TECHNISCHE UNIVERSITÄT WIEN

DIPLOMARBEIT

Fractional Gaussian Fields and the Gaussian Multiplicative Chaos

zur Erlangung des akademischen Grades

Diplom-Ingenieur

im Rahmen des Studiums

Technische Mathematik

ausgeführt am

Institut für Stochastik und Wirtschaftsmathematik

unter der Anleitung von

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Wien, am 22.8.2023

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Kurzfassung

In dieser Arbeit führen wir die Ein-Parameter Familie von Gauss-verteilten Prozessen - fractional Gaussian fields - ein, welche auch die Brownsche Bewegung, die fraktionale Brownsche Bewegung und das Gaussian free field inkludiert. Dieser Parameter wird auch Hurst parameter genannt. Wir zeigen hier die Existenz und einige Eigenschaften, wie zum Beispiel die Kovarianzstruktur, welche bereits einiges an Arbeit und Wissen in Fourier Analysis und Funktionalanalysis benötigen. Wir konstruieren diese Familie Gauss-verteilter Zufallsvariablen mit Hilfe des Satzes von Bochner-Minlos als ein zufälliges Element des topologischen Dualraums des Schwartz-Raums. Eine in diesem Kontext sehr interessante Frage ist wie man die Markov-Eigenschaft, welche für die Brownsche Bewegung ein bekanntes Konzept ist, für das fractional Gaussian field verallgemeinern kann. Dies benötigt allerdings Vorsicht, da die Umsetzung in diesem Setting nicht so einfach ist. Im Fall des Gaussian free fields führen diese Überlegungen zu einem neuen Konzept, den sogenannten lokalen Mengen.

Im zweiten Teil der Arbeit führen wir das Gaussian multiplicative chaos ein, welches derzeit ein in einigen Bereichen noch nicht gut erforschtes Objekt ist und im Bereich der Finanzmathematik wichtige Anwendungen hat. Wir konstruieren das Gaussian multiplicative Chaos direkt mit Hilfe des fractional Gaussian field. Der interessanteste Teil ist hier die Frage, was passiert, wenn man den Hurst parameter gegen 0 gehen lässt. Wir stellen hier ein Konvergenzergebnis, welches von Paul Hager und Eyal Neuman im Jahr 2020 entdeckt wurde, vor. Wie im ersten Teil ist der Beweis sehr lange, wovon wir hier den Großteil zeigen. Zuletzt führen wir noch ein paar Beispiele an, für welche man Konvergenz des Gaussian multiplicative chaos zeigen kann.

Abstract

In this work we introduce the fractional Gaussian field as a large one-parameter family of Gaussian processes including many important examples, as the Brownian motion, the fractional Brownian motion and the Gaussian free field. This parameter is well known as the Hurst parameter. The existence and first properties, for example the covariance structure, are shown here, what already takes a lot of effort and knowledge in Fourier analysis and functional analysis. We construct the fractional Gaussian field via the Bochner-Minlos theorem, as a random element of the topological dual space of the Schwartz space. One very interesting question is how to generalize the Markov property, which is for the Brownian motion a well known concept, that needs in this general setting some care. In the case of the Gaussian free field we obtain particularly interesting results that lead to a new concept, the so called local sets.

In the second part we introduce the Gaussian multiplicative chaos, which is a still very unknown object, but has some important applications in financial mathematics. It can be directly constructed out of a fractional Gaussian field. The interesting part here is what happens if one lets the Hurst parameter go to 0. Here we prove a convergence result by Paul Hager and Eyal Neuman discovered in 2020. Again this takes a very long proof, that we will show the most parts of. At the very end we show some examples one can apply this result on.

Acknowledgement

I want to thank my supervisor Univ.Prof. Dipl.-Ing. Dr.techn. Stefan Gerhold for the great cooperation and inspiration of the topic, that aroused a deep fascination about the generalization of Brownian Motion. He always was open for all kinds of questions and encouraged me diving deeper into topics that were unknown to me.

Moreover, I want to thank my partner, my family and colleagues for their personal support.

Eidesstattliche Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Diplomarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Wien, am 22.8.2023

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1 Introduction

The most important object in stochastic calculus is the Brownian motion $(B_t)_{t\in[0,\infty)}$, that was already long ago introduced in physics. Later mathematicians described it in an rigorous way, proved existence, observed properties and found many applications in various fields. Thus, one interesting question is how to generalize this Gaussian process. In this work we introduce the one-parameter family of fractional Gaussian fields, denoted by $FGF_s(\mathbb{R}^d)$, which is a Gaussian process, that generalizes the Brownian motion in terms of the dimension of the domain and an additional parameter $s \geqslant -\frac{d}{2}$. Here we define the so called Hurst parameter

$$H := s - \frac{d}{2},$$

where d is the dimension of the domain. As $s \geqslant -\frac{d}{2}$, we obtain $H \geqslant 0$. The Hurst parameter indicates the amount of dependence of the field in different regions. For $H = 0$ the fractional Gaussian field is at every point independent of all other parts. This special case is called white noise. The tricky part is that the fractional Gaussian field can not be defined point-wise for all $H \geqslant 0$. Here we need to make a compromise and find another way. In our case we define the fractional Gaussian fields as Gaussian processes with index set being the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Despite the big cost of being unable to evaluate point-wise, we obtain another interesting property. Using the Fourier transformation $\mathcal{F}$, which is bijective on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, we define the fractional Laplacian operator $(-\Delta)^s$ for $s \in \mathbb{R}$. As the notation indicates, it is indeed a generalization of the common Laplacian operator, i.e. $(-\Delta)^1 = -\Delta$. Furthermore, $(-\Delta)^{-s}$ is the inverse of $(-\Delta)^s$ and it holds that $(-\Delta)^s(-\Delta)^t = (-\Delta)^{s+t}$. The interesting part is now, that we can couple all fractional Gaussian fields to one big random object. We can choose a white noise W and get a fractional Gaussian field $h \sim FGF_s(\mathbb{R}^d)$ by

$$h := (-\Delta)^{-s/2}W.$$

By the properties of the fractional Laplacian operator, we get the following relation. For $s, t \geqslant -\frac{d}{2}$, $h_s \sim FGF_s(\mathbb{R}^d)$ and $h_t \sim FGF_t(\mathbb{R}^d)$, we obtain

$$h_s \overset{d}{=} (-\Delta)^{\frac{t-s}{2}} h_t$$

in distribution.

One further generalization is that we consider Gaussian free fields in domains $D \subseteq \mathbb{R}^d$ unequal to $\mathbb{R}^d$. This corresponds, in the case of the Brownian motion, to the Brownian bridge. Let $t_0 > 0$ and $(B_t)_{t\in[0,\infty)}$ be a standard Brownian motion. Then the Brownian bridge is defined by $(B_t)_{t\in[0,t_0]}$ conditioned on $B_{t_0} = 0$. We want to define an analogue on

$D \subsetneq \mathbb{R}^d$ as a fractional Gaussian field $h \sim FGF_s(\mathbb{R}^d)$ restricted to D and conditioned on the event that $h = 0$ on ∂D. This step requires more care, as the equation $h = 0$ on ∂D is, due to the fact that we are not able to evaluate point-wise, not well defined.

The rigorous proof of all these thoughts is presented in chapter 2 of this work. We follow here mostly the paper [LSSW16], but need to mention that there are still no educational books on this topic, that present these thoughts starting from a standard level of knowledge in stochastic calculus. The idea of this work is to arouse interest and understanding for the fascinating idea of fractional Gaussian fields in more people.

In the second part of this work we want to show an application of the fractional Gaussian field. Despite connections to Liouville quantum gravity, there are many ways to use the fractional Gaussian field for observing new ways of modeling prices in financial mathematics. One important tool is the so called Gaussian multiplicative chaos. One can think about it as the stochastic exponential of a fractional Brownian field B_H. Formally, it is defined as a random distribution, defined via its density that is given by

$$M_\gamma^H(dx) := \exp\Big(\gamma B_H(x) - \frac{\gamma^2}{2}\mathbb{E}\big[B_H(x)^2\big]\Big)\,dx,$$

where $\gamma > 0$ is a constant. The big question is now, what happens if one lets the Hurst parameter H go to 0. Inspired by the work of Nathanaël Berestycki [Ber17], Paul Hager and Eyal Neuman proved in [HN20] a statement of convergence of the Gaussian multiplicative chaos. In particular they proved that for a family of fractional Brownian fields, that own a particular form of covariance structure, there exists a constant $\gamma^* < \sqrt{\frac{7d}{4}}$, that only depends on the dimension, such that for all $\gamma \leqslant \gamma^*$ the associated Gaussian multiplicative chaos M_γ^H converges as $H \to 0$ to a Borel measure M_γ on D in probability with respect to the weak topology of measures. This result will be given as Theorem 3.1.3 in this work. We will show most parts of the proof, that requires much effort and is given in the subsections 3.2 and 3.3.

Finally, we discuss some examples of families of fractional Brownian fields, that satisfy the assumption of Theorem 3.1.3. The main part is the proof of Theorem 3.4.4 that shows different ways of constructing families of fractional Brownian fields, that we can apply Theorem 3.1.3 on. This is done in the sections 3.4 and 3.5.

2 Definition and properties of the Fractional Gaussian Field

The goal of this section is to rigorously introduce the fractional Gaussian field. For this purpose we look at tempered distributions, define a generalization of the Laplace differential operator and prove the Bochner-Minlos theorem. Furthermore we discuss the covariance structure of the fractional Gaussian field and define the fractional Brownian motion as a special case of the fractional Gaussian field. Moreover, we shortly reflect about continuous and differentiable versions of the fractional Gaussian field. In section 2.8 we discuss the generalization of the Markov property. Finally, we reflect about two special cases of the fractional Gaussian field.

2.1 Tempered distributions

We start introducing the Fourier transformation and the tempered distributions on the real valued Schwartz space. This section follows from chapter 2.1 in [LSSW16].

Definition 2.1.1 (Schwartz space)**.** *A multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^d$ is a n-tupel of nonnegative integers with order $|\alpha| := \sum_{i=1}^d \alpha_i$. The differential operator D^α is defined as $D^\alpha f := \frac{\partial x^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_d^{\alpha_d}} f$ and the monom $m_\alpha(x) := \prod_{i=1}^d x_i^{\alpha_i}$. For two multi-indices α, β we define the semi-norm $\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} |m_\beta(x) D^\alpha f(x)|$. Let $\mathcal{A}$ denote the set of all multi-indices. The real valued Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is now defined as*

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C_c^\infty(\mathbb{R}^d, \mathbb{R}) : \|f\|_{\alpha,\beta} < \infty \quad \forall \alpha, \beta \in \mathcal{A} \right\}$$

equipped with the topology induced by the semi-norms $\| \cdot \|_{\alpha,\beta}$. The complex valued Schwartz space $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ is defined as the space of complex valued functions, such that their real and imaginary part are in the real valued Schwartz space.

Remark 2.1.2. The elements ϕ of $\mathcal{S}(\mathbb{R}^d)$ are called Schwartz functions. From the definition it follows directly, that if ϕ is a Schwartz function and α, β are multi-indices, then $m_\beta D^\alpha \phi$ is again a Schwartz function. Furthermore, $\mathcal{S}(\mathbb{R}^d)$ equipped with its topology is a complete metric space. Clearly it holds that $\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d, \mathbb{C}) \subseteq L^2(\mathbb{R}^d)$ as sets, but the topologies are different. In particular, the topology on $\mathcal{S}(\mathbb{R}^d)$ induced by the semi-norms $\| \cdot \|_{\alpha,\beta}$ is finer as the one inherited by the L^2-norm. As $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ is dense with respect to the L^2-norm in $L^2(\mathbb{R}^d)$ and $C_c^\infty(\mathbb{R}^d, \mathbb{C}) \subseteq \mathcal{S}(\mathbb{R}^d, \mathbb{C})$, we get that also $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ is dense with respect to the L^2-norm in $L^2(\mathbb{R}^d)$. Let us recall the definition of the Fourier transformation $\mathcal{F}$.

Definition 2.1.3 (Fourier transformation)**.** *For a function $\phi \in L^1(\mathbb{R}^d)$ we define the Fourier transformation by*

$$\mathcal{F}\phi(\xi) := \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(x) e^{-ix\cdot\xi} dx$$

which is a linear operation on $\mathcal{S}(\mathbb{R}^d)$. We often denote $\mathcal{F}\phi$ as $\hat{\phi}$. The inverse Fourier transformation $\mathcal{F}^{-1}$ on $L^1(\mathbb{R}^d)$ is defined as

$$\mathcal{F}^{-1}\phi(x) := \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(\xi) e^{ix\cdot\xi} dx.$$

Remark 2.1.4. One can easily see that $\phi \in L^1(\mathbb{R}^d)$ is real valued if and only if for all $\xi \in \mathbb{R}^d$ it holds that $\hat{\phi}(-\xi) = \overline{\hat{\phi}(\xi)}$. Consider

$$\hat{\phi}(-\xi) = \tfrac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \phi(x) e^{ix\cdot\xi} dx = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \overline{\overline{\phi(x)} e^{-ix\cdot\xi}} dx = \overline{\hat{\overline{\phi}}(\xi)}.$$

Now if $\hat{\phi}(-\xi) = \overline{\hat{\phi}(\xi)}$ then also $\overline{\hat{\overline{\phi}}(\xi)} = \overline{\hat{\phi}(\xi)}$ so $\hat{\overline{\phi}}(\xi) = \hat{\phi}(\xi)$ and therefore $\overline{\phi} = \phi$. On the other hand if ϕ is real valued it follows that $\hat{\phi}(-\xi) = \overline{\hat{\overline{\phi}}(\xi)} = \overline{\hat{\phi}(\xi)}$.

