PLEXUS
Lisbon, May 2023

From Producer-Consumer Games to Substructural Calculi

Chris Fermüller[1]
(based of joint work with Robert Freiman[1] and Timo Lang[2])
[1]**Vienna University of Technology**
[2]**University College London**

Plan of the talk

- Motivation: From loose metaphors to formal games
- Game Type 1 [F & Lang, Tableaux 2017], [F, Lorenzen-Vol. 2021]
  Proponent claims that **C**lient can extract desired packages of information from the information that **S**erver provides, while **O**pponent seeks to refute **P**'s claim

  Type 1 games interpret substructural sequent calculi
- Game Type 2 (work in progress, tentative)
  directly model the ongoing interaction of **P**roducer and **C**onsumer
  P wins if **C**'s demands can be served, potentially forever

  Games first! — Which one connect to calculi? How?
- Open issues: A research agenda

Motivation

- substructural logics are often motivated by resource consciousness
- usually only metaphorical – think of Girard's cigarette example:

  "For $1 you get a pack of Camels, but also a pack of Marlboro"

  "but also": multiplicative in contrast to additive conjunction
- to breathe life into the resource metaphor, we need dynamics

  $\Longrightarrow$ games modeling strategic interaction
- traditional paradigm of game semantics: 'formulas as games'
  In contrast, we consider 'formulas as items (of information)'
- Two types of games:

  Type 1: Information extraction games
  Type 2: Models of **P**roducer-**C**onsumer interaction

Information Extraction Games

Guiding Ideas:

Information: Lorenzen's 'assertions', rather than 'formulas as games'
resource conciousness: information can be stored/accessed/consumed

Different ways of accessing information:

A **S**erver may provide information to a **C**lient e.g. as follows:

- **C** can have all of those once: $\{a, b, c, \ldots\}$ (multiset)
- **C** can have any one of those: $(a, b, c, \ldots)$ (**C**'s choice)
- **S** gives **C** (just) one of those: $[a, b, c, \ldots]$ (**S**'s choice)
- **C** can have the first of those: $\langle a, b, c, \ldots |$ (stack)
- **C** can have the first or last of those: $\langle a, b, \ldots e, f \rangle$ (deque)
- **C** can have these as often as **C** wants: $\| a, b, c, \ldots \|$ ('protected')

Arbitrary nestings results in information packages:

- for example: $\{\|a\|, (\langle b, c|, [d, e, f])\}$

Accessing/extracting information as a game

Information extraction game(s) (formerly **C**/**S** game – [F/Lang, 2017]):

> **P**roponent claims that **S** provides the information that **C** wants,
> while **O**pponent seeks to refute **P**'s claim

Remark:

In [F/Lang 2017] we identified **C** with **P** and **S** with **O** ('**C**/**S** game')
We now prefer to keep **S** passive and let **P** act in **C**'s behalf, opposed by **O**

States of the game: $\boxed{\Gamma \triangleright F}$

Γ ... bundle of information provided by **S**

F ... information wanted by **C** (possibly structured, as explained below)

Two possible interpretations:

Strict reading: F is equivalent to Γ

Affine reading: F is (modulo equivalence) contained in Γ

NB: 'equivalence' is (implicitly) defined by the rules of the game

Where are the logical connectives?

(Some) logical connectives directly correspond to access structures.
Following tradition, we formulate rules for binary and unary connectives.

Compound (information) items offered by **S**:

- $\{a, b\}$ multiset $\implies a \otimes b$ (multiplicative conjunction)
- (a, b) any – **P**'s choice (for **C**) $\implies a \wedge b$ (additive conjunction)
- $[a, b]$ any – **O**'s choice (against **C**) $\implies a \vee b$ (additive disjunction)
- $\langle a, b|$ first a, then b $\implies a; b$ (a new connective)
- $\|a\|$ 'protected' a $\implies !a$ ('bang', 'of course')

We speak of Information Packages (IPs), rather than formulas.

In order to make these corespondences precise and make them work in full generality we need to provide precise specifications of game rules!

