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Identities in Finite Taylor Algebras

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Kurzfassung

Eine endliche, idempotente Algebra heißt *Taylor*, wenn sie ein System nichttrivialer Gleichungen erfüllt. Dabei wird eine Algebra *idempotent* genannt, wenn für jede ihrer Termoperationen f die Identität $f(x, \ldots, x) = x$ gilt. Ein tiefliegendes Ergebnis von M. Maróti und R. McKenzie besagt, dass eine endliche, idempotente Algebra genau dann Taylor ist, wenn sie eine *weak near-unanimity- (WNU-)* Termoperation besitzt, also eine Termoperation w, welche die Identitäten $w(x, \ldots, x, y) = w(x, \ldots, y, x) = \cdots = w(y, x, \ldots, x)$ erfüllt. WNU-Termoperationen spielten eine Schlüsselrolle beim Nachweis der lange ungelösten Dichotomie-Vermutung für *Constraint Satisfaction Problems (CSPs)* über einer endlichen Menge Γ von Relationen auf einer endlichen Domäne: CSP(Γ) ist genau dann in polynomieller Zeit lösbar, wenn Γ von einer WNU-Termoperation erhalten wird.

Die Theorie der *loop conditions* fußt auf der Beobachtung, dass die Erfüllung von Identitäten der Form $f(x_{1,1}, \ldots, x_{1,n}) = f(x_{2,1}, \ldots, x_{2,n}) = \cdots = f(x_{k,1}, \ldots, x_{k,n})$ durch eine Termoperation f einer Algebra äquivalent zur Existenz eines konstanten Tupels in einer zugehörigen Relation ist. Die durch eine WNU-Termoperation induzierte Relation erweist sich als symmetrisch. Die Charakterisierung von endlichen, idempotenten Taylor-Algebren durch die Existenz von WNU-Termoperationen ergibt sich nun durch die Tatsache, dass jede nichtleere, symmetrische und invariante Relation einer geeigneten Arität, die auf einer endlichen, idempotenten Taylor-Algebra definiert ist, ein konstantes Tupel enthält.

Von diesem Punkt aus beginnen wir mit der systematischen Untersuchung von k-WNU-Termoperationen, einer Verallgemeinerung von WNU-Termoperationen. Die Relation, die durch die eine k-WNU-Termoperation der Arität n definierende loop condition hervorgeht, besitzt die Invarianzeigenschaft der (n, k)-*Symmetrie*. Das Hauptziel dieser Arbeit ist, hinreichende Bedingungen an n zu finden, die die Existenz eines konstanten Tupels in jeder nichtleeren, (n, k)-symmetrischen und invarianten Relation auf einer endlichen, idempotenten Taylor-Algebra garantieren und somit die Existenz einer k-WNU-Termoperation der Arität n implizieren.

Abstract

A finite idempotent algebra is said to be *Taylor* if it satisfies a set of nontrivial identities. Here, an algebra is called *idempotent* if the identity $f(x, \ldots, x) = x$ holds for any of its term operations f. By a deep theorem of M. Maróti and R. McKenzie, a finite idempotent algebra is Taylor if and only if it possesses a *weak near-unanimity (WNU)* term operation, i.e., an operation w satisfying the identities $w(x, \ldots, x, y) = w(x, \ldots, y, x) = \cdots = w(y, x, \ldots, x)$. WNU term operations played a key role in proving the long-unsolved dichotomy conjecture for finite-domain *Constraint Satisfaction Problems (CSPs)*: $\mathrm{CSP}(\Gamma)$ is tractable if and only if Γ is preserved by a WNU term operation.

The theory of *loop conditions* is based on the observation that the satisfaction of identities of the form $f(x_{1,1}, \ldots, x_{1,n}) = f(x_{2,1}, \ldots, x_{2,n}) = \cdots = f(x_{k,1}, \ldots, x_{k,n})$ by a term operation f of an algebra is equivalent to the existence of a constant tuple in an associated relation. The relation associated with a WNU term operation turns out to be symmetric. The characterisation of finite idempotent Taylor algebras by the existence of WNU term operations now arises from the fact that any nonempty, symmetric, and invariant relation of an appropriate arity defined on a finite idempotent Taylor algebra contains a constant tuple.

From this point of origin we initiate the systematic study of k-WNU term operations, which generalise the notion of WNU term operations. The relation associated with the loop condition defining a k-WNU term operation of arity n has the invariance property of being (n, k)-*symmetric*. The main goal of this thesis is to find sufficient conditions on n that guarantee the existence of a constant tuple in any nonempty, (n, k)-symmetric, and invariant relation on a finite idempotent Taylor algebra, and thus imply the existence of a k-WNU term operation of arity n.

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Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

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Contents

1 Introduction

We begin by considering the well-known *k-colouring problem*. Given a graph $\mathbb{G}$, we want to decide whether or not it is possible to colour each of its vertices in such a way that no two adjacent vertices are given the same colour, while no more than k colours are used. In terms of computational complexity, it is folklore knowledge that this decision problem is NP-complete whenever $k \geq 3$. Finding a 2-colouring of a graph, on the other hand, is the same as asking if the graph is bipartite, and can be solved in linear time. It is possible to put even more constraints on the admissibility of colourings by considering colourings that are in some sense compatible with a fixed template. Namely, let $\mathbb{H}$ be a finite graph. The $\mathbb{H}$*-colouring problem* is defined as the following decision problem:

Problem: $\mathbb{H}$-colouring
Input: A graph $\mathbb{G}$
Question: Does there exist a homomorphism $\mathbb{G} \to \mathbb{H}$?

The vertices of $\mathbb{H}$ are called *colours*. An $\mathbb{H}$-colouring of $\mathbb{G}$ can then be understood as an assignment of colours to the vertices of $\mathbb{G}$ such that adjacent vertices of $\mathbb{G}$ receive adjacent colours. The k-colouring problem is precisely the $\mathbb{H}$-colouring problem for $\mathbb{H} = \mathbb{K}_k$, where $\mathbb{K}_k$ denotes the complete graph with k vertices.

It is a remarkable result that the complexity properties known for the k-colouring problem extend to the wider class of $\mathbb{H}$-colouring problems. Namely, the computational complexity of the $\mathbb{H}$-colouring problem enjoys a prominent dichotomy, first discovered by P. Hell and J. Nešetřil in 1990. Here, a problem is said to be *tractable* if and only if it belongs to the class P of all decision problems that are solvable in polynomial time.

Theorem 1.1 (Hell-Nešetřil)**:** *[HN90] Let $\mathbb{H}$ be a finite graph that does not contain loops. If $\mathbb{H}$ is bipartite, then the $\mathbb{H}$-colouring problem is tractable. Otherwise, it is* NP*-complete.*

If $\mathbb{H}$ contains a loop, then any graph $\mathbb{G}$ admits an $\mathbb{H}$-colouring by simply mapping all vertices of $\mathbb{G}$ to the vertex with the loop, and the $\mathbb{H}$-colouring problem is solvable in constant time.

The $\mathbb{H}$-colouring problem can be represented in the form of a *Constraint Satisfaction Problem (CSP)*. Formally, let A be a set and let Γ be a finite set of relations on A. Γ will also be called a *constraint language*. A sentence is said to be a *primitive positive (pp-)* sentence over Γ if it is an existentially quantified conjunction of relations from Γ. Given a pp-sentence ϕ over a constraint language Γ, the Constraint Satisfaction Problem CSP(Γ) of Γ is defined as:

Problem: CSP(Γ) **Input:** A primitive positive formula ϕ over Γ **Question:** Does ϕ hold true in $(A; \Gamma)$?

Suppose that $\mathbb{H} = (V_{\mathbb{H}}; E_{\mathbb{H}})$ is a graph. Then the $\mathbb{H}$-colouring problem is equivalent to CSP$(E_{\mathbb{H}})$. Namely, for $\mathbb{G} = (V_{\mathbb{G}}; E_{\mathbb{G}})$ where $|V_{\mathbb{G}}| = n$, let ϕ be the primitive positive formula

$$\exists v_1, \ldots, v_n : \bigwedge_{(v_i,v_j) \in E_{\mathbb{G}}} (v_i, v_j) \in E_{\mathbb{H}}.$$

The values of the variables $v_1, \ldots, v_n$ correspond to the colours of the vertices. The question whether or not ϕ is satisfiable in $\mathbb{H}$ is equivalent to asking if $\mathbb{G}$ is $\mathbb{H}$-colourable. Conversely, let ϕ be a pp-sentence over the constraint language $\Gamma = E_{\mathbb{H}}$. Let $\mathbb{D}(\phi)$ be the structure whose elements are the variables that occur in ϕ. We define a tuple $(x_1, \ldots, x_m)$ to belong to the relation $E_{\mathbb{H}}^{\mathbb{D}(\phi)}$ of $\mathbb{D}(\phi)$ if and only if $R(x_1, \ldots, x_m)$ is used in ϕ. We have that $\mathbb{H}$ satisfies ϕ if and only if there exists a homomorphism $\mathbb{D}(\phi) \to \mathbb{H}$. The structure $\mathbb{D}(\phi)$ is also called *canonical database* of ϕ.

The complexity of CSPs has been the subject of thorough enquiry. In accordance with the dichotomy for the $\mathbb{H}$-colouring problem, the following conjecture was first formulated by T. Feder and M. Y. Vardi in their seminal paper from 1993 [FV93; FV98].

Conjecture 1.2 (Dichotomy Conjecture)**:** *Let Γ be a finite-domain constraint language. Then either* CSP(Γ) *is tractable, or it is* NP*-complete.*

This conjecture was independently proved in 2017 by A. Bulatov [Bul17] and D. Zhuk [Zhu17; Zhu20a]. The key part of both proofs was to find a polynomial algorithm that solves CSP(Γ) in the tractable case. Leading up to this achievement was the recognition that the complexity of CSP(Γ) can also be studied from an algebraic point of view.

To a relational structure $\mathbb{A} = (A; (R_i)_{i \in I})$ we assign its *polymorphism clone* Pol($\mathbb{A}$), the set consisting of all multivariate operations on A that leave all relations R_i, $i \in I$ invariant. Elements of Pol($\mathbb{A}$) are called *polymorphisms* of $\mathbb{A}$. We set CSP($\mathbb{A}$) := CSP($\{R_i : i \in I\}$). A *clone homomorphism* is a mapping that preserves identities. It is an important result obtained by A. Bulatov, P. Jeavons, and A. Krokhin that the computational complexity of CSP($\mathbb{A}$) only depends on the identities satisfied by the polymorphisms of $\mathbb{A}$:

Theorem 1.3: *[BJK05] Let $\mathbb{A}$ and $\mathbb{B}$ be finite relational structures. If there exists a clone homomorphism* Pol($\mathbb{A}$) $\to$ Pol($\mathbb{B}$)*, then* CSP($\mathbb{B}$) *is reducible to* CSP($\mathbb{A}$) *in polynomial time.*

The algebraic approach of analysing identities satisfied by the polymorphism clone Pol($\mathbb{A}$) of a relational structure $\mathbb{A}$ offered a new tool for studying the complexity of CSP($\mathbb{A}$). Firstly, it was shown that it is enough to restrict oneself to the case of relational structures $\mathbb{A}$ all of whose polymorphisms are *idempotent*, that is, every polymorphism $f \in$ Pol($\mathbb{A}$) satisfies the identity $f(x, \ldots, x) = x$.

Theorem 1.4: *[BJK05] Let $\mathbb{A}$ be a finite relational structure. Then there exists a finite relational structure $\mathbb{A}'$ all of whose polymorphisms are idempotent such that* $\mathrm{CSP}(\mathbb{A}')$ *is reducible to* $\mathrm{CSP}(\mathbb{A})$ *in polynomial time.*

Moreover, the following relation between the complexity of $\mathrm{CSP}(\mathbb{A})$ and identities satisfied by $\mathrm{Pol}(\mathbb{A})$ was obtained:

Theorem 1.5: *[BJK05] Let $\mathbb{A}$ be a finite relational structure all of whose polymorphisms are idempotent. If* $\mathrm{Pol}(\mathbb{A})$ *does not admit a polymorphism t of some arity $k \geq 2$ satisfying for all $i \in [k]$ an identity of the form*

$$t(\square_1, \ldots, \square_k) = t(\triangle_1, \ldots, \triangle_k), \tag{1.1}$$

where $\square_i = x$, $\triangle_i = y$, and $\square_j, \triangle_j \in \{x, y\}$ for all $j \in [k]$, then $\mathrm{CSP}(\mathbb{A})$ *is* NP*-complete.*

Operations t satisfying a set of identities of the form (1.1) are called *Taylor terms*. Notice that in particular, the Taylor identities fail to be satisfied by projections. Theorem 1.5 suggested a criterion for distinguishing between the tractable and the NP-complete case of $\mathrm{CSP}(\mathbb{A})$ for finite relational structures $\mathbb{A}$ all of whose polymorphisms are idempotent. It was conjectured by A. Bulatov, P. Jeavons, and A. Krokhin that $\mathrm{CSP}(\mathbb{A})$ is tractable if $\mathrm{Pol}(\mathbb{A})$ has a Taylor term. Proving this was the remaining part in order to solve Conjecture 1.2.

Long before the algebraic approach to CSPs had begun, W. Taylor analysed algebras that admit Taylor terms. For an algebra $\mathbf{A}$, let $\mathrm{Clo}(\mathbf{A})$ denote the its *clone of term operations*, that is, the smallest clone of operations that contains all the basic operations of $\mathbf{A}$. An algebra $\mathbf{A}$ is said to be *idempotent* if all of its term operations are idempotent. A well-known result obtained by W. Taylor in 1977 states that a finite idempotent algebra has a Taylor term if and only if there does not exist an mapping $\mathrm{Clo}(\mathbf{A}) \to \mathrm{Proj}$ that preserves identities, where Proj denotes the trivial clone consisting only of the projections on a two-element set (Theorem 1.9). Algebras satisfying the latter condition are called *Taylor algebras*. Mappings that preserve identities are also called *clone homomorphisms*.

A new proof of Hell and Nešetřil's Dichotomy Theorem formulated in terms of the algebraic approach to CSPs was provided in 2005 by A. Bulatov in [Bul05]. As a main result, a link between bipartiteness of loopless graphs and Taylor algebras was derived. The proof used methods which have later been formalised by the notion of *pp-constructions*. If $\mathbb{B}$ is a structure that can be pp-constructed from another structure $\mathbb{A}$, then $\mathrm{CSP}(\mathbb{B})$ can be reduced to $\mathrm{CSP}(\mathbb{A})$ in polynomial time. Furthermore, height 1 identities, i.e., non-nested identities of the form $f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)$, are preserved by pp-constructions [BOP18]. The following theorem offers a connection between the satisfaction of height 1 identities and pp-constructions.

Theorem 1.6: *[BOP18] Let $\mathbb{A}$ be a finite relational structure. Then $\mathbb{A}$ pp-constructs $\mathbb{K}_3$ if and only if there exists a mapping* $\mathrm{Pol}(\mathbb{A}) \to \mathrm{Proj}$ *that preserves height* 1 *identities.*

In other words, a relational structure $\mathbb{A}$ pp-constructs $\mathbb{K}_3$ if and only if all identities of height 1 that are satisfied by the polymorphisms of $\mathbb{A}$ are also satisfied by projections.

Mappings that preserve height 1 identities are also called *h*1 *clone homomorphisms* or *minion homomorphisms*. It is a well-known result that follows easily from [BOP18] that if a structure pp-constructs $\mathbb{K}_3$, then it also pp-constructs any finite structure. Note that if $\mathbb{H}$ is a graph that pp-constructs $\mathbb{K}_3$, then the $\mathbb{H}$-colouring problem is as hard as the 3-colouring problem and thus NP-complete. Bringing together these results, we are now able to state a structural counterpart to Theorem 1.1:

Theorem 1.7 (Loop Theorem, [Bul05])**:** *Let $\mathbb{H}$ be a finite graph. Then one of the following holds:*

(i) $\mathbb{H}$ pp-constructs $\mathbb{K}_3$ (and consequently, the $\mathbb{G}$-colouring problem is NP-complete), or

(ii) $\mathbb{H}$ is bipartite (and consequently, the $\mathbb{G}$-colouring problem is tractable), or

(iii) $\mathbb{H}$ contains a loop (and consequently, the $\mathbb{G}$-colouring problem is tractable).

In 2010 – two decades after the first proof of Hell and Nešetřil's Dichotomy Theorem had appeared – M. Siggers used Theorem 1.7 to obtain a surprising algebraic invariance property. We include this proof in the introduction, though the reader unfamiliar with universal algebra may refer to section 2 for relevant definitions.

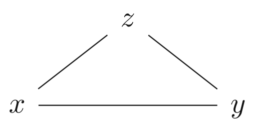
Theorem 1.8: *[Sig10] Let* $\mathbf{A}$ *be a finite algebra and assume that* $\mathrm{Clo}(\mathbf{A})$ *has a Taylor term. Then* $\mathrm{Clo}(\mathbf{A})$ *also has a* 6*-ary term operation s that satisfies the identity*

$$s(x,x,y,y,z,z) = s(y,z,z,x,x,y). \tag{1.2}$$

Proof. Let $\mathbf{F} := \mathcal{F}_{\mathbf{A}}(x,y,z)$ be the free algebra over the set $\{x,y,z\}$ in the variety generated by $\mathbf{A}$. Note that $\mathbf{F}$ is finite and admits a Taylor term. Consider the relation $R \subseteq \mathbf{F} \times \mathbf{F}$ generated by the columns of identity (1.2), i.e.,

$$R := \left\langle \left\{ \binom{x}{y}, \binom{x}{z}, \binom{y}{z}, \binom{y}{x}, \binom{z}{x}, \binom{z}{y} \right\} \right\rangle_{\mathbf{F}}.$$

As R is symmetric, it is in fact a graph and satisfies the conditions of Theorem 1.7. Since R contains the triangle



it is not bipartite. By definition of R, it holds that $R \in \mathrm{Inv}(\mathbf{F})$. Consequently, we have $\mathrm{Pol}(R) \supseteq \mathrm{Clo}(\mathbf{F})$. Since $\mathrm{Clo}(\mathbf{F})$ has a Taylor term, there does not exist an h1 clone homomorphism $\mathrm{Clo}(\mathbf{F}) \to \mathrm{Proj}$. In particular, there does not exist an h1 clone homomorphism $\mathrm{Pol}(R) \to \mathrm{Proj}$. Thus, by Theorem 1.6, R does not pp-construct $\mathbb{K}_3$. By Theorem 1.7, there therefore exists a loop $(a,a) \in R$. By definition of R, there exists $s \in \mathrm{Clo}(\mathbf{A})$ such that

$$s\left(\binom{x}{y}, \binom{x}{z}, \binom{y}{z}, \binom{y}{x}, \binom{z}{x}, \binom{z}{y} \right) = \binom{a}{a}.$$

In other words, we have the identity

$$s(x, x, y, y, z, z) = a = s(y, z, z, x, x, y)$$

on the generators x, y, z of $\mathbf{F}$. Since $\mathbf{F}$ is free in the variety generated by $\mathbf{A}$, the identity also holds on $\mathbf{A}$. □

Terms satisfying identity (1.2) are called *Siggers terms*. Note that in particular, Siggers terms are Taylor terms. Namely, the variable z can be substituted arbitrarily by either x or y in order to see that each of the 6 required identities hold. The existence of a Siggers term operation therefore characterises finite idempotent Taylor algebras. The following theorem collects other characterisations that have been obtained.

Theorem 1.9: *[Tay77; MM08; Sig10; BK12; KMM15] Let* $\mathbf{A}$ *be a finite idempotent algebra. The following are equivalent:*

(i) *There does not exist a clone homomorphism* $\mathrm{Clo}(\mathbf{A}) \to \mathrm{Proj}$.

(ii) (W. Taylor, 1977) $\mathrm{Clo}(\mathbf{A})$ *has a Taylor term* t*, i.e.,* t *satisfies identities of the form*

$$\begin{aligned} t(x, *, *, \dots, *) &= t(y, *, *, \dots, *) \\ t(*, x, *, \dots, *) &= t(*, y, *, \dots, *) \\ &\vdots \\ t(*, *, *, \dots, x) &= t(*, *, *, \dots, y). \end{aligned}$$

(iii) (M. Maróti and R. McKenzie, 2008) *For each prime* $p > |A|$, $\mathrm{Clo}(\mathbf{A})$ *has a WNU term operation* w *of arity* p*, i.e.,* w *satisfies the identities*

$$w(x, \dots, x, y) = w(x, \dots, y, x) = \dots = w(y, x, \dots, x).$$

(iv) (M. Siggers, 2010) $\mathrm{Clo}(\mathbf{A})$ *has a Siggers term* s*, i.e.,* s *satisfies the identity*

$$s(x, x, y, y, z, z) = s(y, z, z, x, x, y).$$

(v) (L. Barto and M. Kozik, 2012) *For every prime* $p > |A|$, $\mathrm{Clo}(\mathbf{A})$ *has a cyclic term* c*, i.e.,* c *satisfies the identity*

$$c(x_1, \dots, x_p) = c(x_2, \dots, x_p, x_1).$$

(vi) (K. Kearnes, P., Marković, and R. McKenzie, 2014) $\mathrm{Clo}(\mathbf{A})$ *has a* 4*-ary term* s *that satisfies the identity*

$$s(a, r, e, a) = s(r, a, r, e).$$

In his proof of the Dichotomy Conjecture D. Zhuk provides an algorithm that solves CSP(Γ) in polynomial time if Γ is preserved by a WNU term operation. Formally, he proved the following reformulation of Conjecture 1.2:

Theorem 1.10: *[Zhu17; Zhu20a] Let* Γ *be a finite-domain constraint language. Then* $\mathrm{CSP}(\Gamma)$ *is tractable if* Γ *is preserved by a WNU term operation. Otherwise,* $\mathrm{CSP}(\Gamma)$ *is* NP*-complete.*

The WNU identities offer more information about the term operation satisfying it than the Taylor identities do. We are interested in the study of structural implications that arise from the existence of a WNU term operation in an algebra. In particular, we are interested in the structure of a particular relation associated with the WNU identities. Inspired by M. Sigger's proof of Theorem 1.8 using the Loop Theorem 1.7, the theory of *loop conditions* evolved. It provides a method of showing the validity of identities in an algebra via the existence of a loop in an associated relation. A series of statements have been made or reformulated in the language of loop conditions. We refer for instance to [MM08; BK12; BP16; Olš18; GJP19; Zhu20b; BP20; MP22; Bar+23].