We quickly recall some direct properties of the Fourier transformation.

Proposition 2.1.5. *a) For all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and multi-indices α it holds that*

$$D^\alpha \hat{\phi} = (-i)^{|\alpha|} \widehat{m_\alpha \phi} \qquad \text{and} \qquad \widehat{D^\alpha \phi} = i^{|\alpha|} m_\alpha \hat{\phi}$$

b) $\mathcal{F}$ maps the complex valued Schwartz functions $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ into itself and is bijective. The inverse function of the Fourier transformation on the complex valued Schwartz functions is the inverse Fourier transformation $\mathcal{F}^{-1}$.

c) The Fourier transformation on $\mathcal{S}(\mathbb{R}^d)$ is an isometry with respect to the L^2-scalar product, i.e. $(f, g)_{L^2(\mathbb{R}^d)} = (\hat{f}, \hat{g})_{L^2(\mathbb{R}^d)}$.

Proof. See Proposition 3.2.2 for a), Proposition 3.3.1 for b) and Theorem 3.3.2 for c) in [Bl7]. ■

For $r \in \mathbb{R}$ we define

$$\mathcal{S}_r(\mathbb{R}^d) := \Big\{\phi \in \mathcal{S}(\mathbb{R}^d) : D^\alpha \hat{\phi}(0) = 0 \text{ for all } \quad |\alpha| \leqslant r\Big\}.$$

Using Proposition 2.1.5, we get $D^\alpha \hat{\phi}(0) = (-i)^{|\alpha|} \widehat{m_\alpha \phi}(0) = \frac{(-i)^{|\alpha|}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} m_\alpha(x)\phi(x) dx$. Thus, $\mathcal{S}_r(\mathbb{R}^d)$ is the space of all Schwartz functions such that $\int_{\mathbb{R}^d} m_\alpha(x)\phi(x)dx = 0$ for all multi indices $|\alpha| \leqslant r$. For $r < 0$ we set $\mathcal{S}_r(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$ and the case $r = 0$ indicates all Schwartz functions with zero mean, i.e. $\mathcal{S}_0(\mathbb{R}^d) = \{\phi \in \mathcal{S}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi(x)dx = 0\}$. Since $r \in \mathbb{R}$, multiple r satisfy the inequality for one specific α, so several spaces $\mathcal{S}_r(\mathbb{R}^d)$ for different r are the same. Despite this ambiguousity we still want to keep the structure like that, as this will work well for the later introduced Hilbert spaces the fractional Gaussian field lives on. Let us consider the image of $\mathcal{S}_r(\mathbb{R}^d)$ under the inverse Fourier transformation and denote that space by $\tilde{\mathcal{S}}_r(\mathbb{R}^d)$. From Proposition 2.1.5 we know that the Fourier transformation is bijective. Thus, it follows that

$$\tilde{\mathcal{S}}_r(\mathbb{R}^d) = \Big\{\phi \in \mathcal{S}(\mathbb{R}^d) : D^\alpha\phi(0) = 0 \quad \text{for all} \quad |\alpha| \leqslant r\Big\}.$$

Remark 2.1.6. Let us consider the topological dual space $\mathcal{S}'(\mathbb{R}^d)$, that consists of all linear and continuous functionals on $\mathcal{S}(\mathbb{R}^d)$. Here we equip the functionals with the weak topology on $\mathcal{S}'(\mathbb{R}^d)$, the topology of point-wise convergence. So we have for a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^d)$

$$f_n \longrightarrow f \quad \iff \quad \forall\phi \in \mathcal{S}(\mathbb{R}^d) : (f_n, \phi) \longrightarrow (f, \phi).$$

For every linear functional f it follows that $f(\phi_n) \to 0$ for every sequence $(\phi_n)_{n\in\mathbb{N}} \to 0$ in $\mathcal{S}(\mathbb{R}^d)$. We observe that one can canonically embed $\mathcal{S}'(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$ via

$$\iota : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d), \quad f \mapsto \Big(\phi \mapsto (f,\phi)_{L^2(\mathbb{R}^d)}\Big) \qquad \text{with} \qquad (f,\phi)_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\overline{\phi(x)}dx,$$

whereby, in our case, an embedding indicates an injective and continuous map. We want to point out, that one can view ι as a function into a bigger codomain. Furthermore the Fourier transformation on the topological dual space can readily be defined by $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \quad (\hat{f}, \phi) := (f, \mathcal{F}^{-1}\phi)_{L^2(\mathbb{R}^d)}$. This definition makes sense as it is consistent with the embedding ι. Indeed with Fubini follows for $f, g \in \mathcal{S}(\mathbb{R}^d)$

$$\begin{aligned}
(\hat{f}, g)_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \hat{f}(x)\overline{g(x)}dx \\
&= \int_{\mathbb{R}^d} \left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(\xi)e^{-i\xi\cdot x}d\xi\right)\overline{g(x)}dx \\
&= \int_{\mathbb{R}^d} f(\xi)\left(\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \overline{e^{i\xi\cdot x}g(x)}dx\right)d\xi \\
&= \int_{\mathbb{R}^d} f(\xi)\overline{\mathcal{F}^{-1}g(\xi)}d\xi \\
&= (f, \mathcal{F}^{-1}g)_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

With our previous definitions we can conclude that $f \in \mathcal{S}_r'(\mathbb{R}^d) \Leftrightarrow f \in \mathcal{S}'(\mathbb{R}^d) \quad \text{and} \quad (f, m_\alpha) = 0$ for all $|\alpha| \leqslant r$. Therefore, if we denote the polynomials on $\mathbb{R}^d$ with degree $\leqslant r$ by $\mathcal{P}_r(\mathbb{R}^d)$, we see, that there exists a canonical isomorphism between $\mathcal{S}_r'(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}_r(\mathbb{R}^d)$.

Definition 2.1.7 (Tempered distributions)**.** *The elements of the space $\mathcal{S}'(\mathbb{R}^d)$ equipped with the weak topology are called the tempered distributions on $\mathbb{R}^d$.*

2.2 The Hilbert space the fractional Gaussian field lives on

Our goal is now to define a Hilbert space where the fractional Gaussian field lives on. Therefore we consider the space

$$\mathring{H}^s(\mathbb{R}^d) := \Big\{f \in \mathcal{S}(\mathbb{R}^d) : \xi \mapsto |\xi|^s\hat{f}(\xi) \in L^2(\mathbb{R}^d)\Big\}$$

and equip it with an inner product for $f, g \in \overset{\circ}{H}{}^s(\mathbb{R}^d)$ defined by

$$(f,g)_{\overset{\circ}{H}{}^s(\mathbb{R}^d)} := \Big(\xi \mapsto |\xi|^s \hat{f}(\xi), \xi \mapsto |\xi|^s \hat{g}(\xi)\Big)_{L^2(\mathbb{R}^d)}.$$

We want $\overset{\circ}{H}{}^s(\mathbb{R}^d)$ to become a Hilbert space, thus, we need to find a suitable completion for it. In order to do that we embed it into $\mathcal{S}'_H(\mathbb{R}^d)$ with $H := s - d/2$ being the Hurst parameter. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\overset{\circ}{H}{}^s(\mathbb{R}^d)$ and $\phi \in \mathcal{S}_H(\mathbb{R}^d)$. With Proposition 2.1.5 c) and the Cauchy Schwarz inequality it follows for $n, m \in \mathbb{N}$

$$\begin{aligned}|(f_n - f_m, \phi)_{L^2(\mathbb{R}^d)}| &= |(\hat{f}_n - \hat{f}_m, \hat{\phi})_{L^2(\mathbb{R}^d)}| \\ &\leqslant \Big(\int_{\mathbb{R}^d} |\hat{f}_n(\xi) - \hat{f}_m(\xi)|^2 |\xi|^{2s} d\xi\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^d} |\hat{\phi}(\xi)|^2 |\xi|^{-2s} d\xi\Big)^{\frac{1}{2}}.\end{aligned}$$

We need to explain why the second integral is finite. Since $\hat{\phi}$ is again a complex Schwartz function and $|\xi|^2$ is, outside a small ball around zero, either bounded or of polynomial growth, we get that, outside this ball, the integral is finite. We have to take care of the integral near 0. As $\hat{\phi}$ has a zero of degree $\lfloor H \rfloor$ at the origin, we get, using a Taylor approximation, that $|\hat{\phi}(\xi)| \leqslant |\xi|^{\lfloor H \rfloor + 1}$ near zero. Now using a polar coordinate transformation one can see that

$$\int_{B_\epsilon(0)} |\hat{\phi}(\xi)|^2 |\xi|^{-2s} d\xi \leqslant C \int_0^\epsilon \frac{r^{2\lfloor H \rfloor + 2}}{r^{2s}} r^{d-1} dr = \int_0^\epsilon r^{2\lfloor H \rfloor - 2s + d + 1}.$$

The integral on the right hand side is finite if and only if the exponent is greater than -1. We have that $\lfloor H \rfloor = H - \delta$ with $\delta \in [0,1)$. Therefore we get for the exponent $2\lfloor H \rfloor - 2s + d + 1 = 2s - d - 2\delta - 2s + d + 1 = 1 - 2\delta > -1$. Thus, the integral on the right hand side is finite. All together we conclude that $(f_n, \phi)_{L^2(\mathbb{R}^d)}$ is a Cauchy sequence in $\mathbb{R}$. So we define $f : \mathcal{S}_H(\mathbb{R}^d) \to \mathbb{R}$ as the point-wise limit $(f, \phi) := \lim_{n\to\infty}(f_n, \phi)$ for all $\phi \in \mathcal{S}_H(\mathbb{R}^d)$. It readily follows that f is continuous. Indeed, whenever $\phi_k \to 0$ in $\mathcal{S}_H(\mathbb{R}^d)$, we get with Cauchy Schwarz

$$\limsup_{k\to\infty} |(f, \phi_k)_{L^2(\mathbb{R}^d)}|^2 \leqslant \limsup_{k\to\infty} \limsup_{n\to\infty} \Big(\int_{\mathbb{R}^d} |\hat{f}_n(\xi)|^2 |\xi|^{2s} d\xi\Big)\Big(\int_{\mathbb{R}^d} |\hat{\phi}_k(\xi)|^2 |\xi|^{-2s} d\xi\Big).$$

The integral over the pole is finite for all $k \in \mathbb{N}$. As for all multi-indices α, it holds that $D^\alpha \hat{\phi}(0) \to 0$, the integral over a small neighbourhood of 0 gets arbitrarily small. Outside the neighbourhood the function $|\xi|^{-2s}$ is of polynomial growth so the convergence to zero follows from the convergence in the Schwartz sense of ϕ. In conclusion, f is linear and continuous, and thus, we can embed $\overset{\circ}{H}{}^s(\mathbb{R}^d)$ into $\mathcal{S}'_H(\mathbb{R}^d)$.

Definition 2.2.1 (The space $\dot{H}^s(\mathbb{R}^d)$)**.** *Finally we define $\dot{H}^s(\mathbb{R}^d)$ as the completion of $\iota(\overset{\circ}{H}{}^s(\mathbb{R}^d)) \subseteq \mathcal{S}'_H(\mathbb{R}^d)$.*

Remark 2.2.2. Let us consider $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$, the set of measurable functions that are quadratic integrable with respect to the measure with density $|\xi|^{2s}$. Let $f \in \dot{H}^s(\mathbb{R}^d)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathring{H}^s(\mathbb{R}^d)$ that converges in $\dot{H}^s(\mathbb{R}^d)$ to f. As $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ is complete, $(f_n)_{n\in\mathbb{N}}$ converges in $\dot{H}^s(\mathbb{R}^d)$ if and only if the sequence $(\hat{f}_n)_{n\in\mathbb{N}}$ converges in $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$. Therefore there exists a $g \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ such that $(\hat{f}_n)_{n\in\mathbb{N}} \to g$ in $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$. Using Cauchy-Schwartz, we obtain for $\phi \in \tilde{\mathcal{S}}_r(\mathbb{R}^d)$

$$\Big|\int_{\mathbb{R}^d} g(\xi)\phi(\xi) d\xi\Big| \leqslant \Big(\int_{\mathbb{R}^d} |g(\xi)|^2 |\xi|^{2s} d\xi\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^d} |\phi(\xi)|^2 |\xi|^{-2s} d\xi\Big)^{\frac{1}{2}} < \infty$$

and therefore $g \in \tilde{\mathcal{S}}'_H(\mathbb{R}^d)$. With Proposition 2.1.5 c) it follows for $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ that $(f, \phi) = (g, \hat{\phi})$. We get $\hat{f} = g$ whereby $\hat{f}$ and g are viewed as elements in $\tilde{\mathcal{S}}'_H(\mathbb{R}^d)$. Thus, we can identify the space $\dot{H}^s(\mathbb{R}^d)$ as

$$\dot{H}^s(\mathbb{R}^d) = \Big\{ f \in \mathcal{S}'_H(\mathbb{R}^d) : \hat{f} \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \Big\}.$$

2.3 The fractional Laplacian

This section follows from chapter 2.1 in [LSSW16].

