DIPLOMARBEIT

# Do entangled states correspond to entangled measurements under local transformations?

zur Erlangung des akademischen Grades

**Diplom-Ingenieur**

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# Preface

The research for this diploma thesis has been conducted by Florian Pimpel, Martin J. Renner and Armin Tavakoli and has been submitted to "arXiv" as “Correspondence between entangled states and entangled bases under local transformations” [1]. It has recently been accepted for publication as a Regular Article in “Physical Review A”.

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# Abstract

The entanglement of quantum states is amongst the most studied features of quantum mechanics. It is integral for the majority quantum information protocols and describes a non-classical type of correlation, that links the states of physical systems in a way that is different from our intuitive understanding. Some of the most relevant techniques for quantum technologies such as teleportation, dense coding and entanglement swapping however, depend on the entanglement of joint quantum measurements, which has been seeing much less scientific attention. In general, the established Bell state measurement is used, whose measurement basis consists of the four maximally entangled states. This is the reason why we can consider it as the measurement corresponding to the maximally entangled state. In this thesis we want to address the question whether all entangled states can generally be related to a corresponding iso-entangled measurement in which all measurement basis vectors are local unitary transformations of the original state. In the process of analysing some quantum systems we prove that a corresponding basis exists for every bipartite state with a local dimension of either two, four or eight. Furthermore, we find strong numerical evidence that the same is true for two qutrits and three qubits. Nevertheless, we conjecture that there are quantum states without a basis, since the same numerics cannot find a measurement basis for some four qubit states. More restrictively, we examine whether there exist local unitaries that generate a basis from any state, independently of the specific state. We prove that such a state-independent basis construction does not exist for general quantum states, but we show that it is possible for real valued composite qubit states if and only if the amount of qubits is smaller than four and that it cannot exit for multipartite states with an odd local dimension. Additionally, we give explicit constructions for some specific n-qubit entangled states. The results presented in this thesis suggest that similarly to the entanglement of quantum states, the entanglement of iso-entangled joint measurements show a strong dependence on particle numbers and dimension.

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# Zusammenfassung

Die Verschränkung von Quantenzuständen ist eine der am meisten untersuchten Eigenschaften der Quantenmechanik. Sie ist für den Großteil von Quanteninformationsprotokollen unerlässlich und beschreibt eine nicht-klassische Art der Korrelation, die die Zustände von physikalischen Systemen auf eine Weise verknüpft, die sich von unserem intuitiven Verständnis abhebt. Viele der wichtigsten Techniken für Quantentechnologien wie Teleportation, Dense Coding und Entanglement Swapping beruhen jedoch auf der Verschränkung von Quantenmessungen, die bisher weitaus weniger wissenschaftliche Aufmerksamkeit erfahren hat. Im Allgemeinen wird die etablierte Bell-Zustandsmessung verwendet, deren Messbasis aus den vier maximal verschränkten Zuständen besteht. Aus diesem Grund können wir sie als die Messung betrachten, die dem maximal verschränkten Zustand entspricht. In dieser Arbeit wollen wir der Frage nachgehen, ob sich alle verschränkten Zustände allgemein auf eine entsprechende iso-verschränkte Messung beziehen lassen, bei der alle Messbasisvektoren lokale unitäre Transformationen des ursprünglichen Zustands sind. Bei der Analyse einiger Quantensysteme beweisen wir, dass für jeden zweiseitigen Zustand mit einer lokalen Dimension von entweder zwei, vier oder acht eine entsprechende Basis existiert. Außerdem finden wir starke numerische Hinweise dafür, dass dies auch für zwei Qutrits und drei Qubits gilt. Dennoch stellen wir die Vermutung auf, dass es Quantenzustände ohne Basis gibt, da dieselbe Numerik keine Messbasis für einige Vier-Qubit-Zustände finden kann. Außerdem untersuchen wir restriktiver, ob es lokale unitäre Transformationen gibt, die aus jedem Zustand eine Basis erzeugen können, unabhängig von dem spezifischen Zustand selbst. Wir beweisen, dass eine solche zustandsunabhängige Basiskonstruktion für allgemeine Quantenzustände nicht existiert, aber wir zeigen, dass sie für reellwertige zusammengesetzte Qubit-Zustände genau dann möglich ist, wenn die Anzahl der Qubits kleiner als vier ist, und dass sie für mehrteilige Zustände mit ungerader lokaler Dimension nicht existieren kann. Zusätzlich geben wir explizite Konstruktionen für einige spezifische verschränkte n-Qubit-Zustände an. Die in dieser Arbeit vorgestellten Ergebnisse legen nahe, dass ähnlich wie die Verschränkung von Quantenzuständen auch die Verschränkung von iso-verschränkten Messungen eine starke Abhängigkeit von der Teilchenzahl und der Dimension aufweist.



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# Acknowledgments

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# 1 Introduction

Since its establishment in the early twentieth century, quantum mechanics has grown to one of the most successful theories physics can offer. Based on a few repeatedly validated postulates, it provides the mathematical framework that is used in various disciplines of physics to describe systems consisting of the smallest particles. One of such fields of research that has been gaining more and more attention in the last years is quantum information science. While addressing some of the most foundational questions, it also enables possible protocols which, in some specific situations, are capable of outperforming any computation that relies on classical information theory. In contrast to its classical counterpart, quantum information theory describes information processing in systems, that obey the laws of quantum mechanics and can therefore make use of some of its bizarre properties. One of these key features is a non-classical type of correlation that is fundamental to quantum mechanics, called *entanglement*. With its partly counterintuitive implications, that gained a lot of scientific and public attention, it is not only broadly useful for applications in quantum information science, but also an intensively studied research field itself.

Some of the most relevant techniques for quantum technologies such as teleportation [2], dense coding [3] and entanglement swapping [4] however, depend on the entanglement of joint quantum measurements, which has been seeing much less scientific attention. With *entangled measurements* we refer to projective joint measurements, with an eigenbasis that consists of equally entangled states. In general the established Bell state measurement is used, whose measurement basis consists of the four maximally entangled states $(|00\rangle \pm |11\rangle)/\sqrt{2}$ and $(|01\rangle \pm |10\rangle)/\sqrt{2}$. This is the reason why we can consider it as the measurement corresponding to the maximally entangled state. Similarly, iso-entangled basis states are known to exist for other special states, like GHZ-, W- , and some Dicke states [5, 6], which we will introduce later. To broaden the understanding of entangled measurements we therefore want to address the question whether all entangled states can generally be related to a corresponding iso-entangled measurement in which all measurement basis vectors are local unitary transformations of the original state. Studying this connections may not only be an interesting aspect of quantum mechanics. It also has the potential to open up new possibilities in quantum information applications, since entangled measurements beyond the bell basis are increasingly interesting for topics such as network nonlocality [7] and entanglement-assisted quantum communication [8, 9].

Considering that we are given a joint pure quantum state $|\psi\rangle$ of $n$ subsystems, each



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of dimension $d$, we examine, whether it is possible to find a measurement, namely an orthonormal basis of the global $d^n$-dimensional Hilbert space, in which all the basis states have the same degree of entanglement. Specifically, we want to decide the existence of $d^n$ strings of $n$ local unitary transformations, such that when applied to the joint quantum state, each of the strings generates a vector of an orthonormal basis. The resulting basis vectors are all equivalent under local unitary transformations and hence equally entangled. After that, we investigate, whether there exist classes of states, such that there are strings of local unitary operators, that map any state of the class to an orthonormal basis.

The basic concepts, that are necessary for the scope of this work, will be introduced in the following sections [10].

## 1.1 Postulates of Quantum Mechanics

**Postulate 1**: Any isolated physical system can be associated to a complex vector space endowed with an inner product (Hilbert space $\mathcal{H}$), called the *state space*. The properties of the system are fully described by its *state vector*.

The state vectors, called *kets* $|\psi\rangle$, are unit vectors in the Hilbert space and can be depicted as a linear combination of the orthonormal basis states $|e_i\rangle \in \mathcal{H}$. In the case of an $N$-dimensional Hilbert space, this is realized by a finite sum.

$$|\psi\rangle = \sum_{i=1}^{N} \alpha_i |e_i\rangle \in \mathcal{H}, \quad \alpha_i \in \mathbb{C}, \sum_{i=1}^{N} |\alpha_i| = 1, \langle e_i|e_j\rangle = \delta_{ij} \tag{1.1}$$

There is a bijective map between the Hilbert space $\mathcal{H}$ and its dual space $\mathcal{H}^*$, which elements are called *bras*.

$$(.)^\dagger : \mathcal{H} \to \mathcal{H}^*, \; |\psi\rangle \mapsto (|\psi\rangle)^\dagger = \langle\psi| = \sum_{i=1}^{N} \alpha_i^* \langle e_i| \tag{1.2}$$

The inner product assigns to every pair of state vectors $|\phi\rangle, |\psi\rangle$ a complex number $\langle\phi|\psi\rangle$ and has the following properties:

$$\begin{aligned} \langle\psi|\phi\rangle &= \langle\phi|\psi\rangle^* \\ \langle\psi|\psi\rangle &= 1 \quad \forall\, |\psi\rangle, \end{aligned} \tag{1.3}$$

where the second property is equivalent to the restriction of the kets to be unit vectors. In a finite dimensional Hilbert space we can calculate the inner product by using the coordinate representation with respect to the orthonormal basis:

$$\langle\psi|\phi\rangle = \sum_{i=1}^{N} \alpha_i^* \beta_i, \tag{1.4}$$



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where $|\psi\rangle = \sum_{i=1}^{N} \alpha_i |e_i\rangle$ and $|\phi\rangle = \sum_{i=1}^{N} \beta_i |e_i\rangle$.

The simplest quantum mechanical system is a two dimensional Hilbert space. One example of such a system is the spin of a single spin-1/2 particle, e. g. an electron. The orthonormal basis states are then referred to as $|0\rangle$ and $|1\rangle$ and every state vector can be written as

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle . \tag{1.5}$$

**Postulate 2**: The evolution of a closed quantum system is described by a unitary transformation. The state $|\psi_2\rangle$ at time $t_2$ is related to the state $|\psi_1\rangle$ at time $t_1$ by a unitary operator, that depends only on the times $t_1$ and $t_2$:

$$|\psi_2\rangle = U(t_2, t_1) |\psi_1\rangle . \tag{1.6}$$

In the case of the two dimensional space any possible unitary transformation can be obtained in the laboratory. Of particular interest are the both unitary and hermitian Pauli Operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{1.7}$$

With the help of them any $2 \times 2$ unitary operator can be created up to a global phase, which is irrelevant for our purposes. These are all elements of the $SU(2)$-group and can be generated by a matrix exponential using three real parameters:

$$\begin{gathered} U(\alpha, \theta, \phi) = e^{i\alpha\vec{r}\cdot\vec{\sigma}} \\ \text{with } \vec{r} = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ \sin\theta \end{pmatrix} \text{ and } \vec{\sigma} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} . \end{gathered} \tag{1.8}$$

The X- and Z-operators are often referred to as *bit flip* and *phase flip* operators.