This thesis attempts to line up in the recent results on loop conditions by examining conditions for the existence of k-WNU term operations in finite idempotent Taylor algebras. Here, a term operation w of some arity $n \geq k$ is called *k-weak near unanimity (k-WNU)* if it satisfies all identities of the form

$$w(\square_1, \ldots, \square_n) = w(\triangle_1, \ldots, \triangle_n),$$

where $\square_i, \triangle_i \in \{x, y\}$ for all $i \in [n]$, and $|\{i \in [n] : \square_i = y\}| = k = |\{i \in [n] : \triangle_i = y\}|$. Evidently, a WNU term operation is a k-WNU term operation for $k = 1$.

Analysing the properties of a particular relation associated with the k-WNU identities, we are to consider $\binom{n}{k}$-ary relations that are invariant under a certain action of the symmetric group S_n on $[n]$. Namely, we index the tuples' components with all k-element subsets of $[n]$. A group action of S_n on tuples of size $\binom{n}{k}$ is given by applying a permutation $\pi \in S_n$ to the indices of every component:

$$\pi\left(\left(r_{\{i_1,\ldots,i_k\}}\right)_{\{i_1,\ldots,i_k\}\in\binom{n}{k}}\right) := \left(r_{\{\pi(i_1),\ldots,\pi(i_k)\}}\right)_{\{i_1,\ldots,i_k\}\in\binom{n}{k}}.$$

We say that an $\binom{n}{k}$-ary relation R is *(n,k)-symmetric* if it is invariant under this action, i.e., if for any $\mathbf{r} \in R$ and any $\pi \in S_n$ it holds that also $\pi(\mathbf{r}) \in R$.

For all finite algebras $\mathbf{A}$, the existence of a k-WNU term operation of arity n in $\mathrm{Clo}(\mathbf{A})$ is equivalent to the existence of a constant tuple in some particular $\binom{n}{k}$-ary relation that turns out to be (n,k)-symmetric. Our study is restricted to the case of finite idempotent Taylor algebras. The foundation for our attempts of finding such a constant tuple is thus set by a major result obtained by D. Zhuk in his proof of the Dichotomy Conjecture. Namely, he showed the existence of either some particular subalgebra or of some particular factor whenever an algebra is finite, idempotent, and Taylor (Corollary 3.1.3). Given a finite idempotent Taylor algebra $\mathbf{A}$, this theorem allows us to reduce our enquiry to find a constant tuple in an invariant relation on A to the consideration of two cases: Either $\mathbf{A}$ possesses a nontrivial strong subuniverse, or it factors into a p-affine algebra for some prime number $p \in \mathbb{P}$. If an algebra is such that any nontrivial subalgebra admits a nontrivial strong subuniverse, the existence of k-WNU term operations is always guaranteed (Lemma 5.3). However, number theoretic conditions on the integers n and k appear in the p-affine case.

We call a tuple (n, k, p) of integers $n, k \in \mathbb{N}$ and $p \in \mathbb{P}$ *loop-friendly* if any nonempty (n, k)-symmetric relation that is preserved by all p-affine operations contains a constant tuple. A characterisation of all loop-friendly tuples (n, k, p) for $k = 1$ has been obtained by D. Zhuk in [Zhu20b]. The case $k = 2$ has been solved by L. Barto, Z. Brady, M. Pinsker, and D. Zhuk in unpublished work. In this thesis, we try to generalise the methods used to arbitrary $k \in \mathbb{N}$.

In order for a finite idempotent Taylor algebra to have a k-WNU term operation of arity n, it is sufficient that the tuples (n, k, p) are loop-friendly for all primes $p \leq |A|$ (Corollary 6.3). We offer some necessary conditions for a tuple (n, k, p) to be loop-friendly, which gives rise to the question whether or not these conditions are already strong enough to characterise loop-friendly tuples, and thus suffice to show the existence of a k-WNU term operation of arity n.

Question 1.11: *Let* $\mathbf{A}$ *be a finite idempotent Taylor algebra, and let* $k \in \mathbb{N}$*. Assume that* $n \in \mathbb{N}$ *is such that for all prime numbers* $p \in \mathbb{P}$ *with* $p \leq |A|$ *we have* $p \nmid n\binom{n}{k}$ *and for all* $1 \leq j \leq k-1$ *and* $k \leq M \leq n$ *at least one of the following conditions holds:*

$$\binom{n-j}{k-j} \equiv 0 \mod p, \text{ or}$$
$$\binom{M}{k} \not\equiv 1 \mod p, \text{ or}$$
$$\binom{M-j}{k-j} \not\equiv 0 \mod p.$$

Do these conditions imply that all tuples (n, k, p) *for* $p \leq |A|$ *are loop-friendly and consequently imply that* $\mathrm{Clo}(\mathbf{A})$ *has a* k*-WNU term operation of arity* n*?*

The organisation of this thesis is as follows. In Section 2 we introduce basic notions used in universal algebra. Section 3 is devoted to the special class of Taylor algebras dealt with in this thesis. Namely, we introduce the notion of strong subuniverses and p-affine algebras and we state Zhuk's Cases Theorem for finite idempotent Taylor algebras. Section 4 gives a short insight into the theory of loop conditions and thus links the satisfaction of identities to the finding of constant tuples. Section 5 collects the main results about the existence of constant tuples in (n, k)-symmetric relations defined on finite idempotent Taylor algebras. Finally, Section 6 uses these results to derive k-WNU term operations under certain conditions. Questions concerning the existence of k-WNU term operations are formulated in Section 7.

2 Preliminaries

In this section we provide well-known notions from universal algebra.

2.1 Notation

The set of natural numbers is $\mathbb{N} = \{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$ we denote the set $\{1, 2, \dots, n\}$ by $[n]$. Given an integer $n > 1$ and $a, b \in \mathbb{Z}$ we write $a \equiv b \mod n$ if n is a divisor of $a - b$, that is, there is an integer $k \in \mathbb{Z}$ such that $a - b = kn$. If $n, m \in \mathbb{Z}$ are integers such that n is a divisor of m, we write $n \mid m$. Otherwise, if n is not a divisor of m, we write $n \nmid m$. For $n > 1$, $\mathbb{Z}_n$ denotes the ring of integers modulo n. Recall that $\mathbb{Z}_n$ is a field if and only if n is a prime number. The set consisting of all prime numbers is denoted by $\mathbb{P}$. Two integers are *coprime* if they do not share any common divisors $n \geq 2$.

2.2 Algebras and H, S, P

Let A be a set and $n \in \mathbb{N}$. An *n-ary operation* on A is a mapping $f : A^n \to A$. If $n = 0$, $n = 1$, or $n = 2$ we also say *constant, unary,* or *binary* operation, respectively. The set of all operations on A is denoted by $\mathcal{O}_A$. Let I be finite a set, and for any $i \in I$ assume that $f_i : A^{n_i} \to A$ is an n_i-ary operation on A. Then the pair $\mathbf{A} = (A, (f_i)_{i \in I})$ is called an *algebra* with *universe* A of *type* $(n_i)_{i \in I}$. The operations f_i are called *basic operations* of $\mathbf{A}$. We will denote algebras by bold letters and the corresponding universes by letters in normal font. If A is finite, then $\mathbf{A}$ is a *finite algebra*.

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras of the same type. A *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a map $\phi : A \to B$ that is compatible with all basic operations, meaning that if $f_i^{\mathbf{A}}$ is an n_i-ary basic operation of $\mathbf{A}$ and $f_i^{\mathbf{B}}$ is the corresponding basic operation on $\mathbf{B}$, then

$$\phi(f_i^{\mathbf{A}}(a_1, \dots, a_{n_i})) = f_i^{\mathbf{B}}(\phi(a_1), \dots, \phi(a_{n_i}))$$

for all tuples $(a_1, \dots, a_{n_i}) \in A^{n_i}$. If ϕ is bijective, then it is also called an *isomorphism*, and we write $\mathbf{A} \cong \mathbf{B}$. If $\phi : A \to A$ is a homomorphism on an algebra $\mathbf{A}$, then ϕ is called an *endomorphism* of $\mathbf{A}$.

Let $\mathbf{A}$ be an algebra. A subset $B \subseteq A$ that is closed under all basic operations of $\mathbf{A}$ is called a *subuniverse*, i.e., if f is an n-ary basic operation of $\mathbf{A}$ and $(b_1, \dots, b_n) \in B^n$ then also $f(b_1, \dots, b_n) \in B$. A subuniverse B is made into an algebra $\mathbf{B}$ by restricting all basic functions of $\mathbf{A}$ to the set B. The algebra $\mathbf{B}$ is then called *subalgebra* of $\mathbf{A}$ and we write $\mathbf{B} \leq \mathbf{A}$. A subalgebra is *nontrivial* if it is proper and nonempty. If $\mathcal{K}$ is a class of algebras of the same type, we denote the class of algebras $\mathbf{A}$ that are isomorphic to a subalgebra of an algebra $\mathbf{A}' \in \mathcal{K}$ by $\mathrm{S}(\mathcal{K})$.

Given a family of algebras $(\mathbf{A}_j)_{j\in J}$ of the same type, we define an algebraic structure on the product $\Pi_{j\in J} A_j$ by applying the basic functions separately to each component of a tuple. The resulting algebra $\Pi_{j\in J}\mathbf{A}_j$ is called the *(direct) product* of $(\mathbf{A}_j)_{j\in J}$. Let $\mathcal{K}$ be a class consisting of same-type algebras. The class of all algebras $\mathbf{A}$ that are isomorphic to a direct product of a family $(\mathbf{A}_j)_{j\in J}$ of algebras $\mathbf{A}_j,\ j \in J$ that belong to $\mathcal{K}$ is denoted by $\mathrm{P}(\mathcal{K})$.

If θ is an equivalence relation on the universe of an algebra $\mathbf{A}$ such that θ is preserved by all basic operations of $\mathbf{A}$, then θ is called a *congruence*. Here, an equivalence relation θ is said to be *preserved* by an n-ary operation f if for all $a_i, b_i \in A,\ i \in [n]$ it holds that $(a_1, b_1), \ldots, (a_n, b_n) \in \theta$ implies $(f(a_1, \ldots, a_n)), f(b_1, \ldots, b_n)) \in \theta$ (see also the definition given in section 2.7). For any $a \in A$ we denote its block of a congruence θ by $[a]_\theta := \{b \in A : (a, b) \in \theta\}$. The *factor algebra* $\mathbf{A}/\theta$ is then obtained by defining the basic operations on the quotient set $A/\theta = \{[a]_\theta : a \in A\}$ in the natural way. A congruence is *nontrivial* if it is neither the equality relation nor A^2. Given a class $\mathcal{K}$ of algebras of the same type, the class of all algebras $\mathbf{B}$ such that $\mathbf{B} \cong \mathbf{A}/\theta$ for some $\mathbf{A} \in \mathcal{K}$ and some congruence θ on $\mathbf{A}$ is denoted by $\mathrm{H}(\mathcal{K})$.

If the operators H, S, P are applied where $\mathcal{K} = \{\mathbf{A}\}$ we write $\mathrm{H}(\mathbf{A}),\ \mathrm{S}(\mathbf{A}),\ \mathrm{P}(\mathbf{A})$, respectively. The operators can be composed. For example, $\mathrm{HSP}(\mathbf{A})$ is the set of all algebras $\mathbf{B}$ such that $\mathbf{B} \cong \mathbf{S}/\theta$ for some subpower $\mathbf{S} \leq \Pi_{i\in I}\mathbf{A}$ of $\mathbf{A}$ and a congruence θ on $\mathbf{S}$. A class of algebras of the same type is called *variety* if it is closed under substructures, products, and homomorphic images. By Birkhoff's HSP Theorem, this definition coincides with the one given in Section 2.3. Namely, the Theorem states that a class of same-type algebras is closed under the application of the operators H, S, and P if and only if there exists some set of identities so that an algebra belongs to the class if and only if it satisfies these identities.

2.3 Identities and varieties

Let us fix a type $\tau = (n_i)_{i\in I}$ of algebras and a set X. For every $i \in I$ let f_i be an operation symbol. Assume that all elements $x \in X$ and $f_i, i \in I$ are distinct in pairs. The set of terms $T = T(X, \tau)$ of type τ over X is defined as the smallest set that contains X and for elements $t_1, \ldots, t_{n_i} \in T$ also all strings $f_i(t_1, \ldots, t_{n_i})$ for each $i \in I$. For every $i \in I$ let f_i^T be the n_i-ary operation on T defined by $f_i^T(t_1, \ldots, t_{n_i}) \mapsto f_i(t_1, \ldots, t_{n_i})$. Then $\mathbf{T}(X, \tau) := (T, (f_i^T)_{i\in I})$ is an algebra of type τ. We call $\mathbf{T}(X, \tau)$ the *term algebra* of type τ over X, and X the set of *variables*.

It is folklore knowledge that if $\mathbf{A}$ is any algebra of type τ and $\phi : X \to A$ is a map, then there exists a unique homomorphism $\bar{\phi} : \mathbf{T}(X, \tau) \to \mathbf{A}$ such that $\bar{\phi}\big|_X = \phi$. This is also known as the *universal property* of the term algebra. For terms $s, t \in T$ we write $s^{\mathbf{A}} = t^{\mathbf{A}}$ if for each assignment of variables $\phi : X \to A$ it holds that $\bar{\phi}(s) = \bar{\phi}(t)$.

A pair $(t_1, t_2) \in T^2$ is called an *identity*, we also write $t_1 = t_2$. An algebra $\mathbf{A}$ *satisfies* the identity $t_1 = t_2$ if $t_1^{\mathbf{A}} = t_2^{\mathbf{A}}$. In this case, we write $\mathbf{A} \models t_1 = t_2$. A *variety* of type τ is a class consisting of algebras of type τ with the property that there exists a set $\Gamma \subseteq T^2$ of identities so that an algebra belongs to the class if and only if it satisfies all identities in Γ. Given $\Gamma \subseteq T^2$ the variety induced by Γ is denoted by $\mathcal{V}(\Gamma)$, i.e., for an algebra $\mathbf{A}$ of type τ it holds that

$$\mathbf{A} \in \mathcal{V}(\Gamma) \leftrightarrow \mathbf{A} \models t_1 = t_2 \text{ for all } (t_1, t_2) \in \Gamma.$$

The variety generated by $\mathbf{A}$ is the class of all type τ algebras that satisfy all identities that $\mathbf{A}$ does, and is denoted by $\mathcal{V}(\mathbf{A})$. By Birkhoff's HSP Theorem, it holds that $\mathcal{V}(\mathbf{A}) = \mathrm{HSP}(\mathbf{A})$.

In this thesis we consider algebras whose basic operations f satisfy the identity

$$f(x, x, \ldots, x) = x.$$

An algebra is said to be *idempotent* if this condition is satisfied. An immediate consequence of the definition is given by the following lemma.

Lemma 2.3.1: *Let $\mathbf{A}$ be an idempotent algebra and θ a congruence on $\mathbf{A}$. Then every congruence block of θ is a subuniverse of $\mathbf{A}$.*

Proof. Let $a \in A$ and consider $[a]_\theta = \{b \in A : (a, b) \in \theta\}$. We need to show that $[a]_\theta$ is closed under all basic operations of $\mathbf{A}$. Let f be an n-ary basic operation of $\mathbf{A}$ and take $b_1, \ldots, b_n \in [a]_\theta$. Since θ is a congruence, it is by definition preserved by basic operations, and we have $(f(b_1, \ldots, b_n), a) = (f(b_1, \ldots, b_n), f(a, \ldots, a)) \in \theta$, i.e., $f(b_1, \ldots, b_n) \in [a]_\theta$. □

A term operation w on an algebra $\mathbf{A}$ is called *weak near-unanimity (WNU)* term operation if the identities

$$w(y, x, x, \ldots, x) = w(x, y, x, \ldots, x) = \cdots = w(x, \ldots, x, y)$$

hold in $\mathbf{A}$. A generalisation yields the notion of *k-weak near-unanimity (k-WNU)* term operations. For an integer $k \in \mathbb{N}$, a k-WNU term operation w of arity $n \geq k$ satisfies all identities of the form

$$w(\square_1, \ldots, \square_n) = w(\triangle_1, \ldots, \triangle_n),$$

where $\square_i, \triangle_i \in \{x, y\}$ for all $i \in [n]$, and

$$|\{i \in [n] : \square_i = y\}| = k = |\{i \in [n] : \triangle_i = y\}|.$$

Evidently, there exist $\binom{n}{k}$ different tuples of size n that have exactly k instances of y and $n - k$ instances of x. Therefore, when simplified to a non-redundant system of identities, a k-WNU term operation of arity n is defined by a system of $\binom{n}{k} - 1$ identities.

2.4 Free algebras

Let $\mathcal{V}$ be a variety of type τ, X a set, and $\mathbf{T} = \mathbf{T}(X, \tau)$ the corresponding term algebra. On its universe T we consider the following congruence relation:

$$\Sigma_{\mathcal{V}} := \bigcap_{\mathbf{A} \in \mathcal{V}} \left\{(s,t) : s^{\mathbf{A}} = t^{\mathbf{A}}\right\} \subseteq T^2.$$

We set the *free algebra* $\mathcal{F}_{\mathcal{V}}(X)$ over X in the variety $\mathcal{V}$ to be the factor algebra $\mathbf{T}/\Sigma_{\mathcal{V}}$. If $X = \{x_1, \ldots, x_n\}$, we write $\mathcal{F}_{\mathcal{V}}(x_1, \ldots, x_n)$ and call it the *free algebra with* n *generators*. All identities that hold in $\mathcal{F}_{\mathcal{V}}(X)$ also hold in any $\mathbf{A} \in \mathcal{V}$.

2.5 Clones of operations

Let $\mathcal{A}$ be a set of operations on a set A. $\mathcal{A}$ is called a *clone* if

- $\mathcal{A}$ contains all projections $\mathrm{pr}_i^n : A^n \to A : (x_1, \ldots, x_n) \mapsto x_i$ for all $n \in \mathbb{N}$ and $i \in [n]$, and
- $\mathcal{A}$ is closed under composition, i.e., if $f \in \mathcal{A}$ is n-ary and $g_1, \ldots, g_n \in \mathcal{A}$ are m-ary, then also $f \circ (g_1, \ldots, g_n) \in \mathcal{A}$ where

$$f \circ (g_1, \ldots, g_n)(x_1, \ldots, x_m) := f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)).$$

For an algebra $\mathbf{A}$, by $\mathrm{Clo}(\mathbf{A})$ we denote the clone generated by all basic operations of $\mathbf{A}$, i.e., $\mathrm{Clo}(\mathbf{A})$ is the smallest clone consisting of operations on A that contains all the basic operations of $\mathbf{A}$. Elements of $\mathrm{Clo}(\mathbf{A})$ are called *term operations*.

Let $\mathbf{A}$ be an algebra and for $n \in \mathbb{N}$ let $\mathrm{Clo}_n(\mathbf{A})$ be the clone of all n-ary term operations of $\mathbf{A}$. We endow $\mathrm{Clo}_n(\mathbf{A})$ with an algebraic structure of the same type as $\mathbf{A}$. Namely, if f is a basic operation of $\mathbf{A}$ of some arity m, then we set

$$f(g_1, \ldots, g_m) := f \circ (g_1, \ldots, g_m)$$

for all $g_1, \ldots, g_m \in \mathrm{Clo}_n(\mathbf{A})$. By definition of a clone, this assignment is well-defined, i.e., it holds that $f(g_1, \ldots, g_m) \in \mathrm{Clo}_n(\mathbf{A})$. The following lemma justifies our terminology for elements of $\mathrm{Clo}(\mathbf{A})$: It states that any term operation can be defined by a term over the basic operations of $\mathbf{A}$. Furthermore, since $\mathrm{Clo}_n(\mathbf{A}) \leq \mathbf{A}^{A^n} \in \mathrm{SP}(\mathbf{A})$, it yields a way to regard the free algebra $\mathcal{F}_{\mathcal{V}(\mathbf{A})}(x_1, \ldots, x_n)$ with n generators in the variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$ as a finite subpower of $\mathbf{A}$. In particular, this implies that if $\mathbf{A}$ is finite, idempotent, and Taylor, then so is $\mathcal{F}_{\mathcal{V}(\mathbf{A})}(x_1, \ldots, x_n)$ for any $n \in \mathbb{N}$.

