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Automation and its Implication for Education and Fertility

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Abstract

The complexity of the relationship between automation, fertility, and education is setting the context for the thesis: As fertility rates decrease, automation is identified as a solution to address the shrinking workforce, although it indirectly influences fertility through changes in employment patterns. According to this, job displacement, altered income levels, and instability in the labor market can affect family planning decisions. Automation's impact on skill groups and wage distribution is discussed in this work, noting how higher wages may lead to reduced fertility rates as career priorities shift. The thesis aims to analyze this complex relationship by incorporating automation and fertility into an overlapping generation (OLG) model, examining how these factors interact in different skill groups. It begins by introducing the OLG model including automation based on Gasteiger and Prettner (2020). This model is the foundation for subsequent chapters. Chapter 3 presents an OLG model following Lankisch (2017), focusing on "skilled" and "unskilled" worker groups. Chapter 4 builds on the previous model, introducing endogenous fertility. Chapter 5, based on Chen (2007), extends the analysis by including endogenous skill investments and endogenous fertility, studying how these factors interact. The thesis concludes that none of the presented models exhibit growth, and it can be inferred that an increase in automation leads to a reduction in birth rates. Furthermore, it is demonstrated that while the skill premium generally rises with increasing automation, the influx of workers into skilled positions due to automation eventually reduces this premium.

Contents

Chapter 1

Introduction

In recent decades, many developed countries have experienced a decrease in fertility rates. Researchers have highlighted the decline in fertility rate as a significant demographic phenomenon. Doepke et al. (2022) state that the factors influencing people's decisions regarding fertility in modern developed economies are different compared to what they used to be in previous decades. For example, Prettner and Abeliansky (2023) note that one of the factors associated with declining fertility rates is the increase of the education level. As individuals pursue higher education and spend more time on career development, they may choose to delay starting a family or have fewer children as a consequence. This relationship between education and fertility rates has important implications for population dynamics, as countries with higher education levels often experience lower birth rates and an aging population. In addition, Prettner and Abeliansky (2023) argue that these declines in fertility rates have consequences for the working-age population in industrialized countries, ultimately slowing down its growth rate.

The combination of increasing longevity and low fertility rates presents challenges that impact economic factors. As the proportion of older individuals in the population increases and simultaneously the workforce becomes smaller, the overall economic output per person may decrease. This can have severe implications for economic growth, as a smaller working-age population may struggle to generate the same level of productivity and output as a larger one. In response to this challenge, there is an increasing reliance on automation to fill this gap and maintain productivity levels.

It seems that automation, as a technological advancement, may not have a direct impact on birth rates. However, its influence on fertility can be observed through indirect means. For example, automation leads to changes in employment patterns. In the OECD countries, it was estimated that around 10–14% of jobs will be fully replaced by robots. Furthermore, an estimated 25-32% of jobs will become 50-70% automated over the next two decades (Arntz et al. (2017), Nedelkoska and Quintini (2018) cited in Matysiak et al. (2022)). Job displacement and changes in the labor market affect income levels and job stability, which, in turn, might influence people's decisions about family planning and having children.

Considering an aging population, Prettner and Bloom (2020) highlight that a diminishing workforce leads to a rising labor demand, resulting in increased wages due to a scarcity of available workers. Consequently, companies seek alternatives to reduce labor dependency and explore cost-effective solutions. Additionally, Prettner and Strulik (2019) propose that advancing technology amplifies the value of high-skilled labor, making it less replaceable to automation compared to low-skilled labor, which emphasizes the conclusion that automation has varying effects on distinct skill groups in the production process. While automation and technological progress contribute to higher productivity and economic growth (as demonstrated by Stähler, 2020), they also lead to shifts in the labor market. Irmen (2021) asserts that in the context of relatively high labor costs, automation becomes increasingly attractive as a method of saving costs. Moreover, if the expenses associated with implementing automation are lower compared to other alternatives, such as hiring more workers, businesses are more likely to invest in automation to enhance efficiency and productivity. However, this progress may also lead to changes in job availability and wage distribution. According to Prettner and Bloom (2020), individuals displaced by automation may experience difficulties in finding employment in other sectors of the economy, and if they do find positions, they often come with reduced wages, potentially leading to wage inequality.

Lower wages for unskilled workers due to automation can also have an impact on the opportunity costs faced by women in the context of fertility decisions. According to Doepke et al. (2022), the cost of having children is not just monetary but also includes the time and effort invested by women, when women are primarily responsible for child-rearing. When women's wages rise, the opportunity cost of their time also increases. This means that potential earnings from working become more valuable, and the foregone income from choosing to raise children becomes more significant. Consequently, higher wages may lead to reduced fertility rates as women prioritize their careers over having more children. Moreover, since automation affects the wages of women differently based on their skill levels ("skilled" and "unskilled" workers), it also leads to varied fertility decisions among these groups. Automation-induced changes in wages impact the fertility decisions of different groups of women. An increase in the level of automation reduces wages for unskilled work, therefore the tradeoff between pursuing work and starting or expanding a family may shift. Lower potential income from work decreases the opportunity costs of having children for women in lower skilled job positions. As a result, women may evaluate the cost of raising a child as lower, potentially influencing their decisions regarding fertility, whereas highly skilled women evaluate the cost of giving birth as higher. Therefore, it can be assumed that fertility might be higher among less skilled women due to the comparatively lower foregone wages, whereas highly skilled women might exhibit lower fertility rates.

To understand how changes in automation *(economic trends)* and fertility *(demographic trends)* influence the economy, it is important to analyze the relationship between these factors. It has to be noted that the relationship between automation, fertility, and education is complex, and there may be additional factors to consider. The thesis aims to contribute to the existing analysis by studying how automation affects fertility and education. It takes into account considerations like changes in factor returns.

The thesis first introduces the canonical overlapping generation model (OLG model) in Chapter 2, which incorporates automation based on the work of Gasteiger and Prettner (2020). This model serves as the foundation for the subsequent chapters. In Chapter 3, the work of Lankisch (2017) is presented, who developed an OLG model focusing on two different skill levels: "skilled" labor and "unskilled" labor. These skill levels are determined exogenously. Building upon the foundations laid out in Chapter 2, Chapter 4 expands the existing OLG model by introducing endogenous fertility. This means that fertility decisions are no longer predetermined but are influenced by the model's variables and mechanisms. In the following, Chapter 5, based on Chen (2007) takes the analysis a step further by considering the aspect of endogenous skill investments in addition to endogenous fertility. This allows for a more intense analysis of the interplay between automation and fertility in different skill groups. The chapter studies how individuals' decisions regarding skill investments and fertility shape the dynamics of the model. The thesis concludes by reviewing and comparing the key findings from each chapter.

Chapter 2

OLG model and automation

2.1 Model assumptions

In this chapter, the focus is on an Overlapping Generations (OLG) model with automation, based on the research by Gasteiger and Prettner (2020). The model aims to analyze the implications of automation for the economy. Since the model is extended in further chapters with endogenous fertility and endogenous skill investment, the chosen framework follows Gasteiger and Prettner (2020, page 4) and is an OLG model with discrete time. Continuous time models, like the Ramsey model, can be more complex and less suitable when focusing on endogenous fertility and skill investment, as the decisions related to family planning and education actually occur within one single period. Therefore the OLG model with discrete time aligns better with the objectives of this thesis.

2.2 The model

2.2.1 Households

In this discrete-time economic model, i.e., $t = 0, 1, 2...$ following Gasteiger and Prettner (2020, page 4), households progress through three distinct stages: youth, adulthood, and retirement. During the youth stage, parents fulfill their children's needs by allocating resources to consumption expenditures. Children are assumed to be dependent on their parents and they lack the ability to make economic decisions.

Later during adulthood, they offer their available time in exchange for a market-clearing wage denoted as w_t. Individuals contribute their labor to the labor market and simultaneously save for their retirement, the wage splits into

$$w_t = c_{1,t} + s_t. \tag{2.1}$$

Once individuals enter the retirement stage, they cease working and no longer engage in labor market activities. Instead, they rely on their accumulated savings from their working years to finance their consumption during old age

$$c_{2,t+1} = (1 + r_{t+1})s_t. \tag{2.2}$$

Furthermore, the impact of exogenous population growth is taken into account in this model using the variable n, which represents a growth rate greater than -1. As a result, the population size in the next period, N_{t+1}, is calculated by $N_{t+1} = (1 + n)N_t$.

The household's lifetime utility is calculated by summing up the utility levels at each stage of life, discounted by an appropriate discount rate to account for the time value of money. It is then maximized by choosing the optimal allocation of resources across different periods of life, considering factors such as income, savings, and inter generational transfers. In this model and according to Diamond (1965) household's lifetime utility is given by

$$U_t = \log(c_{1,t}) + \beta \log(c_{2,t+1}).$$

$c_{1,t}$ represents the utility derived from consumption during adulthood, and $c_{2,t+1}$ represents the utility obtained from consumption during retirement. To ensure analytical feasibility, a logarithmic utility function is used, and households are assumed to discount the future at a rate $\rho > 0$. This discounting is represented by the discount factor β, which is calculated as $\beta = 1/(1 + \rho)$.

Additionally, the model incorporates the real interest rate on savings from time t to time $t + 1$, denoted as r_{t+1}. The standard budget constraint for households can be expressed as follows

$$c_{1,t} + \frac{c_{2,t+1}}{1 + r_{t+1}} = w_t,$$

where the left-hand side represents the present value of the lifetime consumption expenditure, and the right-hand side denotes the lifetime employment income. This equation illustrates the balance between the sum of consumption during adulthood and retirement, discounted by the real interest rate, and the lifetime income earned by the household.

The intertemporal maximization problem of the household can be solved by setting up a Lagrangian

$$\mathcal{L}(.) = \log(c_{1,t}) + \lambda_t(w_t - s_t - c_{1,t}) + \beta\left[\log(c_{2,t+1}) + \lambda_{t+1}((1+r_{t+1})s_t - (c_{2,t+1}))\right].$$

The first-order conditions (FOCs) are

$$\frac{\partial \mathcal{L}(.)}{\partial c_{1,t}} \overset{!}{=} 0 \quad \Leftrightarrow \quad \frac{1}{c_{1,t}} - \lambda_t \overset{!}{=} 0, \tag{2.3}$$

$$\frac{\partial \mathcal{L}(.)}{\partial c_{2,t+1}} \overset{!}{=} 0 \quad \Leftrightarrow \quad \beta(\frac{1}{c_{2,t+1}} - \lambda_{t+1}) \overset{!}{=} 0, \tag{2.4}$$

$$\frac{\partial \mathcal{L}(.)}{\partial s_t} \overset{!}{=} 0 \quad \Leftrightarrow \quad -\lambda_t + \beta(1+r_{t+1})\lambda_{t+1} \overset{!}{=} 0, \tag{2.5}$$

$$\frac{\partial \mathcal{L}(.)}{\partial \lambda_t} \overset{!}{=} 0 \quad \Leftrightarrow \quad w_t - s_t - c_{1,t} \overset{!}{=} 0, \tag{2.6}$$

$$\frac{\partial \mathcal{L}(.)}{\partial \lambda_{t+1}} \overset{!}{=} 0 \quad \Leftrightarrow \quad \beta((1+r_{t+1})s_t - c_{2,t+1}) \overset{!}{=} 0. \tag{2.7}$$

With the FOCs the Keynes-Ramsey rule can be derived

$$\begin{aligned} \lambda_t &= \beta(1+r_{t+1})\lambda_{t+1} && \Leftrightarrow \\ \frac{1}{c_{1,t}} &= \beta(1+r_{t+1})\frac{1}{c_{2,t+1}} && \Leftrightarrow \\ \frac{c_{2,t+1}}{c_{1,t}} &= \beta(1+r_{t+1}). \end{aligned} \tag{2.8}$$

Combining the Keynes-Ramsey rule and the budget constraint for adulthood and retirement (2.1) and (2.2), respectively, by first solving (2.1) for s_t and inserting:

$$\begin{aligned} \frac{(1+r_{t+1})(w_t - c_{1,t})}{c_{1,t}} &= \beta(1+r_{t+1}) && \Leftrightarrow \\ w_t - c_{1,t} &= \beta c_{1,t} && \Leftrightarrow \\ w_t &= c_{1,t}(\beta+1), \end{aligned}$$

the optimal consumption and savings of adults can be written as

$$c_{1,t} = \frac{1}{1+\beta}w_t, \qquad s_t = \frac{\beta}{1+\beta}w_t. \tag{2.9}$$

Comparing the consumption side with the standard OLG model from Acemoglu (2009, Chap 9.3) without automation, the budget constraint for optimal consumption and the equation for savings of the adults are the same. The two equations in (2.9) illustrate that individuals allocate a constant proportion of their income for consumption and saving, and the specific proportion depends on the factor β. A higher time preference factor β reflects a greater preference for saving, leading to a higher proportion of income being saved and a smaller proportion being used for consumption.

2.2.2 Production

As mentioned above, the consumption side remains unaffected. The production side however changes in response to automation. Compared to the standard OLG model there is now one more factor of production: labor, traditional physical and the new factor automation capital. Through the use of automation technologies such as robots and artificial intelligence, human labor can increasingly be replaced by automation capital. Traditional physical capital, although an important resource for production, cannot fully substitute for human labor (*imperfect substitute*). On the other hand, automation capital enables almost complete substitution of labor, as it is able to perform the tasks and functions of human workers (*perfect substitute*) (see Gasteiger and Prettner, 2020).

These changes on the production side have implications for labor markets. Sachs and Kotlikoff (2012), Benzell et al. (2015), and Sachs et al. (2015) (cited in Gasteiger and Prettner (2020)) argue that there is a concern about a trend of declining wages and decreasing welfare, although automation increases productivity and efficiency. Furthermore, the need to retrain the workforce accordingly to the automation rises.

The aggregate production sector and total output are represented by the following production function as described by Prettner (2019)

$$Y_t = K_t^{\alpha}(N_t + P_t)^{(1-\alpha)}$$

where Y_t describes aggregate output (real GDP), N_t denotes aggregate labor supply, K_t represents physical capital, and P_t stands for automation capital like robots. Robots together with labor supply form a composite factor of production for the Cobb-Douglas function. Due to linearity, labor and robots are perfect substitutes, but machines can imperfectly substitute robots and workers.

$\alpha \in (0,1)$ is the elasticity of production of output with respect to physical capital. It indicates how much the output changes in response to a change in the quantity of physical capital used for production. If α is closer to 1, it suggests that a small increase in physical capital results in a relatively larger increase in output. $1-\alpha$ is the elasticity of production of the composite factor of production, which includes automation capital and labor supply. The individual production elasticities of labor and robots are thus smaller than $1-\alpha$.

Gasteiger and Prettner (2020) assume perfect competition in the goods and factor markets, hence all three factors of production are paid their marginal value product. In a perfectly competitive market, each firm in the market is a price taker, meaning it has to accept the prevailing market price. The profit of the firm is given by

$$\Pi_t = \underbrace{p}_{=1} Y_t - C_t(Y_t),$$

with $C_t(.)$ representing the cost function. The Y_t term in the profit equation does not have any multiplier or coefficient attached to it, since the price of the final good in this setting is normalized to 1 (numéraire good).

With the rate of return on traditional physical capital R_t^K and R_t^P describing the rate of return on automation capital, the profits of the representative firm are given by

$$\Pi_t = K_t^{\alpha}(N_t + P_t)^{(1-\alpha)} - w_t N_t - R_t^K K_t - R_t^P P_t.$$

$K_t^{\alpha}(N_t + P_t)^{(1-\alpha)}$ denotes the revenue of the representative firm. The other three terms account for the costs of production, including the wage sum $w_t N_t$, the expenses for traditional physical capital $R_t^K K_t$, and the expenses for automation capital $R_t^P P_t$.

The representative firm maximizes its profits, therefore the following first-order conditions can be obtained

$$\begin{aligned}
\frac{\partial \Pi_t}{\partial N_t} \overset{!}{=} 0 \quad &\Leftrightarrow \quad (1-\alpha)K_t^{\alpha}(N_t + P_t)^{-\alpha} - w_t \overset{!}{=} 0 \\
\frac{\partial \Pi_t}{\partial P_t} \overset{!}{=} 0 \quad &\Leftrightarrow \quad (1-\alpha)K_t^{\alpha}(N_t + P_t)^{-\alpha} - R_t^P \overset{!}{=} 0 \\
\frac{\partial \Pi_t}{\partial K_t} \overset{!}{=} 0 \quad &\Leftrightarrow \quad \alpha K_t^{\alpha-1}(N_t + P_t)^{1-\alpha} - R_t^K \overset{!}{=} 0.
\end{aligned}$$

This yields the following factor rewards:

$$w_t \overset{!}{=} R_t^P = (1-\alpha)\left(\frac{K_t}{N_t+P_t}\right)^{\alpha}, \tag{2.10}$$

$$R_t^K = \alpha\left(\frac{N_t+P_t}{K_t}\right)^{1-\alpha}. \tag{2.11}$$

The analysis of these factor rewards based on changes in the factors of production leads to the following observations (see Gasteiger and Prettner (2020) page 5):

There is a correlation between the level of physical capital and the wage rate: A rise in traditional physical capital K_t is associated with an increase in the wage rate w_t. This occurs because the higher presence of physical capital enhances worker productivity through increased utilization of machinery in the economy.

An increase in automation capital P_t decreases the wage rate: Automation capital replaces workers and therefore leads to a reduction in their productivity, which in turn lowers the wage rate.

