Dancing rod problem in the context of Lagrangian mechanics

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Motivation

Dancing rod problem

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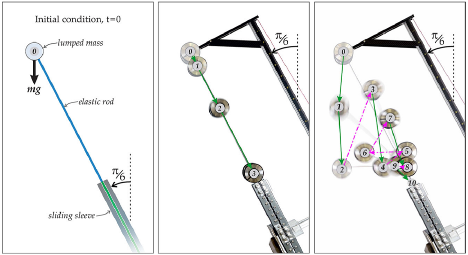
Configurational forces and nonlinear structural dynamics

C. Armanini, F. Dal Corso, D. Misseroni, D. Bigoni

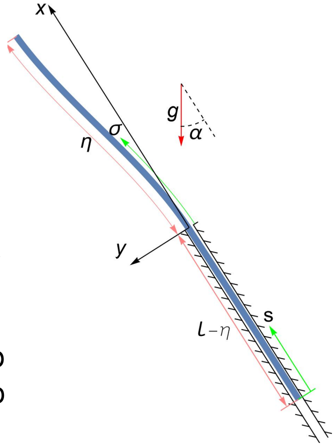


- Flexible rod, low friction, concentrated mass
- Configurational force at the tip of the sleeve is related to energy release rate and prevents full injection.

Mathematical model

Flexible rod partially sliding in a rigid sleeve

- Inextensible unshearable rod
- Distributed mass
- No friction
- Lagrangian (material) description inefficient
- Non-material mixed Eulerian-Lagrangian model
- Material coordinate s, length l
- Length of the free part $\eta(t)$
- Configurational force at the tip of the sleeve is proportional to local curvature and is work conjugate to η.



Mathematical model

Kinematic description

- Parametrization of free part:

$$\boldsymbol{x} = x\boldsymbol{e}_x + y\boldsymbol{e}_y = \boldsymbol{x}(\sigma, t),$$

$$0 \leq \sigma \leq 1 \qquad (1)$$

- Mapping:

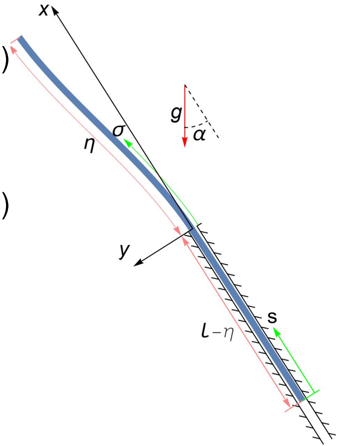
$$s = l - \eta + \eta\sigma,$$

$$\sigma = 1 - (l - s)/\eta \qquad (2)$$

- Strain energy requires 2nd order derivatives:

$$\partial_s \boldsymbol{x} = \partial_\sigma \boldsymbol{x}\, \partial_s \sigma = \frac{1}{\eta} \partial_\sigma \boldsymbol{x},$$

$$\partial_s^2 \boldsymbol{x} = \frac{1}{\eta^2} \partial_\sigma^2 \boldsymbol{x} \qquad (3)$$



Mathematical model

Kinematic description

- Material velocity of a particle:

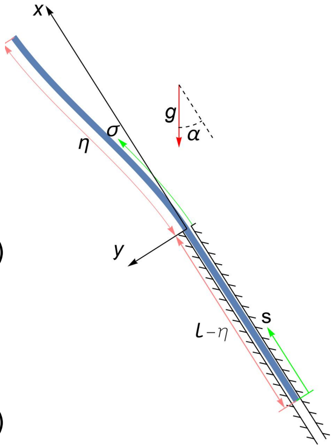
$$\dot{\boldsymbol{x}} = \partial_t \boldsymbol{x}|_{s=\mathrm{const}} =$$

$$= \partial_t \boldsymbol{x}|_{\sigma=\mathrm{const}} + \dot{\sigma}\, \partial_\sigma \boldsymbol{x}$$

$$\dot{s} = 0 \quad \Rightarrow \quad \dot{\sigma} = \frac{(1-\sigma)\dot{\eta}}{\eta} \qquad (4)$$