Lemma 2.5.1: *Let* $\mathbf{A}$ *be an algebra of size* $|A| \geq 2$, $\mathcal{V}(\mathbf{A})$ *its generated variety, and* $X = \{x_1, \ldots, x_n\}$ *a finite set. Then there is an isomorphism between* $\mathrm{Clo}_n(\mathbf{A})$ *and the free algebra* $\mathcal{F}_{\mathcal{V}(\mathbf{A})}(x_1, \ldots, x_n)$ *with* n *generators in the variety* $\mathcal{V}(\mathbf{A})$.

Proof. Recall the definition of the free algebra in the variety $\mathcal{V}(\mathbf{A})$ as the term algebra $\mathbf{T}$ over the variable set $\{x_1, \dots, x_n\}$ factorised by the congruence

$$\Sigma_{\mathcal{V}(\mathbf{A})} := \bigcap_{\mathbf{B} \in \mathcal{V}(\mathbf{A})} \{(s,t) \in T^2 : s^{\mathbf{B}} = t^{\mathbf{B}}\}$$

as given in Section 2.4. We claim that every $g \in \mathrm{Clo}_n(\mathbf{A})$ is induced by an element of T, and that every term $t \in T$ is induced by some n-ary term operation. We show the claim by induction of the complexity of g and t, respectively. If g is a projection pr_i^n for some $i \in [n]$, then $g(x_1, \dots, x_n)$ is just the variable x_i, hence, g is induced by a term. Conversely, the variables x_i, $i \in [n]$ are induced by the projections pr_i^n. For the inductive step, first assume that g is of the form $f \circ (g_1, \dots, g_m)$, where $f \in \mathrm{Clo}(\mathbf{A})$ is m-ary and $g_1 \dots, g_m \in \mathrm{Clo}_n(\mathbf{A})$ are induced by terms $t_1, \dots, t_m$, respectively. It then follows from the definitions of $\mathrm{Clo}(\mathbf{A})$ and of the term algebra $\mathbf{T}$ that $f \circ (g_1, \dots, g_m)(x_1, \dots, x_n) \in T$. Secondly, assume that t is of the form $f(t_1, \dots, t_m)$ for some m-ary basic operation f of $\mathbf{A}$ and terms $t_1, \dots, t_m \in T$ that are induced by n-ary term operations $g_1, \dots, g_m \in \mathrm{Clo}_n(\mathbf{A})$, respectively. Then $f \circ (g_1, \dots, g_m)(x_1, \dots, x_n) = t$ and $f \circ (g_1, \dots, g_m) \in \mathrm{Clo}(\mathbf{A})$. This concludes the proof of the claim. Therefore, for $g \in \mathrm{Clo}_n(\mathbf{A})$ we may consider the block of its induced term $[g(x_1, \dots, x_n)]_{\Sigma_{\mathcal{V}(\mathbf{A})}}$ under the congruence $\Sigma_{\mathcal{V}(\mathbf{A})}$.

By definition of $\Sigma_{\mathcal{V}(\mathbf{A})}$, the map that sends $g \in \mathrm{Clo}_n(\mathbf{A})$ to the block $[g(x_1, \dots, x_n)]_{\Sigma_{\mathcal{V}(\mathbf{A})}}$ of its induced term $g(x_1, \dots, x_n)$ is an injective homomorphism from $\mathrm{Clo}_n(\mathbf{A})$ to the free algebra $\mathcal{F}_{\mathcal{V}(\mathbf{A})}(x_1, \dots, x_n)$. Namely, two operations $g_1, g_2 \in \mathrm{Clo}_n(\mathbf{A})$ are different if and only if there exists a tuple $(a_1, \dots, a_n) \in A^n$ such that $g_1(a_1, \dots, a_n) \neq g_2(a_1, \dots, a_n)$. Thus, if $g_1 \neq g_2$, it follows that $[g(x_1, \dots, x_n)]_{\Sigma_{\mathcal{V}(\mathbf{A})}} \neq [g_2(x_1, \dots, x_n)]_{\Sigma_{\mathcal{V}(\mathbf{A})}}$. For surjectivity of the mapping, take $[t]_{\Sigma_{\mathcal{V}(\mathbf{A})}}$ arbitrary. Let $g \in \mathrm{Clo}_n(\mathbf{A})$ such that g induces t. We then have that $g \mapsto [g(x_1, \dots, x_n)]_{\Sigma_{\mathcal{V}(\mathbf{A})}} = [t]_{\Sigma_{\mathcal{V}(\mathbf{A})}}$.

□

The following notion allows us to compare identities satisfied by algebras.

Let $\mathcal{A}, \mathcal{B}$ be clones. A map $\xi : \mathcal{A} \to \mathcal{B}$ is called a *clone homomorphism* if

- ξ preserves arities, i.e., it sends every operation from $\mathcal{A}$ to an operation of the same arity in $\mathcal{B}$,
- ξ preserves projections, i.e., for all $n \in \mathbb{N}$ and all $i \in [n]$ it sends the n-ary projection onto the i-th coordinate in $\mathcal{A}$ to the same projection in $\mathcal{B}$ [1], and
- ξ preserves compositions, i.e., if $f \in \mathcal{A}$ is n-ary and $g_1, \dots, g_n \in \mathcal{A}$ have the same arity, then it holds that $\xi(f \circ (g_1, \dots, g_n)) = \xi(f) \circ (\xi(g_1), \dots, \xi(g_n))$.

It is easy to see that clone homomorphisms preserve identities. Namely, if $\mathbf{A}$ and $\mathbf{B}$ are algebras, and $\xi : \mathrm{Clo}(\mathbf{A}) \to \mathrm{Clo}(\mathbf{B})$ is a clone homomorphism, then any identity satisfied by term operations from $\mathbf{A}$ are also satisfied by their images in $\mathrm{Clo}(\mathbf{B})$ under the clone

[1] In the following, we will abuse notation and will not distinguish between projections on different sets, i.e., the condition that a clone homomorphism ξ preserves projections may be written as $\xi(\mathrm{pr}_i^n) = \mathrm{pr}_i^n$.

homomorphism ξ.

Example 2.5.2: For example, let $\mathbf{A}$ be an algebra and assume that $\mathbf{A}$ has a term operation $m \in \text{Clo}(\mathbf{A})$ satisfying the identities

$$m(x, y, y) = m(y, y, x) = x.$$

Let $\mathbf{B}$ be an algebra, possibly of different type, and let $\xi : \text{Clo}(\mathbf{A}) \to \text{Clo}(\mathbf{B})$ be a clone homomorphism. Since ξ preserves arities, compositions, and projections we have that

$$\xi(m)(x, y, y) = \xi(m) \circ (\pi_1^3, \pi_2^3, \pi_2^3)(x, y, z) = \xi(m \circ (\pi_1^3, \pi_2^3, \pi_2^3)(x, y, z)) = \xi(m(x, y, y)),$$

i.e., $\xi(m) \in \text{Clo}(\mathbf{B})$ also satisfies the identity $\xi(m)(x, y, y) = \xi(m)(y, y, x) = x$. Operations m satisfying the identities $m(x, y, y) = m(y, y, x) = x$ are called *Mal'cev operations.*

For any set U, by Proj^U we denote the clone consisting merely of all projections on U. Proj denotes the clone of projections on a two-element set. An immediate consequence of the definition of clone homomorphisms is given by the following lemma.

Lemma 2.5.3: *Let U be any set of size $|U| \geq 2$. Then there exist clone homomorphisms* $\text{Proj}^U \to \text{Proj}$ *and* $\text{Proj} \to \text{Proj}^U$*.*

Proof. For $\pi_i^n : U^n \to U : (u_1, \ldots, u_n) \mapsto u_i$ let $\xi(\pi_i^n) : \{0, 1\}^n \to \{0, 1\} : (x_1, \ldots, x_n) \to x_i$ and vice versa. □

2.6 Relational structures

Let A be a set. A *relation* R on A is a subset $R \subseteq A^n$ for some $n \in \mathbb{N}$. We call n the *arity* of the relation. By $\mathcal{R}_A$ we denote the set of all relations on A. If $(a_1, \ldots, a_n) \in R$ for some $R \in \mathcal{R}_A$, we also write $R(a_1, \ldots, a_n)$. A *relational structure* with *universe* A is a tuple $\mathbb{A} = (A; (R_i)_{i \in I})$, where $(R_i)_{i \in I}$ is a finite family of relations on A. We denote a relational structure with double-struck letters, and the corresponding universe in normal font.

A *graph* is relational structure $\mathbb{G} = (V; E)$, where V is a finite set whose elements are called *vertices*, and $E \subseteq V^2$ is a binary symmetric relation on V whose elements are called *edges*. A graph is *bipartite* if its vertices can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex from V_1 to one in V_2. It is a well-known fact that a graph is bipartite if and only if it does not contain any cycles of odd length. For the purpose of this thesis, a *hypergraph* is a relational structure $\mathbb{G} = (V; E)$ where $E \in \mathcal{R}_V$ is a relation on V of arbitrary arity $k \in \mathbb{N}$, i.e., a hypergraph is just a relational structure with a single relation that is not necessarily symmetric. $\mathbb{G}$ is called *complete* if $E = V^k$, and *empty* if $E = \emptyset$. A *loop* in a hypergraph $\mathbb{G} = (V; E)$ is a constant tuple $(c, \ldots, c) \in E$.

Let $\mathbb{A} = (A; R_1, \ldots, R_m)$ and $\mathbb{B} = (B; S_1, \ldots, S_m)$ be two relational structures such that for all $i \in [m]$ the arities of R_i and S_i coincide and are given by n_i. A *homomorphism* from $\mathbb{A}$ to $\mathbb{B}$ is a map $\phi : A \to B$ with the property that

$$(a_1, \ldots, a_{n_i}) \in R_i \Rightarrow (\phi(a_1), \ldots, \phi(a_{n_i})) \in S_i$$

for all $i \in [m]$ and $(a_1, \dots, a_{n_i}) \in A^{n_i}$. Two structures $\mathbb{A}$ and $\mathbb{B}$ are *homomorphically equivalent* if there exist homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$.

Let $\mathbb{A} = (A, (R_i)_{i \in I})$ be a relational structure. Assume that for every $i \in I$ the arity of R_i is given by r_i. We say that an n-ary relation R on A is *pp-definable* from $(R_i)_{i \in I}$ if there exists a finite family $(R_j)_{j \in J}$ of relations from $(R_i)_{i \in I}$ and some $k \in \mathbb{N}$ such that for any $(x_1, \dots, x_n) \in A^n$ it holds that

$$(x_1, \dots, x_n) \in R \Leftrightarrow \exists y_1 \in A, \dots, \exists y_k \in A : \bigwedge_{j \in J} R_j(z_{j,1}, \dots, z_{j,r_j}),$$

where $z_{j,l} \in \{y_1, \dots, y_k, x_1, \dots, x_n\}$ for any $j \in J$ and $l \in [r_j]$. Formulas of the form $\exists y_1 \dots \exists y_k \Phi$ where Φ is a conjunction of relational symbols and equalities are called *primitive positive formulas (pp-formulas)*.

Let $\mathbb{A}$ and $\mathbb{B}$ be relational structures. We say that $\mathbb{B}$ is a *pp-power* of $\mathbb{A}$ if there exists some $n \geq 1$ such that $\mathbb{B}$ is isomorphic to a structure with domain A^n whose relations are definable using only primitive positive formulas over $\mathbb{A}$. Here, a k-ary relation on A^n is being considered as a kn-ary relation on A. Furthermore, we say that $\mathbb{A}$ *pp-constructs* $\mathbb{B}$ if $\mathbb{B}$ is homomorphically equivalent to a pp-power of $\mathbb{A}$.

2.7 The Pol-Inv correspondence

Let A be a set. An m-ary operation $f \in \mathcal{O}_A$ on A is said to *preserve* an n-ary relation $R \in \mathcal{R}_A$ if for all $r_1, \dots, r_m \in R$ it holds that $f(r_1, \dots, r_m) \in R$, where f is applied to tuples coordinatewisely. Explicitly, this means

$$\begin{pmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{pmatrix}, \dots, \begin{pmatrix} x_1^m \\ \vdots \\ x_n^m \end{pmatrix} \in R \Rightarrow f\left(\begin{pmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{pmatrix}, \dots, \begin{pmatrix} x_1^m \\ \vdots \\ x_n^m \end{pmatrix}\right) = \begin{pmatrix} f(x_1^1, \dots, x_1^m) \\ \vdots \\ f(x_n^1, \dots, x_n^m) \end{pmatrix} \in R.$$

For a set $F \subseteq \mathcal{O}_A$ of operations on A we let

$$\mathrm{Inv}(F) := \{R \in \mathcal{R}_A : R \text{ is invariant under each operation from } F\}.$$

Conversely, for a set $\Gamma \subseteq \mathcal{R}_A$ of relations on A we define

$$\mathrm{Pol}(\Gamma) := \{f \in \mathcal{O}_A : f \text{ preserves each relation from } \Gamma\}.$$

The operators Pol and Inv form a Galois-correspondence between the sets $\mathcal{R}_A$ and $\mathcal{O}_A$.

Let $\mathbb{A} = (A; (R_i)_{i \in I})$ be a finite relational structure. The *polymorphism clone* of $\mathbb{A}$ is the set $\mathrm{Pol}(\mathbb{A}) := \mathrm{Pol}(\{R_i : i \in I\})$. Its elements are called *polymorphisms* of $\mathbb{A}$. Note that the polymorphism clone of a relational structure is in fact a clone in the sense of Section 2.5.

For an algebra $\mathbf{A} = (A; (f_i)_{i \in I})$, by $\mathrm{Inv}(\mathbf{A})$ we denote the set $\mathrm{Inv}(\mathbf{A}) := \mathrm{Inv}(\{f_i : i \in I\})$ of relations on A that are preserved by all basic operations of $\mathbf{A}$. Observe that a relation

$R \subseteq A^n$ is a subuniverse of $\mathbf{A}^n$ if and only if it holds that $R \in \mathrm{Inv}(\mathbf{A})$. By $\langle R \rangle_{\mathbf{A}}$ we denote the smallest subuniverse of $\mathbf{A}^n$ that contains R. In other words, $\langle R \rangle_{\mathbf{A}}$ is the subuniverse of $\mathbf{A}^n$ generated by R.

2.8 (n,k)-symmetric relations

If $(G,+,-)$ is a group with identity element $e \in G$ and X is a set, then a *group action* of G on X is a map $\alpha : G \times X \to X$ such that $\alpha(e,x) = x$ and $\alpha(g,\alpha(h,x)) = \alpha(g+h,x)$ for all $g,h \in G$ and $x \in X$. A group action of a group G on a set X is also denoted by $G \overset{\alpha}{\curvearrowright} X$. If α is clear from the context, we just write $G \curvearrowright X$.

Let $k \leq n$ be positive integers. We will identify the binomial coefficient $\binom{n}{k}$ with the set of all subsets of $\{1,\dots,n\}$ that contain exactly k elements. Let S_n denote the symmetric group on $\{1,\dots,n\}$. For $E = \{i_1,\dots,i_k\} \in \binom{n}{k}$ and $\sigma \in S_n$ we set $\sigma(E) := \{\sigma(i_1),\dots,\sigma(i_k)\}$. This defines a group action $S_n \overset{\alpha}{\curvearrowright} \binom{n}{k}$ given by $\alpha(\sigma,E) := \sigma(E)$. For any set A, this yields a group action $S_n \overset{\bar\alpha}{\curvearrowright} A^{\binom{n}{k}}$ on the product $A^{\binom{n}{k}}$ by setting [2]

$$\bar\alpha(\sigma,(a_E)_{E\in\binom{n}{k}}) := (a_{\alpha(\sigma,E)})_{E\in\binom{n}{k}}.$$

A relation $R \subseteq A^{\binom{n}{k}}$ is said to be (n,k)-*symmetric* if it is invariant under the action $S_n \overset{\bar\alpha}{\curvearrowright} A^{\binom{n}{k}}$, meaning that for all $\sigma \in S_n$ and for all $\mathbf{r} \in R$ it holds that also $\bar\alpha(\sigma,\mathbf{r}) \in R$. If $k = 1$, we say that R is *symmetric*.

Remark 2.8.1: Observe that if R is an (n,k)-symmetric relation and

$$(\square_1,\dots,\square_{i-1},r_i,\square_{i+1},\dots,\square_{j-1},r_j,\square_{j+1},\dots,\square_{\binom{n}{k}}) \in R \tag{2.1}$$

is a tuple from R such that r_i and r_j are its i-th and its j-th component, respectively, then there exists a tuple

$$(\triangle_1,\dots,\triangle_{i-1},r_j,\triangle_{i+1},\dots,\triangle_{j-1},r_i,\triangle_{j+1},\dots,\triangle_{\binom{n}{k}}) \in R \tag{2.2}$$

such that r_j is its i-th and r_i is its j-th coordinate. Here, the components $\triangle_l$, $l \in [\binom{n}{k}]$ of the tuple (2.2) are given by an appropriate reordering of the components $\square_l$, $l \in [\binom{n}{k}]$ of the tuple (2.1). In particular, if for every $i \in [\binom{n}{k}]$ we denote the projection on the i-th coordinate by pr_i, we have

$$\mathrm{pr}_1(R) = \mathrm{pr}_2(R) = \cdots = pr_{\binom{n}{k}}(R).$$

[2]Note that equivalently, the product $A^{\binom{n}{k}}$ can be regarded as the set of all functions $f : \binom{n}{k} \to A$. The group action $S_n \overset{\bar\alpha}{\curvearrowright} A^{\binom{n}{k}}$ is then given by

$$\bar\alpha(\sigma,f) := g : \begin{cases} \binom{n}{k} & \to A \\ E & \mapsto f(\alpha(\sigma,E)) \end{cases}.$$

Recall the definition of pp-definable relations from Section 2.6. A basic fact about pp-definable relations on an algebra $\mathbf{A}$ is given by the following lemma.

Lemma 2.8.2: *Let $\mathbf{A}$ be an algebra. If R is a relation on A that is pp-definable from a family of invariant relations, then it holds that $R \in \operatorname{Inv}(\mathbf{A})$.*

Proof. Let the arity of R be given by n. Let $(R_i)_{i\in I}$ be a family of invariant relations on A and $k \in \mathbb{N}$ be such that a tuple $(x_1, \dots, x_n) \in A^n$ belongs to R if and only if it holds that

$$\exists y_1 \in A, \dots, \exists y_k \in A : \bigwedge_{i\in I} R_i(z_{i,1}, \dots, z_{i,r_i}),$$

where for all $i \in I$ the arity of R_i is given by r_i, and $z_{i,j} \in \{y_1, \dots, y_k, x_1, \dots, x_n\}$ for all $j \in [r_i]$. Let f be a m-ary basic operation of $\mathbf{A}$. For each $j \in [m]$ take some tuple $\mathbf{r}_j = (x_1^j, \dots, x_n^j) \in R$. We have to show that

$$f(\mathbf{r}_1, \dots, \mathbf{r}_m) = \begin{pmatrix} f(x_1^1, \dots, x_1^m) \\ \vdots \\ f(x_n^1, \dots, x_n^m) \end{pmatrix} \in R.$$

By definition, for any $j \in [m]$ there exist $y_1^j, \dots, y_k^j \in A$ witnessing $(x_1^j, \dots, x_n^j) \in R$. For $i \in [N]$ let $\mathbf{z}_i^j$ denote the tuple $(z_{i,1}^j, \dots, z_{i,r_i}^j) \in \{y_1^j, \dots, y_k^j, x_1^j, \dots, x_n^j\}^{r_i}$ such that $R_i(\mathbf{z}_i^j)$ is satisfied. Since $R_i \in \operatorname{Inv}(\mathbf{A})$, it holds that $f(\mathbf{z}_i^1, \dots, \mathbf{z}_i^m) \in R_i$. If for $l \in [k]$ we set $y_l := f(y_l^1, \dots, y_l^m)$, then by the tuple $(y_1, \dots, y_k)$ we have found a witness that $f(\mathbf{r}_1, \dots, \mathbf{r}_m) \in R$. ☐

A relation $R \subseteq A^n$ is said to be *subdirect* if $\operatorname{pr}_i(R) = A$ for every $i \in [n]$. If $\mathbf{A}$ is an algebra and it additionally holds that R is a subuniverse of the product $\mathbf{A}^n$, we write $\mathbf{R} \leq_{\text{sd}} \mathbf{A}^n$.

A relation $R \subseteq A^n$ on an algebra $\mathbf{A}$ is said to satisfy the *restriction-property* if there exists a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ such that $R \cap B^n \neq \emptyset$.