**Postulate 3**: A *measurement* is described by a set of *measurement operators* $\{M_a\}$ acting on the system being measured, where the index $a$ stands for the *measurement outcome*. The probability for the outcome $a$ when measuring a system in the state $|\psi\rangle$ is

$$p(a) = \langle\psi|M_a^\dagger M_a|\psi\rangle \tag{1.9}$$

and the *post-measurement state* after the system is:

$$|\psi_{\text{post}}\rangle = \frac{M_a |\psi\rangle}{\sqrt{\langle\psi|M_a^\dagger M_a|\psi\rangle}} . \tag{1.10}$$

The measurement operators satisfy the completeness equation,

$$\sum_a M_a^\dagger M_a = \mathbb{1} . \tag{1.11}$$



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The completeness equation guarantees that the probabilities sum to $1$:

$$\sum_a \langle\psi|M_a^\dagger M_a|\psi\rangle = \langle\psi|\sum_a M_a^\dagger M_a\,|\psi\rangle = 1. \tag{1.12}$$

An important special class of measurements are the *projective measurements*. For this the measurement operators have to be orthogonal projectors, hence they have to satisfy $M_a M_{a'} = M_a \delta_{aa'}$. Such a measurement is called an observable. It is a hermitian operator and the projective measurement operators are the projections onto the eigenspace of the various measurement outcomes:

$$M = \sum_a a M_a = \sum_a a\,|a\rangle\,\langle a|\,. \tag{1.13}$$

The states $|a\rangle$ are the eigenstates of the measurement and form an orthonormal basis. As a consequence of the projection, the states $|a\rangle$ are also the post-measurement states for the respective measurement outcome. The quantum-physical expectation value of an observable is the probabilistic average of the measurement outcomes and can be written as

$$\langle M\rangle_\psi = \langle\psi|M|\psi\rangle = \sum_a a\,|\langle\psi|a\rangle|^2\,. \tag{1.14}$$

**Postulate 4**: The state space of the composition of physical systems is the tensor product of the state spaces of the constituent systems. If the partial systems are numbered from $1$ to $n$, the state space of the whole is $\mathcal{H} = \bigotimes_{i=1}^n \mathcal{H}_i$. If the state of the $i$-th system is $|\psi_i\rangle$, the joint state is $|\psi\rangle = \bigotimes_{i=1}^n |\psi_i\rangle$, or alternatively $|\psi_1\psi_2...\psi_n\rangle$.

To this point we have only considered pure quantum states. However, if we want to describe a non-isolated system, hence a subsystem where we do not possess information of the entire composite system, we use statistical ensembles in form of a density operator:

$$\rho = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|\,. \tag{1.15}$$

The density operator states that the system is in the state $|\psi_i\rangle$ with the probability $p_i$ and has the following properties:

- The trace is normalized: $\mathrm{tr}\{\rho\} = 1$
- It is positive-semidefinite: $\rho \geq 0$, and therefore hermitian: $\rho^\dagger = \rho$.

Also, the density operator represents a pure state $\rho_{\text{pure}} = |\psi\rangle\,\langle\psi|$ iff $\mathrm{tr}\{\rho^2\} = 1$. If it is not a pure state, it is called a mixed state where the maximally mixed state for a $d$-dimensional system is $\rho_m = \mathbb{1}/d$.



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The evolution by a unitary operator is then described by

$$\rho_2 = U\rho_1 U^\dagger, \tag{1.16}$$

the probability for a measurement outcome $a$ is

$$p\,(a) = \mathrm{tr}\Big\{M_a^\dagger M_a \rho\Big\}, \tag{1.17}$$

and the post-measurement state is

$$\rho_{\mathrm{post}} = \frac{M_a \rho M_a^\dagger}{\mathrm{tr}\Big\{M_a^\dagger M_a \rho\Big\}}. \tag{1.18}$$

The quantum-physical expectation value of an observable $M = \sum_a a\,|a\rangle\,\langle a|$ can be evaluated by

$$\langle M\rangle_\rho = \mathrm{tr}\{\rho M\}. \tag{1.19}$$

Given the state $\rho_{\mathrm{AB}}$ of the joint system of subsystems $\mathrm{A}$ and $\mathrm{B}$, it is possible to obtain a description of the state of one subsystem by building the *reduced density operator*. It is defined by the partial trace:

$$\rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}\{\rho_{\mathrm{AB}}\}. \tag{1.20}$$

This is a valid description of the subsystem, since it fulfils the correct measurement statistics for measurements performed on the respective subsystem.

## 1.2 Qubits and Bloch representation

Similarly to the term *bit* that is used for the simplest form of classical information, the two-dimensional Hilbert space is referred to as the system of one *qubit* – one *quantum bit*. It is spanned by two orthonormal state vectors, e.g. $\{|0\rangle\,,|1\rangle\}$. Alternatively to the form in Eq. (1.5) we can parametrize the pure state vector of a qubit by using the angles $\theta$ and $\phi$:

$$|\psi\rangle = \cos(\theta/2)\,|0\rangle + \mathrm{e}^{i\phi}\sin(\theta/2)\,|1\rangle\,, \tag{1.21}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. This corresponds to the geometrical representation of the qubit-state space called the *Bloch sphere* that is depicted in Fig. 1.1. Every pure state of a single qubit can be associated to a point on the surface of the Bloch sphere. Two states that are mutually orthogonal are represented by antipodal points on the sphere. Notice that a global phase is neglected in this representation, hence $|\phi_2\rangle = \mathrm{e}^{i\alpha}(\cos(\theta/2)\,|0\rangle + \mathrm{e}^{i\phi}\sin(\theta/2)\,|1\rangle) \widehat{=} \cos(\theta/2)\,|0\rangle + \mathrm{e}^{i\phi}\sin(\theta/2)\,|1\rangle = |\phi_1\rangle$. This is because there is no physical difference between $|\phi_1\rangle$ and $|\phi_2\rangle$. We can emphasize this by having a look at any possible measurement we can perform on the system. The probability for a specific measurement outcome is independent of the global phase, since

$$\langle\psi_2|M_a^\dagger M_a|\psi_2\rangle = \langle\psi_1|\mathrm{e}^{-i\alpha}M_a^\dagger M_a \mathrm{e}^{i\alpha}|\psi_1\rangle = \langle\psi_1|M_a^\dagger \mathrm{e}^{-i\alpha}\mathrm{e}^{i\alpha}M_a|\psi_1\rangle = \langle\psi_1|M_a^\dagger M_a|\psi_1\rangle.$$

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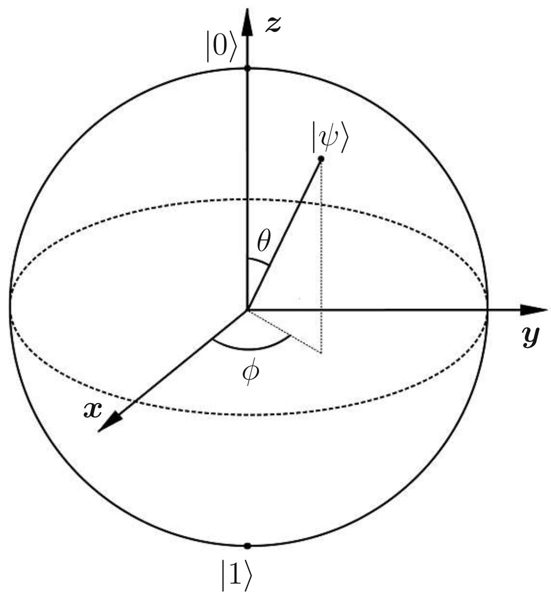


Figure 1.1: The Bloch sphere is a geometric representation of the qubit-state space. Every pure qubit-state is associated to a point an the surface of the sphere. Mixed states are located within in the sphere.

The action of the Pauli operators $(X, Y, Z)$ on a qubit-state results in rotations of the state in the Bloch ball, namely $\pi$-rotations around the respective axes $(x, y, z)$. This can be verified by checking the action of $(X, Y, Z)$ on a state $|\psi\rangle$, e.g. for $Z$:

$$\begin{aligned} Z|\psi\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\cos(\theta/2)|0\rangle - \mathrm{e}^{i\phi}\sin(\theta/2)|1\rangle\right) \\ &= \cos(\theta/2)|0\rangle + \mathrm{e}^{i(\phi+\pi)}\sin(\theta/2)|1\rangle\,. \end{aligned} \tag{1.22}$$

When regarded as projective measurement operators, it is clear that the two eigenvectors of $(X, Y, Z)$ are the two antipodal vectors that lie on the respective axes $(x, y, z)$, e.g. the eigenvectors of $Z$ are $|0\rangle$ and $|1\rangle$. The coordinates of a given qubit state with respect to the axes $(x, y, z)$ can also be evaluated by calculating the expectation values of the Pauli operators. This means that the unit vector $\vec{r}$ of a state $|\psi\rangle$ in the Bloch sphere is given by

$$\vec{r} = \begin{pmatrix} \langle X\rangle_\psi \\ \langle Y\rangle_\psi \\ \langle Z\rangle_\psi \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}, \tag{1.23}$$

where $(\theta, \phi)$ are the parameters of the state in the representation that we used before. This is also possible for density operators of a qubit, which also includes mixed states,



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hence $\vec{r} = (\langle X \rangle_\rho, \langle Y \rangle_\rho, \langle Z \rangle_\rho)^T$. Therefore we can write every qubit state in its *Bloch decomposition*:

$$\rho = \frac{1}{2} \left( \mathbb{1} + \vec{a} \cdot \vec{\sigma} \right), \tag{1.24}$$

where $\vec{a} \in \mathbb{R}^3$, $||\vec{a}|| \leq 1$ and $\vec{\sigma} = (X, Y, Z)^T$. The Bloch sphere also provides a geometric interpretation of the general unitary operator $U(\alpha, \theta, \phi) = e^{i\alpha \vec{r} \cdot \vec{\sigma}}$ as defined in Eq. (1.8). It rotates through an angle of $\alpha$ about the axis that is given by the $(\theta, \phi)$ in the Bloch sphere.

## 1.3 Quantum entanglement

Quantum entanglement is one of the most studied aspects of quantum mechanics, since it is one of the essential properties that distinguishes quantum from classical theories. It is also one of the most potent resources for quantum information processing and communication applications. For the sake of simplicity, this section will only cover the entanglement of pure states, although they are not fully attainable in a realistic scenario. As we will see below, the entanglement in pure states of a composite system is a distinct quantum property, that makes it impossible to describe the information of the whole system by local properties in the subsystems. To analyse entanglement, it is therefore obligatory to define all the operations that can be done locally.

### 1.3.1 Local operations assisted by classical communication

All non-selective operations that map a valid quantum state $\rho$ to another quantum state $\rho'$ can be described by completely positive, trace preserving operators [11, 12]:

$$\rho' = \sum_i L_i \rho L_i^\dagger, \quad \sum_i L_i^\dagger L_i = \mathbb{1}. \tag{1.25}$$

Operations where the operators $L_i$ are spatially separable, i.e. $L_i = L_i^A \otimes L_i^B \otimes L_i^C \otimes ...$, are called separable operations ($SEP$) [13]. All operations which can be performed by spatially separated parties that share a joint state form a specific class, called *local operations assisted by classical communication* ($LOCC$). Every such operation has to be separable, but the class is not equivalent to $SEP$, hence $LOCC \subsetneq SEP$. While the above physical definition of $LOCC$ is rather intuitive, the simple mathematical characterization for $SEP$ does not apply [14].

A special class of $LOCC$ are local unitary operators LU:

$$\begin{aligned} \rho' &= U_{\text{local}} \rho U_{\text{local}}^\dagger \\ U_{\text{local}} &= U^A \otimes U^B \otimes U^C \otimes .... \end{aligned} \tag{1.26}$$

They are reversible and preserve the entanglement.

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If two states $\rho$ and $\rho'$ are related by LOCC as in (1.25), they are called *equivalent under LOCC*. If they are related by (1.26), they are called *local unitary equivalent* or shorter *LU-equivalent*. It is known that two states are equivalent under LOCC if and only if they are LU-equivalent [12].

There is also the possibility to perform *stochastic* LOCC (SLOCC), which means that a state is not transformed to another deterministically, but with a certain non-vanishing probability. States that are connected via SLOCC are called to be in the same *SLOCC-equivalence class*.

### 1.3.2 Bipartite entanglement

A bipartite pure state is the easiest setting to discuss entanglement. If the joint quantum state of the composition of two systems $|\psi_{AB}\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ can be written as the tensor product of a state of system $\mathcal{H}_A$ and a state of system $\mathcal{H}_B$, the joint state is called *separable* or *not entangled*:

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \, . \tag{1.27}$$

If this is not possible, the state is called *entangled*.