An increase of the population size N_t leads to a reduction in R_t^P. Hence, workers' wages also decrease, which in return reduces the incentives to invest in automation capital.

It's important to consider that despite the differing effects on the wage rate, both traditional physical capital and automation capital contribute to improving overall labor productivity. The output is determined by $Y_t = K_t^{\alpha}(N_t + P_t)^{(1-\alpha)}$ and labor productivity is measured in output per worker Y_t/N_t with a fixed population size N_t

$$\frac{Y_t}{N_t} = y_t = k_t^{\alpha}(1+p_t)^{(1-\alpha)}. \tag{2.12}$$

Both types of capital enable a higher output for a given amount of labor input, leading to improved labor productivity.

2.3 Results

In order to discuss the model equilibrium, the Inada conditions are introduced following Acemoglu (2009, Chap 2.1.3):

Inada Conditions. *Y_t satisfies the Inada conditions if*

$$\lim_{K_t \to 0} \frac{\partial Y_t(K_t, P_t, N_t)}{\partial K_t} = \infty \quad \text{and} \lim_{K_t \to \infty} \frac{\partial Y_t(K_t, P_t, N_t)}{\partial K_t} = 0 \quad \forall N_t, P_t > 0$$
$$\lim_{P_t \to 0} \frac{\partial Y_t(K_t, P_t, N_t)}{\partial P_t} = \infty \quad \text{and} \lim_{P_t \to \infty} \frac{\partial Y_t(K_t, P_t, N_t)}{\partial P_t} = 0 \quad \forall N_t, K_t > 0$$
$$\lim_{N_t \to 0} \frac{\partial Y_t(K_t, P_t, N_t)}{\partial N_t} = \infty \quad \text{and} \lim_{N_t \to \infty} \frac{\partial Y_t(K_t, P_t, N_t)}{\partial N_t} = 0 \quad \forall K_t, P_t > 0.$$

These conditions suggest that the initial units of physical capital, automation capital and labor are characterized by high productivity. Contrarily, when there is an abundance of both types of capital or labor, their marginal products tend to approach zero.

Based on the Inada conditions, it becomes evident that every factor input is essential. If either capital or labor is absent, no output can be produced. Therefore, a production function that satisfies the Inada conditions does not allow corner solutions. This implies that a factor input cannot vanish or grow infinitely at the point of maximum profit.

Taking now the limits at the factor rewards towards zero, equations (2.10) and (2.11), imply that the Inada conditions for automation capital, are not met, because:

$$\lim_{P_t \to 0} R_t^P = (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha} \quad \text{and} \quad \lim_{K_t \to 0} R_t^K = \infty.$$

Gasteiger and Prettner (2020) point out that when the stock of both traditional physical capital and automation capital is close to zero, individuals would prefer to invest in traditional physical capital accumulation due to its higher rate of return. As a result, a corner solution arises where traditional physical capital becomes the dominant choice for investment.

2.3.1 Steady state

To achieve an inner equilibrium, a certain threshold of traditional physical capital needs to be reached in the future. In other words, the capital market can only achieve an inner equilibrium once a sufficient level of physical capital is available to support it.

According to Gasteiger and Prettner (2020, page 6), for a large enough physical capital stock, investments are then made in both traditional physical capital and automation capital, resulting in the same rate of return. This means that an inner equilibrium on the capital market is reached whenever $R_t^K = R_t^P$ and there is no-arbitrage between traditional physical capital and automation capital.

Consequently, in the case of an inner capital market equilibrium, P_t can be represented as follows

$$P_t = \left(\frac{1-\alpha}{\alpha}\right)K_t - N_t.$$

The above equation is derived by the FOCs of the firm, assuming $R_t^K = R_t^P$

$$\begin{aligned}
\alpha\left(\frac{N_t + P_t}{K_t}\right)^{(1-\alpha)} &= (1-\alpha)\left(\frac{K_t}{N_t + P_t}\right)^{\alpha} && \Leftrightarrow \\
\alpha\left(\frac{N_t + P_t}{K_t}\right) &= (1-\alpha) && \Leftrightarrow \\
P_t &= \left(\frac{1-\alpha}{\alpha}\right)K_t - N_t.
\end{aligned}$$

Abeliansky and Prettner (2017); Acemoglu and Restrepo (2018) support empirical evidence, that a higher stock of traditional physical capital K_t leads to an increase in the rate of return R_t^P (see equation (2.10)) and a decrease in the rate of return R_t^K (see equation (2.11)). However, since there is equality between the returns in the equilibrium, the stock of automation capital must increase.

To take into account that households do not invest in automation capital when P_t is negative, the stock of automation capital is specified by

$$P_t = \max\left\{0, \left(\frac{1-\alpha}{\alpha}\right)K_t - N_t\right\}.$$

If $P_t = 0$ the production function simplifies to the standard form of the OLG model $Y_t = K_t^{\alpha} N_t^{1-\alpha}$. When $P_t = 0$ the per worker capital stock stays constant and does not change over time:

According to Blanchard and Fischer (1990, page 95), the capital accumulation is described by the following equation

$$(1+n)k_{t+1} = s_t. \tag{2.13}$$

It follows

$$\begin{aligned} k_{t+1} &= \frac{s_t}{1+n} && \overset{(1)}{\Leftrightarrow} \\ &= \frac{\beta w_t}{(1+n)(1+\beta)} && \overset{(2)}{\Leftrightarrow} \\ &= \frac{\beta(1-\alpha)k_t^\alpha}{(1+n)(1+\beta)} && \Leftrightarrow \end{aligned}$$

with inserting the equations (2.9) and (2.10) with $P_t = 0$ for (1) and (2) respectively. So in total the steady state is given by

$$k^* = \left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}}.$$

As stated by Gasteiger and Prettner (2020, page 7), to find the steady state associated with an interior equilibrium of the capital market for $P_t > 0$, the no-arbitrage relationship is plugged into the production function. This leads to

$$\begin{aligned} Y_t &= K_t^\alpha\big(N_t + P_t\big)^{(1-\alpha)} && \Leftrightarrow \\ Y_t &= K_t^\alpha\big(N_t + \left(\frac{1-\alpha}{\alpha}\right)K_t - N_t\big)^{(1-\alpha)} && \Leftrightarrow \\ Y_t &= \left(\frac{1-\alpha}{\alpha}\right)^{(1-\alpha)} K_t. \end{aligned}$$

The equation describes an AK-type production function in equilibrium which means that the output Y_t is only determined by the level of capital stock K_t. The parameter α determines the elasticity of output with respect to physical capital.

Steigum (2011) and Prettner (2019) state, that an AK-type production function implies sustained economic growth for the neoclassical growth model because it exhibits constant returns to physical capital accumulation. If the economy invests more in capital, output and income will increase proportionally. According to Gasteiger and Prettner (2020) however, in the context of the OLG model, sustained economic growth is not guaranteed despite the presence of an AK-type production function.

Gasteiger and Prettner (2020, page 7) offer two key assumptions to explain this: the closed economy assumption and the assumption that both traditional physical capital K_t and automation capital P_t fully depreciate over a generation.

The first one, the closed economy assumption, implies that the economy does not engage in trade or interaction with other economies. This assumption restricts the potential for growth through trade.

In the OLG model, the second assumption means that any capital accumulation, whether it is in the form of traditional physical capital or automation capital, is completely annihilated within a generation. As a result, the economy cannot rely on the accumulation of capital to drive sustained growth.

So the law of motion for the aggregate stock of assets and furthermore the capital accumulation equation is derived by

$$\begin{aligned} S_t = s_t N_t &\stackrel{!}{=} K_{t+1} + P_{t+1} && \Leftrightarrow \\ \frac{\beta}{1+\beta} w_t N_t &= K_{t+1} + P_{t+1} && \Leftrightarrow \\ \frac{\beta}{1+\beta}(1-\alpha)\left(\frac{K_t}{N_t + P_t}\right)^{\alpha} N_t &= K_{t+1} + P_{t+1}. && (2.14) \end{aligned}$$

Inserting the non-arbitrage condition into (2.14) yields

$$\begin{aligned} K_{t+1} + (\frac{1-\alpha}{\alpha})K_{t+1} - N_{t+1} &= \frac{\beta}{1+\beta}(1-\alpha)\left(\frac{K_t}{N_t + (\frac{1-\alpha}{\alpha})K_t - N_t}\right)^{\alpha} N_t \Leftrightarrow \\ K_{t+1} + (\frac{1-\alpha}{\alpha})K_{t+1} - N_{t+1} &= \frac{\beta}{1+\beta}(1-\alpha)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} N_t. \qquad (2.15) \end{aligned}$$

Dividing equation 2.15 by the size of the adult population N_{t+1} finally results in the capital accumulation equation per worker

$$\begin{aligned} k_{t+1} + \left(\frac{1-\alpha}{\alpha}\right)k_{t+1} - 1 &= \frac{\beta}{1+\beta}(1-\alpha)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\frac{1}{1+n} && \Leftrightarrow \\ k_{t+1}\left(1 + (\frac{1-\alpha}{\alpha})\right) &= \left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+n}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + 1 && \Leftrightarrow \\ k_{t+1} &= \frac{\left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+n}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + 1}{\frac{1}{\alpha}} && \Leftrightarrow \\ k_{t+1} &= \alpha + \alpha\left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+n}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}. && (2.16) \end{aligned}$$

The economy is in a steady state since the capital-labor ratio is constant $k_{t+1} = k_t = k$.

2.3.2 Dynamics

The capital-labor ratio remains constant and therefore the level of output per worker also remains constant over time. Hence there is no economic growth.

Gasteiger and Prettner (2020, page 8) state that the lack of long-term economic growth in this economy is due to the absence of transitional dynamics. Transitional dynamics refer to changes in factor inputs (k_t, p_t). In this case, since the capital-labor ratio remains constant and does not experience any changes, the economy cannot develop the necessary dynamics for sustained economic growth.

So in total, there are no indicators that the economy will grow in the long term. Instead, the economy is expected to stagnate, with no significant changes or improvements in the economy.

Proposition 2.3.1 (Gasteiger and Prettner (2020) page 8)**.**

"In the canonical OLG model with automation and an interior capital market equilibrium in which both traditional physical capital and automation capital are accumulated:

(i) the production structure resembles the properties of an AK type of growth model;

(ii) the accumulation of automation capital reduces wages and therefore the savings/investments of households;

(iii) the economy is trapped in a stagnation equilibrium because of the feedback effect between automation and wage income."

It can be summarized: In the standard OLG model with automation, the economy is expected to remain stagnant, even with investments in both types of capital. This is in contrast to the neoclassical growth model that incorporates automation. Furthermore, in the OLG model, investments are funded exclusively by wage income, as observed in the equations for optimal consumption and savings of adults (equations (2.9)). However, automation leads to a reduction in wage income. As a result, automation undermines its own success in the OLG model because the reduced wage income limits the capacity for investments.

Chapter 3

OLG model with automation and two skill levels

3.1 Model assumptions

The chapter is based on the master's thesis of Lankisch (2017). Again an OLG model with automation is discussed. In addition, the model has two worker categories: "skilled" and "unskilled". The idea is that robots are better at replacing less skilled workers than more skilled ones. The model in Lankisch (2017, Chap. 5, page 45) assumes that robots and unskilled workers are perfect substitutes, but robots and skilled workers are only imperfect substitutes. The degree of their substitutability can be adjusted through an exogenous model parameter.

Lankisch (2017, Chap. 4, page 21) states, that the assumption with two skill groups in the model is just a simplification. While robots have certain tasks that they can perform more easily than others, there are also tasks that are currently beyond their capabilities. In addition, the following model does not consider investment in human capital. Lankisch (2017, Chap. 4, page 21) asserts that human capital or education increases productivity. Therefore, economies with greater investments in the workforce are more likely to experience higher productivity, which allows to advance from a less skilled worker to a more skilled one.

Individuals experience three life stages: youth, adulthood, and retirement. Children don't make economic decisions and rely on their parents for consumption. Their parents only work in the second stage and aim to consume in adulthood and retirement. They can spend their wages on consumption or invest them in physical capital or robots to fund consumption in the second life stage. In line with the previous model, the model assumes that there are two types of investment: traditional physical capital and automation capital like robots. Households aim to maximize their lifetime utility. To maximize their utility the model follows the no-arbitrage condition, introduced in Chapter 2, meaning the returns or interest rates for physical capital and automation capital must be equal.

If the returns or interest rates on physical capital and robots were unequal, this would open up an arbitrage opportunity. Households invest in the asset with higher returns leading to an imbalance. To prevent this and ensure a consistent solution, the model follows the no-arbitrage condition.

It is also assumed that the initial capital is fully depreciated or used up by the end of the current period in adulthood before transitioning to retirement. So the existing capital in adulthood has been fully consumed and is no longer contributing to the savings in retirement.

3.2 The model

3.2.1 Households

Similar to Chapter 2 the household's lifetime utility is given by

$$U_t = \log(c_{1,t}) + \beta \log(c_{2,t+1}).$$

$c_{1,t}$ describes the utility from consumption in adulthood and $c_{2,t+1}$ the utility from consumption in retirement. $\beta \in (0,1)$ denotes the discount factor. Individuals must adhere to the budget constraint and cannot incur debt. The budget constraint is defined by

$$w_t = c_{1,t} + s_t. \tag{3.1}$$

A worker's wage during period t, is divided into consumption, $c_{1,t}$, and savings, s_t. The budget constraint allows the worker to allocate their wage to either consumption or savings. The level of savings, s_t, is also determined by the discount factor β, which reflects the worker's preference for consumption in the next period $t+1$.

Following Lankisch (2017, Chap. 5, page 46), individuals, whether they are "skilled" or "unskilled" workers, maximize their utility while complying with the budget constraint (3.1). But the distinction between "skilled" and "unskilled" workers leads to differences in their wages, which in turn affect their respective levels of utility or satisfaction. Also, individuals operate as price takers. This suggests that workers do not have control over determining their wages and must accept the prevailing wage rates in the market. They then base their decisions and utility maximization on these given wage levels.[1]

As already shown in Chapter 2, the following expressions for optimal consumption and savings and the dynamics of population also apply here

$$c_{1,t} = \frac{1}{1+\beta} w_t, \tag{3.2}$$

$$s_t = \frac{\beta}{1+\beta} w_t, \tag{3.3}$$

$$N_{t+1} = N_t(1+n). \tag{3.4}$$

Again with the budget constraint of households

$$c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = w_t \tag{3.5}$$

and the Keynes-Ramsey rule from Chapter 2 the above equations can be derived

$$\begin{aligned} c_{1,t} + \frac{\beta(1+r_{t+1})c_{1,t}}{1+r_{t+1}} &= w_t && \Leftrightarrow \\ c_{1,t} + \beta c_{1,t} &= w_t && \Leftrightarrow \\ c_{1,t}(1+\beta) &= w_t. \end{aligned} \tag{3.6}$$

3.2.2 Production

It is now necessary to consider the proportions of "skilled" and "unskilled" workers in the total population, l_s and l_u, which are given through

$$\frac{L_{s,t}}{N_t} = l_s \quad \text{and} \quad \frac{L_{u,t}}{N_t} = l_u. \tag{3.7}$$

In this chapter, it is assumed that the shares of "skilled" and "unskilled" workers in the population are constant and thus exogenous variables.[2]

[1] For better readability, indices for the different types of workers will initially be omitted.

[2] In Chapter 5 the ratio between "skilled" and "unskilled" is then determined within the model and therefore endogenous.

The quantity of "skilled workers" in generation t is represented by $L_{s,t}$ and that of "unskilled workers" by $L_{u,t}$. The notation follows Lankisch (2017, chap. 5, page 47): The first index denotes variables that distinguish between "skilled" and "unskilled" classes (e.g. different wages), indicating the type of labor class type. The second index pertains to the generation (adulthood or retirement).

The production function from Chapter 2 is now changed into

$$Y_t = A_t K_t^{\alpha} \left[(1-\theta) L_{s,t}^{\gamma} + \theta (P_t + L_{u,t})^{\gamma}\right]^{\frac{1-\alpha}{\gamma}}.$$

Exogenous growth effects are excluded in this model, hence the technological level A_t is normalized to 1.

The production function describes a nested CES production function assuming that inputs are combined in a nested manner. In the first nested group, "unskilled" workers and automation capital are perfect substitutes for each other, and "skilled" labor forms another nested group. The final stage integrates the factor inputs from the prior layers. This combined composition, besides traditional physical capital, constitutes another CES production function.

Following Lankisch (2017, Chap 4., page 22) the parameter γ determines the substitution between robots and "skilled" workers. Its value ranges between 0 and 1, indicating whether they are substitutes rather than complements. When $\gamma = 1$, the elasticity of substitution becomes infinitely high, implying that robots and "skilled" workers are entirely interchangeable.

Lankisch (2017, Chap 4, page 22) also introduces the parameter θ to measure the efficiency of utilization of "skilled" or "unskilled workers" (and robots), with a range of 0 to 1. The higher it is, the more effective "unskilled workers" are compared to "skilled workers". If θ is omitted, both worker types are assumed to have equal efficiency.

Firms aim to maximize their profit and as in Chapter 2 perfect competition in the goods and factor markets is assumed.

