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RESEARCH ARTICLE

The stationary AKPZ equation: Logarithmic superdiffusivity

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Abstract
We study the two-dimensional Anisotropic KPZ equation (AKPZ) formally given by

$$\partial_t H = \frac{1}{2}\Delta H + \lambda((\partial_1 H)^2 - (\partial_2 H)^2) + \xi,$$

where ξ is a space-time white noise and λ is a strictly positive constant. While the classical two-dimensional KPZ equation, whose nonlinearity is $|\nabla H|^2 = (\partial_1 H)^2 + (\partial_2 H)^2$, can be linearised via the Cole-Hopf transformation, this is not the case for AKPZ. We prove that the stationary solution to AKPZ (whose invariant measure is the Gaussian Free Field (GFF)) is superdiffusive: its diffusion coefficient diverges for large times as $\sqrt{\log t}$ up to $\log\log t$ corrections, in a Tauberian sense. Morally, this says that the correlation length grows with time like $t^{1/2} \times (\log t)^{1/4}$. Moreover, we show that if the process is rescaled diffusively ($t \to t/\varepsilon^2, x \to x/\varepsilon, \varepsilon \to 0$), then it evolves non-trivially already on time-scales of order approximately $1/\sqrt{|\log \varepsilon|} \ll 1$. Both claims hold as soon as the coefficient λ of the nonlinearity is non-zero. These results are in contrast with the belief, common in the mathematics community, that the AKPZ equation is diffusive at large scales and, under simple diffusive scaling, converges to the two-dimensional

Stochastic Heat Equation (2dSHE) with additive noise (i.e., the case $\lambda = 0$).

CONTENTS

1 | INTRODUCTION

The KPZ equation is a stochastic PDE that formally is written as

$$\partial_t H = \nu \Delta H + \langle \nabla H, Q \nabla H \rangle + \sqrt{D}\xi, \tag{1.1}$$

where $H = H(t, x)$ depends on time $t \geq 0$ and x, the spatial d-dimensional coordinate (e.g., $x \in \mathbb{R}^d$ or $\mathbb{T}^d$), ξ is a space-time (white) noise, ν, D are two positive constants, and Q is a $d \times d$ matrix. The KPZ equation was originally derived as a description for $(d+1)$-dimensional stochastic growth: the Laplacian is a smoothing term that overall flattens the interface, the noise models the microscopic local randomness, while the non-linear term encodes the slope-dependence of the growth mechanism. Indeed, at a heuristic level, the connection between a specific (microscopic) growth model and the KPZ equation is that Q is proportional to the Hessian $D^2 v$ of the average speed of growth v of the microscopic model, seen as a function of the average interface slope.

The SPDE (1.1) is well known to be analytically ill-posed if ξ is a white noise, due to the nonlinear term, so that in order to study the large-scale properties of its solution, a standard approach is to focus on a regularised version of it obtained by smoothing either the noise or the nonlinearity (or both). In the spirit of Renormalisation Group, one would like to determine whether the nonlinearity is *relevant* or not, that is, if it affects the asymptotic behaviour in a qualitative way, in particular by changing the growth and roughness exponents with respect to those of the linear equation obtained by setting $Q \equiv 0$. Note that the latter is just the d-dimensional Stochastic Heat Equation (SHE) with additive noise. Already in the seminal paper [25] it was predicted that, if $d \geq 3$ and the nonlinearity is small enough (say if the norm of Q is small), then the nonlinearity is irrelevant and the scaling limit is given by the solution of SHE (up to a finite renormalisation of ν, D). A recent series of works (see [16, 18, 21, 30, 31]) has confirmed this prediction mathematically (with the important restriction that Q is assumed to be proportional to the identity matrix: only in this case one can linearise (1.1) via the Cole-Hopf transform, and map it to a problem of directed polymers in random environment). As for $d = 1$, [25] conjectures, and it is by now well established (see [1, 3, 17, 32]), that the nonlinearity, no matter its strength provided it is non-zero, is relevant and changes the growth exponent from $\beta = 1/4$ to $\beta = 1/3$. In dimension $d = 2$, the situation is subtler since finer details of the equation, and in particular the structure of the matrix Q, might affect the relevance claim. Indeed, it was predicted in [4, 45] that if $\det Q > 0$ (Isotropic KPZ equation) then the nonlinearity is relevant and gives rise to non-trivial and model-independent growth and roughness exponents. In view of the above-mentioned connection between Q and the Hessian of v, the condition $\det Q > 0$ corresponds to growth models with strictly convex or concave speed of growth. In the complementary case, $\det Q \leq 0$ (Anisotropic KPZ or AKPZ equation), the physicists' prediction, based on non-rigourous, one-loop Renormalisation Group computations (see [4, 45]), states that the equation has the same scaling limit as the 2dSHE.

A first clear indication that the isotropic and anisotropic versions of the equation have a radically different behaviour is obtained by looking at the equation where the nonlinearity parameter Q is scaled to zero together with ε (the noise regularisation parameter). In the case of the isotropic KPZ equation with $Q = \lambda \mathrm{Id}$, $\lambda > 0$, that is, nonlinearity $\lambda|\nabla H|^2$, it was found in [13] that, taking $\lambda = \hat{\lambda}/\sqrt{|\log \varepsilon|}$, H tends as $\varepsilon \to 0$ to the solution of the linear equation with renormalised coefficients if $\hat{\lambda}$ is smaller than a precisely identified threshold $\hat{\lambda}_c$, and the noise strength in the limiting linear equation diverges as $\hat{\lambda} \to \hat{\lambda}_c$. In contrast, for the (stationary) AKPZ equation, the findings of [10] imply that there is no phase transition in this scaling.

In the present work, we study the regularised AKPZ equation at stationarity with the specific choice $Q = \lambda\, \mathrm{diag}\,(+1, -1)$ (in which case the stationary state is given by the Gaussian Free Field, that, from now on, we will abbreviate with GFF [10]), and *we do not scale λ down to zero*. As remarked in [19], this choice of Q is the only one, modulo rotations, for which the stationary state is Gaussian. Our main results state that in contrast with the stochastic heat equation the AKPZ equation *is not even asymptotically invariant under diffusive scaling*. In fact, while the former is scale invariant under diffusive scaling, that is, time scaled as t/ε^2 and space as x/ε, we find that as soon as $\lambda > 0$, the stationary and diffusively rescaled process $H^\varepsilon(t, x) \stackrel{\text{def}}{=} H(t/\varepsilon^2, x/\varepsilon)$ evolves non-trivially already on time-scales of order $|\log \varepsilon|^{-1/2} \ll 1$, up to corrections polynomial in $\log|\log \varepsilon|$. By "evolves non-trivially" we mean for instance that, if φ is a test function of zero total mass, the normalised covariance at different times of the locally averaged field $H^\varepsilon[t](\varphi) \stackrel{\text{def}}{=} \int \varphi(x) H^\varepsilon(t, x)\mathrm{d}x$,

$$\frac{\mathrm{Cov}(H^\varepsilon[t](\varphi), H^\varepsilon[0](\varphi))}{\mathrm{Var}(H^\varepsilon[0](\varphi))}, \tag{1.2}$$

is strictly smaller than 1 uniformly in ε, for $t \approx |\log \varepsilon|^{-1/2}$ (see Theorem 1.2 and the subsequent comments). Moreover, we show that the diffusion coefficient $D(t)$, which (once multiplied by t) measures the mean square distance of spreading of correlations as a function of time, grows in time as $\approx |\log t|^{1/2}$ for t large as soon as $\lambda > 0$ (see Theorem 1.1 for the precise formulation), thus excluding diffusive behaviour since the linear equation instead is known to diffuse at constant rate $D(t) = 1$ (which immediately follows from the representation of $D(t)$ in Equation 1.6, see also Appendix A). We emphasise that logarithmic super-diffusivity for the AKPZ equation was *not expected* in the mathematical literature [7, 8], and we are not aware of predictions in this sense even in the relevant physics literature [4, 24, 45]. Based on the "mode-coupling" heuristics we give in Appendix B, it is reasonable to expect that, once the logarithmic corrections to the scaling are taken into account, the large-scale behaviour of the equation is Gaussian. A first result in this direction was recently obtained by the authors in [11]: in the case where the strength λ of the nonlinearity is suitably scaled to zero, the AKPZ equation scales to the stochastic heat equation with renormalised coefficients.

Finally, it is also interesting to look at more local quantities, such as the time-dependence of the variance of the height increment at a single point, $H(t,0) - H(0,0)$. Since H fails to be a function, we will study the variance of the height tested against a fixed test function of compact support. According to the physicists' predictions [4, 45] and to numerical simulations [24], this should grow asymptotically like $\log t$, as for the linear equation. In Theorem 1.5 we prove an upper bound of this order (implying that the growth exponent β is zero); as we explain in Remark 1.6, this is not in contradiction with our finding of anomalous diffusivity.

To put our result into a wider context, let us mention that $\sqrt{\log t}$-behaviour for the diffusion coefficient has been conjectured also for a whole universality class of two-dimensional (self)-interacting diffusions, including tracer particles in non-ideal fluids [44], self-repelling random walks and Brownian polymers [2, 34, 35, 42] and the diffusion of a tracer particle in the curl of the two-dimensional GFF [42]. The best rigourous result we are aware of in this context are super-diffusivity lower and upper bounds of order $\log \log t$ and $\log t$, respectively, obtained in [27] for lattice gas models and in [42] for self-repelling polymers and for the diffusion in the curl of the GFF. We believe that the tools developed in the present paper (and in particular Theorem 3.4) will help to significantly improve the estimates for these models.

The crucial ingredient of the proof is a control of the variance of the time integral of the nonlinearity, that is obtained via an iterative argument inspired by the works [28, 46], where the authors study the super-diffusivity of the asymmetric simple exclusion process (ASEP) in dimensions $d = 1, 2$. In particular, [46] proves $(\log t)^{2/3}$ super-diffusion for $2d$-ASEP. Let us emphasise that, while the iterative method of [46] gives a logarithmic correction to diffusivity at any finite step k of the iteration, and the limit $k \to \infty$ is needed to pin down the power of the logarithm to $2/3$, in our case at step k we get only a $|\log \log t|^k/k!$ correction and we need to take a k diverging with t to get the $\sqrt{\log t}$ result. This difference is not a technical limitation of our estimates but rather it reflects a different structure of the operators involved in the two problems. The different symmetry properties of $2d$-ASEP and the AKPZ equation are also responsible for the different exponents, $2/3$ versus $1/2$, in the logarithmic super-diffusivity corrections; this was already pointed out in [27, 42] in the context of lattice gases and self-repelling polymers.

To conclude this introduction, let us recall that there are several microscopic $(2+1)$-dimensional growth models that are known to belong to the AKPZ universality class, in the sense that their speed of growth satisfies $\det(D^2 v) \le 0$. These include the Gates-Westcott model [29, 36], certain two-dimensional arrays of interlaced particle systems [9] and the domino shuffling algorithm [15] just to mention a few (other growth processes like the 6-vertex dynamics of [5] and the q-Whittaker particle system [6] should belong to this class, but an explicit computation of v is not possible since their stationary measures are non-determinantal; see also [40] for further references). Typical results that have been proven for such models are the scaling of stationary fluctuations (at fixed time) to a GFF, a logarithmic upper bound on height fluctuation growth [14, 29, 41] (similar to Theorem 1.5 below) and CLTs for height fluctuations on the scale $\sqrt{\log t}$ for certain non-stationary, "integrable" initial conditions [9]. However, the more challenging issue of studying the large-scale diffusivity (or super-diffusivity) properties of these models is entirely unexplored. While logarithmic super-diffusivity effects are quite hard to be observed numerically, the $(\log t)^{2/3}$ behaviour for two-dimensional asymmetric simple exclusion has been very recently exhibited in simulations [26]. It would be extremely interesting to study the super-diffusivity phenomenon we determine for the continuum AKPZ equation also for discrete growth models in the same universality class.

1.1 | The AKPZ equation and the main results

In order to avoid integrability issues arising in the infinite volume regime (that are anyway addressed in [12]) we study the solution H_N of the regularised AKPZ equation on a large torus of size $N \in \mathbb{N}$, which is given by

$$\partial_t H_N = \frac{1}{2}\Delta H_N + \lambda \tilde{\mathcal{N}}[H_N] + \xi, \qquad H_N(0) = \tilde{\eta} \tag{1.3}$$

where[1] $H_N = H_N(t,x)$ for $t \geq 0$ and $x \in \mathbb{T}_N^2$, the two-dimensional torus of side length $2\pi N$,

- $\tilde{\eta}$ is a GFF on $\mathbb{T}_N^2$ with covariance

$$\mathbb{E}[\tilde{\eta}(\varphi)\tilde{\eta}(\psi)] = \langle(-\Delta)^{-1}\varphi, \psi\rangle_{L^2(\mathbb{T}_N^2)}, \qquad \text{for all } \varphi, \psi \in H^{-1}(\mathbb{T}_N^2),$$

  so that in particular, the 0 Fourier mode of φ and ψ is 0,
- ξ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}_N^2$ independent of $\tilde{\eta}$ with covariance

$$\mathbb{E}[\xi(\varphi)\xi(\psi)] = \langle\varphi, \psi\rangle_{L^2(\mathbb{R}_+\times\mathbb{T}_N^2)}, \qquad \text{for all } \varphi, \psi \in L^2(\mathbb{R}_+ \times \mathbb{T}_N^2),$$

- the "nonlinearity" $\tilde{\mathcal{N}} \stackrel{\text{def}}{=} \tilde{\mathcal{N}}^1$ is defined as

$$\tilde{\mathcal{N}}^1[H_N] \stackrel{\text{def}}{=} \Pi_1\big((\Pi_1\partial_1 H_N)^2 - (\Pi_1\partial_2 H_N)^2\big), \tag{1.4}$$

  and, for $M \in \mathbb{N}$, Π_M is the operator acting in Fourier space by cutting the modes larger than M, that is,

$$\widehat{\Pi_M w}(k) \stackrel{\text{def}}{=} \hat{w}(k)\mathbb{1}_{|k|\leq M}, \tag{1.5}$$

  $\hat{w}(k)$ is the k-th Fourier component of w (see below for our conventions on Fourier transforms) and $|k|$ denotes the Euclidean norm of k.[2]
- $\lambda > 0$ is a constant that regulates the strength of the nonlinearity.

As was proven in [10] (see also Lemma 2.1 below), the periodic GFF $\tilde{\eta}$ is a stationary state for the process *independently* of λ and of the cut-off parameter which above is set to be equal to 1. From now on, $\mathbf{P} = \mathbf{P}^N$ and $\mathbf{E} = \mathbf{E}^N$ will respectively denote the law and expectation of the stationary space-time process H_N, while $\mathbb{P} = \mathbb{P}^N$ and $\mathbb{E} = \mathbb{E}^N$ will be used for the law and expectation with respect to the stationary measure (the GFF).

The goal of the present paper is to understand the large-scale properties of H_N as a space-time process in comparison with the linear case $\lambda = 0$, that is simply the stochastic heat equation with additive noise.

[1] The tildas on $\tilde{\mathcal{N}}, \tilde{\eta}$ are there because we will actually work with analogous quantities that are denoted by the same symbols, without tildas.

[2] In [10], the r.h.s. of (1.5) was defined with $\mathbb{1}_{|k|_\infty \leq M}$ instead; however, all results proven in [10] hold true with the definition (1.5); in this respect, it is important that both norms have the symmetries of $\mathbb{Z}^2$. In this work we prefer to work with the Euclidean norm because it slightly simplifies certain technical steps.

The first observable we consider is the *bulk diffusivity* which can be thought of as a measure of how the correlations of a process spread in space as a function of time. The definition we will work with is in terms of the following Green-Kubo formula

$$D_N(t) = 1 + 2\frac{\lambda^2}{t}\int_0^t \int_0^s \int_{\mathbb{T}_N^2} \mathbf{E}\big[\tilde{\mathcal{N}}[H_N](r,x)\tilde{\mathcal{N}}[H_N](0,0)\big]\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s \tag{1.6}$$

which has the advantage of being well-defined since our regularisation of the nonlinearity ensures that $\tilde{\mathcal{N}}[H_N]$ is smooth even if H_N is not, so that point-wise evaluation is allowed. Note that in the above integral, the parameter r is integrated until s so that a direct evaluation of the integral is not possible. The heuristics connecting the spread of the correlations of H_N to the formula above is given in Appendix A. For now, we simply remark that (1.6) is the analog in the present context of the definition used in [28, 38, 46] for the bulk diffusion coefficient of the asymmetric exclusion processes on $\mathbb{Z}^d$, or in [3] for the bulk diffusion coefficient of the one-dimensional KPZ equation.

A crucial feature of the bulk diffusivity is that it provides a way to discern if a process behaves diffusively or not. Indeed, while for the linear equation, which is diffusive, D_N is constant in time (in case of (1.3) with $\lambda = 0$, clearly $D_N \equiv 1$), an indication of superdiffusive behaviour can be obtained by showing that D_N diverges in time as $t \to \infty$. For technical reasons, we will work with the Laplace transform of $t\,D_N(t)$, defined for $\mu > 0$ as

$$\mathcal{D}_N(\mu) = \mu \int_0^\infty e^{-\mu t}\, t\, D_N(t)\,\mathrm{d}t \tag{1.7}$$

The expression above differs from the usual Laplace transform in that we weighted the exponential in such a way that $t \mapsto \mu e^{-\mu t}$ is a probability density, which will make some expressions later on more pleasant.

Before stating our first result on the bulk diffusivity of H_N, let us set for lightness of notation

$$\mathrm{L}(x,0) := 1 + \lambda^2 \log(1 + x^{-1})$$

(the second argument of L is there just for coherence with the notation introduced in (3.10) below) and note that

$$\mathrm{L}(x,0) \overset{x\to 0^+}{\sim} \lambda^2 |\log x|.$$

Theorem 1.1. *Let $\lambda > 0$ and, for $N \in \mathbb{N}$, D_N be defined according to (1.6) and $\mathcal{D}_N$ be its Laplace transform as in (1.7). Then, for every $\delta > 0$ there exists a constant $0 < c_{\mathrm{bulk}} < \infty$ such that for any $\mu > 0$ sufficiently small*

$$\limsup_{N\to\infty} \mathcal{D}_N(\mu) \le \frac{c_{\mathrm{bulk}}}{\mu}\sqrt{\mathrm{L}(\mu,0)}(\log \mathrm{L}(\mu,0))^{5+\delta} \tag{1.8}$$

and

$$\liminf_{N\to\infty} \mathcal{D}_N(\mu) \ge \frac{1}{\mu\, c_{\mathrm{bulk}}}\sqrt{\mathrm{L}(\mu,0)}(\log \mathrm{L}(\mu,0))^{-5-\delta}. \tag{1.9}$$

Let us point out that by translating [37, Lemma 1] into our setting, the upper bound (1.8) can be turned into $D_N(t) \lesssim (1+\lambda^2\log(1+t))^{1/2+o(1)}$. In general the same cannot be said for the lower bound but, thanks to [20, Ch. XIII.5], $\mu\int_0^\infty e^{-\mu t} t\, f(t)\mathrm{d}t \sim \frac{1}{\mu}(\log(1/\mu))^{1/2}$ as $\mu\to 0$ implies $\frac{1}{T}\int_0^T t\, f(t)\mathrm{d}t \sim T(\log T)^{1/2}$ as $T\to\infty$. Thus, Theorem 1.1 says that, contrary to the linear stochastic heat equation, *the bulk diffusivity of the Anisotropic KPZ equation grows essentially as the square root of the logarithm of time*, at least in a weak Tauberian sense, thus suggesting a superdiffusive behaviour. Note that instead for the KPZ equation in $d=1$, [3] showed that the bulk diffusion coefficient grows in time as $t^{1/3}$.

As a side remark, the control of the sub-dominant corrections in Theorem 1.1 is sharper than the one obtained in [46] for $2d$-ASEP.

A natural question to ask when analysing stochastic PDEs of the form (1.3) is what happens when the regularisation is removed and this is closely related to the large-scale properties of H_N. To understand this point, let us pretend for a moment that the equation is defined on the whole plane instead of the torus, and note that rescaling the solution $H \stackrel{\text{def}}{=} H_\infty$ of (1.3) on $\mathbb{R}^2$, diffusively, that is, $H^\varepsilon(t,x) \stackrel{\text{def}}{=} H(t/\varepsilon^2, x/\varepsilon)$ for $\varepsilon>0$, one obtains the equation

$$\partial_t H^\varepsilon = \frac{1}{2}\Delta H^\varepsilon + \lambda\tilde{\mathcal{N}}^{1/\varepsilon}[H^\varepsilon] + \xi^\varepsilon \tag{1.10}$$

where the nonlinearity is now smoothed via a Fourier cut-off at ε^{-1}, that is,

$$\tilde{\mathcal{N}}^{1/\varepsilon}[H^\varepsilon] \stackrel{\text{def}}{=} \Pi_{1/\varepsilon}\big((\Pi_{1/\varepsilon}\partial_1 H^\varepsilon)^2 - (\Pi_{1/\varepsilon}\partial_2 H^\varepsilon)^2\big),$$

and the rescaled noise ξ^ε is equal in distribution to the original noise ξ.

Since H^ε (and H) are merely distributions (even for $\varepsilon>0$ fixed since the noise is not regularised), the random variables to be considered in this context are

$$H^\varepsilon(t)[\varphi] \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \varphi(x)H^\varepsilon(t,x)\mathrm{d}x = H(t/\varepsilon^2)[\varphi^{(\varepsilon)}], \qquad t\geq 0$$

for φ a smooth real-valued test function [from now on, for technical simplicity, φ is assumed to be at least C^1 and of compact support], and $\varphi^{(\varepsilon)}(\cdot) \stackrel{\text{def}}{=} \varepsilon^2\varphi(\varepsilon\cdot)$. Again, we want to avoid integrability issues, so we will be actually looking at the periodic version of the quantity above, namely

$$H_N^\varepsilon(t)[\varphi] \stackrel{\text{def}}{=} H_N(t/\varepsilon^2)[\varphi^{(\varepsilon)}] \tag{1.11}$$

in the regime when $N \gg \varepsilon^{-1}$ (morally, we are sending $N\to\infty$ first and then $\varepsilon\to 0$). For any fixed time t the distribution of $H_N(t)$ (and $H_N^\varepsilon(t)$) is the same for both $\lambda>0$ and $\lambda=0$ and is given by the GFF $\tilde\eta$; therefore, in order to set apart the behaviour in the two cases, we will focus on the covariance between $H_N^\varepsilon(t)[\varphi]$ and $H_N^\varepsilon(s)[\varphi]$, which depends only on $t-s$ by stationarity, or equivalently on the variance

$$V_\varphi^{\varepsilon,N}(t) = \mathbf{E}\big[H_N^\varepsilon(t)[\varphi] - H_N^\varepsilon(0)[\varphi]\big]^2, \qquad t>0, \tag{1.12}$$

whose Laplace transform is

$$\mathcal{V}_\varphi^{\varepsilon,N}(\mu) = \mu \int_0^\infty e^{-\mu t} V_\varphi^{\varepsilon,N}(t)\,\mathrm{d}t, \qquad \mu > 0. \tag{1.13}$$

To motivate the next result, let us recall that the linear equation ($\lambda = 0$) in the whole plane (i.e., for $N = \infty$) is invariant in law under diffusive scaling, that is, $H^\varepsilon|_{\lambda=0} \overset{\text{law}}{=} H|_{\lambda=0}$, as is apparent from (1.10). Equivalently, the random variable $H(t)[\varphi]|_{\lambda=0} - H(0)[\varphi]|_{\lambda=0}$ has the same law as $H(t/\varepsilon^2)[\varphi^{(\varepsilon)}]|_{\lambda=0} - H(0)[\varphi^{(\varepsilon)}]|_{\lambda=0}$. In fact, an explicit computation shows that for any $t > 0$

$$\lim_{N\to\infty} V_\varphi^{\varepsilon,N}(t)\Big|_{\lambda=0} = V_\varphi^\infty(t)\Big|_{\lambda=0} \overset{\text{def}}{=} \frac{1}{2\pi^2}\int_{\mathbb{R}^2} \frac{|\hat\varphi(k)|^2}{|k|^2}\left(1 - e^{-\frac{|k|^2}{2}t}\right)\mathrm{d}k$$

and consequently, the Laplace transform satisfies

$$\lim_{N\to\infty} \mathcal{V}_\varphi^{\varepsilon,N}(\mu)\Big|_{\lambda=0} = \mathcal{V}_\varphi^\infty(\mu)\Big|_{\lambda=0} = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{|\hat\varphi(k)|^2}{\mu + \frac{1}{2}|k|^2}\mathrm{d}k$$

for any $\mu > 0$. Note the following:

- if $\int \varphi(x)\mathrm{d}x \neq 0$ (so that $\hat\varphi(k)$ tends to a non-zero constant for $k \to 0$) then $t \mapsto V_\varphi^\infty(t)|_{\lambda=0}$ is a strictly increasing function that starts from 0 and grows as $\log t$ for $t \to \infty$, or equivalently, its Laplace transform, $\mu \mapsto \mathcal{V}_\varphi^\infty(\mu)|_{\lambda=0}$, is a strictly positive function that tends to zero as $\mu \to \infty$ and to $+\infty$ as $\mu \to 0$;
- if instead $\int \varphi(x)\mathrm{d}x = 0$ (so that $\hat\varphi(k) = O(k)$ as $k \to 0$, due to the smoothness of φ), then $t \mapsto V_\varphi^\infty(t)|_{\lambda=0}$ is again a strictly increasing function that starts from 0 but this time tends to a positive constant v_φ as $t \to \infty$ (v_φ equals twice the variance of $H[\varphi]$). For the Laplace transform we then have that $\mathcal{V}_\varphi^\infty(\mu)|_{\lambda=0}$ is strictly positive, uniformly bounded above and tends to zero as $\mu \to \infty$ and to v_φ for $\mu \to 0$.

It is now natural to ask if the AKPZ equation is at least asymptotically diffusively scale invariant, that is, if scale invariance holds asymptotically when first $N \to \infty$ and then $\varepsilon \to 0$. Our next result (see in particular Corollary 1.3 and the subsequent discussion) corroborates Theorem 1.1 and again strongly indicates that *this is not the case*. More precisely, it suggests that, in order to stand any chance for H_N^ε to converge to some limit, one should rescale time as $t \mapsto t/(\varepsilon^2|\log\varepsilon|^{1/2})$ (possibly up to corrections polynomial in $\log|\log\varepsilon|$).