To study bipartite entanglement it is useful to introduce the *Schmidt decomposition*. A general pure state of two subsystem, $\mathcal{H}_A$ of dimension $n$ and $\mathcal{H}_B$ of dimension $m$, where we assume that $n \geq m$, can be described by using the respective orthonormal bases $\{|i_A\rangle\}$ and $\{|i_B\rangle\}$:

$$|\psi_{AB}\rangle = \sum_i \sum_j \alpha_{ij} |i_A\rangle |j_B\rangle \, . \tag{1.28}$$

Similar to the singular value decomposition, it is proven that such a state can be written as

$$|\psi_{AB}\rangle = U^A \otimes U^B \sum_{i=0}^{m-1} \lambda_i |i_A i_B\rangle \, , \tag{1.29}$$

where $\forall i \, : \, \lambda_i \in \mathbb{R}, \, \lambda_i \geq 0$ and $\sum_{i=0}^{m-1} \lambda_i^2 = 1$ applies. The state that the local unitaries act on is called the *Schmidt decomposition* of $|\psi_{AB}\rangle$, which is therefore LU-equivalent to the state and possesses the same entanglement. The strictly positive factors $\{\lambda_i \, : \, \lambda_i > 0\}$ are called the *Schmidt coefficients* and the amount of them is called the *Schmidt rank*. We can easily see that therefore a bipartite pure state is entangled if and only if it's Schmidt rank is greater than one. Note that for multipartite systems, meaning the composition of more than two subsystems, such a decomposition and hence also the definition of a Schmidt rank is not as easily applicable, however a generalisation is possible [15].



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A commonly used entanglement measure for the bipartite setup is the entanglement entropy, which is the von Neumann entropy of the reduced state.

$$S(\rho_A) = -\operatorname{tr}\{\rho_A \log_2 \rho_A\}, \quad \rho_A = \operatorname{tr}_B\{\rho\} \tag{1.30}$$

It does not matter, to what subsystem the state is reduced, which can easily be seen by using the Schmidt decomposition. Using the Schmidt coefficients $\lambda_i$ the entropy is given by

$$S(\rho_A) = S(\rho_B) = -\sum_i \lambda_i^2 \log_2 \lambda_i^2. \tag{1.31}$$

The entanglement entropy is maximized by the *maximally entangled states*, which for two qubits are the four Bell states:

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned} \tag{1.32}$$

They build an orthonormal basis of the joint space of two qubits.

### 1.3.3 Tripartite entanglement

The structure of entanglement of three parties is already much more complicated than the bipartite case that we have studied before. It was shown that for three qubits, there are different SLOCC classes, displaying that two different types of entanglement exist [16].

We now consider the joint Hilbert space of three qubits $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B} \otimes \mathcal{H}_\mathcal{C}$. A maximally entangled representative of the *GHZ class* is the *GHZ state*

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{1.33}$$

The GHZ state can be regarded as the maximally entangled three qubit state in many aspects, e.g. it maximizes the tripartite entanglement measure called 3-tangle [16, 17]. If two subsystem are traced out, the reduced state is the maximally mixed state

$$\rho_A = \rho_B = \rho_C = \frac{1}{2}\mathbb{1}. \tag{1.34}$$



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It also has the property that any two-particle reduced state is separable:

$$\rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) = \frac{1}{2}\left(|0\rangle\langle 0|\otimes|0\rangle\langle 0| + |1\rangle\langle 1|\otimes|1\rangle\langle 1|\right). \quad (1.35)$$

The maximally entangled state of the SLOCC class, that possesses a different kind of entanglement, is the W state

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right). \quad (1.36)$$

While this state has zero 3-tangle, it maximizes the residual bipartite entanglement [16]. The two-particle reduced state for any two subsystems is

$$\rho_{AB} = \frac{2}{3}\left|\Psi^{+}\right\rangle\left\langle\Psi^{+}\right| + \frac{1}{3}|00\rangle\langle 00|, \quad (1.37)$$

which can be shown to be as high in bipartite entanglement as possible in the case of such a reduction.

As a consequence of the separable two-particle reduced states, the entanglement in a GHZ state is fragile under particle losses, whereas the W state is maximally robust in this regard.

## 1.4 Generalized Schmidt decomposition for three qubits

The Schmidt decomposition we used in the previous section is only applicable to bipartite systems. There is also a generalization for three qubits, as introduced by Acin et.al. [18], and a canonical form for even more general systems as shown by Carteret et.al [19]. In the following chapter, we make use of the canonical form for three qubits, which reads:

$$|\psi\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle + e|111\rangle, \quad (1.38)$$

where $(b, c, d, e)$ are real numbers and $a$ is a complex number. As shown by Perdomo [20], every real three-qubit state

$$|\psi_r\rangle = \sum_i \lambda_i |i\rangle, \quad (1.39)$$

with $i = i_1 i_2 i_3 \in \{0, 1\}^3$ and $\lambda_i \in \mathbb{R}$ is local unitary equivalent to the form

$$|\psi_r'\rangle = a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle + e|111\rangle, \quad (1.40)$$



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where all parameters $(a, b, c, d, e)$ are real numbers. Since every pair of state that does not have the exact same coefficients in its canonical form as is (1.38) are *not* local unitary equivalent [19], the result of Perdomo implies that there are states where we can not get rid of a complex factor by local unitary transformations. This is true because if a state in the form of (1.38) with $a \notin \mathbb{R}$ is LU-equivalent to a real superposition of the product states, it would also be LU-equivalent to a real canonical form, which cannot be true.

## 1.5 Locally maximally entanglable states

Kruszynska and Kraus introduced the class of locally maximally entanglable states (LME states) [21]. They are characterized by the following properties: We consider an $n$-qubit pure state $|\psi\rangle$ and supplement each qubit with a qubit ancilla. We use controlled operations $C_l = \sum_{i=0}^{1} U_l^{(i)} \otimes |i\rangle_{l_a} \langle i|$, where $U_l^{(i)}$ are unitary operators for the system $l$ and $|i\rangle_{l_a} \langle i|$ acts on the ancilla-qubit of system $l$. We call the state $|\psi\rangle$ locally maximally entanglable (LME) iff there exist local controlled operators $C_l$ such that $C_1 \otimes C_2 \otimes ... \otimes C_n |\psi\rangle |+\rangle^{\otimes n}$ is maximally entangled with regards to the separation of the system and the auxiliary system. The state of the ancilla-qubits is defined by $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. Examples for LME states are the GHZ state or graph states [22].

Kruszynska and Kraus also prove that an $n$-qubit state $|\psi\rangle$ is LME iff there exists a unitary operator $U_l$ for each qubit $l$ such that $\{U_1^{i_1} \otimes U_2^{i_2} \otimes ... \otimes U_n^{i_n} |\psi\rangle \,|i_l = 0, 1\}$ forms an orthonormal basis.

## 1.6 Joint quantum measurements

In this thesis we want to study the entanglement of joint quantum measurements, which are used to measure the joint state of two ore more subsystem. The most famous and commonly used entangled joint measurement is the *Bell state measurement* that is a projection onto the four maximally entangled Bell states in Eq. (1.32). This type of entangled measurement is crucial for a lot of quantum information protocols, such as teleportation [2], dense coding [3] and entanglement swapping [4]. As already mentioned, we can view the Bell state measurement as the measurement corresponding to the maximally entangled two-qubit state. A projective measurement with the product-state basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is an example for a measurement with zero entanglement and corresponds to the product state $|00\rangle$. In a similar way, we want to ask, whether all entangled states can be associated to a measurement basis. Specifically, we want to find a basis, where all the basis states are LU-equivalent. The subject of LU-equivalence has already been studied by investigating whether two quantum states are LU-equivalent and by introducing a method to determine the connecting unitary [23, 24]. We want to



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explore a related but different scenario. For a joint quantum state $|\psi\rangle$ of $n$ subsystems of local dimension $d$, we examine the existence of $d^n$ strings, $\{V_j\}_{j=1}^{d^n}$, of local unitary transformations,

$$V_j = \bigotimes_{k=1}^{n} U_k^{(j)} \tag{1.41}$$

where $U_k^{(j)}$ is a $d$-dimensional unitary operator, such that the set of states $|\psi_j\rangle \equiv V_j |\psi\rangle$ form an orthonormal measurement basis, i.e. $|\langle\psi_j|\psi_{j'}\rangle| = \delta_{jj'}$. If this is possible, we say that $|\psi\rangle$ admits a basis and we call the set of basis vectors $\{|\psi_j\rangle\}_{j=1}^{d^n}$ a $|\psi\rangle$-basis. The eigenstates of this projective joint measurement are therefore equally entangled and we can ascribe a specific amount of entanglement to the measurement.

In the previous sections we introduced some classifications for the different kinds of entanglement in pure states. In the following chapters we want to investigate whether something similar is possible for entangled measurements.



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# 2 Are there states without a basis?

In this chapter we want to investigate, whether we can find an entangled state $|\psi\rangle$ that cannot be associated to a measurement basis in a sense that all the basis states of the measurement basis are LU-equivalent to $|\psi\rangle$. We therefore analyse some of the simplest systems with low dimensionality.

## 2.1 Bipartite systems

We begin with bipartite systems, specifically systems that are composed of two equally dimensional subsystems. The simplest example is the composition of two qubits, hence $(n, d) = (2, 2)$.

### 2.1.1 Two qubits

In the previous chapter, we saw that the maximally entangled two qubit states are given by the four Bell states in (1.32), which form the basis of the commonly used Bell state measurement. We can easily check that they indeed are local unitary equivalent by only using the Pauli operators:

$$\begin{aligned} \left|\Phi^{+}\right\rangle &= \mathbb{1} \otimes \mathbb{1} \left|\Phi^{+}\right\rangle \\ \left|\Phi^{-}\right\rangle &= Z \otimes \mathbb{1} \left|\Phi^{+}\right\rangle \\ \left|\Psi^{+}\right\rangle &= \mathbb{1} \otimes X \left|\Phi^{+}\right\rangle \\ \left|\Psi^{-}\right\rangle &= Z \otimes X \left|\Phi^{+}\right\rangle \end{aligned} \tag{2.1}$$

The question is, if this is possible for every two qubit state $|\psi\rangle$. We constructively show that every such state admits a basis. We therefore first apply the state-dependent local unitary transformations $W_\psi^A \otimes W_\psi^B$ that map $|\psi\rangle$, via a Schmidt decomposition, into the computational basis,

$$|\psi_S\rangle = \lambda |00\rangle + \sqrt{1-\lambda^2}\, |11\rangle \tag{2.2}$$

for some coefficient $0 \leq \lambda \leq 1$. One can easily verify that the following construction



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transforms $|\psi_S\rangle$ into an orthonormal $|\psi\rangle$-basis:

$$\begin{array}{c||c} V^j & V^j |\psi_S\rangle \\ \hline \mathbb{1}\otimes\mathbb{1} & \lambda|00\rangle + \sqrt{1-\lambda^2}|11\rangle \\ \mathbb{1}\otimes XZ & \lambda|01\rangle - \sqrt{1-\lambda^2}|10\rangle \\ XZ\otimes Z & \lambda|10\rangle + \sqrt{1-\lambda^2}|01\rangle \\ XZ\otimes X & \lambda|11\rangle - \sqrt{1-\lambda^2}|00\rangle \end{array} \tag{2.3}$$

Notice that once the state has been rotated into the Schmidt form $|\psi_S\rangle$, the subsequent unitaries in Eq. (2.3) do not depend on $\lambda$.

### 2.1.2 Four- and eight-dimensional subsystems

The construction for two qubits can be extended to four- and eight-dimensional subsystems, hence $(n, d) = (2, 4)$ and $(n, d) = (2, 8)$. Again via Schmidt decomposition, we can find state-dependent local unitaries that transform $|\psi\rangle$ into $|\psi_S\rangle = \sum_{l=0}^{d-1} \lambda_l |ll\rangle$ for some Schmidt coefficients $\sum_l \lambda_l^2 = 1$, $\lambda_l \geq 0$. We can now show that there is a set of local unitaries that indeed leads to a $|\psi\rangle$-basis independently of the specific Schmidt coefficients.