The following lemma states that for an (n, k)-symmetric invariant relation R, the failure of the restriction-property already implies R to be subdirect.

Lemma 2.8.3: *Let $\mathbf{A}$ be an algebra. Assume that $R \subseteq A^{\binom{n}{k}}$ is a nonempty, invariant, and (n, k)-symmetric relation on A for some $n, k \in \mathbb{N}$. If R does not satisfy the restriction-property, then $\mathbf{R} \leq_{\text{sd}} \mathbf{A}^{\binom{n}{k}}$.*

Proof. Since R does not satisfy the restriction-property, for all proper subalgebras $\mathbf{B} \lneq \mathbf{A}$ it holds that $R \cap B^n = \emptyset$. By (n, k)-symmetry of R, we have

$$\operatorname{pr}_1(R) = \operatorname{pr}_2(R) = \cdots = \operatorname{pr}_{\binom{n}{k}}(R).$$

Notice that $\operatorname{pr}_1(R)$ is pp-definable from R by

$$x \in \operatorname{pr}_1(R) \leftrightarrow \big(\exists a_2 \in A, \dots, \exists a_{\binom{n}{k}} \in A : (x, a_2, \dots, a_{\binom{n}{k}}) \in R\big),$$

and is therefore a subuniverse of $\mathbf{A}$ by Lemma 2.8.2. Since $R \subseteq \operatorname{pr}_1(R)^{\binom{n}{k}}$, we must therefore have $\operatorname{pr}_1(R) = A$. Hence, $\mathbf{R} \leq_{\text{sd}} \mathbf{A}^{\binom{n}{k}}$. ☐

3 Taylor Algebras

In this thesis, we will only consider finite idempotent *Taylor algebras*. These have played a major role in the proof of the Dichotomy Conjecture 1.2. A fundamental result obtained by D. Zhuk in the course of his proof (see [Zhu17; Zhu20a]) states the occurence of one of two cases whenever an algebra is finite, idempotent, and Taylor. Namely, he showed that any finite idempotent Taylor algebra possesses either a *strong subuniverse* or a *p-affine* factor for some prime number $p \in \mathbb{P}$ (Corollary 3.1.3). We will formulate Zhuk's Theorem in section 3.1 and provide the required definitions and some consequences in the subsequent sections.

We will use the following definition of Taylor algebras:

Definition 3.1: Let $\mathbf{A}$ be a finite algebra. $\mathbf{A}$ is called *Taylor* if there does not exist a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathrm{Proj}$.

Since for any $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$ there exists a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathrm{Clo}(\mathbf{B})$, it suffices to ask that the variety generated by $\mathbf{A}$ does not contain a two-element algebra $\mathbf{B}$ whose every operation is a projection, i.e., an algebra $\mathbf{B}$ such that $\mathrm{Clo}(\mathbf{B}) = \mathrm{Proj}$. In fact, for a finite idempotent algebra $\mathbf{A}$ it is even enough to require that $\mathrm{HS}(\mathbf{A})$ does not contain such an algebra [BJ01].

For finite idempotent algebras there exist many different characterisations. Namely, finite idempotent Taylor algebras can be characterised by the existence of term operations satisfying certain sets of identities. Theorem 1.9 in the introduction collects some of these characterisations.

Example 3.2: As an example of a Taylor algebra consider an algebra $\mathbf{A}$ that has a Mal'cev term operation $m \in \mathrm{Clo}(\mathbf{A})$, i.e., m satisfies the identities $m(x,y,y) = m(y,y,x) = x$. In Example 2.5.2 we showed that if $\mathbf{B}$ is an algebra such that there exists a clone homomorphism $\xi : \mathrm{Clo}(\mathbf{A}) \to \mathrm{Clo}(\mathbf{B})$, then $\mathrm{Clo}(\mathbf{B})$ admits the Mal'cev operation $\xi(m)$. In particular, since the Mal'cev identities fail to be satisfied by projections, there does not exist a clone homomorphism $\xi : \mathrm{Clo}(\mathbf{A}) \to \mathrm{Proj}$. As any group $\mathbf{A} = (A; +, -)$ has the Mal'cev operation given by $m(x,y,z) := x - y + z$, groups are examples of Taylor algebras.

Example 3.3: Another simple example of a Taylor algebra is the three-element rock-paper-scissors algebra

$$\big(\{\text{rock, paper, scissors}\}; \mathrm{winner}(x,y)\big).$$

Namely, it satisfies the identity $\mathrm{winner}(x,y) = \mathrm{winner}(y,x)$, which cannot be satisfied by projections.

3.1 Zhuk's Theorem

Zhuk's Cases Theorem yielded a crucial ingredient for his algebraic approach to CSPs. The theorem is stated below, the required concepts will be provided in the sections thereafter. For the proof of the theorem we refer to [Zhu17; Zhu20a; Zhu20b].

An algebra is *essentially unary* if for any of its basic operations f, say of arity n, there exists a unary operation g and $i \in [n]$ such that $f(x_1, \ldots, x_n) = g(x_i)$, i.e., f has at most one non-dummy variable. The notions of strong subuniverses and p-affine algebras are given in Definition 3.2.1 and Definition 3.3.1, respectively.

Theorem 3.1.1 (Zhuk's Cases)**:** *[Zhu20b, Theorem 3.3] Let* $\mathbf{A}$ *be a finite idempotent algebra of size at least* 2*. Then at least one of the following holds:*

(i) $\mathbf{A}$ *has a nontrivial strong subuniverse.*

(ii) There exist $p \in \mathbb{P}$ *and a congruence* θ *on* A *such that* $\mathbf{A}/\theta$ *is* p*-affine.*

(iii) There exists an essentially unary algebra $\mathbf{U} \in \mathrm{HS}(\mathbf{A})$ *of size at least* 2*.*

For the purpose of this thesis, however, essentially unary algebras do not show up:

Lemma 3.1.2: *Let* $\mathbf{A}$ *be a Taylor algebra. Then there does not exist an essentially unary algebra* $\mathbf{U} \in \mathrm{HS}(\mathbf{A})$ *of size at least* 2*.*

Proof. For any $\mathbf{U} \in \mathrm{HS}(\mathbf{A})$ there exists a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathrm{Clo}(\mathbf{U})$. If $\mathbf{U}$ is essentially unary of size at least 2, the mapping that sends every $f \in \mathrm{Clo}(\mathbf{U})$ to the projection onto its non-dummy variable is a clone homomorphism $\mathrm{Clo}(\mathbf{U}) \to \mathrm{Proj}^U$. By Lemma 2.5.3, there exists a clone homomorphism $\mathrm{Proj}^U \to \mathrm{Proj}$. Thus, if $\mathbf{U} \in \mathrm{HS}(\mathbf{A})$ is essentially unary, we obtain a clone homomorphism $\mathrm{Clo}(\mathbf{A}) \to \mathrm{Proj}$, i.e., $\mathbf{A}$ is non-Taylor. □

In case $\mathbf{A}$ is Taylor, Zhuk's Cases Theorem therefore reduces to the following corollary:

Corollary 3.1.3: *Let* $\mathbf{A}$ *be a finite idempotent Taylor algebra of size at least* 2*. Then at least one of the following holds:*

(i) $\mathbf{A}$ *has a nontrivial strong subuniverse.*

(ii) There exist $p \in \mathbb{P}$ *and a congruence* θ *on* A *such that* $\mathbf{A}/\theta$ *is* p*-affine.*

3.2 Strong subuniverses

The notion of a strong subuniverse is a collective term referring to subalgebras of one of three types. Even though we will not need the exact definitions for the purpose of this thesis, we provide them for completeness.

Let $\mathbf{A}$ be an algebra and let $\mathbf{B} \leq \mathbf{A}$ be a subalgebra of $\mathbf{A}$.

B is called an *absorbing subuniverse* if there exists a term operation $t \in \mathrm{Clo}(\mathbf{A})$, say of arity n, such that for all $(a_1, \ldots, a_n) \in A^n$ it holds that

$$|\{i \in [n] : a_i \notin B\}| \leq 1 \Rightarrow t(a_1, \ldots, a_n) \in B.$$

If t can be chosen binary, then B is called *binary absorbing subuniverse*.

B is *central* if it is an absorbing subuniverse and for every $a \in A \setminus B$ it holds that

$$(a, a) \notin \langle(\{a\} \times B) \cup (B \times \{a\})\rangle_{\mathbf{A}}.$$

B is said to be a *projective subuniverse* if for every $n \in \mathbb{N}$ and every n-ary basic operation f of $\mathbf{A}$ there exists a coordinate $i \in [n]$ such that for all $(a_1, \ldots, a_n) \in A^n$ it holds that $f(a_1, \ldots, a_n) \in B$ whenever $a_i \in B$.

An algebra $\mathbf{A}$ is *polynomially complete (PC)* if the clone of all operations is generated by all basic operations of $\mathbf{A}$ and all constant operations. A nontrivial subuniverse $B \subseteq A$ is called a *PC subuniverse* if there exists a nontrivial congruence θ on $\mathbf{A}$ and some $a \in A$ such that $B = [a]_\theta$ and $A/\theta \cong D_1 \times \cdots \times D_s$ where each D_i is a PC algebra that has no nontrivial binary absorbing subuniverses, or no nontrivial central subuniverses, or no nontrivial projective subuniverses.

The definition of a strong subuniverse is now given as follows:

Definition 3.2.1: Let $\mathbf{A}$ be an algebra and let $B \subseteq A$ be a subuniverse of $\mathbf{A}$. Then B is called a *strong subuniverse* if it is a binary absorbing subuniverse, a central subuniverse, or a PC subuniverse.

3.3 p-affine algebras

The other type of algebras that appears in Corollary 3.1.3 is given by p-affine algebras. We will show that the clone of term operations of a p-affine algebra contains all p-affine operations.

Definition 3.3.1: Let $\mathbf{A}$ be an idempotent algebra and $p \in \mathbb{P}$. $\mathbf{A}$ is called *p-affine* if there exist operations $\oplus$ and $\ominus$ on A such that $(\mathbf{A}, \oplus, \ominus) \cong ((\mathbb{Z}_p)^n, +, -)$ for some $n \geq 1$,

(A1) $m(x, y, z) := x \ominus y \oplus z \in \mathrm{Clo}(\mathbf{A})$, and

(A2) $\{(x, y, z, w) : x \oplus y = z \oplus w\} \in \mathrm{Inv}(\mathbf{A})$.

Remark 3.3.2: Let $\mathbf{A}$ be p-affine. Condition (A2) implies that any m-ary term operation $f \in \mathrm{Clo}(\mathbf{A})$ fulfills $f(\mathbf{x} \ominus \mathbf{y} \oplus \mathbf{z}) = f(\mathbf{x}) \ominus f(\mathbf{y}) \oplus f(\mathbf{z})$ for m-tuples $\mathbf{x}, \mathbf{y}, \mathbf{z}$, where $\oplus$ and $\ominus$ are applied separately to each coordinate. Namely, take any $\mathbf{x} = (x_1, \ldots, x_m)$, $\mathbf{y} = (y_1, \ldots, y_m)$, $\mathbf{z} = (z_1, \ldots, z_m) \in A^m$, and for $i \in [m]$ set $w_i := x_i \ominus y_i \oplus z_i$. Since by definition, $x_i \oplus z_i = y_i \oplus w_i$ for all $i \in [m]$, all tuples (x_i, z_i, y_i, w_i) belong to the relation that is invariant by condition (A2). Thus, we have that the tuple

$$\begin{pmatrix} f(x_1, \dots, x_m) \\ f(z_1, \dots, z_m) \\ f(y_1, \dots, y_m) \\ f(w_1, \dots, w_m) \end{pmatrix}$$

fulfills $f(x_1, \dots, x_m) \oplus f(z_1, \dots, z_m) = f(y_1, \dots, y_m) \oplus f(w_1, \dots, w_m)$. In other words, it holds that $f(\mathbf{x}) \ominus f(\mathbf{y}) \oplus f(\mathbf{z}) = f(\mathbf{x} \ominus \mathbf{y} \oplus \mathbf{z})$. From now on, we will only use the symbols $+$ and $-$ to denote the operations $\oplus$ and $\ominus$, respectively.

Remark 3.3.3: [Bra22b] If $\mathbf{A}$ is p-affine, we obtain a nice normal form for its term operations. Namely, assuming that $A = (\mathbb{Z}_p)^n$, condition (A2) is equivalent to asking that any term operation $f \in \mathrm{Clo}(\mathbf{A})$ can be written in the form

$$f(x_1, \dots, x_m) = \sum_{i=1}^{m} A_i x_i \tag{3.1}$$

with $A_i \in (\mathbb{Z}_p)^{n \times n}$ such that $\sum_{i=1}^{m} A_i = I_n$, where I_n denotes the identity matrix of size n. In order to see this, take $f \in \mathrm{Clo}(\mathbf{A})$ and assume that its arity is given by m. For each $i \in [m]$ let $A_i : A \to A$ be the operation defined by

$$A_i(x) := f(0, \dots, 0, x, 0 \dots, 0),$$

where x is on the i-th place. Observe that by Remark 3.3.2, $A_i(x + y) = A_i(x) + A_i(y)$. Namely, since $\mathbf{A}$ is idempotent, we have $f(0, \dots, 0) = 0$. Thus, we have

$$A_i(x+y) = f\left(\begin{pmatrix}0\\ \vdots\\ 0\\ x\\ 0\\ \vdots\\ 0\end{pmatrix} - \begin{pmatrix}0\\ \vdots\\ 0\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix} + \begin{pmatrix}0\\ \vdots\\ 0\\ y\\ 0\\ \vdots\\ 0\end{pmatrix}\right) = f\left(\begin{pmatrix}0\\ \vdots\\ 0\\ x\\ 0\\ \vdots\\ 0\end{pmatrix}\right) - f\left(\begin{pmatrix}0\\ \vdots\\ 0\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix}\right) + f\left(\begin{pmatrix}0\\ \vdots\\ 0\\ y\\ 0\\ \vdots\\ 0\end{pmatrix}\right) =$$
$$= A_i(x) + A_i(y).$$

Thus, A_i is in fact an endomorphism of $((\mathbb{Z}_p)^n, +, -)$ and can thus be represented by a matrix in $(\mathbb{Z}_p)^{n \times n}$. We claim that equality (3.1) holds for all tuples $(x_1, \dots, x_m) \in A^m$. Take some $(x_1, \dots, x_m) \in A^m$ and let k be the number of non-zero values among $x_1, \dots, x_m$. We proceed by induction of k. By idempotence, $f(0, \dots, 0) = 0$. It follows immediately by the definition of A_i that the equality holds if $k = 1$. Now assume that the representation holds true for all tuples $(x_1, \dots, x_m) \in A^m$ such that $|\{i \in [n] : x_i \neq 0\}| = k$. Let $(x_1, \dots, x_m) \in A^m$ be such that the number of non-zero values among $x_1, \dots, x_m$ is equal to $k+1$. Let $j \in [n]$ be such that $x_j \neq 0$. It again follows from condition (A2) and Remark 3.3.2 that

$$f(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_m) = f(x_1,\dots,x_{j-1},0,x_{j+1},\dots,x_m)+f(0,\dots,0,x_j,0,\dots,0),$$

where x_j is in the j-th place. By inductive hypothesis, we then have

$$f(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_m) = \sum_{\substack{i\in[m],\\ i\neq j}} A_i x_i + A_j x_j.$$

By idempotence of $\mathbf{A}$, we must have $x = f(x,\dots,x) = \sum_{i=1}^m A_i x$ for all $x \in A$, and therefore $\sum_{i=1}^m A_i = I_n$.

The following proposition states that if f is an operation on a p-affine algebra $\mathbf{A}$ of the form (3.1) such that each of the endomorphisms A_i is simply the multiplication with some constant, then f must already be a term operation of $\mathbf{A}$. We refer to such an operation f of the form

$$f(x_1,\dots,x_n) = \sum_{i=1}^n \lambda_i x_i,$$

where $n \in \mathbb{N}$ and $\lambda_1,\dots,\lambda_n \in \mathbb{Z}_p$ with $\sum_{i=1}^n \lambda_i = 1$ as *p-affine operation.*

Proposition 3.3.4: *Let $\mathbf{A}$ be a p-affine algebra. If f is any p-affine operation on A, then $f \in \mathrm{Clo}(\mathbf{A})$.*

Proof. By definition of $\mathbf{A}$, there exists a term operation m of the form

$$m(x,y,z) = x - y + z \in \mathrm{Clo}(\mathbf{A}).$$

Let $n \in \mathbb{N}$ and $\lambda_1,\dots,\lambda_n \in \mathbb{Z}_p$ be such that $\sum_{i=1}^n \lambda_i = 1$. Without loss of generality, we can assume that $\lambda_i \neq 0$ for all $i \in [n]$. Let f be the operation given by setting $f(x_1,\dots,x_n) := \sum_{i=1}^n \lambda_i x_i$ for all $(x_1,\dots,x_n) \in A^n$. In order to see that $f \in \mathrm{Clo}(\mathbf{A})$, we proceed by induction on n. If $n = 1$, i.e., f is unary, then it is in fact the projection pr_1^1 and therefore contained in $\mathrm{Clo}(\mathbf{A})$ by definition of $\mathrm{Clo}(\mathbf{A})$.

Observe that if $p = 2$, there do not exist 2-affine combinations of even length since $2 \mid n$ implies that $\sum_{i=1}^n \lambda_i = 0$. Therefore, n must be odd, i.e., there exists some $k \in \mathbb{N}$ such that $n = 2k+1$, and we have

$$f(x_1,\dots,x_{2k+1}) = x_1 + \dots + x_{2k+1}.$$

In this case, for the inductive step assume that any 2-affine operation of arity $2k+1$ is contained in $\mathrm{Clo}(\mathbf{A})$. If now $n = 2k+3$, we obtain $f \in \mathrm{Clo}(\mathbf{A})$ as

$$x_1 + \dots + x_{2k+3} = m(\underbrace{x_1 + \dots + x_{2k+1}}_{\in \mathrm{Clo}(\mathbf{A})}, x_{2k+2}, x_{2k+3}).$$

Assume now that $p \neq 2$. Before we proceed to the inductive step, we consider $n = 2$ and a special case of $n = 3$. Subsequently, we inductively prove that $f \in \mathrm{Clo}(\mathbf{A})$ for all $n \geq 3$. First, let $n = 2$, i.e., f is of the form

$$f(x_1, x_2) = \lambda x_1 + (1 - \lambda)x_2. \tag{3.2}$$

For $\lambda = 2$ we have $\lambda x_1 + (1 - \lambda)x_2 = m(x_1, x_2, x_1)$, and for $\lambda \geq 3$

$$(\lambda + 1)x_1 + (1 - (\lambda + 1))x_2 = \lambda x_1 + (1 - \lambda)x_2 + x_1 - x_2 = m(\lambda x_1 + (1 - \lambda)x_2, x_2, x_1).$$

In both cases, f is generated by m. Hence, we get $f \in \mathrm{Clo}(\mathbf{A})$. Secondly, if $n = 3$ and f is of the form

$$f(x_1, x_2, x_3) = x_1 - \lambda x_2 + \lambda x_3,$$

it is generated by m and term operations of the form (3.2). Namely, if $\lambda = 1$, then $f = m$. Otherwise, by setting $\mu := 2 + (\lambda - 1)^{-1}$ we have that

$$x_1 - \lambda x_2 + \lambda x_3 = \lambda \underbrace{(\lambda^{-1}x_1 + (1 - \lambda^{-1})x_2)}_{\in \mathrm{Clo}(\mathbf{A})} + (1 - \lambda) \underbrace{(\mu x_2 + (1 - \mu)x_3)}_{\in \mathrm{Clo}(\mathbf{A})}.$$

Consequently, $f \in \mathrm{Clo}(\mathbf{A})$.

For the inductive step, assume now that any n-ary p-affine function belongs to $\mathrm{Clo}(\mathbf{A})$ and let an $n+1$-ary p-affine operation be given by $f(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} \lambda_i x_i$. If we have that $\lambda := \sum_{i=1}^{n} \lambda_i \neq 0$, then by the inductive hypothesis it follows that

$$\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i \in \mathrm{Clo}(\mathbf{A}).$$

But then also

$$f(x_1, \ldots, x_{n+1}) = \lambda \cdot \underbrace{\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} x_i}_{\in \mathrm{Clo}(\mathbf{A})} + (1 - \lambda) \underbrace{x_{n+1}}_{\in \mathrm{Clo}(\mathbf{A})} \in \mathrm{Clo}(\mathbf{A}),$$

being of the form $\lambda f_1 + (1 - \lambda)f_2$ with $f_1, f_2 \in \mathrm{Clo}(\mathbf{A})$. In case $\lambda := \sum_{i=1}^{n} \lambda_i = 0$, we have $\sum_{i=2}^{n} \lambda_i = -\lambda_1$. As before, the inductive hypothesis yields

$$\sum_{i=2}^{n} \frac{\lambda_i}{-\lambda_1} x_i \in \mathrm{Clo}(\mathbf{A}).$$

Thus,

$$f(x_1, \ldots, x_{n+1}) = \lambda_1 \underbrace{x_1}_{\in \mathrm{Clo}(\mathbf{A})} - \lambda_1 \underbrace{\sum_{i=2}^{n} \frac{\lambda_i}{-\lambda_1} x_i}_{\in \mathrm{Clo}(\mathbf{A})} + \underbrace{x_{n+1}}_{\in \mathrm{Clo}(\mathbf{A})} \in \mathrm{Clo}(\mathbf{A}),$$

being of the form $f_1 - \lambda f_2 + \lambda f_3$ with $f_1, f_2, f_3 \in \mathrm{Clo}(\mathbf{A})$. □

For $p \in \mathbb{P}$, a relation R defined on the domain of some p-affine algebra is said to be *p-affine* if it is preserved by any p-affine operation. If $\mathbf{A}$ is a p-affine algebra, then by Lemma 3.3.4 we have that any invariant relation $R \in \mathrm{Inv}(\mathbf{A})$ is p-affine. The following definition will be essential for our endeavours of finding constant tuples in (n,k)-symmetric relations.