We now want to introduce the fractional Laplacian, which extends the usual Laplacian differential operator. For this purpose we consider for $\phi \in \mathcal{S}_1(\mathbb{R}^d)$ using Proposition 2.1.5 and the linearity of $\mathcal{F}$ and $\mathcal{F}^{-1}$

$$-\Delta\phi(\xi) = -\sum_{i=1}^{d} \frac{\partial^2}{\partial \xi_i^2}\phi(\xi) = -\mathcal{F}^{-1}\sum_{i=1}^{d} \mathcal{F}\frac{\partial^2}{\partial \xi_i^2}\phi(\xi) = -\mathcal{F}^{-1}\sum_{i=1}^{d}(-i)^2 x_i^2 \hat{\phi}(\xi) = \mathcal{F}^{-1}|x|^2\hat{\phi}(\xi).$$

The inverse Fourier transformation in this case is well defined, because $\phi \in \mathcal{S}_1(\mathbb{R}^d)$ and therefore $|x|^2\hat{\phi}(\xi) \in L^1(\mathbb{R}^d)$. This fact gives rise to the following definition:

Definition 2.3.1 (Fractional Laplacian)**.** *For an integer $k \geqslant -1$, $\phi \in \mathcal{S}_k(\mathbb{R}^d)$ and $s > -\frac{d+k+1}{2}$ we define the fractional Laplacian $(-\Delta)^s$ as*

$$(-\Delta)^s\phi(\xi) := \mathcal{F}^{-1}\Big(\xi \mapsto |\xi|^{2s}\hat{\phi}(\xi)\Big).$$

This is well defined due to $\xi \mapsto |\xi|^{2s}\hat{\phi}(\xi) \in L^1(\mathbb{R}^d)$. Furthermore it clearly holds that $(-\Delta)^0\phi = \phi$ and for $s_1, s_2 > -\frac{d+k+1}{2}$

$$\begin{aligned}(-\Delta)^{s_1}(-\Delta)^{s_2}\phi(\xi) &= (-\Delta)^{s_1}\mathcal{F}^{-1}\Big(\xi \mapsto |\xi|^{2s_2}\hat{\phi}(\xi)\Big)\\ &= \mathcal{F}^{-1}\Big(\xi \mapsto |\xi|^{2(s_1+s_2)}\hat{\phi}(\xi)\Big)\\ &= (-\Delta)^{s_1+s_2}\phi(\xi).\end{aligned}$$

Remark 2.3.2. For an integer $k \geqslant -1$, $\phi \in \mathcal{S}_k(\mathbb{R}^d)$ and $s > -\frac{d+k+1}{2}$ follows that $(-\Delta)^s\phi \in C^\infty(\mathbb{R}^d)$. Indeed if we use Proposition 2.1.5 a) we see that for a multi-index α it holds that $D^\alpha \mathcal{F}^{-1}(\phi) = i^{|\alpha|}\mathcal{F}^{-1}(m_\alpha \phi)$. We conclude that

$$\begin{aligned} D^\alpha(-\Delta)^s\phi &= D^\alpha \mathcal{F}^{-1}\Big(\xi \mapsto |\xi|^{2s}\hat{\phi}(\xi)\Big) \\ &= i^{|\alpha|}\mathcal{F}^{-1}\Big(\xi \mapsto m_\alpha(\xi)|\xi|^{2s}\hat{\phi}(\xi)\Big) \\ &= \mathcal{F}^{-1}\Big(\xi \mapsto |\xi|^{2s} m_\alpha(\xi)\widehat{D^\alpha\phi}\Big) \\ &= (-\Delta)^s(D^\alpha\phi). \end{aligned}$$

Furthermore with Remark 2.1.4 we get that $(-\Delta)^s\phi$ is real valued.

Lemma 2.3.3. *For an integer $k \geqslant -1$, $\phi \in \mathcal{S}_k(\mathbb{R}^d)$, $s > -\frac{d+k+1}{2}$ and a multi-index α it holds that*

$$\sup_{\xi\in\mathbb{R}^d}\big(1 + |\xi|^{d+2s+k+1}\big)|\partial^\alpha(-\Delta)^s\phi(\xi)| < \sup_{|\beta|\leqslant \max|\alpha|,k+1}\|\phi\|_{\beta,0}.$$

Proof. See Proposition 2.1 in [LSSW16]. ■

Definition 2.3.4 (The space $\mathcal{U}_s(\mathbb{R}^d)$)**.** *For $s > -\frac{d}{2}$, a multi-index α and $\phi \in C^\infty(\mathbb{R}^d)$ we define the semi-norm $\|\phi\|_{\mathcal{U}_s(\mathbb{R}^d),\alpha} := \sup_{\xi\in\mathbb{R}^d}(1 + |\xi|^{d+2s})|\partial^\alpha\phi(\xi)|$ and the corresponding space*

$$\mathcal{U}_s(\mathbb{R}^d) := \Big\{\phi \in C^\infty(\mathbb{R}^d) : \|\phi\|_{\mathcal{U}_s(\mathbb{R}^d),\alpha} < \infty \quad \forall\alpha \in \mathcal{A}\Big\}$$

equipped with the topology induced by the semi-norms $\|\cdot\|_{\mathcal{U}_s(\mathbb{R}^d),\alpha}$. It clearly follows that $\mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{U}_s(\mathbb{R}^d) \subseteq \mathcal{U}_{s'}(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$ whenever $-\frac{d}{2} < s' < s$.

Remark 2.3.5. Together with Lemma 2.3.3 we conclude that the operator $(\Delta)^s : \mathcal{S}_k(\mathbb{R}^d) \to \mathcal{U}_{s+(k+1)/2}(\mathbb{R}^d)$ is continuous. In addition, if we assume for $\phi \in \mathcal{S}_k(\mathbb{R}^d)$ that $(-\Delta)^s\phi = 0$, it follows due to the injectivity of the inverse Fourier transformation, that $\hat{\phi}$ vanishes everywhere except at the origin. As $\hat{\phi}$ is smooth it has to be zero and with the injectivity of the Fourier transformation it follows that $\phi = 0$. Therefore $(-\Delta)^s$ is injective for all $s > -\frac{d}{2}$.

Remark 2.3.6. Analogous to the Fourier transformation we can easily define the fractional Laplacian $(-\Delta)^s$ on the topological dual space of the image $(-\Delta)^s\mathcal{S}_k(\mathbb{R}^d) \subseteq \mathcal{U}_{s+(k+1)/2}(\mathbb{R}^d)$. For $f \in ((-\Delta)^s\mathcal{S}_k(\mathbb{R}^d))'$ and $\phi \in \mathcal{S}_k(\mathbb{R}^d)$ we define

$$((-\Delta)^s f, \phi) := (f, (-\Delta)^s\phi)$$

which is well defined. For the embedding ι in Remark 2.1.6 and $f \in \mathcal{S}(\mathbb{R}^d)$ it readily follows with the definition of the Fourier transformation on the topological dual space that

$\iota((-\Delta)^s f) = (-\Delta)^s \iota(f)$. To see that the element $(-\Delta)^s f$ can in fact be interpreted as an element of the dual space $((-\Delta)^s \mathcal{S}_k(\mathbb{R}^d))'$, we consider for a $f, g \in \mathcal{S}_k(\mathbb{R}^d)$

$$\begin{aligned}\iota((-\Delta)^s f)(g) &= ((-\Delta)^s f, g) = (\hat{f}|\xi|^{2s}, \hat{g}) \\ &= (\hat{f}, \hat{g}|\xi|^{2s}) = (f, (-\Delta)^s g) \\ &= ((-\Delta)^s f, g) = (-\Delta)^s \iota(f)(g).\end{aligned}$$

Therefore, the extension of the fractional Laplacian on the topological dual space makes sense. Furthermore, from this consideration it directly follows that the fractional Laplacian on the topological dual space still satisfies the property $(-\Delta)^{s_1}(-\Delta)^{s_2} = (\Delta)^{s_1+s_2}$ for suitable s_1, s_2.

We want to define the fractional Laplacian operator on the space $\dot{H}^s(\mathbb{R}^d)$. Therefore we embed it into the space $((-\Delta)^s \mathcal{S}_H(\mathbb{R}^d))'$ via the map

$$\iota : \begin{cases} \dot{H}^s(\mathbb{R}^d) \longrightarrow ((-\Delta)^s \mathcal{S}_k(\mathbb{R}^d))' \\ f \mapsto (\phi \mapsto (\hat{f}, \hat{\phi})) \end{cases}.$$

First we show injectivity. Clearly ι is linear, so it suffices to show that the kernel of ι is trivial. As $(-\Delta)^s$ is injective, we get for $\phi \in (-\Delta)^s \mathcal{S}_k(\mathbb{R}^d)$ that there exists a unique $g \in \mathcal{S}_k(\mathbb{R}^d)$ with $(-\Delta)^s g = \phi$. Now follows that

$$\iota(f)(\phi) = (\hat{f}, \hat{\phi}) = (\hat{f}, \widehat{(-\Delta)^s g}) = (f, |\xi|^{2s}\hat{g}) \stackrel{!}{=} 0.$$

As $|\xi|^{2s}\hat{g}$ are dense in $\mathcal{S}_H(\mathbb{R}^d)$, we get that $f = 0$. Due to the linearity, it suffices to show continuity at the origin. Consider a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \dot{H}^s(\mathbb{R}^d)$ that converges to zero. As $|\xi|^{2s}\hat{g} \in \mathcal{S}_H(\mathbb{R}^d)$ we have $\iota(f_n)(\phi) = (f_n, |\xi|^{2s}\hat{g}) \to 0$ as $n \to 0$. Therefore ι is a well defined embedding.

Now we can make use of the definition of $(-\Delta)^s$ on the space $((-\Delta)^s \mathcal{S}_k(\mathbb{R}^d))'$ and define

$$(-\Delta)^s : \begin{cases} \dot{H}^{s_0}(\mathbb{R}^d) \to \dot{H}^{s_0-2s}(\mathbb{R}^d) \\ f \mapsto (-\Delta)^s \iota(f) \end{cases}. \tag{2.1}$$

It follows that the fractional Laplacian operator is an isometric isomorphism. This is more easy to see if one interprets the space $\dot{H}^s(\mathbb{R}^d)$ as in Remark 2.2.2. Here the fractional Laplacian operator turns out to be the usual fractional Laplacian defined for topological dual spaces as above. Then it clearly follows for $f \in \dot{H}^s_0(\mathbb{R})$ that

$$\begin{aligned}\|(-\Delta)^s f\|_{\dot{H}^s_0(\mathbb{R}^d)} &= \|\widehat{(-\Delta)^s f}\|_{L^2(\mathbb{R}^d, |\xi|^{2s_0-4s} d\xi)} \\ &= \|\hat{f}|\xi|^{2s}\|_{L^2(\mathbb{R}^d, |\xi|^{2s_0-4s} d\xi)} \\ &= \|\hat{f}\|_{L^2(\mathbb{R}^d, |\xi|^{2s_0} d\xi)} = \|f\|_{\dot{H}^s_0-2s(\mathbb{R}^d)}.\end{aligned}$$

From that the bijectivity easily follows as well. Therefore we can define for $s < 0$ the fractional Laplacian operator as the inverse of $(-\Delta)^{|s|}$. For $s = 0$ we set $(\Delta)^0 = id_{\dot{H}^{s_0}(\mathbb{R}^d)}$. In total we arrive at the following Proposition.

Proposition 2.3.7. *For $s \in \mathbb{R}$ the fractional Laplacian operator $(-\Delta)^s : \dot{H}^{s_0}(\mathbb{R}^d) \to \dot{H}^{s_0-2s}(\mathbb{R}^d)$ defined in equation 2.1 is a well defined isometric isomorphism.*

Proof. See Remark 2.3.6. ■

The following Lemma gives us an easier way of interpreting the fractional Laplacian for $s \in (0,1)$ and is following the idea of Proposition 3.3 of [NPV11].

