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Adaptive FEM with quasi-optimal overall cost for nonsymmetric linear elliptic PDEs

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We consider a general nonsymmetric second-order linear elliptic partial differential equation in the framework of the Lax–Milgram lemma. We formulate and analyze an adaptive finite element algorithm with arbitrary polynomial degree that steers the adaptive meshrefinement and the inexact iterative solution of the arising linear systems. More precisely, the iterative solver employs, as an outer loop, the so-called Zarantonello iteration to symmetrize the system and, as an inner loop, a uniformly contractive algebraic solver, for example, an optimally preconditioned conjugate gradient method or an optimal geometric multigrid algorithm. We prove that the proposed inexact adaptive iteratively symmetrized finite element method leads to full linear convergence and, for sufficiently small adaptivity parameters, to optimal convergence rates with respect to the overall computational cost, i.e., the total computational time. Numerical experiments underline the theory.

Keywords: adaptive finite element method; iterative solver; nonsymmetric PDEs; optimal convergence rates; cost-optimality.

1. Introduction

The mathematical understanding of optimal adaptivity for finite element methods (AFEMs) has reached a high level of maturity; see, e.g., Binev *et al.* (2004); Stevenson (2007); Cascón *et al.* (2008); Kreuzer & Siebert (2011); Cascón & Nochetto (2012); Carstensen *et al.* (2014); Feischl *et al.* (2014) for some contributions to linear partial differential equations (PDEs). While the focus is usually on optimal convergence rates with respect to the degrees of freedom (Binev *et al.*, 2004; Cascón *et al.*, 2008; Kreuzer & Siebert, 2011; Cascón & Nochetto, 2012; Carstensen *et al.*, 2014; Feischl *et al.*, 2014), the cumulative nature of adaptivity should rather ask for optimal convergence rates with respect to the overall computational cost, i.e., the overall elapsed computational time. This, usually called *optimal complexity*, has been thoroughly analyzed for adaptive wavelet methods (Cohen *et al.*, 2001, 2003), and it has also been addressed in the seminal work (Stevenson, 2007) on AFEM for the Poisson model problem. Recent works (Gantner *et al.*, 2021; Haberl *et al.*, 2021; Heid *et al.*, 2021) considered optimal complexity for energy minimization problems and, in particular, for symmetric linear elliptic PDEs. In contrast to this, optimal complexity for nonsymmetric linear elliptic PDEs remained an open question due to the lack of a contractive algebraic solver that is compatible with the variational structure of the PDE. Closing this gap is the topic of the present work. While the canonical candidate for solving the nonsymmetric discrete systems would be GMRES, we take a different path that is motivated by up-to-date proofs of the Lax–Milgram lemma and closely related to the Richardson iteration used in the context of optimal adaptive wavelet methods. Some comments on the challenges presented by GMRES and related future work are given below.

As a model problem, we consider the nonsymmetric second-order linear elliptic PDE

$$-\mathrm{div}(\boldsymbol{A}\nabla u^\star) + \boldsymbol{b}\cdot\nabla u^\star + cu^\star = f - \mathrm{div}\boldsymbol{f} \quad \text{in } \Omega \quad \text{subject to} \quad u^\star = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

on a polyhedral Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \geq 1$, where $\boldsymbol{A} \in [L^\infty(\Omega)]^{d\times d}_{\mathrm{sym}}$ is a symmetric diffusion matrix, $\boldsymbol{b} \in [L^\infty(\Omega)]^d$ is a convection coefficient, $c \in L^\infty(\Omega)$ is a reaction coefficient and $f \in L^2(\Omega)$ and $\boldsymbol{f} \in [L^2(\Omega)]^d$ are the given data.

With $b(u, v) := \langle \boldsymbol{A}\nabla u, \nabla v\rangle_\Omega + \langle \boldsymbol{b}\cdot\nabla u + cu, v\rangle_\Omega$ and $F(v) := \langle f, v\rangle_\Omega + \langle \boldsymbol{f}, \nabla v\rangle_\Omega$, where $\langle\cdot,\cdot\rangle_\Omega$ denotes the usual $L^2(\Omega)$-scalar product, the weak formulation of (1.1) reads:

$$\text{Find } u^\star \in \mathcal{X} := H^1_0(\Omega) \quad \text{such that} \quad b(u^\star, v) = F(v) \quad \text{for all } v \in \mathcal{X}. \tag{1.2}$$

To ensure the existence and uniqueness of $u^\star \in H^1_0(\Omega)$, we assume that the bilinear form $b(\cdot,\cdot)$ is continuous and elliptic on $H^1_0(\Omega)$ so that the Lax–Milgram lemma applies.

To discretize (1.2), we employ a conforming finite element method based on a conforming simplicial triangulation $\mathcal{T}_\ell$ of Ω and a fixed polynomial degree $m \in \mathbb{N}$. With

$$\mathcal{X}_\ell := \big\{v_\ell \in H^1_0(\Omega) : v_\ell|_T \text{ is a polynomial of degree} \leq m, \text{ for all } T \in \mathcal{T}_\ell\big\},$$

the finite element formulation reads:

$$\text{Find } u^\star_\ell \in \mathcal{X}_\ell \quad \text{such that} \quad b\left(u^\star_\ell, v_\ell\right) = F(v_\ell) \quad \text{for all } v_\ell \in \mathcal{X}_\ell. \tag{1.3}$$

Existence and uniqueness of $u^\star_\ell$ follow again from the Lax–Milgram lemma. Note that (1.3) leads to a *nonsymmetric*, yet *positive definite* linear system of equations. To derive an optimal nonsymmetric algebraic solver, we follow the constructive proof of the Lax–Milgram lemma and reduce the discrete formulations (1.3) to symmetric problems by employing the so-called Zarantonello symmetrization (sometimes referred to as Banach–Picard fixed-point iteration). To this end, we define the bilinear form associated with the principal part of the PDE by

$$a(u, v) := \langle \boldsymbol{A}\nabla u, \nabla v\rangle_\Omega \quad \text{for all } u, v \in \mathcal{X}. \tag{1.4}$$

Note that $a(\cdot,\cdot)$ is continuous and elliptic on $\mathcal{X}$ and consult Section 2 for details. For a given damping parameter $\delta > 0$, define the Zarantonello mapping $\Phi_\ell(\delta;\cdot)\colon \mathcal{X}_\ell \to \mathcal{X}_\ell$ by

$$a(\Phi_\ell(\delta; u_\ell), v_\ell) = a(u_\ell, v_\ell) + \delta\left[F(v_\ell) - b(u_\ell, v_\ell)\right] \quad \text{for all } v_\ell \in \mathcal{X}_\ell; \tag{1.5}$$

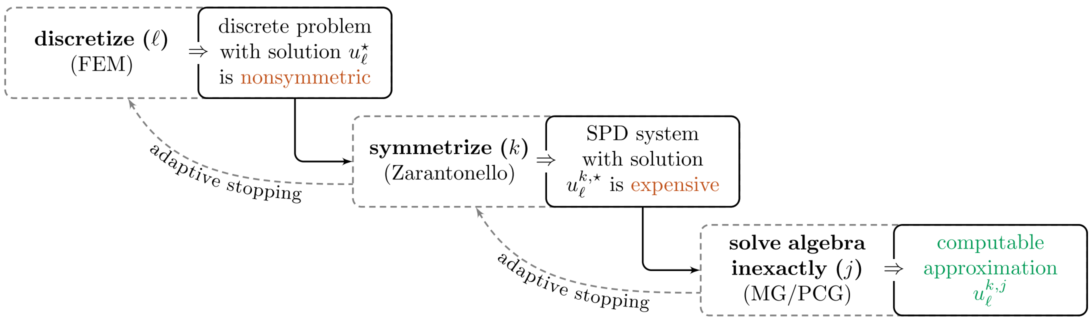

FIG. 1. Schematic view of the AISFEM algorithm components.

see Zarantonello (1960) or Zeidler (1990, Section 25.4). The Riesz–Fischer theorem (and also the Lax–Milgram lemma) proves existence and uniqueness of $\Phi_\ell(\delta; u_\ell) \in \mathscr{X}_\ell$, i.e., the Zarantonello operator is well-defined. In particular, $u_\ell^\star = \Phi(\delta; u_\ell^\star)$ is the only fixpoint of $\Phi(\delta; \cdot)$ for any $\delta > 0$. Moreover, choosing δ suitably will lead to a contractive method to approximate $u_\ell^\star$ in the spirit of the Banach fixpoint theorem with respect to the $a(\cdot,\cdot)$-induced energy norm $|||v||| := a(v,v)^{1/2}$. At this point, it thus remains to treat a symmetric, positive definite (SPD) linear system of equations corresponding to (1.5), which can be solved iteratively in practice for instance by the use of either a conjugate gradient (CG) method with an optimal preconditioner, see e.g., Chen *et al.* (2012), or an optimal geometric multigrid (MG) solver, see e.g., Jinbiao & Zheng (2017); Innerberger *et al.* (2022).

The proposed adaptive strategy of this work, hereafter referred to as adaptive iteratively symmetrized finite element method (AISFEM), begins with the initial guess $u_0^{0,0} := u_0^{0,\underline{j}} := u_0^{0,\star} := 0 \in \mathscr{X}_0$ associated to a coarse mesh $\mathscr{T}_0$. Finite element approximations $u_\ell^{k,j} \in \mathscr{X}_\ell$ are successively computed, where $\ell \in \mathbb{N}_0$ is the mesh-refinement index of the ℓth adaptively refined mesh. More precisely, $u_\ell^{k,j}$ is obtained after j algebraic solver steps in the kth step of the Zarantonello symmetrization approximating the unique $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}}) \in \mathscr{X}_\ell$, where $u_\ell^{k-1,\underline{j}} \in \mathscr{X}_\ell$ denotes the final approximation of $u_\ell^{k-1,\star}$ when the algebraic solver is *adaptively* terminated. In particular, our analysis provides stopping criteria for the algebraic solver as well as the (perturbed) Zarantonello symmetrization. We give a schematic view of our approach in Fig. 1; see Algorithm A in Section 3 below for the formal statement.

Overall, the adaptive strategy thus leads to a triple index set

$$\mathscr{Q} := \big\{(\ell,k,j) \in \mathbb{N}_0^3 : u_\ell^{k,j} \text{ is used by the AISFEM Algorithm A}\big\}, \tag{1.6}$$

equipped with the natural lexicographic ordering $|\cdot,\cdot,\cdot|$. This enables us to present the main contributions of this work: first, in the spirit of Gantner *et al.* (2021); Haberl *et al.* (2021), we prove that the quasi-error

$$\Delta_\ell^{k,j} := |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell\big(u_\ell^{k,j}\big) \quad \text{for all } (\ell,k,j) \in \mathscr{Q}, \tag{1.7}$$

which is the sum of the overall error plus the algebraic solver error plus the residual error estimator, is linearly convergent with respect to the ordering of $\mathscr{Q}$, i.e., $|\ell',k',j'| \le |\ell,k,j|$ means that $u_{\ell'}^{k',j'}$ is computed earlier than $u_\ell^{k,j}$ within the (sequential) adaptive loop and $|\ell,k,j| - |\ell',k',j'| \in \mathbb{N}_0$ is the overall number of discretization, symmetrization and algebraic solver steps in between. In explicit terms, Theorem 4.1 proves the existence of constants $C_{\rm lin} > 0$ and $0 < q_{\rm lin} < 1$ as well as an index $\ell_0 \in \mathbb{N}_0$ such that, for all $(\ell, k, j), (\ell', k', j') \in \mathcal{Q}$ with $|\ell, k, j| > |\ell', k', j'|$ and $\ell' \geq \ell_0$, there holds that

$$\Delta_\ell^{k,j} \leq C_{\rm lin} q_{\rm lin}^{|\ell,k,j|-|\ell',k',j'|} \, \Delta_{\ell'}^{k',j'}. \tag{1.8}$$

The threshold level $\ell_0 \in \mathbb{N}_0$ arises from the lack of Galerkin orthogonality with respect to the $a(\cdot,\cdot)$-induced energy norm leading to a more involved analysis. Secondly, as shown in Corollary 4.2, this implies that, for any $s > 0$, there holds the equivalence

$$\sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \, \Delta_\ell^{k,j} < \infty \quad \Longleftrightarrow \quad \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q} \\ |\ell',k',j'|\leq|\ell,k,j|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^{k,j} < \infty. \tag{1.9}$$

The interpretation of (1.9) is that the AISFEM algorithm leads to algebraic convergence rate $s > 0$ with respect to the degrees of freedom (finite left-hand side) if and only if it leads to algebraic convergence rate s with respect to the overall computational cost (finite right-hand side), i.e., with respect to the computational time. Thirdly, extending available results from the literature (Cascón & Nochetto, 2012; Feischl *et al.*, 2014; Bespalov *et al.*, 2017), Theorem 4.3 proves that, for sufficiently small adaptivity parameters, the proposed algorithm has optimal complexity (which follows from optimal rates with respect to the degrees of freedom and (1.9)). Finally, we admit that the proposed strategy hinges crucially on the appropriate (sufficiently small) choice of the Zarantonello parameter $\delta > 0$ in (1.5) as well as on the parameter $\lambda_{\rm alg} > 0$ in the stopping criterion for the algebraic solver in Algorithm A(i.b.II) below. If these parameters are chosen too large, the proposed method may fail to converge. Besides this restriction, linear convergence (1.8) is guaranteed for any choice of the other adaptivity parameters $\lambda_{\rm sym}, \theta, C_{\rm mark}$ (see Algorithm A below).

Outline

The remainder of the work is organized as follows. Section 2 focuses on the setting and underlying assumptions. In Section 3, we present the AISFEM algorithm in full detail and highlight some of its properties. The main results of this work are presented in Section 4, the proofs of which are given in Section 5. Numerical experiments in Section 6 underline the theoretical results, before the short Section 7 concludes our results and outlines future work.

2. Preliminaries

In this section, we state all prerequisites to formulate the AISFEM algorithm (Algorithm A in Section 3 below). In particular, we collect the contraction properties of the Zarantonello symmetrization, the algebraic solver, the mesh-refinement strategy and the required properties of the *a posteriori* error estimator.

2.1 *Abstract formulation of the model problem*

According to the Rellich compactness theorem (Kufner *et al.*, 1977, Theorem 5.8.2), $\langle \mathcal{K} u, v\rangle := \langle \boldsymbol{b} \cdot \nabla u + cu, v\rangle_\Omega$ defines a compact linear operator $\mathcal{K} : \mathcal{X} \to \mathcal{X}'$, where we recall that

$\mathcal{X}' = H^{-1}(\Omega)$ is the dual space of $\mathcal{X} = H^1_0(\Omega)$. With this notation, the weak formulation (1.2) takes the more abstract form

$$b(u^\star, v) = a(u^\star, v) + \langle \mathcal{K} u^\star, v \rangle = F(v) \quad \text{for all } v \in \mathcal{X}. \tag{2.1}$$

Since $b(\cdot, \cdot)$ is continuous and elliptic on $\mathcal{X}$, i.e., there exists $\alpha_0 > 0$ such that

$$\alpha_0 \, \|u\|^2_{\mathcal{X}} \le b(u, u) \quad \text{for all } u \in \mathcal{X}, \tag{2.2}$$

a simple compactness argument proves that also the principal part $a(\cdot, \cdot)$ is elliptic, i.e., there exists $\alpha'_0 > 0$ such that

$$\alpha'_0 \, \|u\|^2_{\mathcal{X}} \le a(u, u) \quad \text{for all } u \in \mathcal{X}; \tag{2.3}$$

see, e.g., Bespalov *et al.* (2017, Remark 3). In particular, $a(\cdot, \cdot)$ is a scalar product on $\mathcal{X}$ and the $a(\cdot, \cdot)$-induced energy norm $|||v|||^2 = a(v, v)$ is an equivalent norm on $\mathcal{X}$, i.e., $|||v||| \simeq \|v\|_{\mathcal{X}}$ for all $v \in \mathcal{X}$. Consequently, $b(\cdot, \cdot)$ is also elliptic and continuous with respect to $|||\cdot|||$, i.e., there exist (in practice unknown) constants $0 < \alpha \le L < \infty$ such that

$$\alpha \, |||u|||^2 \le b(u, u) \quad \text{and} \quad |b(u, v)| \le L \, |||u||| \, |||v||| \quad \text{for all } u, v \in \mathcal{X}. \tag{2.4}$$

While this setting already guarantees the Céa-type quasi-optimality of Galerkin solutions $u^\star_\ell \in \mathcal{X}_\ell \subset \mathcal{X}$ to (1.3), i.e.,

$$\left|\left|\left| u^\star - u^\star_\ell \right|\right|\right| \le C_{\text{Céa}} \min_{v_\ell \in \mathcal{X}_\ell} \left|\left|\left| u^\star - v_\ell \right|\right|\right| \quad \text{with} \quad C_{\text{Céa}} := L/\alpha, \tag{2.5}$$

we recall from Bespalov *et al.* (2017, Theorem 20) that adaptivity improves the constant $C_{\text{Céa}}$ in the Céa-type estimate (2.5): if $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ and $\left|\left|\left| u^\star - u^\star_\ell \right|\right|\right| \to 0$ as $\ell \to \infty$, then (2.5) holds with a constant $1 \le C_\ell \le L/\alpha$ and $C_\ell \to 1$ as $\ell \to \infty$.

Remark 2.1. The contractive Zarantonello symmetrization and hence the results of this work hold in an abstract framework beyond that of the introduction in Section 1. More precisely, the analysis allows for an abstract separable Hilbert space $\mathcal{X}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|_{\mathcal{X}}$ and a weak formulation (2.1), where $a(\cdot, \cdot)$ is a Hermitian and continuous sesquilinear form on $\mathcal{X}$ and $\mathcal{K} : \mathcal{X} \to \mathcal{X}'$ is a compact linear operator such that $b(\cdot, \cdot)$ is elliptic and continuous on $\mathcal{X}$. Provided that a contractive algebraic solver is used (see Section 2.5), the analysis thus also applies to other boundary conditions (e.g., mixed Dirichlet–Neumann–Robin instead of homogeneous Dirichlet boundary conditions used in the introduction).

2.2 *Mesh refinement*

From now on, let $\mathcal{T}_0$ be a given conforming triangulation of $\Omega \subset \mathbb{R}^d$ with $d \ge 1$, which is admissible in the sense of Stevenson (2008) for $d \ge 3$. For mesh refinement, we employ newest vertex bisection (NVB); see Aurada *et al.* (2015) for $d = 1$, Stevenson (2008) for $d \ge 1$ and Karkulik *et al.* (2013) for $d = 2$ with nonadmissible $\mathcal{T}_0$. For each triangulation $\mathcal{T}_H$ and marked elements $\mathcal{M}_H \subseteq \mathcal{T}_H$, let $\mathcal{T}_h := \texttt{refine}(\mathcal{T}_H, \mathcal{M}_H)$ be the coarsest triangulation where all $T \in \mathcal{M}_H$ have been refined, i.e., $\mathcal{M}_H \subseteq \mathcal{T}_H \backslash \mathcal{T}_h$. We write $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ if $\mathcal{T}_h$ results from $\mathcal{T}_H$ by finitely many steps of refinement and, for $N \in \mathbb{N}_0$, we write $\mathcal{T}_h \in \mathbb{T}_N(\mathcal{T}_H)$ if $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ and $\#\mathcal{T}_h - \#\mathcal{T}_H \le N$. To abbreviate notation, let $\mathbb{T} := \mathbb{T}(\mathcal{T}_0)$. Throughout, each triangulation $\mathcal{T}_H \in \mathbb{T}$ is associated with a finite-dimensional finite element space $\mathcal{X}_H \subset \mathcal{X}$, and we assume that $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$ implies nestedness $\mathcal{X}_H \subseteq \mathcal{X}_h \subset \mathcal{X}$.