Definition 3.3.5: Let $n, k \in \mathbb{N}$ and $p \in \mathbb{P}$. The tuple (n, k, p) is said to be *loop-friendly* if every nonempty (n,k)-symmetric p-affine relation contains a constant tuple.

We are able to immediately exclude tuples (n, k, p) that satisfy $p \mid \binom{n}{k}$.

Counterexample 3.3.6: A necessary condition for the loop-friendliness of a tuple (n, k, p) is that $p \nmid \binom{n}{k}$. To see this, consider the relation

$$R := \{(x_1, \dots, x_{\binom{n}{k}}) \in (\mathbb{Z}_p)^{\binom{n}{k}} : \sum_{i=1}^{\binom{n}{k}} x_i = 1\}.$$

Then R is a nonempty and (n,k)-symmetric relation that is preserved by all p-affine operations. However, if $p \mid \binom{n}{k}$, it fails to contain a constant tuple.

The following lemma allows us to represent any p-affine algebra in $\mathrm{HSP}(\mathbf{A})$ as one in in $\mathrm{HS}(\mathbf{A})$, and thus limits the possible values of p. Namely, if $\mathbf{B} \in \mathrm{HS}(\mathbf{A})$ is p-affine, we must have $p \leq |A|$. The proof is found in [Zhu20b].

Lemma 3.3.7: *[Zhu20b, Corollary 4.3.1] Let $\mathbf{A}$ be a finite idempotent algebra and let $\mathbf{B} \in \mathrm{HSP}(\mathbf{A})$ be p-affine for some prime $p \in \mathbb{P}$. Then there exists a p-affine algebra $\mathbf{B}' \leq \mathbf{B}$ such that $\mathbf{B}' \in \mathrm{HS}(\mathbf{A})$.*

4 Loop Conditions

Our attempts of proving the existence of a k-WNU term operation in $\mathrm{Clo}(\mathbf{A})$ for any finite idempotent Taylor algebra $\mathbf{A}$ and all $k \in \mathbb{N}$ are primarily inspired by M. Sigger's proof of the existence of a 6-ary Siggers term operation that is provided in the introductory section of this thesis (Theorem 1.8). Namely, we consider the relation R generated by the columns of the k-WNU identities in the free algebra and try to argue the existence of a constant tuple in R. The following section provides a short insight into the underlying concept of *loop conditions*. It is based on [GJP19].

4.1 Definition

A *loop condition* on a finite set X is a set L of identities of the form

$$f(x_{1,1}, \dots, x_{1,n}) = f(x_{2,1}, \dots, x_{2,n}) = \dots = f(x_{k,1}, \dots, x_{k,n}), \tag{4.1}$$

where f is an n-ary function symbol, $k \geq 2$, and $x_{i,j} \in X$ for all $i \in [k]$ and $j \in [n]$. The numbers k and n are referred to as the *width* and the *arity* of the loop condition L, respectively. The k-ary *relation associated with* L is

$$R_L := \{(x_{1,i}, \dots, x_{k,i}) : 1 \leq i \leq n\} \subseteq X^k.$$

Given a relation $R = \{(r_{1,1}, \dots, r_{k,1}), \dots, (r_{1,n}, \dots, r_{k,n})\} \subseteq X^k$ for some $k \geq 2$, we assign to it the loop condition L_R defined by the identities

$$f(r_{1,1}, \dots, r_{1,n}) = f(r_{2,1}, \dots, r_{2,n}) = \dots = f(r_{k,1}, \dots, r_{k,n}).$$

Technically, these identities depend on the enumeration of the tuples in R. But since we are interested in the satisfaction of identities in clones of operations, which are closed under the permutation of variables, we may ignore this dependency and call L_R *the loop condition associated with* R.

4.2 Satisfaction of loop conditions

An algebra $\mathbf{A}$ is said to *satisfy* a loop condition L of the form (4.1) if the function symbol that appears in L can be assigned a term operation $f \in \mathrm{Clo}(\mathbf{A})$ in such a way that the resulting identity is true for all values of the variables $x_{i,j}$ in the domain of $\mathbf{A}$. A loop condition is called *trivial* if it can be satisfied by an algebra of size at least 2 whose only basic operations are projections. Note that a loop condition L is trivial if and only if the associated hypergraph $(X; R_L)$ contains a loop, i.e., if R_L contains a constant tuple. Namely, let L be a loop condition of some arity $n \in \mathbb{N}$ and width $k \in \mathbb{N}$ and assume that L is trivial.

We want to show that R_L contains a constant tuple. Let $i \in [n]$ be such that pr_i^n is the projection satisfying the loop condition. Then we have $(x_{1,i}, x_{2,i}, \ldots, x_{k,i}) \in R_L$ by definition of R_L. As the identities $\mathrm{pr}_i(x_{1,1}, \ldots, x_{1,n}) = \mathrm{pr}_i(x_{2,1}, \ldots, x_{2,n}) = \cdots = \mathrm{pr}_i(x_{k,1}, \ldots, x_{k,n})$ are satisfied by an algebra of size at least two, we must have $x_{1,i} = x_{2,i} = \cdots = x_{k,i}$. Thus, the hypergraph $(X; R_L)$ contains a loop. The converse is clear.

Furthermore, the following lemma states a connection between the satisfaction of loop conditions and loops in an associated hypergraph and thus justifies the terminology. It represents a generalisation of the methods used in the proof of Theorem 1.8 in the introduction.

Lemma 4.2.1: *Let L be a loop condition of width $k \geq 2$ and arity n on a finite set X. Let $R_L \subseteq X^k$ be the relation associated with L, and let $\mathbf{A}$ be an algebra. Then the following are equivalent:*

(i) L is satisfied by $\mathbf{A}$.

(ii) The relation

$$\langle R_L \rangle_{\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X)}$$

generated by R_L contains a constant tuple.

Proof. Let $R := \langle R_L \rangle_{\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X)}$ and assume that L is of the form

$$f(x_{1,1}, \ldots, x_{1,n}) = f(x_{2,1}, \ldots, x_{2,n}) = \cdots = f(x_{k,1}, \ldots, x_{k,n}).$$

First, we prove the implication $(i) \Rightarrow (ii)$. Assume that $\mathbf{A}$ satisfies L and let $f \in \mathrm{Clo}(\mathbf{A})$ witness this fact. R_L consists of all tuples $x_i := (x_{1,i}, \ldots, x_{k,i})$ for $i \in [n]$. As, by definition, $R \in \mathrm{Inv}(\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X))$, we have $f(x_1, \ldots, x_n) \in R$. Since f witnesses L, $f(x_1, \ldots, x_n)$ is a constant tuple.

For the implication $(ii) \Rightarrow (i)$, assume that $(c, \ldots, c) \in R$ is a constant tuple in R. Since R is generated by R_L, there exists an n-ary term operation $f \in \mathrm{Clo}(\mathbf{A})$ such that

$$\begin{pmatrix} c \\ \vdots \\ c \end{pmatrix} = f(x_1, \ldots, x_n) = \begin{pmatrix} f(x_{1,1} \ldots, x_{1,n}) \\ \vdots \\ f(x_{k,1} \ldots, x_{k,n}) \end{pmatrix}.$$

This means that the identities are satisfied on the generators $x_{i,j},\ i \in [k],\ j \in [n]$. Since $\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X)$ is the free algebra with these generators, the loop condition is also satisfied by $\mathbf{A}$. □

Consequently, we obtain the following result for the satisfaction of loop conditions whose associated relational structures are homomorphic.

Corollary 4.2.2: *Let L and L' be loop conditions of the same width $k \geq 2$ on a finite set X. If there exists a homomorphism $(X; R_L) \to (X; R_{L'})$, then any algebra that satisfies L also satisfies L'.*

Proof. Let $\mathbf{A}$ be an algebra that satisfies L. By Lemma 4.2.1, it suffices to show that the relation $R' := \langle R_{L'} \rangle_{\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X)}$ generated by $R_{L'}$ contains a constant tuple. Let $R := \langle R_L \rangle_{\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X)}$ denote the relation generated by R_L. Since $\mathbf{A}$ satisfies L, there exists a constant tuple $(c, \ldots, c) \in R$ by Lemma 4.2.1. Let $\xi : X \to X$ be the homomorphism that sends tuples from R_L to tuples from $R_{L'}$. We can extend this mapping to a homomorphism $\bar{\xi}$ on $\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X)$ by freeness. Inevitably, $\bar{\xi}$ sends tuples from R to tuples from R'. It follows that R' contains the constant tuple $(\bar{\xi}(c), \ldots, \bar{\xi}(c))$. □

4.3 Application to k-WNU

In the course of this thesis, we will consider the loop condition L on the variable set $X = \{x, y\}$ arising from the defining identities satisfied by a k-WNU term operation of arity n. Recall that an operation w of some arity $n \in \mathbb{N}$ is a k-WNU term operation if for every n-tuple $\mathbf{z} \in \{x, y\}^n$ that has exactly k many instances of y, w satisfies the identity

$$w(\underbrace{y, \ldots, y, y}_{k}, \underbrace{x, x, \ldots, x}_{n-k}) = w(\mathbf{z}).$$

Note that k-WNU terms operations are Taylor terms. In particular, the k-WNU identities fail to be satisfied by projections.

As there exist $\binom{n}{k}$ different tuples of size n that have exactly k instances of y and $n - k$ instances of x, L has arity n and width $\binom{n}{k}$. Written out explicitly, the n-ary k-WNU identities are of the form

$$\begin{aligned} &w(\overbrace{y, \ldots, y, y}^{k}, x, x, \ldots, x) = \\ =&w(y, \ldots, y, x, y, x, \ldots, x) = \\ &\vdots \\ =&w(\underbrace{x, \ldots, x, x, x, y, \ldots, y}_{n}). \end{aligned}$$

The $\binom{n}{k}$-ary relation R_L associated with this loop condition is given by the n columns of the identities above. Clearly, these columns depend on the order of the identities. However, for any order of the identities, R_L turns out to be (n, k)-symmetric in the sense of Section 2.8. In other words, a permutation of the components of the n-tuple

$$(\underbrace{y, \ldots, y}_{k}, \underbrace{x, \ldots, x}_{n-k})$$

induces a permutation of the components of elements of R_L, and R_L is invariant under all permutations that arise in this way.

In light of Lemma 4.2.1, we wish to find a constant tuple in $R := \langle R_L \rangle_{\mathcal{F}_{\mathcal{V}(\mathbf{A})}(X)}$, where $\mathbf{A}$ is finite, idempotent, and Taylor, and R_L is as above. The idea is to go by induction on the size of the algebra $\mathbf{A}$, using Zhuk's result Corollary 3.1.3. In the strong subalgebra case, we wish to restrict R to a smaller relation whose domain is a proper subset of the domain of R and to find a constant tuple there. However, a restriction of R will not be of the same form as R, i.e., generated by the columns of the identities of L. Therefore, we prove a stronger statement by induction. Namely, we prove that in the strong subalgebra case, any non-empty, invariant, and (n, k)-symmetric relation contains a constant tuple (Lemma 5.3). R inherits its (n, k)-symmetry from R_L.

As opposed to the property of R to be generated by a certain set of generators, the property of R to be (n, k)-symmetric is preserved by the restriction of R to a proper substructure. To see this, let $B \subsetneq A$ be a proper subset of A such that $R \cap B^{\binom{n}{k}} \neq \emptyset$. Take $\mathbf{r} \in R \cap B^{\binom{n}{k}}$ and $\pi \in S_n$. By (n, k)-symmetry of R, we have $\pi(\mathbf{r}) \in R$. But since the tuple $\pi(\mathbf{r})$ is given by an appropriate reordering of the components of $\mathbf{r}$ that belong to B by assumption, it follows that $\pi(\mathbf{r}) \in R \cap B^{\binom{n}{k}}$, and the restriction $R \cap B^{\binom{n}{k}}$ is (n, k)-symmetric.

Thus, in the strong subalgebra case, the only challenge to inductively restrict the finding of a constant tuple in R to a smaller relation remains to show non-emptiness of an appropriate restriction of R (Lemma 5.2). The p-affine case of Zhuk's Corollary 3.1.3 requires different methods. The Lemmata 5.1.1, 5.2.1, and 5.2.2 solve the p-affine case for the values $k = 1$ and $k = 2$. Lemma 5.3.1 and Conjecture 5.4.1 try to generalise the statements to arbitrary $k \geq 3$, however, definite results remain to be achieved.

5 Constant Tuples

This section examines conditions under which an (n,k)-symmetric invariant relation defined on the domain of a finite idempotent Taylor algebra contains a constant tuple. We will start by examining such relations for $k = 1, 2, 3$, and apply the same ideas for arbitrary $k \in \mathbb{N}$. Corollary 3.1.3 (Zhuk's cases) splits the process into the consideration of algebras that admit strong subuniverses and those which factor to a p-affine algebra for prime numbers $p \in \mathbb{P}$.

When employed for the purpose of finding a constant tuple in some relation R, the strength of strong subuniverses lies in the fact that they can be used to inductively restrict the search to a smaller relation that is contained in R and whose domain is a proper subset of the domain of R. We show that if an algebra $\mathbf{A}$ is such that any nontrivial subalgebra has a nontrivial strong subuniverse, then the property of an invariant relation $R \in \mathrm{Inv}(\mathbf{A})$ to be (n,k)-symmetric is sufficient to ensure a constant tuple in R (Lemma 5.3). This is done by reducing the statement to the existence of a constant tuple in a proper subalgebra. To this end, we collect connections between the existence of nontrivial strong subuniverses and (n,k)-symmetric relations.

Let A be a set and let $B \subseteq A$. A relation $R \subseteq A^n$ of some arity $n \in \mathbb{N}$ is called B-*essential* if it holds that $R \cap B^n = \emptyset$ and $R \cap (B^{i-1} \times A \times B^{n-i}) \neq \emptyset$ for all $i \in [n]$.

First, we state some algebraic properties of subdirect subpowers of an algebra admitting a strong subalgebra. For the proof we refer to [Zhu20b].

Theorem 5.1: *[Zhu20b, follows from Theorem 3.5] Let* $\mathbf{A}$ *be a finite idempotent algebra,* $\mathbf{R} \leq_{\mathrm{sd}} \mathbf{A}^n$ *a subdirect subalgebra for some* $n \geq 2$*, and* $\mathbf{B} \leq \mathbf{A}$ *a strong subuniverse of* $\mathbf{A}$*.*

(i) If $\mathbf{A}$ *has no nontrivial central subuniverses or* $\mathbf{B}$ *is central, then for all* $j \in [n]$ *the set*

$$\mathrm{pr}_1(R \cap (A \times \cdots \times A \times B \times A \times \cdots \times A)),$$

where B *occurs at the* j*-th position, is a strong subuniverse of* $\mathbf{A}$*.*

(ii) If R *is* B*-essential, then* $n = 2$*.*

Following [Zhu20b], we show how to use Theorem 5.1 in order to prove that the existence of a nontrivial strong subuniverse allows us to restrict our search of a constant tuple to a proper subalgebra. Recall that a relation R defined on the domain of an algebra $\mathbf{A}$ is said to satisfy the *restriction-property* if there exists a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ such that $B^n \cap R \neq \emptyset$.

Lemma 5.2: *[Zhu20b] Let* $\mathbf{A}$ *be a finite idempotent algebra and let* $R \subseteq A^{\binom{n}{k}}$ *be a nonempty, invariant, and* (n,k)*-symmetric relation for some* $n > 2$, $k < n$. *If* $\mathbf{A}$ *admits a nontrivial strong subuniverse, then* R *satisfies the restriction-property.*

Proof. We have to show that there exists a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ such that $B^{\binom{n}{k}} \cap R \neq \emptyset$. If R is not subdirect, then R satisfies the restriction-property by Lemma 2.8.3. On the other hand, assume that R is a subdirect relation, i.e., we have $\mathbf{R} \leq_{\text{sd}} \mathbf{A}^{\binom{n}{k}}$. Let $\mathbf{B} \lneq \mathbf{A}$ be a nontrivial subuniverse of $\mathbf{A}$. If $R \cap B^{\binom{n}{k}} \neq \emptyset$, then we have found a witness that R does satisfies the restriction-property. Otherwise, let us agree on an order of the elements of $[\binom{n}{k}]$ so that we can write

$$\mathbf{r} = (r_{E_1}, \ldots, r_{E_{\binom{n}{k}}})$$

for all elements $\mathbf{r}$ of R. Note that since $n > 2$ and $k < n$, it holds that $\binom{n}{k} \geq 3$. Take $j \in [\binom{n}{k}]$ such that $|E_1 \setminus E_j| = 1$. We let

$$C := \text{pr}_1(R \cap (A \times \cdots \times A \times B \times A \cdots \times A)),$$

where B is on the j-th place. Observe that if there exists a nontrival central subuniverse $\mathbf{S} \lneq \mathbf{A}$, then we can simply replace $\mathbf{B}$ by $\mathbf{S}$. In any case, the requirements of Theorem 5.1 *(i)* are fulfilled and we obtain that C is a strong subuniverse of $\mathbf{A}$. We claim that C is nontrivial and that $C \cap R^{\binom{n}{k}} \neq \emptyset$. Thus, C witnesses the fact that R does satisfies the restriction-property.

First, we must have $C \neq A$. Otherwise, $C = A$ implies that $\text{pr}_{\{1,j\}}(R) \cap B^2 \neq \emptyset$, where for any subset $M \subseteq [\binom{n}{k}]$ we denote by $\text{pr}_M(R)$ the $|M|$-ary relation consisting of all $|M|$-tuples obtained by projecting tuples from R onto their components that belong to M. Since $R \cap B^{\binom{n}{k}} = \emptyset$, we can choose a subset $\{1, j\} \subseteq M \subseteq [\binom{n}{k}]$ that is minimal with respect to the partial order $\subseteq$ on the power set of $[\binom{n}{k}]$, and with the property that $\text{pr}_M(R) \cap B^{|M|} = \emptyset$. But then, $\text{pr}_M(R)$ is a B-essential relation of arity $|M| > 2$ in contradiction to item *(ii)* of Theorem 5.1. Therefore, C is a nontrivial strong subuniverse.

Now, suppose towards a contradiction that $C \cap R^{\binom{n}{k}} = \emptyset$ and take $c \in C$. By definition, there exists a tuple $\mathbf{r} = (r_{E_1}, \ldots, r_{E_{\binom{n}{k}}}) \in R$ such that $r_{E_1} = c$ and $r_{E_j} \in B$. We claim that there exists $M \subseteq [\binom{n}{k}]$ of size $|M| > 2$ such that $\text{pr}_M(R)$ is a C-essential relation. Then, this yields a contradiction to item *(ii)* of Theorem 5.1. In order to prove the claim, observe that since $|E_1 \setminus E_j| = 1$ and $\binom{n}{k} \geq 3$, there exists a k-element subset $F \in \binom{n}{k}$ such that the following conditions are satisfied:

$$|F \setminus E_j| = |F \setminus E_1| = 1$$
$$|F \cap E_j| = |F \cap E_1| = k - 1$$
$$F \neq E_1, E_j.$$

These condition guarantee that there exist permutations $\pi, \tau \in S_n$ such that

$$\text{pr}_1(\pi(\mathbf{r})) = r_F,$$

$$\mathrm{pr}_j(\pi(\mathbf{r})) = r_{E_j},$$
$$\mathrm{pr}_1(\tau(\mathbf{r})) = r_{E_1}, \text{ and}$$
$$\mathrm{pr}_j(\tau(\mathbf{r})) = r_F.$$

Namely, if $k < n-1$, we set

$$F := (E_1 \cap E_j) \cup \{i\},$$

where $i \notin E_1 \cup E_j$. Then π is the transposition that swaps i with the element of $E_1 \setminus E_j$, and τ is the transposition that swaps i with the element of $E_j \setminus E_1$. Otherwise, if $k = n-1$, we set

$$F := (E_1 \setminus E_j) \cup (E_j \setminus E_1) \cup ((E_1 \cap E_j) \setminus \{i\}),$$

where $i \in E_1 \cap E_j$. Then π is the transposition that swaps the element of $E_j \setminus E_1$ with the element of $E_1 \setminus F$, and τ is the transposition that swaps the element of $E_1 \setminus E_j$ with the element of $E_j \setminus F$. By (n,k)-symmetry of R, $\mathrm{pr}_1(\pi(\mathbf{r})) = r_F$ and $\mathrm{pr}_j(\pi(\mathbf{r})) = r_{E_j} \in B$ yield that $r_F \in C$. Moreover, as $\mathrm{pr}_1(\tau(\mathbf{r})) = r_{E_1} \in C$ and $\mathrm{pr}_j(\tau(\mathbf{r})) = r_F \in C$, by (n,k)-symmetry of R we get $\mathrm{pr}_{\{1,j\}}(R) \cap C^2 \neq \emptyset$. Since we have assumed that $C \cap R^{\binom{n}{k}} = \emptyset$, there exists a subset $\{1,j\} \subseteq M \subseteq [\binom{n}{k}]$ that is minimal with respect to the partial order $\subseteq$ on the power set of $[\binom{n}{k}]$, and with the property that $\mathrm{pr}_M(R) \cap B^{|M|} = \emptyset$. But then $\mathrm{pr}_M(R)$ is C-essential with arity $|M| > 2$, again contradicting item *(ii)* of Theorem 5.1. Therefore, we must have $C \cap R^{\binom{n}{k}} \neq \emptyset$. Hence, R satisfies the restriction-property.