Theorem 1.2. *For $N \in \mathbb{N}$ and $\lambda > 0$, let H_N be the solution of (1.3) started from the invariant measure and let $\varphi : \mathbb{R}^2 \mapsto \mathbb{R}$ be compactly supported and C^∞. For every $\delta > 0$ there exists $c_\delta > 0$ independent of φ such that the following statements hold for some constants $a_\varphi, b > 0$:*

- *defining $\mathcal{V}_\varphi^{\varepsilon,N}$ according to (1.13),*

$$\limsup_{N\to\infty} \mathcal{V}_\varphi^{\varepsilon,N}(\mu) \leq \frac{c_\delta}{\mu}\sqrt{\mathrm{L}(\mu\varepsilon^2,0)}(\log \mathrm{L}(\mu\varepsilon^2,0))^{5+\delta}\|\varphi\|_{L^2(\mathbb{R}^2)}^2\,; \tag{1.14}$$

- *if* $\mu = \mu(\varepsilon) \in [a_\varphi, (1/c_\delta)\sqrt{\mathrm{L}(\varepsilon,0)}(\log \mathrm{L}(\varepsilon,0))^{-5-\delta}]$, *then*

$$\liminf_{\varepsilon\to 0}\liminf_{N\to\infty} \mathcal{V}^{\varepsilon,N}_\varphi(\mu) \geq b\|\varphi\|^2_{-1} \stackrel{def}{=} b\int_{\mathbb{R}^2}\frac{|\hat\varphi(p)|^2}{|p|^2}\mathrm{d}p \tag{1.15}$$

(the integral is finite iff $\int_{\mathbb{R}^2}\varphi(x)\mathrm{d}x = 0$*).*

The restriction $\mu(\varepsilon) \geq a_\varphi$ in the lower bound is purely technical; at any rate, the interesting regime for our purposes (see the proof of Corollary 1.3) corresponds to $\mu(\varepsilon)$ diverging as $\approx \sqrt{|\log\varepsilon|}$. To appreciate the meaning of Theorem 1.2, note that, defining

$$t_-(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{\sqrt{|\log\varepsilon|}}(\log|\log\varepsilon|)^{-5-\delta}, \quad t_+(\varepsilon) \stackrel{\text{def}}{=} \frac{1}{\sqrt{|\log\varepsilon|}}(\log|\log\varepsilon|)^{5+\delta}, \tag{1.16}$$

$\mathcal{V}^{\varepsilon,N}_\varphi(\mu)$ is essentially zero if $\mu \gtrsim 1/t_-(\varepsilon)$ while it is strictly positive (or exploding, if $\int_{\mathbb{R}^2}\varphi(x)\mathrm{d}x \neq 0$) if $\mu \lesssim 1/t_+(\varepsilon)$. Using the scaling relation

$$\mu\int_0^\infty e^{-\mu t}V^{\varepsilon,N}_\varphi(t/\tau)\mathrm{d}t = \mathcal{V}^{\varepsilon,N}_\varphi(\mu\tau), \quad \tau > 0,$$

we see that the "correct" time scale to observe non-trivial correlations of the process H^ε_N is $\approx 1/(\varepsilon^2\sqrt{|\log\varepsilon|})$.

This observation can be made sharper in the case φ has zero average. In fact, with little extra work, we will deduce from Theorem 1.2 the following corollary.

Corollary 1.3. *Let φ be a compactly supported, C^∞ test function of zero mean, and let $\delta > 0$. One has, with $t_\pm(\varepsilon)$ defined as in (1.16),*

$$\inf_{t\leq t_-(\varepsilon)}\liminf_{N\to\infty}\frac{\mathrm{Cov}(H^\varepsilon_N(t)[\varphi], H^\varepsilon_N(0)[\varphi])}{\mathrm{Var}(H^\varepsilon_N(0)[\varphi])} = 1. \tag{1.17}$$

On the other hand, there exists $t = t(\varepsilon) \in (t_-(\varepsilon), t_+(\varepsilon))$ and $a < 1$ independent of ε, φ such that

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\frac{\mathrm{Cov}(H^\varepsilon_N(t)[\varphi], H^\varepsilon_N(0)[\varphi])}{\mathrm{Var}(H^\varepsilon_N(0)[\varphi])} \leq a. \tag{1.18}$$

In other words, $H^\varepsilon_N(t)[\varphi]$ and $H^\varepsilon_N(0)[\varphi]$ are almost perfectly correlated for times smaller than $t_-(\varepsilon)$ but, contrary to what happens in the linear case, they decorrelate non-trivially already on a time-scale of order $t_-(\varepsilon) \leq t(\varepsilon) \leq t_+(\varepsilon) \ll 1$. To see the relation with Theorem 1.2, note first that if $\int\varphi(x)\mathrm{d}x = 0$ then by stationarity the variance of $H^\varepsilon_N(0)[\varphi]$ is finite uniformly in both N and ε (for $N \to \infty$, it tends to $(2\pi)^{-2}\|\varphi\|^2_{-1}$). Note also that

$$\frac{\mathrm{Cov}(H^\varepsilon_N(t)[\varphi], H^\varepsilon_N(0)[\varphi])}{\mathrm{Var}(H^\varepsilon_N(0)[\varphi])} = 1 - \frac{V^{\varepsilon,N}_\varphi(t)}{2\mathrm{Var}(H^\varepsilon_N(0)[\varphi])}. \tag{1.19}$$

Remark 1.4. The existence of the $N \to \infty$ limits is shown in [12] but with a slightly different regularisation (the cut-off chosen is smooth in Fourier space) so we preferred to state the above results with liminf and limsup. Actually, as will appear from the proof, Theorem 1.2 holds in the more general setting where $\varepsilon \to 0$ and $N \to \infty$ jointly, with $N\varepsilon \to \infty$.

Our last result is a bit different in spirit and our main motivation here is to establish a connection with similar statements proven for *discrete* growth models in the AKPZ universality class, as for instance in [14, 29, 41]. In the discrete setting, one natural viewpoint is to look at the large-time behaviour of the height at a single point, and in particular at the growth of its variance. Since, as remarked above, point evaluation is not possible in the present context, we look at the locally averaged field, that is, we test H_N against a *fixed* test function φ and obtain an upper bound on $V_\varphi^{1,N}(t)$ in the $N \to \infty$ limit, for t arbitrarily large.

Theorem 1.5. *For $N \in \mathbb{N}$, let H_N be the solution of (1.3), started from the invariant measure. For any compactly supported test function φ on $\mathbb{R}^2$ there exists $c_\varphi > 0$ such that, for every $t > 0$,*

$$\limsup_{N\to\infty} V_\varphi^{1,N}(t) \le c_\varphi(1+\lambda^2)\max(\log t, 1). \tag{1.20}$$

Remark 1.6. It is well known (and it can be checked using the explicit solution) that for the linear equation, one has

$$\lim_{N\to\infty} V_\varphi^{1,N}(t)\Big|_{\lambda=0} \overset{t\to\infty}{\sim} c_\varphi \log t, \quad \lambda = 0. \tag{1.21}$$

While (1.20) and (1.21) show the same large-time behaviour (at least as an upper bound), this *is not in contradiction with the fact that the relevant scaling for the process with $\lambda > 0$ is different from diffusive* as shown in Theorems 1.1 and 1.2. Indeed, the $\log t$ behaviour is not a distinguishing feature of the two-dimensional stochastic heat equation. For instance consider the fractional stochastic heat equation

$$\partial_t Z = -\tfrac{1}{2}(-\Delta)^\theta Z + (-\Delta)^{\frac{\theta-1}{2}}\xi$$

with $Z = Z(t,x)$, $t \ge 0$, $x \in \mathbb{R}^2$, $\theta \in (0,1)$, ξ a space-time white noise as above and $(-\Delta)^\theta$ acting in Fourier space as a multiplication by $|k|^{2\theta}$. It is easily checked that the GFF on the whole plane is stationary for this equation and Z is scale invariant under the superdiffusive scaling $t \to t/\varepsilon^{2\theta}$, $x \to x/\varepsilon$. Nonetheless, an explicit computation shows that for the stationary process

$$\mathbf{E}[z(t)[\varphi] - z(0)[\varphi]]^2 = \frac{1}{2\pi^2}\int_{\mathbb{R}^2} \frac{|\hat\varphi(k)|^2}{|k|^2}\left(1 - e^{-\frac{t}{2}|k|^{2\theta}}\right)\mathrm{d}k \overset{t\to\infty}{\sim} c_{\varphi,\theta}\log t, \tag{1.22}$$

as is the case for the usual stochastic heat equation where $\theta = 1$. Note that, in contrast, in dimension $d = 1$, the large-time behaviour of (1.22) is power-law for large t, with a θ-dependent exponent $1/(2\theta)$.

Organisation of the article

The rest of this work is organised as follows. In Section 2 we turn the equation (1.3) into a regularised Burgers equation, we introduce some preliminary formalism and we recall some basic results from [10]. Section 3 is the core of the work and the main outcome are upper and lower bounds on the variance of the time integral of the nonlinearity. Given those bounds, Theorems 1.1 and 1.2 are proven in Section 4. The proof of Theorem 1.5 is instead based on different (simpler) tools and it is contained in Section 4.3. Finally, in the appendix we provide a heuristic for the Green-Kubo formula (1.6), a heuristic argument explaining our main result of $\sqrt{\log t}$ diffusivity, and collect some technical results.

Notation

For $N > 0$, let $\mathbb{Z}_N \stackrel{\text{def}}{=} \mathbb{Z}/N$ and $\mathbb{T}^2_N$ be the two-dimensional torus of side length $2\pi N$. If $N = 1$ then we simply write $\mathbb{T}^2$ instead of $\mathbb{T}^2_N$. We denote by $\{e_k\}_{k\in\mathbb{Z}^2_N}$ the Fourier basis defined via $e_k(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}e^{ik\cdot x}$ which, for all $j,\ k \in \mathbb{Z}^2_N$, satisfies $\langle e_k, e_{-j}\rangle_{L^2(\mathbb{T}^2_N)} = \delta_{k,j}N^2$.

The Fourier transform of a given function $\varphi \in L^2(\mathbb{T}^2_N)$ will be represented as $\mathcal{F}(\varphi)$ or by $\hat{\varphi}$ and, for $k \in \mathbb{Z}^2_N$ is given by the formula

$$\mathcal{F}(\varphi)(k) = \hat{\varphi}(k) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2_N} \varphi(x)e_{-k}(x)\mathrm{d}x, \tag{1.23}$$

so that in particular

$$\varphi(x) = \frac{1}{N^2}\sum_{k\in\mathbb{Z}^2_N} \hat{\varphi}(k)e_k(x), \qquad \text{for all } x \in \mathbb{T}^2_N. \tag{1.24}$$

For any real valued distribution $\eta \in \mathcal{D}'(\mathbb{T}^2_N)$ and $k \in \mathbb{Z}^2_N$, we will denote its Fourier transform by

$$\hat{\eta}(k) \stackrel{\text{def}}{=} \eta(e_{-k}) \tag{1.25}$$

and note that $\overline{\eta(e_k)} = \eta(e_{-k})$. Moreover, we recall that the Laplacian Δ on $\mathbb{T}^2_N$ has eigenfunctions $\{e_k\}_{k\in\mathbb{Z}^2_N}$ with eigenvalues $\{-|k|^2 \ :\ k \in \mathbb{Z}^2_N\}$, so that, for $\theta > 0$, we can define the operator $(-\Delta)^\theta$ by its action on the basis elements

$$(-\Delta)^\theta e_k \stackrel{\text{def}}{=} |k|^{2\theta} e_k, \tag{1.26}$$

for $k \in \mathbb{Z}^2_N$.

Throughout the paper, we will write $a \lesssim b$ if there exists a constant $C > 0$ such that $a \le Cb$ and $a \sim b$ if $a \lesssim b$ and $b \lesssim a$. We will adopt the previous notations only in case in which the hidden constants do not depend on any quantity which is relevant for the result.

2 | PRELIMINARIES

The aim of this section is twofold. On the one hand, we will state some basic tools from Wiener space analysis that we will need in the rest of the paper while on the other hand we will reduce the analysis of (1.3) to the torus of length size 1, that is, to the setting of [10], and recall some of the results on the Anisotropic KPZ equation obtained therein.

Notice at first that an immediate scaling argument guarantees that for any $N \in \mathbb{N}$,

$$H_N(t,x) \stackrel{\text{law}}{=} h^N(t/N^2, x/N), \qquad t \geq 0 \text{ and } x \in \mathbb{T}^2_N \tag{2.1}$$

where h^N is the solution of

$$\partial_t h^N = \tfrac{1}{2}\Delta h^N + \lambda \tilde{\mathcal{N}}^N(h^N) + \xi, \qquad h^N(0) = \tilde{\eta} \tag{2.2}$$

in which $h^N = h^N(x,t)$ for $t \geq 0$, $x \in \mathbb{T}^2 \stackrel{\text{def}}{=} \mathbb{T}^2_1$, and all the other quantities are defined as in the discussion after (1.3) (with the only change that all quantities are now defined on $\mathbb{T}^2$). Therefore, even though all the statements in the introduction as well as the results we aim for are formulated (and ultimately proved) for the solution H_N of (1.3), (2.1) guarantees that we can focus instead on h^N since whatever is shown for the latter can then be translated back to H_N.

As in [10], it turns out to be convenient to work with the Stochastic Burgers equation instead of AKPZ, which can be derived from (2.2) by setting $u^N \stackrel{\text{def}}{=} (-\Delta)^{\frac{1}{2}} h^N$ so that u^N solves

$$\partial_t u^N = \tfrac{1}{2}\Delta u^N + \lambda \mathcal{M}^N[u^N] + (-\Delta)^{\frac{1}{2}}\xi, \quad u^N(0) = \eta \stackrel{\text{def}}{=} (-\Delta)^{\frac{1}{2}}\tilde{\eta} \tag{2.3}$$

where $u^N = u^N(t,x)$, $t \geq 0$, $x \in \mathbb{T}^2$, and the nonlinearity $\mathcal{M}^N$ is given by

$$\mathcal{M}^N[u^N] \stackrel{\text{def}}{=} (-\Delta)^{\frac{1}{2}} \Pi_N \Big((\Pi_N \partial_1 (-\Delta)^{-\frac{1}{2}} u^N)^2 - (\Pi_N \partial_2 (-\Delta)^{-\frac{1}{2}} u^N)^2 \Big). \tag{2.4}$$

Note that, since $\tilde{\eta}$ is a standard GFF, η is a (spatial) white noise on $\mathbb{T}^2$ whose basic properties are recalled in the next section (for more on it see [33, Chapter 1], or [22, 23] and [10, Section 2]).

2.1 | Elements of Wiener space analysis

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and η be a mean-zero spatial white noise on the two-dimensional torus $\mathbb{T}^2$, that is, η, defined in Ω, is a centred isonormal Gaussian process (see [33, Definition 1.1.1]), on $H \stackrel{\text{def}}{=} L^2_0(\mathbb{T}^2)$, the space of square-integrable functions with 0 total mass, whose covariance function is given by

$$\mathbb{E}[\eta(\varphi)\eta(\psi)] = \langle \varphi, \psi \rangle_{L^2(\mathbb{T}^2)} \tag{2.5}$$

where $\varphi, \psi \in H$ and $\langle \cdot, \cdot \rangle_{L^2(\mathbb{T}^2)}$ is the usual scalar product in $L^2(\mathbb{T}^2)$. For $n \in \mathbb{N}$, let $\mathscr{H}_n$ be the *n-th homogeneous Wiener chaos*, that is, the closed linear subspace of $L^2(\eta) \stackrel{\text{def}}{=} L^2(\Omega)$ generated by the random variables $H_n(\eta(h))$, where H_n is the n-th Hermite polynomial, and $h \in H$ has norm

1. By [33, Theorem 1.1.1], $\mathscr{H}_n$ and $\mathscr{H}_m$ are orthogonal whenever $m \neq n$ and $L^2(\eta) = \bigoplus_{n\geq 0} \mathscr{H}_n$. Moreover, there exists a canonical contraction $I : \bigoplus_{n\geq 0} L^2(\mathbb{T}^{2n}) \to L^2(\eta)$, which restricts to an isomorphism $I : \Gamma L^2 \to L^2(\eta)$ on the Fock space $\Gamma L^2 := \bigoplus_{n\geq 0} \Gamma L^2_n$, where ΓL^2_n denotes the space $L^2_{\mathrm{sym}}(\mathbb{T}^{2n})$ of functions in $L^2(\mathbb{T}^{2n})$ which are symmetric with respect to permutation of variables. The restriction of I to ΓL^2_n, I_n, called n-th (iterated) Wiener-Itô integral with respect to η, is itself an isomorphism from ΓL^2_n to $\mathscr{H}_n$ so that by [33, Theorem 1.1.2], for every $F \in L^2(\eta)$ there exists $f = (f_n)_{n\in\mathbb{N}} \in \Gamma L^2$ such that

$$F = \sum_{n\geq 0} I_n(f_n) \qquad \text{and} \qquad \|F\|^2_\eta = \sum_{n\geq 0} n!\|f_n\|^2_{L^2(\mathbb{T}^{2n})} \tag{2.6}$$

and we take the right hand side as the definition of the scalar product on ΓL^2, that is,

$$\langle f, g\rangle_{\Gamma L^2} \stackrel{\text{def}}{=} \sum_{n\geq 0}\langle f_n, g_n\rangle_{\Gamma L^2_n} \stackrel{\text{def}}{=} \sum_{n\geq 0} n!\langle f_n, g_n\rangle_{L^2(\mathbb{T}^{2n})}. \tag{2.7}$$

We conclude this paragraph by mentioning that we will mainly work with the Fourier representation $\{\hat\eta(k)\}_k$ of η, which is a family of complex valued, centred Gaussian random variables such that

$$\hat\eta(0) = 0, \quad \overline{\hat\eta(k)} = \hat\eta(-k) \quad \text{and} \quad \mathbb{E}[\hat\eta(k)\hat\eta(j)] = \mathbb{1}_{k+j=0}. \tag{2.8}$$

2.2 | Stochastic burgers equation and its generator

The properties of equation (2.3) which will be important for us were obtained in [10, Section 3]. In order to fix the relevant notations, below we recall the Fourier representation of (2.3) and summarise some of its features referring to [10] for the proofs.

The Fourier representation of (2.3) is equivalent to an infinite system of (complex-valued) SDEs given by

$$\mathrm{d}\hat u^N(k) = \left(-\frac{1}{2}|k|^2\hat u^N(k) + \lambda\mathcal{M}^N_k[u^N]\right)\mathrm{d}t + |k|\mathrm{d}B_k(t), \qquad k \in \mathbb{Z}^2\setminus\{0\}. \tag{2.9}$$

The k-th Fourier component of the nonlinearity is

$$\mathcal{M}^N_k[u^N] \stackrel{\text{def}}{=} \mathcal{M}^N[u^N](e_{-k}) = |k|\sum_{\ell+m=k}\mathcal{K}^N_{\ell,m}\hat u^N(\ell)\hat u^N(m), \tag{2.10}$$

$$\mathcal{K}^N_{\ell,m} \stackrel{\text{def}}{=} \frac{1}{2\pi}\frac{c(\ell,m)}{|\ell||m|}\mathbb{J}^N_{\ell,m}, \qquad c(\ell,m) \stackrel{\text{def}}{=} \ell_2 m_2 - \ell_1 m_1 \tag{2.11}$$

where $\ell = (\ell_1,\ell_2)$, $m = (m_1, m_2) \in \mathbb{Z}^2$ and

$$\mathbb{J}^N_{\ell,m} \stackrel{\text{def}}{=} \mathbb{1}_{0<|\ell|\leq N, 0<|m|\leq N, |\ell+m|\leq N} \tag{2.12}$$

and all the variables in the sum (2.10) range over $\mathbb{Z}^2 \setminus \{0\}$ (the value 0 is automatically excluded by the definition of $\mathbb{J}^N$).

In (2.9), the B_k's are complex valued Brownian motions defined via $B_k(t) \stackrel{\text{def}}{=} \int_0^t \hat{\xi}(s,k)\,ds$, $\hat{\xi}(k) = \xi(e_{-k})$, so that (recalling that ξ is a space-time white noise)

$$\overline{B_k} = B_{-k}, \qquad \text{and} \qquad \mathrm{d}\langle B_k, B_\ell \rangle_t = \mathbb{1}_{\{k+\ell=0\}}\,\mathrm{d}t.$$

Since eventually we are interested in h^N rather than in u^N, note that $(-\Delta)^{\frac{1}{2}}$ is an invertible linear bijection on functions with zero mass, so that we can recover all the non-zero Fourier components of h^N via

$$\hat{h}^N(k) = \frac{\hat{u}^N(k)}{|k|}, \quad k \neq 0 \tag{2.13}$$

On the other hand, the zero-mode $\hat{h}^N(0)$ is also a function of u^N and of an independent Brownian motion, since it satisfies

$$\mathrm{d}\hat{h}^N(0) = \lambda \mathcal{N}_0^N + \mathrm{d}B_0(t) \tag{2.14}$$

where

$$\mathcal{N}_0^N = \sum_{\ell+m=0} \mathcal{K}_{\ell,m}^N \hat{u}^N(\ell)\hat{u}^N(m).$$

Proposition 3.4 of [10] guarantees that, for any $N \in \mathbb{N}$, the process $t \mapsto \hat{u}^N(t) = \{\hat{u}^N(t,k)\}_{k\in\mathbb{Z}^2\setminus\{0\}}$ solution to (2.9) is a strong Markov process and we denote its generator by $\mathcal{L}^N$. If the initial condition is white noise, the law of the process is also translation invariant. Let F be a cylinder function acting on the space of distributions $\mathcal{D}'(\mathbb{T}^2)$ and depending only on finitely many Fourier components, that is, F is such that there exists a smooth function $f = f((x_k)_{k\in\mathbb{Z}^2\setminus\{0\}})$ with all derivatives growing at most polynomially and depending only on finitely many variables, for which $F(\eta) = f((\hat{\eta}(k))_{k\in\mathbb{Z}^2\setminus\{0\}})$. Then, $\mathcal{L}^N$ can be written as the sum of $\mathcal{L}_0$ and $\mathcal{A}^N$, whose action on F as above is given by

$$(\mathcal{L}_0 F)(v) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^2} \frac{1}{2}|k|^2(-\hat{v}(-k)D_k + D_{-k}D_k)F(v) \tag{2.15}$$

$$(\mathcal{A}^N F)(v) = \lambda \sum_{m,\ell\in\mathbb{Z}^2\setminus\{0\}} |m+\ell|\mathcal{K}_{m,\ell}^N \hat{v}(m)\hat{v}(\ell)D_{-m-\ell}F(v). \tag{2.16}$$

Here, for $k \in \mathbb{Z}^2$ and F as above, $D_k F$ is defined as[3]

$$D_k F \stackrel{\text{def}}{=} (\partial_{x_k} f)((\hat{\eta}(k))_{k\in\mathbb{Z}^2\setminus\{0\}}). \tag{2.17}$$

[3] For more on the actual definition of cylinder function, Malliavin derivative and the formula below, we address the reader to [10, Section 2 and Lemma 2.1]

In the following lemma and throughout the remainder of the paper, we will slightly abuse notations and use the same symbol to denote an operator acting on (a subspace of) $L^2(\eta)$ and its Fock space version.

Lemma 2.1. [10, Lemmas 3.1 and 3.5] *For any $N \in \mathbb{N}$, the spatial white noise η on $\mathbb{T}^2$ defined in (2.5) is invariant for the solution $\hat{u}^N$ of (2.9) and, with respect to η, the symmetric and anti-symmetric part of $\mathcal{L}^N$ are given by $\mathcal{L}_0$ and $\mathcal{A}^N$, respectively.*

Moreover, for all $n \in \mathbb{N}$ the operator $\mathcal{L}_0$ leaves $\mathscr{H}_n$ invariant, while $\mathcal{A}^N$ can be written as the sum of two operators $\mathcal{A}_+^N$ and $\mathcal{A}_-^N$ which respectively map $\mathscr{H}_n$ into $\mathscr{H}_{n+1}$ and $\mathscr{H}_{n-1}$ and are such that $-\mathcal{A}_+^N$ is the adjoint of $\mathcal{A}_-^N$.

Finally, on the Fock space ΓL^2, we have that $\mathcal{L}_0 = -\frac{1}{2}\Delta$ and, in Fourier variables, the action $\mathcal{L}_0$, $\mathcal{A}_-^N$ and $\mathcal{A}_+^N$ on $\varphi_n \in \Gamma L^2_n$ is given by

$$\mathcal{F}(\mathcal{L}_0\varphi_n)(k_{1:n}) = \frac{1}{2}|k_{1:n}|^2\hat{\varphi}_n(k_{1:n}) \tag{2.18}$$

$$\mathcal{F}(\mathcal{A}_+^N\varphi_n)(k_{1:n+1}) = n\lambda|k_1+k_2|\mathcal{K}^N_{k_1,k_2}\hat{\varphi}_n(k_1+k_2,k_{3:n+1}) \tag{2.19}$$

$$\mathcal{F}(\mathcal{A}_-^N\varphi_n)(k_{1:n-1}) = 2n(n-1)\lambda\sum_{\ell+m=k_1}|m|\mathcal{K}^N_{k_1,-\ell}\hat{\varphi}_n(\ell,m,k_{2:n-1}), \tag{2.20}$$

where the functions on the right hand side need to be symmetrised with respect to all permutations of their arguments (see, e.g. (3.23)). In (2.18)–(2.20), all the variables belong to $\mathbb{Z}^2\setminus\{0\}$ and we adopted the short-hand notations $k_{1:n} \stackrel{def}{=} (k_1,k_2,\dots,k_n)$ and $|k_{1:n}|^2 \stackrel{def}{=} |k_1|^2+\cdots+|k_n|^2$.

For later purposes, let us introduce the following definition:

Definition 2.2. An operator $\mathcal{Z}$ on ΓL^2 is said to be diagonal if for every n it maps ΓL^2_n into itself, and it acts in Fourier space as a multiplier, that is there exists a sequence of symmetric functions $\zeta = (\zeta_n)_{n\geq 1}$ such that for all n and $\varphi \in \Gamma L^2_n$, one has $\mathcal{F}(\mathcal{Z}\varphi)(k_{1:n}) = \zeta_n(k_{1:n})\hat{\varphi}(k_{1:n})$.

In this sense, the operator $\mathcal{L}_0$ is diagonal while $\mathcal{A}_+^N, \mathcal{A}_-^N$ are clearly not.

3 | THE VARIANCE OF THE NONLINEARITY

The present section represents the bulk of the paper and focuses on the term in (1.3) that distinguishes SHE and AKPZ, that is, the nonlinearity. In particular, we aim at estimating from above and below the Laplace transform of the second moment of the time integral of $\tilde{\mathcal{N}}^N(h^N)$ tested against a suitable test function, see Proposition 3.13 for the result we are after. We will then see later in Section 4 that the nonlinearity gives the dominant contribution to the bulk diffusivity D_N (Lemma 4.1) and to $h^N(t)[\varphi] - h^N(0)[\varphi]$ (Proposition 4.2).

Let φ be a sufficiently regular test function and, for $t \geq 0$, denote the time integral of the nonlinearity against φ by

$$B^N_\varphi(t) \stackrel{\text{def}}{=} \int_0^t \lambda\tilde{\mathcal{N}}^N[h^N(s)][\varphi]\mathrm{d}s = \int_0^t \lambda\mathcal{N}^N[u^N(s)][\varphi]\mathrm{d}s \tag{3.1}$$

where the second equality is an immediate consequence of (2.11) and (2.13) once we set

$$\mathcal{N}^N[u^N][\varphi] \stackrel{\text{def}}{=} \sum_{\ell,m\in\mathbb{Z}^2} \mathcal{K}^N_{\ell,m}\hat{u}^N(\ell)\hat{u}^N(m)\hat{\varphi}(-\ell-m). \tag{3.2}$$

In the stationary process, the random variable B^N_φ is centred. This follows from (3.2) and from the anti-symmetry of $\mathcal{K}^N_{\ell,m}$ under $\ell=(\ell_1,\ell_2)\mapsto(\ell_2,\ell_1), m=(m_1,m_2)\mapsto(m_2,m_1)$. Its variance then coincides with its second moment.