Since the final good is again numéraire with price normalized to 1, the profits of the firm are given by

$$\Pi_t = \underbrace{K_t^{\alpha} \left[(1-\theta) L_{s,t}^{\gamma} + \theta (P_t + L_{u,t})^{\gamma}\right]^{\frac{1-\alpha}{\gamma}}}_{Y_t} - w_{s,t} L_{s,t} - w_{u,t} L_{u,t} - R_t^K K_t - R_t^P P_t.$$

Similar to Chapter 2, Y_t denotes the revenue of the firm, while the other four terms are the costs for producing including the wage bill for the "skilled" and "unskilled" workers and the expenses for traditional physical capital as well as the expenses for the automation capital.

The following first-order conditions of the representative firm can be obtained

$$\begin{aligned}
\frac{\partial \Pi_t}{\partial L_{s,t}} \overset{!}{=} 0 \quad &\Leftrightarrow \quad \frac{\partial Y_t}{\partial L_{s,t}} - w_{s,t} \overset{!}{=} 0, \\
\frac{\partial \Pi_t}{\partial L_{u,t}} \overset{!}{=} 0 \quad &\Leftrightarrow \quad \frac{\partial Y_t}{\partial L_{u,t}} - w_{u,t} \overset{!}{=} 0, \\
\frac{\partial \Pi_t}{\partial K_t} \overset{!}{=} 0 \quad &\Leftrightarrow \quad \frac{\partial Y_t}{\partial K_t} - R_t^K \overset{!}{=} 0.
\end{aligned}$$

This yields the following factor rewards, since all production factors are paid their marginal product and no profits remain.

$$\begin{aligned}
w_{s,t} = \frac{\partial Y_t}{\partial L_{s,t}} &= K_t^\alpha \left(\frac{1-\alpha}{\gamma}\right) \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}-1} \\
&\cdot \frac{\partial}{\partial L_{s,t}} \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right] \\
&= K_t^\alpha \left(\frac{1-\alpha}{\gamma}\right) \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}-1} \cdot \gamma(1-\theta)L_{s,t}^{\gamma-1} \\
&= \underbrace{K_t^\alpha \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}}}_{Y_t} \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{-1} \\
&\cdot (1-\alpha)(1-\theta)L_{s,t}^{\gamma-1} \\
&= (1-\alpha)Y_t \frac{(1-\theta)L_{s,t}^{\gamma-1}}{(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma},
\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
w_{u,t} = R^P_{t+1} = \frac{\partial Y_t}{\partial L_{u,t}} = \frac{\partial Y_t}{\partial P_t} &= K_t^\alpha \left(\frac{1-\alpha}{\gamma}\right) \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}-1} \\
&\cdot \frac{\partial}{\partial L_{u,t}} \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right] \\
&= K_t^\alpha \left(\frac{1-\alpha}{\gamma}\right) \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}-1} \cdot \gamma\theta(P_t + L_{u,t})^{\gamma-1} \\
&= \underbrace{K_t^\alpha \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}}}_{Y_t} \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{-1} \\
&\cdot (1-\alpha)\theta(P_t + L_{u,t})^{\gamma-1} \\
&= (1-\alpha)Y_t \frac{\theta(P_t + L_{u,t})^{\gamma-1}}{(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma},
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
R^K_{t+1} = \frac{\partial Y_t}{K_t} &= \alpha K_t^{\alpha-1} \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}} \\
&= \alpha \underbrace{K_t^\alpha \left[(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma\right]^{\frac{1-\alpha}{\gamma}}}_{Y_t} K_t^{-1} \\
&= \alpha \frac{Y_t}{K_t},
\end{aligned}
\tag{3.10}
$$

and

$$
Y_t = w_{s,t}L_{s,t} + w_{u,t}L_{u,t} + R^K_{t+1}K_t + R^P_{t+1}P_t. \tag{3.11}
$$

In general, "skilled" workers tend to earn higher wages than "unskilled" workers. The *wage differential* between these two groups is denoted by $\frac{w_{s,t}}{w_{u,t}}$ and with equation (3.8) and (3.9) it follows

$$
\frac{w_{s,t}}{w_{u,t}} = \frac{(1-\theta)L_{s,t}^{\gamma-1}}{\theta(P_t + L_{u,t})^{\gamma-1}}. \tag{3.12}
$$

Given that the profit is zero, equation (3.11) states that the output Y_t must be equal to the costs of the inputs. The contributions of each input are weighted by their respective prices or rate of returns $(w_{s,t}, w_{u,t}, R^K_{t+1}, R^P_{t+1})$ and multiplied by their quantities $(L_{s,t}, L_{u,t}, K_t, P_t)$. The summation of these terms represents the total output Y_t of the production.

As mentioned earlier, the no-arbitrage condition is assumed: In a perfectly efficient and competitive market, the expected return from investing in physical capital R^K_{t+1} should be equal to the rate of return from automation capital

R^P_{t+1}, and from the first-order conditions equality it is obtained

$$R^K_{t+1} = R^P_{t+1} \overset{(1)}{=} w_{u,t}.$$

This implies that physical capital K_t can be represented as a function of the automation capital P_t:

$$\begin{aligned}
\alpha \frac{Y_t}{K_t} &= (1-\alpha) Y_t \frac{\theta (P_t + L_{u,t})^{\gamma-1}}{(1-\theta) L_{s,t}^{\gamma} + \theta (P_t + L_{u,t})^{\gamma}} \qquad \Leftrightarrow \\
K_t &= \frac{\alpha}{(1-\alpha)} \frac{(1-\theta) L_{s,t}^{\gamma} + \theta (P_t + L_{u,t})^{\gamma}}{\theta (P_t + L_{u,t})^{\gamma-1}},
\end{aligned} \tag{3.13}$$

and respectively in per worker terms,

$$k_t = \frac{\alpha}{(1-\alpha)} \frac{(1-\theta) l_s^{\gamma} + \theta (p_t + l_u)^{\gamma}}{\theta (p_t + l_u)^{\gamma-1}}. \tag{3.14}$$

Considering equation (3.9) and (3.10) it follows

$$\lim_{P_t \to 0} R_t^p = (1-\alpha) Y_t \frac{\theta L_{u,t}^{\gamma-1}}{(1-\theta) L_{s,t}^{\gamma} + \theta L_{u,t}^{\gamma}} \quad \text{and} \quad \lim_{K_t \to 0} R_t^k = \infty.$$

The above equation demonstrates that only K_t fullfills the Inada condition. Therefore, the no-arbitrage condition may not hold for all parameters.

When traditional physical capital and automation capital stocks are low, the interest rate for physical capital is higher than that for robots

$$R^P_{t+1} < R^K_{t+1}. \tag{3.15}$$

Accordingly, no investments will be made in automation capital. By substituting the expressions (3.9) and (3.10) into the above inequality (3.15) with $p_t = 0$ it follows

$$\begin{aligned}
(1-\alpha) y_t \frac{\theta (l_u)^{\gamma-1}}{(1-\theta) l_s^{\gamma} + \theta (l_u)^{\gamma}} &< \alpha \frac{y_t}{k_t} \qquad \Leftrightarrow \\
(1-\alpha) \frac{\theta (l_u)^{\gamma-1}}{(1-\theta) l_s^{\gamma} + \theta l_u^{\gamma}} &< \frac{\alpha}{k_t}.
\end{aligned}$$

In particular, the rate of return of physical capital R^K_{t+1} will always be higher as the rate of return of automation capital R^P_{t+1} if

$$k_t < \frac{\alpha}{1-\alpha} \frac{1-\theta}{\theta} l_s^{\gamma} l_u^{1-\gamma} + l_u \tag{3.16}$$

holds.

While physical capital is necessary for production under a Cobb-Douglas production function, automation capital (such as robots) can be substituted with labor. Investment in automation capital only occurs when the capital stock of physical capital is high enough.

Economic considerations support the validity of the above inequality. When the inequality (3.16) holds, it implies that investment in automation capital does not take place. However, when the right-hand side of the inequality becomes smaller, and the inequality is no longer satisfied, it indicates that investment in robots becomes more likely. This can happen, for instance, if the parameter α decreases, indicating that firms rely less on physical capital. Similarly, it can happen when the parameter θ increases, meaning higher substitutability between labor and automation capital (see Lankisch, 2017, Chap. 5, page 48).

3.3 Results

3.3.1 Steady state

As discussed in Chapter 2, savings are composed of traditional physical capital K_t and automation capital P_t in the next period, $S_{t+1} = K_{t+1} + P_{t+1}$. In order to find the steady state associated with an inner equilibrium of the capital market with two skill levels, it is important to consider the different wages that "skilled" and "unskilled" workers receive

$$S_{t+1} = K_{t+1} + P_{t+1} = s_{s,t}L_{s,t} + s_{u,t}L_{u,t}.$$

According to Lankisch (2017, Chap. 5, page 48), the equation above describes the total stock of physical capital and automation capital in period $t+1$. The capital stock of period t no longer remains, as it is completely depreciated. To calculate the per worker capital stock it is important to note that the capital was saved by the population size N_t, but is divided by N_{t+1} people

$$\begin{aligned} s_{t+1} = \frac{S_{t+1}}{N_{t+1}} &= k_{t+1} + p_{t+1} \\ &= \frac{s_{s,t}L_{s,t}}{N_{t+1}} + \frac{s_{u,t}L_{u,t}}{N_{t+1}} \overset{(1)}{=} s_{s,t}\frac{l_s}{1+n} + s_{u,t}\frac{l_u}{1+n}, \end{aligned} \tag{3.17}$$

with (1) $\frac{L_{s,t}}{N_t} = l_s$ and $\frac{L_{u,t}}{N_t} = l_u$ respectively.

Rewriting equation (3.17) with the optimal savings derived from the FOC conditions (equation (3.3)) and in the following with the factor rewards for the two different types of workers (3.8) and (3.9) respectively, leads to

$$\begin{aligned} s_{t+1} &= \frac{\beta}{1+\beta} w_{s,t} \frac{l_s}{1+n} + \frac{\beta}{1+\beta} w_{u,t} \frac{l_u}{1+n} \\ &= \frac{\beta}{1+\beta}\frac{1}{1+n}(l_s w_{s,t} + l_u w_{u,t}) \\ &= \frac{\beta}{1+\beta}\frac{1}{1+n}(1-\alpha)Y_t \frac{1}{(1-\theta)L_s^\gamma + \theta(P_t+L_u)^\gamma} \\ &\cdot [(1-\theta)L_s^{\gamma-1} l_s + \theta(P_t+L_u)^{\gamma-1} l_u]. \end{aligned} \tag{3.18}$$

After several manipulations of terms, like expanding with $\frac{N^\gamma N^{1-\gamma}}{N^\gamma N^{1-\gamma}}$ to rewrite the factor inputs in equation (3.18) in per worker terms, it follows

$$\begin{aligned} s_{t+1} &= \underbrace{\frac{\beta}{1+\beta}\frac{1-\alpha}{1+n}}_{:=h_1} \frac{Y_t}{N^\gamma N^{1-\gamma}} \frac{N^\gamma}{(1-\theta)L_s^\gamma + \theta(P_t+L_u)^\gamma} \\ &\cdot N^{1-\gamma}[(1-\theta)L_s^{\gamma-1} l_s + \theta(P_t+L_u)^{\gamma-1} l_u] \\ &= h_1 y_t \frac{1}{(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma}[(1-\theta)l_s^{\gamma-1} l_s + \theta(p_t+l_u)^{\gamma-1} l_u]. \end{aligned} \tag{3.19}$$

The substitution of y_t as well as k_t from equation (3.14) results in

$$\begin{aligned} s_{t+1} &= h_1 \underbrace{k_t^\alpha [(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma]^{\frac{1-\alpha}{\gamma}}}_{y_t} \frac{1}{(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma}[(1-\theta)l_s^\gamma + \theta(p_t+l_u)^{\gamma-1} l_u] \\ &= h_1 \underbrace{\left(\frac{\alpha}{1-\alpha}\frac{(1-\theta)ls^\gamma + \theta(p_t+l_u)^\gamma}{\theta(p_t+l_u)^{\gamma-1}}\right)^\alpha}_{k_t} [(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma]^{\frac{1-\alpha}{\gamma}} \\ &\cdot \frac{1}{(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma}[(1-\theta)l_s^\gamma + \theta(p_t+l_u)^{\gamma-1} l_u] \\ &= \underbrace{h_1\left(\frac{\alpha}{(1-\alpha)\theta}\right)^\alpha}_{:=h} \theta(l_u+p_t)^{(1-\gamma)\alpha}\left[(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma\right]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \\ &\cdot \left[(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma \frac{l_u}{l_u+p_t}\right] \\ &= h(l_u+p_t)^{(1-\gamma)\alpha-1}[(1-\theta)l_s^\gamma + \theta(p_t+l_u)^\gamma]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}[(l_u+p_t)(1-\theta)l_s^\gamma + l_u\theta(p_t+l_u)^\gamma]. \end{aligned} \tag{3.20}$$

Furthermore, from the market equilibrium, (3.17), where capital demand of firms equals capital supplies of consumers, and recalling equation (3.14), it can be derived

$$s_{t+1} = p_{t+1} + \underbrace{\frac{\alpha}{(1-\alpha)} \frac{(1-\theta)l_s^\gamma + \theta(p_{t+1}+l_u)^\gamma}{\theta(p_{t+1}+l_u)^{\gamma-1}}}_{k_{t+1}}. \tag{3.21}$$

Combining equations (3.20) and (3.21), the supply of capital and the demand of capital respectively, yields an implicit equation for p_{t+1} as a function of p_t. As p_t changes over time, it influences the subsequent value of p_{t+1}, indicating a dynamic relationship.

A possible steady state is

$$\begin{aligned} p^* + k^* &= p_{t+1} + k_{t+1} \\ &= p^* + \frac{\alpha}{(1-\alpha)} \frac{(1-\theta)l_s^\gamma + \theta(p^*+l_u)^\gamma}{\theta(p^*+l_u)^{\gamma-1}} \\ &\overset{!}{=} h(l_u+p^*)^{(1-\gamma)\alpha-1}[(1-\theta)l_s^\gamma + \theta(p^*+l_u)^\gamma]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \\ &\cdot [(l_u+p^*)(1-\theta)l_s^\gamma + l_u\theta(p^*+l_u)^\gamma] \\ &= s^* \end{aligned} \tag{3.22}$$

The equation represents a possible steady state where the no-arbitrage condition holds.

Lankisch (2017, Chap. 5, page 50) states that there is another possibility for a steady state where only physical capital is invested since the interest rate for physical capital is always higher than the interest rate for automation capital.

The steady state without automation capital is determined by

$$k^* = \left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}} [(1-\theta)l_s^\gamma + \theta l_u^\gamma]^{\frac{1}{\gamma}}. \tag{3.23}$$

To show the above equation (3.23), equation (3.18) and the equality from the market equilibrium $s_{t+1} = k_{t+1} + p_{t+1}$ are assumed with $p_t = 0$

$$k_{t+1} = \frac{\beta}{1+\beta}\frac{1}{1+n}\frac{(1-\alpha)Y_t}{(1-\theta)L_s^\gamma + \theta L_u^\gamma}[(1-\theta)L_s^{\gamma-1}l_s + \theta L_u^{\gamma-1}l_u].$$

As seen before, the above equation is extended with $\frac{N^\gamma N^{1-\gamma}}{N^\gamma N^{1-\gamma}}$ to describe Y_t and the proportions L_s and L_u of "skilled" and "unskilled" workers in per worker terms. Subsequently replacing the output per worker y_t with the production function yields

$$\begin{aligned}
k_{t+1} &= \frac{\beta}{1+\beta}\frac{1}{1+n}(1-\alpha)\frac{Y}{N^{\gamma}N^{1-\gamma}}\frac{N^{\gamma}}{(1-\theta)L_s^{\gamma}+\theta L_u^{\gamma}} \\
&\cdot N^{1-\gamma}[(1-\theta)L_s^{\gamma-1}L_s+\theta L_u^{\gamma-1}L_u] \\
&= \frac{\beta}{1+\beta}\frac{1}{1+n}(1-\alpha)y\frac{1}{(1-\theta)l_s^{\gamma}+\theta l_u^{\gamma}}[(1-\theta)l_s^{\gamma-1}l_s+\theta l_u^{\gamma-1}l_u] \\
&= \frac{\beta}{1+\beta}\frac{1}{1+n}(1-\alpha)k_t^{\alpha}[(1-\theta)l_s^{\gamma}+\theta l_u^{\gamma}]^{\frac{1-\alpha}{\gamma}}.
\end{aligned}$$

And in total equation (3.23) follows, since at the steady state $k_{t+1} = k_t = k^*$ holds,

$$k^{(1-\alpha)*} = \left(\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right)[(1-\theta)l_s^{\gamma}+\theta l_u^{\gamma}]^{\frac{1-\alpha}{\gamma}}.$$

It is clear that depending on the parameterization of the model, particularly whether (3.16) holds, only one of the two steady states may occur.

3.3.2 Dynamics

If investments are made exclusively in traditional physical capital without considering automation capital, the economy will reach a steady state as defined by equation (3.23). In this steady state, there will be no economic growth, meaning that the economy will not experience an increase in its overall output or productivity over time.

Lankisch (2017, chap. 5, page 50) makes the following considerations: If an investment is made in both physical and automation capital and the no-arbitrage condition holds, the growth rate g of the capital stock $(p_t + k_t)$ will become negative as p_t and thus also k_t increase.

In the following, it will be demonstrated that

$$\frac{p_{t+1}+k_{t+1}}{p_t+k_t} \tag{3.24}$$

converges to zero. Utilizing the equation $p_{t+1} + k_{t+1} = (1 + g)(p_t + k_t)$ it consequently implies that the growth rate g must converge towards -1.