□

It now follows that if an algebra is such that every nontrivial subalgebra admits a nontrivial strong subuniverse, then the existence of a constant tuple is guaranteed for any (n,k)-symmetric invariant relation.

Lemma 5.3: *[Zhu20b] Let* $\mathbf{A}$ *be a finite idempotent algebra with the property that any subalgebra* $\mathbf{B} \leq \mathbf{A}$ *with* $|B| \geq 2$ *has a nontrivial strong subuniverse. Assume that* $R \subseteq A^{\binom{n}{k}}$ *for some* $n > 2$, $k \leq n$ *is a nonempty, invariant, and* (n,k)*-symmetric relation. Then* R *contains a constant tuple.*

Proof. If $n = k$, the statement is trivial, so we can suppose that $k < n$. We go by induction on $|A|$. The base case $|A| = 1$ is clear. Assume the statement holds for all algebras of size less than $|A|$. Since by assumption, $\mathbf{A}$ admits a nontrivial strong subuniverse, we have that R does satisfies the restriction-property by Lemma 5.2. Thus, there exists a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ with $R \cap B^{\binom{n}{k}} \neq \emptyset$. Now, $R \cap B^{\binom{n}{k}}$ is a nonempty, invariant, and (n,k)-symmetric relation on $\mathbf{B}$. If $|B| = 1$, we have found a constant tuple. Otherwise, since a subalgebra of a subalgebra is a subalgebra, $\mathbf{B}$ satisfies the assumptions of the inductive hypothesis, and $R \cap B^n$ contains a constant tuple. In particular, R contains a constant tuple. □

Recall that a tuple (n,k,p) where $n,k \in \mathbb{N}$ and $p \in \mathbb{P}$ is called *loop-friendly* if every nonempty (n,k)-symmetric p-affine relation contains a constant tuple. If (n,k,p) is a loop-friendly tuple, then the existence of a p-affine factor also allows us to restrict the search of a constant tuple to a smaller relation.

Lemma 5.4: *Let $n, k \in \mathbb{N}$ and $p \in \mathbb{P}$ be such that the tuple (n, k, p) is loop-friendly. Assume that $\mathbf{A}$ is an algebra and that θ is a congruence on $\mathbf{A}$ such that $\mathbf{A}/\theta$ is p-affine. Then every nonempty, (n, k)-symmetric, and invariant relation on A satisfies the restriction-property.*

Proof. Let $R \in \mathrm{Inv}(\mathbf{A})$ be a nonempty and (n, k)-symmetric relation. We want to construct a proper subuniverse B of $\mathbf{A}$ such that $B^{\binom{n}{k}} \cap R \neq \emptyset$. Let

$$R^\theta := \left\{([a_1]_\theta, \dots, [a_{\binom{n}{k}}]_\theta) : (a_1, \dots, a_{\binom{n}{k}}) \in R\right\}$$

be the relation on A/θ consisting of the equivalence classes of elements of R under the congruence θ. Then R^θ is a nonempty, (n, k)-symmetric, and invariant relation on the domain of the p-affine algebra $\mathbf{A}/\theta$. By loop-friendliness of the tuple (n, k, p), R^θ contains a constant tuple $([a]_\theta, \dots, [a]_\theta)$. In other words, R has a tuple all of whose components belong to the same block under θ. By Lemma 2.3.1, the block $[a]_\theta$ is a subuniverse of $\mathbf{A}$. Since $A/\theta = \mathbb{Z}_p \times \dots \times \mathbb{Z}_p$, we have $|A/\theta| \geq 2$. Thus, $\theta \neq A^2$ and $[a]_\theta \lneq \mathbf{A}$, i.e., $[a]_\theta$ is a proper subuniverse of $\mathbf{A}$ that satisfies $[a]_\theta^{\binom{n}{k}} \cap R \neq \emptyset$. Hence, R satisfies the restriction-property. □

5.1 The symmetric case

Any symmetric relation $R \in \mathrm{Inv}(\mathbf{A})$ on a finite idempotent Taylor algebra has a constant tuple if only its arity satisfies some number theoretic conditions. In case $\mathbf{A}$ is a p-affine algebra, its existence can be shown constructively as follows.

Lemma 5.1.1: *[Zhu20b] Let $p \in \mathbb{P}$ and let $n \in \mathbb{N}$. Then the tuple $(n, 1, p)$ is loop-friendly if and only if $p \nmid n$.*

Proof. The necessity of the condition $p \nmid n$ has been shown in Counterexample 3.3.6. For the converse, let $\mathbf{A}$ be a p-affine algebra and let $R \subseteq A^n$ be a nonempty, p-affine, and symmetric relation. We will find a term operation $f \in \mathrm{Clo}(\mathbf{A})$ such that f applied to any tuple $\mathbf{r} \in R$ and appropriate permutations of $\mathbf{r}$, which are elements of R by symmetry, yields a constant tuple. To this end, by Lemma 3.3.4 it is sufficient to provide a linear combination $f(x_1, \dots, x_m) := \sum_{i=1}^m \lambda_i x_i$ with $\sum_{i=1}^m \lambda_i = 1$ for some $m \in \mathbb{N}$ and with the mentioned property.

Consider the action $S_n \curvearrowright R$ and take the cyclic permutation σ on $[n]$ given by

$$\sigma : 1 \mapsto 2 \mapsto \dots \mapsto n \mapsto 1$$

Then

$$\sigma^0(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}) + \sigma^1(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}) + \dots + \sigma^{n-1}(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}) = \begin{pmatrix} a_1 + a_2 + \dots + a_n \\ \vdots \\ a_1 + a_2 + \dots + a_n \end{pmatrix}.$$

Since $p \nmid n$ by assumption, the sum n of the coefficients fulfills $n \neq 0$. Using that R is p-affine we get that

$$\sum_{i=0}^{n-1} \frac{1}{n} \sigma^i(\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}) = \frac{1}{n} \begin{pmatrix} a_1 + a_2 + \cdots + a_n \\ \vdots \\ a_1 + a_2 + \cdots + a_n \end{pmatrix} \in R.$$

Hence, we have found a constant tuple in R. Therefore, the tuple $(n, 1, p)$ is loop-friendly. □

Putting together Lemma 5.1.1 and the results we have obtained in the beginning of Section 5, we are now able to show the existence of a constant tuple in any symmetric relation of an appropriate arity as in [Zhu20b].

Theorem 5.1.2: *Let* $\mathbf{A}$ *be a finite idempotent Taylor algebra and let* $n \in \mathbb{N}$ *be such that* $p \nmid n$ *for all* $p \in \mathbb{P}$ *with* $p \leq |A|$. *If* $R \subseteq A^n$ *is a nonempty, invariant, and symmetric relation, then* R *contains a constant tuple.*

Proof. If $n = 1$, the statement is trivial. Since $2 \nmid n$, we can therefore assume that $n > 2$. We prove by induction on $|A|$. The base case $|A| = 1$ is clear. Assume that the statement holds for all algebras of size less than $|A|$. We claim that there exists a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ with $R \cap B^n \neq \emptyset$, i.e., we claim that R satisfies the restriction-property. Then, by the inductive hypothesis, there exists a constant tuple in $R \cap B^n$. In particular, R has a constant tuple.

By Corollary 3.1.3, either $\mathbf{A}$ has a nontrivial strong subuniverse, or $\mathbf{A}$ factors to a p-affine algebra for some prime $p \in \mathbb{P}$. If $\mathbf{A}$ has a nontrivial strong subuniverse, then R satisfies the restriction-property by Lemma 5.2. If on the other hand, there exists a congruence θ on $\mathbf{A}$ such that $\mathbf{A}/\theta$ is p-affine, we must have $p \leq |A|$ and thus $p \nmid n$ by assumption. By Lemma 5.1.1, the tuple $(n, 1, p)$ is loop-friendly, and by Lemma 5.4, R satisfies the restriction-property. Therefore, in both cases of Corollary 3.1.3, there exists a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ with $R \cap B^n \neq \emptyset$ and we can apply the inductive hypothesis to obtain a constant tuple in R.

□

Remark 5.1.3: Observe that actually, the number theoretic conditions on the arity n are slightly stronger than necessary. It suffices to ask that n is not divisible by $p \in \mathbb{P}$ if there exists a p-affine algebra in $\mathrm{HS}(\mathbf{A})$.

5.2 The $(n, 2)$-symmetric case

When trying to generalise the statement of Theorem 5.1.2 to $(n, 2)$-symmetric relations, difficulties arise in the p-affine case. Namely, the construction of a constant tuple in an $(n, 2)$-symmetric invariant relation on a p-affine algebra now substantially depends on the value of p. The proof for the case $p = 2$ was given by Z. Brady in a personal correspondence with L. Barto, M. Pinsker, and D. Zhuk. The proof for $p \neq 2$ goes back to M. Pinsker and D. Zhuk.

Lemma 5.2.1: *Let $p \in \mathbb{P} \setminus \{2\}$ and let $n \in \mathbb{N}$ be such that $p \nmid n\binom{n}{2}$. Then the tuple $(n, 2, p)$ is loop-friendly.*

Proof. Let $\mathbf{A}$ be a p-affine algebra and assume that $R \subseteq A^{\binom{n}{2}}$ is a nonempty, p-affine, and $(n,2)$-symmetric relation. We have to show that R contains a constant tuple. For a two-element subset $G = \{i, j\} \in \binom{n}{2}$ we write $t_G \in S_n$ for the transposition that swaps i and j. The idea of the proof is to find a p-affine combination of transpositions t_G that yields a constant tuple when applied to any element $\mathbf{r} \in R$. To this end, we consider the following operators:

$$\begin{aligned} T_0(\mathbf{r}) &:= \mathbf{r} \\ T_1(\mathbf{r}) &:= \sum_{G \in \binom{n}{2}} t_G(\mathbf{r}) \\ T_2(\mathbf{r}) &:= \sum_{\substack{G_1, G_2 \in \binom{n}{2}, \\ G_1 \cap G_2 = \emptyset}} t_{G_1} \circ t_{G_2}(\mathbf{r}). \end{aligned}$$

Note that the number of addends in the operators T_0, T_1, and T_2 are given by 1, $\binom{n}{2}$, and $\binom{n}{2}\binom{n-2}{2}$, respectively. We claim that there exist elements $c_0, c_1 \in \mathbb{Z}_p$ such that for any $\mathbf{r} \in A^{\binom{n}{2}}$,

$$T_2(\mathbf{r}) + c_1 T_1(\mathbf{r}) + c_0 T_0(\mathbf{r}) \tag{5.1}$$

gives a constant tuple. Furthermore, we claim that c_0 and c_1 can be chosen such that the operation $A^3 \to A$ given by

$$(x_0, x_1, x_2) \mapsto \binom{n-2}{2}\binom{n-4}{2} x_2 + \binom{n}{2} c_1 x_1 + c_0 x_0 \tag{5.2}$$

is p-affine. By Lemma 3.3.4, proving this claim suffices to show the loop-friendliness of the tuple $(n, 2, p)$. Namely, by $(n,2)$-symmetry of R, all addends of the tuple (5.1) belong to R, and as the coefficients used add up to 1, the tuple belongs to R by p-affinity of R and Lemma 3.3.4.

In order to prove the claim, given $E \in \binom{n}{2}$ we want to determine the sizes of the sets

$$\begin{aligned} T_{0,F}(E) &:= \begin{cases} \{E\}, & E = F \\ \emptyset, & \text{else} \end{cases} \\ T_{1,F}(E) &:= \left\{ G \in \binom{n}{2} : t_G(E) = F \right\} \\ T_{2,F}(E) &:= \left\{ (G_1, G_2) \in \binom{n}{2} \times \binom{n}{2} : t_{G_1} \circ t_{G_2}(E) = F,\ G_1 \cap G_2 = \emptyset \right\} \end{aligned}$$

for any $F \in \binom{n}{2}$. The sizes of these sets express the number of addends used in the definition of the operators T_0, T_1, and T_2 that move the set E to the set F. Consequently, as the components of any $\mathbf{r} = (r_E)_{E \in \binom{n}{2}} \in R$ are indexed by two-element subsets of $[n]$, they determine how often the F-th component r_F of $\mathbf{r}$ appears in the E-th component of the tuple $T_2(\mathbf{r}) + c_1 T_1(\mathbf{r}) + c_0 T_0(\mathbf{r})$. Our goal now is to find constants c, c_0 and c_1 independent of E and F such that

$$|T_{2,F}(E)| + c_1|T_{1,F}(E)| + c_0|T_{0,F}(E)| = c. \tag{5.3}$$

It then follows that for every $\mathbf{r} = (r_E)_{E\in\binom{n}{2}} \in R$, every component r_E of $\mathbf{r}$ appears exactly c times in every component of $T_2(\mathbf{r}) + c_1 T_1(\mathbf{r}) + c_0 T_0(\mathbf{r})$. In other words, expression (5.1) yields the constant tuple

$$T_2(\mathbf{r}) + c_1 T_1(\mathbf{r}) + c_0 T_0(\mathbf{r}) = c \begin{pmatrix} \sum_{E\in\binom{n}{2}} r_E \\ \vdots \\ \sum_{E\in\binom{n}{2}} r_E \end{pmatrix}.$$

In relation to any $E \in \binom{n}{2}$ there are essentially three types of edges in the complete graph with vertices $\{1, \ldots, n\}$ that are of interest: There is E itself, edges $F \in \binom{n}{2}$ such that $|E \cap F| = 1$, and edges $F \in \binom{n}{2}$ such that $E \cap F = \emptyset$. The size of $T_{i,F}(E)$ depends on the type of F in relation to E. The following table collects all values of $|T_{i,F}(E)|$ for $i = 0, 1, 2$.

	$\|T_{0,F}(E)\|$	$\|T_{1,F}(E)\|$	$\|T_{2,F}(E)\|$
$E = F$	1	$\binom{n-2}{2} + 1$	$\binom{n-2}{2}\binom{n-4}{2} + 2\binom{n-2}{2}$
$\|E \cap F\| = 1$	0	1	$2\binom{n-3}{2}$
$E \cap F = \emptyset$	0	0	4

The conditions on the constants c, c_0, and c_1 in equation (5.3) now give a system of linear equations:

$$\begin{cases} 4 & = c \\ 2\binom{n-3}{2} + c_1 & = c \\ \binom{n-2}{2}\binom{n-4}{2} + 2\binom{n-2}{2} + c_1(\binom{n-2}{2} + 1) + c_0 & = c. \end{cases}$$

We obtain $c = 4$, $c_1 = -(n^2 - 7n + 8)$, and $c_0 = \frac{n^4 - 10n^3 + 27n^2 - 18n}{4}$. Plugging in these values, the sum of all coefficients in operation (5.2) is equal to

$$\binom{n}{2}\binom{n-2}{2} + c_1\binom{n}{2} + c_0 = 2n(n-1). \tag{5.4}$$

Since $p \neq 2$ and $p \nmid \binom{n}{2} = \frac{n(n-1)}{2}$ by assumption, we have $2n(n-1) \neq 0$. Therefore, the operation

$$(x_0, x_1, x_2) \mapsto \frac{1}{2n(n-1)}\left(\binom{n}{2}\binom{n-2}{2}x_2 + \binom{n}{2}c_1 x_1 + c_0 x_0\right)$$

is p-affine. As $R \in \mathrm{Inv}(\mathbf{A})$, R is preserved by any p-affine operation by Lemma 3.3.4. Thus, for any $\mathbf{r} \in R$ we have

$$\frac{1}{2n(n-1)}\big(T_2(\mathbf{r}) + c_1 T_1(\mathbf{r}) + c_0\mathbf{r}\big) \in R.$$

By our choice of c_0 and c_1, the expression above yields a constant tuple. Hence, we have now shown that if we take any $\mathbf{r} = (r_E)_{E\in\binom{n}{2}} \in R$, then R also contains the constant tuple

$$\frac{4}{2n(n-1)}\Big(\sum_{E\in\binom{n}{2}} r_E, \dots, \sum_{E\in\binom{n}{2}} r_E\Big).$$

Therefore, the tuple $(n, 2, p)$ is loop-friendly. □

Since the number of all permutations (5.4) used in the construction above will always be divisible by 2, we have to choose a different construction for $p = 2$.

Lemma 5.2.2: *[Bra22a] Let $n \in \mathbb{N}$ be such that $2 \nmid n\binom{n}{2}$. Then the tuple $(n, 2, 2)$ is loop-friendly.*

Proof. Let $\mathbf{A}$ be a 2-affine algebra and assume that $R \subseteq A^{\binom{n}{2}}$ is a nonempty, 2-affine, and $(n, 2)$-symmetric relation. Since $2 \nmid n\binom{n}{2}$, we must have $n \equiv 3 \mod 4$, so there exists $k \in \mathbb{N}$ such that $n = 4k+3$. To any $\mathbf{r} = (r_E)_{E\in\binom{n}{2}} \in R$ we assign the graph $\mathbb{G}_{\mathbf{r}} := ([n]; E_{\mathbf{r}})$ where $(i, j) \in E_{\mathbf{r}}$ if and only if $r_{\{i,j\}} = 1$. We will find a 2-affine term operation such that when applied to any $\mathbf{r} \in R$ and appropriate permutations of $\mathbf{r}$ it gives a constant tuple, i.e., the resulting graph is either the empty or the complete graph.

Let $\mathbf{r} \in R$. Let $\sigma \in S_n$ be the cyclic permutation given by

$$\sigma : 1 \mapsto 2 \mapsto \cdots \mapsto n \mapsto 1,$$

and set

$$S(\mathbf{r}) := \sum_{i=0}^{n-1} \sigma^i(\mathbf{r}).$$

By $(n, 2)$-symmetry and 2-affinity of R, we have $S(\mathbf{r}) \in R$. The graph $\mathbb{G}_{S(\mathbf{r})}$ contains an edge between vertices j and k if and only if the number of graphs $\mathbb{G}_{\sigma^i(\mathbf{r})}$ containing the edge is odd, where $i = 0, \dots, n-1$. Observe that, in fact, $\mathbb{G}_{S(\mathbf{r})}$ contains an edge between vertices j and k if and only if there is an odd number of edges in $\mathbb{G}_{\mathbf{r}}$ between vertices of distance $\pm(k-j) \mod n$. $S(\mathbf{r})$ is therefore determined by the tuple

$$s(\mathbf{r}) := (s_1, s_3, s_5, \dots, s_{n-2}) \in (\mathbb{Z}_2)^{2k+1},$$

where s_i counts modulo 2 the number of edges between vertices of distance i in $\mathbb{G}_{\mathbf{r}}$. Let T denote the permutation on $[n]$ given by $1 \mapsto 2 \mapsto 3 \mapsto 1$ and which fixes all other elements.

We claim that the following identity holds:

$$s(S(T(S(\mathbf{r})))) = (s_3, s_1 + s_3 + s_5, s_3 + s_5 + s_7, \dots, s_{n-6} + s_{n-4} + s_{n-2}, s_{n-2}) \tag{5.5}$$

In order to prove the claim, we will examine for every $i = 0, \dots, 2k$ how vertices of distance $2i + 1$ in $\mathbb{G}_{S(T(S(\mathbf{r})))}$ derive from edges in $\mathbb{G}_{\mathbf{r}}$.

Edges in $\mathbb{G}_{S(T(S(\mathbf{r})))}$ between vertices of distance 1 can arise from edges in $\mathbb{G}_{\mathbf{r}}$ in the following ways:

- edges in $\mathbb{G}_{\mathbf{r}}$ between distinct vertices i and j of distance 1 where $\{i, j\} \cap \{1, 2, 3\} = \emptyset$,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 1 and 3,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 3 and n,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 1 and 2, or
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 2 and 4.

The number of edges in $\mathbb{G}_{S(T(S(\mathbf{r})))}$ between vertices of distance 1 is modulo 2 therefore equal to $s_1 + s_{n-2} + s_3 + s_1 + s_{n-2} \equiv s_3 \mod 2$. A symmetric argument shows that the number of edges between vertices of distance $n - 2$ is modulo 2 equal to s_{n-4}.

