DISSERTATION

Hierarchical Techniques in the Discretization of Elliptic Boundary Value Problems

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Kurzfassung

In dieser Dissertation betrachten wir die folgenden Multilevel-Aspekte bei elliptische Randwertproblemen:

- Multilevel Darstellung von Besov-Normen und ihre Anwendung auf die Vorkonditionierung des fractionalen Laplace.
- Verwendung hierarchischer Matrizen (d. h. $\mathcal{H}$-Matrizen) für die Kopplung von Finite-Element- und Randelementmethoden.
- $\mathcal{H}-$Matrix-Approximation der Inversen der Steifigkeitsmatrizen, die bei FEM-Diskretisierung der zeitharmonischen Maxwell-Gleichungen entstehen.

Wir zeigen, dass lokal $L^2(\Omega)$-stabile Operatoren, welche in Räume von stetigen, stückweise polynomiellen Funktionen auf formregulären Gittern abbilden und gewisse $L^2(\Omega)$ Approximationseigenschaften haben, stabile Abbildungen von $H^{3/2}(\Omega) \to B^{3/2}_{2,\infty}(\Omega)$ sind ($H^s(\Omega)$ und $B^s_{2,q}(\Omega)$ sind Sobolev- und Besov Räume). Die klassischen Operatoren vom Typ Scott-Zhang sind in dieser Klasse enthalten. Interpolation liefert Stabilität $B^{3\theta/2}_{2,q}(\Omega) \to B^{3\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1), q \in [1,\infty]$. Ein analoges Ergebnis gilt für stückweise Polynome: lokal L^2-stabile Operatoren wie die elementweise L^2-Projektion sind stabil $B^{\theta/2}_{2,q}(\Omega) \to B^{\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1), q \in [1,\infty]$.

Für Räume stückweiser Polynome auf adaptiv verfeinerten Netzen, die durch Newest Vertex Bisection (NVB) erzeugt wurden, konstruieren wir eine Multilevel-Zerlegung mit Normäquivalenz im Besov-Raum $B^{3\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1)$, $q \in [1,\infty]$.

Als Anwendung präsentieren wir einen multilevel diagonalen Vorkonditionierer für den integralen fractionalen Laplace $(-\Delta)^s$ für $s \in (0,1)$ auf lokal verfeinerten Gittern. Es wird gezeigt, dass dieser Vorkonditionierer zu einer gleichmäßig beschränkten Konditionszahl führt.

Darüber hinaus erzielt diese Arbeit Approximationsergebnisse für die Inverse Matrix der Steifigkeitsmatrizen bei zwei Problem erlassen: FEM-BEM-Kopplungsprobleme und die zeitharmonische Maxwell-Gleichung.

$\mathcal{H}$-Matrizen sind eine Klasse von Matrizen, die aus blockweise Niedrigrang-Matrizen vom Rang r bestehen. Hier sind die Blöcke in einem Baum $\mathbb{T}_{\mathcal{I}}$ so organisiert, dass der Speicherbedarf normalerweise $O(Nr\,\mathrm{depth}(\mathbb{T}_{\mathcal{I}}))$ ist (N ist die Problemgröße). Eine wesentliche Frage im Zusammenhang mit $\mathcal{H}$-Matrizen ist, ob Matrizen und ihre Inversen im gewählten Format gut dargestellt werden können.

Wir betrachten drei verschiedene Methoden zur Kopplung von FEM und BEM, nämlich die Bielak-MacCamy-Kopplung, die symmetrische Kopplung und die Johnson-Nédélec-Kopplung jeweils für Galerkindiskretisierung niedrigster Ordnung. Wir beweisen die Exis-

tenz von exponentiell im Blockrang konvergenten $\mathcal{H}$-Matrix-Approximationen an die inversen Matrizen.

Wir zeigen auch, dass die Inverse der Steifigkeitsmatrizen, die zu den zeitharmonischen Maxwell-Gleichungen mit perfekt leitenden Randbedingungen gehören, im $\mathcal{H}$-Matrix-Format mit exponentieller Genauigkeit im Blockrang approximiert werden kann.

Um diese $\mathcal{H}$-Matrix-Approximationsresultate zu beweisen, nutzen wir Caccioppoli-Ungleichungen für die diskreten Probleme. Für die FEM-BEM-Kopplung ermöglicht die Caccioppoli-Ungleichung die Kontrolle von Funktionen und induzierten Potentialen in stärkeren Normen durch schwächere Normen, wenn bestimmte Orthogonalitätsbedingungen erfüllt sind. Für die Maxwell-Gleichungen hat die Caccioppoli-Schätzung die Form einer Kontrolle der $\mathbf{H}(\text{curl})$-Norm durch die $\mathbf{L^2}$-norm.

Abstract

In this thesis, we analyze the following multilevel aspects in elliptic boundary value problems:

- Multilevel representation of Besov norms and application to preconditioning of the fractional Laplacian.
- Use of hierarchical matrices ($\mathcal{H}$-matrices) for the coupling of Finite- and Boundary Element Methods (FEM-BEM couplings).
- $\mathcal{H}$-matrix approximability of inverses of matrices corresponding to the discretization of the time-harmonic Maxwell equations using Finite Element Method (FEM).

We show that locally $L^2(\Omega)$-stable operators mapping into spaces of continuous piecewise polynomial set on shape regular meshes with certain approximation properties in $L^2(\Omega)$ are stable mappings $H^{3/2}(\Omega) \to B^{3/2}_{2,\infty}(\Omega)$, where $H^s(\Omega)$ and $B^s_{2,q}(\Omega)$ are Sobolev and Besov spaces. The classical Scott-Zhang type operators are included in the setting. Interpolation gives stability $B^{3\theta/2}_{2,q}(\Omega) \to B^{3\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1), q \in [1,\infty]$. An analogous result allows for spaces of discontinuous piecewise polynomials: locally L^2-stable operators such as the elementwise L^2-projection are stable $B^{\theta/2}_{2,q}(\Omega) \to B^{\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1), q \in [1,\infty]$.

For spaces of piecewise polynomials on adaptively refined meshes generated by *Newest Vertex Bisection* (NVB), we construct a multilevel decomposition with norm equivalence in the Besov space $B^{3\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1)$, $q \in [1,\infty]$.

As an application, we present a multilevel diagonal preconditioner for the integral fractional Laplacian $(-\Delta)^s$ for $s \in (0,1)$ on locally refined meshes. This preconditioner is shown to lead to uniformly bounded condition numbers.

This work is also concerned with approximation results for the inverses of stiffness matrices corresponding to the FEM and FEM-BEM discretizations in the $\mathcal{H}$-matrix format for the time-harmonic Maxwell equation and a scaler transmission problem.

$\mathcal{H}$-matrices are a class of matrices that consists of blockwise low-rank matrices of rank r where the blocks are organized in a tree $\mathbb{T}_{\mathcal{I}}$ so that the memory requirement is typically $O(Nr\,\mathrm{depth}(\mathbb{T}_{\mathcal{I}}))$, where N is the problem size. A basic question in connection with the $\mathcal{H}$-matrix arithmetic is whether matrices, and their inverses can be represented well in the chosen format.

We consider three different methods for the coupling of the FEM and the BEM, namely, the Bielak-MacCamy coupling, the symmetric coupling, and the Johnson-Nédélec coupling for the lowest order Galerkin discretization of each of these coupling techniques, and we prove the existence of root exponentially convergent $\mathcal{H}$-matrix approximants to the inverse matrices.

We also show that the inverse of the stiffness matrices corresponding to the time-harmonic Maxwell equations with perfectly conducting boundary conditions can be approximated in the format of $\mathcal{H}$-matrices, at a root exponential rate in the block rank.

In order to prove these $\mathcal{H}$-matrix approximability results, we provide interior regularity results known as Caccioppoli estimates for the discrete problems. For the FEM-BEM coupling, the Caccioppoli inequality allows for control of functions and induced potentials in stronger norms by weaker norms, if certain orthogonality conditions are satisfied. For Maxwell equations, the Caccioppoli estimate takes the form of control of the $\mathbf{H}(\mathrm{curl})$-norm by the $\mathbf{L}^2$-norm.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Dissertation selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 15. Juni 2021

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Contents

1 Introduction

Multilevel methods have proved to be among the most efficient techniques for numerically solving partial differential equations (PDEs). The judicious use of multilevel structures allows for complexity reduction from algebraic to log-linear. As well-known examples of such techniques, we recall multilevel preconditioning (BPX [BPX91], MDS [Zha92]), $\mathcal{H}$-matrices, fast multipole method [GR97], matrix compression by wavelets [DPS93], panel-clustering [HN89].

In this thesis, we study the following aspects

- multilevel decompositions and their stability,
- multilevel preconditioning of the fractional Laplacian,
- $\mathcal{H}$-matrix approximation of the inverse of the stiffness matrix corresponding to the coupling of Finite- and Boundary Element Methods (FEM-BEM couplings).
- $\mathcal{H}$-matrix approximation of the inverse of the stiffness matrix corresponding to the time-harmonic Maxwell equations.

Multilevel decompositions and their stability

For a given Hilbert space $(V, \|\cdot\|_V)$, the standard multilevel algorithms for finite element discretizations are based on a hierarchy of finite element spaces $V_0 \subset \cdots \subset V_L \subset \cdots \subset V$ associated with a sequence of nested meshes $(\mathcal{T}_\ell)_{\ell \geq 0}$ such that for the functions $u_L \in V_L$, we have the following decomposition

$$u_L = \mathcal{I}_0 u_L + \sum_{\ell=1}^{L} (\mathcal{I}_\ell - \mathcal{I}_{\ell-1}) u_L, \tag{1.0.1}$$

where $\mathcal{I}_\ell : V \to V_\ell$ denotes a linear projector. Let $(\beta_\ell)_{\ell \geq 1}$ be a non-decreasing sequence of positive real numbers. One of the crucial aspects of the multilevel algorithms is to find equivalent norms for the following discrete norm

$$|||u|||^2 := \|\mathcal{I}_0 u\|_V^2 + \sum_{\ell \geq 1} \beta_\ell \left\| (\mathcal{I}_\ell - \mathcal{I}_{\ell-1})\, u \right\|_V^2 \qquad \forall u \in V. \tag{1.0.2}$$

A characterization of Sobolev and Besov spaces in terms of such multilevel representations allows us to design preconditioners and estimate condition numbers of preconditioned systems. For $V := L^2(\Omega)$, $\beta_\ell := h_\ell^{-2}$ and the spaces of continuous piecewise linear polynomials $V_1 \subset \cdots \subset V_L$, the idea of replacing $\mathcal{I}_\ell$ by the L^2-orthogonal projection onto V_ℓ,

$\ell \in \{0, \cdots, L\}$ proposed in [BPX91, Xu89], was a break-through in the construction of additive methods. The norm equivalence

$$\|u_L\|^2_{H^1(\Omega)} \simeq \|\mathcal{I}_0 u_L\|^2_{L^2(\Omega)} + \sum_{\ell=1}^{L} h_\ell^{-2} \|(\mathcal{I}_\ell - \mathcal{I}_{\ell-1}) u_L\|^2_{L^2(\Omega)} \qquad \forall u_L \in V_L, \tag{1.0.3}$$

for uniformly refined meshes and a special class of non-uniform triangulations was proven in [BPX91] with the constants of equivalence depending linearly on L^{-1} (for the lower bound) and L (for the upper bound). Zhang in [Zha92] improved their result with constants of equivalence independence of the mesh size and the number of levels. Oswald in [Osw91] proved a similar result using the fact that Sobolev space $H^1(\Omega)$ coincides with a certain Besov space. Multilevel representations of Sobolev spaces based on sequences of uniformly refined meshes are available in the literature; see, e.g., [Osw94, Sch98, BPV00], and the references therein. For fractional Sobolev spaces $H^s(\Omega)$ and general meshes (with certain restrictions on s), we mention [Ste98], where wavelet bases are employed.

In this thesis, for a given γ-shape regular mesh $\mathcal{T}$, we consider operators from $L^2(\Omega)$ to the space of piecewise polynomials on $\mathcal{T}$ satisfying the following properties: L^2-stability, quasi-locality, and certain approximation properties. Then, for such operators, we prove the following stability results:

- For the space of continuous piecewise polynomials on $\mathcal{T}$, we prove that such operators are stable mappings $H^{3/2}(\Omega) \to B^{3/2}_{2,\infty}(\Omega)$.
- If the mesh $\mathcal{T}$ is additionally quasi-uniform, for the space of continuous piecewise polynomials on $\mathcal{T}$, we prove a sharper stability estimate $B^{3/2}_{2,\infty}(\Omega) \to B^{3/2}_{2,\infty}(\Omega)$.
- For the space of elementwise polynomials on $\mathcal{T}$, we prove that the mentioned operators are stable mappings from $H^{1/2}(\Omega)$ into $B^{1/2}_{2,\infty}(\Omega)$.
- We also show that for the quasi–uniform meshes, we have the stability $B^{1/2}_{2,\infty}(\Omega) \to B^{1/2}_{2,\infty}(\Omega)$, for the space of elementwise polynomials on $\mathcal{T}$.
- By interpolation arguments, we derive the stability estimate $B^{(m-1)\theta/2}_{2,q}(\Omega) \to B^{(m-1)\theta/2}_{2,q}(\Omega)$ where $\theta \in (0,1)$, $q \in [1, \infty]$, and for the space of continuous piecewise polynomials $m = 2$ and for the space of elementwise polynomials $m = 1$.

The Scott-Zhang operators are local, $L^2(\Omega)$-stable operators with certain approximation properties in $L^2(\Omega)$, therefore these operators admit the first two stability results.

For the spaces of continuous piecewise polynomials on adaptively refined meshes, we develop a multilevel decomposition based on modified Scott-Zhang operators defined on a hierarchy of meshes generated by the *finest common coarsening* (`fcc`) of two meshes. Given a mesh $\mathcal{T}$ obtained by *Newest Vertex Bisection* (NVB) refinement from a regular triangulation $\widehat{\mathcal{T}}_0$ and $\widehat{\mathcal{T}}_\ell$ as the sequence of uniformly refined NVB-generated meshes, we denote $\widetilde{\mathcal{T}}_\ell := \texttt{fcc}(\mathcal{T}, \widehat{\mathcal{T}}_\ell)$ as the finest common coarsening of $\mathcal{T}$ and $\widehat{\mathcal{T}}_\ell$. For the space of continuous piecewise polynomials defined on the mesh hierarchy $\widetilde{\mathcal{T}}_\ell$, the modified Scott-Zhang operator $\widetilde{I}^{SZ}_\ell$ is constructed in such a way that for the functions belonging to the space of continuous piecewise polynomials on $\mathcal{T}$, it coincides with the Scott-Zhang operator $\widehat{I}_\ell^{SZ}$ on $\widehat{\mathcal{T}}_\ell$.

Taking advantage of the mentioned stability results and the property of the modified Scott-Zhang operators defined on the mesh hierarchy $\widetilde{\mathcal{T}}_\ell$, we present multilevel norm equivalences in the Besov spaces $B^{3\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1)$, $q \in [1,\infty]$, for the standard discrete spaces of globally continuous piecewise polynomials on $\mathcal{T}$.

Multilevel preconditioning of the fractional Laplacian

An important application of multilevel decompositions is the design of multilevel additive Schwarz preconditioners. We recall [BPX91] as one of the earliest works applying an additive multilevel operator (BPX) to develop preconditioners for second order elliptic boundary value problems and prove that it is nearly optimal. For the case of non-uniform meshes, the optimal complexity of BPX is proved in [DK92, BY93].

In this work, we propose a local multilevel diagonal preconditioner for the integral fractional Laplacian $(-\Delta)^s$ for $s \in (0,1)$ on adaptively refined meshes. Using the additive Schwarz framework, we prove this multilevel diagonal scaling gives rise to uniformly bounded condition number for the integral fractional Laplacian.

The need for a preconditioner arises from the observation that the condition number of the stiffness matrix $\mathbf{A}^\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ corresponding to a FEM discretization by piecewise linears of the integral fractional Laplacian grows like $\kappa(\mathbf{A}^\ell) \sim N_\ell^{2s/d}\left(\frac{h^\ell_{\max}}{h^\ell_{\min}}\right)^{d-2s}$, where $h^\ell_{\max}, h^\ell_{\min}$ denote the maximal and minimal mesh width of $\mathcal{T}_\ell$, respectively, see, e.g., [AMT99, AG17]. Since the fractional Laplacian on bounded domains features singularities at the boundary, typical meshes are strongly refined towards the boundary so that the quotient $h^\ell_{\max}/h^\ell_{\min}$ is large (see, e.g., [AG17, BBN+18, FMP19] for adaptively generated meshes). While the impact of the variation of the element size can be controlled by diagonal scaling (see, e.g., [BS89, AMT99]), the factor $N_\ell^{2s/d}$ persists. A good preconditioner is therefore required for an efficient iterative solution for large problem sizes N_ℓ.

Preconditioning for fractional differential operators has attracted attention recently. We mention multigrid preconditioners [AG17], based on uniformly refined mesh hierarchies and operator preconditioning, [Hip06, GSUT19, SvV19], which requires one to realize an operator of the opposite order.

The framework of additive Schwarz preconditioners is analyzed in a BPX-setting with Fourier techniques in [BLN19]. For a different definition of the fractional Laplacian via spectral and PDE theory, [CS07], locally refined FEMs have been studied in [CNOS15] and [CNOS16] provides an almost optimal multilevel method for this interpretation. We also mention [BKM19], where optimal additive Schwarz preconditioners on quasi-uniform meshes for the spectral fractional Laplacian are proposed.

In this thesis, using the additive Schwarz framework, we provide an optimal local multilevel diagonal preconditioner for two types of mesh hierarchies: $\widetilde{\mathcal{T}}_\ell$ and the meshes generated by an adaptive algorithm.

$\mathcal{H}$-matrices

The second part of this thesis is concerned with $\mathcal{H}$-matrices introduced in [Hac99] and analysed in [HK00, GH03a, Hac15, Gra01]. This class of matrices consists of blockwise low-rank matrices of rank r where the blocks are organized in a multilevel structure, i.e., a tree $\mathbb{T}_{\mathcal{I}}$, so that the memory requirement is typically $O(Nr\,\mathrm{depth}(\mathbb{T}_{\mathcal{I}}))$, where N is the problem size. This format comes with an (approximate) arithmetic that allows for addition, multiplication, inversion, and LU-factorization in logarithmic-linear complexity. Therefore, computing an (approximate) inverse in the $\mathcal{H}$-format can be considered a serious alternative to a direct solver or it can be used as a "black box" preconditioner in iterative solvers. A basic question in connection with the $\mathcal{H}$-matrix arithmetic is whether matrices and their inverses or factors in an LU-factorization can be represented well in the chosen format. While stiffness matrices arising from differential operators are sparse and are thus easily represented exactly in the standard $\mathcal{H}$-matrix formats, the situation is more involved for the inverse.

The works [Bör10, BH03, Beb07, FMP15, AFM20] prove that the inverse of the stiffness matrix corresponding to the finite element discretization of the scalar elliptic operators can be approximated in the $\mathcal{H}$-matrix format and the error decays exponentially in the block rank. The works [FMP16, FMP17] show similar results for the boundary element method. The underlying mechanism in these works is that ellipticity of the operator allows one to prove a discrete Caccioppoli inequality where a higher order norm (e.g., the H^1-norm, the $\mathbf{H}(\mathrm{curl})$-norm) is controlled by a lower order norm (e.g., the L^2-norm, the $\mathbf{L}^2$-norm) on a slightly larger region. A consequence of Caccioppoli-type estimates is the existence of blockwise low-rank approximants to inverses of FEM or BEM matrices [BH03, Bör10, FMP15, FMP16, FMP17].

$\mathcal{H}$-matrix approximation of the inverse of the stiffness matrix corresponding to FEM-BEM couplings

In this work, we consider three different FEM-BEM coupling techniques, namely, the Bielak-MacCamy coupling [BM84], Costabel's symmetric coupling [Cos88, CES90], and the Johnson-Nédélec coupling [JN80], for the transmission problems posed on unbounded domains. We present an approximation result for the inverses of stiffness matrices corresponding to the lowest order FEM-BEM discretizations in the $\mathcal{H}$-matrix format.

A crucial step in the proof of the existence of such $\mathcal{H}$-matrix approximations is to provide the discrete Caccioppoli-type inequalities. The Caccioppoli-type inequalities control the stronger norm of the weak solutions of elliptic PDEs with locally zero right-hand sides by a weaker norm on a (slightly) enlarged regions. To see examples of the Caccioppoli-type estimates for the continuous solutions of elliptic PDEs, we refer to [Hac15, Lem. 11.17] and [BH03, Lem. 2.4]. For the finite element solutions, the Caccioppoli estimate has the following form

$$\|\nabla u_h\|_{L^2(B_{R_1})} \leq C \left(\frac{h}{\mathrm{dis}(\partial B_{R_1}, \partial B_{R_2})} \|\nabla u_h\|_{L^2(B_{R_2})} + \frac{1}{\mathrm{dis}(\partial B_{R_1}, \partial B_{R_2})} \|u_h\|_{L^2(B_{R_2})} \right),$$

where $B_{R_1} \subset B_{R_2}$ are two subdomains of the domain Ω and h is the mesh size. The above estimate can be found in [FMP15]. Analogously, in a BEM context, we refer to [FMP16, FMP17].

In this work, we provide the Caccioppoli-type estimates for the finite element solution of the transmission problem on regions that are not supported by the right-hand side functions as well as for the single- and double-layer potentials of the boundary element solution. i.e., we simultaneously control a stronger norm of the interior solution and both layer potentials by a weaker norm on a larger domain.

As a consequence of such Caccioppoli-type estimates, we prove that root exponential convergence can be achieved in the rank employed.

$\mathcal{H}$-matrix approximation of the inverse of the stiffness matrix corresponding to the time-harmonic Maxwell equations

The last part of the thesis deals with the $\mathcal{H}$-matrix approximability of the inverse of the stiffness matrix corresponding to the time-harmonic Maxwell equation with perfectly conducting boundary conditions. We restrict ourself to the case that the domain is filled with a homogeneous isotropic material.

We provide a Caccioppoli inequality that controls the $\mathbf{H}(\mathrm{curl})$-norm of the discrete solution by the $\mathbf{L}^2$-norm. However, since $\mathbf{H}(\mathrm{curl})$ is not compactly embedded in $\mathbf{L}^2$, this Caccioppoli inequality is insufficient for approximation purposes. We therefore combine this Caccioppoli inequality with a local discrete Helmholtz-type decomposition. The gradient part can be treated with techniques established in [FMP15] for Poisson problems, whereas the remaining part can, up to a small perturbation, be controlled in $\mathbf{H}^1$. As a result of the Caccioppoli inequalities, we prove the existence of $\mathcal{H}$-matrix approximations to the inverse of the stiffness matrices corresponding to the time-harmonic Maxwell equations that converges root exponentially in the block-rank.

1.1 Outline and contributions

Chapter 2

In this chapter, we introduce Sobolev spaces on domains $\Omega \subset \mathbb{R}^d$ as well as on the corresponding boundaries $\partial\Omega$, the interpolation spaces and some vector-valued function spaces. Also, we briefly mention the trace operators and their properties. Then, we present some basic results on the discretization of the domain Ω, the classical H^1-conforming and low-order $\mathbf{H}(\mathrm{curl})$-conforming finite element methods and discretization of the boundary $\partial\Omega$. This is followed by a short introduction to the abstract additive Schwarz theory and the hierarchical matrices.

Chapter 3

In this chapter, considering local $L^2(\Omega)$-stable operators mapping into the spaces of continuous piecewise polynomial on shape regular meshes with certain approximation properties in $L^2(\Omega)$, we prove that such operators are stable mappings $H^{3/2}(\Omega) \to B^{3/2}_{2,\infty}(\Omega)$. Analogously, we show locally L^2-stable operators such as the elementwise L^2-projection are stable $B^{1/2}_{2,\infty}(\Omega) \to B^{1/2}_{2,\infty}(\Omega)$. An interpolation argument, for the space of continuous piecewise polynomials, gives stability $B^{3\theta/2}_{2,q}(\Omega) \to B^{3\theta/2}_{2,q}(\Omega)$ and for the space of elementwise polynomials, gives stability $B^{\theta/2}_{2,q}(\Omega) \to B^{\theta/2}_{2,q}(\Omega)$ where $\theta \in (0,1), q \in [1,\infty]$.

We then go on to present some extensions such as inverse estimates in Besov norms (Lemma 3.1.8) or an interpolation result for discrete spaces in Besov norms (Corollary 3.1.10).

The rest of this chapter is devoted to present a multilevel decomposition based on a modified Scott-Zhang operator on $\widetilde{\mathcal{T}}_\ell := \texttt{fcc}(\mathcal{T}, \widehat{\mathcal{T}}_\ell)$ where $\widetilde{\mathcal{T}}_\ell$ is generated by the finest common coarsening of a fixed mesh $\mathcal{T}$ and the sequence of uniformly refined meshes $\widehat{\mathcal{T}}_\ell$. In order to construct the multilevel decomposition, first we develop properties of the finest common coarsening of two given meshes obtained by NVB refinement. Then, we introduce modified Scott-Zhang operators on the mesh hierarchy $\widetilde{\mathcal{T}}_\ell$ such that for the space of continuous piecewise polynomials on $\mathcal{T}$, these operators coincide with the Scott-Zhang operators on the mesh hierarchy $\widehat{\mathcal{T}}_\ell$.

Finally, since the space of continuous piecewise polynomials on $\widetilde{\mathcal{T}}_\ell$ is a subset of the space of continuous piecewise polynomials on $\mathcal{T}$ and due to the fact that the modified Scott-Zhang operators on the mesh hierarchy $\widetilde{\mathcal{T}}_\ell$ coincide with the Scott-Zhang operators on the mesh hierarchy $\widehat{\mathcal{T}}_\ell$, we prove multilevel norm equivalences in the Besov space $B^{3\theta/2}_{2,q}(\Omega)$, $\theta \in (0,1)$, $q \in [1,\infty]$, with the aid of mentioned stability results.

Chapter 4

On a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, we consider the integral fractional Laplacian $(-\Delta)^s$ for $s \in (0,1)$ on adaptively refined meshes $\mathcal{T}_\ell$. An optimal local multilevel diagonal preconditioner for the fractional Laplacian for two types of mesh hierarchies are presented. The first one is assumed to be generated by an adaptive algorithm and discussed in Theorem 4.3.1. The second one is based on the sequence $\widetilde{\mathcal{T}}_\ell$ and analysed in Theorem 4.3.4.

As the main result of this chapter, using an abstract additive Schwarz framework, we show that, in the presence of adaptively refined meshes, multilevel diagonal scaling leads to uniformly bounded condition numbers for the integral fractional Laplacian. To prove the main result, we apply the norm equivalence of the multilevel decomposition in Chapter 3 and combine it with an inverse estimate in fractional Sobolev norms.

For the space of piecewise linear polynomials, the inverse inequality for the Laplacian operator $(-\Delta)^s$ is proven in [FMP19, Thm. 2.8]. In this chapter, for $0 < s < 1/2$, we generalize this inverse estimate to the space of piecewise constants.

Chapter 5

On a Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$ with polygonal (for $d = 2$) or polyhedral (for $d = 3$) boundary Γ, we consider a transmission problem and study three different FEM-BEM couplings, the Bielak-MacCamy coupling [BM84], Costabel's symmetric coupling [Cos88, CES90], and the Johnson-Nédélec coupling [JN80]. In this chapter, we prove the existence of exponentially convergent $\mathcal{H}$-matrix approximants to the inverses of the stiffness matrices of the FEM-BEM couplings.

To prove the $\mathcal{H}$-matrix approximability, we show that for the interior finite element solution and for the single-layer and double-layer potentials of the boundary element solution, a Caccioppoli type estimate holds, i.e., the stronger H^1-seminorm can be estimated by a weaker h-weighted H^1-norm on a larger domain.

Analyzing the procedure in [FMP15, FMP16, AFM20] shows structural similarities in the derivation of $\mathcal{H}$-matrix approximations based on low-dimensional spaces of functions: A single-step approximation is obtained by using a Scott-Zhang operator on a coarse grid. Iterating this argument is made possible by a Caccioppoli inequality, resulting in a multi-step approximation. Finally, with the aid of the approximated solutions from low-dimensional spaces, we prove the existence of the $\mathcal{H}$-matrix approximants with the exponential convergence in the block rank.

Chapter 6

On $\Omega \subset \mathbb{R}^3$, a simply connected open polyhedral domain with boundary $\Gamma := \partial\Omega$, we consider the time-harmonic Maxwell equations and their discretization with lowest order Nédélec's curl-conforming elements. We prove the existence of $\mathcal{H}$-matrix approximations to the inverse of corresponding stiffness matrix and we show exponential convergence of the error in the $\mathcal{H}$-matrix block-rank r.

In order to prove the $\mathcal{H}$-matrix approximability result, we introduce a local discrete Helmholtz decomposition and provide stability and approximation properties of this decomposition. We also present a Caccioppoli-type inequality for discrete $\boldsymbol{\mathcal{L}}$-harmonic functions with $\boldsymbol{\mathcal{L}}$ being the Maxwell operator. Then, we combine this Caccioppoli inequality with the local discrete Helmholtz-type decomposition and treat gradient part with techniques established in [FMP15] for Poisson problems whereas the remaining part can, up to a small perturbation, be controlled in $\mathbf{H}^1$. So that approximation becomes feasible and one may proceed structurally similarly to the scalar case in the previous chapter.

In this thesis, Chapters 3 and 4 are the results of the paper [FMP21b] which will appear in ESAIM Math. Model. Numer. Anal. (M2AN). Chapter 5 is contained in the paper[FMP20] which is submitted to Numerische Mathematik. Finally, Chapter 5 is presented in the paper [FMP21a] which is submitted to Advances in Computational Mathematics.

2 Background

This chapter is devoted to introduce some basic notations and definitions. In Subsection 2.1.1, we define Sobolev spaces on domain $\Omega \subset \mathbb{R}^d$ as well as on the boundary $\partial\Omega$. Subsection 2.1.2 deals with interpolation spaces. Subsection 2.1.3 recalls some vector-valued function spaces. Subsection 2.1.4 is devoted to trace operators and their properties. In Section 2.2, we briefly introduce triangulation of the domain Ω as well as the classical H^1-conforming and low-order $\mathbf{H}(\mathrm{curl})$-conforming finite element methods. Section 2.3 recalls triangulation of the boundary $\partial\Omega$. Section 2.4 is concerned with the definition of some (quasi-) interpolation operators and their properties. A short introduction to the abstract Schwarz theory is given in Section 2.5. The final Section 2.6 deals with the hierarchical matrices.

Throughout this chapter, $\Omega \subset \mathbb{R}^d, \quad d \geq 1$ denotes a Lipschitz domain with the boundary $\Gamma := \partial\Omega$ and $\Omega^{\mathrm{ext}} := \mathbb{R}^d \setminus \overline{\Omega}$ denotes the exterior of Ω.

Additionally, the notation $\lesssim$ abbreviates $\leq$ up to a constant $C > 0$. Moreover, we use $\simeq$ to indicate that both estimates $\lesssim$ and $\gtrsim$ hold where C is positive constant independent of the mesh parameters except the γ-shape regularity.

2.1 Function spaces

2.1.1 Sobolev spaces

In this section, we define Sobolev spaces of integer and real orders for both positive and negative cases; see, e.g [Ada75, BS02, Mon03, SS11]. For $p \geq 1$, let $L^p(\Omega)$ be the usual Lebesgue spaces on Ω with corresponding norm $\|\cdot\|_{L^p(\Omega)}$. Analogously, Lebesgue spaces on the boundary Γ are denoted by $L^p(\Gamma)$ with the norm $\|\cdot\|_{L^p(\Gamma)}$. We denote $C^k(\Omega)$, $k \in \mathbb{N}_0$, as the space of k times continuously differentiable functions on Ω and $C_0^k(\Omega)$ as the space of compactly supported functions belonging to $C^k(\Omega)$. Let $C^\infty(\Omega)$ denote the space of infinitely differentiable functions on Ω and $C_0^\infty(\Omega)$ denote the space of compactly supported functions in $C^\infty(\Omega)$.

Sobolev spaces on Ω

Let $L^1_{\mathrm{loc}}(\Omega)$ denote the space of absolutely integrable functions on every compact subset of Ω. For a multi-index $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}_0^d$, we set $|\boldsymbol{\alpha}| := \sum_{i=1}^d \alpha_i$, and the classical derivative is denoted by

$$D^{\boldsymbol{\alpha}} u := \frac{\partial^{|\boldsymbol{\alpha}|} u}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

For $u \in L^1_{\text{loc}}(\Omega)$, we call a function in $L^1_{\text{loc}}(\Omega)$, which we denote $D^{\boldsymbol{\alpha}} u$, the weak derivative of order $\boldsymbol{\alpha}$ if

$$\int_\Omega D^{\boldsymbol{\alpha}} u \, \varphi \, d\boldsymbol{x} = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u \, D^{\boldsymbol{\alpha}} \varphi \, d\boldsymbol{x} \qquad \forall \varphi \in C_0^\infty(\Omega).$$

The Sobolev space $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$ and $p \in [1, \infty]$ is defined as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \qquad : \qquad D^{\boldsymbol{\alpha}} u \in L^p(\Omega), \, \forall \, |\boldsymbol{\alpha}| \in \mathbb{N}_0^d, \, 0 \leq \boldsymbol{\alpha} \leq k\},$$

with the corresponding norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^d \\ 0 \leq |\boldsymbol{\alpha}| \leq k}} \|D^{\boldsymbol{\alpha}} u\|^p_{L^p(\Omega)} \right)^{1/p} & \text{for } p \in [1, \infty), \\ \max_{|\boldsymbol{\alpha}| \leq k} \|D^{\boldsymbol{\alpha}} u\|_{L^\infty(\Omega)} & \text{for } p = \infty. \end{cases}$$

For $p = 2$ we use the standard notation $H^k(\Omega) := W^{k,2}(\Omega)$. This space is a separable Hilbert space with the scalar product

$$\langle u, v \rangle_{H^k(\Omega)} := \sum_{\substack{\boldsymbol{\alpha} \in \mathbb{N}_0^d \\ 0 \leq |\boldsymbol{\alpha}| \leq k}} \langle D^{\boldsymbol{\alpha}} u, D^{\boldsymbol{\alpha}} v \rangle_{L^2(\Omega)},$$

which induces the norm $\|u\|_{H^k(\Omega)} = \sqrt{\langle u, u \rangle_{H^k(\Omega)}}$. We also define the following semi-norm $|u|_{H^k(\Omega)} := \left(\sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} \int_\Omega (D^{\boldsymbol{\alpha}} u)^2 \right)^{1/2}$. For $0 < \beta < 1$, we define the following Slobodeckij semi-norm

$$|u|^2_{H^\beta(\Omega)} := \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{|x - y|^{d + 2\beta}} dx dy.$$

For $s \in \mathbb{R}^+ \backslash \mathbb{N}$, we define the Slobodeckij norm as

$$\|u\|^2_{H^s(\Omega)} := \|u\|^2_{H^{\lfloor s \rfloor}(\Omega)} + \sum_{|\alpha| = \lfloor s \rfloor} \int_\Omega \int_\Omega \frac{(D^\alpha u(x) - D^\alpha u(y))^2}{|x - y|^{d + 2\beta}} dx dy,$$

where $\beta := s - \lfloor s \rfloor$. Then, the Sobolev spaces of fractional order $s \in \mathbb{R}^+ \backslash \mathbb{N}$ are defined as

$$H^s(\Omega) := \{u \in L^2(\Omega) \qquad : \qquad \|u\|_{H^s(\Omega)} < \infty\}.$$

In the above definitions of Sobolev spaces, we are allowed to replace Ω with $\mathbb{R}^d$. In the following, we define the space of functions in $H^s(\Omega)$ with zero boundary conditions. For $s > 0$, we denote

$$H^s_0(\Omega) := \overline{C_0^\infty(\Omega)} \qquad \text{closure with respect to the } H^s(\Omega)\text{-norm},$$

For $s \in \mathbb{R}^+$, we also define the following Hilbert space

$$\widetilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^d) \,:\, u \equiv 0 \text{ on } \Omega^c\}, \qquad \|v\|^2_{\widetilde{H}^s(\Omega)} := \|v\|^2_{H^s(\Omega)} + \left\|\operatorname{dist}(\cdot, \partial\Omega)^{-s} v\right\|^2_{L^2(\Omega)}.$$

The Sobolev spaces with negative orders are defined as the topological dual of positive order Sobolev spaces: According to [McL00, Thm. 3.30], we have $\widetilde{H}^{-s}(\Omega) = (H^s(\Omega))'$ and $H^{-s}(\Omega) = (\widetilde{H}^s(\Omega))'$, for $s \in \mathbb{R}$. In the following chapters, we need to extend the Sobolev spaces from Ω to $\mathbb{R}^d$ in a stable way. This can be done using the following lemma.

Lemma 2.1.1. *[Ada75, Thm. 4.32] Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then, there exists a bounded linear extension operator $\mathcal{E} : H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that $\mathcal{E}u = u$ on Ω and*

$$\|u\|_{H^1(\mathbb{R}^d)} \leq C \|u\|_{H^1(\Omega)},$$

where the constant $C > 0$ only depends on d and Ω.

Sobolev spaces on the boundary Γ

In this subsection, we shall extend the definition of Sobolev spaces to the boundary Γ. For $0 < s < 1$, we define

$$H^s(\Gamma) := \{u \in L^2(\Gamma) \quad : \quad \|u\|_{H^s(\Gamma)} < \infty\},$$

where $\|\cdot\|_{H^s(\Gamma)}$ is the Aronstein-Slobodeckij (semi-)norm defined as

$$\|u\|^2_{H^s(\Gamma)} := \|u\|^2_{L^2(\Gamma)} + |u|^2_{H^s(\Gamma)} \qquad \text{with} \quad |u|^2_{H^s(\Gamma)} := \int_\Gamma \int_\Gamma \frac{(u(x) - u(y))^2}{|x - y|^{d-1+2s}} ds(x) ds(y),$$

where ds is the surface measure on Γ; see, e.g [SS11, Def. 2.4.1]. For $0 < s < 1$, we denote $H^{-s}(\Gamma) := (H^s(\Gamma))'$ as the negative order Hilbert space and we can equip the dual space with the following norm

$$\|u\|_{H^{-s}(\Gamma)} := \sup_{v \in H^s(\Gamma)} \frac{\langle u, v\rangle_\Gamma}{\|v\|_{H^s(\Gamma)}},$$

where $\langle u, v\rangle_\Gamma$ denote the duality paring.

2.1.2 Interpolation spaces

In this subsection, we briefly introduce the interpolation spaces and overview some of the key results regarding the interpolation with the K-method for Banach and Hilbert spaces. We also note that the fractional order Sobolev spaces can equivalently be obtained by interpolating between integer order spaces and it is useful since working with the norms of integer order spaces are easier than the Slobodeckij norms. For more details, we refer to [Tar07] and [McL00, Apendix B].

Let $(X_0, \|\cdot\|_{X_0})$ and $(X_1, \|\cdot\|_{X_1})$ be two normed spaces with continuous embedding $X_1 \subseteq X_0$. The K-functional on X_0 is defined by

$$K(t,u) := \inf_{u_t \in X_1} \left(\|u - u_t\|_0 + t\|u_t\|_1\right).$$

For $\theta \in (0,1)$ and $q \in [1,\infty]$, we define the interpolation space $X_{\theta,q}$ as

$$X_{\theta,q} := (X_0, X_1)_{\theta,q} := \{u \in X_0 \quad : \quad \|u\|_{\theta,q} < \infty\},$$

equipped with the norm

$$\|u\|_{\theta,q} := \begin{cases} \left(\int_{t=0}^{\infty} \left(t^{-\theta} K(t,u)\right)^q \frac{dt}{t}\right)^{1/q} & q \in [1,\infty), \\ \operatorname{esssup}_{t>0} \left(t^{-\theta} K(t,u)\right) & q = \infty. \end{cases}$$

In the following lemma, we mention two simple but important properties of the interpolation spaces.

Lemma 2.1.2. *[Tri95, Sec. 1.3.3] There exists a positive constant $C_{\theta,q}$ such that for $u \in X_1$, we have the following estimation for the interpolation norm*

$$\|u\|_{\theta,q} \leq C_{\theta,q} \|u\|_{X_0}^{1-\theta} \|u\|_{X_1}^{\theta} \qquad \theta \in (0,1) \quad q \in [1,\infty].$$

Furthermore, for q, $q' \in [1,\infty]$ and $0 < \theta < \theta' < 1$, it holds the following continuous embedding

$$X_{\theta,q} \subset X_{\theta',q'}.$$

There are two additional results regarding the interpolation spaces that we require later. The first one is called the "reiteration theorem" and tells us about the possibility of interpolating between the interpolation spaces. The second one is concerned with interpolation between finite dimensional spaces.

Lemma 2.1.3. *(Reiteration Theorem, [Tar07, Thm. 26.3]) For $0 < \theta_0 < \theta_1 < 1$, $1 \leq p_0, p_1, q \leq \infty$ and $0 < \lambda < 1$, there holds*

$$\left(\left((X_0, X_1)_{\theta_0,p_0}, \|\cdot\|_{\theta_0,p_0}\right), \left((X_0, X_1)_{\theta_1,p_1}, \|\cdot\|_{\theta_1,p_1}\right)\right)_{\lambda,q} = \left((X_0, X_1)_{(1-\lambda)\theta_0+\lambda\theta_1,q}, \|\cdot\|_{(1-\lambda)\theta_0+\lambda\theta_1,q}\right),$$

with equivalent norms.

Let $\mathcal{L}(A,B)$ denotes the space of continuous linear operators from A to B. Then, we have the following lemma.

Lemma 2.1.4. *[AL09, Lem. 2.2] Let $(X_{0,N}, \|\cdot\|_{X_0}) \subseteq (X_0, \|\cdot\|_{X_0})$ and $(X_{1,N}, \|\cdot\|_{X_1}) \subseteq (X_1, \|\cdot\|_{X_1})$ be two finite dimensional Hilbert spaces with $N = \dim X_{0,N} = X_{1,N}$. Also, let there exists an operator $\pi_N \in \mathcal{L}(X_0, X_{0,N}) \cap \mathcal{L}(X_1, X_{1,N})$ such that $\pi_N u = u$ for all $u \in X_{0,N}$. Then, it holds*

$$\left((X_{0,N}, \|\cdot\|_{X_0}), (X_{1,N}, \|\cdot\|_{X_1})\right)_{\theta,q} = \left((X_{0,N}, X_{1,N})_{\theta,q}, \|\cdot\|_{\theta,q}\right),$$

with equivalent norms for $\theta \in (0,1)$ and $q \in [1,\infty]$.

In the following, we consider the interpolation between two weighted L^2-spaces.

Lemma 2.1.5. *[Tar07, Lem. 23.1] Let ω be a (measurable) positive function on Ω and*

$$\mathcal{E}(\omega) = \left\{ v \mid \int_\Omega |v(x)|^2 \, \omega(x) \, dx < \infty \right\} \quad \text{with} \quad \|v\|_\omega = \left(\int_\Omega |v(x)|^2 \, \omega(x) \, dx \right)^{1/2}.$$

If ω_0 and ω_1 are two such functions, then for $0 < \theta < 1$, we have

$$(\mathcal{E}(\omega_0), \mathcal{E}(\omega_1))_{\theta,2} = \mathcal{E}(\omega_\theta),$$

with equivalent norms, where $\omega_\theta = \omega_0^{1-\theta} \omega_1^\theta$.

Besov spaces are defined as suitable interpolation between Sobolev spaces, cf. [Tar07, Ch. 34. 23.1]. In particular, we have the following definition of Besov spaces.

Definition 2.1.6. For $s > 0$, $s \notin \mathbb{N}_0$, $q \in [1, \infty]$, the Besov spaces $B^s_{2,q}(\Omega)$ are defined as the interpolation spaces

$$B^s_{2,q}(\Omega) := (H^\sigma(\Omega), H^{\sigma+1}(\Omega))_{\theta,q},$$

where $\sigma = \lfloor s \rfloor$ and $\theta = s - \sigma \in (0, 1)$.

It often is convenient to characterize fractional Sobolev spaces as interpolation spaces.

Lemma 2.1.7. *[McL00, Thm. B.8] For s_1, $s_2 \in \mathbb{R}$ and $\theta \in (0, 1)$, the following equivalence holds*

$$H^{(1-\theta)s_1+\theta s_2}(\Omega) = (H^{s_1}(\Omega), H^{s_2}(\Omega))_{\theta,2}.$$

2.1.3 Vector-valued function spaces

In this section, we recall standard notations and definitions for vector-valued function spaces. For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and an arbitrary function space, we apply bold letter for the corresponding vector valued version, e.g., $\mathbf{W} := (W)^3$. In particular, we use the following notations for vector-valued L^p- and Sobolev spaces

$$\mathbf{L}^p(\Omega) := (L^p(\Omega))^3 \qquad \mathbf{H}^k(\Omega) := (H^k(\Omega))^3, \quad \forall k \in \mathbb{N}_0.$$

For a scalar function u, the gradient operator is defined as

$$\nabla u := \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \right)^T.$$

For a smooth vector field $\mathbf{U} = (U_1, U_2, U_3)^T$, we recall the curl and the divergence operators as

$$\nabla \times \mathbf{U} := \left(\frac{\partial U_3}{\partial x_2} - \frac{\partial U_2}{\partial x_3}, \frac{\partial U_1}{\partial x_3} - \frac{\partial U_3}{\partial x_1}, \frac{\partial U_2}{\partial x_1} - \frac{\partial U_1}{\partial x_2} \right)^T \qquad \nabla \cdot \mathbf{U} := \sum_{i=1}^3 \left(\frac{\partial U_i}{\partial x_i} \right).$$

Next, using partial integration, we mention the definition of the derivatives in the weak sense, see [Mon03, Sec. 3.5]

Definition 2.1.8. (Generalized differential operators).

1. For $\mathbf{U} \in \mathbf{L}^2(\Omega)$, we call $\mathbf{F}_1 \in \mathbf{L}^1_{\text{loc}}(\Omega)$ the (generalized) curl of $\mathbf{U}$, if there holds
$$\int_\Omega \mathbf{F}_1 \cdot \mathbf{V}\, d\mathbf{x} = \int_\Omega \mathbf{U} \cdot \nabla \times \mathbf{V}\, d\mathbf{x} \qquad \forall \mathbf{V} \in \mathbf{C}_0^\infty(\Omega),$$
and we write $\nabla \times \mathbf{U} = \mathbf{F}_1$.
2. For $u \in L^2(\Omega)$, we call $\mathbf{F}_2 \in \mathbf{L}^1_{\text{loc}}(\Omega)$ the (generalized) gradient of u, if there holds
$$\int_\Omega \mathbf{F}_2 \cdot \mathbf{V}\, d\mathbf{x} = -\int_\Omega u \nabla \cdot \mathbf{V}\, d\mathbf{x} \qquad \forall \mathbf{V} \in \mathbf{C}_0^\infty(\Omega),$$
and we write $\nabla u = \mathbf{F}_2$.
3. For $\mathbf{U} \in \mathbf{L}^2(\Omega)$, we call $F_3 \in L^1_{\text{loc}}(\Omega)$ the (generalized) divergence of $\mathbf{U}$, if there holds
$$\int_\Omega F_3\, v\, d\mathbf{x} = -\int_\Omega \mathbf{U} \cdot \nabla v\, d\mathbf{x} \qquad \forall v \in C_0^\infty(\Omega),$$
and we write $\nabla \cdot \mathbf{U} = F_3$.

This leads to the definition of the following $\mathbf{H}(\text{curl}, \Omega)$ and $\mathbf{H}(\text{div}, \Omega)$ spaces.

Definition 2.1.9. We define the following function spaces
$$\begin{aligned} \mathbf{H}(\text{curl}, \Omega) &:= \left\{ \mathbf{U} \in \mathbf{L}^2(\Omega) \;:\; \nabla \times \mathbf{U} \in \mathbf{L}^2(\Omega) \right\}, \\ \mathbf{H}(\text{div}, \Omega) &:= \left\{ \mathbf{U} \in \mathbf{L}^2(\Omega) \;:\; \nabla \cdot \mathbf{U} \in L^2(\Omega) \right\}, \end{aligned}$$
with the following scalar products
$$\begin{aligned} \langle \mathbf{U}, \mathbf{V} \rangle_{\mathbf{H}(\text{curl},\Omega)} &:= \langle \mathbf{U}, \mathbf{V} \rangle_{\mathbf{L}^2(\Omega)} + \langle \nabla \times \mathbf{U}, \nabla \times \mathbf{V} \rangle_{\mathbf{L}^2(\Omega)}, \\ \langle \mathbf{U}, \mathbf{V} \rangle_{\mathbf{H}(\text{div},\Omega)} &:= \langle \mathbf{U}, \mathbf{V} \rangle_{\mathbf{L}^2(\Omega)} + \langle \nabla \cdot \mathbf{U}, \nabla \cdot \mathbf{V} \rangle_{\mathbf{L}^2(\Omega)}. \end{aligned}$$
Moreover, the induced norms are denoted by
$$\begin{aligned} \|\mathbf{U}\|^2_{\mathbf{H}(\text{curl},\Omega)} &:= \|\mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \times \mathbf{U}\|^2_{\mathbf{L}^2(\Omega)}, \\ \|\mathbf{U}\|^2_{\mathbf{H}(\text{div},\Omega)} &:= \|\mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla \cdot \mathbf{U}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Definition 2.1.10. The spaces $\mathbf{H}_0(\text{curl}, \Omega) \subset \mathbf{H}(\text{curl}, \Omega)$ and $\mathbf{H}_0(\text{div}, \Omega) \subset \mathbf{H}(\text{div}, \Omega)$ are defined as
$$\begin{aligned} \mathbf{H}_0(\text{curl}, \Omega) &:= \overline{\mathbf{C}_0^\infty(\Omega)} \qquad \text{closure with respect to the } \mathbf{H}(\text{curl}, \Omega)\text{-norm}, \\ \mathbf{H}_0(\text{div}, \Omega) &:= \overline{\mathbf{C}_0^\infty(\Omega)} \qquad \text{closure with respect to the } \mathbf{H}(\text{div}, \Omega)\text{-norm}. \end{aligned}$$

For a simply connected domain Ω with connected boundary Γ, the connection between the function spaces $\mathbf{H}(\mathrm{curl}, \Omega)$, $\mathbf{H}(\mathrm{div}, \Omega)$, $H^1(\Omega)$ can be collected in the following so-called de Rham diagram

$$\mathbb{R} \to H^1(\Omega) \xrightarrow{\nabla} \mathbf{H}(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} \mathbf{H}(\mathrm{div}, \Omega) \xrightarrow{\nabla\cdot} L^2(\Omega) \xrightarrow{0} 0,$$

see e.g. [Mon03, eq. (3.59)]. This sequence is exact which means that the image of one operator coincides with the kernel of the following operator. Moreover, considering boundary conditions we have the following exact sequence

$$\mathbb{R} \to H^1_0(\Omega) \xrightarrow{\nabla} \mathbf{H}_0(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} \mathbf{H}_0(\mathrm{div}, \Omega) \xrightarrow{\nabla\cdot} L^2(\Omega)/\mathbb{R} \xrightarrow{0} 0,$$

where $L^2(\Omega)/\mathbb{R} := \left\{u \in L^2(\Omega) \quad : \quad \int_\Omega u\,dx = 0\right\}$, see e.g. [Mon03, eq. (3.60)].

In the following, we mention the existence of a scalar potential for the curl-free vector fields.

Theorem 2.1.11. *[Mon03, Thm. 3.37] Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected Lipschitz domain and $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$. Then, $\nabla \times \boldsymbol{u} = 0$ in Ω if and only if there exists a scalar potential $\psi \in H^1(\Omega)$ such that $\boldsymbol{u} = \nabla\psi$. Moreover, ψ is unique up to an additive constant.*

2.1.4 Trace operators

In this subsection, we collect some standard notations and results regarding the interior and exterior trace operators as well as the corresponding conormal derivatives. Furthermore, we present the traces of functions in $\mathbf{H}(\mathrm{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$. Most of the results in this section can be found in [SS11, Mon03].

Trace operators for the space $H^s(\Omega)$

We introduce the space of functions with distributional Laplacian in L^2 as

$$H^1_\Delta(\Omega) := \{u \in H^1(\Omega) \quad : \quad \nabla u \in \mathbf{H}(\mathrm{div}, \Omega)\},$$

with the corresponding norm

$$\|u\|^2_{H^1_\Delta(\Omega)} := \|u\|^2_{H^1(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)}.$$

Lemma 2.1.12. *[SS11, Thm. 2.6.8, Thm. 2.7.7] Let $1/2 < s < 3/2$. Then there exists a linear and continuous interior trace operator*

$$\gamma_0^{\mathrm{int}} : H^s(\Omega) \to H^{s-1/2}(\Gamma) \quad \text{such that} \quad \gamma_0^{\mathrm{int}} u = u|_\Gamma \quad \forall u \in C^\infty(\overline{\Omega}).$$

Analogously, there exists a linear and continuous exterior trace operator

$$\gamma_0^{\mathrm{ext}} : H^s(\Omega^{\mathrm{ext}}) \to H^{s-1/2}(\Gamma) \quad \text{such that} \quad \gamma_0^{\mathrm{ext}} u = u|_\Gamma \quad \forall u \in C^\infty(\overline{\Omega^{\mathrm{ext}}}),$$

Let ν be the outward normal vector of Ω, then there exists a bounded linear exterior conormal derivative operator

$$\gamma_1^{\mathrm{int}} : H^1_\Delta(\Omega) \to H^{-1/2}(\Gamma) \quad \text{such that} \quad \gamma_1^{\mathrm{int}} u = \gamma_0^{\mathrm{int}} \nabla u \cdot \nu \quad \forall u \in C^\infty(\overline{\Omega}).$$

Analogously, there exists a linear and continuous exterior conormal derivative

$$\gamma_1^{\mathrm{ext}} : H^1_\Delta(\Omega^{\mathrm{ext}}) \to H^{-1/2}(\Gamma) \quad \text{such that} \quad \gamma_1^{\mathrm{ext}} u = \gamma_0^{\mathrm{ext}} \nabla u \cdot \nu \quad \forall u \in C^\infty(\overline{\Omega}).$$

The traces and conormal derivatives are generally discontinuous across Γ. Therefore, if a function u has an interior and an exterior trace, then we define the following jump term

$$[\gamma_0 u] := \gamma_0^{\text{ext}} u - \gamma_0^{\text{int}} u,$$

and if u has an interior and an exterior conormal derivative, then the corresponding jump is defined as

$$[\gamma_1 u] := \gamma_1^{\text{ext}} u - \gamma_1^{\text{int}} u,$$

see, e.g., [SS11, Subsec. 2.7]. The trace operators are used to incorporate the boundary conditions to the function spaces, therefore the following lemma helps us to have a more explicit definition of $H_0^s(\Omega)$, $0 \leq s \leq 1$, using the trace operators.

Lemma 2.1.13. *[McL00, Thm. 3.10] Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*

1. *For $0 \leq s \leq 1/2$, it holds $H_0^s(\Omega) = H^s(\Omega)$.*
2. *For $1/2 < s \leq 1$, it holds $H_0^s(\Omega) := \{u \in H^s(\Omega) \quad : \quad \gamma_0^{\text{int}} u = 0\}$.*

Trace operators for the spaces $\mathbf{H}(\text{curl}, \Omega)$ and $\mathbf{H}(\text{div}, \Omega)$

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. In this subsection, we introduce the trace of functions in $\mathbf{H}(\text{curl}, \Omega)$ and $\mathbf{H}(\text{div}, \Omega)$.

Lemma 2.1.14. *[Mon03, Thm. 3.24, Thm. 3.29] Let $\boldsymbol{n}$ denotes the outward normal vector on Γ. Then*

1. *There exists a linear, continuous operator $tr_{\boldsymbol{\tau}} : \boldsymbol{H}(\text{curl}, \Omega) \to \boldsymbol{H}^{-1/2}(\Gamma)$ such that*

$$tr_{\boldsymbol{\tau}}(\boldsymbol{u}) = (\boldsymbol{u} \times \boldsymbol{n})|_\Gamma \qquad \forall \boldsymbol{u} \in \mathbf{C}^\infty(\overline{\Omega}),$$

and

$$\|tr_{\boldsymbol{\tau}}(\boldsymbol{u})\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\text{curl},\Omega)}.$$

2. *There exists a linear, continuous operator $tr_{\boldsymbol{n}} : \boldsymbol{H}(\text{div}, \Omega) \to \boldsymbol{H}^{-1/2}(\Gamma)$ such that*

$$tr_{\boldsymbol{n}}(\boldsymbol{u}) = (\boldsymbol{u} \cdot \boldsymbol{n})|_\Gamma \qquad \forall \boldsymbol{u} \in \mathbf{C}^\infty(\overline{\Omega}),$$

and

$$\|tr_{\boldsymbol{n}}(\boldsymbol{u})\|_{\boldsymbol{H}^{-1/2}(\Gamma)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}(\text{div},\Omega)}.$$

Theorem 2.1.15 (Integration by parts for $\mathbf{H}(\text{curl}, \Omega)$ and $\mathbf{H}(\text{div}, \Omega)$)**.** *[Mon03, Theorems 3.24, 3.29]*

1. *The following integration by parts formula holds for $\boldsymbol{u} \in \boldsymbol{H}(\text{curl}, \Omega)$ and $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$*

$$\int_\Omega \nabla \times \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x} = \int_\Omega \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} \, d\boldsymbol{x} - \int_\Gamma tr_{\boldsymbol{\tau}}(\boldsymbol{u}) \cdot \boldsymbol{v} \, ds.$$

2. *The following version of Green's theorem holds for* $\boldsymbol{u} \in \boldsymbol{H}(\text{div}, \Omega)$ *and* $v \in H^1(\Omega)$

$$\int_\Omega \nabla \cdot \boldsymbol{u}\, v \; d\boldsymbol{x} = -\int_\Omega \boldsymbol{u} \cdot \nabla v \; d\boldsymbol{x} + \int_\Gamma tr_{\boldsymbol{n}}(\boldsymbol{u})\, v \; ds.$$

Lemma 2.1.16. *[Mon03, Thm. 3.25, Thm. 3.33] The spaces* $\boldsymbol{H}_0(\text{curl}, \Omega)$ *and* $\boldsymbol{H}_0(\text{div}, \Omega)$ *can be defined equivalently as*

$$\boldsymbol{H}_0(\text{curl}, \Omega) := \{\boldsymbol{U} \in \boldsymbol{L}^2(\Omega) \;:\; \nabla \times \boldsymbol{U} \in \boldsymbol{L}^2(\Omega),\; tr_{\boldsymbol{\tau}}(\boldsymbol{U}) = 0 \text{ on } \Gamma\},$$
$$\boldsymbol{H}_0(\text{div}, \Omega) := \{\boldsymbol{U} \in \boldsymbol{L}^2(\Omega) \;:\; \nabla \cdot \boldsymbol{U} \in L^2(\Omega),\; tr_{\boldsymbol{n}}(\boldsymbol{U}) = 0 \text{ on } \Gamma\}.$$

2.2 Triangulation of Ω

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a polygonal or polyhedral Lipschitz domain. In the following, we present a definition of a triangulation $\mathcal{T}$ on Ω.

Definition 2.2.1. A set $\mathcal{T}$ is called a conforming triangulation of Ω, if it satisfies the following properties

1. Each element $T \in \mathcal{T}$ is an open d-simplex.
2. $\mathcal{T}$ is regular in the Ciarlet sense, i.e., for two elements T, $T' \in \mathcal{T}$, the intersection of $\overline{T} \cap \overline{T'}$ is either empty, a common vertex, a joint edge or a joint facet ($d = 3$).
3. The union of all elements covers Ω, i.e., $\overline{\Omega} = \bigcup_{T \in \mathcal{T}} \overline{T}$.

Definition 2.2.2. Let $\mathcal{T}$ be a triangulation of Ω. Then,

1. $\mathcal{T}$ is called γ-shape regular if

$$\max_{T \in \mathcal{T}} (\text{diam}(T)/\,|T|^{1/d}) \leq \gamma < \infty,$$

   where $\text{diam}(T) := \sup_{x,y \in T} |x - y|$ and $|T|$ denotes the volume ($d = 3$) or the area ($d = 2$) of T.

2. A γ-shape regular triangulation $\mathcal{T}$ is called quasi-uniform, if there exists $C > 0$ such that

$$\max_{T \in \mathcal{T}} (\text{diam}(T)) \leq C \min_{T \in \mathcal{T}} (\text{diam}(T))$$

3. $h = \max_{T \in \mathcal{T}} (\text{diam}(T))$ is called the mesh size of $\mathcal{T}$.

Let $\mathcal{T}$ denote a regular (in the sense of Ciarlet) and γ-shape regular triangulation of $\Omega \subset \mathbb{R}^d$. In the following, we define the space of piecewise polynomials ($d = 2, 3$) and we consider the low-order finite element spaces and the lowest order local Nédélec spaces of first kind ($d = 3$).

Definition 2.2.3. Let $P_p(T)$ be the space of polynomials of (maximal) degree $p \geq 0$ on the element $T \in \mathcal{T}$. The spaces of $\mathcal{T}$-piecewise polynomials of degree $p \in \mathbb{N}_0$ and regularity $m \in \mathbb{N}_0$ are defined by

$$S^{p,m}(\mathcal{T}) := \{u \in H^m(\Omega) \colon u|_T \in P_p(T) \quad \forall T \in \mathcal{T}\},$$
$$S_0^{p,m}(\mathcal{T}) := S^{p,m}(\mathcal{T}) \cap H_0^m(\Omega) \qquad m = 0, 1.$$

Considering the fact that the results of the last two chapters of this thesis are mainly formulated for matrices, we need to impose assumptions on the basis of $S^{p,1}(\mathcal{T})$. To do that it will be convenient to use Lagrange bases of the space $S^{p,1}(\mathcal{T})$ defined on a mesh $\mathcal{T}$.

Definition 2.2.4. On the reference d-simplex $\widehat{T} = \operatorname{conv}\{z_1, \ldots, z_{d+1}\}$, let the $\dim P_p$ nodes $\mathcal{N}_p(\widehat{T})$ be the regularly spaced nodes as described in [Cia78, Sec. 2.2] (called "principal lattice" there),

$$\mathcal{N}_p(\widehat{T}) := \Big\{ x = \sum_{j=1}^{d+1} \lambda_j z_j \ : \ \sum_{j=1}^{d+1} \lambda_j = 1,\ \lambda_j \in \Big\{ \frac{i}{p}, i = 0, \ldots, p \Big\} \Big\}.$$

The nodes $\mathcal{N}_p(\mathcal{T}) \subset \overline{\Omega}$ for the mesh $\mathcal{T}$ are the push-forward of the nodes of $\mathcal{N}_p(\widehat{T})$ under the element maps. The Lagrange basis $\mathcal{B}_{\mathcal{T},p} := \{\varphi_{z,\mathcal{T}} \mid z \in \mathcal{N}_p(\mathcal{T})\}$ of $S^{p,1}(\mathcal{T})$ (with respect to the nodes $\mathcal{N}_p(\mathcal{T})$) is characterized by $\varphi_{z,\mathcal{T}}(z') = \delta_{z,z'}$ for all z, $z' \in \mathcal{N}_p(\mathcal{T})$; here, $\delta_{z,z'}$ is the Kronecker Delta defined as $\delta_{z,z'} = 1$ if $z = z'$ and $\delta_{z,z'} = 0$ for $z \neq z'$.

For $p = 1$, we abbreviate $\mathcal{N}(\mathcal{T}) := \mathcal{N}_1(\mathcal{T})$ and if the triangulation $\mathcal{T}$ is additionally quasi-uniform with the mesh size h, then we abbreviate $\mathcal{B}_h := \mathcal{B}_{\mathcal{T},1}$. For a quasi-uniform triangulation $\mathcal{T}$ of Ω with the mesh size h, let $\mathcal{N}(\mathcal{T}) = \{z_1, \cdots, z_n\}$ be the set of the nodes of $\mathcal{T}$ and $\xi_j := \varphi_{z_j,\mathcal{T}}$, $j = 1, \cdots, n$. Then $\mathcal{B}_h$ features the following norm equivalences:

$$c_1 h^{d/2} \|\mathbf{x}\|_2 \leq \|\Phi \mathbf{x}\|_{L^2(\Omega)} \leq c_2 h^{d/2} \|\mathbf{x}\|_2 \qquad \forall \mathbf{x} \in \mathbb{R}^n, \tag{2.2.1a}$$

for the isomorphism $\Phi : \mathbb{R}^n \to S^{1,1}(\mathcal{T})$, $\mathbf{x} \mapsto \sum_{j=1}^n \mathbf{x}_j \xi_j$.

Definition 2.2.5. On $T \in \mathcal{T}$, the lowest order local Nédélec spaces of first kind is defined as

$$\mathcal{N}_1^I(T) = \{\mathbf{a} + \mathbf{b} \times \mathbf{x} \ : \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}.$$

Let $\boldsymbol{\tau}$ be a unit vector parallel to the edge e, then the edge-based degrees of freedom are given by

$$M_e(\mathbf{U}) := \int_e \mathbf{U} \cdot \boldsymbol{\tau} \, de \quad \forall \text{ edges } e \text{ of } T,$$

i.e., the line integrals of the tangential component over the edges of T, see e.g., [Mon03, Sec. 5.5.1], [BBF13, Sec. 2.3.2].

Definition 2.2.6. Let $T \in \mathcal{T}$ be an arbitrary tetrahedron and e be an edge of T with endpoints V_1, V_2. Then, the nodal basis of $\mathcal{N}_1^I(T)$ is defined based on the following edge-based shape functions

$$\Phi_e = \lambda_{V_1} \nabla \lambda_{V_2} - \lambda_{V_2} \nabla \lambda_{V_1},$$

where λ_{V_i} is the barycentric coordinate function associated with vertex V_i, see, e.g., [Mon03, Sec. 5.5.1].

We set

$$\begin{aligned} \mathbf{X}_h(\mathcal{T}, \Omega) &:= \{\mathbf{U}_h \in \mathbf{H}(\mathrm{curl}, \Omega) : \ \mathbf{U}_h|_T \in \mathcal{N}_1^I(T) \quad \forall T \in \mathcal{T}\}, \\ \mathbf{X}_{h,0}(\mathcal{T}, \Omega) &:= \mathbf{X}_h(\mathcal{T}, \Omega) \cap \mathbf{H}_0(\mathrm{curl}, \Omega). \end{aligned}$$

Since the standard degrees of freedom of $\mathbf{X}_h(\mathcal{T}, \Omega)$ are the line integrals of the tangential component of $\mathbf{U}_h$ on the edges of $\mathcal{T}$, the dimension of $\mathbf{X}_h(\mathcal{T}, \Omega)$ is the number of edges of $\mathcal{T}$. The standard basis of $\mathbf{X}_h(\mathcal{T}, \Omega)$ consists of the so-called (lowest order) edge elements $\mathcal{X}_h := \{\Psi_e\}$, where for each edge e, the function $\Psi_e \in \mathbf{X}_h(\mathcal{T}, \Omega)$ is defined on the tetrahedra sharing e as an edge, as in Definition 2.2.6 and is supported by the union of the tetrahedra sharing edge e.

A basis $\mathcal{X}_{h,0} := \{\Psi_1, \ldots, \Psi_N\}$ of $\mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ with $N := \dim \mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ is obtained by taking the $\Psi_e \in \mathcal{X}_h$ whose edge e satisfies $e \subset \Omega$; that is, $\mathcal{X}_{h,0}$ is obtained from $\mathcal{X}_h$ by removing the shape functions associated with edges lying on Γ.

2.3 Triangulation of $\partial\Omega$

We additionally need to define triangulations of the boundary Γ.

Definition 2.3.1. A set $\mathcal{K}$ is called a regular triangulation of Γ if

1. Each $K \in \mathcal{K}$ is an open line segment ($d = 2$) or an open triangle ($d = 3$) in $\mathbb{R}^d$.
2. $\mathcal{K}$ is regular in the Ciarlet sense, i.e., for two elements K, $K' \in \mathcal{K}$, the intersection of $\overline{K} \cap \overline{K'}$ is either empty, a common vertex ($d \geq 2$) or a joint edge ($d = 3$).
3. The union of all elements cover Γ, i.e., $\overline{\Gamma} = \bigcup_{K \in \mathcal{K}} \overline{K}$.

Definition 2.3.2. Let $\mathcal{K}$ be a triangulation of Γ. For $d = 2$, $\mathcal{K}$ is called γ-shape regular if

$$\max_{K \in \mathcal{K}} \max_{K' \in \mathcal{K}} \frac{|K|}{|K'|} \leq \gamma,$$

and for $d = 3$, $\mathcal{K}$ is called γ-shape regular if

$$\max_{K \in \mathcal{K}} (\mathrm{diam}(K)^2 / |K|) \leq \gamma.$$

Furthermore, a γ-shape regular triangulation $\mathcal{K}$ is called quasi-uniform, if there exists $C > 0$ such that

$$\max_{K \in \mathcal{K}} (\mathrm{diam}(K)) \leq C \min_{K \in \mathcal{K}} (\mathrm{diam}(K)).$$

Let $\mathcal{K}$ denote a regular (in the sense of Ciarlet) and γ-shape regular triangulation of Γ. Then, we have the following definition for the space of piecewise constant functions on $\mathcal{K}$.

Definition 2.3.3. Let $\mathcal{K}$ denote a regular (in the sense of Ciarlet) and γ-shape regular triangulation of Γ. Let $P_p(K)$ be the space of polynomials of (maximal) degree $p \geq 0$ on the element $K \in \mathcal{K}$. The space of piecewise constant functions on $\mathcal{K}$ is defined as

$$S^{0,0}(\mathcal{K}) := \{u \in L^2(\Gamma) : u|_K \in P_0(K) \quad \forall K \in \mathcal{K}\}.$$

For a quasi-uniform triangulation $\mathcal{K}$ of Γ with the mesh size h, since the results of Chapter 5 are devised for matrices, assumptions on the basis of $S^{0,0}(\mathcal{K})$ are required to be imposed. Therefore, we let $\mathcal{W}_h := \{\chi_j : j = 1, \ldots, m\}$ be the basis of $S^{0,0}(\mathcal{K})$ that consists of the characteristic functions of the surface elements. This basis features the following norm equivalences:

$$c_3 h^{(d-1)/2} \|\mathbf{y}\|_2 \leq \|\Psi \mathbf{y}\|_{L^2(\Gamma)} \leq c_4 h^{(d-1)/2} \|\mathbf{y}\|_2 \quad \forall \mathbf{y} \in \mathbb{R}^m, \tag{2.3.1a}$$

for the isomorphism $\Psi : \mathbb{R}^m \to S^{0,0}(\mathcal{K})$, $\mathbf{y} \mapsto \sum_{j=1}^m \mathbf{y}_j \chi_j$.

2.4 (quasi-) interpolation

Let V_h be a finite dimensional subspace of $L^2(\Omega)$. Then the $L^2(\Omega)$-orthogonal projection $\Pi_h^{L^2} : L^2(\Omega) \to V_h$ is defined by

$$\left\langle \Pi_h^{L^2} u - u, \varphi_h \right\rangle_{L^2(\Omega)} = 0 \qquad \forall \varphi_h \in V_h.$$

Using $\varphi_h = \Pi_h^{L^2} u$ as the test function and applying the Cauchy-Schwarz inequality give rise to the following stability estimate

$$\left\| \Pi_h^{L^2} u \right\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}.$$

Analogously, the $L^2(\Gamma)$-orthogonal projection $I_h^\Gamma : L^2(\Gamma) \to W_h$, for a finite dimensional space $W_h \subset L^2(\Gamma)$, is defined as

$$\left\langle I_h^\Gamma v - v, \psi_h \right\rangle_{L^2(\Gamma)} = 0 \qquad \forall \psi_h \in W_h. \tag{2.4.1}$$

Let $\mathcal{K}$ denote a γ-shape regular quasi-uniform triangulation of Γ, $u \in L^2(\Gamma)$ and $u|_K \in H^1(K)$, for all $K \in \mathcal{K}$. The $L^2(\Gamma)$-orthogonal projection $I_h^\Gamma : L^2(\Gamma) \to S^{0,0}(\mathcal{K})$ has the following approximation property

$$\left\| u - I_h^\Gamma u \right\|_{L^2(K)} \leq Ch \, |u|_{H^1(K)}, \tag{2.4.2}$$

where $C > 0$ depends only on the shape-regularity of the triangulation $\mathcal{K}$, see e.g. [SS11, Eq. 4.51].

In the following, we mention a classical approximation result, so-called super-approximation, see, e.g., [NS74, Wah91].