The subsequent calculations derive the statement mentioned above. The procedure used to demonstrate this result is attributed to Lankisch (2017, Chap. 5, page 50).

It holds

$$\begin{aligned} p_t + k_t &= p_t + \frac{\alpha}{(1-\alpha)} \frac{(1-\theta)l_s^\gamma + \theta(p_t + l_u)^\gamma}{\theta(p_t + l_u)^{\gamma-1}} \\ &> \frac{\alpha}{(1-\alpha)} \frac{\theta(p_t + l_u)^\gamma}{\theta(p_t + l_u)^{\gamma-1}} = \frac{\alpha}{(1-\alpha)}(l_u + p_t). \end{aligned} \tag{3.25}$$

Furthermore, to give an upper approximation for the term

$$[(1-\theta)l_s^\gamma + \theta(p_t + l_u)^\gamma]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \tag{3.26}$$

in equation (3.22) the following equation is used

$$x^q + y^q \leq (x+y)^q \leq 2^{q-1}(x^q + y^q) \quad \text{for} \quad x, y \geq 0 \quad \text{and} \quad q \in [1, \infty],$$

with $x = (1-\theta)l_s^\gamma$ and $y = \theta(p_t + lu)^\gamma$ and $q := \frac{1}{\gamma}$. As described by Lankisch (2017, Chap. 5, page 50), this equation is applicable, because γ lies in the range (0, 1), making $\frac{1}{\gamma}$ fall within the range $[1, \infty]$.

Therefore it follows

$$\begin{aligned} [(1-\theta)l_s^\gamma + \theta(p_t + l_u)^\gamma]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} &\leq 2^{\frac{1}{\gamma}-1} \left[(1-\theta)^{\frac{1}{\gamma}} l_s + \theta^{\frac{1}{\gamma}} (p_t + l_u) \right]^{(1-\alpha)(1-\gamma)} \\ &\overset{(1)}{\leq} 2^{\frac{1}{\gamma}-1} \Big[(1-\theta)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} l_s^{(1-\alpha)(1-\gamma)} \\ &\quad + \theta^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} (p_t + l_u)^{(1-\alpha)(1-\gamma)} \Big]. \end{aligned} \tag{3.27}$$

At (1), the property of subadditivity for the function $f(x) := x^{(1-\alpha)(1-\gamma)}$ is utilized. A function is said to be subadditive if the following condition holds

$$f(x+y) \leq f(x) + f(y).$$

The above defined function is subadditive, because of the concavity and the fact that this function is non-negative, given that both α and γ lie in the range (0, 1).

With equation (3.22) and the usage of inequality (3.25) at (2) the ratio can be estimated

$$\begin{aligned}\frac{p_{t+1}+k_{t+1}}{p_t+k_t} &= h(l_u+p_t)^{(1-\gamma)\alpha-1}[(1-\theta)l_s^\gamma+\theta(p_t+l_u)^\gamma]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \\ &\cdot [(l_u+p_t)(1-\theta)l_s^\gamma+l_u\theta(p_t+l_u)^\gamma]\frac{1}{p_t+k_t} \\ &\overset{(2)}{\leq} h(l_u+p_t)^{(1-\gamma)\alpha-1}[(1-\theta)l_s^\gamma+\theta(p_t+l_u)^\gamma]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \\ &\cdot [(l_u+p_t)(1-\theta)l_s^\gamma+l_u\theta(p_t+l_u)^\gamma]\frac{(1-\alpha)}{\alpha(l_u+p_t)}.\end{aligned}$$

The ratio can be further estimated using inequality (3.27)

$$\begin{aligned}\frac{p_{t+1}+k_{t+1}}{p_t+k_t} \leq{}& \tilde{h}(l_u+p_t)^{(1-\gamma)\alpha-2}2^{\frac{1}{\gamma}-1}\Big[(1-\theta)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}l_s^{(1-\alpha)(1-\gamma)} \\ &+\theta^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}(p_t+l_u)^{(1-\alpha)(1-\gamma)}\Big]\Big[(l_u+p_t)(1-\theta)l_s^\gamma+l_u\theta(p_t+l_u)^\gamma\Big],\end{aligned}$$

with $\tilde{h} := h\frac{(1-\alpha)}{\alpha}$.

For the last step, the four constants $h_1, \cdots, h_4$ are introduced, which result from expanding terms and encompass all constants that are not dependent on p_t

$$\begin{aligned}\frac{p_{t+1}+k_{t+1}}{p_t+k_t} \leq{}& h_1(l_u+p_t)^{(1-\gamma)\alpha-1}+h_2(l_u+p_t)^{(1-\gamma)-1}+h_3(l_u+p_t)^{(1-\gamma)+\gamma-2} \\ &+h_4(l_u+p_t)^{(1-\gamma)\alpha+\gamma-2}.\end{aligned}$$

As mentioned earlier, the parameters α and γ both fall within the range (0,1), resulting in negative exponents in the above equation. Consequently, as p_t approaches infinity, the expression converges to zero. This implies that the growth rate of the capital stock will also converge to zero.

When there is an increase in investment in automation capital, it has two consequences. The wages earned by "unskilled" workers decrease (see equation (3.9)), and in addition a reduction in savings follows. As a combined outcome of these effects, the capital stock per worker, $p_t + k_t$, remains stagnant and there is no opportunity for long-term economic growth.

Chapter 4

OLG model with automation and endogenous fertility

4.1 Model assumptions

In recent years, demographers have noticed a decline in fertility rates in industrialized countries. This trend leads to a smaller workforce. As fewer people are born, there are fewer individuals entering the workforce. To counter this, automation is being increasingly adopted to fill the gap and maintain productivity.

In addition, Matysiak et al. (2022) indicate that in areas where a significant number of workers are at risk of losing their jobs due to automation, such as highly industrialized regions and regions with lower levels of education, the presence of robots tends to have a negative impact on fertility. When people in these regions are facing job insecurity due to automation, it appears that their decision to have children is negatively affected (see Matysiak et al. (2022)).

These developments raise the question of the role of technology in explaining fertility changes. In this chapter, the relationship between fertility and automation is examined. To accomplish this, fertility is introduced endogenously. Households make their own decisions regarding fertility, functioning as a key component expanding the framework of the OLG model presented in Chapter 2. The methodology used in this chapter is following the approach outlined by Chen (2007).

4.2 The model

4.2.1 Households

This model enables an analysis of how individuals' choices influence the demographic structure over time. Fertility decisions, an example of individual decision-making, play a crucial role in demographic changes.

Building upon the research shown in the previous chapters, which assumed a constant fertility rate, the discussion in this chapter explores a more complex perspective. Chen (2007, page 42) points out economic considerations, such as the cost and time of raising children, to model these individual choices, which significantly impact fertility rates. In the following section, the choices individuals make regarding family planning determine the population size in the future.

The model based on Chen (2007, page 42) divides the population into three distinct stages: youth, adulthood, and retirement. Within this framework, individuals make decisions during their adulthood, including choices about consumption and savings for retirement. The household's lifetime utility is given by

$$U_t = \nu \log(n_t) + \log(c_{1,t}) + \beta \log(c_{2,t+1}). \tag{4.1}$$

In this setting, n_t describes the number of children, $\beta \in (0, 1)$ is the discount factor and $\nu > 0$ expresses the level of altruism towards children.

Altruism refers to the willingness of parents to make both financial and non-financial sacrifices for the benefit of their children, often without expecting any direct rewards or compensation in return. This can involve choices like devoting time and money to their children's education or supporting their children financially in a way that might have a negative effect on their own financial condition.

As in Chen (2007, page 42), individuals are given one unit of time as their initial endowment for each time period. In this model, adults need to work to earn wages, denoted as w_t, which they can then divide between saving and consumption. Additionally, adults who are also parents need to allocate some of their time to raise their children. Each child consumes a fixed portion of their parent's time. This portion is represented by $z \in (0, 1)$, indicating that it is a fraction of the parent's time.

The budget constraint with a fixed portion of the time required for child care is then

$$c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = (1 - zn_t)w_t. \tag{4.2}$$

To solve the intertemporal maximization problem of the household the Lagrangian function is set up:

$$\begin{aligned}\mathcal{L}(.) = \nu \log(n_t) + \log(c_{1,t}) + \lambda_t(w_t(1 - zn_t) - s_t - c_{1,t}) \\ + \beta\big[\log(c_{2,t+1}) + \lambda_{t+1}((1+r_{t+1})s_t - (c_{2,t+1}))\big].\end{aligned} \tag{4.3}$$

The first-order conditions (FOCs) are

$$\frac{\partial \mathcal{L}(.)}{\partial c_{1,t}} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \frac{1}{c_{1,t}} - \lambda_t \stackrel{!}{=} 0, \tag{4.4}$$

$$\frac{\partial \mathcal{L}(.)}{\partial c_{2,t+1}} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \beta(\frac{1}{c_{2,t+1}} - \lambda_{t+1}) \stackrel{!}{=} 0, \tag{4.5}$$

$$\frac{\partial \mathcal{L}(.)}{\partial s_t} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad -\lambda_t + \beta(1+r_{t+1})\lambda_{t+1} \stackrel{!}{=} 0, \tag{4.6}$$

$$\frac{\partial \mathcal{L}(.)}{\partial n_t} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \frac{\nu}{n_t} - \lambda_t(w_t z) \stackrel{!}{=} 0, \tag{4.7}$$

$$\frac{\partial \mathcal{L}(.)}{\partial \lambda_t} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad w_t(1 - zn_t) - s_t - c_{1,t} \stackrel{!}{=} 0, \tag{4.8}$$

$$\frac{\partial \mathcal{L}(.)}{\partial \lambda_{t+1}} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \beta((1+r_{t+1})s_t - c_{2,t+1}) \stackrel{!}{=} 0. \tag{4.9}$$

With the FOCs the Keynes-Ramsey rule for the OLG model with automation and endogenous fertility can be derived

$$\begin{aligned}\lambda_t &= \beta(1+r_{t+1})\lambda_{t+1} && \Leftrightarrow \\ \frac{1}{c_{1,t}} &= \beta(1+r_{t+1})\frac{1}{c_{2,t+1}} && \Leftrightarrow \\ \frac{c_{2,t+1}}{c_{1,t}} &= \beta(1+r_{t+1})\end{aligned} \tag{4.10}$$

and is equivalent to the Keynes-Ramsey rule from Chapter 2, equation (2.8).

The optimal number of children n_t is also derived from the first-order conditions of the household. From (4.7) and in combination with (4.4) one obtains

$$\begin{aligned} \frac{\nu}{n_t} &= \lambda_t w_t z && \Leftrightarrow \\ \frac{\nu}{n_t} &= \frac{1}{c_{1,t}} w_t z && \Leftrightarrow \\ \nu c_{1,t} &= n_t w_t z. && (4.11) \end{aligned}$$

The consumption in adulthood $c_{1,t}$ can be expressed from the budget constraint, equation (4.2) and substituting this term into equation (4.11) results in

$$\begin{aligned} \nu\big[(1 - zn_t)w_t - \frac{c_{2,t+1}}{1 + r_{t+1}}\big] &= n_t z w_t && \Leftrightarrow \\ \big[(1 - zn_t)w_t - \frac{c_{2,t+1}}{1 + r_{t+1}}\big] &= \frac{n_t z w_t}{\nu} && \Leftrightarrow \\ w_t - w_t z n_t - \frac{c_{2,t+1}}{1 + r_{t+1}} &= \frac{1}{\nu}\big(n_t z w_t\big) && \Leftrightarrow \\ w_t - \frac{c_{2,t+1}}{1 + r_{t+1}} &= \left(\frac{1}{\nu} + 1\right)(n_t z w_t) && \Leftrightarrow \\ w_t - \left(\frac{1 + \nu}{\nu}\right) n_t z w_t &= \frac{c_{2,t+1}}{1 + r_{t+1}}. && (4.12) \end{aligned}$$

Rewriting the Keynes-Ramsey rule yields

$$\frac{c_{2,t+1}}{1 + r_{t+1}} = c_{1,t}\beta,$$

and again substituting the expression for consumption in adulthood, $c_{1,t}$ derived from the budget constraint (4.2), delivers another expression for equation (4.12)

$$\begin{aligned} \frac{c_{2,t+1}}{1 + r_{t+1}} &= \big[(1 - zn_t)w_t - \frac{c_{2,t+1}}{1 + r_{t+1}}\big]\beta && \Leftrightarrow \\ \frac{c_{2,t+1}}{1 + r_{t+1}} &= (1 - zn_t)w_t\beta - \frac{c_{2,t+1}}{1 + r_{t+1}}\beta && \Leftrightarrow \\ \frac{c_{2,t+1}}{1 + r_{t+1}}\big(1 + \beta\big) &= (1 - zn_t)w_t\beta && \Leftrightarrow \\ \frac{c_{2,t+1}}{1 + r_{t+1}} &= \frac{\beta(1 - zn_t)}{1 + \beta} w_t. && (4.13) \end{aligned}$$

By equating equations (4.12) and (4.13) and solving them for n_t, the optimal number of children for the household can be determined.

This allows for a deeper understanding of how fertility decisions are influenced by the given parameters ν, z and β

$$\begin{aligned}
w_t - \left(\frac{1+\nu}{\nu}\right) n_t z w_t &= \frac{\beta}{1+\beta}(1 - z n_t) w_t && \Leftrightarrow \\
w_t - \left(\frac{1+\nu}{\nu}\right) n_t z w_t &= \frac{\beta}{1+\beta} w_t - z n_t w_t \frac{\beta}{1+\beta} && \Leftrightarrow \\
w_t \left(1 - \frac{\beta}{1+\beta}\right) &= n_t z w_t \left(\frac{1+\nu}{\nu} - \frac{\beta}{1+\beta}\right) && \Leftrightarrow \\
\frac{1}{1+\beta} &= n_t z \left(\frac{1+\nu}{\nu} - \frac{\beta}{1+\beta}\right) && \Leftrightarrow \\
n_t &= \frac{1}{1+\beta} \frac{1}{z} \frac{\nu(1+\beta)}{(1+\nu)(1+\beta) - \nu\beta} && \Leftrightarrow \\
n_t &= \frac{\nu}{z(1+\nu+\beta)}. && (4.14)
\end{aligned}$$

n_t thus depends only on the parameters ν, z, and β and is therefore constant.

As mentioned before ν represents the altruism among parents, so ν can be seen as the weight households place on having children in their utility function. If ν increases, it implies that households value having children more, and therefore, n_t also increases. Conversely, if ν decreases, households may assign less importance to have children, leading to a lower optimal number of children.

If z increases, it indicates that raising one child has become more time-consuming, which may discourage households from having more children. In this case, n_t decreases. On the other hand, if z decreases, the cost regarding the time of raising children reduces, potentially leading to a higher optimal number of children.

As β represents the discount factor, a rise in β indicates that households exhibit a greater preference for current consumption compared to future consumption. So n_t decreases as households prioritize immediate consumption rather than investing in raising children. If β decreases, households may have a lower time preference, giving more weight to future generations, potentially leading to a higher optimal number of children.

Combining the Keynes Ramsey rule from (4.10) and the budget constraint with a fixed portion of the time required for child care (4.2) yields as a result the optimal consumption in adulthood $c_{1,t}$

$$\begin{aligned}
\frac{\big((1-zn_t)w_t - c_{1,t}\big)(1+r_{t+1})}{c_{1,t}} &= \beta(1+r_{t+1}) && \Leftrightarrow \\
(1-zn_t)w_t - c_{1,t} &= \beta c_{1,t} && \Leftrightarrow \\
(1-zn_t)w_t &= \beta c_{1,t} + c_{1,t} && \Leftrightarrow \\
(1-zn_t)w_t &= c_{1,t}(1+\beta).
\end{aligned}$$

Inserting the optimal number for children n_t gives

$$\begin{aligned}
\left(1 - z\left(\frac{\nu}{z(1+\nu+\beta)}\right)\right)w_t &= c_{1,t}(1+\beta) && \Leftrightarrow \\
\left(\frac{1+\nu+\beta-\nu}{1+\nu+\beta}\right)w_t &= c_{1,t}(1+\beta) && \Leftrightarrow \\
c_{1,t} &= \frac{1}{1+\nu+\beta}w_t, && (4.15)
\end{aligned}$$

and from the first-order conditions, equation (4.8), the optimal saving rate of adults follows

$$\begin{aligned}
0 &= w_t\big(1-zn_t\big) - s_t - c_{1,t} \\
s_t &= w_t\left(1 - z\left(\frac{\nu}{z(1+\nu+\beta)}\right)\right) - c_{1,t} \\
&= w_t\left(1 - \frac{\nu}{1+\nu+\beta}\right) - \left(\frac{1}{1+\nu+\beta}\right)w_t \\
&= w_t\left(\frac{1+\beta}{1+\nu+\beta}\right) - w_t\left(\frac{1}{1+\nu+\beta}\right) \\
&= \frac{\beta}{1+\nu+\beta}w_t. && (4.16)
\end{aligned}$$

Comparing the consumption side with the OLG model with automation from Chapter 2, the difference between the equations for the budget constraint for optimal consumption $c_{1,t}$ and the savings of adults s_t is the inclusion of the altruism parameter ν.

4.2.2 Production

Parents invest a portion of their time $z \in (0,1)$ in childcare. Therefore, they do not allocate their entire time to the labor market, instead only the labor force $L_t = (1 - zn_t)N_t$ is available. As shown before n_t is a constant, therefore the time index t will be omitted in the following.