Rules of the (standard) Information Extraction Game

The rules stepwise reduce states to simpler states in round:

Step 1 **P**, as scheduler, chooses an IP F of the state $\Gamma \triangleright H$

Step 2 two cases:

- F in $\Gamma \Longrightarrow$ UNPACK the IP provided by **S**
- $F = H \Longrightarrow$ CHECK the IP wanted by **C**

corresponding choices by **P** or by **O** determine the next state

We focus on the case, where the IPs of **S** form a multiset $\Gamma = [G_1, \ldots, G_n]$
(More general 'deep inference' style rules could be obtained analogously)

UNPACK-rules (F among **S**'s IPs)

($U_\vee$) $F = F_1 \vee F_2$: **O** chooses i, F_i replaces F in Γ

($U_\wedge$) $F = F_1 \wedge F_2$: **P** chooses i, F_i replaces F in Γ

($U_\otimes$) $F = F_1 \otimes F_2$: F_1 and F_2 replace F in Γ

CHECK-rules (F is **C**'s current IP — rules are dual)

($C_\vee$) $F = F_1 \vee F_2$: **P** chooses i, F_i replaces F as **C**'s wanted IP

($C_\wedge$) $F = F_1 \wedge F_2$: **O** chooses i, F_i replaces F as **C**'s wanted IP

($C_\otimes$) $F = F_1 \otimes F_2$: **P** has declares which part of Γ will be used for extracting F_1 and F_2, respectively; **O** chooses correspondingly

Did we loose implication?

$F_1 \to F_2$ is interpreted as conditional information: F_2 given F_1

The corresponding CHECK-rules (state $\Gamma \rhd F_1 \to F_2$) is obvious :

$(C_\to)$ F_2 becomes **C**'s current IP, F_1 is added to Γ

For $F_1 \to F_2$ provided by **S**, the following is obvious too:

$(U_\to)$ If F_1 as well as $F_1 \to F_2$ are in Γ, the **P** may choose to replace these two IP-occurrences by F_2

More generally, F_1 only needs to be contained in information in Γ:

$(U_\to)$ **P** has declares which part (Γ_1) of Γ is to be used for extracting F_1 and which part (Γ_2), augmented by F_2, allows to extract **C**'s wanted IP; **O** chooses correspondingly

Written in sequent style:

$$\frac{F_1, \Gamma \rhd F_2}{\Gamma \rhd F_1 \to F_2}\ (C_\to) \qquad\qquad \frac{\Gamma_1 \vdash F_1 \qquad F_2, \Gamma_2 \rhd H}{F_1 \to F_2, \Gamma_1, \Gamma_2 \rhd H}\ (U_\to)$$

Rules for Protected IP's

Simple form of interpreting 'protection':

- only relevant for information provided by **S**
- $\|F_1,\ldots,F_n\|$: F_i remain in Γ for each reduction, no splitting necessary

Reflective form of interpreting 'protection': (linear logic style)

- we also allow $\|\|\|F\|\|\|$ and $[F,\|F\|]$, etc, also for **C**'s wanted IP
- the corresponding connective is ! ('bang' of linear logic)
  - ($U_!$) if **P** picks $!F$, then **P** may either replace if by F, delete it or add another copy of F in Γ, as wished
  - ($C_!$) If **C**'s wanted $!F$ is picked, **P** may replace it by F if all formulas in Γ are protected

Final States (Winning Conditions)

recall the two possible interpretations of $\Gamma \triangleright F$: equivalent / contained in
corresponding winning states for **P**: $F,\Gamma \triangleright F$ or $F \triangleright F$
Adding the clearly contradictory IP $\perp$ renders $\perp,\Gamma \triangleright F$ winning for **P**, too

Instances of the game matching well known calculi

Full completeness and soundness is straightforward for some calculi.

For the game with reflective form of protection (strict and affine reading):

Theorem

Each of **P**'s winning strategies for $G_1, \ldots, G_n \triangleright F$ translates into a cut-free proof of $G_1, \ldots, G_n \vdash F$ in **(affine) ILL**, and vice versa.

For the game with simple form of protection (**S**: ‖...‖ instead of [...]):

Theorem

Each of **P**'s winning strategies for $G_1, \ldots, G_n \triangleright F$ translates into a cut-free proof of $G_1, \ldots, G_n \vdash F$ in Gentzen's **LI**, and vice versa.

Note: explicit weakening corresponds to dismissal of information by **P**

In a similar vain, many other calculi, eg. Lambek's, can modelled
Moreover, also new calculi with new connectives arise!

Types 2 games: Modeling Producer-Consumer Interaction

A slogan borrowed from Grigori Japaridze:
Games first! (Even if they call for other/new types of calculi)

Like in information extraction games:

- **P**roducer(**S**erver) provides packages of items (possibly information)
- **C**onsumer requests/consumes such items
- focus on **P**'s winning strategies

Unlike in information extraction games:

- **P**roducer = **S**erver as **P**roponent / **C**onsumer = **C**lient as **O**pponent
- states given by complex requests/produce packages (multitasks)
- **P** wins if **C**'s atomic requests are satisfied (potententially) forever
- asynchronous request and produce moves!
  P does not regulate (= pick the next task to be processed)

(Multi)tasks / **P-C**-Game Moves

Game states are fully specified by multitasks

A multitask is a multiset of tasks

Let $X \in \{\mathbf{C}, \mathbf{P}\}$

- Atomic tasks: $a^!$ (produce), $a^?$ (request)
- If S, T are multitasks then $S \vee_X T$ is a task (X-choice)
- If S is a multitask then $\texttt{rep}_X(S)$ is a task (X-repeat)

P-C-game Moves:

- $S \vee_X T$: X can replace this by all tasks in S, or by all tasks in T
- $\texttt{rep}_X(S)$: X can add all tasks in S to the multitask
- **P** can remove a matching pair $(a^!, a^?)$ from the multitask

NB: The game is not yet fully specified!
Who wins? Who moves when?

