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Analytic Proof Theory for Deontic Mīmāṃsā Logic

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by

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to the Faculty of Informatics

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Vienna, 4th September, 2023

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Declaration of Authorship

Stella Mahler, B.Sc.

I hereby declare that I have written this work independently, have given full details of the sources and aids used, and have marked places in the work—including tables, maps and illustrations—which are taken from other works or from the Internet, either verbatim or in spirit, as borrowed, in any case indicating the source.

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Kurzfassung

Die deontische Logik ist der Zweig der Logik, der sich mit Verpflichtungen, Verboten, Erlaubnissen und anderen normativen Konzepten beschäftigt. Sie hat in Bereichen wie Ethik, Recht und KI zunehmend an Bedeutung gewonnen. Die deontische Logik ist ein relativ junges Forschungsgebiet, dessen Anfänge auf von Wrights Arbeit aus dem Jahr 1951 zurückgehen. Im Gegensatz dazu hat Mīmāṃsā - eine bedeutende Schule der Sanskrit-Philosophie - seit mehr als zwei Jahrtausenden deontische Konzepte in einer fast formalen Weise gründlich diskutiert. Als solche bietet Mīmāṃsā eine Fundgrube von 2000 Jahren deontischer Untersuchungen. Diese Arbeit konzentriert sich auf die Mīmāṃsā-Autoren Prabhākara und Kumārila und diskutiert zwei Modallogiken, die ihre deontischen Theorien formalisieren. Die Analytizität ihrer jeweiligen Beweistheorien, die auf dem Hypersequentenkalkül basieren, wird wird durch den Prozess der Schnitteliminierung nachgewiesen. Darüber hinaus wird die Korrektheit und Vollständigkeit bewiesen.

Abstract

Deontic logic is the branch of logic that deals with obligations, prohibitions, permissions, and other normative concepts. It has become increasingly relevant in domains such as Ethics, Law and AI. Deontic Logic is a relatively recent field of study, with its inception traced back to von Wright's 1951 paper. By constrast, Mīmāṃsā – a prominent school of Sanskrit philosophy – has thoroughly discussed for more than two Millennia deontic concepts almost in a formal way. As such, Mīmāṃsā offers a treasure trove of 2000 years of deontic investigations. This thesis focuses on the Mīmāṃsā authors Prabhākara and Kumārila and discusses two modal logics that formalize their deontic theories. The analyticity of their respective proof theories, based on the hypersequent calculus, is established through the process of cut-elimination. Further, soundness and completeness is proven.

Contents

CHAPTER 1

Introduction

Deontic logic has become increasingly relevant in domains such as Ethics, Legal Reasoning and Artificial Intelligence. It is the branch of logic that deals with obligations, permissions, prohibitions, and other normative concepts. As one of the earliest axiomatically specified deontic logics, Von Wright's introduction of the standard deontic logic SDL in 1951 [1] marked a significant milestone. It features monadic deontic operators representing unconditional norms and has firmly established itself as one of the most frequently cited and widespread systems today [2]. However, the limitations of SDL become evident when confronted with various deontic "paradoxes."

In the context of deontic logic, a "paradox" refers to a set of sentences that appear intuitively coherent but cannot be consistently formalized within certain logical frameworks, or lead to the deduction of sentences that seem counterintuitive based on common-sense reading [3].

An illustrative example, introduced by Forrester [4] in 1984, is the "Paradox of the Gentle Murder." This paradox states that while it is obligatory not to kill, if killing occurs, it must be done gently. Although these statements may seem consistent at first glance, they defy consistent formalization within SDL. This emphasizes the limitations of SDL in dealing with Contrary-to-Duty obligations, which refer to obligations that arise when a norm has been violated. This has sparked active and ongoing discussions within the academic community (see [5] for an early overview) and has resulted in the introduction of dyadic deontic logics, as already outlined in [3]. Through the incorporation of dyadic deontic operators, these logics are able to capture conditional deontic commands such as *"If X is the case, then it is obligatory to do Y."*

While the "Paradox of the Gentle Murder" has been formulated more recently, a similar controversy can be traced back over two millennia. The historical debate, known as the *śyena* controversy, captivated scholars from the Mīmāṃsā school of Sanskrit philosophy for an extended period of time. *Śyena* refers to a ritual that leads to the death of the performer's enemy. While the Vedas, the fundamental scriptures of Hinduism, are regarded as consistent due to their sacred nature, they present a seeming contradiction by prohibiting harm to any living being while also appearing to prescribe the *śyena* ritual in cases where someone desires to kill an enemy. In the face of this apparent paradox, different Mīmāṃsā scholars have proposed and discussed various approaches to resolving this dilemma. The resemblance between the *śyena* controversy and the "Paradox of the Gentle Murder" is quite evident. Therefore, the solutions proposed by Mīmāṃsā authors continue to hold significance even today. For a comprehensive exploration of this subject, you can refer to [6], [7], and [8].

The Mīmāṃsā scholars did not exclusively address the *śyena* controversy. Over the course of more than two millennia, the Mīmāṃsā school of Indian philosophy was broadly concerned with how to navigate apparent contradictions in the prescriptive portions of the Vedas. To address this, Mīmāṃsā authors formulated a comprehensive set of principles and rules. Through the application of these so-called *nyāyas*, they contend that any apparent conflicts within the Vedas can be resolved.

The Mīmāṃsā scholars Prabhākara and Kumārila, both active around the 7th century CE, had several common viewpoints. They agreed on the actions prescribed by the Vedic commands, as well as those that were prohibited. However, they differed in their reasoning behind interpreting these commands. This distinct interpretation of the Vedas resulted in the formation of two separate sub-schools: the Prabhākara and Bhāṭṭa sub-schools, respectively.

In the study conducted by van Berkel et al. [8], formal logic is utilized to reconstruct the solutions proposed by Prabhākara, Kumārila, and other scholars in response to the *śyena* dilemma. Moreover, the study investigates how the introduced logical frameworks behave when faced with other deontic paradoxes, such as the "Paradox of the Gentle Murder." That could potentially encourage a discussion about traditional paradigms in deontic logic, due to the distinct characteristics of Mīmāṃsā deontic logic, such as the non-interdefinability of the concepts of obligations and prohibitions. As a result, the use of formal logic yields a more profound understanding of the diverse Mīmāṃsā authors from an indiological point of view and offers novel perspectives on contemporary deontic logic.

However, there is more than just a shared interest in the formalization of Mīmāṃsā reasoning across different fields; a multidisciplinary collaboration between Sanskritists and Logicians is actually essential to construct appropriate logical frameworks. The goal is to create logics that accurately represent the original concepts of various Mīmāṃsā scholars without introducing any external ideas or assumptions. This process encompasses several crucial steps. Firstly, the relevant *nyāyas* must be identified; subsequently, they need to be translated into English, interpreted, and ultimately formalized using Hilbert axioms.

The incorporation of Hilbert Axioms into existing systems is a common approach for introducing non-classical logics. However, a notable limitation of Hilbert systems is their lack of analyticity. An analytic proof relies solely on formulas already present in the final conclusion.

The absence of this property makes formal systems not usable for automated proof search. This is primarily because it requires examining a variety of formulas beyond those present within the theorem under consideration. Additionally, analyticity plays a crucial role in establishing properties such as decidability, consistency, and interpolation within the formulated logics.

In 1935, Gerhard Gentzen introduced the sequent calculus framework [9], [10], which has since become a favored formalism for developing analytic calculi for various logics. Consequently, initial proof theories for Mīmāṃsā deontic logic were based on Gentzen-style sequent systems [11]. The underlying logics were based on modal logic S4. However, recent developments have introduced two distinct logics for Prabhākara and Kumārila, based on modal logic S5 [7]. The decision to transition from S4 to S5 was driven by the necessity to articulate facts that hold universally, regardless of the situation. Given that there is no cut-free analytic sequent calculus for modal logic S5, this shift necessitates significant modifications in the associated proof theory.

This limitation of the sequent calculus has led to various generalizations, one of which is known as the hypersequent calculus, introduced by Avron [12] and independently by Pottinger [13]. The additional expressive power of the hypersequent calculus allows to capture logics that go beyond what can be accommodated within the conventional sequent framework.

1.1 Aim of the Thesis

The primary objective of this thesis is to define analytic proof systems for the logics of Prabhākara and Kumārila [7]. As these logics use S5 as their base logic, the hypersequent framework will be employed. The corresponding rule set includes hypersequent rules which cover S5, and rules which incorporate the deontic operators within the logics of Prabhākara and Kumārila. The rules related to the deontic operators are derived from the sequent calculus rules introduced in [14], and will be transformed into a hypersequent version.

The soundness of the resulting calculi will be established through a semantic proof, while completeness will be syntactically demonstrated by proving all axioms of their respective Hilbert systems, as well as by simulating Modus Ponens and Necessity. To simulate Modus Ponens within the calculi for Prabhākara and Kumārila, the use of the so-called *cut* rule becomes necessary. However, the *cut* rule is the only one that lacks analyticity, making the entire calculi non-analytic. Therefore, to establish the analyticity of the calculi in this thesis we will eliminate any instance of the cut rule from proofs within the introduced calculi.

1.2 Thesis Overview

The structure of the thesis is outlined as follows:

Chapter 2: Mīmāṃsā Deontic Logics

In this chapter, the syntax and semantics of the logics for Prabhākara and Kumārila are introduced. The primary objective is to offer a comprehensive overview of the Mīmāṃsā school and accentuate the unique attributes of Prabhākara and Kumārila. Furthermore, the chapter outlines the relationship between the *nyāyas* and their formalizations as Hilbert axioms.

Chapter 3: Sequent and Hypersequent Calculus

This chapter serves as an introduction to the sequent calculus framework. It also motivates the need for hypersequent calculi for logics extending S5. Finally, the chapter introduces a hypersequent calculus for the logic S5.

Chapter 4: Cut-free Hypersequent Calculi for Mīmāṃsā Logics

Chapter 4 constitutes the core section of the thesis and presents original contributions. Here, an analytic hypersequent calculus for the logics of Prabhākara and Kumārila is defined. The chapter includes a semantic proof of soundness and a syntactic proof of completeness. Lastly, the analyticity of the introduced calculi is demonstrated through a cut-elimination proof.

Chapter 5: Conclusion

The final chapter of the thesis summarizes the main findings and delves into potential avenues for future research.

CHAPTER 2

Mīmāṃsa Deontic Logics

Mīmāṃsā, a prominent school of Sanskrit philosophy, has thrived for over two millennia, spanning from the last centuries B.C. to the 20th century. The Sanskrit word "Mīmāṃsā" translates to "profound thought", "reflection", "consideration", "investigation", "examination" or "discussion" [15], aligning with the school's primary focus on analyzing the prescriptive parts of the Vedas. The Vedas are a vast collection of Sanskrit texts, consisting of the four sacred canonical texts in Hinduism *Ṛigveda*, *Yajurveda*, *Sāmaveda* and *Atharvaveda*.

In contrast to other philosophical schools, Mīmāṃsā is primarily concerned with the study of norms in the Vedas. As the Vedas are assumed to be consistent due to their sacred nature, Mīmāṃsā authors focus on the question of what ought to be done in the presence of apparent contradictions. Each Veda consists of different layers or classes of text. As the school's attention is particularly directed towards the prescriptive portions of the Vedas, they focus on the Brāhamaṇa layer, which deals with the interpretation of texts related to sacrifices and rituals.

One of the distinguishing features of the Mīmāṃsā school, setting it apart from other religious or moral traditions, is its belief that Vedic commands do not need legitimization or enforcement through superior ethical concepts or mediation by any divine or human authority. They are considered authorless [6]. As a result of this view, the Vedic commands are perceived as a consistent and self-sufficient corpus of laws, with no external entity capable of altering their meaning or implications.

The commands in the prescriptive parts of the Vedas can take on different forms. This thesis primarily focuses on prescriptions and prohibitions. Prescriptions pertain to the performance of sacrifices, while prohibitions encompass commands related to specific sacrifices, like the instruction not to dress informally during a particular ritual. Additionally, prohibitions include commands that apply to individuals throughout their lives, such as the prohibition against causing harm.

As some commands may appear contradictory at first glance, Mīmāṃsā authors have devised a comprehensive framework of metarules known as *nyāyas*. With this framework, all authors agree that any alleged conflicts in the Vedas can be resolved.

In general, there was agreement on what the Vedic commands dictated to do and not to do. However, there was disagreement about the principles of why a command should be interpreted in a specific way [16]. Despite differing views, they share the same basic ideas and authorities, as all later Mīmāṃsā authors refer to the text *Pūrva Mīmāṃsā Sūtra*[1] by Jaimini (approximately 250 B.C.) and its most ancient commentary *Bhāṣya* by Śabara (around 5th century C.E.) as a common foundation.

This thesis will focus on the Mīmāṃsā authors Prabhākara and Kumārila (both ca. 7th c. C.E.). Among others, both are highly influential, and their interpretation of the Vedas led to the establishment of the Prabhākara and Bhāṭṭa subschools, respectively. While they share many similarities, they differ regarding a subcategory of prescriptions, namely the elective sacrifices. These sacrifices are only required to be performed if someone desires a specific result. For Prabhākara, they are considered a form of obligation, whereas Kumārila sees them more as recommendations. This chapter will further elaborate on these differences.

Developing appropriate logics for Mīmāṃsā reasoning is a challenging task, albeit *nyāya*s lend themselves through a formalization through logic. Since no Indian philosophical school uses mathematical formulations, abstraction from the texts has to be done very carefully. This task requires collaboration among experts from different disciplines. It involves identifying and understanding relevant *nyāyas* and translating them from Sanskrit to English. Additionally, Hilbert axioms must be extracted from the translated and parsed *nyāyas*. The resulting logics directly reflect the principles derived from the Mīmāṃsā texts, without introducing any external properties or assumptions. This ensures a faithful representation of Mīmāṃsā reasoning without undue influence from outside sources.

The development of the "Basic Mīmāṃsā Deontic Logic" (bMDL) was an initial step towards formalizing Mīmāṃsā reasoning and considers only obligation [11]. As different Mīmāṃsā authors provide different interpretations, a variety of logics are needed. Consequently, two distinct logics, LPr and LKu, were developed for Prabhākara's and Kumārila's systems, respectively, building upon bMDL [7]. These logics extend bMDL by incorporating a specific prohibition operator and, in the case of LKu, an additional operator representing elective sacrifices. Moreover, they utilize S5 as a base logic instead of S4 as previously employed. The logics LPr and LKu will be presented in this chapter.

[1]The *Pūrva Mīmāṃsā Sūtra* is organized into books, chapters, and aphorisms. The notation of references will indicate the book, chapter, and aphorism in that order. For example, *Pūrva Mīmāṃsā Sūtra* 1.2.3 refers to the third aphorism of the second chapter in the first book. You will find references to commentaries by other authors.

2.1 The logics of Prabhākara and Kumārila

When introducing or describing a logic, it is essential to consider two levels: syntax and semantics. The syntax deals with the formal language of the logic and provides rules for constructing its formulae. However, it does not assign any meaning to these symbols. The semantics, on the other hand, gives meaning to the formulae and determines their behavior in relation to a notion of "truth." Once the syntax is established, the semantics of the logic is defined by adding properties in form of Hilbert axioms to the base system. This approach is the one we adopt here.

The logics LPr and LKu for Prabhākara's and Kumārila's systems are based on classical logic rather than intuitionistic logic [11]. This choice is due to the legitimacy of the excluded middle and the reductio ad absurdum law, which is implied by various examples proposed by Mīmāṃsā authors. For instance, Jayanta's book *Nyāyamañjarī* states that *"When there is a contradiction (φ and not φ), at the denial of one (alternative), the other is known (to be true)"*. Hence, if we deny the negation of a proposition $\neg\varphi$, then it implies that φ holds, leading to reductio ad absurdum (see [6] for more details).

In contrast to bMDL, LPr and LKu utilize the modal logic S5 as a base logic instead of S4 as previously employed [7]. Although not explicitly stated by Mīmāṃsā authors, necessity is primarily used in these logics to express facts that hold in all possible situations. Hence, these can be seen as global assumptions. The transition from S4 to S5 is not trivial, as this will lead to a more complex proof theory, which we will explore in greater detail later on.

In *Pūrva Mīmāṃsā Sūtra* and *Bhāṣya*, prescriptive and prohibitive commands are treated as distinct concepts [14], [17]. For instance, following obligations leads to the accumulation of good karma which will lead to felicity, while obeying a prohibition has no immediate consequence. However, violating a prohibition results in the accumulation of bad karma. Hence, unlike prescriptions, following a prohibition does not yield a result, but violating it will yield a sanction. This means that adherence to a negative obligation gives a desired outcome, while compliance with a corresponding prohibition avoids a sanction. As highlighted in [14], prescriptions encourage action, while prohibitions deter action. The logics LPr and LKu effectively model these concepts using the primitive deontic operators $\mathcal{O}(./.)$ for obligation and $\mathcal{F}(./.)$ for prohibition. This distinctive feature sets it apart from common approaches in deontic logic, where obligation and prohibition are often interdefinable. Furthermore, recent findings, not covered in this thesis, extend the non-interdefinability of deontic concepts to the concept of permission [18].

The prescriptive commands in the Vedas can further be distinguished based on their deontic strength. The category of "fixed sacrifices" encompasses those that must be performed every single day, while "occasional sacrifices" are reserved for certain occasions. Both cannot be omitted. In contrast, "elective sacrifices" are only required if one desires their specific outcome, and their omission does not have any negative consequences.

While both Prabhākara and Kumārila share a common view regarding prohibitions and fixed/occasional sacrifices, which are understood as obligations, their views differ regarding elective sacrifices. For Prabhākara, an elective sacrifice is considered a type of obligation. Eligible agents are identified based on their desires for a specific worldly result. Once the eligibility conditions of an elective sacrifice are satisfied, it becomes mandatory to perform it. Therefore, the distinction between elective sacrifices and fixed/occasional sacrifices is solely determined by their eligibility conditions. For a fixed sacrifice, the eligibility conditions apply to all agents (e.g. the conditions of being alive or desiring eternal happiness), while the eligibility conditions of an elective sacrifice only hold for agents with a specific desire.

On the other hand, Kumārila views elective sacrifices differently. As they represent a guaranteed way for obtaining the desired results, eligible agents are inclined, but not obligated to perform it. Hence, elective sacrifices can be omitted without risk. Consequently, while Prabhākara considers elective sacrifices as conditional obligations, Kumārila views them as a distinct type of Vedic command. This distinction is formalized in [7] and leads to the logic LKu, which extends the logic LPr by introducing the primitive deontic operator $\mathcal{R}(./.)$. Therefore, the logic for Prabhākara is a proper subset of Kumārila's logic.

Axiomatization

Definition 2.1.1. The languages $\mathcal{L}_{\mathsf{LPr}}$ for Prabhākaras logic and $\mathcal{L}_{\mathsf{LKu}}$ for Kumārilas logic are defined by the following Backus-Naur-Form:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \mathcal{X}(\varphi/\varphi)$$

with $p \in \mathsf{Atom}$ and $\mathcal{X} \in \{\mathcal{O}, \mathcal{F}\}$ for $\mathcal{L}_{\mathsf{LPr}}$ and $\mathcal{X} \in \{\mathcal{O}, \mathcal{F}, \mathcal{R}\}$ for $\mathcal{L}_{\mathsf{LKu}}$. Atom is the set of atomic propositions, $\neg$ and $\to$ are primitive connectives. $\Box\varphi$ reads as "it is universally necessary that φ" and $\mathcal{O}(\varphi/\psi), \mathcal{F}(\varphi/\psi), \mathcal{R}(\varphi/\psi)$ as "given ψ, φ is obligatory/forbidden/recommended".