The production function has to be adjusted accordingly

$$Y_t = K_t^\alpha \Big(L_t + P_t\Big)^{(1-\alpha)}.$$

To calculate the factor rewards the same procedure as in Chapter 2 is used. The profits of the representative firm are given by

$$\Pi_t = K_t^\alpha \Big(L_t + P_t\Big)^{(1-\alpha)} - w_t L_t - R_t^K K_t - R_t^P P_t.$$

As in Chapter 2, the revenue of the representative firm is denoted by $K_t^\alpha \big((1 - zn_t)N_t + P_t\big)^{(1-\alpha)}$. Accounting for production costs, the remaining three terms encompass the wage sum $w_t L_t$, expenses for traditional physical capital $R_t^K K_t$, and expenditures for automation capital $R_t^P P_t$.

It has to hold

$$\begin{aligned}
\frac{\partial \Pi_t}{\partial L_t} \overset{!}{=} 0 \quad &\Leftrightarrow \quad (1-\alpha)K_t^\alpha \Big(L_t + P_t\Big)^{-\alpha} - w_t \overset{!}{=} 0 \\
\frac{\partial \Pi_t}{\partial P_t} \overset{!}{=} 0 \quad &\Leftrightarrow \quad (1-\alpha)K_t^\alpha \Big(L_t + P_t\Big)^{-\alpha} - R_t^P \overset{!}{=} 0 \\
\frac{\partial \Pi_t}{\partial K_t} \overset{!}{=} 0 \quad &\Leftrightarrow \quad \alpha K_t^{\alpha-1} \Big(L_t + P_t\Big)^{(1-\alpha)} - R_t^K \overset{!}{=} 0.
\end{aligned}$$

This yields the following factor rewards with $L_t = (1 - zn_t)N_t$:

$$w_t \overset{!}{=} R_t^P = (1-\alpha)\left(\frac{K_t}{(1-zn)N_t + P_t}\right)^\alpha \tag{4.17}$$

$$R_t^K = \alpha\left(\frac{(1-zn)N_t + P_t}{K_t}\right)^{(1-\alpha)}. \tag{4.18}$$

4.3 Results

4.3.1 Steady state

When determining the steady state, the assumption of the no-arbitrage condition $R_t^K = R_t^P$ is made. P_t can be represented as follows

$$
\begin{aligned}
R_t^K &= R_t^P && \Leftrightarrow \\
\alpha\left(\frac{(1-zn)N_t + P_t}{K_t}\right)^{(1-\alpha)} &= (1-\alpha)\left(\frac{K_t}{(1-zn)N_t + P_t}\right)^{\alpha} && \Leftrightarrow \\
\alpha\left(\frac{(1-zn)N_t + P_t}{K_t}\right) &= (1-\alpha) && \Leftrightarrow \\
P_t &= \left(\frac{1-\alpha}{\alpha}\right)K_t - (1-zn)N_t,
\end{aligned}
$$

with $n_t = n$ from equation (4.14).

This leads to an AK-type production function

$$
\begin{aligned}
Y_t &= K_t^{\alpha}\Big((1-zn)N_t + P_t\Big)^{(1-\alpha)} \\
&= K_t^{\alpha}\left((1-zn)N_t + \left(\frac{1-\alpha}{\alpha}\right)K_t - (1-zn)N_t\right)^{(1-\alpha)} \\
&= \left(\frac{1-\alpha}{\alpha}\right)^{(1-\alpha)} K_t.
\end{aligned}
$$

Unlike the preceding chapters, individuals in this context also make decisions regarding the desired number of children they plan to have, therefore endogenous fertility directly determines the future population size.

Again it has to hold

$$S_t = s_t N_t \stackrel{!}{=} K_{t+1} + P_{t+1},$$

because it follows from the market equilibrium that the capital supplied by consumers must equal the capital demanded by firms.

Rewriting this equation with the expression for the optimal savings rate s_t

and the wage w_t results in

$$\begin{aligned}\frac{\beta}{1+\nu+\beta}w_t N_t &= K_{t+1}+P_{t+1} \qquad \Leftrightarrow\\ \frac{\beta}{1+\nu+\beta}\underbrace{(1-\alpha)\left(\frac{K_t}{(1-zn)N_t+P_t}\right)^{\alpha}}_{w_t}N_t &= K_{t+1}+P_{t+1}.\end{aligned}$$

Substituting the no-arbitrage condition $R^K_{t+1} = R^P_{t+1}$ which implies $P_{t+1} = \left(\frac{1-\alpha}{\alpha}\right)K_{t+1} - (1-zn)N_{t+1}$, yields

$$\begin{aligned}K_{t+1} + (\frac{1-\alpha}{\alpha})K_{t+1} - (1-zn)N_{t+1} &= \frac{\beta(1-\alpha)}{1+\nu+\beta}\\ &\cdot\left(\frac{K_t}{(1-zn)N_t + (\frac{1-\alpha}{\alpha})K_t - (1-zn)N_t}\right)^{\alpha} N_t \Leftrightarrow\\ K_{t+1} + (\frac{1-\alpha}{\alpha})K_{t+1} - (1-zn)N_{t+1} &= \frac{\beta(1-\alpha)}{1+\nu+\beta}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} N_t \qquad \Leftrightarrow\\ K_{t+1} &= \frac{\alpha\beta(1-\alpha)}{1+\nu+\beta}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} N_t + \alpha(1-zn)N_{t+1}.\end{aligned} \tag{4.19}$$

Equation (4.19) can be rewritten in terms of capital per worker, leading to the derivation of the capital accumulation equation

$$\begin{aligned}k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{K_{t+1}}{nN_t} &= \frac{\alpha\beta(1-\alpha)}{n(1+\nu+\beta)}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + \alpha(1-zn) \qquad \Leftrightarrow\\ k_{t+1} &= \frac{z(1+\nu+\beta)}{\nu}\frac{\alpha\beta(1-\alpha)}{1+\nu+\beta}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + \alpha(1-zn) \quad \Leftrightarrow\\ k_{t+1} &= \frac{z\alpha\beta(1-\alpha)}{\nu}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + \alpha(1-zn).\end{aligned} \tag{4.20}$$

With equation (4.14) the above equation only depends on the parameters of the household's utility function and the parameters of the firm's production function α, β, ν, and z

$$k_{t+1} = \frac{z\alpha\beta(1-\alpha)}{\nu}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + \alpha\left(\frac{1+\beta}{1+\nu+\beta}\right). \tag{4.21}$$

The level of output per worker remains constant and does not exhibit any growth or improvement since k_{t+1} does not depend on t, so the economy is in a steady state. It implies that the economy has reached a long-term equilibrium where the amount of capital available per worker stays the same. Consequently, the level of output produced per worker also remains constant, indicating that there is no increase in productivity or output growth in the economy.

4.3.2 Dynamics

When comparing equation (4.21) with the capital accumulation equation from Chapter 2, equation (2.16) which does not consider endogenous fertility,

$$k_{t+1} = \alpha + \alpha\left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+n}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha},$$

it becomes evident that the capital stock in the model with endogenous fertility is larger if the inequality

$$n > \frac{\alpha\left(\frac{\beta}{1+\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}(1-\alpha)}{\left(\frac{z\alpha\beta(1-\alpha)}{\nu}\left(\frac{\alpha}{(1-\alpha)}\right)^{\alpha} + \alpha\left(\frac{1+\beta}{1+\nu+\beta}\right) - \alpha\right)} - 1 \tag{4.22}$$

holds.

The following steps demonstrate the derivation of this inequality

$$\frac{z\alpha\beta(1-\alpha)}{\nu}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + \alpha\left(\frac{1+\beta}{1+\nu+\beta}\right) > \alpha + \alpha\left(\frac{\beta}{1+\beta}\right)\left(\frac{1-\alpha}{1+n}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}$$

$$\frac{\left[\frac{z\alpha\beta(1-\alpha)}{\nu}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + \alpha\left(\frac{1+\beta}{1+\nu+\beta}\right) - \alpha\right]}{\alpha\left(\frac{\beta}{1+\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}} > \frac{1-\alpha}{1+n}$$

$$\left[\left(\frac{z\alpha\beta(1-\alpha)}{\nu}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} + \alpha\left(\frac{1+\beta}{1+\nu+\beta}\right) - \alpha\right](1+n) > (1-\alpha)\alpha\left(\frac{\beta}{1+\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}$$

If n exceeds this value, a higher capital stock can be achieved in the model with endogenous fertility.

Similar to the results in Chapter 2, the model with endogenous fertility also does not experience any growth in output per worker over time. The level of output per worker also remains constant.

Chapter 5

OLG model with automation and endogenous fertility and endogenous skill investment

5.1 Model assumptions

An OLG model with automation and endogenous fertility and endogenous skill investment refers to a framework where the determination of worker categories, specifically "skilled" and "unskilled," is considered endogenous. In Chapter 3, the categories to which each worker belonged to were fixed and considered exogenous. However, in this model, the skill levels of individuals are not predetermined but are determined by their decision influenced by market-driven wages. Individuals have the ability to invest in their human capital, which includes knowledge, skills and abilities acquired through education, training and experience. These investments in human capital can impact an individual's skill level, wage and overall well-being.

Additionally, the model considers endogenous fertility, as discussed in Chapter 4. The choice of fertility has implications for intergenerational transfers, population growth and the economic dynamics of the model because it affects the size and composition of future generations. When individuals choose to have less children, the size of the younger generation decreases relative to the older generation.

The OLG model with automation and endogenous fertility and endogenous skill investment follows the typical structure of an OLG model. The economy is divided into three generations: youth, adulthood and retirement.

Individuals within each generation make choices regarding saving, consumption, labor supply, human capital investment and fertility based on their preferences and available resources.

5.2 The model

5.2.1 Households

The following model is based on Chen (2007, page 42). Let i ($i = s, u$), s for "skilled" and u for "unskilled", denote the category of workers. Individuals make decisions based on their desired levels of consumption during adulthood and old age. The lifetime utility function, which is the same for all adults, is expressed as

$$U_t^i = \nu \log(n_t^i) + \log(c_{1,t}^i) + \beta \log(c_{2,t+1}^i), \quad i = s, u, \tag{5.1}$$

with the level of altruism towards children $\nu > 0$ and the discount factor $\beta \in (0, 1)$.

Following Chapter 4, adults spend a fixed fraction of their initial unit of time z for child-rearing. In addition, "skilled" workers spend a fixed proportion $\sigma \in (0, 1)$ of their time acquiring education. As a result, the budget constraint for "skilled" workers is described as follows

$$c_{1,t}^s + s_t^s = (1 - \sigma - zn_t^s)w_t^s. \tag{5.2}$$

If adults decide not to pursue education, they will be categorized as "unskilled" workers and the budget constraint as well as the optimal consumption and the saving rate stay the same as in Chapter 4 with endogenous fertility

$$c_{1,t}^u + s_t^u = (1 - zn_t^u)w_t^u, \tag{5.3}$$

$$c_{1,t}^u = \frac{1}{1 + \nu + \beta} w_t^u, \tag{5.4}$$

$$s_t^u = \frac{\beta}{1 + \nu + \beta} w_t^u. \tag{5.5}$$

The optimal number of children for "unskilled" workers is also calculated based on Chapter 4 and equal to equation (4.14)

$$n_t^u = \frac{\nu}{z(1 + \nu + \beta)}. \tag{5.6}$$

To derive these equations for the "skilled" workers the Lagrangian function is set up:

$$\mathcal{L}(.) = \nu \log(n_t^s) + \log(c_{1,t}^s) + \lambda_t(w_t^s(1-\sigma - zn_t^s) - s_t^s - c_{1,t}^s) + \beta\big[\log(c_{2,t+1}^s) + \lambda_{t+1}((1+r_{t+1})s_t^s - (c_{2,t+1}^s))\big]. \quad (5.7)$$

The first-order conditions (FOCs) are

$$\frac{\partial \mathcal{L}(.)}{\partial c_{1,t}^s} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \frac{1}{c_{1,t}^s} - \lambda_t \stackrel{!}{=} 0, \quad (5.8)$$

$$\frac{\partial \mathcal{L}(.)}{\partial c_{2,t+1}^s} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \beta(\frac{1}{c_{2,t+1}^s} - \lambda_{t+1}) \stackrel{!}{=} 0, \quad (5.9)$$

$$\frac{\partial \mathcal{L}(.)}{\partial s_t^s} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad -\lambda_t + \beta(1+r_{t+1})\lambda_{t+1} \stackrel{!}{=} 0, \quad (5.10)$$

$$\frac{\partial \mathcal{L}(.)}{\partial n_t^s} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \frac{\nu}{n_t^s} - \lambda_t(w_t^s z) \stackrel{!}{=} 0, \quad (5.11)$$

$$\frac{\partial \mathcal{L}(.)}{\partial \lambda_t} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad w_t^s(1-\sigma - zn_t^s) - s_t^s - c_{1,t}^s \stackrel{!}{=} 0, \quad (5.12)$$

$$\frac{\partial \mathcal{L}(.)}{\partial \lambda_{t+1}} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \beta\big[1+r_{t+1})s_t^s - c_{2,t+1}^s\big] \stackrel{!}{=} 0. \quad (5.13)$$

The Keynes-Ramsey rule for the OLG model for "skilled" worker is derived as in Chapter 4 and is equivalent to the Keynes-Ramsey rule from chapter 2, equation (2.8)

$$\frac{c_{2,t+1}}{c_{1,t}} = \beta(1+r_{t+1}). \quad (5.14)$$

To calculate the optimal number of children for the "skilled" worker, the same procedure is followed as in Chapter 4: λ_t is derived from equation (5.8) and subsequently substituted into equation (5.11)

$$\frac{\nu}{n_t^s} - \frac{1}{c_{1,t}^s}(w_t^s z) = 0 \quad \Leftrightarrow$$

$$\nu c_{1,t}^s = n_t^s w_t^s z.$$

The consumption in adulthood $c_{1,t}^s$ is expressed by the budget constraint

(5.2) and inserted into the equation:

$$
\begin{aligned}
\nu\big[(1-\sigma-zn_t^s)w_t^s - \frac{c_{2,t+1}^s}{1+r_{t+1}}\big] &= n_t^s w_t^s z && \Leftrightarrow \\
w_t^s - w_t^s\sigma - zn_t^s w_t^s - \frac{c_{2,t+1}^s}{1+r_{t+1}} &= \frac{1}{\nu}\big(n_t^s w_t^s z\big) && \Leftrightarrow \\
w_t^s(1-\sigma) - \frac{c_{2,t+1}^s}{1+r_{t+1}} &= \left(\frac{1+\nu}{\nu}\right)\big(n_t^s w_t^s z\big) && \Leftrightarrow \\
w_t^s(1-\sigma) - \left(\frac{1+\nu}{\nu}\right)n_t^s w_t^s z &= \frac{c_{2,t+1}^s}{1+r_{t+1}}. && (5.15)
\end{aligned}
$$

By rewriting the Keynes-Ramsey rule (5.14) and inserting $c_{1,t}^s$, expressed by the budget constraint (5.2), another expression for (5.15) is derived

$$
\begin{aligned}
\frac{c_{2,t+1}^s}{1+r_{t+1}} &= \big[(1-\sigma-zn_t^s)w_t^s - \frac{c_{2,t+1}^s}{1+r_{t+1}}\big]\beta && \Leftrightarrow \\
\frac{c_{2,t+1}^s}{1+r_{t+1}}(1+\beta) &= (1-\sigma-zn_t^s)w_t^s\beta && \Leftrightarrow \\
\frac{c_{2,t+1}^s}{1+r_{t+1}} &= \frac{\beta}{1+\beta}(1-\sigma-zn_t^s)w_t^s. && (5.16)
\end{aligned}
$$

The optimal number of children for the household is now determined by plugging $\frac{c_{2,t+1}^s}{1+r_{t+1}}$ from (5.16) into (5.15)

$$
\begin{aligned}
w_t^s(1-\sigma) - \left(\frac{1+\nu}{\nu}\right)n_t^s w_t^s z &= \frac{\beta}{1+\beta}(1-\sigma-zn_t^s)w_t^s && \Leftrightarrow \\
w_t^s\left((1-\sigma) - \frac{\beta}{1+\beta} + \frac{\beta\sigma}{1+\beta}\right) &= n_t^s w_t^s z\left(\frac{1+\nu}{\nu} - \frac{\beta}{1+\beta}\right) && \Leftrightarrow \\
\frac{1-\sigma}{1+\beta} &= n_t^s z\left(\frac{1+\nu}{\nu} - \frac{\beta}{1+\beta}\right) && \Leftrightarrow \\
n_t^s &= \frac{\nu(1-\sigma)}{z(1+\nu+\beta)}. && (5.17)
\end{aligned}
$$

An alternative formulation of the budget constraint for the "skilled" worker (5.2) is

$$
c_{1,t}^s + \frac{c_{2,t+1}^s}{1+r_{t+1}} = (1-\sigma-zn_t^s)w_t^s. \qquad (5.18)
$$

Solving the rewritten budget constraint for $c_{2,t+1}^s$ yields

$$
c_{2,t+1}^s = \big((1-\sigma-zn_t^s)w_t^s - c_{1,t}^s\big)(1+r_{t+1}) \qquad (5.19)
$$

Inserting this expression into the Keynes Ramsey rule from equation (5.14) the optimal consumption rate for the "skilled" worker $c^s_{1,t}$ is obtained

$$\begin{aligned}
\frac{\big((1-\sigma-zn^s_t)w^s_t - c^s_{1,t}\big)(1+r_{t+1})}{c^s_{1,t}} &= \beta(1+r_{t+1}) && \Leftrightarrow \\
(1-\sigma-zn^s_t)w^s_t - c^s_{1,t} &= \beta c^s_{1,t} && \Leftrightarrow \\
(1-\sigma-zn^s_t)w^s_t &= c^s_{1,t}(1+\beta) && \overset{(1)}{\Leftrightarrow} \\
\left(1-\sigma-z\left(\frac{\nu(1-\sigma)}{z(1+\nu+\beta)}\right)\right)w^s_t &= c^s_{1,t}(1+\beta) && \Leftrightarrow \\
c^s_{1,t} &= \frac{1-\sigma}{1+\nu+\beta}w^s_t, && (5.20)
\end{aligned}$$

with substituting the optimal number of children from (5.17) at (1).