Now, fix some $l \in [2k - 1]$. The number of edges in $\mathbb{G}_{S(T(S(\mathbf{r})))}$ between vertices of distance $\bar{l} := 2l + 1$ can arise from edges in $\mathbb{G}_{\mathbf{r}}$ in the following ways:

- edges in $\mathbb{G}_{\mathbf{r}}$ between distinct vertices i and j of distance $\bar{l}$ where $\{i, j\} \cap \{1, 2, 3\} = \emptyset$,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 3 and $1 + \bar{l}$,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 3 and $n - \bar{l} + 1$,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 2 and $3 + \bar{l}$,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 2 and $n - \bar{l} + 3$,
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 1 and $2 + \bar{l}$, or
- an edge in $\mathbb{G}_{\mathbf{r}}$ between the vertices 1 and $n - \bar{l} + 2$.

It is therefore modulo 2 equal to

$$s_{\bar{l}} + s_{\bar{l}-2} + s_{\bar{l}+2} + s_{n-(\bar{l}+1)} + s_{n-\bar{l}+1} + s_{n-(\bar{l}+1)} + s_{n-\bar{l}+1} \equiv s_{\bar{l}} + s_{\bar{l}-2} + s_{\bar{l}+2} \mod 2.$$

Hence, $S(T(S(\mathbf{r})))$ is indeed determined by the tuple claimed in (5.5). We now inductively define operators A_i for $i \in [k]$ by

$$A_1(\mathbf{r}) := S(T(S(\mathbf{r})))$$
$$A_{i+1}(\mathbf{r}) := A_1(A_i(\mathbf{r})) + A_i(\mathbf{r}) + A_{i-1}(\mathbf{r}) \mod 2$$

Plugging in identity (5.5), it follows that

$$\begin{aligned} s(A_i(\mathbf{r})) =&(s_{2i+1}, s_{2i-1}+s_{2i+1}+s_{2i+3}, s_{2i-3}+s_{2i-1}+s_{2i+1}+s_{2i+3}+s_{2i+5}, \dots, \\ &s_1+s_3+\dots+s_{4i-1}+s_{4i+1}, s_3+s_5+\dots+s_{4i+1}+s_{4i+3}, \dots, \\ &s_{n-4i-2}+s_{n-4i}+\dots+s_{n-4}+s_{n-2}, \dots, s_{n-2i-4}+s_{n-2i-2}+s_{n-2i}, s_{n-2i-2}) \end{aligned}$$

For $i = k$ we have that the tuple $s(A_k(\mathbf{r})) := (a_1, a_2, \dots, a_{k+1}, \dots, a_{2k}, a_{2k+1})$ fulfills $a_l = a_{2k+2-l}$ for all $l \in [k]$, i.e., $s(A_k(\mathbf{r}))$ is left-right symmetric. Applying A_k once more we obtain the constant tuple $s(A_k(A_k(\mathbf{r}))) = (a_{k+1}, a_k+a_{k+1}+a_{k+2}, \dots, a_k+a_{k+1}+a_{k+2}, a_{k+1})$. Hence, $\mathbb{G}_{A_k(A_k(\mathbf{r}))}$ is either the empty or the full graph. Since the sum of all coefficients appearing in $A_k \circ A_k$ is odd, by $(n, 2)$-symmetry of R and Lemma 3.3.4 it holds that the constant tuple $A_k(A_k(\mathbf{r}))$ is indeed an element of R for any $\mathbf{r} \in R$. □

The condition that $2 \nmid n\binom{n}{2}$ in Theorem 5.2.5 is not only sufficient but also necessary to guarantee that the tuple $(n, 2, 2)$ is loop-friendly. The necessity of the condition $2 \nmid \binom{n}{2}$ has been shown in Counterexample 3.3.6. This counterexample was found by L. Barto.

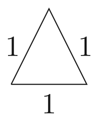
Counterexample 5.2.3: Assume that $2|n$. Take any subset $P \subseteq [n]$ such that $|P| = \frac{n}{2}$, and let $P^C := [n] \setminus P$. By $\binom{P}{2}$ and $\binom{P^C}{2}$ we denote the set of all 2-element subsets of P and P^C, respectively. Consider the relation R_P defined by

$$R_P := \left\{ \mathbf{x} \in (\mathbb{Z}_2)^{\binom{n}{2}} : \sum_{E \in \binom{P}{2}} x_E + \sum_{F \in \binom{P^C}{2}} x_F = 1 \right\},$$

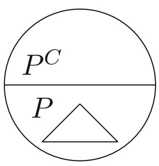
and let

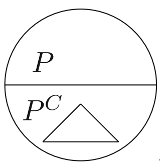
$$R := \bigcap_{P \subseteq [n],\ |P| = \frac{n}{2}} R_P.$$

Then R is invariant under the action of S_n. Being a solution set of a system of linear equations, R is 2-affine. In order to see that R is also nonempty, consider a tuple $\mathbf{x} \in (\mathbb{Z}_2)^{\binom{n}{2}}$ induced by a triangle



i.e., in the notation of the proof of Lemma 5.2.2, the graph $\mathbb{G}_\mathbf{x}$ assigned to $\mathbf{x}$ has exactly 3 edges and these edges form a triangle. We claim that $\mathbf{x}$ belongs to any R_P. Let $P \subseteq [n]$ with $|P| = \frac{n}{2}$. If all 3 vertices with non-zero adjacent edges of $\mathbb{G}_\mathbf{x}$ lie within P or if all 3 vertices with non-zero adjacent edges of $\mathbb{G}_\mathbf{x}$ lie within P^C, i.e.,



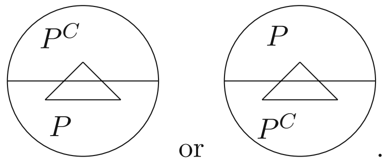
or


,

then

$$\sum_{E\in\binom{P}{2}} x_E + \sum_{F\in\binom{P^C}{2}} x_F = 1+1+1 = 1$$

and thus $\mathbf{x} \in R_P$. Otherwise, either exactly 1 edge lies within P and no edge lies within P^C, or exactly 1 edge lies within P^C and no edge lies within P:



In these cases, we also have

$$\sum_{E\in\binom{P}{2}} x_E + \sum_{F\in\binom{P^C}{2}} x_F = 1.$$

Therefore, in all cases $\mathbf{x} \in R_P$. However, R cannot contain a constant tuple as for all $c \in \mathbb{Z}_2$ and $P \subseteq [n]$ with $|P| = \frac{n}{2}$ we have $\sum_{E\in\binom{P}{2}} c + \sum_{F\in\binom{P^C}{2}} c = 0$

We have now shown the following characterisation of loop-friendly tuples.

Corollary 5.2.4: *Let $n \in \mathbb{N}$. Then the tuple $(n, 2, 2)$ is loop-friendly if and only if $2 \nmid n\binom{n}{2}$.*

A combination of the Lemmata 5.2, 5.4, 5.2.1, and 5.2.2 now gives a constant tuple in any appropriate $(n, 2)$-symmetric relation.

Theorem 5.2.5: *Let $\mathbf{A}$ be a finite idempotent Taylor algebra and let $n \in \mathbb{N}$ be such that $p \nmid n\binom{n}{2}$ for all $p \leq |A|$ with $p \in \mathbb{P}$. If $R \subseteq A^{\binom{n}{2}}$ is a nonempty, invariant, and $(n, 2)$-symmetric relation, then R contains a constant tuple.*

Proof. We proceed by induction on $|A|$. The base case $|A| = 1$ is clear. We can assume that $n > 2$. We apply Corollary 3.1.3 and show that in both cases, $\mathbf{A}$ admits a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ such that $R \cap B^{\binom{n}{2}} \neq \emptyset$, i.e., R satisfies the restriction-property. It then follows by the inductive hypothesis that there exists a constant tuple in $R \cap B^{\binom{n}{2}}$. Hence, R contains a constant tuple. If $\mathbf{A}$ admits a strong subuniverse, then R satisfies the restriction-property by Lemma 5.2. Otherwise, there exists a congruence θ on $\mathbf{A}$ such that $\mathbf{A}/\theta$ is p-affine for some $p \in \mathbb{P}$. In particular, we have $p \leq |A|$. Lemma 5.2.1 and Lemma 5.2.2 now yield the loop-friendliness of the tuple $(n, 2, p)$. By Lemma 5.4, R satisfies the restriction-property. □

A way to obtain appropriate $n \in \mathbb{N}$ that satisfy the number theoretic conditions of Theorem 5.2.5 is given in Lemma 8.1 in Appendix A.

5.3 The $(n,3)$-symmetric case

In order to motivate a proof of the existence of constant tuples in (n,k)-symmetric relations for arbitrary $k \in \mathbb{N}$, we constructively show the loop-friendliness of tuples $(n,3,p)$ for $p \neq 2,3$ and appropiate $n \in \mathbb{N}$. To this end, we generalise the ideas by D. Zhuk and M. Pinsker for the case $k = 2$.

Lemma 5.3.1: *Let $p \in \mathbb{P}$ with $p > 3$. If $n \in \mathbb{N}$ is such that $p \nmid n\binom{n}{3}$, then the tuple $(n,3,p)$ is loop-friendly.*

Proof. Let $\mathbf{A}$ be a p-affine algebra and let $R \subseteq A^{\binom{n}{3}}$ a nonempty, p-affine, and $(n,3)$-symmetric relation. We proceed as in the proof of Lemma 5.2.1. Again, for a two-element subset $G \in \binom{n}{2}$ we write t_G for the transposition on $[n]$ of the elements of G. We claim that there exist elements $c_0, c_1, c_2 \in \mathbb{Z}_p$ such that for any $\mathbf{r} \in A^{\binom{n}{3}}$ the expression

$$T_3(\mathbf{r}) + c_2 T_2(\mathbf{r}) + c_1 T_1(\mathbf{r}) + c_0 \mathbf{r}$$

gives a constant tuple, where

$$\begin{aligned} T_1(\mathbf{r}) &:= \sum_{G \in \binom{n}{2}} t_G(\mathbf{r}) \\ T_2(\mathbf{r}) &:= \sum_{\substack{G_1, G_2 \in \binom{n}{2}, \\ G_1 \cap G_2 = \emptyset}} t_{G_1} \circ t_{G_2}(\mathbf{r}) \\ T_3(\mathbf{r}) &:= \sum_{\substack{G_1, G_2, G_3 \in \binom{n}{2}, \\ G_1, G_2, G_3 \text{ disjoint in pairs}}} t_{G_1} \circ t_{G_2} \circ t_{G_3}(\mathbf{r}). \end{aligned}$$

Furthermore, we claim that we can choose c_0, c_1, and c_2 in such a way that the operation $A^4 \to A$ given by

$$(x_0, x_1, x_2, x_3) \mapsto \binom{n}{2}\binom{n-2}{2}\binom{n-4}{2} x_3 + c_2 \binom{n}{2}\binom{n-2}{2} x_2 + c_1 \binom{n}{2} x_1 + c_0 x_0 \tag{5.6}$$

is p-affine and therefore preserves R. By $(n,3)$-symmetry and p-affinity of R, it then follows that R contains a constant tuple.

Again, for $i = 0,1,2,3$ and $E, F \in \binom{n}{3}$ we let

$$\begin{aligned} T_{0,F}(E) &:= \begin{cases} \{E\}, & E = F \\ \emptyset, & \text{else} \end{cases} \\ T_{1,F}(E) &:= \left\{ G \in \binom{n}{2} : t_G(E) = F \right\} \\ T_{2,F}(E) &:= \left\{ (G_1, G_2) \in \binom{n}{2} \times \binom{n}{2} : t_{G_1} \circ t_{G_2}(E) = F,\ G_1 \cap G_2 = \emptyset \right\} \end{aligned}$$

$$T_{3,F}(E) := \left\{ (G_1, G_2, G_3) \in \binom{n}{2}^3 : t_{G_1} \circ t_{G_2} \circ t_{G_3}(E) = F,\ G_1, \ldots, G_l \text{ disjoint in pairs} \right\}.$$

The table for the values of $|T_{i,F}(E)|,\ i = 0, 1, 2, 3$ now looks as follows:

	$\lvert T_{0,F}(E)\rvert$	$\lvert T_{1,F}(E)\rvert$	$\lvert T_{2,F}(E)\rvert$	$\lvert T_{2,F}(E)\rvert$
$E = F$	1	$3 + \binom{n-3}{2}$	$6\binom{n-3}{2} + 6\binom{n-3}{4}$	$54\binom{n-3}{4} + 90\binom{n-3}{6}$
$\lvert E \cap F\rvert = 2$	0	1	$2 + 2\binom{n-4}{2}$	$6\binom{n-4}{2} + 18\binom{n-4}{4}$
$\lvert E \cap F\rvert = 1$	0	0	4	$12\binom{n-5}{2}$
$E \cap F = \emptyset$	0	0	0	36

Our aim is to find $c, c_0, c_1, c_2 \in \mathbb{Z}_p$ such that for all $E, F \in \binom{n}{3}$

$$|T_{3,F}(E)| + c_2|T_{2,F}(E)| + c_1|T_{1,F}(E)| + c_0|T_{0,F}(E)| = c.$$

This yields the following linear equation:

$$\begin{pmatrix} 1 & 3 + \binom{n-3}{2} & 6\binom{n-3}{2} + 6\binom{n-3}{4} & 54\binom{n-3}{4} + 90\binom{n-3}{6} \\ 0 & 1 & 2 + 2\binom{n-4}{2} & 6\binom{n-4}{2} + 18\binom{n-4}{4} \\ 0 & 0 & 4 & 12\binom{n-5}{2} \\ 0 & 0 & 0 & 36 \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \\ c \end{pmatrix} \tag{5.7}$$

In particular, we have that $c = 36$. Let c_0, c_1, c_2 be such that $(c_0, c_1, c_2, 1)$ solves (5.7). Plugging in these values, the sum of all coefficients that appear in operation (5.6) is given by

$$\begin{pmatrix} 1 \\ \binom{n}{2} \\ \binom{n}{2}\binom{n-2}{2} \\ \binom{n}{2}\binom{n-2}{2}\binom{n-4}{2} \end{pmatrix}^T \cdot \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ 1 \end{pmatrix} = 6n(n-1).$$

Since $p \neq 2, 3$, we have $6n(n-1) \neq 0$ by assumption. Therefore, the operation that maps a tuple (x_0, x_1, x_2, x_3) to

$$\frac{1}{6n(n-1)} \left(\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2} x_3 + c_2 \binom{n}{2}\binom{n-2}{2} x_2 + c_1 \binom{n}{2} x_1 + c_0 x_0 \right)$$

is p-affine. By definition of c_0, c_1, and c_2,

$$\frac{1}{6n(n-1)} (T_3(\mathbf{r}) + c_2 T_2(\mathbf{r}) + c_1 T_1(\mathbf{r}) + c_0 \mathbf{r})$$

gives a constant tuple for every $\mathbf{r} \in R$. By $(n, 3)$-symmetry and p-affinity of R, we obtain a constant tuple in R. Thus, the tuple $(n, 3, p)$ is loop-friendly.

□

Clearly, the above proof fails if $p = 2, 3$. Different constructions are necessary for these values of p in order to obtain a similar result to Theorem 5.2.5. However, to the best of our knowledge, this problem remains open. In the general case of p-affine (n, k)-symmetric relations the same issues arise for all $p \leq k$.

5.4 The (n, k)-symmetric case

In this section, we finally consider (n, k)-symmetric relations $R \in \mathrm{Inv}(\mathbf{A})$ for arbitrary $n \geq k$. Difficulties when trying to find a constant tuple in an (n, k)-symmetric relation $R \in \mathrm{Inv}(\mathbf{A})$ only occur in the case where $\mathbf{A}$ is a p-affine algebra. Generalising the methods of the previous sections, Conjecture 5.4.1 shows in principle how to construct a constant tuple in all p-affine cases with $p > k$. Some evidence supporting that this construction works has been carried out numerically.

Conjecture 5.4.1: *Let $k \in \mathbb{N}$ and $p \in \mathbb{P}$ such that $p > k$. If $n \in \mathbb{N}$ is such that $p \nmid n\binom{n}{k}$, then the tuple (n, k, p) is loop-friendly.*

Proof. Let $\mathbf{A}$ be a p-affine algebra and assume that $R \subseteq A^{\binom{n}{k}}$ is a nonempty, p-affine, and (n, k)-symmetric relation. As before, let t_G denote the transposition of the elements of G for any $G \in \binom{n}{2}$. For $l \in [k]$ and $\mathbf{r} \in A^{\binom{n}{k}}$ we set

$$T_l(\mathbf{r}) := \sum_{\substack{G_1,\ldots,G_l \in \binom{n}{2}, \\ G_1,\ldots,G_l \text{ disjoint in pairs}}} t_{G_1} \circ \cdots \circ t_{G_l}(\mathbf{r}).$$

We claim that there exist elements $c_0, \ldots, c_{k-1} \in \mathbb{Z}_p$ such that the expression

$$T_k + c_{k-1}T_{k-1} + \cdots + c_1T_1 + c_0. \tag{5.8}$$

yields a constant tuple for any $\mathbf{r} \in R$. Moreover, $c_0, \ldots, c_{k-1}$ can be chosen such that the operation $A^{k+1} \to A$ given by

$$(x_0, \ldots, x_k) \mapsto \prod_{j=0}^{k-1} \binom{n-2j}{2} x_k + c_{k-1} \prod_{j=0}^{k-2} \binom{n-2j}{2} x_{k-1} + \cdots + c_1 \binom{n}{2} x_1 + c_0x_0$$

is p-affine. It then follows by p-affinity and (n, k)-symmetry of R that R contains a constant tuple. As before, $c_0, \ldots, c_{k-1}$ will be defined according to the size of the sets

$$T_{l,F}(E) := \left\{ (G_1, \ldots, G_l) \in \binom{n}{2}^l : t_{G_1} \circ \cdots \circ t_{G_l}(E) = F,\ G_1, \ldots, G_l \text{ disjoint in pairs} \right\}$$

for all subsets $E, F \in \binom{n}{k}$ depending on the size of their intersection $E \cap F$, where $l \in [k]$. For $l = 0$ we let

$$|T_{0,F}(E)| := \begin{cases} 1 & E = F \\ 0 & \text{else} \end{cases}$$

Assume that $E, F \in \binom{n}{k}$ are such that $|E \cap F| = r$, where $0 \leq r \leq k$. Clearly, it holds that

$$|T_{l,F}(E)| = 0 \text{ for } 0 \leq l < k - r.$$

With the convention that $\binom{a}{b} := 0$ if $a < b$, we define for $k-r \leq l \leq k$ and $0 \leq i \leq l-k+r$:

$$M_{i,l,r} := \underbrace{\binom{r}{2i}\prod_{j=1}^{i}(2j-1)}_{\substack{\text{number of ways to}\\ \text{have } i \text{ pairwisely disjoint}\\ \text{transpositions within } E\cap F}} \underbrace{\binom{n-2(k-r)-r}{2(l-k+r-i)}\prod_{j=1}^{l-k+r-i}(2j-1)}_{\substack{\text{number of ways to}\\ \text{have } l-k+r-i \text{ pairwisely disjoint}\\ \text{transpositions within } \{1,\dots,n\}\setminus(E\cup F)}} \underbrace{(k-r)!}_{\substack{\text{number of bijections}\\ E\setminus(E\cap F)\to F\setminus(E\cap F)}} \quad l!$$

$$\alpha_{l,r} := \sum_{i=0}^{l-k+r} M_{i,l,r}.$$

We then have

$$|T_{l,F}(E)| = \alpha_{l,r} = \sum_{i=0}^{l-k+r} M_{i,r,l}$$

Since we want to find $c, c_0, \dots, c_{k-1}$ such that

$$|T_{k,F}(E)| + c_{k-1}|T_{k-1,F}(E)| + \dots + c_0|T_{0,F}(E)| = c$$

for all $E, F \in \binom{n}{k}$, we obtain the linear equations

$$\begin{pmatrix} 1 & \alpha_{1,k} & \alpha_{2,k} & \alpha_{3,k} & \dots & \alpha_{k,k} \\ 0 & 1 & \alpha_{2,k-1} & \alpha_{3,k-1} & \dots & \alpha_{k,k-1} \\ 0 & 0 & 4 & \alpha_{3,k-2} & \dots & \alpha_{k,k-2} \\ 0 & 0 & 0 & 36 & \dots & \alpha_{k,k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \alpha_{k,0} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \\ c \\ \vdots \\ c \end{pmatrix}.$$

Observe that the diagonal entries fulfill $\alpha_{l,k-l} = (l!)^2$ for all $0 \leq l \leq k$. Since $p > k$, it thus holds that $\alpha_{l,k-l} \neq 0$ for all $0 \leq l \leq k$, i.e., the matrix has full rank. In other words, we can recursively define $c_l,\ 0 \leq l \leq k-1$ by

$$c_l := \frac{1}{(l!)^2}\left((k!)^2 - \alpha_{k,k-l} - \sum_{j=l+1}^{k-1} \alpha_{j,k-l} c_j\right).$$

If we plug in those values, operation (5.8) now gives a constant tuple when applied to any $\mathbf{r} \in R$. We have tested numerically that the sum of all coefficients appearing in operation (5.8) satisfies

$$\prod_{j=0}^{k-1}\binom{n-2j}{2} + c_{k-1}\prod_{j=0}^{k-2}\binom{n-2j}{2} + \dots + c_1\binom{n}{2} + c_0 = k!n(n-1)\cdots(n-k+1). \quad (5.9)$$

The Python code providing some evidence for identity (5.9) can be found in the appendix. As $p > k$ and $p \nmid \binom{n}{k}$, it holds that $k!n(n-1)\cdots(n-k+1) \neq 0$. By p-affinity and (n,k)-symmetry of R, for any $\mathbf{r} \in R$ we obtain that R also contains the constant tuple

$$\frac{1}{k!n(n-1)\dots(n-k+1)}\big(T_k(\mathbf{r}) + c_{k-1}T_{k-1}(\mathbf{r}) + \dots + c_1T_1(\mathbf{r}) + c_0\mathbf{r}\big) \in R.$$

□

The construction above fails if $p \leq k$. However, in order to inductively prove the existence of a constant tuple in an invariant (n,k)-symmetric relation defined on a finite idempotent Taylor algebra $\mathbf{A}$, the loop-friendliness of all tuples (n,k,p) for $p \leq |A|$ needs to be guaranteed. The proof of the following generalisation of the symmetric and the $(n,2)$-symmetric cases is practically identical to the ones of Theorem 5.1.2 and Theorem 5.2.5.