Within the setting of AFEM, we will work with a hierarchy $\{\mathcal{T}_\ell\}_{\ell\in\mathbb{N}_0}$ generated by NVB refinements from the initial mesh $\mathcal{T}_0$.

2.3 A posteriori *error estimator and axioms of adaptivity*

For $\mathcal{T}_H \in \mathbb{T}$, let

$$\eta_H(T;\cdot)\colon \mathcal{X}_H \to \mathbb{R}_{\ge 0} \quad \text{for all } T \in \mathcal{T}_H \tag{2.6}$$

be the local contributions of some computable error estimator. We define

$$\eta_H(\mathcal{U}_H; v_H) := \Big(\sum_{T\in\mathcal{U}_H} \eta_H(T; v_H)^2 \Big)^{1/2} \quad \text{for all } \mathcal{U}_H \subseteq \mathcal{T}_H \text{ and } v_H \in \mathcal{X}_H.$$

To abbreviate notation, let $\eta_H(v_H) := \eta_H(\mathcal{T}_H; v_H)$. Furthermore, we suppose that η_H satisfies the following *axioms of adaptivity* from Carstensen *et al.* (2014) with constants $C_{\rm stab}, C_{\rm rel}, C_{\rm drel} > 0$ and $0 < q_{\rm red} < 1$ only depending on the dimension d, the polynomial degree m and γ-shape regularity of $\mathcal{T}_0$:

(A1) **stability:** For all $\mathcal{T}_H \in \mathbb{T}$ and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, all $v_h \in \mathcal{X}_h$ and all $v_H \in \mathcal{X}_H$ and every $\mathcal{U}_H \subseteq \mathcal{T}_H \cap \mathcal{T}_h$, it holds that

$$|\eta_h(\mathcal{U}_H, v_h) - \eta_H(\mathcal{U}_H, v_H)| \le C_{\rm stab} \, |||v_h - v_H|||;$$

(A2) **reduction:** For all $\mathcal{T}_H \in \mathbb{T}$ and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, and all $v_H \in \mathcal{X}_H$, it holds that

$$\eta_h\big(\mathcal{T}_h \setminus \mathcal{T}_H, v_H\big) \le q_{\rm red}\, \eta_H\big(\mathcal{T}_H \setminus \mathcal{T}_h, v_H\big);$$

(A3) **reliability:** For all $\mathcal{T}_H \in \mathbb{T}$, the exact solutions $u^\star \in \mathcal{X}$ of (1.2) and $u_H^\star \in \mathcal{X}_H$ of (1.3) satisfy that

$$\big|\big|\big| u^\star - u_H^\star \big|\big|\big| \le C_{\rm rel}\, \eta_H\left(u_H^\star\right);$$

(A4) **discrete reliability:** For all $\mathcal{T}_H \in \mathbb{T}$ and $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, the corresponding exact discrete solutions satisfy that

$$\big|\big|\big| u_h^\star - u_H^\star \big|\big|\big| \le C_{\rm drel}\, \eta_H\left(\mathcal{T}_H \backslash \mathcal{T}_h, u_H^\star\right).$$

We note that these axioms (A1)–(A4) are satisfied for the standard residual error estimators; see Section 6 below for the model problem (1.1) from the introduction.

2.4 *Contractive Zarantonello symmetrization*

It is well known (Zeidler, 1990, Section 25.4) that the Zarantonello mapping $\Phi_H(\delta;\cdot)$ introduced in (1.5) is a contraction for sufficiently small $\delta > 0$, i.e., for $0 < \delta < 2\alpha/L^2$ and all $u_H, w_H \in \mathcal{X}_H$, there holds

$$|||\Phi_H(\delta;u_H) - \Phi_H(\delta;w_H)||| \le q[\delta]\, |||u_H - w_H||| \quad \text{with} \quad q[\delta] := 1 - \delta(2\alpha - \delta L^2) < 1. \tag{2.7}$$

Theoretically, $\delta^\star := \alpha/L^2$ minimizes the expression in (2.7) resulting in $q[\delta^\star] = 1 - \alpha^2/L^2$; see, e.g., Heid & Wihler (2020).

2.5 *Contractive algebraic solver*

We assume that we have at hand an iterative algebraic solver with iteration step $\Psi_H\colon \mathcal{X}' \times \mathcal{X}_H \to \mathcal{X}_H$. This means, given a linear and continuous functional $G \in \mathcal{X}'$ and an approximation $w_H \in \mathcal{X}_H$ of the unique solution $w_H^\star \in \mathcal{X}_H$ to

$$a\left(w_H^\star, v_H\right) = G(v_H) \quad \text{for all } v_H \in \mathcal{X}_H, \tag{2.8}$$

the algebraic solver returns an improved $\Psi_H(G; w_H) \in \mathcal{X}_H$ in the sense that there exists a constant $0 < q_{\rm alg} < 1$, which is independent of G and $\mathcal{X}_H$, such that

$$\left|\left|\left| w_H^\star - \Psi_H(G; w_H)\right|\right|\right| \le q_{\rm alg} \left|\left|\left| w_H^\star - w_H\right|\right|\right|. \tag{2.9}$$

To simplify notation when the right-hand side G is complicated or lengthy (as for the Zarantonello iteration (1.5)), we shall write $\Psi_H(w_H^\star;\cdot)$ instead of $\Psi_H(G;\cdot)$, even though $w_H^\star$ is unknown and will never be computed.

In the framework of AFEM, possible examples for such contractive solvers include optimally preconditioned conjugate gradient methods or optimal geometric multigrid methods, see, e.g., Chen *et al.* (2012) or Jinbiao & Zheng (2017), respectively, for approaches focused on lowest-order discretizations and Innerberger *et al.* (2022) for an optimal multigrid method, which is also robust with respect to the polynomial degree.

3. Completely adaptive algorithm

In the following, we formulate an inexact AISFEM in the spirit of Haberl *et al.* (2021). For ease of presentation, we make the following conventions: Algorithm A defines certain terminal indices $\underline{\ell}$, $\underline{k}[\ell]$, $\underline{j}[\ell,k]$, indicated by underlining. We shall omit the arguments of $\underline{k}$ and $\underline{j}$ if these are clear from the context, e.g., we simply write

$$u_\ell^{k,\underline{j}} := u_\ell^{k,\underline{j}[\ell,k]} \quad \text{and} \quad u_\ell^{\underline{k},\underline{j}} := u_\ell^{\underline{k}[\ell],\underline{j}[\ell,\underline{k}[\ell]]}, \quad \text{etc.}$$

A similar convention will be used for triple indices, e.g., $(\ell,k,\underline{j}) = (\ell,k,\underline{j}[\ell,k])$, etc.

Remark 3.1. To give an interpretation of the stopping criteria in Step (i.b.II) and Step (i.d.) of Algorithm A, we note the following: since the algebraic solver is contractive (2.9), the term $|||u_\ell^{k,j} - u_\ell^{k,j-1}|||$ provides

Algorithm A: adaptive iteratively symmetrized finite element method (AISFEM)

Input: initial triangulation $\mathscr{T}_0$, initial guess $u_0^{0,0} := u_0^{0,\underline{j}} := 0$, marking parameters $0 < \theta \leq 1$ and $C_{\rm mark} \geq 1$, solver parameters $\lambda_{\rm sym}, \lambda_{\rm alg} > 0$ and damping parameter $\delta > 0$.

Loop: For $\ell = 0, 1, 2, \ldots$, repeat the following steps (i)–(iv):

- (i) For all $k = 1, 2, 3, \ldots$, repeat the following steps (a)–(d):
  - (a) Define $u_\ell^{k,0} := u_\ell^{k-1,\underline{j}}$ and, for purely theoretical reasons, $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$.
  - (b) For all $j = 1, 2, 3, \ldots$ repeat the following steps (*I*)–(*II*):
    - (I) Compute $u_\ell^{k,j} := \Psi_\ell(u_\ell^{k,\star}; u_\ell^{k,j-1})$ and $\eta_\ell(T; u_\ell^{k,j})$ for all $T \in \mathscr{T}_\ell$.
    - (II) Terminate j-loop if $|||u_\ell^{k,j} - u_\ell^{k,j-1}||| \leq \lambda_{\rm alg} \big[\lambda_{\rm sym}\, \eta_\ell\big(u_\ell^{k,j}\big) + |||u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}|||\big]$.
  - (c) Upon termination of the j-loop, define $\underline{j}[\ell, k] := j$.
  - (d) Terminate k-loop if $|||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| \leq \lambda_{\rm sym}\, \eta_\ell\big(u_\ell^{k,\underline{j}}\big)$.
- (ii) Upon termination of the k-loop, define $\underline{k}[\ell] := k$.
- (iii) Determine $\mathscr{M}_\ell \subseteq \mathscr{T}_\ell$ of up to the constant $C_{\rm mark}$ minimal cardinality, satisfying $\theta\, \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 \leq \eta_\ell(\mathscr{M}_\ell; u_\ell^{\underline{k},\underline{j}})^2$.
- (iv) Generate $\mathscr{T}_{\ell+1} := \texttt{refine}(\mathscr{T}_\ell, \mathscr{M}_\ell)$ and define $u_{\ell+1}^{0,0} := u_{\ell+1}^{0,\underline{j}} := u_{\ell+1}^{0,\star} := u_\ell^{\underline{k},\underline{j}}$.

Output: discrete approximations $u_\ell^{k,j}$ and corresponding error estimators $\eta_\ell(u_\ell^{k,j})$.

a posteriori error control on the algebraic error $|||u_\ell^{k,\star} - u_\ell^{k,j}|||$, i.e.,

$$|||u_\ell^{k,\star} - u_\ell^{k,j}||| \leq \frac{q_{\rm alg}}{1 - q_{\rm alg}}\, |||u_\ell^{k,j} - u_\ell^{k,j-1}|||.$$

Moreover, for sufficiently small $\lambda_{\rm alg} > 0$, also the perturbed Zarantonello symmetrization is a contraction; see Lemma 5.1 below. With the same reasoning as for the algebraic solver, the term $|||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| = |||u_\ell^{k,\underline{j}} - u_\ell^{k,0}|||$ thus provides *a posteriori* error control of the symmetrization error $|||u_\ell^\star - u_\ell^{k,\underline{j}}|||$. With this understanding and the interpretation that the error estimator $\eta_\ell(u_\ell^{k,j})$ controls the discretization error $|||u^\star - u_\ell^\star|||$ (which is indeed true for $u_\ell^{k,j} = u_\ell^{\underline{k},\underline{j}}$), the heuristics behind the stopping criteria is as follows: We stop the algebraic solver in Algorithm A(i.b.II) provided that the algebraic error $|||u_\ell^{k,\star} - u_\ell^{k,j}|||$ is of the level of the discretization error plus the symmetrization error. Moreover, we stop the (perturbed) Zarantonello symmetrization in Algorithm A(i.d.) provided that the symmetrization error $|||u_\ell^\star - u_\ell^{k,\underline{j}}|||$ is of the level of the discretization error. Up to the factors $\lambda_{\rm alg}$ and $\lambda_{\rm sym}$, this ensures that all three error sources of $|||u^\star - u_\ell^{\underline{k},\underline{j}}|||$ are equibalanced.

For the analysis of Algorithm A, we recall that the set $\mathscr{Q}$ from (1.6) is given by

$$\mathscr{Q} := \big\{(\ell, k, j) \in \mathbb{N}_0^3 : u_\ell^{k,j} \text{ is used in Algorithm A}\big\}.$$

Together with this set, we define

$$\underline{\ell} := \sup\{\ell \in \mathbb{N}_0 : (\ell, 0, 0) \in \mathscr{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, \tag{3.1a}$$

$$\underline{k}[\ell] := \sup\{k \in \mathbb{N}_0 : (\ell, k, 0) \in \mathscr{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, \quad \text{whenever } (\ell, 0, 0) \in \mathscr{Q}, \tag{3.1b}$$

$$\underline{j}[\ell, k] := \sup\{j \in \mathbb{N}_0 : (\ell, k, j) \in \mathscr{Q}\} \in \mathbb{N}_0 \cup \{\infty\}, \quad \text{whenever } (\ell, k, 0) \in \mathscr{Q}. \tag{3.1c}$$

Note that these definitions are consistent with that of Algorithm A, but also cover the cases that the ℓ-loop, the k-loop or the j-loop in the algorithm do not terminate, respectively. We note that formally $\#\mathscr{Q} = \infty$ and hence either $\underline{\ell} = \infty$ or $\underline{k}[\underline{\ell}] = \infty$ or $\underline{j}[\underline{\ell}, \underline{k}[\underline{\ell}]] = \infty$, where the latter case is excluded by Lemma 3.2.

On $\mathscr{Q}$, we define an ordering by

$$(\ell', k', j') \le (\ell, k, j) \quad \Longleftrightarrow \quad u_{\ell'}^{k',j'} \text{ is computed earlier in Algorithm A than } u_\ell^{k,j}.$$

Furthermore, we introduce the total step counter $|\cdot,\cdot,\cdot|$, defined for all $(\ell, k, j) \in \mathscr{Q}$, by

$$|\ell, k, j| := \#\{(\ell', k', j') \in \mathscr{Q} : (\ell', k', j') \le (\ell, k, j)\} \in \mathbb{N}_0. \tag{3.2}$$

Our first observation is that the algebraic solver in the innermost loop of Algorithm A always terminates.

Lemma 3.2. Independently of the adaptivity parameters θ, λ_{sym} and λ_{alg}, the j-loop of Algorithm A always terminates, i.e., $\underline{j}[\ell, k] < \infty$ for all $(\ell, k, 0) \in \mathscr{Q}$.

Proof. Let $(\ell, k, 0) \in \mathscr{Q}$. We argue by contradiction and assume that the stopping criterion in Algorithm A(i.b.II) always fails and hence $\underline{j}[\ell, k] = \infty$. By assumption (2.9), the algebraic solver is contractive and hence convergent with limit $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$. Moreover, by failure of the stopping criterion in Algorithm A(i.b.II), we thus obtain that

$$\eta_\ell\big(u_\ell^{k,j}\big) + \big|\!\big|\!\big| u_\ell^{k,j} - u_\ell^{k-1,\underline{j}} \big|\!\big|\!\big| \lesssim \big|\!\big|\!\big| u_\ell^{k,j} - u_\ell^{k,j-1} \big|\!\big|\!\big| \xrightarrow{j\to\infty} 0.$$

This yields $\big|\!\big|\!\big| u_\ell^{k,\star} - u_\ell^{k-1,\underline{j}} \big|\!\big|\!\big| = 0$. Consequently, $u_\ell^{k-1,\underline{j}}$ is a fixpoint of $\Phi_\ell(\delta; \cdot)$, cf. Algorithm A(i.a), and hence $u_\ell^{k-1,\underline{j}} = u_\ell^\star$ by uniqueness of the fixpoint. In particular, the initial guess $u_\ell^{k,0} = u_\ell^{k-1,\underline{j}} = u_\ell^{k,\star}$ is already the exact solution of the linear Zarantonello system and hence the algebraic solver guarantees that $u_\ell^{k,j} = u_\ell^{k,\star}$ for all $j \in \mathbb{N}_0$. Consequently, the stopping criterion in Algorithm A(i.b.II) will be satisfied for $j = 1$. This contradicts our assumption, and hence we conclude that $\underline{j}[\ell, k] < \infty$. □

Remark 3.3. For the mathematical tractability, we formulated Algorithm A in a way that $\#\mathscr{Q} = \infty$. Any practical implementation will aim to provide a sufficiently accurate approximation $u_\ell^{k,j}$ in finite time. More precisely, Algorithm A will then be terminated after Algorithm A(i.b.II) if

$$\eta_{\underline{\ell}}(u_{\underline{\ell}}^{\underline{k},\underline{j}}) + |||u_{\underline{\ell}}^{\underline{k},\underline{j}} - u_{\underline{\ell}}^{\underline{k},0}||| + |||u_{\underline{\ell}}^{\underline{k},\underline{j}} - u_{\underline{\ell}}^{\underline{k},\underline{j}-1}||| \le \tau, \tag{3.3}$$

where $\tau > 0$ is a user-specified tolerance. For $\tau = 0$, finite termination yields that $u_{\underline{\ell}}^{\underline{k},\underline{j}} = u^\star$ with $\eta_{\underline{\ell}}(u_{\underline{\ell}}^{\underline{k},\underline{j}}) = 0$. To see this, note that (3.3) implies $u_{\underline{\ell}}^{\underline{k},\star} = u_{\underline{\ell}}^{\underline{k},\underline{j}} = u_{\underline{\ell}}^{\underline{k},\underline{j}-1}$ and $u_{\underline{\ell}}^\star = u_{\underline{\ell}}^{\underline{k},\underline{j}} = u_{\underline{\ell}}^{\underline{k}-1,\underline{j}}$ by uniqueness of the fixpoint of the contractive solver and the contractive Zarantonello symmetrization, respectively. Finally, the first summand in (3.3) states $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = \eta_{\underline{\ell}}(u_{\underline{\ell}}^{\underline{k},\underline{j}}) = 0$ and hence $u_{\underline{\ell}}^{\underline{k},\underline{j}} = u_{\underline{\ell}}^\star = u^\star$ by reliability (A3) of the estimator.

REMARK 3.4. Up to the algebraic stopping criterion in Algorithm A(i.b.II), the AISFEM algorithm coincides with the adaptive algorithm from Haberl *et al.* (2021), where the (perturbed) Zarantonello iteration is employed for an adaptive iteratively linearized finite element method for the solution of an energy minimization problem with strongly monotone nonlinearity. However, the present analysis is much more refined than that of Haberl *et al.* (2021):

(i) To guarantee full linear convergence, Haberl *et al.* (2021, Theorem 4) requires θ sufficiently small, $\lambda_{\rm sym}$ sufficiently small with respect to θ and $\lambda_{\rm alg}$ sufficiently small with respect to $\lambda_{\rm sym}$. In contrast, we require $\lambda_{\rm alg}$ to be sufficiently small with respect to $0 < q_{\rm alg} < 1$ and $0 < q_{\rm sym} < 1$ to preserve the contraction of the perturbed Zarantonello iteration (see Lemma 5.1 below in comparison to Haberl *et al.*, 2021, Lemma 6), while θ and $\lambda_{\rm sym}$ can be arbitrary.