Lemma 2.4.1. *Let $\mathcal{K}$ be a quasi-uniform triangulation of Γ and $I_h^\Gamma : L^2(\Gamma) \to S^{0,0}(\mathcal{K})$ be the $L^2(\Gamma)$-orthogonal projection. Then, there is $C > 0$ depending only on the shape-regularity of the triangulation and Γ such that for any discrete function $\psi_h \in S^{0,0}(\mathcal{K})$ and any $\eta \in W^{1,\infty}(\Gamma)$*

$$\left\|\eta\psi_h - I_h^\Gamma(\eta\psi_h)\right\|_{H^{-1/2}(\Gamma)} \leq Ch^{3/2} \left\|\nabla\eta\right\|_{L^\infty(\Gamma)} \left\|\psi_h\right\|_{L^2(\Gamma\cap \mathrm{supp}(\eta))}. \tag{2.4.3}$$

Proof. The main observation is that, on each element $K \in \mathcal{K}$, we have $\nabla\psi_h|_K \equiv 0$. Therefore, the standard approximation result (2.4.2) reduces to

$$\left\|\eta\psi_h - I_h^\Gamma(\eta\psi_h)\right\|_{L^2(K)} \lesssim h \left\|\nabla(\eta\psi_h)\right\|_{L^2(K)} \lesssim h \left\|\nabla(\eta)\psi_h\right\|_{L^2(K)}. \tag{2.4.4}$$

Since I_h^Γ is the L^2-projection, one can write

$$\left\|\varphi - I_h^\Gamma\varphi\right\|_{L^2(\Gamma)} \lesssim h \left\|\varphi\right\|_{H^{1/2}(\Gamma)}, \tag{2.4.5}$$

see e.g. [SS11, Eq. 4.58]. Combining the above equations, we obtain the following approximation in the $H^{-1/2}(\Gamma)$-norm

$$\begin{aligned}
\left\|\eta\psi_h - I_h^\Gamma(\eta\psi_h)\right\|_{H^{-1/2}(\Gamma)} &\leq \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\langle\eta\psi_h - I_h^\Gamma(\eta\psi_h),\varphi\rangle_{L^2(\Gamma)}}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\
&\overset{Eq.\,(2.4.1)}{=} \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\langle\eta\psi_h - I_h^\Gamma(\eta\psi_h),\varphi - I_h^\Gamma\varphi\rangle_{L^2(\Gamma)}}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\
&\lesssim \left\|\eta\psi_h - I_h^\Gamma(\eta\psi_h)\right\|_{L^2(\Gamma)} \sup_{\varphi\in H^{1/2}(\Gamma)} \frac{\left\|\varphi - I_h^\Gamma\varphi\right\|_{L^2(\Gamma)}}{\|\varphi\|_{H^{1/2}(\Gamma)}} \\
&\overset{Eq.\,(2.4.5)}{\lesssim} h^{1/2} \left\|\eta\psi_h - I_h^\Gamma(\eta\psi_h)\right\|_{L^2(\Gamma)} \\
&\overset{Eq.\,(2.4.4)}{\lesssim} h^{3/2} \sqrt{\sum_{K\in\mathcal{K}} \|\nabla(\eta\psi_h)\|_{L^2(K)}^2},
\end{aligned}$$

which gives us the desired result. □

Another main tool that we need in the following chapters is the nodal interpolation operator $I_h^\Omega : C(\overline{\Omega}) \to S^{1,1}(\mathcal{T})$. For γ-shape regular quasi-uniform triangulation $\mathcal{T}$ of $\Omega \subset \mathbb{R}^d$, we denote $H^2_{\mathrm{pw}}(\Omega) := \{u \in L^2(\Omega) \quad : \quad u|_T \in H^2(T) \quad T \in \mathcal{T}\}$. Since $d/2 < 2$ for $d \in \{1, 2, 3\}$, the nodal interpolation operator has the following local approximation property [BS02, Thm. 4.4.4]

$$\left\|u - I_h^\Omega u\right\|_{H^k(T)}^2 \leq Ch^{2(2-k)} \left|u\right|_{H^2(T)}^2 \qquad \forall u \in C(\overline{\Omega}) \cap H^2_{\mathrm{pw}}(\Omega), \quad 0 \leq k \leq 2, \tag{2.4.6}$$

where $C > 0$ depends only on the shape-regularity of the triangulation $\mathcal{T}$.

Another super-approximation result holds for the nodal interpolation operators.

Lemma 2.4.2. *Let $\mathcal{T}$ be a quasi-uniform triangulation of Ω and $I_h^\Omega : C(\overline{\Omega}) \to S^{1,1}(\mathcal{T})$ be the nodal interpolation operator, then the following super-approximation result holds*

$$\left\|\eta v_h - I_h^\Omega(\eta v_h)\right\|_{H^k(\Omega)} \lesssim h^{2-k} \Big(\|\nabla\eta\|_{L^\infty(\Omega)} \|\nabla v_h\|_{L^2(\Omega\cap\mathrm{supp}(\eta))} + \left\|D^2\eta\right\|_{L^\infty(\Omega)} \|v_h\|_{L^2(\Omega\cap\mathrm{supp}(\eta))}\Big), \tag{2.4.7}$$

for any discrete function $v_h \in S^{1,1}(\mathcal{T})$, any $\eta \in W^{2,\infty}(\Omega)$, and $k = 0, 1$, where $H^0(\Omega) := L^2(\Omega)$.

Proof. On each element $T \in \mathcal{T}$, we have $D^2 v_h|_T \equiv 0$. Therefore, the standard approximation result (2.4.6) reduces to

$$\left\|\eta v_h - I_h^\Omega(\eta v_h)\right\|_{H^k(T)} \lesssim h^{2-k} |\eta v_h|_{H^2(T)} \lesssim h^{2-k} \left|D^2(\eta v_h)\right|_{L^2(T)},$$

which concludes the proof. □

2.4.1 Scott-Zhang operators on Ω

Let $\mathcal{T}$ be a regular (in the sense of Ciarlet) and γ-shape regular triangulation of Ω. The Scott-Zhang projection $I^{SZ} : H^1(\Omega) \to S^{p,1}(\mathcal{T})$ is a quasi-interpolation operator that preserves homogeneous boundary conditions naturally, i.e., $u|_\Gamma = 0$ implies $I^{SZ}u|_\Gamma = 0$. In this subsection, we recall the basic construction of the Scott-Zhang operator of [SZ90a] or [BS02, Sec. 4.8]. It will be convenient to use Lagrange bases of the space $S^{p,1}(\mathcal{T})$ from Definition 2.2.4.

1. The basis functions $\varphi_{z,\mathcal{T}}$ have the following support properties: a) if $z \in T$ for some $T \in \mathcal{T}$, then $\mathrm{supp}\,\varphi_{z,\mathcal{T}} \subset \overline{T}$; b) if $z \in f$ for some j-dimensional face $(j < d)$ of T, then $\mathrm{supp}\,\varphi_{z,\mathcal{T}} \subset \omega_f$, where $\omega_f = \mathrm{int} \bigcup\{\overline{T} \colon f \text{ is } j\text{-face of } T \in \mathcal{T}'\}$. In particular, if $z \notin \overline{T}$, then $\mathrm{supp}\,\varphi_{z,\mathcal{T}} \cap T = \emptyset$.
2. For each element $T \in \mathcal{T}$, one has a dual basis $\{\varphi^*_{z,T} : z \in \overline{T}\} \subset P_p(T)$ of $P_p(T)$, i.e., $\int_T \varphi^*_{z,T}\varphi_{z',\mathcal{T}} = \delta_{z,z'}$ for all nodes z, $z' \in \overline{T}$. In particular, this gives
$$\int_T \varphi^*_{z,T} u\,dx = u(z) \qquad \forall T \in \mathcal{T} \quad \forall u \in P_p(T). \tag{2.4.8}$$
3. For each node $z \in \mathcal{N}_p(\mathcal{T})$, define the *admissible set of averaging elements* as $\mathcal{A}(z,\mathcal{T}) := \{T \in \mathcal{T} \colon z \in \overline{T}\}$. A Scott-Zhang operator is then defined by selecting, for each z, a $T_z \in \mathcal{A}(z,\mathcal{T})$ and setting
$$I^{SZ}u := \sum_{z\in\mathcal{N}_p(\mathcal{T})} \varphi_{z,\mathcal{T}} \left(\int_{T_z} \varphi^*_{z,T_z} u\,dx\right). \tag{2.4.9}$$

For nodes z that are on the boundary of an element, the admissible set $\mathcal{A}(z,\mathcal{T})$ has more than one element. However, from (2.4.8), we get that the values of the functionals coincide on $S^{p,1}(\mathcal{T})$:

$$\int_{T_z} \varphi^*_{z,T_z} u\,dx = u(z) = \int_{T'_z} \varphi^*_{z,T'_z} u\,dx \qquad \forall T_z, T'_z \in \mathcal{A}(z,\mathcal{T}) \quad \forall u \in S^{p,1}(\mathcal{T}). \tag{2.4.10}$$

We also highlight that (2.4.8) implies that I^{SZ} is a projection onto $S^{p,1}(\mathcal{T})$. In order to state the properties of the Scott-Zhang projections, first we need to define the element patches. For $T \in \mathcal{T}$ and $k \in \mathbb{N}$, we inductively define the element patches

$$\omega^0(T) := T, \qquad \omega^k(T) := \text{interior}\left(\bigcup\{\overline{T'}\colon T' \in \mathcal{T}, \overline{T'} \cap \overline{\omega^{k-1}(T)} \neq \emptyset\}\right),$$

and for the first order patch, we abbreviate $\omega(T) := \omega^1(T)$.

Lemma 2.4.3. *[EG17, Lem. 1.130] The Scott–Zhang projection has the following properties:*

1. *The stability in L^2-norm and H^1-semi-norm*

$$\left|I^{SZ}u\right|_{H^\ell(T)} \leq Ch^2 \left|u\right|_{H^\ell(\omega(T))}, \qquad \ell \in \{0,1\}.$$

2. *The local approximation property*

$$\left\|u - I^{SZ}u\right\|^2_{H^\ell(T)} \leq Ch^{2(m-\ell)} \left|u\right|^2_{H^m(\omega(T))}, \qquad 0 \leq \ell \leq 1, \quad \ell \leq m \leq p+1. \quad (2.4.11)$$

where the constant $C > 0$ depends only on γ-shape regularity of the triangulation $\mathcal{T}$.

2.4.2 Scott-Zhang operators on $\mathbb{R}^d$

Given Ω, let $\mathcal{R}_H$ be a quasi-uniform (infinite) triangulation of $\mathbb{R}^d$ (into open simplices $R \in \mathcal{R}_H$) with mesh width H that conforms to Ω, i.e., every $R \in \mathcal{R}_H$ satisfies either $R \subset \Omega$ or $R \subset \Omega^{\text{ext}}$ and the restrictions $\mathcal{R}_H|_\Omega$ and $\mathcal{R}_H|_{\Omega^{\text{ext}}}$ are regular and γ-shape regular triangulations of Ω and Ω^{ext} of mesh size H, respectively.

For $s > 0$, we define the space $H^s(\mathbb{R}^d\backslash\Gamma)$ as

$$H^s(\mathbb{R}^d\backslash\Gamma) := \left\{u \in L^2(\mathbb{R}^d) \quad : \quad u|_\Omega \in H^s(\Omega),\, u|_{\Omega^{\text{ext}}} \in H^s(\Omega^{\text{ext}})\right\}.$$

Furthermore, the space $L^2(\mathbb{R}^d\backslash\Gamma)$ is defined as

$$L^2(\mathbb{R}^d\backslash\Gamma) := \left\{u \in L^1_{\text{loc}}(\mathbb{R}^d) \quad : \quad u|_\Omega \in L^2(\Omega),\, u|_{\Omega^{\text{ext}}} \in L^2(\Omega^{\text{ext}})\right\}.$$

With the Scott-Zhang projections I_H^{int}, I_H^{ext} for the grids $\mathcal{R}_H|_\Omega$ and $\mathcal{R}_H|_{\Omega^c}$, we define the operator $I_H^{\text{pw}} : H^1(\mathbb{R}^d\backslash\Gamma) \to S^{1,1}_{\text{pw}}(\mathcal{R}_H) := \{v : v|_\Omega \in S^{1,1}(\mathcal{R}_H|_\Omega) \text{ and } v|_{\Omega^{\text{ext}}} \in S^{1,1}(\mathcal{R}_H|_{\Omega^{\text{ext}}})\}$ in a piecewise fashion by

$$I_H^{\text{pw}} v = \begin{cases} I_H^{\text{int}} v & \text{on } \Omega, \\ I_H^{\text{ext}} v & \text{on } \Omega^{\text{ext}}. \end{cases} \qquad (2.4.12)$$

We denote the patch of an element $R \in \mathcal{R}_H$ by

$$\omega_R^\Omega := \text{interior}\left(\bigcup\{\overline{R'} :\ R' \in \mathcal{R}_H|_\Omega \quad \text{s.t.} \quad \overline{R} \cap \overline{R'} \neq \emptyset\}\right),$$
$$\omega_R^{\Omega^{\text{ext}}} := \text{interior}\left(\bigcup\{\overline{R'} :\ R' \in \mathcal{R}_H|_{\Omega^{\text{ext}}} \quad \text{s.t.} \quad \overline{R} \cap \overline{R'} \neq \emptyset\}\right).$$

Lemma 2.4.4. *The Scott-Zhang projection reproduces piecewise affine functions and has the following local approximation property for piecewise H^s functions:*

$$\|v - I_H^{\mathrm{pw}} v\|_{H^t(R)}^2 \leq C H^{2(s-t)} \begin{cases} |v|_{H^s(\omega_R^\Omega)}^2 & \text{if } R \subset \Omega \\ |v|_{H^s(\omega_R^{\Omega^{\mathrm{ext}}})}^2 & \text{if } R \subset \Omega^{\mathrm{ext}} \end{cases} \quad t, s \in \{0,1\}, \quad 0 \leq t \leq s \leq 1, \tag{2.4.13}$$

with a constant C depending only on the shape-regularity of $\mathcal{R}_H$ and d.

2.5 Abstract Additive Schwarz Framework

The additive Schwarz technique creates an abstract framework to design preconditioners based on stable decompositions of a finite dimensional Hilbert space V, using the subspaces V_ℓ , $\ell = 0, \cdots, L$. The main references for this section are [Osw94] and [TW05].

Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be a symmetric positive definite bilinear form and $\langle \cdot,\cdot \rangle_V$ denote the inner product in V. We consider the problem of finding $u \in V$ such that

$$a(u, v) = \phi(v) \qquad \forall v \in V, \tag{2.5.1}$$

where $\phi(v) := (f, v)_V$ is a linear functional on V and $f \in V$ is given. This equation is equivalent to the following linear operator equation

$$\mathcal{A}u = f,$$

where $\mathcal{A} : V \longrightarrow V$ is a symmetric positive definite operator defined by $\langle \mathcal{A}u, v\rangle_V = a(u, v)$. Let $\{V_\ell, \quad \ell = 0, \cdots, L\}$ be a family of finite dimensional spaces and $R_\ell^T : V_\ell \to V$ be the natural inclusion which we call prolongation. We assume that V can be decomposed as

$$V = R_0^T V_0 + \sum_{\ell=1}^{L} R_\ell^T V_\ell.$$

Let $\widetilde{a}_\ell(\cdot,\cdot) : V_\ell \times V_\ell \to \mathbb{R}$ be a symmetric positive definite bilinear form defined as

$$\widetilde{a}_\ell(u_\ell, v_\ell) = \langle \mathcal{A}_\ell u_\ell, v_\ell \rangle_V \qquad \forall u_\ell, v_\ell \in V_\ell$$

where $\mathcal{A}_\ell : V_\ell \to V_\ell$ is a linear symmetric positive definite operator. Moreover, we introduce the linear operator $\widetilde{P}_\ell : V \to V_\ell$ given by

$$\widetilde{a}_\ell(\widetilde{P}_\ell u, v_\ell) = a(u, R_\ell^T v_\ell) \qquad v_\ell \in V_\ell.$$

Let $R_\ell : V \to V_\ell$ be the corresponding adjoint of R_ℓ^T with respect to the inner product $\langle \cdot,\cdot \rangle_V$, i.e.,

$$\langle R_\ell^T u_\ell, v\rangle_V = \langle u_\ell, R_\ell v\rangle_V \qquad \forall u_\ell \in V_\ell,\ v \in V.$$

We should note that the symmetry and positive definiteness of $\widetilde{a}_\ell$ ensure that the operator $\widetilde{P}_\ell$ is well defined and can be written equivalently as

$$\widetilde{P}_\ell = \mathcal{A}_\ell^{-1} R_\ell \mathcal{A}. \tag{2.5.2}$$

Then, *Schwarz operators* are defined in terms of projection-like operators

$$\mathcal{P}_\ell := R_\ell^T \widetilde{P}_\ell : V \to R_\ell^T V_\ell \subset V, \quad \ell = 0, \cdots, L,$$

and the *additive Schwarz operator* is defined by $\mathcal{P}_{AS}^L := \sum_{\ell=0}^L \mathcal{P}_\ell$. Considering the definition of $\mathcal{P}_\ell$ from (2.5.2), $\mathcal{P}_{AS}^L$ can also be written in the following form

$$\mathcal{P}_{AS}^L = \sum_{\ell=0}^L R_\ell^T \mathcal{A}_\ell^{-1} R_\ell \mathcal{A}. \tag{2.5.3}$$

Definition 2.5.1. We define the condition number of $\mathcal{P}_{AS}^L$ as

$$\kappa(\mathcal{P}_{AS}^L) = \frac{\lambda_{\max}(\mathcal{P}_{AS}^L)}{\lambda_{\min}(\mathcal{P}_{AS}^L)},$$

where

$$\lambda_{\max}(\mathcal{P}_{AS}^L) = \sup_{u\in V} \frac{a(\mathcal{P}_{AS}^L u, u)}{a(u,u)}, \qquad \lambda_{\min}(\mathcal{P}_{AS}^L) = \inf_{u\in V} \frac{a(\mathcal{P}_{AS}^L u, u)}{a(u,u)}.$$

In order to provide an upper bound for the condition number of the additive Schwarz operator, first we need to mention the following assumptions.

Assumption 2.5.2. (Stable decomposition) *There exists a constant C_0, such that each $u \in V$ has the decomposition*

$$u = \sum_{\ell=0}^L R_\ell^T u_\ell \qquad u_\ell \in V_\ell \quad \ell = 0, \cdots, L,$$

that satisfies

$$\sum_{\ell=0}^L \widetilde{a}_\ell(u_\ell, u_\ell) \le C_0^2 a(u,u).$$

Assumption 2.5.3. (Strengthened Cauchy-Schwarz inequalities) *There exists constants $0 \le \varepsilon_{i,j} \le 1$, $1 \le i, j \le L$ such that*

$$\left|a(R_i^T u_i, R_j^T u_j)\right| \le \varepsilon_{i,j} a(R_i^T u_i, R_i^T u_i)^{1/2} a(R_j^T u_j, R_j^T u_j)^{1/2} \qquad \forall u_i \in V_i, \quad u_j \in V_j.$$

Assumption 2.5.4. (Local stability) *There exist $\zeta > 0$ such that*

$$a(R_\ell^T u_\ell, R_\ell^T u_\ell) \le \zeta \widetilde{a}_\ell(u_\ell, u_\ell) \qquad u_\ell \in range(\widetilde{P}_\ell) \subset V_\ell \qquad 0 \le \ell \le L.$$

Theorem 2.5.5. *[TW05, Theorem 2.7] Let Assumptions 2.5.2-2.5.4 be satisfied. Then, the condition number of the additive Schwarz operator is bounded by*

$$\kappa(\mathcal{P}_{AS}^L) \le C_0^2 \zeta(\rho(\varepsilon) + 1),$$

where $\rho(\varepsilon)$ is the spectral radius of $\varepsilon := \{\varepsilon_{i,j}\}$.

2.6 Hierarchical Matrices

The main idea of $\mathcal{H}$-matrices is to store certain far field blocks of the matrix efficiently as low-rank matrices. This can be done by appropriate partitioning of the product index set into a so-called block cluster tree such that the restriction of the matrix to the blocks of this partitioning is either small or a low rank matrix. In order to choose blocks that are suitable for compression, we need to introduce the concept of admissibility.

Let V_h be a finite dimensional Hilbert space and $\{\zeta_j : j = 1, \dots, M\}$ be a basis for V_h. Let $a(\cdot, \cdot) : V_h \times V_h \to \mathbb{C}$ be a bilinear form and $\mathbf{A} \in \mathbb{C}^{M \times M}$ be the corresponding Galerkin matrix with $\mathbf{A}_{i,j} = a(\zeta_j, \zeta_i)$. Let $|\tau|$ denote the cardinality of the finite set τ. Then, we have the following definitions:

Definition 2.6.1 (Cluster, cluster tree)**.** A *cluster* τ is a subset of the index set $\mathcal{I} = \{1, 2, ..., M\}$. A *cluster tree* with *leaf size* $n_{\text{leaf}} \in \mathbb{N}$ is a binary tree $\mathbb{T}_\mathcal{I}$ with root $\mathcal{I}$ such that each cluster $\tau \in \mathbb{T}_\mathcal{I}$ is either a leaf of the tree and satisfies $|\tau| \leq n_{\text{leaf}}$, or there exist disjoint subsets $\tau_1, \tau_2 \in \mathbb{T}_\mathcal{I}$ of τ, so-called sons, with $\tau = \tau_1 \cup \tau_2$. We denote the set of sons of τ by $\mathcal{S}(\tau) := \{\tau_1, \tau_2\}$.

Definition 2.6.2 (Level function, depth of a cluster tree and balanced tree)**.** The *level function* $\text{level} : \mathbb{T}_\mathcal{I} \to \mathbb{N}_0$ is inductively defined by $\text{level}(\mathcal{I}) = 0$ and $\text{level}(\tau') := \text{level}(\tau) + 1$ for τ' a son of τ. The *depth* of a cluster tree is $\text{depth}(\mathbb{T}_\mathcal{I}) := \max_{\tau \in \mathbb{T}_\mathcal{I}} \text{level}(\tau)$. We call a tree *balanced* if the sons of each cluster possess the same number of indices.

Definition 2.6.3 (Bounding boxes and η-admissibility)**.** For a cluster $\tau \subset \mathcal{I}$, the axis-parallel $B_{R_\tau} \subseteq \mathbb{R}^d$ is called a bounding box if B_{R_τ} is a hypercube with side length R_τ and $\cup_{i \in \tau} \operatorname{supp} \zeta_i \subseteq B_{R_\tau}$.
For $\eta > 0$, a pair of clusters (τ, σ) with $\tau, \sigma \subset \mathcal{I}$ is called η-admissible if there exist bounding boxes B_{R_τ} and B_{R_σ} such that

$$\max\{\text{diam}(B_{R_\tau}), \text{diam}(B_{R_\sigma})\} \leq \eta \ \text{dist}(B_{R_\tau}, B_{R_\sigma}), \tag{2.6.1}$$

where $\text{dist}(B_{R_\tau}, B_{R_\sigma}) := \inf \{\|x - y\|_2 \quad : \quad x \in B_{R_\tau},\ y \in B_{R_\sigma}\}$.

Remark 2.6.4. If $\mathbf{A}$ is a symmetric matrix, then we are allowed to use a weaker admissibility condition, i.e., $\min\{\text{diam}(B_{R_\tau}), \text{diam}(B_{R_\sigma})\} \leq \eta \ \text{dist}(B_{R_\tau}, B_{R_\sigma})$. ■

Definition 2.6.5 (block cluster tree, sparsity constant and partition)**.** Let $\mathbb{T}_\mathcal{I}$ be a cluster tree with root $\mathcal{I}$ and $\eta > 0$ be a fixed admissibility parameter. The block cluster tree $\mathbb{T}_{\mathcal{I} \times \mathcal{I}}$ is a tree constructed recursively from the root $\mathcal{I} \times \mathcal{I}$ such that for each block $\tau \times \sigma \in \mathbb{T}_{\mathcal{I} \times \mathcal{I}}$ with τ, $\sigma \in \mathbb{T}_\mathcal{I}$, the set of sons of $\tau \times \sigma$ is defined as

$$\mathcal{S}(\tau \times \sigma) := \begin{cases} \emptyset & \text{if } \tau \times \sigma \text{ is } \eta\text{-admissible or } \mathcal{S}(\tau) = \emptyset \text{ or } \mathcal{S}(\sigma) = \emptyset, \\ \mathcal{S}(\tau) \times \mathcal{S}(\sigma) & \text{else.} \end{cases}$$

Replacing the largest possible matrix blocks by low-rank approximations allows us to keep the computational complexity and memory requirements low. One possible way to achieve this goal is to apply the admissibility condition and identify the admissible cluster pairs, which gives rises to the following definitions of the far-field and near-field sets and the sparsity constant.

Definition 2.6.6 (Far-field, near-field, and sparsity constant)**.** The leaves of the block cluster tree induce a partition P of the set $\mathcal{I} \times \mathcal{I}$. For such a partition P and a fixed admissibility parameter $\eta > 0$, we define the *far-field* and the *near-field* as

$$P_{\text{far}} := \{(\tau, \sigma) \in P \;:\; (\tau, \sigma) \text{ is } \eta\text{-admissible}\}, \quad P_{\text{near}} := P \setminus P_{\text{far}}. \tag{2.6.2}$$

The *sparsity constant* C_{sp} of such a partition was introduced in [Gra01, Def. 5.3] as

$$C_{\text{sp}} := \max\left\{ \max_{\tau \in \mathbb{T}_{\mathcal{I}}} |\{\sigma \in \mathbb{T}_{\mathcal{I}} \;:\; \tau \times \sigma \in \mathbb{T}_{\mathcal{I} \times \mathcal{I}}\}|\,, \max_{\sigma \in \mathbb{T}_{\mathcal{I}}} |\{\tau \in \mathbb{T}_{\mathcal{I}} \;:\; \tau \times \sigma \in \mathbb{T}_{\mathcal{I} \times \mathcal{I}}\}| \right\}. \tag{2.6.3}$$

A partition P of $\mathcal{I} \times \mathcal{I}$ is said to be based on the cluster tree $\mathbb{T}_{\mathcal{I}}$ if it satisfies the conditions of Def. 2.6.5.

Definition 2.6.7. Let P be a partition of $\mathcal{I} \times \mathcal{I}$ based on the cluster tree $\mathbb{T}_{\mathcal{I}}$. Then, P is called *sparse* if $\operatorname{depth}(\mathbb{T}_{\mathcal{I}}) \lesssim \log(M)$ and $C_{\text{sp}} \lesssim 1$.

Now, we need to define the notion of the concentric boxes.

Definition 2.6.8. (Concentric boxes) Two (quadratic) boxes B_R and $B_{R'}$ of side length R and R' are said to be concentric if they have the same barycenter and B_R can be obtained by stretching of $B_{R'}$ by the factor R/R' taking their common barycenter as the origin.

For clusters τ, $\sigma \subset \mathcal{I}$, we adopt the notation

$$\mathbb{C}^{\tau} := \{\mathbf{x} \in \mathbb{C}^M \colon \mathbf{x}_i = 0 \quad \text{if } i \notin \tau\},$$
$$\mathbb{C}^{\tau \times \sigma} := \{\mathbf{A} \in \mathbb{C}^{M \times M} \colon \mathbf{A}_{ij} = 0 \quad \text{if } i \notin \tau \text{ or } j \notin \sigma\}.$$

For $\mathbf{x} \in \mathbb{C}^M$ and $\mathbf{A} \in \mathbb{C}^{M \times M}$, the restrictions $\mathbf{x}|_\tau$ and $\mathbf{A}_{\tau \times \sigma}$ are understood as $(\mathbf{x}|_\tau)_i = \chi_\tau(i)\mathbf{x}_i$ and $(\mathbf{A}|_{\tau \times \sigma})_{ij} = \chi_\tau(i)\chi_\sigma(j)\mathbf{A}_{ij}$, where χ_τ and χ_σ are the characteristic functions of the sets τ, σ. For integers $r \in \mathbb{N}$, matrices $\mathbb{C}^{\tau \times r}$ are understood as matrices in $\mathbb{C}^{M \times r}$ such that each column is in $\mathbb{C}^{\tau}$. In the following, we present the definition of $\mathcal{H}$-matrices.

Definition 2.6.9 ($\mathcal{H}$-matrices)**.** Let P be a partition of $\mathcal{I} \times \mathcal{I}$ based on a cluster tree $\mathbb{T}_{\mathcal{I}}$ and $\eta > 0$. A matrix $\mathbf{B}_{\mathcal{H}} \in \mathbb{C}^{M \times M}$ is an $\mathcal{H}$-matrix, if for every admissible block $(\tau, \sigma) \in P_{\text{far}}$, we have a rank r factorization

$$\mathbf{B}_{\mathcal{H}}|_{\tau \times \sigma} = \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H,$$

where $\mathbf{X}_{\tau\sigma} \in \mathbb{C}^{\tau \times r}$ and $\mathbf{Y}_{\tau\sigma} \in \mathbb{C}^{\sigma \times r}$.

There are several ways to construct a cluster tree $\mathbb{T}_{\mathcal{I}}$. The *cardinality balanced* clustering divides the index cluster into a specific number of sons with the same size with respect to the number of indices, i.e., the bounding boxes are divided such that the new boxes contain the same number of degrees of freedom.

In the *geometric* clustering, the bounding boxes are divided into two boxes by connecting the midpoints of the largest side lengths and the new set of indices are stored as the sons. For details on clustering techniques we refer to [GHLB04], [Hac13, Appendix D] and [Hac15, Sec. 5.4.2]. The

The low-rank structure of the far-field blocks allows for efficient storage of $\mathcal{H}$-matrices as the memory requirement to store an $\mathcal{H}$-matrix is $\mathcal{O}(C_{\mathrm{sp}}\,\mathrm{depth}(\mathbb{T}_{\mathcal{I}})rM)$, i.e, [GH03b, Lem. 2.4]. For the quasi-uniform grids, the standard clustering methods, such as the geometric clustering lead to balanced cluster trees, i.e., $\mathrm{depth}(\mathbb{T}_{\mathcal{I}}) \sim \log(M)$ (see, e.g., [Hac15, Remark 5.19]) and a uniformly (in the mesh size h) bounded sparsity constant. In total this gives a storage complexity of $\mathcal{O}(rM\log(M))$ to construct the matrix $\mathbf{B}_{\mathcal{H}}$ from Definition 2.6.9.

One of the main advantages of $\mathcal{H}$-matrices to other matrix compression techniques is the ability to perform matrix operations such as addition, inversion, LU-factorization and multiplication with logarithmic-linear storage complexity. The mentioned matrix operations exploit the properties of the low-rank blocks and apply a truncation strategy based on the singular value decomposition (SVD) to achieve the storage complexity of $\mathcal{O}(M\log^{\alpha}(M))$. For details, see [BGH03] and [Hac15, Chapter 7].

Finally, the question of approximating the whole block-wise arbitrary matrix $\mathbf{M}$ can be reduced to the question of blockwise approximation.

Lemma 2.6.10. *([Hac15, Lem. 6.32], [Bör10, Lem. 5]) Let $\mathbf{M} \in \mathbb{C}^{M\times M}$ and P be a sparse partition of $\mathcal{I}\times\mathcal{I}$ based on the cluster tree $\mathbb{T}_{\mathcal{I}}$. Moreover, let P be a level-conserving partition, i.e., for all $d := (\tau,\sigma) \in P$, it holds $\mathrm{level}(d) = \mathrm{level}(\tau) = \mathrm{level}(\sigma)$. Then, the following inequalities hold:*

$$\|\mathbf{M}\|_2 \leq C_{\mathrm{sp}} \left(\sum_{\ell=0}^{\infty} \max\{\|\mathbf{M}|_{\tau\times\sigma}\|_2 : (\tau,\sigma)\in P,\ \mathrm{level}(\tau) = \mathrm{level}(\sigma) = \ell\} \right), \tag{2.6.4}$$

$$\|\mathbf{M}\|_2 \leq C_{\mathrm{sp}}\,\mathrm{depth}(\mathbb{T}_{\mathcal{I}}) \max\{\|\mathbf{M}|_{\tau\times\sigma}\|_2 : (\tau,\sigma)\in P,\ \mathrm{level}(\tau) = \mathrm{level}(\sigma)\}, \tag{2.6.5}$$

where $\|\cdot\|_2$ denotes the spectral norm.

Throughout this thesis, we always assume P is a level-conserving partition.

3 A multilevel decomposition based on mesh hierarchies generated by NVB

The Scott-Zhang projection, originally introduced in [SZ90a], is a very important tool in numerical analysis and has been generalized in various ways, [BG98, GS02, Car99, CH09, Ape99b, Aco01, Ran12, Cia13, FW15, AFF+15, KM15, EG17]. In its classical form, it is quasi-local, it is a projection onto the space of globally continuous, piecewise polynomials, it is stable in both L^2 and H^1 (and thus, by interpolation also in H^s, $s \in (0,1)$), and has optimal approximation properties. Therefore, it is well-suited for the analysis of classical finite element methods (FEMs), [BS02], and plays a key role in the analyses of, e.g., anisotropic finite elements, [Ape99a], adaptive finite element methods, [AFK+13], or mixed methods, [Bad12].

As globally continuous piecewise linear functions are not only in the Sobolev space $H^1(\Omega)$, but also in (fractional) Sobolev spaces $H^{3/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ — in fact, they are in the Besov space $B^{3/2}_{2,\infty}(\Omega)$ — a natural question is whether the operator is also stable in the stronger norms imposed on these spaces.

In this chapter, we prove the stability of local, $L^2(\Omega)$-stable operators with certain approximation properties in $L^2(\Omega)$ on shape-regular meshes in the norm $\|\cdot\|_{B^{3/2}_{2,\infty}(\Omega)}$ including the case of Scott-Zhang operator. We also provide an endpoint stability result for the operators such as the elementwise L^2-projection that map into spaces of piecewise constants, where the corresponding endpoint space is $B^{1/2}_{2,\infty}(\Omega)$.

In this chapter, we develop properties of the finest common coarsening of two given meshes obtained by NVB refinement. We also introduce a modified Scott-Zhang operator for the hierarchy $\widetilde{\mathcal{T}_\ell}$ generated by the finest common coarsening of a fixed mesh $\mathcal{T}$ and the sequence of uniformly refined meshes $\widehat{\mathcal{T}_\ell}$. Finally, based on these modified Scott-Zhang operators and using the mentioned stability result for Scott-Zhang type operators we develop multilevel norm equivalences in Besov spaces up to the endpoint case for standard discrete spaces of globally continuous piecewise polynomials on $\mathcal{T}$.

3.1 Stability of (quasi-) interpolation operators in Besov spaces

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and for the discretization, we assume that a regular (in the sense of Ciarlet) triangulation $\mathcal{T}$ of Ω consisting of open simplices is given. Additionally, $\mathcal{T}$ is assumed to be γ-shape regular. By $h \in L^\infty(\Omega)$, we denote the piecewise constant mesh size function satisfying $h|_T := h_T := |T|^{1/d}$ for $T \in \mathcal{T}$. In the following, we study (quasi-) interpolation operators I_h^m satisfying the following locality, stability and approximation properties.

Assumption 3.1.1. *Let* $m \geq 1$ *and* I_h^m *be an operator* $I_h^m : L^2(\Omega) \to S^{p,m-1}(\mathcal{T})$ *that satisfies:*

(i) Quasi-locality: For every $T \in \mathcal{T}$ *the restriction* $(I_h^m u)|_T$ *depends solely on* $u|_{\omega(T)}$.

(ii) Stability in L^2*: For* $u \in L^2(\Omega)$*, there holds*

$$\|I_h^m u\|_{L^2(T)} \leq C\|u\|_{L^2(\omega(T))}.$$

(iii) Approximation properties of m-th order: For $u \in H^m(\Omega)$*, there holds*

$$\|u - I_h^m u\|_{L^2(T)} \leq C h_T^m \|u\|_{H^m(\omega(T))}.$$

The constants in (ii) and (iii) depend only on Ω*,* d*,* m*,* p*, and the* γ*-shape regularity of* $\mathcal{T}$*.*

We will need mollifiers with certain local approximation properties. Essentially, such operators are given by those classical mollifiers that reproduce, or at least approximate to high order, polynomials of degree p. The following proposition, which is taken from [KM15], provides such operators. Our primary reason for working with this particular class of approximation operators is that the technical complications associated with the boundary of $\partial\Omega$ have been taken care of.

Proposition 3.1.2 ([KM15, Thm. 2.3])**.** *Let* Ω *be a bounded Lipschitz domain and* $p \in \mathbb{N}_0$ *be fixed. For open* $\omega \subset \Omega$ *and* $\varepsilon > 0$ *denote by* $\omega_\varepsilon := \Omega \cap \cup_{x\in\omega} B_\varepsilon(x)$ *the "*ε*-neighbourhood" of* ω*. Then, there exists a constant* $C > 0$ *such that for every* $\varepsilon > 0$ *there is a linear operator* $\mathcal{A}_\varepsilon : L^1_{loc}(\Omega) \to C^\infty(\overline{\Omega})$ *with the following stability and approximation properties for arbitrary open* $\omega \subset \Omega$*:*

(i) If $u \in H^k(\omega_\varepsilon)$ *with* $k \leq p+1$*, then* $\|\mathcal{A}_\varepsilon u\|_{H^\ell(\omega)} \leq C\varepsilon^{-\ell+k}\|u\|_{H^k(\omega_\varepsilon)}$*,* $\quad \ell = k, \dots, p+1$.

(ii) If $u \in H^k(\omega_\varepsilon)$ *with* $k \leq p+1$*, then* $\|u - \mathcal{A}_\varepsilon u\|_{H^\ell(\omega)} \leq C\varepsilon^{k-\ell}\|u\|_{H^k(\omega_\varepsilon)}$*,* $\quad \ell = 0, \dots, k$.

Proof. The proof for the much more technical case of a *variable* length scale function $\varepsilon = \varepsilon(x)$ is given in [KM15, Thm. 2.3]. We give the idea of the proof: in the interior of Ω, the operator $\mathcal{A}_\varepsilon$ has the form $\mathcal{A}_\varepsilon u = u * \rho_\varepsilon$, where the mollifier ρ_ε is such that it reproduces polynomials of degree p (the "classical" mollifier reproduces merely constant functions). Near the boundary, this standard averaging is modified such that $\mathcal{A}_\varepsilon u(x)$ is not obtained by averaging u on $B_\varepsilon(x)$ but by averaging u on the ball $B_\varepsilon(x+\varepsilon b)$ and evaluating the Taylor polynomial of degree p of this averaged function at the point x of interest; the vector b is suitable of size $O(1)$ and it ensures that the averaging is performed inside Ω. □

With the mollifiers from Proposition 3.1.2, we can prove stability and approximation properties for operators satisfying Assumption 3.1.1 in stronger norms.

Lemma 3.1.3. *Let $m \in \{1,2\}$ and $p \geq m-1$. Assume that the linear operator $I_h^m : H^m(\Omega) \to S^{p,m-1}(\mathcal{T})$ satisfies Assumption 3.1.1. Then, there is a constant $C > 0$ depending solely on d, m, p, and the γ-shape-regularity of $\mathcal{T}$ such that for all $T \in \mathcal{T}$ the following stability and approximation properties hold:*

$$\|I_h^m u\|_{H^r(T)} \leq C\|u\|_{H^r(\omega^2(T))}, \qquad r = 0,\dots,m, \tag{3.1.1}$$
$$\|u - I_h^m u\|_{H^r(T)} \leq C h_T^{k-r}\|u\|_{H^k(\omega^2(T))}, \qquad r = 0,\dots,\min\{k,m\}, \quad k = 0,\dots,p+1. \tag{3.1.2}$$

Proof. Let $T \in \mathcal{T}$ be arbitrary. We use the operator $\mathcal{A}_\varepsilon$ of Proposition 3.1.2 with $\omega = \omega(T)$ and $\varepsilon \sim h_T$, such that $\omega_\varepsilon \subset \omega^2(T)$. We write using the triangle inequality

$$\begin{aligned}\|u - I_h^m u\|_{H^r(T)} &\leq \|u - \mathcal{A}_\varepsilon u\|_{H^r(T)} + \|\mathcal{A}_\varepsilon u - I_h^m \mathcal{A}_\varepsilon u\|_{H^r(T)} + \|I_h^m(u - \mathcal{A}_\varepsilon u)\|_{H^r(T)} \\ &=: T_1 + T_2 + T_3.\end{aligned}$$

By Proposition 3.1.2, we have $T_1 \lesssim h_T^{k-r}\|u\|_{H^k(\omega^2(T))}$. A polynomial inverse estimate, see, e.g., [DFG+04], the stability property ii of Assumption 3.1.1, and Proposition 3.1.2 give

$$T_3 \lesssim h_T^{-r}\|u - \mathcal{A}_\varepsilon u\|_{L^2(\omega(T))} \lesssim h_T^{-r} h_T^k \|u\|_{H^k(\omega^2(T))}.$$

In order to estimate T_2, we use a piecewise polynomial $q \in S^{p,m-1}(\mathcal{T})$ with approximation properties in the H^r-norm (e.g., a Clément or Scott-Zhang type interpolation) as given by [BS02, Thm. 4.8.12]. Then,

$$T_2 \leq \|\mathcal{A}_\varepsilon u - u\|_{H^r(T)} + \|u - q\|_{H^r(T)} + \|I_h^m \mathcal{A}_\varepsilon u - q\|_{H^r(T)} =: T_{2,1} + T_{2,2} + T_{2,3}.$$

We have already estimated $T_{2,1} = T_1$. By [BS02, Thm. 4.8.12] (and inspection of the procedure there), we obtain $T_{2,2} \lesssim h_T^{k-r}\|u\|_{H^k(\omega^2(T))}$. Finally, for $T_{2,3}$, we use an inverse estimate

$$T_{2,3} \lesssim h_T^{-r}\|I_h^m \mathcal{A}_\varepsilon u - q\|_{L^2(T)} \lesssim h_T^{-r}\left[\|I_h^m \mathcal{A}_\varepsilon u - \mathcal{A}_\varepsilon u\|_{L^2(T)} + \|\mathcal{A}_\varepsilon u - u\|_{L^2(T)} + \|u - q\|_{L^2(T)}\right].$$

The last two terms have the desired form due to Proposition 3.1.2 and [BS02, Thm. 4.8.12]. For the remaining term, we write with Assumption 3.1.1 iii and Proposition 3.1.2

$$\|I_h^m \mathcal{A}_\varepsilon u - \mathcal{A}_\varepsilon u\|_{L^2(T)} \lesssim h_T^m \|\mathcal{A}_\varepsilon u\|_{H^m(\omega(T))} \lesssim h_T^m h_T^{k-m}\|u\|_{H^k(\omega^2(T))}.$$

Finally, (3.1.1) follows from (3.1.2) by selecting $r = k$. □

The generalization of Proposition 3.1.2 to the case of variable length scale functions from [KM15, Thm. 2.3] can also be used to derive a smooth operator with approximation and stability properties for h-weighted and fractional norms.

Corollary 3.1.4. *With the mesh size function h of $\mathcal{T}$ and $t > 0$, define the function $\overline{h} := \max\{t, h\}$. Let m, $n \in \mathbb{N}_0$ be fixed and $u \in H^m(\Omega)$. Then, for every $t > 0$ there* *exists a linear operator* $J_t : L^2(\Omega) \to C^\infty(\overline{\Omega})$ *with the following stability and approximation properties:*

$$\|\overline{h}^n \nabla^{m+n} J_t u\|_{L^2(\Omega)} \leq C_{m,n} \|u\|_{H^m(\Omega)}, \tag{3.1.3}$$

$$\sum_{j=0}^{m} \|\overline{h}^{-(j-m)} \nabla^j (u - J_t u)\|_{L^2(\Omega)} \leq C_m \|u\|_{H^m(\Omega)}. \tag{3.1.4}$$

In particular, interpolation arguments give

$$\|\overline{h}^{1/2} \nabla J_t u\|_{L^2(\Omega)} + \|\overline{h}^{-1/2}(u - J_t u)\|_{L^2(\Omega)} \leq C \|u\|_{H^{1/2}(\Omega)}, \tag{3.1.5}$$

$$\|\overline{h}^{1/2} \nabla^2 J_t u\|_{L^2(\Omega)} + \|\overline{h}^{-3/2}(u - J_t u)\|_{L^2(\Omega)} + \|\overline{h}^{-1/2} \nabla (u - J_t u)\|_{L^2(\Omega)} \leq C \|u\|_{H^{3/2}(\Omega)}. \tag{3.1.6}$$

The constants $C_{m,n}$ *and* C_m *depend on* m *and* n *as indicated, as well as on* Ω *and the* γ*-shape regularity of* $\mathcal{T}$*. The constant* C *depends only on* Ω *and the* γ*-shape regularity of* $\mathcal{T}$*.*

Proof. 1. step: For $t \geq \operatorname{diam} \Omega$, one may select $J_t = 0$.

2. step: For $t \leq \operatorname{diam} \Omega$, one constructs a length scale function ε with $\varepsilon \sim \overline{h}$ in the following way: First, by mollification of the piecewise constant function h (see [KM15, Lemma 3.1] for details), one obtains a function $\widetilde{h} \in C^\infty(\overline{\Omega})$, whose Lipschitz constant $\mathcal{L}$ depends solely on the γ-shape regularity of $\mathcal{T}$ and Ω. Next, one defines the auxiliary length scale function $\widetilde{\varepsilon}(x) := \widetilde{h}(x) + t$. We note that the Lipschitz constant of $\widetilde{\varepsilon}$ is still $\mathcal{L}$. From [KM15, Lemma 5.7], there are parameters $0 < \alpha < \beta$ (depending on $\mathcal{L}$) and $N_d \in \mathbb{N}$ (depending only on the spatial dimension d) as well as closed balls $B_{ij} := \overline{B}_{\alpha\widetilde{\varepsilon}(x_{ij})}(x_{ij})$, $i = 1, \ldots, N_d$, $j \in \mathbb{N}$ such that the following holds:

(a) $\Omega \subset \cup_{i=1}^{N_d} \cup_{j \in \mathbb{N}} B_{ij}$;

(b) There is a constant $C_{\text{big}} > 0$, such that, for each $i \in \{1, \ldots, N_d\}$, the stretched balls $\widehat{B}_{ij} := \overline{B}_{\beta\widetilde{\varepsilon}(x_{ij})}(x_{ij})$ satisfy an overlap condition: $\#\{j' \,|\, \widehat{B}_{ij'} \cap \widehat{B}_{ij} \neq \emptyset\} \leq C_{\text{big}}$ for all $j \in \mathbb{N}$.

(c) For pairs (i,j) and (i',j') with $\widehat{B}_{ij} \cap \widehat{B}_{i'j'} \neq \emptyset$, there holds $\widetilde{\varepsilon}(x_{ij}) \sim \widetilde{\varepsilon}(x_{i'j'})$ with implied constant depending solely on $\mathcal{L}$ and β. This implies *a fortiori* that for pairs (i,j) and (i',j') with $B_{ij} \cap B_{i'j'} \neq \emptyset$ there holds $\widetilde{\varepsilon}(x_{ij}) \sim \widetilde{\varepsilon}(x_{i'j'})$ with implied constant depending solely on $\mathcal{L}$ and β (which follows by inspection of the proof of [KM15, Lemma 5.7]).

Denoting by χ_A the characteristic function of the set A, we define the desired length scale function ε as

$$\varepsilon := \sum_{i=1}^{N_d} \sum_{j \in \mathbb{N}} \widetilde{\varepsilon}(x_{ij}) (\chi_{B_{ij}} * \rho_{(\beta-\alpha)\widetilde{\varepsilon}(x_{ij})}), \tag{3.1.7}$$

where ρ_δ is a standard non-negative mollifier supported by $B_\delta(0)$. Let $x \in \Omega$. Due to (a) there is (i,j) with $x \in B_{ij}$. The non-negativity of the mollifier ρ_δ gives $\varepsilon(x) \gtrsim \widetilde{\varepsilon}(x_{ij})$.

Furthermore, (b), (c) imply that the sum (3.1.7) is locally finite (with at most $N_d C_{\mathrm{big}}$ non-zero terms). In view of (c), we get $\varepsilon(x) \lesssim \varepsilon(x_{ij})$. By studying derivatives of ε, we recognize that it is a length scale function in the sense of [KM15, Def. 2.1].

3. step: The upshot of [KM15, Lemma 5.7] is that, once a length scale function ε is available, then a covering argument can be employed. That is, the operator $\mathcal{A}_\varepsilon$ of [KM15, Thm. 2.3] yields

$$\sum_{j=0}^{m} \|\varepsilon^{m-j}\nabla^j(u - \mathcal{A}_\varepsilon u)\|_{L^2(\Omega)} \lesssim \|u\|_{H^m(\Omega)}, \qquad \|\varepsilon^n \nabla^{m+n}\mathcal{A}_\varepsilon u\|_{L^2(\Omega)} \lesssim \|u\|_{H^m(\Omega)},$$

which proves (3.1.3) and (3.1.4) since $\varepsilon \sim \overline{h}$.

4. step: Using Eq. (3.1.4), for $m = 0$ and $m = 1$, the following estimates hold

$$\|u - J_t u\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}, \tag{3.1.8}$$

$$\|\overline{h}^{-1}(u - J_t u)\|_{L^2(\Omega)} \lesssim \|u\|_{H^1(\Omega)}. \tag{3.1.9}$$

Applying Lemma 2.1.5 to interpolate between the above inequalities with $\theta = 1/2$ gives us

$$\|\overline{h}^{-1/2}(u - J_t u)\|_{L^2(\Omega)} \leq C\|u\|_{H^{1/2}(\Omega)}. \tag{3.1.10}$$

Moreover, for $m = 1$ and $n = 0$, Eq. (3.1.3) leads to

$$\|\nabla J_t u\|_{L^2(\Omega)} \lesssim \|u\|_{H^1(\Omega)},$$

and for $m = 0$ and $n = 1$, we get

$$\|\overline{h}\nabla J_t u\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)}.$$

Interpolation between the above inequalities using Lemma 2.1.5 with $\theta = 1/2$, results in

$$\|\overline{h}^{1/2}\nabla J_t u\|_{L^2(\Omega)} \lesssim \|u\|_{H^{1/2}(\Omega)}. \tag{3.1.11}$$

Combining (3.1.10) and (3.1.11) gives us (3.1.5). Similarly, Eq. (3.1.4) with $m = 1$ and $m = 2$ yields

$$\|\overline{h}^{-1}(u - J_t u))\|_{L^2(\Omega)} \lesssim \|u\|_{H^1(\Omega)}, \tag{3.1.12}$$

$$\|\overline{h}^{-2}(u - J_t u)\|_{L^2(\Omega)} \lesssim \|u\|_{H^2(\Omega)}. \tag{3.1.13}$$

Applying Lemma 2.1.5 with $\theta = 1/2$, leads to

$$\|\overline{h}^{-3/2}(u - J_t u)\|_{L^2(\Omega)} \lesssim \|u\|_{H^{3/2}(\Omega)}. \tag{3.1.14}$$

Also, for $m = 1$ and $n = 1$, Eq. (3.1.3) leads to

$$\|\overline{h}\nabla^2 J_t u\|_{L^2(\Omega)} \lesssim \|u\|_{H^1(\Omega)},$$

and for $m = 2$ and $n = 0$, we obtain

$$\|\nabla^2 J_t u\|_{L^2(\Omega)} \lesssim \|u\|_{H^2(\Omega)}.$$

Applying Lemma 2.1.5 with $\theta = 1/2$, we get

$$\|\overline{h}^{1/2}\nabla^2 J_t u\|_{L^2(\Omega)} \lesssim \|u\|_{H^{3/2}(\Omega)}. \tag{3.1.15}$$

Using Eq. (3.1.4) with $m = 1$ and $m = 2$, one has

$$\|\nabla(u - J_t u)\|_{L^2(\Omega)} \lesssim \|u\|_{H^1(\Omega)}, \tag{3.1.16}$$
$$\|\overline{h}^{-1}\nabla(u - J_t u)\|_{H^1(\Omega)} \lesssim \|u\|_{H^2(\Omega)}. \tag{3.1.17}$$

Interpolation between the above inequalities from Lemma 2.1.5 with $\theta = 1/2$, then gives us

$$\|\overline{h}^{-1/2}\nabla(u - J_t u)\|_{L^2(\Omega)} \leq C\|u\|_{H^{3/2}(\Omega)}. \tag{3.1.18}$$

Combination of (3.1.14), (3.1.15) and (3.1.18) concludes the bound (3.1.6) □

The following theorem states a stability result in the Besov space $B^{m-1/2}_{2,\infty}(\Omega)$ for operators satisfying Assumption 3.1.1.

Theorem 3.1.5. *Fix $m \in \{1, 2\}$ and $p \in \mathbb{N}_0$ with $p \geq m - 1$. Let $\mathcal{T}$ be a γ-shape regular triangulation. Let an operator I_h^m satisfying Assumption 3.1.1 be given. Then,*

$$\|I_h^m u\|_{B^{m-1/2}_{2,\infty}(\Omega)} \leq C\|u\|_{H^{m-1/2}(\Omega)} \qquad \forall u \in H^{m-1/2}(\Omega), \tag{3.1.19}$$

where the constant $C > 0$ depends solely on Ω, d, m, p, and the γ-shape regularity of $\mathcal{T}$.

If the mesh $\mathcal{T}$ is additionally quasi-uniform*, then, the following sharper estimate holds:*

$$\|I_h^m u\|_{B^{m-1/2}_{2,\infty}(\Omega)} \leq C\|u\|_{B^{m-1/2}_{2,\infty}(\Omega)} \qquad \forall u \in B^{m-1/2}_{2,\infty}(\Omega). \tag{3.1.20}$$

Proof. The function $I_h^m u$ is piecewise smooth on a finite mesh. Hence, it is an element of $B^{m-1/2}_{2,\infty}(\Omega)$, so that only the stability estimate has to be proved. This is achieved by constructing an element $u_t := \mathcal{A}_{\delta t}(I_h^m u)$ for an appropriate $\delta > 0$ such that the K-functional can be estimated by the $H^{m-1/2}$-norm of u. We have

$$\begin{aligned}\|I_h^m u\|_{B^{m-1/2}_{2,\infty}(\Omega)} &= \sup_{t>0} t^{-1/2} K(t, I_h^m u) \\ &\lesssim \sup_{t>0} t^{-1/2} \left(\|I_h^m u - \mathcal{A}_{\delta t}(I_h^m u)\|_{H^{m-1}(\Omega)} + t\,\|\mathcal{A}_{\delta t}(I_h^m u)\|_{H^m(\Omega)}\right). \end{aligned}\tag{3.1.21}$$

With the operator J_t from Corollary 3.1.4, we further decompose $u = (u - J_t u) + J_t u =: u_0 + u_1$ into an element of $H^{m-1}(\Omega)$ and one in $H^m(\Omega)$. By the triangle inequality, we have to control the right-hand side of (3.1.21) for both contributions separately.

1. step: For fixed $t > 0$, we split the mesh into elements of size smaller than t and larger than t:

$$\mathcal{T}_{\le t} := \{T \in \mathcal{T}\colon \operatorname{diam} T \le t\}, \qquad \mathcal{T}_{>t} := \{T \in \mathcal{T}\colon \operatorname{diam} T > t\}$$

and define the regions covered by these elements by

$$\Omega_{\le t} := \operatorname{interior}\Big(\bigcup_{T\in\mathcal{T}_{\le t}} \overline{T}\Big), \qquad \Omega_{>t} := \operatorname{interior}\Big(\bigcup_{T\in\mathcal{T}_{>t}} \overline{T}\Big). \tag{3.1.22}$$

There is a constant $\delta > 0$, depending solely on the γ-shape regularity of $\mathcal{T}$, such that the "δt-neighborhood" $T_{\delta t} := \Omega \cap \cup_{x\in T} B_{\delta t}(x)$ of each element in $\mathcal{T}_{>t}$ is contained in the patch of the element, i.e., $T_{\delta t} \subset \omega(T)$ for all $T \in \mathcal{T}_{>t}$. Moreover, for each $T \in \mathcal{T}_{>t}$, we define the *inside strip* $S_{T,\delta t}$ at the boundary ∂T of T by

$$S_{T,\delta t} := \{x \in T\colon \operatorname{dist}(x, \partial T) < \delta t\}. \tag{3.1.23}$$

For the set $\mathcal{T}_{\le t}$, the γ-shape regularity of $\mathcal{T}$ implies the existence of $\eta \ge \delta$ and $C > 0$ depending only on the γ-shape regularity such that the extended set $\Omega_{\eta t} := \Omega\cap\bigcup_{x\in\Omega_{\le t}} B_{\eta t}(x)$ satisfies the conditions

$$T \in \mathcal{T}_{\le t} \Longrightarrow \omega^2(T) \subset \Omega_{\eta t}, \tag{3.1.24}$$
$$T \in \mathcal{T} \text{ with } T \subset \Omega_{\eta t} \Longrightarrow \operatorname{diam} T \le Ct, \tag{3.1.25}$$
$$T \in \mathcal{T} \text{ with } T \cap \Omega_{\eta t} \neq \emptyset \Longrightarrow \omega(T) \subset \Omega_{c\eta t}. \tag{3.1.26}$$

where $c > 0$ is a constant depending solely on the γ-shape regularity of $\mathcal{T}$. The choice of η is dictated by the requirement (3.1.24). We note that the γ-shape regularity of $\mathcal{T}$ ensures that for all $T \in \mathcal{T}_{\le t}$ the diameters of all elements $T' \subset \omega(T)$ are bounded by $\widehat{C}t$ for some $\widehat{C} > 0$ depending only on γ. This implies (3.1.24) if η is chosen sufficiently large.

To see (3.1.25), it suffices to consider elements $T \in \mathcal{T}$ with $T \subset \Omega_{\eta t} \setminus \Omega_{\le t}$. Let m_T be the center of the largest inscribed sphere in T and note that the radius ρ_T of that sphere is comparable to the element diameter h_T. Let $\widetilde{m}_T \in \overline{\Omega_{\le t}}$ satisfy $\operatorname{dist}(m_T, \Omega_{\le t}) = \operatorname{dist}(m_T, \widetilde{m}_T)$. By definition of $\Omega_{\eta t}$, we have $m_T \in B_{\eta t}(\widetilde{m}_T)$ and by $T \subset \Omega_{\eta t} \setminus \Omega_{\le t}$ that $B_{\rho_T}(m_T) \subset \Omega_{\eta t} \setminus \Omega_{\le t}$. Thus,

$$h_T \sim \rho_T \le \operatorname{dist}(m_T, \Omega_{\le t}) = \operatorname{dist}(m_T, \widetilde{m}_T) \le \eta t,$$

which proves (3.1.25).

With the sets from (3.1.22) and (3.1.23), we decompose for $k \in \mathbb{N}_0$ and $v \in H^k(\Omega)$

$$\|v\|^2_{H^k(\Omega)} \lesssim \|v\|^2_{H^k(\Omega_{\le t})} + \|v\|^2_{H^k(\Omega_{>t})} \lesssim \|v\|^2_{H^k(\Omega_{\le t})} + \sum_{T\in\mathcal{T}_{>t}} \|v\|^2_{H^k(T\setminus S_{T,\delta t})} + \sum_{T\in\mathcal{T}_{>t}} \|v\|^2_{H^k(S_{T,\delta t})}. \tag{3.1.27}$$

We employ this decomposition in (3.1.21) for $k = m - 1$ and $v = I_h^m u_i - \mathcal{A}_{\delta t}(I_h^m u_i)$ as well as for $k = m$ and $v = \mathcal{A}_{\delta t}(I_h^m u_i)$ and $i \in \{0, 1\}$. In the following, we estimate all these contributions separately by the desired $H^{m-1/2}(\Omega)$-norm of u. The main ideas are that, a) on $\Omega_{\leq t}$, we exploit that elements are small; and b) on $T\backslash S_{T,\delta t}$, we may exploit that a sufficiently small neighborhood of this set is still contained in T; c) we can use the smoothness of $I_h^m u_i$ inside T; d) for $S_{T,\delta t}$, we exploit the thinness of the strip.

2. step: We estimate $I_h^m u_i - \mathcal{A}_{\delta t}(I_h^m u_i)$ on $\Omega_{\leq t}$, where $\delta \leq \eta$ is given by step 1.

For $i = 0$, we use the stability estimates of Proposition 3.1.2 and Lemma 3.1.3 and finally Corollary 3.1.4 (using $\overline{h} \sim t$ due to (3.1.25)) to obtain

$$\begin{aligned}\|I_h^m u_0 - \mathcal{A}_{\delta t}(I_h^m u_0)\|_{H^{m-1}(\Omega_{\leq t})} &\leq \|I_h^m u_0\|_{H^{m-1}(\Omega_{\leq t})} + \|\mathcal{A}_{\delta t}(I_h^m u_0)\|_{H^{m-1}(\Omega_{\leq t})} \\ &\lesssim \|I_h^m u_0\|_{H^{m-1}(\Omega_{\leq t})} + \|I_h^m u_0\|_{H^{m-1}(\Omega_{\eta t})} \\ &\overset{(3.1.1)}{\lesssim} \|u_0\|_{H^{m-1}(\Omega_{c\eta t})} = \|u - J_t u\|_{H^{m-1}(\Omega_{c\eta t})} \\ &\overset{\text{Cor. } 3.1.4}{\lesssim} t^{1/2}\, \|u\|_{H^{m-1/2}(\Omega)}\,.\end{aligned}$$

For $i = 1$, we use the approximation property of I_h^m (cf. (3.1.2) with $r = m-1$ and $k = m$) together with the fact that the element size of elements in $\Omega_{\leq t}$ is bounded by t as well as the local stability and approximation properties of $\mathcal{A}_{\delta t}$ from Proposition 3.1.2 to get

$$\begin{aligned}&\|I_h^m u_1 - \mathcal{A}_{\delta t}(I_h^m u_1)\|_{H^{m-1}(\Omega_{\leq t})} \\ &\quad\leq \|I_h^m u_1 - u_1\|_{H^{m-1}(\Omega_{\leq t})} + \|u_1 - \mathcal{A}_{\delta t} u_1\|_{H^{m-1}(\Omega_{\leq t})} + \|\mathcal{A}_{\delta t}(u_1 - I_h^m u_1)\|_{H^{m-1}(\Omega_{\leq t})} \\ &\quad\overset{h\lesssim t}{\lesssim} t\|u_1\|_{H^m(\Omega_{\eta t})} + t\|u_1\|_{H^m(\Omega_{\eta t})} + \|u_1 - I_h^m u_1\|_{H^{m-1}(\Omega_{\eta t})} \overset{h\lesssim t}{\lesssim} t\|u_1\|_{H^m(\Omega_{c\eta t})} \\ &\overset{\text{Cor. } 3.1.4}{\lesssim} t^{1/2}\, \|u\|_{H^{m-1/2}(\Omega)}\,.\end{aligned}$$

3. step: We estimate $\mathcal{A}_{\delta t}(I_h^m u_i)$ on $\Omega_{\leq t}$. For $i = 0$, using the stability properties of the smoothing operator from Proposition 3.1.2, the stability of I_h^m, and Corollary 3.1.4, we get

$$t\|\mathcal{A}_{\delta t}(I_h^m u_0)\|_{H^m(\Omega_{\leq t})} \lesssim \|I_h^m u_0\|_{H^{m-1}(\Omega_{\eta t})} \overset{(3.1.1)}{\lesssim} \|u_0\|_{H^{m-1}(\Omega_{c\eta t})} \overset{\text{Cor. } 3.1.4}{\lesssim} t^{1/2}\|u\|_{H^{m-1/2}(\Omega)}.$$

Similarly, for $u_1 \in H^m(\Omega)$, we obtain with Proposition 3.1.2

$$\begin{aligned}t\|\mathcal{A}_{\delta t}(I_h^m u_1)\|_{H^m(\Omega_{\leq t})} &\lesssim t\|\mathcal{A}_{\delta t}(I_h^m u_1 - u_1)\|_{H^m(\Omega_{\leq t})} + t\|\mathcal{A}_{\delta t} u_1\|_{H^m(\Omega_{\leq t})} \\ &\lesssim \|I_h^m u_1 - u_1\|_{H^{m-1}(\Omega_{\eta t})} + t\|u_1\|_{H^m(\Omega_{\eta t})} \overset{(3.1.2),h\lesssim t}{\lesssim} t\|u_1\|_{H^m(\Omega_{c\eta t})} \\ &\overset{\text{Cor. } 3.1.4}{\lesssim} t^{1/2}\|u\|_{H^{m-1/2}(\Omega)}.\end{aligned}$$

4. step: We derive estimates on $T\backslash S_{T,\delta t}$ for $T \in \mathcal{T}_{>t}$. Since the "δt-neighborhood" $(T\backslash S_{T,\delta t})_{\delta t}$ of $T\backslash S_{T,\delta t}$ satisfies $(T\backslash S_{T,\delta t})_{\delta t} \subseteq T$, Proposition 3.1.2 and an inverse inequality imply

$$\begin{aligned}\|I_h^m u_0 - \mathcal{A}_{\delta t}(I_h^m u_0)\|_{H^{m-1}(T\backslash S_{T,\delta t})} &\lesssim t\|I_h^m u_0\|_{H^m(T)} \lesssim t h_T^{-1}\|I_h^m u_0\|_{H^{m-1}(T)} \\ &\overset{(3.1.1)}{\lesssim} t h_T^{-1}\|u_0\|_{H^{m-1}(\omega^2(T))}.\end{aligned}$$

Summation over all elements $T \in \mathcal{T}_{>t}$ and Corollary 3.1.4, (3.1.5)–(3.1.6) (noting that $t < h_T$ implies $\overline{h} = h$ on $\mathcal{T}_{>t}$) give the desired estimate

$$\sum_{T\in\mathcal{T}_{>t}} \|I_h^m u_0 - \mathcal{A}_{\delta t}(I_h^m u_0)\|^2_{H^{m-1}(T\setminus S_{T,\delta t})} \lesssim t^2 \sum_{T\in\mathcal{T}_{>t}} h_T^{-2}\|u_0\|^2_{H^{m-1}(\omega^2(T))}$$
$$\overset{t<h_T}{\lesssim} t \sum_{j=0}^{m-1} \left\|\overline{h}^{-1/2}\nabla^j(u - J_t u)\right\|^2_{L^2(\Omega)} \lesssim t\,\|u\|^2_{H^{m-1/2}(\Omega)}. \tag{3.1.28}$$

Similarly, the approximation properties of $\mathcal{A}_{\delta t}$, the stability of I_h^m, and Corollary 3.1.4 give

$$\sum_{T\in\mathcal{T}_{>t}} \|I_h^m u_1 - \mathcal{A}_{\delta t}(I_h^m u_1)\|^2_{H^{m-1}(T\setminus S_{T,\delta t})} \lesssim t^2 \sum_{T\in\mathcal{T}_{>t}} \|I_h^m u_1\|^2_{H^m(T)} \overset{(3.1.1)}{\lesssim} t^2 \sum_{T\in\mathcal{T}_{>t}} \|u_1\|^2_{H^m(\omega^2(T))}$$
$$\overset{t<h_T}{\lesssim} t \sum_{T\in\mathcal{T}_{>t}} h_T\|J_t u\|^2_{H^m(\omega^2(T))} \overset{\text{Cor. 3.1.4}}{\lesssim} t\,\|u\|^2_{H^{m-1/2}(\Omega)}. \tag{3.1.29}$$

Using the stability instead of the approximation properties of $\mathcal{A}_{\delta t}$ from Proposition 3.1.2, the same arguments and an inverse estimate lead to

$$t\|\mathcal{A}_{\delta t}(I_h^m u_0)\|_{H^m(T\setminus S_{T,\delta t})} \lesssim t\|I_h^m u_0\|_{H^m(T)} \lesssim t h_T^{-1}\|u_0\|_{H^{m-1}(\omega^2(T))},$$
$$t\|\mathcal{A}_{\delta t}(I_h^m u_1)\|_{H^m(T\setminus S_{T,\delta t})} \lesssim t\|I_h^m u_1\|_{H^m(T)} \lesssim t\|u_1\|_{H^m(\omega^2(T))}.$$

Summation and employing Corollary 3.1.4 gives the desired estimates as in (3.1.28) and (3.1.29).

5. step: We derive approximation results for I_h^m on the strip $S_{T,\delta t}$ for $T \in \mathcal{T}_{>t}$. For $v \in H^m(\Omega)$, we claim

$$\|v - I_h^m v\|_{H^{m-1}(S_{T,\delta t})} \lesssim \sqrt{t h_T}\|v\|_{H^m(\omega^2(T))}. \tag{3.1.30}$$

On the reference element $\widehat{T}$, with the aid of [LMWZ10, Eq. 6] for arbitrary $\widehat{t} > 0$, it follows

$$\|\widehat{v}\|^2_{L^2(S_{\widehat{T},\delta t})} \lesssim \delta t \left(\widehat{t}\,\|\widehat{v}\|^2_{L^2(\widehat{T})} + \frac{1}{\widehat{t}}\|\widehat{v}\|^2_{H^1(\widehat{T})}\right) \qquad \forall \widehat{v} \in H^1(\widehat{T}).$$

We select $\widehat{t} = \|\widehat{v}\|_{H^1(\widehat{T})}/\|\widehat{v}\|_{L^2(\widehat{T})}$ and arrive at

$$\|\widehat{v}\|^2_{L^2(S_{\widehat{T},\delta t})} \lesssim \delta t\,\|\widehat{v}\|_{L^2(\widehat{T})}\|\widehat{v}\|_{H^1(\widehat{T})} \le \delta t\left(\|\widehat{v}\|^2_{L^2(\widehat{T})} + \|\widehat{v}\|_{L^2(\widehat{T})}\|\nabla\widehat{v}\|_{L^2(\widehat{T})}\right).$$

If $\|\widehat{v}\|_{L^2(\widehat{T})} = 0$, then we select $\widehat{t} = 1$ and the above estimate holds easily. Let $\Phi_T : \widehat{T} \to T$ be an affine parametrization of T and $\widehat{v} := v \circ \Phi_T$. Applying a scaling argument, one can show for $v \in H^1(T)$ and $T \in \mathcal{T}_{>t}$

$$\|v\|^2_{L^2(S_{T,\delta t})} \lesssim h_T^d\,\|\widehat{v}\|^2_{L^2(S_{\widehat{T},\delta' t/h_T})} \lesssim h_T^{d-1}\left(t\|\widehat{v}\|^2_{L^2(\widehat{T})} + t\|\widehat{v}\|_{L^2(\widehat{T})}\|\nabla\widehat{v}\|_{L^2(\widehat{T})}\right)$$
$$\lesssim \frac{t}{h_T}\|v\|^2_{L^2(T)} + t\|v\|_{L^2(T)}\|\nabla v\|_{L^2(T)}, \tag{3.1.31}$$

for some $\delta' \sim \delta$. For polynomials $v \in P_p(T)$, an inverse estimate and (3.1.31) furthermore lead to

$$\|v\|^2_{L^2(S_{T,\delta t})} \lesssim \frac{t}{h_T}\|v\|^2_{L^2(T)}. \tag{3.1.32}$$

To see (3.1.30), we estimate

$$\begin{aligned}\|v - I_h^m v\|^2_{L^2(S_{T,\delta t})} &\overset{(3.1.31)}{\lesssim} \frac{t}{h_T}\|v - I_h^m v\|^2_{L^2(T)} + t\|v - I_h^m v\|_{L^2(T)}\|\nabla(v - I_h^m v)\|_{L^2(T)} \\ &\overset{(3.1.2)}{\lesssim} h_T t\|v\|^2_{H^1(\omega^2(T))}.\end{aligned}$$

This show (3.1.30) for $m = 1$. For $m = 2$, we apply (3.1.31) to $\nabla(u - I_h^m u)$ and proceed similarly.

6. step: We derive an estimate for $I_h^m u_i - \mathcal{A}_{\delta t}(I_h^m u_i)$ on the strip $S_{T,\delta t}$ for $T \in \mathcal{T}_{>t}$. Here, we need the "δt-neighborhood" $(S_{T,\delta t})_{\delta t}$ of the strip $S_{T,\delta t}$. Our assumption on δ implies that $(S_{T,\delta t})_{\delta t} \subset \omega(T)$. Moreover, we note that the strip $(S_{T,\delta t})_{\delta t}$ is contained in the inside strip $S_{T,2\delta t}$ of T and in parts of the inside strip of width δt of the elements $T' \in \omega(T)$.

Using the triangle inequality, Proposition 3.1.2 and (3.1.32) on each element of the patch $\omega(T)$ separately for $v = I_h^m u_0$ in the case $m = 1$ or $v = \nabla I_h^m u_0$ for $m = 2$, we get, since $h_{T'} \sim h_T$ for $T' \in \omega(T)$,

$$\begin{aligned}\|I_h^m u_0 - \mathcal{A}_{\delta t}(I_h^m u_0)\|_{H^{m-1}(S_{T,\delta t})} \le \|I_h^m u_0\|_{H^{m-1}((S_{T,\delta t})_{\delta t})} &\overset{(3.1.32)}{\lesssim} t^{1/2}h_T^{-1/2}\|I_h^m u_0\|_{H^{m-1}(\omega(T))} \\ &\lesssim t^{1/2}h_T^{-1/2}\|u_0\|_{H^{m-1}(\omega^3(T))}.\end{aligned} \tag{3.1.33}$$

Summing over all elements $T \in \mathcal{T}_{>t}$ and employing the arguments from (3.1.28), we get the desired bound by $t^{1/2}\,\|u\|_{H^{m-1/2}(\Omega)}$. For u_1, we use the triangle inequality, Proposition 3.1.2, and (3.1.30)

$$\begin{aligned}&\|I_h^m u_1 - \mathcal{A}_{\delta t}(I_h^m u_1)\|_{H^{m-1}(S_{T,\delta t})} \\ &\quad\le \|I_h^m u_1 - u_1\|_{H^{m-1}(S_{T,\delta t})} + \|u_1 - \mathcal{A}_{\delta t} u_1\|_{H^{m-1}(S_{T,\delta t})} + \|\mathcal{A}_{\delta t}(u_1 - I_h^m u_1)\|_{H^{m-1}(S_{T,\delta t})} \\ &\quad\overset{\text{Prop. 3.1.2}}{\lesssim} \|I_h^m u_1 - u_1\|_{H^{m-1}((S_{T,\delta t})_{\delta t})} + \|u_1 - \mathcal{A}_{\delta t} u_1\|_{H^{m-1}(S_{T,\delta t})} \\ &\quad\overset{(3.1.30),\text{Prop. 3.1.2}}{\lesssim} \sqrt{th_T}\|u_1\|_{H^m(\omega^3(T))} + t\|u_1\|_{H^m(\omega(T))} \overset{t\le h_T}{\lesssim} \sqrt{th_T}\|u_1\|_{H^m(\omega^3(T))}.\end{aligned}$$

Summing over all elements $T \in \mathcal{T}_{>t}$ and employing the arguments from (3.1.29), we get the desired bound.