By (3.1), (2.6) and [10, Lemma 5.1], for any $\mu>0$, the Laplace transform of the second moment of B^N_φ satisfies

$$\mathcal{B}^N_\varphi(\mu) \stackrel{\text{def}}{=} \mu\int_0^\infty e^{-\mu t}\mathbf{E}\Big[\big(B^N_\varphi(t)\big)^2\Big]\mathrm{d}t = \frac{2}{\mu}\left\langle \mathfrak{n}^N_\varphi, (\mu-\mathcal{L}^N)^{-1}\mathfrak{n}^N_\varphi\right\rangle_{\Gamma L^2} \tag{3.3}$$

where $\mathcal{L}^N$ is the generator of u^N and $\mathfrak{n}^N_\varphi$ is the representation in Fock space of $\lambda\mathcal{N}^N[\eta](\varphi)$, that is,

$$\lambda\mathcal{N}^N[\eta](\varphi) = I_2(\mathfrak{n}^N_\varphi) \quad\text{with}\quad \hat{\mathfrak{n}}^N_\varphi(\ell,m) = \lambda\mathcal{K}^N_{\ell,m}\hat{\varphi}(-\ell-m), \quad \ell,m\in\mathbb{Z}^2 \tag{3.4}$$

as can be read off (3.2).

Now, in order to control B^N_φ we need to improve our understanding of the scalar product at the right hand side of (3.3), which in particular means that we need to invert $\mu-\mathcal{L}^N$, which is though not feasible in view of the singularity induced by the antisymmetric part of operator $\mathcal{L}^N$, that is, $\mathcal{A}^N$. To overcome this difficulty we will exploit a technique first established in [28] and explored in full strength in [46], where the authors studied the superdiffusivity of the asymmetric simple exclusion process in dimension $d=1,2$, and which essentially consists in truncating the resolvent equation. To be more precise for $n\in\mathbb{N}$, let $I_{\le n}$ be the projection onto $\Gamma L^2_{\le n} \stackrel{\text{def}}{=} \bigoplus_{k=0}^n \Gamma L^2_k$ and $\mathcal{L}^N_n = I_{\le n}\mathcal{L}^N I_{\le n}$. Then, let $\mathfrak{h}^{N,n} \stackrel{\text{def}}{=} (\mathfrak{h}^{N,n}_j)_{j\le n}\in\Gamma L^2_{\le n}$ be the solution of the *truncated generator equation*

$$(\mu-\mathcal{L}^N_n)\mathfrak{h}^{N,n} = \mathfrak{n}^N_\varphi \tag{3.5}$$

(which will be given explicitly below), and further write $\mathfrak{h}^N=(\mu-\mathcal{L}^N)^{-1}\mathfrak{n}^N_\varphi$. The property of $\mathfrak{h}^{N,n}$ that allows one to reduce the analysis to that of the truncated resolvent equation is stated in the following lemma, derived in [28, Lemma 2.1].

Lemma 3.1. *Let $\mu>0$. Then, for every $n\in\mathbb{N}$, we have that*

$$\left\langle \mathfrak{n}^N_\varphi, \mathfrak{h}^{N,2n+1}\right\rangle_{\Gamma L^2} \le \left\langle \mathfrak{n}^N_\varphi, (\mu-\mathcal{L}^N)^{-1}\mathfrak{n}^N_\varphi\right\rangle_{\Gamma L^2} \le \left\langle \mathfrak{n}^N_\varphi, \mathfrak{h}^{N,2n}\right\rangle_{\Gamma L^2}.$$

Moreover, the sequence $\{\langle\mathfrak{n}^N_\varphi,\mathfrak{h}^{N,2n+1}\rangle_{\Gamma L^2}\}_n$ is increasing while the sequence $\{\langle\mathfrak{n}^N_\varphi,\mathfrak{h}^{N,2n}\rangle_{\Gamma L^2}\}_n$ is decreasing and they both converge to $\langle\mathfrak{n}^N_\varphi,\mathfrak{h}^N\rangle_{\Gamma L^2}$ as $n\to\infty$.

Notice that, thanks to Lemma 3.1, on the one hand we have reduced the problem of studying the solution of the full generator equation (which is the same as (3.5) with $\mathcal{L}^N$ replacing $\mathcal{L}^N_n$) to that of its truncated version given in (3.5). On the other hand, by orthogonality

$$\left\langle \mathfrak{n}^N_\varphi, \mathfrak{h}^{N,n}\right\rangle_{\Gamma L^2} = \left\langle \mathfrak{n}^N_\varphi, \mathfrak{h}^{N,n}_2\right\rangle_{\Gamma L^2_2}, \qquad \text{for all } n\in\mathbb{N},$$

so that we only need to determine the component of $\mathfrak{h}^{N,n}$ in ΓL^2_2.

Getting back to (3.5), by Lemma 2.1 $\mathcal{L}^N$ can be decomposed in the sum of $\mathcal{L}_0$, $\mathcal{A}_+^N$ and $\mathcal{A}_-^N$, the first of which leaves the order of the Wiener chaos component invariant, whereas the others respectively increase and decrease it by 1. Thus, the truncated generator equation coincides with the following hierarchical system

$$\begin{cases} (\mu-\mathcal{L}_0)\mathfrak{h}_n^{N,n} - \mathcal{A}_+^N\mathfrak{h}_{n-1}^{N,n} = 0, \\ (\mu-\mathcal{L}_0)\mathfrak{h}_{n-1}^{N,n} - \mathcal{A}_+^N\mathfrak{h}_{n-2}^{N,n} - \mathcal{A}_-^N\mathfrak{h}_n^{N,n} = 0, \\ \dots \\ (\mu-\mathcal{L}_0)\mathfrak{h}_2^{N,n} - \mathcal{A}_+^N\mathfrak{h}_1^{N,n} - \mathcal{A}_-^N\mathfrak{h}_3^{N,n} = \mathfrak{n}_\varphi^N, \\ (\mu-\mathcal{L}_0)\mathfrak{h}_1^{N,n} - \mathcal{A}_-^N\mathfrak{h}_2^{N,n} = 0, \end{cases} \tag{3.6}$$

where, in the last equation we exploited the fact that $\mathcal{A}_+^N$ is 0 on constants by Lemma 2.1. Let us introduce the operators $\mathcal{H}_k^N$, $k \geq 2$, which are recursively defined via

$$\begin{cases} \mathcal{H}_2^N \equiv 0, \\ \mathcal{H}_k^N = -\mathcal{A}_-^N\left[(\mu-\mathcal{L}_0)+\mathcal{H}_{k-1}^N\right]^{-1}\mathcal{A}_+^N, \qquad k\geq 3 \end{cases} \tag{3.7}$$

and which satisfy the properties stated in the following lemma.

Lemma 3.2. *For any $k \geq 3$, the operators $\mathcal{H}_k^N$ in (3.7) are positive definite and such that, for all $n \in \mathbb{N}$, $\mathcal{H}_k^N(\mathscr{H}_n) \subset \mathscr{H}_n$.*

Proof. We first consider the case $k = 3$. Since $\mu - \mathcal{L}_0$ is positive for every $\mu \geq 0$, we have

$$\begin{aligned}\langle \mathcal{H}_3^N\psi,\psi\rangle_{\Gamma L^2} &= \left\langle -\mathcal{A}_-^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_+^N\psi,\psi\right\rangle_{\Gamma L^2} \\ &= \langle (\mu-\mathcal{L}_0)^{-1}\mathcal{A}_+^N\psi, \mathcal{A}_+^N\psi\rangle_{\Gamma L^2} = \left\| (\mu-\mathcal{L}_0)^{-\frac{1}{2}}\mathcal{A}_+^N\psi\right\|_{\Gamma L^2}^2 \geq 0\end{aligned}$$

while by Lemma 2.1, $\mathcal{H}_3^N(\mathscr{H}_n) \subset \mathscr{H}_n$. For $k > 3$, the result can be proved inductively using the recursive definition of $\mathcal{H}_k^N$. □

Now, upon solving (3.6) for $\mathfrak{h}^{N,n}$, starting from the first equation in (3.6) and using the definition of $\mathcal{H}_k^N$ in (3.7), we see that $\mathfrak{h}_2^{N,n}$ can be written as[4]

$$\mathfrak{h}_2^{N,n} = \left((\mu-\mathcal{L}_0)+\mathcal{H}_n^N - \mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N\right)^{-1}\mathfrak{n}_\varphi^N, \tag{3.8}$$

and consequently, for every $n \in \mathbb{N}$, we have

$$\langle \mathfrak{n}_\varphi^N, \mathfrak{h}^{N,n}\rangle_{\Gamma L^2} = \left\langle \mathfrak{n}_\varphi^N, \left((\mu-\mathcal{L}_0)+\mathcal{H}_n^N - \mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N\right)^{-1}\mathfrak{n}_\varphi^N\right\rangle_{\Gamma L^2_2}. \tag{3.9}$$

Therefore, to take advantage of Lemma 3.1, it suffices to derive suitable bounds on the operators $\mathcal{H}_n^N$ and $\mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N$ and the rest of the section is indeed devoted to this purpose.

[4] The other components $\mathfrak{h}_j^{N,n}$, $j \neq 2$ can also be written down explicitly in terms of $\mathfrak{h}_2^{N,n}$ and of the operators $\mathcal{H}_n^N$, but we do not need their explicit expression.

3.1 | An iterative approach for the operators $\mathcal{H}_n^N$'s

The advantage of dealing with diagonal (positive) operators, in the sense of Definition 2.2, is that their inverse is fully explicit and easily computable. The difficulty in getting upper and lower bounds for the operators $\mathcal{H}_n^N$ is that, even though they are diagonal with respect to the chaos, they are definitely not in Fourier space. Hence, it is a priori hard to determine any bound on their inverse, which is though essential given that, by (3.7), for all k, $\mathcal{H}_k^N$ is *defined* in terms of the inverse of $\mathcal{H}_{k-1}^N$. Therefore, the goal of this section is to show that it is possible to recursively estimate the $\mathcal{H}_k^N$'s in terms of diagonal operators $\mathcal{S}_k$ (see Theorem 3.4). Let us begin by giving some preliminary definitions, necessary to rigorously introduce the $\mathcal{S}_k$'s.

Let $k \in \mathbb{N}$ and $\lambda > 0$. For $x > 0$ and $z \geq 1$, we define the functions L, LB_k and UB_k (here LB stands for lower bound and UB for upper bound) on $\mathbb{R}_+ \times [1, \infty)$ as follows

$$\mathrm{L}(x,z) \stackrel{\mathrm{def}}{=} \lambda^2(z + \log(1 + x^{-1})) + 1, \tag{3.10}$$

$$\mathrm{LB}_k(x,z) \stackrel{\mathrm{def}}{=} \sum_{j\leq k} \frac{\left(\frac{1}{2}\log \mathrm{L}(x,z)\right)^j}{j!} \quad \text{and} \quad \mathrm{UB}_k(x,z) \stackrel{\mathrm{def}}{=} \frac{\mathrm{L}(x,z)}{\mathrm{LB}_k(x,z)}. \tag{3.11}$$

For $n \in \mathbb{N}$ and any $\delta > 0$, set

$$z_k(n) \stackrel{\mathrm{def}}{=} K_1(\lambda^2 \vee 1)(n+k)^{3+2\delta}, \qquad f_k(n) \stackrel{\mathrm{def}}{=} K_2\sqrt{z_k(n)} \tag{3.12}$$

where K_1, K_2 are sufficiently large absolute positive constants which will be fixed below (they will be chosen in such a way that (3.35) and (3.53) hold). As we will need the following fact below, note that for any $n,\ k \in \mathbb{N}$, z_k and f_k trivially satisfy

$$f_k(n+1) = f_{k+1}(n) \qquad \text{and} \qquad z_k(n+1) = z_{k+1}(n). \tag{3.13}$$

We are now ready to introduce the operators $\mathcal{S}_k$.

Definition 3.3. Let $\lambda, \mu > 0$. Let $\mathcal{S}_2^N$ be the operator which is identically equal to 0 and, for $k \in \mathbb{N}$, $k \geq 3$ and $N \in \mathbb{N}$, define $\mathcal{S}_k^N$ via

$$\mathcal{S}_k^N \stackrel{\mathrm{def}}{=} \begin{cases} f_k(\mathcal{N})\,\sigma_k^N(\mu - \mathcal{L}_0, z_k(\mathcal{N})), & \text{if } k \text{ is odd,} \\ \frac{1}{f_k(\mathcal{N})}\left[\sigma_k^N(\mu - \mathcal{L}_0, z_k(\mathcal{N})) - f_k(\mathcal{N})\right], & \text{if } k \text{ is even,} \end{cases} \tag{3.14}$$

where f_k and z_k are given in (3.12), $\mathcal{N}$ is the *number operator* – the operator such that, for any $g : \mathbb{N} \to \mathbb{R}$, the action of $g(\mathcal{N})$ in Fock space is $g(\mathcal{N})\varphi_n = g(n)\varphi_n$, $\varphi_n \in \Gamma L_n^2$ - and $\mathcal{L}_0$ is given as in (2.18). At last, the Fourier multiplier

$$\sigma_k^N(x,z) \stackrel{\mathrm{def}}{=} \sigma_k(x/N^2, z)$$

is such that

$$\sigma_k(x,z) \stackrel{\text{def}}{=} \begin{cases} \mathrm{UB}_{\frac{k-3}{2}}(x,z), & \text{if } k \text{ is odd,} \\ \mathrm{LB}_{\frac{k}{2}-1}(x \vee 1, z), & \text{if } k \text{ is even.} \end{cases} \tag{3.15}$$

In what follows we will write $\mathrm{L}^N(x,\cdot)$, $\mathrm{LB}_k^N(x,\cdot)$, $\mathrm{UB}_k^N(x,\cdot)$ for $\mathrm{L}(x/N^2,\cdot)$, $\mathrm{LB}_k(x/N^2,\cdot)$, $\mathrm{UB}_k(x/N^2,\cdot)$, respectively.

The following theorem is the main result of this section and establishes the previously announced bounds on the operators $\mathcal{H}_k^N$.

Theorem 3.4. *Let $\lambda > 0$, $\mu > 0$, $N \in \mathbb{N}$ and $\{\mathcal{H}_n^N\}_{n\geq 2}$ be the family of operators on $L^2(\eta)$ recursively defined according to (3.7). Then, for all $k \in \mathbb{N}$ and any $\delta > 0$ there exist constants $c_{2k+1}^+ = c_{2k+1}^+(\delta) > 1$ and $c_{2k+2}^- = c_{2k+2}^-(\delta) < 1$ independent of μ and N such that the following bounds hold*

$$\mathcal{H}_{2k+1}^N \leq c_{2k+1}^+ (-\mathcal{L}_0)\mathcal{S}_{2k+1}^N, \tag{3.16}$$

$$\mathcal{H}_{2k+2}^N \geq c_{2k+2}^- (-\mathcal{L}_0)\mathcal{S}_{2k+2}^N, \tag{3.17}$$

where the operators $\{\mathcal{S}_m\}_{m\geq 2}$ are given in Definition 3.3.[5] *Furthermore, the constants c_{2k+1}^+, c_{2k+2}^- can be chosen as $c_2^- = \frac{1}{2}$,*

$$c_{2k+1}^+ \stackrel{def}{=} \frac{1}{\pi c_{2k}^-}\left(1 + \frac{1}{2k^{1+\delta}}\right) \text{ and } c_{2k+2}^- \stackrel{def}{=} \frac{1}{\pi\left(1+\frac{1}{f_{2k+2}(1)}\right)c_{2k+1}^+}\left(1 - \frac{1}{2k^{1+\delta}}\right) \tag{3.18}$$

so that, in particular, the sequences $\{c_{2k+1}^+\}_{k\geq 1}$ and $\{c_{2k+2}^-\}_{k\geq 1}$ tend to two positive and finite limits as $k \to \infty$.

Remark 3.5. To obtain a better understanding of the importance of the above result note that since $\lim_{k\to\infty} \mathrm{LB}_k(x,z) = \lim_{k\to\infty} \mathrm{UB}(x,z) = \sqrt{\mathrm{L}(x,z)}$ Equations (3.16) and (3.17) essentially state that for k sufficiently large $\mathcal{H}_k^N \approx (-\mathcal{L}_0^N)\sqrt{\log(\mu - \mathcal{L}_0^N)}$, which is very much in the form needed to conclude Theorems 1.1 and 1.2.

The rest of the section is devoted to the proof of the previous statement and its application in estimating (3.3) (see Proposition 3.13). We will first make some observations and prove some statements that will streamline the subsequent analysis.

We will show Theorem 3.4 inductively on k, which, modulo the initial inductive step, reduces to prove that if (3.16) holds then so does (3.17) with c_{2k+2}^- as in (3.18) and viceversa. Note that thanks to the definition of $\{\mathcal{H}_n^N\}_{n\geq 2}$ in (3.7), (3.16) implies

$$\mathcal{H}_{2k+2}^N \geq -\mathcal{A}_-^N\left(\mu - \mathcal{L}_0\left(1 + c_{2k+1}^+\mathcal{S}_{2k+1}^N\right)\right)^{-1}\mathcal{A}_+^N,$$

[5] For any two operators $\mathcal{Z}_1\mathcal{Z}_2$ on ΓL^2, $\mathcal{Z}_1 \leq \mathcal{Z}_2$ if and only if for all $\varphi \in \Gamma L^2$, $\langle \mathcal{Z}_1\varphi,\varphi\rangle_{\Gamma L^2} \leq \langle \mathcal{Z}_2\varphi,\varphi\rangle_{\Gamma L^2}$.

while (3.17) implies

$$\mathcal{H}^N_{2k+3} \le -\mathcal{A}^N_-\Big(\mu - \mathcal{L}_0\Big(1 + c^+_{2k+2}\mathcal{S}^N_{2k+2}\Big)\Big)^{-1}\mathcal{A}^N_+.$$

Here we use the fact that for any two positive operators A, B,

$$0 < A \le B \quad \text{if and only if} \quad 0 < B^{-1} \le A^{-1}. \tag{3.19}$$

Now, the operators at the right hand sides above are of the form $-\mathcal{A}^N_-\mathcal{Z}\mathcal{A}^N_+$ for some diagonal operator $\mathcal{Z}$. Therefore, the quantity we need to bound is

$$\langle -\mathcal{A}^N_-\mathcal{Z}\mathcal{A}^N_+\varphi, \varphi\rangle_{\Gamma L^2} = \langle \mathcal{Z}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\rangle_{\Gamma L^2}$$

where we used that $-\mathcal{A}^N_- = (\mathcal{A}^N_+)^*$ by Lemma 2.1. Let us derive a decomposition of the latter which will be useful in highlighting the relevant contributions to the scalar product.

Lemma 3.6. *Let $\mathcal{Z}$ be a diagonal operator on ΓL^2 with Fourier multiplier $\zeta = (\zeta_n)_{n\in\mathbb{N}}$. Then, for every $\varphi \in \Gamma L^2_n$, the following decomposition holds*

$$\langle \mathcal{Z}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\rangle_{\Gamma L^2_{n+1}} = \langle \mathcal{Z}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\rangle_{\text{Diag}} + \sum_{i=1}^{2}\langle \mathcal{Z}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\rangle_{\text{off}_i}$$

where the first summand will be referred to as the "diagonal part" and is given by

$$\langle \mathcal{Z}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\rangle_{\text{Diag}} \stackrel{def}{=} n!\, n\, 2\lambda^2 \sum_{k_{1:n}} |k_1|^2 |\hat\varphi(k_{1:n})|^2 \sum_{\ell+m=k_1} \zeta_{n+1}(\ell, m, k_{2:n})\Big(\mathcal{K}^N_{\ell,m}\Big)^2 \tag{3.20}$$

while the other two terms will be referred to as the "off-diagonal part of type 1 and 2" and are respectively given by

$$\begin{aligned}\langle \mathcal{Z}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\rangle_{\text{off}_1} &\stackrel{def}{=} n!\, c_{\text{off}_1}(n)\,\lambda^2 \\ &\times \sum_{k_{1:n+1}} \zeta_{n+1}(k_{1:n+1})|k_1+k_2|\mathcal{K}^N_{k_1,k_2}|k_1+k_3|\mathcal{K}^N_{k_1,k_3} \\ &\times \hat\varphi(k_1+k_2, k_3, k_{4:n+1})\overline{\hat\varphi(k_1+k_3, k_2, k_{4:n+1})},\end{aligned} \tag{3.21}$$

$$\begin{aligned}\langle \mathcal{Z}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\rangle_{\text{off}_2} &\stackrel{def}{=} n!\, c_{\text{off}_2}(n)\,\lambda^2 \\ &\times \sum_{k_{1:n+1}} \zeta_{n+1}(k_{1:n+1})|k_1+k_2|\mathcal{K}^N_{k_1,k_2}|k_3+k_4|\mathcal{K}^N_{k_3,k_4} \\ &\times \hat\varphi(k_1+k_2, k_3, k_4, k_{5:n+1})\overline{\hat\varphi(k_3+k_4, k_1, k_2, k_{5:n+1})}\end{aligned} \tag{3.22}$$

where, for $i = 1, 2$, $c_{\text{off}_i}(n)$ is an explicit positive constant only depending on n and such that $c_{\text{off}_i}(n) = O(n^{i+1})$.

Proof. As stated in Lemma 2.1, the right hand side of (2.19) still needs to be symmetrised, so that, to be precise, we have

$$\begin{aligned}
&\mathcal{F}(\mathcal{A}_+^N\varphi)(k_{1:n+1})\\
&\quad= \frac{n\lambda}{(n+1)!}\sum_{s\in S_{n+1}} |k_{s(1)}+k_{s(2)}|\mathcal{K}^N_{k_{s(1)},k_{s(2)}}\hat{\varphi}(k_{s(1)}+k_{s(2)},k_{s(3):s(n+1)})\\
&\quad= \frac{2\lambda}{n+1}\sum_{i<j}|k_i+k_j|\mathcal{K}^N_{k_i,k_j}\hat{\varphi}(k_i+k_j,k_{\{1:n+1\}\setminus\{i,j\}})\\
&\quad=: \frac{2\lambda}{n+1}\sum_{i<j}(A_+^N\varphi_n)_{i,j}(k_{1:n+1})
\end{aligned}\tag{3.23}$$

where S_{n+1} is the set of permutations of $\{1,\dots,n+1\}$. Then, by the definition of $\langle\cdot,\cdot\rangle_{\Gamma L^2_n}$ in (2.7),

$$\langle \mathcal{Z}\mathcal{A}_+^N\varphi,\mathcal{A}_+^N\varphi\rangle_{\Gamma L^2_{n+1}} = n!\frac{4\lambda^2}{n+1}\sum_{k_{1:n+1}}\zeta_{n+1}(k_{1:n+1})\sum_{\underline{i},\underline{j}\in I}\prod_{\ell=1}^{2}(A_+^N\varphi_n)_{i_\ell,j_\ell}(k_{1:n+1})$$

where $\underline{i}=(i_1,i_2)$, $\underline{j}=(j_1,j_2)\in\{1,\dots,n+1\}^2$ and I is the set $\{(\underline{i},\underline{j})\,:\,i_1<j_1 \text{ and } i_2<j_2\}$. We now split I into the disjoint union of I_m, $m=0,1,2$, containing those $(\underline{i},\underline{j})\in I$ such that $|\{i_1,j_1\}\cap\{i_2,j_2\}|=m$. By using basic combinatorics (see also [46, Section 4.1]), it is not hard to see that

$$|I_2|=\binom{n+1}{2},\quad |I_1|=2\binom{n+1}{2}(n-1),\quad\text{and}\quad |I_0|=\tfrac{(n+1)n(n-1)(n-2)}{4}.\tag{3.24}$$

Then,

$$\begin{aligned}
&\langle \mathcal{Z}\mathcal{A}_+^N\varphi,\mathcal{A}_+^N\varphi\rangle_{\Gamma L^2_{n+1}}\\
&\quad= n!\frac{4\lambda^2}{n+1}\sum_{m=0}^{2}\sum_{k_{1:n+1}}\zeta_{n+1}(k_{1:n+1})\sum_{\underline{i},\underline{j}\in I_m}\prod_{\ell=1}^{2}(A_+^N\varphi_n)_{i_\ell,j_\ell}(k_{1:n+1})
\end{aligned}$$

which, by relabelling the variables and using (3.24) to derive the value of the prefactor of (3.20) and of the constants c_{off_1} and c_{off_2}, gives the decomposition we were after – the diagonal term (3.20) is the summand corresponding to $m=2$, the off-diagonal term of the first type (3.21) that with $m=1$ and the off-diagonal of the second type (3.22) that with $m=0$. □

The following lemma provides a condition under which we can easily deduce operator bounds for the diagonal term.

Lemma 3.7. *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be diagonal operators on ΓL^2 with Fourier multipliers $\zeta^i=(\zeta^i_n)_n$, $i=1,2$. If for every $k_{1:n}\in\mathbb{Z}^{2n}$*

$$\sum_{\ell+m=k_1}\left(\mathcal{K}^N_{\ell,m}\right)^2\zeta^1_{n+1}(\ell,m,k_{2:n})\le\zeta^2_n(k_{1:n}),\tag{3.25}$$

where the sum is over ℓ, $m \in \mathbb{Z}^2$, then for every $\varphi \in \Gamma L^2_n$

$$\langle \mathcal{Z}_1 \mathcal{A}^N_+ \varphi, \mathcal{A}^N_+ \varphi \rangle_{\mathrm{Diag}} \leq 4\lambda^2 \langle (-\mathcal{L}_0) \mathcal{Z}_2 \varphi, \varphi \rangle_{\Gamma L^2_n}. \tag{3.26}$$

If instead (3.25) holds with the opposite inequality, then so does (3.26).

Proof. By (3.20), we have

$$\begin{aligned}
&\langle \mathcal{Z}_1 \mathcal{A}^N_+ \varphi, \mathcal{A}^N_+ \varphi \rangle_{\mathrm{Diag}} \\
&\quad \stackrel{\text{def}}{=} n!\, n\, 2\lambda^2 \sum_{k_{1:n}} |k_1|^2 |\hat{\varphi}(k_{1:n})|^2 \sum_{\ell+m=k_1} \zeta^1_{n+1}(\ell, m, k_{2:n}) (\mathcal{K}^N_{\ell,m})^2 \\
&\quad \leq n!\, n\, 2\lambda^2 \sum_{k_{1:n}} |k_1|^2 \zeta^2_n(k_{1:n}) |\hat{\varphi}(k_{1:n})|^2 \\
&\quad = n! 4\lambda^2 \sum_{k_{1:n}} \frac{1}{2} |k_{1:n}|^2 \zeta^2_n(k_{1:n}) |\hat{\varphi}(k_{1:n})|^2
\end{aligned}$$

where the first inequality is a consequence of (3.25), while in the second equality we simply symmetrised the arguments. Then, (3.26) is an immediate consequence of the definition of the scalar product on ΓL^2_n (see (2.7)). □

In view of the results above, we are ready to state and prove the three lemmas which represent the core of the proof of Theorem 3.4. In the first two we respectively show a lower and an upper bound for the diagonal term, while in the last we focus on the off-diagonal terms.

Lemma 3.8. *Let $\lambda, \mu > 0$, n, $k \in \mathbb{N}$ and $c > 1$. Then, for any $\varphi \in \Gamma L^2_n$*

$$\begin{aligned}
&\left\langle \left(\mu - \mathcal{L}_0 \left(1 + c \mathcal{S}^N_{2k+1} \right) \right)^{-1} \mathcal{A}^N_+ \varphi, \mathcal{A}^N_+ \varphi \right\rangle_{\mathrm{Diag}} \\
&\qquad \geq \frac{1}{\pi c \left(1 + \frac{1}{f_{2k+2}(1)} \right)} \left\langle (-\mathcal{L}_0) \tilde{\mathcal{S}}^N_{2k+2} \varphi, \varphi \right\rangle_{\Gamma L^2_n}
\end{aligned} \tag{3.27}$$

where the operator $\tilde{\mathcal{S}}^N_{2k+2}$ is defined as

$$\tilde{\mathcal{S}}^N_{2k+2} \stackrel{\text{def}}{=} \frac{1}{f_{2k+2}(\mathcal{N})} \left[\sigma^N_{2k+2}(\mu - \mathcal{L}_0, z_{2k+2}(\mathcal{N})) \left(1 - \frac{4\pi\lambda C_{\mathrm{Diag}}}{\sqrt{z_{2k+2}(1)}} \right) - \frac{f_{2k+2}(\mathcal{N})}{2} \right] \tag{3.28}$$

and the constant $C_{\mathrm{Diag}} > 0$ depends just on the constant K of Lemmas C.5 and C.7.