- Boundary conditions at the tip of the sleeve:

$$x(0,t) = y(0,t) = 0, \qquad (5)$$
$$\partial_\sigma y(0,t) = 0$$



Finite element approximation

- We divide the domain $0 \leq \sigma \leq 1$ into n finite elements.
- Generalized coordinates: nodal unknowns $\boldsymbol{x}_i$, $(\partial_\sigma \boldsymbol{x})_i$
- Cubic approximation on an element e

$$\boldsymbol{x}(\sigma) = S_1(\sigma)\boldsymbol{x}_e + S_2(\sigma)(\partial_\sigma \boldsymbol{x})_e + S_3(\sigma)\boldsymbol{x}_{e+1} + S_4(\sigma)(\partial_\sigma \boldsymbol{x})_{e+1} \tag{6}$$

  guarantees C^1 interelement continuity.
- Further generalized coordinate is length of free part η.
- Axial strain

$$\varepsilon = \frac{1}{2}(\partial_s \boldsymbol{x} \cdot \partial_s \boldsymbol{x} - 1) \tag{7}$$

  is penalized and $\varepsilon \to 0$ when $n \to \infty$.
- Bending strain is curvature

$$\kappa = \partial_s^2 \boldsymbol{x} \cdot (\boldsymbol{e}_z \times \partial_s \boldsymbol{x}), \quad |\kappa| = |\partial_s^2 \boldsymbol{x}|. \tag{8}$$

Finite element approximation

Energies and equations of motion

- Total energy of bending + penalty for axial strain:

$$U = \int_0^1 \frac{1}{2}(a\kappa^2 + b\varepsilon^2)\partial_\sigma s \, \mathrm{d}\sigma \tag{9}$$

- Total potential of gravity:

$$W = -\int_0^1 \rho g \boldsymbol{x} \cdot (\boldsymbol{e}_x \cos\alpha + \boldsymbol{e}_y \sin\alpha)\partial_\sigma s \, \mathrm{d}\sigma - \\ -\frac{1}{2}\rho g \cos\alpha (l - \eta)^2 \tag{10}$$

- Total kinetic energy:

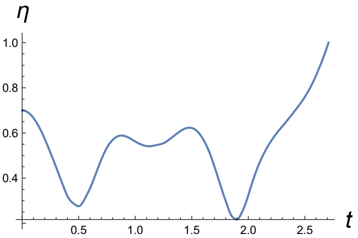
$$T = \int_0^1 \frac{1}{2}\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} \, \partial_\sigma s \, \mathrm{d}\sigma + \frac{1}{2}\rho(l-\eta)\dot{\eta}^2 \tag{11}$$

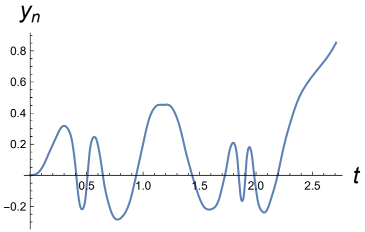
- Dissipation function R proportional to $\dot{\varepsilon}^2$ to damp out high frequency axial vibrations
- Lagrange's equations of motion for nodal d.o.f. and η

Simulation results

Very flexible rod with small initial injection

- Parameters (SI units):
  - Length $l = 1$, thickness $h = 0.5 \cdot 10^{-3}$
  - Material $E = 2 \cdot 10^{11}$, $\rho_3 = 7800$
  - Gravity $g = 9.8$, $\alpha = \pi/4$
  - Initial length of free part $\eta_0 = 0.7l$
- Results for $n = 8$ f.e. almost converged
- Length and tip deflection until full ejection at $t \approx 2.71$





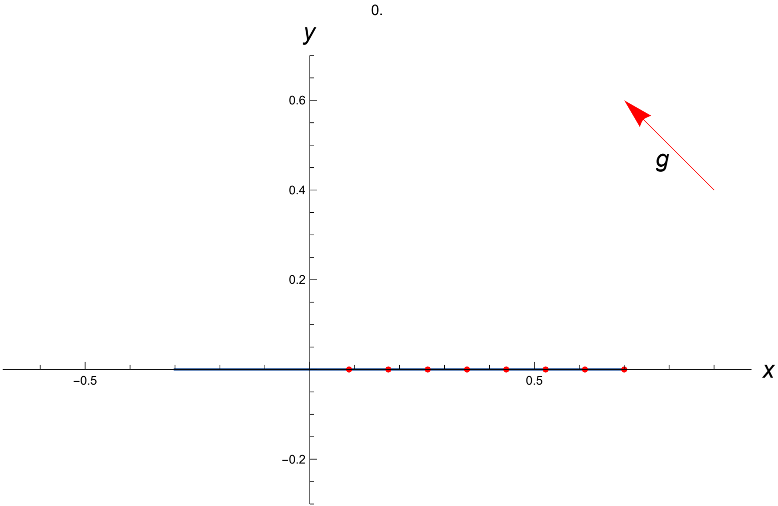
- Simulation validated against

  Han, S.; Bauchau, O.A.: Configurational forces in variable-length beams for flexible multibody dynamics. Multibody System Dynamics, pp. 1-24, 2022.