Definition 2.1.2. Let φ and ψ be $\mathcal{L}_{\mathsf{LPr}}$ or $\mathcal{L}_{\mathsf{LKu}}$ formulas, respectively. The symbols $\vee$, $\wedge$, $\Diamond$, $\top$ and $\bot$ are defined as:

$\varphi \vee \psi := (\varphi \to \psi) \to \psi$

$\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$

$\Diamond\varphi := \neg\Box\neg\varphi$

$\top := \varphi \vee \neg\varphi$

$\bot := \neg\top$

We begin the axiomatization of LPr and LKu by establishing the axioms of modal logic S5. After that, we extend this system to incorporate the properties of the deontic operators $\mathcal{O}$, $\mathcal{F}$ and $\mathcal{R}$.

Definition 2.1.3. The axiomatisation of the modal logic $\mathsf{S5}$ consists of the axioms of some Hilbert system of classical propositional logic and the following axioms and rules:

Axioms:

K . $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

T . $\Box\varphi \rightarrow \varphi$

$\mathsf{4}$. $\Box\varphi \rightarrow \Box\Box\varphi$

$\mathsf{5}$. $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$

Rules:

(MP) If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$

(Nec) If $\vdash \varphi$, then $\vdash \Box\varphi$

Definition 2.1.4. Prabhākara's logic LPr extends $\mathsf{S5}$ from Def. 2.1.3 with the following axioms:

$\mathsf{A_{Pr}}1$. $(\Box(\varphi \rightarrow \psi) \wedge \mathcal{O}(\varphi/\Theta)) \rightarrow \mathcal{O}(\psi/\Theta)$

$\mathsf{A_{Pr}}2$. $(\Box(\varphi \rightarrow \psi) \wedge \mathcal{F}(\psi/\Theta)) \rightarrow \mathcal{F}(\varphi/\Theta)$

$\mathsf{A_{Pr}}3$. $\neg(\mathcal{X}(\varphi/\Theta) \wedge \mathcal{X}(\neg\varphi/\Theta))$ for $\mathcal{X} \in \{\mathcal{O}, \mathcal{F}\}$

$\mathsf{A_{Pr}}4$. $\neg(\mathcal{O}(\varphi/\Theta) \wedge \mathcal{F}(\varphi/\Theta))$

$\mathsf{A_{Pr}}5$. $(\Box((\psi \rightarrow \Theta) \wedge (\Theta \rightarrow \psi)) \wedge \mathcal{X}(\varphi/\psi)) \rightarrow \mathcal{X}(\varphi/\Theta)$ for $\mathcal{X} \in \{\mathcal{O}, \mathcal{F}\}$

Kumārila's logic LKu extends LPr with the following axioms:

$\mathsf{A_{Ku}}6$. $(\Box(\varphi \rightarrow \psi) \wedge \mathcal{R}(\varphi/\Theta)) \rightarrow \mathcal{R}(\psi/\Theta)$

$\mathsf{A_{Ku}}7$. $\Box(\neg\varphi) \rightarrow \neg\mathcal{R}(\varphi/\psi)$

$\mathsf{A_{Ku}}8$. $(\Box((\psi \rightarrow \Theta) \wedge (\Theta \rightarrow \psi)) \wedge \mathcal{R}(\varphi/\psi)) \rightarrow \mathcal{R}(\varphi/\Theta)$

Note that $\mathsf{A_{Ku}}8$ relates to axiom $\mathsf{A_{Pr}}5$ using $\mathcal{R}$ instead of $\mathcal{O}$ or $\mathcal{F}$.

The properties of the deontic operators $\mathcal{O}(./.)$, $\mathcal{F}(./.)$, and $\mathcal{R}(./.)$ in Def. 2.1.4 are derived from *nyāyas* (meta-rules) introduced by Mīmāṃsā authors to interpret the Vedas independently. It is worth noting that these principles were not explicitly defined by the Mīmāṃsā authors but were used in various contexts. As a result, the principles below are abstractions of the original texts [7]. [6] provides a detailed description of how the first three principles were transformed into axioms (only for axioms regarding $\mathcal{O}$). For further discussions, refer to [11], [14], and [7]. Moreover, recently identified *nyāyas*, not covered here, lead to additional principles and axioms related to the concept of permission, as discussed in [18] and to a property called aggregation [8].

(P1) "If the accomplishment of an action presupposes the accomplishment of another connected but different action, the obligation to perform the first action prescribes also the second one. Conversely, if an action necessarily implies a prohibited action, it will also be prohibited."

This principle is formalized by the axioms $\mathsf{A}_{\mathsf{Pr}}1$, $\mathsf{A}_{\mathsf{Pr}}2$ and $\mathsf{A}_{\mathsf{Ku}}6$. It corresponds to various *nyāya*s (see [6]), for example one in *Tantrarahasya* IV.4.3.3 which can be abstracted to "if a certain action is obligatory but it implies other activities, then these other activities are also obligatory". For obligations, this implies that any sub-actions of an obligation are obligatory as well, and similarly, the sub-actions of a prohibition are prohibited as well. Formally, this corresponds to the property of monotonicity. Note that the usage of the operator $\Box$ in the axioms $\mathsf{A}_{\mathsf{Pr}}1$, $\mathsf{A}_{\mathsf{Pr}}2$, and $\mathsf{A}_{\mathsf{Ku}}6$ ensures that the correlation between the truth of φ and the truth of ψ is not accidental but represents a permanent relationship between the two.

(P2) "Two actions that exclude each other can neither be prescribed nor prohibited simultaneously to the same group of eligible people under the same conditions."

The axioms $\mathsf{A}_{\mathsf{Pr}}3$ and $\mathsf{A}_{\mathsf{Pr}}4$ arise from principle (P2), which is an abstraction of the so-called "principle of the half-hen" [2] found in various Mīmāṃsā contexts, such as in *Tantravārtika* ad 1.3.3. This principle highlights the discrepancy of commanding someone to act (or not to act) in opposition to themselves on the same matter. In other words, an obligation to perform an act cannot be simultaneously prohibited, nor can its negation be obligatory. It should be noted that this principle does not lead to $\mathsf{A}_{\mathsf{Ku}}7$, as Kumārila considers elective sacrifices differently than Prabhākara. Therefore, this principle does not apply to elective sacrifices in Kumārila's sense. However, to ensure self-consistency regarding elective sacrifices, the axiom $\mathsf{A}_{\mathsf{Ku}}7$ ensures that something logically impossible cannot be enjoined.

(P3) "If two sets of conditions always identify the same group of eligible agents, then a command valid under the conditions in one of those sets is also enforceable under the conditions in the other set."

The third principle (P3) is formalized by the axioms $\mathsf{A}_{\mathsf{Pr}}5$ and $\mathsf{A}_{\mathsf{Ku}}8$. It originates from a discussion about the eligibility to perform sacrifices in *Śābarabhāṣya* on *Pūrva Mīmāṃsā Sūtra* 6.1.25 and emphasizes the generality of sacrifices concerning logically equivalent conditions.

[2] The "principle of the half-hen" refers to the idea of fully acknowledging or completely rejecting an authority, as opposed to a partial acknowledgement.

Semantics

Recall that the modal logic S5 is chosen to express facts that hold in all possible situations (global assumptions). The semantics for LPr and LKu are built on the standard semantics for modal logic S5, which uses Kripke frames with a reflexive, transitive, and symmetric accessibility relation. In other words, each world is accessible from every other world [19]. To capture the additional modalities $\mathcal{O}$, $\mathcal{F}$, and $\mathcal{R}$, neighborhood semantics are used [20]. Unlike Kripke semantics, which employs a world-to-world accessibility relation, neighborhood semantics uses a neighborhood function $\mathcal{N}$ that maps worlds to sets of sets of worlds. Informally, the neighborhood function identifies a set of deontically best sets of worlds for certain possible situations.

Definition 2.1.5. An LPr-frame $\mathfrak{F}_{\mathsf{LPr}} = \langle W, R_\Box, \mathcal{N}_\mathcal{O}, \mathcal{N}_\mathcal{F}\rangle$ is a tuple where $W \neq \emptyset$ is a set of worlds w, v, u, ... $R_\Box = W \times W$ is the universal relation and $\mathcal{N}_\mathcal{X} : W \mapsto \wp(\wp(W) \times \wp(W)))$ with $\mathcal{X} \in \{\mathcal{O}, \mathcal{F}\}$ is a neighbourhood function. Let $X, Y, Z \subseteq W$ then $\mathfrak{F}_{\mathsf{LPr}}$ satisfies:

(i) if $(X, Z) \in \mathcal{N}_\mathcal{O}(w)$ and $X \subseteq Y$, then $(Y, Z) \in \mathcal{N}_\mathcal{O}(w)$

(ii) if $(X, Z) \in \mathcal{N}_\mathcal{F}(w)$ and $Y \subseteq X$, then $(Y, Z) \in \mathcal{N}_\mathcal{F}(w)$

(iii) if $(X, Y) \in \mathcal{N}_\mathcal{X}(w)$, then $(\overline{X}, Y) \notin \mathcal{N}_\mathcal{X}(w)$ for $\mathcal{X} \in \{\mathcal{O}, \mathcal{F}\}$

(iv) if $(X, Z) \in \mathcal{N}_\mathcal{O}(w)$, then $(X, Z) \notin \mathcal{N}_\mathcal{F}(w)$

An LPr-*model* extends and LPr-frame with a *valuation* function V which maps propositional variables to subsets of W.

Definition 2.1.6. An LKu-frame $\mathfrak{F}_{\mathsf{LKu}} = \langle W, R_\Box, \mathcal{N}_\mathcal{O}, \mathcal{N}_\mathcal{F}, \mathcal{N}_\mathcal{R}\rangle$ is a LPr- frame extended with a neighbourhood function $\mathcal{N}_\mathcal{R} : W \mapsto \wp(\wp(W) \times \wp(W)))$ which satisfies:

(v) if $(X, Z) \in \mathcal{N}_\mathcal{R}(w)$ and $X \subseteq Y$, then $(Y, Z) \in \mathcal{N}_\mathcal{R}(w)$

(vi) if $(X, Y) \in \mathcal{N}_\mathcal{R}(w)$, then $X \neq \emptyset$

An LKu-*model* extends and LKu-frame with a *valuation* function V which maps propositional variables to subsets of W.

Properties (i), (ii), and (vi) correspond to principle (P1), expressing the monotonicity of the first argument of the deontic operators. Similarly, (iii), (iv), and (vii) correspond to principle (P2). Note that principle (P3) is not explicitly represented in the frame conditions for LPr and LKu, as it corresponds to the minimal property of neighbourhood models, meaning that equivalent formulas are satisfied in the same worlds [20].

Definition 2.1.7. Let $\mathfrak{M}_{\mathsf{Pr}}$ be an LPr- model and let $\|\varphi\|$ be the truth set $\{w \in W \mid \mathfrak{M}_{\mathsf{Pr}}, w \vDash \varphi\}$ of the formula $\varphi \in \mathcal{L}_{\mathsf{Pr}}$. The *satisfaction of a formula* $\varphi \in \mathcal{L}_{\mathsf{Pr}}$ at any w of $\mathfrak{M}_{\mathsf{Pr}}$ is inductively defined as follows:

$$
\begin{array}{lll}
\mathfrak{M}_{\mathsf{Pr}}, w \vDash p & \text{iff} & w \in V(p), \text{ for } p \in \mathsf{Atom} \\
\mathfrak{M}_{\mathsf{Pr}}, w \vDash \neg\varphi & \text{iff} & \mathfrak{M}_{\mathsf{Pr}}, w \nvDash \varphi \\
\mathfrak{M}_{\mathsf{Pr}}, w \vDash \varphi \rightarrow \psi & \text{iff} & \mathfrak{M}_{\mathsf{Pr}}, w \nvDash \varphi \text{ or } \mathfrak{M}_{\mathsf{Pr}}, w \vDash \psi \\
\mathfrak{M}_{\mathsf{Pr}}, w \vDash \Box\varphi & \text{iff} & \text{for all } w_i \in W : \mathfrak{M}_{\mathsf{Pr}}, w_i \vDash \varphi \\
\mathfrak{M}_{\mathsf{Pr}}, w \vDash \mathcal{X}(\varphi/\psi) & \text{iff} & (\|\varphi\|, \|\psi\|) \in \mathcal{N}_{\mathcal{X}}(w) \text{ for } \mathcal{X} \in \{\mathcal{O}, \mathcal{F}\}
\end{array}
$$

The satisfaction of a formula in a LKu- model $\mathfrak{M}_{\mathsf{Ku}}$ adds to the clauses above:

$$
\mathfrak{M}_{\mathsf{Ku}}, w \vDash \mathcal{R}(\varphi/\psi) \quad \text{iff} \quad (\|\varphi\|, \|\psi\|) \in \mathcal{N}_{\mathcal{R}}(w)
$$

Definition 2.1.8. A formula $\varphi \in \mathcal{L}_{\mathsf{Pr}}$ is *valid* in an LPr- model $\mathfrak{M}_{\mathsf{Pr}}$, if $\forall w \in W$: $\mathfrak{M}_{\mathsf{Pr}}, w \models \varphi$. This is denoted by $\mathfrak{M}_{\mathsf{Pr}} \models \varphi$.

Definition 2.1.9. A formula $\varphi \in \mathcal{L}_{\mathsf{Ku}}$ is *valid* in an LKu- model $\mathfrak{M}_{\mathsf{Ku}}$, if $\forall w \in W$: $\mathfrak{M}_{\mathsf{Ku}}, w \models \varphi$. This is denoted by $\mathfrak{M}_{\mathsf{Ku}} \models \varphi$.

Theorem 2.1.1. *A formula is a theorem of LPr (LKu, respectively) iff it is valid in all LPr-models (LKu-models, respectively).*

Proof. In [8]. □

CHAPTER 3

Sequent and Hypersequent Calculus

Non-classical logics are typically defined by augmenting known systems with Hilbert axioms as we did in the previous chapter. However, these additions often prove to be quite cumbersome when employing automated reasoning methods or when proving essential properties of the formalized logics. A better proof calculus is the sequent calculus for classical and intuitionistic logic (LK and LJ respectively). Both systems were introduced by Gerhard Gentzen in 1935 in his famous papers *Über das logische Schliessen I + II* [9] [10]. To establish the completeness of a sequent calculus, it is necessary to demonstrate the validity of all axioms in their corresponding Hilbert systems, along with simulating Modus Ponens and Necessity. To simulate Modus Ponens, the use of the so-called *cut* rule becomes necessary, making the *cut* rule a crucial component in any sequent calculus. Gentzen's cut-elimination theorem, also known as *Gentzen's Hauptsatz*, asserts that all proofs utilizing the cut-rule in LK and LJ can be transformed into cut-free proofs. This result is significant as it yields analytic proofs that exhibit the subformula property, indicating that each formula in the end-sequent of a cut-free proof is a subformula of one of its premises. Analytic calculi can be used to prove for example decidability, consistency, interpolation theorems, Gentzen's midsequent theorem and Herbrand's theorem. They are more feasible for automated proof search, as non-analytic calculi introduce non-determinism into the process through the cut-rule.

Some logics, including the modal logic S5, do not have a cut-free analytic sequent calculus. This limitation has lead to various generalizations of the sequent calculus. This thesis focuses on one of these generalizations known as the hypersequent calculus as introduced by Avron [12] and independently by Pottinger [13]. The basic idea is that, instead of single sequents, a rule operates on a multiset of sequents. This approach offers additional expressive power through rules that manipulate various components of one or more hypersequents simultaneously. By doing so, it becomes possible to capture logics that go beyond what can be accommodated within the conventional sequent framework.

In this chapter we will recall the sequent and hypersequent framework.

3.1 Sequent Calculus

Gentzen's sequent calculi are based on the concept of sequents, which consist of sequences of formulas on the left and right sides of a sequent sign. In our case, we have chosen to use an alternative definition that uses multisets instead of sequences. This deliberate choice allows us to omit permutation rules from our calculus.

Definition 3.1.1. Let Γ and Δ be finite multisets of formulas and $\vdash$ be a symbol not belonging to the logical language. Then $\Gamma \vdash \Delta$ is called a *sequent*. Γ and Δ are called the *antecedent* and *succedent* respectively.

Definition 3.1.2. Let $S : A_1, ..., A_n \vdash B_1, ...B_m$ be a sequent, then the *semantic interpretation*, denoted by $I(S)$, stands for

$$I(S) : \bigwedge_{i=1}^{n} A_i \to \bigvee_{j=1}^{m} B_j.$$

If there are no formulas in the antecedent, hence $n = 0$, we assign $\top$ to $\bigwedge_{i=1}^{n} A_i$. Respectively, if $m = 0$ we assign $\bot$ to $\bigvee_{j=1}^{m} B_j$.

In the following we write $\bigwedge \Gamma$ instead of $\bigwedge_{i=1}^{n} A_i$ with $\Gamma = A_1, ..., A_n$, and $\bigvee \Delta$ instead of $\bigvee_{j=1}^{m} B_j$ with $\Delta = B_1, ..., B_m$.

Sequent calculi are characterised by their axioms (initial sequents) and inference rules, these are typically various logical and structural rules and the cut rule. Those rules are represented as general schemata, which become a concrete instance by instantiating the rule variables with concrete formulae.

Definition 3.1.3. Let $S, S_1, ..., S_n$ be sequents. An *inference* is an expression of the form

$$\frac{S_1 \quad ... \quad S_n}{S}$$

$S_1, ..., S_n$ are called *upper sequents* and S is called the *lower sequent*.

We build upon Gentzen's sequent calculus for classical logic LK, specifically focusing on its propositional fragment, which serves as the foundation for our sequent systems later in this work.

As mentioned before, we use a modified version of LK that employs multisets instead of sequences of formulae. This variant omits the need for permutation rules, which manipulate the order of formulae within a sequent. The rule set is divided into logical and structural rules, each with left and right versions. However, the cut rule is treated as a separate category, even though it falls under the structural rules. It is distinguished by not having a left and right version and plays a significant role in the following. Additionally, we exclude the right and left logical rules ($\wedge$) and ($\vee$) from Gentzen's original, as both can be simulated using the left and right rules ($\neg$) and ($\rightarrow$). For the simulation in a hypersequent setting, refer to Lemma 4.1.2.

Definition 3.1.4. (LK)

Let A, B denote formulae and $\Gamma, \Delta, \Pi, \Lambda$ denote multisets of formulae. System LK consists of the following axioms and rules:

Axiom:

$$A \vdash A, \text{ for } \textit{atomic} \text{ formulae } A.$$

$$\bot \vdash \Delta$$

The cut rule:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\,(cut)$$

A is called the *cut formula* of this inference.

The structural rules:

weakening

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,(w-l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,(w-r)$$

contraction

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,(c-l) \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,(c-r)$$

The logical rules

$\neg$ - introduction

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\,(\neg l) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,(\neg r)$$

$\rightarrow$ - introduction

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Gamma \vdash \Delta}{A \rightarrow B, \Gamma \vdash \Delta}\,(\rightarrow l) \qquad \frac{A, \Gamma \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}\,(\rightarrow r)$$

Definition 3.1.5. A *derivation* (in LK) of a sequent S from sequents $S_1, ..., S_n$ is a finite directed labelled tree, where nodes are labelled by sequents and edges by corresponding rule applications. S is the root node and $S_1, ..., S_n$ are the leaf nodes. Any internal nodes are labelled by sequents obtained by applying the corresponding inference rule to its immediate predecessors. S is called the *end-sequent* and $S_1, ..., S_n$ are called *initial sequents* of this derivation.