Proof. Thanks to Lemma 3.7 applied to the operator $\mathcal{Z}_1 \stackrel{\text{def}}{=} (\mu - \mathcal{L}_0(1 + c\mathcal{S}^N_{2k+1}))^{-1}$, it is sufficient to focus on

$$\sum_{\ell+m=k_1} \frac{(\mathcal{K}^N_{\ell,m})^2}{\mu + \Gamma(\ell, m, k_{2:n})(1 + c f_{2k+2} \mathrm{UB}^N_{k-1}(\mu + \Gamma(\ell, m, k_{2:n}), z_{2k+2}))}$$

where we recall the definition of σ_{2k+1}^N in (3.15), and we set

$$\Gamma(\ell, m, k_{2:n}) \stackrel{\text{def}}{=} \tfrac{1}{2}(|\ell|^2 + |m|^2 + |k_{2:n}|^2), \qquad \ell, m, k_2, \dots, k_n \in \mathbb{Z}^2. \tag{3.29}$$

We omitted the dependence on n of f_{2k+2} and z_{2k+2} since n will be fixed throughout, and used (3.13) to justify the subscript of f_{2k+2} and z_{2k+2} (note that $\mathcal{Z}_1$ is applied to $\mathcal{A}_+^N\varphi$ which is in the $n+1$-th chaos). Since c, f_{2k+2} and UB_{k-1} are all bigger than 1, the previous is lower bounded by

$$\frac{1}{cf_{2k+2}\left(1+\frac{1}{f_{2k+2}}\right)} \sum_{\ell+m=k_1} \frac{(\mathcal{K}_{\ell,m}^N)^2}{\mu + \Gamma(\ell, m, k_{2:n})\mathrm{UB}_{k-1}^N(\mu + \Gamma(\ell, m, k_{2:n}), z_{2k+2})}. \tag{3.30}$$

Now, upon choosing $F^N(x, z) = \mathrm{UB}_{k-1}^N(x, z)$ and introducing the short-hand notation

$$\alpha_N \stackrel{\text{def}}{=} \mu/N^2 + \tfrac{1}{2}|k_{1:n}/N|^2, \tag{3.31}$$

Lemmas C.5 and C.7 imply that there exists a constant $K > 0$ such that the sum in (3.30) is bounded from below by

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\left(\mathcal{K}_{xN,-xN}^N\right)^2}{(|x|^2+\alpha_N)(1+|x|^2+\alpha_N)\mathrm{UB}_{k-1}(|x|^2+\alpha_N, z_{2k+2})}\mathrm{d}x - K\frac{\mathrm{LB}_{k-1}\left(\alpha_N \vee \frac{1}{N^2}, z_{2k+2}\right)}{\lambda\sqrt{z_{2k+2}(n)}} \\
&\geq \int_{\mathbb{R}^2} \frac{\left(\mathcal{K}_{xN,-xN}^N\right)^2}{(|x|^2+\alpha_N)(1+|x|^2+\alpha_N)\mathrm{UB}_{k-1}(|x|^2+\alpha_N, z_{2k+2})}\mathrm{d}x - K\frac{\mathrm{LB}_{k}\left(\alpha_N \vee \frac{1}{N^2}, z_{2k+2}\right)}{\lambda\sqrt{z_{2k+2}(1)}}
\end{aligned} \tag{3.32}$$

where we used the definition of UB_{k-1} and LB_{k-1} in (3.11), the monotonicity of the latter in k and that $z_{2k+2}(n) \geq z_{2k+2}(1)$. To control the first term above, notice that, in polar coordinates, that is, setting $x = r(\cos\theta, \sin\theta)$, the Fourier coefficient of the non-linearity (2.11) reads

$$\mathcal{K}_{xN,-xN}^N = -\frac{1}{2\pi}\cos(2\theta)\mathbb{1}_{r\in[1/N,1]} \tag{3.33}$$

so that the integral in (3.32) factorises and we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\left(\mathcal{K}_{xN,-xN}^N\right)^2}{(|x|^2+\alpha_N)(1+|x|^2+\alpha_N)\mathrm{UB}_{k-1}(|x|^2+\alpha_N, z_{2k+2})}\mathrm{d}x \\
&= \int_0^{2\pi}\left(\frac{\cos(2\theta)}{2\pi}\right)^2\mathrm{d}\theta \int_{\frac{1}{N}}^1 \frac{r\mathrm{d}r}{(r^2+\alpha_N)(1+r^2+\alpha_N)\mathrm{UB}_{k-1}(r^2+\alpha_N, z_{2k+2})} \\
&= \frac{1}{8\pi}\int_{\frac{1}{N^2}+\alpha_N}^{1+\alpha_N} \frac{\mathrm{d}\varrho}{\varrho(\varrho+1)\mathrm{UB}_{k-1}(\varrho, z_{2k+2})} \\
&= \frac{1}{4\pi\lambda^2}\left[\mathrm{LB}_k\left(\tfrac{1}{N^2}+\alpha_N, z_{2k+2}\right) - \mathrm{LB}_k(1+\alpha_N, z_{2k+2})\right]
\end{aligned} \tag{3.34}$$

where in the last step we exploited (C.4). Notice now that, by using the same identity,

$$\begin{aligned}
&\mathrm{LB}_k\left(\tfrac{1}{N^2}+\alpha_N, z_{2k+2}\right)\\
&\quad= \mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}\right) - \frac{\lambda^2}{2}\int_{\frac{1}{N^2}\vee\alpha_N}^{\frac{1}{N^2}+\alpha_N} \frac{\mathrm{d}\varrho}{\varrho(\varrho+1)\mathrm{UB}_{k-1}(\varrho, z_{2k+2})}\\
&\quad= \mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}\right) - \frac{\lambda^2}{2}\int_{\frac{1}{N^2}\vee\alpha_N}^{\frac{1}{N^2}+\alpha_N} \frac{\mathrm{LB}_{k-1}(\varrho, z_{2k+2})\mathrm{d}\varrho}{\varrho(\varrho+1)\mathrm{L}(\varrho, z_{2k+2})}\\
&\quad\geq \mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}\right)\left(1 - \frac{\lambda^2}{2}\int_{\frac{1}{N^2}\vee\alpha_N}^{\frac{1}{N^2}+\alpha_N} \frac{\mathrm{d}\varrho}{\varrho\mathrm{L}(\varrho, z_{2k+2})}\right)\\
&\quad\geq \mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}\right)\left(1 - \frac{\lambda^2}{2z_{2k+2}}\int_{\frac{1}{N^2}\vee\alpha_N}^{\frac{1}{N^2}+\alpha_N} \frac{\mathrm{d}\varrho}{\varrho}\right)\\
&\quad\geq \mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}\right)\left(1 - \frac{\lambda^2}{2z_{2k+2}}\frac{\frac{1}{N^2}+\alpha_N-\frac{1}{N^2}\vee\alpha_N}{\frac{1}{N^2}\vee\alpha_N}\right)\\
&\quad\geq \mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}\right)\left(1 - \frac{\lambda^2}{\sqrt{z_{2k+2}(1)}}\right)
\end{aligned}$$

where in the first step we exploited the definition of UB_k in (3.11), in the second the fact that $\mathrm{LB}_{k-1}(\varrho, z)$ is decreasing in ϱ and increasing in k by Lemma C.2 and in the last that the fraction involving α_N is bounded above by 2 and $z_{2k_2}(n)$ is increasing in n and greater or equal to 1. Moreover,

$$\begin{aligned}
\mathrm{LB}_k(1+\alpha_N, z_{2k+2}) &\leq \mathrm{LB}_k(1, z_{2k+2}) \leq \sqrt{\mathrm{L}(1, z_{2k+2})}\\
&= \sqrt{\lambda^2(z_{2k+2}+\log 2)+1} \leq \tfrac{1}{2} f_{2k+2}
\end{aligned} \tag{3.35}$$

which in turn is a consequence of the fact that LB_k is decreasing in the first variable (see Lemma C.2), (C.2) and a choice of a sufficiently large K_2 in (3.12).

As a consequence, (3.34) is lower bounded by

$$\frac{1}{4\pi\lambda^2}\left[\mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}\right) - \frac{f_{2k+2}}{2}\right]$$

so that, in conclusion, there exists a C_{Diag} so that (3.30) is lower bounded by

$$\begin{aligned}
&\frac{1}{4\lambda^2\pi c\left(1+\frac{1}{f_{2k+2}(1)}\right)}\frac{1}{f_{2k+2}(n)}\\
&\quad\times\left[\mathrm{LB}_k\left(\tfrac{1}{N^2}\vee\alpha_N, z_{2k+2}(n)\right)\left(1-\frac{4\pi\lambda C_{\mathrm{Diag}}}{\sqrt{z_{2k+2}(1)}}\right) - \frac{f_{2k+2}(n)}{2}\right],
\end{aligned}$$

where we additionally used that $n \mapsto f_{2k+2}(n)$ is increasing to replace $f_{2k+2}(n)$ by $f_{2k+2}(1)$. Applying Lemma 3.7 is sufficient to conclude the proof. □

Lemma 3.9. *Let $\lambda, \mu > 0$, $n, k \in \mathbb{N}$ and $c < 1$. Then, for any $\varphi \in \Gamma L^2_n$*

$$\langle (\mu - \mathcal{L}_0(1 + c\mathcal{S}^N_{2k+2}))^{-1}\mathcal{A}^N_+\varphi_n, \mathcal{A}^N_+\varphi_n\rangle_{\text{Diag}} \leq \frac{1}{\pi c}\left(1 + \frac{4\lambda\pi C_{\text{Diag}}}{\sqrt{z_{2k+2}(1)}}\right)\langle (-\mathcal{L}_0)\mathcal{S}^N_{2k+3}\varphi, \varphi\rangle_{\Gamma L^2_n}. \tag{3.36}$$

Proof. Applying Lemma 3.7 to the operator $\mathcal{Z}_1 \stackrel{\text{def}}{=} (\mu - \mathcal{L}_0(1 + c\mathcal{S}^N_{2k+2}))^{-1}$, we see that we can focus on

$$\sum_{\ell+m=k_1} \frac{(\mathcal{K}^N_{\ell,m})^2}{\mu + \Gamma(\ell, m, k_{2:n})(1 + \frac{c}{f_{2k+3}}[\text{LB}^N_k((\mu + \Gamma(\ell, m, k_{2:n})) \vee 1, z_{2k+3}) - f_{2k+3}])} \tag{3.37}$$

where, as in the proof of Lemma 3.8, we recalled the definition of σ^N_{2k+1} in (3.15), defined Γ as in (3.29), omitted the dependence on n of f_{2k+2} and z_{2k+2} since n will be fixed throughout, and used (3.13). Note that by assumption $1 - c > 0$ so that the sum above can be upper bounded by

$$\frac{f_{2k+3}}{c} \sum_{\ell+m=k_1} \frac{(\mathcal{K}^N_{\ell,m})^2}{\mu + \Gamma(\ell, m, k_{2:n})\text{LB}^N_k(\mu + \Gamma(\ell, m, k_{2:n}), z_{2k+3})} \tag{3.38}$$

where we removed the $\vee$ at the denominator of (3.37) as, by the definition of $\mathcal{K}^N_{\ell,m}$ in (2.11) and of $\mathbb{J}^N_{\ell,m}$ in (2.12), both ℓ and m are such that $|\ell|, |m| \geq 1$ thus ensuring $\mu + \Gamma(\ell, m, k_{2:n}) \geq 1$. Now, upon choosing $F^N(x, z) = \text{LB}^N_k(x, z)$ and defining α_N according to (3.31), Lemmas C.5 and C.7, imply that the sum in (3.38) is bounded from above by

$$\int_{\mathbb{R}^2} \frac{\left(\mathcal{K}^N_{xN,-xN}\right)^2}{(|x|^2 + \alpha_N)(1 + |x|^2 + \alpha_N)\text{LB}_k(|x|^2 + \alpha_N, z_{2k+3})}\text{d}x + C_{\text{Diag}}\frac{\text{UB}_k(\alpha_N, z_{2k+3})}{\lambda\sqrt{z_{2k+3}(1)}} \tag{3.39}$$

where we used the definition of LB_k and UB_k in (3.11) and again that $z_{2k+3}(n) \geq z_{2k+3}(1)$. To control the integral, we proceed as in the proof of Lemma 3.8 – we pass to polar coordinates and use (3.33), so that we obtain

$$\int_{\mathbb{R}^2} \frac{\left(\mathcal{K}^N_{xN,-xN}\right)^2}{(|x|^2 + \alpha_N)(1 + |x|^2 + \alpha_N)\text{LB}_k(|x|^2 + \alpha_N, z_{2k+3})}\text{d}x$$
$$= \int_0^{2\pi}\left(\frac{\cos(2\theta)}{2\pi}\right)^2\text{d}\theta\int_{\frac{1}{N}}^1 \frac{r\text{d}r}{(r^2 + \alpha_N)(1 + r^2 + \alpha_N)\text{LB}_k(r^2 + \alpha_N, z_{2k+3})}$$

$$
\begin{aligned}
&= \frac{1}{8\pi}\int_{\frac{1}{N^2}+\alpha_N}^{1+\alpha_N} \frac{\mathrm{d}\varrho}{\varrho(\varrho+1)\mathrm{LB}_k(\varrho, z_{2k+3})}\\
&\leq \frac{1}{4\pi\lambda^2}\left[\mathrm{UB}_k\left(\tfrac{1}{N^2}+\alpha_N, z_{2k+3}\right) - \mathrm{UB}_k(1+\alpha_N, z_{2k+3})\right]\\
&\leq \frac{1}{4\pi\lambda^2}\mathrm{UB}_k(\alpha_N, z_{2k+3})
\end{aligned}
$$

where in the second to last step we exploited (C.5) and in the last the monotonicity of UB as stated in Lemma C.2.

Summarising what done so far, we have showed that (3.38) is bounded above by

$$
\frac{1}{4\lambda^2\pi c}\left(1 + C_{\mathrm{Diag}}\frac{4\lambda\pi}{\sqrt{z_{2k+2}(1)}}\right) f_{2k+3}(n)\mathrm{UB}_k(\alpha_N, z_{2k+3}(n))
$$

so that (3.36) follows at once by Lemma 3.7 and the definition of $\mathcal{S}_{2k+3}$ in (3.14). □

Lemma 3.10. *For $m\in\mathbb{N}$, $m\geq 2$, let c_m be a constant such that if m is odd then $c_m>1$, while if m is even $c_m<1$. Then, there exists a constant $C_{\mathrm{off}}>1$ independent of μ, λ, k and N such that for all $m\in\mathbb{N}$, $m>2$, and any $\varphi\in\Gamma L^2_n$*

$$
\sum_{i=1,2}\left|\left\langle(\mu-\mathcal{L}_0(1+c_m\mathcal{S}^N_m))^{-1}\mathcal{A}^N_+\varphi, \mathcal{A}^N_+\varphi\right\rangle_{\mathrm{off}_i}\right| \leq \lambda^2 C_{\mathrm{off}}\left\langle(-\mathcal{L}_0)\,\mathcal{S}^{N,\mathrm{off}}_{m+1}\varphi,\varphi\right\rangle_{\Gamma L^2_n} \tag{3.40}
$$

where $\mathcal{S}^{N,\mathrm{off}}_{m+1}$ is the operator defined by

$$
\mathcal{S}^{N,\mathrm{off}}_{m+1} \stackrel{def}{=} \frac{\mathcal{N}^2}{\tilde{c}_m z_{m+1}(\mathcal{N})}\sigma_{m+1}(\mu-\mathcal{L}_0, z_{m+1}(\mathcal{N})) \tag{3.41}
$$

and $\tilde{c}_m \stackrel{def}{=} c_m f_{m+1}(\mathcal{N})$ if m is odd, while $\tilde{c}_m \stackrel{def}{=} c_m/f_{m+1}(\mathcal{N})$ if m is even. If $m=2$, (3.51) still holds but with $\mathcal{S}^{N,\mathrm{off}}_{m+1}$ replaced by $\mathcal{N}^2$.

Proof. We begin with the off-diagonal term of the first type, which, by (3.21) and (2.11), is

$$
\begin{aligned}
n!\frac{c_{\mathrm{off}_1}(n)\lambda^2}{4\pi^2}\sum_{j_{1:3},k_{3:n}}&\frac{c(j_1,j_2)}{|j_1||j_2|}\frac{c(j_1,j_3)}{|j_1||j_3|}\hat{\varphi}(j_1+j_2,j_3,k_{3:n})\hat{\varphi}(j_1+j_3,j_2,k_{3:n})\\
&\times\frac{|j_1+j_2||j_1+j_3|\mathbb{J}^N_{j_1,j_2}\mathbb{J}^N_{j_2,j_3}}{\mu+\Gamma(j_{1:3},k_{3:n})(1+\tilde{c}_m[\sigma^N_m(\mu+\Gamma(j_{1:3},k_{3:n}),z_{m+1})-a_m])}
\end{aligned} \tag{3.42}
$$

where a_m is 0 if m is odd and f_{m+1} otherwise, and, as in the proof of Lemmas 3.8 and 3.9, σ^N_m is as in (3.15), we adopted the same convention for Γ as in (3.29), omitted the dependence on n of z_{m+1} since n will be fixed throughout, and used (3.13). In the rest of the proof we will omit the subscript of z since it will not change.

In order to control the absolute value of the previous, we first bound the factors $c(j_1,j_2)/(|j_1||j_2|)$, $c(j_1,j_3)/(|j_1||j_3|)$ by 1 in absolute value, neglect $1-\tilde{c}_m a_m>0$ inside the parenthesis in the denominator (recall that for m even $c_m<1$) and denote the product of the indicators by $\mathbb{J}^N_{j_{1:3}}$. Furthermore, define $\Phi(\ell_{1:n}) \stackrel{\text{def}}{=} \hat{\varphi}(\ell_{1:n}) \prod_{i=1}^n |\ell_i|$, so that the sum in (3.42) can be upper bounded by

$$
\begin{aligned}
&\sum_{j_{1:3},k_{3:n}} \frac{|\Phi(j_1+j_2,j_3,k_{3:n})||\Phi(j_1+j_3,j_2,k_{3:n})|\mathbb{J}^N_{j_{1:3}}}{|j_2||j_3|(\mu+\tilde{c}_m\Gamma(j_{1:3},k_{3:n})\sigma^N_m(\mu+\Gamma(j_{1:3},k_{3:n}),z))\prod_{i=3}^n|k_i|^2}\\
&\le \sum_{j_{1:3},k_{3:n}} \frac{|\Phi(j_1+j_2,j_3,k_{3:n})|^2}{\prod_{i=3}^n|k_i|^2} \frac{\mathbb{J}^N_{j_{1:2}}}{|j_2||j_3|(\mu+\tilde{c}_m\Gamma(j_{1:3},k_{3:n}))\sigma^N_m(\mu+\Gamma(j_{1:3},k_{3:n}),z))}\\
&= \sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2|k_1|^2|k_2| \sum_{j_1+j_2=k_1} \frac{\mathbb{J}^N_{j_{1:2}}}{|j_2|(\mu+\tilde{c}_m\Gamma(j_{1:2},k_{2:n})\sigma^N_m(\mu+\Gamma(j_{1:2},k_{2:n})),z))}
\end{aligned}
$$

where in the first passage we estimated the product of the Φ_n's by half the sum of their squares and in the second we renamed the variables ($j_3 = k_2$). Let us point out that in case $m = 2$, in all the inequalities above there is no σ^N_m in the denominator.

Concerning the inner sum, by (C.1), $\frac{1}{8}(|j_2|^2 + |k_{1:n}|^2) \le \Gamma(j_{1:2},k_{2:n}) \le 2(|j_2|^2 + |k_{1:n}|^2)$, and, since σ_m is monotonically decreasing in the first variable, we have

$$
\begin{aligned}
&\sum_{j_1+j_2=k_1} \frac{\mathbb{J}^N_{j_{1:2}}}{|j_2|(\mu+\tilde{c}_m\Gamma(j_{1:2},k_{2:n})\sigma^N_m(\mu+\Gamma(j_{1:2},k_{2:n})),z))}\\
&\le \frac{1}{N}\sum_{|j_2/N|\le 1} \frac{1}{N^2}\frac{1}{|j_2/N|} \frac{1}{\left(\mu_N + \frac{\tilde{c}_m}{8}(|j_2/N|^2+|k_{1:n}/N|^2)\sigma^N_m(2|j_2/N|^2+4\alpha_N,z)\right)}\\
&\lesssim \frac{1}{N}\int_0^1 \frac{\mathrm{d}\varrho}{\mu_N+\frac{\tilde{c}_m}{8}(\varrho^2+|k_{1:n}/N|^2)\sigma_m(2\varrho^2+4\alpha_N,z)}
\end{aligned}
\tag{3.43}
$$

where the last bound follows by Riemann sum approximation and the use of polar coordinates.

Now, if $m = 2$, modulo constants, (3.43) is bounded above by

$$
\frac{1}{N}\int_0^1 \frac{\mathrm{d}\varrho}{\varrho^2+\alpha_N} \lesssim \frac{1}{N}\frac{1}{\sqrt{\alpha_N}} \le \frac{1}{\sqrt{|k_{1:n}|^2}}.
$$

For $m > 2$, notice that, if $\mu_N \le |k_{1:n}/N|^2$, then the denominator in the integral can be trivially bounded from below by

$$
\begin{aligned}
\mu_N + \frac{\tilde{c}_m}{8}(\varrho^2+|k_{1:n}/N|^2)\sigma_m &\ge \frac{\tilde{c}_m}{8}(\varrho^2+|k_{1:n}/N|^2)\sigma_m\\
&\gtrsim \tilde{c}_m(2\varrho^2+4\alpha_N)\sigma_m
\end{aligned}
\tag{3.44}
$$

where we omitted the arguments of σ_m since they do not change. Thus, modulo constants, (3.43) can be bounded from above by

$$\begin{aligned}
&\frac{1}{\tilde{c}_m N}\int_0^1 \frac{\mathrm{d}\varrho}{(2\varrho^2+4\alpha_N)\sigma_m(2\varrho^2+4\alpha_N,z)} \\
&\quad\le \frac{1}{\tilde{c}_m N}\int_0^\infty \frac{\mathrm{d}\varrho}{(\varrho^2+4\alpha_N)\sigma_m(\varrho^2+4\alpha_N,z)} \\
&\quad\lesssim \frac{1}{\tilde{c}_m N}\frac{1}{\sqrt{\alpha_N}}\frac{1}{\sigma_m(8\alpha_N,z)} \lesssim \frac{1}{\tilde{c}_m}\frac{1}{\sqrt{|k_{1:n}|^2}}\frac{1}{\sigma_m^N\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z\right)}
\end{aligned} \tag{3.45}$$

where we applied Lemma C.4, which holds since σ_m satisfies its assumptions.

If instead $\mu_N > \alpha_N$, then we split the integral in (3.43) according to $\varrho > \sqrt{\mu_N}$ and $\varrho \le \sqrt{\mu_N}$. In the former case, we first bound the denominator as in (3.44) and then extend the integral to the interval [0,1] so that we can exploit (3.45). For the latter, we exploit the monotonicity of σ_m which ensures that

$$\sigma_m(4(\mu_N+\tfrac{1}{2}(\varrho^2+|k_{1:n}/N|^2)),z) \ge \sigma_m(8\alpha_N,z). \tag{3.46}$$

Hence,

$$\begin{aligned}
&\frac{1}{N}\int_0^{\sqrt{\mu_N}} \frac{\mathrm{d}\varrho}{\mu_N+\frac{\tilde{c}_m}{8}(\varrho^2+|k_{1:n}/N|^2)\sigma_m(4(\mu_N+\frac{1}{2}(\varrho^2+|k_{1:n}/N|^2)),z)} \\
&\qquad\lesssim \frac{1}{\tilde{c}_m N}\frac{1}{\sigma_m(8\alpha_N,z)}\int_0^\infty \frac{\mathrm{d}\varrho}{\varrho^2+|k_{1:n}/N|^2} \lesssim \frac{1}{\tilde{c}_m}\frac{1}{\sqrt{|k_{1:n}|^2}}\frac{1}{\sigma_m(8\alpha_N,z)}
\end{aligned}$$

where in the first passage we neglected μ_N and extended the integral to the positive real line. Overall, we have shown that there exists a constant $C>0$ such that, for any $m \ge 2$, (3.42) is bounded above by

$$\begin{aligned}
&\left|\langle(\mu-\mathcal{L}_0(1+c_m\mathcal{S}_m^N))^{-1}\mathcal{A}_+^N\varphi,\mathcal{A}_+^N\varphi\rangle_{\mathrm{off}_i}\right| \\
&\quad\le n!\frac{c_{\mathrm{off}_1}(n)\lambda^2}{4\pi^2}\frac{C}{\tilde{c}_m}\sum_{k_{1:n}}|\hat{\varphi}(k_{1:n})|^2|k_1|^2\frac{|k_2|}{\sqrt{|k_{1:n}|^2}}\frac{1}{\sigma_m^N\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z\right)} \\
&\quad\le n!C\sum_{k_{1:n}}|\hat{\varphi}(k_{1:n})|^2|k_{1:n}|^2\frac{\lambda^2 n}{\tilde{c}_m\sigma_m^N\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z\right)},
\end{aligned} \tag{3.47}$$

and the n at the numerator follows by $c_{\mathrm{off}_1}(n)=O(n^2)$ (see Lemma 3.6) and by replacing $|k_1|^2$ by $|k_{1:n}|^2$.

For the off-diagonal term of second type, we proceed similarly. Adopting the same notations as in (3.42), (3.22) equals

$$\frac{n!\, c_{\mathrm{off}_2}(n)\,\lambda^2}{4\pi^2} \sum_{j_{1:4},k_{4:n}} \frac{c(j_1,j_2)}{|j_1||j_2|}\frac{c(j_3,j_4)}{|j_3||j_4|}\hat{\varphi}(j_1+j_2, j_{3:4}, k_{4:n})$$
$$\times \hat{\varphi}(j_{1:2}, j_3+j_4, k_{4:n}) \frac{|j_1+j_2||j_3+j_4|\mathbb{J}^N_{j_1,j_2}\mathbb{J}^N_{j_3,j_4}}{\mu+\Gamma(j_{1:4},k_{4:n})(1+\tilde{c}_m[\sigma^N_m(\mu+\Gamma(j_{1:4},k_{4:n}),z)-a_m])}. \tag{3.48}$$

By retracing the same steps as in the proof of the bound on the off-diagonal term of the first type, we see that the sum above is upper bounded by

$$\sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2|k_1|^2|k_2||k_3| \sum_{j_1+j_2=k_1} \frac{\mathbb{J}^N_{j_1,j_2}}{|j_1||j_2|(\mu+\tilde{c}_m\Gamma(j_{1:2},k_{2:n})\sigma^N_m(\mu+\Gamma(j_{1:2},k_{2:n}),z))}$$
$$\le 2\sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2|k_1||k_2||k_3| \sum_{j_1+j_2=k_1} \frac{\mathbb{J}^N_{j_1,j_2}}{|j_2|(\mu+\tilde{c}_m\Gamma(j_{1:2},k_{2:n})\sigma^N_m(\mu+\Gamma(j_{1:2},k_{2:n}),z))}$$

where we used that, since $j_1+j_2=k_1$, the modulus of at least one of the two must be bigger than $|k_1|/2$. Hence, we can proceed as in (3.43) so that, again, modulo constants, the previous is bounded above by

$$\frac{1}{\tilde{c}_m}\sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2 \frac{|k_1||k_2||k_3|}{\sqrt{|k_{1:n}|^2}} \frac{1}{\sigma^N_m\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z\right)}$$
$$\le \frac{1}{\tilde{c}_m}\frac{1}{n}\sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2|k_{1:n}|^2 \frac{1}{\sigma^N_m\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z\right)}. \tag{3.49}$$

Hence, it follows that there exists a constant $C>0$ such that

$$\left|\langle \mathcal{A}^N_-(\mu-\mathcal{L}_0(1+\tilde{c}_m\mathcal{S}^N_m))^{-1}\mathcal{A}^N_+\varphi,\varphi\rangle_{\mathrm{off}_2}\right|$$
$$\le \frac{n!\, c_{\mathrm{off}_2}(n)\,\lambda^2}{4\pi^2 n}\frac{C}{\tilde{c}_m}\sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2|k_{1:n}|^2 \frac{1}{\sigma^N_m\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z\right)}$$
$$\le n!\, C\sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2|k_{1:n}|^2 \frac{\lambda^2 n^2}{\tilde{c}_m\sigma^N_m\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z\right)} \tag{3.50}$$

and, again, the n^2 comes from $c_{\mathrm{off}_2}(n)=O(n^3)$ (see Lemma 3.6).