(ii) Despite the linear model problem, our analytical setting is more involved: the compact perturbation in (2.1) prevents the use of energy arguments that guarantee a Pythagorean-type identity in terms of the energy error (see, e.g., Haberl *et al.*, 2021; Heid *et al.*, 2021). Instead, we first need to show plain convergence of Algorithm A (see Proposition 5.3) to deduce a quasi-Pythagorean estimate in Lemma 5.4, which then allows proving linear convergence (Theorem 4.1). As a consequence (and beyond the results of Haberl *et al.*, 2021), this finally yields that, for arbitrary θ and $\lambda_{\rm sym}$, the convergence rates with respect to the number of the degrees of freedom and with respect to the overall computational work coincide (Corollary 4.2).

The following proposition provides a computable upper bound for the energy error $|||u^\star - u_\ell^{k,j}|||$. Since Algorithm A follows the structure of Haberl *et al.* (2021, Algorithm 1), the proof can be obtained analogously to Haberl *et al.* (2021, Proposition 2) and is thus omitted here.

PROPOSITION 3.5. (reliable error control) Suppose that the estimator satisfies (A1) and (A3). Then, for all $(\ell, k, j) \in \mathcal{Q}$, it holds that

$$|||u^\star - u_\ell^{k,j}||| \le C'_{\rm rel} \begin{cases} \eta_\ell(u_\ell^{k,j}) + |||u_\ell^{k,j} - u_\ell^{k,0}||| + |||u_\ell^{k,j} - u_\ell^{k,j-1}||| & \text{if } 1 \le k \le \underline{k}[\ell] \text{ and } 1 \le j < \underline{j}[\ell,k], \\ \eta_\ell(u_\ell^{k,\underline{j}}) + |||u_\ell^{k,\underline{j}} - u_\ell^{k,0}||| & \text{if } 1 \le k \le \underline{k}[\ell] \text{ and } j = \underline{j}[\ell,k], \\ \eta_\ell(u_\ell^{\underline{k},\underline{j}}) & \text{if } k = \underline{k}[\ell] \text{ and } j = \underline{j}[\ell,\underline{k}], \\ \eta_{\ell-1}(u_{\ell-1}^{\underline{k},\underline{j}}) & \text{if } \ell > 0 \text{ and } k = 0. \end{cases} \tag{3.4}$$

The constant $C'_{\rm rel} > 0$ depends only on $C_{\rm rel}$, $C_{\rm stab}$, $q_{\rm alg}$, $\lambda_{\rm alg}$, $q_{\rm sym}$ and $\lambda_{\rm sym}$.

4. Main results

In the following, we formulate the main results of the present work. We refer to Section 5 for the proofs and Section 6 for numerical experiments, which underline these theoretical results. First, recall from (2.7) that a sufficiently small parameter $\delta > 0$ ensures contraction of the Zarantonello mapping and hence

$$|||u_\ell^\star - u_\ell^{k,\star}||| \le q_{\rm sym}\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \quad \text{for all } (\ell,k,0) \in \mathcal{Q} \tag{4.1}$$

with $0 < q_{\rm sym} < 1$. The following theorem states full linear convergence of the quasi-error.

THEOREM 4.1. (full linear convergence of AISFEM) Suppose that $\delta > 0$ is sufficiently small and that the estimator satisfies (A1)–(A3). Choose $\lambda_{\rm alg}^\star > 0$ depending only on $q_{\rm alg}$ from (2.9) and $q_{\rm sym}$ from (4.1) such that

$$0 < \overline{q}_{\rm sym} := \frac{q_{\rm sym} + 2\,\frac{q_{\rm alg}}{1-q_{\rm alg}}\,\lambda_{\rm alg}^\star}{1 - 2\,\frac{q_{\rm alg}}{1-q_{\rm alg}}\,\lambda_{\rm alg}^\star} < 1. \tag{4.2}$$

Then, for arbitrary $0 < \theta \le 1$ and $\lambda_{\rm sym} > 0$, but sufficiently small $\lambda_{\rm alg}$, satisfying $0 < \lambda_{\rm alg} \le \lambda_{\rm alg}^\star$, Algorithm A guarantees full linear convergence: there exist constants $C_{\rm lin} > 0$ and $0 < q_{\rm lin} < 1$ as well as an index $\ell_0 \in \mathbb{N}_0$ with $\ell_0 \le \underline{\ell}$ such that the quasi-error

$$\Delta_\ell^{k,j} := |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell\big(u_\ell^{k,j}\big) \quad \text{for all } (\ell,k,j) \in \mathcal{Q} \tag{4.3}$$

satisfies that, for all $(\ell,k,j), (\ell',k',j') \in \mathcal{Q}$ with $|\ell,k,j| > |\ell',k',j'|$ and $\ell' \ge \ell_0$,

$$\Delta_\ell^{k,j} \le C_{\rm lin} q_{\rm lin}^{|\ell,k,j|-|\ell',k',j'|}\, \Delta_{\ell'}^{k',j'}. \tag{4.4}$$

The constants $C_{\rm lin}$ and $q_{\rm lin}$ as well as the index ℓ_0 depend only on $C_{\rm stab}$, $C_{\rm rel}$, $q_{\rm red}$, $q_{\rm sym}$, $q_{\rm alg}$, θ, $\lambda_{\rm sym}$, $\lambda_{\rm alg}$ and $C_{\rm Céa} = L/\alpha$.

While the proof of Theorem 4.1 is postponed to Section 5.5, we shall immediately prove the following important consequence of Theorem 4.1: Algorithm A guarantees that rates with respect to the number of degrees of freedom coincide with rates with respect to the overall computational cost.

COROLLARY 4.2. Let $s > 0$. Under the assumptions of Theorem 4.1, the output of Algorithm A guarantees that

$$M(s) := \sup_{\substack{(\ell,k,j)\in\mathcal{Q}\\ \ell\ge\ell_0}} (\#\mathcal{T}_\ell)^s\, \Delta_\ell^{k,j} \le \sup_{\substack{(\ell,k,j)\in\mathcal{Q}\\ \ell\ge\ell_0}} \Bigg(\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|\\ \ell'\ge\ell_0}} \#\mathcal{T}_{\ell'} \Bigg)^s \Delta_\ell^{k,j} \le \frac{C_{\rm lin}}{\big(1-q_{\rm lin}^{1/s}\big)^s}\, M(s). \tag{4.5}$$

This yields the equivalence

$$\sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\,\Delta_\ell^{k,j} < \infty \quad\Longleftrightarrow\quad \sup_{(\ell,k,j)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|}} \#\mathcal{T}_{\ell'}\Big)^s\,\Delta_\ell^{k,j} < \infty. \tag{4.6}$$

Proof. The lower bound in (4.5) is obvious. To prove the upper bound, without loss of generality, we may assume that $M(s) < \infty$. By definition of $M(s)$, it follows that

$$\#\mathcal{T}_{\ell'} \le M(s)^{1/s}\big[\Delta_{\ell'}^{k',j'}\big]^{-1/s} \quad \text{for } (\ell',k',j') \in \mathcal{Q} \text{ with } \ell' \ge \ell_0. \tag{4.7}$$

For $|\ell,k,j| \ge |\ell',k',j'|$ and $\ell' \ge \ell_0$, full linear convergence (4.4) can be rewritten as

$$\big[\Delta_{\ell'}^{k',j'}\big]^{-1/s} \le C_{\rm lin}^{1/s}\big[q_{\rm lin}^{1/s}\big]^{|\ell,k,j|-|\ell',k',j'|}\,\big[\Delta_\ell^{k,j}\big]^{-1/s}. \tag{4.8}$$

The geometric series yields that

$$\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|\\ \ell'\ge\ell_0}} \#\mathcal{T}_{\ell'} \overset{(4.7)}{\le} M(s)^{1/s} \sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|\\ \ell'\ge\ell_0}} \big[\Delta_{\ell'}^{k',j'}\big]^{-1/s} \overset{(4.8)}{\le} M(s)^{1/s}\,C_{\rm lin}^{1/s}\,\frac{1}{1-q_{\rm lin}^{1/s}}\,\big[\Delta_\ell^{k,j}\big]^{-1/s}.$$

Rearranging this estimate, we see that

$$\Big(\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\le|\ell,k,j|\\ \ell'\ge\ell_0}} \#\mathcal{T}_{\ell'}\Big)^s\,\Delta_\ell^{k,j} \le M(s)\,C_{\rm lin}\,\frac{1}{\big(1-q_{\rm lin}^{1/s}\big)^s}.$$

Taking the supremum over all $(\ell,k,j) \in \mathcal{Q}$ with $\ell \ge \ell_0$, we prove the second estimate in (4.5). Moreover,

$$\mathcal{Q}\backslash\{(\ell,k,j)\in\mathcal{Q} : \ell \ge \ell_0\} = \{(\ell,k,j)\in\mathcal{Q} : \ell < \ell_0\} \quad \text{is finite,}$$

i.e., the sets over which we compute the suprema in (4.5)–(4.6) differ only by finitely many index triples. This and (4.5) thus prove the equivalence in (4.6). □

To present our second main result, we first introduce the notion of approximation classes. For $\mathcal{T} \in \mathbb{T}$ and $s > 0$, define

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T})} := \sup_{N\in\mathbb{N}_0}\Big((N+1)^s \min_{\mathcal{T}_{\rm opt}\in\mathbb{T}_N(\mathcal{T})} \big[|\!|\!|u^\star - u^\star_{\rm opt}|\!|\!| + \eta_{\rm opt}\big(u^\star_{\rm opt}\big)\big]\Big), \tag{4.9}$$

with $\eta_{\rm opt}$ denoting the estimator on the optimal triangulation $\mathcal{T}_{\rm opt} \in \mathbb{T}_N(\mathcal{T})$. When (4.9) is finite, this means that a decrease of the error plus estimator with rate s is possible along optimal meshes obtained by refining $\mathcal{T}$.

THEOREM 4.3. (optimal computational complexity) Suppose that $\delta > 0$ is sufficiently small and that the estimator satisfies (A1)–(A4). Consider $\lambda_{\rm alg}^\star > 0$ and $0 < \overline{q}_{\rm sym} < 1$ as in Theorem 4.1. Let $0 < \lambda_{\rm alg} \leq \lambda_{\rm alg}^\star$. Define $\lambda_{\rm sym}^\star := (1 - \overline{q}_{\rm sym})/(\overline{q}_{\rm sym}\, C_{\rm stab})$. Let $0 < \theta < \theta^\star := (1 + C_{\rm stab}^2\, C_{\rm rel}^2)^{-1} < 1$ and choose $0 < \lambda_{\rm sym} < \lambda_{\rm sym}^\star$ sufficiently small such that

$$0 < \theta_{\rm mark} := \left(\frac{\theta^{1/2} + \lambda_{\rm sym}/\lambda_{\rm sym}^\star}{1 - \lambda_{\rm sym}/\lambda_{\rm sym}^\star}\right)^2 < \theta^\star. \tag{4.10}$$

Then, Algorithm A guarantees, for all $s > 0$, that

$$c_{\rm opt}\, \|u^\star\|_{\mathbb{A}_s(\mathscr{T}_0)} \leq \sup_{(\ell,k,j)\in\mathscr{Q}} \Bigg(\sum_{\substack{(\ell',k',j')\in\mathscr{Q} \\ |\ell',k',j'|\leq|\ell,k,j|}} \#\mathscr{T}_{\ell'} \Bigg)^s \Delta_\ell^{k,j}, \tag{4.11a}$$

$$\sup_{\substack{(\ell,k,j)\in\mathscr{Q} \\ \ell\geq\ell_0}} \Bigg(\sum_{\substack{(\ell',k',j')\in\mathscr{Q} \\ |\ell',k',j'|\leq|\ell,k,j| \\ \ell'\geq\ell_0}} \#\mathscr{T}_{\ell'} \Bigg)^s \Delta_\ell^{k,j} \leq C_{\rm opt}\, \max\big\{\|u^\star\|_{\mathbb{A}_s(\mathscr{T}_{\ell_0})}, \Delta_{\ell_0}^{0,0}\big\}, \tag{4.11b}$$

where $\ell_0 \in \mathbb{N}$ is the index from Theorem 4.1. The constant $c_{\rm opt} > 0$ depends only on $C_{\rm Céa} = L/\alpha$, $C_{\rm stab}$, $C_{\rm rel}$, $C_{\rm child}$ and s; $C_{\rm opt} > 0$ depends only on $C_{\rm stab}$, $C_{\rm rel}$, $C_{\rm mark}$, $C_{\rm Céa} = L/\alpha$, $C'_{\rm rel}$, $C_{\rm mesh}$, $C_{\rm lin}$, $q_{\rm lin}$, $\#\mathscr{T}_{\ell_0}$, $q_{\rm red}$, $\lambda_{\rm sym}$, $\overline{q}_{\rm sym}$, θ and s. In particular, this proves the equivalence

$$\|u^\star\|_{\mathbb{A}_s(\mathscr{T}_0)} < \infty \quad \Longleftrightarrow \quad \sup_{(\ell,k,j)\in\mathscr{Q}} \Bigg(\sum_{\substack{(\ell',k',j')\in\mathscr{Q} \\ |\ell',k',j'|\leq|\ell,k,j|}} \#\mathscr{T}_{\ell'} \Bigg)^s \Delta_\ell^{k,j} < \infty, \tag{4.12}$$

which yields optimal complexity of Algorithm A.

The proof is postponed to Section 5.6.

5. Proofs

5.1 *Contraction of perturbed Zarantonello symmetrization*

Recall that, for $\delta < 2\,\delta^\star$, the Zarantonello mapping is a contraction (2.7). However, Algorithm A does not compute $u_\ell^{k,\star} := \Phi_\ell(\delta; u_\ell^{k-1,\underline{j}})$ exactly, but relies on an approximation $u_\ell^{k,\underline{j}} \approx u_\ell^{k,\star}$. The next lemma states that, for a sufficiently small stopping parameter $\lambda_{\rm alg} > 0$ in Algorithm A, the Zarantonello symmetrization remains a contraction under this perturbation. Its proof essentially follows along the lines of Haberl *et al.* (2021, Lemma 6). However, the present work considers a stopping criterion of the algebraic solver in Algorithm A(i.b.II), which allows to choose $\lambda_{\rm alg}$ independently of $\lambda_{\rm sym}$.

LEMMA 5.1. Let $\lambda_{\rm alg}^\star > 0$ and $0 < \overline{q}_{\rm sym} < 1$ as in Theorem 4.1. Then, for all stopping parameters $0 < \lambda_{\rm alg} \le \lambda_{\rm alg}^\star$ and $\lambda_{\rm sym} > 0$, it holds that

$$|\!|\!| u_\ell^\star - u_\ell^{k,\underline{j}} |\!|\!| \le \overline{q}_{\rm sym}\, |\!|\!| u_\ell^\star - u_\ell^{k-1,\underline{j}} |\!|\!| \quad \text{for all } (\ell, k, 0) \in \mathcal{Q} \text{ with } 1 \le k < \underline{k}[\ell]. \tag{5.1}$$

Proof. By using the triangle inequality and the contraction (4.1) of the unperturbed Zarantonello iteration, we obtain that

$$|\!|\!| u_\ell^\star - u_\ell^{k,\underline{j}} |\!|\!| \le |\!|\!| u_\ell^\star - u_\ell^{k,\star} |\!|\!| + |\!|\!| u_\ell^{k,\star} - u_\ell^{k,\underline{j}} |\!|\!| \stackrel{(4.1)}{\le} q_{\rm sym}\, |\!|\!| u_\ell^\star - u_\ell^{k-1,\underline{j}} |\!|\!| + |\!|\!| u_\ell^{k,\star} - u_\ell^{k,\underline{j}} |\!|\!|. \tag{5.2}$$

It remains to treat the algebraic error term and to show that it is sufficiently contractive. We use the contraction (2.9) of the algebraic solver, i.e.,

$$|\!|\!| u_\ell^{k,\star} - u_\ell^{k,\underline{j}} |\!|\!| \le q_{\rm alg}\, |\!|\!| u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1} |\!|\!|, \tag{5.3}$$

the met algebraic stopping criterion of Algorithm A(i.b.II), and the not met stopping criterion of Algorithm A(i.d.), to obtain that

$$\begin{aligned} |\!|\!| u_\ell^{k,\star} - u_\ell^{k,\underline{j}} |\!|\!| &\stackrel{(5.3)}{\le} \frac{q_{\rm alg}}{1-q_{\rm alg}}\, |\!|\!| u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1} |\!|\!| \stackrel{\text{(i.b.II)}}{\le} \lambda_{\rm alg} \frac{q_{\rm alg}}{1-q_{\rm alg}} \big[\lambda_{\rm sym}\, \eta_\ell(u_\ell^{k,\underline{j}}) + |\!|\!| u_\ell^{k,\underline{j}} - u_\ell^{k,0} |\!|\!|\big] \\ &< 2\,\lambda_{\rm alg} \frac{q_{\rm alg}}{1-q_{\rm alg}}\, |\!|\!| u_\ell^{k,\underline{j}} - u_\ell^{k,0} |\!|\!| \le 2\,\lambda_{\rm alg} \frac{q_{\rm alg}}{1-q_{\rm alg}} \big[|\!|\!| u_\ell^\star - u_\ell^{k,\underline{j}} |\!|\!| + |\!|\!| u_\ell^\star - u_\ell^{k-1,\underline{j}} |\!|\!| \big]. \end{aligned}$$

Combining the last estimate with (5.2) and rearranging the terms lead us to

$$|\!|\!| u_\ell^\star - u_\ell^{k,\underline{j}} |\!|\!| \le \frac{q_{\rm sym} + 2\,\lambda_{\rm alg} \frac{q_{\rm alg}}{1-q_{\rm alg}}}{1 - 2\,\lambda_{\rm alg} \frac{q_{\rm alg}}{1-q_{\rm alg}}}\, |\!|\!| u_\ell^\star - u_\ell^{k-1,\underline{j}} |\!|\!| \le \overline{q}_{\rm sym}\, |\!|\!| u_\ell^\star - u_\ell^{k-1,\underline{j}} |\!|\!|.$$

This concludes the proof. □

An important consequence of the contraction (5.1) of the perturbed Zarantonello iteration is that $\underline{k}[\underline{\ell}] = \infty$ implies that the exact solution is already discrete $u^\star = u_{\underline{\ell}}^\star \in \mathcal{X}_{\underline{\ell}}$.

LEMMA 5.2. Suppose that the estimator satisfies stability (A1) and reliability (A3), and that the perturbed Zarantonello iteration is contractive (5.1). Then, $\underline{\ell} < \infty$ implies that $\underline{k}[\underline{\ell}] = \infty$ as well as $u^\star = u_{\underline{\ell}}^\star$ with $\eta_{\underline{\ell}}(u_{\underline{\ell}}^\star) = 0$.