Definition 3.1.6. A *proof* (in LK) of a sequent S is a derivation of S in which all leaf nodes are axioms (of LK).

All rules in the calculus above, except for the cut rule, possess the subformula property. This property guarantees that the formulas in the upper sequents of a rule are subformulas of the formulas occurring in the lower sequent. A rule that adheres to the subformula property is considered analytic. Therefore, a sequent calculus can be classified as analytic if the (cut) rule can be eliminated and the other rules satisfy the subformula property. Hence, Gentzen's cut-elimination theorem holds great significance in his work as it enables the transformation of a proof involving a cut into a cut-free proof. To handle the contraction rules, Gentzen replaced (cut) with a rule known as (mix) in his proof of the cut-elimination theorem:

$$\frac{\Gamma \vdash \Delta \qquad \Pi \vdash \Lambda}{\Gamma, \Pi^* \vdash \Delta^*, \Lambda}\,(mix)(A)$$

where A occurs at least once in Δ and Π, and Δ^* and Π^* are obtained by removing all occurrences of A in Δ and Π, respectively. Note that (cut) is a special instance of (mix) and (mix) can be simulated by (multiple applications of) contraction and (cut).

Gentzen showed that any derivation in whose lowest inference is (mix) can be transformed into a derivation without (mix). The fundamental idea of his proof is to shift the (mix) up the derivation, leading to a (mix) formula with lower complexity (fewer logical symbols). Eventually, the (mix) formula will be atomic and the (mix) can be completely eliminated from the proof. Consider the following example, where the right derivation is replaced by the left derivation. The application of (mix) moved upwards and the (mix) formula is of lower complexity.

$$\frac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\,(\neg r) \qquad \dfrac{\Pi \vdash \Lambda, A}{\neg A, \Pi \vdash \Lambda}\,(\neg l)}{\Gamma, \Pi \vdash \Delta, \Lambda}\,(mix) \qquad\qquad \frac{\Gamma, A \vdash \Delta \qquad \Pi \vdash \Lambda, A}{\Gamma, \Pi \vdash \Delta, \Lambda}\,(mix)$$

3.1.1 Sequent Calculus for Modal Logics

To define a sequent calculus for modal logics, sequent calculus systems as introduced before are often extended by rules regarding the operator $\Box$. However, the specific modal rules vary depending on the particular modal logic being considered. As this thesis focuses solely on systems based on the modal logic S5, this section is restricted to discussing this particular case. For a broader overview, refer to [21], [22], or [23].

To obtain a sequent calculus for the modal logic S5, the previously introduced system LK has to be extended with rules that are linked to the axioms K, T, S4, and S5, which were introduced in Def. 2.1.3. To get a sequent calculus for the modal logic S4 (also called KT4 in reference to its axioms), the system LK has to be extended with the rules:

$$\frac{A, \Gamma \vdash \Delta}{\Box A, \Gamma \vdash \Delta} (\Box l) \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A} (\Box r)$$

Note that the subformula property holds for these rules, and together with the establishment of cut-elimination, the calculus above is analytic [24].

Difficulties arise for the modal logic S5. The calculus for S5 replaces $(\Box r)$ with the following rule:

$$\frac{\Box\Gamma \vdash \Box\Delta, A}{\Box\Gamma \vdash \Box\Delta, \Box A} (\Box r)'$$

Onishi and Matsumoto [25] showed that the sequent calculus for S5 using $(\Box r)'$ does not admit cut-elimination, as demonstrated by the following example:

$$\frac{\dfrac{\dfrac{\Box A \vdash \Box A}{\vdash \Box A, \neg\Box A} (\neg r)}{\vdash \Box A, \Box\neg\Box A} (\Box r)' \qquad \dfrac{A \vdash A}{\Box A \vdash A} (\Box l)}{\vdash \Box\neg\Box A, A} (mix)$$

In this case, the standard Gentzen-style proof, which was sketched in the section above, does not work anymore. By shifting the cut upwards, one obtains:

$$\frac{\dfrac{\Box A \vdash \Box A}{\vdash \Box A, \neg\Box A} (\neg r) \qquad \dfrac{A \vdash A}{\Box A \vdash A} (\Box l)}{\vdash \neg\Box A, A} (mix)$$

But another application $(\Box r)'$ is not possible since A is not boxed.

This is just one example, but similar problems occur in various proposed calculi. The formula $\vdash \Box\neg\Box A, A$ does not have a cut-free proof. Various approaches have been explored in the literature to address this issue, often they are missing analyticity. The sequent calculi proposed by Blamey and Humberstone [26] is not cut-free and others do not satisfy the subformula property [27], [28].

3.2 Hypersequent Calculus

Certain logics do not have a cut-free analytic sequent calculus, as demonstrated in the case of the modal logic S5 discussed earlier. This limitation has lead to various generalizations of the sequent framework (e.g., [29], [28], [30], [31]). Avron [12] and Pottinger [13] addressed the problem by adopting a hypersequent framework. Adaptations of the hypersequent approach can be found in works by Restall [32] and Poggiolesi [33] and yielded calculi for many non-classical logics (e.g., [34], [35], [36]).

The fundamental idea behind the hypersequent approach is that, instead of operating on single sequents, rules are applied to multisets of sequents, known as hypersequents. This extension provides additional expressive power to the calculus w.r.t. the sequent calculus, as the hypersequent structure opens the possibility of defining rules that operate on sequents, rather than just formulas.

Definition 3.2.1. Let $\Gamma_i \vdash \Delta_i$ with $i = 1, ..., n$ be sequents and $|$ be a symbol not belonging to the logical language. Then

$$\Gamma_1 \vdash \Delta_1 \mid ... \mid \Gamma_n \vdash \Delta_n$$

is called a *hypersequent* and each $\Gamma_i \vdash \Delta_i$ is called a *component*.

Definition 3.2.2. Let $HS : \Gamma_1 \vdash \Delta_1 \mid ... \mid \Gamma_n \vdash \Delta_n$ be a hypersequent, then the *semantic interpretation* in the calculus for modal logics, denoted by $I(HS)$, stands for

$$I(HS) : \Box(\bigwedge \Gamma_1 \rightarrow \bigvee \Delta_1) \vee ... \vee \Box(\bigwedge \Gamma_n \rightarrow \bigvee \Delta_n).$$

Similar to regular sequent calculi as introduced in the previous section, the distinguishing features of a hypersequent calculus are its axioms and inference rules. In the hypersequent version of LK, the logical rules and internal structural rules are largely similar to those in Def. 3.1.4. However, what distinguishes them is the inclusion of contexts G, H, which represent (possibly empty) side hypersequents. Furthermore, external contraction and external weakening are introduced which manipulate the components of a hypersequent.

Definition 3.2.3. Let A, B denote formulae, $\Gamma, \Delta, \Pi, \Lambda$ denote multisets of formulae and G denote multisets of hypersequents. System HSLK consists of the following axioms and rules:

Axioms:

$G \mid \Gamma, A \vdash A, \Delta$, for *atomic* formulae A.

$$G \mid \bot \vdash \Delta$$

The cut rule:

$$\frac{G \mid \Gamma \vdash \Lambda, A \qquad G \mid A, \Pi \vdash \Delta}{G \mid \Gamma, \Pi \vdash \Lambda, \Delta} \ (cut)$$

A is called the *cut-formula* of this inference.

The internal structural rules:

internal weakening

$$\frac{G \mid \Gamma \vdash \Delta}{G \mid A, \Gamma \vdash \Delta}\,(iw-l) \qquad \frac{G \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta, A}\,(iw-r)$$

internal contraction

$$\frac{G \mid A, A, \Gamma \vdash \Delta}{G \mid A, \Gamma \vdash \Delta}\,(ic-l) \qquad \frac{G \mid \Gamma \vdash \Delta, A, A}{G \mid \Gamma \vdash \Delta, A}\,(ic-r)$$

The external structural rules:

external weakening

$$\frac{G}{G \mid \Gamma \vdash \Delta}\,(ew)$$

external contraction

$$\frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta}\,(ec)$$

The logical rules:

$\neg$ - introduction

$$\frac{G \mid \Gamma \vdash A, \Delta}{G \mid \Gamma, \neg A \vdash \Delta}\,(\neg l) \qquad \frac{G \mid \Gamma, A \vdash \Delta}{G \mid \Gamma \vdash \Delta, \neg A}\,(\neg r)$$

$\rightarrow$ - introduction

$$\frac{G \mid \Gamma \vdash \Delta, A \qquad G \mid B, \Gamma \vdash \Delta}{G \mid A \rightarrow B, \Gamma \vdash \Delta}\,(\rightarrow l) \qquad \frac{G \mid A, \Gamma \vdash B, \Delta}{G \mid \Gamma \vdash A \rightarrow B, \Delta}\,(\rightarrow r)$$

3.2.1 Hypersequent Calculus for Modal Logics

To obtain a hypersequent calculus for S5 we consider the hypersequent version of the sequent rules for S4 as introduced in the previous section and then include the *modal splitting rule*. The modal splitting rule is an external structural rule that enables the exchange or communication of information between components. This rule, along with similar rules in other calculi (cf. [36]), enhances the expressive power of hypersequent calculi. Even though the modal splitting rule is part of the external structural rules, we treat it as a separate category.

Definition 3.2.4. Let A, B denote formulae, $\Gamma_{1/2}, \Delta$ denote multisets of formulae and G, H multisets of hypersequents. System HSS5 consists of the axioms and all the rules from Def. 3.2.3 and:

The S4 modal rules:

$$\frac{G \mid A, \Gamma \vdash \Delta}{G \mid \Box A, \Gamma \vdash \Delta}\,(\Box l) \qquad\qquad \frac{G \mid \Box\Gamma \vdash A}{G \mid \Box\Gamma \vdash \Box A}\,(\Box r)$$

The modal splitting rule:

$$\frac{G \mid \Box\Gamma_1, \Gamma_2 \vdash \Delta}{G \mid \Box\Gamma_1 \vdash \mid \Gamma_2 \vdash \Delta}\,(MS)$$

The (MS) rule is necessary to capture modal logic $\mathsf{S5}$. With this rule, it becomes possible to prove the $\mathsf{S5}$ axiom as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\varphi \vdash \Box\neg\Box\varphi, \Box\varphi}{\Box\varphi \vdash \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(MS)}{\vdash \neg\Box\varphi \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(\neg r)}{\vdash \Box\neg\Box\varphi \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(\Box r)}{\vdash \Box\neg\Box\varphi, \Box\varphi \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(iw-r)}{\vdash \Box\neg\Box\varphi, \Box\varphi}\,(ec)}{\neg\Box\varphi \vdash \Box\neg\Box\varphi}\,(\neg l)}{\vdash \neg\Box\varphi \to \Box\neg\Box\varphi} \to r$$

Note that all rules of $\mathsf{HSS5}$, except for (cut), adhere to the subformula property. The proof of cut-elimination will be presented in the next chapter.

CHAPTER 4

Cut-free Hypersequent Calculi for Mīmāṃsā Logics

In the previous chapter we have recalled the hypersequent calculus for the modal logic S5. Here we extend this calculus to accommodate the specific features of LPr and LKu, as discussed in Chapter 2. We introduce the hypersequent calculi $\mathsf{HS}_{\mathsf{LPr}}$ and $\mathsf{HS}_{\mathsf{LKu}}$, which incorporate hypersequent rules that account for the concepts of obligation, prohibition, and recommendation. The calculi for the S4 version of these rules were originally introduced in [14], utilizing the sequent calculus. However, due to the switch to S5, a transformation into a hypersequent framework becomes necessary.

Furthermore, this chapter focuses on proving the soundness and completeness of the hypersequent calculi for LPr and LKu concerning their respective Hilbert Axioms. An essential objective of this chapter is to establish the analyticity of the introduced calculi. This is accomplished through the process of cut-elimination, demonstrating how every application of the cut rule can be eliminated. This not only leads to cut-free proofs but also ensures that the calculi possess the desired subformula property.

4.1 Hypersequent Calculi $\mathsf{HS}_{\mathsf{LPr}}$ and $\mathsf{HS}_{\mathsf{LKu}}$

To construct sequent calculi for the logics LPr and LKu [14] utilized a method from [35] to translate axioms into sequent rules. To incorporate these rules into a hypersequent framework, a hypersequent context is added to each of them.

Let $A, ..., D$ denote formulae, Γ, Δ denote multisets of formulae and G a (possibly empty) hypersequent.

Definition 4.1.1. The system $\mathsf{HS}_{\mathsf{LPr}}$ consists of the axioms and all the rules from Def. 3.2.4 and:

$$\frac{G \mid \Gamma^{\Box}, A \vdash C \qquad G \mid \Gamma^{\Box}, B \vdash D \qquad G \mid \Gamma^{\Box}, D \vdash B}{G \mid \Gamma, \mathcal{O}(A/B) \vdash \mathcal{O}(C/D), \Delta}\,(Mon_{\mathcal{O}}) \qquad \frac{G \mid \Gamma^{\Box}, A \vdash}{G \mid \Gamma, \mathcal{O}(A/B) \vdash \Delta}\,(D_1)$$

$$\frac{G \mid \Gamma^{\Box}, C \vdash A \qquad G \mid \Gamma^{\Box}, B \vdash D \qquad G \mid \Gamma^{\Box}, D \vdash B}{G \mid \Gamma, \mathcal{F}(A/B) \vdash \mathcal{F}(C/D), \Delta}\,(Mon_{\mathcal{F}}) \qquad \frac{G \mid \Gamma^{\Box} \vdash A}{G \mid \Gamma, \mathcal{F}(A/B) \vdash \Delta}\,(P_{\mathcal{F}})$$

$$\frac{G \mid \Gamma^{\Box}, A, C \vdash \qquad G \mid \Gamma^{\Box}, B \vdash D \qquad G \mid \Gamma^{\Box}, D \vdash B}{G \mid \Gamma, \mathcal{O}(A/B), \mathcal{O}(C/D) \vdash \Delta}\,(D_2)$$

$$\frac{G \mid \Gamma^{\Box}, A \vdash C \qquad G \mid \Gamma^{\Box}, B \vdash D \qquad G \mid \Gamma^{\Box}, D \vdash B}{G \mid \Gamma, \mathcal{O}(A/B), \mathcal{F}(C/D) \vdash \Delta}\,(D_{\mathcal{OF}})$$

$$\frac{G \mid \Gamma^{\Box} \vdash A, B \qquad G \mid \Gamma^{\Box}, C \vdash D \qquad G \mid \Gamma^{\Box}, D \vdash C}{G \mid \Gamma, \mathcal{F}(A/C), \mathcal{F}(B/D) \vdash \Delta}\,(D_{\mathcal{F}})$$

Note that $\Gamma^{\Box}$ denotes Γ in which all formulae not of the form $\Box A$ are deleted.

The correspondence between the Axioms from Def. 2.1.4 and their respective hypersequent rules is as follows: Axioms $\mathsf{A_{Pr}}1$ and $\mathsf{A_{Pr}}5$ were transformed into $(Mon_{\mathcal{O}})$, while $\mathsf{A_{Pr}}2$ and $\mathsf{A_{Pr}}5$ became $(Mon_{\mathcal{F}})$. Axiom $\mathsf{A_{Pr}}3$ corresponds to (D_1), (D_2), $(D_{\mathcal{F}})$, and $(P_{\mathcal{F}})$, respectively.

Definition 4.1.2. The system $\mathsf{HS_{LKu}}$ consists of the axioms and all the rules from Def. 4.1.1 and:

$$\frac{G \mid \Gamma^{\Box}, A \vdash C \qquad G \mid \Gamma^{\Box}, B \vdash D \qquad G \mid \Gamma^{\Box}, D \vdash B}{G \mid, \mathcal{R}(A/B) \vdash \mathcal{R}(C/D), \Delta}\,(Mon_{\mathcal{R}}) \qquad \frac{G \mid \Gamma^{\Box}, A \vdash}{G \mid \Gamma, \mathcal{R}(A/B) \vdash \Delta}\,(P_{\mathcal{R}})$$

Note that $\Gamma^{\Box}$ denotes Γ in which all formulae not of the form $\Box A$ are deleted.

$\mathsf{A_{Ku}}6$ and $\mathsf{A_{Ku}}8$ were transformed into $(Mon_{\mathcal{R}})$, and $(P_{\mathcal{R}})$ arises from $\mathsf{A_{Ku}}7$.

Definition 4.1.3 (Grade of a Formula)**.** The *grade of a formula* A, denoted by $g(A)$, is defined as:

- $g(A) = 0$ if A is atomic
- $g(\neg A) = g(A) + 1$
- $g(A \to B) = g(A) + g(B) + 1$
- $g(\Box A) = g(A) + 1$

- $g(\mathcal{O}(A/B)) = g(A) + g(B) + 1$
- $g(\mathcal{F}(A/B)) = g(A) + g(B) + 1$
- $g(\mathcal{R}(A/B)) = g(A) + g(B) + 1$

The *grade of a cut*, is the grade of its cut formula.

We will now demonstrate that the restriction to atomic initial sequents, necessary for the invertibility of $(\neg l)$ and $(\rightarrow l)$ (which we will prove later in this chapter), does not limit the derivability of initial sequents with arbitrary principal formulas in $\mathsf{HS_{LKu}}$.

Lemma 4.1.1. *Every instance of* $G \mid \Gamma, A \vdash A, \Delta$ *is derivable in* $\mathsf{HS_{LKu}}$.

Proof. Induction on the grade of A.

- Base case: $g(A) = 0$

  As A is atomic, $G \mid \Gamma, A \vdash A, \Delta$ is an initial sequent.

- Inductive step.

  For $g(A) \geq 1$ we distinguish cases according to the shape of A.

  – For $A = \neg B$ consider the following derivation:

$$\dfrac{\dfrac{\dfrac{}{G \mid \Gamma, B \vdash B, \Delta}\,(IH)}{G \mid \Gamma \vdash \neg B, B, \Delta}\,(\neg r)}{G \mid \Gamma, \neg B \vdash \neg B, \Delta}\,(\neg l)$$

  The case where $A = \Box B$ is similar and uses $(\Box r)$ and $(\Box l)$.

  – For $A = B \rightarrow C$ consider the following derivation:

$$\dfrac{\dfrac{\dfrac{}{G \mid \Gamma, B \vdash C, \Delta, B}\,(IH)}{G \mid \Gamma \vdash B \rightarrow C, \Delta, B}\,(\rightarrow r) \qquad \dfrac{\dfrac{}{G \mid C, \Gamma, B \vdash C, \Delta}\,(IH)}{G \mid C, \Gamma \vdash B \rightarrow C, \Delta}\,(\rightarrow r)}{G \mid \Gamma, B \rightarrow C \vdash B \rightarrow C, \Delta}\,(\rightarrow l)$$

  – For $A = \mathcal{O}(B/C)$ consider the following derivation:

$$\dfrac{\dfrac{}{G \mid \Gamma^{\Box}, B \vdash B}\,(IH) \qquad \dfrac{}{G \mid \Gamma^{\Box}, C \vdash C}\,(IH) \qquad \dfrac{}{G \mid \Gamma^{\Box}, C \vdash C}\,(IH)}{G \mid \Gamma, \mathcal{O}(B/C) \vdash \mathcal{O}(B/C), \Delta}\,(Mon_{\mathcal{O}})$$

  The cases where A is $\mathcal{F}(B/C)$ or $\mathcal{R}(B/C)$ are similar and use $(Mon_{\mathcal{F}})$ and $Mon_{\mathcal{R}}$ respectively.