By (3.47) and (3.50), we obtained

$$\sum_{i=1,2}\left|\langle(\mu-\mathcal{L}_0(1+c_m\mathcal{S}^N_m))^{-1}\mathcal{A}^N_+\varphi,\mathcal{A}^N_+\varphi\rangle_{\mathrm{off}_i}\right|$$
$$\le n!\, C\sum_{k_{1:n}} |\hat{\varphi}(k_{1:n})|^2|k_{1:n}|^2 \frac{\lambda^2 n^2}{\tilde{c}_m\sigma^N_m\left(8\left(\mu+\frac{1}{2}|k_{1:n}|^2\right),z_{m+1}(n)\right)} \tag{3.51}$$

for some constant $C > 0$ independent of m, N, μ. Now, to conclude the proof it suffices to control the fraction inside the sum. If m is even by (3.15), we have

$$\frac{\lambda^2 n^2}{\tilde{c}_m \sigma_m^N(8(\mu + \frac{1}{2}|k_{1:n}|^2), z_{m+1}(n))} = \frac{\lambda^2 n^2}{\tilde{c}_m \mathrm{LB}^N_{\frac{m}{2}-1}\left(8\left(\mu + \frac{1}{2}|k_{1:n}|^2\right) \vee 1, z_{m+1}(n)\right)}$$

$$= \frac{n^2\lambda^2}{\tilde{c}_m \mathrm{L}^N(8(\mu + \frac{1}{2}|k_{1:n}|^2) \vee 1, z_{m+1}(n))} \mathrm{UB}^N_{\frac{m}{2}-1}\left(8\left(\mu + \frac{1}{2}|k_{1:n}|^2\right) \vee 1, z_{m+1}(n)\right)$$

$$\leq \frac{n^2}{\tilde{c}_m z_{m+1}(n)} \mathrm{UB}^N_{\frac{m}{2}-1}\left(\mu + \frac{1}{2}|k_{1:n}|^2, z_{m+1}(n)\right)$$

$$= \frac{n^2}{\tilde{c}_m z_{m+1}(n)} \sigma_{m+1}\left(\mu + \frac{1}{2}|k_{1:n}|^2, z_{m+1}(n)\right),$$

while il m is odd

$$\frac{\lambda^2 n^2}{\tilde{c}_m \sigma_m^N\left(8\left(\mu + \frac{1}{2}|k_{1:n}|^2\right), z_{m+1}(n)\right)}$$

$$= \frac{\lambda^2 n^2}{\tilde{c}_m \mathrm{UB}^N_{\frac{m-3}{2}}\left(8\left(\mu + \frac{1}{2}|k_{1:n}|^2\right), z_{m+1}(n)\right)}$$

$$\leq \frac{\lambda^2 n^2}{\tilde{c}_m \mathrm{UB}^N_{\frac{m-3}{2}}\left(8\left(\mu + \frac{1}{2}|k_{1:n}|^2\right) \vee 1, z_{m+1}(n)\right)}$$

$$= \frac{n^2\lambda^2}{\tilde{c}_m \mathrm{L}^N\left(8\left(\left(\mu + \frac{1}{2}|k_{1:n}|^2\right) \vee 1\right), z_{m+1}(n)\right)} \mathrm{LB}^N_{\frac{m-3}{2}}\left(8\left(\left(\mu + \frac{1}{2}|k_{1:n}|^2\right) \vee 1\right), z_{m+1}(n)\right)$$

$$\leq \frac{n^2 \mathrm{LB}^N_{\frac{m-1}{2}}\left(\left(\mu + \frac{1}{2}|k_{1:n}|^2\right) \vee 1, z_{m+1}(n)\right)}{\tilde{c}_m z_{m+1}(n)}$$

$$= \frac{n^2}{\tilde{c}_m z_{m+1}(n)} \sigma_{m+1}\left(\mu + \frac{1}{2}|k_{1:n}|^2, z_{m+1}(n)\right)$$

where, in both cases, we exploited the definition of $\mathrm{UB}^N, \mathrm{LB}^N, \mathrm{L}^N$ in (3.11), (3.10) and their monotonicity properties in Lemma C.2. □

We have now all the ingredients we need for the proof of Theorem 3.4.

Proof of Theorem 3.4. As mentioned above, the proof goes by induction. Since by definition (3.7) $\mathcal{H}_2^N \equiv 0 \equiv \mathcal{S}_2^N$, (3.17) clearly holds for $k = 0$ with an arbitrary c_2^- that we can pick to be equal to $\frac{1}{2}$. Now, we show that if (3.16) holds for $2k+1$, then (3.17) holds for $2k+2$ with c_{2k+2}^- as in (3.18). Using the definition of $\mathcal{H}_{2k+2}^N$ in (3.7), the inductive hypothesis (3.16) and (3.19), we have

$$\mathcal{H}_{2k+2}^N \geq -\mathcal{A}_-^N\left(\mu - \mathcal{L}_0\left(1 + c_{2k+1}^+ \mathcal{S}_{2k+1}^N\right)\right)^{-1} \mathcal{A}_+^N.$$

Let $\varphi = (\varphi_n)_{n\in\mathbb{N}}$. Thanks to the decomposition of Lemma 3.6, we see that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\langle \mathcal{H}^N_{2k+2}\varphi_n, \varphi_n\rangle_{\Gamma L^2_n} &\geq \langle \left(\mu - \mathcal{L}_0\left(1 + c^+_{2k+1}\mathcal{S}^N_{2k+1}\right)\right)^{-1}\mathcal{A}^N_+\varphi_n, \mathcal{A}^N_+\varphi_n\rangle_{\Gamma L^2_{n+1}}\\ &\geq \langle \left(\mu - \mathcal{L}_0\left(1 + c^+_{2k+1}\mathcal{S}^N_{2k+1}\right)\right)^{-1}\mathcal{A}^N_+\varphi_n, \mathcal{A}^N_+\varphi_n\rangle_{\mathrm{Diag}}\\ &\quad - \sum_{i=1,2}\left|\langle(\mu - \mathcal{L}_0(1 + c^+_{2k+1}\mathcal{S}^N_{2k+1}))^{-1}\mathcal{A}^N_+\varphi_n, \mathcal{A}^N_+\varphi_n\rangle_{\mathrm{off}_i}\right|. \end{aligned} \tag{3.52}$$

We are now in a position to apply Lemma 3.8 for the diagonal term and Lemma (3.10) to the off-diagonal, which together give the lower bound for (3.52)

$$\frac{1}{\pi c^+_{2k+1}\left(1 + \frac{1}{f_{2k+2}(1)}\right)}\left\langle(-\mathcal{L}_0)\left[\tilde{\mathcal{S}}^N_{2k+2} - \lambda^2 C_{\mathrm{off}}\pi c^+_{2k+1}\left(1 + \frac{1}{f_{2k+2}(1)}\right)\mathcal{S}^{N,\mathrm{off}}_{2k+2}\right]\varphi_n, \varphi_n\right\rangle_{\Gamma L^2_n}$$

where $\tilde{\mathcal{S}}^N_{2k+2}$ and $\mathcal{S}^{N,\mathrm{off}}_{2k+2}$ are respectively defined in (3.28) and (3.41). It remains to look at the difference of the operators in brackets, which is

$$\begin{aligned}&\tilde{\mathcal{S}}^N_{2k+2} - \lambda^2 C_{\mathrm{off}}\pi c^+_{2k+1}\left(1 + \frac{1}{f_{2k+2}(1)}\right)\mathcal{S}^{N,\mathrm{off}}_{2k+2}\\ &= \frac{1}{f_{2k+2}(\mathcal{N})}\Bigg[\sigma^N_{2k+2}(\mu - \mathcal{L}_0, z_{2k+2}(\mathcal{N}))\\ &\times\left(1 - \frac{4\pi\lambda C_{\mathrm{Diag}}}{\sqrt{z_{2k+2}(1)}} - \frac{\lambda^2 C_{\mathrm{off}}\pi\left(1 + \frac{1}{f_{2k+2}(1)}\right)\mathcal{N}^2}{z_{2k+2}(\mathcal{N})}\right) - \frac{f_{2k+2}(\mathcal{N})}{2}\Bigg]\end{aligned}$$

We now choose K_1 in (3.12) big enough so that

$$\begin{aligned}&\frac{4\pi\lambda C_{\mathrm{Diag}}}{\sqrt{z_{2k+2}(1)}} + \frac{\lambda^2\pi\left(1 + \frac{1}{f_{2k+2}(1)}\right)C_{\mathrm{off}}\mathcal{N}^2}{z_{2k+2}(\mathcal{N})}\\ &= \frac{4\pi\lambda C_{\mathrm{Diag}}}{\sqrt{K_1(\lambda^2\vee 1)}(2k+3)^{\frac{3}{2}+\delta}} + \frac{\pi\left(1 + \frac{1}{f_{2k+2}(1)}\right)C_{\mathrm{off}}\mathcal{N}^2}{K_1(\lambda^2\vee 1)(\mathcal{N} + 2k + 2)^{3+2\delta}}\\ &\leq \frac{4\pi C_{\mathrm{Diag}}}{\sqrt{K_1}}k^{-\frac{3}{2}-\delta} + \frac{2\pi C_{\mathrm{off}}}{K_1}k^{-1-2\delta} \leq \frac{1}{2k^{1+\delta}}.\end{aligned} \tag{3.53}$$

where we recalled that the operator $\mathcal{N}$ takes values in $\mathbb{N}$, so that in particular it is positive. Since the right hand side of (3.53) is smaller than $1/2$, by defining the constant c^-_{2k+2} according to (3.18) it is immediate to see that (3.52) is lower bounded by

$$\frac{1}{\pi c^{+}_{2k+1}\left(1+\frac{1}{f_{2k+2}(1)}\right)}\left\langle(-\mathcal{L}_0)\frac{1}{f_{2k+2}(\mathcal{N})}\right.$$
$$\left.\times\left[\sigma^N_{2k+2}(\mu-\mathcal{L}_0,z_{2k+2}(\mathcal{N}))\left(1-\frac{1}{2k^{1+\delta}}\right)-\frac{f_{2k+2}(\mathcal{N})}{2}\right]\varphi_n,\varphi_n\right\rangle_{\Gamma L^2_n}$$
$$\geq c^{-}_{2k+2}\left\langle(-\mathcal{L}_0)\frac{1}{f_{2k+2}(\mathcal{N})}\left[\sigma^N_{2k+2}(\mu-\mathcal{L}_0,z_{2k+2}(\mathcal{N}))-f_{2k+2}(\mathcal{N})\right]\varphi_n,\varphi_n\right\rangle_{\Gamma L^2_n}$$
$$=c^{-}_{2k+2}\,\langle(-\mathcal{L}_0)\mathcal{S}_{2k+2}\varphi_n,\varphi_n\rangle_{\Gamma L^2_n},$$

which is what we wanted to show.

We now assume (3.17) and prove that (3.16) holds for $2k+3$. Exploiting (3.19), the above and the decomposition in diagonal and off diagonal parts in Lemma 3.6, for $\varphi=(\varphi_n)_{n\in\mathbb{N}}\in\Gamma L^2$ and any $n\in\mathbb{N}$, we have

$$\langle\mathcal{H}^N_{2k+3}\varphi_n,\varphi_n\rangle_{\Gamma L^2_n}\leq\left\langle\left(\mu-\mathcal{L}_0\left(1+c^{-}_{2k+2}\mathcal{S}^N_{2k+2}\right)\right)^{-1}\mathcal{A}^N_+\varphi_n,\mathcal{A}^N_+\varphi_n\right\rangle_{\Gamma L^2_{n+1}}$$
$$\leq\left\langle\left(\mu-\mathcal{L}_0\left(1+c^{-}_{2k+2}\mathcal{S}^N_{2k+2}\right)\right)^{-1}\mathcal{A}^N_+\varphi_n,\mathcal{A}^N_+\varphi_n\right\rangle_{\text{Diag}}$$
$$+\sum_{i=1,2}\left|\left\langle(\mu-\mathcal{L}_0(1+c^{-}_{2k+2}\mathcal{S}^N_{2k+2}))^{-1}\mathcal{A}^N_+\varphi_n,\mathcal{A}^N_+\varphi_n\right\rangle_{\text{off}_i}\right|. \tag{3.54}$$

for which Lemmas 3.9 and 3.10 provide an upper bound of the form

$$\frac{1}{\pi c^{-}_{2k+2}}\left\langle(-\mathcal{L}_0)\left[\left(1+\frac{4\lambda\pi C_{\text{Diag}}}{\sqrt{z_{2k+2}(1)}}\right)\mathcal{S}^N_{2k+3}+\lambda^2C_{\text{off}}\pi c^{-}_{2k+2}\mathcal{S}^{N,\text{off}}_{2k+3}\right]\varphi_n,\varphi_n\right\rangle_{\Gamma L^2_n}.$$

Note that, by (3.41) and (3.14), we have

$$\mathcal{S}^{N,\text{off}}_{2k+3}=\frac{1}{c^{-}_{2k+2}}\frac{\mathcal{N}^2}{z_{2k+2}(\mathcal{N})}\mathcal{S}^N_{2k+3}$$

which, together with the equation above, implies that (3.54) is upper bounded by

$$\frac{1}{\pi c^{-}_{2k+2}}\left\langle(-\mathcal{L}_0)\left(1+\frac{4\lambda\pi C_{\text{Diag}}}{\sqrt{z_{2k+2}(1)}}+\frac{\lambda^2C_{\text{off}}\mathcal{N}^2}{z_{2k+2}(\mathcal{N})}\right)\mathcal{S}^N_{2k+3}\varphi_n,\varphi_n\right\rangle_{\Gamma L^2_n}$$
$$\leq c^{+}_{2k+3}\,\langle(-\mathcal{L}_0)\mathcal{S}_{2k+3}\varphi_n,\varphi_n\rangle_{\Gamma L^2_n}$$

where in the last step we exploited both (3.53) and the definition of c^{+}_{2k+3} in (3.18), and the proof is completed. ☐

3.2 | The operator $-\mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N$

We now come to the other operator we need to estimate in order to control (3.9), namely $-\mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N$. In view of (3.19) we need an upper bound on this operator (as lower bound we will simply use that $-\mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N$ is positive). Note, however, that we only need to analyse its action on elements of the second Wiener chaos. This is because, in (3.9), $\mathfrak{n}_\varphi^N$ belongs to the second chaos, and $\mathcal{L}_0$ and $\mathcal{H}_n^N$ (and also $-\mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N$) leave the order of the chaos unchanged. We have the following lemma.

Lemma 3.11. *There exists a constant $c>0$ such that for any $\varphi\in\Gamma L_2^2$ and any function $G:\mathbb{R}_+\mapsto[1,\infty)$,*

$$-\langle\mathcal{A}_+^N\,(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N\varphi,\varphi\rangle_{\Gamma L_2^2}\le c\lambda^2\langle(-\mathcal{L}_0)\mathcal{S}^{+-}\varphi,\varphi\rangle_{\Gamma L_2^2},\tag{3.55}$$

where the operator $\mathcal{S}^{+-}$ acts in Fourier space on $\varphi\in\Gamma L_2^2$ as

$$\mathcal{F}(\mathcal{S}^{+-}\varphi)(\ell,m)=\mathbb{J}_{\ell,m}^N g(\ell+m)G\Big(\mu+\frac{1}{2}(|\ell|^2+|m|^2)\Big)\hat\varphi(\ell,m),$$

where

$$g(k)\stackrel{def}{=}\frac{|k|^2}{\mu+\frac{1}{2}|k|^2}\sum_{\ell+m=k}\frac{\mathbb{J}_{\ell,m}^N}{\frac{1}{2}(|\ell|^2+|m|^2)G(\mu+\frac{1}{2}(|\ell|^2+|m|^2))}.\tag{3.56}$$

Proof. Notice that, by Lemma 2.1,

$$\begin{aligned}&\langle(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N\varphi_2,\mathcal{A}_-^N\varphi_2\rangle_{\Gamma L_2^2}\\&\quad=\frac{4\lambda^2}{\pi^2}\sum_k\frac{1}{\mu+\frac{1}{2}|k|^2}\left(\sum_{\ell+m=k}|m|\frac{c(k,-\ell)}{|\ell||k|}\mathbb{J}_{\ell,m}^N\hat\varphi(\ell,m)\right)^2.\end{aligned}\tag{3.57}$$

We begin by analysing the inner sum and we treat differently the small and the large values of ℓ. For lightness of notation, we write

$$\sum_{\ell+m=k}'\cdots\stackrel{\text{def}}{=}\sum_{\ell+m=k}\mathbb{J}_{\ell,m}^N\cdots.$$

We consider first the case $|\ell|\le2|k|$. Note that $|m|=|\ell-k|\le3|k|$, hence

$$\left|\sum_{\substack{\ell+m=k\\|\ell|\le2|k|}}'|m|\frac{c(k,-\ell)}{|\ell||k|}\hat\varphi(\ell,m)\right|\le3|k|\sum_{\substack{\ell+m=k\\|\ell|\le2|k|}}'|\hat\varphi(\ell,m)|.\tag{3.58}$$

To continue, we multiply and divide by $(\frac{1}{2}(|\ell|^2+|m|^2))^{\frac{1}{2}}G(\mu+\frac{1}{2}(|\ell|^2+|m|^2))^{\frac{1}{2}}$ and apply the Cauchy-Schwarz inequality. This readily gives the desired contribution.

Next we consider the case $|\ell| > 2|k|$. Since φ is symmetric, so is $\Phi(\ell, m) \stackrel{\text{def}}{=} |\ell||m|\hat{\varphi}(\ell, m)$, hence, the summand in the inner sum at the right hand side of (3.57) can be rewritten as

$$\sum_{\substack{\ell+m=k\\|\ell|>2|k|}} \frac{1}{|k|}\frac{c(k,-\ell)}{|\ell|^2}\Phi(\ell, k-\ell) = \sum_{\substack{\ell+m=k\\|\ell|>2|k|}} \frac{1}{2|k|}\left(\frac{c(k,-\ell)}{|\ell|^2} + \frac{c(k,\ell-k)}{|k-\ell|^2}\right)\Phi(\ell, k-\ell).$$

A direct computation using the definition of $c(k,\ell)$ shows that the summand equals

$$-\frac{c(k,k)}{2|k|}\frac{|\ell|}{|k-\ell|}\hat{\varphi}(\ell, k-\ell) + \frac{1}{2|k|}c(k,\ell)\left(\frac{1}{|k-\ell|^2} - \frac{1}{|\ell|^2}\right)\Phi(\ell, k-\ell).$$

Since $|\ell| > 2|k|$, $3|\ell|/2 \geq |k-\ell| \geq |\ell|/2$. Therefore,

$$\left|\frac{c(k,k)}{2|k|}\sideset{}{'}\sum_{\substack{\ell+m=k\\|\ell|>2|k|}} \frac{|\ell|}{|k-\ell|}\hat{\varphi}(\ell, k-\ell)\right| \leq |k| \sideset{}{'}\sum_{\ell+m=k, |\ell|\geq 2|k|} |\hat{\varphi}(\ell, k-\ell)| \tag{3.59}$$

that can be estimated as (3.58). To estimate the second summand above we note that, since $|c(k,\ell)| \leq |k||\ell|$, we have

$$|c(k,\ell)||k-\ell||\ell|\left|\frac{1}{|k-\ell|^2} - \frac{1}{|\ell|^2}\right| \leq |k|\frac{\left||\ell|^2 - |k-\ell|^2\right|}{|k-\ell|} \leq \frac{|k|^2|2\ell-k|}{|k-\ell|} \lesssim |k|^2.$$

Thus,

$$\left|\frac{1}{2|k|}\sideset{}{'}\sum_{\substack{\ell+m=k\\|\ell|>2|k|}} c(k,\ell)\left(\frac{1}{|k-\ell|^2} - \frac{1}{|\ell|^2}\right)\Phi(\ell, k-\ell)\right| \lesssim |k| \sideset{}{'}\sum_{\substack{\ell+m=k\\|\ell|>2|k|}} |\hat{\varphi}(\ell, k-\ell)|, \tag{3.60}$$

which once again can be bounded as (3.58). Hence, the result follows. □

In view of Theorem 3.4 and the definition of the operators $\{\mathcal{S}^N_{2k+1}\}_k$ in (3.14), a special role will be played by the case in which the function G is chosen to depend on $k \in \mathbb{N}$ and is of the form $G(x) \stackrel{\text{def}}{=} \mathrm{UB}^N_{k-1}(x, z_{2k+1}(2))$.

Lemma 3.12. *In the setting of Lemma 3.11, choose G as* $G(x) \stackrel{\text{def}}{=} \mathrm{UB}^N_{k-1}(x, z_{2k+1}(2))$. *Then, there exists a constant* $c_0 > 0$ *independent of* μ, k *and* N *for which*

$$g(j) \leq c_0 \frac{|j|^2}{\mu + \frac{1}{2}|j|^2} \times \begin{cases} \dfrac{\log(\mu/|j|^2)}{\mathrm{UB}^N_{k-1}(8\mu, z_{2k+1}(2))} + \dfrac{1+\mu_N}{\lambda^2}\mathrm{LB}^N_k(8\mu, z_{2k+1}(2)), & \text{if } |j|^2 \leq \mu \\ \dfrac{1+\mu_N}{\lambda^2}\mathrm{LB}^N_k\left(4\left(\mu + \frac{1}{2}|j|^2\right), z_{2k+1}(2)\right), & \text{if } |j|^2 > \mu. \end{cases} \tag{3.61}$$

Proof. Note that with our choice of G it is enough to estimate

$$\sum_{\ell+m=j} \frac{\mathbb{J}^N_{\ell,m}}{(|\ell|^2+|m|^2)\mathrm{UB}^N_{k-1}\left(\mu+\frac{1}{2}(|\ell|^2+|m|^2), z_{2k+1}(2)\right)}.$$

Thanks to (C.1), and by an immediate extension of Lemma C.7, the previous is upper bounded by

$$\begin{aligned}
&\sum_{1/N\leq|\ell/N|\leq 1} \frac{1}{N^2} \frac{1}{\frac{1}{4}(|\ell/N|^2+|j/N|^2)\mathrm{UB}_{k-1}(4\left(\mu_N+\frac{1}{2}(|\ell/N|^2+|j/N|^2), z_{2k+1}(2)\right)} \\
&\quad\lesssim \int_0^1 \frac{\varrho\,\mathrm{d}\varrho}{(\varrho^2+|j/N|^2)\mathrm{UB}_{k-1}\left(4\left(\mu_N+\frac{1}{2}(\varrho^2+|j/N|^2)\right), z_{2k+2}(2)\right)} \\
&\quad\lesssim \int_{|j/N|^2}^1 \frac{\mathrm{d}\varrho}{\varrho\mathrm{UB}_{k-1}(4(\mu_N+\varrho), z_{2k+1}(2))}
\end{aligned}$$

where in the last line we enlarged the integration interval by using that $|j/N|\leq 1$, which holds for all values of j appearing above, because by definition $\mathbb{J}^N_{\ell,m}$ is zero if $|j|=|\ell+m|>N$. We now distinguish two cases, depending on the relation between μ and $|j|^2$. If $|j|^2\leq\mu$, then we split the integral as

$$\left(\int_{|j/N|^2}^{\mu_N}+\int_{\mu_N}^1\right)\frac{\mathrm{d}\varrho}{\varrho\mathrm{UB}_{k-1}(4(\mu_N+\varrho), z_{2k+1}(2))} =: I_1+I_2.$$

For I_1 we exploit the fact that UB is decreasing, so that

$$I_1 \leq \frac{1}{\mathrm{UB}_{k-1}(8\mu_N, z_{2k+1}(2))}\int_{|j/N|^2}^{\mu_N}\frac{\mathrm{d}\varrho}{\varrho} = \frac{\log(\mu/|j|^2)}{\mathrm{UB}^N_{k-1}(8\mu, z_{2k+1}(2))}.$$

For the other integral we have

$$\begin{aligned}
I_2 &\lesssim \int_{\mu_N}^1 \frac{\mathrm{d}\varrho}{4(\mu_N+\varrho)\mathrm{UB}_{k-1}(4(\mu_N+\varrho), z_{2k+1}(2))} \\
&\lesssim (1+\mu_N)\int_{8\mu_N}^{4(1+\mu_N)} \frac{\mathrm{d}\varrho}{(\varrho^2+\varrho)\mathrm{UB}_{k-1}(\varrho, z_{2k+1}(2))} \\
&\lesssim \frac{1+\mu_N}{\lambda^2}\mathrm{LB}^N_k(8\mu, z_{2k+1}(2)),
\end{aligned}$$

where we used (C.4). If instead $|j/N|^2 > \mu_N$, then, proceeding as in the bound for I_2 we get

$$\begin{aligned}\int_{|j/N|^2}^1 &\frac{\mathrm{d}\varrho}{\varrho \mathrm{UB}_{k-1}(4(\mu_N+\varrho), z_{2k+1}(2))}\\ &\lesssim \int_{|j/N|^2}^1 \frac{\mathrm{d}\varrho}{4(\mu_N+\varrho)\mathrm{UB}_{k-1}(4(\mu_N+\varrho), z_{2k+1}(2))}\\ &\lesssim \frac{1+\mu_N}{\lambda^2}\mathrm{LB}_k^N(4(\mu+\tfrac{1}{2}|j|^2), z_{2k+1}(2)),\end{aligned}$$

and the statement follows. ☐

3.3 | Estimating $\mathcal{B}_\varphi^N(\mu)$

Based on the results obtained above, we are ready to formulate and prove the main result of this section. In the next proposition, we provide both an upper and a lower bound on the Laplace transform of the second moment of $B_\varphi^N(t)$ given in (3.1). We will adopt the same notations and conventions introduced at the beginning of Section 3.1.

