 67.21, 71.16, 73.26)
		56.18	M	(45.64, 50, 52.42, 55.57)			72.65	L	(65.11, 68.98, 73.67, 76.74)
		95.78	VL	(95.88, 96.93, 98.55, 99.6)			9.61	VS	(8.89, 10.82, 15.43, 16.48)
		56.67	M	(54.28, 58, 60.58, 64.14)			90.78	VL	(84.33, 88.45, 92.33, 94.75)
		9.71	S	(8.56, 9.94, 12.12, 13.25)			4.31	VS	(1.7, 6.46, 10.58, 11.71)
		98.14	VL	(93.7, 96.53, 97.98, 99.84)			30.2	S	(26.98, 29.24, 31.83, 36.19)
		65.88	M	(67.37, 69.55, 71.16, 77.87)			70	L	(64.7, 65.51, 68.9, 72.7)
							80.78	L	(72.94, 78.43, 82.47, 85.78)
							69.41	L	(62.12, 64.62, 68.26, 73.99)
							34.61	S	(27.71, 31.74, 35.95, 37.64)

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