□

To abbreviate some subsequent derivations, we will now introduce additional rules based on Def. 2.1.2.

Definition 4.1.4. The rules $(\wedge l)$, $(\wedge r)$, $(\vee l)$ and $(\vee r)$ are defined as follows:

$(\wedge l)$:

$$\frac{G \mid A, \Gamma \vdash \Delta}{G \mid A \wedge B, \Gamma \vdash \Delta} (\wedge l)$$

$(\wedge r)$:

$$\frac{G \mid \Gamma \vdash \Delta, A \qquad G \mid \Gamma \vdash \Delta, B}{G \mid \Gamma \vdash \Delta, A \wedge B} (\wedge r)$$

$(\vee l)$:

$$\frac{G \mid A, \Gamma \vdash \Delta \qquad G \mid B, \Gamma \vdash \Delta}{G \mid A \vee B, \Gamma \vdash \Delta} (\vee l)$$

$(\vee r)$:

$$\frac{G \mid \Gamma \vdash A, \Delta}{G \mid \Gamma \vdash \Delta, A \vee B} (\vee r)$$

Lemma 4.1.2. *The rules $(\wedge l)$, $(\wedge r)$, $(\vee l)$ and $(\vee r)$ are derivable in $\mathsf{HS}_{\mathsf{LKu}}$.*

Proof.

$(\wedge l)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{G \mid A, \Gamma \vdash \Delta}{G \mid A, \Gamma, B \vdash \Delta} (w-l)}{G \mid A, \Gamma \vdash \neg B, \Delta} (\neg r)}{G \mid \Gamma \vdash A \to \neg B, \Delta} (\to r)}{G \mid \neg(A \to \neg B), \Gamma \vdash \Delta} (\neg l)$$

$(\wedge r)$:

$$\cfrac{\cfrac{G \mid \Gamma \vdash \Delta, A \qquad \cfrac{G \mid \Gamma \vdash \Delta, B}{G \mid \neg B, \Gamma \vdash \Delta} (\neg l)}{G \mid A \to \neg B, \Gamma \vdash \Delta} (\to l)}{G \mid \Gamma \vdash \Delta, \neg(A \to \neg B)} (\neg r)$$

$(\vee l)$:

$$\cfrac{\cfrac{\cfrac{G \mid A, \Gamma \vdash \Delta}{G \mid A, \Gamma \vdash B, \Delta} (w-r)}{G \mid \Gamma \vdash \Delta, A \to B} (\to r) \qquad G \mid B, \Gamma \vdash \Delta}{G \mid (A \to B) \to B, \Gamma \vdash \Delta} (\to l)$$

$(\vee r)$:

$$\cfrac{\cfrac{\cfrac{G \mid \Gamma \vdash A, \Delta}{G \mid \Gamma \vdash A, \Delta, B} (w-r) \qquad \cfrac{}{G \mid B, \Gamma \vdash B, \Delta} (Ax)}{G \mid A \to B, \Gamma \vdash B, \Delta} (\to l)}{G \mid \Gamma \vdash \Delta, (A \to B) \to B} (\to r)$$

□

4.2 Soundness and Completeness

Theorem 4.2.1 (Soundness)**.** *If there is a proof in $\mathsf{HS}_{\mathsf{LKu}}$ of $HS := \Gamma_1 \vdash \Pi_1 \mid ... \mid \Gamma_n \vdash \Pi_n$, then $I(HS) := \Box(\bigwedge \Gamma_1 \to \bigvee \Pi_1) \vee ... \vee \Box(\bigwedge \Gamma_n \to \bigvee \Pi_n)$ is valid.*

Proof. By induction on the length of a cut-free proof of $\mathsf{HS}_{\mathsf{LKu}}$. Let (r) be the last applied rule. If (r) is an axiom, then the claim holds trivially.

The structural rules:

- If (r) is a structural rule, the cases are straightforward. (Note that we consider the rule (MS) as its own category. The case for (MS) will be presented as part of the modal rules.) Here, we only present the case where $(r) = (ec)$.

Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\bigwedge\Gamma \to \bigvee\Delta) \vee \Box G$.

Hence, there exist u, v such that

(1) $M, u \not\models \bigwedge\Gamma \to \bigvee\Delta$

(2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

From (1) conclude $M, u \not\models \Box(\bigwedge\Gamma \to \bigvee\Delta)$. But by validity of the premise and (2) we have $M, u \models \Box(\bigwedge\Gamma \to \bigvee\Delta) \vee \Box(\bigwedge\Gamma \to \bigvee\Delta)$. Contradiction.

The logical rules:

- $(r) = (\to r)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\bigwedge\Gamma \to (A \to B) \vee \bigvee\Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \not\models \bigwedge\Gamma \to (A \to B) \vee \bigvee\Delta$
  $\Rightarrow M, u \models \bigwedge\Gamma$
  $M, u \not\models A \to B \Rightarrow M, u \models A$ and $M, u \not\models B$
  $M, u \not\models \bigvee\Delta$

  (2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

  From (1) conclude $M, u \models A \wedge \bigwedge\Gamma$ and $M, u \not\models B \vee \bigvee\Delta$. But by validity of the premise and (2) we have $M, u \models A \wedge \bigwedge\Gamma \to B \vee \bigvee\Delta$. Contradiction.

- $(r) = (\to l)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box((A \to B) \wedge \bigwedge\Gamma \to \bigvee\Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \not\models (A \to B) \wedge \bigwedge\Gamma \to \bigvee\Delta$
  $\Rightarrow M, u \models A \to B$
  $M, u \models \bigwedge\Gamma$
  $M, u \not\models \bigvee\Delta$

  (2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

  By (1) and semantics of $\to$ we have $M, u \models \neg A \vee B$. Consider the following two cases:

Case 1: Assume $M, u \models \neg A$.

By validity of the first premise and (2) we have $M, u \models \bigwedge \Gamma \rightarrow \bigvee \Delta \vee A$. This contradicts (1) together with the assumption of the case.

Case 2: Assume $M, u \models B$.

By validity of the second premise and (2) we have $M, u \models B \wedge \bigwedge \Gamma \rightarrow \bigvee \Delta$. This contradicts (1) together with the assumption of the case.

Both cases falsify a premise, which contradicts the initial assumption.

- $(r) = (\neg l)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\bigwedge \Gamma \wedge \neg A \rightarrow \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v, such that

  (1) $M, u \not\models \bigwedge \Gamma \wedge \neg A \rightarrow \bigvee \Delta$

  $\Rightarrow$ $M, u \models \bigwedge \Gamma \wedge \neg A$

  $M, u \not\models \bigvee \Delta$

  (2) $M, v \not\models G \Rightarrow M, v \not\models \Box G$

  From (1) and semantics of $\wedge$ we conclude that $M, u \models \bigwedge \Gamma$ and $M, u \models \neg A$. By the latter we have that $\forall_{\alpha \in A} : M, u \not\models \alpha$ and therefore $M, u \not\models \bigvee A$ by semantics of $\vee$. But by validity of the premise, global assumption and (2) we have $M, u \models \bigwedge \Gamma \rightarrow \bigvee A \vee \bigvee \Delta$. Contradiction.

  The case where (r) is $(\neg r)$ is similar.

The modal rules:

- $(r) = (\Box l)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\Box A \wedge \bigwedge \Gamma \rightarrow \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v, such that

  (1) $M, u \not\models \Box A \wedge \bigwedge \Gamma \rightarrow \bigvee \Delta$

  $\Rightarrow$ $M, u \models \Box A$

  $M, u \models \bigwedge \Gamma$

  $M, u \not\models \bigvee \Delta$

  (2) $M, v \not\models G \Rightarrow M, v \not\models \Box G$

By validity of the premise and (2) we have $M, u \vDash A \wedge \bigwedge \Gamma \rightarrow \bigvee \Delta$. Contradiction by (1).

- $(r) = (\Box r)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \nvDash \Box(\bigwedge \Box\Gamma \rightarrow \Box A) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \nvDash \bigwedge \Box\Gamma \rightarrow \Box A$
  $\Rightarrow M, u \vDash \bigwedge \Box\Gamma$
  $M, u \nvDash \Box A$

  (2) $M, v \nvDash G \Rightarrow M, v \nvDash \Box G$

  By validity of the premise and (2) we have $M, u \vDash \bigwedge \Box\Gamma \rightarrow A$. Contradiction by (1).

- $(r) = (MS)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \nvDash \Box(\bigwedge \Box\Gamma_1 \rightarrow \bot) \vee \Box(\bigwedge \Gamma_2 \rightarrow \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v, w such that

  (1) $M, u \nvDash \bigwedge \Box\Gamma_1 \rightarrow \bot$
  $\Rightarrow M, u \vDash \bigwedge \Box\Gamma_1$

  (2) $M, v \nvDash \bigwedge \Gamma_2 \rightarrow \bigvee \Delta$
  $\Rightarrow M, v \vDash \bigwedge \Gamma_2$
  $M, v \nvDash \bigvee \Delta$

  (3) $M, w \nvDash G$

  Due to global necessity of truths and (1) we have $M, v \vDash \bigwedge \Box\Gamma_1$ and by (3) $M, v \nvDash \Box G$. Hence, by (2) and semantics of $\wedge$ we have $M, v \vDash \bigwedge \Box\Gamma_1 \wedge \bigwedge \Gamma_2$. But by validity of the premise $M, v \vDash \Box(\bigwedge \Box\Gamma_1 \wedge \bigwedge \Gamma_2 \rightarrow \bigvee \Delta) \vee \Box G$. Contradiction.

The deontic rules:

- $(r) = (Mon_{\mathcal{O}})$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \nvDash \Box(\bigwedge \Gamma \wedge \mathcal{O}(A/B) \rightarrow \mathcal{O}(C/D) \vee \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v such that

(1) $M, u \not\models (\bigwedge\Gamma \wedge \mathcal{O}(A/B) \rightarrow \mathcal{O}(C/D) \vee \bigvee\Delta)$

$\Rightarrow M, u \models \bigwedge\Gamma \Rightarrow M, u \models \bigwedge\Gamma^{\Box}$

$M, u \models \mathcal{O}(A/B) \Leftrightarrow (\|A\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$

$M, u \not\models \mathcal{O}(C/D) \Leftrightarrow (\|C\|, \|D\|) \notin \mathcal{N}_{\mathcal{O}}(u)$

(2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

Due to global necessities of truths and (1) we have $\forall w \in W : M, w \models \Gamma^{\Box}$. Together with validity of the first premise and (2) we have $\forall w \in W : M, w \models A \rightarrow C$ and conclude if $M, w \models A$ then $M, w \models C$. Hence, $\|A\| \subseteq \|C\|$. Using Condition (i) in Def. 2.1.5 we conclude $(\|C\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$. But from the validity of the second and third premise and (2) we conclude $\|B\| = \|D\|$, hence $(\|C\|, \|B\|) \notin \mathcal{N}_{\mathcal{O}}(u)$. Contradiction.

The case where $(r) = (Mon_{\mathcal{R}})$ is the same. The case where $(r) = (Mon_{\mathcal{F}})$ is similar and uses Condition (ii) in Def. 2.1.5.

- $(r) = (D_2)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\bigwedge\Gamma \wedge \mathcal{O}(A/B) \wedge \mathcal{O}(C/D) \rightarrow \bigvee\Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \not\models (\bigwedge\Gamma \wedge \mathcal{O}(A/B) \wedge \mathcal{O}(C/D) \rightarrow \bigvee\Delta)$

  $\Rightarrow M, u \models \bigwedge\Gamma \Rightarrow M, u \models \bigwedge\Gamma^{\Box}$

  $M, u \models \mathcal{O}(A/B) \Leftrightarrow (\|A\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$

  $M, u \models \mathcal{O}(C/D) \Leftrightarrow (\|C\|, \|D\|) \in \mathcal{N}_{\mathcal{O}}(u)$

  (2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

  Due to global necessities of truths and (1) we have $\forall w \in W : M, w \models \Gamma^{\Box}$. Together with validity of the first premise and (2) we have $\forall w \in W : M, w \models A \wedge C \rightarrow \bot$ and conclude if $M, w \models A$ then $M, w \models \overline{C}$. Hence, $\|A\| \subseteq \|\overline{C}\|$. Using Condition (i) in Def. 2.1.5 we conclude $(\|\overline{C}\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$. But from the validity of the second and third premise and (2) we conclude $\|B\| = \|D\|$, hence $(\|C\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$. Using this and Condition (iii) in Def. 2.1.5 we have $(\|\overline{C}\|, \|B\|) \notin \mathcal{N}_{\mathcal{O}}(u)$. Contradiction.

- $(r) = (D_{\mathcal{O}\mathcal{F}})$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\bigwedge\Gamma \wedge \mathcal{O}(A/B) \wedge \mathcal{F}(C/D) \rightarrow \bigvee\Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \not\models (\bigwedge\Gamma \wedge \mathcal{O}(A/B) \wedge \mathcal{F}(C/D) \rightarrow \bigvee\Delta)$

  $\Rightarrow M, u \models \bigwedge\Gamma \Rightarrow M, u \models \bigwedge\Gamma^{\Box}$

$M, u \models \mathcal{O}(A/B) \Leftrightarrow (\|A\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$
$M, u \models \mathcal{F}(C/D) \Leftrightarrow (\|C\|, \|D\|) \in \mathcal{N}_{\mathcal{F}}(u)$

(2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

Due to global necessities of truths and (1) we have $\forall w \in W : M, w \models \Gamma^{\Box}$. Together with validity of the first premise and (2) we have $\forall w \in W : M, w \models A \to C$ and conclude if $M, w \models A$ then $M, w \models C$. Hence, $\|A\| \subseteq \|C\|$. Using Condition (i) in Def. 2.1.5 we conclude $(\|C\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$. From the validity of the second and third premise and (2) we conclude $\|B\| = \|D\|$, hence $(\|C\|, \|D\|) \in \mathcal{N}_{\mathcal{O}}(u)$. Using this and Condition (iv) in Def. 2.1.5 we have $(\|C\|, \|D\|) \notin \mathcal{N}_{\mathcal{F}}(u)$. Contradiction with (1).

- $(r) = (D_{\mathcal{F}})$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\bigwedge \Gamma \wedge \mathcal{F}(A/C) \wedge \mathcal{F}(B/D) \to \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \not\models (\bigwedge \Gamma \wedge \mathcal{F}(A/C) \wedge \mathcal{F}(B/D) \to \bigvee \Delta)$
  $\Rightarrow M, u \models \bigwedge \Gamma \Rightarrow M, u \models \bigwedge \Gamma^{\Box}$
  $M, u \models \mathcal{F}(A/C) \Leftrightarrow (\|A\|, \|C\|) \in \mathcal{N}_{\mathcal{F}}(u)$
  $M, u \models \mathcal{F}(B/D) \Leftrightarrow (\|B\|, \|D\|) \in \mathcal{N}_{\mathcal{F}}(u)$

  (2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

  Due to global necessities of truths and (1) we have $\forall w \in W : M, w \models \Gamma^{\Box}$. Together with validity of the first premise and (2) we have $\forall w \in W : M, w \models A \vee B$ and conclude if $M, w \models \overline{A}$ then $M, w \models B$. Hence, $\|\overline{A}\| \subseteq \|B\|$. Using Condition (ii) in Def. 2.1.5 we conclude $(\|\overline{A}\|, \|D\|) \in \mathcal{N}_{\mathcal{F}}(u)$. But from the validity of the second and third premise and (2) we conclude $\|C\| = \|D\|$, hence $(\|A\|, \|D\|) \in \mathcal{N}_{\mathcal{F}}(u)$. Using this and Condition (iii) in Def. 2.1.5 we have $(\|\overline{A}\|, \|D\|) \notin \mathcal{N}_{\mathcal{F}}(u)$. Contradiction.

- $(r) = (D_1)$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \not\models \Box(\bigwedge \Gamma \wedge \mathcal{O}(A/B) \to \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \not\models (\bigwedge \Gamma \wedge \mathcal{O}(A/B) \to \bigvee \Delta)$
  $\Rightarrow M, u \models \bigwedge \Gamma \Rightarrow M, u \models \bigwedge \Gamma^{\Box}$
  $M, u \models \mathcal{O}(A/B) \Leftrightarrow (\|A\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$

  (2) $M, v \not\models G \Rightarrow \forall w \in W : M, w \not\models \Box G$

Due to global necessities of truths and (1) we have $\forall w \in W : M, w \vDash \Gamma^{\Box}$. Together with validity of the premise and (2) we have $\forall w \in W : M, w \vDash A \to \bot$ and conclude $M, w \nvDash A$. Hence, $\|A\| = \emptyset$ and $(\emptyset, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$. By Condition (i) in Def. 2.1.5 and the fact that $\emptyset \subseteq \|\overline{A}\|$ we then have $(\|\overline{A}\|, \|B\|) \in \mathcal{N}_{\mathcal{O}}(u)$. But using Condition (iii) in Def. 2.1.5 and (1) we have $(\|\overline{A}\|, \|B\|) \notin \mathcal{N}_{\mathcal{O}}(u)$. Contradiction.

- $(r) = (P_{\mathcal{R}})$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \nvDash \Box(\bigwedge \Gamma \wedge \mathcal{R}(A/B) \to \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \nvDash (\bigwedge \Gamma \wedge \mathcal{R}(A/B) \to \bigvee \Delta)$
  $\Rightarrow M, u \vDash \bigwedge \Gamma \Rightarrow M, u \vDash \bigwedge \Gamma^{\Box}$
  $M, u \vDash \mathcal{R}(A/B) \Leftrightarrow (\|A\|, \|B\|) \in \mathcal{N}_{\mathcal{R}}(u)$

  (2) $M, v \nvDash G \Rightarrow \forall w \in W : M, w \nvDash \Box G$

  Due to global necessities of truths and (1) we have $\forall w \in W : M, w \vDash \Gamma^{\Box}$. Together with validity of the premise and (2) we have $\forall w \in W : M, w \vDash A \to \bot$ and conclude $M, w \nvDash A$. Hence, $\|A\| = \emptyset$. But by Condition (vi) in Def. 2.1.6 and (1) we have $\|A\| \neq \emptyset$. Contradiction.

- $(r) = (P_{\mathcal{F}})$

  Suppose validity of the premise but not of the conclusion. Let M be an arbitrary model and let x be an arbitrary world in M such that $M, x \nvDash \Box(\bigwedge \Gamma \wedge \mathcal{F}(A/B) \to \bigvee \Delta) \vee \Box G$.

  Hence, there exist u, v such that

  (1) $M, u \nvDash (\bigwedge \Gamma \wedge \mathcal{F}(A/B) \to \bigvee \Delta)$
  $\Rightarrow M, u \vDash \bigwedge \Gamma \Rightarrow M, u \vDash \bigwedge \Gamma^{\Box}$
  $M, u \vDash \mathcal{F}(A/B) \Leftrightarrow (\|A\|, \|B\|) \in \mathcal{N}_{\mathcal{F}}(u)$

  (2) $M, v \nvDash G \Rightarrow \forall w \in W : M, w \nvDash \Box G$

  Due to global necessities of truths and (1) we have $\forall w \in W : M, w \vDash \Gamma^{\Box}$. Together with validity of the premise and (2) we have $\forall w \in W : M, w \vDash \Gamma^{\Box} \to A$ and conclude $M, w \vDash A$. Hence, $\|\overline{A}\| = \emptyset$. By Condition (ii) in Def. 2.1.5 and the fact that $\emptyset \subseteq \|A\|$ we then have $(\|\overline{A}\|, \|B\|) \in \mathcal{N}_{\mathcal{F}}(u)$. But using Condition (iii) in Def. 2.1.5 and (1) we have $(\|\overline{A}\|, \|B\|) \notin \mathcal{N}_{\mathcal{F}}(u)$. Contradiction.