7. step: We estimate $\mathcal{A}_{\delta t}(I_h^m u_i)$ on the strip $S_{T,\delta t}$ for $T \in \mathcal{T}_{>t}$. The inverse estimate for $\mathcal{A}_{\delta t}$ of Proposition 3.1.2, (3.1.32) employed on the patch $\omega(T)$ as in the previous step, and the stability (3.1.1) of I_h^m imply

$$\begin{aligned}t\|\mathcal{A}_{\delta t}(I_h^m u_0)\|_{H^m(S_{T,\delta t})} \lesssim \|I_h^m u_0\|_{H^{m-1}((S_{T,\delta t})_{\delta t})} &\lesssim t^{1/2}h_T^{-1/2}\|I_h^m u_0\|_{H^{m-1}(\omega(T))} \\ &\lesssim t^{1/2}h_T^{-1/2}\|u_0\|_{H^{m-1}(\omega^3(T))}.\end{aligned} \tag{3.1.34}$$

Summing over all elements $T \in \mathcal{T}_{>t}$ and employing the arguments from (3.1.28), we get the desired bound by $t^{1/2}\,\|u\|_{H^{m-1/2}(\Omega)}$. For u_1, Proposition 3.1.2 and (3.1.30) on the patch $\omega(T)$ give

$$
\begin{aligned}
t\|\mathcal{A}_{\delta t}(I_h^m u_1)\|_{H^m(S_{T,\delta t})} &\leq t\|\mathcal{A}_{\delta t}(u_1 - I_h^m u_1)\|_{H^m(S_{T,\delta t})} + t\|\mathcal{A}_{\delta t} u_1\|_{H^m(S_{T,\delta t})} \\
&\lesssim \|u_1 - I_h^m u_1\|_{H^{m-1}((S_{T,\delta t})_{\delta t})} + t\|u_1\|_{H^m((S_{T,\delta t})_{\delta t})} \\
&\overset{(3.1.30)}{\lesssim} (th_T)^{1/2}\|u_1\|_{H^m(\omega^3(T))} + t\|u_1\|_{H^m(\omega^3(T))} \\
&\overset{t<h_T}{\lesssim} (th_T)^{1/2}\|u_1\|_{H^m(\omega^3(T))}.
\end{aligned}
$$

Summing over all elements $T \in \mathcal{T}_{>t}$ and employing the argument from (3.1.29), we get the desired bound.

Combining the estimates of steps 2–7, where all relevant terms are bounded by $t^{1/2}\,\|u\|_{H^{m-1/2}(\Omega)}$, gives the desired bound for (3.1.21), which proves (3.1.19).

Final step: We show (3.1.20) with similar arguments as in steps 2–7. Let $u = u_0 + u_1$ be an arbitrary decomposition with $u_0 \in H^{m-1}(\Omega)$ and $u_1 \in H^m(\Omega)$. We distinguish the cases $t \leq h$ and $t > h$, where h is the maximal mesh size of the quasi-uniform triangulation. We note that in the decomposition (3.1.27) the sums $\sum_{T\in\mathcal{T}_{>t}}$ are not present in the case $t > h$ and the terms involving $\|\cdot\|_{H^{m-1}(\Omega_{\leq t})}$ or $\|\cdot\|_{H^m(\Omega_{\leq t})}$ in the converse case. Inspection of the above arguments therefore gives:

- For $t > h$: As in steps 2–3, we get
$$
\begin{aligned}
t^{-1}\|I_h^m u_0 - \mathcal{A}_{\delta t}(I_h^m u_0)\|^2_{H^{m-1}(\Omega)} + t\|\mathcal{A}_{\delta t}(I_h^m u_0)\|^2_{H^m(\Omega)} &\lesssim t^{-1}\|u_0\|^2_{H^{m-1}(\Omega)}, \\
t^{-1}\|I_h^m u_1 - \mathcal{A}_{\delta t}(I_h^m u_1)\|^2_{H^{m-1}(\Omega)} + t\|\mathcal{A}_{\delta t}(I_h^m u_1)\|^2_{H^m(\Omega)} &\lesssim t\|u_1\|^2_{H^m(\Omega)}.
\end{aligned}
$$
This implies $t^{-1/2}K(t, I_h^m u) \lesssim t^{-1/2}\|u_0\|_{H^{m-1}(\Omega)} + t^{1/2}\|u_1\|_{H^m(\Omega)}$. Infimizing over all possible decompositions $u = u_0 + u_1$ yields $t^{-1/2}K(t, I_h^m u) \lesssim t^{-1/2}K(t,u) \lesssim \|u\|_{B^{m-1/2}_{2,\infty}(\Omega)}$.

- For $t \leq h$: As in steps 4–7, we get
$$
\begin{aligned}
t^{-1}\|I_h^m u_0 - \mathcal{A}_{\delta t}(I_h^m u_0)\|^2_{H^{m-1}(\Omega)} + t\|\mathcal{A}_{\delta t}(I_h^m u_0)\|^2_{H^m(\Omega)} &\lesssim h^{-1}\|u_0\|^2_{H^{m-1}(\Omega)}, \\
t^{-1}\|I_h^m u_1 - \mathcal{A}_{\delta t}(I_h^m u_1)\|^2_{H^{m-1}(\Omega)} + t\|\mathcal{A}_{\delta t}(I_h^m u_1)\|^2_{H^m(\Omega)} &\lesssim h\|u_1\|^2_{H^m(\Omega)}.
\end{aligned}
$$
This implies $t^{-1/2}K(t, I_h^m u) \lesssim h^{-1/2}\|u_0\|_{H^{m-1}(\Omega)} + h^{1/2}\|u_1\|_{H^m(\Omega)}$. Infimizing over all possible decompositions $u = u_0 + u_1$ yields $t^{-1/2}K(t, I_h^m u) \lesssim h^{-1/2}K(h,u) \lesssim \|u\|_{B^{m-1/2}_{2,\infty}(\Omega)}$.

Combining the above two cases yields $\sup_{t>0} K(t, I_h^m u) \lesssim \|u\|_{B^{m-1/2}_{2,\infty}(\Omega)}$, as claimed. □

Remark 3.1.6. For $m = 1$, a possible choice for I_h^m is the $L^2(\Omega)$-orthogonal projection that trivially satisfies Assumption 3.1.1. For $m = 2$, the Scott-Zhang projection, introduced in [SZ90a] and defined below, is an example of an operator I_h^m satisfying Assumption 3.1.1. Therefore, Theorem 3.1.5 provides a novel stability estimates for these projection operators in Besov spaces. ■

While, for *finite* meshes, we have the continuous embeddings $S^{p,1}(\mathcal{T}) \subset B^{3/2}_{2,\infty}(\Omega)$ and $S^{p,0}(\mathcal{T}) \subset B^{1/2}_{2,\infty}(\Omega)$, this is not necessarily the case for infinite meshes. As a consequence, one cannot expect that on general K-meshes a stability $I_h^m : B^{1/2}_{2,\infty}(\Omega) \to B^{1/2}_{2,\infty}(\Omega)$ can hold. The following example illustrates this.

Example 3.1.7. Let $\Omega = (0,1)$. Set $I_1 = (0,1/2)$ and $I_2 = (1/2,1)$. Let $\varphi \in C^\infty(\mathbb{R})$ be a 1-periodic function, whose averages $\overline{\varphi}_1 := 1/|I_1| \int_{I_1} \varphi(x)\, dx$ and $\overline{\varphi}_2 := 1/|I_2| \int_{I_2} \varphi(x)\, dx$ are *different*. Define the function $u \in C^\infty((0,\infty))$ by

$$u(x) := \varphi(\ln x).$$

Define the (infinite) mesh $\mathcal{T}$ on Ω, whose elements are given by the break points $x_j = e^{-2j}$, $j \in \mathbb{N}_0$. Let $m = 1$ and let $I_h^m : L^2(\Omega) \to S^{0,0}(\mathcal{T})$ be the L^2-projection onto the piecewise constant functions. By the periodicity of φ, the piecewise constant function $I_h^m u$ takes only the values $\overline{\varphi}_1$ and $\overline{\varphi}_2$

$$(I_h^m u)|_{(x_{j+1},x_j)} = \begin{cases} \overline{\varphi}_1 & \text{if } j \text{ is even} \\ \overline{\varphi}_2 & \text{if } j \text{ is odd} . \end{cases}$$

The computation of Besov norms is conveniently done in terms of the modulus of smoothness as defined in, e.g., [DL93, Chap. 2, Sec. 7]. For an interval $[a,b]$ and a function v defined on $A := [a,b]$, and $t > 0$, we define the difference operator Δ_h by $(\Delta_h v)(x) := v(x+h) - v(x)$ on $A_h := [a, b-h]$. the modulus of smoothness $\omega_1(v,t)_2$ is then given by $\omega_1(v,t)_2 := \sup_{0<h\le t} \|\Delta_h(v,\cdot)\|_{L^2(A_h)}$. Let $t > 0$. Consider all elements with diameter $> t$. For the region covered by these elements, $\Omega_{>t}$, we can compute the modulus of smoothness ω_1 in view of the fact that $I_h^m u$ is piecewise constant

$$\omega_1(I_h^m u, t)^2_{2,\Omega_{>t}} \gtrsim \sum_{x_j : x_j > t} t |[I_h^m u](x_j)|^2,$$

where $[I_h^m u](x_j)$ denotes the jump of $I_h^m u$ at the break point x_j. We conclude

$$\omega_1(I_h^m u, t)_2^2 \ge \omega_1(I_h^m u, t)^2_{2,\Omega_{>t}} \gtrsim \sum_{x_j : x_j > t} t |[I_h^m u](x_j)|^2 = \sum_{x_j : x_j > t} |\overline{\varphi}_1 - \overline{\varphi}_2|^2 t \sim |\overline{\varphi}_1 - \overline{\varphi}_2|^2 t |\ln t|.$$

Next, we claim that $\omega_1(u,t)_2^2 \lesssim t$. Since u is bounded, we compute for $0 < h \le t$

$$\begin{aligned} \int_0^{1-h} |\Delta_h u|^2\, dx &= \int_0^{1-h} |u(x+h) - u(x)|^2\, dx = \int_0^h |u(x+h) - u(x)|^2\, dx \\ &\quad + \int_h^{1-h} |u(x+h) - u(x)|^2\, dx \le 4h \|u\|^2_{L^\infty(\Omega)} + \int_h^1 \left| \int_x^{x+h} u'(\xi)\, d\xi \right|^2 dx \\ &\le 4h \|u\|^2_{L^\infty(\Omega)} + \|\varphi'\|^2_{L^\infty(\Omega)} h^2 \int_h^1 \left(\frac{1}{x}\right)^2 dx \\ &\le 4h \|u\|^2_{L^\infty(\Omega)} + \|\varphi'\|^2_{L^\infty(\Omega)} h. \end{aligned}$$

This implies $\omega_1(u,t)_2 \leq Ct^{1/2}$ and therefore $u \in B^{1/2}_{2,\infty}(\Omega)$, since, by [DL93, Chap. 6, Thm. 2.4], $\omega(u,t)_2 \sim K(t,u) = \inf_{v\in H^1(I)} \|u-v\|_{L^2(\Omega)} + t\|v\|_{H^1(\Omega)}$. However, the above calculation shows that $I^m_h u \notin B^{1/2}_{2,\infty}(\Omega)$, which implies that I^m_h cannot be a bounded linear map $B^{1/2}_{2,\infty}(\Omega) \to B^{1/2}_{2,\infty}(\Omega)$. ∎

3.1.1 Some generalizations and applications

For quasi-uniform meshes, there also holds the following inverse estimate for the limiting case.

Lemma 3.1.8. *Let $\mathcal{T}$ be a quasi-uniform mesh on Ω of mesh size h and $m \in \{1,2\}$. Then, for $m' \in (0, m-1/2]$ and $q \in [1,\infty]$, there holds for a constant $C > 0$ depending only on Ω, d, the γ-shape-regularity of $\mathcal{T}$, and p:*

$$\|u\|_{B^{m'}_{2,q}(\Omega)} \leq Ch^{-m'}\|u\|_{L^2(\Omega)} \qquad \forall u \in S^{p,1}(\mathcal{T}). \tag{3.1.35}$$

Proof. To fix ideas, we only prove the case $m=2$ as the case $m=1$ is handled with similar arguments. By definition, we have

$$\|u\|_{B^{3/2}_{2,\infty}(\Omega)} = \sup_{t>0} t^{-1/2}K(t,u)$$

with the K-functional $K(t,u) = \inf_{v\in H^2(\Omega)} \|u-v\|_{H^1(\Omega)} + t\|v\|_{H^2(\Omega)}$. For $t > h$, we estimate

$$t^{-1/2}K(t,u) = t^{-1/2} \inf_{v\in H^2(\Omega)} \|u-v\|_{H^1(\Omega)} + t\|v\|_{H^2(\Omega)} \leq t^{-1/2}\|u\|_{H^1(\Omega)} \lesssim h^{-1/2}\|u\|_{H^1(\Omega)}, \tag{3.1.36}$$

by choosing $v \equiv 0$ to estimate the K-functional.

For $t \leq h$, we estimate the K-functional more carefully. For a suitably small $\delta > 0$, we set $v := \mathcal{A}_{\delta t}u$ with the smoothing operator $\mathcal{A}_{\delta t}$ of Proposition 3.1.2. As in the proof of Theorem 3.1.5, we decompose an element into $T = T\setminus S_{T,\delta t} \cup S_{T,\delta t}$, where $S_{T,\delta t}$ is the inside strip defined in the first step of the proof of Theorem 3.1.5. Employing Proposition 3.1.2 and a classical polynomial inverse estimate, we obtain

$$\|v\|_{H^2(T\setminus S_{T,\delta t})} \overset{\text{Prop. 3.1.2}}{\lesssim} \|u\|_{H^2(T)} \lesssim h^{-1}\|u\|_{H^1(T)}, \tag{3.1.37a}$$

$$\|u-v\|_{H^1(T\setminus S_{T,\delta t})} \overset{\text{Prop. 3.1.2}}{\lesssim} t\|u\|_{H^2(T)} \lesssim th^{-1}\|u\|_{H^1(T)}. \tag{3.1.37b}$$

As in steps 6–7 in the proof of Theorem 3.1.5, using Proposition 3.1.2 to obtain (3.1.34), (3.1.33), we get

$$\|v\|_{H^2(S_{T,\delta t})} \overset{(3.1.34)}{\lesssim} (th)^{-1/2}\|u\|_{H^1(\omega(T))}, \tag{3.1.38a}$$

$$\|u-v\|_{H^1(S_{T,\delta t})} \overset{(3.1.33)}{\lesssim} t^{1/2}h^{-1/2}\|u\|_{H^1(\omega(T))}. \tag{3.1.38b}$$

Summation over all elements, using (3.1.37)–(3.1.38) leads to

$$t^{-1/2}K(t,u) \lesssim \left(t^{1/2}h^{-1} + h^{-1/2}\right)\|u\|_{H^1(\Omega)} \overset{t\leq h}{\lesssim} h^{-1/2}\|u\|_{H^1(\Omega)}. \tag{3.1.39}$$

Combining (3.1.36) and (3.1.39) yields $\|u\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim h^{-1/2}\|u\|_{H^1(\Omega)}$. A further polynomial inverse estimate gives the desired result for $m' = 3/2$.

Finally, (3.1.35) follows from interpolation between the case $m' = 3/2$ and the trivial inequality $\|u\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)}$ noting that by the reinterpolation theorem (see, e.g., [Tar07, Chap. 26]), we have $B^{\theta(m-1/2)}_{2,q}(\Omega) = (L^2(\Omega), B^{m-1/2}_{2,\infty}(\Omega))_{\theta,q}$ (with equivalent norms) for $\theta \in (0,1)$. □

The operator I_h^m is stable in $L^2(\Omega)$ (by Assumption 3.1.1) and is stable as an operator $H^{m-1/2}(\Omega) \to B^{m-1/2}_{2,\infty}(\Omega)$ by Theorem 3.1.5. Interpolation therefore yields a stability for intermediate spaces.

Corollary 3.1.9. *Let $\mathcal{T}$ be a finite shape-regular mesh, $m \in \{1,2\}$, and let $I_h^m : L^2(\Omega) \to S^{p,m-1}(\mathcal{T})$ satisfy Assumption 3.1.1. Fix $q \in [1,\infty]$ and $\theta \in (0,1)$. Then, there is a constant $C > 0$ depending only on Ω, p, q, θ, and the γ-shape regularity of $\mathcal{T}$ such that*

$$\|I_h^m u\|_{B^{\theta(m-1/2)}_{2,q}(\Omega)} \leq C\|u\|_{B^{\theta(m-1/2)}_{2,q}(\Omega)}. \tag{3.1.40}$$

Proof. The assumed L^2-stability and the stability proved in Theorem 3.1.5 imply the result using the reinterpolation theorem (see, e.g., [Tar07, Chap. 26]) as in the proof of Lemma 3.1.8. □

Furthermore, Corollary 3.1.9 allows one to assert that interpolating between the discrete space $S^{p,m-1}(\mathcal{T})$ equipped with the L^2-norm and the H^s-norm yields the same space equipped with the $H^{s\theta}$-norm.

Corollary 3.1.10. *Let $m \in \{1,2\}$, $q \in [1,\infty]$, and $\theta \in (0,1)$. Then, there holds*

$$\Big((S^{p,m-1}(\mathcal{T}), \|\cdot\|_{L^2(\Omega)}), (S^{p,m-1}(\mathcal{T}), \|\cdot\|_{B^{m-1/2}_{2,\infty}(\Omega)})\Big)_{\theta,q} = \Big(S^{p,m-1}(\mathcal{T}), \|\cdot\|_{B^{\theta(m-1/2)}_{2,q}(\Omega)}\Big),$$

with equivalent norms. The norm equivalence constants depend only on Ω, p, q, θ, and the γ-shape regularity of $\mathcal{T}$. More generally, for any $B^{m'-1/2}_{2,q'}(\Omega)$ with $1/2 < m' < m$ and $q' \in [1,\infty]$, there holds, with equivalent norms,

$$\Big((S^{p,m-1}(\mathcal{T}), \|\cdot\|_{L^2(\Omega)}), (S^{p,m-1}(\mathcal{T}), \|\cdot\|_{B^{m'-1/2}_{2,q'}(\Omega)})\Big)_{\theta,q} = \Big(S^{p,m-1}(\mathcal{T}), \|\cdot\|_{B^{\theta(m'-1/2)}_{2,q}(\Omega)}\Big).$$

Proof. The proof follows from the existence of projection operators as presented in [AL09]. One needs a (stable) projection onto $S^{p,m-1}(\mathcal{T})$ satisfying Assumption 3.1.1, then Corollary 3.1.9 also provides the needed stability in the Besov-spaces. For $m = 1$, one may simply use the L^2-projection, which trivially satisfies Assumption 3.1.1. For $m = 2$, one employs the Scott-Zhang operator I^{SZ} of [SZ90a] without treating the boundary in a special way as it is done there. Then, I^{SZ} satisfies Assumption 3.1.1 by, e.g., [BS02, Sec. 4.8]. □

3.2 The finest common coarsening

Let $\mathcal{T}$ and $\mathcal{T}'$ be two regular triangulations obtained by NVB from the same triangulation $\widehat{\mathcal{T}}_0$. For a discussion of properties of NVB meshes, we refer to [KPP13] for the case $d = 2$ and to [Ste08] for the case $d \geq 3$. We define the *finest common coarsening* as

$$\texttt{fcc}(\mathcal{T},\mathcal{T}') := \underbrace{\{T \in \mathcal{T} \,:\, \exists T' \in \mathcal{T}' \text{ s.t. } T' \subsetneq T\}}_{=:\mathfrak{T}_1} \cup \underbrace{\{T' \in \mathcal{T}' \,:\, \exists T \in \mathcal{T} \text{ s.t. } T \subsetneq T'\}}_{=:\mathfrak{T}_2} \cup \underbrace{(\mathcal{T} \cap \mathcal{T}')}_{=:\mathfrak{T}_3}. \tag{3.2.1}$$

Figure 3.2.1 provides two examples for this concept. We refer to Lemma 3.2.1 for the proofs that the three sets in the definition (3.2.1) are pairwise disjoint and that $\texttt{fcc}(\mathcal{T},\mathcal{T}')$ is indeed a regular triangulation of Ω.

Let $\widehat{\mathcal{T}}_\ell$ be the ℓ-th uniform refinement of $\widehat{\mathcal{T}}_0$. We call $\text{level}(T) := \ell$ the *level* of an element $T \in \widehat{\mathcal{T}}_\ell$. Given a regular triangulation $\mathcal{T}$ that is obtained by NVB from $\mathcal{T}_0$ we will consider

$$\widetilde{\mathcal{T}}_\ell := \texttt{fcc}(\mathcal{T}, \widehat{\mathcal{T}}_\ell),$$

which is, in general, a coarser mesh than the uniform triangulation $\widehat{\mathcal{T}}_\ell$.

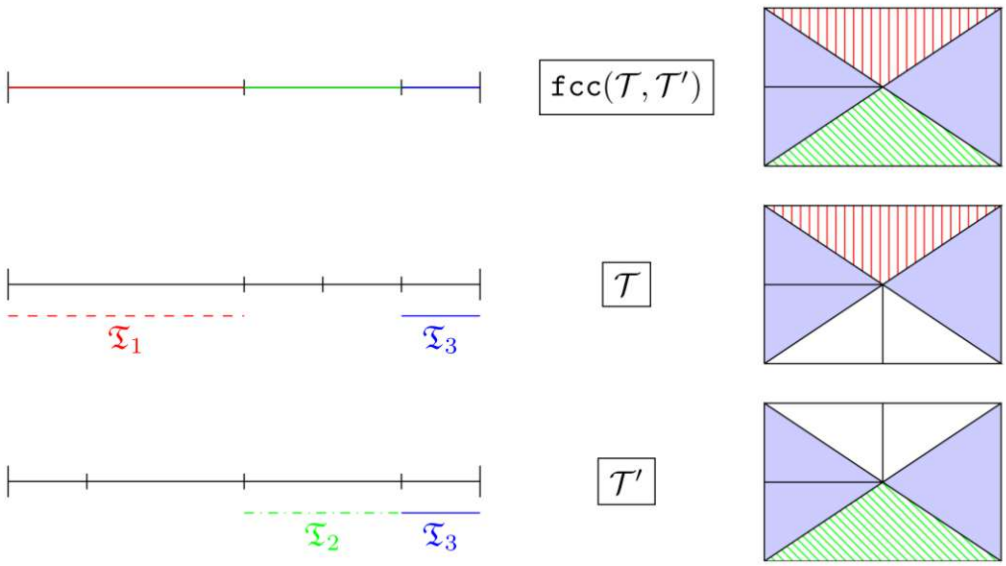

Figure 3.2.1: Example of the finest common coarsening of $\mathcal{T}$ and $\mathcal{T}'$ and the sets $\mathfrak{T}_1$ (coarser elements of $\mathcal{T}$, red), $\mathfrak{T}_2$ (coarser elements of $\mathcal{T}'$, green), $\mathfrak{T}_3$ (common elements, blue) in (3.2.1).

3.2.1 Properties of the finest common coarsening (fcc)

The following Lemma 3.2.1 shows that the finest common coarsening of two NVB meshes obtained from the same coarse regular triangulation is indeed a regular triangulation.

Lemma 3.2.1. *Let $\mathcal{T}$, $\mathcal{T}'$ be NVB refinements of the same common triangulation $\widehat{\mathcal{T}}_0$ of Ω. Then:*

1. *$\texttt{fcc}(\mathcal{T},\mathcal{T}') = \texttt{fcc}(\mathcal{T}',\mathcal{T})$. The three sets $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$ in the definition of $\texttt{fcc}(\mathcal{T},\mathcal{T}')$ are pairwise disjoint.*
2. *$\texttt{fcc}(\mathcal{T},\mathcal{T}')$ consists of simplices that cover Ω.*
3. *If $\mathcal{T}$ and $\mathcal{T}'$ are regular triangulations, then $\texttt{fcc}(\mathcal{T},\mathcal{T}')$ is a regular triangulation of Ω.*

Proof. Proof of 1: The symmetry of $\texttt{fcc}$ is obvious. To see that the sets $\mathfrak{T}_1$, $\mathfrak{T}_2$, $\mathfrak{T}_3$ are pairwise disjoint, let $T \in \mathfrak{T}_1$. Then $T \in \mathcal{T}$ but not in $\mathcal{T}'$. Hence, $T \notin \mathfrak{T}_2$ and $T \notin \mathfrak{T}_3$. By symmetry, $T \in \mathfrak{T}_2$ also implies $T \notin \mathfrak{T}_1$ and $T \notin \mathfrak{T}_3$. Finally, if $T \in \mathfrak{T}_3$, then it cannot be in $\mathfrak{T}_1$ or $\mathfrak{T}_2$.

Proof of 2: Let $x \in \Omega$ (but not on the skeleton of $\mathcal{T}$ or $\mathcal{T}'$). Since $\mathcal{T}$, $\mathcal{T}'$ cover Ω, there are $T \in \mathcal{T}$ and $T' \in \mathcal{T}'$ with $x \in T$, $x \in T'$. Since both T and T' are obtained by NVB and $T \cap T' \neq \emptyset$, we must have $T = T'$ or $T \subsetneq T'$ or $T' \subsetneq T$. In the first case $T = T' \in \mathfrak{T}_3$, in the second one $T' \in \mathfrak{T}_2$, and in the third one $T \in \mathfrak{T}_1$. Hence, x is in an element of $\texttt{fcc}(\mathcal{T},\mathcal{T}')$.

Proof of 3: Let T, T' be two elements of $\texttt{fcc}(\mathcal{T},\mathcal{T}')$ with $f := \overline{T} \cap \overline{T'} \neq \emptyset$. We have to show that for some j, the intersection $\overline{T} \cap \overline{T'} \neq \emptyset$ is a full j-face of both T and T'. If both T, T' are in $\mathcal{T}$ (or both are in $\mathcal{T}'$), then, by the regularity of $\mathcal{T}$ (or the regularity of $\mathcal{T}'$), their intersection is indeed a *full* j-face of either element. Assume therefore $T \in \mathcal{T} \setminus \mathcal{T}'$ and $T' \in \mathcal{T}'$ (or, similarly, $T \in \mathcal{T}$ and $T' \in \mathcal{T}' \setminus \mathcal{T}$). Since $T, T' \in \texttt{fcc}(\mathcal{T},\mathcal{T}')$, we obtain $T \in \mathfrak{T}_1$ and $T' \in \mathfrak{T}_2$. Since both T and T' are created by NVB from the same initial triangulation, the intersection $f = \overline{T} \cap \overline{T'}$ is a full j-face of either T or T'.

Let us assume that f is a full j-face of T, and, by contradiction, that f is not a full j-face of T'. Then, f is a proper subset of a j-face f' of T'. Since $T \in \mathfrak{T}_1$, it contains elements of $\mathcal{T}'$. Hence, there is an element $T_1' \in \mathcal{T}'$ with $T_1' \subset T$ that has a j-face f_1' with $f_1' \subset f$. Thus, we have found elements T', $T_1' \in \mathcal{T}'$ with j-faces $f_1' \subset f \subsetneq f'$, contradicting the regularity of $\mathcal{T}'$. Hence, f is also a full j-face of T'. Thus, $\texttt{fcc}(\mathcal{T},\mathcal{T}')$ is a regular triangulation. □

A completion of an (NVB-generated) mesh is any NVB refinement of it that is regular. We next show that the minimal completion is unique.

Lemma 3.2.2. *Let $\mathcal{T}$ be a NVB refinement of $\widehat{\mathcal{T}}_0$ and let $\mathcal{T}_1$, $\mathcal{T}_2$ be two completions of $\mathcal{T}$. Then $\texttt{fcc}(\mathcal{T}_1,\mathcal{T}_2)$ is a completion of $\mathcal{T}$. The completion of minimal cardinality is unique.*

Proof. Let $\mathcal{T}_3 := \texttt{fcc}(\mathcal{T}_1,\mathcal{T}_2)$. We claim that $\mathcal{T}_3$ is a completion of $\mathcal{T}$. Since $\mathcal{T}_3$ is regular by Lemma 3.2.1, we have to assert that each element of $\mathcal{T}_3$ is contained in an element of $\mathcal{T}$. Suppose not. Then there is $T_3 \in \mathcal{T}_3$ and a $T \in \mathcal{T}$ with $T \subsetneq T_3$. (We use that these meshes are obtained by NVB from a common $\mathcal{T}_0$.). By definition, T_3 is either in $\mathcal{T}_1$ or $\mathcal{T}_2$, which are both completions of $\mathcal{T}$, i.e., their elements are contained in elements of $\mathcal{T}$. This is a contradiction.

To see the uniqueness of the minimal completion, let $\mathcal{T}_1 \neq \mathcal{T}_2$ be two completions of minimal cardinality N. Note that $\mathcal{T}_3 := \texttt{fcc}(\mathcal{T}_1,\mathcal{T}_2)$ is also a completion. However, in view of $\mathcal{T}_1 \neq \mathcal{T}_2$, at least one element of, say, $\mathcal{T}_1$ is a refinement of an element of $\mathcal{T}_2$ so that we have by definition of $\texttt{fcc}(\mathcal{T}_1, \mathcal{T}_2)$ that $\operatorname{card} \mathcal{T}_3 \leq N - 1$, which contradicts the minimality. □

Lemma 3.2.3. *Let $\widehat{\mathcal{T}}_\ell$, $\ell = 0, 1, \dots,$ be a sequence of uniform refinements of a regular mesh $\widehat{\mathcal{T}}_0$ and $\widetilde{\mathcal{T}}_\ell = \texttt{fcc}(\mathcal{T}, \widehat{\mathcal{T}}_\ell)$. Then:*

(i) If $T \in \widetilde{\mathcal{T}}_\ell \cap \mathcal{T}$ then $T \in \widetilde{\mathcal{T}}_{\ell+m}$ for all $m \geq 0$.

(ii) If $T \in \widetilde{\mathcal{T}}_\ell \setminus \mathcal{T}$ then $T \notin \widetilde{\mathcal{T}}_{\ell+1}$.

(iii) Denote by $\widetilde{\mathcal{N}}^1_\ell$ the set of nodes of $\widetilde{\mathcal{T}}_\ell$. Then $\widetilde{\mathcal{N}}^1_{\ell+1} \supset \widetilde{\mathcal{N}}^1_\ell$ for all ℓ.

(iv) Let $\widetilde{\mathcal{M}}^1_\ell = \widetilde{\mathcal{N}}^1_\ell \setminus \widetilde{\mathcal{N}}^1_{\ell-1} \cup \{z \in \widetilde{\mathcal{N}}^1_\ell \cap \widetilde{\mathcal{N}}^1_{\ell-1} \,|\, \omega_\ell(z) \subsetneq \omega_{\ell-1}(z)\}$. Then, we have $\operatorname{card} \widetilde{\mathcal{M}}^1_\ell \leq C \operatorname{card} \widetilde{\mathcal{N}}^1_\ell \setminus \widetilde{\mathcal{N}}^1_{\ell-1}$ for a $C > 0$ depending only on the shape regularity of the triangulations.

Proof. For statement i, we only show the case $m = 1$ as the general case follows by induction. We note that $T \in \widetilde{\mathcal{T}}_\ell \cap \mathcal{T}$ implies $T \notin \mathfrak{T}_{2,\ell}$, where $\mathfrak{T}_{i,\ell}, \in \{1, 2, 3\}$ are the three sets given in (3.2.1). If $T \in \mathfrak{T}_{3,\ell}$, then $T \in \mathfrak{T}_{1,\ell+1}$. If $T \in \mathfrak{T}_{1,\ell}$, then, $T \in \mathfrak{T}_{1,\ell+1}$. For statement ii, we note that $T \in \widetilde{\mathcal{T}}_\ell \setminus \mathcal{T}$ implies $T \in \widehat{\mathcal{T}}_\ell \setminus \mathcal{T}$ and hence T is neither in $\widehat{\mathcal{T}}_{\ell+1}$ nor in $\mathcal{T}$. Hence $T \notin \widetilde{\mathcal{T}}_{\ell+1}$.

For statement iii, let $z \in \widetilde{\mathcal{N}}^1_\ell$ and $T \in \widetilde{\mathcal{T}}_\ell$ be an element such that z is a node of T. We consider two cases. First, if $T \in \mathcal{T} \cap \widetilde{\mathcal{T}}_\ell$, then, by statement i, we have $T \in \mathcal{T}_{\ell+1}$ so that $z \in \widetilde{\mathcal{N}}^1_{\ell+1}$. Second, let $T \in \widetilde{\mathcal{T}}_\ell \setminus \mathcal{T}$. Then $T \in \widehat{\mathcal{T}}_\ell$ and in fact in $\mathfrak{T}_{2,\ell}$. The node z is the node of an element $T' \in \widehat{\mathcal{T}}_{\ell+1}$. This element T' is either in $\mathcal{T}$, which implies $z \in \widetilde{\mathcal{N}}^1_{\ell+1}$, or $T' \in \mathfrak{T}_{2,\ell+1}$, which also implies $z \in \widetilde{\mathcal{N}}^1_{\ell+1}$.

For statement iv, one observes that for a node $z \in \{z \in \widetilde{\mathcal{N}}^1_\ell \cap \widetilde{\mathcal{N}}^1_{\ell-1} \,|\, \omega_\ell(z) \subsetneq \omega_{\ell-1}(z)\}$, there are elements $T \in \widetilde{\mathcal{T}}_{\ell-1}$ and $T' \in \widetilde{\mathcal{T}}_\ell$ with $T' \subsetneq T$ and z is a node of T. Hence $T' \in \widetilde{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_{\ell-1}$, and it has a node $z' \in \widetilde{\mathcal{N}}^1_\ell \setminus \widetilde{\mathcal{N}}^1_{\ell-1}$. We conclude $\operatorname{card}\{z \in \widetilde{\mathcal{N}}^1_\ell \cap \widetilde{\mathcal{N}}^1_{\ell-1} \,|\, \omega_\ell(z) \subsetneq \omega_{\ell-1}(z)\} \leq \operatorname{card} \widetilde{\mathcal{N}}^1_\ell \setminus \widetilde{\mathcal{N}}^1_{\ell-1}$. □

Remark 3.2.4. If the shape-regular mesh $\mathcal{T}$ is obtained by repeated NVB from a coarse grid $\mathcal{T}_0$, then a simpler proof is possible for Corollary 3.1.4: one may take a quasi-uniform mesh $\mathcal{T}_t$ of mesh size $\sim t$ and consider $\widetilde{\mathcal{T}} := \texttt{fcc}(\mathcal{T}, \mathcal{T}_t)$. Then, J_t can be taken as a mollifier of the standard Scott-Zhang operator associated with $\widetilde{\mathcal{T}}$. ■

3.3 Adapted Scott-Zhang operators

The scott-Zhang operators defined in Section 2.4.1 satisfy the stability and approximation properties of Assumption 3.1.1 with constants that solely depend on p, the specific polynomial basis, the shape-regularity of the underlying triangulation, and Ω. In particular, the constants are independent of the specific choice of averaging region T_z.

The freedom in the choice of the averaging element T_z can be exploited to ensure additional properties, see also [CNX12, Sec. 4], [DKS16, Sec. 3],[FFPS17a, Sec. 4.3]. For the Scott-Zhang operator on general NVB meshes, the mesh decomposition of [CNX12] can be employed to transfer information between the refinement levels. In the following, we define a modified Scott-Zhang operator for the hierarchy $\big(\texttt{fcc}(\mathcal{T},\widehat{\mathcal{T}}_\ell)\big)_\ell$, where a guiding principle is that in the definition of $\widetilde{I}^{SZ}$ one selects the averaging element T_z from the mesh $\mathcal{T}$ whenever possible:

Definition 3.3.1 (adapted Scott-Zhang operators)**.** For given $\mathcal{T}$ that is obtained by NVB-refinement from a regular triangulation $\widehat{\mathcal{T}}_0$ and $\widetilde{\mathcal{T}}_\ell = \texttt{fcc}(\mathcal{T},\widehat{\mathcal{T}}_\ell)$, the operators $\widetilde{I}^{SZ}_\ell : L^2(\Omega) \to \widetilde{V}_\ell = S^{p,1}(\widetilde{\mathcal{T}}_\ell)$ and $\widehat{I}^{SZ}_\ell : L^2(\Omega) \to \widehat{V}_\ell = S^{p,1}(\widehat{\mathcal{T}}_\ell)$ are Scott-Zhang operators as defined in (2.4.9) with the following choice of averaging element T_z for $\widetilde{I}^{SZ}_\ell$ and $\widehat{I}^{SZ}_\ell$:

(1) First, loop through all $T \in \widehat{\mathcal{T}}_\ell \cap \widetilde{\mathcal{T}}_\ell$ (in any fixed order) and select the averaging sets T_z for the nodes $z \in \overline{T}$ as follows:
   (a) If $z \in T$, then select $T_z = T$ for both $\widehat{I}^{SZ}_\ell$ and $\widetilde{I}^{SZ}_\ell$.
   (b) If $z \in \partial T$ and the node z has not been assigned an averaging set T_z yet, then:
      (i) If $\mathcal{A}(z,\widehat{\mathcal{T}}_\ell)$ contains an element $T' \in \widehat{\mathcal{T}}_\ell$ that is a proper subset of an element $\widetilde{T} \in \mathcal{T}$, then select this T' to define $\widehat{I}^{SZ}_\ell$ and select $\widetilde{T}$ for the definition of $\widetilde{I}^{SZ}_\ell$.
      (ii) Else select T for both $\widehat{I}^{SZ}_\ell$ and $\widetilde{I}^{SZ}_\ell$.

(2) Next, loop through all $T \in \widetilde{\mathcal{T}}_\ell \setminus \widehat{\mathcal{T}}_\ell$ (in any fixed order). Select, for the construction of $\widetilde{I}^{SZ}_\ell$, this T as the averaging element for all nodes z with $z \in \overline{T}$ that have not already been fixed in step (1) or in a previous step of the loop. This completes the definition of $\widetilde{I}^{SZ}_\ell$.

(3) Finally, loop through all $T \in \widehat{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_\ell$ (in any fixed order). Select, for the construction of $\widehat{I}^{SZ}_\ell$, this T as the averaging element for all nodes z with $z \in \overline{T}$ that have not already been fixed in step (1) or in a previous step of the loop. This completes the definition of $\widehat{I}^{SZ}_\ell$.

We note, that this definition of the adapted Scott-Zhang operators is exploited to show $\widehat{I}^{SZ}_\ell u = \widetilde{I}^{SZ}_\ell u$ for all $u \in S^{p,1}(\mathcal{T})$, which is proven in Lemma 3.3.2 below.

The following lemma shows that the adapted Scott-Zhang operators for the meshes $\widetilde{\mathcal{T}}_\ell$ and $\widehat{\mathcal{T}}_\ell$ coincide on piecewise polynomials on the mesh $\mathcal{T}$.

Lemma 3.3.2. *Let $\mathcal{T}$ be generated by NVB from $\widehat{\mathcal{T}}_0$. Let $\widetilde{I}^{SZ}_\ell : L^2(\Omega) \to S^{p,1}(\widetilde{\mathcal{T}}_\ell)$ and $\widehat{I}^{SZ}_\ell : L^2(\Omega) \to S^{p,1}(\widehat{\mathcal{T}}_\ell)$ be the Scott-Zhang operators defined in Definition 3.3.1. Then, there holds*

$$\widetilde{I}^{SZ}_\ell u = \widehat{I}^{SZ}_\ell u \qquad \forall u \in S^{p,1}(\mathcal{T}).$$

Proof. 1. step: Let $T \in \widehat{\mathcal{T}}_\ell \cap \widetilde{\mathcal{T}}_\ell$. We claim that $(\widetilde{I}^{SZ}_\ell u)|_T = (\widehat{I}^{SZ}_\ell u)|_T$. The nodes $z \in \overline{T}$ and the shape functions $\varphi_{z,\widehat{\mathcal{T}}_\ell}$, $\varphi_{z,\widetilde{\mathcal{T}}_\ell}$ for the meshes $\widehat{\mathcal{T}}_\ell$ and $\texttt{fcc}(\mathcal{T},\widehat{\mathcal{T}}_\ell)$ coincide on T. For the averaging element T_z associated with $z \in \overline{T}$, two cases can occur:

1. The two averaging sets for the two operators coincide. This happens in the following three cases: a) if $z \in T$ (case 1a of Def. 3.3.1); b) if $z \in \partial T$ and (case 1(b)ii of Def. 3.3.1) arose for T in the loop; c) (case 1(b)ii of Def. 3.3.1) arose for an element $T' \in \widehat{\mathcal{T}}_\ell \cap \widetilde{\mathcal{T}}_\ell$ with $z \in \overline{T'}$ that appeared earlier in the loop than T. Since the averaging sets coincide, the value of the linear functionals are the same.

2. Case 1(b)i of Def. 3.3.1 arose. Then, both averaging sets are contained in an element $\widetilde{T} \in \mathcal{T}$. Since $u|_{\widetilde{T}} \in P_p$, we obtain from (2.4.8) that both linear functionals equal $u(z)$.

Hence, in all cases the values of the linear functionals coincide so that indeed the Scott-Zhang operators on the element T are equal.

2. step: In the region not covered by elements in $\widehat{\mathcal{T}}_\ell \cap \widetilde{\mathcal{T}}_\ell$ we show $\widetilde{I}_\ell^{SZ} u = u$ and $\widehat{I}_\ell^{SZ} u = u$ for $u \in S^{p,1}(\mathcal{T})$. For $\widetilde{I}_\ell^{SZ}$ this is shown in step 3 and for $\widehat{I}_\ell^{SZ}$ in step 4. This completes the proof of the lemma.

3. step: We start by noting that the definition of the finest common coarsening implies

$$\text{for any } T' \in \widehat{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_\ell \text{ there exists } \widetilde{T} \in \mathcal{T} \text{ with } T' \subset \widetilde{T}. \tag{3.3.1}$$

Consider now $T \in \widehat{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_\ell$. By (3.3.1) there exists $\widetilde{T} \in \mathcal{T}$ such that $T \subset \widetilde{T}$. For $u \in S^{p,1}(\mathcal{T})$ we have $u|_{\widetilde{T}} \in P_p(\widetilde{T})$. Moreover, $(\widetilde{I}_\ell^{SZ} u)|_T = \sum_{z \in \mathcal{N}_p(T)} \varphi_{z,\widehat{\mathcal{T}}_\ell} l_z(u)$ with the linear functional $l_z(u) = \int_{T_z} \varphi^*_{z,T} u$. For the interior nodes $z \in T$ we have $T_z = T$ and, since $u|_T \in P_p(T)$, $l_z(u) = u(z)$ by (2.4.8). For $z \in \partial T$, the following cases may occur:

(1) If $T_z = T$, then again $l_z(u) = u(z)$ by (2.4.8).

(2) If T_z is a neighbouring element of T, then the following cases can occur:
   (a) $T_z \in \widehat{\mathcal{T}}_\ell \cap \widetilde{\mathcal{T}}_\ell$: Then, $z \in \partial T$ and hence also in ∂T_z. The construction of the averaging sets in Def. 3.3.1 is such that the averaging set T_z for node z is chosen such that it is contained in an element $T' \in \mathcal{T}$ if possible. Since $T \subset \widetilde{T} \in \mathcal{T}$ is possible by (3.3.1), we conclude that also $T_z \subset T'' \in \mathcal{T}$ for some $T'' \in \mathcal{T}$. Hence, $u|_{T_z} \in P_p(T_z)$, and the value of the linear functional is $u(z)$.
   (b) $T_z \in \widehat{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_\ell$. Then, by (3.3.1) we get $u|_{T_z} \in P_p(T_z)$ so that again by (2.4.8) $l_z(u) = u(z)$.

In total, we have arrived at $(\widehat{I}_\ell^{SZ} u)|_T = \sum_{z \in \mathcal{N}_p(T)} \varphi_{z,\widehat{\mathcal{T}}_\ell} u(z) = u|_T$, since $u|_T \in P_p(T)$.

4. step: Consider $T \in \widetilde{\mathcal{T}}_\ell \setminus \widehat{\mathcal{T}}_\ell$. Then $T \in \mathcal{T}$. We have $(\widetilde{I}_\ell^{SZ} u)|_T = \sum_{z \in \mathcal{N}_p(T)} \varphi_{z,\widetilde{\mathcal{T}}_\ell} l_z(u)$ with the linear functional $l_z(u) = \int_{T_z} \varphi^*_{z,T} u$. For the interior nodes $z \in T$ we have $T_z = T$ and, since $u|_T \in P_p(T)$, the property (2.4.8) gives $l_z(u) = u(z)$.

For $z \in \partial T$, two cases may occur: If $T_z = T$, then again $l_z(u) = u(z)$ by (2.4.8). If T_z is a neighboring element of T, then either $T_z \in \widehat{\mathcal{T}}_\ell \cap \widetilde{\mathcal{T}}_\ell$, which means $l_z(u) = u(z)$ by the same reasoning as in step 3, item 2a, or $T_z \in \widetilde{\mathcal{T}}_\ell \setminus \widehat{\mathcal{T}}_\ell \subset \mathcal{T}$ so that $u|_{T_z} \in P_p(T_z)$ and thus by (2.4.8) $l_z(u) = u(z)$. In total, we have arrived at $(\widetilde{I}_\ell^{SZ} u)|_T = \sum_{z \in \mathcal{N}_p(T)} \varphi_{z,\widetilde{\mathcal{T}}_\ell} u(z) = u|_T$, since $u|_T \in P_p(T)$. □

3.4 Multilevel decomposition based on mesh hierarchies generated by NVB

With the use of the adapted Scott-Zhang operators $\widetilde{I}_\ell^{SZ}$ and a mesh hierarchy based on the finest common coarsening between NVB meshes and uniformly refined meshes, we obtain a multilevel decomposition with norm equivalence in the Besov space $B^{3\theta/2}_{2,q}(\Omega)$ as a consequence of the stability estimate of Theorem 3.1.5.

Theorem 3.4.1. *Let $\mathcal{T}$ be a mesh obtained by NVB refinement of a triangulation $\widehat{\mathcal{T}}_0$ with mesh size $\widehat{h}_0$. Let $\widehat{\mathcal{T}}_\ell$ be the sequence of uniformly refined meshes starting from $\widehat{\mathcal{T}}_0$ with mesh size $\widehat{h}_\ell = \widehat{h}_0 2^{-\ell}$. Set $\widetilde{\mathcal{T}}_\ell := \texttt{fcc}(\mathcal{T}, \widehat{\mathcal{T}}_\ell)$. Let $\widetilde{I}_\ell^{SZ} : L^2(\Omega) \to S^{p,1}(\widetilde{\mathcal{T}}_\ell)$ be the adapted Scott-Zhang operator defined in Definition 3.3.1. Then, on the space $S^{p,1}(\mathcal{T})$ the following three norms are equivalent with equivalence constants depending only on $\widehat{\mathcal{T}}_0$, p, $\theta \in (0,1)$, and $q \in [1,\infty]$:*

$$\|u\|_{B^{3\theta/2}_{2,q}(\Omega)}, \tag{3.4.1}$$

$$\|\widetilde{I}_0^{SZ} u\|_{L^2(\Omega)} + \|(2^{3\theta\ell/2}\|u - \widetilde{I}_\ell^{SZ} u\|_{L^2(\Omega)})_{\ell\geq 0}\|_{\ell^q}, \tag{3.4.2}$$

$$\|\widetilde{I}_0^{SZ} u\|_{L^2(\Omega)} + \|(2^{3\theta\ell/2}\|\widetilde{I}_{\ell+1}^{SZ} u - \widetilde{I}_\ell^{SZ} u\|_{L^2(\Omega)})_{\ell\geq 0}\|_{\ell^q}. \tag{3.4.3}$$

Proof. We apply [Coh03, Thm. 3.5.3] for the spaces $X = \left(S^{p,1}(\mathcal{T}), \|\cdot\|_{L^2(\Omega)}\right)$, $Y = \left(S^{p,1}(\mathcal{T}), \|\cdot\|_{B^{3/2}_{2,\infty}(\Omega)}\right)$ noting that we have $S^{p,1}(\widetilde{\mathcal{T}}_\ell) \subset S^{p,1}(\mathcal{T})$. Then, [Coh03, Thm. 3.5.3] provides the equivalence of the second and third norm to the norm on the interpolation space $(X,Y)_{\theta,q}$, which by Corollary 3.1.10 is the $B^{3/2\theta}_{2,q}(\Omega)$-norm, provided a Jackson-type and a Bernstein-type estimate holds.

1. step (Jackson-type inequality): Using Lemma 3.3.2, we compute for $u \in S^{p,1}(\mathcal{T})$ and arbitrary $w \in S^{p,1}(\widehat{\mathcal{T}}_\ell)$

$$\begin{aligned}\inf_{v\in S^{p,1}(\widetilde{\mathcal{T}}_\ell)} \|u - v\|_{L^2(\Omega)} &\leq \|u - \widetilde{I}_\ell^{SZ} u\|_{L^2(\Omega)} = \|u - \widehat{I}_\ell^{SZ} u\|_{L^2(\Omega)} \\ &= \|u - w - \widehat{I}_\ell^{SZ}(u-w)\|_{L^2(\Omega)} \lesssim \|u - w\|_{L^2(\Omega)}.\end{aligned}$$

Hence, standard approximation results from [Wid77, p. 332] on the quasi-uniform meshes $\widehat{\mathcal{T}}_\ell$ of mesh size $\widehat{h}_\ell = \widehat{h}_0 2^{-\ell}$ provide

$$\inf_{v\in S^{p,1}(\widetilde{\mathcal{T}}_\ell)} \|u - v\|_{L^2(\Omega)} \lesssim \inf_{w\in S^{p,1}(\widehat{\mathcal{T}}_\ell)} \|u - w\|_{L^2(\Omega)} \lesssim \widehat{h}_\ell^{3/2}\|u\|_{B^{3/2}_{2,\infty}(\Omega)} \lesssim 2^{-3\ell/2}\|u\|_{B^{3/2}_{2,\infty}(\Omega)}. \tag{3.4.4}$$

We note that this estimate also implies the additional assumption [Coh03, Eqn.(3.5.29)] on the projection operators $\widetilde{I}_\ell^{SZ}$.

2. step (Bernstein-type inequality): Using the projection property of the Scott-Zhang operators and Lemma 3.3.2, we get for arbitrary $v \in S^{p,1}(\widetilde{\mathcal{T}}_\ell)$

$$\begin{aligned}\|v\|_{B^{3/2}_{2,\infty}(\Omega)} &= \|\widetilde{I}_\ell^{SZ} v\|_{B^{3/2}_{2,\infty}(\Omega)} = \|\widehat{I}_\ell^{SZ} v\|_{B^{3/2}_{2,\infty}(\Omega)} \overset{\text{Lem. 3.1.8}}{\lesssim} \widehat{h}_\ell^{-3/2}\|\widehat{I}_\ell^{SZ} v\|_{L^2(\Omega)} \\ &= \widehat{h}_\ell^{-3/2}\|\widetilde{I}_\ell^{SZ} v\|_{L^2(\Omega)} = \widehat{h}_\ell^{-3/2}\|v\|_{L^2(\Omega)}.\end{aligned} \tag{3.4.5}$$

As the family of operators $\widetilde{I}_\ell^{SZ} : X \to S^{p,1}(\widetilde{\mathcal{T}}_\ell)$ is also uniformly bounded in the $L^2(\Omega)$-norm, all assumptions of [Coh03, Thm. 3.5.3] are valid and consequently the norm equivalences are proven. □

3.5 Boundary conditions

Let the Hilbert space $\widetilde{H}^s(\Omega)$ be defined as

$$\widetilde{H}^s(\Omega) := \{u \in H^s(\mathbb{R}^d) \,:\, u \equiv 0 \text{ on } \Omega^c\}, \qquad \|v\|^2_{\widetilde{H}^s(\Omega)} := \|v\|^2_{H^s(\Omega)} + \left\|\mathrm{dist}(\cdot, \partial\Omega)^{-s} v\right\|^2_{L^2(\Omega)}.$$

The previous results do not consider (homogeneous) Dirichlet boundary conditions. For the application we have in mind (cf. (4.2.1)), an interpolation result similar to Corollary 3.1.10 for the spaces $L^2(\Omega)$, $H^1_0(\Omega)$ and $\widetilde{H}^s(\Omega)$ for $s \in (0,1)$ is of interest. Such results are already available in the literature, see, e.g., [AFF$^+$15], where the proof uses stability properties of the Scott-Zhang projection and the abstract result from [AL09], similarly to Corollary 3.1.10. For sake of completeness, we state the result in the following corollary.

Corollary 3.5.1. *Let $s \in (0,1)$. Then, there holds*

$$\Big((S^{p,1}_0(\mathcal{T}), \|\cdot\|_{L^2(\Omega)}), (S^{p,1}_0(\mathcal{T}), \|\cdot\|_{H^1(\Omega)})\Big)_{s,2} = \Big(S^{p,1}_0(\mathcal{T}), \|\cdot\|_{\widetilde{H}^s(\Omega)}\Big),$$

with equivalent norms.

As done, for example, in [AFF$^+$15], the Scott-Zhang operators $\widetilde{I}^{SZ}_\ell$ and $\widehat{I}^{SZ}_\ell$ can be modified by simply dropping the contributions from the shape functions associated with nodes on $\partial\Omega$ and thus map into the spaces $S^{p,1}_0(\widetilde{\mathcal{T}}_\ell)$ and $S^{p,1}_0(\widehat{\mathcal{T}}_\ell)$, respectively. We denote these operators by $\widetilde{I}^{SZ}_{0,\ell}$ and $\widehat{I}^{SZ}_{0,\ell}$, and they are still stable in $L^2(\Omega)$ and $H^1_0(\Omega)$. Therefore, Theorem 3.4.1 also provides a lower bound for the multilevel decomposition based on the Scott-Zhang operator in the $\widetilde{H}^s(\Omega)$-norm.

Corollary 3.5.2. *Let $\mathcal{T}$ be a mesh obtained by NVB refinement of a triangulation $\widehat{\mathcal{T}}_0$. Let $\widehat{\mathcal{T}}_\ell$ be the sequence of uniformly refined meshes starting from $\widehat{\mathcal{T}}_0$ with mesh size $\widehat{h}_\ell = \widehat{h}_0 2^{-\ell}$. Set $\widetilde{\mathcal{T}}_\ell := \texttt{fcc}(\mathcal{T}, \widehat{\mathcal{T}}_\ell)$. Let $\widetilde{I}^{SZ}_{0,\ell} : \widetilde{H}^s(\Omega) \to S^{p,1}_0(\widetilde{\mathcal{T}}_\ell)$ be the Scott-Zhang operator defined as above. Then, we have*

$$\sum_{\ell=0}^{\infty} \widehat{h}_\ell^{-2s} \left\|u - \widetilde{I}^{SZ}_{0,\ell} u\right\|^2_{L^2(\Omega)} \le C_s \|u\|^2_{\widetilde{H}^s(\Omega)} \qquad \forall u \in S^{p,1}_0(\mathcal{T}), \quad 0 < s < 1. \tag{3.5.1}$$

Proof. We note that Jackson-type and Bernstein-type estimates (3.4.4) and (3.4.5) in the proof of Theorem 3.4.1 also hold for the variant of the Scott-Zhang projection that preserves homogeneous boundary conditions, if we replace $\widehat{h}_\ell^{3/2}\|u\|_{B^{3/2}_{2,\infty}(\Omega)}$ with $\widehat{h}_\ell\|u\|_{H^1_0(\Omega)}$ in (3.4.4), and if we replace in (3.4.5) the norms $\|\cdot\|_{B^{3/2}_{2,\infty}(\Omega)}$ with $\|\cdot\|_{H^1(\Omega)}$ and correspondingly $\widehat{h}^{-3/2}$ with $\widehat{h}^{-1}$. Therefore, the norm equivalences of Theorem 3.4.1 are still valid if one replace $B^{3\theta/2}_{2,\infty}(\Omega)$ with $H^\theta_0(\Omega)$, $\widetilde{I}^{SZ}_\ell$ with $\widetilde{I}^{SZ}_{0,\ell}$, and $2^{3\theta\ell/2}$ with $2^{\theta\ell}$. $\square$

4 An optimal multilevel preconditioner for the fractional Laplacian

In this chapter, we present a multilevel diagonal preconditioner with uniformly bounded condition number on locally refined triangulations for the fractional Laplacian. We also adopt the additive Schwarz framework and show that, also in the presence of adaptively refined meshes, multilevel diagonal scaling leads to uniformly bounded condition numbers for the integral fractional Laplacian. The norm equivalence of the multilevel decomposition in Chapter 3 provides the lower bound for the eigenvalues and an inverse estimate in fractional Sobolev norms, similar to [FMP19], gives the upper bound for the eigenvalues. We mention that very closely related to preconditioning of discretizations of the fractional differential operators is earlier work on preconditioning for the hypersingular integral equation (e.g., the operators coincide for the case $s = 1/2$ for screen problems) in boundary element methods (BEMs), [TS96, TSM97, TSZ98, AM03, Mai09, FFPS17a].

4.1 Fractional Laplacian

4.1.1 Singular integral representation

We denote the principal value of the integral as

$$\text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy,$$

where B_ε is a ball of radius ε. One representation for the fractional Laplacian is a pointwise characterization based on the principal value integral, i.e.,

$$(-\Delta)^s u(x) := C(d, s)\, \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x - y|^{d+2s}} dy \qquad C(d, s) := 2^{2s} s \frac{\Gamma(s + d/2)}{\pi^{d/2}\Gamma(1 - s)} \qquad s \in (0, 1),$$

where $\Gamma(\,\cdot\,)$ denotes the Gamma function, see [Kwa17].

4.1.2 The Caffarelli-Silvestre extension

One of the main difficulties in the study of fractional differential equations is the nonlocality nature of theses derivatives. To overcome this, Caffarelli-Silvestre [CS07] proved that the fractional Laplacian in $\mathbb{R}^d$ can be written as an operator mapping a Dirichlet boundary condition to a Neumann-type condition using an extension problem on the half-space $\mathbb{R}^{d+1}_+ := \{(\mathbf{x}, y) \,|\, \mathbf{x} \in \mathbb{R}^d,\ y > 0\}$.

Definition 4.1.1. Let $\alpha := 1 - 2s \in (-1, 1)$ and $\mathcal{S} \subset \mathbb{R}^{d+1}_+$ be a measurable set, then we define the following weighted L^2-norm

$$\|u\|^2_{L^2_\alpha(\mathcal{S})} := \int_{\mathcal{S}} y^\alpha \, |u(\mathbf{x}, y)|^2 \, d\mathbf{x}dy.$$

Also, we introduce

$$L^2_\alpha(\mathcal{S}) := \left\{ u \in L^2(\mathcal{S}) \mid \|u\|^2_{L^2_\alpha(\mathcal{S})} < \infty \right\}.$$

Let $D'(\mathbb{R}^{d+1}_+)$ denote the space of all distributions. Then, the Beppo-Levi space is defined as

$$\dot{H}^s(\mathbb{R}^{d+1}_+) := \left\{ u \in D'(\mathbb{R}^{d+1}_+) \,|\, \nabla u \in L^2_\alpha(\mathbb{R}^{d+1}_+) \right\}.$$

The fractional Laplacian can be written as the Neumann data of the extension problem, i.e.,

$$(-\Delta)^s u(x) = -d_s \lim_{y \to 0^+} y^{1-2s} \frac{\partial}{\partial y} (\mathcal{L}_s u)(\mathbf{x}, y) \qquad \mathbf{x} \in \mathbb{R}^d,$$

where $d_s := 2^{(1-2s)} \, |\Gamma(s)| \, / \Gamma(1 - s)$ and $(\mathcal{L}_s u) \in \dot{H}^s(\mathbb{R}^{d+1}_+)$ is a solution to the following extension problem by Caffarelli-Silvestre

$$\begin{aligned} - \operatorname{div} \left(y^{1-2s} \nabla \mathcal{L}_s u \right) &= 0 && \text{in} \quad \mathbb{R}^{d+1}_+, \\ \operatorname{tr} \mathcal{L}_s \mathrm{u} &= u && \text{in} \quad \mathbb{R}^d \times \{y = 0\}. \end{aligned}$$

4.2 Model problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. In this section, we consider the equation

$$\begin{aligned} (-\Delta)^s u &= f && \text{in} \quad \Omega, \\ u &= 0 && \text{in} \quad \Omega^c, \end{aligned} \tag{4.2.1}$$

for a given right-hand side $f \in H^{-s}(\Omega)$.

The weak formulation of (4.2.1) is given by finding $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u, v) := \frac{C(d, s)}{2} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2s}} \, dx \, dy = \int_\Omega f v \, dx \quad \forall v \in \widetilde{H}^s(\Omega). \tag{4.2.2}$$

Existence and uniqueness of $u \in \widetilde{H}^s(\Omega)$ follow from the Lax–Milgram lemma.

With a given regular triangulation $\mathcal{T}_0$, we consider two hierarchical sequence of meshes $\mathcal{T}_\ell$, $\widetilde{\mathcal{T}}_\ell$, $\ell = 0, \dots, L$:

1. (sequence $(\mathcal{T}_\ell)_\ell$): The meshes $\mathcal{T}_\ell$ are generated by an adaptive algorithm (see, e.g., [Dör96]) of the form `SOLVE` − `ESTIMATE` − `MARK` − `REFINE`, where the step `REFINE` is done by newest vertex bisection. In the following, both for the case of piecewise linear and piecewise constant basis function, we always assume that the meshes $\mathcal{T}_\ell$ are regular in the sense of Ciarlet.

2. (sequence $(\widetilde{\mathcal{T}}_\ell)_\ell$): From a given triangulation $\mathcal{T}_L$ obtained by NVB refinement of $\mathcal{T}_0$, which may, e.g., be obtained from an adaptive algorithm, the finest common coarsening of $\mathcal{T}_L$ with the uniform refinements of $\mathcal{T}_0$ (denoted by $\widehat{\mathcal{T}}_\ell$) provides a hierarchy of meshes $\widetilde{\mathcal{T}}_\ell = \texttt{fcc}(\mathcal{T}_L, \widehat{\mathcal{T}}_\ell)$.

4.3 Local multilevel diagonal preconditioners

4.3.1 A local multilevel diagonal preconditioner for adaptively refined meshes

We start with the case of the adaptively generated mesh hierarchy $(\mathcal{T}_\ell)_\ell$. On the mesh $\mathcal{T}_\ell$, we discretize with piecewise constants (for $0 < s < 1/2$) as the space $V_\ell^0 = S^{0,0}(\mathcal{T}_\ell)$ and piecewise linears (for $0 < s < 1$) as the space $V_\ell^1 = S_0^{1,1}(\mathcal{T}_\ell)$. If the distinction between V_ℓ^0 and V_ℓ^1 is not essential, we write V_ℓ meaning $V_\ell \in \{V_\ell^0, V_\ell^1\}$. The Galerkin discretization (4.2.1) in V_ℓ of reads as: Find $u_\ell \in V_\ell$, such that

$$a(u_\ell, v_\ell) = \langle f, v_\ell \rangle_{L^2(\Omega)} \qquad \forall v_\ell \in V_\ell. \tag{4.3.1}$$

Moreover, on the uniformly refined meshes $\widehat{\mathcal{T}}_\ell$, in the same way, we define the discrete spaces $\widehat{V}_\ell^0 = S^{0,0}(\widehat{\mathcal{T}}_\ell)$, $\widehat{V}_\ell^1 = S_0^{1,1}(\widehat{\mathcal{T}}_\ell)$, and $\widehat{V}_\ell \in \{\widehat{V}_\ell^0, \widehat{V}_\ell^1\}$.

We define sets of "characteristic" points $\mathcal{N}_\ell^i$, $i = 0, 1$, representing the degrees of freedom of V_ℓ. For the piecewise constant case V_ℓ^0, the set $\mathcal{N}_\ell^0$ comprises all barycenters of elements of the mesh $\mathcal{T}_\ell$. For the piecewise linear case V_ℓ^1, we denote the set of all interior vertices of the mesh $\mathcal{T}_\ell$ by $\mathcal{N}_\ell^1$. If the distinction between $\mathcal{N}_\ell^0$ and $\mathcal{N}_\ell^1$ is not essential, we will write $\mathcal{N}_\ell$ meaning $\mathcal{N}_\ell \in \{\mathcal{N}_\ell^0, \mathcal{N}_\ell^1\}$ is either $\mathcal{N}_\ell^0$ if $V_\ell = V_\ell^0$ or $\mathcal{N}_\ell^1$ if $V_\ell = V_\ell^1$. The points $z \in \mathcal{N}_\ell$ are called nodes.

We choose a basis of $V_\ell = \operatorname{span}\{\varphi_{z_j}^\ell \,:\, z_j \in \mathcal{N}_\ell, j = 1, \dots, N_\ell\}$: for the piecewise constants we take the characteristic functions $\varphi_{z_j}^\ell = \chi_{T_j}$ of the element satisfying $z_j \in T_j \in \mathcal{T}_\ell$, and for the piecewise linears we take hat functions corresponding to the interior nodes defined by $\varphi_{z_j}^\ell(z_i) = \delta_{j,i}$ for all nodes $z_i \in \mathcal{N}_\ell$. With these bases, we can write $u_\ell = \sum_{j=1}^{N_\ell} \mathbf{x}_j^\ell \varphi_{z_j}^\ell$, and (4.3.1) is equivalent to solving the linear system

$$\mathbf{A}^\ell \mathbf{x}^\ell = \mathbf{b}^\ell \tag{4.3.2}$$

with the stiffness matrix $\mathbf{A}^\ell$ and load vector $\mathbf{b}^\ell$

$$\mathbf{A}_{kj}^\ell := a(\varphi_{z_j}^\ell, \varphi_{z_k}^\ell), \qquad \mathbf{b}_k^\ell := \left\langle f, \varphi_{z_k}^\ell \right\rangle_{L^2(\Omega)}. \tag{4.3.3}$$

Again, we mention that the ℓ^2-condition number of the unpreconditioned Galerkin matrix grows like $\kappa(\mathbf{A}^\ell) \sim N_\ell^{2s/d} \left(\frac{h_{\max}^\ell}{h_{\min}^\ell}\right)^{d-2s}$, which stresses the need for a preconditioner in order to use an iterative solver.

For fixed $L \in \mathbb{N}_0$, we introduce a *local multilevel diagonal preconditioner* $(\mathbf{B}^L)^{-1}$ of BPX-type for the stiffness matrix $\mathbf{A}^L$ from (4.3.2) in the same way as in [FFPS17a, AM03]. That is, following [FFPS17a], we define the patch of a node $z \in \mathcal{N}_\ell$ as

$$\omega_\ell(z) := \text{interior} \bigcup \{\overline{T} \,:\, T \in \mathcal{T}_\ell, z \in \overline{T}\}.$$

The sets $\mathcal{M}_\ell^i$, $i = 0, 1$, defined in the following, describe the changes in the mesh hierarchy between the levels ℓ and $\ell - 1$ and are crucial for the definition of the local diagonal scaling. For the case of piecewise linears, we define the sets $\mathcal{M}_\ell^1$ as the sets of new vertices and their direct neighbours in the mesh $\mathcal{T}_\ell$: We set $\mathcal{M}_0^1 := \mathcal{N}_0^1$ and

$$\mathcal{M}_\ell^1 := \mathcal{N}_\ell^1 \backslash \mathcal{N}_{\ell-1}^1 \cup \{z \in \mathcal{N}_\ell^1 \cap \mathcal{N}_{\ell-1}^1 \,:\, \omega_\ell(z) \subsetneq \omega_{\ell-1}(z)\}, \quad \ell \geq 1. \tag{4.3.4}$$

For the case of a piecewise constant discretization, we define the set $\mathcal{M}_\ell^0$ simply as the barycenters corresponding to the new elements, i.e., $\mathcal{M}_\ell^0 := \mathcal{N}_\ell^0 \backslash \mathcal{N}_{\ell-1}^0$ for $\ell \geq 1$. In the same way as for the nodes $\mathcal{N}_\ell$, we write $\mathcal{M}_\ell$ to either be $\mathcal{M}_\ell^0$ and $\mathcal{M}_\ell^1$, which should be clear from context.

The local multilevel diagonal preconditioner is given by

$$(\mathbf{B}^L)^{-1} := \sum_{\ell=0}^{L} \mathbf{I}^\ell \, \mathbf{D}_{\text{inv}}^\ell (\mathbf{I}^\ell)^T, \tag{4.3.5}$$

where, with $N_\ell := \#\mathcal{N}_\ell$, the appearing matrices are defined as

- $\mathbf{I}^\ell \in \mathbb{R}^{N_L \times N_\ell}$ denotes the identity matrix correspond to the embedding $\mathcal{I}^\ell : V_\ell \to V_L$.
- $\mathbf{D}_{\text{inv}}^\ell \in \mathbb{R}^{N_\ell \times N_\ell}$ is a diagonal matrix with entries $(\mathbf{D}_{\text{inv}}^\ell)_{jk} = \begin{cases} (\mathbf{A}_{jj}^\ell)^{-1}\delta_{jk} & j \,:\, z_j \in \mathcal{M}_\ell \\ 0 & \text{otherwise} \end{cases}$.
  That is, the entries of the diagonal matrix are the reciprocals of the diagonal entries of the matrix $\mathbf{A}^\ell$ corresponding to the degrees of freedom in $\mathcal{M}_\ell$.

Moreover, we define the additive Schwarz matrix $\mathbf{P}_{AS}^L := (\mathbf{B}^L)^{-1}\mathbf{A}^L$. Instead of solving (4.3.2) for $\ell = L$, we solve the following preconditioned linear systems

$$\mathbf{P}_{AS}^L \mathbf{x}^L = (\mathbf{B}^L)^{-1} \mathbf{b}^L. \tag{4.3.6}$$

The following theorem is the main result of this section and provides optimal bounds to the eigenvalues of the preconditioned matrix and the proof is given in Section 4.4.

Theorem 4.3.1. *The minimal and maximal eigenvalues of the additive Schwarz matrix* $\mathbf{P}_{AS}^L$ *are bounded by*

$$c \leq \lambda_{\min}\left(\mathbf{P}_{AS}^L\right) \quad \text{and} \quad \lambda_{\max}\left(\mathbf{P}_{AS}^L\right) \leq C, \tag{4.3.7}$$

where the constants c, $C > 0$ *depend only on* Ω, d, s, *and the initial triangulation* $\mathcal{T}_0$.

Remark 4.3.2. The preconditioner $(\mathbf{B}^L)^{-1}$ is a symmetric positive definite matrix and the preconditioned matrix $\mathbf{P}_{AS}^L$ is symmetric and positive definite with respect to the inner product induced by $\mathbf{B}^L$. Therefore, Theorem 4.3.1 leads to $\kappa(\mathbf{P}_{AS}^L) \leq C/c$. ∎

Remark 4.3.3. The cost to apply the preconditioner is proportional to $\sum_{\ell=0}^{L} \operatorname{card} \mathcal{M}_\ell = O(N_L)$ by [FFPS17a, Sec. 3.1]. ∎

4.3.2 A local multilevel diagonal preconditioner using a finest common coarsening mesh hierarchy

In this subsection, we provide a result similar to Theorem 4.3.1 for the meshes $\widetilde{\mathcal{T}}_\ell = \texttt{fcc}(\mathcal{T}_L, \widehat{\mathcal{T}}_\ell)$, where $\ell = 0, \dots, L$. With $\widetilde{V}_\ell^0 = S^{0,0}(\widetilde{\mathcal{T}}_\ell)$, $\widetilde{V}_\ell^1 = S_0^{1,1}(\widetilde{\mathcal{T}}_\ell)$, and $\widetilde{V}_\ell \in \{\widetilde{V}_\ell^0, \widetilde{V}_\ell^1\}$ being either the piecewise constants or piecewise linears on $\widetilde{\mathcal{T}}_\ell$, the Galerkin discretization of finding $\widetilde{u}_\ell \in \widetilde{V}_\ell$ such that

$$a(\widetilde{u}_\ell, \widetilde{v}_\ell) = \langle f, \widetilde{v}_\ell \rangle_{L^2(\Omega)} \qquad \forall\, \widetilde{v}_\ell \in \widetilde{V}_\ell \tag{4.3.8}$$

is equivalent to solving the linear system

$$\widetilde{\mathbf{A}}^\ell \widetilde{\mathbf{x}}^\ell = \widetilde{\mathbf{b}}^\ell \tag{4.3.9}$$

by choosing a nodal basis as in the previous subsection. The set of nodes $\widetilde{\mathcal{N}}_\ell^i$, $i = 0, 1$, and $\widetilde{\mathcal{N}}_\ell$ as well as the sets $\widetilde{\mathcal{M}}_\ell^i$, $i = 0, 1$, and $\widetilde{\mathcal{M}}_\ell$ can be defined in exactly the same way as in the previous subsection by just replacing the meshes $\mathcal{T}_\ell$ with $\widetilde{\mathcal{T}}_\ell$. Therefore, in exactly the same way as in (4.3.5), we can define the local multilevel diagonal preconditioner

$$(\widetilde{\mathbf{B}}^L)^{-1} := \sum_{\ell=0}^{L} \mathbf{I}^\ell\, \widetilde{\mathbf{D}}_{\mathrm{inv}}^\ell (\mathbf{I}^\ell)^T.$$

The following theorem then gives optimal bounds for the smallest and largest eigenvalues of the preconditioned matrix $\widetilde{\mathbf{P}}_{AS}^L := \big(\widetilde{\mathbf{B}}^L\big)^{-1} \widetilde{\mathbf{A}}^L$ and the proof is given in Section 4.4.

Theorem 4.3.4. *The minimal and maximal eigenvalues of the additive Schwarz matrix $\widetilde{\mathbf{P}}_{AS}^L$ are bounded by*

$$c \le \lambda_{\min}\left(\widetilde{\mathbf{P}}_{AS}^L\right) \qquad \text{and} \qquad \lambda_{\max}\left(\widetilde{\mathbf{P}}_{AS}^L\right) \le C, \tag{4.3.10}$$

where the constants c, $C > 0$ depend only on Ω, d, s, and the initial triangulation $\mathcal{T}_0$.