Proof. Since $\underline{j}[\underline{\ell},\underline{k}] < \infty$ by virtue of Lemma 3.2, it follows for $\underline{\ell} < \infty$ that $\underline{k}[\underline{\ell}] = \infty$ and hence

$$\eta_{\underline{\ell}}\big(u_{\underline{\ell}}^{k,\underline{j}}\big) < \lambda_{\mathrm{sym}}^{-1} \,|||u_{\underline{\ell}}^{k,\underline{j}} - u_{\underline{\ell}}^{k,0}||| \quad \text{for all } k \in \mathbb{N}.$$

Since the perturbed Zarantonello iteration is convergent with limit $u_{\underline{\ell}}^{\star}$ (and thus $(u_{\underline{\ell}}^{k,\underline{j}})_{k\in\mathbb{N}_0}$ is a Cauchy sequence), we infer that

$$\eta_{\underline{\ell}}(u_{\underline{\ell}}^{\star}) \overset{\text{(A1)}}{\leq} \eta_{\underline{\ell}}\big(u_{\underline{\ell}}^{k,\underline{j}}\big) + C_{\mathrm{stab}}\,|||u_{\underline{\ell}}^{\star} - u_{\underline{\ell}}^{k,\underline{j}}||| \xrightarrow{k\to\infty} 0.$$

This proves $\eta_{\underline{\ell}}(u_{\underline{\ell}}^{\star}) = 0$, whence with reliability (A3), we conclude $u_{\underline{\ell}}^{\star} = u^{\star}$. □

5.2 *Plain convergence*

For general second-order linear elliptic PDEs, a plain convergence result (for the exact Galerkin solutions) is required to ensure that there holds a quasi-Pythagorean estimate; see Lemma 5.4 below.

PROPOSITION 5.3. (plain convergence) Suppose that the perturbed Zarantonello iteration is contractive (5.1) and that the estimator satisfies (A1)–(A3). Then, it follows that

$$\begin{aligned} |||u^{\star} - u_{\underline{\ell}}^{\star}||| + |||u^{\star} - u_{\underline{\ell}}^{k,\underline{j}}||| + \eta_{\ell}\big(u_{\underline{\ell}}^{k,\underline{j}}\big) &\xrightarrow{k\to\underline{k}} 0 \quad \text{for } \underline{\ell} < \infty,\\ |||u^{\star} - u_{\ell}^{\star}||| + |||u^{\star} - u_{\ell}^{k,\underline{j}}||| + \eta_{\ell}\big(u_{\ell}^{k,\underline{j}}\big) &\xrightarrow{\ell\to\infty} 0 \quad \text{for } \underline{\ell} = \infty. \end{aligned} \tag{5.4}$$

Proof. **Case 1 ($\underline{\ell} < \infty$).** According to Lemma 5.2, it follows that $\underline{k}[\underline{\ell}] = \infty$ and $u^{\star} = u_{\underline{\ell}}^{\star}$, thus leading us to

$$|||u^{\star} - u_{\underline{\ell}}^{\star}||| + |||u^{\star} - u_{\underline{\ell}}^{k,\underline{j}}||| = |||u_{\underline{\ell}}^{\star} - u_{\underline{\ell}}^{k,\underline{j}}||| \xrightarrow{k\to\underline{k}} 0,$$

where convergence follows from contraction (5.1). Estimator convergence follows from the not met stopping criterion in Algorithm A(i.d.), yielding that

$$\eta_{\underline{\ell}}\big(u_{\underline{\ell}}^{k,\underline{j}}\big) < \lambda_{\mathrm{sym}}^{-1}\,|||u_{\underline{\ell}}^{k,\underline{j}} - u_{\underline{\ell}}^{k,0}||| \xrightarrow{k\to\underline{k}} 0.$$

This concludes the first case.

Case 2 ($\underline{\ell} = \infty$). The proof for the remaining case is split into four steps.

Step 1. We introduce the discrete limit space $\mathcal{X}_{\infty} := \mathrm{closure}\big(\bigcup_{\ell=0}^{\infty} \mathcal{X}_{\ell}\big)$. The Lax–Milgram lemma guarantees the existence and uniqueness of $u_{\infty}^{\star} \in \mathcal{X}_{\infty}$ such that

$$b\left(u_{\infty}^{\star}, v_{\infty}\right) = F(v_{\infty}) \quad \text{for all } v_{\infty} \in \mathcal{X}_{\infty}.$$

Since $u_\ell^\star \in \mathscr{X}_\ell \subseteq \mathscr{X}_\infty$ is a Galerkin approximation of $u_\infty^\star$, the Céa lemma (2.5) holds with $u^\star$ being replaced by $u_\infty^\star$, and the definition of $\mathscr{X}_\infty$ proves that

$$|||u_\infty^\star - u_\ell^\star||| \overset{(2.5)}{\leq} C_{\text{Céa}} \min_{v_\ell \in \mathscr{X}_\ell} |||u_\infty^\star - v_\ell||| \xrightarrow{\ell \to \infty} 0. \tag{5.5}$$

Step 2. Note that $\underline{k}[\ell] \geq 1$ for all $\ell \in \mathbb{N}_0$, i.e., at least one step of the perturbed Zarantonello iteration is performed. With contraction (5.1) and nested iteration $u_{\ell+1}^{0,\underline{j}} = u_\ell^{\underline{k},\underline{j}}$, we obtain that

$$\begin{aligned}|||u_{\ell+1}^\star - u_{\ell+1}^{\underline{k},\underline{j}}||| \overset{(5.1)}{\leq} \overline{q}_{\text{sym}}^{\underline{k}} |||u_{\ell+1}^\star - u_{\ell+1}^{0,\underline{j}}||| &= \overline{q}_{\text{sym}}^{\underline{k}} |||u_{\ell+1}^\star - u_\ell^{\underline{k},\underline{j}}||| \\ &\leq \overline{q}_{\text{sym}} |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + \overline{q}_{\text{sym}} |||u_{\ell+1}^\star - u_\ell^\star|||.\end{aligned}$$

With the definitions $a_\ell := |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}|||$ and $b_\ell := \overline{q}_{\text{sym}} |||u_{\ell+1}^\star - u_\ell^\star|||$, and $0 < \overline{q}_{\text{sym}} < 1$, the last estimate is of the form

$$0 \leq a_{\ell+1} \leq \overline{q}_{\text{sym}} a_\ell + b_\ell \quad \text{for all } \ell \in \mathbb{N}_0 \quad \text{with} \quad \lim_{\ell \to \infty} b_\ell \overset{(5.5)}{=} 0. \tag{5.6}$$

Elementary calculus (see, e.g., Carstensen *et al.*, 2014, Lemma 4.7) then proves that

$$a_\ell = |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| \xrightarrow{\ell \to \infty} 0. \tag{5.7}$$

Therefore, (5.5) and (5.7) yield convergence

$$|||u_\infty^\star - u_\ell^{\underline{k},\underline{j}}||| \leq |||u_\infty^\star - u_\ell^\star||| + |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| \xrightarrow{\ell \to \infty} 0. \tag{5.8}$$

Step 3. With stability (A1), reduction (A2) and the Dörfler marking in Algorithm A(iii), we see that

$$\begin{aligned}\eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2 &= \eta_{\ell+1}(\mathscr{T}_{\ell+1} \cap \mathscr{T}_\ell; u_\ell^{\underline{k},\underline{j}})^2 + \eta_{\ell+1}(\mathscr{T}_{\ell+1} \backslash \mathscr{T}_\ell; u_\ell^{\underline{k},\underline{j}})^2 \\ &\overset{(A1)}{=} \eta_\ell(\mathscr{T}_{\ell+1} \cap \mathscr{T}_\ell; u_\ell^{\underline{k},\underline{j}})^2 + \eta_{\ell+1}(\mathscr{T}_{\ell+1} \backslash \mathscr{T}_\ell; u_\ell^{\underline{k},\underline{j}})^2 \\ &\overset{(A2)}{\leq} \eta_\ell\left(\mathscr{T}_{\ell+1} \cap \mathscr{T}_\ell; u_\ell^{\underline{k},\underline{j}}\right)^2 + q_{\text{red}}^2 \eta_\ell\left(\mathscr{T}_\ell \backslash \mathscr{T}_{\ell+1}; u_\ell^{\underline{k},\underline{j}}\right)^2 \\ &= \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 - (1 - q_{\text{red}}^2)\, \eta_\ell(\mathscr{T}_\ell \backslash \mathscr{T}_{\ell+1}; u_\ell^{\underline{k},\underline{j}})^2 \\ &\leq \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 - (1 - q_{\text{red}}^2)\, \eta_\ell(\mathscr{M}_\ell; u_\ell^{\underline{k},\underline{j}})^2 \leq \left[1 - (1 - q_{\text{red}}^2)\,\theta\right] \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2.\end{aligned} \tag{5.9}$$

With stability (A1), the Young inequality proves, for arbitrary $\varepsilon > 0$, that

$$\eta_{\ell+1}(u_{\ell+1}^{k,\underline{j}})^2 \overset{\text{(A1)}}{\le} (1+\varepsilon)\,\eta_{\ell+1}(u_\ell^{k,\underline{j}})^2 + (1+\varepsilon^{-1})C_{\text{stab}}^2\,|||u_{\ell+1}^{k,\underline{j}} - u_\ell^{k,\underline{j}}|||^2$$
$$\overset{(5.9)}{\le} (1+\varepsilon)\big[1-\big(1-q_{\text{red}}^2\big)\,\theta\big]\,\eta_\ell(u_\ell^{k,\underline{j}})^2 + (1+\varepsilon^{-1})C_{\text{stab}}^2\,|||u_{\ell+1}^{k,\underline{j}} - u_\ell^{k,\underline{j}}|||^2.$$

Since $(1-q_{\text{red}}^2)\,\theta < 1$, we can choose $\varepsilon > 0$ sufficiently small such that $0 < q := (1+\varepsilon)\,[1-(1-q_{\text{red}}^2)\,\theta] < 1$. By defining $a_\ell := \eta_\ell(u_\ell^{k,\underline{j}})^2$ and $b_\ell := (1+\varepsilon^{-1})\,C_{\text{stab}}^2\,|||u_{\ell+1}^{k,\underline{j}} - u_\ell^{k,\underline{j}}|||^2$, which tends to zero as $\ell \to \infty$ by (5.8), the last estimate takes the form of equation (5.6). Therefore, elementary calculus proves convergence

$$a_\ell = \eta_\ell(u_\ell^{k,\underline{j}})^2 \xrightarrow{\ell\to\infty} 0. \tag{5.10}$$

Step 4. With the triangle inequality, reliability (A3) and stability (A1), it holds that

$$|||u^\star - u_\ell^\star||| + |||u^\star - u_\ell^{k,\underline{j}}||| + \eta_\ell(u_\ell^{k,\underline{j}}) \le 2\,|||u^\star - u_\ell^\star||| + |||u_\ell^\star - u_\ell^{k,\underline{j}}||| + \eta_\ell(u_\ell^{k,\underline{j}})$$
$$\overset{\text{(A3)}}{\le} 2\,C_{\text{rel}}\,\eta_\ell\,(u_\ell^\star) + |||u_\ell^\star - u_\ell^{k,\underline{j}}||| + \eta_\ell(u_\ell^{k,\underline{j}})$$
$$\overset{\text{(A1)}}{\le} (2\,C_{\text{rel}}+1)\,\eta_\ell(u_\ell^{k,\underline{j}}) + (1+2\,C_{\text{rel}}\,C_{\text{stab}})\,|||u_\ell^\star - u_\ell^{k,\underline{j}}||| \xrightarrow{\ell\to\infty} 0,$$

where we used (5.10) and (5.7) in the last step. This concludes the proof. □

5.3 *Quasi-Pythagorean estimate*

While symmetric PDEs satisfy a Pythagorean identity in the energy norm (with $\varepsilon = 0$ and $\ell_0 = 0$ in (5.11)), the situation is more involved for nonsymmetric PDEs. The following result generalizes Bespalov *et al.* (2017, Lemma 18) by considering general $v_\ell \in \mathcal{X}_\ell$ and by additionally proving the lower bound. Although the proof follows essentially that of Bespalov *et al.* (2017), we include it for the sake of completeness.

LEMMA 5.4. (quasi-Pythagorean estimate) Suppose that the estimator satisfies the axioms (A1)–(A3). Suppose that the exact Galerkin approximations satisfy convergence $|||u^\star - u_\ell^\star||| \to 0$ along the sequence of nested spaces $\mathcal{X}_\ell \subseteq \mathcal{X}_{\ell+1}$ as $\ell \to \underline{\ell}$. Then, for all $0 < \varepsilon < 1$, there exists an index $\ell_0 \in \mathbb{N}_0$ with $\ell_0 \le \underline{\ell}$ such that, for all $\ell_0 \le \ell \le \underline{\ell}$,

$$\frac{1}{1+\varepsilon}\,|||u^\star - v_\ell|||^2 \le |||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - v_\ell|||^2 \le \frac{1}{1-\varepsilon}\,|||u^\star - v_\ell|||^2 \quad \text{for all } v_\ell \in \mathcal{X}_\ell. \tag{5.11}$$

Proof. The proof is split into four steps.

Step 1. If $\underline{\ell} < \infty$, Lemma 5.2 proves that $u^\star = u_{\underline{\ell}}^\star$. We choose $\ell_0 = \underline{\ell}$ and obtain that (5.11) holds with equality and $\varepsilon = 0$, since $\ell = \underline{\ell}$ and hence $u^\star = u_{\underline{\ell}}^\star$. Consequently, (5.11) holds also for all $0 < \varepsilon < 1$. Therefore, it only remains to prove (5.11) for $\underline{\ell} = \infty$.

Step 2. Let $\ell \in \mathbb{N}_0$ and $v_\ell \in \mathscr{X}_\ell$. The weak formulation (2.1) yields that

$$\left\|\left\|u^\star - v_\ell\right\|\right\|^2 = \left\|\left\|u^\star\right\|\right\|^2 + \left\|\left\|v_\ell\right\|\right\|^2 - 2\,\mathrm{Re}\,a\left(u^\star, v_\ell\right) \overset{(2.1)}{=} \left\|\left\|u^\star\right\|\right\|^2 + \left\|\left\|v_\ell\right\|\right\|^2 - 2\,\mathrm{Re}\left[F(v_\ell) - \langle \mathscr{K} u^\star, v_\ell\rangle\right]. \tag{5.12}$$

Analogously, from the discrete counterpart (1.3) of formulation (2.1) and the linearity of $\mathscr{K}$, we obtain that

$$\begin{aligned}\left\|\left\|u_\ell^\star - v_\ell\right\|\right\|^2 &= \left\|\left\|u_\ell^\star\right\|\right\|^2 + \left\|\left\|v_\ell\right\|\right\|^2 - 2\,\mathrm{Re}\,a\left(u_\ell^\star, v_\ell\right)\\ &\overset{(1.3)}{=} \left\|\left\|u_\ell^\star\right\|\right\|^2 + \left\|\left\|v_\ell\right\|\right\|^2 - 2\,\mathrm{Re}\left[F(v_\ell) - \langle \mathscr{K} u_\ell^\star, v_\ell\rangle\right]\\ &= \left\|\left\|u_\ell^\star\right\|\right\|^2 + \left\|\left\|v_\ell\right\|\right\|^2 - 2\,\mathrm{Re}\left[F(v_\ell) - \langle \mathscr{K} u^\star, v_\ell\rangle + \langle \mathscr{K}\left(u^\star - u_\ell^\star\right), v_\ell\rangle\right]\end{aligned} \tag{5.13}$$

as well as

$$F(u_\ell^\star) \overset{(1.3)}{=} a\left(u_\ell^\star, u_\ell^\star\right) + \langle \mathscr{K} u_\ell^\star, u_\ell^\star\rangle = \left\|\left\|u_\ell^\star\right\|\right\|^2 + \langle \mathscr{K} u_\ell^\star, u_\ell^\star\rangle. \tag{5.14}$$

For $v_\ell = u_\ell^\star$, we see that

$$\begin{aligned}\left\|\left\|u^\star - u_\ell^\star\right\|\right\|^2 &\overset{(5.12)}{=} \left\|\left\|u^\star\right\|\right\|^2 + \left\|\left\|u_\ell^\star\right\|\right\|^2 - 2\,\mathrm{Re}\left[F\left(u_\ell^\star\right) - \langle \mathscr{K} u^\star, u_\ell^\star\rangle\right]\\ &\overset{(5.14)}{=} \left\|\left\|u^\star\right\|\right\|^2 - \left\|\left\|u_\ell^\star\right\|\right\|^2 + 2\,\mathrm{Re}\langle \mathscr{K}\left(u^\star - u_\ell^\star\right), u_\ell^\star\rangle.\end{aligned} \tag{5.15}$$

Summing (5.13) and (5.15), we obtain that

$$\begin{aligned}\left\|\left\|u^\star - u_\ell^\star\right\|\right\|^2 + \left\|\left\|u_\ell^\star - v_\ell\right\|\right\|^2 &= \left\|\left\|u^\star\right\|\right\|^2 + \left\|\left\|v_\ell\right\|\right\|^2 - 2\,\mathrm{Re}\left[F(v_\ell) - \langle \mathscr{K} u^\star, v_\ell\rangle - \langle \mathscr{K}\left(u^\star - u_\ell^\star\right), u_\ell^\star - v_\ell\rangle\right]\\ &\overset{(5.12)}{=} \left\|\left\|u^\star - v_\ell\right\|\right\|^2 + 2\,\mathrm{Re}\langle \mathscr{K}\left(u^\star - u_\ell^\star\right), u_\ell^\star - v_\ell\rangle.\end{aligned} \tag{5.16}$$

Step 3. We recall from Bespalov *et al.* (2017, Lemma 17) that plain convergence (5.4) of Proposition 5.3 yields that

$$e_\ell := \begin{cases} \dfrac{u^\star - u_\ell^\star}{\left\|\left\|u^\star - u_\ell^\star\right\|\right\|} & \text{if } u^\star \neq u_\ell^\star,\\ 0 & \text{otherwise}\end{cases}$$

defines a weakly convergent sequence in $\mathscr{X}$ with $e_\ell \rightharpoonup 0$ as $\ell \to \infty$. We recall that compact operators turn weak convergence into norm convergence. With the operator norm $\|\|\phi\|\|' := \sup_{v \in \mathscr{X}\setminus\{0\}} |\phi(v)| \,/\, \|\|v\|\|$ of $\phi \in \mathscr{X}'$, it thus follows that

$$\left|\langle \mathscr{K}(u^\star - u_\ell^\star), u_\ell^\star - v_\ell\rangle\right| \le \left\|\left\|\mathscr{K} e_\ell\right\|\right\|' \left\|\left\|u^\star - u_\ell^\star\right\|\right\| \left\|\left\|u_\ell^\star - v_\ell\right\|\right\| \quad \text{and} \quad \left\|\left\|\mathscr{K} e_\ell\right\|\right\|' \xrightarrow{\ell\to\infty} 0.$$

Given $\varepsilon > 0$, this provides an index $\ell_0 \in \mathbb{N}$ such that $|\!|\!| \mathscr{K} e_\ell |\!|\!|' \le \varepsilon$ for all $\ell \ge \ell_0$ and hence

$$2 \left|\mathrm{Re}\langle \mathscr{K}\left(u^\star - u_\ell^\star\right), u_\ell^\star - v_\ell\rangle\right| \le 2\varepsilon\, |\!|\!| u^\star - u_\ell^\star |\!|\!| \, |\!|\!| u_\ell^\star - v_\ell |\!|\!| \le \varepsilon \left[|\!|\!| u^\star - u_\ell^\star |\!|\!|^2 + |\!|\!| u_\ell^\star - v_\ell |\!|\!|^2 \right]. \tag{5.17}$$

Step 4. Rearranging the identity (5.16) and estimating the compact perturbation via (5.17), we obtain that

$$|\!|\!| u^\star - v_\ell |\!|\!|^2 \overset{(5.16)}{=} |\!|\!| u^\star - u_\ell^\star |\!|\!|^2 + |\!|\!| u_\ell^\star - v_\ell |\!|\!|^2 - 2\,\mathrm{Re}\langle \mathscr{K}\left(u^\star - u_\ell^\star\right), u_\ell^\star - v_\ell\rangle \overset{(5.17)}{\le} (1+\varepsilon)\left[|\!|\!| u^\star - u_\ell^\star |\!|\!|^2 + |\!|\!| u_\ell^\star - v_\ell |\!|\!|^2 \right].$$

This proves the lower estimate in (5.11), and the upper estimate is proved analogously. □

5.4 *Auxiliary contraction estimates*

The following lemma extends (Gantner *et al.*, 2021, Lemma 10) to the present setting with a quasi-Pythagorean estimate.