Lemma 5.4.2: *Let $\mathbf{A}$ be a finite idempotent Taylor algebra. Let $n,k \in \mathbb{N}$ be such that for all $p \in \mathbb{P}$ with $p \leq |A|$ the tuple (n,k,p) is loop-friendly. If R is a nonempty, invariant, and (n,k)-symmetric relation on A, then R contains a constant tuple.*

Proof. We proceed as in the proofs of Theorem 5.1.2 and Theorem 5.2.5. The statement is trivial for $|A| = 1$. Assume that the statement holds for all algebras of size less than $|A|$. If $\mathbf{A}$ has a nontrivial strong subuniverse, then R satisfies the restriction-property by Lemma 5.2. Otherwise, by Corollary 3.1.3, there exists a congruence θ on $\mathbf{A}$ such that $\mathbf{A}/\theta$ is p-affine for some $p \in \mathbb{P}$. Evidently, $p \leq |A|$. By assumption, the tuple (n,k,p) is therefore loop-friendly. By Lemma 5.4, R satisfies the restriction-property. Thus, in both cases of Corollary 3.1.3, there exists a proper subalgebra $\mathbf{B} \lneq \mathbf{A}$ such that $R \cap B^{\binom{n}{2}} \neq \emptyset$. By the inductive hypothesis, $R \cap B^{\binom{n}{2}}$ has a constant tuple. Hence, there exists a constant tuple in R. $\square$

Recall that by Corollary 5.2.4, a tuple $(n,2,p)$ is loop-friendly if and only if $2 \nmid n\binom{n}{2}$. However, the following counterexample shows that for arbitrary $k \geq 3$ a characterisation of all loop-friendly tuples (n,k,p) is more difficult. Namely, there exist $n,k \in \mathbb{N}$ and $p \in \mathbb{P}$ that satisfy $p \nmid n\binom{n}{k}$, but the tuple (n,k,p) fails to be loop-friendly. This counterexample was found by L. Barto.

Counterexample 5.4.3: Consider $n = 7,\ k = 3,\ p = 3$. Then $\binom{n}{k} \equiv 2 \mod p$, and $n \equiv 1 \mod p$, i.e., it holds that $p \nmid n\binom{n}{k}$. For a tuple $\mathbf{x} = (x_E)_{E \in \binom{n}{k}} \in \mathbb{Z}_p{}^{\binom{n}{k}}$ and $u, v \in [n]$ we define

$$\mathbf{x}_\emptyset := \sum_{E \in \binom{n}{k}} x_E$$

$$\mathbf{x}_v := \sum_{\substack{E \in \binom{n}{k} \\ v \in E}} x_E$$

$$\mathbf{x}_{uv} := \sum_{\substack{E \in \binom{n}{k} \\ u,v \in E}} x_E.$$

Let R be the relation given by

$$R := \left\{ \mathbf{r} \in \mathbb{Z}_p{}^{\binom{n}{k}} : \mathbf{r}_\emptyset = 1,\ \forall u,v \in [n]\ \mathbf{r}_v = \mathbf{r}_{uv} = 0 \right\}.$$

R is p-affine and symmetric. In order to see that $R \neq \emptyset$, take any $M \subseteq [n]$ of size $|M| = 5$.

Let $\mathbf{x} = (x_E)_{E \in \binom{n}{k}}$ be defined by

$$x_E := \begin{cases} 1 & \text{if } E \subseteq M \\ 0 & \text{else} \end{cases}.$$

We have

$$\begin{aligned} \mathbf{x}_\emptyset &= \sum_{E \subseteq M} x_E = \binom{5}{3} = 10 \equiv 1 \mod p \\ \mathbf{x}_v &= 0 \text{ if } v \notin M \\ \mathbf{x}_v &= \binom{4}{2} = 6 \equiv 0 \mod p \text{ if } v \in M \\ \mathbf{x}_{uv} &= 0 \text{ if } \{u,v\} \not\subseteq M \\ \mathbf{x}_{uv} &= 3 \equiv 0 \mod p \text{ if } \{u,v\} \subseteq M. \end{aligned}$$

Hence, $\mathbf{x} \in R$, and we have shown $R \neq \emptyset$. But if R contained a constant tuple $\mathbf{c} = (c, \ldots, c)$ then we would have

$$1 = \mathbf{c}_\emptyset = 35 \cdot c \equiv -c \mod p, \text{ hence } c = 2,$$

but also

$$0 = \mathbf{c}_{uv} = 5 \cdot c \equiv -c \mod p, \text{ hence } c = 0.$$

This yields a contradiction. Therefore, the tuple $(7,3,3)$ is not loop-friendly.

Note that Counterexample 5.4.3 does not contradict Conjecture 5.4.1 as Conjecture 5.4.1 only states the loop-friendliness of tuples (n,k,p) that satisfy $p \nmid n\binom{n}{k}$ and $p > k$.

6 Existence of WNU

Applying our results from Section 5, we show that if $\mathbf{A}$ is any finite idempotent Taylor algebra, then it contains a k-WNU term operation for $k = 1$ and $k = 2$.

Recall that to any loop condition L we can assign the relation R_L associated with L. Let $\mathbf{A}$ be a finite algebra, and $\mathbf{F} = \mathcal{F}_{\mathcal{V}(\mathbf{A})}(x, y)$ be the free algebra over the set $\{x, y\}$ in the variety $\mathcal{V}(\mathbf{A})$ generated by $\mathbf{A}$. Let R_L be the $\binom{n}{k}$-ary relation associated with the loop condition given by the n-ary k-WNU identities

$$\begin{aligned} &w(\overbrace{y, \dots, y, y}^{k}, x, x, \dots, x) = \\ =&w(y, \dots, y, x, y, x, \dots, x) = \\ &\vdots \\ =&w(\underbrace{x, \dots, x, x, x, y, \dots, y}_{n}). \end{aligned} \tag{6.1}$$

We obtain R_L by taking the columns of the identities in (6.1), i.e.,

$$R_L = \left\{ \begin{pmatrix} y \\ y \\ \vdots \\ x \end{pmatrix}, \dots, \begin{pmatrix} y \\ x \\ \vdots \\ \vphantom{x} \end{pmatrix}, \begin{pmatrix} x \\ y \\ \vdots \\ \vphantom{x} \end{pmatrix}, \dots, \begin{pmatrix} x \\ x \\ \vdots \\ y \end{pmatrix} \right\} \subseteq V^{\binom{n}{k}}.$$

Note that we have $|R_L| = n$. By Lemma 4.2.1, the existence of a k-WNU term operation of arity n in $\mathrm{Clo}(\mathbf{A})$ is equivalent to asking that the $\binom{n}{k}$-ary relation

$$\langle R_L \rangle_{\mathbf{F}} = \{s(r_1, \dots, r_n) : s \in \mathrm{Clo}(\mathbf{A}),\ r_1, \dots, r_n \in R_L\}$$

generated by R_L contains a constant tuple. R_L is (n, k)-symmetric and so is $\langle R_L \rangle_{\mathbf{F}}$, for it holds that

$$\pi(s(r_1, \dots, r_n)) = s(\pi(r_1), \dots, \pi(r_n))$$

for any $\pi \in S_n,\ s \in \mathrm{Clo}(\mathbf{A}),\ r_i \in R_L$.

The existence of a WNU term operation in any finite idempotent Taylor algebra was first shown by M. Maróti and R. McKenzie in [MM08].

Corollary 6.1: *[[MM08]; see also [Zhu17; Zhu20a]] Let $\mathbf{A}$ be a finite idempotent Taylor algebra of size at least 2. If $n \in \mathbb{N}$ is such that $p \nmid n$ for all $p \in \mathbb{P}$ with $p \leq |A|$, then $\mathrm{Clo}(\mathbf{A})$ has a WNU term operation of arity n.*

Proof. Let $\mathbf{F}$ be the free algebra with 2 generators in the variety generated by $\mathbf{A}$. By Lemma 2.5.1, we have that $\mathbf{F} \cong \mathrm{Clo}(\mathbf{A})^{(2)}$. Since $\mathrm{Clo}(\mathbf{A})^{(2)} \leq \mathbf{A}^{\mathbf{A}^2} \in SP(\mathbf{A})$, it follows that $\mathbf{F}$ is finite, idempotent, and Taylor. Let $R \subseteq \mathbf{F}^n$ be the relation

$$R := \left\langle \left\{ \begin{pmatrix} x \\ x \\ \vdots \\ y \end{pmatrix}, \begin{pmatrix} x \\ \vdots \\ y \\ x \end{pmatrix}, \ldots, \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix} \right\} \right\rangle_{\mathbf{F}}.$$

By definition, R is symmetric and $R \in \mathrm{Inv}(\mathbf{F})$. By Lemma 3.3.7, if there exists a p-affine algebra in $\mathrm{HSP}(\mathbf{A})$, then there is also one in $\mathrm{HS}(\mathbf{A})$. In particular, if θ is a congruence on $\mathbf{F}$ such that $\mathbf{F}/\theta$ is p-affine, we must have $p \leq |A|$, and it holds that $p \nmid n$ by assumption. By Remark 5.1.3, we may therefore apply Theorem 5.1.2 to $\mathbf{F}$ and R. This implies that R contains a constant tuple $(c, \ldots, c)$. In other words, there exists $w \in \mathrm{Clo}(\mathbf{A})$ such that

$$w\left(\begin{pmatrix} x \\ x \\ \vdots \\ y \end{pmatrix}, \begin{pmatrix} x \\ \vdots \\ y \\ x \end{pmatrix}, \ldots, \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix} \right) = \begin{pmatrix} c \\ c \\ \vdots \\ c \end{pmatrix},$$

i.e., $w(x, x, \ldots, y) = w(x, \ldots, y, x) = \cdots = w(y, x, \ldots, x)$.

□

The proof of Corollary 6.1 can easily be extended to show the existence of a 2-WNU term operation.

Corollary 6.2: *Let $\mathbf{A}$ be a finite idempotent Taylor algebra of size at least 2. If $n \in \mathbb{N}$ is such that $p \nmid n\binom{n}{2}$ for all $p \leq |A|$ with $p \in \mathbb{P}$, then $\mathrm{Clo}(\mathbf{A})$ has a 2-WNU-term of arity n.*

Proof. Let again $\mathbf{F}$ be the free algebra over the set $\{x, y\}$ in the variety generated by $\mathbf{A}$. It is finite, idempotent and Taylor. Let $R \subseteq \mathbf{F}^{\binom{n}{2}}$ be the relation generated by the columns of the n-ary 2-WNU identities, i.e.,

$$R := \left\langle \left\{ \begin{pmatrix} x \\ x \\ \vdots \\ y \end{pmatrix}, \begin{pmatrix} x \\ x \\ \vdots \\ y \end{pmatrix}, \ldots, \begin{pmatrix} x \\ y \\ \vdots \\ x \end{pmatrix}, \begin{pmatrix} y \\ x \\ \vdots \\ x \end{pmatrix}, \begin{pmatrix} y \\ y \\ \vdots \\ x \end{pmatrix} \right\} \right\rangle_{\mathbf{F}}.$$

Then R is $(n, 2)$-symmetric and $R \in \mathrm{Inv}(\mathbf{F})$. Theorem 5.2.5 implies that R contains a constant tuple. Thus, there exists a 2-WNU term operation of arity n. □

In order to have that for every $k \in \mathbb{N}$ and for every finite idempotent Taylor algebra $\mathbf{A}$ there exists a k-WNU term operation of some arity $n \in \mathbb{N}$, it is sufficient to find conditions under which all tuples (n, k, p) for $p \in \mathbb{P}$ with $p \leq |A|$ are loop-friendly.

Corollary 6.3: *Let $\mathbf{A}$ be a finite idempotent algebra and let $k \in \mathbb{N}$. If $n \in \mathbb{N}$ is such that the tuples (n, k, p) are loop-friendly for all $p \in \mathbb{P}$ with $p \leq |A|$, then $\mathbf{A}$ has a k-WNU term operation of arity n.*

Proof. The proof is practically identical to the ones above. Namely, we consider the relation R generated by the columns of the n-ary k-WNU identities in the free algebra over the set $\{x, y\}$ in the variety generated by $\mathbf{A}$. By invariance and (n, k)-symmetry of R, Lemma 5.4.2 yields a constant tuple in R. $\square$

7 Open Questions

In light of Lemma 5.4.2 and Corollary 6.3, our open questions concern conditions under which a tuple (n, k, p) is loop-friendly. Analysing Counterexample 5.4.3, we see that a tuple (n, k, p) will never be loop-friendly as long as there exist some $1 \leq j \leq k-1$ and some $k \leq M \leq n$ such that

$$\binom{n-j}{k-j} \not\equiv 0 \mod p, \text{ and} \tag{7.1}$$

$$\binom{M}{k} \equiv 1 \mod p, \text{ and} \tag{7.2}$$

$$\binom{M-j}{k-j} \equiv 0 \mod p. \tag{7.3}$$

We raise the question whether or not these are the only conditions under which a tuple fails to be loop-friendly.

Question 7.1: *Let $n, k \in \mathbb{N}$ and $p \in \mathbb{P}$. Assume that $p \nmid n\binom{n}{k}$, and there do not exist any $1 \leq j \leq k-1$ and $k \leq M \leq n$ such that all of the conditions (7.1)-(7.3) hold. Do these assumptions imply that the tuple (n, k, p) is loop-friendly?*

Given a finite idempotent Taylor algebra $\mathbf{A}$ and $k \in \mathbb{N}$, Lemma 5.4.2 and Corollary 6.3 require $n \in \mathbb{N}$ to be such that for all $p \in \mathbb{P}$ with $p \leq |A|$ the tuples (n, k, p) are loop-friendly. Consequently, we are confronted with the question whether or not for all $k \in \mathbb{N}$ and all prime numbers up to a given bound there exists $n \in \mathbb{N}$ so that the number theoretic conditions that appear in Question 7.1 are simultaneously satisfied. For $j \in \mathbb{N}$ let $p_j \in \mathbb{P}$ denote the j-th prime number.

Question 7.2: *Let $k, J \in \mathbb{N}$. Does there exist $n \in \mathbb{N}$ such that the following conditions hold for all $j \in [J]$:*

(i) $p_j \nmid n\binom{n}{k}$

(ii) for all $1 \leq j \leq k-1$ and $k \leq M \leq n$ it holds that either

- $\binom{n-j}{k-j} \equiv 0 \mod p_j$, *or*
- $\binom{M}{k} \not\equiv 1 \mod p_j$, *or*
- $\binom{M-j}{k-j} \not\equiv 0 \mod p_j$

So far, we are only able to provide a partial answer to Question 7.2. Namely, for any $k, J \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $j \in [J]$ we have $p_j \nmid n\binom{n}{k}$. A proof of this statement is given in Appendix A.

In combination with Corollary 6.3, a positive answer to the questions above exhibits a sufficient condition for the existence of k-WNU term operations for all finite idempotent Taylor algebras.

Proposition 7.3: *If Question 7.1 and Question 7.2 can be answered positively, then for any finite idempotent Taylor algebra* $\mathbf{A}$ *and any* $k \in \mathbb{N}$ *there exists* $n \in \mathbb{N}$ *such that* $\mathrm{Clo}(\mathbf{A})$ *contains a* k*-WNU term operation of arity* n*.*

Proof. Let $\mathbf{A}$ be a finite idempotent Taylor algebra and let $k \in \mathbb{N}$. If $n \in \mathbb{N}$ is such that the conditions *(i)* and *(ii)* formulated in Question 7.2 hold for all $p \in \mathbb{P}$ with $p \leq |A|$, then a positive answer to Question 7.1 yields the loop-friendliness of all tuples (n, k, p). By Corollary 6.3, this implies that $\mathbf{A}$ has a k-WNU term operation of arity n. □

In view of Proposition 7.3, future lines of enquiry may include the comprehensive characterisation of all loop-friendly tuples (n, k, p).

8 Appendix A

The following lemma provides a partial answer to Question 7.2. It shows that the first condition formulated in the question is always satisfiable. In the following, for $j \in \mathbb{N}$ we denote the j-th prime number by p_j.

Lemma 8.1: *For any $k, J \in \mathbb{N}$ there exist infinitely many $n \in \mathbb{N}$ such that for all $j \in [J]$ it holds that $p_j \nmid n\binom{n}{k}$.*

Proof. For every $l \in [k]$ let its prime factorisation be given by the tuple $(l_j)_{j\in\mathbb{N}}$ where l_j is the multiplicity of the j-th prime number p_j in the prime factorisation of l. For every $j \in [J]$ we set

$$k_j := \max_{1\leq l\leq k} l_j + 1.$$

Now, define

$$n := p_1^{k_1} \cdots p_J^{k_J} - 1.$$

It follows that $p_j \nmid n$ for all $1 \leq j \leq J$ since otherwise we would have $p_j \mid 1$. In order to see that also $p_j \nmid \binom{n}{k}$, observe that

$$\binom{n}{k} = \frac{(p_1^{k_1} \cdots p_J^{k_J} - 1) \cdots (p_1^{k_1} \cdots p_J^{k_J} - k)}{k!} =$$
$$= \left(\frac{p_1^{k_1} \cdots p_J^{k_J}}{1} - 1\right)\left(\frac{p_1^{k_1} \cdots p_J^{k_J}}{2} - 1\right) \cdots \left(\frac{p_1^{k_1} \cdots p_J^{k_J}}{k} - 1\right).$$

For all $j \in [J]$ and $l \in [k]$ the multiplicity of p_j in l is not greater than $k_j - 1$, hence $p_j \mid \frac{p_1^{k_1} \cdots p_J^{k_J}}{l}$. It follows that $p_j \nmid \frac{p_1^{k_1} \cdots p_J^{k_J}}{l} - 1$ and therefore $p_j \nmid \binom{n}{k}$ as required. By the same arguments, all numbers of the form $m(n+1) - 1$ for positive integers m satisfy our assumptions. □

9 Appendix B

The following code checks identity (5.9) for values $k \leq 10$ and $n \leq 30$. It can, however, easily be adapted for higher values.

```
from math import comb
from math import factorial
from math import prod
import numpy as np

def M(i,l,r):
    L1=list()
    L2=list()

    if n-2*(k-r)-r < 0:
        return 0
    else:
        for j in range (1,i+1):
            L1.append(2*j-1)
        for j in range(1,l-k+r-i+1):
            L2.append(2*j-1)
        return factorial(k-r)*prod(L1)*prod(L2)*comb(r, 2*i)*
            comb(n-2*(k-r)-r, 2*(l-k+r-i))*factorial(l)

def a(l,r):
    L=list()
    for i in range(l-k+r+1):
        L.append(M(i,l,r))
    return sum(L)

def T(l):
    if l==0:
        return 1
    else:
        L=list()
        for j in range(l):
            L.append(comb(n-2*j, 2))
        return prod(L)

def res(x):
```

```
    L=list()
    for j in range (k):
        L.append(x[j]*T(j))
    L.append(T(k))
    return sum(L)

def solve_upper_tri_matrix(M, b, k):
    x = np.zeros(k+1, dtype=object)
    for i in range(k, -1, -1):
        if i == k:
            x[k] = 1
        else:
            tmp = np.dot(M[i], x)
            x[i] = (b[i]-tmp) // M[i][i]
    return x

for k in range(7, 11):
    for n in range (2*k+1, 31):

        C=np.zeros((k+1, k+1), dtype=object)
        for i in range(k+1):
            for j in range(i, k+1):
                C[i][j]=a(j, k-i)

        b=np.array([factorial(k)*factorial(k)]*(k+1), dtype=object)
        x = solve_upper_tri_matrix(C,b, k)

        print('n='+str(n)+", k="+ str(k)+", res="+
              str(res(x)-(comb(n,k)*factorial(k)*factorial(k))))
```

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