Lemma 2.3.8. *For $f \in \mathcal{S}(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$ and $0 < s < 1$ it holds that*

$$(-\Delta)^s f(\xi) = -\tfrac{1}{2}C(d,s)\int_{\mathbb{R}^d}\frac{f(\xi+x)-2f(\xi)+f(\xi-x)}{|x|^{d+2s}}dx$$
$$\text{with } C(d,s) = \Big(\int_{\mathbb{R}^d}\frac{(1-cos(\xi_1))}{|\xi|^{d+2s}}d\xi\Big)^{-1}.$$

Proof. First we give an argument why the integrals exist. At 0 the numerator $f(\xi+x)-2f(\xi)+f(\xi-x)$ and its first derivatives D^α $|\alpha| \leqslant 1$ are 0. Therefore, using multidimensional Taylor we can estimate the integrand by $C|x|^{2-2s-d}$ for some constant C, so the integral near 0 exists, because $s > 1$. For the integral over $\mathbb{R}^d \backslash B_1(0)$ we can estimate the integrand by $C|x|^{-d-2s}$ for some constant. Again, the integral exists as this time $s > 0$. The constant $C(d,s)$ is well defined as we can see in an analogue way. First we consider

$$\begin{aligned}
\mathcal{F}f(\xi+x) &= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(y+x)e^{-iy\cdot\xi}dy \\
&= \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(z)e^{-i(z-x)\cdot\xi}dz \\
&= e^{ix\cdot\xi}\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(z)e^{-iz\cdot\xi}dz \\
&= e^{ix\cdot\xi}\mathcal{F}f(\xi).
\end{aligned}$$

Now using the linearity of the Fourier transformation we calculate

$$\begin{aligned}
&\mathcal{F}\Big(-\frac{1}{2}C(d,s)\int_{\mathbb{R}^d}\frac{f(\xi+x)-2f(\xi)+f(\xi-x)}{|x|^{d+2s}}dx\Big) \\
&= -\frac{1}{2}C(d,s)\int_{\mathbb{R}^d}\frac{(e^{ix\cdot\xi}-2+e^{-ix\cdot\xi})\mathcal{F}f(\xi)}{|x|^{d+2s}}dx \\
&= C(d,s)\int_{\mathbb{R}^d}\frac{1-\frac{e^{ix\cdot\xi}+e^{-ix\cdot\xi}}{2}}{|x|^{2+2s}}dx\mathcal{F}f(\xi)
\end{aligned}$$

$$= C(d,s) \underbrace{\int_{\mathbb{R}^d} \frac{1 - cos(x \cdot \xi)}{|x|^{2+2s}} dx}_{=:I(\xi)} \mathcal{F}f(\xi).$$

Let us show that $I(\xi)$ is radially symmetric in ξ. For $d = 1$ this is clear as cos is even. For $d \geqslant 2$ let R be a rotation such that $R(|\xi|e_1) = \xi$. Then it holds that

$$\begin{aligned} I(\xi) &= \int_{\mathbb{R}^d} \frac{1 - cos(x \cdot R(|\xi|e_1))}{|x|^{d+2s}} dx \\ &= \int_{\mathbb{R}^d} \frac{1 - cos(R^{-1}(x) \cdot |\xi|e_1)}{|x|^{d+2s}} dx \\ &= \int_{\mathbb{R}^d} \frac{1 - cos(x \cdot |\xi|e_1)}{|x|^{d+2s}} dx = I(|\xi|e_1). \end{aligned}$$

Now using the symmetry and the substitution $z = \frac{x}{|\xi|}$

$$\begin{aligned} I(\xi) &= \int_{\mathbb{R}^d} \frac{1 - cos(x \cdot |\xi|e_1)}{|x|^{d+2s}} dx \\ &= \int_{\mathbb{R}^d} |\xi|^{-d} \frac{1 - cos(z_1)}{\frac{|z|^{d+2s}}{|\xi|^{d+2s}}} dx \\ &= |\xi|^{2s} \int_{\mathbb{R}^d} \frac{1 - cos(z_1)}{|z|^{d+2s}} dz = |\xi|^{2s} C(d,s)^{-1}. \end{aligned}$$

Putting all together we get

$$-\tfrac{1}{2} C(d,s) \int_{\mathbb{R}^d} \frac{f(\xi + x) - 2f(\xi) + f(\xi - x)}{|x|^{d+2s}} dx = \mathcal{F}^{-1} |\xi|^{2s} \mathcal{F}f(\xi) = (-\Delta)^s f(\xi).$$

■

Remark 2.3.9. For $s \geqslant 0$ let $n := \lfloor s \rfloor$ and let us decompose $(-\Delta)^s = (-\Delta)^{s-n}(-\Delta)^n$. As $(-\Delta)^n$ results from applying the usual differential $-\Delta$ operator n times, we now have, together with Lemma 2.3.8 using $0 \leqslant s - n < 1$, an explicit way of calculating the fractional Laplacian.

2.4 White noise

In this section we want to define the so-called white noise, which is an analogue of a standard Gaussian random variable on a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)_{\mathcal{H}}$. In the case of a finite dimensional Hilbert space this is an easy procedure. For this purpose we just choose an orthonormal basis $h_1, ..., h_n \in \mathcal{H}, n \in \mathbb{N}$ and independent standard Gaussian random variables $Y_1, ...Y_n \sim N(0,1)$ on Ω and define the standard Gaussian on $\Omega \times \mathcal{H}$ in the following way

$$Y : \begin{cases} \Omega \times \mathcal{H} \to \mathbb{R} \\ (\omega, x) \mapsto \sum_{i=1}^{n} Y_i(\omega)(x, h_i)_{\mathcal{H}} \end{cases} .$$

This random variable on $\mathcal{H}$ has now the characteristic property of a standard Gaussian variable, that for $x, y \in \mathcal{H}$ it holds that

$$Cov(Y(x), Y(y)) = \sum_{i,j=1}^{n} (x, h_i)_{\mathcal{H}}(y, h_j)_{\mathcal{H}}\mathbb{E}[Y_iY_j] = \sum_{i=1}^{n}(x, h_i)_{\mathcal{H}}(y, h_i)_{\mathcal{H}} = (x, y)_{\mathcal{H}}$$

and therefore $Y(x) \sim N(0, \|x\|_{\mathcal{H}}^2)$. However, for an infinite dimensional Hilbert space $\mathcal{H}$ this is not possible. If we tried to define the standard Gaussian in an analogue way and choose an infinite orthonormal basis $(h_i)_{i\in\mathbb{N}}$ and a sequence of independent standard Gaussian random variables $(Y_i)_{i\in\mathbb{N}}$ and set

$$Y : \begin{cases} \Omega \times \mathcal{H} \to \mathbb{R} \\ (\omega, x) \mapsto \sum_{i\in\mathbb{N}} Y_i(\omega)(x, h_i)_{\mathcal{H}} \end{cases}$$

we would again get $Cov(Y(x), Y(y)) = (x, y)_{\mathcal{H}}$ for $x, y \in \mathcal{H}$. The problem now is that for the Borel sigma algebra $\mathcal{A}$ the function $\mathbb{P} : \mathcal{A} \to [0, 1], A \mapsto \mathbb{P}[Y \in A]$ is not a probability measure on $\mathcal{H}$ any more. To see this, for $x \in \mathcal{H}, r > 0$ consider the balls $B(x, r) := \{y \in \mathcal{H} : \|x - y\|_{\mathcal{H}} < r\}$. Computing $\mathbb{P}[B(x, r)]$ we see that this is smaller or equal the probability, that infinitely many independent standard Gaussian random variables take values in the compact set $[-r, r]$, whereby the latter turns out to be 0. As the Hilbert space $\mathcal{H}$ is a countable union of such balls but $\mathbb{P}[\mathcal{H}] = 1$, we see that $\mathbb{P}$ is not sigma sub-additive anymore and therefore not a probability measure.[Jan97]

In conclusion, we need to find another way of defining a standard Gaussian on an infinite dimensional Hilbert space, namely as a random object on the topological dual space of $\mathcal{H}$. The tool that will help us with that is the Bochner-Minlos theorem. First, we introduce the term of a cylinder set measure.

Definition 2.4.1 (Cylinder set measure)**.** *Let X be a real separable topological vector space and $\mathcal{U}$ be the set of finite dimensional subspaces $U \subseteq X$. Furthermore we define*

$$\mathcal{T}(X) := \Big\{T : X \to U : T \text{ linear, surjective}, U \in \mathcal{U}\Big\}.$$

Then the cylinder sets are the sets of the form $T^{-1}(B) \subseteq X$ with $T \in \mathcal{T}(X)$ and B being a Borel set in U, i.e. as U is finite dimensional we have that it is isomorphic to $\mathbb{R}^{dim(U)}$ and we define the Borel sets on U via the Borel sets on $\mathbb{R}^{dim(U)}$. The set of all cylinder sets is denoted by $Cyl(X)$ and it holds that $\sigma(Cyl(X)) = \mathcal{B}(X)$.

Now a cylinder set measure is a family of probability measures $(\mu_T)_{T\in\mathcal{T}(X)}$, where μ_T is a probability measure on $T(X) \in \mathcal{U}$ endowed with the Borel sigma-algebra, that fulfills the following condition. For every projection $P_{VU} : V \to U$ with $U, V \in \mathcal{U}$, $U \subseteq V$ it holds that

$$\mu_U = \mu_V P_{VU}^{-1},$$

i.e. μ_U is the push forward measure of μ_V under the map P_{VU}.

For every probability measure μ on a real separable topological vector space and $T \in \mathcal{T}$ we can consider the measure $\mu_T := \mu T^{-1}$. This collection forms a cylinder set measure. The important part to notice is that not every cylinder set measure can be constructed by a probability measure on X. We give an easy example of that case.

Example 2.4.2 (The canonical Gaussian cylinder set measure). Let $\mathcal{H}$ be a Hilbert space equipped with an inner product $(\cdot, \cdot)$. We endow every finite dimensional subspace U with the inner product of $\mathcal{H}$ restricted to U and consider a standard Gaussian probability measure P_U on U exactly constructed like in the beginning of this subsection. Then these probability measures fulfill the requirements of a cylinder set measure. However they cannot be extended to a standard Gaussian probability measure on $\mathcal{H}$ as we have seen in the first part of this subsection.

There are Hilbert spaces such that for every cylinder set measure there exists an extension to a probability measure on the whole space. One example are dual spaces of nuclear Hilbert spaces. The Schwartz space is indeed a nuclear space. We don't want to introduce the term of nuclear spaces here, but directly show, that one can extend every cylinder set measure on $\mathcal{S}'(\mathbb{R})$ to a probability measure. Let us state a useful characterization of the Schwartz space via the Hermite basis, see Theorem 2.3 in [LSSW16].

Remark 2.4.3. For $n \in \mathbb{N}$ we define

$$\psi_n(x) : \begin{cases} \mathbb{R} \to \mathbb{R} \\ x \mapsto \dfrac{(-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}[e^{-x^2}]}{\pi^{1/4}\sqrt{2^n n!}} \end{cases}.$$

Then the sequence $(\psi_n)_{n\in\mathbb{N}}$ is an orthogonal basis of $L^2(\mathbb{R}, \mathbb{R})$, the so called Hermite basis.

Proposition 2.4.4. *Let $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. We define for an integer m the semi-norm on $\mathbb{R}^{\mathbb{N}}$ by $\|(x_n)_{n\in\mathbb{N}}\|_m := \sum_{n\in\mathbb{N}}(1+n^2)^m|x_n|$. Let s_m be the space of sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that all the norms $\|\cdot\|_m$ are finite, i.e.*

$$s_m := \Big\{(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \|(x_n)_{n\in\mathbb{N}}\|_m < \infty\Big\}.$$

Then we set

$$s := \bigcap_{m\in\mathbb{Z}} s_m,$$

equipped with the topology induced by all the semi norms $\|\cdot\|_m$. It follows that the one-dimensional Schwartz space $\mathcal{S}(\mathbb{R})$ is isomorphic to the space s. Furthermore, the topological dual space is of the form

$$s' = \bigcup_{m \in \mathbb{Z}} s_m.$$

Proof. See Theorem 2.3 in [LSSW16]. ∎

Remark 2.4.5. In a very analogue way one can show for the higher dimensional Schwartz space $\mathcal{S}(\mathbb{R}^d)$ the existence of a Hermite basis. There one considers the so called Hilbert Schmidt operator H and defines for all integers n and finite linear combinations of Hermite basis elements ϕ, ψ the inner product $(\phi, \psi)_n := (\phi, (H+I)^n \psi)_{L^2(\mathbb{R}^d)}$. The induced norm is denoted by $\|\phi\|_n := (\phi, \phi)_n^{1/2}$. If one completes this space of finite linear combinations one gets a Hilbert space $\mathcal{H}_n$ for every $n \in \mathbb{Z}$. As H is a positive definite operator one gets for all linear combinations that $\|\phi\|_n \leqslant \|\phi\|_{n+1}$. Furthermore $\mathcal{H}_0 = L^2(\mathbb{R}^d)$. Finally, we arrive at the following equalities

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{n \in \mathbb{Z}} \mathcal{H}_n \text{ and } \mathcal{S}'(\mathbb{R}^d) = \bigcup_{n \in \mathbb{Z}} \mathcal{H}_n.$$

Furthermore, for $r > 0$ and $n \in \mathbb{Z}$ we define the ball around the origin with respect to the norm $\|\cdot\|_n$ as $B_r^n(0) := \{\phi \in \mathcal{S}'(\mathbb{R}^d) : \|\phi\|_n \leqslant r\}$. For more details see Lemma A.3.2 in [GJ87].