Proposition 3.13. *Let $\lambda > 0$ and, for $N \in \mathbb{N}$, let h^N be the solution of (2.2) and $\varphi \in L^2(\mathbb{T}^2)$ be a test function. Let $\mathcal{B}_\varphi^N$ be defined as in (3.3).*

(UB) For every $\delta > 0$ there exists a constant $c_\delta > 0$ (depending also on λ) such that for all $N \in \mathbb{N}$ and $\mu > 0$

$$\mathcal{B}_\varphi^N(\mu) \le \frac{c_\delta}{\mu}\left[\mathrm{L}^N(\mu,0)\right]^{\frac{1}{2}}\left(\log \mathrm{L}^N(\mu,0)\right)^{5+\delta}\|\varphi\|^2_{L^2(\mathbb{T}^2)} \tag{3.62}$$

where $\mathrm{L}^N(\mu,0)$ is defined according to (3.10) (see also Definition 3.3);

(LB) There exists a constant C such that for all $N \in \mathbb{N}, k \in \mathbb{N}$ and $0 < \mu \le N^2$,

$$\begin{aligned}\mathcal{B}_\varphi^N(\mu) \ge \frac{1}{C\mu}\sum_{|j|^2\le\mu} &\frac{|\hat{\varphi}(j)|^2}{c_{2k+1}^+ f_{2k+1}(2) + \frac{|j|^2}{\mu}\mathrm{LB}_k^N(4\mu, z_{2k+1}(2))}\\ &\times\Big[\mathrm{LB}_k^N\Big(\Big(\mu+\tfrac{1}{2}|j|^2\Big)\vee 1, z_{2k+2}(2)\Big) - f_{2k+1}(2)\Big].\end{aligned} \tag{3.63}$$

Proof. We use (3.3) and we focus on the scalar product at its right hand side. Throughout the proof, in order to lighten the notation we omit the subscript "ΓL^2" in all scalar products appearing below.

Let us begin with (UB). By Lemma 3.1 together with the fact that $-\mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N$ is a positive operator, for any $k \in \mathbb{N}$ we have

$$\begin{aligned}\langle \mathfrak{n}_\varphi^N, (\mu-\mathcal{L}^N)^{-1}\mathfrak{n}_\varphi^N\rangle &\le \langle \mathfrak{n}_\varphi^N, (\mu-\mathcal{L}_0+\mathcal{H}_{2k+2}^N-\mathcal{A}_+^N(\mu-\mathcal{L}_0)^{-1}\mathcal{A}_-^N)^{-1}\mathfrak{n}_\varphi^N\rangle\\ &\le \langle \mathfrak{n}_\varphi^N, (\mu-\mathcal{L}_0+\mathcal{H}_{2k+2}^N)^{-1}\mathfrak{n}_\varphi^N\rangle\\ &\le \langle \mathfrak{n}_\varphi^N, (\mu-\mathcal{L}_0(1+c_{2k+2}^-\mathcal{S}_{2k+2}^N))^{-1}\mathfrak{n}_\varphi^N\rangle,\end{aligned} \tag{3.64}$$

where in the last passage we applied Theorem 3.4. Recalling the definition of $\mathfrak{n}_\varphi^N$ in (3.2) and (3.4), and observing that the operator $(\mu - \mathcal{L}_0(1 + c_{2k+2}\mathcal{S}_{2k+2}^N))^{-1}$ is diagonal in Fourier space, the right hand side equals

$$2\lambda^2 \sum_{j\in\mathbb{Z}^2} |\hat\varphi(j)|^2 \times$$
$$\times \sum_{\ell+m=j} \frac{(\mathcal{K}_{\ell,m}^N)^2}{\mu + \Gamma(\ell,m)\Big(1 + \frac{c_{2k+2}^-}{f_{2k+2}(2)}\Big[\mathrm{LB}_k^N(\mu+\Gamma(\ell,m), z_{2k+2}(2)) - f_{2k+2}(2)\Big]\Big)},$$

where we adopted the same convention as in (3.29) for Γ and used that, by the definition of $\mathcal{K}^N$ and $\mathbb{J}^N$, $|\ell|, |m| \geq 1$, to remove the $\vee 1$ that appears in the definition of $\mathcal{S}_{2k+2}^M$. Note that the inner sum (for fixed j) is precisely of the form (3.37), except that $|k_{2:n}|$ is set to zero. Therefore, proceeding as in the proof of Lemma 3.9, we obtain

$$\begin{aligned}
&\langle \mathfrak{n}_\varphi^N, (\mu - \mathcal{L}^N)^{-1}\mathfrak{n}_\varphi^N\rangle \\
&\leq \frac{f_{2k+2}(2)}{2\pi c_{2k+2}^-}\left(1 + \frac{4\lambda\pi C_{\mathrm{Diag}}}{\sqrt{z_{2k+2}(1)}}\right)\sum_{j\in\mathbb{Z}^2} |\hat\varphi(j)|^2 \mathrm{UB}_k^N\left(\mu + \tfrac{1}{2}|j|^2, z_{2k+3}(2)\right) \\
&\leq \frac{c_{2k+3}^+}{2} f_{2k+2}(2)\mathrm{UB}_k^N(\mu, z_{2k+3}(2))\|\varphi\|_{L^2(\mathbb{T}^2)}^2 \\
&\lesssim \frac{k^{\frac{9}{2}+3\delta}}{\mathrm{LB}_k^N(\mu,0)}\mathrm{L}^N(\mu,0)\|\varphi\|_{L^2(\mathbb{T}^2)}^2
\end{aligned} \tag{3.65}$$

where the second inequality holds in view of (3.53) and the monotonicity of UB^N in the first variable, while the last by the definition of UB^N, the monotonicity of LB^N in its second argument, the definition of L^N and f_{2k+3}, z_{2k+3} in (3.11) and in (3.12), respectively, and the fact that, in view of Theorem 3.4 the sequence $\{c_{2k+1}^+\}_k$ converges to a finite constant depending on δ, so that the constant hidden in $\lesssim$ is an absolute positive constant depending only on δ, λ but on neither k nor N.

At this point, it remains to optimise over k in order to obtain the smallest possible upper bound. By Stirling's formula

$$\frac{k^{\frac{9}{2}+3\delta}}{\mathrm{LB}_k^N(\mu,0)} \leq \frac{k^{\frac{9}{2}+3\delta} k!}{(\frac{1}{2}\log \mathrm{L}^N(\mu,0))^k} \lesssim e k^{5+3\delta} \exp\left[k \log\left(\frac{2k}{e\log \mathrm{L}^N(\mu,0)}\right)\right]. \tag{3.66}$$

We choose then $k = k(\mu/N^2)$ as

$$k(\mu/N^2) \stackrel{\text{def}}{=} \left\lfloor \frac{1}{2}\log \mathrm{L}^N(\mu,0)\right\rfloor = \left\lfloor \frac{1}{2}\log \mathrm{L}(\mu/N^2,0)\right\rfloor. \tag{3.67}$$

With this choice of k and (3.66), we obtain

$$\frac{k^{\frac{9}{2}+3\delta}}{\mathrm{LB}_k^N(\mu,0)} \lesssim \left[\mathrm{L}^N(\mu,0)\right]^{-\frac{1}{2}}\left(\log \mathrm{L}^N(\mu,0)\right)^{5+3\delta} \tag{3.68}$$

from which, plugging (3.68) into (3.65) and recalling (3.3), (3.62) follows (with δ replaced by 3δ).

We now turn to (LB). Arguing as in (3.64), for any $k \in \mathbb{N}$ we have

$$\langle \mathfrak{n}_\varphi^N, (\mu - \mathcal{L}^N)^{-1} \mathfrak{n}_\varphi^N \rangle \geq \langle \mathfrak{n}_\varphi^N, (\mu - \mathcal{L}_0 + \mathcal{H}_{2k+1}^N - \mathcal{A}_+^N(\mu - \mathcal{L}_0)^{-1}\mathcal{A}_-^N)^{-1} \mathfrak{n}_\varphi^N \rangle. \tag{3.69}$$

For $\mathcal{H}_{2k+1}^N$ we use the upper bound provided by Theorem 3.4 while for $-\mathcal{A}_+^N(\mu - \mathcal{L}_0)^{-1}\mathcal{A}_-^N$ we use Lemma 3.11 with the choice

$$G(x) \stackrel{\text{def}}{=} \mathrm{UB}_{k-1}^N(x, z_{2k+1}(2)).$$

Hence, (3.69) is bounded below by

$$\begin{aligned} &2\lambda^2 \sum_j |\hat{\varphi}(j)|^2 \sum_{\ell+m=j} \frac{(\mathcal{K}_{\ell,m}^N)^2}{\mu + \Gamma(\ell,m)(1+F_k(j))\mathrm{UB}_{k-1}^N(\mu+\Gamma(\ell,m), z_{2k+1}(2)))} \\ &\geq \lambda^2 \sum_j \frac{|\hat{\varphi}(j)|^2}{F_k(j)} \sum_{\ell+m=j} \frac{(\mathcal{K}_{\ell,m}^N)^2}{\mu + \Gamma(\ell,m)\mathrm{UB}_{k-1}^N(\mu+\Gamma(\ell,m), z_{2k+1}(2))} \end{aligned} \tag{3.70}$$

where we introduced $F_k(j) \stackrel{\text{def}}{=} c_{2k+1}^+ f_{2k+1}(2) + c\lambda^2 g(j) \geq 1$ for $k \in \mathbb{N}$ and $j \in \mathbb{Z}^2$, in which c is the constant that appears in (3.55) and g is defined in (3.56). Also in this case, the inner sum in (3.70) has the same structure as in (3.30), so that, proceeding as in the proof of Lemma 3.8, we obtain

$$\begin{aligned} &\sum_{\ell+m=j} \frac{(\mathcal{K}_{\ell,m}^N)^2}{\mu + \Gamma(\ell,m)\mathrm{UB}_{k-1}^N(\mu+\Gamma(\ell,m), z_{2k+1}(2))} \\ &\geq \frac{1}{4\lambda^2\pi}\left(1 - \frac{4\pi\lambda C_{\mathrm{Diag}}}{\sqrt{z_{2k+1}(1)}}\right)\left[\mathrm{LB}_k^N\left(1+\mu+\tfrac{1}{2}|j|^2, z_{2k+1}(2)\right) - f_{2k+1}(2)\right] \\ &\geq \frac{1}{8\lambda^2\pi}\left[\mathrm{LB}_k^N\left(1+\mu+\tfrac{1}{2}|j|^2, z_{2k+1}(2)\right) - f_{2k+1}(2)\right] \end{aligned} \tag{3.71}$$

where we further exploited that by our choice of z_{2k+1} in (3.12) with K_1 large, the quantity the constant in the rounded brackets in the second line is bigger than $1/2$ (see (3.53)).

Restricting to $|j|^2 \leq \mu$ and plugging the expression for $F_k(j)$ back in, we see that, for all $k \in \mathbb{N}$ (modulo constants independent of μ, k and N) (3.70) is lower bounded by

$$\sum_{|j|^2 \leq \mu} \frac{|\hat{\varphi}(j)|^2}{c_{2k+1}^+ f_{2k+1}(2) + \lambda^2 g(j)}\left[\mathrm{LB}_k^N(1+\mu+\tfrac{1}{2}|j|^2, z_{2k+1}(2)) - f_{2k+1}(2)\right]. \tag{3.72}$$

We are left to deal with the denominator in the sum in (3.72) for which we need Lemma 3.12. Since $|j|^2 \leq \mu \leq N^2$ (in particular $\mu_N \leq 1$) for any $k \in \mathbb{N}$ we have

$$\begin{aligned} g(j) &\lesssim \frac{|j|^2}{\mu}\left(\frac{\log(\mu/|j|^2)}{\mathrm{UB}_{k-1}^N(8\mu, z_{2k+1}(2))} + \frac{1}{\lambda^2}\mathrm{LB}_k^N(8\mu, z_{2k+1}(2))\right) \\ &\lesssim \frac{|j|^2}{\lambda^2\mu}\left(\mathrm{LB}_{k-1}^N(8\mu, z_{2k+1}(2)) + \mathrm{LB}_k^N(8\mu, z_{2k+1}(2))\right) \\ &\lesssim \frac{|j|^2}{\lambda^2\mu}\mathrm{LB}_k^N(4\mu, z_{2k+1}(2)) \end{aligned}$$

where, in the passage from the first to the second line we exploited the definition of UB^N_{k-1} and of L^N, while in the last the monotonicity of LB_k with respect to its first argument and the fact that for all k, $\mathrm{LB}_{k-1} \leq \mathrm{LB}_k$. Recalling (3.3), we deduce (3.63). □

4 | PROOF OF THE MAIN RESULTS

This section is devoted to the proofs of the main theorems. We begin with the bulk diffusivity, since, as we will see, the bounds we aim at follow directly from Proposition 3.13.

4.1 | The bulk diffusivity: Proof of Theorem 1.1

At first we provide an equivalent formulation of the bulk diffusivity D_N defined in (1.6), which shows that D_N represents the average speed at which the mass of the solution H_N spreads in time.

Lemma 4.1. *For $N \in \mathbb{N}$, let D_N be the bulk diffusivity defined in (1.6). Then, for all $N \in \mathbb{N}$ and $t > 0$, the following equality holds*

$$t\, D_N(t) = t + N^2 \mathbf{E}[B^N_{e_0}(t/N^2)^2] \tag{4.1}$$

where B^N was defined in (3.1) and $e_0 \equiv \frac{1}{2\pi}$ is the 0-th Fourier basis element.

Proof. Notice at first, that in view of the scaling relation (2.1) and the definition of $\mathcal{N}^N$ in (3.2), it is immediate to see that for any $N \in \mathbb{N}$ and $t \geq 0$

$$tD_N(t) = t + 2\lambda^2 N^2 \int_0^{t/N^2} \int_0^s \int_{\mathbb{T}^2} \mathbf{E}\big[\mathcal{N}^N[u^N](r,0)\mathcal{N}^N[u^N](0,x)\big]\mathrm{d}x\mathrm{d}r\mathrm{d}s. \tag{4.2}$$

Now, the process u^N with white noise initial condition is translation invariant in law, which implies that, for every $r \geq 0$, the spatial integral in the second summand on the right hand equals

$$\begin{aligned}
&\frac{1}{4\pi^2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} \mathbf{E}\left[\mathcal{N}^N\left[u^N\right](r,y)\mathcal{N}^N\left[u^N\right](0,x+y)\right]\mathrm{d}x\mathrm{d}y\\
&\quad = \mathbf{E}\left[\mathcal{N}^N[u^N(r)](e_0)\mathcal{N}^N[u^N(0)](e_0)\right]
\end{aligned}$$

where the last passage can be obtained by integrating first in x and then in y. To conclude it is sufficient to note that for any $t \geq 0$

$$\int_0^t \int_0^s \mathbf{E}\big[\mathcal{N}^N[u^N(r)](e_0)\mathcal{N}^N[u^N(0)](e_0)\big]\mathrm{d}r\mathrm{d}s = \frac{1}{2}\mathbf{E}\left[\left(\int_0^t \mathcal{N}^N[u^N(s)](e_0)\mathrm{d}s\right)^2\right]. \qquad \square$$

The advantage of (4.1) is that the bulk diffusivity D_N is expressed in terms of an observable of the form (3.1) so that the results in the previous section are directly applicable. We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to (4.1) and (3.3), it is immediate to show that for every $N \in \mathbb{N}$

$$\mathcal{D}_N(\mu) = \mu \int_0^\infty e^{-\mu t} t\, D_N(t) \mathrm{d}t = \frac{1}{\mu} + N^2 \mathcal{B}^N_{e_0}(\mu N^2).$$

Therefore, it remains to bound the second summand, for which we exploit Proposition 3.13. We begin with the upper bound. Notice that (3.62) gives

$$\begin{aligned} N^2 \mathcal{B}^N_{e_0}(\mu N^2) &\le \frac{C}{\mu} [\mathrm{L}^N(\mu N^2, 0)]^{\frac{1}{2}} \left(\log \mathrm{L}^N(\mu N^2, 0)\right)^{5+\delta} \\ &= \frac{C}{\mu} [\mathrm{L}(\mu, 0)]^{\frac{1}{2}} (\log \mathrm{L}(\mu, 0))^{5+\delta} \end{aligned}$$

from which (1.8) follows.

For the lower bound instead, (3.63) implies that for all $k \in \mathbb{N}$, $\mu > 0$ we have

$$\liminf_{N\to\infty} N^2 \mathcal{B}^N_{e_0}(\mu N^2) \ge \frac{1}{C\mu} \left[\frac{\mathrm{LB}_k(\mu, z_{2k+2}(2))}{c^+_{2k+1} f_{2k+1}(2)} - \frac{1}{c^+_{2k+1}} \right], \tag{4.3}$$

and we are left to optimise over k. But we have already done so in the proof of Proposition 3.13(UB). Upon choosing $k = k(\mu)$ as in (3.67), by (3.68), we have

$$\frac{\mathrm{LB}_k(\mu, z_{2k+2}(2))}{c^+_{2k+1} f_{2k+1}(2)} \gtrsim [\mathrm{L}(\mu, 0)]^{\frac{1}{2}} (\log \mathrm{L}(\mu, 0))^{-5-\delta} \tag{4.4}$$

and, since the right hand side diverges as $\mu \to 0$, at the price of an absolute constant, we can reabsorb the $-1/c^+_{2k+1}$ in (4.3), so that the proof is complete. □

4.2 | The diffusive scaling: Proof of Theorem 1.2 and Corollary 1.3

In order to prove Theorem 1.2, we first consider the weak formulation of AKPZ on the torus of side length 1 and separately analyse each of the three summands appearing on the right hand side of (2.2). More precisely, let φ be a smooth test function on $\mathbb{T}^2$ and N fixed, then, for all $t \ge 0$, h^N satisfies

$$h^N(t)[\varphi] - h^N(0)[\varphi] = \underbrace{\frac{1}{2} \int_0^t h^N(s)[\Delta\varphi] \mathrm{d}s}_{A^N_\varphi(t)} + B^N_\varphi(t) + \underbrace{\int_0^t \xi(\mathrm{d}s)[\varphi]}_{C^N_\varphi(t)}, \tag{4.5}$$

where $B^N_\varphi(t)$, the integral in time of the nonlinearity, was defined in (3.1). We recall that B^N_φ is a centred random variable, and the same can be easily verified for A^N_φ, C^N_φ. Now, B^N_φ, or more precisely the Laplace transform of its second moment, has been thoroughly studied in the previous section. In the following proposition, we derive suitable bounds on the second moments of A^N_φ and C^N_φ and on their Laplace transforms. To that end we define

$$\mathcal{A}^N_\varphi(\mu) \stackrel{\text{def}}{=} \mu \int_0^\infty e^{-\mu t} \mathbf{E} A^N_\varphi(t)^2 \,\mathrm{d}t \quad \text{and} \quad \mathcal{C}^N_\varphi(\mu) \stackrel{\text{def}}{=} \mu \int_0^\infty e^{-\mu t} \mathbf{E} C^N_\varphi(t)^2 \,\mathrm{d}t \tag{4.6}$$

for $\mu > 0$.

Proposition 4.2. *Let $N \in \mathbb{N}$, h^N be the solution of (2.2) and $\varphi \in L^2(\mathbb{T}^2)$ be a test function. Let $A^N_\varphi(t)$ and $C^N_\varphi(t)$ be defined according to (4.5). Then, there exists a constant $c > 0$ independent of N and φ such that for every t, $\mu > 0$ we have*

$$\mathbf{E}A^N_\varphi(t)^2 \le ct\|\varphi\|^2_{L^2(\mathbb{T}^2)} \quad \textit{and} \quad \mathcal{A}^N_\varphi(\mu) \le \frac{c}{\mu}\|\varphi\|^2_{L^2(\mathbb{T}^2)}, \tag{4.7}$$

$$\mathbf{E}C^N_\varphi(t)^2 = t\|\varphi\|^2_{L^2(\mathbb{T}^2)} \quad \textit{and} \quad \mathcal{C}^N_\varphi(\mu) = \frac{1}{\mu}\|\varphi\|^2_{L^2(\mathbb{T}^2)}. \tag{4.8}$$

Proof. The result on the Laplace transform is an immediate consequence of that on A^N_φ and C^N_φ. The first identity in (4.8) is straightforward and follows from an explicit computation that uses the correlation structure of the white noise ξ. To estimate $A^N_\varphi(t)$ we make use of [10, Lemma 4.3, Equation (4.11)], which states that for any $T > 0$

$$\mathbb{E}\left[\sup_{t\le T}\left|\int_0^t u^N(s)[\Delta\varphi]\mathrm{d}s\right|^2\right]^{1/2} \lesssim T^{1/2}\|\varphi\|_{1,2}, \tag{4.9}$$

where

$$\|\varphi\|^2_{1,2} = \sum_{k\in\mathbb{Z}^2}(1+|k|^2)|\hat{\varphi}(k)|^2.$$

Here, we recall that u^N is the solution to the stochastic Burgers equation (2.3) so that, upon noting

$$A^N_\varphi(t) = \frac{1}{2}\int_0^t u^N(s)[(-\Delta)^{1/2}\varphi]\mathrm{d}s, \tag{4.10}$$

the result follows at once. □

The previous statement was the missing ingredient needed in the proof of Theorem 1.2.

Proof of Theorem 1.2. Notice at first that, given any test function φ, by the definition of $H^\varepsilon_N[\varphi]$ in (1.11) and the equality in law (2.1), it follows that

$$\mathcal{V}^{\varepsilon,N}_\varphi(\mu) = \mathcal{V}^{N^{-1},1}_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu)$$

where, for any $a > 0$, $\varphi^{(a)}$ is given as in the introduction, that is, $\varphi^{(a)}(\cdot) = a^2\varphi(a\cdot)$. In view of the decomposition (4.5), we have

$$\mathcal{V}^{N^{-1},1}_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu) \lesssim \mathcal{A}^N_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu) + \mathcal{B}^N_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu) + \mathcal{C}^N_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu)$$

so that, since $\|\varphi^{(N\varepsilon)}\|^2_{L^2(\mathbb{T}^2)} = (N\varepsilon)^2\|\varphi\|^2_{L^2(\mathbb{R}^2)}$, the bound (1.14) follows immediately from Propositions 4.2 and 3.13(UB).

We now turn to the lower bounds. To that end note that for any $\mu > 0$, we have

$$\mathcal{V}^{N^{-1},1}_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu) \gtrsim \mathcal{B}^N_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu) - \frac{\|\varphi\|_{L^2(\mathbb{R}^2)}}{\sqrt{\mu}}\sqrt{\mathcal{B}^N_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu)} - \frac{\|\varphi\|^2_{L^2(\mathbb{R}^2)}}{\mu} \tag{4.11}$$

where we exploited once more the decomposition (4.5), the Cauchy-Schwarz inequality to control the cross products as well as Proposition 4.2, to bound the occurrences of $\mathcal{A}^N_\varphi$ and $\mathcal{C}^N_\varphi$. Now, by Proposition 3.13(LB), for any integer $k \in \mathbb{N}$, we have

$$\liminf_{N\to\infty} \mathcal{B}^N_{\varphi^{(\varepsilon N)}}(\varepsilon^2 N^2\mu) \gtrsim \left[1 - \frac{1}{c^+_{2k+1} Y_k}\right] \lim_{N\to\infty} \frac{1}{(N\varepsilon)^2} \sum_{|j|^2 \le (N\varepsilon)^2\mu} \frac{|\widehat{\varphi^{(N\varepsilon)}}(j)|^2}{\mu/Y_k + |j|^2/(N\varepsilon)^2} \tag{4.12}$$

where

$$Y_k \stackrel{\text{def}}{=} \frac{\mathrm{LB}_k(4\varepsilon^2\mu, z_{2k+1}(2))}{c^+_{2k+1} f_{2k+1}(2)} \tag{4.13}$$

and we used the fact that LB_k is continuous and decreasing in its first argument. Since $\widehat{\varphi^{(N\varepsilon)}}(j) = \hat\varphi(p)$ for $p \stackrel{\text{def}}{=} j/(N\varepsilon)$, the limit in (4.12) reduces to

$$\lim_{N\to\infty} \frac{1}{(N\varepsilon)^2} \sum_{\substack{p\in(\mathbb{Z}/N\varepsilon)^2\\ |p|^2\le\mu}} \frac{|\hat\varphi(p)|^2}{\mu/Y_k + |p|^2} = \int_{\mathbb{R}^2} \mathbb{1}_{|p|^2\le\mu} \frac{|\hat\varphi(p)|^2}{\mu/Y_k + |p|^2} \mathrm{d}p. \tag{4.14}$$

Now we proceed similarly to the proof of the lower bound in Theorem 1.1. Namely, we fix $k(\varepsilon^2\mu)$ as in (3.67) and we note that, since we are assuming $\mu \le \varepsilon^{-1}$, $k(\varepsilon^2\mu)$ diverges as $\varepsilon \to 0$. Moreover, arguing once more as in the proof of Proposition 3.13(UB) we see that also $Y_{k(\varepsilon^2\mu)}$ diverges since

$$Y_{k(\varepsilon^2\mu)} \gtrsim \sqrt{\mathrm{L}(\varepsilon^2\mu,0)}(\log \mathrm{L}(\varepsilon^2\mu,0))^{-5-\delta}. \tag{4.15}$$

In particular, the negative term in (4.12) can be neglected. Also, since we are assuming $\mu \le (\log(1/\varepsilon))^{\frac12}(\log\log(1/\varepsilon))^{-5-\delta}$, the ratio μ/Y_k tends to zero as $\varepsilon \to 0$. Altogether, we get

$$\liminf_{\varepsilon\to 0} \liminf_{N\to\infty} \mathcal{B}^N_{\varphi^{(\varepsilon N)}}(\varepsilon^2N^2\mu) \gtrsim \int_{\mathbb{R}^2} \mathbb{1}_{|p|^2\le\mu} \frac{|\hat\varphi(p)|^2}{|p|^2}\mathrm{d}p. \tag{4.16}$$

If φ has non-zero average then the integral is infinite. If instead the (smooth) function φ has zero average, the integral is finite: in this case, assuming that μ is sufficiently large (larger than some constant a_φ), the integral is arbitrarily close to $\|\varphi\|^2_{-1}$. In either case, (1.15) is proven. □

Proof of Corollary 1.3. Throughout the proof we fix $\delta > 0$ as in the formulation of Corollary 1.3. We start with (1.19) so that

$$0 \le 1 - \frac{\mathrm{Cov}(H^\varepsilon_N(t)[\varphi], H^\varepsilon_N(0)[\varphi])}{\mathrm{Var}(H^\varepsilon_N(0)[\varphi])} = \frac{V^{\varepsilon,N}_\varphi(t)}{2\mathrm{Var}(H^\varepsilon_N(0)[\varphi])} \tag{4.17}$$

and we recall that the denominator is uniformly positive and finite: since the stationary measure is the GFF on the plane and $\int_{\mathbb{R}^2}\varphi(x)\mathrm{d}x = 0$,

$$\lim_{N\to\infty} \mathrm{Var}(H^\varepsilon_N(0)[\varphi]) = \frac{1}{2\pi^2}\int_{\mathbb{R}^2} \frac{|\hat\varphi(k)|^2}{|k|^2}\mathrm{d}k < \infty. \tag{4.18}$$

To prove (1.17) we need to show that $V_\varphi^{\varepsilon,N}(t)$ is uniformly small for $t < t_-(\varepsilon)$. Recall the definition (1.12) of $V_\varphi^{\varepsilon,N}(t)$, together with $H_N^\varepsilon(t)[\varphi] = h^N(t/(N\varepsilon)^2)[\varphi^{(\varepsilon N)}]$. One has then (with the notations of (4.5))

$$V_\varphi^{\varepsilon,N}(t) \lesssim \mathbf{E}A_{\varphi^{(\varepsilon N)}}(t/(\varepsilon N)^2)^2 + \mathbf{E}B_{\varphi^{(\varepsilon N)}}(t/(\varepsilon N)^2)^2 + \mathbf{E}C_{\varphi^{(\varepsilon N)}}(t/(\varepsilon N)^2)^2.$$

Thanks to Proposition 4.2 and $\|\varphi^{(\varepsilon N)}\|^2_{L^2(\mathbb{T}^2)} = (N\varepsilon)^2\|\varphi\|^2_{L^2(\mathbb{R}^2)}$, the terms involving A and C are upper bounded by a constant times $t\|\varphi\|^2_{L^2(\mathbb{R}^2)}$ which is uniformly small in $\varepsilon \to 0$ if $t \le t_-(\varepsilon) \ll 1$. As for the term involving B, recall the definition of $\mathcal{B}_\varphi^N(\mu)$ in (3.3). With the same argument as in the proof of [37, Lemma 1] one can show that there exists a universal positive constant C such that, for $t > 0$ and letting $\mu_t \stackrel{\text{def}}{=} 1/t$,

$$\mathbf{E}[B_\varphi^N(t)^2] \le C\mathcal{B}_\varphi^N(\mu_t).$$

Thanks to (3.62) and recalling that $\mathrm{L}^N(x,z) = L(x/N^2, z)$, we deduce that for any $\delta' > 0$ there exists a constant $c_{\delta'} > 0$ such that

$$\mathbf{E}[B^N_{\varphi^{(\varepsilon N)}}(t)^2] \le c_{\delta'}t\sqrt{\mathrm{L}(\varepsilon^2/t,0)}(\log \mathrm{L}(\varepsilon^2/t,0))^{5+\delta'}\|\varphi\|^2_{L^2(\mathbb{R}^2)}.$$

Choosing $\delta' < \delta$, the claim then follows from the fact that

$$\lim_{\varepsilon\to 0} t_-(\varepsilon)\sqrt{\mathrm{L}(\varepsilon^2/t_-(\varepsilon),0)}(\log \mathrm{L}(\varepsilon^2/t_-(\varepsilon),0))^{5+\delta'} = 0.$$

Finally, let us prove (1.18). We choose $\mu = \mu(\varepsilon)$ that satisfies $1/t_+(\varepsilon) \ll \mu \ll 1/t_-(\varepsilon)$ as $\varepsilon \to 0$. By definition,

$$\mathcal{V}_\varphi^{\varepsilon,N}(\mu)$$
$$= \mu\int_0^{t_-(\varepsilon)} e^{-\mu t}V_\varphi^{\varepsilon,N}(t)\mathrm{d}t + \mu\int_{t_-(\varepsilon)}^{t_+(\varepsilon)} e^{-\mu t}V_\varphi^{\varepsilon,N}(t)\mathrm{d}t + \mu\int_{t_+(\varepsilon)}^{\infty} e^{-\mu t}V_\varphi^{\varepsilon,N}(t)\mathrm{d}t.$$

Since $V_\varphi^{\varepsilon,N}$ is uniformly bounded and $\mu t_+(\varepsilon) \gg 1$, the third integral is negligible. The first integral is also negligible, as follows recalling (4.17) and (1.17). On the other hand, from (1.15), we have that $\liminf_\varepsilon \liminf_N \mathcal{V}_\varphi^{\varepsilon,N}(\mu) \ge b\|\varphi\|^2_{-1}$. Since the function $t \mapsto \mu e^{-\mu t}$ has integral 1, we deduce that

$$\liminf_{\varepsilon\to 0}\liminf_{N\to\infty}\sup_{t\in[t_-(\varepsilon),t_+(\varepsilon)]} V_\varphi^{\varepsilon,N}(t) \ge b\|\varphi\|^2_{-1}.$$

The conclusion now follows recalling (4.17) and (4.18). ☐

4.3 | Large time behaviour: Proof of Theorem 1.5

In order to prove Theorem 1.5, we need a refined version of the bound obtained in [10, Lemma 4.3] on observables of the form in (3.1).