Remark 4.3.5. By Lemma 3.2.3 the cost of the preconditioner are, up to a constant, $\operatorname{card}\widetilde{\mathcal{M}}_0 + \sum_{\ell=1}^L \operatorname{card}\widetilde{\mathcal{M}}_\ell \lesssim \operatorname{card}\widetilde{\mathcal{M}}_0 + \sum_{\ell=0}^L \operatorname{card}\widetilde{\mathcal{N}}_\ell - \operatorname{card}\widetilde{\mathcal{N}}_{\ell-1} \lesssim \operatorname{card}\widetilde{\mathcal{N}}_L = \operatorname{card}\mathcal{T}_L$. ∎

4.4 Optimal additive Schwarz preconditioning for the fractional Laplacian on locally refined meshes

In this section, we prove the optimal bounds on the eigenvalues of the preconditioned matrices $\mathbf{P}_{AS}^L$ of Theorem 4.3.1 and $\widetilde{\mathbf{P}}_{AS}^L$ of Theorem 4.3.4. The key steps are done in Proposition 4.4.2 or Proposition 4.4.1, which state a spectral equivalence of the corresponding additive Schwarz operator and the identity in the energy scalar product.

4.4.1 Abstract analysis of the additive Schwarz method: The mesh hierarchy $\widetilde{\mathcal{T}}_\ell = \texttt{fcc}(\mathcal{T}_L, \widehat{\mathcal{T}}_\ell)$

The additive Schwarz method is based on a local subspace decomposition. For the mesh hierarchy $\widetilde{\mathcal{T}}_\ell = \texttt{fcc}(\mathcal{T}_L, \widehat{\mathcal{T}}_\ell)$, we recall that $\widetilde{V}_\ell \in \{S^{0,0}(\widetilde{\mathcal{T}}_\ell), S_0^{1,1}(\widetilde{\mathcal{T}}_\ell)\}$ is either the space of piecewise constants or piecewise linears on the mesh $\widetilde{\mathcal{T}}_\ell$. We follow the abstract setting of [TW05] and decompose $\widetilde{V}_L = \sum_{\ell=0}^{L} \widetilde{\mathcal{V}}_\ell$ with

$$\widetilde{\mathcal{V}}_\ell := \operatorname{span}\left\{\widetilde{\varphi}_z^\ell \ : \ z \in \widetilde{\mathcal{M}}_\ell\right\}, \tag{4.4.1}$$

where $\widetilde{\varphi}_z^\ell$ denotes the basis function associated with the node $z \in \widetilde{\mathcal{N}}_\ell$. We recall that these functions are either characteristic functions of elements (for the piecewise constant case) or nodal hat functions (for the case of piecewise linears). We note that $\widetilde{\mathcal{V}}_\ell \subset \widehat{V}_\ell$ and, since $\widetilde{\mathcal{M}}_\ell$ only contains new nodes and direct neighbors, this space effectively is a discrete space on a uniform submesh (cf. Lemma 4.4.6). On the subspaces $\widetilde{\mathcal{V}}_\ell$, we introduce the symmetric, positive definite bilinear form $\widetilde{a}_\ell(\cdot,\cdot) : \widetilde{\mathcal{V}}_\ell \times \widetilde{\mathcal{V}}_\ell$ (also known as local solvers) with

$$\widetilde{a}_\ell(u_\ell, u_\ell) := \sum_{z \in \widetilde{\mathcal{M}}_\ell} \left\| \widehat{h}_\ell^{-s} u_\ell(z) \widetilde{\varphi}_z^\ell \right\|_{L^2(\Omega)}^2 \simeq \sum_{z \in \widetilde{\mathcal{M}}_\ell} \widehat{h}_\ell^{d-2s} \left| u_\ell(z) \right|^2.$$

The following proposition, c.f., e.g., [Zha92, MN85], gives bounds on the minimal and maximal eigenvalues of the preconditioned matrix $\widetilde{\mathbf{P}}_{AS}^L$ based on the abstract additive Schwarz theory.

Proposition 4.4.1. *(i) Assume that every $u \in \widetilde{V}_L$ admits a decomposition $u = \sum_{\ell=0}^{L} u_\ell$ with $u_\ell \in \widetilde{\mathcal{V}}_\ell$ satisfying $\sum_{\ell=0}^{L} \widetilde{a}_\ell(u_\ell, u_\ell) \le C_0\, a(u,u)$ with a constant $C_0 > 0$. Then, we have $\lambda_{\min}(\widetilde{\mathbf{P}}_{AS}^L) \ge C_0^{-1}$.*

(ii) Assume that there exists a constant $C_1 > 0$ such that for every decomposition $u = \sum_{\ell=0}^{L} u_\ell$ with $u_\ell \in \widetilde{\mathcal{V}}_\ell$, we have $a(u,u) \le C_1 \sum_{\ell=0}^{L} \widetilde{a}_\ell(u_\ell, u_\ell)$. Then, $\lambda_{\max}(\widetilde{\mathbf{P}}_{AS}^L) \le C_1$.

The first part of Proposition 4.4.1 is sometimes called Lions' Lemma and follows from the existence of a stable decomposition proven in Lemma 4.4.5 below. The assumption of the second statement follows directly from our strengthened Cauchy-Schwarz inequality (Lemma 4.4.7) and local stability (Lemma 4.4.9).

4.4.2 Abstract analysis of the additive Schwarz method: The mesh hierarchy $\mathcal{T}_\ell$ provided by an adaptive algorithm

For the case of a mesh hierarchy $\mathcal{T}_\ell$ generated by an adaptive algorithm, similar definitions can be made and analyzed. However, here, we follow the notation of [FFPS17a], where the additive Schwarz operator consisting of a sum of projections onto one dimensional spaces is analyzed. With the spaces $V_z^\ell := \operatorname{span}\{\varphi_z^\ell\}$ one may define local projections $\mathcal{P}_z^\ell : \widetilde{H}^s(\Omega) \to V_z^\ell$ in the energy scalar product as

$$a(\mathcal{P}_z^\ell u, v_z^\ell) = a(u, v_z^\ell) \qquad \text{for all } v_z^\ell \in V_z^\ell,$$

and define the additive Schwarz operator as

$$\mathcal{P}_{AS}^{L} := \sum_{\ell=0}^{L} \sum_{z \in \mathcal{M}_\ell} \mathcal{P}_z^\ell.$$

Moreover, for $u, v \in V_L$ and their expansions $u = \sum_{j=1}^{N_L} \mathbf{x}_j \varphi_{z_j}^L$, $v = \sum_{j=1}^{N_L} \mathbf{y}_j \varphi_{z_j}^L$, we have

$$a(\mathcal{P}_{AS}^L u, v) = \left\langle \mathbf{P}_{AS}^L \mathbf{x}, \mathbf{y} \right\rangle_{\mathbf{A}^L}, \tag{4.4.2}$$

where $\langle \cdot, \cdot \rangle_{\mathbf{A}^L} := \left\langle \mathbf{A}^L \cdot, \cdot \right\rangle_2$. Therefore, the multilevel diagonal scaling is a multilevel additive Schwarz method, and we may analyze the additive Schwarz operator instead of the preconditioned matrix.

Proposition 4.4.2. *The operator $\mathcal{P}_{AS}^L$ is linear, bounded and symmetric in the energy scalar product. Moreover, for $u \in V_L$, we have the spectral equivalence*

$$c \left\| u \right\|_{\widetilde{H}^s(\Omega)}^2 \leq a(\mathcal{P}_{AS}^L u, u) \leq C \left\| u \right\|_{\widetilde{H}^s(\Omega)}^2, \tag{4.4.3}$$

where the constants c, $C > 0$ only depend on Ω, d, s, and $\mathcal{T}_0$.

As in [FFPS17a], Proposition 4.4.2 directly implies Theorem 4.3.1.

Proof of Theorem 4.3.1. Combining the bounds of Proposition 4.4.2 with (4.4.2) gives

$$c \left\| \mathbf{x} \right\|_{\mathbf{A}^L}^2 \leq \left\langle \mathbf{P}_{AS}^L \mathbf{x}, \mathbf{x} \right\rangle_{\mathbf{A}^L} \leq C \left\| \mathbf{x} \right\|_{\mathbf{A}^L}^2$$

for all $\mathbf{x} \in \mathbb{R}^{N_L}$, and therefore the bounds for the minimal and maximal eigenvalues. □

4.4.3 Inverse estimates for the fractional Laplacian

For the proof of a strengthened Cauchy Schwarz inequality, we employ an inverse inequality for the operator $(-\Delta)^s$ of the form

$$\| h^s (-\Delta)^s v \|_{L^2(\Omega)} \lesssim \| v \|_{\widetilde{H}^s(\Omega)}. \tag{4.4.4}$$

For the piecewise linear case $v \in S_0^{1,1}(\mathcal{T})$, this inverse estimate is proven in [FMP19, Thm. 2.8]. We stress that (4.4.4) only holds for $s < 3/4$, since in the converse case the left-hand side is not well defined for $v \in S_0^{1,1}(\mathcal{T})$. To obtain an estimate for $s \in [3/4, 1)$, one has to introduce a weight function $w(x) := \inf_{T \in \mathcal{T}} \operatorname{dist}(x, \partial T)$. Then, [FMP19, Thm. 2.8] provides the inverse estimate

$$\left\| h^{1/2} w^{s-1/2} (-\Delta)^s v \right\|_{L^2(\Omega)} \lesssim \| v \|_{\widetilde{H}^s(\Omega)}. \tag{4.4.5}$$

For the case of piecewise constants, similar inverse estimates are stated in the lemma below. Here, we additionally stress that for $v \in S^{0,0}(\mathcal{T})$ and $x \in T \in \mathcal{T}$, the estimate

$$\begin{aligned} |(-\Delta)^s v(x)| &= \left| C(d,s) \int_{\mathbb{R}^d \setminus B_{\mathrm{dist}(x,\partial T)}(x)} \frac{v(x)-v(y)}{|x-y|^{d+2s}} dy \right| \\ &\lesssim \|v\|_{L^\infty(\Omega)} \int_{B_{\mathrm{dist}(x,\partial T)}(x)^c} \frac{1}{|x-y|^{d+2s}} dy \\ &= \|v\|_{L^\infty(\Omega)} \int_{\nu \in \partial B_1(0)} \int_{r=\mathrm{dist}(x,\partial T)}^{\mathrm{diam}\,\Omega} r^{-2s-1} dr d\nu \\ &\lesssim \|v\|_{L^\infty(\Omega)} \, \mathrm{dist}(x,\partial T)^{-2s}, \end{aligned} \tag{4.4.6}$$

gives

$$w^\beta (-\Delta)^s v \in L^2(\Omega) \qquad \text{if } \beta > 2s - 1/2.$$

For $s < 1/4$, we may choose $\beta = 0$ and for $1/4 \le s < 1/2$, we may choose, e.g., $\beta = s$ or $\beta = 3/2s - 1/4$ (to additionally ensure $\beta < s$) to fulfill this requirement.

Lemma 4.4.3. *Let $\mathcal{T}$ be a regular and γ-shape regular mesh generated by NVB refinement of a mesh $\mathcal{T}_0$. Let $v \in S^{0,0}(\mathcal{T})$, h be the piecewise constant mesh width function of the triangulation $\mathcal{T}$, and set $w(x) := \inf_{T\in\mathcal{T}} \mathrm{dist}(x,\partial T)$. Let $\beta > 2s - 1/2$. Then, the inverse estimates*

$$\|h^s(-\Delta)^s v\|_{L^2(\Omega)} \le C\|v\|_{\widetilde{H}^s(\Omega)} \qquad 0 < s < 1/4, \tag{4.4.7}$$

$$\|h^{s-\beta} w^\beta (-\Delta)^s v\|_{L^2(\Omega)} \le C\|v\|_{\widetilde{H}^s(\Omega)} \qquad 1/4 \le s < 1/2 \tag{4.4.8}$$

hold, where the constant $C > 0$ depends only on Ω, d, s, and the γ-shape regularity of $\mathcal{T}$.

Proof. If we set $\beta = 0$ for $s < 1/4$, we can prove both statements of the lemma at once by estimating the L^2-norms with the weight $h^{s-\beta} w^\beta$.

Considering the nonlocality of the fractional operator, we need to split it into two parts, a localized near-field part and a smoother far-field part. For this purpose, we follow the lines of [FMP19, Thm. 2.8], starting with a splitting into a near-field and a far-field part. The estimates of the near-field and the far-field are rather similar to the case of piecewise linears from [FMP19, Lem. 4.1–4.5]. Therefore, we quote the identical parts of the proof and outline the necessary modifications for the piecewise constant case.

For each $T \in \mathcal{T}$, we choose a cut-off function $\chi_T \in C_0^\infty(\mathbb{R}^d)$ with the following properties: 1) $\mathrm{supp}\,\chi_T \cap \Omega \subset \omega(T)$; 2) $\chi_T \equiv 1$ on a set B satisfying $T \subset B \subset \omega(T)$ and $\mathrm{dist}\,(B, \partial\omega(T) \cap \Omega) \sim h_T$; 3) $\|\chi_T\|_{W^{1,\infty}(\omega(T))} \lesssim h_T^{-1}$; 4) $0 \le \chi_T \le 1$. Moreover, for each $T \in \mathcal{T}$, we denote the average of v on the patch $\omega^2(T)$ by $c_T \in \mathbb{R}$, i.e.,

$$c_T := \begin{cases} 0 & \text{if } \overline{T} \cap \partial\omega^2(T) \neq 0 \\ \frac{1}{|\omega^2(T)|} \int_{\omega^2(T)} v\, dx & \text{otherwise.} \end{cases}$$

Since c_T is a constant, we have $(-\Delta)^s c_T \equiv 0$. Therefore, we can decompose v into the near-field $v_{\mathrm{near}}^T := \chi_T(v - c_T)$ and the far-field $v_{\mathrm{far}}^T := (1-\chi_T)(v - c_T)$, and obtain

$(-\Delta)^s v = (-\Delta)^s v^T_{\text{near}} + (-\Delta)^s v^T_{\text{far}}.$

We start with the near-field, where compared to the result for the case of piecewise linears, we do not need to distinguish cases for s. The definition of the fractional Laplacian leads to

$$\begin{aligned}\frac{1}{C(d,s)^2}\left\|w^\beta(-\Delta)^s v^T_{\text{near}}\right\|^2_{L^2(T)} &= \int_T w(x)^{2\beta}\left(\text{P.V.}\int_{\mathbb{R}^d}\frac{(v(x)-c_T)\chi_T(x)-(v(y)-c_T)\chi_T(y)}{|x-y|^{d+2s}}dy\right)^2 dx\\ &\lesssim \int_T w(x)^{2\beta}(v(x)-c_T)^2\left(\text{P.V.}\int_{\mathbb{R}^d}\frac{\chi_T(x)-\chi_T(y)}{|x-y|^{d+2s}}dy\right)^2 dx\\ &\quad+\int_T w(x)^{2\beta}\left(\text{P.V.}\int_{\mathbb{R}^d}\chi_T(y)\frac{v(x)-v(y)}{|x-y|^{d+2s}}dy\right)^2 dx.\end{aligned} \tag{4.4.9}$$

The first term on the right-hand side can be estimated using the Lipschitz continuity of χ_T and a Poincaré inequality on the patch $\omega(T)$ in the same way as in the proof of [FMP19, Lem. 4.3]. Considering $x \in T$ and $\chi_T \equiv 1$, we get

$$\begin{aligned}&\int_T w(x)^{2\beta}(v(x)-c_T)^2\left(\text{P.V.}\int_{\mathbb{R}^d}\frac{\chi_T(x)-\chi_T(y)}{|x-y|^{d+2s}}dy\right)^2 dx\\ &=\int_T w(x)^{2\beta}(v(x)-c_T)^2\left(\int_{B_{\text{dist}(x,\partial T)}(x)^c}\frac{\chi_T(x)-\chi_T(y)}{|x-y|^{d+2s}}dy\right)^2 dx.\end{aligned} \tag{4.4.10}$$

Let B' be an arbitrary set defined such that $T \subset B'$ and $\text{dist}\left((,T),\partial B'\right) \sim h_T$ and it still satisfies $\chi_T \equiv 1$ on B'. Therefore, applying polar coordinates $y = x + r\nu$, $\nu \in \partial B_1(0)$, where $\partial B_1(0)$ is the $(d-1)$-dimensional unit sphere, gives us

$$\left|\int_{B_{\text{dist}(x,\partial T)}(x)^c}\frac{\chi_T(x)-\chi_T(y)}{|x-y|^{d+2s}}dy\right| = \left|\int_{B'^c}\frac{1-\chi_T(y)}{|x-y|^{d+2s}}dy\right| \lesssim \int_{ch_T}^\infty \frac{1}{r^{1+2s}}dy \simeq h_T^{-2s}. \tag{4.4.11}$$

Substituting (4.4.11) into (4.4.10), using $\omega|_T \le h_T$ and [FMP19, Lem. 4.1], we can write

$$\begin{aligned}&\int_T w(x)^{2\beta}(v(x)-c_T)^2\left(\text{P.V.}\int_{\mathbb{R}^d}\frac{\chi_T(x)-\chi_T(y)}{|x-y|^{d+2s}}dy\right)^2 dx\\ &\lesssim \int_T w(x)^{2\beta}(v(x)-c_T)^2 h_T^{-4s}dx \lesssim h_T^{2\beta-4s}\int_T (v(x)-c_T)^2 dx\\ &\lesssim h_T^{2\beta-2s}\left\|v\right\|^2_{H^s(\omega^2(T))}.\end{aligned}$$

For the second term in (4.4.9), we split it into two parts, a smoother, integrable part and a principal value integral part:

$$\begin{aligned}
&\int_T w(x)^{2\beta} \left(\text{P.V.} \int_{\mathbb{R}^d} \chi_T(y) \frac{v(x)-v(y)}{|x-y|^{d+2s}} dy \right)^2 dx \\
&\quad \lesssim \int_T w(x)^{2\beta} \left(\text{P.V.} \int_{B_{\text{dist}(x,\partial T)}(x)} \chi_T(y) \frac{v(x)-v(y)}{|x-y|^{d+2s}} dy \right)^2 dx \\
&\quad + \int_T w(x)^{2\beta} \left(\text{P.V.} \int_{B_{\text{dist}(x,\partial T)}(x)^c} \chi_T(y) \frac{v(x)-v(y)}{|x-y|^{d+2s}} dy \right)^2 dx := S_1 + S_2.
\end{aligned}$$

We observe that the integrand in S_1 vanishes for $y \in T$, since v is piecewise constant and for S_2, we employ the same estimate as for (4.4.6) to obtain

$$\int_T w(x)^{2\beta} \left(\text{P.V.} \int_{\mathbb{R}^d} \chi_T(y) \frac{v(x)-v(y)}{|x-y|^{d+2s}} dy \right)^2 dx \lesssim \|v - c_T\|^2_{L^\infty(\omega(T))} \int_T w(x)^{2\beta-4s} dx;$$

here, we added and subtracted the constant c_T in the integrand and used the support properties of χ_T to obtain the L^∞-norm on the patch.

As, by choice of β, we always have $2\beta - 4s > -1$, the last integral exists, and we can further estimate using a classical inverse estimate and a Poincaré inequality

$$\begin{aligned}
\|v - c_T\|^2_{L^\infty(\omega(T))} \int_T w(x)^{2\beta-4s} dx &\lesssim h_T^{2\beta-4s+d} \|v - c_T\|^2_{L^\infty(\omega(T))} \\
&\lesssim h_T^{2\beta-4s} \|v - c_T\|^2_{L^2(\omega(T))} \lesssim h_T^{2\beta-2s} \|v\|^2_{H^s(\omega^2(T))}.
\end{aligned}$$

Inserting everything into (4.4.9), multiplying with $h_T^{2s-2\beta}$ and summing over all elements $T \in \mathcal{T}$ gives the desired estimate for the near-field.

The far-field can be estimated using the Caffarelli-Silvestre extension, cf. [CS07], combined with a Caccioppoli-type inverse estimate for the solution of the extension problem with boundary data $(1 - \chi_T)(v - c_T)$ as in [FMP19]. In fact, we observe that [FMP19, Lem. 4.5] holds for arbitrary $v \in \widetilde{H}^s(\Omega)$ and weight functions w with non-negative exponent. This directly gives

$$\sum_{T \in \mathcal{T}} \|h^{s-\beta} w^\beta (-\Delta)^s v^T_{\text{far}}\|^2_{L^2(T)} \lesssim \|v\|^2_{\widetilde{H}^s(\Omega)},$$

and combining the estimates for near- and far-field proves the lemma. □

4.4.4 Proof of the assumptions of Proposition 4.4.1

In order to apply Proposition 4.4.1, we show the existence of a stable decomposition (Lemma 4.4.5) and a strengthened Cauchy-Schwarz inequality (Lemma 4.4.7).

The following result relates the Scott-Zhang operators on two consecutive levels, similarly to [CNX12], and is a key ingredient of the proof of Lemma 4.4.5.

Lemma 4.4.4. *Let $p = 1$ and let $\widetilde{\mathcal{N}}_\ell^1$, $\widetilde{\mathcal{M}}_\ell^1$ be defined in Section 4.3.2. The Scott-Zhang operators $\widetilde{I}_\ell^{SZ} : L^2(\Omega) \to S^{1,1}(\widetilde{\mathcal{T}}_\ell)$ can be constructed such that, additionally, they satisfy for all $\ell \in \mathbb{N}$ and all $u \in L^2(\Omega)$*

$$(\widetilde{I}_\ell^{SZ} - \widetilde{I}_{\ell-1}^{SZ})u(z) = 0 \qquad \forall z \in \widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{M}}_\ell^1. \tag{4.4.12}$$

Also the Scott-Zhang operators $\widetilde{I}_{0,\ell}^{SZ} : L^2(\Omega) \to S_0^{1,1}(\widetilde{\mathcal{T}}_\ell)$ can be constructed such that (4.4.12) holds with $\widetilde{I}_\ell^{SZ}$ and $\widetilde{I}_{\ell-1}^{SZ}$ replaced with $\widetilde{I}_{0,\ell}^{SZ}$ and $\widetilde{I}_{0,\ell-1}^{SZ}$, respectively.

Proof. We only consider the case of the operators $\widetilde{I}_\ell^{SZ}$. We also recall that for the present case $p = 1$ the nodes coincide with the nodes of the triangulations.

1. step: $z \in \widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{M}}_\ell^1$ implies $z \in \widetilde{\mathcal{N}}_\ell^1 \cap \widetilde{\mathcal{N}}_{\ell-1}^1$. To see $z \in \widetilde{\mathcal{N}}_{\ell-1}^1$, we note $\widetilde{\mathcal{N}}_{\ell-1}^1 \subset \widetilde{\mathcal{N}}_\ell^1$ by Lemma 3.2.3 and therefore that $z \in \widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{M}}_\ell^1 \subset \widetilde{\mathcal{N}}_\ell^1 \setminus (\widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{N}}_{\ell-1}^1) = \widetilde{\mathcal{N}}_{\ell-1}^1$.

2. step: $z \in \widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{M}}_\ell^1 \subset \widetilde{\mathcal{N}}_\ell^1 \cap \widetilde{\mathcal{N}}_{\ell-1}^1$ implies that all elements of the patches $\omega_\ell(z)$ and $\omega_{\ell-1}(z)$ are in $\mathcal{T}$. To see this, we note $z \in \widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{M}}_\ell^1 \subset \widetilde{\mathcal{N}}_\ell^1 \setminus \{z \in \widetilde{\mathcal{N}}_\ell^1 \cap \widetilde{\mathcal{N}}_{\ell-1}^1 \,|\, \omega_\ell(z) \subsetneq \omega_{\ell-1}(z)\}$. The condition $\omega_{\ell-1}(z) = \omega_\ell(z)$ implies that all elements of $\omega_{\ell-1}(z) = \omega_\ell(z)$ must be elements of $\mathcal{T}$.

3. step: The basic idea for the choice of averaging sets T_z in the construction of $\widetilde{I}_{\ell-1}^{SZ}$ and $\widetilde{I}_\ell^{SZ}$ in Def. 3.3.1 is to select an element of $\mathcal{T}$ whenever possible. Our modified construction of the operators $\widetilde{I}_\ell^{SZ}$ is by induction on ℓ and carefully exploits the freedom left in the choice of the averaging sets T_z in Def. 3.3.1. We start with an $\widetilde{I}_0^{SZ}$ as constructed in Def. 3.3.1. Suppose the averaging sets T_z for $\widetilde{\mathcal{T}}_{\ell-1}$ have been fixed. Effectively, Def. 3.3.1 performs a loop over all nodes of $\widetilde{\mathcal{T}}_\ell$. When assigning an averaging set T_z to a node $z \in \widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{M}}_\ell^1$, we select as T_z the element that has already been selected on the preceding level $\ell - 1$. This is possible since $z \in \widetilde{\mathcal{N}}_\ell^1 \setminus \widetilde{\mathcal{M}}_\ell^1$ implies $z \in \widetilde{\mathcal{N}}_{\ell-1}^1$ by Step 1, and by Step 2 we know that all elements of both $\widetilde{\mathcal{T}}_{\ell-1}$ and $\widetilde{\mathcal{T}}_\ell$ having z as a vertex are elements of $\mathcal{T}$.

The same construction can also be applied to the operators $\widetilde{I}_{0,\ell}^{SZ}$. □

The following lemma provides the existence of a stable decomposition for the mesh hierarchy generated by the finest common coarsening. Rather than analyzing the L^2-orthogonal projection onto a space of piecewise polynomials on a uniform mesh, as in [FFPS17a], we use the result of Corollary 3.5.2.

Lemma 4.4.5. *(Stable decomposition for the mesh hierarchy $(\widetilde{\mathcal{T}}_\ell)_\ell$). For every $u \in \widetilde{V}_L$, there is a decomposition $u = \sum_{\ell=0}^L u_\ell$ with $u_\ell \in \widetilde{\mathcal{V}}_\ell$ satisfying the stability estimate*

$$\sum_{\ell=0}^{L} \widetilde{a}_\ell(u_\ell, u_\ell) = \sum_{\ell=0}^{L} \sum_{z \in \widetilde{\mathcal{M}}_\ell} \left\| \widehat{h}_\ell^{-s} u_\ell(z) \widetilde{\varphi}_z^\ell \right\|_{L^2(\Omega)}^2 \leq C_{\mathrm{stab}}^2 \|u\|_{\widetilde{H}^s(\Omega)}^2,$$

with a constant $C_{\mathrm{stab}} > 0$ depending only on Ω, d, s, and the initial triangulation $\mathcal{T}_0$.

Proof. We only show the case of piecewise linears, the piecewise constant case is even simpler as the basis functions are L^2-orthogonal. Let $\widetilde{I}_{0,\ell}^{SZ} : \widetilde{H}^s(\Omega) \to S_0^{1,1}(\widetilde{\mathcal{T}}_\ell)$ be the adapted Scott-Zhang projection from Definition 3.3.1 in the form given by Lemma 4.4.4. Set $\widetilde{I}^{SZ}_{0,-1} = 0$. Then, we define

$$u_\ell := \sum_{z \in \widetilde{\mathcal{M}}_\ell} (\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u(z)\widetilde{\varphi}^\ell_z.$$

Since $(\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u \in \widetilde{\mathcal{V}}_\ell$, we may decompose using a telescoping series and (4.4.12)

$$u = \widetilde{I}^{SZ}_{0,L}u = \sum_{\ell=0}^{L} (\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u = \sum_{\ell=0}^{L} \sum_{z \in \widetilde{\mathcal{M}}_\ell} (\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u(z)\widetilde{\varphi}^\ell_z = \sum_{\ell=0}^{L} u_\ell. \tag{4.4.13}$$

We next prove the stability of the decomposition (4.4.13). The standard scaling of the hat functions in L^2 provides $\|\widetilde{\varphi}^\ell_z\|^2_{L^2(\Omega)} \simeq h_\ell(z)^d$, with $h_\ell(z)$ denoting the maximal mesh width on the patch corresponding to the node z. With (4.4.12) and an inverse estimate – cf. [DFG+04, Proposition 3.10], which provides an estimate for the nodal value of a piecewise linear function on the mesh $\widetilde{\mathcal{T}}_\ell$ by its L^2-norm on the patch – this gives

$$\begin{aligned}
\sum_{\ell=0}^{L} \sum_{z \in \widetilde{\mathcal{M}}_\ell} \left\| \widehat{h}_\ell^{-s} (\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u(z)\widetilde{\varphi}^\ell_z \right\|^2_{L^2(\Omega)} &\lesssim \sum_{\ell=0}^{L} \widehat{h}_\ell^{-2s} \sum_{z \in \widetilde{\mathcal{M}}_\ell} h_\ell(z)^d |(\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u(z)|^2 \\
&\lesssim \sum_{\ell=0}^{L} \widehat{h}_\ell^{-2s} \sum_{z \in \widetilde{\mathcal{N}}_\ell} \left\| (\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u \right\|^2_{L^2(\omega_\ell(z))} \\
&\lesssim \sum_{\ell=0}^{L} \widehat{h}_\ell^{-2s} \sum_{T \in \widetilde{\mathcal{T}}_\ell} \left\| (\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u \right\|^2_{L^2(T)}.
\end{aligned} \tag{4.4.14}$$

Finally, we can use Corollary 3.5.2 to obtain

$$\sum_{\ell=0}^{L} \widetilde{a}_\ell(u_\ell, u_\ell) \lesssim \sum_{\ell=0}^{L} \widehat{h}_\ell^{-2s} \left\| (\widetilde{I}^{SZ}_{0,\ell} - \widetilde{I}^{SZ}_{0,\ell-1})u \right\|^2_{L^2(\Omega)} \lesssim \|u\|^2_{\widetilde{H}^s(\Omega)}, \tag{4.4.15}$$

which proves the existence of a stable decomposition. □

The following lemma shows that the submesh consisting of the elements corresponding to the points in $\widetilde{\mathcal{M}}_\ell$ is indeed quasi-uniform in that all elements have size $O(\widehat{h}_\ell)$.

Lemma 4.4.6. *Let $\widetilde{\mathcal{M}}_\ell$ be defined in Section 4.3.2 and let $z \in \widetilde{\mathcal{M}}_\ell$, then it holds $h_\ell(z) \simeq \widehat{h}_\ell$, where $h_\ell(z)$ denotes the maximal mesh width on the patch $\omega_\ell(z)$. In particular, we have $\widetilde{\mathcal{V}}_\ell \subset \widehat{V}_\ell$, meaning $\widetilde{\mathcal{V}}_\ell \subset \widehat{V}^0_\ell$ if $\widetilde{\mathcal{M}}_\ell = \widetilde{\mathcal{M}}^0_\ell$ and $\widetilde{\mathcal{V}}_\ell \subset \widehat{V}^1_\ell$ if $\widetilde{\mathcal{M}}_\ell = \widetilde{\mathcal{M}}^1_\ell$.*

Proof. We first note that if $T \in \widetilde{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_{\ell-1}$, then $h_T \simeq \widehat{h}_\ell$. If $T \notin \mathfrak{T}_{1,\ell}$ for the first set in the definition of the finest common coarsening (3.2.1), then $T \in \widehat{\mathcal{T}}_\ell$ and $h_T \simeq \widehat{h}_\ell$ follows since the mesh $\widehat{\mathcal{T}}_\ell$ is quasi-uniform. Now, let $T \in \mathfrak{T}_{1,\ell}$, which implies $T \in \mathcal{T}$, and that

T is a proper superset of an element $\widehat{T}_\ell \in \widehat{\mathcal{T}}_\ell$, i.e., $h_T \geq \widehat{h}_\ell$. Since $\mathcal{T}$ and $\widehat{T}_{\ell-1}$ are NVB refinements of the same mesh, we either have $T \subset \widehat{T}_{\ell-1}$, $T = \widehat{T}_{\ell-1}$ or $T \supset \widehat{T}_{\ell-1}$ for some element $\widehat{T}_{\ell-1} \in \widehat{\mathcal{T}}_{\ell-1}$. For the first two cases, we have $h_T \lesssim \widehat{h}_{\ell-1} \simeq 2\widehat{h}_\ell$, which gives $h_T \simeq \widehat{h}_\ell$. The third case $T \supset \widehat{T}_{\ell-1}$ implies that $T \in \mathfrak{T}_{1,\ell-1}$ and therefore $T \in \widetilde{\mathcal{T}}_{\ell-1}$, which contradicts the assumption $T \in \widetilde{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_{\ell-1}$.

This immediately proves the case $\widetilde{\mathcal{M}}_\ell = \widetilde{\mathcal{M}}_\ell^0$, since new points in $\widetilde{\mathcal{M}}_\ell^0$ (barycenters) correspond to new elements in $\widetilde{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_{\ell-1}$.

For the case $\widetilde{\mathcal{M}}_\ell = \widetilde{\mathcal{M}}_\ell^1$, let $z \in \widetilde{\mathcal{M}}_\ell$. By definition, this implies that there exists (at least) one element $T = T(z)$ with $T(z) \subset \omega_\ell(z)$ and $T(z) \in \widetilde{\mathcal{T}}_\ell \setminus \widetilde{\mathcal{T}}_{\ell-1}$. The previous discussion gives $h_{T(z)} \simeq \widehat{h}_\ell$. By shape-regularity this gives that $h_\ell(z) = \max_{T \in \omega_\ell(z)} h_T \simeq \widehat{h}_\ell$. □

With the inverse estimate of the previous subsection we now prove a strengthened Cauchy-Schwarz inequality.

Lemma 4.4.7. *(Strengthened Cauchy-Schwarz inequality for the mesh hierarchy $(\widetilde{\mathcal{T}}_\ell)$:) Let $u_\ell \in \widetilde{\mathcal{V}}_\ell$ for $\ell = 0, 1, ..., L$. Then, we have*

$$a(u_m, u_k) \leq \mathcal{E}_{km} \|u_m\|_{\widetilde{H}^s(\Omega)} \left\| \widehat{h}_k^{-s} u_k \right\|_{L^2(\Omega)} \qquad 0 \leq m \leq k \leq L,$$

with $\mathcal{E}_{km} = C_{\mathrm{CS}} (\widehat{h}_k / \widehat{h}_m)^{s-\beta}$. Here, β is given as $\beta = \begin{cases} 0 & \text{for } 0 < s < \frac{1}{4} \\ \frac{3}{2}s - \frac{1}{4} & \text{for } \frac{1}{4} \leq s < \frac{1}{2} \end{cases}$ for the piecewise constant case and $\beta = \max\{s - 1/2, 0\}$ for the piecewise linear case. Moreover, the appearing constant $C_{\mathrm{CS}} > 0$ depends only on Ω, d, s and the initial mesh $\mathcal{T}_0$.

Proof. We define a modified mesh size function $\widetilde{h}_m^s$ as $\widetilde{h}_m^s := h_m^{s-\beta} w_m^\beta$ with the weight function w_m defined such that the inverse estimates of (4.4.4), (4.4.5) or Lemma 4.4.3 (either for the piecewise linears or the piecewise constants) hold. Moreover, we note that this choice of β fulfills the assumptions of Lemma 4.4.3 as well as $\beta < s$. Therefore, the classical Cauchy-Schwarz inequality implies

$$\begin{aligned} a(u_m, u_k) &= \langle (-\Delta)^s u_m, u_k \rangle_{L^2(\Omega)} = \left\langle \widetilde{h}_m^s (-\Delta)^s u_m, \widetilde{h}_m^{-s} u_k \right\rangle_{L^2(\Omega)} \\ &\leq \left\| \widetilde{h}_m^s (-\Delta)^s u_m \right\|_{L^2(\Omega)} \left\| \widetilde{h}_m^{-s} u_k \right\|_{L^2(\Omega)}. \end{aligned} \tag{4.4.16}$$

A scaling argument as in [FMP19, Lem. 3.2.] yields

$$\left\| w_k^{-\beta} u_k \right\|_{L^2(T)} \lesssim h_k^{s-\beta}(T) \|u_k\|_{H^s(T)} + h_k^{-\beta}(T) \|u_k\|_{L^2(T)}.$$

Together with $w_k \leq w_m$, since $\widetilde{\mathcal{T}}_k$ is a refinement of $\widetilde{\mathcal{T}}_m$, and $h_m(T) := h_m|_T \geq \widehat{h}_m$ this gives

$$\begin{aligned} \left\| \widetilde{h}_m^{-s} u_k \right\|_{L^2(T)} &\lesssim h_m^{\beta-s}(T) \left\| w_k^{-\beta} u_k \right\|_{L^2(T)} \lesssim h_m^{\beta-s}(T) \left(h_k^{s-\beta}(T) \|u_k\|_{H^s(T)} + h_k^{-\beta}(T) \|u_k\|_{L^2(T)} \right) \\ &\lesssim \widehat{h}_m^{\beta-s} h_k^{s-\beta}(T) \|u_k\|_{H^s(T)} + \widehat{h}_m^{\beta-s} h_k^{-\beta}(T) \|u_k\|_{L^2(T)} \\ &\lesssim \widehat{h}_m^{\beta-s} h_k^{-\beta}(T) \|u_k\|_{L^2(T)} + (\widehat{h}_k / \widehat{h}_m)^{s-\beta} \left\| \widehat{h}_k^{-s} u_k \right\|_{L^2(T)} \\ &\lesssim (\widehat{h}_k / \widehat{h}_m)^{s-\beta} \left\| \widehat{h}_k^{-s} u_k \right\|_{L^2(T)}. \end{aligned}$$

Summation over all the elements of $\widetilde{\mathcal{T}}_m$ gives

$$\left\|\widetilde{h}_m^{-s} u_k\right\|_{L^2(\Omega)} \lesssim (\widehat{h}_k/\widehat{h}_m)^{s-\beta} \left\|\widehat{h}_k^{-s} u_k\right\|_{L^2(\Omega)}. \tag{4.4.17}$$

Combining (4.4.16) and (4.4.17) with the inverse estimate

$$\left\|\widetilde{h}_m^{s}(-\Delta)^s u_m\right\|_{L^2(\Omega)} \lesssim \|u_m\|_{\widetilde{H}^s(\Omega)}$$

of (4.4.4), (4.4.5) or Lemma 4.4.3 proves the strengthened Cauchy-Schwarz inequality. □

Remark 4.4.8.

1. Since $(\widehat{h}_k/\widehat{h}_m)^{s-\beta} = 2^{-(k-m)(s-\beta)}$ for $0 \leq m \leq k \leq L$, we get – following the notation of [TW05] – that the symmetric matrix $\mathcal{E}$ with upper triangular part given by $\mathcal{E}_{km} = C_{\mathrm{CS}}(\widehat{h}_k/\widehat{h}_m)^{s-\beta}$ satisfies $\rho(\mathcal{E}) < C_{\mathrm{spr}}$, with a constant depending only on Ω, d, s, and the initial triangulation $\mathcal{T}_0$.
2. There is some freedom in the choice of the parameter β in Lemma 4.4.7: the proof shows that the essential conditions are $2s - 1/2 < \beta < s$. ■

Lemma 4.4.9. *(Local stability). For all $u_\ell \in \widetilde{\mathcal{V}}_\ell$, we have*

$$\|u_\ell\|^2_{\widetilde{H}^s(\Omega)} \leq C_{\mathrm{loc}}\, \widetilde{a}_\ell(u_\ell, u_\ell),$$

with a constant $C_{\mathrm{loc}} > 0$ depending only on Ω, d, s, and the initial triangulation $\mathcal{T}_0$.

Proof. Since $u_\ell \in \widetilde{\mathcal{V}}_\ell$, we have $u_\ell = \sum_{z\in\widetilde{\mathcal{M}}_\ell} u_\ell(z)\widetilde{\varphi}_z^\ell$. With an inverse estimate, which can be applied, since due to Lemma 4.4.6 u_ℓ only lives on a quasi-uniform submesh, we can estimate using that the number of overlapping basis functions $\widetilde{\varphi}_z^\ell$ is bounded by a constant depending only on the γ-shape regularity of the initial triangulation

$$\|u_\ell\|^2_{\widetilde{H}^s(\Omega)} \lesssim \left\|\widehat{h}_\ell^{-s} u_\ell\right\|^2_{L^2(\Omega)} = \widehat{h}_\ell^{-2s}\left\|\sum_{z\in\widetilde{\mathcal{M}}_\ell} u_\ell(z)\widetilde{\varphi}_z^\ell\right\|^2_{L^2(\Omega)} \lesssim \widehat{h}_\ell^{-2s}\sum_{z\in\widetilde{\mathcal{M}}_\ell} |u_\ell(z)|^2 \left\|\widetilde{\varphi}_z^\ell\right\|^2_{L^2(\Omega)}.$$

By definition of $\widetilde{a}_\ell(\cdot,\cdot)$, this finishes the proof. □

For $0 \leq k \leq \ell \leq L$, let $\mathcal{E}$ be a symmetric matrix with upper triangular part given by $\mathcal{E}_{\ell k} = C_{\mathrm{CS}}(\widehat{h}_\ell/\widehat{h}_k)^{s-\beta}$. Now, the assumptions of Proposition 4.4.1 follow directly from Lemma 4.4.5 (lower bound) and Lemma 4.4.7 together with Lemma 4.4.9 (upper bound) by writing $u = \sum_k u_k$ and

$$a(u,u) = \sum_{k,\ell=1}^{L} a(u_k,u_\ell) \leq 2\sum_{\ell=1}^{L}\sum_{k=1}^{\ell} a(u_k,u_\ell) \overset{\text{Lemma 4.4.7}}{\leq} 2\sum_{\ell=1}^{L}\sum_{k=1}^{\ell}\mathcal{E}_{\ell k}\sqrt{a(u_k,u_k)\,\widetilde{a}_\ell(u_\ell,u_\ell)}$$
$$\overset{\text{Lemma 4.4.9}}{\leq} 2C_{\mathrm{loc}}^{1/2}\sum_{\ell=1}^{L}\sum_{k=1}^{\ell}\mathcal{E}_{\ell k}\sqrt{\widetilde{a}_k(u_k,u_k)\,\widetilde{a}_\ell(u_\ell,u_\ell)} \leq 2C_{\mathrm{loc}}^{1/2}\rho(\mathcal{E})\sum_{\ell=0}^{L}\widetilde{a}_\ell(u_\ell,u_\ell),$$

and the appearing constants are independent of L.

The following remark discusses the proof of Theorem 4.3.1:

Remark 4.4.10. (Stable decomposition and strengthened Cauchy-Schwarz inequality of mesh hierarchy $(\mathcal{T}_\ell)_\ell$ generated by an adaptive algorithm—Proof of Theorem 4.3.1): The existence of a stable decomposition and consequently the lower bound in Proposition 4.4.2 follows essentially verbatim as in [FFPS17a, Sec. 4.5], where instead of Corollary 3.5.2 an L^2-orthogonal projection onto a uniform mesh is used.

Analysing the proof of Lemma 4.4.7, we observe that the choice of mesh hierarchy is not crucial for the arguments, one only needs an inverse estimate and a Poincaré-type inequality. Both hold for the case of the decomposition into one dimensional spaces V_z^ℓ instead of $\widetilde{\mathcal{V}}^\ell$ as well, and, therefore, we directly obtain a strengthened Cauchy-Schwarz inequality for $(\mathcal{T}_\ell)_\ell$ as well. The algebraic arguments of [FFPS17a, Sec. 4.6] then give the upper bound for Proposition 4.4.2. ∎

Remark 4.4.11. In the same way as in [FFPS17a], it is possible to define a *global multilevel diagonal preconditioner* by taking the whole diagonal of the matrix $\mathbf{A}^\ell$ instead of only the diagonal corresponding to the nodes in $\mathcal{M}_\ell$. However, compared to the local multilevel diagonal preconditioner, the preconditioner is not optimal in the sense that the condition number of the preconditioned system grows (theoretically) by a logarithmic factor of N_L. We refer to [FFPS17a] for numerical observations of the sharpness of this bound for the hyper-singular integral operator in the BEM, which essentially corresponds to the case $s = 1/2$ here. ∎

4.4.5 Numerical example

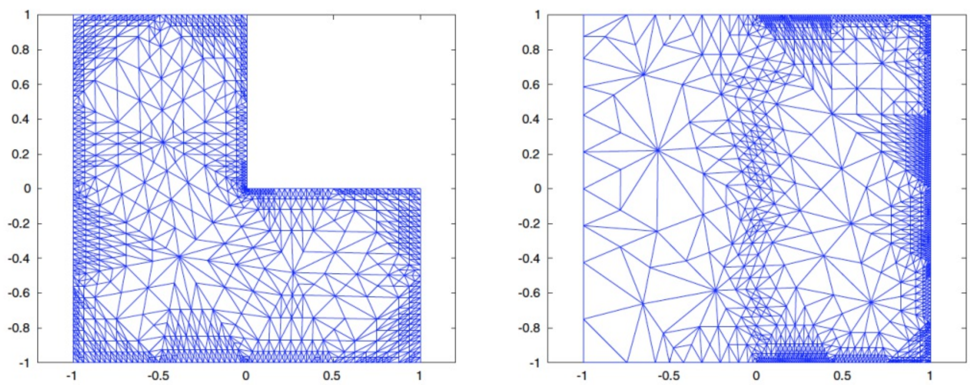

Figure 4.4.1: Adaptively generated NVB mesh on L-shaped domain and square.

We consider two examples: the L-shaped domain $\Omega = (-1,1)^2 \backslash [0,1]^2$ with $f \equiv 1$ and the square $\Omega = (-1,1)^2$ with discontinuous $f = \chi_{x>0}$. We discretize (4.2.1) by piecewise linear functions in $S_0^{1,1}(\mathcal{T}_\ell)$ on adaptively generated NVB meshes $\mathcal{T}_\ell$ that are generated by the adaptive algorithm proposed in [FMP19] and are depicted in Figure 4.4.1. This adaptive algorithm is steered by local error indicators given by

$$\eta_\ell = \left(\sum_{T\in\mathcal{T}_\ell} \left\| \widetilde{h}_\ell^s \big(f - (-\Delta)^s u_\ell\big) \right\|_{L^2(T)}^2 \right)^{1/2} \quad \text{with} \quad \widetilde{h}_\ell^s := \begin{cases} h_\ell^s & \text{for } 0 < s \le 1/2, \\ h_\ell^{1/2} w_\ell^{s-1/2} & \text{for } 1/2 < s < 1, \end{cases}$$

where u_ℓ is the solution of (4.3.1). We note that by [FMP19, Theorem 2.3] theses indicators are reliable and for $s < 1/2$ efficient in some weak sense. Moreover, [FMP19, Theorem 2.6] proves optimal convergence rates for the adaptive algorithm based on these estimators. Our implementation of the classical `SOLVE-ESTIMATE-MARK-REFINE` adaptive algorithm uses the `MATLAB` code from [ABB17] for the module `SOLVE` and adapted the `MATLAB` code for the local multilevel preconditioner from [FFPS17a] to our model problem. Figure 4.4.2 gives the estimated condition numbers for the Galerkin matrix $\mathbf{A}^L$ and the preconditioned matrix $\mathbf{P}_{AS}^L$, where the condition number has been estimated using power iteration and inverse power iteration (with random initial vectors) to compute approximations to the smallest and largest eigenvalues.

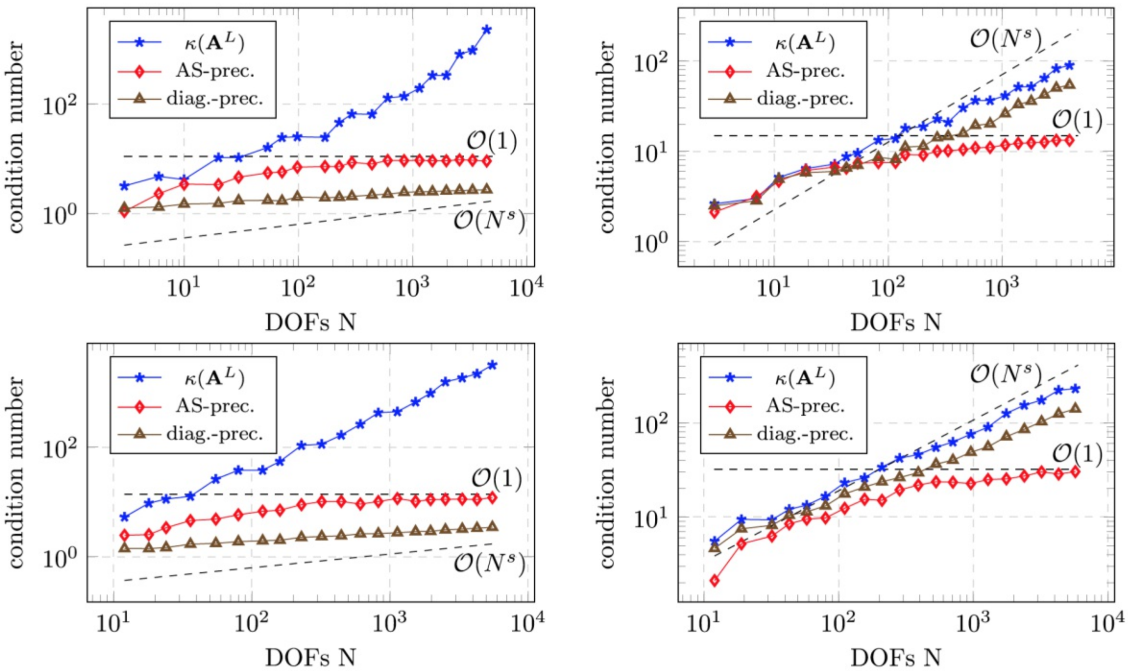

Figure 4.4.2: Estimated condition numbers for $\mathbf{A}^L$, the preconditioned matrices $\mathbf{P}_{AS}^L$, and $\operatorname{diag}(\mathbf{A}^L)^{-1}\mathbf{A}^L$. Top: L-shaped domain, bottom: square; left: $s = 0.25$, right: $s = 0.75$.

We observe that, as expected, the condition number of the unpreconditioned system grows with the problem size, whereas the preconditioner leads to uniformly bounded condition numbers for the preconditioned system. Moreover, diagonal scaling eliminates the dependence on the quotient of maximal and minimal mesh size, which is the dominant part in the case $s = 0.25$. While there is still dependence on the problem size, the growth with respect to the number of degrees of freedom is very moderate, and for the problem sizes considered here, diagonal scaling performs very well for the case $s = 0.25$, but not for the case $s = 0.75$.

As the preconditioner is structurally similar to the one used in [FFPS17a] for the hypersingular integral equation, we refer to the numerical results there for the confirmation that the preconditioner can also be realized efficiently.

5 $\mathcal{H}$-Matrix approximations to inverses for FEM-BEM couplings

Transmission problems are usually posed on unbounded domains, where a (possibly nonlinear) equation is given on some bounded domain, and another linear equation is posed on the complement of the bounded domain. While the interior problem can be treated numerically by the finite element method, the unbounded nature of the exterior problem makes this problematic. A suitable method to treat unbounded problems is provided by the boundary element method, where the differential equation in the unbounded domain is reformulated via an integral equation posed just on the boundary. In order to combine both methods for transmission problems, additional conditions on the interface have to be fulfilled, which leads to different approaches for the coupling of the FEM and the BEM. We study three different FEM-BEM couplings, the Bielak-MacCamy coupling [BM84], Costabel's symmetric coupling [Cos88, CES90], and the Johnson-Nédélec coupling [JN80]. Well-posedness and unique solvability of these formulations have been studied in, e.g., [Ste11, Say13, AFF+13], where a main observation is that the couplings are equivalent to an elliptic problem.

Elliptic problems typically feature interior regularity known as Caccioppoli estimates, where stronger norms can be estimated by weaker norms on larger domains. In this chapter, we provide Caccioppoli-type estimates for the discrete problem. Using the Caccioppoli-type estimates, we prove the existence of low-rank approximants to the inverses of stiffness matrices corresponding to the lowest order FEM-BEM discretizations and we show the error converges exponentially in the rank employed.

5.1 Model problem

On a Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with polygonal (for $d = 2$) or polyhedral (for $d = 3$) boundary $\Gamma := \partial\Omega$, we study the transmission problem

$$-\operatorname{div}(\mathbf{C} \cdot \nabla u) = f \qquad \text{in } \Omega, \tag{5.1.1a}$$

$$-\Delta u^{\text{ext}} = 0 \qquad \text{in } \Omega^{\text{ext}}, \tag{5.1.1b}$$

$$u - u^{\text{ext}} = u_0 \qquad \text{on } \Gamma, \tag{5.1.1c}$$

$$\left(\mathbf{C}\nabla u - \nabla u^{\text{ext}}\right) \cdot \nu = \varphi_0 \qquad \text{on } \Gamma, \tag{5.1.1d}$$

$$u^{\text{ext}} = \begin{cases} \mathcal{O}(|x|^{-1}) & \text{as } |x| \to \infty \text{ if } d = 3 \\ b \log|x| + \mathcal{O}(|x|^{-1}) & \text{for some } b \in \mathbb{R} \text{ as } |x| \to \infty \text{ if } d = 2 \end{cases} \tag{5.1.1e}$$

Here, $\Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}$ denotes the exterior of Ω, and ν denotes the outward normal vector. For the data, we assume $f \in L^2(\Omega)$, $u_0 \in H^{1/2}(\Gamma)$, $\varphi_0 \in H^{-1/2}(\Gamma)$, and $\mathbf{C} \in L^\infty(\Omega; \mathbb{R}^d)$ to be pointwise symmetric and positive definite, i.e., there is a constant $C_{\text{ell}} > 0$ such that

$$\langle \mathbf{C}x, x\rangle_2 \geq C_{\text{ell}} \|x\|_2^2 . \tag{5.1.2}$$

For $d = 2$, we assume $\operatorname{diam} \Omega < 1$ for the single-layer operator V introduced below to be elliptic.

Remark 5.1.1. The radiation condition (5.1.1e) is such that the representation form $u^{\text{ext}} = -\widetilde{V}\varphi + \widetilde{K}u^{\text{ext}}$ holds in Ω^{ext} with $\varphi = \nabla u^{\text{ext}} \cdot \nu$ (see, e.g., [SS11, Chap. 3.1]). For $d = 2$, the compatibility condition $\langle f, 1\rangle_{L^2(\Omega)} + \langle \varphi_0, 1\rangle_{L^2(\Gamma)} = 0$ ensures $b = 0$ in (5.1.1e). See also [McL00, Thm. 8.9] for more on the radiation condition. ■

5.2 Layer potential and boundary integral operators

In this section, we define the volume potential operators $\widetilde{V}$, $\widetilde{K}$ and the boundary integral operators V, K, K', W and mention some of their properties . For details, we refer to [SS11, Ch. 3] and [Ste07, Ch. 6].

Definition 5.2.1. With the Green's function for the Laplacian $G(x) = -\frac{1}{2\pi}\log|x|$ for $d = 2$ and $G(x) = \frac{1}{4\pi}\frac{1}{|x|}$ for $d = 3$, we introduce the single-layer boundary integral operator $V \in L(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$ by

$$V\phi(x) := \int_\Gamma G(x-y)\phi(y)ds_y, \quad x \in \Gamma.$$

The double-layer operator $K \in L(H^{1/2}(\Gamma), H^{1/2}(\Gamma))$ has the form

$$K\phi(x) := \int_\Gamma (\partial_{\nu(y)} G(x-y))\phi(y)ds_y, \quad x \in \Gamma,$$

where $\partial_{\nu(y)}$ denotes the normal derivative at the point y. The adjoint of K is denoted by K'. The hyper-singular operator $W \in L(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ is given by

$$W\phi(x) := -\partial_{\nu(x)} \int_\Gamma (\partial_{\nu(y)} G(x-y))\phi(y)ds_y, \quad x \in \Gamma.$$

In addition to the boundary integral operators, we introduce the volume potentials $\widetilde{V}$ and $\widetilde{K}$ by

$$\begin{aligned} \widetilde{V}\phi(x) &:= \int_\Gamma G(x-y)\phi(y)ds_y, & x \in \mathbb{R}^d\backslash\Gamma, \\ \widetilde{K}\phi(x) &:= \int_\Gamma \partial_{\nu(y)} G(x-y)\phi(y)ds_y, & x \in \mathbb{R}^d\backslash\Gamma. \end{aligned}$$

Remark 5.2.2. The single-layer operator V is elliptic for $d = 3$ and for $d = 2$ provided $\operatorname{diam}(\Omega) < 1$. The hyper-singular operator W is semi-elliptic with a kernel of dimension being the number of components of connectedness of Γ.

For $D \subseteq \mathbb{R}^d$ and $s > 0$, we introduce the space $H^s_{\text{loc}}(\Omega)$ as

$$H^s_{\text{loc}}(D) := \left\{u \in (C_0^\infty(D))' \quad : \quad \varphi u \in H^s(D) \quad \forall \varphi \in C_0^\infty(D)\right\}.$$

We also denote

$$H^s_{\text{loc}}(\mathbb{R}^d \backslash \Gamma) := \left\{ u \in L^2(\mathbb{R}^d) \quad : \quad u|_\Omega \in H^s_{\text{loc}}(\Omega),\, u|_{\Omega^{\text{ext}}} \in H^s_{\text{loc}}(\Omega^{\text{ext}}) \right\}.$$

In the following, we state some well-known facts about these operators.

- With the interior trace operator γ_0^{int} (for Ω) and exterior trace operator γ_0^{ext} (for $\mathbb{R}^d \backslash \overline{\Omega}$), we have

$$\begin{aligned} &\gamma_0^{\text{int}} \widetilde{V}\varphi = V\varphi = \gamma_0^{\text{ext}} \widetilde{V}\varphi, \\ &\gamma_0^{\text{int}} \widetilde{K}u = (-1/2 + K)u \qquad \text{and} \qquad \gamma_0^{\text{ext}} \widetilde{K}u = (1/2 + K)u, \end{aligned} \tag{5.2.1}$$

  which implies the jump conditions across Γ

$$[\gamma_0 \widetilde{V}\varphi] := \gamma_0^{\text{ext}} \widetilde{V}\varphi - \gamma_0^{\text{int}} \widetilde{V}\varphi = 0, \qquad [\gamma_0 \widetilde{K}u] = u. \tag{5.2.2}$$

- Similarly, with the interior $\gamma_1^{\text{int}} u := \gamma_0^{\text{int}} \nabla u \cdot \nu$ and exterior conormal derivative $\gamma_1^{\text{ext}} u := \gamma_0^{\text{ext}} \nabla u \cdot \nu$ (ν is the outward normal vector of Ω), we have

$$\begin{aligned} &\gamma_1^{\text{int}} \widetilde{V}\varphi = (1/2 + K')\varphi \qquad \text{and} \qquad \gamma_1^{\text{ext}} \widetilde{V}\varphi = (-1/2 + K')\varphi, \\ &\gamma_1^{\text{int}} \widetilde{K}u = -Wu = \gamma_1^{\text{ext}} \widetilde{K}u, \end{aligned} \tag{5.2.3}$$

  and consequently the jump conditions

$$[\gamma_1 \widetilde{V}\varphi] := \gamma_1^{\text{ext}} \widetilde{V}\varphi - \gamma_1^{\text{int}} \widetilde{V}\varphi = -\varphi, \qquad [\gamma_1 \widetilde{K}u] = 0. \tag{5.2.4}$$

- The potentials $\widetilde{V}\varphi$ and $\widetilde{K}u$ are harmonic in $\mathbb{R}^d \backslash \Gamma$ and are bounded operators (see [SS11, Ch. 3.1.2])

$$\widetilde{V} : H^{-1/2+s}(\Gamma) \to H^{1+s}_{\text{loc}}(\mathbb{R}^d), \qquad \widetilde{K} : H^{1/2+s}(\Gamma) \to H^{1+s}_{\text{loc}}(\mathbb{R}^d \backslash \Gamma), \qquad |s| \leq 1/2. \tag{5.2.5}$$

  Consequently, we have the boundedness for the boundary integral operators as

$$V : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \quad K : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \tag{5.2.6}$$

$$W : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma) \tag{5.2.7}$$

  for $s \in \mathbb{R}$ with $|s| \leq 1/2$.

5.3 FEM-BEM coupling techniques

In the following, we consider three different variational formulations, namely, the symmetric coupling, the Bielak-MacCamy coupling, and the Johnson-Nédélec coupling for our model problem. All three are well-posed without compatibility assumptions on the data. The compatibility condition $\langle f, 1\rangle_{L^2(\Omega)} + \langle \varphi_0, 1\rangle_{L^2(\Gamma)} = 0$ for $d = 2$ ensures the radiation condition (5.1.1e); lifting the compatibility condition yields a solution that satisfies a different radiation condition, namely, $u^{\text{ext}} = b \log|x| + \mathcal{O}(|x|^{-1})$ as $|x| \to \infty$ for some $b \in \mathbb{R}$ for the three coupling strategies considered. Our analysis requires only the unique solvability of the variational formulations. In this section, we study discretizations of weak solutions of the model problem reformulated via three different FEM-BEM couplings: the Bielak-MacCamy coupling, Costabel's symmetric coupling, and the Johnson-Nédélec coupling. All these couplings lead to a variational formulation of finding $(u, \varphi) \in H^1(\Omega) \times H^{-1/2}(\Gamma) =: \mathbf{X}$ such that

$$a(u, \varphi; \psi, \zeta) = g(\psi, \zeta) \qquad \forall (\psi, \zeta) \in \mathbf{X}, \tag{5.3.1}$$

where $a : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ is a bilinear form and $g : \mathbf{X} \to \mathbb{R}$ is continuous linear functional.

For the discretization, we assume that Ω is triangulated by a quasi-uniform mesh $\mathcal{T} = \{T_1, \dots, T_{\widehat{n}}\}$ of mesh width $h := \max_{T_j \in \mathcal{T}} \operatorname{diam}(T_j)$. The elements $T_j \in \mathcal{T}$ are open triangles ($d = 2$) or tetrahedra ($d = 3$). Additionally, we assume that the mesh $\mathcal{T}$ is regular in the sense of Ciarlet and γ-shape regular in the sense that we have $\operatorname{diam}(T_j) \leq \gamma\, |T_j|^{1/2}$ for all $T_j \in \mathcal{T}$, where $|T_j|$ denotes the Lebesgue measure of T_j. By $\mathcal{K} := \{K_1, \dots, K_{\widehat{m}}\}$, we denote the restriction of $\mathcal{T}$ to the boundary Γ, which is a regular and shape-regular triangulation of the boundary.

In this Chapter, we consider lowest order Galerkin discretizations in $S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$. We let $\mathcal{B}_h := \{\xi_j \,:\, j = 1, \dots, n\}$ be the basis of $S^{1,1}(\mathcal{T})$ consisting of the standard hat functions, and we let $\mathcal{W}_h := \{\chi_j \,:\, j = 1, \dots, m\}$ be the basis of $S^{0,0}(\mathcal{K})$ that consists of the characteristic functions of the surface elements.

5.3.1 The Bielak–MacCamy coupling

The Bielak–MacCamy coupling is derived by making a single-layer ansatz for the exterior solution, i.e., $u^{\text{ext}} = \widetilde{V}\varphi$ in Ω^{ext} with an unknown density $\varphi \in H^{-1/2}(\Gamma)$. For more details, we refer to [BM84]. This approach leads to the bilinear form

$$a_{\text{bmc}}(u, \varphi; \psi, \zeta) := \langle \mathbf{C}\nabla u, \nabla \psi\rangle_{L^2(\Omega)} + \big\langle (1/2 - K')\varphi, \psi\big\rangle_{L^2(\Gamma)} - \langle u, \zeta\rangle_{L^2(\Gamma)} + \langle V\varphi, \zeta\rangle_{L^2(\Gamma)}, \tag{5.3.2a}$$

$$g_{\text{bmc}}(\psi, \zeta) := \langle f, \psi\rangle_{L^2(\Omega)} + \langle \varphi_0, \psi\rangle_{L^2(\Gamma)} - \langle u_0, \zeta\rangle_{L^2(\Gamma)}. \tag{5.3.2b}$$

Replacing $H^1(\Omega) \times H^{-1/2}(\Gamma)$ by the finite dimensional subspace $S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$, we arrive at the Galerkin discretization of (5.3.2) of finding $(u_h, \varphi_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ such that

$$\langle \mathbf{C}\nabla u_h, \nabla \psi_h \rangle_{L^2(\Omega)} + \big\langle (1/2 - K')\varphi_h, \psi_h \big\rangle_{L^2(\Gamma)} = \langle f, \psi_h \rangle_{L^2(\Omega)} + \langle \varphi_0, \psi_h \rangle_{L^2(\Gamma)} \;\; \forall \psi_h \in S^{1,1}(\mathcal{T}), \tag{5.3.3a}$$

$$\langle u_h, \zeta_h \rangle_{L^2(\Gamma)} - \langle V\varphi_h, \zeta_h \rangle_{L^2(\Gamma)} = \langle u_0, \zeta_h \rangle_{L^2(\Gamma)} \;\; \forall \zeta_h \in S^{0,0}(\mathcal{K}). \tag{5.3.3b}$$

If the ellipticity constant of $\mathbf{C}$ satisfies $C_{\text{ell}} > 1/4$, then [AFF+13, Thm. 9] shows that the Bielak-MacCamy coupling is equivalent to an elliptic problem with the use of a (theoretical) implicit stabilization. Therefore, (5.3.3) is uniquely solvable.

With the bases $\mathcal{B}_h$ of $S^{1,1}(\mathcal{T})$ and $\mathcal{W}_h$ of $S^{0,0}(\mathcal{K})$, the Galerkin discretization (5.3.3) leads to a block matrix $\mathbf{A}_{\text{bmc}} \in \mathbb{R}^{(n+m)\times(n+m)}$

$$\mathbf{A}_{\text{bmc}} := \begin{pmatrix} \mathbf{A} & \frac{1}{2}\mathbf{M}^T - \mathbf{K}^T \\ \mathbf{M} & -\mathbf{V} \end{pmatrix}, \tag{5.3.4}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is given by $\mathbf{A}_{ij} = \langle \mathbf{C}\nabla \xi_j, \nabla \xi_i \rangle_{L^2(\Omega)}$, $\mathbf{M} \in \mathbb{R}^{m\times n}$ by $\mathbf{M}_{ij} = \langle \xi_i, \chi_j \rangle_{L^2(\Gamma)}$, $\mathbf{K} \in \mathbb{R}^{m\times n}$ by $\mathbf{K}_{ij} = \langle K\xi_i, \chi_j \rangle_{L^2(\Gamma)}$, and $\mathbf{V} \in \mathbb{R}^{m\times m}$ by $\mathbf{V}_{ij} = \langle V\chi_j, \chi_i \rangle_{L^2(\Gamma)}$. As mentioned in the introduction, we omitted the trace operators, i.e., in $\mathbf{M}$ and $\mathbf{K}$, ξ_i is understood as $\gamma_0^{\text{int}}\xi_i$.

5.3.2 Costabel's symmetric coupling

The coupling is based on the representation formula $u^{\text{ext}} = -\widetilde{V}\varphi + \widetilde{K}u^{\text{ext}}$ in Ω^{ext} with $\varphi = \nabla u^{\text{ext}} \cdot \nu$ (see, e.g., [SS11, Chap. 3.1]). . Coupling the interior and exterior solution in a symmetric way (which uses all four boundary integral operators), this leads to Costabel's symmetric coupling, introduced in [Cos88] and [Han90]. Here, the bilinear form and right-hand side are given by

$$\begin{aligned} a_{\text{sym}}(u, \varphi; \psi, \zeta) &:= \langle \mathbf{C}\nabla u, \nabla \psi \rangle_{L^2(\Omega)} + \big\langle (K' - 1/2)\varphi, \psi \big\rangle_{L^2(\Gamma)} + \langle Wu, \psi \rangle_{L^2(\Gamma)} \\ &\quad + \langle (1/2 - K)u, \zeta \rangle_{L^2(\Gamma)} + \langle V\varphi, \zeta \rangle_{L^2(\Gamma)}, \end{aligned} \tag{5.3.5a}$$

$$\begin{aligned} g_{\text{sym}}(\psi, \zeta) &:= \langle f, \psi \rangle_{L^2(\Omega)} + \langle \varphi_0 + Wu_0, \psi \rangle_{L^2(\Gamma)} + \langle (1/2 - K)u_0, \zeta \rangle_{L^2(\Gamma)} \\ &=: \langle f, \psi \rangle_{L^2(\Omega)} + \langle v_0, \psi \rangle_{L^2(\Gamma)} + \langle w_0, \zeta \rangle_{L^2(\Gamma)}. \end{aligned} \tag{5.3.5b}$$

The Galerkin discretization leads to the problem of finding $(u_h, \varphi_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ such that

$$\langle \mathbf{C}\nabla u_h, \nabla \psi_h \rangle_{L^2(\Omega)} + \big\langle (K' - 1/2)\varphi_h, \psi_h \big\rangle_{L^2(\Gamma)} + \langle Wu_h, \psi_h \rangle_{L^2(\Gamma)} = \langle f, \psi_h \rangle_{L^2(\Omega)} + \langle v_0, \psi_h \rangle_{L^2(\Gamma)} \tag{5.3.6a}$$

$$\langle (1/2 - K)u_h, \zeta_h \rangle_{L^2(\Gamma)} + \langle V\varphi_h, \zeta_h \rangle_{L^2(\Gamma)} = \langle w_0, \zeta_h \rangle_{L^2(\Gamma)} \tag{5.3.6b}$$

for all $(\psi_h, \zeta_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$.

With similar arguments as for the Bielak-MacCamy coupling, [AFF+13] prove unique solvability for the symmetric coupling for any $C_{\text{ell}} > 0$.

With the bases $\mathcal{B}_h$ of $S^{1,1}(\mathcal{T})$ and $\mathcal{W}_h$ of $S^{0,0}(\mathcal{K})$, the Galerkin discretization (5.3.6) leads to a block matrix $\mathbf{A}_{\text{sym}} \in \mathbb{R}^{(n+m)\times(n+m)}$

$$\mathbf{A}_{\text{sym}} := \begin{pmatrix} \mathbf{A}+\mathbf{W} & \mathbf{K}^T - \frac{1}{2}\mathbf{M}^T \\ \frac{1}{2}\mathbf{M}-\mathbf{K} & \mathbf{V} \end{pmatrix}, \tag{5.3.7}$$

where $\mathbf{A}$, $\mathbf{M}$, $\mathbf{K}$ are defined in (5.3.4), and $\mathbf{W} \in \mathbb{R}^{n\times n}$ is given by $\mathbf{W}_{ij} = \langle W\xi_j, \xi_i\rangle_{L^2(\Gamma)}$. As mentioned in the introduction, we omitted the trace operators. Thus, the matrix $\mathbf{W}$ is assembled with respect to the traces of basis functions in the volume Ω.

5.3.3 The Johnson-Nédélec coupling

The Johnson-Nédélec coupling, introduced in [JN80] again uses the representation formula for the exterior solution, but differs from the symmetric coupling in the way how the interior and exterior solutions are coupled on the boundary. Instead of all four boundary integral operators, only the single-layer and the double-layer operator are needed. The bilinear form for the Johnson-Nédélec coupling is given by

$$a_{\text{jn}}(u,\varphi;\psi,\zeta) := \langle \mathbf{C}\nabla u, \nabla\psi\rangle_{L^2(\Omega)} - \langle\varphi,\psi\rangle_{L^2(\Gamma)} + \langle(1/2-K)u,\zeta\rangle_{L^2(\Gamma)} + \langle V\varphi,\zeta\rangle_{L^2(\Gamma)}, \tag{5.3.8a}$$

$$\begin{aligned} g_{\text{jn}}(\psi,\zeta) &:= \langle f,\psi\rangle_{L^2(\Omega)} + \langle\varphi_0,\psi\rangle_{L^2(\Gamma)} + \langle(1/2-K)u_0,\zeta\rangle_{L^2(\Gamma)} \\ &=: \langle f,\psi\rangle_{L^2(\Omega)} + \langle\varphi_0,\psi\rangle_{L^2(\Gamma)} + \langle w_0,\zeta\rangle_{L^2(\Gamma)}. \end{aligned} \tag{5.3.8b}$$

The Galerkin discretization in $S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ leads to the problem of finding $(u_h,\varphi_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ such that

$$\langle \mathbf{C}\nabla u_h, \nabla\psi_h\rangle_{L^2(\Omega)} - \langle\varphi_h,\psi_h\rangle_{L^2(\Gamma)} = \langle f,\psi_h\rangle_{L^2(\Omega)} + \langle\varphi_0,\psi_h\rangle_{L^2(\Gamma)} \quad \forall\psi_h \in S^{1,1}(\mathcal{T}), \tag{5.3.9a}$$

$$\langle(1/2-K)u_h,\zeta_h\rangle_{L^2(\Gamma)} + \langle V\varphi_h,\zeta_h\rangle_{L^2(\Gamma)} = \langle(1/2-K)u_0,\zeta_h\rangle_{L^2(\Gamma)} \quad \forall\zeta_h \in S^{0,0}(\mathcal{K}). \tag{5.3.9b}$$

As in the case of the Bielak-MacCamy coupling, the Johnson-Nédélec coupling has an unique solution provided $C_{\text{ell}} > 1/4$, see [AFF⁺13].

With the bases $\mathcal{B}_h$ of $S^{1,1}(\mathcal{T})$ and $\mathcal{W}_h$ of $S^{0,0}(\mathcal{K})$, the Galerkin discretization (5.3.9) leads to a matrix $\mathbf{A}_{\text{jn}} \in \mathbb{R}^{(n+m)\times(n+m)}$

$$\mathbf{A}_{\text{jn}} := \begin{pmatrix} \mathbf{A} & -\mathbf{M}^T \\ \frac{1}{2}\mathbf{M}-\mathbf{K} & \mathbf{V} \end{pmatrix}, \tag{5.3.10}$$

where $\mathbf{A}$, $\mathbf{M}$, $\mathbf{K}$, $\mathbf{V}$ are defined in (5.3.4).

5.4 Main results

Due to the low-rank structure on far-field blocks, the memory requirement to store an $\mathcal{H}$ matrix is given by $\sim C_{\text{sp}}\,\text{depth}(\mathbb{T}_{\mathcal{I}})r(n+m)$. Provided C_{sp} is bounded and the cluster tree is balanced, i.e., $\operatorname{depth}(\mathbb{T}_{\mathcal{I}}) \sim \log(n+m)$, which can be ensured by suitable clustering methods (e.g. geometric clustering, [Hac09]), we get a storage complexity of $\mathcal{O}(r(n+m)\log(n+m))$.