The optimal saving rate for the "skilled" worker follows from the first-order condition (5.12)

$$\begin{aligned}
0 &= w^s_t(1-\sigma-zn^s_t) - s^s_t - c^s_{1,t} && \overset{(1)}{\Leftrightarrow} \\
s^s_t &= w^s_t\left(1-\sigma-z\frac{(1-\sigma)\nu}{z(1+\nu+\beta)}\right) - c^s_{1,t} && \overset{(2)}{\Leftrightarrow} \\
&= w^s_t\left(1-\sigma-z\frac{(1-\sigma)\nu}{z(1+\nu+\beta)}\right) - \frac{1-\sigma}{1+\nu+\beta}w^s_t && \Leftrightarrow \\
&= w^s_t\left(\frac{(1-\sigma)(1+\nu+\beta)-(1-\sigma)\nu-(1-\sigma)}{1+\nu+\beta}\right) && \Leftrightarrow \\
&= w^s_t\left(\frac{(1-\sigma)(1+\nu+\beta-\nu-1)}{1+\nu+\beta}\right) && \Leftrightarrow \\
&= \frac{\beta(1-\sigma)}{1+\nu+\beta}w^s_t, && (5.21)
\end{aligned}$$

with the optimal number of children n^s_t from equation (5.17) and the consumption rate $c^s_{1,t}$ from equation (5.20) at (1) and (2) respectively.

The following equations provide a summary of the optimal consumption, savings rate, and number of children of "skilled" and "unskilled" workers

$$
\begin{aligned}
n_t^s &= \frac{\nu(1-\sigma)}{z(1+\nu+\beta)}, & n_t^u &= \frac{\nu}{z(1+\nu+\beta)}, \\
c_{1,t}^s &= \frac{1-\sigma}{1+\nu+\beta} w_t^s, & c_{1,t}^u &= \frac{1}{1+\nu+\beta} w_t^u, \\
s_t^s &= \frac{\beta(1-\sigma)}{1+\nu+\beta} w_t^s, & s_t^u &= \frac{\beta}{1+\nu+\beta} w_t^u.
\end{aligned}
$$

Comparing the equations for "skilled" and "unskilled" workers, it becomes evident that the fertility rate of "skilled" workers will always be lower than that of "unskilled" workers. However, when considering optimal consumption and saving rates, this might not necessarily hold true. While $1-\sigma < 1$ the wages of "skilled" workers can potentially surpass those of "unskilled" workers. Equation (5.23), as derived on the following pages, will confirm this assumption.

Workers are able to move freely between "skilled" and "unskilled" labor markets, and at equilibrium, they become indifferent between choosing to become "skilled" workers or "unskilled" workers. This equilibrium state of labor market choices means that workers are not showing a preference for either "skilled" or "unskilled" work.

Next, the threshold, at which workers become indifferent to investing in skills, is calculated.

$$U_t^s = U_t^u, \tag{5.22}$$

denotes the equilibrium, where U_t^s represents the utility of "skilled" workers and U_t^u represents the utility of "unskilled" workers. This equation implies that the level of utility experienced by "skilled" workers is equal to the level of utility experienced by "unskilled" workers in the equilibrium state.

To obtain the ratio of the wage rate for "unskilled" labor and "skilled" labor, the expressions for optimal consumption c_t^i and optimal fertility n_t^i are substituted into the utility function for both "skilled" and "unskilled" labor. The last term of the utility function, the utility obtained from consumption during retirement $c_{2,t+1}^i$, is expressed by the Keynes-Ramsey rule for "skilled" and "unskilled" workers, equation (5.16) and (4.13), respectively.

So it has to hold

$$U_t^s \stackrel{!}{=} U_t^u.$$

$$\nu \log\left(\frac{\nu(1-\sigma)}{z(1+\nu+\beta)}\right) + \log\left(\frac{(1-\sigma)w_t^s}{1+\nu+\beta}\right) + \beta\log\left(\frac{\beta(1-\sigma-zn_t^s)}{1+\beta}w_t^s(1+r_{t+1})\right) \stackrel{!}{=}$$
$$\nu \log\left(\frac{\nu}{z(1+\nu+\beta)}\right) + \log\left(\frac{w_t^u}{1+\nu+\beta}\right) + \beta\log\left(\frac{\beta(1-zn_t^u)}{1+\beta}w_t^u(1+r_{t+1})\right) \Leftrightarrow$$

$$\nu\log(1-\sigma) + \nu \log\left(\frac{\nu}{z(1+\nu+\beta)}\right) + \log\left(\frac{1}{1+\nu+\beta}\right) + \log\left((1-\sigma)w_t^s\right)$$
$$+\beta\log\left(\frac{\beta}{1+\beta}(1+r_{t+1})\right) + \beta\log\left((1-\sigma-zn_t^s)w_t^s\right) \stackrel{!}{=}$$
$$\nu \log\left(\frac{\nu}{z(1+\nu+\beta)}\right) + \log\left(\frac{1}{1+\nu+\beta}\right) + \log(w_t^u) + \beta\log\left(\frac{\beta}{1+\beta}(1+r_{t+1})\right)$$
$$+\beta\log\left((1-zn_t^u)w_t^u\right) \Leftrightarrow$$

$$\nu\log(1-\sigma) + \log(1-\sigma) + \log(w_t^s) + \beta\log(1-\sigma-zn_t^s) + \beta\log(w_t^s) \stackrel{!}{=}$$
$$\log(w_t^u) + \beta\log(1-zn_t^u) + \beta\log(w_t^u) \Leftrightarrow$$

$$(1+\nu)\log(1-\sigma) + \log(w_t^s)(1+\beta) + \beta\log(1-\sigma-zn_t^s) \stackrel{!}{=}$$
$$\log(w_t^u)(1+\beta) + \beta\log(1-zn_t^u) \Leftrightarrow$$

$$(1+\nu)\log(1-\sigma) + \beta\big(\log(1-\sigma-zn_t^s) - \log(1-zn_t^u)\big) \stackrel{!}{=}$$
$$(1+\beta)(\log(w_t^u) - \log(w_t^s)).$$

Rewriting and substituting the optimal number of children n_t^i yields the ratio of the wage rate that agrees with Chen (2007, page 45)

$$\log\left[(1-\sigma)^{1+\nu}\left(\frac{1-\sigma-zn_t^s}{1-zn_t^u}\right)^{\beta}\right] = \log\left[\left(\frac{w_t^u}{w_t^s}\right)^{1+\beta}\right] \Leftrightarrow$$
$$\frac{w_t^u}{w_t^s} = \left((1-\sigma)^{1+\nu}\left(\frac{1-\sigma-zn_t^s}{1-zn_t^u}\right)^{\beta}\right)^{\frac{1}{1+\beta}} \Leftrightarrow$$
$$= \left((1-\sigma)^{1+\nu+\beta}\right)^{\frac{1}{1+\beta}} \Leftrightarrow$$
$$= (1-\sigma)^{\frac{1+\nu+\beta}{1+\beta}}. \quad (5.23)$$

The individual would be in a position of indifference if equation (5.23) holds, i.e. if the wage differential is equal to $(1-\sigma)^{\frac{1+\nu+\beta}{1+\beta}}$. The utility they derive from either choice is the same and there is no incentive for workers to switch between the two categories so the above equation describes the equilibrium.

From equation (5.23), it also follows that the wage of the unskilled worker is lower than that of the skilled worker, i.e. $w_t^u < w_t^s$, since the exponent is greater than 1, but $1-\sigma$ is less than 1, thus making the ratio overall smaller than 1.

5.2.2 Production

As in Chapter 4, N_t represents the total population of *workers* in period t. The ratios of "skilled" workers and "unskilled" workers to the adult population in period t are now denoted as ϕ_t and $(1-\phi_t)$ respectively

$$\phi_t = \frac{N_t^s}{N_t} \qquad \text{and} \qquad (1-\phi_t) = \frac{N_t^u}{N_t}. \tag{5.24}$$

In contrast to Chapter 3, the labor forces of "skilled" and "unskilled" workers are no longer assumed as exogenous. The aggregate "skilled" labor $L_{s,t}$ and aggregate "unskilled" labor $L_{u,t}$ result from the following equations

$$L_{s,t} = (1-\sigma-zn_t^s)\phi_t N_t \qquad \text{and} \tag{5.25}$$
$$L_{u,t} = (1-zn_t^u)(1-\phi_t)N_t. \tag{5.26}$$

Substituting (5.24) into equations (5.25) and (5.26) yields

$$L_{s,t} = (1-\sigma-zn_t^s)\phi_t N_t = (1-\sigma-zn_t^s)N_t^s,$$
$$L_{u,t} = (1-zn_t^u)(1-\phi_t)N_t = (1-zn_t^u)N_t^u.$$

This implies that the adult population, N_t, does not represent the labor force, as time is also dedicated to child-rearing, as well as to skill investment if an individual decides to become "skilled".

The total output Y_t is produced by utilizing traditional physical capital K_t, automation capital P_t, "skilled" labor $L_{s,t}$ and "unskilled" labor $L_{u,t}$ following the production function

$$Y_t = K_t^{\alpha}\left[(1-\theta){L_{s,t}}^{\gamma} + \theta(P_t + L_{u,t})^{\gamma}\right]^{\frac{1-\alpha}{\gamma}}. \tag{5.27}$$

The production function is a nested CES function, with "skilled" labor forming one subgroup, and "unskilled" workers and automation capital forming another subgroup.

Since the production function remains unchanged from Chapter 3, the factor rewards stay the same and are derived from (3.8) and (3.9)

$$w_t^s = (1-\alpha)Y_t \frac{(1-\theta)L_{s,t}^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma} + \theta(P_t + L_{u,t})^{\gamma}}, \tag{5.28}$$

$$w_t^u = (1-\alpha)Y_t \frac{\theta(P_t + L_{u,t})^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma} + \theta(P_t + L_{u,t})^{\gamma}}. \tag{5.29}$$

Therefore, the ratio of the wage rate of "unskilled" workers to the "skilled" workers v_t is

$$v_t = \frac{w_{u,t}}{w_{s,t}} = \frac{\theta}{1-\theta} \frac{(P_t + L_{u,t})^{\gamma-1}}{L_{s,t}^{\gamma-1}}. \tag{5.30}$$

5.3 Results

The main objective of the following calculations is to calculate the factor inputs that determine the ratio of "skilled" workers to the adult population. Understanding which factors affect the ratio enables to analyse the dynamics of the model.

In order to determine this dependence, equation (5.23), which results from the utility maximization of the household side, and equation (5.30), which results from the profit maximization of the production side, are set equal to each other

$$\begin{aligned} (1-\sigma)^{\frac{1+\nu+\beta}{1+\beta}} &= \frac{\theta}{1-\theta} \frac{(P_t + L_{u,t})^{\gamma-1}}{L_{s,t}^{\gamma-1}} \qquad \Leftrightarrow \\ \underbrace{\left[\frac{1-\theta}{\theta}(1-\sigma)^{\frac{1+\nu+\beta}{1+\beta}}\right]^{\frac{1}{\gamma-1}}}_{:=\Lambda} &= \frac{P_t + L_{u,t}}{L_{s,t}}. \end{aligned}$$

The aggregate automation capital P_t is determined by the per potential worker automation capital times the total population of workers $P_t = p_t N_t$. Using this equation as well as equation (5.25) and (5.26), the subsequent steps are derived

$$\begin{aligned} \Lambda &= \frac{p_t N_t + (1 - z n_t^u)(1-\phi_t)N_t}{(1-\sigma - z n_t^s)\phi_t N_t} \qquad \Leftrightarrow \\ p_t &= \Lambda(1-\sigma - z n_t^s)\phi_t - (1 - z n_t^u)(1-\phi_t). \end{aligned}$$

By considering the distinct fertility rates for "skilled" and "unskilled" workers and rewriting terms yields

$$\begin{aligned}
\Lambda(1-\sigma-zn_t^s)\phi_t-(1-\phi_t)(1-zn_t^u) &= p_t && \Leftrightarrow\\
\Lambda\phi_t\left(1-\sigma-\frac{\nu(1-\sigma)}{1+\nu+\beta}\right)-(1-\phi_t)\left(1-\frac{\nu}{1+\nu+\beta}\right) &= p_t && \Leftrightarrow\\
\Lambda\phi_t\Bigg((1-\sigma)\underbrace{\Big(1-\frac{\nu}{1+\nu+\beta}\Big)}_{:=\Omega}\Bigg)-(1-\phi_t)\underbrace{\Bigg(1-\frac{\nu}{1+\nu+\beta}\Bigg)}_{\Omega} &= p_t.
\end{aligned}$$

The factor inputs that influence the ratio of the "skilled" workers to the adult population ϕ_t are obtained

$$\begin{aligned}
\Lambda\phi_t(1-\sigma)\Omega-(1-\phi_t)\Omega &= p_t && \Leftrightarrow\\
\phi_t\big(\Lambda(1-\sigma)\Omega+\Omega\big) &= p_t+\Omega && \Leftrightarrow\\
\phi_t &= \frac{p_t+\Omega}{\Omega(1+\Lambda(1-\sigma))}, && (5.31)
\end{aligned}$$

with

$$\Lambda=\left(\frac{1-\theta}{\theta}(1-\sigma)^{\frac{1+\nu+\beta}{1+\beta}}\right)^{\frac{1}{\gamma-1}} \quad \text{and} \quad \Omega=\left(1-\frac{\nu}{1+\nu+\beta}\right).$$

Thus, it can be seen that the ratio of "skilled" workers ϕ_t to the total population N_t depends only on p_t. Other factor inputs are not involved in determining this ratio. This means ϕ_t is a function of p_t: $\phi_t(p_t)$. The reason for the dependence of the ratio solely on the automation capital per worker p_t can be attributed to the impact of automation on the skill requirements in the workforce.

Analyzing the derivative with respect to p_t

$$\frac{d\phi_t}{dp_t}=\frac{1}{\Omega(1+\Lambda(1-\sigma))} \quad > 0 \tag{5.32}$$

reveals that the ratio of "skilled" workers ϕ_t is increasing as the level of automation capital p_t increases.

When automation capital per worker p_t increases, it generally indicates a higher level of technological advancement and automation in the production processes. This increased automation often leads to a higher demand for "skilled" workers who possess the knowledge and expertise to operate, maintain or optimize these advanced technologies.

Conversely, when automation capital per worker p_t decreases, it suggests a lower level of automation and technological advancement. As a result, the demand for "skilled" workers may be relatively lower, and the workforce composition may include a larger proportion of "unskilled" workers.

In summary, the level of automation capital per worker p_t serves as an essential determinant of the skill requirements within the workforce. As the automation capital per worker changes, it directly influences the ratio of "skilled" workers to the total population, as "skilled" workers become more or less necessary based on the level of automation.

5.3.1 Steady States and Dynamics

In this section, the discussion focuses on three distinct levels of automation capital

(i) $p_t = 0$,

(ii) $p_t > 0$, and

(iii) $p_t \geq \Omega(1-\sigma)\Omega := \overline{p}$.

In the first case, there is no automation capital investment. This suggests that the level of automation remains stagnant or does not exist, indicating a lack of technological advancement or automation in the production processes. For $p_t = 0$ the ratio of "skilled" workers is exogenously given as follows

$$\phi = \frac{\Omega}{\Omega(1+\Lambda(1-\sigma))}. \tag{5.33}$$

The second case, where $p_t > 0$, is straightforward. It implies that there is some level of investment in automation capital.

The third case arises from the assumption that the ratio of "skilled" workers ϕ_t to the total number of workers cannot exceed 1. This implies that the proportion of "skilled" workers in the workforce cannot be higher than the total number of workers.

Equation (5.31) combined with $\phi_t = 1$, allows determining this value of p_t at which the transition occurs, resulting in a scenario where all workers in the population possess the category "skilled".

$$\begin{aligned} p_t + \Omega &= \Omega\big(1 + \Lambda(1-\sigma)\big) && \Leftrightarrow \\ p_t &= \Omega\big(1 + \Lambda(1-\sigma) - 1\big) && \Leftrightarrow \\ p_t &= \Omega(1-\sigma)\Omega := \overline{p}, \end{aligned}$$

so when $p_t \geq \Omega(1-\sigma)\Omega$ the ratio of "skilled" worker $\phi_t = 1$. This threshold is a condition for the population to consist exclusively of "skilled" workers.