Our next step to our goal of proving the Bochner-Minlos theorem is to give a sufficient condition for the extension of the cylinder set measure to exist on the whole space $\mathcal{S}'(\mathbb{R}^d)$. For the purpose we introduce the term of a vanishing measure at infinity. This part follows chapters A.3, A.4 and A.6 in [GJ87].

Definition 2.4.6 (vanishing measure at infinity)**.** *Let $\mathcal{S}'(\mathbb{R}^d)$ be the topological dual space characterized as in Remark 2.4.5. Then we say for a finitely additive cylinder set measure μ on $\mathcal{S}'(\mathbb{R}^d)$ it has vanishing measure at infinity if for every $\epsilon > 0$ there exist $n \in \mathbb{Z}$ and $r > 0$ such that for all $C \in Cyl(\mathcal{S}'(\mathbb{R}^d))$ with $C \cap B_r^n(0) = \emptyset$ we have $\mu(C) \leqslant \epsilon$. If we can find for $n \in \mathbb{Z}$ and all $\epsilon > 0$ an $r > 0$, such that the inequality is fulfilled, we say that μ has vanishing measure at infinity on $\mathcal{H}_n$.*

Lemma 2.4.7. *Let μ be a finitely additive and regular probability measure on the cylinder sets $Cyl(\mathcal{S}'(\mathbb{R}^d))$ that has vanishing measure at infinity on $\mathcal{H}_n$. Then μ is a countably additive measure on $\mathcal{H}_n$.*

Remark 2.4.8. Here the regularity of the measure μ is defined with respect to the weak topology on $\mathcal{S}'(\mathbb{R}^d)$ discussed in Remark 2.1.6. In particular, μ is regular if for all $C \in Cyl(\mathcal{S}'(\mathbb{R}))$ it holds that

$$\mu(C) = \inf\{\mu(O) : C \subseteq O \in Cyl(\mathcal{S}'(\mathbb{R}^d), O \text{ is open w.r.t the weak topology}\}.$$

Proof of Lemma 2.4.7. Let $(C_k)_{k \in \mathbb{N}}$ be a sequence of pairwise disjoint cylinder sets, $C := \bigcup_{k=1}^{\infty} C_k$ and define $C_0 := \mathcal{S}'(\mathbb{R}^d) \backslash C$. It suffices to show

$$\sum_{k=0}^{\infty} \mu(C_k) = 1.$$

Indeed, if we have this equality we get

$$1 - \mu(C_0) = \mu\Big(\bigcup_{k=0}^{\infty} C_k\Big) = \sum_{k=0}^{\infty} \mu(C_k).$$

Therefore μ is countably additive. By the finite additivity of μ we get

$$\sum_{k=0}^{\infty} \mu(C_k) = \lim_{K\to\infty} \sum_{k=0}^{K} \mu(c_k) = \lim_{K\to\infty} \mu\Big(\bigcup_{k=0}^{K} C_k\Big) \leqslant \mu\Big(\bigcup_{k=0}^{\infty} C_k\Big).$$

It remains to prove the other inequality. As μ is regular, it suffices to show the inequality for weakly open sets. Indeed, for all $\epsilon > 0$ we find weakly open sets $O_k \supseteq C_k$ such that $\mu(C_k) + \frac{\epsilon}{2^{k+1}} > \mu(O_k)$ and therefore, if μ is countably additive on weakly open sets we have

$$\sum_{k=0}^{\infty} \mu(C_k) + \epsilon > \sum_{K=0}^{\infty} \mu(O_k) = \mu\Big(\bigcup_{k=0}^{\infty} O_k\Big) \geqslant \mu\Big(\bigcup_{k=0}^{\infty} C_k\Big).$$

Therefore let all C_k be weakly open. Let $\epsilon > 0$. As μ has vanishing measure at infinity on $\mathcal{H}_n$, there exists $r > 0$ such that for all $C \in Cyl(\mathcal{S}'(\mathbb{R}^d))$ with $C \cap B_r^n(0)$ we have $\mu(C) \leqslant \epsilon$. Now we use that the ball $B_r^n(0)$ is, according to Banach-Anaoglu (Theorem 5.5.6 in [HWB20]), weakly compact. We get a finite open cover $C_{k_1}, ..., C_{k_L}$, $L \in \mathbb{N}$ of $B_r^n(0)$. It follows

$$\sum_{k=0}^{\infty} \mu(c_k) \geqslant \sum_{l=1}^{L} \mu(C_{k_l}) = \mu\Big(\bigcup_{l=1}^{L} C_{k_l}\Big) \geqslant \mu(B_r^n(0)) \geqslant 1 - \epsilon.$$

As this inequality holds for all $\epsilon > 0$, the countably additivy of μ follows. ■

Definition 2.4.9 (Characteristic function of a probability measure)**.** *Let μ be a probability measure on $\mathcal{S}'(\mathbb{R}^d)$ and $\Phi : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. Then Φ is called the characteristic function of μ, if for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have*

$$\Phi(\phi) = \int_{\mathcal{S}'(\mathbb{R}^d)} e^{i(f,\phi)} d\mu(f). \tag{2.2}$$

Definition 2.4.10 (positive definite functional)**.** *A $\mathbb{C}$-valued linear function Φ on a Hilbert space $\mathcal{H}$ is called positive semi-definite, if for all $n \in \mathbb{N}$, $h_1, ..., h_n \in \mathcal{H}$ and $c_1, ..., c_n \in \mathbb{C}$ it holds that*

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \Phi(h_i - h_j) \geqslant 0.$$

Theorem 2.4.11 (Bochner-Minlos Theorem)**.** *Let $\Phi : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. Then there exists a probability measure μ on $\mathcal{S}'(\mathbb{R}^d)$ such that Φ is the characteristic function of μ, if and only if Φ is continuous, positive semi-definite and $\Phi(0) = 1$.*

Remark 2.4.12. For finite dimensional subspaces of $\mathcal{S}'(\mathbb{R}^d)$ the Bochner-Minlos theorem reduces to an easier case, which can be seen as a special case of Bochner's theorem. In general, Bochner's theorem is proven for locally compact abelian groups.

Sketch of the proof of theorem 2.4.11. Let μ be a probability measure such that Φ is its characteristic function. Clearly, it holds that $\Phi(0) = \mathbb{E}[e^{i(f,0)}] = \mathbb{E}[1] = 1$. Furthermore, for $c_1, ..., c_n \in \mathbb{C}$ and $\phi_1, ..., \phi_n \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\begin{aligned}
\sum_{i,j=1}^{n} c_i \bar{c_j} \Phi(\phi_i - \phi_j) &= \int_{\mathcal{S}'(\mathbb{R}^d)} \sum_{i,j=1}^{n} c_i \bar{c_j} e^{i(f,\phi_i - \phi_j)} d\mu(f) \\
&= \int_{\mathcal{S}'(\mathbb{R}^d)} \left(\sum_{i=1}^{n} c_i e^{i(f,\phi_i)} \right) \left(\sum_{j=1}^{n} \overline{c_j e^{i(f,\phi_j)}} \right) d\mu(f) \\
&= \int_{\mathcal{S}'(\mathbb{R}^d)} \underbrace{\left| \sum_{i=1}^{n} c_i e^{i(f,\phi_i)} \right|^2}_{\geqslant 0} d\mu(f) \geqslant 0
\end{aligned}$$

and therefore Φ is positive definite. Finally, with the dominant convergence theorem, we get, due to the continuity of every $f \in \mathcal{S}'(\mathbb{R}^d)$, for a sequence $\phi_n \to \phi$ in $\mathcal{S}(\mathbb{R}^d)$, that

$$\Phi(\phi_n) = \int_{\mathcal{S}'(\mathbb{R}^d)} e^{i(f,\phi_n)} d\mu(f) \longrightarrow \int_{\mathcal{S}'(\mathbb{R}^d)} e^{i(f,\phi)} d\mu(f) = \Phi(\phi)$$

and therefore the continuity of Φ.

Now we take a function Φ with the three properties above and further need to construct a probability measure μ on $\mathcal{S}'(\mathbb{R}^d)$ such that Φ is its characteristic function. The main idea is to use Bochner's theorem for all finite dimensional cases and then put them all together using Kolmogorov's extension theorem (Theorem 2.9.2). For more details of that part of the proof see Theorem A.6 in [GJ87]. For every finite dimensional linear subspace $\mathcal{V}$ $of \mathcal{S}(\mathbb{R}^d)$ we consider the restriction of Φ to that space $\Phi_{\mathcal{V}} := \Phi|_{\mathcal{V}}$. Clearly, $\Phi_{\mathcal{V}}$ inherits all three properties from above. Now we can use Bochner's theorem as described in remark 2.4.12 and obtain for every $\mathcal{V}$ a probability measure $\mu_{\mathcal{V}}$ on $\mathcal{V}'$. As there exists a projection $P_{\mathcal{V}}$ for all $\mathcal{V}$, one can view $\mu_{\mathcal{V}}$ as a probability measure on the $\mathcal{V}$-cylinder sets, i.e. on $\sigma(\{P_{\mathcal{V}}^{-1}(A) : A \in \mathcal{B}(\mathcal{V})\})$ where $\mathcal{B}(\mathcal{V})$ denotes the Borel sets in the subspace $\mathcal{V}$. In particular, for a set $A \in P^{-1}(\mathcal{B}(\mathcal{V}))$ one considers the measure $A \mapsto \mu_{\mathcal{V}}(P(A))$. We want to show that this collection of probability measures is a cylinder set measure. Therefore let $\mathcal{V} \subseteq \mathcal{U}$ be linear subspaces of $\mathcal{S}$ and $P : \mathcal{U} \to \mathcal{V}$ the projection operator. We need to show

$$\mu_{\mathcal{V}} = \mu_{\mathcal{U}} P^{-1},$$

which directly follows using the uniqueness part of Bochner's theorem. Hence, we have a projective family of cylinder set measures and therefore get a unique cylider set measure μ such that for every projection $P_{\mathcal{V}} \mathcal{S}'(\mathbb{R}^d) \to \mathcal{V}'$ we have

$$\mu_{\mathcal{V}} = \mu P_{\mathcal{V}}^{-1}.$$

By construction, μ has Φ as characteristic function. It remains to show that μ can indeed be extended to a measure on the Borel sets of $\mathcal{S}'(\mathbb{R}^d)$. For that purpose it is sufficient to show that μ is countably additive. We therefore want to apply Lemma 2.4.7. Hence, we need to prove that μ has vanishing measure at infinity on $\mathcal{H}_n$ for every $n \in \mathbb{N}$. Let $B_r^n(0)$ be defined as in remark 2.4.5 and C be a $\mathcal{V}$-cylinder set such that $C \cap B_r^{-n}(0) = \varnothing$. Denoting by P the projection $\mathcal{S}'(\mathbb{R}^d) \to \mathcal{V}'$, we get

$$\mu(C) \leqslant \int_{\mathcal{V}'\setminus P(B_r^{-n}(0))} 1 d\mu \leqslant 2\int_{\mathcal{V}'} \left(1 - e^{-\frac{ln(2)}{r^2}\|f\|_{-n}^2}\right) d\mu_{\mathcal{V}}.$$

The second inequality follows from the fact that on $\mathcal{V}'\setminus P(B_r^{-n}(0))$ the exponential term is less than one half. Denote by Q the projection from $\mathcal{S}$ onto $\mathcal{V}$ and by I the identity, and let $A := (Q(H+I)^{-n}Q)^{-1}$. Then we get

$$\begin{aligned}
&\int_{\mathcal{V}'} \exp\left(-\frac{\ln(2)}{r^2}\|f\|_{-n}^2\right) d\mu_{\mathcal{V}}(f) \\
&= \int_{\mathcal{V}'} \exp\left(-\frac{\ln(2)}{r^2}(f, A^{-1}f)\right) d\mu_{\mathcal{V}}(f) \\
&= C\int_{\mathcal{V}}\int_{\mathcal{V}'} \exp\Big(i(Qf,\phi)\Big) \exp\left(-\frac{r^2(\phi, A\phi)}{4\ln(2)}\right) d\mu(f)d\phi,
\end{aligned}$$

with

$$C := \left(\int_{\mathcal{V}} \exp\left(-\frac{r^2(\phi, A\phi)}{4\ln(2)}\right) d\phi\right)^{-1}.$$