LEMMA 5.5. (combined discretization-symmetrization error) Suppose that the perturbed Zarantonello iteration satisfies contraction (5.1) and that the estimator satisfies (A1)–(A3). Then, for all $0 < \theta \le 1$ and $\lambda_{\mathrm{sym}} > 0$, there exists an index $\ell_0 \in \mathbb{N}_0$ with $\ell_0 \le \underline{\ell}$ and scalars $\nu > 0$ and $0 < q_{\mathrm{lin}} < 1$ such that

$$\Lambda_\ell^k := \left[|\!|\!| u^\star - u_\ell^{k,\underline{j}} |\!|\!|^2 + \nu\, \eta_\ell\big(u_\ell^{k,\underline{j}}\big)^2 \right]^{1/2} \quad \text{for all } (\ell, k, \underline{j}) \in \mathcal{Q} \tag{5.18}$$

satisfies

$$\Lambda_\ell^{k+1} \le q_{\mathrm{lin}}\, \Lambda_\ell^k \qquad \text{for all } (\ell, k+1, 0) \in \mathcal{Q} \text{ with } \ell \ge \ell_0 \text{ and } k+1 < \underline{k}[\ell], \tag{5.19a}$$

$$\Lambda_{\ell+1}^0 \le q_{\mathrm{lin}}\, \Lambda_\ell^{\underline{k}-1} \qquad \text{for all } (\ell+1, 0, 0) \in \mathcal{Q} \text{ with } \ell \ge \ell_0. \tag{5.19b}$$

Proof. Let $0 < \varepsilon < 1$ as well as $\nu, \omega > 0$ be free parameters to be fixed below. The proof consists of seven steps.

Step 1. Lemma 5.4 provides an index $\ell_0 = \ell_0(\varepsilon)$ such that for all $\ell_0 \le \ell \le \underline{\ell}$ the quasi-Pythagorean estimate (5.11) holds true. For $(\ell, k+1, 0) \in \mathcal{Q}$ with $\ell_0 \le \ell$, we get that

$$\big(\Lambda_\ell^{k+1}\big)^2 = |\!|\!| u^\star - u_\ell^{k+1,\underline{j}} |\!|\!|^2 + \nu\, \eta_\ell\big(u_\ell^{k+1,\underline{j}}\big)^2 \overset{(5.11)}{\le} (1+\varepsilon)\, |\!|\!| u^\star - u_\ell^\star |\!|\!|^2 + (1+\varepsilon)\, |\!|\!| u_\ell^\star - u_\ell^{k+1,\underline{j}} |\!|\!|^2 + \nu\, \eta_\ell\big(u_\ell^{k+1,\underline{j}}\big)^2. \tag{5.20}$$

Analogously, for $(\ell+1,0,0) \in \mathcal{Q}$ with $\ell \geq \ell_0$, nested iteration $u_{\ell+1}^{0,j} = u_\ell^{\underline{k},\underline{j}}$ shows that

$$\begin{aligned}(\Lambda_{\ell+1}^0)^2 &= |||u^\star - u_\ell^{\underline{k},\underline{j}}|||^2 + \nu\, \eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2 \\ &\overset{(5.11)}{\leq} (1+\varepsilon)\, |||u^\star - u_\ell^\star|||^2 + (1+\varepsilon)\, |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}|||^2 + \nu\, \eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2. \end{aligned} \tag{5.21}$$

Step 2. Define $C_1 := 6C_{\mathrm{rel}}^2$ and $C_2 := 6C_{\mathrm{rel}}^2 C_{\mathrm{stab}}^2$. Then, stability (A1) and reliability (A3) prove that, for all $v_\ell \in \mathcal{X}_\ell$,

$$\begin{aligned}3\, |||u^\star - u_\ell^\star|||^2 &\overset{(A3)}{\leq} 3\, C_{\mathrm{rel}}^2\, \eta_\ell\,(u_\ell^\star)^2 \overset{(A1)}{\leq} 6\, C_{\mathrm{rel}}^2\, \eta_\ell(v_\ell)^2 + 6\, C_{\mathrm{rel}}^2\, C_{\mathrm{stab}}^2\, |||u_\ell^\star - v_\ell|||^2 \\ &= C_1\, \eta_\ell(v_\ell)^2 + C_2\, |||u_\ell^\star - v_\ell|||^2 .\end{aligned} \tag{5.22}$$

Step 3. This step concerns estimator reduction via mesh refinement and thus applies only to the case $(\ell+1,0,0) \in \mathcal{Q}$. Stability (A1) and reduction (A2) in combination with the Dörfler marking criterion in Algorithm A(iii) as in Step 3 of the proof of Proposition 5.3 show that

$$\eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2 \overset{(5.9)}{\leq} \big[1 - (1-q_{\mathrm{red}}^2)\theta\big]\, \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 =: q_\theta\, \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 \quad \text{with } 0 < q_\theta < 1. \tag{5.23}$$

Step 4. For $(\ell,k+1,0) \in \mathcal{Q}$, contraction (5.1) of the perturbed Zarantonello iteration proves that

$$|||u_\ell^{k+1,\underline{j}} - u_\ell^{k,\underline{j}}||| \leq |||u_\ell^\star - u_\ell^{k+1,\underline{j}}||| + |||u_\ell^\star - u_\ell^{k,\underline{j}}||| \leq (1+\overline{q}_{\mathrm{sym}})\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||.$$

Define $C_3 := \lambda_{\mathrm{sym}}^{-2}(1+\overline{q}_{\mathrm{sym}})^2$. Using this with the not met stopping criterion in Algorithm A(i.d.) for $(\ell,k+1,0) \in \mathcal{Q}$ with $k+1 < \underline{k}[\ell]$ shows that

$$\eta_\ell(u_\ell^{k+1,\underline{j}})^2 < \lambda_{\mathrm{sym}}^{-2} |||u_{big\ell}^{k+1,\underline{j}} - u_\ell^{k,\underline{j}}|||^2 \leq C_3\, |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2. \tag{5.24}$$

Analogously, for $(\ell+1,0,0) \in \mathcal{Q}$ it holds that

$$|||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k}-1,\underline{j}}||| \leq |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}||| + |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}||| \leq (1+\overline{q}_{\mathrm{sym}})\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||.$$

Define $C_4 := C_{\mathrm{stab}}^2(1+\overline{q}_{\mathrm{sym}})^2$. Stability (A1) and the Young inequality in the form $(a+b)^2 \leq (1+\omega)a^2 + (1+\omega^{-1})b^2$ for $a, b \in \mathbb{R}$ and $\omega > 0$ show that

$$\begin{aligned}\eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 &\overset{(A1)}{\leq} (1+\omega)\, \eta_\ell(u_\ell^{\underline{k}-1,\underline{j}})^2 + (1+\omega^{-1})\, C_{\mathrm{stab}}^2\, |||u_\ell^{\underline{k},\underline{j}} - u_\ell^{\underline{k}-1,\underline{j}}|||^2 \\ &\leq (1+\omega)\, \eta_\ell(u_\ell^{\underline{k}-1,\underline{j}})^2 + (1+\omega^{-1})\, C_4\, |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2.\end{aligned} \tag{5.25}$$

Step 5. For $(\ell, k+1, 0) \in \mathcal{Q}$ with $\ell \geq \ell_0$ and $k+1 < \underline{k}[\ell]$, we have

$$
\begin{aligned}
(\Lambda_\ell^{k+1})^2 &\overset{(5.20)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + 3\varepsilon\ |||u^\star - u_\ell^\star|||^2 + (1+\varepsilon)\ |||u_\ell^\star - u_\ell^{k+1,\underline{j}}|||^2 + \nu\, \eta_\ell(u_\ell^{k+1,\underline{j}})^2 \\
&\overset{(5.22)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + (\nu + \varepsilon\, C_1)\, \eta_\ell(u_\ell^{k+1,\underline{j}})^2 + (1+\varepsilon(1+C_2))\ |||u_\ell^\star - u_\ell^{k+1,\underline{j}}|||^2 \\
&\overset{(5.1)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + (\nu + \varepsilon\, C_1)\, \eta_\ell(u_\ell^{k+1,\underline{j}})^2 + (1+\varepsilon(1+C_2))\, \overline{q}_{\mathrm{sym}}^2\ |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2 \\
&\overset{(5.24)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + \big[(\nu + \varepsilon\, C_1)C_3 + (1+\varepsilon(1+C_2))\, \overline{q}_{\mathrm{sym}}^2\big]\ |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2.
\end{aligned}
$$

Provided that

$$
(\nu + \varepsilon\, C_1)C_3 + (1+\varepsilon(1+C_2))\, \overline{q}_{\mathrm{sym}}^2 = \overline{q}_{\mathrm{sym}}^2 + \nu C_3 + \varepsilon\, [C_1 C_3 + (1+C_2)\, \overline{q}_{\mathrm{sym}}^2] \leq 1 - 2\varepsilon,
$$

the quasi-Pythagorean estimate (5.11) proves that

$$
\big(\Lambda_\ell^{k+1}\big)^2 \leq (1-2\varepsilon)\ \Big[|||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{k,\underline{j}}|||^2\Big] \overset{5.11}{\leq} \frac{1-2\varepsilon}{1-\varepsilon}\ |||u^\star - u_\ell^{k,\underline{j}}|||^2 \leq \frac{1-2\varepsilon}{1-\varepsilon}\ \big(\Lambda_\ell^k\big)^2.
$$

This proves (5.19a) up to the choice of the parameters ε, and ν.

Step 6. For $(\ell+1, 0, 0) \in \mathcal{Q}$ with $\ell \geq \ell_0$, we have that

$$
\begin{aligned}
\big(\Lambda_{\ell+1}^0\big)^2 &\overset{(5.21)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + 3\varepsilon\ |||u^\star - u_\ell^\star|||^2 + (1+\varepsilon)\ |||u_\ell^\star - u_\ell^{\underline{k},\underline{j}}|||^2 + \nu\, \eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2 \\
&\overset{(5.22)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + \varepsilon\, C_1\, \eta_\ell(u_\ell^{\underline{k}-1,\underline{j}})^2 + \big[\varepsilon\, C_2 + (1+\varepsilon)\, \overline{q}_{\mathrm{sym}}^2\big]\ |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2 \\
&\quad + \nu\, \eta_{\ell+1}(u_\ell^{\underline{k},\underline{j}})^2 \\
&\overset{(5.23)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + \varepsilon\, C_1\, \eta_\ell(u_\ell^{\underline{k}-1,\underline{j}})^2 + \big[\varepsilon\, C_2 + (1+\varepsilon)\, \overline{q}_{\mathrm{sym}}^2\big]\ |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2 \\
&\quad + q_\theta \nu\, \eta_\ell(u_\ell^{\underline{k},\underline{j}})^2 \\
&\overset{(5.25)}{\leq} (1-2\varepsilon)\ |||u^\star - u_\ell^\star|||^2 + \big[\varepsilon\, C_2 + (1+\varepsilon)\, \overline{q}_{\mathrm{sym}}^2 + C_4 q_\theta \nu (1+\omega^{-1})\big]\ |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2 \\
&\quad + \big[\varepsilon\, C_1 \nu^{-1} + q_\theta(1+\omega)\big]\nu\, \eta_\ell(u_\ell^{\underline{k}-1,\underline{j}})^2.
\end{aligned}
$$

Provided that

$$
\varepsilon\, C_1 \nu^{-1} + q_\theta(1+\omega) \leq 1 - 2\varepsilon
$$

and

$$
\varepsilon\, C_2 + (1+\varepsilon)\overline{q}_{\mathrm{sym}}^2 + C_4 q_\theta \nu(1+\omega^{-1}) = \overline{q}_{\mathrm{sym}}^2 + \nu C_4 q_\theta (1+\omega^{-1}) + \varepsilon\big(C_2 + \overline{q}_{\mathrm{sym}}^2\big) \leq 1 - 2\varepsilon,
$$

the quasi-Pythagorean estimate (5.11) shows that

$$(\Lambda_{\ell+1}^{0})^2 \le (1-2\varepsilon)\Big[|||u^\star - u_\ell^\star|||^2 + |||u_\ell^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2 + \nu\, \eta_\ell\big(u_\ell^{\underline{k}-1,\underline{j}}\big)^2 \Big]$$
$$\overset{(5.11)}{\le} \frac{1-2\varepsilon}{1-\varepsilon}\, |||u^\star - u_\ell^{\underline{k}-1,\underline{j}}|||^2 + (1-2\varepsilon)\nu\, \eta_\ell\big(u_\ell^{\underline{k}-1,\underline{j}}\big)^2 \le \frac{1-2\varepsilon}{1-\varepsilon}\, \big(\Lambda_\ell^{\underline{k}-1}\big)^2.$$

This proves (5.19b) up to the choice of the parameters ω, ν and ε in the following step.

Step 7. A suitable choice of the parameters ω, ν and ε can be obtained as follows:

- first, we choose ω such that $(1+\omega)q_\theta < 1$;
- second, we choose ν such that $\overline{q}_{\mathrm{sym}}^2 + \nu C_3 < 1$ and $\overline{q}_{\mathrm{sym}}^2 + \nu q_\theta C_4(1+\omega^{-1}) < 1$;
- finally, we choose $\varepsilon > 0$ sufficiently small so that all constraints in Step 5 and Step 6 are satisfied.

This concludes the proof with $q_{\mathrm{lin}}^2 := \frac{1-2\varepsilon}{1-\varepsilon} < 1$. □

5.5 *Proof of Theorem* 1

The proof is split into five steps. Recall the definitions

$$\Delta_\ell^{k,j} \overset{(4.3)}{=} |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell\big(u_\ell^{k,j}\big) \quad \text{and} \quad \Lambda_\ell^{k} \overset{(5.18)}{=} \Big[|||u^\star - u_\ell^{k,\underline{j}}|||^2 + \nu\, \eta_\ell\big(u_\ell^{k,\underline{j}}\big)^2 \Big]^{1/2}.$$

Step 1. In the first step, we prove that

$$\Delta_\ell^{k,j} \lesssim |||u_\ell^{k,\star} - u_\ell^{k,j-1}||| \quad \text{for all } (\ell,k,j) \in \mathcal{Q} \text{ with } 1 \le k \le \underline{k}[\ell] \text{ and } 1 \le j < \underline{j}[\ell,k]. \tag{5.26}$$

Together with reliability (A3) and stability (A1), the definition of $\Delta_\ell^{k,j}$ shows that

$$\begin{aligned}
\Delta_\ell^{k,j} &\overset{(4.3)}{=} |||u^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell\big(u_\ell^{k,j}\big)\\
&\le |||u^\star - u_\ell^\star||| + |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell\big(u_\ell^{k,j}\big)\\
&\overset{(A3)}{\le} C_{\mathrm{rel}}\, \eta_\ell\big(u_\ell^\star\big) + |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| + \eta_\ell\big(u_\ell^{k,j}\big)\\
&\overset{(A1)}{\le} (1+C_{\mathrm{rel}})\, \eta_\ell\big(u_\ell^{k,j}\big) + (1+C_{\mathrm{stab}}C_{\mathrm{rel}})\, |||u_\ell^\star - u_\ell^{k,j}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}|||.
\end{aligned}$$

The contraction of the (unperturbed) Zarantonello iteration (4.1) proves that

$$\begin{aligned}
|||u_\ell^\star - u_\ell^{k,j}||| \le |||u_\ell^\star - u_\ell^{k,\star}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}||| &\overset{(4.1)}{\le} \frac{q_{\mathrm{sym}}}{1-q_{\mathrm{sym}}}\, |||u_\ell^{k,\star} - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k,j}|||\\
&\lesssim |||u_\ell^{k,\star} - u_\ell^{k,j}||| + |||u_\ell^{k,j} - u_\ell^{k-1,\underline{j}}|||.
\end{aligned}$$

Furthermore, the contraction of the algebraic solver (5.3) proves that

$$|||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| \overset{(5.3)}{\leq} \frac{q_{\rm alg}}{1-q_{\rm alg}} \, |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}|||.$$

Combining the last three estimates with the not met stopping criterion of the algebraic solver in Algorithm A(i.b.II) for $1 \leq j < \underline{j}[\ell,k]$, we conclude that

$$\Delta_\ell^{k,\underline{j}} \lesssim \eta_\ell(u_\ell^{k,\underline{j}}) + |||u_\ell^{k,\underline{j}} - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| \overset{\text{(i.b.II)}}{\lesssim} |||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}|||.$$

Finally, the triangle inequality and the contraction (2.9) imply (5.26).

Step 2. Next, we show that

$$\Delta_\ell^{k,\underline{j}} \lesssim \Delta_\ell^{k,j} \quad \text{for all } (\ell,k,j) \in \mathcal{Q}, \tag{5.27}$$

which is trivial for $j = \underline{j}[\ell,k]$. To deal with $j = \underline{j}[\ell,k]-1$, note that the definition of $\Delta_\ell^{k,j}$ shows that

$$\begin{aligned} \Delta_\ell^{k,\underline{j}} &\overset{(4.3)}{=} |||u^\star - u_\ell^{k,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}}||| + \eta_\ell(u_\ell^{k,\underline{j}}) \\ &\leq |||u^\star - u_\ell^{k,\underline{j}-1}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1}||| + 2\,|||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| + \eta_\ell(u_\ell^{k,\underline{j}}). \end{aligned}$$

Stability (A1) and the algebraic solver contraction (5.3) lead us to

$$\begin{aligned} 2\,|||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| + \eta_\ell(u_\ell^{k,\underline{j}}) &\overset{(A1)}{\leq} (2 + C_{\rm stab})\,|||u_\ell^{k,\underline{j}} - u_\ell^{k,\underline{j}-1}||| + \eta_\ell(u_\ell^{k,\underline{j}-1}) \\ &\overset{(5.3)}{\leq} (2 + C_{\rm stab})(1+q_{\rm alg})\,|||u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1}||| + \eta_\ell(u_\ell^{k,\underline{j}-1}). \end{aligned}$$

Combining the last two estimates verifies (5.27) for $j = \underline{j}[\ell,k]-1$, i.e.,

$$\Delta_\ell^{k,\underline{j}} \lesssim |||u^\star - u_\ell^{k,\underline{j}-1}||| + |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}-1}||| + \eta_\ell(u_\ell^{k,\underline{j}-1}) = \Delta_\ell^{k,\underline{j}-1}. \tag{5.28}$$

We prove the remaining case $j < \underline{j}[\ell,k]-1$ by (5.26) from Step 1 and the algebraic solver contraction (5.3), i.e.,

$$\Delta_\ell^{k,\underline{j}} \overset{(5.28)}{\lesssim} \Delta_\ell^{k,\underline{j}-1} \overset{(5.26)}{\lesssim} |||u_\ell^{k,\star} - u_\ell^{k,\underline{j}-2}||| \overset{(5.3)}{\leq} q_{\rm alg}^{(\underline{j}[\ell,k]-2)-j}\,|||u_\ell^{k,\star} - u_\ell^{k,j}||| \leq \Delta_\ell^{k,j}.$$

This concludes the proof of (5.27).