Automation capital $p_t = 0$

This scenario focuses on investments only in physical capital K_t. As mentioned in previous chapters this case occurs, when the physical capital stock K_t is too low. Furthermore, this case also arises because the interest rate for physical capital is higher than the interest rate for automation capital.

The equilibrium is determined by (5.46), as derived on the following pages, with $p_t = 0$ and the ratio of the "skilled" workers ϕ_t, as given in equation (5.33) is constant.

$$\begin{aligned} k_{t+1} &= \frac{\beta z(1-\alpha)}{\nu(1-\sigma\phi)} k_t^{\alpha} \big[(1-\theta)A_1^{\gamma} + \theta B_1^{\gamma}\big]^{\frac{1-\alpha}{\gamma}} \left(\frac{\phi(1-\sigma)(1-\theta)A_1^{\gamma-1} + (1-\phi)\theta B_1^{\gamma-1}}{(1-\theta)A_1^{\gamma} + \theta B_1^{\gamma}}\right) \\ &= \frac{\beta z(1-\alpha)}{\nu(1-\sigma\phi)} k_t^{\alpha} \big[(1-\theta)A_1^{\gamma} + \theta B_1^{\gamma}\big]^{\frac{1-\alpha-\gamma}{\gamma}} \big(\phi(1-\sigma)(1-\theta)A_1^{\gamma-1} + (1-\phi)\theta B_1^{\gamma-1}\big),. \end{aligned}$$

with

$$A_1 = \frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi \qquad \text{and} \qquad B_1 = \frac{\beta+1}{1+\nu+\beta}(1-\phi),$$

A possible steady state is therefore

$$\begin{aligned} k_t^{1-\alpha} &= \frac{\beta z(1-\alpha)}{\nu(1-\sigma\phi)} \big[(1-\theta)A_1^{\gamma} + \theta B_1^{\gamma}\big]^{\frac{1-\alpha-\gamma}{\gamma}} \big(\phi(1-\sigma)(1-\theta)A_1^{\gamma-1} + (1-\phi)\theta B_1^{\gamma-1}\big) \Leftrightarrow \\ k^* &= \left(\frac{\beta z(1-\alpha)}{\nu(1-\sigma\phi)} \big(\phi(1-\sigma)(1-\theta)A_1^{\gamma-1} + (1-\phi)\theta B_1^{\gamma-1}\big)\right)^{\frac{1}{1-\alpha}} \\ &\quad \cdot \big[(1-\theta)A_1^{\gamma} + \theta B_1^{\gamma}\big]^{\frac{1-\alpha-\gamma}{\gamma(1-\alpha)}}. \end{aligned} \tag{5.34}$$

The wages derived in Chapter 3, from equations (3.8) and (3.9) can be expressed as a function of k_t as well

$$\begin{aligned} w_t^s &= (1-\alpha)Y_t \frac{(1-\theta)L_{s,t}^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma} + \theta L_{u,t}^{\gamma}} \\ &= (1-\alpha)Y_t \frac{(1-\theta)\big[(1-\sigma-zn_t^s)\phi N_t\big]^{\gamma-1}}{(1-\theta)\big[(1-\sigma-zn_t^s)\phi N_t\big]^{\gamma} + \theta\big[(1-zn_t^u)(1-\phi)N_t\big]^{\gamma}} \\ &= (1-\alpha)y_t \frac{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi\big]^{\gamma-1}}{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi\big]^{\gamma} + \theta\big[\frac{\beta+1}{1+\nu+\beta}(1-\phi)\big]^{\gamma}} \\ &= (1-\alpha)y_t \frac{(1-\theta)A_1^{\gamma-1}}{(1-\theta)A_1^{\gamma} + \theta B_1^{\gamma}}, \end{aligned} \tag{5.35}$$

$$\begin{aligned} w_t^u &= (1-\alpha)Y_t \frac{\theta L_{u,t}^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma} + \theta L_{u,t}^{\gamma}} \\ &= (1-\alpha)Y_t \frac{\theta\big[(1-zn_t^u)(1-\phi)N_t\big]^{\gamma-1}}{(1-\theta)\big[(1-\sigma-zn_t^s)\phi N_t\big]^{\gamma} + \theta\big[(1-zn_t^u)(1-\phi)N_t\big]^{\gamma}} \\ &= (1-\alpha)y_t \frac{\theta\big[\frac{\beta+1}{1+\nu+\beta}(1-\phi)\big]^{\gamma-1}}{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi\big]^{\gamma} + \theta\big[\frac{\beta+1}{1+\nu+\beta}(1-\phi)\big]^{\gamma}} \\ &= (1-\alpha)y_t \frac{\theta B_1^{\gamma-1}}{(1-\theta)A_1^{\gamma} + \theta B_1^{\gamma}}, \end{aligned} \tag{5.36}$$

with $y_t = k_t^{\alpha}[(1-\theta)l_s^{\gamma} + \theta l_u^{\gamma}]^{\frac{1-\alpha}{\gamma}}$.

In this context, the expression

$$\frac{w_{s,t}}{w_{u,t}} = \frac{(1-\theta)A_1^{\gamma-1}}{\theta B_1^{\gamma-1}}, \tag{5.37}$$

represents the *skill premium*. The skill premium refers to the difference in wages between "skilled" workers and "unskilled" workers, and it is often an essential indicator of income inequality within an economy. When the skill premium is high, it means that "skilled" workers earn significantly more than "unskilled" workers. Conversely, a lower skill premium suggests a tighter wage gap between the two groups.

Automation capital $p_t > 0$

To calculate the dynamics of p_t and k_t the factor rewards from Chapter 3, specifically equations (3.9) and (3.10) are resolved. In the case where $p_t > 0$, the no-arbitrage condition $R_t^P = R_t^K$ must be satisfied. This assumption leads to the next calculations, as seen in equation (3.13)

$$\begin{aligned}
\alpha\frac{Y_t}{K_t} &= (1-\alpha)Y_t\frac{\theta(P_t+L_{u,t})^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma}+\theta(P_t+L_{u,t})^{\gamma}} \qquad &\Leftrightarrow\\
K_t &= \frac{\alpha}{(1-\alpha)}\frac{(1-\theta)L_{s,t}^{\gamma}+\theta(P_t+L_{u,t})^{\gamma}}{\theta(P_t+L_{u,t})^{\gamma-1}} \qquad &\Leftrightarrow\\
&= \frac{\alpha}{(1-\alpha)}\frac{(1-\theta)\big[(1-\sigma-zn_t^s)\phi_t N_t\big]^{\gamma}+\theta\big[(P_t+(1-zn_t^u)(1-\phi_t)N_t)\big]^{\gamma}}{\theta\big[P_t+(1-zn_t^u)(1-\phi_t)N_t\big]^{\gamma-1}},
\end{aligned}$$

and in per worker terms with the fertility rates inserted,

$$\begin{aligned}
k_t = \frac{K_t}{N_t} &= \frac{\alpha}{(1-\alpha)}\frac{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi_t\big]^{\gamma}+\theta\big[p_t+\frac{\beta+1}{1+\nu+\beta}(1-\phi_t)\big]^{\gamma}}{\theta\big[p_t+\frac{\beta+1}{1+\nu+\beta}(1-\phi_t)\big]^{\gamma-1}} \qquad \Leftrightarrow\\
&= \frac{\alpha}{(1-\alpha)}\frac{(1-\theta)A^{\gamma}+\theta(p_t+B)^{\gamma}}{\theta(p_t+B)^{\gamma-1}} \qquad (5.38)
\end{aligned}$$

with

$$A = \frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi_t \qquad \text{and} \qquad B = \frac{\beta+1}{1+\nu+\beta}(1-\phi_t).$$

Therefore k_t is a function that is determined by p_t, $k_t(p_t)$. Comparing (5.38) with (3.14), it becomes evident that with endogenous fertility and endogenous skill investment, the per worker capital depends also on the parameters of the utility function and the production function, and has a similar structure as in Chapter 3 with A replacing l_s and B replacing l_u. That is, the share of "skilled" and "unskilled" population is no longer constant.

Considering the derivation $\frac{\partial k_t}{\partial p_t} > 0$, results in the conclusion that an increase in automation capital corresponds to an increase in traditional physical capital.

The wages derived in Chapter 3 (equations (3.8) and (3.9)) can be expressed as a function of k_t and p_t

$$
\begin{aligned}
w_t^s &= (1-\alpha)Y_t\frac{(1-\theta)L_{s,t}^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma}+\theta(P_t+L_{u,t})^{\gamma}}\\
&= (1-\alpha)Y_t\frac{(1-\theta)\big[(1-\sigma-zn_t^s)\phi_t N_t\big]^{\gamma-1}}{(1-\theta)\big[(1-\sigma-zn_t^s)\phi_t N_t\big]^{\gamma}+\theta\big[p_t N_t+(1-zn_t^u)(1-\phi_t)N_t\big]^{\gamma}}\\
&= (1-\alpha)y_t\frac{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi_t\big]^{\gamma-1}}{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi_t\big]^{\gamma}+\theta\big[p_t+\frac{\beta+1}{1+\nu+\beta}(1-\phi_t)\big]^{\gamma}}\\
&= (1-\alpha)y_t\frac{(1-\theta)A^{\gamma-1}}{(1-\theta)A^{\gamma}+\theta(p_t+B)^{\gamma}},
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
w_t^u &= (1-\alpha)Y_t\frac{\theta(P_t+L_{u,t})^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma}+\theta(P_t+L_{u,t})^{\gamma}}\\
&= (1-\alpha)Y_t\frac{\theta\big[p_t N_t+(1-zn_t^u)(1-\phi_t)N_t\big]^{\gamma-1}}{(1-\theta)\big[(1-\sigma-zn_t^s)\phi_t N_t\big]^{\gamma}+\theta\big[p_t N_t+(1-zn_t^u)(1-\phi_t)N_t\big]^{\gamma}}\\
&= (1-\alpha)y_t\frac{\theta\big[p_t+\frac{\beta+1}{1+\nu+\beta}(1-\phi_t)\big]^{\gamma-1}}{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi_t\big]^{\gamma}+\theta\big[p_t+\frac{\beta+1}{1+\nu+\beta}(1-\phi_t)\big]^{\gamma}}\\
&= (1-\alpha)y_t\frac{\theta(p_t+B)^{\gamma-1}}{(1-\theta)A^{\gamma}+\theta(p_t+B)^{\gamma}},
\end{aligned}
\tag{5.40}
$$

with $y_t = k_t^{\alpha}[(1-\theta)l_s^{\gamma}+\theta(p_t+l_u)^{\gamma}]^{\frac{1-\alpha}{\gamma}}$.

Since y_t is a function of both k_t and p_t, and k_t is shown to be only dependent on p_t due to the arbitrage condition, it follows that wages for both "skilled" and "unskilled" workers are determined solely by the automation capital per worker p_t.

In the case $p_t > 0$ the skill premium is

$$
\frac{w_{s,t}}{w_{u,t}} = \frac{(1-\theta)A^{\gamma-1}}{\theta(p_t+B)^{\gamma-1}}. \tag{5.41}
$$

As automation increases, the skill premium tends to rise. This is likely because automation often requires specialized skills to operate, leading to higher wages for "skilled" workers. However, when more workers transition into "skilled" positions due to the demand created by automation, the influx of "skilled" workers reduces the skill premium. This could happen as the supply of skilled workers increases, potentially leading to a more competitive job market and reducing the wage differences between "skilled" and "unskilled" workers.

Compared to the *wage differential* from Chapter 3, which represents the same ratio,

$$\frac{w_{s,t}}{w_{u,t}} = \frac{(1-\theta)L_{s,t}^{\gamma-1}}{\theta(P_t + L_{u,t})^{\gamma-1}}. \tag{5.42}$$

it can be inferred that in both models, the magnitude of the difference depends on the automation level and the proportion of "skilled" and "low-skilled" workers.

The study of the dynamics of the economy starts with the analysis of the market clearing condition of the capital market

$$S_{t+1} = K_{t+1} + P_{t+1} = \left[\phi_t s_t^s + (1-\phi_t) s_t^u\right] N_t. \tag{5.43}$$

This clearing condition refers to the equilibrium state where the demand and supply of capital in the market are balanced. It ensures that the quantity of capital demanded equals the quantity of capital supplied in the economy.

Unlike the previous chapters, the size of the adult population in the subsequent period N_t is determined through endogenous fertility and endogenous skill investment

$$\begin{aligned} N_{t+1} &= m_t N_t \\ &= \left[\phi_t n_t^s + (1-\phi_t) n_t^u\right] N_t. \end{aligned}$$

The share of "skilled" workers ϕ_t is also endogenous. Therefore the overall population growth is now endogenous: While n_t^s and n_t^u are still exogenous, i.e. depend only on parameters, the share of "skilled" and "unskilled" workers is endogenous and depends on p_t, so the total growth rate m_t is endogenous. This endogenous determination of population size introduces a dynamic element to the analysis, as changes in fertility rates and skill investments can have long-term effects that in turn have an impact on the equilibrium.

Rewriting equation (5.43) in terms of capital per worker we obtain the following expression

$$
\begin{aligned}
s_{t+1} = k_{t+1} + p_{t+1} &= \left[\phi_t s_t^s + (1-\phi_t) s_t^u\right] \frac{N_t}{N_{t+1}} \qquad \Leftrightarrow \\
= k_{t+1} + p_{t+1} &= \frac{\phi_t s_t^s + (1-\phi_t) s_t^u}{m_t}. \qquad (5.44)
\end{aligned}
$$

Inserting the savings rate for the "skilled" and "unskilled" worker s_t^s and s_t^u respectively, the population growth factor m_t and in addition the optimal number of children for both worker categories yields

$$
\begin{aligned}
k_{t+1} + p_{t+1} &= \frac{\phi_t \frac{\beta(1-\sigma)}{1+\nu+\beta} w_t^s + (1-\phi_t)\frac{\beta}{1+\nu+\beta} w_t^u}{\phi_t n_t^s + (1-\phi_t) n_t^u} \\
&= \frac{\phi_t \frac{\beta(1-\sigma)}{1+\nu+\beta} w_t^s + (1-\phi_t)\frac{\beta}{1+\nu+\beta} w_t^u}{\phi_t \frac{(1-\sigma)\nu}{z(1+\nu+\beta)} + (1-\phi_t)\frac{\nu}{z(1+\nu+\beta)}} \\
&= \frac{z\big(\phi_t \beta(1-\sigma) w_t^s + (1-\phi_t)\beta w_t^u\big)}{\phi_t(1-\sigma)\nu + (1-\phi_t)\nu} \\
&= \frac{\beta z}{\nu(1-\sigma\phi_t)} \big[\phi_t(1-\sigma) w_t^s + (1-\phi_t) w_t^u\big]. \qquad (5.45)
\end{aligned}
$$

With the expressions for the wages w_t^s and w_t^u from (5.39) and (5.40) it follows

$$
\begin{aligned}
k_{t+1} + p_{t+1} = \\
&= \frac{\beta z}{\nu(1-\sigma\phi_t)} \\
&\cdot \left(\frac{\phi_t(1-\sigma)(1-\alpha)y_t(1-\theta)A^{\gamma-1} + (1-\phi_t)(1-\alpha)y_t\theta(p_t+B)^{\gamma-1}}{(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma}} \right) \\
&= \underbrace{\frac{\beta z(1-\alpha)}{\nu(1-\sigma\phi_t)}}_{:=h_1} y_t \left(\frac{\phi_t(1-\sigma)(1-\theta)A^{\gamma-1} + (1-\phi_t)\theta(p_t+B)^{\gamma-1}}{(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma}} \right) \\
&= h_1 \underbrace{\underbrace{\left[\frac{\alpha}{(1-\alpha)} \frac{(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma}}{\theta(p_t+B)^{\gamma-1}} \right]^{\alpha}}_{k_t} \left[(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma} \right]^{\frac{1-\alpha}{\gamma}}}_{y_t} \\
&\cdot \left(\frac{\phi_t(1-\sigma)(1-\theta)A^{\gamma-1} + (1-\phi_t)\theta(p_t+B)^{\gamma-1}}{(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma}} \right) \\
&= \underbrace{h_1 \left(\frac{\alpha}{(1-\alpha)\theta} \right)^{\alpha}}_{:=h} \left[(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma} \right]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \frac{1}{(p_t+B)^{(\gamma-1)\alpha}} \\
&\cdot \left(\phi_t(1-\sigma)(1-\theta)A^{\gamma-1} + (1-\phi_t)\theta(p_t+B)^{\gamma-1} \right) \\
&= h \left[(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma} \right]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \frac{1}{(p_t+B)^{(\gamma-1)\alpha}} \\
&\cdot \left(\phi_t(1-\sigma)(1-\theta)A^{\gamma-1} + (1-\phi_t)\theta(p_t+B)^{\gamma-1} \right).
\end{aligned} \tag{5.46}
$$

Furthermore, from the market equilibrium, where capital demand of firms equals capital supplies of consumers and recalling that the non-arbitrage condition (5.38) has to hold it follows

$$
s_{t+1} = p_{t+1} + \underbrace{\frac{\alpha}{(1-\alpha)} \frac{(1-\theta)A^{\gamma} + \theta(p_t+B)^{\gamma}}{\theta(p_t+B)^{\gamma-1}}}_{k_{t+1}}. \tag{5.47}
$$

Combining both equations (5.46) and (5.47) implies that for a possible steady state for the case that the automation capital $p_t > 0$ it has to hold

$$\begin{aligned}p^* + k^* &= p_{t+1} + k_{t+1} \\ &= p^* + \frac{\alpha}{(1-\alpha)} \frac{(1-\theta)A^\gamma + \theta(p^* + B)^\gamma}{\theta(p^* + B)^{\gamma-1}} \\ &\overset{!}{=} h\Big[(1-\theta)A^\gamma + \theta(p^* + B)^\gamma\Big]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} \frac{1}{(p_t + B)^{(\gamma-1)\alpha}} \\ &\cdot \Big(\phi_t(1-\sigma)(1-\theta)A^{\gamma-1} + (1-\phi_t)\theta(p^* + B)^{\gamma-1}\Big).\end{aligned} \tag{5.48}$$

Automation capital $p_t \geq \overline{p}$

As previously mentioned, when the capital stock of automation capital p_t reaches the threshold $\overline{p} = \Omega(1-\sigma)\Omega$, it results in an economy that only requires "skilled" workers. In this scenario, the ratio of "skilled" workers ϕ_t to the total workforce becomes equal to 1.