$\Box$

Corollary 4.2.1.1. *If there is a proof in* $\mathsf{HS}_{\mathsf{LPr}}$ *of* $HS := \Gamma_1 \vdash \Pi_1 \mid \ldots \mid \Gamma_n \vdash \Pi_n$*, then* $I(HS) := \Box(\bigwedge \Gamma_1 \to \bigvee \Pi_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \to \bigvee \Pi_n)$ *is valid.*

Proof. The soundness theorem for $\mathsf{HS}_{\mathsf{LKu}}$, excluding the cases demonstrating $(r) = (Mon_{\mathcal{R}})$ and $(r) = (P_{\mathcal{R}})$, serves as a proof of soundness for $\mathsf{HS}_{\mathsf{LPr}}$. □

Theorem 4.2.2 (Completeness)**.** *If φ is valid, then $\vdash \varphi$ is derivable in $\mathsf{HS}_{\mathsf{LKu}}$.*

Proof. We first demonstrate the derivability of all axioms from Def. 2.1.3 and Def. 2.1.4. The proof for the propositional axioms is straightforward. Here, we only show the modal and deontic axioms. After establishing the derivability of the axioms, we proceed to show that Modus Ponens and Necessity can be simulated within $\mathsf{HS}_{\mathsf{LKu}}$.

- The modal axioms

Axiom K:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \psi, \varphi \qquad \psi, \varphi \vdash \psi}{\varphi \to \psi, \varphi \vdash \psi}\,(\to l)}{\Box(\varphi \to \psi), \Box\varphi \vdash \psi}\,2x(\Box l)}{\Box(\varphi \to \psi), \Box\varphi \vdash \Box\psi}\,(\Box r)}{\Box(\varphi \to \psi) \vdash \Box\varphi \to \Box\psi}\,(\to r)}{\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)}\,(\to r)$$

Axiom T:

$$\dfrac{\dfrac{\varphi \vdash \varphi}{\Box\varphi \vdash \varphi}\,(\Box l)}{\vdash \Box\varphi \to \varphi}\,(\to r)$$

Axiom $\mathsf{4}$:

$$\dfrac{\dfrac{\Box\varphi \vdash \Box\varphi}{\Box\varphi \vdash \Box\Box\varphi}\,(\Box r)}{\vdash \Box\varphi \to \Box\Box\varphi}\,(\to r)$$

Axiom $\mathsf{5}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box\varphi \vdash \Box\neg\Box\varphi, \Box\varphi}{\Box\varphi \vdash \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(MS)}{\vdash \neg\Box\varphi \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(\neg r)}{\vdash \Box\neg\Box\varphi \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(\Box r)}{\vdash \Box\neg\Box\varphi, \Box\varphi \mid \vdash \Box\neg\Box\varphi, \Box\varphi}\,(iw - r)}{\vdash \Box\neg\Box\varphi, \Box\varphi}\,(ec)}{\neg\Box\varphi \vdash \Box\neg\Box\varphi}\,(\neg l)}{\vdash \neg\Box\varphi \to \Box\neg\Box\varphi}\to r$$

- The deontic axioms

Axiom $\mathsf{A}_{\mathsf{Pr}}1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi, \psi \qquad \psi, \varphi \vdash \psi}{\varphi \to \psi, \varphi \vdash \psi}\,(\to l)}{\Box(\varphi \to \psi), \varphi \vdash \psi}\,(\Box l) \qquad \Box(\varphi \to \psi), \Theta \vdash \Theta \qquad \Box(\varphi \to \psi), \Theta \vdash \Theta}{\Box(\varphi \to \psi), \mathcal{O}(\varphi/\Theta) \vdash \mathcal{O}(\psi, \Theta)}\,(Mon_{\mathcal{O}})}{\Box(\varphi \to \psi) \wedge \mathcal{O}(\varphi/\Theta), \Box(\varphi \to \psi) \wedge \mathcal{O}(\varphi/\Theta) \vdash \mathcal{O}(\psi, \Theta)}\,2x(\wedge l)}{\Box(\varphi \to \psi) \wedge \mathcal{O}(\varphi/\Theta) \vdash \mathcal{O}(\psi, \Theta)}\,(ic - l)}{\vdash (\Box(\varphi \to \psi) \wedge \mathcal{O}(\varphi/\Theta)) \to \mathcal{O}(\psi/\Theta)}\,(\to r)$$

Axiom $\mathsf{A}_{\mathsf{Pr}}2$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi, \psi \qquad \psi, \varphi \vdash \psi}{\varphi \to \psi, \varphi \vdash \psi}\,(\to l)}{\Box(\varphi \to \psi), \varphi \vdash \psi}\,(\Box l) \qquad \Box(\varphi \to \psi), \Theta \vdash \Theta \qquad \Box(\varphi \to \psi), \Theta \vdash \Theta}{\Box(\varphi \to \psi), \mathcal{F}(\psi/\Theta) \vdash \mathcal{F}(\varphi/\Theta)}\,(Mon_{\mathcal{F}})}{\Box(\varphi \to \psi) \wedge \mathcal{F}(\psi/\Theta), \Box(\varphi \to \psi) \wedge \mathcal{F}(\psi/\Theta) \vdash \mathcal{F}(\varphi/\Theta)}\,2x(\wedge l)}{\Box(\varphi \to \psi) \wedge \mathcal{F}(\psi/\Theta) \vdash \mathcal{F}(\varphi/\Theta)}\,(ic-l)}{\vdash (\Box(\varphi \to \psi) \wedge \mathcal{F}(\psi/\Theta)) \to \mathcal{F}(\varphi/\Theta)}\,(\to r)$$

Axiom $\mathsf{A}_{\mathsf{Ku}}3$ with $\mathcal{X} = \mathcal{O}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi}{\varphi, \neg\varphi \vdash}\,(\neg l) \qquad \Theta \vdash \Theta \qquad \Theta \vdash \Theta}{\mathcal{O}(\varphi/\Theta), \mathcal{O}(\neg\varphi/\Theta) \vdash}\,(D_2)}{\mathcal{O}(\varphi/\Theta) \wedge \mathcal{O}(\neg\varphi/\Theta), \mathcal{O}(\varphi/\Theta) \wedge \mathcal{O}(\neg\varphi/\Theta) \vdash}\,2x(\wedge l)}{\mathcal{O}(\varphi/\Theta) \wedge \mathcal{O}(\neg\varphi/\Theta) \vdash}\,(ic-l)}{\vdash \neg(\mathcal{O}(\varphi/\Theta) \wedge \mathcal{O}(\neg\varphi/\Theta))}\,(\neg r)$$

Axiom $\mathsf{A}_{\mathsf{Pr}}3$ with $\mathcal{X} = \mathcal{F}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi}{\vdash \varphi, \neg\varphi}\,(\neg r) \qquad \Theta \vdash \Theta \qquad \Theta \vdash \Theta}{\mathcal{F}(\varphi/\Theta), \mathcal{F}(\neg\varphi/\Theta) \vdash}\,(D_{\mathcal{F}})}{\mathcal{F}(\varphi/\Theta) \wedge \mathcal{F}(\neg\varphi/\Theta), \mathcal{F}(\varphi/\Theta) \wedge \mathcal{F}(\neg\varphi/\Theta) \vdash}\,2x(\wedge l)}{\mathcal{F}(\varphi/\Theta) \wedge \mathcal{F}(\neg\varphi/\Theta) \vdash}\,(ic-l)}{\vdash \neg(\mathcal{F}(\varphi/\Theta) \wedge \mathcal{F}(\neg\varphi/\Theta))}\,(\neg r)$$

Axiom $\mathsf{A}_{\mathsf{Pr}}4$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi \qquad \Theta \vdash \Theta \qquad \Theta \vdash \Theta}{\mathcal{O}(\varphi/\Theta), \mathcal{F}(\varphi, \Theta) \vdash}\,(D_{\mathcal{OF}})}{\mathcal{O}(\varphi/\Theta) \wedge \mathcal{F}(\varphi, \Theta), \mathcal{O}(\varphi/\Theta) \wedge \mathcal{F}(\varphi, \Theta) \vdash}\,2x(\wedge l)}{\mathcal{O}(\varphi/\Theta) \wedge \mathcal{F}(\varphi, \Theta) \vdash}\,(ic-l)}{\vdash \neg(\mathcal{O}(\varphi/\Theta) \wedge \mathcal{F}(\varphi/\Theta))}\,\neg r$$

Axiom $\mathsf{A}_{\mathsf{Pr}}5$ for $\mathcal{X} = \mathcal{O}$:

Due to its size, the proof tree is split into parts.

Part A:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Theta \to \psi, \psi \vdash \Theta, \psi \qquad \Theta, \Theta \to \psi, \psi \vdash \Theta}{\psi \to \Theta, \Theta \to \psi, \psi \vdash \Theta}\,(\to l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), (\psi \to \Theta) \wedge (\Theta \to \psi), \psi \vdash \Theta}\,(2x(\wedge l))}{(\psi \to \Theta) \wedge (\Theta \to \psi), \psi \vdash \Theta}\,(ic-l)}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \psi \vdash \Theta}\,(\Box l)$$

Part B:

$$\dfrac{\dfrac{\dfrac{\dfrac{\psi \to \Theta, \Theta \vdash \psi, \Theta \qquad \psi, \psi \to \Theta, \Theta \vdash \psi}{\psi \to \Theta, \Theta \to \psi, \Theta \vdash \psi}\,(\to l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), (\psi \to \Theta) \wedge (\Theta \to \psi), \Theta \vdash \psi}\,2x(\wedge l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), \Theta \vdash \psi}\,(ic-l)}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \Theta \vdash \psi}\,(\Box l)$$

Final proof tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \varphi \vdash \varphi \qquad \overset{\text{Part A}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \psi \vdash \Theta} \qquad \overset{\text{Part B}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \Theta \vdash \psi}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \mathcal{O}(\varphi/\psi) \vdash \mathcal{O}(\varphi/\Theta)}\,(Mon_{\mathcal{O}})}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi), \Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi) \vdash \mathcal{O}(\varphi/\Theta)}\,(2x(\wedge l))}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi) \vdash \mathcal{O}(\varphi/\Theta)}\,(ic-l)}{\vdash (\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{O}(\varphi/\psi)) \to \mathcal{O}(\varphi/\Theta)}\,(\to r)
$$

Axiom $\mathsf{A_{Pr}}5$ for $\mathcal{X} = \mathcal{F}$:

Due to its size, the proof tree is split into parts.

Part A:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Theta \to \psi, \psi \vdash \Theta, \psi \qquad \Theta, \Theta \to \psi, \psi \vdash \Theta}{\psi \to \Theta, \Theta \to \psi, \psi \vdash \Theta}\,(\to l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), (\psi \to \Theta) \wedge (\Theta \to \psi), \psi \vdash \Theta}\,(2x(\wedge l))}{(\psi \to \Theta) \wedge (\Theta \to \psi), \psi \vdash \Theta}\,(ic-l)}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \psi \vdash \Theta}\,(\Box l)
$$

Part B:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\psi \to \Theta, \Theta \vdash \psi, \Theta \qquad \psi, \psi \to \Theta, \Theta \vdash \psi}{\psi \to \Theta, \Theta \to \psi, \Theta \vdash \psi}\,(\to l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), (\psi \to \Theta) \wedge (\Theta \to \psi), \Theta \vdash \psi}\,2x(\wedge l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), \Theta \vdash \psi}\,(ic-l)}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \Theta \vdash \psi}\,(\Box l)
$$

Final proof tree:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \varphi \vdash \varphi \qquad \overset{\text{Part A}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \psi \vdash \Theta} \qquad \overset{\text{Part B}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \Theta \vdash \psi}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \mathcal{F}(\varphi/\psi) \vdash \mathcal{F}(\varphi/\Theta)}\,(Mon_{\mathcal{F}})}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{F}(\varphi/\psi), \Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{F}(\varphi/\psi) \vdash \mathcal{F}(\varphi/\Theta)}\,(2x(\wedge l))}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{F}(\varphi/\psi) \vdash \mathcal{F}(\varphi/\Theta)}\,(ic-l)}{\vdash (\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{F}(\varphi/\psi)) \to \mathcal{F}(\varphi/\Theta)}\,(\to r)
$$

Axiom $\mathsf{A_{Ku}}6$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \psi, \varphi \qquad \psi, \varphi \vdash \psi}{\varphi \to \psi, \varphi \vdash \psi}\,(\to l)}{\Box(\varphi \to \psi), \varphi \vdash \psi}\,(\Box l) \qquad \Box(\varphi \to \psi), \Theta \vdash \Theta \qquad \Box(\varphi \to \psi), \Theta \vdash \Theta}{\Box(\varphi \to \psi), \mathcal{R}(\varphi/\Theta) \vdash \mathcal{R}(\psi/\Theta)}\,(Mon_{\mathcal{R}})}{\Box(\varphi \to \psi) \wedge \mathcal{R}(\varphi/\Theta), \Box(\varphi \to \psi) \wedge \mathcal{R}(\varphi/\Theta) \vdash \mathcal{R}(\psi/\Theta)}\,2x(\wedge l)}{\Box(\varphi \to \psi) \wedge \mathcal{R}(\varphi/\Theta) \vdash \mathcal{R}(\psi/\Theta)}\,(ic-l)}{\vdash (\Box(\varphi \to \psi) \wedge \mathcal{R}(\varphi/\Theta)) \to \mathcal{R}(\psi/\Theta)}\,(\to r)
$$

Axiom $\mathsf{A_{Ku}}7$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \vdash \varphi}{\neg\varphi, \varphi \vdash}\,(\neg l)}{\Box\neg\varphi, \varphi \vdash}\,(\Box l)}{\Box\neg\varphi, \mathcal{R}(\varphi/\psi) \vdash}\,(P_{\mathcal{R}})}{\mathcal{R}(\varphi/\psi) \vdash \neg\Box\neg\varphi}\,(\neg r)}{\vdash \mathcal{R}(\varphi/\psi) \to \neg\Box\neg\varphi}\,(\to r)$$

Axiom $\mathsf{A}_{\mathsf{Ku}}8$:

Due to its size, the proof tree is split into parts.

Part A:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Theta \to \psi, \psi \vdash \Theta, \psi \qquad \Theta, \Theta \to \psi, \psi \vdash \Theta}{\psi \to \Theta, \Theta \to \psi, \psi \vdash \Theta}\,(\to l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), (\psi \to \Theta) \wedge (\Theta \to \psi), \psi \vdash \Theta}\,(2x(\wedge l))}{(\psi \to \Theta) \wedge (\Theta \to \psi), \psi \vdash \Theta}\,(ic-l)}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \psi \vdash \Theta}\,(\Box l)$$

Part B:

$$\dfrac{\dfrac{\dfrac{\dfrac{\psi \to \Theta, \Theta \vdash \psi, \Theta \qquad \psi, \psi \to \Theta, \Theta \vdash \psi}{\psi \to \Theta, \Theta \to \psi, \Theta \vdash \psi}\,(\to l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), (\psi \to \Theta) \wedge (\Theta \to \psi), \Theta \vdash \psi}\,2x(\wedge l)}{(\psi \to \Theta) \wedge (\Theta \to \psi), \Theta \vdash \psi}\,(ic-l)}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \Theta \vdash \psi}\,(\Box l)$$

Final proof tree:

$$\dfrac{\dfrac{\dfrac{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \varphi \vdash \varphi \qquad \overset{\text{Part A}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \psi \vdash \Theta} \qquad \overset{\text{Part B:}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \Theta \vdash \psi}}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)), \mathcal{R}(\varphi/\psi) \vdash \mathcal{R}(\varphi/\Theta)}\,(Mon_{\mathcal{R}})}{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{R}(\varphi/\psi), \Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{R}(\varphi/\psi) \vdash \mathcal{R}(\varphi/\Theta)}\,(2x(\wedge l))}{\dfrac{\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{R}(\varphi/\psi) \vdash \mathcal{R}(\varphi/\Theta)}{\vdash (\Box((\psi \to \Theta) \wedge (\Theta \to \psi)) \wedge \mathcal{R}(\varphi/\psi)) \to \mathcal{R}(\varphi/\Theta)}\,(\to r)}\,(ic-l)$$

It remains to show that Modus Ponens and Necessity can be simulated within $\mathsf{HS}_{\mathsf{LKu}}$.

Modus Ponens:

$$\dfrac{\vdash \varphi \to \psi \qquad \dfrac{\dfrac{\vdash \varphi}{\vdash \varphi, \psi}\,(iw-r) \qquad \psi \vdash \psi}{\varphi \to \psi \vdash \psi}\,(\to l)}{\vdash \psi}\,(cut)$$

Necessity:

$$\frac{\vdash \varphi}{\vdash \Box\varphi}\ (\Box r)$$

□

Corollary 4.2.2.1. *If φ is valid, then $\vdash \varphi$ is derivable in* $\mathsf{HS}_{\mathsf{LPr}}$.

Proof. The completeness theorem for $\mathsf{HS}_{\mathsf{LKu}}$, excluding the cases demonstrating Axiom $\mathsf{A}_{\mathsf{Ku}}7$, Axiom $\mathsf{A}_{\mathsf{Ku}}8$, and Axiom $\mathsf{A}_{\mathsf{Ku}}9$, serves as a proof of completeness for $\mathsf{HS}_{\mathsf{LPr}}$. □

Invertibility

An inference rule in a sequent calculus is a reasoning step in which the derivability of the premises implies derivability of the conclusion through an application of the specific rule. In general, there is no guarantee that the opposite is true, namely that if the conclusion is valid, the premises are valid as well. Consider, for example, the following instance of $(Mon_{\mathcal{O}})$, where the conclusion is derivable, but the premises are not:

$$\frac{A \vdash C \qquad B \vdash D \qquad D \vdash B}{\mathcal{O}(C \wedge B/D), \mathcal{O}(A/B) \vdash \mathcal{O}(C/D)}\ (Mon_{\mathcal{O}})$$

In the above derivation, the principal formula is $\mathcal{O}(A/B)$. Note that the conclusion is derivable through an application of $(Mon_{\mathcal{O}})$ with $\mathcal{O}(C \wedge B/D)$ as the principal formula:

$$\frac{\dfrac{C \vdash C}{C \wedge B \vdash C}\ (\wedge l) \qquad D \vdash D \qquad D \vdash D}{\mathcal{O}(C \wedge B/D), \mathcal{O}(A/B) \vdash \mathcal{O}(C/D)}\ (Mon_{\mathcal{O}})$$

Hence, a derivable conclusion does not necessarily imply a derivable premise for $(Mon_{\mathcal{O}})$. In this case this is due to the fact that the context is eliminated on both sides by its backward application. During proof search, which is generally done bottom-up, one must guess whether $(Mon_{\mathcal{O}})$ is applied to $\mathcal{O}(C \wedge B/D)$ or $\mathcal{O}(A/B)$.

However, there are rules for which a derivable conclusion indeed implies derivable premises. These special rules are known as invertible rules.