Step 3. In this step, we prove that

$$\Lambda_\ell^0 \simeq \Delta_\ell^{0,0} = \Delta_\ell^{0,\underline{j}} \ \text{ and } \ \Lambda_\ell^k \lesssim \Delta_\ell^{k,\underline{j}} \overset{(5.27)}{\lesssim} \Delta_\ell^{k,0} \lesssim \Lambda_\ell^{k-1} \text{ for all } (\ell,k,\underline{j}) \in \mathcal{Q} \text{ with } k \geq 1. \tag{5.29}$$

Together with $u_\ell^{0,\star} = u_\ell^{0,\underline{j}} = u_\ell^{0,0}$, the definition of Λ_ℓ^0 and $\Delta_\ell^{0,0}$ proves that $\Lambda_\ell^0 \simeq \Delta_\ell^{0,0} = \Delta_\ell^{0,\underline{j}}$ as well as $\Lambda_\ell^k \lesssim \Delta_\ell^{k,\underline{j}}$ for all $(\ell,k,\underline{j}) \in \mathcal{Q}$, where the hidden constants depend only on ν. Together with (5.27) from Step 2, it thus only remains to prove $\Delta_\ell^{k,0} \lesssim \Lambda_\ell^{k-1}$ for $k \geq 1$.

To this end, let $(\ell,k,\underline{j}) \in \mathcal{Q}$ with $k \geq 1$. From contraction (4.1) of the unperturbed Zarantonello symmetrization and nested iteration $u_\ell^{k,0} = u_\ell^{k-1,\underline{j}}$, we get that

$$\Delta_\ell^{k,0} = |||u^\star - u_\ell^{k-1,\underline{j}}||| + |||u_\ell^{k,\star} - u_\ell^{k-1,\underline{j}}||| + \eta_\ell(u_\ell^{k-1,\underline{j}})$$
$$\overset{(4.1)}{\leq} |||u^\star - u_\ell^{k-1,\underline{j}}||| + (1+q_{\mathrm{sym}})\, |||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| + \eta_\ell(u_\ell^{k-1,\underline{j}}).$$

The Céa lemma (2.5) proves that

$$|||u_\ell^\star - u_\ell^{k-1,\underline{j}}||| \leq |||u^\star - u_\ell^\star||| + |||u^\star - u_\ell^{k-1,\underline{j}}||| \lesssim |||u^\star - u_\ell^{k-1,\underline{j}}|||.$$

Combining the last two estimates, we arrive at

$$\Delta_\ell^{k,0} \lesssim |||u^\star - u_\ell^{k-1,\underline{j}}||| + \eta_\ell(u_\ell^{k-1,\underline{j}}) \simeq \Lambda_\ell^{k-1}.$$

This concludes the proof of (5.29).

Step 4. In this step, we prove that

$$\sum_{j'=j}^{\underline{j}[\ell,k]} \Delta_\ell^{k,j'} \lesssim \Delta_\ell^{k,\underline{j}} + \Delta_\ell^{k,j} \quad \text{for all } (\ell,k,j) \in \mathcal{Q}. \tag{5.30}$$

According to the right-hand side of (5.30), it remains to consider the sum for $j' = j+1, \ldots, \underline{j}[\ell,k]-1$. With (5.26) and contraction (5.3) of the algebraic solver, we get that

$$\sum_{j'=j+1}^{\underline{j}[\ell,k]-1} \Delta_\ell^{k,j'} \overset{(5.26)}{\lesssim} \sum_{j'=j+1}^{\underline{j}[\ell,k]-1} |||u_\ell^{k,\star} - u_\ell^{k,j'-1}||| \overset{(5.3)}{\leq} |||u_\ell^{k,\star} - u_\ell^{k,j}||| \sum_{j'=j}^{\underline{j}[\ell,k]-2} q_{\mathrm{alg}}^{j'-j}.$$

With the geometric series and $|||u_\ell^{k,\star} - u_\ell^{k,j}||| \leq \Delta_\ell^{k,j}$, this concludes the proof of (5.30).

Step 5. For $(\ell,k,j) \in \mathcal{Q}$ with $\ell \geq \ell_0$, the preceding steps show that

$$\sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ (\ell',k',j') > (\ell,k,j)}} \Delta_{\ell'}^{k',j'} = \sum_{j'=j+1}^{\underline{j}[\ell,k]} \Delta_\ell^{k,j'} + \sum_{\substack{(\ell',k',0) \in \mathcal{Q} \\ (\ell',k',0) > (\ell,k,0)}} \sum_{j'=0}^{\underline{j}[\ell',k']} \Delta_{\ell'}^{k',j'}$$
$$\overset{(5.30)}{\lesssim} \Big[\Delta_\ell^{k,\underline{j}} + \Delta_\ell^{k,j}\Big] + \sum_{\substack{(\ell',k',0) \in \mathcal{Q} \\ (\ell',k',0) > (\ell,k,0)}} \Big[\Delta_{\ell'}^{k',\underline{j}} + \Delta_{\ell'}^{k',0}\Big] \overset{(5.27)}{\lesssim} \Delta_\ell^{k,j} + \sum_{\substack{(\ell',k',0) \in \mathcal{Q} \\ (\ell',k',0) > (\ell,k,0)}} \Delta_{\ell'}^{k',0}.$$

With the linear convergence (5.19) of Λ_ℓ^k from Lemma 5.5 and the geometric series, we thus see that

$$\sum_{\substack{(\ell',k',0)\in\mathcal{Q}\\(\ell',k',0)>(\ell,k,0)}} \Delta_{\ell'}^{k',0} = \sum_{k'=k+1}^{\underline{k}[\ell]} \Delta_\ell^{k',0} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']} \Delta_{\ell'}^{k',0} \overset{(5.29)}{\lesssim} \sum_{k'=k}^{\underline{k}[\ell]-1} \Lambda_\ell^{k'} + \sum_{\ell'=\ell+1}^{\underline{\ell}} \sum_{k'=0}^{\underline{k}[\ell']-1} \Lambda_{\ell'}^{k'}$$
$$\overset{(5.19)}{\lesssim} \Lambda_\ell^k + \Lambda_{\ell+1}^0 \overset{(5.27)}{\lesssim} \Lambda_\ell^k \overset{(5.29)}{\lesssim} \Delta_\ell^{k,\underline{j}} \overset{(5.27)}{\lesssim} \Delta_\ell^{k,j}.$$

Altogether, this proves that

$$\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\(\ell',k',j')>(\ell,k,j)}} \Delta_{\ell'}^{k',j'} \lesssim \Delta_\ell^{k,j} \quad \text{for all } (\ell,k,j)\in\mathcal{Q}.$$

According to basic calculus (see, e.g., Carstensen *et al.*, 2014, Lemma 4.9), this is equivalent to the claimed linear convergence (4.4) with respect to the lexicographic ordering on $\mathcal{Q}$.

5.6 *Proof of Theorem* 2

Thanks to Corollary 4.2, it is sufficient to show that

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} \lesssim \sup_{(\ell,k,j)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \Delta_\ell^{k,j}, \tag{5.31a}$$

$$\sup_{\substack{(\ell,k,j)\in\mathcal{Q}\\ \ell\geq\ell_0}} (\#\mathcal{T}_\ell)^s\, \Delta_\ell^{k,j} \lesssim \max\left\{\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}, \Delta_{\ell_0}^{0,0}\right\}. \tag{5.31b}$$

We split the proof into six steps.

Step 1. We first show (5.31a) for the case $\underline{\ell}=\infty$. Algorithm A ensures that $\#\mathcal{T}_\ell\to\infty$ as $\ell\to\infty$. We recall that in NVB refinement an element is split into at least two, but at most C_{child} child elements. In particular, for all $\ell\geq 0$, we have that

$$\#\mathcal{T}_{\ell+1} \leq C_{\mathrm{child}}\,\#\mathcal{T}_\ell. \tag{5.32}$$

For any given $N\in\mathbb{N}$, we can argue similarly as in the proof of Carstensen *et al.* (2014, Proposition 4.15). Choose the maximal index $\ell'\in\mathbb{N}_0$ such that $\#\mathcal{T}_{\ell'}-\#\mathcal{T}_0\leq N$. The maximality of ℓ' leads us to

$$N+1 \leq \#\mathcal{T}_{\ell'+1}-\#\mathcal{T}_0+1 \leq \#\mathcal{T}_{\ell'+1} \overset{(5.32)}{\leq} C_{\mathrm{child}}\,\#\mathcal{T}_{\ell'}. \tag{5.33}$$

Since $\mathcal{T}_{\ell'}\in\mathbb{T}_N(\mathcal{T}_0)$, we have that

$$\min_{\mathcal{T}_{\mathrm{opt}}\in\mathbb{T}_N(\mathcal{T}_0)} \left[|||u^\star-u^\star_{\mathrm{opt}}||| + \eta_{\mathrm{opt}}(u^\star_{\mathrm{opt}}) \right] \leq |||u^\star-u^\star_{\ell'}||| + \eta_{\ell'}\left(u^\star_{\ell'}\right), \tag{5.34}$$

and stability (A1) and the Céa lemma (2.5) show, for $(\ell', k', j') \in \mathcal{Q}$, that

$$\begin{aligned}
|||u^\star - u^\star_{\ell'}||| + \eta_{\ell'}(u^\star_{\ell'}) &\overset{(A1)}{\le} |||u^\star - u^\star_{\ell'}||| + \eta_{\ell'}(u^{k',j'}_{\ell'}) + C_{\mathrm{stab}}\, |||u^\star_{\ell'} - u^{k',j'}_{\ell'}||| \\
&\le (1 + C_{\mathrm{stab}})\, |||u^\star - u^\star_{\ell'}||| + \eta_{\ell'}(u^{k',j'}_{\ell'}) + C_{\mathrm{stab}}\, |||u^\star - u^{k',j'}_{\ell'}||| \\
&\overset{(2.5)}{\le} \big(C_{\text{Céa}}\,(1 + C_{\mathrm{stab}}) + C_{\mathrm{stab}}\big)\, |||u^\star - u^{k',j'}_{\ell'}||| + \eta_{\ell'}(u^{k',j'}_{\ell'}) \\
&\le \big(C_{\text{Céa}}\,(1 + C_{\mathrm{stab}}) + C_{\mathrm{stab}}\big)\, \Delta^{k',j'}_{\ell'}.
\end{aligned} \tag{5.35}$$

A combination of the previous estimates leads us to

$$\begin{aligned}
&(N+1)^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N(\mathcal{T}_0)} \Big[|||u^\star - u^\star_{\mathrm{opt}}||| + \eta_{\mathrm{opt}}(u^\star_{\mathrm{opt}}) \Big] \overset{(5.34)}{\le} (N+1)^s \big[|||u^\star - u^\star_{\ell'}||| + \eta_{\ell'}\,(u^\star_{\ell'}) \big] \\
&\overset{(5.33)}{\le} C^s_{\mathrm{child}}\, (\#\mathcal{T}_{\ell'})^s \big[|||u^\star - u^\star_{\ell'}||| + \eta_{\ell'}\,(u^\star_{\ell'}) \big] \overset{(5.35)}{\lesssim} (\#\mathcal{T}_{\ell'})^s \Delta^{k',j'}_{\ell'} \le \sup_{(\ell,k,j) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \Delta^{k,j}_\ell.
\end{aligned}$$

Finally, taking the supremum over all N yields the sought result

$$\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} \lesssim \sup_{(\ell,k,j) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \Delta^{k,j}_\ell.$$

Step 2. We proceed to show (5.31a) for the case $\underline{\ell} < \infty$. Recall from Lemma 5.2 that $\eta_{\underline{\ell}}(u^\star_{\underline{\ell}}) = 0$ and $u^\star_{\underline{\ell}} = u^\star$. Without loss of generality, we may assume $\underline{\ell} > 0$, since otherwise $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} = 0$. Combined with reliability (A3), this yields that

$$\begin{aligned}
\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} &\overset{(4.9)}{=} \sup_{N \in \mathbb{N}_0} \left((N+1)^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N(\mathcal{T}_0)} \Big[|||u^\star - u^\star_{\mathrm{opt}}||| + \eta_{\mathrm{opt}}(u^\star_{\mathrm{opt}}) \Big] \right) \\
&\overset{(A3)}{\le} (1 + C_{\mathrm{rel}}) \sup_{0 \le N < \#\mathcal{T}_{\underline{\ell}} - \#\mathcal{T}_0} \left((N+1)^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N(\mathcal{T}_0)} \eta_{\mathrm{opt}}(u^\star_{\mathrm{opt}}) \right).
\end{aligned} \tag{5.36}$$

We argue as in Step 1 above: let $0 \le N < \#\mathcal{T}_{\underline{\ell}} - \#\mathcal{T}_0$. Choose the maximal index $0 \le \ell' < \underline{\ell}$ with $\#\mathcal{T}_{\ell'} - \#\mathcal{T}_0 \le N$. Arguing along the lines of (5.33)–(5.35), we see that

$$\sup_{0 \le N < \#\mathcal{T}_{\underline{\ell}} - \#\mathcal{T}_0} \left((N+1)^s \min_{\mathcal{T}_{\mathrm{opt}} \in \mathbb{T}_N(\mathcal{T}_0)} \eta_{\mathrm{opt}}(u^\star_{\mathrm{opt}}) \right) \lesssim \sup_{(\ell,k,j) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s\, \Delta^{k,j}_\ell.$$

Combining this with (5.36), we conclude the lower bound (5.31a) also in this case.

Step 3. We prove (5.31b) for $\|u^\star\|_{\mathbb{A}_s(\mathscr{T}_{\ell_0})} < \infty$, since the result becomes trivial if $\|u^\star\|_{\mathbb{A}_s(\mathscr{T}_{\ell_0})} = \infty$. First, we show that for all $\ell' \geq \ell_0$ with $(\ell'+1,0,0) \in \mathscr{Q}$, there exists $\mathscr{R}_{\ell'} \subseteq \mathscr{T}_{\ell'}$ such that

$$\#\mathscr{R}_{\ell'} \lesssim \|u^\star\|_{\mathbb{A}_s(\mathscr{T}_{\ell_0})}^{1/s} \left(\Delta_{\ell'+1}^{0,\underline{j}}\right)^{-1/s} \quad \text{and} \quad \theta_{\mathrm{mark}} \eta_{\ell'}\left(u_{\ell'}^\star\right)^2 \leq \eta_{\ell'}\left(\mathscr{R}_{\ell'}, u_{\ell'}^\star\right)^2. \tag{5.37}$$

Since $0 < \theta_{\mathrm{mark}} = (\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star)^2 (1 - \lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star)^{-2} < \theta^\star$, and because there holds (A4), (Carstensen *et al.*, 2014, Lemma 4.14) ensures, for all $\ell' \geq \ell_0$, the existence of a set $\mathscr{R}_{\ell'} \subseteq \mathscr{T}_{\ell'}$, satisfying

$$\#\mathscr{R}_{\ell'} \lesssim \|u^\star\|_{\mathbb{A}_s(\mathscr{T}_{\ell_0})}^{1/s}\, \eta_{\ell'}\left(u_{\ell'}^\star\right)^{-1/s} \quad \text{and} \quad \theta_{\mathrm{mark}} \eta_{\ell'}\left(u_{\ell'}^\star\right)^2 \leq \eta_{\ell'}\left(\mathscr{R}_{\ell'}, u_{\ell'}^\star\right)^2. \tag{5.38}$$

Stability of the estimator (A1), the contraction of the perturbed Zarantonello symmetrization (4.2) and the stopping criterion in Algorithm A(i.d.) show that

$$\begin{aligned}
\eta_{\ell'}\big(u_{\ell'}^{\underline{k},\underline{j}}\big) &\overset{(A1)}{\leq} \eta_{\ell'}(u_{\ell'}^\star) + C_{\mathrm{stab}} \,|\!|\!| u_{\ell'}^\star - u_{\ell'}^{\underline{k},\underline{j}} |\!|\!| \\
&\overset{(4.2)}{\leq} \eta_{\ell'}(u_{\ell'}^\star) + C_{\mathrm{stab}} \frac{\overline{q}_{\mathrm{sym}}}{1-\overline{q}_{\mathrm{sym}}} |\!|\!| u_{\ell'}^{\underline{k},\underline{j}} - u_{\ell'}^{\underline{k},\underline{j}-1} |\!|\!| \leq \eta_{\ell'}\left(u_{\ell'}^\star\right) + \frac{\lambda_{\mathrm{sym}}}{\lambda_{\mathrm{sym}}^\star} \eta_{\ell'}\big(u_{\ell'}^{\underline{k},\underline{j}}\big).
\end{aligned}$$

Since $\lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star < 1$ by assumption, we thus obtain that

$$\left(1 - \lambda_{\mathrm{sym}}/\lambda_{\mathrm{sym}}^\star\right) \eta_{\ell'}\big(u_{\ell'}^{\underline{k},\underline{j}}\big) \leq \eta_{\ell'}\left(u_{\ell'}^\star\right), \tag{5.39}$$

which leads us to

$$\#\mathscr{R}_{\ell'} \overset{(5.38)}{\lesssim} \|u^\star\|_{\mathbb{A}_s(\mathscr{T}_{\ell_0})}^{1/s}\, \eta_{\ell'}\big(u_{\ell'}^{\underline{k},\underline{j}}\big)^{-1/s}.$$

Moreover, thanks to nested iteration, Step 3 of the proof of Theorem 4.1, Step 3 of the proof of Lemma 5.5 and reliability (3.4) of Proposition 3.5, there holds that

$$\begin{aligned}
\Delta_{\ell'+1}^{0,\underline{j}} \overset{(5.29)}{\simeq} \Lambda_{\ell'+1}^{0} &= \left(|\!|\!| u^\star - u_{\ell'}^{\underline{k},\underline{j}} |\!|\!|^2 + \nu\, \eta_{\ell'+1}\big(u_{\ell'}^{\underline{k},\underline{j}}\big)^2\right)^{1/2} \\
&\overset{(5.23)}{\lesssim} \left(|\!|\!| u^\star - u_{\ell'}^{\underline{k},\underline{j}} |\!|\!|^2 + \eta_{\ell'}\big(u_{\ell'}^{\underline{k},\underline{j}}\big)^2\right)^{1/2} \overset{(3.4)}{\lesssim} \eta_{\ell'}\big(u_{\ell'}^{\underline{k},\underline{j}}\big).
\end{aligned} \tag{5.40}$$

By summarizing the last two estimates, we obtain (5.37).