Guarantees Instead of Regulations

Rather than fixing **P**/**C** alternations or choice precedences, we allow for asynchronous moves.

Players give certain guarantees (mutual promises)

Guarantees given by **P**:

- **Success:** Every request is met by a produce eventually
- **Availability:** Every **P**-choice is made eventually

Guarantees given by **C**:

- **Responsiveness:** Every **C**-choice is made eventually

Main question:
Assuming that **C** meets his guarantees, can **P** meet hers?

Note the asymmetry: **P** can wait for **C**, but not vice versa

Soundness with respect to aILL

$$(\varphi_1, \ldots, \varphi_n \Rightarrow \varphi)^\pi = \varphi_1^{\pi-} \cup \ldots \cup \varphi_n^{\pi-} \cup \varphi^{\pi+}$$

$$a^{\pi+} = \{a^?\} \qquad a^{\pi-} = \{a^!\}$$

$$(\varphi \wedge \psi)^{\pi+} = \{\varphi^{\pi+} \vee_{\mathbf{C}} \psi^{\pi+}\} \qquad (\varphi \wedge \psi)^{\pi-} = \{\varphi^{\pi-} \vee_{\mathbf{P}} \psi^{\pi-}\}$$

$$(\varphi \vee \psi)^{\pi+} = \{\varphi^{\pi+} \vee_{\mathbf{P}} \psi^{\pi+}\} \qquad (\varphi \vee \psi)^{\pi-} = \{\varphi^{\pi-} \vee_{\mathbf{C}} \psi^{\pi-}\}$$

$$(\varphi \otimes \psi)^{\pi+} = \varphi^{\pi+} \cup \psi^{\pi+} \qquad (\varphi \otimes \psi)^{\pi-} = \varphi^{\pi-} \cup \psi^{\pi-}$$

$$(\varphi \to \psi)^{\pi+} = \varphi^{\pi-} \cup \psi^{\pi+} \qquad (\varphi \to \psi)^{\pi-} = \varphi^{\pi+} \cup \psi^{\pi-}$$

$$(!\varphi)^{\pi+} = \{\mathtt{rep}_{\mathbf{C}}(\varphi^{\pi+})\} \qquad (!\varphi)^{\pi-} = \{\mathtt{rep}_{\mathbf{P}}(\varphi^{\pi-})\}$$

Theorem

Every cut-free proof of sequent $\Pi \to \Delta$ in aILL translates into a success strategy for **P** for the multitask $(\Pi \to \Delta)^\pi$.

P-winability without linear provability

The sequent

$$a, b \Rightarrow (a \wedge b) \otimes (a \vee b)$$

translates into the multitask

$$\{a^!, b^!, \{a^?\} \vee_{\mathbf{C}} \{b^?\}, \{a^?\} \vee_{\mathbf{P}} \{b^?\}\}$$

Unprovability of the sequent: $\otimes$ calls for splitting $\{a, b\}$ right away

Winning strategy for **P**:
Wait for **C** to choose between $a^?$ and $b^?$.

NB: **P** only needs to be **available** if **C** is **responsive**!

A side remark:
$\neg a \vee \neg b \vee ((a \sqcap b) \wedge (a \sqcup b))$ is unwinable in Japaridze's game

Pure and conditional request/produce packages

Mixtures of requests and productions may be unintended?
Can we avoid them?

Pure request task: atoms are only requests ($a^?$)

Pure produce task: atoms are only productions ($a^!$)

Atomic conditonal task ($a^!|b^?$): **P** produces a, if b is requested by **C**

Conditional task ($A|B$): **P** performs task A if **C** performs task B

NB: We may keep conditional tasks (conditionally) pure
Classifying conditional tasks as **P**-choices preserves fairness

Claim: Conditional tasks correspond to linear implication!

Further Questions / Open Issues

- Is there an adequate calculus for **P**-**C**-games?
- Can one model asynchronous moves by imperfect information games?
- Which (types of) guarantees are natural in **P**-**C**-games?
- Interpreting full (classical) linear logic ('the challenge for $\parr$')
- Game based interpretations of (different types of) subexponentials
- A role for random choice connectives?
- . . . (Many more related topics!) . . .