The following theorem shows that the inverse matrices $\mathbf{A}_{\text{bmc}}^{-1}$, $\mathbf{A}_{\text{sym}}^{-1}$, and $\mathbf{A}_{\text{jn}}^{-1}$ corresponding to the three mentioned FEM-BEM couplings can be approximated in the $\mathcal{H}$-matrix format, and the error converges exponentially in the maximal block rank employed.

Theorem 5.4.1. *For a fixed admissibility parameter $\eta > 0$, let a partition P of $\mathcal{I} \times \mathcal{I}$ that is based on the cluster tree $\mathbb{T}_{\mathcal{I}}$ be given. Then, there exists an $\mathcal{H}$-matrix $\mathbf{B}_{\mathcal{H}}$ with maximal blockwise rank r such that*

$$\left\|\mathbf{A}_{\text{bmc}}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 \leq C_{\text{apx}} C_{\text{sp}} \operatorname{depth}(\mathbb{T}_{\mathcal{I}}) h^{-(2+d)} e^{-br^{1/(2d+1)}}$$

for the Bielak-MacCamy coupling. In the same way, there exists a blockwise rank-r $\mathcal{H}$-matrix $\mathbf{B}_{\mathcal{H}}$ such that

$$\left\|\mathbf{A}_{\text{sym}}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 \leq C_{\text{apx}} C_{\text{sp}} \operatorname{depth}(\mathbb{T}_{\mathcal{I}}) h^{-(2+d)} e^{-br^{1/(3d+1)}}$$

for the symmetric coupling and

$$\left\|\mathbf{A}_{\text{jn}}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 \leq C_{\text{apx}} C_{\text{sp}} \operatorname{depth}(\mathbb{T}_{\mathcal{I}}) h^{-(2+d)} e^{-br^{1/(6d+1)}}$$

for the Johnson-Nédélec coupling. Here, $\|\cdot\|_2$ denotes the spectral norm and the constants $C_{\text{apx}} > 0$ and $b > 0$ depend only on Ω, d, η, and the γ-shape regularity of the quasi-uniform triangulations $\mathcal{T}$ and $\mathcal{K}$.

Remark 5.4.2. The previous approximation result can also be formulated in norms other than the spectral norm, e.g., the Frobenius norm that is commonly used in the $\mathcal{H}$-matrix literature. Using the norm equivalence $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F \leq \sqrt{N}\,\|\mathbf{A}\|_2$ for arbitrary $\mathbf{A} \in \mathbb{R}^{N \times N}$ shows that this simply produces a different (algebraic) prefactor to the exponentials in Theorem 5.4.1. ▪

Remark 5.4.3. Definition 2.6.3 clusters the degrees of freedom associated with triangulation $\mathcal{T}$ of Ω and the triangulation $\mathcal{K}$ of Γ simultaneously. ▪

5.5 The Caccioppoli-type inequalities

Before we can state the interior regularity estimates, we specify the norm we are working with, an h-weighted H^1-equivalent norm. For a box B_R with side length R, an open set $\omega \subset \mathbb{R}^d$, and $v \in H^1(B_R \cap \omega)$, we introduce

$$|||v|||_{h,R,\omega}^2 := h^2 \|\nabla v\|_{L^2(B_R \cap \omega)}^2 + \|v\|_{L^2(B_R \cap \omega)}^2 . \tag{5.5.1}$$

For the case $\omega = \mathbb{R}^d$, we abbreviate $|||\cdot|||_{h,R,\mathbb{R}^d} =: |||\cdot|||_{h,R}$ and for the case $\omega = \mathbb{R}^d \backslash \Gamma$ we write $|||\cdot|||_{h,R,\mathbb{R}^d\backslash\Gamma} =: |||\cdot|||_{h,R,\Gamma^c}$ and understood the norms over $B_R \backslash \Gamma$ as a sum over integrals $B_R \cap \Omega$ and $B_R \cap \Omega^{\text{ext}}$. Moreover, for triples $(u, v, w) \in H^1(B_R \cap \Omega) \times H^1(B_R) \times H^1(B_R \backslash \Gamma)$, we set

$$|||(u, v, w)|||_{h,R}^2 := |||u|||_{h,R,\Omega}^2 + |||v|||_{h,R}^2 + |||w|||_{h,R,\Gamma^c}^2 . \tag{5.5.2}$$

We note that u will be the interior solution, v be chosen as a single-layer potential and w as a double-layer potential (which jumps across Γ), which explains the different requirements for the set ω.

In the proof of the Caccioppoli type inequality, we need the following inverse-type inequalities from [FMP16, Lem. 3.8] and [FMP17, Lem. 3.6].

Lemma 5.5.1 ([FMP16, Lem. 3.8], [FMP17, Lem. 3.6]). *Let $B_R \subset B_{R'}$ be concentric boxes with $\operatorname{dist}(B_R, \partial B_{R'}) \geq 4h$. Then, for every $\psi_h \in S^{0,0}(\mathcal{K})$, we have*

$$\|\psi_h\|_{L^2(B_R\cap\Gamma)} \lesssim h^{-1/2} \left\|\nabla\widetilde{V}\psi_h\right\|_{L^2(B_{R'})}.$$

Moreover, for every $v_h \in S^{1,1}(\mathcal{T})$, we have

$$\left\|\gamma_1\widetilde{K}v_h\right\|_{L^2(B_R\cap\Gamma)} \lesssim h^{-1/2}\left(\left\|\nabla\widetilde{K}v_h\right\|_{L^2(B_{R'})} + \frac{1}{\operatorname{dist}(B_R,\partial B_{R'})}\left\|\widetilde{K}v_h\right\|_{L^2(B_{R'})}\right). \tag{5.5.3}$$

Combining Lemma 2.4.1 with Lemma 5.5.1 (assuming $\operatorname{supp}\eta \subset B_R$), we obtain estimates of the form

$$\left\|\eta\psi_h - I_h^\Gamma(\eta\psi_h)\right\|_{H^{-1/2}(\Gamma)} \lesssim h\,\|\nabla\eta\|_{L^\infty(\Gamma)} \left\|\nabla\widetilde{V}\psi_h\right\|_{L^2(B_{R'})}. \tag{5.5.4}$$

Remark 5.5.2. An inspection of the proof of (5.5.3) ([FMP17, Lem. 3.6]) shows that the main observation is that $\widetilde{K}v_h$ is harmonic. The remaining arguments therein only use mapping properties and jump conditions for the potential $\widetilde{K}$ and can directly be modified such that the same result holds for the single-layer potential as well, i.e., for every $\psi_h \in S^{0,0}(\mathcal{T})$, we have

$$\left\|\gamma_1\widetilde{V}\psi_h\right\|_{L^2(B_R\cap\Gamma)} \lesssim h^{-1/2}\left(\left\|\nabla\widetilde{V}\psi_h\right\|_{L^2(B_{R'})} + \frac{1}{\operatorname{dist}(B_R,\partial B_{R'})}\left\|\widetilde{V}\psi_h\right\|_{L^2(B_{R'})}\right). \tag{5.5.5}$$

5.5.1 The Bielak-MacCamy coupling

The following theorem is one of the main results of this section. It states that for the interior finite element solution and the single-layer potential of the boundary element solution, a Caccioppoli type estimate holds, i.e., the stronger H^1-seminorm can be estimated by a weaker h-weighted H^1-norm on a larger domain.

Theorem 5.5.3. *Assume that $C_{\text{ell}} > 1/4$ in (5.1.2). Let $\varepsilon \in (0,1)$ and $R \in (0, 2\operatorname{diam}(\Omega))$ be such that $\frac{h}{R} < \frac{\varepsilon}{16}$, and let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes. Assume that the data is localized away from $B_{(1+\varepsilon)R}$, i.e., $(\operatorname{supp} f \cup \operatorname{supp}\varphi_0 \cup \operatorname{supp} u_0) \cap B_{(1+\varepsilon)R} = \emptyset$. Then, there exists a constant C depending only on Ω, d, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$, such that for the solution (u_h, φ_h) of (5.3.3) we have*

$$\|\nabla u_h\|_{L^2(B_R\cap\Omega)} + \left\|\nabla\widetilde{V}\varphi_h\right\|_{L^2(B_R)} \leq \frac{C}{\varepsilon R}\left(\|u_h\|_{h,(1+\varepsilon)R,\Omega} + \left\|\widetilde{V}\varphi_h\right\|_{h,(1+\varepsilon)R}\right),$$

where the norms on the right-hand side are defined in (5.5.1).

Proof. In order to reduce unnecessary notation, we write (u, φ) for the Galerkin solution (u_h, φ_h). The assumption on the support of the data implies the local orthogonality

$$a_{\mathrm{bmc}}(u, \varphi; \psi_h, \zeta_h) = 0 \quad \forall (\psi_h, \zeta_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K}) \quad \text{with} \quad \operatorname{supp} \psi_h, \operatorname{supp} \zeta_h \subset B_{(1+\varepsilon)R}. \tag{5.5.6}$$

Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with $\operatorname{supp} \eta \subseteq B_{(1+\delta/4)R}$, $\eta \equiv 1$ on B_R, $0 \le \eta \le 1$, and $\|D^j \eta\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{(\delta R)^j}$ for $j = 1, 2$. Here, $0 < \delta \le \varepsilon$ is such that $\frac{h}{R} \le \frac{\delta}{8}$. We note that this choice of δ implies that $\bigcup\{K \in \mathcal{K} : \operatorname{supp} \eta \cap K \ne \emptyset\} \subset B_{(1+\delta/2)R}$. In the final step of the proof, we will choose two different values for δ $(\le \varepsilon)$ depending on ε - one of them, $\delta = \frac{\varepsilon}{2}$, explains the assumption made on ε in the theorem.

Step 1: We provide a "localized" ellipticity estimate, i.e., we prove an inequality of the form

$$\|\nabla(\eta u)\|_{L^2(\Omega)}^2 + \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\mathbb{R}^d)}^2 \lesssim a_{\mathrm{bmc}}(u, \varphi; \eta^2 u, \eta^2 \varphi) \quad + \quad \text{terms in weaker norms.}$$

(See (5.5.17) for the precise form.) Since the ellipticity constant C_{ell} of $\mathbf{C}$ satisfies $C_{\mathrm{ell}} > 1/4$, we may choose a $\rho > 0$ such that $1/4 < \rho/2 < C_{\mathrm{ell}}$. This implies $C_\rho := \min\{1 - \frac{1}{2\rho}, C_{\mathrm{ell}} - \frac{\rho}{2}\} > 0$, and we start with

$$\begin{aligned} \left(C_{\mathrm{ell}} - \frac{\rho}{2}\right) \|\nabla(\eta u)\|_{L^2(\Omega)}^2 + \left(1 - \frac{1}{2\rho}\right) \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\mathbb{R}^d)}^2 &\le C_{\mathrm{ell}} \|\nabla(\eta u)\|_{L^2(\Omega)}^2 + \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\mathbb{R}^d)}^2 \\ &\quad - \frac{1}{2\rho} \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\Omega)}^2 - \frac{\rho}{2} \|\nabla(\eta u)\|_{L^2(\Omega)}^2 . \end{aligned} \tag{5.5.7}$$

Young's inequality implies

$$\begin{aligned} -\frac{1}{2\rho} \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\Omega)}^2 - \frac{\rho}{2} \|\nabla(\eta u)\|_{L^2(\Omega)}^2 &\le - \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\Omega)} \|\nabla(\eta u)\|_{L^2(\Omega)} \\ &\le - \left\langle \nabla(\eta \widetilde{V} \varphi), \nabla(\eta u) \right\rangle_{L^2(\Omega)} . \end{aligned} \tag{5.5.8}$$

Inserting (5.5.8) into (5.5.7) leads to

$$\begin{aligned} C_\rho \|\nabla(\eta u)\|_{L^2(\Omega)}^2 + C_\rho \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\mathbb{R}^d)}^2 &\le \left\|\nabla(\eta \widetilde{V} \varphi)\right\|_{L^2(\mathbb{R}^d)}^2 + C_{\mathrm{ell}} \|\nabla(\eta u)\|_{L^2(\Omega)}^2 \\ &\quad - \left\langle \nabla(\eta \widetilde{V} \varphi), \nabla(\eta u) \right\rangle_{L^2(\Omega)} . \end{aligned} \tag{5.5.9}$$

An elementary calculation shows

$$\begin{aligned} \left\langle \nabla(\eta \widetilde{V} \varphi), \nabla(\eta u) \right\rangle_{L^2(\Omega)} &= \left\langle \nabla \widetilde{V} \varphi, \nabla(\eta^2 u) \right\rangle_{L^2(\Omega)} \\ &\quad + \left\langle (\nabla \eta) \widetilde{V} \varphi, \nabla(\eta u) \right\rangle_{L^2(\Omega)} - \left\langle \nabla \widetilde{V} \varphi, \eta (\nabla \eta) u \right\rangle_{L^2(\Omega)} . \end{aligned} \tag{5.5.10}$$

Since the single-layer potential is harmonic in Ω, integration by parts (in Ω) and $\gamma_1^{\text{int}}\widetilde{V} = 1/2 + K'$ lead to

$$\left\langle \nabla\widetilde{V}\varphi, \nabla(\eta^2 u)\right\rangle_{L^2(\Omega)} = \left\langle \gamma_1^{\text{int}}\widetilde{V}\varphi, \eta^2 u\right\rangle_{L^2(\Gamma)} = \left\langle (1/2+K')\varphi, \eta^2 u\right\rangle_{L^2(\Gamma)}. \tag{5.5.11}$$

Similarly, with integration by parts (in Ω and Ω^{ext}) and the jump condition of the single-layer potential we obtain

$$\begin{aligned}\left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} &= \left\langle \nabla\widetilde{V}\varphi, \nabla(\eta^2\widetilde{V}\varphi)\right\rangle_{L^2(\mathbb{R}^d)} + \left\langle \nabla\eta\widetilde{V}\varphi, \nabla\eta\widetilde{V}\varphi\right\rangle_{L^2(\mathbb{R}^d)} \\ &= -\left\langle \left[\gamma_1\widetilde{V}\varphi\right], \eta^2 V\varphi\right\rangle_{L^2(\Gamma)} + \left\langle \nabla\eta\widetilde{V}\varphi, \nabla\eta\widetilde{V}\varphi\right\rangle_{L^2(\mathbb{R}^d)} \\ &= \left\langle V\varphi, \eta^2\varphi\right\rangle_{L^2(\Gamma)} + \left\langle \nabla\eta\widetilde{V}\varphi, \nabla\eta\widetilde{V}\varphi\right\rangle_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{5.5.12}$$

Moreover, the symmetry and positive definiteness of $\mathbf{C}$ implies

$$C_{\text{ell}}\left\|\nabla(\eta u)\right\|^2_{L^2(\Omega)} \leq \left\langle \mathbf{C}\nabla(\eta u), \nabla(\eta u)\right\rangle_{L^2(\Omega)} = \left\langle \mathbf{C}\nabla u, \nabla(\eta^2 u)\right\rangle_{L^2(\Omega)} + \left\langle \mathbf{C}\nabla\eta u, \nabla\eta u\right\rangle_{L^2(\Omega)}. \tag{5.5.13}$$

Plugging (5.5.10)–(5.5.13) into (5.5.9), we infer

$$\begin{aligned} C_\rho\left\|\nabla(\eta u)\right\|^2_{L^2(\Omega)} + C_\rho\left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} \leq &\left\langle \mathbf{C}\nabla u, \nabla(\eta^2 u)\right\rangle_{L^2(\Omega)} \\ &+ \left\langle \mathbf{C}\nabla\eta u, \nabla\eta u\right\rangle_{L^2(\Omega)} + \left\langle V\varphi, \eta^2\varphi\right\rangle_{L^2(\Gamma)} \\ &+ \left\|\nabla\eta\widetilde{V}\varphi\right\|^2_{L^2(\mathbb{R}^d)} - \left\langle (1/2+K')\varphi, \eta^2 u\right\rangle_{L^2(\Gamma)} \\ &+ \left\langle \nabla\widetilde{V}\varphi, (\nabla\eta)\eta u\right\rangle_{L^2(\Omega)} - \left\langle \nabla\eta\widetilde{V}\varphi, \nabla(\eta u)\right\rangle_{L^2(\Omega)} \\ =& a_{\text{bmc}}(u,\varphi;\eta^2 u, \eta^2\varphi) + \left\langle \mathbf{C}\nabla\eta u, \nabla\eta u\right\rangle_{L^2(\Omega)} \\ &+ \left\|\nabla\eta\widetilde{V}\varphi\right\|^2_{L^2(\mathbb{R}^d)} + \left\langle \nabla\widetilde{V}\varphi, (\nabla\eta)\eta u\right\rangle_{L^2(\Omega)} \\ &- \left\langle \nabla\eta\widetilde{V}\varphi, \nabla(\eta u)\right\rangle_{L^2(\Omega)}. \end{aligned} \tag{5.5.14}$$

Young's inequality and $\left\|\nabla\eta\right\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{\delta R}$ imply

$$\begin{aligned} \left|\left\langle \nabla\widetilde{V}\varphi, (\nabla\eta)\eta u\right\rangle_{L^2(\Omega)}\right| &\leq \left|\left\langle \nabla(\eta\widetilde{V}\varphi), \nabla\eta u\right\rangle_{L^2(\Omega)}\right| + \left|\left\langle \nabla\eta\widetilde{V}\varphi, \nabla\eta u\right\rangle_{L^2(\Omega)}\right| \\ &\leq \left\|\nabla(\eta\widetilde{V}\varphi)\right\|_{L^2(\Omega)}\left\|\nabla\eta u\right\|_{L^2(\Omega)} \\ &\quad + \frac{C}{(\delta R)^2}\left\|u\right\|_{L^2(B_{(1+\delta)R}\cap\Omega)}\left\|\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})} \\ &\leq \frac{C}{(\delta R)^2}\left(\left\|u\right\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}\right) \\ &\quad + \frac{C_\rho}{4}\left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{5.5.15}$$

as well as

$$\left|\left\langle \nabla\eta\widetilde{V}\varphi, \nabla(\eta u)\right\rangle_{L^2(\Omega)}\right| \leq \left\|\nabla\eta\widetilde{V}\varphi\right\|_{L^2(\Omega)} \|\nabla(\eta u)\|_{L^2(\Omega)} \leq \frac{2C}{(\delta R)^2}\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \frac{C_\rho}{4}\|\nabla(\eta u)\|^2_{L^2(\Omega)}. \tag{5.5.16}$$

Absorbing the gradient terms in (5.5.15)-(5.5.16) into the left-hand side of (5.5.14), we arrive at

$$\|\nabla(\eta u)\|^2_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} \lesssim a_{\mathrm{bmc}}(u,\varphi;\eta^2 u,\eta^2\varphi) + \frac{1}{(\delta R)^2}\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \frac{1}{(\delta R)^2}\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}. \tag{5.5.17}$$

Step 2: We apply the local orthogonality of (u,φ) to piecewise polynomials and use approximation properties.

Let $I_h^\Omega : C(\overline{\Omega}) \to S^{1,1}(\mathcal{T})$ be the nodal interpolation operator and I_h^Γ the $L^2(\Gamma)$-orthogonal projection mapping onto $S^{0,0}(\mathcal{K})$. Then, the orthogonality (5.5.6) leads to

$$\begin{aligned} a_{\mathrm{bmc}}(u,\varphi;\eta^2 u,\eta^2\varphi) &= a_{\mathrm{bmc}}(u,\varphi;\eta^2 u - I_h^\Omega(\eta^2 u),\eta^2\varphi - I_h^\Gamma(\eta^2\varphi)) \\ &= \left\langle \mathbf{C}\nabla u, \nabla(\eta^2 u - I_h^\Omega(\eta^2 u))\right\rangle_{L^2(\Omega)} + \left\langle (1/2-K')\varphi, I_h^\Omega(\eta^2 u) - \eta^2 u\right\rangle_{L^2(\Gamma)} \\ &\quad + \left\langle V\varphi, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)} - \left\langle u, I_h^\Gamma(\eta^2\varphi) - \eta^2\varphi\right\rangle_{L^2(\Gamma)} \\ &=: T_1 + T_2 + T_3 + T_4. \end{aligned} \tag{5.5.18}$$

We mention that the volume term T_1 and the boundary term T_3 involving V were already treated in the works [FMP15] and [FMP16]. However, for sake of completeness, we also provide the estimates in the following. For T_1 in (5.5.18), the assumptions on the cut-off function η, the super-approximation properties of I_h^Ω from Lemma 2.4.2, Young's inequality, and $\frac{h}{\delta R} \leq 1$ lead to

$$\begin{aligned} \left|\left\langle \mathbf{C}\nabla u, \nabla(\eta^2 u - I_h^\Omega(\eta^2 u))\right\rangle_{L^2(\Omega)}\right| &\leq \|\mathbf{C}\nabla u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}\left\|\nabla(\eta^2 u - I_h^\Omega(\eta^2 u))\right\|_{L^2(\Omega)} \\ &\lesssim \|\nabla u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}\left(\frac{h}{(\delta R)^2}\|u\|_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{h}{\delta R}\|\nabla u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}\right) \\ &\lesssim \frac{h}{\delta R}\|\nabla u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{1}{(\delta R)^2}\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}. \end{aligned} \tag{5.5.19}$$

For the term T_3, we mention that the assumption $8h \leq \delta R$ implies that $\operatorname{supp} I_h^\Gamma(\eta^2\varphi) \subseteq B_{(1+\delta/2)R}$. In the following, we employ a second cut-off function $\widetilde{\eta}$ with $0 \leq \widetilde{\eta} \leq 1$, $\widetilde{\eta} \equiv 1$ on $B_{(1+\delta/2)R} \supseteq \operatorname{supp}(\eta^2\varphi - I_h^\Gamma(\eta^2\varphi))$, $\operatorname{supp}\widetilde{\eta} \subseteq \overline{B_{(1+\delta)R}}$ and $\|\nabla\widetilde{\eta}\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{\delta R}$. The trace inequality together with the super-approximation properties of I_h^Γ, expressed in (5.5.4), lead to

$$\begin{aligned}\left|\left\langle V\varphi, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)}\right| &= \left|\left\langle \widetilde{\eta} V\varphi, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)}\right| \leq \|\widetilde{\eta} V\varphi\|_{H^{1/2}(\Gamma)} \left\|\eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\|_{H^{-1/2}(\Gamma)} \\ &\lesssim \left\|\widetilde{\eta}\widetilde{V}\varphi\right\|_{H^1(\Omega)} \frac{h}{\delta R} \left\|\nabla \widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})} \\ &\lesssim \left(\frac{1}{\delta R}\left\|\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})} + \left\|\nabla\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})}\right) \frac{h}{\delta R}\left\|\nabla\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})} \\ &\lesssim \frac{h}{\delta R}\left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \frac{1}{(\delta R)^2}\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}. \end{aligned} \tag{5.5.20}$$

With the same arguments, we obtain an estimate for T_4:

$$\begin{aligned}\left|\left\langle u, I_h^\Gamma(\eta^2\varphi) - \eta^2\varphi\right\rangle_{L^2(\Gamma)}\right| &\lesssim \|\widetilde{\eta} u\|_{H^1(\Omega)} \frac{h}{\delta R}\left\|\nabla\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})} \\ &\lesssim \frac{h}{\delta R}\|\nabla u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{h}{\delta R}\left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \frac{1}{(\delta R)^2}\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}. \end{aligned} \tag{5.5.21}$$

It remains to treat the coupling term T_2 involving the adjoint double-layer operator in (5.5.18). With the support property $\operatorname{supp}(I_h^\Omega(\eta^2 u) - \eta^2 u) \subset B_{(1+\delta/2)R}$, which follows from $8h \leq \delta R$, and $(1/2 - K')\varphi = -\gamma_1^{\text{ext}}\widetilde{V}\varphi$, we obtain

$$\left|\left\langle (1/2 - K')\varphi, I_h^\Omega(\eta^2 u) - \eta^2 u\right\rangle_{L^2(\Gamma)}\right| \leq \left\|\gamma_1^{\text{ext}}\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta/2)R}\cap\Gamma)} \left\|I_h^\Omega(\eta^2 u) - \eta^2 u\right\|_{L^2(\Gamma)}. \tag{5.5.22}$$

The multiplicative trace inequality for Ω, see, e.g., [BS02], the super-approximation property of I_h^Ω from (2.4.7), and $\frac{h}{R} \leq \frac{\delta}{8}$ lead to (see also [FMP15, Eq. (25), (26)] for more details)

$$\begin{aligned} &\left\|I_h^\Omega(\eta^2 u) - \eta^2 u\right\|_{L^2(\Gamma)} \lesssim \left\|I_h^\Omega(\eta^2 u) - \eta^2 u\right\|_{L^2(\Omega)} + \left\|I_h^\Omega(\eta^2 u) - \eta^2 u\right\|^{1/2}_{L^2(\Omega)} \left\|\nabla(I_h^\Omega(\eta^2 u) - \eta^2 u)\right\|^{1/2}_{L^2(\Omega)} \\ &\lesssim \left(\frac{h^2}{(\delta R)^2}\|u\|_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{h^2}{\delta R}\|\nabla u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}\right) \\ &+ \left(\frac{h}{\delta R}\|u\|^{1/2}_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{h}{(\delta R)^{1/2}}\|\nabla u\|^{1/2}_{L^2(B_{(1+\delta)R}\cap\Omega)}\right) \\ &\times \left(\frac{h^{1/2}}{\delta R}\|u\|^{1/2}_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{h^{1/2}}{(\delta R)^{1/2}}\|\nabla u\|^{1/2}_{L^2(B_{(1+\delta)R}\cap\Omega)}\right) \\ &\lesssim \frac{h^{3/2}}{\delta R}\|\nabla u\|_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{h^{3/2}}{(\delta R)^2}\|u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}. \end{aligned} \tag{5.5.23}$$

We use estimate (5.5.5) and (5.5.23) in (5.5.22), which implies

$$\begin{aligned}\left|\langle(1/2-K')\varphi, I_h^\Omega(\eta^2 u)-\eta^2 u\rangle_{L^2(\Gamma)}\right| &\lesssim h^{-1/2}\left(\left\|\nabla\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})}+\frac{1}{\delta R}\left\|\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})}\right)\cdot\\ &\quad\left(\frac{h^{3/2}}{\delta R}\|\nabla u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}+\frac{h^{3/2}}{(\delta R)^2}\|u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}\right)\\ &\lesssim \frac{h}{\delta R}\Big\{\Big(\left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}+\|\nabla u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}\Big)\\ &\quad+\frac{1}{(\delta R)^2}\Big(\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}+\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}\Big)\Big\}.\end{aligned} \tag{5.5.24}$$

Finally, inserting (5.5.20), (5.5.21), (5.5.19), and (5.5.24) into (5.5.18) and further into (5.5.17), and absorbing the term $\frac{1}{4}\|\eta\nabla u\|^2_{L^2(B_{(1+\delta)R})}$ on the left-hand side implies

$$\begin{aligned}\|\nabla u\|^2_{L^2(B_R\cap\Omega)}+\left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_R)} &\le \|\nabla(\eta u)\|^2_{L^2(\Omega)}+\left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)}\\ &\lesssim \frac{h}{\delta R}\left(\|\nabla u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}+\left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}\right)\\ &\quad+\frac{1}{(\delta R)^2}\left(\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}+\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}\right).\end{aligned} \tag{5.5.25}$$

Step 3: We iterate (5.5.25) to improve the powers of h for the gradient terms to finally obtain the result of Theorem 5.5.3.

We set $\delta=\frac{\varepsilon}{2}$, and use (5.5.25) again for the gradient terms on the right-hand side with the boxes $B_{\widetilde{R}}$ and $B_{(1+\widetilde{\delta})\widetilde{R}}$, where $\widetilde{\delta}=\frac{\varepsilon}{\varepsilon+2}$ and $\widetilde{R}=(1+\varepsilon/2)R$. We note that $16h\le\varepsilon R$ implies $8h\le\widetilde{\delta}\widetilde{R}$, so we may apply (5.5.25). Considering $(1+\widetilde{\delta})(1+\frac{\varepsilon}{2})=1+\varepsilon$, we get

$$\begin{aligned}\|\nabla u\|^2_{L^2(B_R\cap\Omega)}+\left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_R)} &\lesssim \frac{h^2}{(\varepsilon R)^2}\left(\|\nabla u\|^2_{L^2(B_{(1+\varepsilon)R}\cap\Omega)}+\left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\varepsilon)R})}\right)\\ &\quad+\left(\frac{h}{(\varepsilon R)^3}+\frac{1}{(\varepsilon R)^2}\right)\left(\|u\|^2_{L^2(B_{(1+\varepsilon)R}\cap\Omega)}+\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\varepsilon)R})}\right),\end{aligned} \tag{5.5.26}$$

and with $\frac{h}{\varepsilon R}<1$, we conclude the proof. □

5.5.2 Costabel's symmetric coupling

The following theorem is similar to Theorem 5.5.3 and provides a simultaneous Caccioppoli-type estimate for the interior solution as well as for the single-layer potential of the boundary solution and the double-layer potential of the trace of the interior solution. Here, the double-layer potential additionally appears since all boundary integral operators, especially the hyper-singular operator appear in the coupling.

Theorem 5.5.4. *Let $\varepsilon \in (0,1)$ and $R \in (0, 2\operatorname{diam}(\Omega))$ be such that $\frac{h}{R} < \frac{\varepsilon}{32}$, and let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes. Assume that the data is localized away from $B_{(1+\varepsilon)R}$, i.e., $(\operatorname{supp} f \cup \operatorname{supp} v_0 \cup \operatorname{supp} w_0) \cap B_{(1+\varepsilon)R} = \emptyset$. Then, there exists a constant C depending only on Ω, d, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$, such that for the solution (u_h, φ_h) of (5.3.6) we have*

$$\|\nabla u_h\|_{L^2(B_R\cap\Omega)} + \left\|\nabla \widetilde{V}\varphi_h\right\|_{L^2(B_R)} + \left\|\nabla \widetilde{K} u_h\right\|_{L^2(B_R\setminus\Gamma)} \leq \frac{C}{\varepsilon R} \left\|(u_h, \widetilde{V}\phi_h, \widetilde{K}u_h)\right\|_{h,(1+\varepsilon)R}, \tag{5.5.27}$$

where the norm on the right-hand side is defined in (5.5.2).

Proof. Again, we write (u, φ) for the Galerkin solution (u_h, φ_h). The assumption on the support of the data implies the local orthogonality

$$a_{\rm sym}(u, \varphi; \psi_h, \zeta_h) = 0 \quad \forall (\psi_h, \zeta_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K}) \quad \text{with} \quad \operatorname{supp}\psi_h, \operatorname{supp}\zeta_h \subset B_{(1+\varepsilon)R}. \tag{5.5.28}$$

As in the proof of Theorem 5.5.3 let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with $\operatorname{supp}\eta \subseteq B_{(1+\delta/4)R}$, $\eta \equiv 1$ on B_R, $0 \leq \eta \leq 1$, and $\left\|D^j\eta\right\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{\delta R}$ for $j = 1, 2$. Here, $0 < \delta \leq \varepsilon$ is given such that $\frac{h}{R} \leq \frac{\delta}{16}$ and will be chosen in the last step of the proof.
Step 1: We start with a local ellipticity estimate. More precisely, we show

$$\|\nabla(\eta u)\|^2_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} + \left\|\nabla(\eta\widetilde{K}u)\right\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} \leq a_{\rm sym}(u, \varphi; \eta^2 u, \eta^2\varphi) \\ + \text{terms in weaker norms}.$$

(See (5.5.33) for the precise statement.) From (5.5.13) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} C_{\rm ell}\|\nabla(\eta u)\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\left\|\nabla(\eta\widetilde{K}u)\right\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} \\ \leq \left\langle \mathbf{C}\nabla u, \nabla(\eta^2 u)\right\rangle_{L^2(\Omega)} + \langle \mathbf{C}\nabla\eta u, \nabla\eta u\rangle_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} \\ + \left\|\nabla(\eta\widetilde{K}u)\right\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} - \left\langle\nabla(\eta\widetilde{V}\varphi), \nabla(\eta\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)}. \end{aligned} \tag{5.5.29}$$

A direct calculation reveals that

$$\|\nabla(\eta\widetilde{K}u)\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} = \|(\nabla\eta)\widetilde{K}u\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} + \left\langle\nabla\widetilde{K}u, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)}.$$

Inserting this and (5.5.12) in (5.5.29) yields

$$\begin{aligned} C_{\mathrm{ell}} \|\nabla(\eta u)\|^2_{L^2(\Omega)} + \frac{1}{2}\left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\left\|\nabla(\eta\widetilde{K}u)\right\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} \\ \leq \left\langle \mathbf{C}\nabla u, \nabla(\eta^2 u)\right\rangle_{L^2(\Omega)} + \left\langle \mathbf{C}\nabla\eta u, \nabla\eta u\right\rangle_{L^2(\Omega)} \\ + \left\langle V\varphi, \eta^2\varphi\right\rangle_{L^2(\Gamma)} + \left\|(\nabla\eta)\widetilde{V}\varphi\right\|^2_{L^2(\mathbb{R}^d)} \\ + \left\langle \nabla\widetilde{K}u, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)} + \left\|(\nabla\eta)\widetilde{K}u\right\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} \\ - \left\langle \nabla(\eta\widetilde{V}\varphi), \nabla(\eta\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)}. \end{aligned} \tag{5.5.30}$$

Integration by parts together with the jump conditions (5.2.2), (5.2.4) for the double-layer potential gives

$$\left\langle \nabla\widetilde{K}u, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)} = \left\langle Wu, \eta^2 u\right\rangle_{L^2(\Gamma)}. \tag{5.5.31}$$

With a calculation analogous to (5.5.10) (in fact, replace u there with $\widetilde{K}u$), we get

$$\left\langle \nabla(\eta\widetilde{V}\varphi), \nabla(\eta\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)} = \left\langle \nabla(\widetilde{V}\varphi), \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)} + \mathrm{l.o.t.},$$

where the omitted terms (cf. (5.5.10))

$$\mathrm{l.o.t.} = \langle(\nabla\eta)\widetilde{V}\varphi, \nabla(\eta\widetilde{K}u)\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)} - \langle\nabla\widetilde{V}\varphi, \eta(\nabla\eta)\widetilde{K}u\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)}$$

can be estimated in weaker norms (i.e., $\|\widetilde{V}\varphi\|_{L^2(B_{(1+\delta/2)R})}$, $\|\widetilde{K}u\|_{L^2(B_{(1+\delta/2)R}\setminus\Gamma)}$) or lead to terms that are absorbed in the left-hand side as in the proof of Theorem 5.5.3 (see (5.5.15), (5.5.16)). With integration by parts on Ω and Ω^{ext}, we get

$$\begin{aligned} \left\langle \nabla\widetilde{V}\varphi, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)} &= \left\langle \gamma_1^{\mathrm{int}}\widetilde{V}\varphi, \gamma_0^{\mathrm{int}}(\eta^2\widetilde{K}u)\right\rangle_{L^2(\Gamma)} \\ &\quad - \left\langle \gamma_1^{\mathrm{ext}}\widetilde{V}\varphi, \gamma_0^{\mathrm{ext}}(\eta^2\widetilde{K}u)\right\rangle_{L^2(\Gamma)} \\ &= \left\langle (K'+1/2)\varphi, \eta^2(K-1/2)u\right\rangle_{L^2(\Gamma)} \\ &\quad - \left\langle (K'-1/2)\varphi, \eta^2(K+1/2)u\right\rangle_{L^2(\Gamma)} \\ &= \left\langle \eta^2\varphi, (K-1/2)u\right\rangle_{L^2(\Gamma)} - \left\langle (K'-1/2)\varphi, \eta^2 u\right\rangle_{L^2(\Gamma)}. \end{aligned} \tag{5.5.32}$$

Putting everything together and using $\|\nabla\eta\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{\delta R}$, we obtain

$$\begin{aligned} \|\nabla(\eta u)\|^2_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} + \left\|\nabla(\eta\widetilde{K}u)\right\|^2_{L^2(\mathbb{R}^d\setminus\Gamma)} \lesssim a_{\mathrm{sym}}(u,\varphi,\eta^2 u,\eta^2\varphi) \\ + \frac{1}{(\delta R)^2}\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{1}{(\delta R)^2}\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} \\ + \frac{1}{(\delta R)^2}\left\|\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}. \end{aligned} \tag{5.5.33}$$

Step 2: Applying the local orthogonality as well as approximation results.

With the $L^2(\Gamma)$-orthogonal projection I_h^Γ and the nodal interpolation operator I_h^Ω, the orthogonality (5.5.28) implies

$$\begin{aligned}
a_{\mathrm{sym}}(u,\varphi;\eta^2 u,\eta^2\varphi) &= a_{\mathrm{sym}}(u,\varphi;\eta^2 u - I_h^\Omega(\eta^2 u),\eta^2\varphi - I_h^\Gamma(\eta^2\varphi)) \\
&= \left\langle \mathbf{C}\nabla u, \nabla(\eta^2 u - I_h^\Omega(\eta^2 u))\right\rangle_{L^2(\Omega)} + \left\langle Wu, \eta^2 u - I_h^\Omega(\eta^2 u)\right\rangle_{L^2(\Gamma)} \\
&\quad + \left\langle (K'-1/2)\varphi, \eta^2 u - I_h^\Omega(\eta^2 u)\right\rangle_{L^2(\Gamma)} + \left\langle V\varphi, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)} \\
&\quad + \left\langle (1/2-K)u, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)} \\
&=: T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned} \tag{5.5.34}$$

The terms T_1, T_3, T_4 can be estimated with (5.5.19), (5.5.24) and (5.5.20) respectively as in the case for the Bielak-MacCamy coupling. We also mention that the term T_2 involving the hyper-singular integral operator W was treated in [FMP17]. For our purpose, a simplified version of the proof is sufficient, which is presented in the following.

For the term T_2, we mention that the assumption $16h \leq \delta R$ implies that $\operatorname{supp} I_h^\Gamma(\eta^2\varphi) \subseteq B_{(1+\delta/2)R}$. We employ equation (5.5.3) from Lemma 5.5.1 for $\widetilde{K}u$ and the boxes $B_{(1+\delta/2)R}$ and $B_{(1+\delta)R}$ satisfying $\operatorname{dist}(B_{(1+\delta/2)R}, \partial B_{(1+\delta)R}) = \frac{\delta}{4} \geq 4h$ due to the assumptions on δ. Together with $Wu = -\gamma_1^{\mathrm{int}}\widetilde{K}u$, (cf. (5.5.23)), and the Young inequality this implies

$$\begin{aligned}
\left|\left\langle Wu, \eta^2 u - I_h^\Omega(\eta^2 u)\right\rangle_{L^2(\Gamma)}\right| &= \left|\left\langle \gamma_1^{\mathrm{int}}\widetilde{K}u, \eta^2 u - I_h^\Omega(\eta^2 u)\right\rangle_{L^2(\Gamma)}\right| \\
&\leq \left\|\gamma_1^{\mathrm{int}}\widetilde{K}u\right\|_{L^2(B_{(1+\delta/2)R}\cap\Gamma)} \left\|\eta^2 u - I_h^\Omega(\eta^2 u)\right\|_{L^2(\Gamma)} \\
&\lesssim h^{-1/2}\left(\left\|\nabla\widetilde{K}u\right\|_{L^2(B_{(1+\delta)R}\cap\Gamma)} + \frac{1}{\delta R}\left\|\widetilde{K}u\right\|_{L^2(B_{(1+\delta)R}\setminus\Gamma)}\right) \\
&\quad \times \left(\frac{h^{3/2}}{\delta R}\|\nabla u\|_{L^2(B_{(1+\delta)R})} + \frac{h^{3/2}}{(\delta R)^2}\|u\|_{L^2(B_{(1+\delta)R})}\right) \\
&\lesssim \frac{h}{\delta R}\left(\|\nabla u\|^2_{L^2(B_{(1+\delta)R})} + \left\|\nabla\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}\right) + \frac{1}{(\delta R)^2}\left(\|u\|^2_{L^2(B_{(1+\delta)R})} + \left\|\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}\right).
\end{aligned}$$

We finish the proof by estimating T_5. To that end, we need another cut-off function $\widetilde{\eta} \in S^{1,1}(\mathcal{T})$ with $0 \leq \widetilde{\eta} \leq 1$, $\widetilde{\eta} \equiv 1$ on $B_{(1+\delta/2)R} \supseteq \operatorname{supp}\left(I_h^\Gamma(\eta^2\varphi) - \eta^2\varphi\right)$, $\operatorname{supp}\widetilde{\eta} \subseteq B_{(1+\delta)R}$ and $\|\nabla\widetilde{\eta}\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{\delta R}$. Since $(1/2-K)u = -\gamma_0^{\mathrm{int}}\widetilde{K}u$, we get with a trace inequality and the approximation properties expressed in (5.5.4) that

$$\begin{aligned}
|T_5| = \left|\left\langle \widetilde{\eta}\gamma_0^{\mathrm{int}}\widetilde{K}u, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)}\right| &\lesssim \left\|\gamma_0^{\mathrm{int}}(\widetilde{\eta}\widetilde{K}u)\right\|_{H^{1/2}(\Gamma)} \left\|\eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\|_{H^{-1/2}(\Gamma)} \\
&\lesssim \frac{h}{\delta R}\left\|\widetilde{\eta}\widetilde{K}u\right\|_{H^1(\Omega\setminus\Gamma)}\left\|\nabla\widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})} \\
&\lesssim \frac{h}{\delta R}\left(\left\|\nabla\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)} + \left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}\right) + \frac{1}{(\delta R)^2}\left\|\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}.
\end{aligned} \tag{5.5.35}$$

Putting everything together in (5.5.34) and further in (5.5.33), and absorbing the terms $\frac{1}{4}\|\eta\nabla u\|_{L^2(\Omega)}$, $\frac{1}{4}\left\|\nabla(\eta\widetilde{K}u)\right\|_{L^2(\mathbb{R}^d)}$ in the left-hand side, finally yields

$$\begin{aligned}\|\nabla u\|^2_{L^2(B_R\cap\Omega)} + \left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_R)} + \left\|\nabla\widetilde{K}u\right\|^2_{L^2(B_R\setminus\Gamma)} \\ \lesssim \frac{h}{\delta R}\left(\|\nabla u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \left\|\nabla\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)} + \left\|\nabla\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}\right) \\ + \frac{1}{(\delta R)^2}\left(\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \left\|\nabla\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}\right).\end{aligned} \tag{5.5.36}$$

Step 3: By reapplying (5.5.36) to the gradient terms with $\delta = \frac{\varepsilon}{2}$ and suitable boxes, we get the desired result exactly as in step 3 of the proof of Theorem 5.5.3. □

5.5.3 The Johnson-Nédélec coupling

In this section, we prove the Caccioppoli-type inequality from Theorem 5.5.6 for the Johnson-Nédélec coupling. Most of the appearing terms have already been treated in the previous sections. The main difference is that the double-layer potential appears naturally due to the boundary coupling terms, but the local orthogonality is not suited to provide an approximation for it, since the hyper-singular operator does not appear in the bilinear form. A remedy for this problem is to localize the double-layer potential by splitting it into a local near-field and a non-local, but smooth far-field. This techniques follows [FM18], where a similar localization using commutators is employed.

Lemma 5.5.5. *Let $\delta \in (0,1)$ and $R \in (0, 2\operatorname{diam}(\Omega))$ and let B_R and $B_{(1+\delta)R}$ be two concentric boxes. Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with $\operatorname{supp}\eta \subseteq B_{(1+\delta/2)R}$, $\eta \equiv 1$ on $B_{(1+\delta/4)R}$, $0 \le \eta \le 1$, and $\left\|D^j\eta\right\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{(\delta R)^j}$ for $j = 1,2$. Then, for $u \in H^1(\Omega)$, we have*

$$\left\|\nabla\widetilde{K}u\right\|_{L^2(B_R\setminus\Gamma)} \lesssim \sqrt{1+1/\delta}\,\|\eta u\|_{H^1(\Omega)} + \frac{1}{\delta R}\|u\|_{L^2(B_{(1+\delta/4)R}\cap\Omega)} + \frac{1}{\delta R}\left\|\widetilde{K}u\right\|_{L^2(B_{(1+\delta/4)R}\setminus\Gamma)}. \tag{5.5.37}$$

Proof. We start with a localized splitting for the double-layer potential. More precisely, with a second cut-off function $\widehat{\eta}$ satisfying $\widehat{\eta} \equiv 1$ on B_R and $\operatorname{supp}\widehat{\eta} \subseteq B_{(1+\delta/4)R}$, $\|\nabla\widehat{\eta}\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{\delta R}$, we write

$$\widehat{\eta}\widetilde{K}u = \widehat{\eta}\widetilde{K}(\eta u) + \widehat{\eta}\widetilde{K}(1-\eta)u =: v_{\text{near}} + v_{\text{far}}.$$

First, we estimate the near-field $v_{\text{near}} := \widehat{\eta}\widetilde{K}(\eta u)$. The mapping properties of the double-layer potential, (5.2.5), together with the fact that $\operatorname{supp}\nabla\widehat{\eta} \subset B_{(1+\delta/4)R}\setminus B_R$ and the trace inequality provide

$$\begin{aligned}\|\nabla v_{\text{near}}\|_{L^2(B_R\setminus\Gamma)} \lesssim \|\eta u\|_{H^{1/2}(\Gamma)} + \frac{1}{\delta R}\|\widetilde{K}(\eta u)\|_{L^2(B_{(1+\delta/4)R}\setminus B_R)} \lesssim \|\eta u\|_{H^1(\Omega)} \\ + \frac{1}{\delta R}\|\widetilde{K}(\eta u)\|_{L^2(B_{(1+\delta/4)R}\setminus B_R)}.\end{aligned}$$

Since $\widehat{\eta}(1-\eta) \equiv 0$, the far field v_{far} is smooth. Integration by parts using $\Delta \widetilde{K}((1-\eta)u) = 0$ as well as $[\gamma_1 \widetilde{K}u] = 0$ and $\widehat{\eta}(1-\eta) \equiv 0$ (therefore no boundary terms appear) leads to

$$\begin{aligned}
\|\nabla v_{\text{far}}\|^2_{L^2(B_R\setminus\Gamma)} &= \left|\left\langle \nabla \widetilde{K}((1-\eta)u), \nabla(\widehat{\eta}^2 \widetilde{K}((1-\eta)u))\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)}\right| + \left\|(\nabla\widehat{\eta})\widetilde{K}((1-\eta)u)\right\|^2_{L^2(\mathbb{R}^d)} \\
&\lesssim \frac{1}{(\delta R)^2}\left\|\widetilde{K}((1-\eta)u)\right\|^2_{L^2(B_{(1+\delta/4)R}\setminus B_R)} \\
&\lesssim \frac{1}{(\delta R)^2}\left\|\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta/4)R}\setminus B_R)} + \frac{1}{(\delta R)^2}\left\|\widetilde{K}(\eta u)\right\|^2_{L^2(B_{(1+\delta/4)R}\setminus B_R)}.
\end{aligned}$$

Here, we used that $\operatorname{supp}(\nabla\widehat{\eta}) \subset B_{(1+\delta/4)R}\setminus B_R$. For the last term, we apply [FMP16, Lem. 3.7,(ii)], which states that

$$\left\|\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R}\setminus B_R)} \lesssim \sqrt{\delta R}\left(\frac{1}{\sqrt{(1+\delta)R}}\left\|\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R}\setminus\Gamma)} + \sqrt{(1+\delta)R}\left\|\nabla\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R}\setminus\Gamma)}\right).$$

[FMP16, Lem. 3.7,(i)] provides the estimate

$$\left\|\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R})} \lesssim \sqrt{R}\left\|\gamma_0^{\text{int}}\widetilde{K}(\eta u)\right\|_{L^2(\Gamma)} + R\left\|\nabla\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R}\setminus\Gamma)}.$$

The combination of these two estimates and the fact that $\gamma_0^{\text{int}}\widetilde{K}u = (-1/2+K)u$ gives us

$$\left\|\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R}\setminus B_R)} \lesssim \sqrt{\delta R}\,\|(1/2-K)(\eta u)\|_{L^2(\Gamma)} + \sqrt{\delta R}\sqrt{(1+\delta)R}\left\|\nabla\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R}\setminus\Gamma)}.$$

With the mapping properties of K, $\widetilde{K}$ from (5.2.5), (5.2.6) and the multiplicative trace inequality this implies

$$\begin{aligned}
\frac{1}{\delta R}\left\|\widetilde{K}(\eta u)\right\|_{L^2(B_{(1+\delta/4)R}\setminus B_R)} &\lesssim \frac{1}{\sqrt{\delta R}}\|\eta u\|_{L^2(\Gamma)} + \sqrt{1+1/\delta}\,\|\eta u\|_{H^1(\Omega)} \\
&\lesssim \frac{1}{\sqrt{\delta R}}\|\eta u\|_{L^2(\Omega)} + \frac{1}{\sqrt{\delta R}}\|\eta u\|^{1/2}_{L^2(\Omega)}\|\nabla(\eta u)\|^{1/2}_{L^2(\Omega)} + \sqrt{1+1/\delta}\,\|\eta u\|_{H^1(\Omega)} \\
&\lesssim \frac{1}{\delta R}\|\eta u\|_{L^2(\Omega)} + \|\nabla(\eta u)\|_{L^2(\Omega)} + \sqrt{1+1/\delta}\,\|\eta u\|_{H^1(\Omega)}.
\end{aligned}$$

Putting the estimates for the near-field and the far-field together, we obtain

$$\begin{aligned}
\left\|\nabla\widetilde{K}u\right\|_{L^2(B_R\setminus\Gamma)} &\leq \|\nabla v_{\text{near}}\|_{L^2(B_R\setminus\Gamma)} + \|\nabla v_{\text{far}}\|_{L^2(B_R\setminus\Gamma)} \\
&\lesssim \sqrt{1+1/\delta}\,\|\eta u\|_{H^1(\Omega)} + \frac{1}{\delta R}\|u\|_{L^2(B_{(1+\delta/4)R}\cap\Omega)} + \frac{1}{\delta R}\left\|\widetilde{K}u\right\|_{L^2(B_{(1+\delta/4)R}\setminus\Gamma)},
\end{aligned}$$

which finishes the proof. □ □

Theorem 5.5.6. *Assume that $C_{\text{ell}} > 1/4$ in (5.1.2). Let $\varepsilon \in (0,1)$ and $R \in (0, 2\operatorname{diam}(\Omega))$ be such that $\frac{h}{R} < \frac{\varepsilon}{32}$, and let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes. Assume that the data is localized away from $B_{(1+\varepsilon)R}$, i.e., $(\operatorname{supp} f \cup \operatorname{supp}\varphi_0 \cup \operatorname{supp}(1/2-K)u_0) \cap B_{(1+\varepsilon)R} = \emptyset$.*

Then, there exists a constant C depending only on Ω, d, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$, such that for the solution (u_h, φ_h) of (5.3.6) we have

$$\|\nabla u_h\|_{L^2(B_R\cap\Omega)} + \left\|\nabla \widetilde{V}\varphi_h\right\|_{L^2(B_R)} + \left\|\nabla \widetilde{K} u_h\right\|_{L^2(B_R\setminus\Gamma)} \leq C\frac{R}{(\varepsilon R)^2}\left|\left|\left|(u_h, \widetilde{V}\varphi_h, \widetilde{K}u_h)\right|\right|\right|_{h,(1+\varepsilon)R}, \tag{5.5.38}$$

where the norm on the right-hand side is defined in (5.5.2).

Proof of Theorem 5.5.6. Once again, we write (u, φ) for the Galerkin solution (u_h, φ_h). The assumption on the support of the data implies the local orthogonality

$$a_{\rm jn}(u, \varphi; \psi_h, \zeta_h) = 0 \quad \forall(\psi_h, \zeta_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K}) \quad \text{with} \quad \operatorname{supp}\psi_h, \operatorname{supp}\zeta_h \subset B_{(1+\varepsilon)R}. \tag{5.5.39}$$

Let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with $\operatorname{supp}\eta \subseteq B_{(1+\delta/2)R}$, $\eta \equiv 1$ on $B_{(1+\delta/4)R}$, $0 \leq \eta \leq 1$, and $\left\|D^j\eta\right\|_{L^\infty(B_{(1+\delta)R})} \lesssim \frac{1}{(\delta R)^j}$ for $j = 1, 2$. Here, $0 < \delta \leq \varepsilon$ is given such that $\frac{h}{R} \leq \frac{\delta}{16}$. We note that the condition $\eta \equiv 1$ on $B_{(1+\delta/4)R}$ is additionally imposed in order to satisfy estimate (5.5.37), as the localization of the double-layer operator is additionally needed in comparison with the other couplings.

Step 1: We provide a localized ellipticity estimate, i.e., we prove

$$\|\nabla(\eta u)\|^2_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} + \left\|\nabla\widetilde{K}u\right\|^2_{L^2(B_R\setminus\Gamma)} \lesssim a_{\rm jn}(u, \varphi; \eta^2 u, \eta^2\varphi) + \text{terms in weaker norms.}$$

(See (5.5.44) for the precise form). We start with (5.5.37) to obtain

$$\begin{aligned}\|\nabla(\eta u)\|^2_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} + \left\|\nabla\widetilde{K}u\right\|^2_{L^2(B_R\setminus\Gamma)} &\lesssim (1 + 1/\delta)\left(\|\nabla(\eta u)\|^2_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)}\right)\\ &\quad + \frac{(1+1/\delta)}{(\delta R)^2}\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}\\ &\quad + \frac{1}{(\delta R)^2}\left\|\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}.\end{aligned} \tag{5.5.40}$$

The last two terms are already in weaker norms, and for the first two terms, we apply (5.5.9). Since we assumed $C_{\rm ell} > 1/4$ for unique solvability, we choose a $\rho > 0$ such that $1/4 < \rho/2 < C_{\rm ell}$ and set $C_\rho := \min\{1 - \frac{1}{2\rho}, C_{\rm ell} - \frac{\rho}{2}\} > 0$. Then, (5.5.9) implies

$$\begin{aligned}C_\rho\|\nabla(\eta u)\|^2_{L^2(\Omega)} + C_\rho\left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} &\leq C_{\rm ell}\|\nabla(\eta u)\|^2_{L^2(\Omega)} + \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)}\\ &\quad - \left\langle\nabla(\eta\widetilde{V}\varphi), \nabla(\eta u)\right\rangle_{L^2(\Omega)} - \left\langle\nabla\widetilde{V}\varphi, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)}\\ &\quad + \left\langle\nabla\widetilde{V}\varphi, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\setminus\Gamma)}.\end{aligned} \tag{5.5.41}$$

The first three terms can be expanded as in Theorem 5.5.3, where (5.5.10) leads to

$$\left\langle\nabla(\eta\widetilde{V}\varphi), \nabla(\eta u)\right\rangle_{L^2(\Omega)} = \left\langle\nabla\widetilde{V}\varphi, \nabla(\eta^2 u)\right\rangle_{L^2(\Omega)} + \text{l.o.t.}, \tag{5.5.42}$$

where the omitted terms (cf. (5.5.10))

$$\text{l.o.t.} = \langle (\nabla\eta)\widetilde{V}\varphi, \nabla(\eta u)\rangle_{L^2(\Omega)} - \langle \nabla\widetilde{V}\varphi, \eta(\nabla\eta)u\rangle_{L^2(\Omega)}$$

can be estimated in weaker norms (i.e., $\|\widetilde{V}\varphi\|_{L^2(B_{(1+\delta/2)R}\cap\Omega)}$, $\|u\|_{L^2(B_{(1+\delta/2)R}\cap\Omega)}$) or lead to terms that are absorbed in the left-hand side as in the proof of Theorem 5.5.3 (see (5.5.15), (5.5.16)). Equations (5.5.32) and (5.5.11) give

$$\left\langle \nabla\widetilde{V}\varphi, \nabla(\eta^2 u)\right\rangle_{L^2(\Omega)} + \left\langle \nabla\widetilde{V}\varphi, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\backslash\Gamma)} = \left\langle (1/2+K)u, \eta^2\varphi\right\rangle_{L^2(\Gamma)}. \tag{5.5.43}$$

Therefore, we only have to estimate the last term in (5.5.41). We write in the same way as in (5.5.42)

$$\left\langle \nabla\widetilde{V}\varphi, \nabla(\eta^2\widetilde{K}u)\right\rangle_{L^2(\mathbb{R}^d\backslash\Gamma)} = \left\langle \nabla(\eta^2\widetilde{V}\varphi), \nabla\widetilde{K}u\right\rangle_{L^2(\mathbb{R}^d\backslash\Gamma)} + \text{l.o.t.},$$

where, again, the omitted terms

$$\text{l.o.t.} = 2\langle (\nabla(\eta\widetilde{V}\varphi), (\nabla\eta)\widetilde{K}u\rangle_{L^2(\mathbb{R}^d\backslash\Gamma)} - 2\langle (\nabla\eta)\widetilde{V}\varphi, \nabla(\eta\widetilde{K}u)\rangle_{L^2(\mathbb{R}^d\backslash\Gamma)}$$

can be estimated in weaker norms (i.e., by $\|\widetilde{K}u\|_{L^2(B_{(1+\delta/2)R}\backslash\Gamma)}$ and $\|\widetilde{V}\varphi\|_{L^2(B_{(1+\delta/2)R})}$ or absorbed in the left-hand side. Integration by parts on $\mathbb{R}^d\backslash\overline{\Omega}$ and Ω together with $\Delta\widetilde{K}u = 0$ and $[\gamma_1\widetilde{K}u] = 0 = [\eta^2\widetilde{V}\varphi]$ implies

$$\left\langle \nabla(\eta^2\widetilde{V}\varphi), \nabla\widetilde{K}u\right\rangle_{L^2(\mathbb{R}^d\backslash\Gamma)} = \left\langle \eta^2\widetilde{V}\varphi, \Delta\widetilde{K}u\right\rangle_{L^2(\mathbb{R}^d\backslash\Gamma)} = 0.$$

Putting everything together into (5.5.41) and in turn into (5.5.40), we obtain

$$\begin{aligned}
\|\nabla(\eta u)\|^2_{L^2(\Omega)} &+ \left\|\nabla(\eta\widetilde{V}\varphi)\right\|^2_{L^2(\mathbb{R}^d)} + \left\|\nabla(\eta\widetilde{K}u)\right\|^2_{L^2(\mathbb{R}^d\backslash\Gamma)} \\
&\lesssim (1+1/\delta)\, a_{\text{jn}}(u,\varphi;\eta^2 u,\eta^2\varphi) + \frac{(1+1/\delta)}{(\delta R)^2}\left\|\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\backslash\Gamma)} \\
&\quad + \frac{(1+1/\delta)}{(\delta R)^2}\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{(1+1/\delta)}{(\delta R)^2}\left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})}.
\end{aligned} \tag{5.5.44}$$

Step 2: We apply the local orthogonality of (u,φ) to piecewise polynomials and use approximation properties.

Let $I_h^\Omega : C(\overline{\Omega}) \to S^{1,1}(\mathcal{T})$ be the nodal interpolation operator and I_h^Γ the $L^2(\Gamma)$-orthogonal projection mapping onto $S^{0,0}(\mathcal{K})$. Then, the orthogonality (5.5.39) leads to

$$\begin{aligned}
a_{\text{jn}}(u,\varphi;\eta^2 u,\eta^2\varphi) &= a_{\text{jn}}(u,\varphi;\eta^2 u - I_h^\Omega(\eta^2 u), \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)) \\
&= \left\langle \nabla u, \nabla(\eta^2 u - I_h^\Omega(\eta^2 u))\right\rangle_{L^2(\Omega)} + \left\langle V\varphi, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)} \\
&\quad - \left\langle \varphi, \eta^2 u - I_h^\Omega(\eta^2 u)\right\rangle_{L^2(\Gamma)} + \left\langle (1/2-K)u, \eta^2\varphi - I_h^\Gamma(\eta^2\varphi)\right\rangle_{L^2(\Gamma)} \\
&=: T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{5.5.45}$$

The terms T_1, T_2 have already been estimated in the proof of Theorem 5.5.3, inequalities (5.5.19), (5.5.20), and T_4 was treated in (5.5.35) in the proof of Theorem 5.5.4.

It remains to estimate T_3. With supp $\left(\eta^2 u - I_h^\Omega(\eta^2 u)\right) \subset B_{(1+\delta/2)R}$ due to $16h \leq \delta R$, we get

$$|T_3| = \left|\left\langle \varphi, \eta^2 u - I_h^\Omega(\eta^2 u)\right\rangle_{L^2(\Gamma)}\right| \leq \|\varphi\|_{L^2(B_{(1+\delta/2)R}\cap\Gamma)} \left\|\eta^2 u - I_h^\Omega(\eta^2 u)\right\|_{L^2(\Gamma)}.$$

Lemma 5.5.1 provides

$$\|\varphi\|_{L^2(B_{(1+\delta/2)R})} \lesssim h^{-1/2} \left\|\nabla \widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})}.$$

Therefore, with (5.5.23), we obtain

$$\begin{aligned}\left|\left\langle \varphi, I_h^\Omega(\eta^2 u) - \eta^2 u\right\rangle_{L^2(\Gamma)}\right| &\lesssim h^{-1/2}\left\|\nabla \widetilde{V}\varphi\right\|_{L^2(B_{(1+\delta)R})}\left(\frac{h^{3/2}}{\delta R}\|\nabla u\|_{L^2(B_{(1+\delta)R}\cap\Omega)} + \frac{h^{3/2}}{(\delta R)^2}\|u\|_{L^2(B_{(1+\delta)R}\cap\Omega)}\right)\\ &\lesssim \frac{h}{\delta R}\left(\left\|\nabla \widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \|\nabla u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)}\right) + \frac{1}{(\delta R)^2}\|u\|^2_{L^2(B_{(1+\delta)R})}.\end{aligned} \tag{5.5.46}$$

Putting the estimates of T_1, T_2, T_3, T_4 together and using $\delta \lesssim 1$ leads to

$$\begin{aligned}&\|\nabla u\|^2_{L^2(B_R\cap\Omega)} + \left\|\nabla \widetilde{V}\varphi\right\|^2_{L^2(B_R)} + \left\|\nabla \widetilde{K}u\right\|^2_{L^2(B_R\setminus\Gamma)}\\ &\quad\lesssim \frac{h}{\delta^2 R}\left(\|\nabla u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \left\|\nabla \widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \left\|\nabla \widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}\right)\\ &\qquad + \frac{1}{\delta^3 R^2}\left(\|u\|^2_{L^2(B_{(1+\delta)R}\cap\Omega)} + \left\|\widetilde{V}\varphi\right\|^2_{L^2(B_{(1+\delta)R})} + \left\|\widetilde{K}u\right\|^2_{L^2(B_{(1+\delta)R}\setminus\Gamma)}\right).\end{aligned} \tag{5.5.47}$$

Step 3. Reapplying (5.5.47) to the gradient terms with $\delta = \frac{\varepsilon}{2}$ and suitable boxes, we get the desired result exactly as in step 3 of the proof of Theorem 5.5.3. □ □

5.6 Abstract setting - low dimensional approximation

In this section, we prove the existence of exponentially convergent $\mathcal{H}$-matrix approximants to the inverses of the stiffness matrices of the FEM-BEM couplings, as stated in Theorem 5.4.1.

Analysing the procedure in [FMP15, FMP16, AFM20] shows structural similarities in the derivation of $\mathcal{H}$-matrix approximations based on low-dimensional spaces of functions: A single-step approximation is obtained by using a Scott-Zhang operator on a coarse grid. Iterating this argument is made possible by a Caccioppoli-inequality, resulting in a multi-step approximation. The key ingredients of the argument are collected in properties (A1)–(A3) below. We mainly follow [AFM20].

5.6.1 From matrices to functions

We start by reformulating the matrix approximation problem as a question of approximating certain functions from low dimensional spaces.

Let $\mathbf{X}$ be a Hilbert space of functions. We consider variational problems of the form: find $\mathbf{u} \in \mathbf{X}$ such that

$$a(\mathbf{u}, \boldsymbol{\psi}) = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \qquad \forall \boldsymbol{\psi} \in \boldsymbol{X}$$

for given $a(\cdot,\cdot) : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$, $\boldsymbol{f} \in \mathbf{X}'$. Here, the bold symbols may denote vectors, e.g., $\mathbf{u} = (u, \varphi)$ in (5.3.1) for $\mathbf{X} = H^1(\Omega) \times H^{-1/2}(\Gamma)$, and $\langle \cdot, \cdot \rangle$ denotes the appropriate duality bracket.

For fixed $k, \ell \in \mathbb{N}$ (given by the formulation of the problem), we define $\mathbf{L}^2 := L^2(\Omega)^k \times L^2(\Gamma)^\ell$.

Definition 5.6.1. Let $\mathbf{X}_N \subset \mathbf{X}$ be a finite dimensional subspace of dimension N that is also a subspace $\mathbf{X}_N \subset \mathbf{L}^2$. Then the linear mapping $\mathcal{S}_N : \mathbf{X}' \to \mathbf{X}_N$ is called the discrete solution operator if for every $\boldsymbol{f} \in \mathbf{X}'$, there exists a unique function $\mathcal{S}_N \boldsymbol{f} \in \mathbf{X}_N$ satisfying

$$a(\mathcal{S}_N \boldsymbol{f}, \boldsymbol{\psi}) = \langle \boldsymbol{f}, \boldsymbol{\psi} \rangle \qquad \forall \boldsymbol{\psi} \in \boldsymbol{X}_N. \tag{5.6.1}$$

Let $\{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N\} \subseteq \mathbf{X}_N$ be a basis of $\mathbf{X}_N$. We denote the Galerkin matrix $\mathbf{A} \in \mathbb{R}^{N \times N}$ by

$$\mathbf{A} = \left(a(\boldsymbol{\phi}_j, \boldsymbol{\phi}_i)\right)_{i,j=1}^N. \tag{5.6.2}$$

The translation of the problem of approximating matrix blocks of $\mathbf{A}^{-1}$ to the problem of approximating certain functions from low dimensional spaces essentially depends on the following crucial property (A1), the existence of a local dual basis.

(A1) There exist dual functions $\{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_N\} \subset \mathbf{L}^2$ satisfying

$$\langle \boldsymbol{\phi}_i, \boldsymbol{\lambda}_j \rangle = \delta_{ij}, \quad \text{and} \quad \Big\| \sum_{j=1}^N \mathbf{x}_j \boldsymbol{\lambda}_j \Big\|_{\mathbf{L}^2} \leq C_{\mathrm{db}}(N) \left\| \mathbf{x} \right\|_2$$

for all $i, j \in \{1, \dots, N\}$ and $\mathbf{x} \in \mathbb{R}^N$. Moreover, we require the $\boldsymbol{\lambda}_i$ to have local support, in the sense that $\#\{j \,:\, \operatorname{supp}(\boldsymbol{\lambda}_i) \cap \operatorname{supp}(\boldsymbol{\lambda}_j) \neq \emptyset\} \lesssim 1$ for all $i \in \{1, \dots, N\}$.

We denote the coordinate mappings corresponding to the basis and the dual basis by

$$\Phi : \left\{ \begin{array}{ccc} \mathbb{R}^N & \longrightarrow & \mathbf{X}_N \\ \mathbf{x} & \longmapsto & \sum_{j=1}^N \mathbf{x}_j \boldsymbol{\phi}_j \end{array} \right., \qquad \Lambda : \left\{ \begin{array}{ccc} \mathbb{R}^N & \longrightarrow & \mathbf{L}^2 \\ \mathbf{x} & \longmapsto & \sum_{j=1}^N \mathbf{x}_j \boldsymbol{\lambda}_j \end{array} \right..$$

The Hilbert space transpose of Λ is denoted by Λ^T. Moreover, for $\tau \subset \{1, \dots, N\}$, we define the sets $D_j(\tau) := \cup_{i \in \tau} \operatorname{supp} \boldsymbol{\lambda}_{i,j}$, where $\boldsymbol{\lambda}_{i,j}$ is the j-th component of $\boldsymbol{\lambda}_i$, and write $\mathbf{L}^2(\tau) := \prod_{j=1}^{k+\ell} L^2(D_j(\tau))$.

In the following lemma, we derive a representation formula for $\mathbf{A}^{-1}$ based on three linear operators Λ^T, $\mathcal{S}_N$ and Λ.

Lemma 5.6.2. *([AFM20, Lem. 3.10], [AFM20, Lem. 3.11]) The restriction of Λ^T to $\mathbf{X}_N$ is the inverse mapping Φ^{-1}. More precisely, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$ and $\mathbf{v} \in \mathbf{X}_N$, we have*

$$\langle \Lambda \mathbf{x}, \Phi \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle_2 , \qquad \Lambda^T \Phi \mathbf{x} = \mathbf{x}, \qquad \Phi \Lambda^T \mathbf{v} = \mathbf{v}.$$

The mappings Λ and Λ^T preserve locality, i.e, for $\tau \subset \{1, \dots, N\}$ and $\mathbf{x} \in \mathbb{R}^N$ with $\{i : \mathbf{x}_i \neq 0\} \subset \tau$, we have $\operatorname{supp}(\Lambda \mathbf{x}) \subset \prod_j D_j(\tau)$. For $\mathbf{v} \in \mathbf{L}^2$, we have

$$\left\| \Lambda^T \mathbf{v} \right\|_{\ell^2(\tau)} \leq \|\Lambda\| \, \|\mathbf{v}\|_{\mathbf{L}^2(\tau)} .$$

Moreover, there holds the representation formula

$$\mathbf{A}^{-1} \boldsymbol{x} = \Lambda^T \mathcal{S}_N \Lambda \boldsymbol{x} \qquad \forall \boldsymbol{x} \in \mathbb{R}^N .$$

Proof. For sake of completeness, we provide the derivation of the representation formula from [AFM20, Lem. 3.11]. Using that $\Lambda^T = \Phi^{-1}|_{\mathbf{X}_N}$ and the definition of the discrete solution operator, we compute

$$\left\langle \mathbf{A} \Lambda^T \mathcal{S}_N \Lambda \mathbf{x}, \mathbf{y} \right\rangle_2 = a(\Phi \Lambda^T \mathcal{S}_N \Lambda \mathbf{x}, \Phi \mathbf{y}) = a(\mathcal{S}_N \Lambda \mathbf{x}, \Phi \mathbf{y}) = \langle \Lambda \mathbf{x}, \Phi \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle_2$$

for arbitrary $\mathbf{y} \in \mathbb{R}^N$. □

This lemma is the crucial step in the proof of the following lemma.

Lemma 5.6.3. *Let $\mathbf{A}$ be the Galerkin matrix, Λ be the coordinate mapping for the dual basis, and $\mathcal{S}_N$ be the discrete solution operator. Let $\tau \times \sigma \subset \{1, \dots, N\} \times \{1, \dots, N\}$ be an admissible block and $\mathbf{W}_r \subseteq \mathbf{L}^2$ be a finite dimensional space. Then, there exist matrices $\mathbf{X}_{\tau\sigma} \in \mathbb{R}^{|\tau| \times r}$, $\mathbf{Y}_{\tau\sigma} \in \mathbb{R}^{|\sigma| \times r}$ of rank $r \leq \dim \mathbf{W}_r$ satisfying*

$$\left\| \mathbf{A}^{-1}|_{\tau \times \sigma} - \mathbf{X}_{\tau\sigma} \mathbf{Y}_{\tau\sigma}^T \right\|_2 \leq \|\Lambda\|^2 \sup_{\substack{\boldsymbol{f} \in \mathbf{L}^2: \\ \operatorname{supp}(\boldsymbol{f}) \subset \prod_j D_j(\sigma)}} \frac{\inf_{\mathbf{w} \in \mathbf{W}_r} \|\mathcal{S}_N \boldsymbol{f} - \mathbf{w}\|_{\mathbf{L}^2(\tau)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2}} .$$

Proof. We use the representation formula from Lemma 5.6.2 to prove the asserted estimate. With the given space $\mathbf{W}_r$, we define $\mathbf{X}_{\tau\sigma} \in \mathbb{R}^{|\tau| \times r}$ columnwise as vectors from an orthonormal basis of the space $\widehat{\mathbf{W}} := (\Lambda^T \mathbf{W}_r)|_\tau$. Then, the product $\mathbf{X}_{\tau\sigma} \mathbf{X}_{\tau\sigma}^T$ is the orthogonal projection onto $\widehat{\mathbf{W}}$. Defining $\mathbf{Y}_{\tau\sigma} := (\mathbf{A}^{-1}|_{\tau \times \sigma})^T \mathbf{X}_{\tau\sigma}$, we can compute for all $\mathbf{x} \in \mathbb{R}^N$ with $\{i \,:\, \mathbf{x}_i \neq 0\} \subset \sigma$ that

$$\begin{aligned}
\left\| (\mathbf{A}^{-1}|_{\tau \times \sigma} - \mathbf{X}_{\tau\sigma} \mathbf{Y}_{\tau\sigma}^T) \mathbf{x}|_\sigma \right\|_{\ell^2(\tau)} &= \left\| (\mathbf{I} - \mathbf{X}_{\tau\sigma} \mathbf{X}_{\tau\sigma}^T)(\mathbf{A}^{-1}\mathbf{x})|_\sigma \right\|_{\ell^2(\tau)} = \inf_{\widehat{\mathbf{w}} \in \widehat{\mathbf{W}}} \left\| (\mathbf{A}^{-1}\mathbf{x})|_\sigma - \widehat{\mathbf{w}} \right\|_{\ell^2(\tau)} \\
&\overset{\text{Lem. 5.6.2}}{=} \inf_{\mathbf{w} \in \mathbf{W}_r} \left\| \Lambda^T (\mathcal{S}_N \Lambda \mathbf{x} - \mathbf{w}) \right\|_{\ell^2(\tau)} \\
&\leq \|\Lambda\| \inf_{\mathbf{w} \in \mathbf{W}_r} \|\mathcal{S}_N \Lambda \mathbf{x} - \mathbf{w}\|_{\mathbf{L}^2(\tau)} .
\end{aligned}$$

Dividing both sides by $\|\mathbf{x}\|_2$, substituting $\boldsymbol{f} := \Lambda \mathbf{x}$ and using that the mapping Λ preserves supports, we get the desired result. □

5.6.2 Low dimensional approximation

We present a general framework that only uses a Caccioppoli type estimate for the construction of exponentially convergent low dimensional approximations.