Let the local dimension be either $d = 4$ or $d = 8$, and index the $d^2$ basis elements as $(\tilde{j}, j)$ where $\tilde{j} = 0, 1, \ldots, d-1$ and $j = 1, 2, \ldots, d$. Let $W_\psi^A \otimes W_\psi^B$ be the state-dependent local unitaries that transform the general state $|\psi\rangle$ into the Schmidt basis, i.e. $|\psi_S\rangle \equiv W_\psi^A \otimes W_\psi^B |\psi\rangle = \sum_{l=0}^{d-1} \lambda_l |l, l\rangle$, with the Schmidt coefficients $\lambda_l \in \mathbb{R}$ satisfying $\sum_l \lambda_l^2 = 1$. We now further decompose the individual $d$-dimensional registers as a string of $m$ qubits, writing $|l\rangle = |l_1 \ldots l_m\rangle$. Thus, the Schmidt decomposed state reads

$$|\psi_S\rangle = \sum_{l_1,\ldots,l_m=0,1} \lambda_l |l_1 \ldots l_m, l_1 \ldots l_m\rangle\,. \tag{2.4}$$

Once the state has been put in the form (2.4), we apply a set of local unitaries that is independent of the Schmidt coefficients. For $d = 4$ and $\tilde{j} = 0$, the two sets of unitaries read as follows:

$$\begin{array}{cc||cc||c} \tilde{j} & j & U_1^{(\tilde{j},j)} & U_2^{(\tilde{j},j)} & U_1^{(\tilde{j},j)} \otimes U_2^{(\tilde{j},j)} |\psi_S\rangle \\ \hline 0 & 1 & \mathbb{1}\otimes\mathbb{1} & \mathbb{1}\otimes\mathbb{1} & \lambda_{00}|00,00\rangle + \lambda_{01}|01,01\rangle \\ & & & & +\lambda_{10}|10,10\rangle + \lambda_{11}|11,11\rangle \\ \hline 0 & 2 & \mathbb{1}\otimes X & \mathbb{1}\otimes XZ & \lambda_{00}|01,01\rangle - \lambda_{01}|00,00\rangle \\ & & & & +\lambda_{10}|11,11\rangle - \lambda_{11}|10,10\rangle \\ \hline 0 & 3 & X\otimes\mathbb{1} & XZ\otimes Z & \lambda_{00}|10,10\rangle - \lambda_{01}|11,11\rangle \\ & & & & -\lambda_{10}|00,00\rangle + \lambda_{11}|01,01\rangle \\ \hline 0 & 4 & X\otimes X & XZ\otimes X & \lambda_{00}|11,11\rangle + \lambda_{01}|10,10\rangle \\ & & & & -\lambda_{10}|01,01\rangle - \lambda_{11}|00,00\rangle \end{array} \tag{2.5}$$



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In addition, we define $U_1^{(\tilde{j},j)} := X_4^{\tilde{j}} \, U_1^{(\tilde{j}=0,j)}$ and $U_2^{(\tilde{j},j)} := U_2^{(\tilde{j}=0,j)}$, where $X_d$ is the $d$-dimensional shift-operator $X_d = \sum_{l=0}^{d-1} |l+1\rangle\langle l|$. Note that, the unitaries $U_2^{(\tilde{j},j)}$ coincide with the set of unitaries for two qubits given in Eq. (2.3) and do not depend on $\tilde{j}$. At the same time, $U_1^{(\tilde{j}=0,j)}$ are the same as $U_2^{(\tilde{j},j)}$ where the $Z$ gates are left out. We now show that $\{U_1^{(\tilde{j},j)} \otimes U_2^{(\tilde{j},j)} \, |\psi_S\rangle\}_{\tilde{j},j}$ is a basis of the bipartite Hilbert space. One can check directly that the four states with $\tilde{j} = 0$ stated in Eq. (2.5) above are pairwise orthogonal. It is worth mentioning that we are exploiting the fact that $U_2^{(\tilde{j}=0,j)}$ are the elements of a state-independent construction for real superpositions, which we will introduce in Chapter 4. To see the connection, note that the calculation for the state-independent construction for an arbitrary real two-qubit state $|\psi_2\rangle = \lambda_{00}\,|00\rangle + \lambda_{01}\,|01\rangle + \lambda_{10}\,|10\rangle + \lambda_{11}\,|11\rangle$ reads as follows:

$$
\begin{aligned}
(\mathbb{1} \otimes \mathbb{1})\,|\psi_2\rangle &= \lambda_{00}\,|00\rangle + \lambda_{01}\,|01\rangle + \lambda_{10}\,|10\rangle + \lambda_{11}\,|11\rangle \ , \\
(\mathbb{1} \otimes XZ)\,|\psi_2\rangle &= \lambda_{00}\,|01\rangle - \lambda_{01}\,|00\rangle + \lambda_{10}\,|11\rangle - \lambda_{11}\,|10\rangle \ , \\
(XZ \otimes Z)\,|\psi_2\rangle &= \lambda_{00}\,|10\rangle - \lambda_{01}\,|11\rangle - \lambda_{10}\,|00\rangle + \lambda_{11}\,|01\rangle \ , \\
(XZ \otimes X)\,|\psi_2\rangle &= \lambda_{00}\,|11\rangle + \lambda_{01}\,|10\rangle - \lambda_{10}\,|01\rangle - \lambda_{11}\,|00\rangle \ .
\end{aligned}
\tag{2.6}
$$

Since these states are pairwise orthogonal for arbitrary real coefficients $\lambda_{l_1 l_2}$, the same holds true for the states in Eq. (2.5). In addition, all of the states where $\tilde{j} = 0$ are elements of the subspace spanned by $|00,00\rangle$, $|01,01\rangle$, $|10,10\rangle$ and $|11,11\rangle$. Hence, they form a basis of this four-dimensional subspace. By shifting now the first system we obtain a basis for the remaining orthogonal subspaces. More precisely, since we defined $U_1^{(\tilde{j},j)} = X_4^{\tilde{j}} \, U_1^{(\tilde{j}=0,j)}$ the states where $\tilde{j} = 1$ are esentially the same states as the ones in Eq. (2.5) but with the first system shifted by one $l \to l \oplus 1 \pmod 4$. For example, $\lambda_{00}\,|11,10\rangle - \lambda_{01}\,|00,11\rangle - \lambda_{10}\,|01,00\rangle + \lambda_{11}\,|10,01\rangle$ is the state that corresponds to $\tilde{j} = 1$ and $j = 3$. In this way, the four states where $\tilde{j} = 1$ form a basis of the subspace spanned by $|01,00\rangle$, $|10,01\rangle$, $|11,10\rangle$ and $|00,11\rangle$ (or all states where $|l+1,l\rangle$). Analogously, the four states where $\tilde{j} = 2$ ($\tilde{j} = 3$) form a basis of the subspaces spanned by the vectors with $|l+2,l\rangle$ ($|l+3,l\rangle$). Altogether, the sixteen states $\{U_1^{(\tilde{j},j)} \otimes U_2^{(\tilde{j},j)} \, |\psi_S\rangle\}_{\tilde{j},j}$ form a basis of the entire sixteen dimensional Hilbert space.

A similar construction can be found for $d = 8$ by using the state-independent con-



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struction of three qubits. The set for $\tilde{j} = 0$ reads as follows:

| $\tilde{j}$ | $j$ | $U_1^{(\tilde{j},j)}$ | $U_2^{(\tilde{j},j)}$ | $U_1^{(\tilde{j},j)} \otimes U_2^{(\tilde{j},j)} \lvert\psi_S\rangle$ |
|---|---|---|---|---|
| 0 | 1 | $\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ | $\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ | $+\lambda_{000} \lvert 000, 000\rangle + \lambda_{001} \lvert 001, 001\rangle$<br>$+\lambda_{010} \lvert 010, 010\rangle + \lambda_{011} \lvert 011, 011\rangle$<br>$+\lambda_{100} \lvert 100, 100\rangle + \lambda_{101} \lvert 101, 101\rangle$<br>$+\lambda_{110} \lvert 110, 110\rangle + \lambda_{111} \lvert 111, 111\rangle$ |
| 0 | 2 | $\mathbb{1} \otimes \mathbb{1} \otimes X$ | $Z \otimes Z \otimes XZ$ | $+\lambda_{000} \lvert 001, 001\rangle - \lambda_{001} \lvert 000, 000\rangle$<br>$-\lambda_{010} \lvert 011, 011\rangle + \lambda_{011} \lvert 010, 010\rangle$<br>$-\lambda_{100} \lvert 101, 101\rangle + \lambda_{101} \lvert 100, 100\rangle$<br>$+\lambda_{110} \lvert 111, 111\rangle - \lambda_{111} \lvert 110, 110\rangle$ |
| 0 | 3 | $\mathbb{1} \otimes X \otimes \mathbb{1}$ | $Z \otimes XZ \otimes \mathbb{1}$ | $+\lambda_{000} \lvert 010, 010\rangle + \lambda_{001} \lvert 011, 011\rangle$<br>$-\lambda_{010} \lvert 000, 000\rangle - \lambda_{011} \lvert 001, 001\rangle$<br>$-\lambda_{100} \lvert 110, 110\rangle - \lambda_{101} \lvert 111, 111\rangle$<br>$+\lambda_{110} \lvert 100, 100\rangle + \lambda_{111} \lvert 101, 101\rangle$ |
| 0 | 4 | $X \otimes \mathbb{1} \otimes \mathbb{1}$ | $XZ \otimes \mathbb{1} \otimes \mathbb{1}$ | $+\lambda_{000} \lvert 100, 100\rangle + \lambda_{001} \lvert 101, 101\rangle$<br>$+\lambda_{010} \lvert 110, 110\rangle + \lambda_{011} \lvert 111, 111\rangle$<br>$-\lambda_{100} \lvert 000, 000\rangle - \lambda_{101} \lvert 001, 001\rangle$<br>$-\lambda_{110} \lvert 010, 010\rangle - \lambda_{111} \lvert 011, 011\rangle$ |
| 0 | 5 | $\mathbb{1} \otimes X \otimes X$ | $Z \otimes X \otimes XZ$ | $+\lambda_{000} \lvert 011, 011\rangle - \lambda_{001} \lvert 010, 010\rangle$<br>$+\lambda_{010} \lvert 001, 001\rangle - \lambda_{011} \lvert 000, 000\rangle$<br>$-\lambda_{100} \lvert 111, 111\rangle + \lambda_{101} \lvert 110, 110\rangle$<br>$-\lambda_{110} \lvert 101, 101\rangle + \lambda_{111} \lvert 100, 100\rangle$ |
| 0 | 6 | $X \otimes \mathbb{1} \otimes X$ | $X \otimes \mathbb{1} \otimes XZ$ | $+\lambda_{000} \lvert 101, 101\rangle - \lambda_{001} \lvert 100, 100\rangle$<br>$+\lambda_{010} \lvert 111, 111\rangle - \lambda_{011} \lvert 110, 110\rangle$<br>$+\lambda_{100} \lvert 001, 001\rangle - \lambda_{101} \lvert 000, 000\rangle$<br>$+\lambda_{110} \lvert 011, 011\rangle - \lambda_{111} \lvert 010, 010\rangle$ |
| 0 | 7 | $X \otimes X \otimes \mathbb{1}$ | $X \otimes XZ \otimes Z$ | $+\lambda_{000} \lvert 110, 110\rangle - \lambda_{001} \lvert 111, 111\rangle$<br>$-\lambda_{010} \lvert 100, 100\rangle + \lambda_{011} \lvert 101, 101\rangle$<br>$+\lambda_{100} \lvert 010, 010\rangle - \lambda_{101} \lvert 011, 011\rangle$<br>$-\lambda_{110} \lvert 000, 000\rangle + \lambda_{111} \lvert 001, 001\rangle$ |
| 0 | 8 | $X \otimes X \otimes X$ | $X \otimes XZ \otimes X$ | $+\lambda_{000} \lvert 111, 111\rangle + \lambda_{001} \lvert 110, 110\rangle$<br>$-\lambda_{010} \lvert 101, 101\rangle - \lambda_{011} \lvert 100, 100\rangle$<br>$+\lambda_{100} \lvert 011, 011\rangle + \lambda_{101} \lvert 010, 010\rangle$<br>$-\lambda_{110} \lvert 001, 001\rangle - \lambda_{111} \lvert 000, 000\rangle$ |

(2.7)

Again, we define $U_1^{(\tilde{j},j)} = X_8^{\tilde{j}}\, U_1^{(\tilde{j}=0,j)}$ and $U_2^{(\tilde{j},j)} = U_2^{(\tilde{j}=0,j)}$. The proof that this forms a basis of the 64-dimension Hilbert space is completely analogous to the case of $d = 4$ before. The eight states for $\tilde{j} = 0$ form a basis of the eight-dimensional subspace spanned by $\lvert l_1 l_2 l_3, l_1 l_2 l_3\rangle$ (for $l_i = 0, 1$). Applying the shift operator $X_8$ to the first system, one obtains bases of the other eight-dimensional orthogonal subspaces spanned by the

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vectors with $|l + \tilde{j}, l\rangle$. This approach cannot (immediately) be generalized to higher dimensions $d = 2^n$, due to the lack of state-independent constructions for $n \geq 4$ qubits (see Chapter 4). However, there is in principle no reason to restrict the unitaries on the second system to tensor products of single qubit Pauli gates as we do here. In principle, we could also consider general permutations with suitably chosen signs such that all terms cancel in this pairwise sense as above. Even when considering this larger class of possibilities, an exhaustive search has been done without finding any additional construction. Due to this, it seems unlikely that a construction exists in which the unitaries do not depend on the Schmidt coefficients.