All together, we get

$$\mu(C) \leqslant 2C\int_{\mathcal{V}} (1 - \Phi(\phi)) \exp\left(-\frac{r^2(\phi, A\phi)}{4\ln(2)}\right) d\phi.$$

Let $\epsilon > 0$ be given. From the continuity of Φ we get $n \in \mathbb{Z}$ and $\delta > 0$ such that from $\|\phi\|_{-n}^2 \leqslant \delta$ it follows that $|1 - \Phi(\phi)| \leqslant \epsilon$. On the other hand, we can in general estimate $|\Phi(\phi)| \leqslant \mathbb{E}[1] = 1$. In total, we get

$$\mu(C) \leqslant 2\epsilon + \tfrac{4C}{\delta}\int_{\mathcal{V}} \|\phi\|_{-n}^2 \exp\left(-\frac{r^2(\phi, A\phi)}{4\ln(2)}\right) df.$$

With some further estimates one can show that $\mu(C) \leqslant 6\epsilon$. Therefore the assumptions of Lemma 2.4.7 are fulfilled and μ is a regular probability measure on $\mathcal{S}'(\mathbb{R}^d)$. ∎

Lemma 2.4.13. *Let $\mathcal{H}$ be a Hilbert space and $\Phi : \mathcal{H} \to \mathbb{C}, \Phi(h) := \exp(-\frac{1}{2}(h,h)_{\mathcal{H}})$. Then Φ is continuous and positive semi-definite on $\mathcal{H}$ with $\Phi(0) = 1$.*

Proof. The continuity is clear as Φ is a composition of continuous functions. Furthermore $\Phi(0) = \exp(-\frac{1}{2}(0,0)_{\mathcal{H}}) = \exp(0) = 1$. So we only need to show that Φ is positive semi-definite. Let $n \in \mathbb{N}$, $h_1, ..., h_n \in \mathcal{H}$ and $c_1, ..., c_n \in \mathbb{C}$. We choose an orthonormal basis $e_1, ..., e_m$ of $span\{h_1, ..., h_n\}$. We want to use the characteristic function of the normal distribution. If $Z \in \mathbb{R}^m$ is standard normal distributed and $v \in \mathbb{R}^m$ then we know

$$\mathbb{E}[e^{ivZ}] = \exp\Big(-\tfrac{1}{2}\sum_{i=1}^{m} v_i^2\Big) = \exp(-\frac{1}{2}(v,v)) = \Phi\Big(\sum_{i=1}^{m} v_i e_i\Big).$$

We define for $i \in \{1, ..., n\} \quad x_i := ((h_i, e_1)_{\mathcal{H}}, ..., (h_i, e_m)_{\mathcal{H}}) \in \mathbb{R}^m$. Then we get

$$\begin{aligned}
\sum_{i,j=1}^{n} c_i\overline{c_j}\Phi(h_i - h_j) &= \sum_{i,j=1}^{n} c_i\overline{c_j}\Phi\Big(\sum_{k=1}^{m}(h_i - h_j, e_k)_{\mathcal{H}} e_k\Big) \\
&= \sum_{i,j=1}^{n} c_i\overline{c_j}\mathbb{E}\Big[\exp\Big(i\sum_{k=1}^{m}(h_i - h_j, e_k)_{\mathcal{H}} Z_k\Big)\Big] \\
&= \sum_{i,j=1}^{n} c_i\overline{c_j}\mathbb{E}\big[\exp\big(i(x_i - x_j)Z\big)\big] = \\
&\sum_{i,j=1}^{n} c_i\overline{c_j}\mathbb{E}\big[\exp\big(ix_i Z\big)\big]\overline{\mathbb{E}\big[\exp\big(ix_j Z\big)\big]} \\
&= \Big|\sum_{i=1}^{n} c_i\mathbb{E}\big[\exp(ix_i Z)\big]\Big|^2 \geqslant 0
\end{aligned}$$

and we see that Φ is positive semi-definite. ∎

We are now ready to define the standard normal distribution on $\mathcal{S}'(\mathbb{R}^d)$.

Definition 2.4.14 (White noise)**.** *Let $\Phi : \mathcal{H} \to \mathbb{C}, \quad \Phi(h) := \exp(-\frac{1}{2}(h,h)_{\mathcal{H}})$. Then we define W as the unique probability measure on $\mathcal{S}'(\mathbb{R}^d)$ such that Φ is its characteristic function. We call W white noise on $\mathcal{S}'(\mathbb{R}^d)$.*

Remark 2.4.15. From the definition of white noise we have for $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathcal{S}'(\mathbb{R}^d)} \exp\Big(i(f,\phi)_{L^2(\mathbb{R}^d)}\Big)dW(f) = \exp\Big(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^d)}\Big).$$

Therefore we can interpret W as a random Gaussian process $(W,\phi)_{\phi\in\mathcal{S}(\mathbb{R}^d)}$ with $(W,\phi) \sim N(0, \|\phi\|^2_{L^2(\mathbb{R}^d)})$. As W is a random element in the topological dual space, the process $W(\phi)_{\phi\in\mathcal{S}'(\mathbb{R}^d)}$ is almost surely continuous.

We would like to find an extension of the process $W(\phi)_{\phi\in\mathcal{S}'(\mathbb{R}^d)}$ to all functions of $L^2(\mathbb{R}^d)$. In order to do that we introduce the term of a Gaussian Hilbert space.

Definition 2.4.16 (Gaussian Hilbert space)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Consider a collection of Gaussian random variables denoted by $\mathcal{H}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and equip it with the inner product $(X,Y)_{\mathcal{H}} := \mathbb{E}[XY]$ for $X, Y \in \mathcal{H}$. Then $\mathcal{H}$ is called a Gaussian Hilbert space if it is closed with respect to the norm $\|X\|_{\mathcal{H}} := (X,X)^{1/2}_{\mathcal{H}}$.*

Let W be a white noise on $\mathcal{S}(\mathbb{R}^d)$ and consider the collection of Gaussian random variables $\{(W,\phi) : \phi \in \mathcal{S}(\mathbb{R}^d)\}$ that are all defined on that one probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the random element W lives on and equip it with the inner product of the definition above. Define $\iota : \mathcal{S}(\mathbb{R}^d) \to \{(W,\phi) : \phi \in \mathcal{S}(\mathbb{R}^d)\}, \phi \mapsto (W,\phi)$. Then ι is an isometry, because, according to Remark 2.4.15, it holds that $\|\iota(\phi)\|_{\mathcal{H}} = \mathbb{E}[(W,\phi)^2] = \|\phi\|_{L^2(\mathbb{R}^d)}$. Therefore, we can extend ι in the following way: Let $f \in L^2(\mathbb{R}^d)$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}(\mathbb{R}^d)$ such that $f_n \to f$ as $n \to \infty$ in $L^2(\mathbb{R}^d)$. Then we define $\iota(f) := \lim_{n\to\infty}(W, f_n)$. We have to show that this definition is well defined. From the dominated convergence theorem we get that for all $\xi \in \mathbb{R}$

$$\lim_{n\to\infty} \mathbb{E}[e^{i\xi(W,f_n)}] = \exp\left(-\frac{1}{2}\xi\|f_n\|^2_{L^2(\mathbb{R}^d)}\right) \longrightarrow \exp\left(-\frac{1}{2}\xi\|f\|^2_{L^2(\mathbb{R}^d)}\right).$$

Thus, there exists an almost surely unique random variable (W,f) on $L^2(\Omega,\mathcal{F},\mathbb{P})$ such that (W,f_n) converges in probability to (W,f) and $(W,f) \sim N(0, \|f\|^2_{L^2(\mathbb{R}^d)})$. Furthermore, the random variable (W,f) is independent of the choice of the sequence $(f_n)_{n\in\mathbb{N}}$. Let now $\mathcal{H}$ be $\iota(L^2(\mathbb{R}^d))$. So we get that $\iota : L^2(\mathbb{R}^d) \to \mathcal{H}$ is an isometry. As $L^2(\mathbb{R}^d)$ is complete, also $\mathcal{H}$ is complete and therefore a Gaussian Hilbert space. As the limit is linear, $\mathcal{H}$ inherits the linear structure of $\{(W,\phi) : \phi \in \mathcal{S}(\mathbb{R}^d)\}$. Here one has to be careful, as point-wise the two elements $(W, f+g)$ and $(W,f) + (W,g)$ for $f, g \in L^2(\mathbb{R}^d)$ do not necessarily coincide. However, in the Hilbert space $\mathcal{H}$ they represent the same element as the convergence in the norm of $\mathcal{H}$ is the convergence in distribution on $L^2(\Omega,\mathcal{F},\mathbb{P})$. Furthermore, from the fact that ι is an isometry we directly get the following property

$$\begin{aligned} Cov\Big((W,f),(W,g)\Big) &= \mathbb{E}[(W,f)(W,g)] \\ &= \frac{1}{4}\Big(\mathbb{E}[(W,f+g)^2] - \mathbb{E}[(W,f-g)^2]\Big) \\ &= \frac{1}{4}\Big(\|f+g\|^2_{L^2(\mathbb{R}^d)} - \|f-g\|^2_{L^2(\mathbb{R}^d)}\Big) \\ &= (f,g)_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{2.3}$$

Definition 2.4.17 (White noise Gaussian Hilbert space)**.** *The Gaussian Hilbert space $\mathcal{H}$ we constructed above is called the white noise Gaussian Hilbert space of $\mathbb{R}^d$.*

One important fact is that the so constructed white noise Hilbert space seen as a stochastic process $(W,f)_{f\in L^2(\mathbb{R}^d)}$ is not continuous any more.

2.5 The Fractional Gaussian Field on $\mathbb{R}^d$

In this section we want to define the fractional Gaussian field with parameter $s \in \mathbb{R}$ on $\mathbb{R}^d$. For this purpose we would like to carry out the same procedure as in the last subsection with the white noise and try to define a standard Gaussian random variable on the space

$\dot{H}^s(\mathbb{R}^d)$ defined in subsection 2.2. Again, as this space is an infinite dimensional Hilbert space, we need to define the random element on the topological dual space. This section follows the ideas of chapter 3 of [LSSW16].

Using the ideas of the construction from the white noise, our goal is a random element h of $\mathcal{S}'_H(\mathbb{R}^d)$ such that for all $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ it holds that

$$(h,\phi)_{\dot{H}^s(\mathbb{R}^d)} \sim N\Big(0, \|\phi\|^2_{\dot{H}^s(\mathbb{R}^d)}\Big). \tag{2.4}$$

We would like to write out this condition in terms of the L^2-scalar product. We compute for $h, \phi \in \mathcal{S}_H(\mathbb{R}^d)$

$$(h,\phi)_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{h}(\xi)\hat{\phi}(\xi)|\xi|^{2s} d\xi = \int_{\mathbb{R}^d} \hat{h}(\xi)\widehat{(-\Delta)^s\phi}(\xi) d\xi = (h, (-\Delta)^s\phi)_{L^2(\mathbb{R}^d)}.$$

Now it follows, with the desired property 2.4,

$$\begin{aligned}
\mathbb{E}\Big[(h,\phi)^2_{L^2(\mathbb{R}^d)}\Big] &= \mathbb{E}\Big[(h,(-\Delta)^{-s}\phi)^2_{\dot{H}^s(\mathbb{R}^d)}\Big] \\
&= \|(-\Delta)^{-s}\phi\|^2_{\dot{H}^s(\mathbb{R}^d)} \\
&= \left(\int_{\mathbb{R}^d} |\xi|^{2s}|\xi|^{-4s}\hat{\phi}(\xi)d\xi\right)^2 \\
&= \left(\int_{\mathbb{R}^d} |\xi|^{-2s}\hat{\phi}(\xi)d\xi\right)^2 = \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}.
\end{aligned}$$

So if we interpret an element $h \in \mathcal{S}_H(\mathbb{R}^d)$ via ι from Remark 2.1.6 as an element of $\mathcal{S}'_H(\mathbb{R}^d)$, it would make sense to demand for all $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ that

$$\mathbb{E}\Big[(h,\phi)_{L^2(\mathbb{R}^d)}\Big] = \|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)}.$$

Therefore, we define the fractional Gaussian field in the following way.

Definition 2.5.1 (Fractional Gaussian Field on $\mathbb{R}^d$)**.** *For $s \in \mathbb{R}$ the factional Gaussian field is defined as a random element h of $\mathcal{S}'_H(\mathbb{R}^d)$ such that for all $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ holds that*

$$(h,\phi)_{L^2(\mathbb{R}^d)} \sim N\Big(0, \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\Big),$$

where $H = s - \frac{d}{2}$ denotes the Hurst-parameter. We will write $h \sim FGF_s(\mathbb{R}^d)$.