Definition 4.2.1 (Height-Preserving Invertible Rule)**.** A rule is called *hp-invertible* in a sequent calculus system if a proof of height h of its conclusion implies the existence of proofs of height $\leq h$ of each of its premises.

We will now demonstrate that the rules $(\neg l)$ and $(\rightarrow l)$ are hp-invertible. To do so, we perform induction on the height of a given derivation. We consider all possible rules that could have been applied last in the derivation, grouped into categories. First, we examine internal and external structural rules, then logical rules and the cut rule, followed by modal rules, and finally, deontic rules.

Theorem 4.2.3. *The* $(\neg l)$ *rule is hp-invertible in* $\mathsf{HS}_{\mathsf{LKu}}$.

Proof. Let A, B, C denote formulae, Γ, Δ denote sets of formulae and G denote a set of hypersequents.

Induction on the height of the derivation.

- Base case. Height $= 1$, so we have:

$$\frac{}{G \mid \Gamma, B, \neg A \vdash B, \Delta} (Ax)$$

  Need to show that the premise of the $\neg l$ rule is derivable. Easy to see that the premise is another instance of the axiom.

$$\frac{}{G \mid \Gamma, B \vdash A, B, \Delta} (Ax)$$

  Note that B is assumed to be propositional in the base case. Hence, this is the only base case that needs to be considered.

- Inductive step.

  We distinguish cases according to the last applied inference rule (r). We assume that (r) is not applied to the hypersequent context (otw. apply IH).

  The structural rules:

  – If (r) is a structural rule, the claim follows by i.h. and an application of (r), weakening or contraction.

  The logical rules:

  – $(r) = (\rightarrow r)$

  Consider the following derivation P:

$$\frac{\frac{\vdots}{G \mid \Gamma, \neg A, B \vdash C, \Delta}}{G \mid \Gamma, \neg A \vdash B \rightarrow C, \Delta} (\rightarrow r)$$

Note that the end sequent of P is an instance of the conclusion of the $\neg l$ rule. The height of P is $n+1$.
Need to show that the hypersequent

$$G \mid \Gamma \vdash A, B \rightarrow C, \Delta$$

is height-preserving derivable as well. Note that it is an instance of the premise of the $\neg l$ rule.
There is a sub-derivation of

$$G \mid \Gamma, \neg A, B \vdash C, \Delta$$

in P with height n. Hence, we apply the inductive hypothesis and obtain derivation P' with height $\leq n+1$:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma, B \vdash A, C, \Delta}\ \text{IH}}{G \mid \Gamma \vdash A, B \rightarrow C, \Delta}\ (\rightarrow r)$$

The cases where (r) is $(\neg r)$ is similar.

– $(r) = (\rightarrow l)$
Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma, \neg A \vdash B, \Delta} \qquad \dfrac{\vdots}{G \mid \Gamma, \neg A, C \vdash \Delta}}{G \mid \Gamma, \neg A, B \rightarrow C \vdash \Delta}\ (\rightarrow l)$$

Note that the end sequent of P is an instance of the conclusion of the $\neg l$ rule. The height of P is $n+1$.
Need to show that the hypersequent

$$G \mid \Gamma, B \rightarrow C \vdash A, \Delta$$

is height-preserving derivable as well. Note that it is an instance of the premise of the $\neg l$ rule.
There are a sub-derivations of

$$G \mid \Gamma, \neg A \vdash B, \Delta$$

$$G \mid \Gamma, \neg A, C \vdash \Delta$$

in P with height $\leq n$. Hence, we apply the inductive hypothesis and obtain derivation P' with height $\leq n+1$:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma \vdash A, B, \Delta}\ \text{IH} \qquad \dfrac{\vdots}{G \mid \Gamma, C \vdash A, \Delta}\ \text{IH}}{G \mid \Gamma, B \rightarrow C \vdash A, \Delta}\ (\rightarrow l)$$

- $(r) = (\neg l)$
  In this case we have to consider two sub-cases:
  (1): $\neg A$ is principle.
  Consider the following derivation P:

$$\cfrac{\cfrac{\vdots}{G \mid \Gamma \vdash A, \Delta}}{G \mid \Gamma, \neg A \vdash \Delta}\,(\neg l)$$

  Note that the end sequent of P is an instance of the conclusion of the $\neg l$ rule. The height of P is $n+1$. The premise of the last application of $(\neg l)$ in P gives the desired derivation of height $\leq n+1$.
  (2): $\neg A$ is not principle.
  Consider the following derivation P:

$$\cfrac{\cfrac{\vdots}{G \mid \Gamma, \neg A, \vdash B, \Delta}}{G \mid \Gamma, \neg A, \neg B \vdash \Delta}\,(\neg l)$$

  Note that the end sequent of P is an instance of the conclusion of the $\neg l$ rule. The height of P is $n+1$.
  Need to show that the hypersequent

$$G \mid \Gamma, \neg B \vdash A, \Delta$$

  is height-preserving derivable as well. Note that it is an instance of the premise of the $\neg l$ rule.
  There is a sub-derivation of

$$G \mid \Gamma, \neg A \vdash B, \Delta$$

  in P with height $\leq n$. Hence, we apply the inductive hypothesis and obtain derivation P' with height $\leq n+1$:

$$\cfrac{\cfrac{\vdots}{G \mid \Gamma \vdash A, B, \Delta}\,\text{IH}}{G \mid \Gamma, \neg B \vdash A, \Delta}\,(\neg l)$$

The cut rule:

- The case where the (r) is (cut) is similar to $(r) = (\rightarrow l)$.

The modal rules:

Note that the case where $(r) = \Box r$ cannot be, as the rule is restricted to formulas of the form $\Box A$ as antecedents.

- $(r) = (\Box l)$
  This case is similar to the case $(r) = (\rightarrow r)$.

- $(r) = (MS)$
  Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Box\Gamma_1, \Gamma_2, \neg A \vdash \Delta}}{G \mid \Box\Gamma_1 \vdash\mid \Gamma_2, \neg A \vdash \Delta}\,(MS)$$

  Note that the end sequent of P is an instance of the conclusion of the $\neg l$ rule. The height of P is $n + 1$.
  Need to show that the hypersequent

$$G \mid \Box\Gamma_1 \vdash\mid \Gamma_2 \vdash A, \Delta$$

  is height-preserving derivable as well. Note that it is an instance of the premise of the $\neg l$ rule.
  There is a sub-derivation of

$$G \mid \Box\Gamma_1, \Gamma_2, \neg A \vdash \Delta$$

  in P with height n. Hence, we apply the inductive hypothesis and obtain derivation P' with height $\leq n + 1$:

$$\dfrac{\dfrac{\vdots}{G \mid \Box\Gamma_1, \Gamma_2 \vdash A, \Delta}\,\text{IH}}{G \mid \Box\Gamma_1 \vdash\mid \Gamma_2 \vdash A, \Delta}\,(MS)$$

The deontic rules

- $(r) = (Mon_{\mathcal{O}})$
  Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma^{\Box}, B \vdash D} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, C \vdash E} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, E \vdash C}}{G \mid \Gamma, \neg A, \mathcal{O}(B/C) \vdash \mathcal{O}(D/E), \Delta}\,(Mon_{\mathcal{O}})$$

  Note that the end sequent of P is an instance of the conclusion of the $\neg l$ rule. A derivation for $G \mid \Gamma, \mathcal{O}(B/C) \vdash \mathcal{O}(D/E), A, \Delta$ with height $\leq n + 1$ follows immediately by an application of $Mon_{\mathcal{O}}$ to the premises.
  The cases where (r) is $(Mon_{\mathcal{R}})$, $(Mon_{\mathcal{F}})$, (D_2) , $(D_{\mathcal{O}\mathcal{F}})$ or $(D_{\mathcal{R}})$ are the same.
- $(r) = (D_1)$
  Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma^{\Box}, B \vdash}}{G \mid \Gamma, \neg A, \mathcal{O}(B/C) \vdash \Delta}\ (D_1)$$

Note that the end sequent of P is an instance of the conclusion of the $\neg l$ rule. A derivation for $G \mid \Gamma, \mathcal{O}(B/C) \vdash A, \Delta$ with height $\leq n+1$ follows immediately by an application of D_1 to the premise.

The cases where (r) is $(P_{\mathcal{R}})$ or $(P_{\mathcal{F}})$ are similar.

□

Theorem 4.2.4. *The* $(\rightarrow l)$ *rule is hp-invertible in* $\mathsf{HS}_{\mathsf{LKu}}$.

Proof. Let $A, ..., F$ denote formulae, Γ, Δ, Λ denote sets of formulae and G denote a set of hypersequents.

Induction on the height of the derivation.

- Base case. Height $= 1$, so we have:

$$\dfrac{}{G \mid \Gamma, C, A \rightarrow B \vdash C, \Delta}\ Ax$$

  Need to show that the premises of the $\rightarrow l$ rule are derivable. It is easy to see that the premises are axiom instances:

$$\dfrac{}{G \mid \Gamma, C \vdash C, \Delta, A}\ Ax$$

$$\dfrac{}{G \mid B, \Gamma, C \vdash C, \Delta}\ Ax$$

  Note that C is assumed to be propositional in the base case. Hence, this is the only base case that needs to be considered.

- Inductive step.

  We distinguish cases according to the last applied inference rule (r). We assume that (r) is not applied to the hypersequent context (otw. apply IH).

  The structural rules:

  – If (r) is a structural rule, the claim follows by (multiple applications of) i.h. and an application of (r), weakening or contraction.

  The logical rules:

– $(r) = (\rightarrow r)$
Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma, A \rightarrow B, C \vdash D, \Delta}}{G \mid \Gamma, A \rightarrow B \vdash C \rightarrow D, \Delta} (\rightarrow r)$$

Note that the end sequent of P is an instance of the conclusion of the $\rightarrow l$ rule. The height of P is $n + 1$.
Need to show that the hypersequents

$$G \mid \Gamma \vdash C \rightarrow D, \Delta, A$$

$$G \mid B, \Gamma \vdash C \rightarrow D, \Delta$$

are height-preserving derivable as well. Note that these are an instance of the premise of the $\rightarrow l$ rule.
There is a sub-derivation of

$$G \mid \Gamma, A \rightarrow B, C \vdash D, \Delta$$

in P with height $\leq n$. Hence, we apply the inductive hypothesis and obtain derivation P' with height $\leq n + 1$:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma, C \vdash D, \Delta, A} \text{IH}}{G \mid \Gamma \vdash C \rightarrow D, \Delta, A} (\rightarrow r)$$

and derivation P'' with height $\leq n + 1$:

$$\dfrac{\dfrac{\vdots}{G \mid B, \Gamma, C \vdash D, \Delta} \text{IH}}{G \mid B, \Gamma \vdash C \rightarrow D, \Delta} (\rightarrow r)$$

The cases where (r) is $(\neg l)$ or $(\neg r)$ are similar.

– $(r) = (\rightarrow l)$
In this case we have to consider two sub-cases:

(1) $A \rightarrow B$ is principle.
Consider the following derivation P:

$$\dfrac{G \mid \Gamma \vdash \Delta, A \qquad G \mid B, \Gamma \vdash \Delta}{G \mid \Gamma, A \rightarrow B \vdash \Delta} (\rightarrow l)$$

Note that the end sequent of P is an instance of the conclusion of the $\rightarrow l$ rule. The height of P is $n+1$. The premises of the last application of $(\rightarrow l)$ in P gives the desired derivations of height $\leq n+1$.

(2) $A \rightarrow B$ is not principle.
Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma, A \rightarrow B \vdash \Delta, C} \qquad \dfrac{\vdots}{G \mid D, \Gamma, A \rightarrow B \vdash \Delta}}{G \mid \Gamma, A \rightarrow B, C \rightarrow D \vdash \Delta}\ (\rightarrow l)$$

Note that the end sequent of P is an instance of the conclusion of the $\rightarrow l$ rule. The height of P is $n+1$.
Need to show that the hypersequents

$$(1)\ G \mid \Gamma, C \rightarrow D \vdash \Delta, A$$
$$(2)\ G \mid B, \Gamma, C \rightarrow D \vdash \Delta$$

are height-preserving derivable as well. Note that these are an instance of the premise of the $\rightarrow l$ rule.
There is a sub-derivation of

$$G \mid \Gamma, A \rightarrow B \vdash \Delta, C$$

in P with height $\leq n$. Hence, we apply the inductive hypothesis and obtain derivations with height $\leq n$ of:

$$(I)\ G \mid \Gamma \vdash \Delta, C, A$$
$$(II)\ G \mid B, \Gamma \vdash \Delta, C$$

There is a sub-derivation of

$$G \mid D, \Gamma, A \rightarrow B \vdash \Delta$$

in P with height $\leq n$. Hence, we apply the inductive hypothesis and obtain derivations of height $\leq n$ of:

$$(III)\ G \mid D, \Gamma \vdash \Delta, A$$
$$(IV)\ G \mid B, D, \Gamma \vdash \Delta$$

Apply $(\rightarrow l)$ to (I) and (III) to obtain a derivation of height $\leq n+1$ of (1). Similarly, apply $(\rightarrow l)$ to (II) and (IV) to obtain a derivation of height $\leq n+1$ of (2).

The cut rule:

– $(r) = (cut)$
Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid A \rightarrow B, \Gamma_1 \vdash \Delta, C} \qquad \dfrac{\vdots}{G \mid C, \Gamma_2 \vdash \Lambda}}{G \mid A \rightarrow B, \Gamma_1, \Gamma_2 \vdash \Delta, \Lambda}\ (cut)$$

Note that the end sequent of P is an instance of the conclusion of the $\rightarrow l$ rule. The height of P is $n+1$.
Need to show that the hypersequents

$$(1)\ G \mid \Gamma_1, \Gamma_2 \vdash \Delta, \Lambda, A$$

$$(2)\ G \mid B, \Gamma_1, \Gamma_2 \vdash \Delta, \Lambda$$

are height-preserving derivable as well. Note that these are instances of the premises of the $\rightarrow l$ rule.
There is a sub-derivation of

$$G \mid A \rightarrow B, \Gamma_1 \vdash \Delta, C$$

in P with height $\leq n$. Hence, we apply the inductive hypothesis and obtain derivation P' with height $\leq n$ of $G \mid \Gamma_1 \vdash \Delta, C, A$ and derivation P'' with height $\leq n$ of $G \mid B, \Gamma_1 \vdash \Delta, C$. Cutting the former with the second premise of the last derivation in P results in (1). Similarly, cutting the latter with the second premise of the last derivation in P results in (2).
The case where $(A \rightarrow B)$ is part of the second premise in the last derivation of P is similar.

The modal rules:

Note that the case where $(r) = (\Box r)$ can't be, as the rule is restricted to formulas of the form $\Box A$ as antecedents.

- $(r) = (\Box l)$
  This case is similar to the case $(r) = (\rightarrow r)$.
- $(r) = (MS)$
  Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Box\Gamma_1, A \rightarrow B, \Gamma_2 \vdash \Delta}}{G \mid \Box\Gamma_1 \vdash\mid A \rightarrow B, \Gamma_2 \vdash \Delta}\ (MS)$$

Note that the end sequent of P is an instance of the conclusion of the $\rightarrow l$ rule. The height of P is $n+1$.
Need to show that the hypersequents

$$(1)\ G \mid \Box\Gamma_1 \vdash\mid \Gamma_2 \vdash \Delta, A$$

$$(2)\ G \mid \Box\Gamma_1 \vdash\mid B, \Gamma_2 \vdash \Delta$$

are height-preserving derivable as well. Note that these are an instance of the premise of the $\rightarrow l$ rule.
There is a sub-derivation of

$$G \mid \Box\Gamma_1, A \to B, \Gamma_2 \vdash \Delta$$

in P with height $\leq n$. Hence, we apply the inductive hypothesis and obtain a derivation of $G \mid \Box\Gamma_1, \Gamma_2 \vdash \Delta, A$ and a derivation of $G \mid \Box\Gamma_1, \Gamma_2, B \vdash \Delta$ both with height $\leq n$. Applying (MS) to each of them results in (1) and (2).

The deontic rules:

– $(r) = (Mon_{\mathcal{O}})$
Consider the following derivation P:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma^{\Box}, C \vdash E} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, D \vdash F} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, F \vdash D}}{G \mid A \to B, \Gamma, \mathcal{O}(C/D) \vdash \mathcal{O}(E/F), \Delta}\ (Mon_{\mathcal{O}})$$

Note that the end sequent of P is an instance of the conclusion of the $\to l$ rule. The height of P is $n+1$.
Need to show that the hypersequents

$$(1)\ G \mid \mathcal{O}(C/D) \vdash \mathcal{O}(E/F), \Delta, A$$

$$(2)\ G \mid B, \mathcal{O}(C/D) \vdash \mathcal{O}(E/F), \Delta$$

are height-preserving derivable as well. Note that these are an instance of the premise of the $\to l$ rule. Both follow immediately through an application of $(Mon_{\mathcal{O}})$ to the premises of the last derivation in P.
The cases where (r) is $(Mon_{\mathcal{F}})$, $(Mon_{\mathcal{R}})$, (D_2), $(D_{\mathcal{OF}})$, $(D_{\mathcal{F}})$, (D_1), $(P_{\mathcal{R}})$ or $(P_{\mathcal{F}})$ are similar.

□

4.3 Cut-elimination

The cut-elimination theorem for $\mathsf{HS}_{\mathsf{LKu}}$ is demonstrated by shifting the uppermost application of the cut rule upwards until the cut-formula is introduced. This is accomplished by first shifting the cut over the premise where the cut-formula appears on the right-hand side of the sequents (Lemma 4.3.3), and then shifting it over the premise where the cut-formula appears on the left-hand side (Lemma 4.3.2). Subsequently, the cut can be replaced by smaller cuts (Lemma 4.3.1), ultimately resulting in the elimination of the cut.

The proof utilises the derivable multiple-cut rule, which allows to cut one component against possibly many components [12]. This rule can be viewed as the hypersequent equivalent, in which the cut-formula might appear in more then one component, of

Gentzen's mix rule, which was discussed in Section 3.1. This is necessary due to the inclusion of the rule (ec) in LKu^+.

Before proceeding with the proof of cut-elimination for $\mathsf{HS}_{\mathsf{LKu}}$, it is necessary to introduce some additional definitions and notations that will be employed in subsequent steps.

Definition 4.3.1 (Grade of a Proof)**.** Let P be proof, then the *grade of* P, denoted as $g(P)$ is the maximum grade of its cut-formulas $+1$. Note that $g(P) = 0$ if P is cut-free.

Definition 4.3.2 (Length of a Proof)**.** The *length of a proof* $\mathcal{D}$, denoted by $\mid \mathcal{D} \mid$, is the maximal number of applications of inference rules $+1$ occurring on any branch of $\mathcal{D}$.

Lemma 4.3.1. *Let A be a compound formula and $\mathcal{D}_l$ and $\mathcal{D}_r$ be $\mathsf{HS}_{\mathsf{LKu}}$ proofs such that:*

- *$\mathcal{D}_l$ is a proof of $G \mid \Gamma, A \vdash \Delta$ ending in a rule introducing A;*
- *$\mathcal{D}_r$ is a proof of $H \mid \Sigma \vdash A, \Pi$ ending in a rule introducing A;*
- *$g(\mathcal{D}_l) \leq g(A)$ and $g(\mathcal{D}_r) \leq g(A)$.*

Then a proof $\mathcal{D}$ can be constructed in $\mathsf{HS}_{\mathsf{LKu}}$ of $G \mid H \mid \Gamma, \Sigma \vdash \Delta, \Pi$ with $g(\mathcal{D}) \leq g(A)$.