### 2.1.3 Two qutrits

The simplest bipartite system that has not been taken into account yet are two qutrits, meaning two three-dimensional subsystems $(n, d) = (2, 3)$. This appears to be considerably different because during the process of research for this thesis, there have not been found strings of local unitaries that bring the Schmidt decomposition $|\psi_S\rangle$ into a basis without explicit dependence on the Schmidt coefficients. Nevertheless, a basis might still be possible to construct by taking the Schmidt coefficients into account when choosing the local unitaries. Actually, this seems to always be possible. To arrive at this, a numerical method has been used. Let $\{|\phi_j\rangle\}_{j=1}^{m}$ be a set of states in a given Hilbert space. These states are pairwise orthogonal if and only if they realise the global minimum (zero) of the following objective function

$$f(\{\phi_j\}) \equiv \sum_{j \neq j'} |\langle \phi_j | \phi_{j'} \rangle|^2. \tag{2.8}$$

It is clear that pairwise orthogonal states realise the minimum of this function by definition. To prove the other direction we consider a set of states $\{|\phi_j\rangle\}_{j=1}^{m}$ that minimizes $f$. Every addend in the sum of $f$ is non-negative, i.e. $|\langle \phi_j | \phi_{j'} \rangle|^2 \geq 0$. Therefore they all have to vanish, which is equivalent to pairwise orthogonality.

For a given state $|\psi\rangle$, we can numerically minimise $f(\{\psi_j\})$ over all possible strings $\{V_j\}_{j=1}^{d^n}$ of local unitaries. To this end, the local unitaries $U_k^{(j)}$ have been parametrized by $d^2 - 1$ real variables using the scheme of Ref. [25]. For the two-qutrit case, 1000 pairs of Schmidt coefficients $(\lambda_1, \lambda_2)$, which (up to local unitaries) fully specify the state $|\psi\rangle$, have been chosen randomly. In each case we initialised the basis states using the parametrized strings $\{\psi_j\}_{j=1}^{d^n} = \{V_j \psi\}_{j=1}^{d^n}$. Setting the first string of unitaries to $V_1 = \mathbb{1}$ and therefore $|\psi_1\rangle = |\psi\rangle$, this results in $2(9-1)$ local unitaries that depend on $2(9-1)^2$ free parameters. Using the MATLAB- nonlinear programming solver “fminunc” with the method “sequential quadratic programming (sqp)”, we minimized $f(\{\psi_j\})$ with respect to the free parameters of the local unitaries. As an alternative, we also used Wolfram Mathematica’s “NMinimize” with the methods “DifferentialEvolution” and “SimulatedAnnealing” for the minimization. Without exception, strings of



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local unitaries that yield a result below a selected precision threshold of $f \leq 10^{-6}$ have been found.

## 2.2 Three qubits

Furthermore, we can also numerically investigate the case of three qubits, $(n, d) = (3, 2)$. This scenario requires a different approach than the previous cases since multipartite states have no Schmidt decomposition. As introduced in Section 1.4, for any given three-qubit state $|\psi\rangle$, there exist local unitary transformations that map it onto the canonical form of (1.38), hence $a|000\rangle + b|011\rangle + c|101\rangle + d|110\rangle + e|111\rangle$ where $(b, c, d, e)$ are real numbers and $a$ is a complex number [18, 19]. Hence, up to local unitaries, the state space (after normalisation) is characterised by five real numbers. From the result of Perdomo [20] in Section 1.4 we also derived that there are states which cannot be described without the complex factor. Later, we will provide an analytical construction of a $|\psi\rangle$-basis for the four-parameter family corresponding to restricting $a$ to be real. However, during the creation of this thesis, there has not been found an analytical basis construction for general three-qubit states. Nevertheless we will conjecture that it exists. To evidence this, the previously introduced numerical search method has been employed. Again, 1000 normalised sets of coefficients $(a, b, c, d, e)$ have been chosen randomly and the minimal value of $f$ has been searched over all the strings of local qubit unitaries with the same methods as for two qutrits in the previous section. In all cases, it has been found that $f$ vanishes up to a selected precision of $f \leq 10^{-6}$.

## 2.3 More than three qubits

Given the above case studies, one might suspect that every pure quantum state admits a basis. Interestingly, this seems not to be true. While some states of four qubits, $(n, d) = (4, 2)$, are found to admit a basis, for example a $W$ state and doubly-excited Dicke state [6], it appears that most four-qubit states do not admit a basis. Many different four-qubit states have been sampled and it has been repeatingly attempted to numerically find a basis via the minimisation of (2.8), also using several different search algorithms. It was regularly found that the estimated minimum is multiple orders of magnitude above our given precision threshold for a basis. For example, the minimum of $f$ has been searched for the state $\frac{2}{\sqrt{6}}|W\rangle + \frac{\sqrt{2}}{\sqrt{6}}|\text{GHZ}_{4,2}\rangle$, with 100 randomised initial points, and it has never reached a value below $f = 10^{-1}$, five orders of magnitude above the previously mentioned precision threshold. It has been attempted to prove that no basis exists by employing semidefinite outer relaxations of $f$ over the set of dimensionally-restricted quantum correlations [26] combined with a modified sampling of the state and measurement space [27] and symmetrisation techniques [28] to efficiently treat the large number of single-qubit unitaries featured in this problem. However, the conjecture



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has resisted these efforts. A guiding intuition for the impossibility of a basis is to note that the number of free parameters is $3n(2^n - 1)$ whereas the number of orthogonality constraints (counting both the real and imaginary part) is $2^{2n} - 2^n$, and the latter is larger than the former only when $n \geq 4$. Also, we numerically minimized $f$ for the same state, with a lesser amount of required orthonormal states than a full basis. The numerics suggest that $N = 12$ orthonormal states are possible to find, while for $N = 13$ we never reached below $f = 10^{-3}$. This is also in line with the above parameter-counting-argument. For $N$ orthonormal states and $n = 4$ qubits, we have $12(N - 1)$ free parameters and $N^2 - N$ orthogonality constraints. The latter is larger than the former for $N \geq 13$.

Furthermore, if an $n$-qubit state $|\psi\rangle$ does not admit a basis, then the $(n + 1)$-qubit state $|\psi'\rangle = |\psi\rangle \otimes |0\rangle$ also does not admit a basis. By contradiction, suppose there are $2^{n+1}$ unitaries $V_j' = V_j \otimes U_{n+1}^{(j)}$ such that $|\langle\psi'|(V_j')^\dagger V_k'|\psi'\rangle| = \delta_{jk}\ \forall j, k \in \{1, ..., 2^{n+1}\}$. Divide the $2^{n+1}$ states $U_{n+1}^{(j)}|0\rangle$ into two sets such that two orthogonal vectors are not in the same set (e.g. the northern and southern hemisphere of the Bloch ball). Consider the set that contains at least as many elements as the other one, hence, at least $2^n$ elements. By construction, these states cannot be distinguished on the last qubit, $|\langle 0|U_{n+1}^{(j)\dagger} U_{n+1}^{(k)}|0\rangle| \neq 0$. Since $|\langle\psi'|(V_j')^\dagger V_k'|\psi'\rangle| = |\langle\psi|V_j^\dagger V_k|\psi\rangle| \cdot |\langle 0|U_{n+1}^{(j)\dagger} U_{n+1}^{(k)}|0\rangle|$, we must have $|\langle\psi|V_j^\dagger V_k|\psi\rangle| = \delta_{jk}$ for all of those pairs, which contradicts that $|\psi\rangle$ does not admit a basis. By induction, this argument shows that if our above conjecture holds, namely that some four-qubit states do not admit a basis, then the same holds for any number of qubits larger than four.



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# 3 Special $n$-qubit states that correspond to a basis

Since not all pure quantum states admit a basis, and this seems to be typical rather than exceptional for four qubits, it is interesting to ask whether some distinguished families of $n$-qubit states can nevertheless admit a basis. This is well-known to be the case for $n$-qubit GHZ-states and graph-states since they are locally maximally entanglable (LME) [21]. As we considered in Section 1.5, every $n$-qubit LME-state can form an orthonormal basis $\{U_1^{i_1} \otimes U_2^{i_2} \otimes ... \otimes U_n^{i_n} |\psi\rangle \, | i_l = 0, 1\}$ by using only $n$ local unitary operators $U_l$. A $GHZ$-state of $n$ subsystems of dimension $d$ can be written as

$$|\mathrm{GHZ}_{n,d}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle^{\otimes n} . \tag{3.1}$$

The corresponding strings of local unitaries are

$$V_j = Z_d^{j_1} \otimes X_d^{j_2} \otimes \ldots \otimes X_d^{j_n} |\mathrm{GHZ}_{n,d}\rangle , \tag{3.2}$$

where $j = j_1 \ldots j_n \in \{0, \ldots, d-1\}^n$ and where $Z_d = \sum_{l=0}^{d-1} e^{\frac{2\pi i}{d} l} |l\rangle\langle l|$ and $X_d = \sum_{l=0}^{d-1} |l+1\rangle\langle l|$ are generalised Pauli operators.

More interestingly, a positive answer is also possible for the other three-qubit maximally entangled states and their $n$-qubit generalisations: the $n$-qubit $W$-state. We can define it similar to Dür et. al. [16] as

$$|W_n\rangle = \frac{1}{\sqrt{n}} \sum_{\sigma} \sigma(|0\rangle^{\otimes n-1} |1\rangle) , \tag{3.3}$$

where $\sigma$ runs over all permutations of the position of "1". A construction for the $n$-qubit $W$-state was given for the purpose of entanglement distillation in Ref. [5] and in Ref. [6] where they also proved the existence of a basis for some Dicke-states, which are defined by

$$|D_n^k\rangle = \binom{n}{k}^{-\frac{1}{2}} \sum_{\sigma} \sigma(|0\rangle^{\otimes n-k} |1\rangle^{\otimes k}) , \tag{3.4}$$

where $\sigma$ runs over all permutations of the positions of $k$ excitations. Besides the $n$-qubit $W$-state they prove the existence of Dicke-state-bases for all systems with $n \leq 5$ and for $(n, k) = (6, 3)$.



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We will now introduce a different, quite convenient, inductive construction scheme for the $n$-qubit $W$-state. Note that $|W_1\rangle = |1\rangle$ and that a $|W_1\rangle$-basis is obtained from the unitaries $\{\mathbb{1}, X\}$. Now we apply induction. Consider that the strings $\{V_j^{(n)}\}_{j=1}^{2^n}$ generate a $|W_n\rangle$-basis. One can then construct a basis for $n+1$ qubits as follows. For half of the basis elements, namely $j = 1, \ldots, 2^n$, define $V_j^{(n+1)} = V_j^{(n)} \otimes \mathbb{1}$ and for the other half, namely $j = 2^n + 1, \ldots, 2^{n+1}$, define $V_j^{(n+1)} = (V_j^{(n)} \otimes X) \bigotimes_{k=1}^{n} Z \otimes \mathbb{1}$. We can now prove that $\{V_j^{(n+1)} |W_{n+1}\rangle\}_j$ is a $W$-basis.