Proposition 2.5.2 (Scaling property)**.** *For $s \in \mathbb{R}$ and $h \sim FGF_s(\mathbb{R}^d)$ the scaling property holds. Let $a > 0$ and consider the random element $h_a(\phi) := h(\phi(a\cdot))$, where $\phi(a\cdot) : \xi \mapsto \phi(a\xi)$ is the scaled Schwartz function. Then it holds that*

$$h_a \overset{d}{=} a^{s-\frac{d}{2}} h.$$

Proof. For $a > 0$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$, using the transformation formula, we calculate

$$\begin{aligned}
\|\phi(a\cdot)\|^2_{\dot{H}^{-s}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |\xi|^{-2s} \hat{\phi}(a\xi) d\xi \\
&= \int_{\mathbb{R}^d} |a\xi|^{-2s} a^{2s} \hat{\phi}(a\xi) d\xi \\
&= a^{2s-d} \int_{\mathbb{R}^d} |\xi|^{-2s} \hat{\phi}(\xi) d\xi \\
&= a^{2s-d} \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}.
\end{aligned}$$

Thus, we get

$$(h_a, \phi) = (h, \phi(a\cdot)) \sim N\Big(0, \underbrace{\|\phi(a\cdot)\|_{\dot{H}^{-s}(\mathbb{R}^d)}}_{=a^{2s-d}\|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}}\Big)$$

and the result follows. ■

The next step is now to prove the existence of $FGF_s(\mathbb{R}^d)$. The tool we will use for that is again the Bochner-Minlos theorem. We would like to apply it on the same functional as for the white noise just on the Hilbert space $\dot{H}^{-s}(\mathbb{R}^d)$. The problem is that in general this functional is not finite for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, but only for functions in $\mathcal{S}_H(\mathbb{R}^d)$. The idea is now to change the functional only on the set $\phi \in \mathcal{S}(\mathbb{R}^d) \backslash \mathcal{S}_H(\mathbb{R}^d)$ such that it fulfills all requirements for the Bochner-Minlos theorem.

Proposition 2.5.3. *For all positive integers $n \in \mathbb{N}$ there exists a family of Schwartz functions $(\phi_\alpha)_{|\alpha| \leqslant n}$ such that for all multi-indices $|\alpha|, |\beta| \leqslant n$ it holds that*

$$\int_{\mathbb{R}^d} m_\alpha(x) \phi_\beta(x) dx = \delta_{\{\alpha=\beta\}}$$

where $\delta_{\{\alpha=\beta\}}$ is 1 if and only if $\alpha = \beta$ and 0 else.

Idea of the proof. We will prove the case $d = 1$. For that purpose we use the Hermite basis $(\psi_n)_{n\in\mathbb{N}}$ given in Remark 2.4.3. Clearly, it is contained in the Schwartz space $\mathcal{S}(\mathbb{R})$. Moreover, it is easy to see that every ψ_n can be written as

$$\psi_n(x) = e^{-\frac{x^2}{2}} P_n(x),$$

where P_n is some polynomial of degree n. As ϕ_n, ϕ_m are orthogonal for $n \neq m$, this is also true for P_n, P_m if $n \neq m$. Thus, we have that

$$span\{\psi_0, ..., \psi_n\} = span\{e^{-x^2/2}, e^{-x^2/2}x, ..., e^{-x^2/2}x^n\}.$$

As those two linear subspaces are finite dimensional, there exists for every $m \in \{1, ..., n\}$ a linear combination of Hermite functions, such that for all $l \in \{1, ..., n\}$

$$\int_{\mathbb{R}} \underbrace{\sum_{k=0}^{n} a_k \psi_k e^{-x^2/2}}_{=:\phi_m(x)} x^l dx = \delta_{\{m=l\}}.$$

∎

Lemma 2.5.4. *Using the family $(\phi_\alpha)_{|\alpha| \leqslant \lfloor H \rfloor}$ which we get from Proposition 2.5.3, we define*

$$P_s : \begin{cases} \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}_H(\mathbb{R}^d) \\ \phi \mapsto \phi - \sum\limits_{|\alpha| \leqslant \lfloor H \rfloor} \phi_\alpha \int_{\mathbb{R}^d} m_\alpha(x)\phi(x)dx \end{cases}.$$

Then P_s is a linear projection from $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}_H(\mathbb{R}^d)$ and therefore continuous.

Proof. P_s is clearly linear. First we show that it is also idempotent. We calculate

$$\begin{aligned} P_s \circ P_s(\phi) = \phi - &\sum_{|\alpha| \leqslant \lfloor H \rfloor} \phi_\alpha \int_{\mathbb{R}^d} m_\alpha(x)\phi(x)dx \\ &- \sum_{|\alpha| \leqslant \lfloor H \rfloor} \phi_\alpha \int_{\mathbb{R}^d} m_\alpha(x) \left(\phi - \sum_{|\beta| \leqslant \lfloor H \rfloor} \phi_\beta \int_{\mathbb{R}^d} m_\beta(x)\phi(x)dx \right) dx \\ &= \phi - 2 \sum_{|\alpha| \leqslant \lfloor H \rfloor} \phi_\alpha \int_{\mathbb{R}^d} m_\alpha(x)\phi(x)dx \\ &+ \sum_{|\alpha| \leqslant \lfloor H \rfloor} \sum_{|\beta| \leqslant \lfloor H \rfloor} \phi_\alpha \underbrace{\int_{\mathbb{R}^d} m_\alpha(x)\phi_\beta(x)dx}_{=\delta_{\{\alpha=\beta\}}} \left(\int_{\mathbb{R}^d} m_\beta(y)\phi(y)dy \right) \\ &= \phi - 2 \sum_{|\alpha| \leqslant \lfloor H \rfloor} \phi_\alpha \int_{\mathbb{R}^d} m_\alpha(x)\phi(x)dx + \sum_{|\alpha| \leqslant \lfloor H \rfloor} \phi_\alpha \int_{\mathbb{R}^d} m_\alpha(x)\phi(x)dx = P_s(\phi). \end{aligned}$$

Furthermore, for all $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ we get per definition

$$P_s(\phi) = \phi - \sum_{|\alpha| \leqslant \lfloor H \rfloor} \phi_\alpha \underbrace{\int_{\mathbb{R}^d} m_\alpha(x)\phi(x)dx}_{=0} = \phi.$$

Finally, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $|\alpha| \leqslant \lfloor H \rfloor$ we get

$$\int_{\mathbb{R}^d} P_s(\phi)(x) m_\alpha(x)dx = \int_{\mathbb{R}^d} \left(\phi(x) - \sum_{|\beta| \leqslant \lfloor H \rfloor} \phi_\beta(x) \int_{\mathbb{R}^d} m_\alpha(y)\phi(y)dy \right) m_\alpha(x)dx$$

$$= \int_{\mathbb{R}^d} \phi(x) m_\alpha(x) dx - \sum_{|\beta| \leq \lfloor H \rfloor} \underbrace{\int_{\mathbb{R}^d} \phi_\beta(x) m_\alpha(x) dx}_{\delta_{\{\alpha=\beta\}}} \left(\int_{\mathbb{R}^d} m_\beta(y) \phi(y) dy \right) = 0.$$

In conclusion, P_s is a projection from $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}_H(\mathbb{R}^d)$. ■

Now we can define our updated functional.

Lemma 2.5.5. *Let* $\Phi_s : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$, $\quad \Phi_s(\phi) := \exp\left(-\frac{1}{2}\|P_s(\phi)\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\right)$. *Then* Φ_s *is continuous, positive semi-definite,* $\Phi_s(0) = 1$ *and for* $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ *it holds that* $\Phi_s(\phi) = \exp(-\frac{1}{2}\|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)})$.

Proof. As P_s is continuous, Φ_s also is continuous. Clearly, $\Phi_s(0) = \exp(0) = 1$. In an analogue way to Lemma 2.4.13, we get, with the linearity of P_s, that Φ_s is positive semi-definite. Finally, for all $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ we get $\Phi_s(\phi) = \exp(-\frac{1}{2}\|P_s(\phi)\|_{\dot{H}^{-s}(\mathbb{R}^d)}) = \exp(-\frac{1}{2}\|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)})$. ■

Now we can again use the Bochner-Minlos theorem to define the fractional Gaussian field with parameter $s \in \mathbb{R}$. Therefore we get the existence of the $FGF_s(\mathbb{R}^d)$.

Remark 2.5.6. Clearly the definition of P_s is dependent on the choice of the family of Schwartz functions $(\phi_\alpha)_{|\alpha| \leq \lfloor H \rfloor}$ and therefore also Φ_s is. Therefore we restrict the $FGF_s(\mathbb{R}^d)$ to the space $\mathcal{S}_H(\mathbb{R}^d)$, where Φ_s coincides with the functional $\phi \mapsto \exp(-\frac{1}{2}\|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)})$.

Remark 2.5.7. Analogue to the white noise Gaussian Hilbert space we want to define a Gaussian Hilbert space to enlarge the domain of the $FGF_s(\mathbb{R}^d)$. The procedure is the same as before. Let $h \sim FGF_s(\mathbb{R}^d)$ and consider $\mathcal{S}_H(\mathbb{R}^d)$ as a subset of $\dot{H}^{-s}(\mathbb{R}^d)$. Define the isometry $\iota : \mathcal{S}_H(\mathbb{R}^d) \to \{(h, \phi) : \phi \in \mathcal{S}_H(\mathbb{R}^d)\} \subseteq L^2(\Omega), \phi \mapsto (h, \phi)$. Indeed $\|\iota(\phi)\|^2_{\mathcal{H}} = \mathbb{E}[(h, \phi)^2] = \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}$. As $L^2(\Omega)$ is complete we can extend the domain of ι to the closure of $\mathcal{S}_H(\mathbb{R}^d) \subseteq \dot{H}^{-s}(\mathbb{R}^d)$ which we denote with $T_s(\mathbb{R}^d)$. In an analogue way we get that this procedure is well defined and that for $\phi \in T_s(\mathbb{R}^d)$ it holds that $(h, \phi) \sim N(0, \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)})$. Again, we lose the continuity of h.

Lemma 2.5.8. *Let* $s \in \mathbb{R}$ *and* W *be a white noise on* $\mathbb{R}^d$.*Then there exists* $h \sim FGF_s(\mathbb{R}^d)$ *such that* $h = (\Delta)^{-s/2} W$.

Proof. Let us define a random element h on $T_s'(\mathbb{R}^d)$. For $\phi \in T_s(\mathbb{R}^d) \subseteq \dot{H}^{-s}(\mathbb{R}^d)$ we get from Remark 2.3.6 that $(-\Delta)^{-s/2}\phi \in \dot{H}^0(\mathbb{R}^d) = \{f \in \mathcal{S}(\mathbb{R}^d) : \hat{f} \in L^2(\mathbb{R}^d)\} = \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{S}'_{-d/2}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d)$. As the white noise Hilbert space lives on $L^2(\mathbb{R}^d)$, we can define a random element h on $T_s(\mathbb{R}^d)'$ in the following way $(h, \phi) := (W, (-\Delta)^{-s/2}\phi)$. This is well defined and therefore the expression $h = (-\Delta)^{-s/2}W$ makes sense. We have to show that $h \sim FGF_s(\mathbb{R}^d)$. As $(-\Delta)^{-s/2} : \dot{H}^{-s}(\mathbb{R}^d) \to \dot{H}^0(\mathbb{R}^d)$ is an isometry we get

$$\begin{aligned}\mathbb{E}[(h,\phi)^2] &= \mathbb{E}[(W, (-\Delta)^{-s/2}h)^2] \\ &= \|(-\Delta)^{-s/2}\phi\|^2_{L^2(\mathbb{R}^d)} \\ &= \|(-\Delta)^{-s/2}\phi\|^2_{\dot{H}^0(\mathbb{R}^d)} = \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\end{aligned}$$

so $h \sim FGF_s(\mathbb{R}^d)$. ■

2.6 The covariance kernel of $FGF_s(\mathbb{R}^d)$

It is a well known fact that every centered Gaussian Process is completely determined by its covariance structure. As all $FGF_s(\mathbb{R}^d)$ are centered Gaussian processes, this leads to the question of how their covariance structure looks like. First, we introduce the term of a covariance kernel. This section follows chapter 3.2 in [LSSW16].

Definition 2.6.1. *Let $(X_f)_{f\in\mathcal{H}}$ be a centered Gaussian process on a function space $\mathcal{H}$ with $f : \mathbb{R}^d \to \mathbb{R}$. If there exists a function $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, such that for all $f, g \in \mathcal{H}$ it holds that*

$$Cov(X_f, X_g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y)f(x)g(y)dxdy,$$

we call G the covariance kernel of $(X_f)_{f\in\mathcal{H}}$.

Remark 2.6.2. Clearly, a covariance kernel is symmetric, i.e. $G(x, y) = G(y, x)$ for all $x, y \in \mathbb{R}^d$ and it determines the Gaussian process uniquely. But for a centered Gaussian process there can be multiple covariance kernels.