Proof. If A is introduced in $\mathcal{D}_l$ (resp. $\mathcal{D}_r$) through an (implicit) form of weakening, the claim follows by an application of the last inference rule in $\mathcal{D}_l$ (resp. $\mathcal{D}_r$) to the upper sequent(s) of the last inference in $\mathcal{D}_l$ (resp. $\mathcal{D}_r$) and (multiple) applications of internal and/or external weakening. We show one example below, the other cases are similar:

A cut in

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, B \vdash}}{H \mid \Sigma, \mathcal{O}(B/C) \vdash A, \Pi}(D_1) \qquad \dfrac{\vdots}{G \mid \Gamma, A \vdash \Delta}}{H \mid G \mid \Sigma, \mathcal{O}(B/C), \Gamma \vdash \Pi, \Delta}(cut)$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, B \vdash}}{H \mid \Sigma, \mathcal{O}(B/C) \vdash \Pi}(D_1)}{\dfrac{\text{Multiple applications of internal and external weakening}}{H \mid G \mid \Sigma, \mathcal{O}(B/C), \Gamma \vdash \Pi, \Delta}}$$

Further, we distinguish cases according to the shape of A.

- For $A = \neg B$ a cut in

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma, B \vdash \Pi}}{H \mid \Sigma \vdash \neg B, \Pi}(\neg r) \qquad \dfrac{\dfrac{\vdots}{G \mid \Gamma \vdash B, \Delta}}{G \mid \Gamma, \neg B \vdash \Delta}(\neg l)}{H \mid G \mid \Sigma, \Gamma \vdash \Pi, \Delta}(cut)$$

is replaced by a proof $\mathcal{P}$

$$\dfrac{\dfrac{\dfrac{\vdots}{G \mid \Gamma \vdash B, \Delta} \qquad \dfrac{\vdots}{H \mid \Sigma, B \vdash \Pi}}{G \mid H \mid \Gamma, \Sigma \vdash \Delta, \Pi}(cut)}{\dfrac{\text{Multiple applications of exchange}}{H \mid G \mid \Sigma, \Gamma \vdash \Pi, \Delta}}$$

with $g(\mathcal{P}) \leq g(A)$. The case where $A = \Box B$ is similar.

- For $A = B \to C$ a cut in

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid B, \Sigma \vdash C, \Pi}}{H \mid \Sigma \vdash B \to C, \Pi}(\to r) \qquad \dfrac{\dfrac{\vdots}{G \mid \Gamma \vdash \Delta, B} \qquad \dfrac{\vdots}{G \mid C, \Gamma \vdash \Delta}}{G \mid B \to C, \Gamma \vdash \Delta} \to l}{H \mid G \mid \Sigma, \Gamma \vdash \Pi, \Delta}(cut)$$

is replaced by a proof $\mathcal{P}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{G \mid \Gamma \vdash \Delta, B} \qquad \dfrac{\vdots}{H \mid B, \Sigma \vdash C, \Pi}}{G \mid H \mid \Gamma, \Sigma \vdash \Delta, C, \Pi}(cut) \qquad \dfrac{\vdots}{G \mid C, \Gamma \vdash \Delta}}{G \mid H \mid G \mid \Gamma, \Sigma, \Gamma \vdash \Delta, \Pi, \Delta}(cut)}{\dfrac{\text{Multiple applications of contraction and weakening}}{H \mid G \mid \Sigma, \Gamma \vdash \Pi, \Delta}}$$

with $g(\mathcal{P}) \leq g(A)$.

- For the case that A is $\mathcal{O}(C/D)$, $\mathcal{F}(C/D)$ or $\mathcal{R}(C/D)$ note that the last applied rule in $\mathcal{D}_r$ has to be $(Mon_{\mathcal{O}})$, $(Mon_{\mathcal{F}})$ or $(Mon_{\mathcal{R}})$ respectively. We group the remaining cases according to the last applied rule in $\mathcal{D}_l$.

  – For the case where $\mathcal{D}_l$ ends with $(Mon_{\mathcal{O}})$, $(Mon_{\mathcal{F}})$, or $(Mon_{\mathcal{R}})$, respectively, we provide one example below:

    For $A = \mathcal{R}(C/D)$ a cut in

    Part A:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, A' \vdash C} \quad \dfrac{\vdots}{H \mid \Sigma^{\Box}, B \vdash D} \quad \dfrac{\vdots}{H \mid \Sigma^{\Box}, D \vdash B}}{H \mid \Sigma, \mathcal{R}(A'/B) \vdash \mathcal{R}(C/D), \Pi}\,(Mon_{\mathcal{R}})$$

    Part B:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma^{\Box}, C \vdash E} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, D \vdash F} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, F \vdash D}}{G \mid \Gamma, \mathcal{R}(C/D) \vdash \mathcal{R}(E/F), \Delta}\,(Mon_{\mathcal{R}})$$

    Complete proof tree:

$$\dfrac{\overset{\text{Part A}}{H \mid \Sigma, \mathcal{R}(A'/B) \vdash \mathcal{R}(C/D), \Pi} \qquad \overset{\text{Part B}}{G \mid \Gamma, \mathcal{R}(C/D) \vdash \mathcal{R}(E/F), \Delta}}{H \mid G \mid \Sigma, \mathcal{R}(A'/B), \Gamma \vdash \Pi, \mathcal{R}(E/F), \Delta}\,(cut)$$

    is replaced by a proof $\mathcal{P}$

    Part A:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, A' \vdash C} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, C \vdash E}}{H \mid G \mid \Sigma^{\Box}, A', \Gamma^{\Box} \vdash E}\,(cut)$$

    Part B:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, B \vdash D} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, D \vdash F}}{H \mid G \mid \Sigma^{\Box}, B, \Gamma^{\Box} \vdash F}\,(cut)$$

    Part C:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, D \vdash B} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, F \vdash D}}{H \mid G \mid \Sigma^{\Box}, F, \Gamma^{\Box} \vdash B}\,(cut)$$

    Complete proof tree:

$$\dfrac{\dfrac{\overset{\text{Part A}}{H \mid G \mid \Sigma^{\Box}, A', \Gamma^{\Box} \vdash E} \quad \overset{\text{Part B}}{H \mid G \mid \Sigma^{\Box}, B, \Gamma^{\Box} \vdash F} \quad \overset{\text{Part C}}{H \mid G \mid \Sigma^{\Box}, F, \Gamma^{\Box} \vdash B}}{Mon_{\mathcal{R}} \text{ and multiple applications of exchange}}}{H \mid G \mid \Sigma, \mathcal{R}(A'/B), \Gamma \vdash \Pi, \mathcal{R}(E/F), \Delta}$$

    with $g(\mathcal{P}) \leq g(A)$.

- For the case where $\mathcal{D}_l$ ends with (D_1), $(P_\mathcal{F})$, or $(P_\mathcal{R})$, respectively, we provide one example below:

  For $A = \mathcal{O}(C/D)$ a cut in

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma^\Box, A' \vdash C} \quad \dfrac{\vdots}{H \mid \Sigma^\Box, B \vdash D} \quad \dfrac{\vdots}{H \mid \Sigma^\Box, D \vdash B}}{H \mid \Gamma, \mathcal{O}(A'/B) \vdash \mathcal{O}(C/D), \Pi}(Mon_\mathcal{O}) \quad \dfrac{\dfrac{\vdots}{G \mid \Gamma^\Box, C \vdash}}{G \mid \Gamma, \mathcal{O}(C/D) \vdash \Delta}(D_1)}{H \mid G \mid \mathcal{O}(A'/B), \Sigma, \Gamma \vdash \Pi, \Delta}(cut)$$

  is replaced by a proof $\mathcal{P}$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma^\Box, A' \vdash C} \quad \dfrac{\vdots}{G \mid \Gamma^\Box, C \vdash}}{H \mid G \mid \Sigma^\Box, \Gamma^\Box, A' \vdash}(cut)}{H \mid G \mid \Sigma, \Gamma, \mathcal{O}(A'/B) \vdash \Pi}(D_1)}{H \mid G \mid \mathcal{O}(A'/B), \Sigma, \Gamma \vdash \Pi, \Delta}\text{Multiple applications of weakening and exchange}$$

  with $g(\mathcal{P}) \leq g(A)$.

For the remaining two cases, A can only have the form $\mathcal{O}(C/D)$ or $\mathcal{F}(C/D)$.

- For the case where $\mathcal{D}_l$ ends with (D_2) or $(D_\mathcal{F})$, respectively, we provide one example below:

  For $A = \mathcal{O}(C/D)$ a cut in

  Part A:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^\Box, A' \vdash C} \quad \dfrac{\vdots}{H \mid \Sigma^\Box, B \vdash D} \quad \dfrac{\vdots}{H \mid \Sigma^\Box, D \vdash B}}{H \mid \Sigma, \mathcal{O}(A'/B) \vdash \mathcal{O}(C/D), \Pi}(Mon_\mathcal{O})$$

  Part B:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma^\Box, C, E \vdash} \quad \dfrac{\vdots}{G \mid \Gamma^\Box, D \vdash F} \quad \dfrac{\vdots}{G \mid \Gamma^\Box, F \vdash D}}{G \mid \Gamma, \mathcal{O}(C/D), \mathcal{O}(E/F) \vdash \Delta}(D_2)$$

  Complete proof tree:

$$\dfrac{\overset{\text{Part A}}{H \mid \Sigma, \mathcal{O}(A'/B) \vdash \mathcal{O}(C/D), \Pi} \quad \overset{\text{Part B}}{G \mid \Gamma, \mathcal{O}(C/D), \mathcal{O}(E/F) \vdash \Delta}}{H \mid G \mid \Sigma, \mathcal{O}(A'/B), \Gamma, \mathcal{O}(E/F), \vdash \Pi, \Delta}(cut)$$

  is replaced by a proof $\mathcal{P}$

  Part A:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, A' \vdash C} \qquad \dfrac{\vdots}{G \mid \Gamma^{\Box}, C, E \vdash}}{H \mid G \mid \Sigma^{\Box}, A', \Gamma^{\Box}, E \vdash}\ (cut)$$

Part B:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, B \vdash D} \qquad \dfrac{\vdots}{G \mid \Gamma^{\Box}, D \vdash F}}{H \mid G \mid \Sigma^{\Box}, B, \Gamma^{\Box} \vdash F}\ (cut)$$

Part C:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, D \vdash B} \qquad \dfrac{\vdots}{G \mid \Gamma^{\Box}, F \vdash D}}{H \mid G \mid \Sigma^{\Box}, F, \Gamma^{\Box} \vdash B}\ (cut)$$

Complete proof tree:

$$\dfrac{\dfrac{\dfrac{\text{Part A}}{H \mid G \mid \Sigma^{\Box}, A', \Gamma^{\Box}, E \vdash} \qquad \dfrac{\text{Part B}}{H \mid G \mid \Sigma^{\Box}, B, \Gamma^{\Box} \vdash F} \qquad \dfrac{\text{Part C}}{H \mid G \mid \Sigma^{\Box}, F, \Gamma^{\Box} \vdash B}}{D_2 \text{ and multiple applications of exchange}}}{H \mid G \mid \Sigma, \mathcal{O}(A'/B), \Gamma, \mathcal{O}(E/F), \vdash \Pi, \Delta}$$

with $g(\mathcal{P}) \leq g(A)$.

– For the case where $\mathcal{D}_l$ ends with $(D_{\mathcal{OF}})$ we provide one example below:
For $A = \mathcal{O}(C/D)$ a cut in

Part A:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, A' \vdash C} \qquad \dfrac{\vdots}{H \mid \Sigma^{\Box}, B \vdash D} \qquad \dfrac{\vdots}{H \mid \Sigma^{\Box}, D \vdash B}}{H \mid \Sigma, \mathcal{O}(A'/B) \vdash \mathcal{O}(C/D), \Pi}\ (Mon_{\mathcal{O}})$$

Part B:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma^{\Box}, C \vdash E} \qquad \dfrac{\vdots}{G \mid \Gamma^{\Box}, D \vdash F} \qquad \dfrac{\vdots}{G \mid \Gamma^{\Box}, F \vdash D}}{G \mid \Gamma, \mathcal{O}(C/D), \mathcal{F}(E/F) \vdash \Delta}\ (D_{\mathcal{OF}})$$

Complete proof tree:

$$\dfrac{\dfrac{\text{Part A}}{H \mid \Sigma, \mathcal{O}(A'/B) \vdash \mathcal{O}(C/D), \Pi} \qquad \dfrac{\text{Part B}}{G \mid \Gamma, \mathcal{O}(C/D), \mathcal{F}(E/F) \vdash \Delta}}{H \mid G \mid \Sigma, \mathcal{O}(A'/B), \Gamma, \mathcal{F}(E/F), \vdash \Pi, \Delta}\ (cut)$$

is replaced by a proof $\mathcal{P}$

Part A:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, A' \vdash C} \qquad \dfrac{\vdots}{G \mid \Gamma^{\Box}, C \vdash E}}{H \mid G \mid \Sigma^{\Box}, A', \Gamma^{\Box} \vdash E}\ (cut)$$

Part B:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, B \vdash D} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, D \vdash F}}{H \mid G \mid \Sigma^{\Box}, B, \Gamma^{\Box} \vdash F}\,(cut)$$

Part C:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box}, D \vdash B} \quad \dfrac{\vdots}{G \mid \Gamma^{\Box}, F \vdash D}}{G \mid H \mid \Gamma^{\Box}, F, \Sigma^{\Box} \vdash B}\,(cut)$$

Complete proof tree:

$$\dfrac{\dfrac{\overset{\text{Part A}}{H \mid G \mid \Sigma^{\Box}, A', \Gamma^{\Box} \vdash E} \quad \overset{\text{Part B}}{H \mid G \mid \Sigma^{\Box}, B, \Gamma^{\Box} \vdash F} \quad \overset{\text{Part C}}{G \mid H \mid \Gamma^{\Box}, F, \Sigma^{\Box} \vdash B}}{D_{\mathcal{OF}} \text{ and multiple applications of exchange}}}{H \mid G \mid \Sigma, \mathcal{O}(A'/B), \Gamma, \mathcal{F}(E/F), \vdash \Pi, \Delta}$$

with $g(\mathcal{P}) \leq g(A)$.

□

Lemma 4.3.2. *Let $\mathcal{D}_l$ and $\mathcal{D}_r$ be $\mathsf{HS}_{\mathsf{LKu}}$ proofs such that:*

- *$\mathcal{D}_l$ is a proof of $G \mid \Gamma_1, A^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n$;*
- *A is a compound formula and $\mathcal{D}_r := H \mid \Sigma \vdash A, \Pi$ ends with a rule introducing an indicated occurrence of A;*
- *$g(\mathcal{D}_l) \leq g(A)$ and $g(\mathcal{D}_r) \leq g(A)$.*

Then a proof $\mathcal{D}$ can be constructed in $\mathsf{HS}_{\mathsf{LKu}}$ of $G \mid H \mid \Gamma_1, \Sigma^{\lambda_1} \vdash \Delta_1, \Pi^{\lambda_1} \mid ... \mid \Gamma_n, \Sigma^{\lambda_n} \vdash \Delta_n, \Pi^{\lambda_n}$ with $g(\mathcal{D}) \leq g(A)$.

Proof. We distinguish cases according to the shape of A.

Case 1: A is of the form $\neg B$. Note that the conclusion of $\mathcal{D}_l$ is then $G \mid \Gamma_1, \neg B^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, \neg B^{\lambda_n} \vdash \Delta_n$. Apply Theorem 4.2.3 to obtain a proof $\mathcal{D}_l'$ ending in $G \mid \Gamma_1 \vdash B^{\lambda_1}, \Delta_1 \mid ... \mid \Gamma_n \vdash B^{\lambda_n}, \Delta_n$. The claim follows by an application of $(\neg l)$ and Lemma 4.3.1 to each component of the sequent. The case where $A = B \to C$ is similar.

Case 2: If A is $\Box B$, $\mathcal{O}(B/C)$, $\mathcal{F}(B/C)$ or $\mathcal{R}(B/C)$ the proof proceeds by induction on $\mid \mathcal{D}_l \mid$.

- Base case: $\mathcal{D}_l$ ends with an initial sequent.

  The conclusion immediately holds.

- Inductive cases: We distinguish cases according to the last applied rule in $\mathcal{D}_l$. Let (r) be the last inference rule applied in $\mathcal{D}_l$.
  - If (r) is only applied to side sequents G, the claim follows by i.h..
  - If (r) is a left rule introducing one of the indicated cut formulas. The claim follows by an application of the induction hypothesis to the premises of (r), followed by an application of (r) and Lemma 4.3.1.

  In the following it is assumed that A is not in the hypersequent context G and that A is not the principal formula of (r).

  - For the case that (r) is an internal or external structural rule, we show the case for $(r) = (ec)$. The remaining cases, namely (r) being $(iw - l)$, $(iw - r)$, $(ic - l)$, $(ic - r)$ or (ew), follow the same pattern.

    For $(r) = (ec)$ a cut in

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash A, \Pi} \qquad \dfrac{\dfrac{\vdots}{G \mid \Gamma_1, A^{\lambda_1} \vdash \Delta_1 \mid \Gamma_1, A^{\lambda_1} \vdash \Delta_1 \mid \dots \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}}{G \mid \Gamma_1, A^{\lambda_1} \vdash \Delta_1 \mid \dots \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}(ec)}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1 \vdash \Pi^{\lambda_1}, \Delta_1 \mid \dots \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(cut)$$

    is replaced by

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash A, \Pi} \qquad \dfrac{\vdots}{G \mid \Gamma_1, A^{\lambda_1} \vdash \Delta_1 \mid \Gamma_1, A^{\lambda_1} \vdash \Delta_1 \mid \dots \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1 \vdash \Pi^{\lambda_1}, \Delta_1 \mid \Sigma^{\lambda_1}, \Gamma_1 \vdash \Pi^{\lambda_1}, \Delta_1 \mid \dots \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(cut)}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1 \vdash \Pi^{\lambda_1}, \Delta_1 \mid \dots \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(ec)$$

    then apply i.h..
  - For the case that (r) is a propositional rule or $(\Box l)$, we show the case for $(r) = (\neg r)$. The remaining cases, namely (r) being $(\neg l)$, $(\rightarrow l)$, $(\rightarrow r)$ or $(\Box l)$, follow the same pattern.

    For $(r) = (\neg r)$ a cut in

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash A, \Pi} \qquad \dfrac{\dfrac{\vdots}{G \mid \Gamma_1, \Gamma_2, A^{\lambda_1} \vdash \Delta_1 \mid \dots \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}}{G \mid \Gamma_1, A^{\lambda_1} \vdash \Delta_1, \neg\Gamma_2 \mid \dots \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}(\neg r)}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1 \vdash \Pi^{\lambda_1}, \Delta_1, \neg\Gamma_2 \mid \dots \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(cut)$$

is replaced by

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash A, \Pi} \quad \dfrac{\vdots}{G \mid \Gamma_1, \Gamma_2, A^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}}{\dfrac{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1, \Gamma_2 \vdash \Pi^{\lambda_1}, \Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1 \vdash \Pi^{\lambda_1}, \Delta_1, \neg\Gamma_2 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(\neg r)}(cut)$$

then apply i.h..

In the remaining cases we have to distinguish between two conditions: when $A \neq \Box B$ and when $A = \Box B$. We first consider the former case. Note that (MS) is the only remaining rule for which the condition $A \neq \Box B$ can hold.