As defined in Eq. (3.3) the $W$-states beginning at the lowest number of subsystems are:

$$
\begin{aligned}
|W_1\rangle &\equiv |1\rangle \\
|W_2\rangle &\equiv \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \\
|W_3\rangle &\equiv \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle) \\
|W_4\rangle &\equiv \frac{1}{2} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \\
&\vdots
\end{aligned}
\tag{3.5}
$$

Note that for one and two qubits, the definition is only introduced for sake of convenience in the induction. The states $|W_1\rangle$ and $|W_2\rangle$ simply describe the *up*-state of a single qubit and the $|\Psi^+\rangle$-Bell-state of two qubits. In addition to the general form in Eq. (3.3), it is also useful for our proof to write the state recursively as

$$
|W_{n+1}\rangle = \sqrt{\frac{n}{n+1}} |W_n\rangle \otimes |0\rangle + \frac{1}{\sqrt{n+1}} |0\rangle^n \otimes |1\rangle \tag{3.6}
$$

We will write the local unitaries in all the strings $V_j^{(n)}$ for a particular number of particles $n$ as $V_j^{(n)} = U_1^{(j)} \otimes U_2^{(j)} \otimes \ldots \otimes U_n^{(j)}$. Clearly, if we apply the local unitaries $V_1^{(1)} = U_1^{(1)} = \mathbb{1}$ and $V_2^{(1)} = U_1^{(2)} = X$ to $|W_1\rangle$ we generate the trivial one-qubit $W$-basis $\{|0\rangle, |1\rangle\}$. Assume now that the local unitaries $\{U_k^{(j)}\}$ for $k = 1, \ldots n$ and $j = 1, \ldots, 2^n$ yield a $|W_n\rangle$-basis. We will now show that under this assumption we can construct a basis for $|W_{n+1}\rangle$ and hence it follows from induction that a $W$-basis exists for any number of qubits.



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We illustrate the induction step as follows,

$$
\begin{array}{ccccccc|cc}
U_1^{(1)} & \otimes & U_2^{(1)} & \otimes & \dots & \otimes & U_n^{(1)} & \otimes & \mathbb{1} \\
U_1^{(2)} & \otimes & U_2^{(2)} & \otimes & \dots & \otimes & U_n^{(2)} & \otimes & \mathbb{1} \\
 & \vdots & & & & \vdots & & \vdots & \\
U_1^{(2^n)} & \otimes & U_2^{(2^n)} & \otimes & \dots & \otimes & U_n^{(2^n)} & \otimes & \mathbb{1} \\
\hline
U_1^{(1)}Z & \otimes & U_2^{(1)}Z & \otimes & \dots & \otimes & U_n^{(1)}Z & \otimes & X \\
U_1^{(2)}Z & \otimes & U_2^{(2)}Z & \otimes & \dots & \otimes & U_n^{(2)}Z & \otimes & X \\
 & \vdots & & & & \vdots & & \vdots & \\
U_1^{(2^n)}Z & \otimes & U_2^{(2^n)}Z & \otimes & \dots & \otimes & U_n^{(2^n)}Z & \otimes & X
\end{array} . \tag{3.7}
$$

We see that for the first $2^n$ basis elements, we extend the unitaries for $n$ qubits by tensoring with $\mathbb{1}$ for qubit number $n+1$. For the latter $2^n$ basis elements, we extend the unitaries for $n$ qubits by multiplying all of them from the right by $Z$ and finally tensoring with $X$ for qubit number $n+1$. As usual, we now write the string of unitaries associated to each row as $V_j^{(n+1)}$ for $j = 1, \dots, 2^{n+1}$. We similarly use $V_j^{(n)}$ for the unitary strings for the case of $n$ qubits.

To see that this yields a basis, we first show that the first $2^n$ basis elements (upper block of table, $j = 1, \dots, 2^n$) are orthogonal. For this purpose, we use the recursion formula (3.6) to write for $j \neq j'$

$$
\begin{aligned}
\langle W_{n+1}|(V_{j'}^{(n+1)})^\dagger V_j^{(n+1)}|W_{n+1}\rangle = & \frac{n}{n+1}\langle W_n 0|(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes \mathbb{1}|W_n 0\rangle \\
& + \frac{1}{n+1}\langle 0\dots 01|(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes \mathbb{1}|0\dots 01\rangle \\
& + \frac{\sqrt{n}}{n+1}\langle W_n 0|(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes \mathbb{1}|0\dots 01\rangle \\
& + \frac{\sqrt{n}}{n+1}\langle 0\dots 01|(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes \mathbb{1}|W_n 0\rangle = 0
\end{aligned}
$$

The first term is zero for all $j' \neq j$ due to the induction hypothesis. The third and fourth terms are zero due to orthogonality in the last qubit register. The second term is zero because for every $j' \neq j$ there exists at least one qubit register $k$ for which $U_k^{(j')}$ and $U_k^{(j)}$ are composed of different numbers of bit-flips ($X$). The latter follows from the initial condition of using $\{\mathbb{1}, X\}$ to construct the $|W_1\rangle$-basis.

The same procedure will analogously show that the latter $2^n$ basis elements (lower block of the table, $j = 2^n + 1, \dots, 2^{n+1}$) are orthogonal. We are left with showing that every overlap between the upper and lower block, i.e. with any $j' = 1, \dots, 2^n$ and any



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$j = 2^n + 1, \dots, 2^{n+1}$, also vanishes. For this we have

$$\begin{aligned}\langle W_{n+1}|(V_{j'}^{(n+1)})^\dagger V_j^{(n+1)}|W_{n+1}\rangle = &\frac{n}{n+1}\langle W_n 0| \left[(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes X\right] \bigotimes_{k=1}^{n} Z \otimes \mathbb{1} |W_n 0\rangle \\ &+\frac{1}{n+1}\langle 0\dots 01| \left[(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes X\right] \bigotimes_{k=1}^{n} Z \otimes \mathbb{1} |0\dots 01\rangle \\ &+\frac{\sqrt{n}}{n+1}\langle W_n 0| \left[(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes X\right] \bigotimes_{k=1}^{n} Z \otimes \mathbb{1} |0\dots 01\rangle \\ &+\frac{\sqrt{n}}{n+1}\langle 0\dots 01| \left[(V_{j'}^{(n)})^\dagger V_j^{(n)} \otimes X\right] \bigotimes_{k=1}^{n} Z \otimes \mathbb{1} |W_n 0\rangle\end{aligned}$$

Note that $\bigotimes_{k=1}^{n} Z \otimes \mathbb{1}\, |W_n 0\rangle = -\, |W_n 0\rangle$ and $\bigotimes_{k=1}^{n} Z \otimes \mathbb{1}\, |0\dots 01\rangle = |0\dots 01\rangle$. The first and second terms are both zero due to orthogonality in the final qubit register. We thus have

$$\begin{aligned}\langle W_{n+1}|(V_{j'}^{(n+1)})^\dagger V_j^{(n+1)}|W_{n+1}\rangle = &\frac{\sqrt{n}}{n+1}\langle W_n|(V_{j'}^{(n)})^\dagger V_j^{(n)}|0\dots 0\rangle \\ &-\frac{\sqrt{n}}{n+1}\langle 0\dots 0|(V_{j'}^{(n)})^\dagger V_j^{(n)}|W_n\rangle \\ = &\frac{\sqrt{n}}{n+1}\, \langle W_n|\, (V_{j'}^{(n)})^\dagger V_j^{(n)} - (V_j^{(n)})^\dagger V_{j'}^{(n)}\, |0\dots 0\rangle = 0.\end{aligned} \tag{3.8}$$

The last equality follows from the fact that it is sufficient, for given $(j, j')$, that there exist some register index $k$ such that $(U^{(j')})_k^\dagger U_k^{(j)} - (U^{(j)})_k^\dagger U_k^{(j')} = 0$ in order for the overlap to vanish. This is always the case because due to our construction (see initial condition and the table), for every two unitaries there is at least one register $k$ where the single-qubit unitaries differ by $X$, meaning that either $(U_k^{(j)}, U_k^{(j')}) = (\mathbb{1}, X)/(Z, XZ)$, or the same with $j \leftrightarrow j'$ is true. The condition above is satisfied by all of these combinations. Hence we conclude that the proposed construction satisfies

$$\langle W_{n+1}|(V_j^{(n+1)})^\dagger V_{j'}^{(n+1)}|W_{n+1}\rangle = \delta_{jj'} \tag{3.9}$$

and therefore yields a $W$-state basis for any number of qubits.

Note that this construction is not limited to the standard $W_n$ states. It also creates an orthonormal basis from less entangled states that consists of the same product states as the $W_n$ state, not in a uniform superposition but with arbitrary real coefficients:

$$\left|\tilde{W}_n\right\rangle = \sum_\sigma \lambda_\sigma \sigma \left(|0\rangle^{\otimes n-1} |1\rangle\right), \quad \lambda_\sigma \in \mathbb{R}, \sum_\sigma \lambda_\sigma^2 = 1\,. \tag{3.10}$$

The proof for the construction works exactly the same for this state. We therefore found unitaries that construct a basis from each state in the family defined by Eq. 3.10, *independently* of the coefficients $\lambda_\sigma$. This realisation leads to the question, if there are



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strings of local unitary operators that, independently of the specific states, generate a basis from every state of the Hilbert space. This will be addressed in the following chapter.

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# 4 State independent constructions for restricted spaces

So far, we have considered whether a specific state can be associated to a specific measurement. In other words, the unitary constructions have been state-dependent. We now go further and introduce a complementary concept, namely whether there exist strings of local unitaries $\{V_j\}$ that can transform *any* state in a space of states $\mathcal{S}$ into a basis, i.e. strings of local unitaries that satisfy

$$\forall\psi \in \mathcal{S}, \qquad |\langle\psi|V_j^\dagger V_{j'}|\psi\rangle| = \delta_{jj'}. \tag{4.1}$$

Naturally, this state-independent notion of basis construction is much stronger than the previously considered state-dependent notion. In the most ambitious case, when we choose the space $\mathcal{S}$ to be the entire Hilbert space of $n$ subsystems of dimension $d$, i.e. $\mathcal{S} \simeq (\mathbb{C}^d)^{\otimes n}$, then a state-independent construction cannot exist. In fact, not even two orthogonal vectors can be state-independently constructed for the full quantum state space. To show this, we can w. l. g. set $V_1 = \mathbb{1}$ and assume that there exist local unitaries $\{U_k\}$ such that $|\psi_1\rangle = |\psi\rangle$ and $|\psi_2\rangle = \bigotimes_{k=1}^n U_k |\psi\rangle$ are orthogonal for all $|\psi\rangle$. Focus now on the particular state $|\psi\rangle = \bigotimes_{k=1}^n |\mu_k\rangle$ where $|\mu_k\rangle$ is some eigenvector of the unitary $U_k$. Since the eigenvalues of a unitary are complex phases, written $e^{i\varphi_k}$ for $U_k$ and $|\mu_k\rangle$, we obtain

$$\begin{aligned} |\psi_1\rangle &= \bigotimes_{k=1}^n |\mu_k\rangle \ , \\ |\psi_2\rangle &= e^{i\sum_{k=1}^n \varphi_k} \bigotimes_{k=1}^n |\mu_k\rangle \ . \end{aligned} \tag{4.2}$$

These two states are evidently not orthogonal and hence we have a contradiction.

Interestingly, the situation changes radically if we limit our state-independent investigation to all quantum states in a real-valued Hilbert space. That is, $\mathcal{S} \simeq (\mathbb{R}^d)^{\otimes n}$. Such real quantum systems have also been contrasted in the literature with their complex counterparts [29, 30, 31]. Let us momentarily ignore the $n$-partition structure of our Hilbert space and simply consider two real states $|\psi_1\rangle = |\psi\rangle$ and $|\psi_2\rangle = U|\psi\rangle$ obtained from a given real target state $|\psi\rangle$ and a fixed ($\psi$-independent) unitary $U$. It holds that $\psi_1$ and $\psi_2$ are orthogonal if and only if $U$ is skew-symmetric. To prove this, assume first the

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skew-symmetry property $U = -U^T$. Since for real states $\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^*$ is equivalent to $\langle\psi|U|\psi\rangle = \langle\psi|U^\dagger|\psi\rangle^* = \langle\psi|U^T|\psi\rangle$, skew-symmetry implies that $\langle\psi_1|\psi_2\rangle = 0$. Conversely, assume that $\langle\psi|U|\psi\rangle = 0$ for all real-valued $|\psi\rangle$. Choosing in particular $|\psi\rangle = |k\rangle$ where $k = 0, \ldots, d^n - 1$ denotes a product state, it follows that all diagonal elements of $U$ must vanish. Then, choose $|\psi\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |j\rangle)$ for any pair $i \neq j$. This yield $U_{ii} + U_{jj} + U_{ij} + U_{ji} = 0$, but since we know that the diagonals vanish we are left with just $U_{ij} = -U_{ji}$ which defines a skew-symmetric operator.