Lemma 4.3. *Let φ be a test function and, for $N \in \mathbb{N}$, let B^N_φ be defined according to (3.1). Then, for all $t \geq 0$ the following bound holds*

$$\mathbf{E}[B^N_\varphi(t)]^2 \lesssim \lambda^2 t \sum_{|k|\leq N} |\hat\varphi(k)|^2 \log\left(\frac{1}{|k/N|^2 \vee N^{-2}}\right). \tag{4.19}$$

Proof. The proof of (4.19) is extremely close to that of [10, Lemma 4.3] so we will adopt the same notations and conventions therein and we will limit ourselves to sketch the main steps, addressing the reader to the above mentioned reference for more details. Let $\mathcal{G}^N$ be the solution of the Poisson equation $\mathcal{L}_0\mathcal{G}^N(\eta)[\varphi] = \lambda\mathcal{N}^N(\eta)[\varphi]$, which is explicitly given by

$$\mathcal{G}^N(\eta)[\varphi] = \lambda \sum_{\ell,m\in\mathbb{Z}^2} \frac{\mathcal{K}^N_{\ell,m}}{|\ell|^2+|m|^2}\hat\eta(\ell)\hat\eta(m)\hat\varphi(-\ell-m),$$

where we recall $\mathcal{K}^N_{\ell,m}$ was defined in (2.11). Then, defining $\mathcal{E}^N$ as in [10, eq. (4.4)], a simple Gaussian computation shows that

$$\mathbb{E}[\mathcal{E}^N(\mathcal{G}^N(\eta)[\varphi])] = 8\lambda^2 \sum_{\ell,m\in\mathbb{Z}^2} \frac{(\mathcal{K}^N_{\ell,m})^2}{|\ell|^2+|m|^2}|\hat\varphi(-\ell-m)|^2.$$

The last sum can be bounded as follows

$$\begin{aligned}
&\sum_{\ell,m\in\mathbb{Z}^2} \frac{(\mathcal{K}^N_{\ell,m})^2}{|\ell|^2+|m|^2}|\hat\varphi(-\ell-m)|^2 \lesssim \sum_{|k|\leq N}|\hat\varphi(k)|^2 \sum_{\ell+m=k}\frac{\mathbb{J}^N_{\ell,m}}{|\ell|^2+|m|^2}\\
&\lesssim \sum_{|k|\leq N}|\hat\varphi(k)|^2 \sum_{\substack{\ell+m=k\\|\ell|\geq|m|}}\frac{\mathbb{J}^N_{\ell,m}}{|\ell|^2} \lesssim \sum_{|k|\leq N}|\hat\varphi(k)|^2 \sum_{N\geq|\ell|>|k|/2\vee 1}\frac{1}{|\ell|^2}\\
&\lesssim \sum_{|k|\leq N}|\hat\varphi(k)|^2 \int_{1\geq|\varrho|>|k/N|/2\vee N^{-2}}\frac{\mathrm{d}\varrho}{|\varrho|^2} \lesssim \sum_{|k|\leq N}|\hat\varphi(k)|^2\log\left(\frac{1}{|k/N|^2\vee N^{-2}}\right)
\end{aligned}$$

where we exploited the symmetry of the summand and the fact that if $\ell + m = k$ and $|\ell| \geq |m|$, then necessarily $|\ell| \geq |k|/2$. The conclusion follows by [10, Lemma 4.1]. □

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let φ be a smooth test function on $\mathbb{R}^2$ and $g : \mathbb{R}^2 \to [0,1]$ be positive, smooth and such that $\int_{\mathbb{R}^2} g(y)\,\mathrm{d}y = 1$. For $n \in \mathbb{N}$ define $g_n(y) \stackrel{\text{def}}{=} g(y/n)$ so that $\int_{\mathbb{R}^2} g_n(y)\,dy = n^2$, and $\psi_n(y) \stackrel{\text{def}}{=} (\varphi * g_n)(y)$. Throughout the proof, we will denote by c_φ a positive constant that may change from line to line and that will depend on φ and possibly g.

Notice at first that

$$H_N(t)[\psi_n] = \int_{\mathbb{R}^2} g_n(y)\{H_N(t)[\varphi(\cdot - y)] - H_N(t)[\varphi]\}\mathrm{d}y + H_N(t)[\varphi]n^2$$

which implies

$$H_N(t)[\varphi] - H_N(0)[\varphi] = \frac{1}{n^2}\big(H_N(t)[\psi_n] - H_N(0)[\psi_n] - v^{(n)}(t) + v^{(n)}(0)\big) \tag{4.20}$$

where

$$v^{(n)}(t) = \int_{\mathbb{R}^2} g_n(y)(H_N(t)[\varphi(\cdot - y)] - H_N(t)[\varphi])\mathrm{d}y.$$

Thanks to the scaling (2.1), it is immediate to see that

$$H_N(t)[\psi_n] - H_N(0)[\psi_n] \stackrel{\text{law}}{=} h^N(t/N^2)[\psi_n^{(N)}] - h^N(0)[\psi_n^{(N)}]$$

where $\psi_n^{(N)}$ is given as in the introduction, that is, $\psi_n^{(N)}(\cdot) \stackrel{\text{def}}{=} N^2\psi_n(N\cdot)$, so that we can focus on the right hand side. Applying the decomposition (4.5), we write

$$h^N(t/N^2)[\psi_n^{(N)}] - h^N(0)[\psi_n^{(N)}] = A^N_{\psi_n^{(N)}}(t/N^2) + B^N_{\psi_n^{(N)}}(t/N^2) + C^N_{\psi_n^{(N)}}(t/N^2). \tag{4.21}$$

By Proposition 4.2 and the fact that $\|\psi_n^{(N)}\|^2_{L^2(\mathbb{T}^2)} = N^2\|\psi_n\|^2_{L^2(\mathbb{R}^2)}$, the variances of A^N and C^N (which are centred) can be bounded by

$$\mathbf{E}[A^N_{\psi_n^{(N)}}(t/N^2)]^2 \lesssim t\|\psi_n\|^2_{L^2(\mathbb{R}^2)} \qquad \mathbf{E}[C^N_{\psi_n^{(N)}}(t/N^2)]^2 \le t\|\psi_n\|^2_{L^2(\mathbb{R}^2)}$$

and, using the fact that $\|\psi_n\|^2_{L^2(\mathbb{R}^2)} \le c_\varphi n^2$, we conclude

$$\frac{1}{n^4}\left(\mathbf{E}[A^N_{\psi_n^{(N)}}(t/N^2)]^2 \vee \mathbf{E}[C^N_{\psi_n^{(N)}}(t/N^2)]^2\right) \le c_\varphi \frac{t}{n^2}. \tag{4.22}$$

Concerning B^N, we exploit Lemma 4.3, which gives

$$\begin{aligned}
\limsup_{N\to\infty} \mathbf{E}[B^N_{\psi_n^{(N)}}(t/N^2)]^2 &\lesssim \lambda^2 t \limsup_{N\to\infty} \sum_{|k/N|\le 1} \frac{1}{N^2}\hat{\psi}_n(k/N)\log\left(\frac{1}{|k/N|^2 \vee N^{-2}}\right)\\
&\lesssim \lambda^2 t \int |\hat{\psi}_n(p)|^2 \log\left(\frac{1}{|p|}\right)\mathbb{1}_{|p|\le 1}\mathrm{d}p\\
&\le \lambda^2 t n^2 \|\varphi\|^2_\infty \int |\hat{g}(p)|^2 \log\left(\frac{n}{|p|}\right)\mathbb{1}_{|p|\le n}\mathrm{d}p \le c_\varphi \lambda^2 t n^2 \log n
\end{aligned} \tag{4.23}$$

where we used that $\hat{\psi}_n(k) = \hat{\varphi}(k)\hat{g}_n(k) = n^2\hat{\varphi}(k)\hat{g}(kn)$. Getting back to (4.20), it remains to control $\mathbf{E}[v^{(n)}(t)]^2 = \mathbb{E}[v^{(n)}(0)]^2$, the equality being due to the stationarity of H_N. Note that

$$\begin{aligned}
\mathbb{E}[v^{(n)}(0)]^2 = &\int g_n(y)g_n(y')\\
&\times\mathbb{E}[(H_N(0)[\varphi(\cdot - y)] - H_N(0)[\varphi])(H_N(0)[\varphi(\cdot - y')] - H_N(0)[\varphi])]\mathrm{d}y\mathrm{d}y'.
\end{aligned} \tag{4.24}$$

Thus, using the Cauchy-Schwarz inequality, that H_N is distributed according to a GFF and that g_n integrates to n^2, it is not hard to see that

$$\limsup_{N\to\infty} \mathbb{E}[v^{(n)}(0)]^2 \le c_\varphi n^4(\log n \vee 1). \tag{4.25}$$

We are now ready to put the bounds (4.22) (4.23) and (4.25) together and, suitably applying the Cauchy-Schwarz inequality to the various terms in (4.20), deduce

$$\limsup_{N\to\infty} V^{1,N}_\varphi(t) \le c_\varphi \left[\frac{t}{n^2} + \lambda^2 t \frac{\log n \vee 1}{n^2} + \log n \vee 1\right]$$

Therefore, choosing $n = \lceil \sqrt{t} \rceil$ concludes the proof. ☐

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APPENDIX A: THE BULK DIFFUSIVITY AND THE GREEN-KUBO FORMULA: A HEURISTIC

In this section we want to provide a heuristic justification for the choice of the definition of the bulk diffusivity given in (1.6). We consider the Stochastic Burgers equation (obtained by (1.3) by formally setting $U_N \stackrel{\text{def}}{=} (-\Delta)^{1/2} H_N$) on the full space ($N = \infty$) with cut-off 1 and initial condition given by a regularised spatial white noise that is independent of ξ, that is,

$$\partial_t U = \frac{1}{2}\Delta U + \lambda \mathcal{M}^1[U] + (-\Delta)^{\frac{1}{2}} \Pi_a \xi, \qquad U(0) = \eta^a \stackrel{\text{def}}{=} \Pi_a \eta, \tag{A.1}$$

where $a \in (1, \infty)$ and $\mathcal{M}$ is defined as in (2.4). Compared to (2.3), in (A.1) also the space-time white noise ξ is smoothened out. The main properties of the solution U remain unaltered, and, with the same techniques adopted in [10], it can be shown that the unique solution U exists globally in time and is a space-time translation invariant strong Markov process with invariant measure η^a. The advantage of (A.1) is that U is smooth so that space-time point evaluation is allowed and well-posed.

The bulk diffusivity serves as a way to measure the spread of the correlations of U and it is classically defined (see for instance [3] for the definition in the context of the 1d KPZ equation) as

$$D^{(a)}(t) \stackrel{\text{def}}{=} \frac{1}{2t} \int_{\mathbb{R}^2} |x|^2 S(t, x) \mathrm{d}x, \tag{A.2}$$

where, for $t \geq 0$ and $x \in \mathbb{R}^2$, S denotes the two-point correlation function

$$S(t, x) \stackrel{\text{def}}{=} \mathbf{E}[U(t, x) U(0, 0)]. \tag{A.3}$$

See for instance [38, Ch. II.2.2] for the analogous definition for interacting particle systems (we put the prefactor $1/2$ simply to ensure that the diffusion coefficient of the linear equation is 1 and, with respect to the interacting particle system references, we omit a prefactor related to the so-called "compressibility"). We now want to formally manipulate the expression on the right hand side of (A.2) in order to connect it to (1.6). Note that if $\lambda = 0$, $S(t, \cdot)$ is explicit and it can be easily shown to integrate to 1 and to decay at ∞ exponentially fast. For the purpose of this section, we will *assume* that $S(t,x)$ decays fast (say, faster than $1/|x|^2$) for $|x|\to\infty$ also for $t>0$. Using integration by parts and that $\mathcal{M}^1[U]=(-\Delta)^{1/2}\mathcal{N}^1[U]$, one then sees that $S(t,\cdot)$ also integrates to 1. Now, upon integrating (A.1) in time and plugging it into the definition of S we see that

$$S(t,x)=S(0,x)+\frac{1}{2}\int_0^t \Delta S(s,x)\mathrm{d}s+\lambda\int_0^t \mathbf{E}[\mathcal{M}^1[U](s,x)u(0,0)]\mathrm{d}s,$$

where the term containing the noise drops out because the initial condition is independent of ξ. Since $\int S(t,x)\mathrm{d}x=1$ and $|S(t,\cdot)|$ decays sufficiently fast, a simple integration by parts gives

$$\frac{1}{4}\int_0^t\int |x|^2\Delta S(s,x)\mathrm{d}x\mathrm{d}s=t.$$

For the term containing the nonlinearity instead, recall that $(-\Delta)^{\frac{1}{2}}\mathcal{N}^1[U]=\mathcal{M}^1[U]$. Then, integrating once more by parts, we get

$$\begin{aligned}&\frac{1}{2}\int |x|^2\mathbf{E}[\mathcal{M}^1[U](s,x)U(0,0)]\mathrm{d}x\\&\quad=-\frac{1}{2}\int(-\Delta)^{\frac{1}{2}}|x|^2\mathbf{E}[\mathcal{N}^1[U](s,x)(U(0,0)-U(s,0))]\mathrm{d}x\\&\quad=-\frac{1}{2}\int(-\Delta)^{\frac{1}{2}}|x|^2\mathbb{E}[\mathcal{N}^1[\eta^a](0)\tilde{\mathbf{E}}_{\eta^a}[\tilde U(s,x)-\tilde U(0,x)]]\mathrm{d}x\end{aligned}$$

where the first passage is a consequence of the fact that $U(s)$ is distributed according to η^a, the latter is Gaussian and $\mathcal{N}^1$ is quadratic, while for the second we further exploited the space-time translation invariance of U and denoted by $\tilde{\mathbf{E}}_{\eta^a}$ the expectation with respect to the process $\tilde U$ starting from η^a and running backward in time, that is, $\tilde U(r,\cdot)=U(s-r,\cdot)$. We point out that $\tilde U$ has the same properties as U and solves (A.1) but with $-\lambda$ replacing λ. Therefore, arguing as above, we write

$$\begin{aligned}&\mathbb{E}[\mathcal{N}^1[\eta^a](0)\tilde{\mathbf{E}}_{\eta^a}[\tilde U(s,x)-\tilde U(0,x)]]\\&\quad=\int_0^s\mathbb{E}[\mathcal{N}^1[\eta^a](0)\tilde{\mathbf{E}}_{\eta^a}[\Delta\tilde U(r,x)-\lambda\mathcal{M}^1[\tilde U](r,x)]]\mathrm{d}r\end{aligned}$$

so that, integrating against $\frac{1}{2}(-\Delta)^{\frac{1}{2}}|x|^2$, we see that the summand containing the Laplacian vanishes while the other becomes

$$\begin{aligned}&2\lambda\int_0^s\int\mathbb{E}[\mathcal{N}^1[\eta^a](0)\tilde{\mathbf{E}}_{\eta^a}[\mathcal{N}^1[\tilde U](r,x)]]\mathrm{d}x\mathrm{d}r\\&\qquad\qquad=2\lambda\int_0^s\int\mathbb{E}[\mathcal{N}^1[\eta^a](0)e^{r\tilde{\mathcal{L}}}\mathcal{N}^1[\eta^a](x)]\mathrm{d}x\mathrm{d}r\end{aligned}$$

with $\tilde{\mathcal{L}}$ the generator of the time reversed process. In conclusion, $D^{(a)}_{\text{bulk}}$ can be rewritten as

$$D^{(a)}(t) = \frac{1}{2t}\int |x|^2 S(0,x)\mathrm{d}x + 1 + 2\frac{\lambda^2}{t}\int_0^t\int_0^s\int \mathbb{E}[\mathcal{N}^1[\eta^a](0)e^{r\tilde{\mathcal{L}}}\mathcal{N}^1[\eta^a](x)]\mathrm{d}x\mathrm{d}r\mathrm{d}s. \tag{A.4}$$

If we let $a \to \infty$, η^a converges to a spatial white noise so that the first term vanishes and (A.4) reduces to

$$D(t) = 1 + 2\frac{\lambda^2}{t}\int_0^t\int_0^s\int \mathbf{E}[\tilde{\mathcal{N}}^1[H](0,0)\tilde{\mathcal{N}}^1[H](r,x)]\mathrm{d}x\mathrm{d}r\mathrm{d}s \tag{A.5}$$

where we used the relation between $\mathcal{N}^1$ and $\tilde{\mathcal{N}}^1$, see (3.1). Now, in case $\lambda = 0$, we recover the well-known result concerning the bulk diffusivity of the linear stochastic heat equation, which is constant in time. On the other hand, for $\lambda > 0$, taking $N \to \infty$, the bulk diffusivity $D_N(t)$ defined in (1.6) formally converges to $D(t)$ given as in (A.5).

APPENDIX B: MODE-COUPLING AND $\sqrt{\log t}$ SUPERDIFFUSIVITY

The ansatz and the calculations in this section are inspired by [43] and Appendix C.2 of [39]. As in the previous appendix we will work with the solution U (and its Fourier transform $\hat{U}$) of (A.1) on the full space and non-regularised noise, that is, $a = \infty$, and we start from a white noise initial condition $\eta = \eta^\infty$. Let $\hat{S}$ be the Fourier transform of the two-point correlation function S in (A.3), which by translation invariance is given by $\hat{S}(t,k) = \frac{1}{(2\pi)^2}\mathbf{E}[\hat{U}(t,k)\hat{U}(0,-k)]$. Formally $\hat{S}$ solves

$$\begin{aligned}\partial_t\hat{S}(t,k) + \frac{1}{2}|k|^2\hat{S}(t,k) &= \frac{\lambda}{(2\pi)^2}\mathbf{E}[\hat{U}(0,-k)\mathcal{M}^1_k(U(t))] \\ &= \frac{\lambda}{(2\pi)^2}\mathbb{E}[\hat{\eta}(-k)e^{\mathcal{L}t}\mathcal{M}^1_k(\eta)].\end{aligned} \tag{B.1}$$

The generator $\mathcal{L}$ of the Markov process U can be written as the sum of $\mathcal{L}_0$ and $\mathcal{A}$, whose definition can be read off (2.18)–(2.20) (the variables now take values in $\mathbb{R}^2$ instead of $\mathbb{Z}^2$ and the sum is replaced by an integral) and whose properties are analogous to those in Lemma 2.1. The semigroup associated to $\mathcal{L}$ satisfies

$$e^{\mathcal{L}t} = e^{\mathcal{L}_0 t} + \int_0^t e^{\mathcal{L}_0(t-s)}\mathcal{A}\,e^{\mathcal{L}}\,\mathrm{d}s.$$

Moreover,

$$\mathbb{E}[\hat{\eta}(-k)e^{\mathcal{L}_0 t}\mathcal{M}^1_k(\eta)] = 0$$

which follows since $e^{\mathcal{L}_0 t}$ corresponds to taking expectation with respect to the Ornstein-Uhlenbeck process and $\mathcal{M}^1_k$ is quadratic. Getting back to (B.1), since the adjoint of $\mathcal{A}$ is $-\mathcal{A}$, the term on the right hand side equals

$$-\frac{\lambda}{(2\pi)^2}\int_0^t e^{-\frac{1}{2}|k|^2(t-s)}\mathbb{E}[(\mathcal{A}\hat{\eta})(-k)e^{\mathcal{L}s}\mathcal{M}^1_k(\eta)]\,\mathrm{d}s\,.$$

Using that $\mathcal{A}\hat{\eta}(-k) = \lambda\mathcal{M}^1_{-k}(\eta)$, and the Fourier representation (2.10) of the non-linearity (with sums replaced by integrals) we see that the above equals

$$-\frac{\lambda^2}{(2\pi)^4}|k|^2\int_0^t e^{-\frac{1}{2}|k|^2(t-s)}\int \mathrm{d}\ell \int \mathrm{d}\ell' \mathcal{K}^1_{\ell,k-\ell}\mathcal{K}^1_{\ell',-k-\ell'}$$
$$\times \mathbf{E}[\hat{U}(s,\ell)\hat{U}(s,k-\ell)\hat{U}(0,\ell')\hat{U}(0,-k-\ell')]\,\mathrm{d}s. \tag{B.2}$$

Now, the "mode-coupling approximation" (see, e.g. [39]) consists in doing a Gaussian approximation of the average of the product of four U variables, which allows to apply Wick's rule. By translation invariance $\mathbb{E}[\hat{U}(s,\ell)\hat{U}(0,m)] = 0$ unless $\ell = -m$. Note also that the Wick contraction $\mathbb{E}[\hat{U}(s,\ell)\hat{U}(s,k-\ell)]\mathbb{E}[\hat{U}(0,\ell')\hat{U}(0,-k-\ell')]$ can be ignored because it vanishes unless $k = 0$, in which case however (B.2) is multiplied by $|k|^2 = 0$. Therefore, summing up the above computations and considerations we see that

$$\left(\partial_t + \frac{1}{2}|k|^2\right)\hat{S}(t,k)$$
$$\approx -2|k|^2\frac{\lambda^2}{(2\pi)^4}\int_0^t e^{-\frac{1}{2}|k|^2(t-s)}\int \mathrm{d}\ell(\mathcal{K}^1_{\ell,k-\ell})^2\hat{S}(s,\ell)\hat{S}(s,k-\ell)\,\mathrm{d}s. \tag{B.3}$$

We now make the ansatz

$$\hat{S}(t,k) = \hat{S}(0,0)e^{-\frac{1}{2}|k|^2t-c|k|^2t(\log t)^\delta} \tag{B.4}$$

for small k and large t, corresponding to a diffusion coefficient of order $(\log t)^\delta$. Our goal is to determine δ such that the left and right hand sides above coincide. According to (B.4), in the regime considered, the left hand side of (B.3) equals

$$\left(\partial_t + \frac{1}{2}|k|^2\right)\hat{S}(t,k) \approx -c|k|^2(\log t)^\delta\hat{S}(0,0). \tag{B.5}$$

Regarding the right hand side instead, we approximate $e^{-\frac{1}{2}|k|^2(t-s)}$ by one and $k-\ell$ by $-\ell$ so that for $k\to 0$ and $t\to\infty$, it gives

$$-|k|^2\lambda^2\int_0^t \mathrm{d}s\int \mathrm{d}\ell(\mathcal{K}^N_{\ell,-\ell})^2e^{-2c|\ell|^2s(\log s)^\delta} \approx -|k|^2\lambda^2(\log t)^{1-\delta}, \tag{B.6}$$

where we used the explicit form of $\mathcal{K}^N_{\ell,-\ell}$ as in (3.33). Equating (B.5) and (B.6) yields that $\delta = \frac{1}{2}$ as desired.

APPENDIX C: SOME TECHNICAL RESULTS

In this section we will state and prove some technical bounds that are needed in Section 3.1.

Lemma C.1. *For any $\ell, m \in \mathbb{Z}^2$ and $k_{1:n} \in (\mathbb{Z}^2)^n$ such that $\ell + m = k_1$ we have*

$$\frac{1}{4}(|\ell|^2 + |k_{1:n}|^2) \le |\ell|^2 + |m|^2 + |k_{2:n}|^2 \le 4(|\ell|^2 + |k_{1:n}|^2). \tag{C.1}$$

Proof. The proof is an application of the triangular inequality, we omit the details. □

In the following lemma, which is used in Lemmas 3.8 and 3.9 we analyse the functions L, LB,, and UB introduced in (3.10) and (3.11), respectively.