Simulation results

Very flexible rod with small initial injection

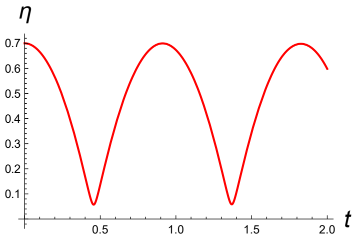
- Animation of the dynamic process

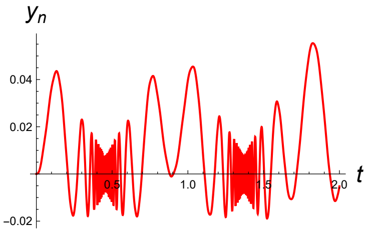


Simulation results

Less flexible rod

- Same process, thickness $h = 2 \cdot 10^{-3}$





- Seemingly periodic process
- Small vibration amplitude, first mode dominating

Analytical solution

Single term Ritz approximation at small vibrations

- Assumptions
  - Small vibration amplitude
  - First mode dominating
  - Same axial motion for all particles
- Approximation with two generalized coordinates:

$$\boldsymbol{x}(\sigma, t) = \sigma\eta(t)\boldsymbol{e}_x + \gamma(t)w(\sigma)\boldsymbol{e}_y \tag{12}$$

- Shape function $w(\sigma)$ is first vibration mode, $w(1) = 1$
- Energy expressions follow by integration:

$$\begin{aligned} U &= ak_U\gamma^2/\eta^3, \\ W &= \rho g(l(l-2\eta)\cos\alpha - 2k_W\gamma\eta\sin\alpha)/2, \\ T &= \rho(\eta\dot{\gamma}^2 + \gamma\dot{\gamma}\dot{\eta} + 4(2k_T\gamma^2 + l\eta)\dot{\eta}^2/\eta)/8 \end{aligned} \tag{13}$$

with $k_U = 1.5453$, $k_T = 0.094385$, $k_W = 0.39150$

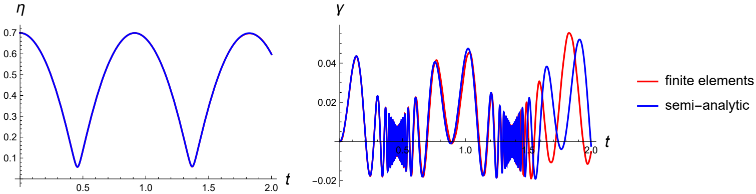
Analytical solution

Equations of motion and results

- Equations of motion take the form

$$\frac{1}{8}\rho\gamma\left(\ddot{\gamma}+\frac{32k_T\dot{\gamma}\dot{\eta}}{\eta}-8gk_W\sin\alpha\right)+l\rho(\ddot{\eta}-g\cos\alpha)=$$
$$=\frac{\gamma^2(3ak_U+k_T\rho\eta^2(\dot{\eta}^2-2\eta\ddot{\eta}))}{\eta^4} \tag{14}$$
$$\frac{2ak_U\gamma}{\eta^3}+\frac{1}{8}\rho\left(2\dot{\gamma}\dot{\eta}+2\eta\ddot{\gamma}+\gamma\left(\ddot{\eta}-\frac{16k_T\dot{\eta}^2}{\eta}\right)\right)=\rho gk_W\eta\sin\alpha$$

- Results of time integration compared to f.e.



- Estimate for eigenfrequency $\omega=2\sqrt{2ak_U/\rho}/\eta^2$ matches very well with numerical data.

Conclusions & Outlook

- Non-material finite element model for large vibrations problem of a rod with solution dependent length of the free part is developed and validated.
- Configurational force results into alternating injection and ejection of the rod, even full ejection is possible.
- Semi-analytical two-d.o.f. model provides good results at small vibrations.
- Closed form estimates for characteristic values like maximal injection length are yet to be found.
- Elaborate investigation on the nature of the configurational force would allow taking friction at the tip of the sleeve into account.
- Further extension to the case of flexible sleeve necessary for approaching practically relevant formulations like concentric tube robots, used in the medicine.

Thank you for attention!