Returning to our $n$-partitioned real Hilbert space, and still w. l. g. taking $V_1 = \mathbb{1}$, the above result demands that we find local unitaries such that

$$U_1 \otimes \ldots \otimes U_n = -U_1^T \otimes \ldots \otimes U_n^T. \tag{4.3}$$

This is only possible if $U_k^T = \pm U_k$. Hence, all local unitaries must be either symmetric or skew-symmetric, and the number of the latter must be odd. When extended from two orthogonal states to a whole basis, we require that this property holds for every pair of distinct labels $(j, j')$ in the basis. In other words, we require that every string $(V_j)^\dagger V_{j'}$ with $j \neq j'$ is skew-symmetric.

The question becomes whether the above condition can be satisfied for a given scenario. Consider it first for qubit systems ($d = 2$). We can show that the set of complex qubit unitaries that are either symmetric or skew-symmetric and whose products are again either symmetric or skew-symmetric, must obey a simple structure; they are essentially equivalent to the four Pauli-type operators $\mathcal{P} \equiv \{\mathbb{1}, X, Z, XZ\}$. First, note that the set is finite since there are exactly $2^n$ basis states. Next, we observe that the identity $\mathbb{1}$ has to be within the set $\mathcal{P}$ since we demand that $V_1 = \mathbb{1}$. Furthermore, we can argue that the gate

$$XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

has to be within the set as well. If we neglect a global phase, it is the only skew symmetric gate, and hence the only one that maps every real qubit state to its orthogonal state. More precisely, if it is not used on the $i$-th qubit at least once, one can choose a real qubit state $|\phi_i\rangle$ such that none of the gates in $\mathcal{P}$ map $|\phi_i\rangle$ to its orthogonal vector. Hence if we apply the state-independent construction to the real-valued product state $|\phi\rangle = |0\rangle_1 \otimes \ldots |0\rangle_{i-1} \otimes |\phi_i\rangle \otimes |0\rangle_{i+1} \otimes \ldots \otimes |0\rangle_n$ none of the resulting $2^n$ states are distinguishable on the $i$-th qubit, which is impossible if these states should form a basis of product states. Therefore, the gate $XZ$ has to be within the set $\mathcal{P}$. Here we can also elaborate a bit on why the problem of state-independent transformations that we showed in (4.2) does not apply, if we restrict the states to be real. The eigenstates of $XZ$ are $|i\pm\rangle = (|0\rangle \pm i\,|1\rangle)/\sqrt{2}$. On of them is shown in in Fig. 4.1 together with the space of real superpositions $|\psi_r\rangle$ of the computational basis. The states $|i\pm\rangle$ are not real, which is why (4.2) is not applicable. All real superpositions on the other hand, are rotated by $XZ$ to their orthogonal state.



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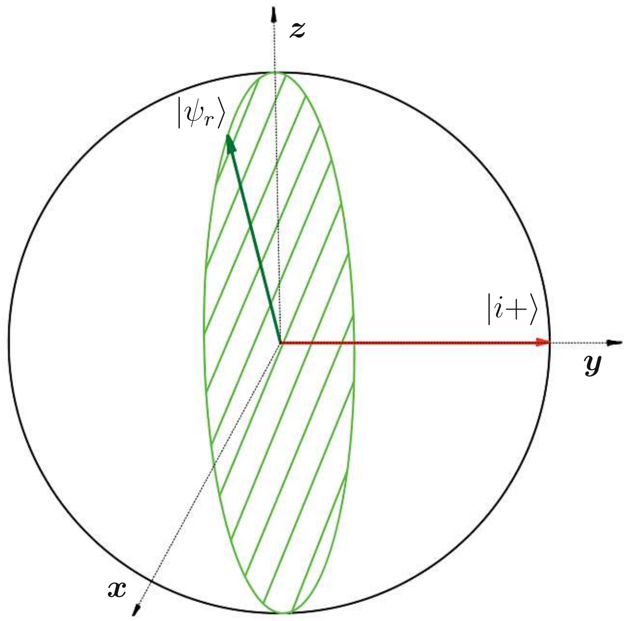


Figure 4.1: The skew-symmetric unitary gate $XZ$ has the eigenstates $|i\pm\rangle$, which represent the rotation axis of the gate. The application of $XZ$ therefore rotates every real superposition about the angle $\pi$ to its orthogonal state.

Apart from the gates $\mathbb{1}$ and $XZ$ we can constrain which other qubit unitaries can be in the set $\mathcal{P}$. We know that if we demand $V_1 = \mathbb{1}$, every string of local unitaries $(V_j)$ and their products $(V_j)^\dagger V_{j'}$ with $j \neq j'$ have to be skew-symmetric. As a result, the local unitaries on each subsystem (hence, the unitaries in the set $\mathcal{P}$) and also all their products have to be either symmetric or skew-symmetric. By neglecting a global phase, the general form of a unitary operator can be written as:

$$U = \begin{pmatrix} \cos(\theta)e^{i\alpha} & \sin(\theta)e^{i\beta} \\ -\sin(\theta)e^{-i\beta} & \cos(\theta)e^{-i\alpha} \end{pmatrix}. \tag{4.4}$$

The only skew-symmetric $2\times 2$ unitary is, up to an irrelevant global phase, the Pauli-type operator $XZ$, which we already found to be necessarily in the set $\mathcal{P}$. All the symmetric matrices of this form can be written as:

$$U = \begin{pmatrix} \cos(\theta)e^{i\alpha} & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta)e^{-i\alpha} \end{pmatrix}. \tag{4.5}$$

If the gate $U$ is in $\mathcal{P}$, it is at some point multiplied with the gate $XZ$ since the operator $XZ$ is used at least once on the $i$-th qubit. Since we know that the result of this product has to be again either symmetric or skew-symmetric, we obtain that $\alpha = \pi/2, 3\pi/2$ due



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to:

$$
\begin{aligned}
(XZ)^{\dagger}U &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta)e^{i\alpha} & i\sin(\theta) \\ i\sin(\theta) & \cos(\theta)e^{-i\alpha} \end{pmatrix} \\
&= \begin{pmatrix} i\sin(\theta) & \cos(\theta)e^{-i\alpha} \\ -\cos(\theta)e^{i\alpha} & -i\sin(\theta) \end{pmatrix}.
\end{aligned} \tag{4.6}
$$

The two possibilities for $\alpha = \pi/2, 3\pi/2$ correspond to the two solutions

$$
U_1 = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}, \quad U_2 = \begin{pmatrix} \sin(\theta) & -\cos(\theta) \\ -\cos(\theta) & -\sin(\theta) \end{pmatrix}. \tag{4.7}
$$

We left the irrelevant global factor $i$ for simplicity. Considering the additional degree of freedom of $\theta$, we can restrict to the first class of solutions $U_1$ since the second class $U_2$ can be obtained by shifting $\theta$ by $\pi/2$. Hence, if we add a gate $U$ to the set $\mathcal{P}$, it has to be of the form given by $U_1$ above. Now if we add two such gates to the set $\mathcal{P}$, the product of $U_1$ with another valid matrix $U_1'$ is

$$
\begin{aligned}
U_1^{\dagger}U_1' &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta') & \sin(\theta') \\ \sin(\theta') & -\cos(\theta') \end{pmatrix} = \\
&= \begin{pmatrix} \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta') & \cos(\theta)\sin(\theta') - \sin(\theta)\cos(\theta') \\ \sin(\theta)\cos(\theta') - \cos(\theta)\sin(\theta') & \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta') \end{pmatrix} = \\
&= \begin{pmatrix} \cos(\theta-\theta') & -\sin(\theta-\theta') \\ \sin(\theta-\theta') & \cos(\theta-\theta') \end{pmatrix}
\end{aligned}
$$

If both, $U_1$ and $U_1'$, are in $\mathcal{P}$, this product has to be again either symmetric, which is true if $\theta = \theta'$ or skew-symmetric, which is true if $\theta = \theta' + \pi/2$. (Note that, also $\theta = \theta' + \pi$ and $\theta = \theta' + 3\pi/2$ are possible solutions but we do not have to consider them since they just differ by an irrelevant global factor of $(-1)$ in one of the two unitaries.) Hence, $U_1'$ is either $U_1$ or the unitary $U_2$ stated above. Hence, for each single-qubit subsystem, we can only use a set of operators $\mathcal{P} \equiv \{\mathbb{1}, U_1, U_2, XZ\}$ for our basis construction.

In a final step, we can show that we can restrict also $\theta$. To see this, suppose a state-independent construction exists where we use the gates from the set $\mathcal{P} \equiv \{\mathbb{1}, U_1, U_2, XZ\}$. Now consider the construction where each gate $U_1$ is replaced with $W^{\dagger}U_1W$, each gate $U_2$ with $W^{\dagger}U_2W$, each gate $XZ$ with $W^{\dagger}XZW$ and each gate $\mathbb{1}$ with $W^{\dagger}\mathbb{1}W$, where:

$$
W = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{4.8}
$$

for some freely chosen parameter $\alpha$. This also has to be a state-independent construction for any state with real coefficients, since $W$ is a map from real states to real states, and all inner products between the basis states remain the same under this local transformation. Hence, if a state-independent construction exists with the gate set



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$\mathcal{P} \equiv \{\mathbb{1}, U_1, U_2, XZ\}$, another state-independent construction with the gate set $\mathcal{P}' \equiv \{W^\dagger \mathbb{1} W, W^\dagger U_1 W, W^\dagger U_2 W, W^\dagger XZW\}$ has to exist as well. Choosing $\alpha = \theta/2$, the set $\mathcal{P}' \equiv \{W^\dagger \mathbb{1} W, W^\dagger U_1 W, W^\dagger U_2 W, W^\dagger XZW\}$ becomes exactly $\mathcal{P}' \equiv \{\mathbb{1}, X, Z, XZ\}$, which concludes the proof.

Thus, if a state-independent construction exists, we can restrict to selecting one of these four operators for each of our local unitaries $U_k^{(j)}$.