Step 4. For $(\ell'+1,0,0) \in \mathscr{Q}$ with $\ell' \geq \ell_0$, we show that

$$\#\mathscr{M}_{\ell'} \leq C_{\mathrm{mark}}\, \#\mathscr{R}_{\ell'}, \tag{5.41}$$

with the constant $C_{\mathrm{mark}} \geq 1$ from Algorithm A. Recall the definition $\theta_{\mathrm{mark}} = (\theta^{1/2} + \lambda_{\mathrm{sym}}/\lambda^{\star}_{\mathrm{sym}})^2\,(1 - \lambda_{\mathrm{sym}}/\lambda^{\star}_{\mathrm{sym}})^{-2}$ with $\lambda^{\star}_{\mathrm{sym}} = (1-\overline{q}_{\mathrm{sym}})/(\overline{q}_{\mathrm{sym}}\,C_{\mathrm{stab}})$. This shows that

$$\begin{aligned}\big\|\!\big| u^{\star}_{\ell'} - u^{\underline{k},\underline{j}}_{\ell'} \big\|\!\big| &\overset{(5.1)}{\leq} \frac{\overline{q}_{\mathrm{sym}}}{1-\overline{q}_{\mathrm{sym}}} \big\|\!\big| u^{\underline{k},\underline{j}}_{\ell'} - u^{\underline{k}-1,\underline{j}}_{\ell'} \big\|\!\big| \leq \frac{\overline{q}_{\mathrm{sym}}}{1-\overline{q}_{\mathrm{sym}}} \lambda_{\mathrm{sym}}\, \eta_{\ell'}\big(u^{\underline{k},\underline{j}}_{\ell'}\big) \\ &= C_{\mathrm{stab}}^{-1} \frac{\lambda_{\mathrm{sym}}}{\lambda^{\star}_{\mathrm{sym}}}\, \eta_{\ell'}\big(u^{\underline{k},\underline{j}}_{\ell'}\big) = C_{\mathrm{stab}}^{-1} \left[\theta_{\mathrm{mark}}^{1/2}\left(1 - \frac{\lambda_{\mathrm{sym}}}{\lambda^{\star}_{\mathrm{sym}}}\right) - \theta^{1/2}\right] \eta_{\ell'}\big(u^{\underline{k},\underline{j}}_{\ell'}\big). \end{aligned} \tag{5.42}$$

Now, we can estimate

$$\begin{aligned}\theta_{\mathrm{mark}}^{1/2}\left(1 - \frac{\lambda_{\mathrm{sym}}}{\lambda^{\star}_{\mathrm{sym}}}\right)\eta_{\ell'}\big(u^{\underline{k}',\underline{j}'}_{\ell'}\big) &\overset{(5.39)}{\leq} \theta_{\mathrm{mark}}^{1/2}\eta_{\ell'}\,\big(u^{\star}_{\ell'}\big) \overset{(5.38)}{\leq} \eta_{\ell'}\,\big(\mathscr{R}_{\ell'}, u^{\star}_{\ell'}\big) \\ &\overset{(A1)}{\leq} \eta_{\ell'}\Big(\mathscr{R}_{\ell'}, u^{\underline{k}',\underline{j}'}_{\ell'}\Big) + C_{\mathrm{stab}}\,\big\|\!\big| u^{\star}_{\ell'} - u^{\underline{k}',\underline{j}'}_{\ell'} \big\|\!\big| \\ &\overset{(5.42)}{\leq} \eta_{\ell'}\Big(\mathscr{R}_{\ell'}, u^{\underline{k}',\underline{j}'}_{\ell'}\Big) + \left[\theta_{\mathrm{mark}}^{1/2}\left(1 - \frac{\lambda_{\mathrm{sym}}}{\lambda^{\star}_{\mathrm{sym}}}\right) - \theta^{1/2}\right]\eta_{\ell'}\Big(u^{\underline{k}',\underline{j}'}_{\ell'}\Big).\end{aligned}$$

Rearranging the terms, we obtain that $\mathscr{R}_{\ell'}$ from Step 3 satisfies the Dörfler marking criterion of Algorithm A(iii) with the same parameter θ, i.e., there holds

$$\theta\, \eta_{\ell'}\Big(u^{\underline{k}',\underline{j}'}_{\ell'}\Big)^2 \leq \eta_{\ell'}\Big(\mathscr{R}_{\ell'}, u^{\underline{k}',\underline{j}'}_{\ell'}\Big)^2. \tag{5.43}$$

Hence, quasi-minimality of the set of marked elements $\mathscr{M}_{\ell'}$ implies (5.41).

Step 5. Consider the case $(\ell,k,j) \in \mathcal{Q}$ with $\ell \geq \ell_0$. Full linear convergence from Theorem 4.1 yields that

$$\sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} \Big(\Delta^{k',j'}_{\ell'}\Big)^{-1/s} \overset{(4.4)}{\lesssim} \big(\Delta^{k,j}_{\ell}\big)^{-1/s} \sum_{\substack{(\ell',k',j')\in\mathcal{Q}\\ |\ell',k',j'|\leq|\ell,k,j|\\ \ell'\geq\ell_0}} \big(q_{\mathrm{lin}}^{1/s}\big)^{|\ell,k,j|-|\ell',k',j'|} \lesssim \big(\Delta^{k,j}_{\ell}\big)^{-1/s}. \tag{5.44}$$

Recall that NVB refinement satisfies the mesh-closure estimate, i.e., there holds that

$$\#\mathscr{T}_{\ell} - \#\mathscr{T}_0 \leq C_{\mathrm{mesh}} \sum_{\ell'=0}^{\ell-1} \#\mathscr{M}_{\ell'} \quad \text{for all } \ell \geq 0, \tag{5.45}$$

where $C_{\text{mesh}} > 1$ depends only on $\mathcal{T}_0$. Thus, for $(\ell, k, j) \in \mathcal{Q}$ with $\ell > \ell_0$, we have by the mesh-closure estimate (5.45), optimality of Dörfler marking (5.41) and full linear convergence (5.44) that

$$\begin{aligned}
\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} &\overset{(5.45)}{\leq} C_{\text{mesh}} \sum_{\ell'=\ell_0}^{\ell-1} \#\mathcal{M}_{\ell'} \overset{(5.41)}{\leq} C_{\text{mesh}} C_{\text{mark}} \sum_{\ell'=\ell_0}^{\ell-1} \#\mathcal{R}_{\ell'} \\
&\overset{(5.37)}{\lesssim} \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}^{1/s} \sum_{\ell'=\ell_0}^{\ell-1} \left(\Delta_{\ell'+1}^{0,j}\right)^{-1/s} \\
&\lesssim \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}^{1/s} \sum_{\substack{(\ell',k',j') \in \mathcal{Q} \\ |\ell',k',j'| \leq |\ell,k,j| \\ \ell' \geq \ell_0}} \left(\Delta_{\ell'}^{k',j'}\right)^{-1/s} \overset{(5.44)}{\lesssim} \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}^{1/s} \left(\Delta_\ell^{k,j}\right)^{-1/s}.
\end{aligned}$$

Rearranging the terms and noting that $\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1 \leq 2\,(\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0})$, we obtain that

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1)^s \Delta_\ell^{k,j} \lesssim \|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})} \quad \text{for} \quad \ell > \ell_0.$$

Trivially, full linear convergence proves that

$$(\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1)^s \Delta_{\ell_0}^{k,j} = \Delta_{\ell_0}^{k,j} \lesssim \Delta_{\ell_0}^{0,0} \quad \text{for} \quad \ell = \ell_0.$$

We recall from Bespalov *et al.* (2017, Lemma 22) that for all $\mathcal{T}_H \in \mathbb{T}$ and all $\mathcal{T}_h \in \mathbb{T}(\mathcal{T}_H)$, it holds that

$$\#\mathcal{T}_h - \#\mathcal{T}_H + 1 \leq \#\mathcal{T}_h \leq \#\mathcal{T}_H\,(\#\mathcal{T}_h - \#\mathcal{T}_H + 1). \tag{5.46}$$

Overall, we have thus shown that

$$(\#\mathcal{T}_\ell)^s \Delta_\ell^{k,j} \overset{(5.46)}{\lesssim} (\#\mathcal{T}_\ell - \#\mathcal{T}_{\ell_0} + 1)^s \Delta_\ell^{k,j} \lesssim \max\left\{\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})}, \Delta_{\ell_0}^{0,0}\right\}$$

for all $(\ell, k, j) \in \mathcal{Q}$ with $\ell \geq \ell_0$. This concludes the proof of the upper bound in (5.31b) and hence that of (4.11).

Step 6. We prove the equivalence in (4.12) by combining the steps above. Recall that

$$\mathcal{Q} \backslash \{(\ell, k, j) \in \mathcal{Q} : \ell \geq \ell_0\} = \{(\ell, k, j) \in \mathcal{Q} : \ell < \ell_0\} \quad \text{is finite}$$

and that $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_0)} < \infty$ is equivalent to $\|u^\star\|_{\mathbb{A}_s(\mathcal{T}_{\ell_0})} < \infty$. Thus, the claim follows immediately by the equivalence in (4.11). This concludes the proof.

6. Numerical experiments

We consider the model problem (1.1) from the introduction. The MATLAB implementation of the following experiments is embedded into the open source software package MooAFEM from

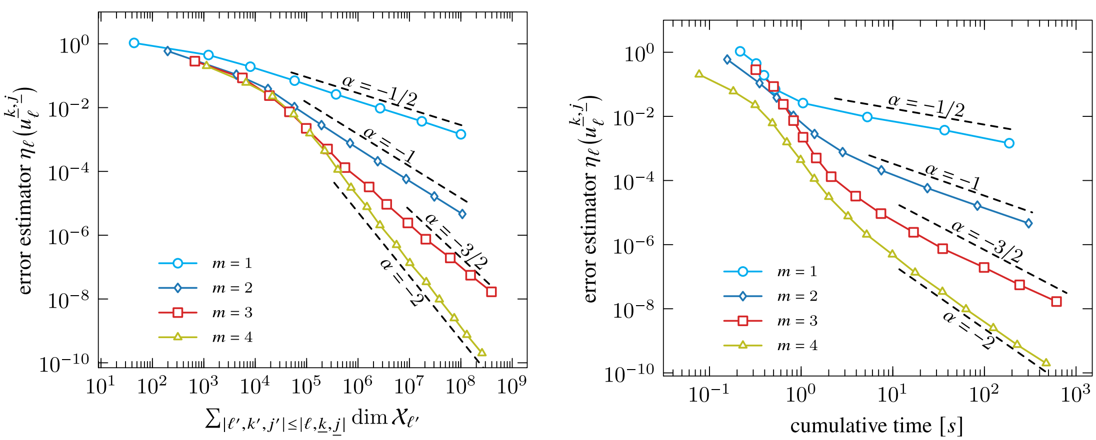

FIG. 2. *Optimality of AISFEM for the diffusion–convection–reaction problem on the L-shaped domain from Section 6.1.* Convergence history plot of the error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}})$ over the computational costs (left) and the elapsed computational time (right) for different polynomial degrees m.

Innerberger & Praetorius (2023). In the following, Algorithm A employs the optimal local *hp*-robust multigrid method from Innerberger *et al.* (2022) as algebraic solver and the standard residual error estimator η_ℓ. Given $T \in \mathscr{T}_\ell$ and $v_\ell \in \mathscr{X}_\ell$, the local contribution of η_ℓ reads

$$\eta_\ell(T; v_\ell)^2 := h_T^2 \,\|-\mathrm{div}(\boldsymbol{A}\,\nabla v_\ell - \boldsymbol{f}) + \boldsymbol{b}\cdot\nabla v_\ell + c\,v_\ell - f\|_{L^2(T)}^2 + h_T\,\|[\![(\boldsymbol{A}\,\nabla v_\ell - \boldsymbol{f})\cdot\boldsymbol{n}]\!]\|_{L^2(\partial T\cap\Omega)}^2.$$

It is well known (see, e.g., Carstensen *et al.*, 2014, Section 6.1) that η_ℓ satisfies the axioms (A1)–(A4) from Section 2.3.

6.1 *Diffusion–convection–reaction on L-shaped domain*

In this subsection, we consider the problem (1.1) on the L-shaped domain $\Omega = (-1,1)^2 \setminus \big([0,1]\times[-1,0]\big) \subset \mathbb{R}^2$ with coefficients $\boldsymbol{A}(x) = \mathrm{Id}$, $\boldsymbol{b}(x) = x$ and $c(x) = 1$, and right-hand side $f(x) = 1$, i.e.,

$$-\Delta u^\star(x) + x\cdot\nabla u^\star(x) + u^\star(x) = 1 \quad \text{for } x \in \Omega \quad \text{subject to} \quad u^\star(x) = 0 \quad \text{for } x \in \partial\Omega.$$

Optimality of AISFEM. We first display the optimality of Algorithm A with respect to the computational cost stated in Theorem 4.3 using the equivalence $\#\mathscr{T}_\ell \simeq \dim \mathscr{X}_\ell$. Numerically, we test with the parameters $\lambda_{\mathrm{sym}} = \lambda_{\mathrm{alg}} = 0.1$, $\delta = 0.5$ and $\theta = 0.5$ and, unless stated explicitly, the stopping criterion $\dim \mathscr{X}_\ell > 10^7$. Note that both the total error and the algebraic error are unknown in all practical purposes. Therefore, we cannot study the decay of the quasi-error, but rather consider the equivalent error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}})$; see (5.40). Figure 2 shows that the proposed algorithm achieves optimal rates $-m/2$ for several polynomial degrees m both with respect to the computational costs and the elapsed computational time after a short preasymptotic phase.

Optimality of the iteratively-symmetrized solver. Optimality of AISFEM is possible when the inherent symmetrization and algebraic procedures are treated efficiently. In Fig. 3, we present the time required for

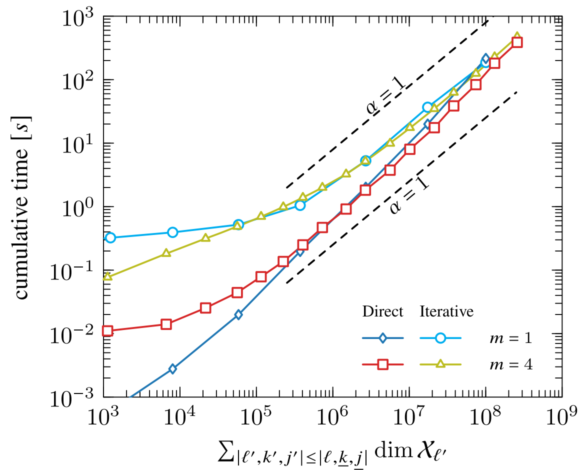

FIG. 3. *Optimality of the combined iterative solver for the diffusion–convection–reaction problem on the L-shaped domain from Section* 6.1. Cumulative time for the direct solve and AISFEM over the computational costs.

our iteratively symmetrized solver compared to the MATLAB built-in direct solver (backslash) of the linear system related to (1.3). We note that the displayed timings are comparing the direct solve time itself with the remaining time (including the setup of the Zarantonello system, computation of the error estimator and mesh refinement). Hence, the presented numbers favor the built-in direct solver over the MATLAB-implemented multigrid code. Nevertheless, the combination of the Zarantonello symmetrization with the optimal local multigrid solver from Innerberger *et al.* (2022) appears to be of comparable speed to the built-in direct solver with the observation that as the dimension of the linear system increases, the backslash performance begins to degrade. Moreover, Fig. 4 shows that the iteration numbers of the solver remain uniformly bounded in the levels for various choices of the parameters λ_{sym} and θ. Note that when λ_{sym} decreases, a higher accuracy of the Zarantonello symmetrization is required. Therefore, the iteration numbers are expected to increase with smaller λ_{sym} as seen in Fig. 4 (left). Moreover, the iteration numbers are also expected to increase as θ becomes larger. This is due to the aggressive refinement leading to hierarchies of low numbers of levels, but with considerable increase in the dimension of the linear systems. This may lead to the conclusion that θ should be chosen very small in order to have less iterations per level, but studying the *cumulative* solver steps in Fig. 4 (right) shows that this is not the best strategy.

Parameter study of AISFEM. We now investigate which parameters yield the best contraction in the iterativelysymmetrized step A(ii)–(iii). Since the parameters depend on the contraction factors q_{alg} from (2.9) and q_{sym} from (4.1), we study a setting where the exact discrete solution $u_\ell^\star$ to (1.3) and the exact Zarantonello solution $u_\ell^{k,\star}$ to (1.5) are computed. Then, we compute $q_{\mathrm{alg}}(\ell,k,j)$ for $(\ell,k,j) \in \mathcal{Q}$ and define the level-wise contraction factors $q_{\mathrm{alg}}(\ell)$ as the maximum over all $q_{\mathrm{alg}}(\ell,k,j)$ for fixed $\ell \in \mathbb{N}_0$ and analogously for q_{sym}. From now on, we fix the polynomial degree $m = 2$ and the parameters $\lambda_{\mathrm{alg}} = 10^{-2}$ for the numerical experiments. We investigate the behavior of the combined solver for various choices of $\lambda_{\mathrm{sym}} \in \{10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$ and $\theta \in \{0.1, 0.2, \ldots, 0.8, 0.9\}$. Figure 6 shows upper bounds $\lambda_{\mathrm{alg}} < \overline{\lambda}_{\mathrm{alg}}^\star = (1 - q_{\mathrm{sym}})(1 - q_{\mathrm{alg}})/(4\,q_{\mathrm{alg}})$ (see the implicit definition of $\overline{\lambda}_{\mathrm{alg}}^\star$ in (4.2))

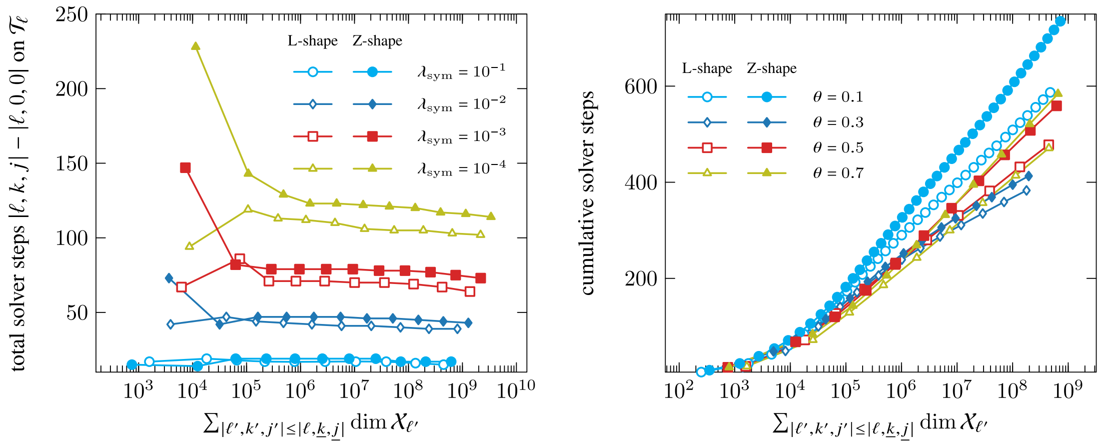

FIG. 4. *Uniform bound on the iteration numbers for the diffusion–convection–reaction problem on the L-shaped domain from Section 6.1 and the strong convection problem on the Z-shaped domain from Section 6.2.* Number of total solver steps $|\ell, k, j| - |\ell, 0, 0|$ on the level ℓ for various selections of the symmetrization stopping parameter λ_{sym} with fixed $\theta = 0.5$ (left) and the cumulative solver steps for different marking parameter θ with fixed $\lambda_{sym} = 0.1$ (right).