Let $M \in \mathbb{N}$ be fixed. For $R > 0$ let $\mathcal{B}_R := \{B_i\}_{i=1}^M$ be a collection of boxes, i.e., $B_i \in \{B_R \cap \Omega, B_R, B_R \backslash \Gamma\}$ for all $i = 1, \dots, M$, where B_R denotes a box of side length R. The choice, which of the three sets is taken for each index i, is determined by the application and fixed.

We write $\mathcal{B} \subset \mathcal{B}' := \{B_i'\}_{i=1}^M$ meaning that $B_i \subset B_i'$ for all $i = 1, \dots, M$. For a parameter $\delta > 0$, we call $\mathcal{B}_R^\delta := \{B_i^\delta\}_{i=1}^M$ a collection of δ-enlarged boxes of $\mathcal{B}_R$, if it satisfies

$$B_i^\delta \in \{B_{R+2\delta} \cap \Omega, B_{R+2\delta}, B_{R+2\delta} \backslash \Gamma\} \;\; \forall i = 1, \dots, M, \qquad \text{and} \qquad \mathcal{B}_R^\delta \supset \mathcal{B}_R,$$

where B_R and $B_{R+2\delta}$ are concentric boxes. Defining $\operatorname{diam}(\mathcal{B}_R) := \max\{\operatorname{diam}(B_i), i = 1, \dots, M\}$, we get

$$\operatorname{diam}(\mathcal{B}_R^\delta) \leq \operatorname{diam}(\mathcal{B}_R) + 2\sqrt{d}\delta. \tag{5.6.3}$$

In order to simplify notation, we drop the subscript R and write $\mathcal{B} := \mathcal{B}_R$ in the following abstract setting.

We use the notation $\mathbf{H}^1(\mathcal{B})$ to abbreviate the product space $\mathbf{H}^1(\mathcal{B}) = \prod_{i=1}^M H^1(B_i)$, and write $\|\mathbf{v}\|_{\mathbf{H}^1(\mathcal{B})}^2 := \sum_{i=1}^M \|\mathbf{v}_i\|_{H^1(B_i)}^2$ for the product norm.

Remark 5.6.4. For the application of the present section, we chose boxes (or suitable subsets of those) for the sets B_i. We also mention that different constructions can be employed as demonstrated in [AFM20], where a construction for non-uniform grids is presented and where the metric is not the Euclidean one but one that is based on the underlying finite element mesh. ∎

In the following, we fix some assumptions on the collections $\mathcal{B}$ of interest and the norm $\|\cdot\|_{\mathcal{B}}$ on $\mathcal{B}$ we derive our approximation result in. In essence, we want a norm weaker than than the classical H^1-norm that has the correct scaling (e.g., an L^2-type norm).

(A2) Assumptions on the approximation norm $\|\cdot\|_{\mathcal{B}}$: For each $\mathcal{B}$, the Hilbertian norm $\|\cdot\|_{\mathcal{B}}$ is a norm on $\mathbf{H}^1(\mathcal{B})$ and such that for any $\delta > 0$ and enlarged boxes $\mathcal{B}^\delta$ and $H > 0$ there is a discrete space $\mathbf{V}_{H,\mathcal{B}^\delta} \subset \mathbf{H}^1(\mathcal{B}^\delta)$ of dimension $\dim \mathbf{V}_{H,\mathcal{B}^\delta} = C(\operatorname{diam}(\mathcal{B}^\delta)/H)^{Md}$ and a linear operator $Q_H : \mathbf{H}^1(\mathcal{B}^\delta) \to \mathbf{V}_{H,\mathcal{B}^\delta}$ such that

$$\|\mathbf{v} - Q_H \mathbf{v}\|_{\mathcal{B}} \leq C_{\text{Qap}} H (\|\nabla \mathbf{v}\|_{L^2(\mathcal{B}^\delta)} + \delta^{-1} \|\mathbf{v}\|_{\mathcal{B}^\delta})$$

with a constant $C_{\text{Qap}} > 0$ that does not depend on $\mathcal{B}, \mathcal{B}^\delta, \delta$, and N.

Finally, we require a Caccioppoli type estimate with respect to the norm from (A2).

(A3) Caccioppoli type estimate: For each $\mathcal{B}$, $\delta > 0$ and collection $\mathcal{B}^\delta$ of δ-enlarged boxes with $\delta \geq C_{\text{Set}}(N)$ with a fixed constant $C_{\text{Set}}(N) > 0$ that may depend on N, there is a subspace $\mathcal{H}_h(\mathcal{B}^\delta) \subset \mathbf{H}^1(\mathcal{B}^\delta)$ such that for all $\mathbf{v} \in \mathcal{H}_h(\mathcal{B}^\delta)$ the inequality

$$\|\nabla \mathbf{v}\|_{L^2(\mathcal{B})} \leq C_{\text{Cac}} \frac{\operatorname{diam}(\mathcal{B})^{\alpha-1}}{\delta^\alpha} \|\mathbf{v}\|_{\mathcal{B}^\delta} \tag{5.6.4}$$

holds. Here, the constants $C_{\rm Cac} > 0$ and $\alpha \geq 1$ do not depend on $\mathcal{B}, \mathcal{B}^\delta$, δ, and N.

We additionally assume the spaces $\mathcal{H}_h(\mathcal{B}^\delta)$ to be finite dimensional and nested, i.e., $\mathcal{H}_h(\mathcal{B}') \subset \mathcal{H}_h(\mathcal{B})$ for $\mathcal{B} \subset \mathcal{B}'$.

By $\Pi_{h,\mathcal{B}}$, we denote the orthogonal projection $\Pi_{h,\mathcal{B}} : \mathbf{H}^1(\mathcal{B}) \to \mathcal{H}_h(\mathcal{B})$ onto that space with respect to the norm $|\!|\!|\cdot|\!|\!|_{\mathcal{B}}$, which is well-defined since, by assumption, $\mathcal{H}_h(\mathcal{B})$ is closed.

Lemma 5.6.5 (single-step approximation)**.** *Let* $2\,\mathrm{diam}(\Omega) \geq \delta \geq 2C_{\rm Set}(N)$ *with the constant* $C_{\rm Set}(N)$ *from (A3),* $\mathcal{B}$ *be a given collections of boxes and* $\mathcal{B} \subset \mathcal{B}^{\delta/2} \subset \mathcal{B}^\delta$ *be enlarged boxes of* $\mathcal{B}$*. Let* $|\!|\!|\cdot|\!|\!|_{\mathcal{B}^\delta}$ *be a norm on* $\mathbf{H}^1(\mathcal{B}^\delta)$ *such that* (A2) *holds for the sets* $\mathcal{B} \subset \mathcal{B}^{\delta/2}$*. Let* $\mathbf{v} \in \mathcal{H}_h(\mathcal{B}^\delta)$ *meaning that* (A3) *holds for the sets* $\mathcal{B}^{\delta/2}, \mathcal{B}^\delta$*. Then, there exists a space* $\mathbf{W}_1$ *of dimension* $\dim \mathbf{W}_1 \leq C_{\rm ssa}\left(\frac{\mathrm{diam}(\mathcal{B}^\delta)}{\delta}\right)^{\alpha M d}$ *such that*

$$\inf_{\mathbf{w}\in\mathbf{W}_1} |\!|\!|\boldsymbol{v} - \mathbf{w}|\!|\!|_{\mathcal{B}} \leq \frac{1}{2}\,|\!|\!|\boldsymbol{v}|\!|\!|_{\mathcal{B}^\delta}\,.$$

Proof. We set $\mathbf{W}_1 := \Pi_{h,\mathcal{B}} Q_H \mathcal{H}_h(\mathcal{B}^\delta) \subset \mathbf{V}_{H,\mathcal{B}^\delta}$. Since $\mathbf{v} \in \mathcal{H}_h(\mathcal{B}^\delta)$, we obtain from (A2) and (A3) that

$$\begin{aligned}
|\!|\!|\mathbf{v} - \Pi_{h,\mathcal{B}} Q_H \mathbf{v}|\!|\!|_{\mathcal{B}} &= |\!|\!|\Pi_{h,\mathcal{B}}(\mathbf{v} - Q_H\mathbf{v})|\!|\!|_{\mathcal{B}} && (5.6.5)\\
&\leq |\!|\!|\mathbf{v} - Q_H\mathbf{v}|\!|\!|_{\mathcal{B}} \leq C_{\rm Qap} H (\|\nabla \mathbf{v}\|_{L^2(\mathcal{B}^{\delta/2})} + 2\delta^{-1}\,|\!|\!|\mathbf{v}|\!|\!|_{\mathcal{B}^{\delta/2}})\\
&\leq C_1 C_{\rm Qap} C_{\rm Cac} \frac{\mathrm{diam}(\mathcal{B}^{\delta/2})^{\alpha-1}}{\delta^\alpha} H\,|\!|\!|\mathbf{v}|\!|\!|_{\mathcal{B}^\delta} && (5.6.6)
\end{aligned}$$

with a constant C_1 depending only on Ω since $\alpha \geq 1$ and $\delta \leq 2\,\mathrm{diam}(\Omega)$. With the choice $H = \frac{\delta^\alpha}{2C_1 C_{\rm Qap} C_{\rm Cac}\,\mathrm{diam}(\mathcal{B}^\delta)^{\alpha-1}}$, we get the asserted error bound. Since $\mathbf{W}_1 \subset \mathbf{V}_{H,\mathcal{B}^\delta}$ and by choice of H, we have

$$\dim \mathbf{W}_1 \leq C\left(\frac{\mathrm{diam}(\mathcal{B}^\delta)}{H}\right)^{Md} \leq C\left(2C_1 C_{\rm Qap} C_{\rm Cac}\frac{\mathrm{diam}(\mathcal{B}^\delta)^\alpha}{\delta^\alpha}\right)^{Md} =: C_{\rm ssa}\left(\frac{\mathrm{diam}(\mathcal{B}^\delta)}{\delta}\right)^{\alpha M d},$$

which concludes the proof. □

Iterating the single-step approximation on concentric boxes leads to exponential convergence.

Lemma 5.6.6 (multi-step approximation)**.** *Let* $L \in \mathbb{N}$ *and* $\delta \geq 2C_{\rm Set}(N)$ *with the constant* $C_{\rm Set}(N)$ *from (A3). Let* $\mathcal{B}$ *be a collection of boxes and* $\mathcal{B}^{\delta L} \supset \mathcal{B}$ *a collection of* δL*-enlarged boxes. Then, there exists a space* $\mathbf{W}_L \subseteq \mathcal{H}_h(\mathcal{B}^{\delta L})$ *such that for all* $\boldsymbol{v} \in \mathcal{H}_h(\mathcal{B}^{\delta L})$ *we have*

$$\inf_{\mathbf{w}\in\mathbf{W}_L} |\!|\!|\boldsymbol{v} - \mathbf{w}|\!|\!|_{\mathcal{B}} \leq 2^{-L}\,|\!|\!|\boldsymbol{v}|\!|\!|_{\mathcal{B}^{\delta L}}\,,$$

and

$$\dim \mathbf{W}_L \leq C_{\rm dim}\Big(L + \frac{\mathrm{diam}(\mathcal{B})}{\delta}\Big)^{\alpha M d+1}.$$

Proof. The assumptions on $\mathcal{B}$ and $\mathcal{B}^{\delta L}$ allow for the construction of a sequence of nested enlarged boxes $\mathcal{B} \subseteq \mathcal{B}^{\delta} \subseteq \mathcal{B}^{2\delta} \subseteq \ldots \subseteq \mathcal{B}^{\delta L}$ satisfying $\operatorname{diam}(\mathcal{B}^{\ell\delta}) \leq \operatorname{diam}(\mathcal{B}) + C\ell\delta$.

We iterate the approximation result of Lemma 5.6.5 on the sets $\mathcal{B}^{\delta\ell}$, $\ell = L, \ldots, 1$. For $\ell = L$, Lemma 5.6.5 applied with the sets $\mathcal{B}^{(L-1)\delta} \subset \mathcal{B}^{\delta L}$ provides a subspace $\mathbf{V}_1 \subset \mathcal{H}_N(\mathcal{B}^{\delta L})$ with $\dim \mathbf{V}_1 \leq C\left(\frac{\operatorname{diam}(\mathcal{B}^{\delta L})}{\delta}\right)^{\alpha M d}$ such that

$$\inf_{\widehat{\mathbf{v}}_1 \in \mathbf{V}_1} \|\mathbf{v} - \widehat{\mathbf{v}}_1\|_{\mathcal{B}^{(L-1)\delta}} \leq 2^{-1} \|\mathbf{v}\|_{\mathcal{B}^{\delta L}} . \tag{5.6.7}$$

For $\widehat{\mathbf{v}}_1 \in \mathbf{V}_1$, we have $(\mathbf{v} - \widehat{\mathbf{v}}_1) \in \mathcal{H}_N(\mathcal{B}^{(L-1)\delta})$, so we can use Lemma 5.6.5 again with the sets $\mathcal{B}^{(L-2)\delta} \subset \mathcal{B}^{(L-1)\delta}$, and get a subspace $\mathbf{V}_2$ of $\mathcal{H}_N(\mathcal{B}^{(L-2)\delta})$ with $\dim \mathbf{V}_2 \leq C\left(\frac{\operatorname{diam}(\mathcal{B}^{(L-1)\delta})}{\delta}\right)^{\alpha M d}$. This implies

$$\inf_{\widehat{\mathbf{v}}_2 \in \mathbf{V}_2} \inf_{\widehat{\mathbf{v}}_1 \in \mathbf{V}_1} \|(\mathbf{v} - \widehat{\mathbf{v}}_1) - \widehat{\mathbf{v}}_2\|_{\mathcal{B}^{(L-2)\delta}} \leq 2^{-1} \inf_{\widehat{\mathbf{v}}_1 \in \mathbf{V}_1} \|\mathbf{v} - \widehat{\mathbf{v}}_1\|_{\mathcal{B}^{(L-1)\delta}} \leq 2^{-2} \|\mathbf{v}\|_{\mathcal{B}^{\delta L}} . \tag{5.6.8}$$

Continuing this process $L-2$ times leads to the subspace $\mathbf{W}_L := \bigoplus_{\ell=1}^{L} \mathbf{V}_\ell$ of $\mathcal{H}_N(\mathcal{B}^{\delta L})$ with dimension

$$\begin{aligned}
\dim \mathbf{W}_L &\leq C \sum_{\ell=1}^{L} \left(\frac{\operatorname{diam}(\mathcal{B}^{\delta\ell})}{\delta}\right)^{\alpha M d} \leq C \sum_{\ell=1}^{L} \left(\frac{\operatorname{diam}(\mathcal{B})}{\delta} + \ell\right)^{\alpha M d} \\
&\leq C_{\dim}\left(L + \frac{\operatorname{diam}(\mathcal{B})}{\delta}\right)^{\alpha M d + 1},
\end{aligned}$$

which finishes the proof. □

5.7 Application of the abstract framework for the FEM-BEM couplings

In this section, we specify the assumptions (A1)–(A3) for the FEM-BEM couplings.

The local dual basis

In the setting of Section 5.6.1, we have $\mathbf{X} = H^1(\Omega) \times H^{-1/2}(\Gamma)$. In order to suitably represent the data f, u_0, φ_0 in (5.1.1), we understand the discrete space $S^{1,1}(\mathcal{T}) \simeq S_0^{1,1}(\mathcal{T}) \times S^{1,1}(\mathcal{K}) \subset L^2(\Omega) \times L^2(\Gamma)$, where $S_0^{1,1}(\mathcal{T}) := S^{1,1}(\mathcal{T}) \cap H_0^1(\Omega)$. Having identified $S^{1,1}(\mathcal{T})$ with $S_0^{1,1}(\mathcal{T}) \times S^{1,1}(\mathcal{K})$, we view the full FEM-BEM coupling problem as one as approximating in $S_0^{1,1}(\mathcal{T}) \times S^{1,1}(\mathcal{K}) \times S^{0,0}(\mathcal{K})$. That is, we set $k = 1$ and $\ell = 2$, and consider $\mathbf{L}^2 = L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma)$ for all three FEM-BEM couplings. The discrete space $\mathbf{X}_N = S_0^{1,1}(\mathcal{T}) \times S^{1,1}(\mathcal{K}) \times S^{0,0}(\mathcal{K}) \subset \mathbf{L}^2$ has dimension $N = n_1 + n_2 + m$, where $n_1 = \dim(S_0^{1,1}(\mathcal{T}))$, $n_2 = \dim(S^{1,1}(\mathcal{K}))$ $(n_1 + n_2 = n)$ and $m = \dim(S^{0,0}(\mathcal{K}))$, and it remains to show (A1).

The dual functions $\boldsymbol{\lambda}_i$ are constructed by use of L^2-dual bases for $S^{1,1}(\mathcal{T})$ and $S^{0,0}(\mathcal{K})$. [AFM20, Sec. 3.3] gives an explicit construction of a suitable dual basis $\{\lambda_i^\Omega : i = 1, \ldots, n_1\}$ for $S_0^{1,1}(\mathcal{T})$. This is done elementwise in a discontinuous fashion, i.e., $\lambda_i^\Omega \in S^{1,0}(\mathcal{T}) \subset L^2(\Omega)$, where each λ_i^Ω is non-zero only on one element of $\mathcal{T}$ (in the patch of the hat function ξ_i), and the function on this element is given by the push-forward of a dual shape function on the reference element. Moreover, the local stability estimate

$$\Big\| \sum_{j=1}^{n} \mathbf{x}_j \lambda_j^\Omega \Big\|_{L^2(\Omega)} \leq h^{-d/2} \|\mathbf{x}\|_2 \tag{5.7.1}$$

holds for all $\mathbf{x} \in \mathbb{R}^n$, and we have $\operatorname{supp} \lambda_i^\Omega \subset \operatorname{supp} \xi_i$. We note that the zero boundary condition is irrelevant for the construction. The same can be done for the boundary degrees of freedom, i.e., there exists a dual basis $\{\lambda_i^\Gamma : i = 1, \ldots, n_2\}$ with the analogous stability and support properties.

For the boundary degrees of freedom in $S^{0,0}(\mathcal{K})$, the dual mappings are given by $\mu_i^\Gamma := \chi_i / \|\chi_i\|_{L^2(\Omega)}^2$, i.e., the dual basis coincides – up to scaling – with the given basis $\{\chi_i : i = 1, \ldots, m\}$ of $S^{0,0}(\mathcal{K})$. With (2.3.1a), this gives

$$\Big\| \sum_{j=1}^{m} \mathbf{y}_j \mu_j^\Gamma \Big\|_{L^2(\Omega)} \leq h^{-(d-1)/2} \|\mathbf{y}\|_2 \tag{5.7.2}$$

for all $\mathbf{y} \in \mathbb{R}^m$.

Now, the dual basis is defined as $\boldsymbol{\lambda}_i := (\lambda_i^\Omega, 0, 0)$ for $i = 1, \ldots, n_1$, $\boldsymbol{\lambda}_{i+n_1} := (0, \lambda_i^\Gamma, 0)$ for $i = 1, \ldots, n_2$ and $\boldsymbol{\lambda}_{i+n} := (0, 0, \mu_i^\Gamma)$ for $i = 1, \ldots, m$, and (5.7.1), (5.7.2) together with the analogous one for the λ_i^Γ show (A1).

Low dimensional approximation

The sets $\mathcal{B}$, $\mathcal{B}^\delta$ and the norm $|||\cdot|||_{\mathcal{B}}$

We take $M = 3$ and choose collections $\mathcal{B} = \mathcal{B}_R := \{B_R \cap \Omega, B_R, B_R \backslash \Gamma\}$, where B_R is a box of side length R. For $\ell \in \mathbb{N}$ the enlarged sets $\mathcal{B}^{\delta\ell}$ then have the form

$$\mathcal{B}^{\delta\ell} = \mathcal{B}_R^{\delta\ell} := \{B_{R+2\delta\ell} \cap \Omega, B_{R+2\delta\ell}, B_{R+2\delta\ell} \backslash \Gamma\} \tag{5.7.3}$$

with the concentric boxes $B_{R+2\delta\ell}$ of side length $R + 2\delta\ell$.

For $\mathbf{v} = (u, v, w)$, we use the norm from (5.5.2)

$$|||\mathbf{v}|||_{\mathcal{B}} := |||(u, v, w)|||_{h,R}$$

in (A2). For the Bielak-MacCamy coupling, taking $M = 2$ and choosing collections $\mathcal{B}_R := \{B_R \cap \Omega, B_R\}$ would suffice, however, in order to keep the notation short, we can use $M = 3$ for this coupling as well by setting the third component to zero, i.e., $\mathbf{v} = (u, v, 0)$.

The operator Q_H and (A2)

For the operator Q_H, we use a combination of localization and Scott-Zhang interpolation, introduced in [SZ90b], on a coarse grid. Since the double-layer potential is discontinuous across Γ, we need to employ a piecewise Scott-Zhang operator. Let $\mathcal{R}_H$ be a quasi-uniform (infinite) triangulation of $\mathbb{R}^d$ (into open simplices $R \in \mathcal{R}_H$) with mesh width H that conforms to Ω, i.e., every $R \in \mathcal{R}_H$ satisfies either $R \subset \Omega$ or $R \subset \Omega^{\text{ext}}$ and the restrictions

$\mathcal{R}_H|_\Omega$ and $\mathcal{R}_H|_{\Omega^{\text{ext}}}$ are γ-shape regular, regular triangulations of Ω and Ω^{ext} of mesh size H, respectively.

With the Scott-Zhang projections I_H^{int}, I_H^{ext} for the grids $\mathcal{R}_H|_\Omega$ and $\mathcal{R}_H|_{\Omega^c}$, we define the operator $I_H^{\text{pw}} : H^1(\mathbb{R}^d \setminus \Gamma) \to S^{1,1}_{\text{pw}}(\mathcal{R}_H) := \{v : v|_\Omega \in S^{1,1}(\mathcal{R}_H|_\Omega)$ and $v|_{\Omega^{\text{ext}}} \in S^{1,1}(\mathcal{R}_H|_{\Omega^{\text{ext}}})\}$ in a piecewise fashion as Eq. (2.4.12).

Let $\eta \in C_0^\infty(B_{R+2\delta})$ be a cut-off function satisfying $\operatorname{supp}\eta \subset B_{R+\delta}$, $\eta \equiv 1$ on B_R and $\|\nabla\eta\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{\delta}$. We define the operator

$$Q_H\mathbf{v} := (I_H^{\text{int}}(\eta\mathbf{v}_1), I_H(\eta\mathbf{v}_2), I_H^{\text{pw}}(\eta\mathbf{v}_3)), \tag{5.7.4}$$

where I_H denotes the classical Scott-Zhang operator for the mesh $\mathcal{R}_H$. We have

$$|\!|\!|\mathbf{v} - Q_H\mathbf{v}|\!|\!|_{\mathcal{B}}^2 = \left\|\mathbf{v}_1 - I_H^{\text{int}}(\eta\mathbf{v}_1)\right\|_{h,R,\Omega}^2 + \|\mathbf{v}_2 - I_H(\eta\mathbf{v}_2)\|_{h,R}^2 + \left\|\mathbf{v}_3 - I_H^{\text{pw}}(\eta\mathbf{v}_3)\right\|_{h,R,\Gamma^c}^2.$$

Each term on the right-hand side can be estimated with the same arguments. We only work out the details for the second component. Assuming $h \le H$, and using approximation properties and stability of the Scott-Zhang projection, we get

$$\begin{aligned}\|\mathbf{v}_2 - I_H(\eta\mathbf{v}_2)\|_{h,R}^2 &= \|\eta\mathbf{v}_2 - I_H(\eta\mathbf{v}_2)\|_{h,R}^2 = h^2\|\nabla(\eta\mathbf{v}_2 - I_H(\eta\mathbf{v}_2))\|_{L^2(B_R)}^2 + \|\eta\mathbf{v}_2 - I_H(\eta\mathbf{v}_2)\|_{L^2(B_R)}^2 \\ &\lesssim (h^2 + H^2)\|\nabla(\eta\mathbf{v}_2)\|_{L^2(\mathbb{R}^d)} \lesssim H^2\left(\|\nabla\mathbf{v}_2\|_{L^2(B_{R+2\delta})}^2 + \delta^{-1}\|\mathbf{v}_2\|_{L^2(B_{R+2\delta})}^2\right),\end{aligned}$$

which shows (A2) for the discrete space $V_{H,\mathcal{B}^\delta} = S^{1,1}(\mathcal{R}_H)|_{B_{R+2\delta}\cap\Omega} \times S^{1,1}(\mathcal{R}_H)|_{B_{R+2\delta}} \times S^{1,1}_{\text{pw}}(\mathcal{R}_H)|_{B_{R+2\delta}}$ of dimension $\dim V_{H,\mathcal{B}^\delta} \le C\left(\frac{\operatorname{diam}(B_{R+2\delta})}{H}\right)^{Md}$.

The Caccioppoli inequalities and (A3)

Theorem 5.5.3–Theorem 5.5.6 provide the Caccioppoli type estimates asserted in (A3) with $\delta = \varepsilon R/2$. For the Bielak-MacCamy coupling we have $\alpha = 1$ and $C_{\text{Set}} = 8h$, for the symmetric coupling $\alpha = 1$ and $C_{\text{Set}} = 16h$. For the Johnson-Nédélec we have to take $\alpha = 2$ and $C_{\text{Set}} = 16h$. For $\mathcal{B}_R = \{B_R \cap \Omega, B_R, B_R\backslash\Gamma\}$, the spaces $\mathcal{H}_h(\mathcal{B}_R)$ can be characterized by

$$\begin{aligned}\mathcal{H}_h(\mathcal{B}_R) := \{&(v, \widetilde{V}\phi, \widetilde{K}v) \in H^1(B_R\cap\Omega) \times H^1(B_R) \times H^1(B_R\backslash\Gamma) : \exists\widetilde{v} \in S^{1,1}(\mathcal{T}), \widetilde{\phi} \in S^{0,0}(\mathcal{K}) : \\ &\widetilde{v}|_{B_R\cap\Omega} = v|_{B_R\cap\Omega}, \quad \widetilde{V}\widetilde{\phi}|_{B_R} = \widetilde{V}\phi|_{B_R}, \quad \widetilde{K}\widetilde{v}|_{B_R\backslash\Gamma} = \widetilde{K}v|_{B_R\backslash\Gamma}, \quad a(v,\phi;\psi_h,\zeta_h) = 0 \\ &\forall(\psi_h,\zeta_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K}), \operatorname{supp}\psi_h,\zeta_h \subset B_R\},\end{aligned}$$

where the bilinear form $a(\cdot,\cdot)$ is either a_{sym} or a_{jn}. For the Bielak-MacCamy coupling, it suffices to require

$$\begin{aligned}\mathcal{H}_h(\mathcal{B}_R) := \{&(v, \widetilde{V}\phi, 0) \in H^1(B_R\cap\Omega) \times H^1(B_R) \times H^1(B_R\backslash\Gamma) : \exists\widetilde{v} \in S^{1,1}(\mathcal{T}), \widetilde{\phi} \in S^{0,0}(\mathcal{K}) : \\ &\widetilde{v}|_{B_R\cap\Omega} = v|_{B_R\cap\Omega}, \quad \widetilde{V}\widetilde{\phi}|_{B_R} = \widetilde{V}\phi|_{B_R}, \quad a_{\text{bmc}}(v,\phi;\psi_h,\zeta_h) = 0 \\ &\forall(\psi_h,\zeta_h) \in S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K}), \operatorname{supp}\psi_h,\zeta_h \subset B_R\}.\end{aligned}$$

With these definitions, the closedness and nestedness of the spaces $\mathcal{H}_h(\mathcal{B}_R)$ clearly holds.

5.7.1 Proof of Theorem 5.4.1

As a consequence of the above discussions, the abstract framework of the previous sections can be applied and it remains to put everything together.

The following theorem constructs the finite dimensional space required from Lemma 5.6.3, from which the Galerkin solution can be approximated exponentially well. We should note that the symmetry of the matrix $\mathbf{A}_{\text{sym}}$ of the symmetric coupling also allows to use the weaker admissibility condition from Remark 2.6.4.

Theorem 5.7.1 (low dimensional approximation for the symmetric coupling)**.** *Let (τ, σ) be a cluster pair with bounding boxes B_{R_τ} and B_{R_σ} that satisfy for given $\eta > 0$*

$$\eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \geq \operatorname{diam}(B_{R_\tau}).$$

Then, for each $L \in \mathbb{N}$, there exists a space $\widehat{\mathbf{W}}_L \subset S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ with dimension $\dim \widehat{\mathbf{W}}_L \leq C_{\text{low}} L^{3d+1}$ such that for arbitrary right-hand sides $f \in L^2(\Omega)$, $v_0 \in L^2(\Gamma)$, and $w_0 \in L^2(\Gamma)$ with $\big(\operatorname{supp} f \cup \operatorname{supp} v_0 \cup \operatorname{supp} w_0\big) \subset B_{R_\sigma}$, the corresponding Galerkin solution (u_h, φ_h) of (5.3.6) satisfies

$$\min_{(\widetilde{u},\widetilde{\varphi})\in\widehat{\mathbf{W}}_L} \Big(\|u_h - \widetilde{u}\|_{L^2(B_{R_\tau}\cap\Omega)} + \|\varphi_h - \widetilde{\varphi}\|_{L^2(B_{R_\tau}\cap\Gamma)}\Big) \leq C_{\text{box}} h^{-2} 2^{-L} \Big(\|f\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Gamma)} + \|w_0\|_{L^2(\Gamma)}\Big).$$

The constants C_{low}, C_{box} depend only on Ω, d, η, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$ and $\mathcal{K}$.

Proof. For given $L \in \mathbb{N}$, we choose $\delta := \frac{R_\tau}{2\eta L}$. Then, we have

$$\operatorname{dist}(B_{R_\tau+2\delta L}, B_{R_\sigma}) \geq \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) - L\delta\sqrt{d} \geq \sqrt{d} R_\tau\Big(\frac{1}{\eta} - \frac{1}{2\eta}\Big) > 0.$$

With $\mathcal{B}_{R_\tau} = \{B_{R_\tau} \cap \Omega, B_{R_\tau}, B_{R_\tau} \backslash \Gamma\}$ and $\mathcal{B}_{R_\tau}^{\delta L} = \{B_{R_\tau+2\delta L} \cap \Omega, B_{R_\tau+2\delta L}, B_{R_\tau+2\delta L} \backslash \Gamma\}$ from (5.7.3), the assumption on the support of the data therefore implies the local orthogonality imposed in the space $\mathcal{H}_h(\mathcal{B}_{R_\tau}^{\delta L})$. In order to define the space $\widehat{\mathbf{W}}_L$, we distinguish two cases.

Case $\delta > 2C_{\text{Set}}$: Then, Lemma 5.6.6 applied with the sets $\mathcal{B}_{R_\tau}^{\delta}$ and $\mathcal{B}_{R_\tau}^{\delta L}$ provides a space $\mathbf{W}_L$ of dimension

$$\dim \mathbf{W}_L \leq C_{\text{dim}} \Big(L - 1 + \frac{\operatorname{diam}(\mathcal{B}_{R_\tau}^{\delta})}{\delta}\Big)^{3d+1} \lesssim \Big(L + \frac{\sqrt{d} R_\tau 2\eta L}{R_\tau}\Big)^{3d+1} \lesssim L^{3d+1}$$

with the approximation properties for $\mathbf{v} = (u_h, \widetilde{V}\varphi_h, \widetilde{K} u_h)$

$$\inf_{\mathbf{w}\in\mathbf{W}_L} \|\mathbf{v} - \mathbf{w}\|_{\mathcal{B}_{R_\tau}^{\delta}} \leq 2^{-(L-1)} \|\mathbf{v}\|_{\mathcal{B}_{R_\tau}^{\delta L}}. \tag{5.7.5}$$

Therefore, it remains to estimate the norm $\|\cdot\|_{\mathcal{B}}$ from above and below.

With $h \lesssim 1$, the mapping properties of $\widetilde{V}$ and $\widetilde{K}$ from (5.2.5), and the trace inequality we can estimate

$$\begin{aligned} \Big\|(u_h, \widetilde{V}\varphi_h, \widetilde{K}u_h)\Big\|_{\mathcal{B}_{R_\tau}^{\delta L}} &\lesssim \|u_h\|_{H^1(\Omega)} + \Big\|\widetilde{V}\varphi_h\Big\|_{H^1(B_{(1+1/(2\eta))R_\tau})} + \Big\|\widetilde{K}u_h\Big\|_{H^1(B_{(1+1/(2\eta))R_\tau}\backslash\Gamma)} \\ &\lesssim \|u_h\|_{H^1(\Omega)} + \|\varphi_h\|_{H^{-1/2}(\Gamma)}. \end{aligned} \tag{5.7.6}$$

The stabilized form

$$\widetilde{a}_{\rm sym}(u,\varphi;\psi,\zeta) := a_{\rm sym}(u,\varphi;\psi,\zeta) + \left\langle 1, V\varphi + (\frac{1}{2}-K)u\right\rangle_{L^2(\Gamma)} \left\langle 1, V\zeta + (\frac{1}{2}-K)\psi\right\rangle_{L^2(\Gamma)},$$

is elliptic, cf. [AFF+13]. Moreover, [AFF+13, Thm. 18] prove that the Galerkin solution also solves $\widetilde{a}_{\rm sym}(u_h,\varphi_h;\psi,\zeta) = g_{\rm sym}(\psi,\zeta) + \langle 1, w_0\rangle_{L^2(\Gamma)} \left\langle 1, (\frac{1}{2}-K)\psi + V\zeta\right\rangle_{L^2(\Gamma)}$. Therefore, we have

$$\begin{aligned}\|\varphi_h\|^2_{H^{-1/2}(\Gamma)} + \|u_h\|^2_{H^1(\Omega)} \lesssim \widetilde{a}_{\rm sym}(u_h,\varphi_h;u_h,\varphi_h) &= \langle f, u_h\rangle_{L^2(\Omega)} + \langle v_0, u_h\rangle_{L^2(\Gamma)} + \langle w_0, \varphi_h\rangle_{L^2(\Gamma)} \\ &\quad + \langle 1, (1/2-K)u_h + V\varphi_h\rangle_{L^2(\Gamma)} \langle 1, w_0\rangle_{L^2(\Gamma)}. \end{aligned} \tag{5.7.7}$$

The stabilization term can be estimated with the mapping properties of V and K from (5.2.6) and the trace inequality by

$$\begin{aligned}\left|\langle 1, (1/2-K)u_h + V\varphi_h\rangle_{L^2(\Gamma)} \langle 1, w_0\rangle_{L^2(\Gamma)}\right| &\lesssim \left(\|(1/2-K)u_h\|_{L^2(\Gamma)} + \|V\varphi_h\|_{L^2(\Gamma)}\right)\|w_0\|_{L^2(\Gamma)} \\ &\lesssim \|w_0\|_{L^2(\Gamma)} \left(\|u_h\|_{H^1(\Omega)} + \|\varphi_h\|_{H^{-1/2}(\Gamma)}\right).\end{aligned}$$

Inserting this in (5.7.7), using the trace inequality and an inverse estimate we further estimate

$$\begin{aligned}\|\varphi_h\|^2_{H^{-1/2}(\Gamma)} + \|u_h\|^2_{H^1(\Omega)} &\lesssim \left(\|f\|_{L^2(\Omega)} + \|v_0\|_{H^{-1/2}(\Gamma)}\right)\|u_h\|_{H^1(\Omega)} \\ &\quad + \|w_0\|_{L^2(\Gamma)} \left(\|\varphi_h\|_{L^2(\Gamma)} + \|u_h\|_{H^1(\Omega)} + \|\varphi_h\|_{H^{-1/2}(\Gamma)}\right) \\ &\leq \left(\|f\|_{L^2(\Omega)} + \|v_0\|_{H^{-1/2}(\Gamma)}\right)\|u_h\|_{H^1(\Omega)} \\ &\quad + h^{-1/2}\|w_0\|_{L^2(\Gamma)} \left(\|u_h\|_{H^1(\Omega)} + \|\varphi_h\|_{H^{-1/2}(\Gamma)}\right).\end{aligned}$$

With Young's inequality, we get

$$\begin{aligned}\|\varphi_h\|^2_{H^{-1/2}(\Gamma)} + \|u_h\|^2_{H^1(\Omega)} &\leq \left(\|f\|_{L^2(\Omega)} + \|v_0\|_{H^{-1/2}(\Gamma)} + h^{-1/2}\|w_0\|_{L^2(\Gamma)}\right)^2 + \frac{1}{4}\|u_h\|^2_{H^1(\Omega)} \\ &\quad + 2h^{-1}\|w_0\|^2_{L^2(\Gamma)} + \frac{1}{8}\left(\|u_h\|_{H^1(\Omega)} + \|\varphi_h\|_{H^{-1/2}(\Gamma)}\right)^2 \\ &\leq \left(\|f\|_{L^2(\Omega)} + \|v_0\|_{H^{-1/2}(\Gamma)} + h^{-1/2}\|w_0\|_{L^2(\Gamma)}\right)^2 \\ &\quad + \frac{1}{2}\left(\|\varphi_h\|^2_{H^{-1/2}(\Gamma)} + \|u_h\|^2_{H^1(\Omega)}\right),\end{aligned}$$

which results in

$$\|\varphi_h\|_{H^{-1/2}(\Gamma)} + \|u_h\|_{H^1(\Omega)} \lesssim \|f\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Gamma)} + h^{-1/2}\|w_0\|_{L^2(\Gamma)}.$$

Inserting this in (5.7.6), we obtain the upper bound

$$\left|\!\left|\!\left|(u_h, \widetilde{V}\varphi_h, \widetilde{K}u_h)\right|\!\right|\!\right|_{\mathcal{B}^{\delta L}_{R_\tau}} \lesssim \left(\|f\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Gamma)} + h^{-1/2}\|w_0\|_{L^2(\Gamma)}\right). \tag{5.7.8}$$

The jump conditions of the single-layer potential and Eq. (5.5.5) provide for arbitrary $\widetilde{\varphi} \in S^{0,0}(\mathcal{K})$

$$\|\varphi_h - \widetilde{\varphi}\|_{L^2(B_{R_\tau}\cap\Gamma)} = \left\|[\gamma_1 \widetilde{V}\varphi_h] - [\gamma_1 \widetilde{V}\widetilde{\varphi}]\right\|_{L^2(B_{R_\tau}\cap\Gamma)} \lesssim h^{-1/2} \left\|\nabla(\widetilde{V}\varphi_h - \widetilde{V}\widetilde{\varphi})\right\|_{L^2(B_{R_\tau+\delta})} \lesssim h^{-3/2} \left\|\widetilde{V}\varphi_h - \widetilde{V}\widetilde{\varphi}\right\|_{h,R_\tau+2\delta}. \tag{5.7.9}$$

For arbitrary $\widetilde{u} \in S^{1,1}(\mathcal{T})$, with $h \lesssim 1$, we can write

$$\|u_h - \widetilde{u}\|_{L^2(B_{R_\tau}\cap\Omega)} \lesssim h^{-3/2} \|u_h - \widetilde{u}\|_{h,R+2\delta}. \tag{5.7.10}$$

Combination of (5.7.9) and (5.7.10) gives us

$$\inf_{(\widetilde{u},\widetilde{\varphi})\in\widehat{\mathbf{W}}_L} \left(\|u_h - \widetilde{u}\|_{L^2(B_{R_\tau}\cap\Omega)} + \|\varphi_h - \widetilde{\varphi}\|_{L^2(B_{R_\tau}\cap\Gamma)}\right) \lesssim h^{-3/2} \inf_{\mathbf{w}\in\mathbf{W}_L} \left\|(u_h, \widetilde{V}\varphi_h, \widetilde{K}u_h) - \mathbf{w}\right\|_{\mathcal{B}^\delta_{R_\tau}}$$

Finally, we define $\widehat{\mathbf{W}}_L := \{(\widetilde{u}, [\gamma_1\widetilde{v}]) : (\widetilde{u}, \widetilde{v}, \widetilde{w}) \in \mathbf{W}_L\}$. Then, the dimension of $\widehat{\mathbf{W}}_L$ is bounded by $\widehat{\mathbf{W}}_L \leq CL^{3d+1}$, and the error estimate follows from (5.7.5) since

$$\inf_{(\widetilde{u},\widetilde{\varphi})\in\widehat{\mathbf{W}}_L} \left(\|u_h - \widetilde{u}\|_{L^2(B_{R_\tau}\cap\Omega)} + \|\varphi_h - \widetilde{\varphi}\|_{L^2(B_{R_\tau}\cap\Gamma)}\right) \lesssim h^{-3/2} \inf_{\mathbf{w}\in\mathbf{W}_L} \left\|(u_h, \widetilde{V}\varphi_h, \widetilde{K}u_h) - \mathbf{w}\right\|_{\mathcal{B}^\delta_{R_\tau}} \lesssim h^{-3/2}2^{-L} \left\|(u_h, \widetilde{V}\varphi_h, \widetilde{K}u_h)\right\|_{\mathcal{B}^{\delta L}_{R_\tau}}.$$

Applying estimate (5.7.8) finishes the proof for the case $\delta \geq 2C_{\text{set}}$.

Case $\delta \leq 2C_{\text{set}} = 32h$: Here, we use the space $\widehat{\mathbf{W}}_L := S^{1,1}(\mathcal{T})|_{B_{R_\tau}} \times S^{0,0}(\mathcal{K})|_{B_{R_\tau}}$. Since $(u_h, \varphi_h)|_{B_{R_\tau}} \in \widehat{\mathbf{W}}_L$ the error estimate holds trivially. For the dimension of $\widehat{\mathbf{W}}_L$, we obtain

$$\dim \widehat{\mathbf{W}}_L \leq C\left(\frac{\operatorname{diam}(B_{R_\tau})}{h}\right)^{2d} \leq C\left(\frac{32\sqrt{d}R_\tau}{\delta}\right)^{2d} \leq C\left(2C_{\text{set}}\sqrt{d}2\eta L\right)^{2d} \lesssim L^{2d},$$

which finishes the proof. □

Theorem 5.7.2 (low dimensional approximation for the Bielak-MacCamy coupling)**.** *Let (τ, σ) be a cluster pair with bounding boxes B_{R_τ} and B_{R_σ} that satisfy for given $\eta > 0$*

$$\eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \geq \operatorname{diam}(B_{R_\tau}).$$

Then, for each $L \in \mathbb{N}$, there exists a space $\widehat{\mathbf{W}}_L \subset S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ with dimension $\dim \widehat{\mathbf{W}}_L \leq C_{\text{low}}L^{2d+1}$ such that for arbitrary right-hand sides $f \in L^2(\Omega)$, $\varphi_0 \in L^2(\Gamma)$, and $u_0 \in L^2(\Gamma)$ with $(\operatorname{supp} f \cup \operatorname{supp} \varphi_0 \cup \operatorname{supp} u_0) \subset B_{R_\sigma}$, the corresponding Galerkin solution (u_h, φ_h) of (5.3.3) satisfies

$$\min_{(\widetilde{u},\widetilde{\varphi})\in\widehat{\mathbf{W}}_L} \left(\|u_h - \widetilde{u}\|_{L^2(B_{R_\tau}\cap\Omega)} + \|\varphi_h - \widetilde{\varphi}\|_{L^2(B_{R_\tau}\cap\Gamma)}\right) \leq C_{\text{box}}h^{-2}2^{-L}\left(\|f\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Gamma)} + \|u_0\|_{L^2(\Gamma)}\right).$$

The constants C_{low}, C_{box} depend only on Ω, d, η, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$ and $\mathcal{K}$.

Proof. The proof is essentially identical to the proof of Theorem 5.7.1. We stress that the bound of the dimension $\dim \widehat{\mathbf{W}}_L \leq C_{\text{low}} L^{2d+1}$ is better, since no approximation for the double-layer potential is needed, i.e., we can choose $M = 2$ in the abstract setting. □

Theorem 5.7.3 (low dimensional approximation for the Johnson-Nédélec coupling)**.** *Let (τ, σ) be a cluster pair with bounding boxes B_{R_τ} and B_{R_σ} that satisfy for given $\eta > 0$*

$$\eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \geq \operatorname{diam}(B_{R_\tau}).$$

Then, for each $L \in \mathbb{N}$, there exists a space $\widehat{\mathbf{W}}_L \subset S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ with dimension $\dim \widehat{\mathbf{W}}_L \leq C_{\text{low}} L^{6d+1}$, such that for arbitrary right-hand sides $f \in L^2(\Omega)$, $\varphi_0 \in L^2(\Gamma)$, and $w_0 \in L^2(\Gamma)$ with $\left(\operatorname{supp} f \cup \operatorname{supp} \varphi_0 \cup \operatorname{supp} w_0\right) \subset B_{R_\sigma}$, the corresponding Galerkin solution (u_h, φ_h) of (5.3.9) satisfies

$$\min_{(\widetilde{u},\widetilde{\varphi})\in\widehat{\mathbf{W}}_L} \left(\|u_h - \widetilde{u}\|_{L^2(B_{R_\tau}\cap\Omega)} + \|\varphi_h - \widetilde{\varphi}\|_{L^2(B_{R_\tau}\cap\Gamma)}\right) \leq C_{\text{box}} h^{-2} 2^{-L} \left(\|f\|_{L^2(\Omega)} + \|\varphi_0\|_{L^2(\Gamma)} + \|w_0\|_{L^2(\Gamma)}\right).$$

The constants C_{low}, C_{box} depend only on Ω, d, η, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$ and $\mathcal{K}$.

Proof. The proof is essentially identical to the proof of Theorem 5.7.1. We stress that the bound of the dimension $\dim \widehat{\mathbf{W}}_L \leq C_{\text{low}} L^{6d+1}$ is worse than for the other couplings, since in the abstract setting, we have to choose $M = 3$ and $\alpha = 2$, and the bound follows from Lemma 5.6.6. □

Finally, we can prove the existence of $\mathcal{H}$-Matrix approximants to the inverse FEM-BEM stiffness matrix.

Proof of Theorem 5.4.1. We start with the symmetric coupling. As $\mathcal{H}$ matrices are low rank only on admissible blocks, we set $\mathbf{B}_{\mathcal{H}}|_{\tau\times\sigma} = \mathbf{A}_{\text{sym}}^{-1}|_{\tau\times\sigma}$ for non-admissible cluster pairs and consider an arbitrary admissible cluster pair (τ, σ) in the following.

With a given rank bound r, we take $L := \lfloor (r/C_{\text{low}})^{1/(3d+1)} \rfloor$. With this choice, we apply Theorem 5.7.1, which provides a space $\widehat{\mathbf{W}}_L \subset S^{1,1}(\mathcal{T}) \times S^{0,0}(\mathcal{K})$ and use this space in Lemma 5.6.3, which produces matrices $\mathbf{X}_{\tau\sigma}, \mathbf{Y}_{\tau\sigma}$ of maximal rank $\dim \widehat{\mathbf{W}}_L$, which is by choice of L bounded by

$$\dim \widehat{\mathbf{W}}_L = C_{\text{low}} L^{3d+1} \leq r.$$

Theorem 5.7.1 can be rewritten in terms of the discrete solution operator of the framework of Section 5.6.1. Let $\boldsymbol{f} = (f, v_0, w_0) \in \mathbf{L}^2$ be arbitrary with $\operatorname{supp}(\boldsymbol{f}) \subset \prod_j D_j(\sigma)$. Then, the locality of the dual functions implies $\left(\operatorname{supp} f \cup \operatorname{supp} v_0 \cup \operatorname{supp} w_0\right) \subset B_{R_\sigma}$, and we obtain

$$\begin{aligned}\inf_{\mathbf{w}\in\widehat{\mathbf{W}}_L} \|\mathcal{S}_N \boldsymbol{f} - \mathbf{w}\|_{\mathbf{L}^2(\tau)} &\leq \inf_{(\widetilde{u},\widetilde{\varphi})\in\widehat{\mathbf{W}}_L} \left(\|u_h - \widetilde{u}\|_{L^2(B_{R_\tau}\cap\Omega)} + \|\varphi_h - \widetilde{\varphi}\|_{L^2(B_{R_\tau}\cap\Gamma)}\right) \\ &\lesssim h^{-2} 2^{-L} \left(\|f\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Gamma)} + \|w_0\|_{L^2(\Gamma)}\right) \lesssim h^{-2} 2^{-L} \|\boldsymbol{f}\|_{\mathbf{L}^2}.\end{aligned}$$

Defining $\mathbf{B}_{\mathcal{H}}|_{\tau\times\sigma} := \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^T$, the estimates (2.6.5) and $\|\Lambda\| \lesssim h^{-d/2}$ together with Lemma 5.6.3 then give the error bound

$$\begin{aligned}\left\|\mathbf{A}_{\text{sym}}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 &\leq C_{\text{sp}}\,\text{depth}(\mathbb{T}_{\mathcal{I}})\max\{\left\|\mathbf{A}^{-1} - \mathbf{B}_{\mathcal{H}}|_{\tau\times\sigma}\right\|_2 : (\tau,\sigma)\in P\}\\ &\leq C_{\text{sp}}\,\text{depth}(\mathbb{T}_{\mathcal{I}})\,\|\Lambda\|^2 \max_{(\tau,\sigma)\in P_{\text{far}}} \sup_{\substack{\boldsymbol{f}\in\mathbf{L}^2:\\ \text{supp}(\boldsymbol{f})\subset\prod_j D_j(\sigma)}} \frac{\inf_{\mathbf{w}\in\widehat{\mathbf{W}}_L}\|\mathcal{S}_N\boldsymbol{f} - \mathbf{w}\|_{\mathbf{L}^2(\tau)}}{\|\boldsymbol{f}\|_{\mathbf{L}^2}}\\ &\lesssim C_{\text{sp}}\,\text{depth}(\mathbb{T}_{\mathcal{I}})h^{-(d+2)}2^{-L}\\ &\leq C_{\text{apx}}C_{\text{sp}}\,\text{depth}(\mathbb{T}_{\mathcal{I}})h^{-(d+2)}\exp(-br^{1/(3d+1)}).\end{aligned}$$

This finishes the proof for the symmetric coupling.

The approximations to $\mathbf{A}_{\text{bmc}}^{-1}$ and $\mathbf{A}_{\text{jn}}^{-1}$ are constructed in exactly the same fashion. The different exponentials appear due to the different dimensions of the low-dimensional space $\widehat{\mathbf{W}}_L$ in Theorem 5.7.2 and Theorem 5.7.3. □

5.8 Numerical results

In this section, we provide a numerical example that supports the theoretical results from Theorem 5.4.1, i.e, we compute an exponentially convergent $\mathcal{H}$-matrix approximant to an inverse FEM-BEM coupling matrix.

If one is only interested in solving a linear system with one (or few) different right-hand sides, rather than computing the inverse – and maybe even its low-rank approximation – it is more beneficial to use an iterative solver. The $\mathcal{H}$-matrix approximability of the inverse naturally allows for black-box preconditioning of the linear system. [Beb07] constructed LU-decompositions in the $\mathcal{H}$-matrix format for FEM matrices by approximating certain Schur-complements under the assumption that the inverse can be approximated with arbitrary accuracy. Theorem 5.4.1 provides such an approximation result and the techniques of [Beb07, FMP15, FMP16, FMP17] can also be employed to prove the existence of $\mathcal{H}$-LU-decompositions for the whole FEM-BEM matrices for each couplings.

Here, we additionally present a different, computationally more efficient approach by introducing a black-box block diagonal preconditioner for the FEM-BEM coupling matrices.

We choose the $3d$-unit cube $\Omega = (0,1)^3$ as our geometry, and we set $\mathbf{C} = \mathbf{I}$. In the following, we only consider the Johnson-Nédélec coupling, the other couplings can be treated in exactly the same way.

In order to guarantee positive definiteness, we study the stabilized system (see [AFF+13, Thm. 15] for the assertion of positive definiteness)

$$\left(\begin{pmatrix}\mathbf{A} & -\mathbf{M}^T\\ \frac{1}{2}\mathbf{M} - \mathbf{K} & \mathbf{V}\end{pmatrix} + \mathbf{s}\mathbf{s}^T\right)\begin{pmatrix}\mathbf{x}\\ \boldsymbol{\phi}\end{pmatrix} = \begin{pmatrix}\mathbf{f}\\ \mathbf{g}\end{pmatrix}, \tag{5.8.1}$$

where the stabilization $\mathbf{s}\in\mathbb{R}^{N+M}$ is given by $\mathbf{s}_i = \langle 1, (1/2 - K)\xi_i\rangle_{L^2(\Gamma)}$ for $i\in\{1,\ldots,N\}$ and $\mathbf{s}_i = \langle 1, V\chi_i\rangle_{L^2(\Gamma)}$ for $i\in\{N+1,\ldots,M\}$.

We stress that [AFF+13] show that solving the stabilized (elliptic) system is equivalent to solving the non stabilized system (with a modified right-hand side). By $\mathbf{A}^{\text{st}} := \mathbf{A} + \mathbf{b}\mathbf{b}^T$, we denote the stabilization of $\mathbf{A}$, where $\mathbf{b}$ contains the degrees of freedom of $\mathbf{s}$ corresponding to the FEM part.

All computations are made using the C-library `HLiB`, [BG99], where we employed a geometric clustering algorithm with admissibility parameter $\eta = 2$ and a leaf-size of 25.

5.8.1 Approximation to the inverse matrix

The $\mathcal{H}$-matrices are computed by using a very accurate blockwise low-rank approximation to

$$\mathbf{B} := \begin{pmatrix} \mathbf{A} & -\mathbf{M}^T \\ \frac{1}{2}\mathbf{M} - \mathbf{K} & \mathbf{V} \end{pmatrix} + \mathbf{s}\mathbf{s}^T. \tag{5.8.2}$$

Then, using $\mathcal{H}$-matrix arithmetics and blockwise projection to rank r, the $\mathcal{H}$-matrix inverse is computed with a blockwise algorithm using $\mathcal{H}$-arithmetics from [Gra01]. In order to not compute the full inverse, we use the upper bound

$$\left\|\mathbf{B}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 \leq \left\|\mathbf{B}^{-1}\right\|_2 \left\|\mathbf{I} - \mathbf{B}\mathbf{B}_{\mathcal{H}}\right\|_2$$

for the error.

We also compute a second approximate inverse by use of the $\mathcal{H}$-LU decomposition, which can be computed using a blockwise algorithm from [Lin04, Beb05]. Hereby, we use $\left\|\mathbf{I} - \mathbf{B}(\mathbf{L}_{\mathcal{H}}\mathbf{U}_{\mathcal{H}})^{-1}\right\|_2$ to measure the error without computing the inverse of $\mathbf{B}$.

Figure 5.8.1 shows convergence of the upper bounds of the error and the growth of the storage requirements with respect to the block-rank r for two different problem sizes. We observe exponential convergence and linear growth in storage for the approximate inverse using $\mathcal{H}$-arithmetics and the approximate inverse using the $\mathcal{H}$-LU decomposition, where the $\mathcal{H}$-LU decomposition performs significantly better. The observed exponential convergence is even better than the asserted bound from Theorem 5.4.1.

5.8.2 Block diagonal preconditioning

Instead of building an $\mathcal{H}$-LU-decomposition of the whole FEM-BEM matrix, it is significantly cheaper to use a block-diagonal preconditioner consisting of $\mathcal{H}$-LU-decompositions for the FEM and the BEM part. The efficiency of block-diagonal preconditioners for the FEM-BEM couplings has been observed in [MS98, FFPS17b].

In the following, we consider block diagonal preconditioners of the form

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_A & 0 \\ 0 & \mathbf{P}_V \end{pmatrix},$$

where $\mathbf{P}_A$ is a good preconditioner for the FEM-block $\mathbf{A}^{\text{st}}$ and $\mathbf{P}_V$ is a good preconditioner of the BEM-block $\mathbf{V}$.

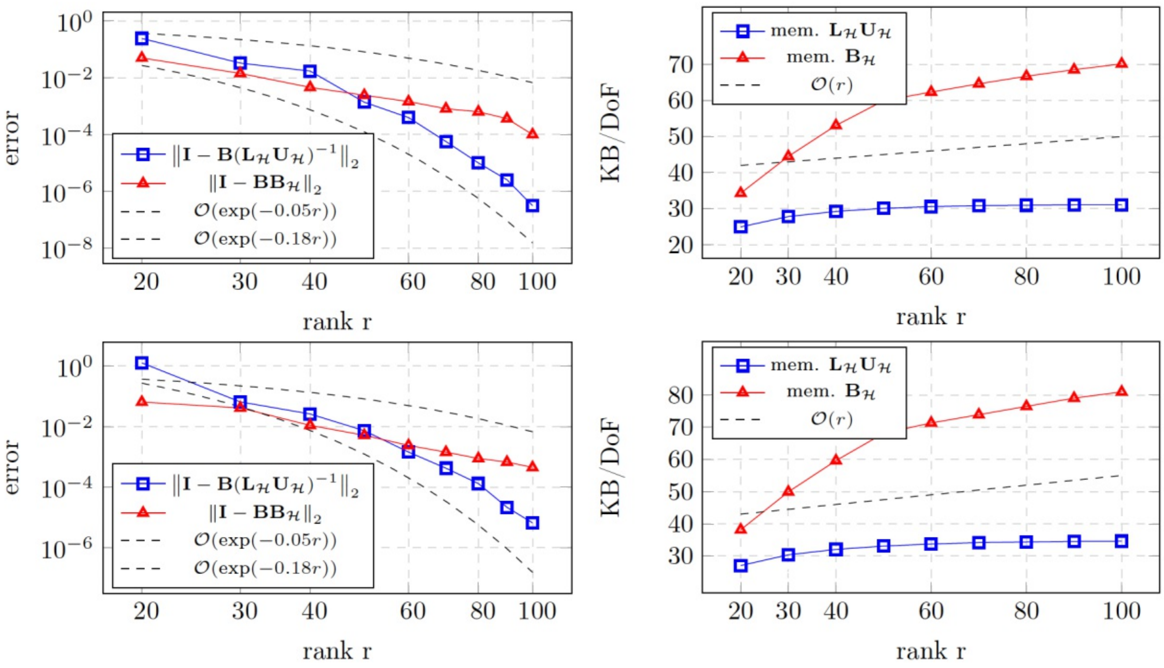

Figure 5.8.1: $\mathcal{H}$-matrix approximation to inverse FEM-BEM matrix; left: error vs. block rank r; right: memory requirement vs. block rank r; top: $N = 6959$ (FEM-dofs), $M = 3888$ (BEM-dofs); bottom: $N = 10648$, $M = 5292$.

The main result of [FFPS17b] is that, provided the preconditioners $\mathbf{P}_A$ and $\mathbf{P}_V$ fulfill the spectral equivalences

$$c_A \mathbf{x}^T \mathbf{P}_A \mathbf{x} \leq \mathbf{x}^T \mathbf{A}^{\text{st}} \mathbf{x} \leq C_A \mathbf{x}^T \mathbf{P}_A \mathbf{x} \tag{5.8.3}$$

$$c_V \mathbf{x}^T \mathbf{P}_V \mathbf{x} \leq \mathbf{x}^T \mathbf{V} \mathbf{x} \quad \leq C_V \mathbf{x}^T \mathbf{P}_V \mathbf{x}, \tag{5.8.4}$$

then, $\mathbf{P}$ is a good preconditioner for the full FEM-BEM system. More precisely, the condition number $\mathbf{P}^{-1}\mathbf{B}$ (with $\mathbf{B}$ from of (5.8.2)) in the spectral norm can be uniformly bounded by

$$\kappa_2(\mathbf{P}^{-1}\mathbf{B}) \leq C \frac{\max\{C_A, C_V\}}{\min\{c_A, c_V\}},$$

where the constant C only depends on the coefficient in the transmission problem. As a consequence, one expects that the number of GMRES iterations needed to reduce the residual by a factor remains bounded independent of the matrix size.

Therefore, we need to provide the preconditioners $\mathbf{P}_A, \mathbf{P}_V$ and prove the spectral equivalences (5.8.3). In the following, we choose hierarchical LU-decompositions as black-box preconditioners, i.e.,

$$\mathbf{P}_A := \mathbf{L}_\mathcal{H}^A \mathbf{U}_\mathcal{H}^A, \quad \mathbf{P}_V := \mathbf{L}_\mathcal{H}^V \mathbf{U}_\mathcal{H}^V,$$

where $\mathbf{A}^{\mathrm{st}} \approx \mathbf{L}_{\mathcal{H}}^{A}\mathbf{U}_{\mathcal{H}}^{A}$ and $\mathbf{V} \approx \mathbf{L}_{\mathcal{H}}^{V}\mathbf{U}_{\mathcal{H}}^{V}$. [FMP15, FMP16] prove that such LU-decompositions of arbitrary accuracy exist for the FEM and the BEM part and the errors, denoted by ε_A and ε_V, converge exponentially in the block-rank of the $\mathcal{H}$-matrices.

With $\left\|\mathbf{P}_A - \mathbf{A}^{\mathrm{st}}\right\|_2 \leq \varepsilon_A \left\|\mathbf{A}^{\mathrm{st}}\right\|_2$, we estimate

$$\left|\mathbf{x}^T\mathbf{P}_A\mathbf{x} - \mathbf{x}^T\mathbf{A}^{\mathrm{st}}\mathbf{x}\right| \leq \|\mathbf{x}\|_2^2 \left\|\mathbf{P}_A - \mathbf{A}^{\mathrm{st}}\right\|_2 \leq \varepsilon_A \|\mathbf{x}\|_2^2 \left\|\mathbf{A}^{\mathrm{st}}\right\|_2 \leq C_1\varepsilon_A h^{-d}\mathbf{x}^T\mathbf{A}^{\mathrm{st}}\mathbf{x}, \quad (5.8.5)$$

where the last step follows from the scaling of the basis of the FEM part and the positive definiteness of $\mathbf{A}^{\mathrm{st}}$. In the same way, for $\mathbf{P}_V$ it follows that

$$\left|\mathbf{x}^T\mathbf{P}_V\mathbf{x} - \mathbf{x}^T\mathbf{V}\mathbf{x}\right| \leq \|\mathbf{x}\|_2^2 \|\mathbf{P}_V - \mathbf{V}\|_2 \leq \varepsilon_V \|\mathbf{x}\|_2^2 \|\mathbf{V}\|_2 \leq C_2\varepsilon_V h^{-d+1}\mathbf{x}^T\mathbf{V}\mathbf{x}. \quad (5.8.6)$$

Choosing the rank of the $\mathcal{H}$-LU-decomposition large enough, such that, e.g., $C_1\varepsilon_A h^{-d} = \frac{1}{2}$ as well as $C_2\varepsilon_V h^{-d+1} = \frac{1}{2}$, then $C_A = C_V = 2$ and $c_A = c_V = \frac{2}{3}$ and the condition number of the preconditioned system is bounded by $\kappa_2(\mathbf{P}^{-1}\mathbf{B}) \leq 3C$.

Finally, we present a numerical simulation that underlines the usefulness of block-diagonal $\mathcal{H}$-LU-preconditioners.

Here, the $\mathcal{H}$-LU decompositions are computed with a recursive algorithm proposed in [Beb05].

The following table provides iteration numbers and computation times for the iterative solution of the system without and with $\mathcal{H}$-LU-block diagonal preconditioner using GM-RES. Here, for the stopping criterion a bound of 10^{-3} for the relative residual is chosen, and the maximal rank of the $\mathcal{H}$-LU decomposition is taken to be $r = 1$.

h	FEM DOF	BEM DOF	Iterations (without **P**)	Iterations (with **P**)	Time solve (without **P**)	Time solve (with **P**)	Time assembly **P**
2^{-3}	729	768	679	3	3.7	0.03	2.6
2^{-4}	4913	3072	3565	4	315	0.9	12.2
2^{-5}	35937	12288	11979	5	35254	30	51.9

Table 5.8.1: Iteration numbers and computation times (in seconds) for the solution with and without preconditioner with block rank $r = 1$.

As expected, the iteration numbers of the preconditioned system is much lower than those of the unpreconditioned system and grow very slowly. The computational cost for the preconditioner is theoretically of order $\mathcal{O}(r^3 N \log^3 N)$. With the choice $r = 1$, we obtain a cheap but efficient preconditioner for the FEM-BEM coupling system.

Table 2 provides the same computations for the case $r = 10$.

h	FEM DOF	BEM DOF	Iterations (without **P**)	Iterations (with **P**)	Time solve (without **P**)	Time solve (with **P**)	Time assembly **P**
2^{-3}	729	768	679	2	3.7	0.02	5.8
2^{-4}	4913	3072	3565	2	315	0.48	24.6
2^{-5}	35937	12288	11979	2	35254	15.7	243.7

Table 5.8.2: Iteration numbers and computation times (in seconds) for the solution with and without preconditioner with block rank $r = 10$.

A higher choice of rank obviously increases the computational time for the assembly of the preconditioner, but leads to lower iteration numbers and faster solution times.

6 $\mathcal{H}$-matrix approximability of inverses of FEM matrices for the time-harmonic Maxwell equations

The Maxwell system consists of four equations describing the behaviour of the electromagnetic fields and was first formulated completely by *James Clark Maxwell* (1831–1879). In this chapter, we start with formulating Maxwell's equations in integral and differential forms. Then, assuming periodicity of the behavior of the electric and magnetic fields with respect to time, we transform the system of first order partial differential equations into a second order partial differential equation to make it easier to solve.

Since the discovery of Nédélec's edge elements (and their higher-order generalizations) finite element methods have become an important discretization technique for these equations with an established convergence theory, [Mon03, Hip02]. While the resulting linear system is sparse, a direct solver cannot achieve linear complexity as one has to expect already for the case of quasi-uniform meshes with problem size N a complexity $O(N^{4/3})$ for the memory requirement and $O(N^2)$ for the solution time of a multifrontal solver, [Liu92]. Iterative solvers such as multigrid or preconditioned Schwarz methods can lead to optimal (or near optimal) complexity for the time-harmonic Maxwell equations, at least in the low-frequency regime, [Hip99, AFW00, GP03].

In this chapter, we investigate whether the inverse of the stiffness matrices arising from the FEM discretization of the time-harmonic Maxwell equations can be represented in the $\mathcal{H}$-matrix format. For its proof, we present a local discrete Helmholtz decomposition and prove the stability and approximation properties of this decomposition. Moreover, we present two types of Caccioppoli inequalities. The first Caccioppoli inequality (Lemma 6.3.16) controls the $\mathbf{H}(\text{curl})$-norm by the $\mathbf{L}^2$-norm. Since this Caccioppoli inequality is insufficient for approximation purposes, applying a local discrete Helmholtz-type decomposition to the discrete solution allows us to control the gradient part, up to a small perturbation, in $\mathbf{H}^1$.

6.1 Model problem

Maxwell's equations are a system of first-order partial differential equations that connect the temporal and spatial rates of change of the electric and magnetic fields possibly in the presence of additional source terms. Also, these equations describe how these fields are related to charge and current. Let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedral domain with boundary $\Gamma := \partial\Omega$ and S be a connected smooth surface with boundary ∂S in the interior of Ω where the electromagnetic waves propagate.

6.1.1 The fundamental equations

Let $\mathcal{E}$ $[V/m]$ denote the electric field intensity, $\mathcal{H}$ $[A/m]$ the magnetic field intensity, $\mathcal{D}$ $[As/m^2]$ the electric displacement field (electric flux) and $\mathcal{B}$ $[Vs/m^2] = Tesla$ magnetic flux density (magnetic flux). Also, we denote the current density function by $\mathcal{G}$ $[A/m^2]$ and the charge density by ρ $[As/m^3]$. We define $\mathbf{n}$ as the unit outward normal vector on Γ and $\boldsymbol{\tau}$ as the unit tangential vector on ∂S.

In Maxwell's equations, two kinds of electric fields can be observed: the electrostatic field produced by an electric charge and the induced electric field generated by a magnetic field. The first one is described by Gauss law and the other one by Faraday's law.

Gauss law for electric fields

This equation describes how electric charges produce an electric field and the electric flux created by this field passing through Ω is proportional to the electric charges inside Ω. The integral form is generally written as:

$$\int_\Gamma \mathcal{D} \cdot n \, ds = \int_\Omega \rho \, d\mathbf{x}. \tag{6.1.1}$$

Faraday's induction law

This equation is the first one that connects electric and magnetic fields. It describes that a changing magnetic flux through the surface S induces a voltage in the boundary of this surface and this voltage produces an electric field. This equation has following the form

$$\int_S \frac{\partial \mathcal{B}}{\partial t} \cdot \mathbf{n} \; ds + \int_{\partial S} \mathcal{E} \cdot \boldsymbol{\tau} \; d\ell = 0. \tag{6.1.2}$$

This equation implies that the electric field is conservative in the absence of a magnetic field or when the magnetic field is constant with respect to time.

Gauss law for magnetic fields

This equation describes the total magnetic flux passing through Ω is zero, i.e.,

$$\int_\Gamma \mathcal{B} \cdot \mathbf{n} \; ds = 0. \tag{6.1.3}$$

Ampère-Maxwell law

In the original form, this law tells us the integral of the magnetic field along a closed path is proportional to the total current thorough the enclosed surface. i.e.,

$$\int_{\partial S} \mathcal{H} \cdot \boldsymbol{\tau} \; d\ell = \int_S \mathcal{G} \cdot \mathbf{n} \; ds. \tag{6.1.4}$$

Then, Maxwell generalized this result by adding another source term, i.e., a changing electric displacement. Presence of this term allowed Maxwell to develop the theory of electromagnetism. The integral form of this equation is written as

$$\int_{\partial S} \mathcal{H} \cdot \boldsymbol{\tau} \, d\ell = \int_S \frac{\partial \mathcal{D}}{\partial t} \cdot \mathbf{n} \, ds + \int_S \mathcal{G} \cdot \mathbf{n} ds. \tag{6.1.5}$$

Applying Gauss' and Stokes' theorems to equations (6.1.1)-(6.1.3) and (6.1.5) gives us the *Maxwell's equations* in the following differential form

$$\nabla \cdot \mathcal{B} = 0, \tag{6.1.6}$$
$$\nabla \cdot \mathcal{D} = -\rho, \tag{6.1.7}$$
$$\frac{\partial}{\partial t}\mathcal{B} + \nabla \times \mathcal{E} = 0, \tag{6.1.8}$$
$$\frac{\partial}{\partial t}\mathcal{D} - \nabla \times \mathcal{H} = -\mathcal{G}. \tag{6.1.9}$$

Taking derivatives of Eq. (6.1.7) w.r.t. t and the divergence of (6.1.9) give rise to the following *equation of continuity*

$$\nabla \cdot \mathcal{G} + \frac{\partial \rho}{\partial t} = 0. \tag{6.1.10}$$

Material properties

The electric and magnetic field intensities and their fluxes are connected thorough the following laws

$$\mathcal{D} = \varepsilon \mathcal{E}, \tag{6.1.11}$$
$$\mathcal{B} = \mu \mathcal{H}, \tag{6.1.12}$$

where the tensor ε $[As/Vm]$ is called the *electric permittivity* and the tensor μ $[Vs/Am]$ is the *magnetic permeability*. The above equations are experimentally derived and called the *material laws*. Also, they depend on the properties of the material filling the domain.

The electric field in conducting media induces a current which is described by Ohm's Law

$$\mathcal{G} = \mathcal{G}_e + \sigma \mathcal{E}, \tag{6.1.13}$$

where $\mathcal{G}_e$ is the external current density, σ $[As]$ is the *electric conductivity*, and $\mathcal{G}$ is the total current density. Generally, ε, μ and σ depend on space, time or even the electromagnetic field. Homogeneous isotropic materials can be characterized by a positive dielectric constant $\varepsilon > 0$, a positive permeability constant $\mu > 0$, and a non-negative electric conductivity constant $\sigma \geq 0$. We will only consider homogeneous isotropic materials in this thesis.

6.1.2 Boundary conditions

A perfect electric conductor is a region with $\sigma \to \infty$. Then according to Ohm's law in this region we have $\mathcal{E} \to 0$. If Ω is surrounded by such a perfectly conducting region, we have the following *perfectly conducting boundary condition* for $\mathcal{E}$

$$\mathbf{n} \times \mathcal{E} = 0 \qquad \text{on} \quad \Gamma. \tag{6.1.14}$$

For a perfect magnetic conductor, i.e., a region with high permeability, we have $\mathcal{H} \to 0$ and if this conductor is situated around Ω, we get the following boundary condition

$$\mathbf{n} \times \mathcal{H} = 0 \qquad \text{on} \quad \Gamma. \tag{6.1.15}$$

6.1.3 Time-Harmonic Fields

Substituting the constitutive equations (6.1.11), (6.1.12) and (6.1.13) into (6.1.6)–(6.1.9) we get

$$\nabla \cdot (\mu \mathcal{H}) = 0 \qquad \text{in } \Omega, \tag{6.1.16a}$$

$$\nabla \cdot (\varepsilon \mathcal{E}) = \rho \qquad \text{in } \Omega, \tag{6.1.16b}$$

$$-\left(\varepsilon \frac{\partial}{\partial t} + \sigma\right) \mathcal{E} + \nabla \times \mathcal{H} = \mathcal{G}_e \qquad \text{in } \Omega, \tag{6.1.16c}$$

$$\mu \frac{\partial}{\partial t} \mathcal{H} + \nabla \times \mathcal{E} = 0 \qquad \text{in } \Omega, \tag{6.1.16d}$$

where $\mathcal{G}_e$ is a known function denoting the applied current. Also, we should notice (6.1.16a) and (6.1.16b) are automatically fulfilled by taking divergence of (6.1.16d)and (6.1.16c) and applying (6.1.10).