Lemma C.2. *For $k \in \mathbb{N}$, let* L, LB_k *and* UB_k *be the functions on $\mathbb{R}_+ \times [1, \infty)$ defined in (3.10) and (3.11). Then,* L, LB_k *and* UB_k *are monotonically decreasing in the first variable and increasing in the second. For any $x > 0$ and $z \geq 1$, we have that* $\mathrm{LB}_k(x,z), \mathrm{UB}_k(x,z) \geq 1$ *and the following inequalities hold*

$$1 \leq \mathrm{LB}_k(x,z) \leq \sqrt{\mathrm{L}(x,z)}, \tag{C.2}$$

$$1 \vee \lambda\sqrt{z} \leq \sqrt{\mathrm{L}(x,z)} \leq \mathrm{UB}_k(x,z) \leq \mathrm{L}(x,z). \tag{C.3}$$

Moreover, for any $0 < a < b$, we have

$$\lambda^2 \int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{UB}_k(x,z)} = 2[\mathrm{LB}_{k+1}(a,z) - \mathrm{LB}_{k+1}(b,z)] \tag{C.4}$$

and

$$\lambda^2 \int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{LB}_k(x,z)} \leq 2[\mathrm{UB}_k(a,z) - \mathrm{UB}_k(b,z)]. \tag{C.5}$$

Finally, one has

$$|\partial_x(xF(x,z))| = |F(x,z) + x\partial_x F(x,z)| \leq (1+\lambda^2)F(x,z) \text{ for every } x \geq 0, \tag{C.6}$$

when F is either LB_k *or* UB_k.

Proof. The two chains of inequalities in (C.2) and (C.3) are a direct consequence of the respective definitions and Taylor's approximation. A computation of the partial derivative with respect to the second variable yields the desired monotonicity. Furthermore, we have that

$$\partial_x \mathrm{L}(x,z) = -\frac{\lambda^2}{x^2+x}, \qquad \partial_x \mathrm{LB}_k(x,z) = -\frac{\lambda^2}{2}\frac{\mathrm{LB}_{k-1}(x,z)}{(x^2+x)\mathrm{L}(x,z)} \tag{C.7}$$

and

$$\begin{aligned}\partial_x \mathrm{UB}_k(x,z) &= -\lambda^2 \frac{\mathrm{LB}_k(x,z) - \frac{1}{2}\mathrm{LB}_{k-1}(x,z)}{(x^2+x)(\mathrm{LB}_k(x,z))^2} \\ &= -\frac{\lambda^2}{2(x^2+x)\mathrm{LB}_k(x,z)}\left[1 + \frac{\frac{(\frac{1}{2}\log \mathrm{L}(x,z))^k}{k!}}{\mathrm{LB}_k(x,z)}\right],\end{aligned} \tag{C.8}$$

which are all strictly negative for any $x > 0$ and $z \geq 1$. The above computation of the partial derivatives moreover reveals that

$$\lambda^2 \int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{UB}_k(x,z)} = 2\int_b^a \partial_x \mathrm{LB}_{k+1}(x,z)\mathrm{d}x = 2[\mathrm{LB}_{k+1}(a,z) - \mathrm{LB}_{k+1}(b,z)],$$

which is (C.4). For (C.5), notice that

$$\lambda^2 \int_a^b \frac{\mathrm{d}x}{(x^2+x)\mathrm{LB}_k(x,z)} = \int_b^a \partial_x \mathrm{UB}_k(x,z)\mathrm{d}x + \frac{\lambda^2}{2}\int_a^b \frac{\mathrm{LB}_{k-1}(x,z)}{(x^2+x)\mathrm{LB}_k(x,z)^2}\mathrm{d}x$$
$$\leq \int_b^a \partial_x \mathrm{UB}_k(x,z)\mathrm{d}x + \frac{\lambda^2}{2}\int_a^b \frac{1}{(x^2+x)\mathrm{LB}_k(x,z)}\mathrm{d}x,$$

where the last inequality follows from the fact that all the terms are positive and for all x we have $\mathrm{LB}_{k-1}(x,z) \leq \mathrm{LB}_k(x,z)$. Bringing the last term to the left hand side gives the required estimate.

Finally, (C.6) follows immediately from (C.7)-(C.8), recalling that $\mathrm{L}(x,z) \geq 1$. □

Remark C.3. For notational convenience, the next three lemmas are formulated for a generic function F satisfying Assumption 1 below. In practice, we will always apply the results when $F(\cdot,z)$ is of the form $a + b\mathrm{UB}_k(\cdot,z)$ or $a + b\mathrm{LB}_k(\cdot,z)$, for some positive constants a, b, possibly depending on k and on z. In this case, the validity of the assumption follows from the definition of UB_k, LB_k and from Lemma C.2 above.

Assumption 1. $F = F(x,z)$ is a function on $\mathbb{R}_+ \times [1,\infty)$ monotonically decreasing in the first variable and such that for all $(x,z) \in \mathbb{R}_+ \times [1,\infty)$, $F(x,z) \geq 1$. We assume further that the function $G = G(x,z)$ given by

$$G(x,z) = \frac{\mathrm{L}(x,z)}{F(x,z)}, \tag{C.9}$$

where L is defined as in (3.10), is also monotonically decreasing in the first variable and satisfies $G(x,z) \geq 1$ for all $(x,z) \in \mathbb{R}_+ \times [1,\infty)$. Finally, we assume that (C.6) holds.

Lemma C.4. *Under Assumption 1, there exists $K > 0$ (independent of F) such that*

$$\int_0^\infty \frac{\mathrm{d}\varrho}{(\varrho^2+\alpha)F(\varrho^2+\alpha,z)} \leq \frac{K}{\sqrt{\alpha}}\frac{1}{F(2\alpha,z)}, \tag{C.10}$$

for all $\alpha > 0$, $\lambda > 0$ and $z \geq 1$.

Proof. We write the integral on the left hand side of (C.10) as the sum of $I_1(\alpha,z)$ and $I_2(\alpha,z)$, where

$$I_1(\alpha,z) = \int_0^{\sqrt{\alpha}} \frac{\mathrm{d}\varrho}{(\varrho^2+\alpha)F(\varrho^2+\alpha,z)},\quad I_2(\alpha,z) = \int_{\sqrt{\alpha}}^\infty \frac{\mathrm{d}\varrho}{(\varrho^2+\alpha)F(\varrho^2+\alpha,z)}.$$

For I_1, we use monotonicity of F w.r.t. its first argument to write

$$I_1(\alpha,z) \le \frac{1}{F(2\alpha,z)}\frac{\sqrt{\alpha}}{\alpha} = \frac{1}{\sqrt{\alpha}F(2\alpha,z)}.$$

Using (C.9), and the fact that L is decreasing w.r.t. its first argument, I_2 can be written as

$$I_2(\alpha,z) = \int_{\sqrt{\alpha}}^{\infty} \frac{G(\varrho^2+\alpha,z)}{(\varrho^2+\alpha)\mathrm{L}(\varrho^2+\alpha,z)} d\varrho \le G(2\alpha,z)\int_{\sqrt{\alpha}}^{\infty} \frac{d\varrho}{\varrho^2 \mathrm{L}(2\varrho^2,z)}$$

and it remains to prove that

$$\int_{\sqrt{\alpha}}^{\infty} \frac{d\varrho}{\varrho^2\mathrm{L}(2\varrho^2,z)} \le \frac{K}{\sqrt{\alpha}\mathrm{L}(2\alpha,z)}. \tag{C.11}$$

If $\alpha \ge 1$, recalling $z \ge 1$, we simply bound

$$\mathrm{L}(2\varrho^2,z) = 1+\lambda^2(z+\log(1+1/(2\varrho^2))) \gtrsim \mathrm{L}(2\alpha,z)$$

and the desired estimate immediately follows. If instead $\alpha \le 1$, we split the integral as

$$\int_{\sqrt{\alpha}}^{\infty} \frac{d\varrho}{\varrho^2\mathrm{L}(2\varrho^2,z)} = I_3(\alpha,z)+I_4(\alpha,z) := \int_{\sqrt{\alpha}}^{\alpha^{1/4}} \frac{d\varrho}{\varrho^2\mathrm{L}(2\varrho^2,z)} + \int_{\alpha^{1/4}}^{\infty} \frac{d\varrho}{\varrho^2\mathrm{L}(2\varrho^2,z)}.$$

For I_3 we simply use $\mathrm{L}(2\varrho^2,z) \ge \mathrm{L}(2\sqrt{\alpha},z) \gtrsim \mathrm{L}(2\alpha,z)$ and then it is upper bounded as the r.h.s. of (C.11). For I_4, instead, we use $\mathrm{L}(2\varrho^2,z) \ge \mathrm{L}(\infty,z)$ so that

$$I_4(\alpha,z) \lesssim \frac{1}{\alpha^{1/4}(1+\lambda^2 z)} \lesssim \frac{1}{\sqrt{\alpha}(1+\lambda^2(z+\log(1+1/(2\alpha)))}$$

as desired. Putting everything together, (C.10) follows. □

Lemma C.5. *Under Assumption 1, define $\Gamma(\ell,m,k_{2:n})$ as in (3.29),*

$$\alpha = \alpha(\mu,k_{1:n}) \stackrel{def}{=} \mu + \tfrac{1}{2}|k_{1:n}|^2, \qquad and \qquad \alpha_N \stackrel{def}{=} \alpha/N^2 \tag{C.12}$$

and $F^N(\cdot,z) := F(\cdot/N^2,z)$, $G^N(\cdot,z) := G(\cdot/N^2,z)$. Then, there exists a positive constant K (depending only on λ) such that

$$\Bigg|\sum_{\ell+m=k_1} \frac{(\mathcal{K}^N_{\ell,m})^2}{\mu+\Gamma(\ell,m,k_{2:n})F^N(\mu+\Gamma(\ell,m,k_{2:n}),z)} - \sum_{\ell} \frac{(\mathcal{K}^N_{\ell,-\ell})^2}{(|\ell|^2+\alpha)(1+|\ell/N|^2+\alpha_N^2)F^N(|\ell|^2+\alpha,z)}\Bigg| \le K\frac{G^N((\mu+\frac{1}{2}|k_{1:n}|^2)\vee 1,z)}{\lambda\sqrt{z}}. \tag{C.13}$$

Remark C.6. Actually, the right hand side of (C.13) could be replaced by a constant depending only on z. However, it is convenient to have the bound in this form since in the iteration, the terms giving the main contribution will be upper or lower bounded by a quantity like $G^N(\mu+\frac{1}{2}|k_{1:n}|^2,z)$ in which case (C.13) (with z being taken suitably large) will be regarded as an error term.

Proof. We proceed in three steps, starting from the first sum in (C.13):

(1) first, we replace the denominator by

$$[\mu+\Gamma(\ell,m,k_{2:n})]F^N(\mu+\Gamma(\ell,m,k_{2:n}),z),$$

(2) then, we replace the denominator by

$$(|\ell|^2+\alpha)(1+|\ell/N|^2+\alpha_N^2)F^N(|\ell|^2+\alpha,z);$$

(3) finally, we replace $\mathcal{K}^N_{\ell,m}$ with $\mathcal{K}^N_{\ell,-\ell}$

and it will turn out that each step produces an error term of the same form as the one at the right hand side of (C.13).

Step 1. Since $|\mathcal{K}^N_{\ell,m}|\leq 1$, it is enough to bound

$$\sum_{\ell+m=k_1}\left|\frac{1}{\mu+\Gamma(\ell,m,k_{2:n})F^N(\mu+\Gamma(\ell,m,k_{2:n}),z)}\right.\\ \left.-\frac{1}{[\mu+\Gamma(\ell,m,k_{2:n})]F^N(\mu+\Gamma(\ell,m,k_{2:n}),z)}\right|. \tag{C.14}$$

Using that $|F^N-1|/F^N\leq 1$, the sum is upper bounded by

$$\mu\sum_{\ell+m=k_1}\frac{1}{[\mu+\Gamma(\ell,m,k_{2:n})F^N(\mu+\Gamma(\ell,m,k_{2:n}),z)](\mu+\Gamma(\ell,m,k_{2:n}))}.$$

We split $\mathbb{Z}^2$ into $\Omega_1=\{\ell\,:\,\mu+\Gamma(\ell,k_1-\ell,k_{2:n})\leq\alpha\vee 1\}$, for α defined as in (C.12), and $\Omega_2=\mathbb{Z}^2\setminus\Omega_1$. In the first case $F^N(\mu+\Gamma(\ell,k_1-\ell,k_{2:n}),z)\geq F^N(\alpha\vee 1,z)$ and thus we obtain the upper bound

$$\begin{aligned}&\frac{\mu}{F^N(\alpha\vee 1,z)}\sum_{\substack{\ell+m=k_1\\ \ell\in\Omega_1}}\frac{1}{\left[\frac{\mu}{(F(\alpha\vee 1,z))}+\Gamma(\ell,m,k_{2:n})\right](\mu+\Gamma(\ell,m,k_{2:n}))}\\ &\leq\frac{\mu}{F^N(\alpha\vee 1,z)}\left(\sum_\ell\frac{1}{(\frac{\mu}{F(\alpha\vee 1,z)}+\frac12|\ell|^2)^2}\right)^{\frac12}\left(\sum_\ell\frac{1}{(\mu+\frac12|\ell|^2)^2}\right)^{\frac12}\\ &\lesssim\frac{\mu}{F^N(\alpha\vee 1,z)}\frac{\sqrt{F^N(\alpha\vee 1,z)}}{\mu}=\frac{1}{\sqrt{F^N(\alpha\vee 1,z)}}\leq\frac{G^N((\mu+\frac12|k_{1:n}|^2)\vee 1,z)}{\lambda\sqrt{z}}\end{aligned} \tag{C.15}$$

where we used the relation between F and G, the assumption $G\geq 1$ and the bound $\mathrm{L}(x,z)\geq\lambda^2 z$. For $\ell\in\Omega_2$, note that

$$F^N(\mu+\Gamma(\ell,m,k_{2:n}),z)=\frac{\mathrm{L}^N(\mu+\Gamma(\ell,m,k_{2:n}),z)}{G^N(\mu+\Gamma(\ell,m,k_{2:n}),z)}\geq\frac{z\lambda^2}{G^N((\mu+\frac12|k_{1:n}|^2)\vee 1,z)}.$$

We can proceed as in (C.15) but with the right hand side above in place of F^N. Hence, also the sum over Ω_2 is upper bounded by

$$\frac{G^N((\mu+\frac{1}{2}|k_{1:n}|^2)\vee 1,z)}{\lambda\sqrt{z}}. \tag{C.16}$$

and consequently so is (C.14).

Step 2. At first, we bound $|\mathcal{K}^N_{\ell,m}|$ by the indicator function of $|\ell|\le N$. The quantity we have to control takes the form (we write for lightness of notation Γ instead of $\Gamma(\ell,m,k_{2:n})$ and A instead of $A(\ell,k_{1:n})$)

$$\begin{aligned}\sum_{\substack{\ell+m=k_1\\|\ell|\le N}}\Big|&\frac{1}{(\mu+\Gamma)F^N(\mu+\Gamma,z)}-\frac{1}{(|\ell|^2+\alpha)F^N(|\ell|^2+\alpha,z)}\\&+\frac{1}{(|\ell|^2+\alpha)F^N(|\ell|^2+\alpha,z)}-\frac{1}{(|\ell|^2+\alpha)(1+|\ell/N|^2+\alpha_N)F^N(|\ell|^2+\alpha,z)}\Big|\\&\le(\mathrm{I})+(\mathrm{II}),\end{aligned} \tag{C.17}$$

where, setting $H(x)\stackrel{\text{def}}{=}xF^N(x,z)$,

$$(\mathrm{I})=\sum_{\substack{\ell+m=k_1\\|\ell|\le N}}\left|\frac{H(|\ell|^2+\alpha)-H(\mu+\Gamma)}{(\mu+\Gamma)(|\ell|^2+\alpha)F^N(\mu+\Gamma,z)F^N(|\ell|^2+\alpha,z)}\right|,$$

$$(\mathrm{II})=\frac{1}{N^2}\sum_{\substack{\ell+m=k_1\\|\ell|\le N}}\frac{1}{F^N(|\ell|^2+\alpha,z)},$$

Recalling assumption (C.6) and that $F(\cdot,z)$ is decreasing, for any $a,b\in\mathbb{R}$ we have

$$|H(a)-H(b)|\le(1+\lambda^2)F^N(a\wedge b,z)|a-b|$$

which we apply in the sum, with $a=|\ell|^2+\alpha$, $b=\mu+\Gamma$, so that $|a-b|\le|\ell||k_1|$. Also, with the same choice of a,b, by (C.1), $a\vee b\le 4a$, so that by the monotonicity of F^N, one has

$$\frac{F^N(a\wedge b,z)}{F^N(a,z)F^N(b,z)}\le\frac{1}{F^N(a\vee b,z)}\le\frac{1}{F^N(4a,z)}.$$

In conclusion, invoking (C.1) once more, (I) can be upper bounded as

$$\begin{aligned}(\mathrm{I})&\lesssim|k_1|\sum_{\ell}\frac{|\ell|}{(|\ell|^2+\alpha)^2F^N(4(|\ell|^2+\alpha),z)}\\&\lesssim|k_1/N|\int_0^\infty\frac{\mathrm{d}\varrho}{(\varrho^2+4\alpha_N)F(\varrho^2+4\alpha_N,z)}.\end{aligned}$$

Using Lemma C.4, the latter expression is bounded by

$$\frac{|k_1/N|}{\sqrt{\alpha_N}F(8\alpha_N,z)} \lesssim \frac{1}{F^N(8(\alpha\vee 1),z)} \leq \frac{G^N((\mu+\frac{1}{2}|k_{1:n}|^2)\vee 1,z)}{\lambda^2 z}.$$

which gives the correct bound on (I). As for (II), the monotonicity of F and G, (C.9) and (C.3) give

$$\frac{1}{N^2}\sum_{\substack{\ell+m=k_1\\|\ell|\leq N}}\frac{1}{F^N(|\ell|^2+\alpha,z)} \leq \frac{1}{N^2}\sum_{\substack{\ell+m=k_1\\|\ell|\leq N}}\frac{1}{F^N((|\ell|^2+\alpha)\vee 1,z)} \leq \frac{G^N(\alpha\vee 1,z)}{\lambda^2 z}.$$

Therefore, the bound on (C.17) is concluded.

Step 3. We need to upper bound

$$\sum_{\ell+m=k_1}\frac{|(\mathcal{K}^N_{\ell,m})^2-(\mathcal{K}^N_{\ell,-\ell})^2|}{(|\ell|^2+\alpha)(1+|\ell/N|^2+\alpha_N)F^N(|\ell|^2+\alpha,z)}$$
$$\leq \sum_{\ell+m=k_1}\frac{|(\mathcal{K}^N_{\ell,m})^2-(\mathcal{K}^N_{\ell,-\ell})^2|}{(|\ell|^2+\alpha)F^N(|\ell|^2+\alpha,z)}. \tag{C.18}$$

We split $\mathbb{Z}^2$ into $\Omega_1=\{\ell : |k_1-\ell|<\frac{1}{2}|k_1|\}$ and its complement Ω_2. In Ω_1, it is immediate to see that one has $\frac{1}{2}|k_1|\leq|\ell|\leq\frac{3}{2}|k_1|$; bounding the term inside the absolute value in (C.18) by a constant, we are left with

$$\sum_{\frac{1}{2}|k_1|\leq|\ell|\leq\frac{3}{2}|k_1|}\frac{1}{(|\ell|^2+\alpha)F^N(|\ell|^2+\alpha,z)}.$$

In the relevant region of summation one has $|\ell|^2+\alpha\leq 6(\alpha\vee 1)$, so that, since F^N is decreasing in the first argument, we can upper bound the sum as

$$\frac{1}{F^N(6(\alpha\vee 1),z)}\sum_{\frac{1}{2}|k_1|\leq|\ell|\leq\frac{3}{2}|k_1|}\frac{1}{|\ell|^2} \lesssim \frac{G^N((\mu+\frac{1}{2}|k_{1:n}|^2)\vee 1,z)}{\lambda^2 z}.$$

In Ω_2, instead, we use the definition (2.11) of $\mathcal{K}^N$ and we note that

$$c(\ell,k_1-\ell)^2-c(\ell,\ell)^2=c(\ell,k_1)c(\ell,k_1-2\ell).$$

Hence,

$$\left|(\mathcal{K}^N_{\ell,k_1-\ell})^2-(\mathcal{K}^N_{\ell,-\ell})^2\right| \leq \frac{c(\ell,\ell)^2}{|\ell|^4}\frac{|k_1\cdot(k_1-2\ell)|}{|k_1-\ell|^2}+\frac{|c(\ell,k_1)c(\ell,k_1-\ell)|}{|\ell|^2|k_1-\ell|^2}$$
$$\lesssim \frac{|k_1|}{|k_1-\ell|}\frac{|k_1-2\ell|}{|k_1-\ell|}+\frac{|k_1|}{|k_1-\ell|} \lesssim \frac{|k_1|}{|k_1-\ell|}\left(1+\frac{|k_1|}{|k_1-\ell|}\right) \lesssim \frac{|k_1|}{|k_1-\ell|}$$

where the last inequality follows from the definition of Ω_2 (note that the denominator cannot vanish therein). Also, one has $|\ell|^2+\alpha \geq \alpha$ so that

$$\frac{1}{F^N(|\ell|^2+\alpha,z)} \leq \frac{1}{F^N((|\ell|^2+\alpha)\vee 1,z)} \leq \frac{G^N((\mu+\frac{1}{2}|k_{1:n}|^2)\vee 1,z)}{z\lambda^2}.$$

Hence, the sum at the right hand side of (C.18) restricted to Ω_2 is bounded from above by

$$\frac{G^N\left(\left(\mu+\frac{1}{2}|k_{1:n}|^2\right)\vee 1,z\right)|k_1|}{\lambda^2 z}\sum_{\ell\neq k_1}\frac{1}{|k_1-\ell|}\frac{1}{|\ell|^2+|k_1-\ell|^2}$$
$$\lesssim \frac{G^N\left(\left(\mu+\frac{1}{2}|k_{1:n}|^2\right)\vee 1,z\right)}{\lambda^2 z}$$

where the restriction $k_1\neq\ell$ comes from the definition of Ω_2 and the last bound is easily obtained by splitting into the region where $|\ell|$ is larger or smaller than $\frac{1}{2}|k_1|$.

Putting together the bounds obtained in Steps 1, 2 and 3, we have bounded the left hand side of (C.13) (modulo absolute constants) by

$$\frac{G^N\left(\left(\mu+\frac{1}{2}|k_{1:n}|^2\right)\vee 1,z\right)}{\lambda\sqrt{z}}+2\frac{G^N\left(\left(\mu+\frac{1}{2}|k_{1:n}|^2\right)\vee 1,z\right)}{\lambda^2 z}.$$

Hence, since $z\geq 1$, there clearly exists a constant K (depending only on λ) for which the statement holds and the proof is completed. □

We conclude this appendix by showing that the bounds on the Riemann-sums performed in the proofs of Lemmas 3.8, 3.9 and 3.10 are uniform in the scale parameter $N\in\mathbb{N}$.

Lemma C.7. *Let F satisfy Assumption 1 and define α and α_N according to (C.12). Then, there exists a constant K (depending only on λ) such that*

$$\left|\sum_{\ell}\frac{\left(\mathcal{K}^N_{\ell,-\ell}\right)^2}{(|\ell|^2+\alpha)(1+|\ell/N|^2+\alpha_N)F^N(|\ell|^2+\alpha,z)}\right.$$
$$\left.-\int_{\mathbb{R}^2}\frac{\left(\mathcal{K}^N_{xN,-xN}\right)^2\mathrm{d}x}{(|x|^2+\alpha_N)(1+|x|^2+\alpha_N)F(|x|^2+\alpha_N,z)}\right|$$
$$\leq K\frac{G^N\left(\left(\mu+\frac{1}{2}|k_{1:n}|^2\right)\vee 1,z\right)}{\lambda^2 z}. \tag{C.19}$$

Proof. A first observation is that, letting $\ell=xN$, the summand is exactly $1/N^2$ times the integrand. The claim is then a Riemann sum approximation statement but some care has to be taken, on the one hand because the integrand is singular at the origin, and on the other because we want the constant K not to depend on F.

Since, by definition (2.11) $\mathcal{K}^N$ contains an indicator function (2.12) which forces $\ell \neq 0$ in the sum and $1/N \leq |x| \leq 1$ in the integral, we can assume these two conditions to be in place, and therefore the difference in (C.19) equals

$$\begin{aligned}\sum_{1\leq|\ell|\leq N} &\int_{Q_\ell^N} |I_1(\ell/N)I_2(\ell/N) - I_1(x)I_2(x)|\mathrm{d}x\\ &\leq \frac{1}{N}\sum_{1\leq|\ell|\leq N}\int_{Q_\ell^N}\mathrm{d}x \sup_{x\in Q_\ell^N}|\nabla(I_1(x)I_2(x))|\\ &= \frac{1}{N^3}\sum_{1\leq|\ell|\leq N}\sup_{x\in Q_\ell^N}|\nabla(I_1(x)I_2(x))|\\ &\leq \frac{1}{N^3}\sum_{1\leq|\ell|\leq N}\sup_{x\in Q_\ell^N}|\nabla I_1(x)||I_2(x)| + \frac{1}{N^3}\sum_{1\leq|\ell|\leq N}\sup_{x\in Q_\ell^N}|I_1(x)||\nabla I_2(x)| \end{aligned}\tag{C.20}$$

where Q_ℓ^N is the square of side-length $1/N$ centred at ℓ, while I_1 and I_2 are the functions defined as

$$I_1(x) \stackrel{\text{def}}{=} \frac{1}{4\pi^2}\frac{c(x,-x)^2}{|x|^2(|x|^2+\alpha_N)(1+|x|^2+\alpha_N)}, \qquad I_2(x) \stackrel{\text{def}}{=} \frac{1}{F(|x|^2+\alpha_N, z)}.$$

We will separately bound the suprema appearing in the sums above. For the first, it is not hard to see that since

$$I_2(x) = \frac{1}{F(|x|^2+\alpha_N, z)} \leq \frac{1}{F((|x|^2+\alpha_N)\vee 1/N^2, z)} \leq \frac{G^N(\alpha\vee 1, z)}{\lambda^2 z},\tag{C.21}$$

which is independent of ℓ, we have

$$\begin{aligned}\sup_{x\in Q_\ell^N}|\nabla I_1(x)||I_2(x)| &\lesssim \frac{G^N(\alpha\vee 1,z)}{\lambda^2 z}\sup_{x\in Q_\ell^N}\frac{1}{|x|(|x|^2+\alpha_N)}\\ &\lesssim \frac{G^N(\alpha\vee 1,z)}{\lambda^2 z}\frac{N^3}{|\ell|(|\ell|^2+\alpha)}.\end{aligned}\tag{C.22}$$

For the other instead, note that

$$\begin{aligned}|\nabla I_2(x)| &\lesssim \frac{|x||F'(|x|^2+\alpha_N,z)|}{F(|x|^2+\alpha_N,z)^2} \leq \frac{(|x|^2+\alpha_N)|F'(|x|^2+\alpha_N,z)|}{(|x|^2+\alpha_N)^{\frac12}F(|x|^2+\alpha_N,z)^2}\\ &\lesssim \frac{1}{(|x|^2+\alpha_N)^{\frac12}F(|x|^2+\alpha_N,z)} \leq \frac{1}{(|x|^2+\alpha_N)^{\frac12}}\frac{G^N(\alpha\vee 1,z)}{\lambda^2 z}\end{aligned}$$

where we exploited assumption (C.6) when passing from the first to the second line, and (C.21) in the last step. Hence,

$$\sup_{x\in Q_\ell^N}|I_1(x)||\nabla I_2(x)| \lesssim \frac{G^N(\alpha\vee 1,z)}{\lambda^2 z}\frac{N^3}{|\ell|(|\ell|^2+\alpha)}.\tag{C.23}$$

We now plug the bounds (C.22) and (C.23) into (C.20), which, consequently, is upper bounded by

$$\frac{G^N(\alpha \vee 1, z)}{\lambda^2 z} \sum_{1 \le |\ell| \le N} \frac{1}{|\ell|(|\ell|^2 + \alpha)} \lesssim \frac{G^N(\alpha \vee 1, z)}{\lambda^2 z},$$

so that the statement follows at once. □