## 4.1 Two and three qubits

Interestingly, for the case of two qubits, $(n, d) = (2, 2)$, a state-independent construction is possible. It is in fact given by the unitary transformations Eq. (2.3). One can straightforwardly verify that the above criterion is satisfied, i.e. all local unitaries are selected from $\mathcal{P}$ and all pairs of products of unitary strings in (2.3) are skew-symmetric. Alternatively, one can easily verify that (2.3) maps every state $\sum_{i,j=0,1} \alpha_{ij} |ij\rangle$ into a basis, for any real coefficients $\alpha_{ij}$.

| $V_j$ | $V_j \|\psi_S\rangle$ |
|---|---|
| $\mathbb{1} \otimes \mathbb{1}$ | $\alpha_{00} \|00\rangle + \alpha_{01} \|01\rangle + \alpha_{10} \|10\rangle + \alpha_{11} \|11\rangle$ |
| $\mathbb{1} \otimes XZ$ | $\alpha_{00} \|01\rangle - \alpha_{01} \|00\rangle + \alpha_{10} \|11\rangle - \alpha_{11} \|10\rangle$ |
| $XZ \otimes Z$ | $\alpha_{00} \|10\rangle - \alpha_{01} \|11\rangle - \alpha_{10} \|00\rangle + \alpha_{11} \|01\rangle$ |
| $XZ \otimes X$ | $\alpha_{00} \|11\rangle + \alpha_{01} \|10\rangle - \alpha_{10} \|01\rangle - \alpha_{11} \|00\rangle$ |

(4.9)

Furthermore, by the same token, a state-independent basis is also possible for every real state of three qubits, $(n, d) = (3, 2)$. One explicit construction that satisfies our necessary and sufficient criterion is the following set of eight strings of local unitaries that maps every real state $\sum_{i,j,k=0,1} \alpha_{ijk} |ijk\rangle$ into a basis



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| $V_j$ | $V_j \vert\psi_S\rangle$ |
|---|---|
| $\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ | $+\alpha_{000}\|000\rangle + \alpha_{001}\|001\rangle + \alpha_{010}\|010\rangle + \alpha_{011}\|011\rangle$ $+\alpha_{100}\|100\rangle + \alpha_{101}\|101\rangle + \alpha_{110}\|110\rangle + \alpha_{111}\|111\rangle$ |
| $Z \otimes Z \otimes XZ$ | $+\alpha_{000}\|001\rangle - \alpha_{001}\|000\rangle - \alpha_{010}\|011\rangle + \alpha_{011}\|010\rangle$ $-\alpha_{100}\|101\rangle + \alpha_{101}\|100\rangle + \alpha_{110}\|111\rangle - \alpha_{111}\|110\rangle$ |
| $Z \otimes XZ \otimes \mathbb{1}$ | $+\alpha_{000}\|010\rangle + \alpha_{001}\|011\rangle - \alpha_{010}\|000\rangle - \alpha_{011}\|001\rangle$ $-\alpha_{100}\|110\rangle - \alpha_{101}\|111\rangle + \alpha_{110}\|100\rangle + \alpha_{111}\|101\rangle$ |
| $XZ \otimes \mathbb{1} \otimes \mathbb{1}$ | $+\alpha_{000}\|100\rangle + \alpha_{001}\|101\rangle + \alpha_{010}\|110\rangle + \alpha_{011}\|111\rangle$ $-\alpha_{100}\|000\rangle - \alpha_{101}\|001\rangle - \alpha_{110}\|010\rangle - \alpha_{111}\|011\rangle$ |
| $Z \otimes X \otimes XZ$ | $+\alpha_{000}\|011\rangle - \alpha_{001}\|010\rangle + \alpha_{010}\|001\rangle - \alpha_{011}\|000\rangle$ $-\alpha_{100}\|111\rangle + \alpha_{101}\|110\rangle - \alpha_{110}\|101\rangle + \alpha_{111}\|100\rangle$ |
| $X \otimes \mathbb{1} \otimes XZ$ | $+\alpha_{000}\|101\rangle - \alpha_{001}\|100\rangle + \alpha_{010}\|111\rangle - \alpha_{011}\|110\rangle$ $+\alpha_{100}\|001\rangle - \alpha_{101}\|000\rangle + \alpha_{110}\|011\rangle - \alpha_{111}\|010\rangle$ |
| $X \otimes XZ \otimes Z$ | $+\alpha_{000}\|110\rangle - \alpha_{001}\|111\rangle - \alpha_{010}\|100\rangle + \alpha_{011}\|101\rangle$ $+\alpha_{100}\|010\rangle - \alpha_{101}\|011\rangle - \alpha_{110}\|000\rangle + \alpha_{111}\|001\rangle$ |
| $X \otimes XZ \otimes X$ | $+\alpha_{000}\|111\rangle + \alpha_{001}\|110\rangle - \alpha_{010}\|101\rangle - \alpha_{011}\|100\rangle$ $+\alpha_{100}\|011\rangle + \alpha_{101}\|010\rangle - \alpha_{110}\|001\rangle - \alpha_{111}\|000\rangle$ |

(4.10)

## 4.2 More than three qubits

Two- and three-qubits are interesting cases because they are exceptional. As we now show, there exists no state-independent construction for real states of four or more qubits. We first prove this for $n = 4$ and then show that this implies impossibility also for $n > 4$. The four-qubit case contains 16 strings of unitaries and we know that each local unitary can w. l. g. be selected from $\mathcal{P}$. Since we seek a state-independent construction, we can momentarily consider only the state $|0000\rangle$. In order for it to be mapped into a basis, we see that $Z$ acts trivially on every register and therefore each one of the 16 combinations of bit-flip or identity operators, $\{X^{c_1} \otimes X^{c_2} \otimes X^{c_3} \otimes X^{c_4}\}$ for $c_1, c_2, c_3, c_4 \in \{0, 1\}$, must be featured in exactly one of the 16 unitary strings $\{V_j\}_{j=1}^{16}$. Let us now look only at six of these strings, namely those corresponding to having zero bit-flips (1 case), one bit-flip (4 cases) and four bit-flips (1 case). W. l. g. fixing $V_1 = \mathbb{1}$ (zero bit-flips), the strings take the form

$$\begin{array}{c||ccccccc} V_1 & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \mathbb{1} & \otimes & \mathbb{1} \\ V_2 & XZ^{r_{11}} & \otimes & Z^{r_{12}} & \otimes & Z^{r_{13}} & \otimes & Z^{r_{14}} \\ V_3 & Z^{r_{21}} & \otimes & XZ^{r_{22}} & \otimes & Z^{r_{23}} & \otimes & Z^{r_{24}} \\ V_4 & Z^{r_{31}} & \otimes & Z^{r_{32}} & \otimes & XZ^{r_{33}} & \otimes & Z^{r_{34}} \\ V_5 & Z^{r_{41}} & \otimes & Z^{r_{42}} & \otimes & Z^{r_{43}} & \otimes & XZ^{r_{44}} \\ V_6 & XZ^{r_{51}} & \otimes & XZ^{r_{52}} & \otimes & XZ^{r_{53}} & \otimes & XZ^{r_{54}} \end{array}, \quad (4.11)$$



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where $r_{ij} \in \{0,1\}$ represent our freedom to insert a $Z$ operator and thus realise the two relevant elements of $\mathcal{P}$. Since every row must be skew-symmetric and the only skew-symmetric element in $\mathcal{P}$ is $XZ$, we must have $r_{11} = r_{22} = r_{33} = r_{44} = 1$ and $r_{51} + r_{52} + r_{53} + r_{54} = 1$ where addition is modulo two. Moreover, every product of two rows must be skew-symmetric, i.e. the product must have an odd number of $XZ$ operations. For the four middle rows, this implies $r_{ij} + r_{ji} = 1$ for distinct indices $i, j \in \{1,2,3,4\}$. For the products $V_6^\dagger V_j$ for $j = 2,3,4,5$, the conditions for skew-symmetry respectively become

$$
\begin{aligned}
r_{12} + r_{13} + r_{14} + r_{52} + r_{53} + r_{54} &= 1 \\
r_{21} + r_{23} + r_{24} + r_{51} + r_{53} + r_{54} &= 1 \\
r_{31} + r_{32} + r_{34} + r_{51} + r_{52} + r_{54} &= 1 \\
r_{41} + r_{42} + r_{43} + r_{51} + r_{52} + r_{53} &= 1. \qquad (4.12)
\end{aligned}
$$

Summing these four equations and using the previously established skew-symmetry conditions, one can cancel out all degrees of freedom $r_{ij}$ and arrive at the contradiction $1 = 0$. Hence, we conclude that the state-independent basis construction for four qubits is impossible.

For the case of five qubits, we can again assume w. l. g. that the 32 combinations of bit-flip or identity operators, $\{X^{c_1} \otimes X^{c_2} \otimes X^{c_3} \otimes X^{c_4} \otimes X^{c_5}\}$ for $c_1, c_2, c_3, c_4, c_5 \in \{0,1\}$ must be featured in exactly one of the 32 unitary strings since the state $|00000\rangle$ has to be mapped into an orthonormal basis. Suppose there is a state-independent construction that maps every real-valued five-qubit state into a basis, in especially any state of the form $|\psi\rangle \otimes |0\rangle$, where $|\psi\rangle$ is an arbitrary real-valued four qubit state. Now consider the 16 strings where $c_5 = 0$. Since the fifth qubit is always mapped to itself, it has to hold that the first four qubits are pairwise distinguishable. However, this implies a state-independent construction for four qubits which is in contradiction to the above. By induction, this implies that no state-independent construction can exist whenever $n \geq 4$.

## 4.3 Odd-dimensional systems

The possibility of state-independent constructions for real-valued bi- and tri-partite systems draws heavily on the simple structure of skew-symmetric qubit unitaries. If we consider real-valued systems of dimension $d > 2$, the situation changes considerably. Using our necessary and sufficient condition, it follows immediately that state-independent constructions are impossible in all odd dimensions, i.e. when $(n, d) = (n, 2m+1)$. This stems from the fact that there exists no skew-symmetric unitary matrix in odd dimensions. To see that, simply note that if $A$ is skew-symmetric then

$$\det(A) = \det\left(A^T\right) = \det(-A) = (-1)^{2m+1} \det(A) = -\det(A)\,, \qquad (4.13)$$



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and hence

$$\det(A) = 0\,, \tag{4.14}$$

but that contradicts unitarity because the determinant of a unitary has unit modulus.

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# 5 Conclusion and Outlook

| | (2,2,$\mathbb{R}$) | (2,2,$\mathbb{C}$) | (3,2,$\mathbb{R}$) | (3,2,$\mathbb{C}$) |
|---|---|---|---|---|
| State-dependent construction | ✓ | ✓ | ✓ | (✓) |
| State-independent construction | ✓ | ✗ | ✓ | ✗ |
| | (4,2,$\mathbb{R}$) | (2,3,$\mathbb{C}$) | (2,4 or 8,$\mathbb{C}$) | ($n$, $2m+1$,$\mathbb{R}$) |
| State-dependent construction | (✗) | (✓) | ✓ | $---$ |
| State-independent construction | ✗ | ✗ | ✗ | ✗ |

Table 5.1: Overview of results. The first row indicates the scenario: $(n, d, \mathcal{S})$ gives particle number, dimension and the type of state space respectively. The symbol ✓indicates the existence of a basis under local unitaries. The symbol ✗indicates that there in general can be no basis under local unitaries, i.e. at least one state admits no basis. Paranthesis indicates that the result is obtained from numerical search. The symbol $---$ indicates that no investigation was made.

In this thesis we have investigated the correspondence between entangled states and entangled joint measurements. We examined, whether there exist sets of local unitary-transformations that map a state to an iso-entangled basis. Furthermore, we explored the possibility of generating an entangled measurement basis without prior knowledge of the specific state. Similar to the entanglement of states, we found that the entanglement of joint LU-equivalent measurements strongly depends on both particle number and local dimension. The analytical and numerical results of our research are summarised in Table 5.1 [1].

The most obvious open problem left by this work is to prove that there exist four-qubit states that do not admit a local unitary-equivalent basis. Since our numerics suggest that nearly all four qubit states fall in that category, it would also be interesting to examine, whether it is possible to bound the relative Hilbert space volume of such states without a basis. Since we proved that it is not possible for general states, we had to restrict the Hilbert space to real state vectors for the state-independent basis constructions. However, maybe there exist state-independent unitaries also for other special spaces. One



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example that could be studied is the set of states with a known entanglement entropy. State-independent constructions evidently exist for product states and maximally entangled states. Another open problem of experimental relevance is to determine which of of the theoretical measurements presented in this work are possible to realise in a lab.

The entanglement of joint quantum measurements is of structural and foundational interest. Exploring them has the potential to broaden the understanding of quantum effects since it may lead to new approaches for observing quantum systems. One example are quantum networks, where entangled joint measurements have played a fundamental role in characterizing new forms of nonlocality that were introduced recently [32, 33, 34]. Measurements that have been used in this context are the so called Elegant Joint Measurements [35, 36]. They also have been realised in various experiments [37, 38, 39] in terms of quantum correlations and communication, which also showcase the relevance of more general entangled measurements beyond the Bell-State measurement for usage in quantum applications. Elegant Joint Measurements can be viewed as a measurement that corresponds to a partially entangled state, but it also has the property that the single qubit reduced states of the four basis states form a tetrahedron in the Bloch-sphere. This goes beyond the requirements in this thesis, but our results may be useful to find measurements with similar symmetric properties.

A known disadvantage of GHZ-based entanglement swapping protocols is that they are very sensible to particle loss, since this causes the reduced state to be separable. Other entangled states which are locally inequivalent to the GHZ-state, like e.g. the W-state, maintain their entanglement under the loss of a particle. Entangled joint measurements, constructed from states with specific properties, may therefore be able to open up new approaches towards more noise-resilient entanglement swapping protocols.



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