We assume the behavior of the electric and magnetic fields are periodic with respect to time, i.e.,

$$\mathcal{E}(x,t) = e^{-i\omega t} \mathbf{E}(x), \tag{6.1.17a}$$

$$\mathcal{H}(x,t) = e^{-i\omega t} \mathbf{H}(x). \tag{6.1.17b}$$

Substituting (6.1.17a) and (6.1.17b) into (6.1.16c) and (6.1.16d), we conclude

$$-\nabla \times \mathbf{H} - i\omega\eta \mathbf{E} = \mathbf{J}(x) \qquad \text{in } \Omega, \tag{6.1.18a}$$

$$\nabla \times \mathbf{E} - i\omega\mu \mathbf{H} = 0 \qquad \text{in } \Omega, \tag{6.1.18b}$$

where $\eta := \varepsilon + i\sigma/\omega$ and $\mathcal{G}_c(x,t) = e^{-i\omega t}\mathbf{J}(x)$. In this chapter, we consider the perfect conducting boundary condition for $\mathcal{E}$, i.e, we assume the domain is surrounded by a perfectly bounded material. Finally, the first order system (6.1.18) can be reduced to a second order equation

$$\mathcal{L}\mathbf{E} := \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \kappa \mathbf{E} = \mathbf{F} \qquad \text{in } \Omega, \tag{6.1.19}$$

where $\kappa := \omega^2 \eta$ and $\mathbf{F} := -i\omega \boldsymbol{J}$. For the sake of simplicity, we assume $\mu = 1$ in the following. However, our arguments can directly be extended for $\mu > 0$ as well.

6.1.4 Discretization by edge elements

Multiplying both sides of (6.1.19) with $\Psi \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ and integrating by parts, we obtain the weak formulation: Find $\mathbf{E} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ such that

$$a(\mathbf{E}, \Psi) := \langle \nabla \times \mathbf{E}, \nabla \times \Psi \rangle_{\mathbf{L}^2(\Omega)} - \kappa \langle \mathbf{E}, \Psi \rangle_{\mathbf{L}^2(\Omega)} = \langle \mathbf{F}, \Psi \rangle_{\mathbf{L}^2(\Omega)} \qquad \forall \Psi \in \mathbf{H}_0(\mathrm{curl}, \Omega). \tag{6.1.20}$$

We assume that κ is not an eigenvalue of the operator $\nabla \times \nabla \times$, see, e.g., [Mon03, Sec. 4]. This implies in particular that $\kappa \neq 0$ since $\nabla H^1_0(\Omega)$ is contained in the kernel of the operator $\nabla \times \nabla \times$. Then, the Fredholm alternative provides the existence of a unique solution to the variational problem, and we have the *a priori* estimate

$$\|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl},\Omega)} \leq C_{\mathrm{stab}} \|\mathbf{F}\|_{\mathbf{L}^2(\Omega)}, \tag{6.1.21}$$

for a constant C_{stab} that depends on Ω and κ.

Let $\mathcal{T} = \{T_1, \dots, T_{N_\mathcal{T}}\}$ be a quasi-uniform triangulation of Ω with the mesh width $h := \max_{T_j \in \mathcal{T}} \mathrm{diam}(T_j)$, where the elements $T_j \in \mathcal{T}$ are open tetrahedra. The mesh $\mathcal{T}$ is assumed to be regular in the sense of Ciarlet, i.e., there are no hanging nodes. The assumption of quasi-uniformity includes the assumption of γ-shape regularity, i.e., there is $\gamma > 0$ such that $\mathrm{diam}(T_j) \leq \gamma |T_j|^{1/3}$ for all $T_j \in \mathcal{T}$. For the Galerkin discretization of (6.1.20), we use Nédélec's $\mathbf{H}(\mathrm{curl}, \Omega)$-conforming elements of the first kind defined in Section 2.2. Let $\mathcal{X}_{h,0} := \{\Psi_1, \dots, \Psi_N\}$ be a basis of $\mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ with $N := \dim \mathbf{X}_{h,0}(\mathcal{T}, \Omega)$. Using $\mathbf{X}_{h,0}(\mathcal{T}, \Omega) \subseteq \mathbf{H}_0(\mathrm{curl}, \Omega)$ as ansatz and test space in (6.1.20), we arrive at the Galerkin discretization of finding $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ such that

$$a(\mathbf{E}_h, \Psi_h) = \langle \mathbf{F}, \Psi_h \rangle_{\mathbf{L}^2(\Omega)} \qquad \forall \Psi_h \in \mathbf{X}_{h,0}(\mathcal{T}, \Omega). \tag{6.1.22}$$

Using the basis $\mathcal{X}_{h,0}$, the Galerkin discretization (6.1.22) can be formulated as a linear system of equations where the system matrix $\mathbf{A} \in \mathbb{C}^{N \times N}$ is given by

$$\mathbf{A}_{ij} := a(\Psi_i, \Psi_j), \qquad \Psi_j, \Psi_i \in \mathcal{X}_{h,0}. \tag{6.1.23}$$

For unique solvability of the discrete problem (6.1.22) or, equivalently, the invertibility of $\mathbf{A}$, we recall the following Lemma 6.1.1. In that result and throughout the chapter, we denote by

$$\mathbf{\Pi}_h^{L^2} : \mathbf{L}^2(\Omega) \to \mathbf{X}_h(\mathcal{T}, \Omega), \tag{6.1.24}$$

the $\mathbf{L}^2(\Omega)$-orthogonal projection onto $\mathbf{X}_h(\mathcal{T}, \Omega)$.

Lemma 6.1.1. *[Hip02, Thm. 5.7] There exists $h_0 > 0$ depending on the parameters of the continuous problem and the γ-shape regularity of $\mathcal{T}$ such that for $h < h_0$, the discrete problem (6.1.22) has a unique solution and there holds the stability estimate*

$$\|\boldsymbol{E}_h\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} \leq C_{\mathrm{stab}} \left\| \mathbf{\Pi}_h^{L^2} \boldsymbol{F} \right\|_{\boldsymbol{L}^2(\Omega)}.$$

Here, $C_{\mathrm{stab}} > 0$ is a constant depending solely on the γ-shape regularity of $\mathcal{T}$ and the parameters of the continuous problems.

6.2 The Main Result

The following theorem is the main result of this chapter. It states that the inverse of the Galerkin matrix $\mathbf{A}$ of (6.1.23) can be approximated at an exponential rate in the block rank by an $\mathcal{H}$-matrix. The proof is given in Section 6.5.

Theorem 6.2.1. *Let $\boldsymbol{A}$ be the stiffness matrix given by (6.1.23) and $\eta > 0$ be a fixed admissibility parameter. Let P be a partition of $\mathcal{I} \times \mathcal{I}$ based on the cluster tree $\mathbb{T}_{\mathcal{I}}$ and the admissibility parameter η (due to the symmetry of the matrix $\boldsymbol{A}$, the admissible cluster pairs are allowed to be identified using the weaker admissibility condition from Remark 2.6.4). Let $h < h_0$ with h_0 defined in Lemma 6.1.1. Then, there exists an $\mathcal{H}$-matrix $\mathbf{B}_{\mathcal{H}}$ with blockwise rank r such that*

$$\left\|\mathbf{A}^{-1} - \mathbf{B}_{\mathcal{H}}\right\|_2 \leq C_{\text{apx}} C_{\text{sp}} \operatorname{depth}(\mathbb{T}_{\mathcal{I}}) h^{-1} e^{-b(r^{1/4}/\ln r)}.$$

The constants C_{apx}, $b > 0$ depend only on κ, Ω, η, the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$. The constant C_{sp} (defined in (2.6.3)) depends only on the partition P.

Remark 6.2.2. The low-rank structure of the far-field blocks allow for efficient storage of $\mathcal{H}$-matrices as the memory requirement to store an $\mathcal{H}$-matrix is $\mathcal{O}(C_{\text{sp}} \operatorname{depth}(\mathbb{T}_{\mathcal{I}}) rN)$. Standard clustering methods such as the geometric clustering (see, e.g., [Hac15, Sec. 5.4.2]) lead to balanced cluster trees, i.e., $\operatorname{depth}(\mathbb{T}_{\mathcal{I}}) \sim \log(N)$ and a uniformly (in the mesh size h) bounded sparsity constant. In total this gives a storage complexity of $\mathcal{O}(rN\log(N))$ for the matrix $\mathbf{B}_{\mathcal{H}}$ rather than $\mathcal{O}(N^2)$ for the fully populated inverse $\mathbf{A}^{-1}$. ■

6.3 Decompositions: continuous and discrete local

The Helmholtz as well as the regular decompositions play a key role in the analysis of $\mathbf{H}(\text{curl})$-problems. In this section, we introduce four different decompositions, the classical, continuous Helmholtz decomposition (see, e.g., [Hip02, Lem. 2.4] and [Hip15, Thm. 11]), its discrete counterpart (see, e.g., [GR86, Corollary 5.1] and [Mon03, Sec. 7.2.1]), the regular decomposition (see, e.g., [Hip02, Lem. 2.4] and [Hip15, Thm. 11]) and a localized discrete version (Definition 6.3.10).

6.3.1 Helmholtz decomposition

The Helmholtz decomposition states that every vector field $\mathbf{E} \in \mathbf{L}^2(\Omega)$ can be decomposed into a gradiant and a divergence-free part, see e.g. [Mon03, Sec. 3.7, Sec. 4.4].

Lemma 6.3.1. *(Helmholtz decomposition) For every vector field $\boldsymbol{E} \in \boldsymbol{L}^2(\Omega)$, there exists the following (unique) orthogonal decomposition*

$$\boldsymbol{E} = \nabla \times \boldsymbol{z} + \nabla\varphi \qquad \boldsymbol{z} \in \boldsymbol{H}(\text{curl}, \Omega),\ \varphi \in H^1(\Omega).$$

Particularly, for $\boldsymbol{E} \in \boldsymbol{H}_0(\text{curl}, \Omega)$ we have the following orthogonal decomposition

$$\boldsymbol{E} = \nabla \times \boldsymbol{z} + \nabla\varphi \qquad \boldsymbol{z} \in \boldsymbol{H}_0(\text{curl}, \Omega),\ \varphi \in H_0^1(\Omega),$$

such that $\boldsymbol{z}$ is a divergence free vector field.

6.3.2 Regular decompositions

The following lemma follows from the seminal paper [CM10]. The notation follows [CM10] in that $\mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3)$, $s \in \mathbb{R}$, denotes the spaces of distributions in $\mathbf{H}^s(\mathbb{R}^3)$ that supported by $\overline{\Omega}$ and that $\mathbf{C}^\infty_{\overline{\Omega}}(\mathbb{R}^3)$ is the space of $\mathbf{C}^\infty(\mathbb{R}^3)$-functions supported by $\overline{\Omega}$. We introduce the space

$$\mathbf{H}^s_{\overline{\Omega}}(\text{curl}) := \{\mathbf{E} \in \mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3) : \nabla \times \mathbf{E} \in \mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3)\}$$

equipped with the norm $\|\mathbf{E}\|_{\mathbf{H}^s_{\overline{\Omega}}(\text{curl})} := \|\mathbf{E}\|_{\mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3)} + \|\nabla \times \mathbf{E}\|_{\mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3)}$

Remark 6.3.2. From [CM10, p. 301], for any $s \in \mathbb{R}$, the space $\mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3)$ is naturally isomorphic to the dual space of $\mathbf{H}^{-s}(\Omega)$. Hence, for $s \geq 0$, we have the alternative norm equivalence $\|\mathbf{v}\|_{\mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3)} \sim \|\mathbf{v}\|_{\widetilde{\mathbf{H}}^s(\Omega)} = \|\mathbf{v}^\star\|_{\mathbf{H}^s(\mathbb{R}^3)}$, where $\mathbf{v}^\star$ is the zero extension of a function $\mathbf{v}$ defined on Ω. ■

Lemma 6.3.3. *Let Ω be a bounded Lipschitz domain. There exist pseudodifferential operators T_1 and $\mathbf{T}_2$, of order -1 and a pseudodifferential operator $\mathbf{L}$ of order $-\infty$ on $\mathbb{R}^3$ with the following properties: For each $s \in \mathbb{Z}$ they have the mapping properties $T_1 : \mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3) \to H^{s+1}_{\overline{\Omega}}(\mathbb{R}^3)$, $\mathbf{T}_2 : \mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3) \to \mathbf{H}^{s+1}_{\overline{\Omega}}(\mathbb{R}^3)$, and $\mathbf{L} : \mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3) \to \mathbf{C}^\infty_{\overline{\Omega}}(\mathbb{R}^3)$ and for any $\mathbf{u} \in \mathbf{H}^s_{\overline{\Omega}}(\text{curl})$ there holds the representation*

$$\mathbf{u} = \nabla T_1 \left(\mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)\right) + \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right) + \mathbf{L}\mathbf{u}. \tag{6.3.1}$$

Proof. In [CM10, Theorem 4.6], operators T_1, $\mathbf{T}_2$, $\mathbf{T}_3$, $\mathbf{L}_1$, $\mathbf{L}_2$ with the mapping properties

$$\begin{aligned} T_1 &: \mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3) \to H^{s+1}_{\overline{\Omega}}\left(\mathbb{R}^3\right), \\ \mathbf{T}_2 &: \mathbf{H}^s_{\overline{\Omega}}(\mathbb{R}^3) \to \mathbf{H}^{s+1}_{\overline{\Omega}}\left(\mathbb{R}^3\right), \\ \mathbf{T}_3 &: H^s_{\overline{\Omega}}(\mathbb{R}^3) \to \mathbf{H}^{s+1}_{\overline{\Omega}}\left(\mathbb{R}^3\right), \\ \mathbf{L}_\ell &: \mathbf{H}^s_{\overline{\Omega}}\left(\mathbb{R}^3\right) \to \mathbf{C}^\infty_{\overline{\Omega}}\left(\mathbb{R}^3\right), \qquad \ell = 1, 2, \end{aligned}$$

are defined, and it is shown that

$$\nabla T_1 \mathbf{v} + \mathbf{T}_2 \left(\nabla \times \mathbf{v}\right) = \mathbf{v} - \mathbf{L}_1 \mathbf{v}, \tag{6.3.2a}$$
$$\nabla \times \mathbf{T}_2 \mathbf{v} + \mathbf{T}_3 \left(\nabla \cdot \mathbf{v}\right) = \mathbf{v} - \mathbf{L}_2 \mathbf{v}. \tag{6.3.2b}$$

Taking $\mathbf{v} = \mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)$ in (6.3.2a), we obtain

$$\begin{aligned} &\nabla T_1 \left(\mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)\right) + \mathbf{T}_2 \left(\nabla \times \left(\mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)\right)\right) \\ &\qquad = \mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right) - \mathbf{L}_1 \left(\mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)\right). \end{aligned} \tag{6.3.3}$$

Since $\nabla \times \mathbf{u}$ is divergence free, we obtain from (6.3.2b) with the choice $\mathbf{v} = \nabla \times \mathbf{u}$

$$\begin{aligned} \mathbf{T}_2 \left(\nabla \times \left(\mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)\right)\right) &= \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right) - \mathbf{T}_2 \left(\nabla \times \mathbf{u} - \mathbf{L}_2 \nabla \times \mathbf{u}\right) \\ &= \mathbf{T}_2 \left(\mathbf{L}_2 \left(\nabla \times \mathbf{u}\right)\right) =: \mathbf{L}_3 \mathbf{u}, \end{aligned}$$

where, again, $\mathbf{L}_3$ is a smoothing operator of order $-\infty$ mapping into $\mathbf{C}^\infty_{\overline{\Omega}}(\mathbb{R}^3)$. Inserting this into (6.3.3) leads to

$$\nabla T_1 \left(\mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)\right) + \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right) = \mathbf{u} - \mathbf{L}_1 \left(\mathbf{u} - \mathbf{T}_2 \left(\nabla \times \mathbf{u}\right)\right) - \mathbf{L}_3 \mathbf{u}.$$

Choosing $\mathbf{L}\mathbf{u} := \left(\mathbf{L}_1 \left(\mathbf{u} - \mathbf{T}_2 \nabla \times \mathbf{u}\right)\right) + \mathbf{L}_3 \mathbf{u}$, we arrive at the representation (6.3.1). □

Corollary 6.3.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then, for every $s \geq 0$ there is a constant C (depending only on Ω and s) such that every $\boldsymbol{u} \in \boldsymbol{H}_0^s(\mathrm{curl}, \Omega)$ can be decomposed as $\boldsymbol{u} = \boldsymbol{z} + \nabla p$ with $\boldsymbol{z} \in \boldsymbol{H}_{\overline{\Omega}}^{s+1}(\mathbb{R}^3)$ and $p \in H_{\overline{\Omega}}^{s+1}(\mathbb{R}^3)$ together with*

$$\|\boldsymbol{z}\|_{\boldsymbol{H}_{\overline{\Omega}}^{s+1}(\mathbb{R}^3)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}_{\overline{\Omega}}^{s}(\mathrm{curl})}, \qquad \|\nabla p\|_{\boldsymbol{H}_{\overline{\Omega}}^{s}(\mathbb{R}^3)} \leq C\|\boldsymbol{u}\|_{\boldsymbol{H}_{\overline{\Omega}}^{s}(\mathbb{R}^3)}, \tag{6.3.4}$$

Proof. From Lemma 6.3.3 we can write $\mathbf{u} = \mathbf{z} + \nabla p$ with

$$\mathbf{z} := \mathbf{T}_2\left(\nabla \times \mathbf{u}\right) + \mathbf{L}\mathbf{u}, \qquad p := T_1\left(\mathbf{u} - \mathbf{T}_2\left(\nabla \times \mathbf{u}\right)\right).$$

The stability estimate for $\mathbf{z}$ follows from the mapping properties of the operators $\mathbf{T}_2$ and $\mathbf{L}$. The mapping properties of $\mathbf{T}_1$ yield

$$\begin{aligned}\|\nabla p\|_{H^s(\Omega)} &\lesssim \|\mathbf{u} - \mathbf{T}_2(\nabla \times \mathbf{u})\|_{\mathbf{H}_{\overline{\Omega}}^{s}(\mathbb{R}^3)} \lesssim \|\mathbf{u}\|_{\mathbf{H}_{\overline{\Omega}}^{s}(\mathbb{R}^3)} + \|\nabla \times \mathbf{u}\|_{\mathbf{H}_{\overline{\Omega}}^{s-1}(\mathbb{R}^3)} \\ &\lesssim \|\mathbf{u}\|_{\mathbf{H}_{\overline{\Omega}}^{s}(\mathbb{R}^3)},\end{aligned}$$

where the last step follows from the mapping property $\nabla\times : \mathbf{H}_{\overline{\Omega}}^{s}(\mathbb{R}^3) \to \mathbf{H}_{\overline{\Omega}}^{s-1}(\mathbb{R}^3)$. □

In the following, we present the regular decomposition. The regular decomposition states that every vector field in $\mathbf{H}_0(\mathrm{curl}, \Omega)$ can be decomposed into two vector fields such that one of them belongs to $\mathbf{H}_0^1(\Omega)$ and the other one is divergence of a function in $H_0^1(\Omega)$.

Lemma 6.3.5 (Regular decomposition)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there is a constant $C > 0$ depending only on Ω such that any $\boldsymbol{E} \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$ can be written as $\boldsymbol{E} = \boldsymbol{z} + \nabla p$ with $\boldsymbol{z} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in H_0^1(\Omega)$ and*

$$\|\boldsymbol{z}\|_{\boldsymbol{H}_0^1(\Omega)} \leq C\|\boldsymbol{E}\|_{\boldsymbol{H}_0(\mathrm{curl},\Omega)}, \qquad \|\boldsymbol{z}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla p\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\boldsymbol{E}\|_{\boldsymbol{L}^2(\Omega)}.$$

Proof. Regular decompositions are available in the literature, see, e.g., [Hip02, Lem. 2.4] and [Hip15, Thm. 11]. The statement that $\|\mathbf{z}\|_{\mathbf{L}^2(\Omega)}$ and $\|\nabla p\|_{\mathbf{L}^2(\Omega)}$ are controlled by $\|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}$ is a variation of these estimates. For a proof, see [CFV20, Thm. B.1] or corollary 6.3.4. □

The function $\mathbf{z}$ of the regular decomposition provided by Lemma 6.3.5 is not necessarily divergence-free. This can be corrected by subtracting a gradient. To that end, we introduce, for a given open set $\widetilde{D} \subseteq \Omega$ and a chosen $\widetilde{\eta} \in L^\infty(\Omega)$ with $\widetilde{\eta} \equiv 1$ on $\widetilde{D}$ the mapping $\mathbf{L}^2(\Omega) \to H_0^1(\Omega)\colon \mathbf{z} \mapsto \varphi_{\mathbf{z}}$ by

$$\langle \nabla \varphi_{\mathbf{z}}, \nabla v\rangle_{L^2(\Omega)} = \langle \widetilde{\eta}\mathbf{z}, \nabla v\rangle_{L^2(\Omega)} \qquad \forall v \in H_0^1(\Omega). \tag{6.3.5}$$

Lemma 6.3.6. *The mapping $\boldsymbol{L}^2(\Omega) \ni \boldsymbol{z} \mapsto \varphi_{\boldsymbol{z}} \in H_0^1(\Omega)$ has the following properties:*

(i) $\|\varphi_{\boldsymbol{z}}\|_{H^1(\Omega)} \leq C\|\widetilde{\eta}\|_{L^\infty(\Omega)}\|\boldsymbol{z}\|_{\boldsymbol{L}^2(\mathrm{supp}\,\widetilde{\eta})}$, *where the constant depends only on Ω.*

(ii) $(\boldsymbol{z} - \nabla\varphi_{\boldsymbol{z}}, \nabla v)_{L^2(\widetilde{D})} = 0$ *for all $v \in H_0^1(\Omega)$.*

Proof. By construction, we have $\|\nabla\varphi_z\|_{\mathbf{L}^2(\Omega)} \leq \|\widetilde{\eta}\mathbf{z}\|_{\mathbf{L}^2(\Omega)}$. The constant C in statement (i) reflects the Poincaré constant of Ω. The property (ii) follows by construction. □

Remark 6.3.7 (classical Helmholtz decomposition)*.* Selecting $\widetilde{D} = \Omega$ and correspondingly $\eta \equiv 1$ yields the decomposition $\mathbf{E} = (\mathbf{z} - \nabla\varphi_{\mathbf{z}}) + \nabla(p + \varphi_{\mathbf{z}})$ with the orthogonality $\langle \mathbf{z} - \nabla\varphi_{\mathbf{z}}, \nabla(p+\varphi_{\mathbf{z}})\rangle_{L^2(\Omega)} = 0$ and $\|\mathbf{z}-\nabla\varphi_{\mathbf{z}}\|_{\mathbf{H}(\mathrm{curl},\Omega)} \lesssim \|\mathbf{E}\|_{\mathbf{H}(\mathrm{curl},\Omega)}$, $\|\mathbf{z}-\nabla\varphi_{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}$, $\|\nabla(p + \varphi_{\mathbf{z}})\|_{L^2(\Omega)} \lesssim \|\mathbf{E}\|_{\mathbf{L}^2(\Omega)}$. ■

6.3.3 Discrete and local discrete Helmholtz decompositions

We introduce the space of discrete divergence-free functions by

$$\mathbf{Z}_0(\mathcal{T}) := \{\mathbf{z}_h \in \mathbf{X}_{h,0}(\mathcal{T},\Omega) \ : \ \langle \mathbf{z}_h, \nabla\zeta_h\rangle_{\mathbf{L}^2(\Omega)} = 0 \quad \forall \zeta_h \in S_0^{1,1}(\mathcal{T})\}.$$

Lemma 6.3.8. *(Discrete Helmholtz decomposition) ([GR86, Corollary 5.1] and [Mon03, Sec. 7.2.1]) For the space $\boldsymbol{X}_{h,0}(\mathcal{T},\Omega)$, we have the following discrete Helmholtz decomposition*

$$\boldsymbol{X}_{h,0}(\mathcal{T},\Omega) = \boldsymbol{Z}_0(\mathcal{T}) \oplus \nabla S_0^{1,1}(\mathcal{T}).$$

Moreover, for $\boldsymbol{E}_h \in \boldsymbol{X}_{h,0}(\mathcal{T},\Omega)$, the decomposition $\boldsymbol{E}_h = \boldsymbol{z}_h + \nabla p_h$ with $\boldsymbol{z}_h \in \boldsymbol{Z}_0(\mathcal{T})$, $p_h \in S_0^{1,1}(\mathcal{T})$ is stable, i.e.,

$$\|\boldsymbol{z}_h\|_{\boldsymbol{H}_0(\mathrm{curl},\Omega)} + \|\nabla p_h\|_{\boldsymbol{L}^2(\Omega)} \leq C \|\boldsymbol{E}_h\|_{\boldsymbol{H}_0(\mathrm{curl},\Omega)}.$$

Regular decompositions as in Lemma 6.3.5 can also be done locally for discrete functions. To that end, we introduce the *localized* spaces of piecewise polynomials:

Definition 6.3.9 (Mesh-conforming region, localized spaces)**.** For $D \subset \mathbb{R}^3$, a simply connected domain, set

$$\mathcal{T}(D) := \{T \in \mathcal{T} : \ |T \cap D| > 0\},$$
$$\widetilde{D} := \mathrm{int}\Big(\bigcup_{T\in\mathcal{T}(D)} \overline{T}\Big).$$

We call $\widetilde{D}$ the *mesh-conforming region* for D. The spaces *localized* to $\widetilde{D}$ are given by

$$S^{1,1}(\mathcal{T},\widetilde{D}) := \{p_h|_{\widetilde{D}} \ : \ p_h \in S_0^{1,1}(\mathcal{T})\}, \tag{6.3.6}$$
$$\mathbf{X}_h(\mathcal{T},\widetilde{D}) := \{\mathbf{E}_h|_{\widetilde{D}} \ : \ \mathbf{E}_h \in \mathbf{X}_{h.0}(\mathcal{T},\Omega)\}. \tag{6.3.7}$$

Definition 6.3.10. (Local discrete regular decomposition) Let $D \subset \Omega$ be a simply connected domain and $\widetilde{D}$ be the corresponding mesh-conforming region. We denote by $\Pi_{\widetilde{D}}^{\nabla}$: $\mathbf{L}^2(\widetilde{D}) \to \nabla S^{1,1}(\mathcal{T},\widetilde{D})$ the $\mathbf{L}^2(\widetilde{D})$-projection onto $\nabla S^{1,1}(\mathcal{T},\widetilde{D})$ given by

$$\langle \mathbf{p} - \Pi_{\widetilde{D}}^{\nabla}\mathbf{p}, \nabla v_h\rangle_{\mathbf{L}^2(\widetilde{D})} = 0 \qquad \forall v_h \in S^{1,1}(\mathcal{T},\widetilde{D}). \tag{6.3.8}$$

Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function with $0 \leq \eta \leq 1$ and $\eta \equiv 1$ on $\widetilde{D}$. Let $\mathbf{E}_h$ be such that $\eta\mathbf{E}_h \in \mathbf{H}_0(\mathrm{curl},\Omega)$ as well as $\mathbf{E}_h|_{\widetilde{D}} \in \mathbf{X}_h(\mathcal{T},\widetilde{D})$. Decompose $\eta\mathbf{E}_h \in \mathbf{H}_0(\mathrm{curl},\Omega)$ as $\eta\mathbf{E}_h = \mathbf{z} + \nabla p$, where $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ and $p \in H_0^1(\Omega)$ are given by Lemma 6.3.5.

Then, the *local discrete regular decomposition* is given by $\mathbf{E}_h = \mathbf{z}_h + \Pi_{\widetilde{D}}^{\nabla}\nabla p$ on $\widetilde{D}$ with $\mathbf{z}_h := \mathbf{E}_h - \Pi_{\widetilde{D}}^{\nabla}\nabla p$. We write $\nabla p_h = \Pi_{\widetilde{D}}^{\nabla}\nabla p$ for some $p_h \in S^{1,1}(\mathcal{T},\widetilde{D})$.

For future reference, we note that

$$\left\|\Pi_{\widetilde{D}}^{\nabla}\mathbf{p}\right\|_{\mathbf{L}^2(\widetilde{D})} \leq \|\mathbf{p}\|_{\mathbf{L}^2(\widetilde{D})}. \tag{6.3.9}$$

Remark 6.3.11.
1. The function $p_h \in S^{1,1}(\mathcal{T}, \widetilde{D})$ that satisfies $\nabla p_h = \Pi^{\nabla}_{\widetilde{D}}\mathbf{p}$, is not unique. However, its gradient ∇p_h is unique.
2. Due to the cut-off function η, the decomposition depends on $\mathbf{E}_h$ on $\operatorname{supp}\eta$ only, which is quanitified in the stability assertions of Lemma 6.3.15.
3. The local regular decomposition provides for a function $\mathbf{E}_h$ that is a discrete function on $\widetilde{D}$ two representations in view of $\eta \equiv 1$ on $\widetilde{D}$, namely, $\mathbf{E}_h = (\mathbf{z} + \nabla p)\,|_{\widetilde{D}} = \mathbf{z}_h + \nabla p_h$.
4. For $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ the decomposition $\mathbf{E}_h = (\mathbf{z} - \nabla\varphi_{\mathbf{z}}) + \nabla(p + \varphi_{\mathbf{z}})$ of Remark 6.3.7 yields, upon setting $\nabla p_h := \Pi^{\nabla}_{\Omega}\nabla(\varphi_{\mathbf{z}} + p) \in \nabla S^{1,1}_0(\mathcal{T}, \Omega) \subset \mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ and $\mathbf{z}_h := \mathbf{E}_h - \nabla p_h \in \mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ the decomposition $\mathbf{E}_h = \mathbf{z}_h + \nabla p_h$ with
$$\langle \mathbf{z}_h, \nabla p_h\rangle_{\mathbf{L}^2(\Omega)} = 0, \qquad \|\mathbf{z}_h\|_{\mathbf{L}^2(\Omega)} + \|\nabla p_h\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)},$$
$$\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl},\Omega)} \lesssim \|\mathbf{E}_h\|_{\mathbf{H}(\mathrm{curl},\Omega)},$$
which is a discrete Helmholtz decomposition as described in Lemma 6.3.8. ■

The following lemma formulates a local exact sequence property.

Lemma 6.3.12. *Let $D \subset \mathbb{R}^3$ be an open set such that $D \cap \Omega$ is a simply connected Lipschitz domain and $\widetilde{D}$ and $\mathcal{T}(D)$ be defined according to Definition 6.3.9. Assume furthermore that $D \cap T$ is simply connected for all $T \in \mathcal{T}(D)$ and that $\overline{\widetilde{D}} \cap \partial\Omega$ is connected. (In particular, the empty set is connected.) Then, for all $\mathbf{v}_h \in \boldsymbol{X}_h(\mathcal{T}, \widetilde{D})$ with $\nabla \times \mathbf{v}_h = 0$ on $D \cap \Omega$, we can find a $\widetilde{\varphi}_h \in S^{1,1}(\mathcal{T}, \widetilde{D})$ such that $\mathbf{v}_h = \nabla\widetilde{\varphi}_h$.*

Proof. We recall from, e.g., [Mon03, Thm. 3.37] the following commuting diagram property: for a simply connected Lipschitz domain ω the condition $\nabla \times \mathfrak{w} = 0$ implies $\mathfrak{w} = \nabla\psi$ for some $\psi \in H^1(\omega)$; furthermore, ψ is unique up to a constant. The discrete commuting diagram property for a tetrahedron T is: if $\mathfrak{w} \in \mathcal{N}_1(T)$ satisfies $\nabla \times \mathfrak{w} = 0$, then there is $\psi_h \in \mathcal{P}_1(T)$ with $\mathfrak{w} = \nabla\psi_h$.

Introduce $S^{1,0}(\mathcal{T}, \widetilde{D}) := \{\psi_h \in L^2(\widetilde{D}) : \psi_h|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}(D)\}$. The condition $\nabla \times \mathbf{v}_h = 0$ on $D \cap \Omega$ implies $\mathbf{v}_h = \nabla\varphi_h$ for some $\varphi_h \in H^1(D \cap \Omega)$. The function φ_h is unique up to a constant, which we fix, for example, by the condition $\int_{D\cap\Omega}\varphi_h = 0$. For each $T \in \mathcal{T}(D)$ the condition $\nabla \times \mathbf{v}_h = 0$ on T implies the existence of $\widetilde{\varphi}_{h,T} \in \mathcal{P}_1(T)$ with $\mathbf{v}_h = \nabla\widetilde{\varphi}_{h,T}$ on T. The polynomial $\widetilde{\varphi}_{h,T}$ is unique up to a constant, which we fix by requiring $\int_{D\cap T}\widetilde{\varphi}_{h,T} = \int_{D\cap T}\varphi_h$. By the uniqueness assertion we have $\varphi_h|_{D\cap T} = \widetilde{\varphi}_{h,T}|_{D\cap T}$.

Define $\widetilde{\varphi}_h \in S^{1,0}(\mathcal{T}, \widetilde{D})$ elementwise by $\widetilde{\varphi}_h|_T = \widetilde{\varphi}_{h,T}$. We note $\widetilde{\varphi}_h|_D = \varphi_h$. Since $\widetilde{\varphi}_h$ is piecewise polynomial (hence smooth), $\varphi_h \in H^1(D \cap \Omega)$ is continuous on $\overline{D \cap \Omega}$. We next show that $\widetilde{\varphi}_h$ is continuous on $\overline{\widetilde{D}}$. Let $\widetilde{\mathcal{V}}_h$ and $\widetilde{\mathcal{E}}_h$ be the sets of vertices and edges of $\mathcal{T}(D)$. Since $\widetilde{\varphi}_h$ is a piecewise polynomial of degree 1, it suffices to assert continuity at the vertices $v \in \widetilde{\mathcal{V}}_h$. As $\widetilde{\varphi}_h$ is continuous at vertices $v \in D$, we have to show the continuity at vertices $v' \in \overline{\widetilde{D}} \setminus \overline{D}$. Given such a vertex v', select an edge $e \in \widetilde{\mathcal{E}}_h$ emanating from v' such that its other endpoint v satisfies $v \in D$. Let T, $T' \in \mathcal{T}(D)$ share this edge e. By the continuity of $\widetilde{\varphi}_h$ on D we conclude $(\widetilde{\varphi}_h|_T)|_{e\cap D} = (\widetilde{\varphi}_h|_{T'})|_{e\cap D}$. Since $\widetilde{\varphi}_h|_T$ and $\widetilde{\varphi}_h|_{T'}$ are linear, we conclude that $(\widetilde{\varphi}_h|_T)|_e = (\widetilde{\varphi}_h|_{T'})|_e$. This implies that $\widetilde{\varphi}_h$ is continuous at v. In total, we have obtained that $\widetilde{\varphi}_h \in S^{1,0}(\mathcal{T}, \widetilde{D})$ is continuous at the vertices of $\mathcal{T}(D)$ and thus is in

$H^1(\widetilde{D})$. By fixing nodal values to be zero for nodes of $\mathcal{T}$ that are not nodes of $\mathcal{T}(D)$, we obtain an element of $S^{1,1}(\mathcal{T})$ that coincides with $\widetilde{\varphi}_h$ on $\widetilde{D}$. If $\overline{\widetilde{D}} \cap \partial\Omega = \emptyset$, then $\widetilde{\varphi}_h$ is in fact in $S_0^{1,1}(\mathcal{T})$. If $\overline{\widetilde{D}} \cap \partial\Omega \neq \emptyset$, then the fact that $\mathbf{v}_h$ satisfies boundary conditions and the fact that $\overline{\widetilde{D}} \cap \partial\Omega$ is connected implies that $\widetilde{\varphi}_h$ is constant on $\overline{\widetilde{D}} \cap \partial\Omega$. This constant can be fixed to be zero and then all other nodal values of $\mathcal{T}$ can be set to zero to obtain a function $\widetilde{\varphi}_h \in S_0^{1,1}(\mathcal{T})$ with $\nabla\widetilde{\varphi}_h = \mathbf{v}_h$ on $\widetilde{D}$. □

In order to prove the following lemmas, we need to introduce some projections and their properties. Let $D \subset \mathbb{R}^3$ be a simply connected Lipschitz domain and $\widetilde{D}$ and $\mathcal{T}(D)$ be defined according to Definition 6.3.9. We define the space

$$\mathbf{H}(\mathrm{div}, \widetilde{D}) := \left\{ \mathbf{U} \in \mathbf{L}^2(\widetilde{D}) \, : \, \nabla \cdot \mathbf{U} \in L^2(\widetilde{D}) \right\}.$$

To define discrete subspace, let $\mathbf{RT}_1(T) := \{\mathbf{p}(\mathbf{x}) + q(\mathbf{x})\mathbf{x} \, : \, \mathbf{p} \in (\mathcal{P}_1(T))^3, \, q \in \mathcal{P}_1(T)\}$ be the classical lowest-order Raviart-Thomas element defined on T and introduce

$$\mathbf{V}_h(\mathcal{T}, \widetilde{D}) := \{\mathbf{U}_h \in \mathbf{H}(\mathrm{div}, \widetilde{D}) \, : \, \mathbf{U}_h|_T \in \mathbf{RT}_1(T) \qquad \forall T \in \mathcal{T}(D)\}.$$

On $\widetilde{D}$ the Raviart-Thomas interpolation operator $\mathfrak{w}_{\widetilde{D}} \colon \mathbf{H}^1(\widetilde{D}) \to \mathbf{V}_h(\mathcal{T}, \widetilde{D})$ is defined element-wise by $\mathfrak{w}_{\widetilde{D}}\mathbf{U}|_T := \mathfrak{w}_T\mathbf{U}$, where the elemental interpolation operator $\mathfrak{w}_T \colon \mathbf{H}^1(T) \to \mathbf{RT}_1(T)$ is characterized by the vanishing of certain moments of $\mathbf{U} - \mathfrak{w}_T\mathbf{U}$, viz.,

$$\int_f (\mathbf{U} - \mathfrak{w}_T\mathbf{U}) \cdot \nu q \, dA = 0 \quad \forall q \in \mathcal{P}_1(f), \; \forall f \text{ faces of } T \in \mathcal{T},$$

where ν is the unit normal to f and dA denotes the surface measure on f. Define the space

$$\mathbf{D}_h(\mathcal{T}, \widetilde{D}) := \{\mathbf{U} \in \mathbf{H}^1(\widetilde{D}) \, : \, \nabla \times \mathbf{U} \in \mathbf{H}^1(T) \quad \forall T \in \mathcal{T}(D)\},$$

and the Nédélec interpolation operator $\mathfrak{r}_{\widetilde{D}} : \mathbf{D}_h(\mathcal{T}, \widetilde{D}) \to \mathbf{X}_h(\mathcal{T}, \widetilde{D})$ elementwise by $\mathfrak{r}_{\widetilde{D}}\mathbf{U}|_T := \mathfrak{r}_T\mathbf{U}$, where the elemental interpolant $\mathfrak{r}_T\mathbf{U} \in \mathcal{N}_1(T)$ is characterized by the vanishing of certain moments of $\mathbf{U} - \mathfrak{r}_T\mathbf{U}$, viz.,

$$\int_e (\mathbf{U} - \mathfrak{r}_T\mathbf{U}) \cdot \boldsymbol{\tau} \, de = 0 \quad \forall \text{ edges } e \text{ of } T \in \mathcal{T};$$

here $\boldsymbol{\tau}$ is a unit vector parallel to the edge e. A key property of the operators $\mathfrak{r}_{\widetilde{D}}$ and $\mathfrak{w}_{\widetilde{D}}$ is that they commute, i.e., (see, e.g., [Mon03, (5.59)])

$$\mathfrak{w}_{\widetilde{D}} \nabla \times \mathbf{U} = \nabla \times \mathfrak{r}_{\widetilde{D}}\mathbf{U} \qquad \forall \, \mathbf{U} \in \mathbf{D}_h(\mathcal{T}, \widetilde{D}). \tag{6.3.10}$$

Lemma 6.3.13. *[Mon03, Thm. 5.41] Let $T \in \mathcal{T}$. Then, for $\boldsymbol{U} \in \boldsymbol{H}^1(T)$ with $\nabla \times \boldsymbol{U} \in \boldsymbol{H}^1(T)$ we have*

$$\begin{aligned} \|\boldsymbol{U} - \mathfrak{r}_T\boldsymbol{U}\|_{\boldsymbol{L}^2(T)} &\lesssim h \left(|\boldsymbol{U}|_{\boldsymbol{H}^1(T)} + \|\nabla \times \boldsymbol{U}\|_{\boldsymbol{H}^1(T)} \right), \\ \|\nabla \times (\boldsymbol{U} - \mathfrak{r}_T\boldsymbol{U})\|_{\boldsymbol{L}^2(T)} &\lesssim h \, \|\nabla \times \boldsymbol{U}\|_{\boldsymbol{H}^1(T)} . \end{aligned}$$

In the following, we show local stability and approximation properties for the local discrete regular decomposition of Definition 6.3.10. This will be based on Lemma 6.3.12 with $D = B_R$, where B_R is a box with side length R. It is an important geometric observation that, due to the assumption that Ω is a Lipschitz polyhedron, the intersection $B_R \cap \Omega$ is a Lipschitz domain and the intersection $B_R \cap \Omega^c$ is connected provided R is sufficiently small. Then, the additional assumptions on $D = \Omega \cap B_R$ in Lemma 6.3.12 can be satisfied. We formulate this as an assumption on R in terms of a number $R_{\max}$ that depends on Ω:

Definition 6.3.14. $R_{\max} > 0$ is such that for any $R \in (0, R_{\max}]$ and any box B_R with $|B_R \cap \Omega| > 0$, the intersection $B_R \cap \Omega$ is a Lipschitz domain and $B_R \cap \Omega^c$ is connected.

Lemma 6.3.15 (stability of local discrete regular decomposition)**.** *Let $\varepsilon \in (0,1)$, $R \in (0, R_{\max}]$ be such that $\frac{h}{R} < \frac{\varepsilon}{2}$, and let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes. Define $\mathcal{T}(B_R)$ and $\widetilde{B}_R$ according to Definition 6.3.9. Let $\eta \in W^{1,\infty}(\Omega)$ be a cut-off function with $\operatorname{supp}\eta \subseteq \overline{B_{(1+\varepsilon)R} \cap \Omega}$, $\eta \equiv 1$ on $\widetilde{B}_R$, $0 \leq \eta \leq 1$, and $\|\nabla \eta\|_{L^\infty(\Omega)} \leq C_\eta \frac{1}{\varepsilon R}$. Let $\boldsymbol{E}_h \in \boldsymbol{H}(\mathrm{curl}, B_{(1+\varepsilon)R} \cap \Omega)$ be such that $\eta \boldsymbol{E}_h \in \boldsymbol{H}_0(\mathrm{curl}, \Omega)$ as well as $\boldsymbol{E}_h \in \boldsymbol{X}_h(\mathcal{T}, \widetilde{B}_R)$. Let $\eta \boldsymbol{E}_h = \boldsymbol{z} + \nabla p$ be the regular decomposition of $\eta \boldsymbol{E}_h$ given by Lemma 6.3.5 and let $\boldsymbol{z}_h$ and ∇p_h be the contributions of the local discrete regular decomposition of Definition 6.3.10 with $D = B_R$ and $\widetilde{D} = \widetilde{B}_R$ there. Then, $\boldsymbol{E}_h = \boldsymbol{z}_h + \nabla p_h$ on $B_R \cap \Omega$, and the following local stability and approximation results hold:*

$$\|\nabla p_h\|_{\boldsymbol{L}^2(B_R \cap \Omega)} + \|\boldsymbol{z}_h\|_{\boldsymbol{H}(\mathrm{curl}, B_R \cap \Omega)} \leq C \left(\|\nabla \times \boldsymbol{E}_h\|_{\boldsymbol{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \frac{1}{\varepsilon R} \|\boldsymbol{E}_h\|_{\boldsymbol{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} \right),$$

$$\begin{aligned} \|\boldsymbol{z} - \boldsymbol{z}_h\|_{\boldsymbol{L}^2(B_R \cap \Omega)} &\leq Ch \|\boldsymbol{z}\|_{\boldsymbol{H}^1(B_{(1+\varepsilon)R} \cap \Omega)} \\ &\leq Ch \left(\|\nabla \times \boldsymbol{E}_h\|_{\boldsymbol{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \frac{1}{\varepsilon R} \|\boldsymbol{E}_h\|_{\boldsymbol{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} \right), \end{aligned}$$

where the constant $C > 0$ depends only on Ω, the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$, and C_η.

Proof. The proof is done in two steps. We note that the condition on the parameter ε ensures that $\widetilde{B}_R \subseteq B_{(1+\varepsilon)R}$.

Step 1: In this step we provide a proof for the stability estimate. Recalling the stability estimate Lemma 6.3.5 and using the product rule for the curl operator, it follows that

$$\begin{aligned} \|\mathbf{z}\|_{\mathbf{H}_0^1(\Omega)} + \|\nabla p\|_{L^2(\Omega)} &\lesssim \|\eta \mathbf{E}_h\|_{\mathbf{H}(\mathrm{curl},\Omega)} \\ &\lesssim \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \|\nabla \eta\|_{L^\infty(B_{(1+\varepsilon)R} \cap \Omega)} \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} \\ &\overset{\varepsilon R \lesssim 1}{\lesssim} \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \frac{1}{\varepsilon R} \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} . \end{aligned} \tag{6.3.11}$$

Since ∇p_h satisfies (6.3.8), we get with the aid of (6.3.11)

$$\|\nabla p_h\|_{L^2(B_R \cap \Omega)} \leq \|\nabla p\|_{L^2(B_R \cap \Omega)} \leq \|\nabla p\|_{L^2(\Omega)} \lesssim \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \frac{1}{\varepsilon R} \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} .$$

The definition of $\mathbf{z}_h$ gives us

$$\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl}, B_R \cap \Omega)} \lesssim \|\nabla \times \mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} + \frac{1}{\varepsilon R} \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R} \cap \Omega)} .$$

The combination of above inequalities gives us the desired local stability result.

Step 2: To prove the approximation property, we first need to ascertain the existence of $\varphi_h \in S^{1,1}(\mathcal{T}, \widetilde{B}_R)$ such that $\mathbf{z}_h - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z} = \nabla\varphi_h$. To that end, we note that $\mathbf{z}_h \in \mathbf{D}(\mathcal{T}, \widetilde{B}_R)$, use the commuting diagram property (6.3.10) of $\mathfrak{r}_{\widetilde{B}_R}$ and $\mathfrak{w}_{\widetilde{B}_R}$, and the fact that $\mathfrak{r}_{\widetilde{B}_R}$ is a projection operator to compute on $\widetilde{B}_R$:

$$\begin{aligned}\nabla\times(\mathbf{z}_h - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}) &= \nabla\times\mathbf{z}_h - \mathfrak{w}_{\widetilde{B}_R}\nabla\times\mathbf{z} = \nabla\times(\mathbf{E}_h|_{\widetilde{B}_R}) - \mathfrak{w}_{\widetilde{B}_R}\nabla\times(\mathbf{E}_h|_{\widetilde{B}_R})\\ &= \nabla\times(\mathbf{E}_h|_{\widetilde{B}_R}) - \nabla\times\mathfrak{r}_{\widetilde{B}_R}(\mathbf{E}_h|_{\widetilde{B}_R}) = 0.\end{aligned}$$

Lemma 6.3.12 then provides the existence of $\varphi_h \in S^{1,1}(\mathcal{T}, \widetilde{B}_R)$ such that $\mathbf{z}_h - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z} = \nabla\varphi_h$. Since p_h satisfies (6.3.8), we get from $\mathbf{z}+\nabla p = \mathbf{E}_h = \mathbf{z}_h + \nabla p_h$ on $\widetilde{B}_R$ and the approximation property of $\mathfrak{r}_{\widetilde{B}_R}$ given in Lemma 6.3.13

$$\begin{aligned}\|\mathbf{z}-\mathbf{z}_h\|^2_{\mathbf{L}^2(\widetilde{B}_R)} &= \left\langle \mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}, \mathbf{z}-\mathbf{z}_h\right\rangle_{\mathbf{L}^2(\widetilde{B}_R)} + \left\langle \mathfrak{r}_{\widetilde{B}_R}\mathbf{z} - \mathbf{z}_h, \mathbf{z}-\mathbf{z}_h\right\rangle_{\mathbf{L}^2(\widetilde{B}_R)}\\ &= \left\langle \mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}, \mathbf{z}-\mathbf{z}_h\right\rangle_{\mathbf{L}^2(\widetilde{B}_R)} - \langle\nabla\varphi_h, \nabla(p_h - p)\rangle_{\mathbf{L}^2(\widetilde{B}_R)}\\ &= \left\langle \mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}, \mathbf{z}-\mathbf{z}_h\right\rangle_{\mathbf{L}^2(\widetilde{B}_R)} \lesssim \left\|\mathbf{z} - \mathfrak{r}_{\widetilde{B}_R}\mathbf{z}\right\|_{\mathbf{L}^2(\widetilde{B}_R)} \|\mathbf{z}-\mathbf{z}_h\|_{\mathbf{L}^2(\widetilde{B}_R)}\\ &\lesssim h\,\|\mathbf{z}\|_{\mathbf{H}^1(B_{(1+\varepsilon)R}\cap\Omega)}\,\|\mathbf{z}-\mathbf{z}_h\|_{\mathbf{L}^2(\widetilde{B}_R)}\,.\end{aligned}$$

The combination of the above inequality and (6.3.11) implies

$$\begin{aligned}\|\mathbf{z}-\mathbf{z}_h\|_{\mathbf{L}^2(B_R\cap\Omega)} \le \|\mathbf{z}-\mathbf{z}_h\|_{\mathbf{L}^2(\widetilde{B}_R)} &\lesssim h\,\|\mathbf{z}\|_{\mathbf{H}^1(B_{(1+\varepsilon)R}\cap\Omega)}\\ &\lesssim h\left(\|\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{1}{\varepsilon R}\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right),\end{aligned}$$

which finishes the proof. □

6.3.4 The Caccioppoli-type inequalities

Caccioppoli inequalities usually estimate higher order derivatives by lower order derivatives on (slightly) enlarged regions. The following discrete Caccioppoli-type inequalities are formulated with an h-weighted $\mathbf{H}(\mathrm{curl})$-norm and an h-weighted H^1-norm. For a box B_R of side length $R > 0$, we define the norms $|||\cdot|||_{c,h,R}$ and $|||\cdot|||_{g,h,R}$ (we note that the subscripts c and g abbreviate 'curl' and 'gradient') as follows:

$$|||\mathbf{U}|||^2_{c,h,R} := \frac{h^2}{R^2}\|\nabla\times\mathbf{U}\|^2_{\mathbf{L}^2(B_R\cap\Omega)} + \frac{1}{R^2}\|\mathbf{U}\|^2_{\mathbf{L}^2(B_R\cap\Omega)} \qquad \forall\mathbf{U}\in\mathbf{H}(\mathrm{curl}, B_R\cap\Omega), \tag{6.3.12}$$

$$|||u|||^2_{g,h,R} := \frac{h^2}{R^2}\|\nabla u\|^2_{\mathbf{L}^2(B_R\cap\Omega)} + \frac{1}{R^2}\|u\|^2_{L^2(B_R\cap\Omega)} \qquad \forall u\in H^1(B_R\cap\Omega). \tag{6.3.13}$$

We say that $\mathbf{E}_h \in \mathbf{X}_h(\mathcal{T}, \widetilde{D})$ is *discrete $\boldsymbol{\mathcal{L}}$-harmonic* on $\widetilde{D}$ if $a(\mathbf{E}_h, \mathbf{v}_h) = 0$ for all $\mathbf{v}_h \in \mathbf{X}_{h,0}(\mathcal{T}, \Omega)$ with $\operatorname{supp}\mathbf{v}_h \subset \overline{\widetilde{D}}$; such a space will be formally introduced as $\mathcal{H}_{c,h}(\widetilde{D})$

below. For any bounded open set $B \subset \mathbb{R}^3$, we define

$$\mathcal{H}_{c,h}(B\cap\Omega) := \{\mathbf{U}_h \in \mathbf{H}(\mathrm{curl}, B\cap\Omega) : \exists \widetilde{\mathbf{U}}_h \in \mathbf{X}_{h,0}(\mathcal{T},\Omega) \text{ s.t. } \mathbf{U}_h|_{B\cap\Omega} = \widetilde{\mathbf{U}}_h|_{B\cap\Omega}, \\ a(\mathbf{U}_h,\Psi_h) = 0 \quad \forall \Psi_h \in \mathbf{X}_{h,0}(\mathcal{T},\Omega), \operatorname{supp}\Psi_h \subset \overline{B\cap\Omega}\},$$

and

$$\mathcal{H}_{g,h}(B\cap\Omega) := \{p_h \in H^1(B\cap\Omega) : \exists \widetilde{p}_h \in S_0^{1,1}(\mathcal{T}) \text{ s.t. } p_h|_{B\cap\Omega} = \widetilde{p}_h|_{B\cap\Omega}, \langle \nabla p_h, \nabla\psi_h\rangle_{\mathbf{L}^2(B\cap\Omega)} = 0, \\ \forall \psi_h \in S_0^{1,1}(\mathcal{T}), \operatorname{supp}\psi_h \subset \overline{B\cap\Omega}\}.$$

The following lemma provides a discrete Caccioppoli-type estimate for functions in $\mathcal{H}_{c,h}(B_{(1+\varepsilon)R}\cap\Omega)$:

Lemma 6.3.16. *Let $\varepsilon \in (0,1)$ and $R \in (0, 2\operatorname{diam}(\Omega))$ be such that $\frac{h}{R} < \frac{\varepsilon}{4}$. Let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes and $\boldsymbol{E}_h \in \mathcal{H}_{c,h}(B_{(1+\varepsilon)R}\cap\Omega)$. Then, there exists a constant C depending only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$ such that*

$$\|\nabla\times\boldsymbol{E}_h\|_{\boldsymbol{L}^2(B_R\cap\Omega)} \leq C\frac{1+\varepsilon}{\varepsilon}|||\boldsymbol{E}_h|||_{c,h,(1+\varepsilon)R}.$$

Proof. For brevity of notation, we write $\mathfrak{r}_h$ instead of $\mathfrak{r}_{\mathcal{T}(\Omega)}$. Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function with $\operatorname{supp}\eta \subseteq B_{(1+\varepsilon/2)R}$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_R \cap \Omega$, and $\|\nabla^j\eta\|_{L^\infty(\Omega)} \lesssim (\varepsilon R)^{-j}$ for $j \in \{0,1,2\}$. We notice $\operatorname{supp}(\eta^2\mathbf{E}_h) \subseteq \overline{B_{(1+\varepsilon/2)R}\cap\Omega}$ and since $4h \leq \varepsilon R$ we have $\operatorname{supp}\mathfrak{r}_h(\eta^2\mathbf{E}_h) \subseteq \overline{B_{(1+\varepsilon)R}\cap\Omega}$. The proof is done in two steps.

Step 1: Using the vector identity

$$\eta^2(\nabla\times\mathbf{E}_h)\cdot(\nabla\times\mathbf{E}_h) = \nabla\times\mathbf{E}_h\cdot\big(\nabla\times(\eta^2\mathbf{E}_h) - \nabla\eta^2\times\mathbf{E}_h\big) \\ = (\nabla\times\mathbf{E}_h)\cdot\nabla\times(\eta^2\mathbf{E}_h) - 2\eta(\nabla\times\mathbf{E}_h)\cdot(\nabla\eta\times\mathbf{E}_h),$$

we get

$$\begin{aligned}\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} &\leq \|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} \\ &= a(\mathbf{E}_h,\eta^2\mathbf{E}_h) + \kappa(\eta\mathbf{E}_h,\eta\mathbf{E}_h)_{\mathbf{L}^2(B_R\cap\Omega)} - 2(\eta\nabla\times\mathbf{E}_h,\nabla\eta\times\mathbf{E}_h)_{\mathbf{L}^2(B_R\cap\Omega)} \\ &\leq \operatorname{Re} a(\mathbf{E}_h,\eta^2\mathbf{E}_h) + \|\kappa\|_{L^\infty}\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} \\ &\quad + 2\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_R\cap\Omega)}\|\nabla\eta\times\mathbf{E}_h\|_{\mathbf{L}^2(B_R\cap\Omega)}.\end{aligned}$$

Young's inequality allows us to have

$$\begin{aligned}\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} &\leq \|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} \\ &\leq \operatorname{Re} a(\mathbf{E}_h,\eta^2\mathbf{E}_h) + \|\kappa\|_{L^\infty}\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} \\ &\quad + \frac{1}{2}\|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} + 2\|\nabla\eta\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)}.\end{aligned} \tag{6.3.14}$$

Kicking back the term $\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_R\cap\Omega)}$ to the left-hand side, we arrive at

$$\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} \leq \|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)} \leq 2\operatorname{Re} a(\mathbf{E}_h,\eta^2\mathbf{E}_h) + 2(\|\kappa\|_{L^\infty} + 2\,\|\nabla\eta\|^2_{L^\infty})\,\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\,. \tag{6.3.15}$$

Since $\|\kappa\|_{L^\infty} + \|\nabla\eta\|^2_{L^\infty} \lesssim (\varepsilon R)^{-2}$ with implied constant depending on κ, we are left with estimating $\operatorname{Re} a(\eta\mathbf{E}_h,\eta\mathbf{E}_h)$.

Step 2: Using the orthogonality relation in the definition of the space $\mathcal{H}_{c,h}(B_{(1+\varepsilon)R}\cap\Omega)$, we get

$$\operatorname{Re} a(\mathbf{E}_h,\eta^2\mathbf{E}_h - \mathfrak{r}_h(\eta^2\mathbf{E}_h)) \lesssim \|\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\left\|\nabla\times\left(\eta^2\mathbf{E}_h - \mathfrak{r}_h(\eta^2\mathbf{E}_h)\right)\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\left\|\eta^2\mathbf{E}_h - \mathfrak{r}_h(\eta^2\mathbf{E}_h)\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\,. \tag{6.3.16}$$

For each element T, Lemma 6.3.13 yields

$$\left\|\eta^2\mathbf{E}_h - \mathfrak{r}_h(\eta^2\mathbf{E}_h)\right\|^2_{\mathbf{L}^2(T)} + \left\|\nabla\times\left(\eta^2\mathbf{E}_h - \mathfrak{r}_h(\eta^2\mathbf{E}_h)\right)\right\|^2_{\mathbf{L}^2(T)} \lesssim h^2\left(|\eta^2\mathbf{E}_h|^2_{\mathbf{H}^1(T)} + |\nabla\times(\eta^2\mathbf{E}_h)|^2_{\mathbf{H}^1(T)}\right). \tag{6.3.17}$$

To proceed further, we observe that $\mathbf{E}_h|_T \in \mathcal{N}_1(T)$ has the form $\mathbf{E}_h = \mathbf{a} + \mathbf{b}\times\mathbf{x}$ so that $\operatorname{curl}\mathbf{E}_h|_T = 2\mathbf{b}$ and hence $\sum_{j=1}^3 |\partial_{x_j}\mathbf{E}_h| \lesssim |\nabla\times\mathbf{E}_h|$ pointwise on T so that we get with an implied constant independent of the function η

$$\sum_{|l|=1}\left\|\eta D^l\mathbf{E}_h\right\|_{\mathbf{L}^2(T)} \lesssim \|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(T)}\,. \tag{6.3.18}$$

Using (6.3.18) we obtain

$$|\eta^2\mathbf{E}_h|_{\mathbf{H}^1(T)} \lesssim \frac{1}{\varepsilon R}\,\|\mathbf{E}_h\|_{\mathbf{L}^2(T)} + \|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(T)}. \tag{6.3.19}$$

Computing $\nabla\times(\eta^2\mathbf{E}_h) = \nabla\eta^2\times\mathbf{E}_h + \eta^2\nabla\times\mathbf{E}_h$, using the product rule and the fact that $D^l(\nabla\times\mathbf{E}_h) = 0$ since $\nabla\times\mathbf{E}_h$ is constant gives again in view of (6.3.18)

$$|\nabla\times(\eta^2\mathbf{E}_h)|_{\mathbf{H}^1(T)} \lesssim \frac{1}{(\varepsilon R)^2}\,\|\mathbf{E}_h\|_{\mathbf{L}^2(T)} + \frac{1}{\varepsilon R}\,\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(T)}\,. \tag{6.3.20}$$

Summing the squares of (6.3.19), (6.3.20) over all elements T with $T\cap\operatorname{supp}\eta\neq\emptyset$, which is ensured if we sum over all T with $T\subset B_{(1+\varepsilon)R}\cap\Omega$, and inserting the result in (6.3.17) yields

$$\operatorname{Re} a(\mathbf{E}_h,\eta^2\mathbf{E}_h - \mathfrak{r}_h(\eta^2\mathbf{E}_h)) \lesssim \left(\|\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right)\times \frac{h}{\varepsilon R}\left(\frac{1}{\varepsilon R}\,\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \|\eta(\nabla\times\mathbf{E}_h)\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right).$$

Using the definition of the norm $\|\!|\cdot|\!\|_{c,h,R}$ and Young's inequality, we obtain

$$\begin{aligned}\operatorname{Re} a(\mathbf{E}_h, \eta^2\mathbf{E}_h - \mathfrak{r}_h(\eta^2\mathbf{E}_h)) &\lesssim \varepsilon^{-2}\left(\frac{1}{R^2}\|\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{h^2}{R^2}\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right)\\ &\quad + \varepsilon^{-1}\left(\frac{h}{R}\|\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right)\\ &\lesssim \varepsilon^{-2}\|\!|\mathbf{E}_h|\!\|^2_{c,h,(1+\varepsilon)R} + \varepsilon^{-1}\|\!|\mathbf{E}_h|\!\|_{c,h,(1+\varepsilon)R}\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.\end{aligned}$$

Inserting this in (6.3.15) produces

$$\begin{aligned}\|\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} &\leq \|\eta\nabla\times\mathbf{E}_h\|^2_{\mathbf{L}^2(\Omega)}\\ &\lesssim \varepsilon^{-2}\|\!|\mathbf{E}_h|\!\|^2{}_{c,h,(1+\varepsilon)R} + \varepsilon^{-1}\|\!|\mathbf{E}_h|\!\|_{c,h,(1+\varepsilon)R}\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.\end{aligned}$$

Using again Young's inequality to kick the term $\|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}$ of the right-hand side back to the left-hand side produces the desired estimate. ☐

For functions in $\mathcal{H}_{g,h}(B_{(1+\varepsilon)R}\cap\Omega)$, a discrete Caccioppoli-type estimate has already been established in [FMP15, Lem. 2], which we, for sake of completeness, state and prove in the following.:

Lemma 6.3.17. *Let $\varepsilon\in(0,1)$ and $R\in(0,2\operatorname{diam}(\Omega))$ be such that $\frac{h}{R}<\frac{\varepsilon}{4}$. Let B_R and $B_{(1+\varepsilon)R}$ be two concentric boxes and $p_h\in\mathcal{H}_{g,h}(B_{(1+\varepsilon)R}\cap\Omega)$. Then, there exists a constant $C>0$ depending only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$ such that*

$$\|\nabla p_h\|_{\boldsymbol{L}^2(B_R\cap\Omega)} \leq C\frac{1+\varepsilon}{\varepsilon}\|\!|p_h|\!\|_{g,h,(1+\varepsilon)R}.$$

Proof. The proof follows from [FMP15, Lem. 2] and we only mention the key parts.

Let $I^{\mathrm{SZ}}\colon H^1_0(\Omega)\to S^{1,1}_0(\mathcal{T})$ be the Scott-Zhang projection given in [SZ90a]. Let $\eta\in S^{1,1}(\mathcal{T})$ be a piecewise linear cut-off function with $\operatorname{supp}\eta\subseteq\overline{B_{(1+\varepsilon/2)R}\cap\Omega}$, $0\leq\eta\leq1$, $\eta\equiv1$ on $B_R\cap\Omega$, and $\|\nabla\eta\|_{L^\infty(B_{(1+\varepsilon)R}\cap\Omega)}\lesssim\frac{1}{\varepsilon R}$. First, we notice $\operatorname{supp}(\eta^2p_h)\subseteq\overline{B_{(1+\varepsilon)R}\cap\Omega}$ and since $4h\leq\varepsilon R$, then we conclude $\operatorname{supp} I^{\mathrm{SZ}}(\eta^2p_h)\subseteq\overline{B_{(1+\varepsilon)R}\cap\Omega}$. Then, in view of $p_h\in\mathcal{H}_{g,h}(B_{(1+\varepsilon)R}\cap\Omega)$, we can estimate

$$\begin{aligned}\|\nabla p_h\|^2_{\mathbf{L}^2(B_R\cap\Omega)} &\lesssim \|\nabla(\eta p_h)\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} \qquad (6.3.21)\\ &\lesssim \left\langle\nabla p_h, \nabla(\eta^2p_h - I^{\mathrm{SZ}}(\eta^2p_h))\right\rangle_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \|(\nabla\eta)p_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\\ &\lesssim \|\nabla p_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\left\|\nabla(\eta^2p_h - I^{\mathrm{SZ}}(\eta^2p_h))\right\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{1}{(\varepsilon R)^2}\|p_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}. \qquad (6.3.22)\end{aligned}$$

The first term on the right-hand side can be estimated in the same way as in [FMP15, Eq. (25)], i.e.,

$$\left\|\nabla(\eta^2p_h - I^{\mathrm{SZ}}(\eta^2p_h))\right\|^2_{\mathbf{L}^2(\Omega)} \lesssim \frac{h^2}{(\varepsilon R)^2}\|\eta\nabla p_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{h^2}{(\varepsilon R)^4}\|p_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}. \qquad (6.3.23)$$

Therefore, applying Young's inequality we have

$$\begin{aligned}\|\nabla(\eta p_h)\|^2_{L^2(B_{(1+\varepsilon)R}\cap\Omega)} \lesssim &\|\nabla p_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} \left(\frac{h}{(\varepsilon R)^2}\|p_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{h}{\varepsilon R}\|\eta\nabla p_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right)\\ \le &C\left(\frac{h^2}{(\varepsilon R)^2}\|\nabla p_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \frac{1}{(\varepsilon R)^2}\|p_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right)\\ &+\frac{1}{2}\|\eta\nabla p_h\|^2_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}. \end{aligned} \tag{6.3.24}$$

Moving the last term in the right hand side of (6.3.24) to the left-hand side and inserting this estimate into (6.3.21), we get the desired result. □

6.4 Low-dimensional approximation of discrete $\mathcal{L}$-harmonic functions

In this subsection, we apply the Caccioppoli-type estimates from Lemmas 6.3.16 and 6.3.17 to find approximations of the Galerkin solutions from low-dimensional spaces. As an important tool in this section, we first start with the Poincaré inequality as given in [GT77, (7.45)]. Let D be an open subset of Ω. Then, for $u \in H^1(\Omega)$ we have

$$\left\|u - \frac{1}{|D|}\int_D u\,dx\right\|_{L^2(\Omega)} \lesssim |D|^{-2/3}(\mathrm{diam}(D))^3\,\|\nabla u\|_{L^2(\Omega)} \tag{6.4.1}$$

In the following, we consider low-dimensional approximation of discrete harmonic functions in Lemma 6.4.1 that generalizes [FMP15, Lem. 4].

Lemma 6.4.1. *Let $\varepsilon \in (0,1)$, $q \in (0,1)$, $R \in (0, 2\,\mathrm{diam}(\Omega))$, and $m \in \mathbb{N}$ satisfy*

$$\frac{h}{R} \le \frac{q\varepsilon}{8m\max\{1, C_{\mathrm{app}}\}}, \tag{6.4.2}$$

where the constant C_{app} is given in [FMP15, Lem. 3, Lem. 4] and depends only on Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$. Let B_R, $B_{(1+\varepsilon)R}$, $B_{(1+2\varepsilon)R}$ be concentric boxes. Then, there exists a subspace W_m of $\mathcal{H}_{g,h}(B_R\cap\Omega)$ of dimension

$$\dim W_m \le C'_{\mathrm{dim}}\left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4,$$

with the following approximation properties:

(i) If $u_h \in \mathcal{H}_{g,h}(B_{(1+\varepsilon)R}\cap\Omega)$ and $\overline{B_{(1+\varepsilon)R}}\cap\Omega^c = \emptyset$ then

$$\min_{\widetilde{u}_m\in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \le C'_{\mathrm{app}} q^m \varepsilon^{-1}\,\|\nabla u_h\|_{\boldsymbol{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}.$$

(ii) If $u_h \in \mathcal{H}_{g,h}(B_{(1+2\varepsilon)R}\cap\Omega)$ and $\overline{B_{(1+\varepsilon)R}}\cap\Omega^c \neq \emptyset$ then

$$\min_{\widetilde{u}_m\in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \le C'_{\mathrm{app}} q^m \varepsilon^{-3}\,\|\nabla u_h\|_{\boldsymbol{L}^2(B_{(1+2\varepsilon)R}\cap\Omega)}.$$

Here, C'_{dim}, C'_{app} depend only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$.