This condition implies that automation capital has reached a level where it can effectively replace the need for unskilled labor. As a result, the economy becomes highly reliant on "skilled" workers who invest in education to posses the qualifications to manage the automated systems. The ratio of "skilled" workers to the total workforce reaching 1 suggests that there is no longer a demand for "unskilled" workers as their roles have been replaced by automation.

This highlights the potential impact of automation on the composition of the workforce, particularly in sectors where automation capital plays a significant role.

Again the wages are derived as a function of k_t and p_t

$$\begin{aligned}w_t^s &= (1-\alpha)Y_t \frac{(1-\theta)L_{s,t}^{\gamma-1}}{(1-\theta)L_{s,t}^\gamma + \theta(P_t + L_{u,t})^\gamma} \\ &= (1-\alpha)Y_t \frac{(1-\theta)\big[(1-\sigma-zn_t^s)N_t\big]^{\gamma-1}}{(1-\theta)\big[(1-\sigma-zn_t^s)\big]^\gamma + \theta\big(p_t N_t\big)^\gamma} \\ &= (1-\alpha)y_t \frac{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\big]^{\gamma-1}}{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\big]^\gamma + \theta p_t^\gamma} \\ &= (1-\alpha)y_t \frac{(1-\theta)A_2^{\gamma-1}}{(1-\theta)A_2^\gamma + \theta p_t^\gamma},\end{aligned} \tag{5.49}$$

$$
\begin{aligned}
w_t^u &= (1-\alpha)Y_t \frac{\theta(P_t+L_{u,t})^{\gamma-1}}{(1-\theta)L_{s,t}^{\gamma}+\theta(P_t+L_{u,t})^{\gamma}} \\
&= (1-\alpha)Y_t \frac{\theta\big(p_tN_t\big)^{\gamma-1}}{(1-\theta)\big[(1-\sigma-zn_t^s)N_t\big]^{\gamma}+\theta\big(p_tN_t\big)^{\gamma}} \\
&= (1-\alpha)y_t \frac{\theta p_t^{\gamma-1}}{(1-\theta)\big[\frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi_t\big]^{\gamma}+\theta\big[p_t+\frac{\beta+1}{1+\nu+\beta}(1-\phi_t)\big]^{\gamma}} \\
&= (1-\alpha)y_t \frac{\theta p_t^{\gamma-1}}{(1-\theta)A_2^{\gamma}+\theta p_t^{\gamma}}, \qquad (5.50)
\end{aligned}
$$

with $y_t = k_t^{\alpha}[(1-\theta)l_s^{\gamma}+\theta(p_t+l_u)^{\gamma}]^{\frac{1-\alpha}{\gamma}}$, and $A_2 = \frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}$.

The skill premium can now be derived

$$
\frac{w_{s,t}}{w_{u,t}} = \frac{(1-\theta)A_2^{\gamma-1}}{\theta p_t^{\gamma-1}}. \qquad (5.51)
$$

The list below presents the three distinct levels of automation and their respective skill premium $\frac{w_{s,t}}{w_{u,t}}$

(i) for $p_t = 0 : \frac{(1-\theta)A_1^{\gamma-1}}{\theta B_1^{\gamma-1}}$,

(ii) $p_t > 0 : \frac{(1-\theta)A^{\gamma-1}}{\theta(p_t+B)^{\gamma-1}}$, and

(iii) $p_t \geq \overline{p} : \frac{(1-\theta)A_2^{\gamma-1}}{\theta p_t^{\gamma-1}}$

with $A = \frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi_t$, $A_1 = \frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}\phi$, $A_2 = \frac{(\beta+1)(1-\sigma)}{1+\nu+\beta}$, and $B = \frac{\beta+1}{1+\nu+\beta}(1-\phi_t)$.

It can be observed that with an increasing level of automation, the skill premium becomes higher. Workers who possess more advanced skills or specialized knowledge tend to command a higher wage compared to "unskilled" workers who can be replaced more easily.

Dynamics

In this model, which considers endogenous fertility and skill investment, economic growth is not possible. Similar to Chapter 3,

$$
\frac{p_{t+1}+k_{t+1}}{p_t+k_t}
$$

converges to zero as both k_t and p_t tend to infinity. Furthermore, the numerator can be rewritten as

$$p_{t+1} + k_{t+1} = (1+g)(p_t + k_t),$$

where g represents the growth rate. If $\frac{p_{t+1}+k_{t+1}}{p_t+k_t}$ converges to 0, then the growth rate g must converges to -1 and becomes negative.

To show that the growth rate is negative, the following inequalities are used

$$\begin{aligned} p_t + k_t &= p_t + \frac{\alpha}{(1-\alpha)} \frac{(1-\theta)A^\gamma + \theta(p_t+B)^\gamma}{\theta(p_t+B)^{\gamma-1}} \\ &> \frac{\alpha}{(1-\alpha)} \frac{(p_t+B)^\gamma}{(p_t+B)^{\gamma-1}} = \frac{\alpha}{(1-\alpha)}(p_t+B), \end{aligned} \tag{5.52}$$

and

$$\begin{aligned} \Big[(1-\theta)A^\gamma + \theta(p_t+B)^\gamma\Big]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}} &\le 2^{\frac{1}{\gamma}-1}\Big[(1-\theta)^{\frac{1}{\gamma}}A + \theta^{\frac{1}{\gamma}}(p_t+B)\Big]^{(1-\alpha)(1-\gamma)} \\ &\le 2^{\frac{1}{\gamma}-1}\Big[(1-\theta)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}A^{(1-\alpha)(1-\gamma)} \\ &\quad + \theta^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}(p_t+B)^{(1-\alpha)(1-\gamma)}\Big]. \end{aligned} \tag{5.53}$$

Inequality (5.53) holds due to the following equation, where $x = (1-\theta)A^\gamma$ and $y = \theta(p_t+B)^\gamma$ and $q := \frac{1}{\gamma}$

$$x^q + y^q \le (x+y)^q \le 2^{q-1}(x^q+y^q) \quad \text{for} \quad x, y \ge 0 \quad \text{and} \quad q \in [1, \infty],$$

and the property of subadditivity for the function $f(x) := x^{(1-\alpha)(1-\gamma)}$.[1]

With equation (5.46) and using inequality (5.52) at (1), the ratio is estimated

[1] For further explanation, see Section 3.3.2.

from above

$$
\begin{aligned}
\frac{p_{t+1}+k_{t+1}}{p_t+k_t} &= h\Big[(1-\theta)A^{\gamma}+\theta(p_t+B)^{\gamma}\Big]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}\frac{1}{(p_t+B)^{(\gamma-1)\alpha}} \\
&\quad\cdot\Big(\phi_t(1-\sigma)(1-\theta)A^{\gamma-1}+(1-\phi_t)\theta(p_t+B)^{\gamma-1}\Big)\frac{1}{p_t+k_t} \qquad (5.54)\\
&\overset{(1)}{\leq} h\Big[(1-\theta)A^{\gamma}+\theta(p_t+B)^{\gamma}\Big]^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}\frac{1}{(p_t+B)^{(\gamma-1)\alpha}} \\
&\quad\cdot\Big(\phi_t(1-\sigma)(1-\theta)A^{\gamma-1}+(1-\phi_t)\theta(p_t+B)^{\gamma-1}\Big)\frac{(1-\alpha)}{\alpha(p_t+B)}. \qquad (5.55)
\end{aligned}
$$

Furthermore, applying inequality (5.53) yields

$$
\begin{aligned}
\frac{p_{t+1}+k_{t+1}}{p_t+k_t} \leq \underbrace{\frac{h(1-\alpha)2^{\frac{1}{\gamma}-1}}{\alpha}}_{:=\tilde{h}}\frac{1}{(p_t+B)}\Big[&(1-\theta)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}A^{(1-\alpha)(1-\gamma)} \\
&+\theta^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}(p_t+B)^{(1-\alpha)(1-\gamma)}\Big]\frac{1}{(p_t+B)^{(\gamma-1)\alpha}} \\
&\cdot\Big(\phi_t(1-\sigma)(1-\theta)A^{\gamma-1}+(1-\phi_t)\theta(p_t+B)^{\gamma-1}\Big).
\end{aligned}
$$

Rearranging and expanding terms lead to

$$
\begin{aligned}
\frac{p_{t+1}+k_{t+1}}{p_t+k_t} \leq\ & \frac{\tilde{h}}{(p_t+B)^{(\gamma-1)\alpha+1}}(1-\theta)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}A^{(1-\alpha)(1-\gamma)}\phi_t(1-\sigma)(1-\theta)A^{\gamma-1} \\
&+\frac{\tilde{h}}{(p_t+B)^{(\gamma-1)\alpha+1}}(1-\theta)^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}A^{(1-\alpha)(1-\gamma)}(1-\phi_t)\theta(p_t+B)^{\gamma-1} \\
&+\frac{\tilde{h}}{(p_t+B)^{(\gamma-1)\alpha+1}}\theta^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}(p_t+B)^{(1-\alpha)(1-\gamma)}\phi_t(1-\sigma)(1-\theta)A^{\gamma-1} \\
&+\frac{\tilde{h}}{(p_t+B)^{(\gamma-1)\alpha+1}}\theta^{\frac{(1-\alpha)(1-\gamma)}{\gamma}}(p_t+B)^{(1-\alpha)(1-\gamma)}(1-\phi_t)\theta(p_t+B)^{\gamma-1}.
\end{aligned}
$$

Taking into account that p_t approaches infinity, it follows that ϕ_t, the ratio of "skilled" workers, converges to 1, as mentioned earlier, indicating that only "skilled" employees are needed. Therefore, the factors A and B for "skilled" and "unskilled" workers respectively, are no longer endogenous and are determined in the model by the exogenous parameters, whereas B completely vanishes.

Overall, this results in

$$\frac{p_{t+1} + k_{t+1}}{p_t + k_t} \leq h_1 p_t^{-1-(\gamma-1)\alpha} + h_3 p_t^{-\gamma},$$

with the constants h_1 and h_3 encompassing all constants being independent of p_t. Both parameters, α and γ, lie within the interval $(0, 1)$, leading to negative exponents in the above equation. As a consequence, when p_t tends towards infinity, the expression converges to zero. This indicates that once a specific threshold of automation capital is reached, the growth rate becomes negative, leading to a decline in the overall capital stock. Consequently, the potential for sustained economic growth over the long term is absent.

5.4 Conclusion

In general, also in a model of endogenous skill investment and endogenous fertility, sustained economic growth, will not be achieved. However, transition dynamics do occur: Investing in automation capital is causing a shift in the composition of the workforce. The proportion of "skilled" workers is increasing due to the higher demand associated with the rise of automation, leading to a scenario where there will ultimately be only "skilled" workers. Less "skilled" workers, who experience negative impacts from automation such as job loss, increased job turnover, and the need to adjust to new requirements, will no longer be present.

Another result that can be obtained is that the overall fertility rate

$$m_t = \phi_t n_t^s + (1 - \phi_t) n_t^u$$

decreases due to an increase in automation. As investments in automation capital become higher, the share of "skilled" workers ϕ_t rises, as there is a higher demand, which in turn reduces $(1 - \phi_t)$ the proportion of "unskilled" workers. Given that the fertility rate among "skilled" workers, denoted as n_t^s, is lower than that of "unskilled" workers n_t^u, it is reasonable to assert that the quantity of children decreases as automation levels increase.

This finding corresponds with the outcome derived in the study conducted by Chen (2007). Chen states that a decrease in mortality leads to a trade-off between quality and quantity of children. This occurs because reduced mortality rates result in lower fertility for both "skilled" and "unskilled" individuals, resulting in fewer children. At the same time, it leads to an improvement in the overall quality of adults by elevating the average level of education in society. When comparing Chen's paper with the present model, which incorporates automation instead of mortality, it can be asserted that this model yields similar outcomes.

Moreover, the OLG model with endogenous fertility and endogenous skill investment serves as a theoretical foundation for the empirical study conducted by Matysiak et al. (2022). Their study focuses on the impact of industrial robots on labor markets and how this influence may extend to fertility rates. It examines whether these long-term changes, driven by the adoption of industrial robots, affect regional fertility rates in six European countries. The findings indicate that regions with lower technological advancement, such as Czechia and Poland, experience less negative effects on fertility due to robot adoption. This is consistent with the theoretical findings of this model, as they indicate that the fertility rate of "unskilled" workers is higher than that of "skilled" workers. Moreover, due to the comparatively lower level of technological advancement, the proportion of "skilled" workers is also lower compared to countries such as Germany.

Matysiak et al. (2022) also state that country differences in fertility effects are observed, with Germany exhibiting the most pronounced negative impact, likely due to its advanced automation. Italy and the UK also experience negative effects, though less severe. Czechia and Poland, with lower labor costs, see fewer disruptive effects on fertility. The study highlights that robot adoption tends to affect fertility more in regions with highly educated populations. These results closely align with the theoretical outcomes derived from this thesis.

Chapter 6

Discussion

The central objective of this master's thesis is to explore how technological advancements impact individuals' decisions regarding family planning and education, and how these decisions, in turn, affect the economy. The thesis intends to achieve this goal through the development and analysis of an OLG model that incorporates automation, fertility decisions, and skill investments. The model is designed to discuss the interactions between these factors and their implications for economic outcomes. The different chapters of the thesis focus on progressively incorporating elements into the model, such as different skill levels, endogenous fertility decisions, and endogenous skill investments and endogenous fertility.

Chapter 2 reveals that investments in automation in the canonical OLG model, following Gasteiger and Prettner (2020) do not lead to economic growth. Instead, the introduction of automation can result in significant job displacements, resulting in challenges in finding alternative employment opportunities. Consequently, this disruption leads to a reduction in wage income, thereby affecting consumption and saving patterns. The economy is not able to grow due to the effect of automation on the wages.

The difference in wages are first introduced in Chapter 3 (Lankisch (2017)) with the distinction between "skilled" and "unskilled" workers. "Skilled" workers experience higher wages compared to those who are still working in lower-paying sectors that have not yet adopted automation. Moreover, automation worsens the wage differential between "skilled" and "unskilled" workers. With a decrease in wages for "unskilled" workers, because of automation, the capital stock per worker remains stagnant and there is no opportunity for growth.

The integration of endogenous fertility in Chapter 4 is used to bring transitional dynamics into the model. Nevertheless, the optimal number of children still remains constant, the total population may shift due to this consideration. When households have more children and fertility is high, the number of adults in the next generation increases. This leads to a larger labor force, which can potentially enhance productivity, since more workers are available to utilize the existing capital. This, in turn, can lead to a higher rate of capital accumulation, as more capital is generated over time. However, this model reveals an interesting insight: Also the model with endogenous fertility does not exhibit growth in output per worker. Depending on the choice of parameters α, β, ν, and z, it determines whether endogenous fertility can contribute to a larger labor force and possibly greater capital accumulation. Endogenous fertility alone does not guarantee a consistent increase in economic output or overall economic growth.

Chapter 5 takes the analysis a step further by incorporating endogenous skill investments and endogenous fertility. This addition highlights the relationship between skill investment, fertility choices, and wages. The distinction between "skilled" and "unskilled" workers becomes even more pronounced, with "skilled" workers allocating a portion of their time to education, thereby reducing the time available for child-rearing. Consequently, fertility rates differ between these two groups, impacting their respective population sizes. On the other hand, "skilled" workers earn higher wages. This wage differential creates a stronger incentive for investing in skills. As skills increase, the opportunity cost of having children rises, often referred to as foregone wages. This means that the fertility rate of "skilled" workers will decrease further, potentially raising the skill premium. A following reduction of the population size could contribute to an increased scarcity of qualified labor, subsequently increasing the demand for such employees. The increasing demand might result in a wider wage gap, further boosting the skill premium. However, if a larger number of workers transition to "skilled" positions due to the higher demand, it could lead to an increase in the supply of "skilled" workers. Thus, this higher supply might contribute to a lower skill premium as the wage differential between the two groups could potentially decrease. Furthermore, an influx of workers into "skilled" positions could foster a more competitive job market for these employees. Despite this transition at the job market, with increasing automation the wages for "unskilled" workers might grow at a slower rate, while wages for "skilled" workers may experience relatively stronger growth (see equations (5.39) and (5.40)). In Chapter 3, where exogenous fertility has been assumed, an increase in automation capital always leads to a rise in the skill premium.

In summary, the interplay between supply and demand for "skilled" labor, the progression of automation, and the difference in fertility rates for the two worker groups can interact in a complex way to influence the skill premium.

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