- For $(r) = (MS)$ and $A \neq \Box B$ a cut in

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash A, \Pi} \quad \dfrac{\dfrac{\vdots}{G \mid \Box\Gamma_1, \Gamma_2, A^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}}{G \mid \Box\Gamma_1 \vdash\mid \Gamma_2, A^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}(MS)}{H \mid G \mid \Box\Gamma_1 \vdash\mid \Sigma^{\lambda_1}, \Gamma_2 \vdash \Pi^{\lambda_1}, \Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(cut)$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash A} \quad \dfrac{\vdots}{G \mid \Box\Gamma_1, \Gamma_2, A^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, A^{\lambda_n} \vdash \Delta_n}}{H \mid G \mid \Sigma^{\lambda_1}, \Box\Gamma_1, \Gamma_2 \vdash \Pi^{\lambda_1}, \Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(cut)}{H \mid G \mid \Box\Gamma_1 \vdash\mid \Sigma^{\lambda_1}, \Gamma_2 \vdash \Pi^{\lambda_1}, \Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(MS)$$

then apply i.h..

For all other cases we assume that $A = \Box B$. Notice that in this case the conclusion of $\mathcal{D}_r$ is $\Sigma \vdash \Box B, \Pi$. To simplify the reasoning process, we can safely use the sequent $\Sigma^{\Box} \vdash \Box B$ instead and apply weakening afterwards. This allows us to shift cuts upwards over all remaining rules.

- For $(r) = (MS)$ and $A = \Box B$ a cut in

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash \Box B, \Pi} \quad \dfrac{\dfrac{\vdots}{G \mid \Box\Gamma_1, \Box B^{\lambda_1}, \Gamma_2 \vdash \Delta_1 \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Delta_n}}{G \mid \Box\Gamma_1, \Box B^{\lambda} \vdash\mid \Gamma_2, \Box B^{\lambda_1 - \lambda} \vdash \Delta_1 \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Delta_n}(MS)}{H \mid G \mid \Sigma^{\lambda}, \Box\Gamma_1 \vdash \Pi^{\lambda} \mid \Sigma^{\lambda_1 - \lambda}, \Gamma_2 \vdash \Pi^{\lambda_1 - \lambda}, \Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}(cut)$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box} \vdash \Box B} \qquad \dfrac{\vdots}{G \mid \Box\Gamma_1, \Box B^{\lambda_1}, \Gamma_2 \vdash \Delta_1 \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Delta_n}}{H \mid G \mid \Sigma^{\lambda_1\Box}, \Box\Gamma_1, \Gamma_2 \vdash \Delta_1 \mid ... \mid \Sigma^{\lambda_n\Box}, \Gamma_n \vdash \Delta_n}\,(cut)}{H \mid G \mid \Sigma^{\lambda\Box}, \Box\Gamma_1 \vdash \mid \Sigma^{\lambda_1-\lambda}, \Gamma_2 \vdash \Delta_1 \mid ... \mid \Sigma^{\lambda_n\Box}, \Gamma_n \vdash \Delta_n}\,(MS)}{H \mid G \mid \Sigma^{\lambda}, \Box\Gamma_1 \vdash \Pi^{\lambda} \mid \Sigma^{\lambda_1-\lambda}, \Gamma_2 \vdash \Pi^{\lambda_1-\lambda}, \Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}\,\text{Multiple applications of weakening}$$

then apply i.h..

- For $(r) = (\Box r)$ and $A = \Box B$ a cut in

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash \Box B, \Pi} \qquad \dfrac{\dfrac{\vdots}{G \mid \Box\Gamma_1, \Box B^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Delta_n}}{G \mid \Box\Gamma_1, \Box B^{\lambda_1} \vdash \Box\Delta_1 \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Delta_n}\,(\Box r)}{H \mid G \mid \Sigma^{\lambda_1}, \Box\Gamma_1 \vdash \Pi^{\lambda_1}, \Box\Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}\,(cut)$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box} \vdash \Box B} \qquad \dfrac{\vdots}{G \mid \Box\Gamma_1, \Box B^{\lambda_1} \vdash \Delta_1 \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Delta_n}}{H \mid G \mid \Sigma^{\lambda_1\Box}, \Box\Gamma_1 \vdash \Delta_1 \mid ... \mid \Sigma^{\lambda_n\Box}, \Gamma_n \vdash \Delta_n}\,(cut)}{H \mid G \mid \Sigma^{\lambda_1\Box}, \Box\Gamma_1 \vdash \Box\Delta_1 \mid ... \mid \Sigma^{\lambda_n\Box}, \Gamma_n \vdash \Delta_n}\,(\Box r)}{H \mid G \mid \Sigma^{\lambda_1}, \Box\Gamma_1 \vdash \Pi^{\lambda_1}, \Box\Delta_1 \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Delta_n}\,\text{Multiple applications of weakening}$$

then apply i.h..

- For $(r) = (Mon_{\mathcal{F}})$ and $A = \Box B$ a cut in

  Part A:

$$\dfrac{\dfrac{\vdots}{G \mid \Gamma_1^{\Box}, \Box B^{\lambda_1}, E \vdash C \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda} \quad \dfrac{\vdots}{G \mid \Gamma_1^{\Box}, \Box B^{\lambda_1}, D \vdash F \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda} \quad \dfrac{\vdots}{G \mid \Gamma_1^{\Box}, \Box B^{\lambda_1}, F \vdash D \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda}}{G \mid \Gamma_1, \Box B^{\lambda_1}, \mathcal{F}(C/D) \vdash \mathcal{F}(E/F), \Delta \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda}\,Mon_{\mathcal{F}}$$

  Complete proof tree:

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash \Box B, \Pi} \qquad \dfrac{\text{Part A}}{G \mid \Gamma_1, \Box B^{\lambda_1}, \mathcal{F}(C/D) \vdash \mathcal{F}(E/F), \Delta \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda}}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1, \mathcal{F}(C/D) \vdash \Pi^{\lambda_1}, \mathcal{F}(E/F), \Delta \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Lambda}\,(cut)$$

is replaced by cutting $H \mid \Sigma^{\Box} \vdash \Box B$ against the premises of the $Mon_{\mathcal{F}}$ rule above to obtain

Formula A: $H \mid G \mid \Sigma^{\lambda_1 \Box}, \Gamma_1^{\Box}, E \vdash C \mid ... \mid \Sigma^{\lambda_n \Box}, \Gamma_n \vdash \Lambda$
Formula B: $H \mid G \mid \Sigma^{\lambda_1 \Box}, \Gamma_1^{\Box}, D \vdash F \mid ... \mid \Sigma^{\lambda_n \Box}, \Gamma_n \vdash \Lambda$
Formula C: $H \mid G \mid \Sigma^{\lambda_1 \Box}, \Gamma_1^{\Box}, F \vdash D \mid ... \mid \Sigma^{\lambda_n \Box}, \Gamma_n \vdash \Lambda$

Complete proof tree:

$$\dfrac{\dfrac{\text{Formula A} \qquad \text{Formula B} \qquad \text{Formula C:}}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1, \mathcal{F}(C/D) \vdash \mathcal{F}(E/F), \Delta \mid ... \mid \Sigma^{\lambda_n \Box}, \Gamma_n \vdash \Lambda} (Mon_{\mathcal{F}})}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1, \mathcal{F}(C/D) \vdash \Pi^{\lambda_1}, \mathcal{F}(E/F), \Delta \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Lambda} \text{Multiple applications of weakening}$$

then apply i.h.. The cases where (r) is $Mon_{\mathcal{O}}$ or $Mon_{\mathcal{R}}$ are the same. The cases where (r) is D_2 , $D_{\mathcal{F}}$ or $D_{\mathcal{OF}}$ are similar.

– For $(r) = D_1$ and $A = \Box B$ a cut in

$$\dfrac{\dfrac{\vdots}{H \mid \Sigma \vdash \Box B, \Pi} \qquad \dfrac{\dfrac{\vdots}{G \mid \Gamma_1^{\Box}, \Box B^{\lambda_1}, D \vdash \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda}}{G \mid \Gamma_1, \Box B^{\lambda_1}, \mathcal{O}(D/C) \vdash \Delta \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda} (D_1)}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1, \mathcal{O}(D/C) \vdash \Pi^{\lambda_1}, \Delta \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Lambda} (cut)$$

is replaced by

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma^{\Box} \vdash \Box B} \qquad \dfrac{\vdots}{G \mid \Gamma_1^{\Box}, \Box B^{\lambda_1}, D \vdash \mid ... \mid \Gamma_n, \Box B^{\lambda_n} \vdash \Lambda}}{H \mid G \mid \Sigma^{\lambda_1 \Box}, \Gamma_1^{\Box}, D \vdash \mid ... \mid \Sigma^{\lambda_n \Box}, \Gamma_n \vdash \Lambda} (cut)}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1, \mathcal{O}(D/C) \vdash \Pi^{\lambda_1}, \Delta \mid ... \mid \Sigma^{\lambda_n \Box}, \Gamma_n \vdash \Lambda} (D_1)}{H \mid G \mid \Sigma^{\lambda_1}, \Gamma_1, \mathcal{O}(D/C) \vdash \Pi^{\lambda_1}, \Delta \mid ... \mid \Sigma^{\lambda_n}, \Gamma_n \vdash \Pi^{\lambda_n}, \Lambda} \text{Multiple applications of weakening}$$

then apply i.h.. The cases where (r) is $(P_{\mathcal{R}})$ or $(P_{\mathcal{F}})$ are the same.

□

Lemma 4.3.3. *Let* $\mathcal{D}_l$ *and* $\mathcal{D}_r$ *be* $\mathsf{HS_{LKu}}$ *proofs such that:*

- $\mathcal{D}_l$ *is a proof of* $G \mid \Gamma, A \vdash \Delta$*;*
- $\mathcal{D}_r$ *is a proof of* $H \mid \Sigma_1 \vdash A^{\lambda_1}, \Pi_1 \mid ... \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n$*;*

- *$g(\mathcal{D}_l) \leq g(A)$ and $g(\mathcal{D}_r) \leq g(A)$.*

Then a proof $\mathcal{D}$ can be constructed in $\mathsf{HS_{LKu}}$ of $G \mid H \mid \Sigma_1, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}$ with $g(\mathcal{D}) \leq g(A)$.

Proof. Let (r) be the last inference rule applied in $\mathcal{D}_r$. If (r) is an axiom, then the claim holds trivially. Otherwise we proceed on induction on $\mid \mathcal{D}_r \mid$.

Case 1: If (r) is only applied to side sequents H, the claim follows by i.h. and an application of (r).

Case 2: If (r) introduces (one of) the indicated occurrence(s) of A, we use Lemma 4.3.2.

In the following it is assumed that (r) is not applied to the hypersequent context H and that (r) does not introduce (one of) the indicated occurrence(s) of A. It should be noted that this implies that (r) does not correspond to any of the deontic rules or the rule $(\Box r)$. Therefore, only the possibilities of (r) being a structural rule, a logical rule, the $\Box l$ rule or the modal splitting rule need to be considered. We group the following cases accordingly.

The structural rules:

- For the case that (r) is an internal or external structural rule, we show the case for $(r) = (iw - l)$. The remaining cases, namely (r) being $(iw - r)$, $(ic - l)$, $(ic - r)$, (ec) or (ew), follow the same pattern.

  For $(r) = (iw - l)$ a cut in

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma_1 \vdash A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n}}{H \mid B, \Sigma_1 \vdash A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n}\,(iw-l) \qquad \dfrac{\vdots}{G \mid \Gamma, A \vdash \Delta}}{H \mid G \mid B, \Sigma_1, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}}\,(cut)$$

  is replaced by

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma_1 \vdash A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n} \qquad \dfrac{\vdots}{G \mid \Gamma, A \vdash \Delta}}{H \mid G \mid \Sigma_1, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}}\,(cut)}{H \mid G \mid B, \Sigma_1, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}}\,(iw-l)$$

then apply i.h.

The logical rules:

- For the case that (r) is a logical rule we show the case for $(r) = (\neg l)$. The remaining cases, namely (r) being $(\neg r)$, $(\rightarrow l)$ or $(\rightarrow r)$, follow the same pattern of first cutting the premise(s) and applying (r) afterwards.

  For $(r) = (\neg l)$ a cut in

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma_1 \vdash B, A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n}}{H \mid \Sigma_1, \neg B \vdash A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n} (\neg l) \quad \dfrac{\vdots}{G \mid \Gamma, A \vdash \Delta}}{H \mid G \mid \Sigma_1, \neg B, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}} (cut)$$

  is replaced by

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Sigma_1 \vdash B, A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n} \quad \dfrac{\vdots}{G \mid \Gamma, A \vdash \Delta}}{H \mid G \mid \Sigma_1, \Gamma^{\lambda_1} \vdash B, \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}} (cut)}{H \mid G \mid \Sigma_1, \neg B, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}} (\neg l)$$

  then apply i.h.

The modal rules:

- The case where $(r) = (\Box l)$ is similar to the case where $(r) = (\neg l)$.
- For $(r) = (MS)$ a cut in

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Box\Sigma_1, \Sigma_2 \vdash A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n}}{H \mid \Box\Sigma_1 \vdash\mid \Sigma_2 \vdash A^{\lambda_1}, \Pi_1 \mid \ldots \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n} (MS) \quad \dfrac{\vdots}{G \mid \Gamma, A \vdash \Delta}}{H \mid G \mid \Box\Sigma_1 \vdash\mid \Sigma_2, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}} (cut)$$

  is replaced by

$$\dfrac{\dfrac{\dfrac{\vdots}{H \mid \Box\Sigma_1, \Sigma_2 \vdash A^{\lambda_1}, \Pi_1 \mid ... \mid \Sigma_n \vdash A^{\lambda_n}, \Pi_n} \quad \dfrac{\vdots}{G \mid \Gamma, A \vdash \Delta}}{H \mid G \mid \Box\Sigma_1, \Sigma_2, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid ... \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}}\,(cut)}{H \mid G \mid \Box\Sigma_1 \vdash \mid \Sigma_2, \Gamma^{\lambda_1} \vdash \Pi_1, \Delta^{\lambda_1} \mid ... \mid \Sigma_n, \Gamma^{\lambda_n} \vdash \Pi_n, \Delta^{\lambda_n}}\,(MS)$$

then apply i.h..

□

Theorem 4.3.4 (Cut Elimination)**.** *Cut-elimination holds for* $\mathsf{HS}_{\mathsf{LKu}}$.

Proof. Let P be a $\mathsf{HS}_{\mathsf{LKu}}$ proof with $g(P) > 0$. We will prove the theorem by double induction on $\langle g(P), ng(P)\rangle$, where $ng(P)$ is the number of cuts in P with grade $g(P)$.

Consider an uppermost application of (cut) in P with grade $g(P)$. By applying Lemma 4.3.3 to its premises, either $g(P)$ or $ng(P)$ decreases. Then we apply the inductive hypothesis.

□

Corollary 4.3.4.1. *Cut-elimination holds for* $\mathsf{HS}_{\mathsf{LPr}}$.

Proof. The cut-elimination theorem for $\mathsf{HS}_{\mathsf{LKu}}$, excluding the cases where $(r) = (Mon_{\mathcal{R}})$ and $(r) = (P_{\mathcal{R}})$, serves as a proof of cut-elimination for $\mathsf{HS}_{\mathsf{LPr}}$. □

CHAPTER 5

Conclusion

The aim of this thesis was to develop analytic sequent-style calculi for the logics of Prabhākara and Kumārila.

We revisited the logics of Prabhākara and Kumārila as introduced in [7], highlighting the reasoning behind specific design choices. Both of these logics are based in classical logic rather than intuitionistic, as principles like the law of the excluded middle and reductio ad absurdum are legimated in Mīmāṃsā reasoning [6]. A significant feature that characterizes both logics is the non-interdefinability of deontic operators representing obligation and prohibition [14], [17]. Further, the logic of Kumārila includes an additional operator representing recommendation, which the logic of Prabhākara does not incorporate. This distinction arises from the differing viewpoints of Prabhākara and Kumārila on elective sacrifices. While Kumārila sees elective sacrifices as something one is inclined to do, Prabhākara sees them more as obligations.

A main alteration introduced in [7] for the logics of Prabhākara and Kumārila, in contrast to previous versions [11], was the transition from using S4 to S5 as the base logic. Proof calculi for previous versions [14], were built upon Gentzen's sequent calculus. However, as there is no cut-free analytic sequent calculus of S5, the necessity emerged to develop new proof calculi tailored to the logics of Prabhākara and Kumārila.

After revisiting the reasons why Gentzen's sequent calculus is insufficient in capturing modal logic S5 while preserving cut-elimination, we became intrigued to revisit the hypersequent framework introduced by Avron [12] and Pottinger [13]. Due to its additional expressive power this calculus is able to capture logics like S5.

The original contribution of this thesis was the development of an analytic sequent-style calculus for the logics of Prabhākara and Kumārila. Expanding on the existing sequent calculi [14] for Prabhākara and Kumārila, which are based on modal logic S4, we introduced a hypersequent version for both logics. This process involved modifying the existing rules by incorporating a hypersequent context and adding external structural rules, along with rules that address the S5 axioms. Subsequently, we established the soundness and completeness of both calculi. To validate the cut-free analytic nature of the resulting calculi, we conducted a cut-elimination proof by eliminating any instance of the cut rule from proofs within the introduced calculi. Hence, through the establishment of soundness, completeness, and cut-elimination, we successfully achieved the intended objective of this thesis, which is to introduce an analytic proof theory for the logics of Prabhākara and Kumārila.

Through successfully establishing soundness, completeness, and cut-elimination, we achieved the primary objective of this thesis: introducing an analytic proof theory for the logics of Prabhākara and Kumārila.

5.1 Further research

The formalization of the deontic theories of Mīmāṃsā is an ongoing interdisciplinary process. While the present logics in this thesis covered certain *nyāya*s, there are many more to discover.

In [8], an additional axiom was incorporated into the logics of Prabhākara and Kumārila used in this thesis:

$$(\Diamond(\varphi \wedge \Theta) \wedge \mathcal{O}(\varphi/\top) \wedge \mathcal{O}(\Theta/\top)) \rightarrow \mathcal{O}(\varphi \wedge \Theta/\top)$$

The axiom corresponds to the logical property known as restricted aggregation and was abstracted from a recently identified *nyāya*. In Mīmāṃsā, different fixed obligations with the same overall goal are handled through "accumulation," meaning one has to do them both as long as the actions are compatible and serve different intermediate results. Otherwise, only one action is performed, chosen based on different criteria or randomly. However, this principle does not apply to elective sacrifices, as they would lead to the same desired result, so it suffices to choose one of them. The underlying principle is:

> "If two fixed duties are prescribed and compatible, their conjunction is obligatory as well."

Future research might cover the incorporation of this new axiom into a proof theory. For this a method to translate modal axioms into cut-free sequent rules from [35] can be employed. This involves breaking down modal axioms into a finite number of sequents. During this process, modal subformulae are treated as propositional variables, and their equality is expressed as premises. These premises can be further resolved using the concept of cuts between rules.

Ciabattoni et al. [18] discovered that the non-interdefinability of deontic concepts in Mīmāṃsā extends not only to the notions of obligation and prohibition but also to the concept of permission. As a result, an additional deontic operator was introduced along with corresponding axioms to include permission in the logics of Prabhākara and Kumārila. Therefore, the development of analytic proof calculi incorporating the concept of permission is also a possible subject of future research.

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