Proof. We distinguish two cases.
Case 1: Let $\overline{B_{(1+\varepsilon)R}} \cap \Omega^c \neq \emptyset$. For the Lipschitz domain Ω in [Ste70, Chap. VI, Sec. 3, Thm. 5'] asserts the existence of a bounded linear extension operator $\mathcal{E}_{\Omega^c} : H^1(\Omega^c) \to H^1(\mathbb{R}^3)$ such that $\mathcal{E}_{\Omega^c} v|_{\Omega^c} = v$ for each $v \in H^1(\Omega^c)$. Then, the fact that Ω^c is Lipschitz (see [HKT08, Thm. 2] for details) implies the existence of a constant $c > 0$ depending only on Ω such that for all $x \in \Omega^c$ and all $r \in (0,1)$ we have $|B_r(x) \cap \Omega^c| \geq cr^3$, where $B_r(x)$ denotes the ball of radius r centered at x. Selecting an $x \in B_{(1+\varepsilon)R} \cap \Omega^c$ and noting that $B_{\varepsilon R/2}(x) \subset B_{(1+2\varepsilon)R}$, we conclude

$$|B_{(1+2\varepsilon)R} \cap \Omega^c| \geq |B_{\varepsilon R}(x) \cap \Omega^c| \geq c(\varepsilon R)^3.$$

Due to (6.4.2), [FMP15, Lem. 4] provides a subspace W_m of $\mathcal{H}_{g,h}(B_R \cap \Omega)$ such that

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \leq q^m |||u_h|||_{g,h,(1+\varepsilon)R}, \tag{6.4.3}$$

with dimension

$$\dim W_m \leq C_{\text{dim}} \left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4,$$

where C_{dim} depends only on Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$. We denote by $\widehat{u}_h$ the extension by zero of u_h to Ω^c. It follows from the Poincaré inequality as given in [GT77, (7.45)] and $\left|B_{(1+2\varepsilon)R} \cap \Omega^c\right| \geq c(\varepsilon R)^3$ that

$$\begin{aligned}
\frac{1}{\varepsilon R} \|u_h\|_{L^2(B_{(1+\varepsilon)R} \cap \Omega)} &\leq \frac{1}{\varepsilon R} \|u_h\|_{L^2(B_{(1+2\varepsilon)R} \cap \Omega)} = \frac{1}{\varepsilon R} \|\widehat{u}_h\|_{L^2(B_{(1+2\varepsilon)R})} \\
&\lesssim \frac{\left|B_{(1+2\varepsilon)R}\right|}{\varepsilon R \left|B_{(1+2\varepsilon)R} \cap \Omega^c\right|^{2/3}} \|\nabla \widehat{u}_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R})} \\
&\lesssim \frac{(1+2\varepsilon)^3 R^3}{(\varepsilon R)^3} \|\nabla \widehat{u}_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R})} \lesssim \varepsilon^{-3} \|\nabla \widehat{u}_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R})}. \qquad (6.4.4)
\end{aligned}$$

Combining (6.4.4) and (6.4.3) leads to

$$\min_{v_m \in W_m} |||u_h - v_m|||_{g,h,R} \lesssim \varepsilon^{-3} q^m \|\nabla u_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R} \cap \Omega)}. \tag{6.4.5}$$

Case 2: Let $\overline{B_{(1+\varepsilon)R}} \cap \Omega^c = \emptyset$. We note that constant functions are in $\mathcal{H}_{g,h}(B_R \cap \Omega)$. Hence, by [FMP15, Lem. 4] there is a subspace $W_m \subset \mathcal{H}_{g,h}(B_R \cap \Omega)$ such that $1 \in W_m$ and

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} = \min_{\widetilde{u}_m \in W_m,\, c \in \mathbb{R}} |||u_h - \widetilde{u}_m + c|||_{g,h,R} \leq q^m \min_{c \in \mathbb{R}} |||u_h - c|||_{g,h,(1+\varepsilon)R} \tag{6.4.6}$$

with dimension

$$\dim W_m \leq C_{\text{dim}} \left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4 + 1 \lesssim \left(\frac{1+\varepsilon^{-1}}{q}\right)^3 m^4.$$

A standard Poincaré inequality (i.e., (6.4.1) with $D = B_{(1+\varepsilon)R}$) implies

$$\begin{aligned}
\min_{c\in\mathbb{R}} |||u_h - c|||_{g,h,(1+\varepsilon)R} &\leq \left|\left|\left| u_h - \frac{1}{|B_{(1+\varepsilon)R}|}\int_{B_{(1+\varepsilon)R}} u_h \right|\right|\right|_{g,h,(1+\varepsilon)R} \\
&\lesssim \frac{|B_{(1+\varepsilon)R}|}{\varepsilon R\,|B_{(1+\varepsilon)R}|^{2/3}} \|\nabla u_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + h\,\|\nabla u_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} \\
&\lesssim \varepsilon^{-1}\,\|\nabla u_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\,. \qquad (6.4.7)
\end{aligned}$$

Combining (6.4.7) and (6.4.6) completes the proof. □

Remark 6.4.2. The factor ε^{-3} instead of ε^{-1} for boxes B_R near the boundary is due to us not assuming a relation between the orientation of the boxes and the boundary. Aligning boxes with the boundary allows one to better exploit boundary conditions and improve the factor ε^{-3}. ■

In the following, we will need a simplified version of Lemma 6.4.1:

Corollary 6.4.3. *Let $R \in (0, 2\operatorname{diam}(\Omega))$, $\varepsilon \in (0,1)$, $q \in (0,1)$. There are constants C''_{dim} and C''_{app} depending only on Ω and the γ-shape regularity of the quasiuniform triangulation $\mathcal{T}$ such that for any concentric boxes B_R, $B_{(1+2\varepsilon)R}$ and any $m \in \mathbb{N}$ there exists a subspace $W_m \subset \mathcal{H}_{g,h}(B_R \cap \Omega)$ of dimension*

$$\dim W_m \leq C''_{\text{dim}}(\varepsilon q)^{-3} m^4$$

such that for any $u_h \in \mathcal{H}_{g,h}(B_{(1+2\varepsilon)R} \cap \Omega)$ there holds

$$\min_{\widetilde{u}_m \in W_m} |||u_h - \widetilde{u}_m|||_{g,h,R} \leq C''_{\text{app}} q^m \varepsilon^{-3}\,\|\nabla u_h\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R}\cap\Omega)}\,. \qquad (6.4.8)$$

Proof. The case that the parameters satisfy (6.4.2) is covered by Lemma 6.4.1. For the converse case $h/R > q\varepsilon/(8m\max\{1, C_{\text{app}}\})$ we take $W_m := \mathcal{H}_{g,h}(B_R \cap \Omega)$ so that the minimum in (6.4.8) is zero and observe in view of the quasi-uniformity of $\mathcal{T}$

$$\dim \mathcal{H}_{g,h}(B_R \cap \Omega) \lesssim \left(\frac{R}{h}\right)^3 \lesssim \left(\frac{m}{\varepsilon q}\right)^3 = (\varepsilon q)^{-3} m^3 \leq (\varepsilon q)^{-3} m^4.$$

□

If $\mathbf{E}_h$ is locally discrete divergence-free, then the function $\nabla(p + \varphi_{\mathbf{z}})$ in the decomposition $\mathbf{E}_h = \mathbf{z} - \nabla\varphi_{\mathbf{z}} + \nabla(p + \varphi_{\mathbf{z}})$ given by Definition 6.3.10 is also locally discrete divergence-free since $\mathbf{z} - \nabla\varphi_{\mathbf{z}}$ is divergence-free. The following lemma shows that also $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(p + \varphi_{\mathbf{z}})$ is discrete divergence-free:

Lemma 6.4.4. *Let $\varepsilon \in (0,1)$, $R \in (0, 2\operatorname{diam}(\Omega))$, and let $B_{(1+j\varepsilon)R}$, $j \in \{0,1,2\}$, be concentric boxes. Introduce $\mathcal{T}(B_{(1+2\varepsilon)R} \cap \Omega)$ and $\widetilde{B}_{(1+2\varepsilon)R}$ according to Definition 6.3.9. Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function with $\eta \equiv 1$ on $\widetilde{B}_{(1+2\varepsilon)R}$. Let $\boldsymbol{E}_h$ be such that $\eta\boldsymbol{E}_h \in \boldsymbol{H}_0(\operatorname{curl}, \Omega)$ and $\boldsymbol{E}_h \in \mathcal{H}_{c,h}(B_{(1+2\varepsilon)R} \cap \Omega)$. Decompose $\eta\boldsymbol{E}_h \in \boldsymbol{H}_0(\operatorname{curl}, \Omega)$ as $\eta\boldsymbol{E}_h = \boldsymbol{z} + \nabla p$*

with $\boldsymbol{z} \in \boldsymbol{H}_0^1(\Omega)$ and $p \in H_0^1(\Omega)$ according to Lemma 6.3.5. Let the mapping $\varphi_{\boldsymbol{z}} : \boldsymbol{H}_0^1(\Omega) \to H_0^1(\Omega)$ be defined according to (6.3.5) taking $\widetilde{\eta} \equiv \eta$ there. Then, $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(p + \varphi_{\boldsymbol{z}})$ is discrete divergence-free on $\widetilde{B}_{(1+2\varepsilon)R}$, i.e.,

$$\langle \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(p+\varphi_{\boldsymbol{z}}), \nabla v_h \rangle_{\boldsymbol{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} = 0 \qquad \forall v_h \in S^{1,1}(\mathcal{T}, \widetilde{B}_{(1+2\varepsilon)R}), \quad \operatorname{supp} v_h \subset \widetilde{B}_{(1+2\varepsilon)R}. \tag{6.4.9}$$

Proof. To see (6.4.9), we use $\mathbf{E}_h \in \mathcal{H}_{c,h}(B_{(1+2\varepsilon)R} \cap \Omega)$ and (6.3.8) so that for $v_h \in S^{1,1}(\mathcal{T}, \widetilde{B}_{(1+2\varepsilon)R})$ with $\operatorname{supp} v_h \subset \overline{\widetilde{B}_{(1+2\varepsilon)R}}$ we have

$$\begin{aligned}
0 = a(\mathbf{E}_h, \nabla v_h) &= \langle \nabla \times \mathbf{E}_h, \nabla \times \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} - \kappa \langle \mathbf{E}_h, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&= -\kappa \langle \mathbf{E}_h, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} = -\kappa \langle \eta \mathbf{E}_h, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} = -\kappa \langle \mathbf{z} + \nabla p, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&= -\kappa \langle \mathbf{z} - \nabla \varphi_{\mathbf{z}} + \nabla \varphi_{\mathbf{z}} + \nabla p, \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&= -\kappa \langle (\mathbf{z} - \nabla \varphi_{\mathbf{z}}) + \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} (\nabla \varphi_{\mathbf{z}} + \nabla p), \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \\
&\overset{\text{Lemma 6.3.6}}{=} -\kappa \langle \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} (\nabla \varphi_{\mathbf{z}} + \nabla p), \nabla v_h \rangle_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})},
\end{aligned}$$

which finishes the proof. □

We will make use of the orthogonal projection

$$\boldsymbol{\Pi}_{B_R} : (\mathbf{H}(\mathrm{curl}, B_R \cap \Omega), |||\cdot|||_{c,h,R}) \to (\mathcal{H}_{c,h}(B_R \cap \Omega), |||\cdot|||_{c,h,R}). \tag{6.4.10}$$

Lemma 6.4.5 (Single-step approximation)**.** *Let $\varepsilon \in (0,1)$, $R > 0$ be such that $(1+4\varepsilon)R \in (0, R_{\max}]$, and $q \in (0,1)$. Let $B_{(1+j\varepsilon)R}$, $j = 0, \dots, 4$, be concentric boxes. Then there exists a family of linear spaces $\boldsymbol{V}_{H,m} \subset \mathcal{H}_{c,h}(B_R \cap \Omega)$ (parameterized by $H > 0$, $m \in \mathbb{N}$) with the following approximation properties: For each $\boldsymbol{E}_h \in \mathcal{H}_{c,h}(B_{(1+4\varepsilon)R} \cap \Omega)$ there is a $\boldsymbol{E}_{1,h} \in \boldsymbol{V}_{H,m} \subset \mathcal{H}_{c,h}(B_R \cap \Omega)$ with*

(i) $(\boldsymbol{E}_h - \boldsymbol{E}_{1,h})|_{B_R \cap \Omega} \in \mathcal{H}_{c,h}(B_R \cap \Omega)$,

(ii) $|||\boldsymbol{E}_h - \boldsymbol{E}_{1,h}|||_{c,h,R} \leq C''_{\mathrm{app}} \left(\frac{H}{R}\varepsilon^{-1} + q^m \varepsilon^{-4}\right) |||\boldsymbol{E}_h|||_{c,h,(1+4\varepsilon)R}$,

(iii) $\dim \boldsymbol{V}_{H,m} \leq C''_{\mathrm{dim}} \left[\left(\frac{R}{H}\right)^3 + (\varepsilon q)^{-3} m^4 \right]$,

where the constants C''_{app} and C''_{dim} depend only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$. Furthermore,

(iv) if $h \geq H$ or $h/R \geq \varepsilon/4$ one may actually take $\boldsymbol{V}_{H,m} = \mathcal{H}_{c,h}(B_R \cap \Omega)$ and $\boldsymbol{E}_{1,h}$ may be taken as $\boldsymbol{E}_{1,h} = \boldsymbol{E}_h|_{B_R \cap \Omega}$.

Proof. **Step 1:** (reduction to $h < H$) As a preliminary step, we show (iv) so that afterwards we may restrict our attention to the case $h < H$ together with $h/R < \varepsilon/4$. If $h \geq H$ or

$h/R \geq \varepsilon/4$, we take $\mathbf{V}_{H,m} := \mathcal{H}_{c,h}(B_R \cap \Omega)$, which implies that the choice $\mathbf{E}_{1,h} = \mathbf{E}_h|_{B_R \cap \Omega}$ is admissible so that $\mathbf{E}_h - \mathbf{E}_{1,h} = 0$. Since either $h \geq H$ or $h/R \geq \varepsilon/4$, we have

$$\dim \mathcal{H}_{c,h}(B_R \cap \Omega) \lesssim \left(\frac{R}{h}\right)^3 \lesssim \left(\frac{R}{H}\right)^3 + \varepsilon^{-3} \lesssim \left(\frac{R}{H}\right)^3 + (\varepsilon q)^{-3}, \tag{6.4.11}$$

which shows that the complexity bound in (iii) is satisfied. We have thus shown (iv) and will assume $h < H$ and $h/R < \varepsilon/4$ for the remainder of the proof.

Step 2: (reduction to $H/R \geq \varepsilon/4$) We assume in the remainder that $\frac{H}{R} \leq \frac{\varepsilon}{4}$. For $\frac{H}{R} > \frac{\varepsilon}{4}$, we may take the space constructed below with the choice $\frac{H}{R} = \frac{\varepsilon}{4}$ since then, the approximation property (ii) and the complexity estimate (iii) are still satisfied.

Step 3: (Scott-Zhang approximation on $\mathbb{R}^3$) Let $\mathcal{M}_H$ be a quasi-uniform infinite triangulation of $\mathbb{R}^3$ with mesh width H. Define further $\mathbf{S}^{1,1}(\mathcal{M}_H) := \{\mathbf{p}_H \in \mathbf{H}^1(\mathbb{R}^3) : \mathbf{p}_H|_M \in (\mathcal{P}_1(M))^3 \quad \forall M \in \mathcal{M}_H\}$. We will use the Scott–Zhang projection operator $\mathbf{I}_H^{\mathrm{SZ}} \colon \mathbf{H}^1(\mathbb{R}^3) \to \mathbf{S}^{1,1}(\mathcal{M}_H)$ introduced in [SZ90a]. Denoting ω_M the element patch of $M \in \mathcal{M}_H$, this operator has the local approximation property

$$\left\|\mathbf{U} - \mathbf{I}_H^{\mathrm{SZ}}\mathbf{U}\right\|^2_{\mathbf{L}^2(M)} \leq CH^2 \left\|\mathbf{U}\right\|^2_{\mathbf{H}^1(\omega_M)} \qquad \forall \mathbf{U} \in \mathbf{H}^1(\omega_M) \tag{6.4.12}$$

with a constant C depending only on Ω and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{M}_H$. Let $\boldsymbol{\mathcal{E}} \colon \mathbf{H}^1(\Omega) \to \mathbf{H}^1(\mathbb{R}^3)$ be an $\mathbf{H}^1$-stable extension operator such as the one from [Ste70, Chap. VI, Sec. 3, Thm. 5'].

Step 4: Let $\mathcal{T}(B_{(1+2\varepsilon)R} \cap \Omega)$ and $\widetilde{B}_{(1+2\varepsilon)R}$ be given according to Definition 6.3.9. Let $\eta \in C^\infty(\overline{\Omega})$ be a cut-off function with $\operatorname{supp} \eta \subseteq \overline{B_{(1+3\varepsilon)R} \cap \Omega}$, $\eta \equiv 1$ on $\widetilde{B}_{(1+2\varepsilon)R}$, $0 \leq \eta \leq 1$ and $\left\|\nabla^\ell \eta\right\|_{L^\infty(\Omega)} \lesssim \frac{1}{(\varepsilon R)^\ell}$ for $\ell \in \{0,1,2\}$. Note that $\eta \mathbf{E}_h \in H_0(\mathrm{curl}, \Omega)$. Decompose $\eta \mathbf{E}_h \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ as $\eta \mathbf{E}_h = \mathbf{z} + \nabla p$ with $\mathbf{z} \in \mathbf{H}_0^1(\Omega)$ and $p \in H_0^1(\Omega)$ according to Lemma 6.3.5. Let $\varphi_{\mathbf{z}}$ be given by (6.3.5) taking $\widetilde{\eta} = \eta$ there. Select representers p_h, $\varphi_{\mathbf{z},h} \in S_0^{1,1}(\mathcal{T})$ such that $\nabla p_h = \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}(\nabla p)$ and $\nabla \varphi_{\mathbf{z},h} = \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla \varphi_{\mathbf{z}}$ on $\widetilde{B}_{(1+2\varepsilon)R}$. By Lemma 6.4.4 we have that $\nabla(p_h + \varphi_{\mathbf{z},h})$ is discrete divergence-free on $\widetilde{B}_{(1+2\varepsilon)R}$ so that $(p_h + \varphi_{\mathbf{z},h}) \in \mathcal{H}_{g,h}(B_{(1+2\varepsilon)R} \cap \Omega)$. We apply Corollary 6.4.3 with the pair (R, ε) replaced with $(\widetilde{R}, \widetilde{\varepsilon}) = (R(1+\varepsilon), \frac{\varepsilon}{2(1+\varepsilon)})$ to get a subspace $W_m \subset \mathcal{H}_{g,h}(B_{(1+\varepsilon)R} \cap \Omega)$ for the box $B_{(1+\varepsilon)R} \cap \Omega$ and an $w_m \in W_m$ such that

$$|||p_h + \varphi_{\mathbf{z},h} - w_m|||_{g,h,(1+\varepsilon)R} \lesssim q^m \varepsilon^{-3} \|\nabla(p_h + \varphi_{\mathbf{z},h})\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R} \cap \Omega)}. \tag{6.4.13}$$

Step 5: Define $\mathbf{z}_H := (\mathbf{I}_H^{SZ} \boldsymbol{\mathcal{E}} \mathbf{z})|_{B_{(1+4\varepsilon)R} \cap \Omega}$. Using Definition 6.3.10 and with the function $\varphi_{\mathbf{z}_H}$ given by (6.3.5) (again, with $\widetilde{\eta} = \eta$ there) we have the representation

$$\begin{aligned}
\mathbf{E}_h|_{\widetilde{B}_{(1+2\varepsilon)R}} &= \mathbf{z}_h + \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}(\nabla p) = (\mathbf{z}_h - \mathbf{z}) + \mathbf{z} - \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla \varphi_{\mathbf{z}} + \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(\varphi_{\mathbf{z}} + p) \\
&= (\mathbf{z}_h - \mathbf{z}) + (\mathbf{z} - \mathbf{z}_H) - \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}(\nabla \varphi_{\mathbf{z}} - \nabla \varphi_{\mathbf{z}_H}) \\
&\quad - \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla \varphi_{\mathbf{z}_H} + \mathbf{z}_H + \Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(\varphi_{\mathbf{z}} + p),
\end{aligned}$$

Of these 6 terms, the first three terms will be seen to be small, the next two terms are from a low-dimensional space, and the last term is exponentially close to ∇w_m by (6.4.13), which is also from a low-dimensional space, namely, ∇W_m. As the approximation of $\mathbf{E}_h$, we thus take

$$\mathbf{E}_{1,h} := \mathbf{\Pi}_{B_R}\left(-\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H} + \mathbf{z}_H + \nabla w_m\right), \tag{6.4.14}$$

with the $|||\cdot|||_{c,h,R}$-orthogonal projection $\mathbf{\Pi}_{B_R}$ of (6.4.10). Property (i) is then satisfied by construction. In order to prove (ii), we compute using the definition of the norm $|||\cdot|||_{c,h,R}$

$$\begin{aligned}
|||\mathbf{E}_h - \mathbf{E}_{1,h}|||_{c,h,R} &= \left|\left|\left|\mathbf{\Pi}_{B_R}\left(\mathbf{E}_h + \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H} - \mathbf{z}_H - \nabla w_m\right)\right|\right|\right|_{c,h,R} \\
&\leq \left|\left|\left|\mathbf{E}_h + \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla\varphi_{\mathbf{z}_H} - \mathbf{z}_H - \nabla w_m\right|\right|\right|_{c,h,R} \\
&\leq |||\mathbf{z}_h - \mathbf{z}|||_{c,h,R} + |||\mathbf{z} - \mathbf{z}_H|||_{c,h,R} + \left|\left|\left|\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}(\nabla\varphi_{\mathbf{z}} - \nabla\varphi_{\mathbf{z}_H})\right|\right|\right|_{c,h,R} \\
&\quad + \left|\left|\left|\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(p + \varphi_{\mathbf{z}}) - \nabla w_m\right|\right|\right|_{c,h,R}.
\end{aligned} \tag{6.4.15}$$

Step 6: The stability estimate (6.3.9) for p_h in the local discrete regular decomposition implies together with Lemma 6.3.5

$$\|\nabla p_h\|_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} + \|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \overset{(6.3.9)}{\lesssim} \|\mathbf{z}\|_{\mathbf{L}^2(\Omega)} + \|\nabla p\|_{\mathbf{L}^2(\Omega)} \lesssim \|\eta\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}, \tag{6.4.16}$$

By Lemma 6.3.15 and the fact that $\frac{1}{\varepsilon R} \lesssim 1$ the Caccioppoli-type estimate of Lemma 6.3.16 (replacing the pairs (R,ε) there with suitably adjusted $(\widetilde{R},\widetilde{\varepsilon})$ as needed), we have

$$\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl},\widetilde{B}_{(1+2\varepsilon)R})} + \|\mathbf{z}\|_{\mathbf{H}^1_0(\Omega)} \lesssim \|\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)} + \frac{1}{\varepsilon R}\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)}, \tag{6.4.17}$$

$$\overset{\text{Lemma 6.3.16}}{\lesssim} \varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}, \tag{6.4.18}$$

Finally, combining Lemmas 6.3.5, 6.3.6, and (6.3.9) leads to

$$\|\nabla\varphi_{\mathbf{z}}\|_{\mathbf{L}^2(\Omega)} + \|\nabla\varphi_{\mathbf{z},h}\|_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \lesssim \|\mathbf{z}\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)} \lesssim \|\eta\mathbf{E}_h\|_{\mathbf{L}^2(\Omega)}, \tag{6.4.19}$$

$$\|\nabla(\varphi_{\mathbf{z}} - \varphi_{\mathbf{z}_H})\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{z} - \mathbf{z}_H\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)}. \tag{6.4.20}$$

Step 7: (controlling $\mathbf{z} - \mathbf{z}_h$) By Lemma 6.3.15 and (6.4.18) we have

$$\frac{1}{R}\|\mathbf{z} - \mathbf{z}_h\|_{\mathbf{L}^2(B_R\cap\Omega)} \lesssim \frac{h}{R}\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}. \tag{6.4.21}$$

Noting $\nabla\times\mathbf{z} = \nabla\times(\eta\mathbf{E}_h)$ together with the definition of $|||\cdot|||_{c,h,R}$ and the estimate (6.4.21), we obtain

$$\begin{aligned}|||\mathbf{z}-\mathbf{z}_h|||_{c,h,R} &\leq \frac{h}{R}\left(\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl},\widetilde{B}_{(1+2\varepsilon)R})} + \|\nabla\times\mathbf{z}\|_{\mathbf{L}^2(B_R\cap\Omega)}\right) + \frac{1}{R}\|\mathbf{z}-\mathbf{z}_h\|_{\mathbf{L}^2(B_R\cap\Omega)} \\ &\leq \frac{h}{R}\left(\|\mathbf{z}_h\|_{\mathbf{H}(\mathrm{curl},\widetilde{B}_{(1+2\varepsilon)R})} + \frac{1}{\varepsilon R}\|\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)} + \|\eta\nabla\times\mathbf{E}_h\|_{\mathbf{L}^2(B_{(1+\varepsilon)R}\cap\Omega)}\right) \\ &\quad + \frac{h}{R}\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R} \end{aligned} \tag{6.4.22}$$

Combining this with Lemma 6.3.16 and the stability estimate (6.4.18) gives rise to

$$|||\mathbf{z}-\mathbf{z}_h|||_{c,h,R} \lesssim \frac{h}{R}\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}. \tag{6.4.23}$$

Step 8: (controlling $\mathbf{z}-\mathbf{z}_H$ and $\nabla(\varphi_\mathbf{z}-\varphi_{\mathbf{z}_H})$) For $\mathbf{z}_H = (\mathbf{I}_H^{SZ}\boldsymbol{\mathcal{E}}\mathbf{z})|_{B_{(1+4\varepsilon)R}\cap\Omega}$ we have by the approximation result (6.4.12), the assumption $H/R\leq\varepsilon$, and the stability properties of $\mathbf{I}_H^{SZ}$

$$\frac{1}{R}\|\mathbf{z}-\mathbf{z}_H\|_{\mathbf{L}^2(B_{(1+j\varepsilon)R}\cap\Omega)} \lesssim \frac{H}{R}\|\boldsymbol{\mathcal{E}}\mathbf{z}\|_{\mathbf{H}^1(B_{(1+(j+1)\varepsilon)R})}, \qquad j=0,\dots,3, \tag{6.4.24}$$

$$\frac{h}{R}\|\mathbf{z}-\mathbf{z}_H\|_{\mathbf{H}^1((B_{(1+j\varepsilon)R}\cap\Omega))} \lesssim \frac{h}{R}\|\boldsymbol{\mathcal{E}}\mathbf{z}\|_{\mathbf{H}^1(B_{(1+(j+1)\varepsilon)R})}, \qquad j=0,\dots,3, \tag{6.4.25}$$

so that, using $\|\boldsymbol{\mathcal{E}}\mathbf{z}\|_{\mathbf{H}_{(1+4\varepsilon)R}} \lesssim \|\mathbf{z}\|_{\mathbf{H}^1(\Omega)}$, we obtain for $j=0,\dots,3$

$$|||\mathbf{z}-\mathbf{z}_H|||_{c,h,(1+j\varepsilon)R} \lesssim \left(\frac{h}{R}+\frac{H}{R}\right)\|\boldsymbol{\mathcal{E}}\mathbf{z}\|_{\mathbf{H}^1(B_{(1+(j+1)\varepsilon)R})} \overset{(6.4.18)}{\lesssim} \left(\frac{h}{R}+\frac{H}{R}\right)\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}. \tag{6.4.26}$$

By the stability properties of the operator $\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}$ given in (6.3.9) and (6.4.20) we infer

$$\begin{aligned}\left|\left|\left|\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(\varphi_\mathbf{z}-\varphi_{\mathbf{z}_H})\right|\right|\right|_{c,h,R} &\leq \frac{1}{R}\|\nabla(\varphi_\mathbf{z}-\varphi_{\mathbf{z}_H})\|_{\mathbf{L}^2(\widetilde{B}_{(1+2\varepsilon)R})} \overset{(6.4.20)}{\leq} \frac{1}{R}\|\mathbf{z}-\mathbf{z}_H\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R}\cap\Omega)} \\ &\overset{(6.4.26)}{\lesssim} \left(\frac{h}{R}+\frac{H}{R}\right)\varepsilon^{-1}|||\mathbf{E}_h|||_{c,h,(1+4\varepsilon)R}. \end{aligned} \tag{6.4.27}$$

Step 9: (Estimate $\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(p+\varphi_\mathbf{z})-\nabla w_m$) By Step 1, we have $p_h+\varphi_{\mathbf{z},h}-w_m\in\mathcal{H}_{g,h}(B_{(1+\varepsilon)R}\cap\Omega)$. Noting $\Pi^\nabla_{\widetilde{B}_{(1+2\varepsilon)R}}\nabla(p+\varphi_\mathbf{z})-\nabla w_m = \nabla(p_h+\varphi_{\mathbf{z},h}-w_m)$ on $B_{(1+\varepsilon)R}\cap\Omega$

we get

$$
\begin{aligned}
\left|\!\left|\!\left| \Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla(p+\varphi_{\mathbf{z}}) - \nabla w_m \right|\!\right|\!\right|_{c,h,R} &= \frac{1}{R} \left\| \nabla(p_h + \varphi_{\mathbf{z},h} - w_m) \right\|_{\mathbf{L}^2(B_R \cap \Omega)} \\
&\overset{\text{Lemma 6.3.17}}{\lesssim} \frac{1+\varepsilon}{\varepsilon R} |\!|\!| (p_h + \varphi_{\mathbf{z},h}) - w_m |\!|\!|_{g,h,(1+\varepsilon)R} \\
&\overset{(6.4.13)}{\lesssim} \frac{q^m \varepsilon^{-3}(1+\varepsilon)}{\varepsilon R} \left\| \nabla(p_h + \varphi_{\mathbf{z},h}) \right\|_{\mathbf{L}^2(B_{(1+2\varepsilon)R} \cap \Omega)} \\
&\overset{(6.4.16),(6.4.19)}{\lesssim} \frac{q^m \varepsilon^{-3}}{\varepsilon R} \left\| \eta \mathbf{E}_h \right\|_{\mathbf{L}^2(\Omega)} \\
&\lesssim \frac{q^m \varepsilon^{-3}}{\varepsilon R} \left\| \mathbf{E}_h \right\|_{\mathbf{L}^2(B_{(1+3\varepsilon)R} \cap \Omega)} \\
&\lesssim q^m \varepsilon^{-4} |\!|\!| \mathbf{E}_h |\!|\!|_{c,h,(1+3\varepsilon)R}. \qquad (6.4.28)
\end{aligned}
$$

Substituting (6.4.23), (6.4.26), (6.4.27) and (6.4.28) into (6.4.15) concludes the proof of (ii).

Step 10: By construction, the approximation $\mathbf{E}_{1,h}$ of (6.4.14) is from the space

$$\mathbf{V}_{H,m} := \{ \mathbf{\Pi}_{B_R} (\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}} \nabla \varphi_{\mathbf{z}_H} + \mathbf{z}_H + \nabla w_m) \,:\, \mathbf{z}_H \in (\mathbf{I}_H^{\mathrm{SZ}} \mathbf{H}^1(\mathbb{R}^3))|_{B_{(1+4\varepsilon)R} \cap \Omega},\ w_m \in \nabla W_m \}.$$

By the linearity of the maps $\mathbf{\Pi}_{B_R}$, $\Pi^{\nabla}_{\widetilde{B}_{(1+2\varepsilon)R}}$, and $\mathbf{z} \mapsto \varphi_{\mathbf{z}}$, the space $\mathbf{V}_{H,m}$ is a linear space. In view of $\dim W_m \lesssim (\varepsilon q)^{-3} m^4$ from Corollary 6.4.3 and $\dim \mathbf{I}_H^{\mathrm{SZ}} \boldsymbol{\mathcal{E}}(\mathbf{H}^1(\Omega))|_{B_{(1+4\varepsilon)R} \cap \Omega} \lesssim \left(\frac{(1+4\varepsilon)R}{H}\right)^3$ we get (iii). □

Lemma 6.4.6 (multi-step approximation)**.** *Let $\zeta \in (0,1)$, $q' \in (0,1)$, $R \in (0, R_{\max}]$. Then, for each $k \in \mathbb{N}$ there exists a subspace $\boldsymbol{V}_k$ of $\mathcal{H}_{c,h}(B_R \cap \Omega)$ of dimension*

$$\dim \boldsymbol{V}_k \le C'''_{\dim} k \left(\frac{k}{\zeta}\right)^3 \left(q'^{-3} + \ln^4 \frac{k}{\zeta}\right), \qquad (6.4.29)$$

such that for $\boldsymbol{E}_{c,h} \in \mathcal{H}_h(B_{(1+\zeta)R} \cap \Omega)$

$$\min_{\widetilde{\boldsymbol{E}}_k \in \boldsymbol{V}_k} \left|\!\left|\!\left| \boldsymbol{E}_h - \widetilde{\boldsymbol{E}}_k \right|\!\right|\!\right|_{c,h,R} \le q'^k |\!|\!| \boldsymbol{E}_h |\!|\!|_{h,(1+\zeta)R}. \qquad (6.4.30)$$

Here, $C'''_{\dim}$ depends only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$.

Proof. The proof relies on iterating the approximation result of Lemma 6.4.5 on boxes $B_{(1+\varepsilon_j)R}$, where $\varepsilon_j = \zeta(1 - \frac{j}{k})$ for $j = 0, \dots, k$. We note that $\zeta = \varepsilon_0 > \varepsilon_1 > \dots > \varepsilon_k = 0$. Define

$$\widetilde{R}_j := R(1+\varepsilon_j), \qquad \widetilde{\varepsilon}_j := \frac{\zeta}{4k(1+\varepsilon_j)} < \frac{1}{4}$$

and note the relationship $B_{(1+4\widetilde{\varepsilon}_j)\widetilde{R}_j} = B_{\widetilde{R}_{j-1}} = B_{R(1+\varepsilon_{j-1})}$ as well as $B_{\widetilde{R}_k} = B_R$ and $B_{\widetilde{R}_0} = B_{R(1+\zeta)}$. Also note

$$\frac{\zeta}{8k} \le \frac{\zeta}{4k(1+\zeta)} \le \widetilde{\varepsilon}_j \le \frac{\zeta}{4k}, \qquad R \le \widetilde{R}_j \le (1+\zeta)R, \qquad j = 0, \dots, k.$$

Select $q \in (0,1)$. With the constant C''_{app} of Lemma 6.4.5 choose

$$H := \frac{1}{8}\frac{q' R\zeta}{k \max\{1, C''_{\text{app}}\}}, \qquad m := \left\lceil \frac{4\ln(\zeta/(4k)) - \ln\max\{1, C''_{\text{app}}\} + \ln(q'/2)}{\ln q} \right\rceil.$$

These constants are chosen such that

$$C''_{\text{app}} \frac{H}{\widetilde{\varepsilon}_j \widetilde{R}_j} \leq \frac{1}{2}q' \qquad \text{and} \qquad C''_{\text{app}} \widetilde{\varepsilon}_j^{-4} q^m \leq \frac{1}{2}q'. \tag{6.4.31}$$

Moreover, the assumption $R \leq R_{\max}$ implies that $(1 + 4\widetilde{\varepsilon}_j)\widetilde{R}_j = R(1 + \varepsilon_{j-1}) \leq R_{\max}$. Therefore, Lemma 6.4.5 provides a space $\mathbf{V}^1_{H,m} \subset \mathcal{H}_{c,h}(B_{\widetilde{R}_1} \cap \Omega)$ and an approximation $\mathbf{E}_{1,h} \in \mathbf{V}^1_{H,m}$ with

$$|||\mathbf{E}_h - \mathbf{E}_{1,h}|||_{c,h,\widetilde{R}_1} \leq C''_{\text{app}} \left(\frac{H}{\widetilde{\varepsilon}_1 \widetilde{R}_1} + \widetilde{\varepsilon}_1^{-4} q^m \right) |||\mathbf{E}_h|||_{c,h,\widetilde{R}_0} \overset{(6.4.31)}{\leq} q' |||\mathbf{E}_h|||_{c,h,\widetilde{R}_0}, \tag{6.4.32}$$

$$\dim \mathbf{V}^1_{H,m} \lesssim \left(\left(\frac{\widetilde{R}_1}{H} \right)^3 + (\widetilde{\varepsilon}_1 q)^{-3} m^4 \right) \leq C \left(\frac{k}{\zeta} \right)^3 \left(q'^{-3} + \ln^4(k/\zeta) \right),$$

where the constant $C > 0$ is independent of $j \in \{0, \dots, k\}$, ζ, k, and q'. Since $\mathbf{E}_h - \mathbf{E}_{1,h} \in \mathcal{H}_{c,h}(B_{\widetilde{R}_1} \cap \Omega)$, we may apply Lemma 6.4.5 again to find a space $\mathbf{V}^2_{H,m} \subset \mathcal{H}_{c,h}(B_{\widetilde{R}_2} \cap \Omega)$ and an approximation $\mathbf{E}_h \in \mathbf{V}^2_{H,m}$ with $\dim \mathbf{V}^2_{H,m} \leq C(k/\zeta)^3 \left(q'^{-3} + \ln^4(k/\zeta) \right)$ such that

$$|||\mathbf{E}_h - \mathbf{E}_{1,h} - \mathbf{E}_{2,h}|||_{c,h,\widetilde{R}_2} \leq q' \, |||\mathbf{E}_h - \mathbf{E}_{1,h}|||_{c,h,\widetilde{R}_1} \leq q'^2 \, |||\mathbf{E}_h|||_{c,h,\widetilde{R}_0}.$$

Repeating this process $k-2$ times leads to the approximation $\widetilde{\mathbf{E}}_k = \sum_{i=1}^k \mathbf{E}_{i,h}$ in the space $\mathbf{V}_k := \sum_{i=1}^k \mathbf{V}^i_{H,m}$ of dimension

$$\dim \mathbf{V}_k \leq Ck(k/\zeta)^3 \left(q'^{-3} + \ln^4(k/\zeta) \right),$$

which concludes the proof. □

6.5 Proof of the main results

The results of the preceding Section 6.4 allow us to show that the Galerkin approximation $\mathbf{E}_h$ of (6.1.22) can be approximated from low-dimensional spaces in regions B_{R_τ} away from the support of the right-hand side $\mathbf{F}$.

Theorem 6.5.1. *Let $h_0 > 0$ be given by Lemma 6.1.1 and let $\mathcal{T}$ be a quasi-uniform mesh with mesh size $h \leq h_0$. Fix $q \in (0,1)$ and $\eta > 0$. Set $\zeta = 1/(1+\eta)$. For every cluster pair (τ, σ) with bounding boxes B_{R_τ} and B_{R_σ} with $\eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \geq \operatorname{diam}(B_{R_\tau})$ and every each $k \in \mathbb{N}$ there exists a space $\boldsymbol{V}_k \subset \boldsymbol{L}^2(B_{R_\tau} \cap \Omega)$ with*

$$\dim \boldsymbol{V}_k \leq \widetilde{C}_{\dim} k(k/\zeta)^3 \left(q^{-3} + \ln^4(k/\zeta) \right) \tag{6.5.1}$$

such that for arbitrary right-hand side $\boldsymbol{F} \in \boldsymbol{L}^2(\Omega)$ with $\operatorname{supp} \boldsymbol{F} \subset B_{R_\sigma} \cap \overline{\Omega}$, the corresponding Galerkin solution $\boldsymbol{E}_h$ of (6.1.22) *can be approximated from $\boldsymbol{V}_k$ such that*

$$\min_{\widetilde{\boldsymbol{E}}_k \in \boldsymbol{V}_k} \left\| \boldsymbol{E}_h - \widetilde{\boldsymbol{E}}_k \right\|_{\boldsymbol{L}^2(B_{R_\tau}\cap\Omega)} \leq C_{\mathrm{box}} q^k \left\| \boldsymbol{\Pi}_h^{L^2} \boldsymbol{F} \right\|_{\boldsymbol{L}^2(\Omega)} \leq C_{\mathrm{box}} q^k \left\| \boldsymbol{F} \right\|_{\boldsymbol{L}^2(B_{R_\sigma}\cap\Omega)}.$$

Here, $\boldsymbol{\Pi}_h^{L^2}$ is the $\boldsymbol{L}^2$-orthogonal projection onto $\boldsymbol{X}_h(\mathcal{T},\Omega)$ and C_{box}, $\widetilde{C}_{\mathrm{dim}}$ are constants depending only on κ, Ω, and the shape-regularity of $\mathcal{T}$.

Proof. From Lemma 6.1.1 we have the *a priori* estimate

$$\|\mathbf{E}_h\|_{\mathbf{H}(\mathrm{curl},\Omega)} \leq C \|\boldsymbol{\Pi}_h^{L^2}\mathbf{F}\|_{\mathbf{L}^2(\Omega)} \leq C\|\mathbf{F}\|_{\mathbf{L}^2(\Omega)} = C\|\mathbf{F}\|_{\mathbf{L}^2(B_\sigma\cap\Omega)}.$$

From $\operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) \geq \eta^{-1} \operatorname{diam} B_{R_\tau}$ the choice $\zeta = 1/(1+\eta)$ implies

$$\operatorname{dist}(B_{(1+\zeta)R_\tau}, B_{R_\sigma}) \geq \operatorname{dist}(B_{R_\tau}, B_{R_\sigma}) - \zeta R_\tau \sqrt{3} \geq \sqrt{3} R_\tau (\eta^{-1} - \zeta) = \sqrt{3} R_\tau \frac{1}{\eta(\eta+1)} > 0.$$

Hence, the Galerkin solution $\mathbf{E}_h$ satisfies $\mathbf{E}_h|_{B_{(1+\zeta)R_\tau}\cap\Omega} \in \mathcal{H}_{c,h}(B_{(1+\zeta)R_\tau}\cap\Omega)$. Since $\frac{h}{R_\tau} \lesssim 1$, it is immediate that

$$|||\mathbf{E}_h|||_{h,(1+\zeta)R} \lesssim \left(1 + \frac{1}{R_\tau}\right) \|\mathbf{E}_h\|_{\mathbf{H}(\mathrm{curl},\Omega)} \lesssim \left(1 + \frac{1}{R_\tau}\right) \left\|\boldsymbol{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)}. \tag{6.5.2}$$

In the following, we employ Lemma 6.4.6. In order to do so, boxes have to have smaller side-length than $R_{\max}/2$, which may not hold for general bounding boxes B_{R_τ}. However, as bounding boxes can always be chosen to satisfy $R_\tau < 2\operatorname{diam}(\Omega)$, there exists a constant $L \in \mathbb{N}$ independent of R_τ, such that $R_\tau/L \leq R_{\max}$ with $R_{\max}$ given in Def. 6.3.14. Consequently, we can decompose a box $B_{R_\tau} = \operatorname{int}\left(\bigcup_{\ell=1}^{C_L} B_{R_{\tau_\ell}}\right)$ into $C_L \in \mathbb{N}$ subboxes $\left\{B_{R_{\tau_\ell}}\right\}_{\ell=1}^{C_L}$ of side-length R_{τ_ℓ} such that $R_{\tau_\ell} \leq R_{\max}$, where C_L does only depend on L. Then, for each $B_{R_{\tau_\ell}}$, Lemma 6.4.6 provides a space $\mathbf{V}_{k,\ell} \subset \mathcal{H}_{c,h}(B_{R_{\tau_\ell}} \cap \Omega)$, whose dimension is bounded by (6.4.29) such that

$$\begin{aligned}\min_{\widetilde{\mathbf{E}}_{k,\ell}\in\mathbf{V}_{k,\ell}} \left\|\mathbf{E}_h - \widetilde{\mathbf{E}}_{k,\ell}\right\|_{\mathbf{L}^2(B_{R_{\tau_\ell}}\cap\Omega)} &\leq R_{\tau_\ell} \min_{\widetilde{\mathbf{E}}_{k,\ell}\in\mathbf{V}_{k,\ell}} \left|\left|\left|\mathbf{E}_h - \widetilde{\mathbf{E}}_{k,\ell}\right|\right|\right|_{c,h,R_{\tau_\ell}} \leq C q^k (R_{\tau_\ell}+1) \left\|\boldsymbol{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)} \\ &\lesssim \operatorname{diam}(\Omega) q^k \left\|\boldsymbol{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

Now, we define the space $\mathbf{V}_k$ as a subspace of $\mathbf{L}^2(B_{R_\tau}\cap\Omega)$ by simply combining all the spaces $\mathbf{V}_{k,\ell}$ of the subboxes, i.e., we extend functions in $\mathbf{V}_{k,\ell}$ by zero to the larger box B_{R_τ} and write $\widehat{\mathbf{V}}_{k,\ell}$ for this space. Then, we can define $\mathbf{V}_k := \sum_{\ell=1}^{C_L} \widehat{\mathbf{V}}_{k,\ell}$ and set $\widetilde{\mathbf{E}}_k|_{B_{R_{\tau_\ell}}} := \widetilde{\mathbf{E}}_{k,\ell} \in \mathbf{V}_{k,\ell}$ for $\widetilde{\mathbf{E}}_k \in \mathbf{V}_k$. This gives

$$\begin{aligned}\min_{\widetilde{\mathbf{E}}_k\in\mathbf{V}_k} \left\|\mathbf{E}_h - \widetilde{\mathbf{E}}_k\right\|_{\mathbf{L}^2(B_{R_\tau}\cap\Omega)} &\leq \sum_{\ell=1}^{C_L} \min_{\widetilde{\mathbf{E}}_{k,\ell}\in\mathbf{V}_{k,\ell}} \left\|\mathbf{E}_h - \widetilde{\mathbf{E}}_{k,\ell}\right\|_{\mathbf{L}^2(B_{R_{\tau_\ell}}\cap\Omega)} \\ &\lesssim C_L q^k \left\|\boldsymbol{\Pi}_h^{L^2}\mathbf{F}\right\|_{\mathbf{L}^2(\Omega)}.\end{aligned}$$

The dimension of $\mathbf{V}_k$ is bounded by

$$\dim \mathbf{V}_k \leq C_L\, C'''_{\mathrm{dim}} k \left(\frac{k}{\zeta}\right)^3 \left(q'^{-3} + \ln^4 \frac{k}{\zeta}\right),$$

which concludes the proof. □

The following result allows us to transfer the approximation result Theorem 6.5.1 to the matrix level. We recall that the system matrix $\mathbf{A}$ is given by (6.1.23).

Lemma 6.5.2. *Let $h \leq h_0$ with h_0 given by Lemma 6.1.1. Then there are constants $\widetilde{C}_{\mathrm{dim}}$, $\widehat{C}_{\mathrm{app}}$ that depend on only on κ, Ω, and the γ-shape regularity of the quasi-uniform triangulation $\mathcal{T}$ such that for $\eta > 0$, $q \in (0,1)$, $k \in \mathbb{N}$, and η-admissible cluster pairs (τ, σ) there exist, for each $k \in \mathbb{N}$, matrices $\mathbf{X}_{\tau\sigma} \in \mathbb{C}^{\tau \times r}$, $\mathbf{Y}_{\tau\sigma} \in \mathbb{C}^{\sigma \times r}$ of rank $r \leq \widetilde{C}_{\mathrm{dim}}\,(1+\eta)^3\, k^4 \left(q^{-3} + \ln^4(k(1+\eta))\right)$ such that*

$$\left\| \mathbf{A}^{-1}|_{\tau\times\sigma} - \mathbf{X}_{\tau\sigma}\mathbf{Y}^H_{\tau\sigma}\right\|_2 \leq \widehat{C}_{\mathrm{app}} h^{-1} q^k.$$

Proof. As a preliminary step, we show that we can reduce the consideration to the case $\operatorname{diam} B_{R_\tau} \leq \eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma})$. Indeed, as $\mathbf{A}$ is symmetric also $\mathbf{A}^{-1}$ is symmetric so that $\mathbf{A}^{-1}|_{\tau\times\sigma} = \mathbf{A}^{-1}|_{\sigma\times\tau}$ and one may approximate either $\mathbf{A}^{-1}|_{\tau\times\sigma}$ or $\mathbf{A}^{-1}|_{\sigma\times\tau}$ by a low-rank matrix. In view of the definition of the admissibility condition (2.6.1), we may therefore assume $\operatorname{diam} B_{R_\tau} \leq \eta \operatorname{dist}(B_{R_\tau}, B_{R_\sigma})$.

The matrices $\mathbf{X}_{\tau\sigma}$ and $\mathbf{Y}_{\tau\sigma}$ will be constructed with the aid of Theorem 6.5.1. In particular, we require in the following the constant $\widetilde{C}_{\mathrm{dim}}$ from Theorem 6.5.1. We distinguish between the cases of "small" blocks and "large" blocks.

Case 1. If $\widetilde{C}_{\mathrm{dim}}(1+\eta)^3 k^4 \left(q^{-3} + \ln^4(k(1+\eta))\right) \geq \min(|\tau|, |\sigma|)$, we use the exact matrix block $\mathbf{X}_{\tau\sigma} = \mathbf{A}^{-1}|_{\tau\times\sigma}$ and we put $\mathbf{Y}_{\tau\sigma} = \mathbf{I}|_{\sigma\times\sigma}$ with $\mathbf{I} \in \mathbb{C}^{N\times N}$ being the identity matrix.

Case 2. If $\widetilde{C}_{\mathrm{dim}}(1+\eta)^3 k^4 \left(q^{-3} + \ln^4(k(1+\eta))\right) < \min(|\tau|, |\sigma|)$, let $\mathbf{V}_k$ be the space constructed in Theorem 6.5.1. From $\mathbf{V}_k$ we construct $\mathbf{X}_{\tau\sigma}$ and $\mathbf{Y}_{\tau\sigma}$ in the following two steps.

Step 1. Let functions $\lambda_i \in \mathbf{L}^2(\Omega)$, $i = 1, \dots, N$, satisfy

$$\operatorname{supp} \lambda_i \subset \operatorname{supp} \Psi_i, \qquad i = 1, \dots, N, \tag{6.5.3a}$$

$$\langle \lambda_i, \Psi_j \rangle_{\mathbf{L}^2(\Omega)} = \delta_{ij}, \qquad i, j = 1, \dots, N, \tag{6.5.3b}$$

$$\|\lambda_i\|_{\mathbf{L}^2(\Omega)} \leq C h^{-1/2}, \qquad i = 1, \dots, N. \tag{6.5.3c}$$

Such a dual basis of $\mathcal{X}_{h,0} := \{\Psi_i \colon i = 1, \dots, N\}$ can be constructed as (discontinuous) piecewise polynomials of degree 1 as described in, e.g., [BS02, Sec. 4.8] for classical Lagrange elements. Let $\mathcal{E}_e := \{\mathbf{e}_1, \cdots, \mathbf{e}_N\}$ be the set of edges corresponding to $\mathcal{X}_{h,0}$. For $\mathbf{e}_i \in \mathcal{E}_e$, $i = 1, \cdots, N$, we define $\widetilde{\mathbf{K}}_i$ as the union of tetrahedra in $\operatorname{supp} \Psi_i$ sharing $\mathbf{e}_i$ as an edge and set $\operatorname{supp} \lambda_i := \widetilde{\mathbf{K}}_i$. Then,

$$\int_{\widetilde{\mathbf{K}}_i} \Psi_j(\mathbf{x}) \lambda_i(\mathbf{x})\, d\mathbf{x} = \delta_{ij} \qquad i,\ j = 1, \dots, N.$$

The constant C depends solely on the γ-shape reguarity of $\mathcal{T}$. We emphasize that our choice of scaling of the functions Ψ_i is reponsible for the factor $h^{-1/2}$.

Define for clusters τ' the mappings

$$\Lambda_{\tau'} : \mathbf{L}^2(\Omega) \to \mathbb{C}^{\tau'}, \quad \mathbf{v} \mapsto \Big(\chi_{\tau'}(i)\langle \lambda_i, \mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}\Big)_{i\in\mathcal{I}},$$

where $\chi_{\tau'}$ is the characteristic function of τ'. For $\mathbf{v} \in \mathbf{L}^2(\Omega)$ and cluster τ' with bounding box $B_{R_{\tau'}}$ we observe for the ℓ^2-norm $\|\cdot\|_2$ on $\mathbb{C}^{\tau'}$ that

$$\|\Lambda_{\tau'}\mathbf{v}\|_2^2 = \sum_{i\in\tau'} |\langle \lambda_i, \mathbf{v}\rangle_{\mathbf{L}^2(\Omega)}|^2 \le \sum_{i\in\tau'} \|\lambda_i\|^2_{\mathbf{L}^2(\Omega)}\|\mathbf{v}\|^2_{\mathbf{L}^2(\operatorname{supp}\lambda_i)} \overset{(6.5.3c)}{\lesssim} h^{-1}\|\mathbf{v}\|^2_{\mathbf{L}^2(B_{R_{\tau'}}\cap\Omega)}. \tag{6.5.4}$$

We observe that for $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T},\Omega)$ expanded as $\mathbf{E}_h = \sum_{i\in\mathcal{I}} \mu_i\Psi_i$, we have $\mu_i = (\Lambda_{\mathcal{I}}(\mathbf{E}_h))_i$. In particular, we have for the coefficients μ_i with $i \in \tau'$

$$\mu_i = (\Lambda_{\tau'}(\mathbf{E}_h))_i. \tag{6.5.5}$$

Step 2: Let $\mathbf{V}_k$ be the space given by Theorem 6.5.1 for the boxes B_{R_τ}, B_{R_σ}. For arbitrary $\mathbf{b} \in \mathbb{C}^\sigma$, define the function $f_{\mathbf{b}} := \sum_{i\in\sigma} \mathbf{b}_i\lambda_i$ and observe:

$$\operatorname{supp} f_{\mathbf{b}} \overset{(6.5.3a)}{\subset} B_{R_\sigma}, \tag{6.5.6a}$$

$$\|f_{\mathbf{b}}\|_{\mathbf{L}^2(\Omega)} \overset{(6.5.4)}{\lesssim} h^{-1/2}\|\mathbf{b}\|_2, \tag{6.5.6b}$$

$$\langle f_{\mathbf{b}}, \Psi_i\rangle_{\mathbf{L}^2(\Omega)} \overset{(6.5.3b)}{=} \mathbf{b}_i, \qquad i = 1,\dots,N. \tag{6.5.6c}$$

Let $\mathbf{E}_h \in \mathbf{X}_{h,0}(\mathcal{T},\Omega)$ be the Galerkin solution corresponding to the right-hand side $f_{\mathbf{b}}$ and $\widetilde{\mathbf{E}}_h \in \mathbf{V}_k$ be the approximation to $\mathbf{E}_h$ asserted in Theorem 6.5.1. Then,

$$\begin{aligned}\left\|\Lambda_\tau\mathbf{E}_h - \Lambda_\tau\widetilde{\mathbf{E}}_k\right\|_2 &\overset{(6.5.4)}{\lesssim} h^{-1/2}\left\|\mathbf{E}_h - \widetilde{\mathbf{E}}_h\right\|_{\mathbf{L}^2(B_{R_\tau}\cap\Omega)} \\ &\overset{\text{Thm. 6.5.1}}{\lesssim} h^{-1/2}q^k\|f_{\mathbf{b}}\|_{\mathbf{L}^2(\Omega)} \overset{(6.5.6b)}{\lesssim} h^{-1}q^k\|\mathbf{b}\|_2.\end{aligned}$$

We define the low-rank factor $\mathbf{X}_{\tau\sigma}$ as an orthogonal basis of the space $\mathcal{V}_\tau := \{\Lambda_\tau(\widetilde{\mathbf{E}}_k) : \widetilde{\mathbf{E}}_k \in \mathbf{V}_k\}$ and set $\mathbf{Y}_{\tau\sigma} := \mathbf{A}^{-1}|^H_{\tau\times\sigma}\mathbf{X}_{\tau\sigma}$. Then, the rank of $\mathbf{X}_{\tau\sigma}$ is bounded by $\dim\mathbf{V}_k \le \widetilde{C}_{\dim}(1+\eta)^3k^4\left(q^{-3}+\ln^4(k(1+\eta))\right)$. Since $\mathbf{X}_{\tau\sigma}\mathbf{X}^H_{\tau\sigma}$ is the orthogonal projection from $\mathbb{C}^N$ onto $\mathcal{V}_\tau$, we conclude that $\mathbf{z} := \mathbf{X}_{\tau\sigma}\mathbf{X}^H_{\tau\sigma}(\Lambda_\tau\mathbf{E}_h)$ is the $\|\cdot\|_2$-best approximation of the Galerkin solution in $\mathcal{V}_\tau$, which results in

$$\|\Lambda_\tau\mathbf{E}_h - \mathbf{z}\|_2 \lesssim \left\|\Lambda_\tau\mathbf{E}_h - \Lambda_\tau\widetilde{\mathbf{E}}_h\right\|_2 \lesssim h^{-1}q^k\|\mathbf{b}\|_2.$$

By (6.5.5) and $\mathbf{b} \in \mathbb{C}^\sigma$, we have

$$\Lambda_\tau\mathbf{E}_h \overset{(6.5.5)}{=} (\Lambda_{\mathcal{I}}\mathbf{E}_h)|_\tau = (\mathbf{A}^{-1}\mathbf{b})|_\tau \overset{\mathbf{b}\in\mathbb{C}^\sigma}{=} (\mathbf{A}^{-1}|_{\tau\times\sigma})\mathbf{b}.$$

Since $\mathbf{z} = \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H\mathbf{b}$, we conclude

$$\left\|(\mathbf{A}^{-1}|_{\tau\times\sigma} - \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H)\mathbf{b}\right\|_2 = \|\Lambda_\tau \mathbf{E}_h - \mathbf{z}\|_2 \lesssim h^{-1}q^k \|\mathbf{b}\|_2 .$$

As $\mathbf{b}$ was arbitrary, we obtain the stated norm bound. □

Proof of Theorem 6.2.1. For each admissible cluster pair (τ,σ), let the matrices $\mathbf{X}_{\tau\sigma}$, $\mathbf{Y}_{\tau\sigma}$ be given by Lemma 6.5.2. Define the $\mathcal{H}$-matrix approximation $\mathbf{B}_\mathcal{H}$ by the conditions

$$\mathbf{B}_\mathcal{H}|_{\tau\times\sigma} = \mathbf{X}_{\tau\sigma}\mathbf{Y}_{\tau\sigma}^H \quad \text{if } (\tau,\sigma)\in P_{\text{far}}, \qquad \mathbf{B}_\mathcal{H}|_{\tau\times\sigma} = \mathbf{A}^{-1}|_{\tau\times\sigma} \quad \text{if } (\tau,\sigma)\in P_{\text{near}}.$$

The blockwise estimate of Lemma 6.5.2 for $q\in(0,1)$ and Lemma 2.6.10 yield

$$\begin{aligned}\left\|\mathbf{A}^{-1} - \mathbf{B}_\mathcal{H}\right\|_2 &\le C_{\text{sp}}\left(\sum_{\ell=0}^{\infty}\max\{\left\|(\mathbf{A}^{-1}-\mathbf{B}_\mathcal{H})|_{\tau\times\sigma}\right\|_2 : (\tau,\sigma)\in P, \text{level}(\tau)=\ell\}\right)\\ &\le \widehat{C}_{\text{app}}C_{\text{sp}}\,\text{depth}(\mathbb{T}_\mathcal{I})h^{-1}q^k.\end{aligned}$$

We next relate k to the blockwise rank r. For $y\ge 0$ the unique (positive) solution k of $k\ln k = y$ has the form

$$k = \frac{y}{\log y}(1+o(1)) \qquad \text{as } y\to\infty \tag{6.5.7}$$

by, e.g., [Olv97, Ex. 5.7, Chap. 1]. In passing, we mention that even higher order asymptotics can directly be inferred from the asymptotics of Lambert's W-function as described in [dB61, p. 25–27]. The asymptotics (6.5.7) implies that the solution k of $k^4\ln^4 k = y$ satisfies $k = y^{1/4}/\ln(y^{1/4})(1+o(1))$ as $y\to\infty$.

From Lemma 6.5.2 we have the rank bound $r \le C_{\text{dim}}(1+\eta)^3k^4\left(q^{-3}+\ln^4(k(1+\eta))\right) \le C_{\text{dim}}\left((1+\eta)q^{-1}\right)^3 k^4\ln^4 k$, so that for suitable b, $C>0$ independent of r we get $q^k \le C\exp(-br^{1/4}/\ln r)$. Consequently, we have

$$\left\|\mathbf{A}^{-1} - \mathbf{B}_\mathcal{H}\right\|_2 \le C_{\text{apx}}C_{\text{sp}}\,\text{depth}(\mathbb{T}_\mathcal{I})h^{-1}e^{-b\left(r^{1/4}/\ln r\right)},$$

which concludes the proof. □

6.6 Numerical results

In this section, in order to validate the theoretical results obtained in this chapter, we study three examples defined on two different geometries.

In order to construct the block partitioning, we use the geometrically balanced cluster tree given in [GHLB04] based on the following modified bounding boxes.

For a basis $\mathcal{X}_{h,0} := \{\Psi_1,\dots,\Psi_N\}$ of $\mathbf{X}_{h,0}(\mathcal{T},\Omega)$ with $N := \dim\mathbf{X}_{h,0}(\mathcal{T},\Omega)$, let $\mathcal{E}_e := \{\mathbf{e}_1,\dots,\mathbf{e}_N\}$ be the set of corresponding edges and $\mathcal{Y}_e := \{\mathbf{y}_1,\dots,\mathbf{y}_N\}$ be the set of midpoints corresponding to each edge. For $h_s>0$, we define $\mathbf{B}_{h_s}(\mathbf{y}_i)\subset\mathbb{R}^d$ $i=1,\dots,N$, as a ball of radius h_s centred at $\mathbf{y}_i$. In the following examples, instead of the definition of bounding boxes from Definition 2.6.3 based on support of the basis functions, we use a slightly modified version:

For a cluster $\tau \subset \mathcal{I} := \{1, ..., N\}$, a bounding box B_{R_τ} is an axis-parallel hypercube with side length R_τ such that $\cup_{i\in\tau}\mathbf{B}_{h_s}(\mathbf{y}_i) \subseteq B_{R_\tau}$.

In the following examples, we select $\eta = 2$ as the admissibility parameter and $n_{\text{leaf}} = 25$ as the leaf size. For the rank bound, we consider the range $r \in \{1, \ldots, 50\}$. In the curl-curl problem (6.1.19), we choose the coefficients to be $\mu = 1$ and $\kappa = 1$.

The $\mathcal{H}$-matrix approximation $\mathbf{B}_\mathcal{H}$ to the inverse of $\mathbf{A}$ is computed by applying a truncated singular valued decomposition of the exact inverse. More precisely, for an admissible block $(\tau, \sigma) \in P_{\text{far}}$, we consider the singular value decomposition $\mathbf{A}^{-1}|_{\tau\times\sigma} = \mathbf{USV}^T \in \mathbb{R}^{\tau\times\sigma}$, where $\mathbf{U} \in \mathbb{R}^{\tau\times\tau}$, $\mathbf{V} \in \mathbb{R}^{\sigma\times\sigma}$ are orthogonal and $\mathbf{S} = \operatorname{diag}(\sigma_1, \cdots, \sigma_m) \in \mathbb{R}^{\tau\times\sigma}$, $m := \min(|\tau|, |\sigma|)$, includes the corresponding singular values $\sigma_1, \cdots, \sigma_m \geq 0$. Then, we set $\mathbf{B}_\mathcal{H}|_{\tau\times\sigma} := \mathbf{U}_r\mathbf{S}_r\mathbf{V}_r^T$ where $\mathbf{U}_r \in \mathbb{R}^{\tau\times r}$, $\mathbf{S}_r \in \mathbb{R}^{r\times r}$ and $\mathbf{V}_r \in \mathbb{R}^{\sigma\times r}$ are the first r columns of $\mathbf{U}$, $\mathbf{S}$ and $\mathbf{V}$, respectively. For $(\tau, \sigma) \in P_{\text{near}}$, we set $\mathbf{B}_\mathcal{H}|_{\tau\times\sigma} := \mathbf{A}^{-1}|_{\tau\times\sigma}$.

The numerical results are implemented in Netgen [Net] and MATLAB, i.e., the stiffness matrix $\mathbf{A}$ is obtained from Netgen and $\mathbf{B}_\mathcal{H}$ is computed in MATLAB.

Example 6.6.1. In this example, we choose $\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \quad : \quad x^2 + y^2 + z^2 < 1\right\}$ as the geometry and take $h_s = h$. The geometry and its mesh configuration are shown in Figure 6.6.1. The block partition including 20 734 admissible, and 52 770 small blocks, is depicted in Figure 6.6.1. In Figure 6.6.2, for a fixed number $N = 16\,971$ of degrees of freedom, we show that $\|\mathbf{I} - \mathbf{AB}_\mathcal{H}\|_2$ decreases as the block-rank increases. Figure 6.6.2 shows the exponential decay of the approximated error. We plot two straight lines obtained by fitting the data (in a least-squares sense) for the computed error values for $1 \leq r \leq 20$ and $r > 20$ (shown by dashed blue and black lines), respectively. For $1 \leq r \leq 20$, the slope of the line is -0.16 and for $r > 20$ is -0.09. The allocated memory is shown in Figure 6.6.2 for different ranks.

Example 6.6.2. We consider the domain $\Omega := (-1, 1) \times (-1, 1) \times [-1, -2) \cup (-2, 2) \times [1, 2) \times (-1, 1) \cup (-2, 2) \times [-1, 1] \times [-1, 1] \cup (-1, 1) \times (-1, 1) \times [1, 2) \cup (-2, 2) \times [-1, 2) \times (-1, 1)$. The geometry is shown in Figure 6.6.3. In Figure 6.6.3, for $h_s = h$, the block partitioning indicates 21 290 admissible blocks and 42 944 small blocks. In Figure 6.6.4, for $h_s = h$, we illustrate the exponential convergence of $\|\mathbf{I} - \mathbf{AB}_\mathcal{H}\|_2$ with respect to the increase of the block-rank for $N = 14\,491$ degrees of freedom and as it is shown in this figure, the decay of the obtained values is exponential. Two straight lines are plotted by fitting the data (in a least-squares sense) for the computed error values for $1 \leq r \leq 20$ and $r > 20$ (shown by dashed blue and black lines), respectively. The slope of the lines are mentioned in the figure as well. For $h_s = h$, we also show the allocated memory [MBytes] in Figure 6.6.4.

Example 6.6.3. For the same geometry as in Example 6.6.2 and a fixed number $N = 15\,491$ of degrees of freedom, Figure 6.6.7 shows that increasing h_s improves convergence of the upper bounds of the error and simultaneously reduces the number of admissible blocks (Figures 6.6.5-6.6.6).

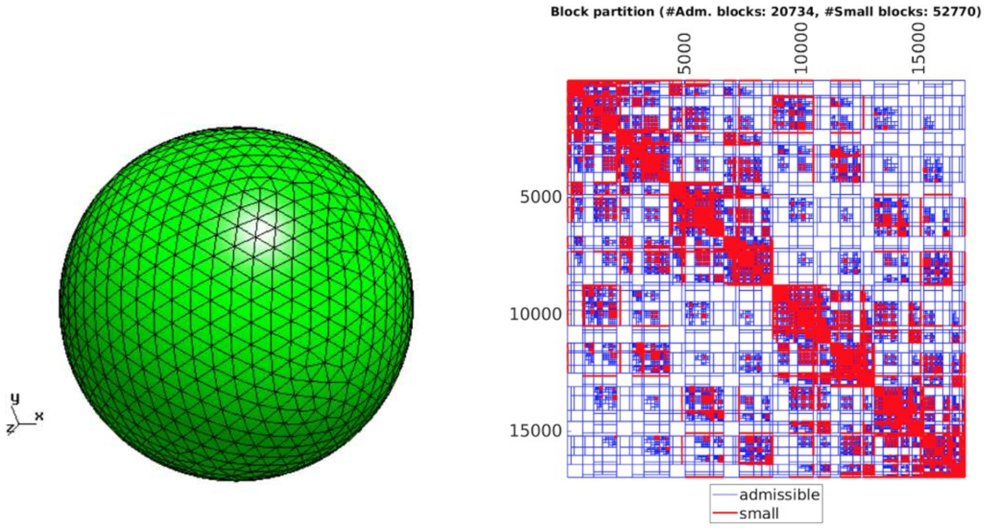

Figure 6.6.1: The mesh $\mathcal{T}$ (left). The block partition for $N = 16\,971$ degrees of freedom and $h_s = h$ (right).

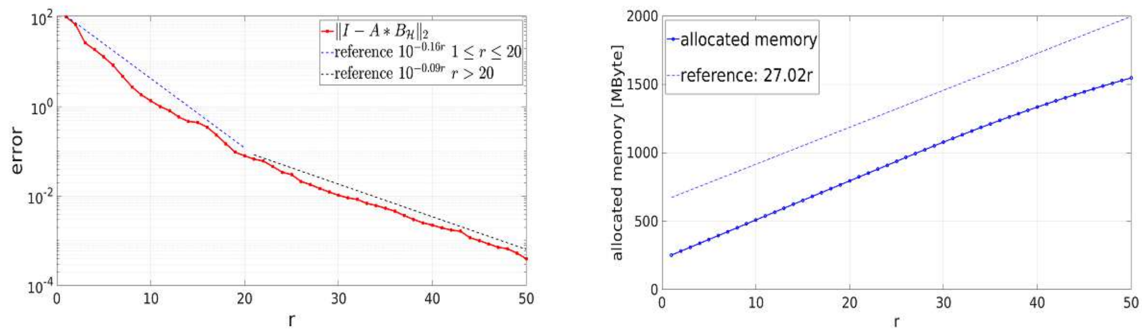

Figure 6.6.2: Approximation error for $N = 16\,971$ degrees of freedom and $h_s = h$ (left). The allocated memory for $N = 16\,971$ degrees of freedom and $h_s = h$ (right).

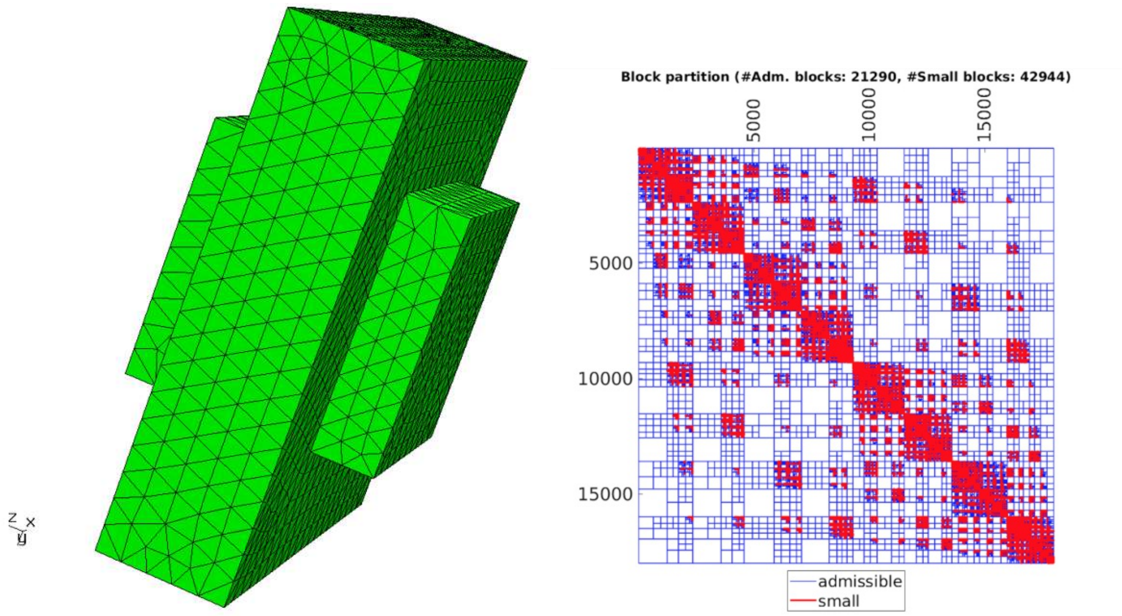

Figure 6.6.3: The mesh $\mathcal{T}$ (left). The block partition for $N = 14\,491$ degrees of freedom and $h_s = h$ (right).

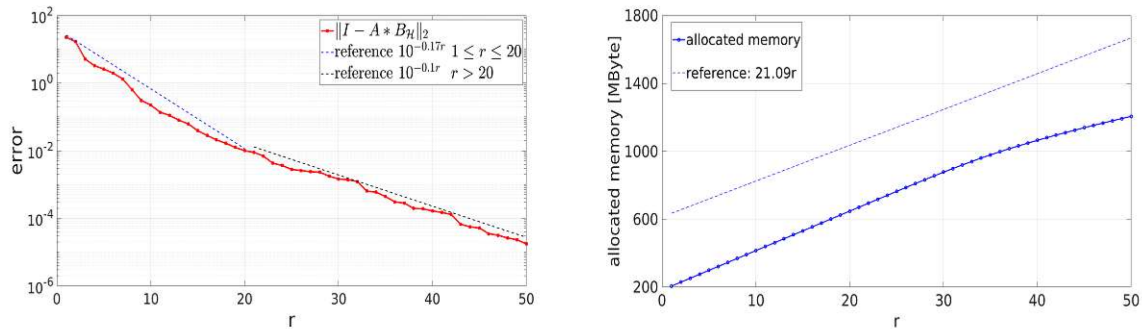

Figure 6.6.4: Approximation error for $N = 14\,491$ degrees of freedom and $h_s = h$ (left) . The allocated memory for $N = 14\,491$ degrees of freedom and $h_s = h$ (right).

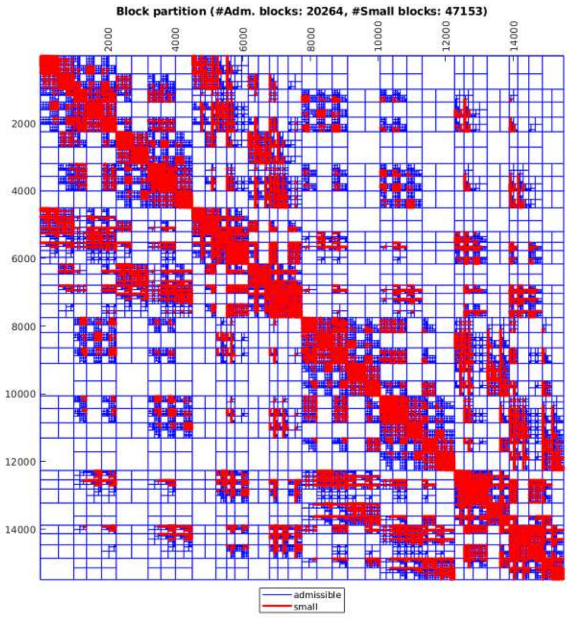

Figure 6.6.5: The block partition for $N = 15\,491$ degrees of freedom, $h = 0.202$ and $h_s = 2h$.

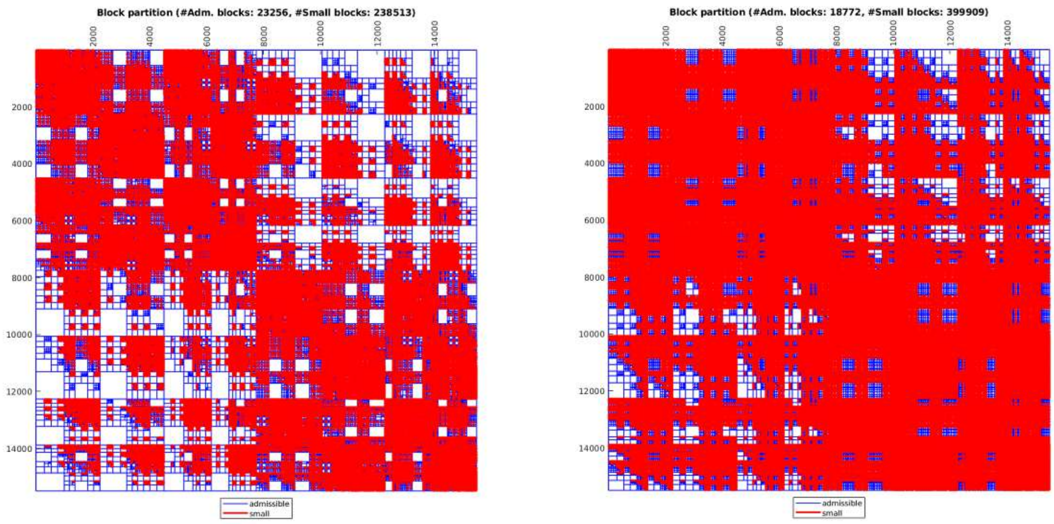

Figure 6.6.6: The block partition for $N = 15\,491$ degrees of freedom; left: $h = 0.202$ and $h_s = 3h$; right: $h = 0.202$ and $h_s = 4h$.

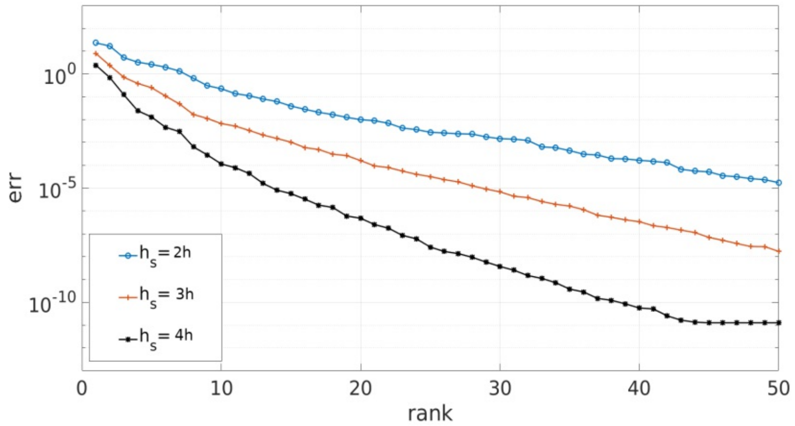

Figure 6.6.7: $\mathcal{H}$-matrix approximation to inverse FEM matrix; error vs. block rank, for $N = 15\,491$ degrees of freedom and $h_s = 2h$, $h_s = 3h$ and $h_s = 4h$.

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Curriculum Vitae

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Selected Aacademic Publications

1. Amirreza Khodadadian, Maryam Parvizi, Mostafa Abbaszadeh, Mehdi Dehghan and Clemens Heitzinger, A direct meshless local collocation method for solving stochastic Cahn-Hilliard and Swift-Hohenberg equations. *Eng. Anal. Bound. Elem.*, 98:253–264, 2019.

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6. Markus Faustmann, Jens Markus Melenk and Maryam Parvizi, On the stability of Scott-Zhang type operators and application to multilevel preconditioning in fractional diffusion. *ESAIM Math. Model. Numer. Anal.*, 55(2):595–625, 2021.

7. Markus Faustmann, Jens Markus Melenk and Maryam Parvizi , Caccioppoli-type estimates and $\mathcal{H}$-Matrix approximations to inverses for FEM-BEM couplings. *submitted to Numer. Math.*, 2020.

8. Markus Faustmann, Jens Markus Melenk and Maryam Parvizi, $\mathcal{H}$-matrix approximability of inverses of FEM matrices for the time-harmonic Maxwell equations., *submitted to Adv. Comput. Math.*, 2021.