TABLE 1 *Optimal selection of parameter with respect to the computational costs for the experiment from Section 6.1. For the comparison, we consider $\sum_{|\ell',k',j'| \le |\ell,\underline{k},\underline{j}|} \dim \mathcal{X}_{\ell'} \times \eta_\ell(u_\ell^{\underline{k},\underline{j}})$ with stopping criterion $\eta_{\underline{\ell}}(u_{\underline{\ell}}^{\underline{k},\underline{j}}) < 10^{-5}$ for various choices of λ_{sym} and θ with the optimal choice highlighted in color*

λ_{sym}/θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
10^{-1}	533	470	402	424	497	608	801	971	1513
10^{-2}	3084	1878	1566	1482	1524	1624	1869	2485	4266
10^{-3}	6543	4490	3478	2831	2894	3371	3826	4729	6956
10^{-4}	10791	6621	5211	4381	4475	4777	5979	7398	10901

and Fig. 5 displays contraction factors $q_{sym} \approx 1/2$ and $\overline{q}_{sym} \approx 1/2$, independently of the choice of θ and λ_{sym}. Note that q_{sym} being close to $\overline{q}_{sym}$ means that the perturbed, i.e., iteratively symmetrized, Zarantonello step is of comparable performance to the unperturbed Zarantonello iteration. Moreover, Table 1 shows that the optimal combination of the parameters with respect to the computational costs is $\theta = 0.3$ and $\lambda_{sym} = 10^{-1}$. Furthermore, it appears that the choice of θ has a stronger impact on the costs than the selection of λ_{sym}.

6.2 *Strong convection on Z-shaped domain*

In this subsection, we consider the problem (1.1) on the Z-shaped domain $\Omega = (-1,1)^2 \setminus \mathrm{conv}\{(0,0),(-1,0),(-1,-1)\} \subset \mathbb{R}^2$ with coefficients $\boldsymbol{A}(x) = \mathrm{Id}$ and $\boldsymbol{b}(x) = (5,5)^\top$, and right-hand side $f(x) = 1$, i.e.,

$$-\Delta u^\star(x) + (5,5)^\top \cdot \nabla u^\star(x) = 1 \quad \text{for } x \in \Omega \quad \text{and} \quad u^\star(x) = 0 \quad \text{for } x \in \partial\Omega.$$

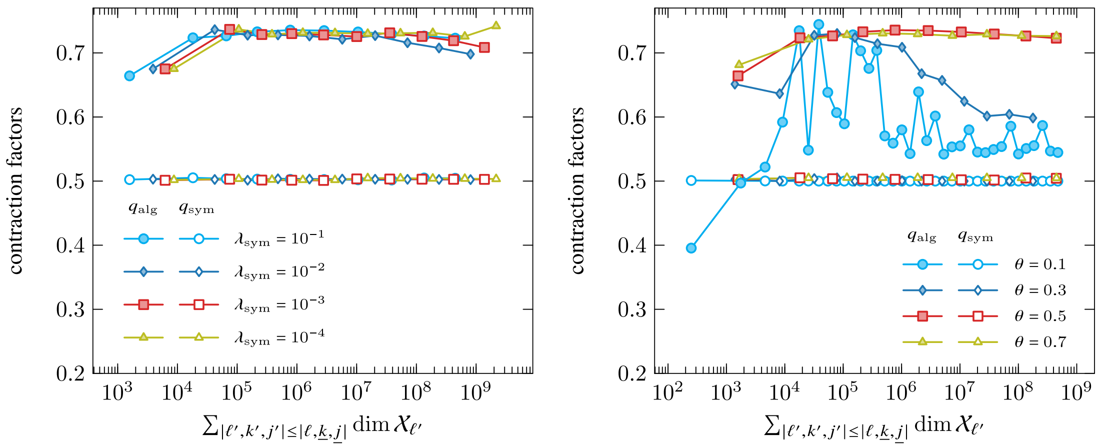

FIG. 5. *Uniform contraction of the iterative solver for the diffusion–convection–reaction problem on the L-shaped domain from Section* 6.1. Experimental contraction factors q_{alg}, q_{sym} and $\overline{q}_{\mathrm{sym}}$ for various choices of the symmetrization stopping parameter λ_{sym} with fixed $\theta = 0.5$ (left) and different marking parameter θ with fixed $\lambda_{\mathrm{sym}} = 0.1$ (right).

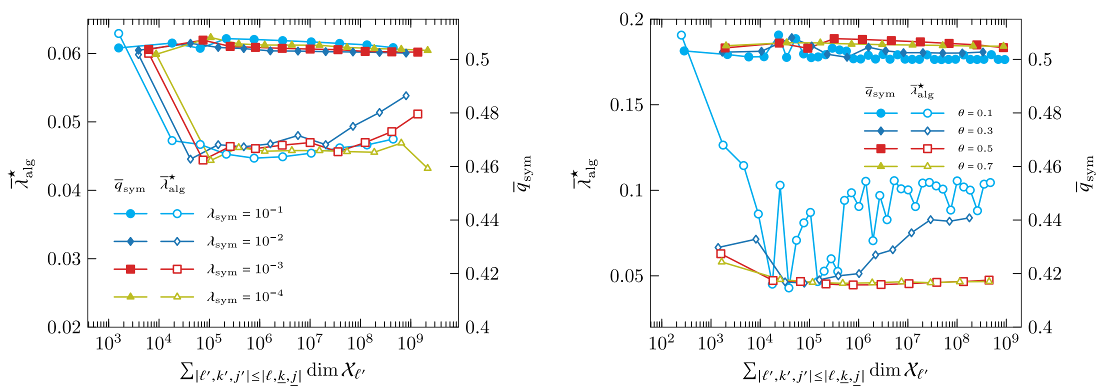

FIG. 6. Computed upper bounds for $\lambda^{\star}_{\mathrm{alg}} < \overline{\lambda}^{\star}_{\mathrm{alg}}$ for various choices of the symmetrization stopping parameter λ_{sym} with fixed $\theta = 0.5$ (left) and different marking parameter θ with fixed $\lambda_{\mathrm{sym}} = 0.1$ (right), where we emphasize the double scaling of the y-axis for $\lambda^{\star}_{\mathrm{alg}}$ resp. $\overline{q}_{\mathrm{sym}}$ in both figures.

Figure 7 shows that even for a strong convection combined with a strong singularity at the origin, the adaptive algorithm recovers the optimal convergence rates $-m/2$ for several polynomial degrees m both with respect to the cumulative costs and computational time. In Fig. 4, we see that the number of solver steps per level ℓ behaves similarly to the diffusion–convection–reaction problem on the L-shape from Section 6.1 with an increase due to the stronger singularity. Furthermore, Fig. 8 displays upper bounds on $\lambda_{\mathrm{alg}} \le \lambda^{\star}_{\mathrm{alg}} < \overline{\lambda}^{\star}_{\mathrm{alg}}$ and the contraction factor $\overline{q}_{\mathrm{sym}} \approx 1/2$ (after an initial phase of reduced contraction) for the perturbed Zarantonello system in (4.2).

7. Conclusion and future work

In this work, we have developed and analyzed an adaptive finite element method for nonsymmetric second-order linear elliptic PDEs (1.1). From a conceptual point of view, the crucial assumption is that

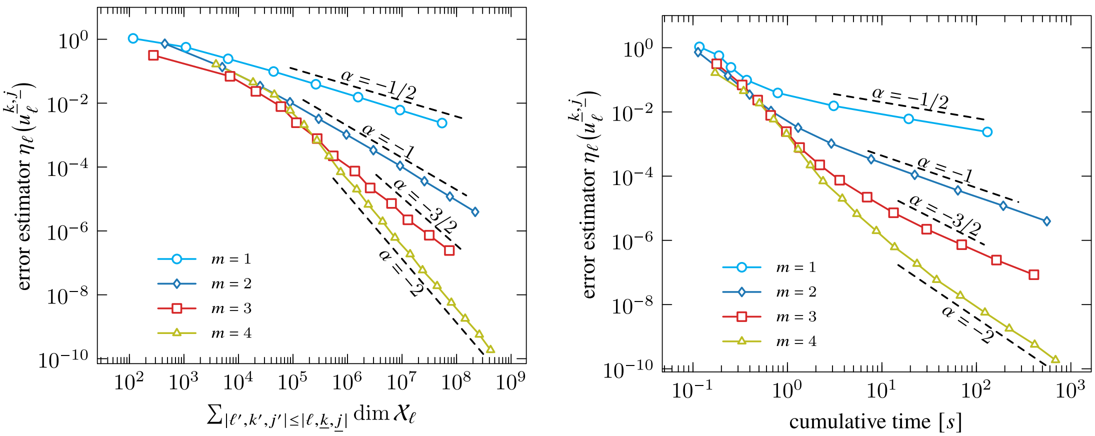

FIG. 7. *Optimality of AISFEM for the strong convection problem on the Z-shaped domain from Section* 6.2. Convergence history plot of the error estimator $\eta_\ell(u_\ell^{\underline{k},\underline{j}})$ over the computational cost (left) and the elapsed computational time (right).

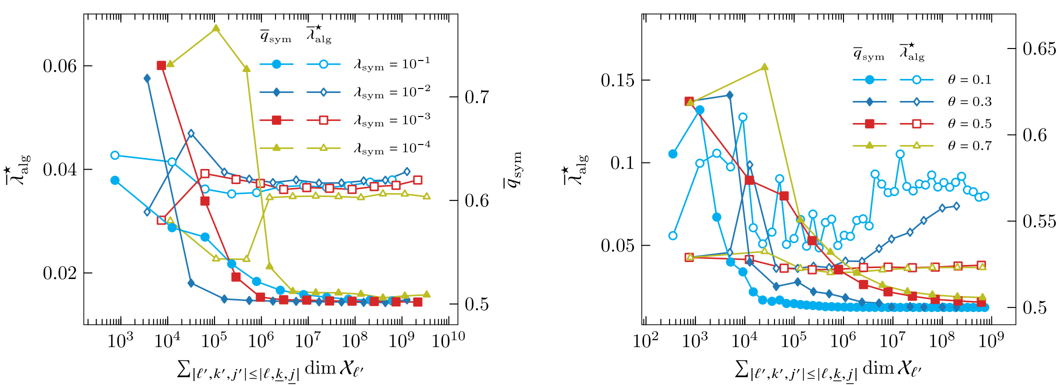

FIG. 8. *Uniform contraction of the combined solver for the strong convection problem on the Z-shaped domain from Section* 6.2. Contraction factor $\overline{q}_{\rm sym}$ and computed upper bound for $\lambda^\star_{\rm alg} < \overline{\lambda}^\star_{\rm alg}$ for various symmetrization stopping parameter $\lambda_{\rm sym}$ with fixed $\theta = 0.5$ (left) and different marking parameter θ with fixed $\lambda_{\rm sym} = 0.1$ (right), where we emphasize the double scaling of the y-axis for $\lambda^\star_{\rm alg}$ resp. $\overline{q}_{\rm sym}$ in both figures.

the weak formulation takes the form

$$b(u^\star, v) := a(u^\star, v) + \langle \mathcal{K} u^\star, v \rangle = F(v) \quad \text{for all } v \in \mathcal{X}, \tag{7.1}$$

where $F \in \mathcal{X}'$ is a linear and continuous functional, $a(\cdot,\cdot)$ is a symmetric, continuous and elliptic bilinear form on $\mathcal{X}$, and $\mathcal{K} : \mathcal{X} \to \mathcal{X}'$ is a compact operator such that the bilinear form $b(\cdot,\cdot)$ is still elliptic on $\mathcal{X}$. Let $|||\cdot|||$ denote the $a(\cdot,\cdot)$-induced energy norm. For the discrete formulation

$$b\left(u_\ell^\star, v_\ell\right) = F(v_\ell) \quad \text{for all } v_\ell \in \mathcal{X}_\ell, \tag{7.2}$$

we require an (abstract) inexact iterative solver whose iteration map is given by $\overline{\Phi}_\ell(F;\cdot)\colon \mathscr{X}_\ell \to \mathscr{X}_\ell$ contracting the *error* in the energy norm, i.e.,

$$|\!|\!| u_\ell^\star - \overline{u}_\ell^{k+1} |\!|\!| \le \overline{q}_{\mathrm{sym}} \, |\!|\!| u_\ell^\star - \overline{u}_\ell^{k} |\!|\!| \quad \text{with } \overline{u}_\ell^{k+1} := \overline{\Phi}_\ell\big(F; \overline{u}_\ell^k\big) \text{ for all } k \in \mathbb{N}, \tag{7.3}$$

where the contraction constant $0 < \overline{q}_{\mathrm{sym}} < 1$ is independent of $\overline{u}_\ell^0 \in \mathscr{X}_\ell$. Under such assumptions and with the usual residual *a posteriori* error estimator $\eta_\ell(\cdot)$ (satisfying the abstract assumptions (A1)–(A4)) on nested conforming discrete spaces $\mathscr{X}_\ell \subseteq \mathscr{X}_{\ell+1} \subset \mathscr{X}$, the present work proves that the analysis from Gantner *et al.* (2021) can be generalized from symmetric PDEs (with $\mathscr{K} = 0$) to the general formulation (7.1): Restricting Algorithm A to the outer ℓ-loop (for mesh refinement) and the inner k-loop (for the solver associated to $\overline{\Phi}_\ell$), we obtain a simplified index set

$$\overline{\mathscr{Q}} := \big\{ (\ell, k) \in \mathbb{N}_0^2 : \overline{u}_\ell^k \text{ is computed by the simplified algorithm} \big\} \tag{7.4}$$

together with the canonical step counter $|\ell, k| \in \mathbb{N}_0$ on $\overline{\mathscr{Q}}$ defined analogously to (3.2). Then, Lemma 5.2 (lucky nontermination of the solver), Proposition 5.3 (plain convergence), Lemma 5.4 (quasi-Pythagorean estimate) and Lemma 5.5 (contraction of weighted discretization and solver error) hold verbatim (and the proof of Lemma 5.4 indeed relies on the compactness of $\mathscr{K}$) if we replace $u_\ell^{k,\underline{j}}$ in the given proofs by $\overline{u}_\ell^k$ in the current solver setting. Therefore, we obtain full linear convergence in the spirit of Theorem 4.1: for arbitrary adaptivity parameters $0 < \theta \le 1$ and $\lambda_{\mathrm{sym}} > 0$, there exist constants $C_{\mathrm{lin}} > 0$ and $0 < q_{\mathrm{lin}} < 1$ as well as an index $\ell_0 \in \mathbb{N}_0$ such that

$$\overline{\Delta}_\ell^k \le C_{\mathrm{lin}} \, q_{\mathrm{lin}}^{|\ell,k| - |\ell',k'|} \, \overline{\Delta}_{\ell'}^{k'} \quad \text{for all } (\ell', k'), (\ell, k) \in \overline{\mathscr{Q}} \text{ with } |\ell', k'| \le |\ell, k| \text{ and } \ell' \ge \ell_0, \tag{7.5}$$

where $\overline{\Delta}_\ell^k := |\!|\!| u^\star - \overline{u}_\ell^k |\!|\!| + \eta_\ell(\overline{u}_\ell^k)$ denotes the corresponding quasi-error. In particular, also Corollary 4.2 holds verbatim with $\mathscr{Q}$ replaced by $\overline{\mathscr{Q}}$ and $\Delta_\ell^{k,\underline{j}}$ replaced by $\overline{\Delta}_\ell^k$, i.e., convergence rates with respect to the number of degrees of freedom coincide with rates with respect to the overall computational cost. Finally, it is easy to check that also Theorem 4.3 holds verbatim and proves that, for sufficiently small adaptivity parameters $0 < \theta \ll 1$ and $0 < \lambda_{\mathrm{sym}} \ll 1$ in the sense of (4.10), it holds that

$$\|u^\star\|_{\mathbb{A}_s(\mathscr{T}_0)} < \infty \quad \Longleftrightarrow \quad \sup_{(\ell,k) \in \mathscr{Q}} \Bigg(\sum_{\substack{(\ell',k') \in \overline{\mathscr{Q}} \\ |\ell',k'| \le |\ell,k|}} \# \mathscr{T}_{\ell'} \Bigg)^s \overline{\Delta}_\ell^k < \infty, \tag{7.6}$$

which yields optimal complexity of the simplified algorithm.

In the current analysis, the combined Zarantonello symmetrization with a contractive SPD algebraic solver is used as one solver module to guarantee (7.3) for $\overline{u}_\ell^k := u_\ell^{k,\underline{j}}$ (see Lemma 5.1), leading to all results being formulated over the triple index set $\mathscr{Q} \subset \mathbb{N}_0^3$ (see Section 3–4).

We note that another choice for solving the arising nonsymmetric FEM systems would be preconditioned GMRES (see, e.g., Saad & Schultz, 1986; Saad, 2003), where an optimal preconditioner for the symmetric part would be employed. Then, it is well-known from the field-of-value analysis (see, e.g., Starke, 1997) that the algebraic solver would satisfy a *generalized* contraction for the algebraic residual (in a discrete vector norm). However, the link between the algebraic residual and the functional setting appears to be open. Moreover, the *a posteriori* error control of the algebraic error for such a GMRES solver is still to be developed.

While these questions are left for future work, we already note some results that can be achieved along the arguments of Gantner *et al.* (2021): if the solver $\overline{\Phi}_\ell(F;\cdot)$ provides iterates $(\overline{u}_\ell^k)_{k\in\mathbb{N}_0}$, satisfying only the generalized contraction

$$|||u_\ell^\star - \overline{u}_\ell^k||| \le \overline{C}_{\rm sym}\,\overline{q}_{\rm sym}^k\,|||u_\ell^\star - \overline{u}_\ell^0||| \quad \text{for all } k \in \mathbb{N} \tag{7.7}$$

together with the *a posteriori* error control

$$|||u_\ell^\star - \overline{u}_\ell^k||| \le \overline{C}'_{\rm sym}\,|||u_\ell^k - \overline{u}_\ell^{k-1}||| \quad \text{for all } k \in \mathbb{N}, \tag{7.8}$$

where $\overline{C}_{\rm sym}, \overline{C}'_{\rm sym} > 0$ and $0 < \overline{q}_{\rm sym} < 1$ are given constants independently of $\overline{u}_\ell^0 \in \mathcal{X}_\ell$, then full linear convergence (7.5) can be proved for all $0 < \theta \le 1$ under the additional assumption that $\lambda_{\rm sym}$ has to be sufficiently small. However, the proof of full linear convergence (7.5) for arbitrary $0 < \theta \le 1$ and arbitrary $\lambda_{\rm sym} > 0$ is open, while optimal complexity (7.6) for sufficiently small $0 < \theta < 1$ and $\lambda_{\rm sym}$ in the sense of (4.10) remains valid (even with the same proof).

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