Remark 2.6.3. For the white noise we can directly derive the covariance kernel from equation 2.3

$$Cov\Big((W, f), (W, g)\Big) = (f, g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \delta_{\{x=y\}} f(x)g(y)dxdy.$$

Therefore the covariance kernel of the white noise is $G(x, y) = \delta_{\{x=y\}}$.

We now compute the covariance kernel of the fractional Gaussian field of $\mathbb{R}^d$ in all other cases.

Theorem 2.6.4. *For $s \in \mathbb{R}$ the covariance kernel of the fractional Gaussian field $h \sim FGF_s(\mathbb{R}^d)$, which we denote by $G_s(\cdot, \cdot)$, has the following structure:*

a) *For $s > 0$ and the Hurst parameter H not being a nonnegative integer, it holds that*

$$G_s(x,y) = C(s,d)|x-y|^{2H} \text{ with } C(s,d) = \frac{2^{-2s}\pi^{-d/2}\Gamma(\frac{d}{2}-s)}{\Gamma(s)}.$$

Note that in some cases the constant $C(s,d)$ is negative.

b) *For $s > 0$ and the Hurst parameter H being a nonnegative integer, it holds*

$$G_s(x,y) = 2c_{-1}^{d/2+H}|x-y|^{2H} log|x-y|$$

where $c_{-1}^{d/2+H}$ is the residue of the function $s \mapsto C(s,d)$ at $\frac{d}{2}+H$

$$c_{-1}^{d/2+H} = \frac{(-1)^{H+1}2^{-2H-d}\pi^{-d/2}}{H!\Gamma(\frac{d}{2}+H)}.$$

Again the constant $c_{-1}^{d/2+H}$ is in some cases negative.

c) *For $s < 0$ not being a negative integer, it holds that*

$$G_s(x,y) = C(s,d)|x-y|^{2H}\left(1 - \sum_{i=0}^{\lfloor -s \rfloor} |x-y|^{2i} H_i \Delta^i \delta_{\{x=y\}}\right),$$

where, $|S^{d-1}|$ being the surface of the unit sphere in $\mathbb{R}^d$ and the empty product is defined to be 1, the constant H_i is given by

$$H_i := \frac{|S^{d-1}|}{2^i i!}\left(\prod_{k=0}^{i-1} d+2k\right)^{-1}$$

d) *For $s < 0$ being a negative integer, it holds that*

$$G_s(x,y) = (-\Delta)^s \delta_{\{x=y\}}.$$

Idea of the Proof. As for $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ and $h \sim FGF_s(\mathbb{R}^d)$ it holds that

$$(h,\phi) \sim N\Big(0, \|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\Big),$$

so we get for $\phi_1, \phi_2 \in \mathcal{C}_H(\mathbb{R}^d)$ that

$$\begin{aligned} Cov\Big((h,\phi_1)(h,\phi_2)\Big) &= \frac{1}{4}\mathbb{E}[(h,\phi_1+\phi_2)^2 - (h,\phi_1-\phi_2)^2] \\ &= (\phi_1,\phi_2)_{\dot{H}^{-s}(\mathbb{R}^d)}. \end{aligned}$$

Ad a) First we assume that $0 < s < \frac{d}{2}$. Then $|\xi|^{-2s}$ is a tempered distribution and it holds that (chapter 1 §1 [LD72])

$$\mathcal{F}^{-1}(|\xi|^{-2s}) = C(s,d)|x|^{2H}.$$

Now it follows that

$$\begin{aligned}
Cov\Big((h,\phi_1),(h,\phi_2)\Big) &= (\phi_1,\phi_2)_{\dot{H}^{-s}(\mathbb{R}^d)} \\
&= \int_{\mathbb{R}^d} \hat{\phi}_1(\xi)\hat{\phi}_2(\xi)|\xi|^{-2s}d\xi \\
&= (\hat{\phi}_1|xi|^{-2s},\hat{\phi}_2)_{L^2(\mathbb{R}^d)} \\
&= (\mathcal{F}^{-1}(|\xi|-2s) * \phi_1,\phi_2)_{L^2(\mathbb{R}^d)} \\
&= C(s,d)\int_{\mathbb{R}^d}(|x|^{2H} * \phi_1)(y)\phi_2(y)dy \\
&= C(s,d)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^{2H}\phi_1(x)\phi_2(y)dxdy.
\end{aligned}$$

For $s \geqslant \frac{d}{2}$ we need to make an argument. Let

$$\psi : \begin{cases} \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R} \\ \psi(x,s) := \int_{\mathbb{R}^d}|x-y|^{2H}\phi_2(y)dy \end{cases}.$$

As in this case $H \geqslant 0$ and ψ is a Schwartz function ψ is well defined. With dominated convergence, it follows that ψ is smooth in x and analytic in s. By an analytic argument, the statement follows (see Chapter 1 §1 in [LD72]).

Ad b) As $\phi_1,\phi_2 \in \mathcal{S}_H(\mathbb{R}^d)$ we get for $y \in \mathbb{R}^d$ that

$$\int_{\mathbb{R}^d}|x-y|^{2H}\phi_i^2(x)dx = 0 \qquad \text{for} \qquad i \in \{1,2\}. \tag{2.5}$$

We want to use a) and a limiting argument. Let $t \in (s-\frac{1}{2}, s-\frac{1}{2})$ and $t \neq s$. Then it holds with 2.5 that

$$\begin{aligned}
(\phi_1,\phi_2)_{\dot{H}^{-t}(\mathbb{R}^d)} &\overset{a)}{=} C(t,d)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^{2t-d}\phi_1(x)\phi_2(y)dxdy \\
&\overset{2.5}{=} C(t,d)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(|x-y|^{2t-d}-|x-y|^{2s-d})\phi_1(x)\phi_2(y)dxdy.
\end{aligned}$$

With multidimensional Taylor we get that

$$|x-y|^{2t-d}-|x-y|^{2s-d} = 2(t-s)|x-y|^{2s-d}ln|x-y| + \mathcal{O}\Big(((t-s)|x-y|^{2s-d}ln|x-y|)^2\Big).$$

Furthermore, for $t \to s$ we see that $(t-s)C(t,d) \to c_{-1}^{d/2+H}$, which is the residue of the function $s \mapsto C(s,d)$ at $\frac{d}{2}+H$. Putting all together we get

$$(\phi_1,\phi_2)_{\dot{H}^{-s}(\mathbb{R}^d)} = 2c_{-1}^{d/2+H}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^{2H}log|x-y|\phi_1(x)\phi_2(y)dxdy$$

and the statement follows. Statements c) and d) are following from equation (1.1.10) in [LD72]. ■

2.7 The Fractional Gaussian Field on a domain

The idea of this section is to define the fractional Gaussian field on proper subsets D of $\mathbb{R}^d$. For that purpose, not all subsets of $\mathbb{R}^d$ are suitable. We therefore introduce the term of an allowable domain. Furthermore there is another distinction to make. We have to choose how the fractional Gaussian field should behave like on the boundary of our domain D. Here we define the fractional Gaussian field with zero boundary conditions. First we need to define an appropriate space, denoted by $\dot{H}_0^s(D)$, where we define the fractional Gaussian field with zero boundary conditions on. This section follows Chapter 4 of [LSSW16].

Definition 2.7.1 (Allowable domain)**.** *Let $D \subseteq \mathbb{R}^d$ and $s \geqslant 0$. Then D is called an allowable domain if for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ there exists a constant C that depends on D and ϕ such that for all $g \in C_c^\infty(D)$ it holds that*

$$|(\phi, g)_{L^2(\mathbb{R}^d)}| \leqslant C\|g\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Remark 2.7.2. $C_c^\infty(D)$ denotes the set of the infinitely often differentiable functions with compact support in D, so we have $C_c^\infty(D) \subseteq \mathcal{S}(\mathbb{R}^d) \subseteq \dot{H}^s$.

The following Lemma gives a better perspective on when a domain D is allowable.

Lemma 2.7.3. *Let $s \geqslant 0$, $D \subseteq \mathbb{R}^d$ and $H = s - \frac{d}{2}$ be the Hurst parameter. If $\mathbb{R}^d \backslash D$ contains an open set, then D is an allowable domain.*

Proof. Let $s \geqslant 0$, $D \subseteq \mathbb{R}^d$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $g \in C_c^\infty(D)$. Using Proposition 2.1.5 c) and Cauchy Schwarz we get

$$\begin{aligned}
|(\phi, g)_{L^2(\mathbb{R}^d)}| &= \left|\int_{\mathbb{R}^d} \hat{\phi}(\xi)\hat{g}(\xi)|\xi|^{-s}|\xi|^s d\xi\right| \\
&\leqslant \left(\int_{\mathbb{R}^d} \hat{\phi}(\xi)|\xi|^{-s}\right)\left(\int_{\mathbb{R}^d} \hat{g}(\xi)|\xi|^s d\xi\right) \\
&= \|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)}\|g\|_{\dot{H}^s(\mathbb{R}^d)}.
\end{aligned}$$

If $0 \leqslant s < \frac{d}{2}$ i.e. $-\frac{d}{2} \leqslant H < 0$, then both norms are finite, we can set $C(D, \phi) := \|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)}$ and D is allowable. If $s \geqslant \frac{d}{2}$ then it is not necessarily true, that $\|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)}$ is finite. We therefore need to change our argumentation a bit. Assume that $\mathbb{R}^d \backslash D$ contains an open set. Then we find an open ball $B \subseteq \mathbb{R}^d \backslash D$. Using a Gram Schmidt procedure we find a function $\psi \in C_c^\infty(B)$ such that for all multi-indices $|\alpha| \leqslant H$ we get

$$\int_{\mathbb{R}^d} m_\alpha(\xi)\psi(\xi)d\xi = \int_{\mathbb{R}^d} m_\alpha(\xi)\phi(\xi)d\xi.$$

Thus, we have that $D^\alpha(\hat{\phi} - \hat{\psi})(0) = 0$ for all $|\alpha| \leqslant H$ and it holds that

$$\|\phi - \psi\|_{\dot{H}^{-s}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{\phi}(\xi) - \hat{\psi}(\xi)|^2 |\xi|^{-2s} d\xi < \infty.$$

Then we conclude, using Cauchy Schwarz as above, that

$$(\phi, g)_{L^2(\mathbb{R}^d)} = (\phi - \psi, g)_{L^2(\mathbb{R}^d)} \leqslant \|\phi - \psi\|_{\dot{H}^{-s}(\mathbb{R}^d)} \|g\|_{\dot{H}^s(\mathbb{R}d)} < \infty$$

and can choose the constant $C(D, \phi) := \|\phi - \psi\|_{\dot{H}^{-s}(\mathbb{R}^d)}$. ∎

Let us now define the space $\dot{H}_0^s(D)$.

Definition 2.7.4 (The space $\dot{H}_0^s(D)$)**.** *Let $s \geqslant 0$ and $D \subseteq \mathbb{R}^d$ be an allowable domain. Then we define $\dot{H}_0^s(D)$ to be the completion of $C_c^\infty(D) \subseteq \dot{H}^s(\mathbb{R}^d)$, which is a Hilbert space by itself, equipped with the inner product of $\dot{H}^s(\mathbb{R}^d)$.*

Remark 2.7.5. We consider for $\phi \in \mathcal{S}(\mathbb{R}^d)$ the linear functional on $\dot{H}_0^s(D)$ given by $g \mapsto (\phi, g)_{L^2(\mathbb{R}^d)}$. From the definition of $\dot{H}_0^s(D)$ and D being an allowable domain, we get, that this functional is continuous. Therefore, we can find with the Riesz representation theorem a unique element $f \in \dot{H}_0^s(D)$ such that for all $g \in \dot{H}_0^s(D)$ we have

$$(\phi, g)_{L^2(\mathbb{R}^d)} = (f, g)_{\dot{H}_0^s(D)}. \tag{2.6}$$

By the definition of the fractional Laplacian operator on the topological dual space in Remark 2.3.6 and with Proposition 2.1.5 c) we get that for all $g \in \dot{H}_0^s(D)$

$$\begin{aligned}
((-\Delta)^s f, g)_{L^2(\mathbb{R}^d)} &= (f, (-\Delta)^s g)_{L^2(\mathbb{R}^d)} \\
&= (\hat{f}, |\xi|^{2s} \hat{g})_{L^2(\mathbb{R}^d)} \\
&= (\hat{f}|\xi|^s, \hat{g}|\xi|^s)_{L^2(\mathbb{R}^d)} \\
&= (f, g)_{\dot{H}_0^s(D)} \\
&= (\phi, g)_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

With the injectivity of the fractional Laplacian, we get that f is the unique solution to the distributional equation

$$(-\Delta)^s f = \phi, \qquad f \in \dot{H}_0^s(D). \tag{2.7}$$

Definition 2.7.6 (The semi-norm $\|\cdot\|_{\dot{H}^{-s}(D)}$)**.** *For $s > 0$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$ we choose the unique solution f of the distributional equation 2.7 and define the map*

$$\|\cdot\|_{\dot{H}^{-s}(D)} : \begin{cases} \mathcal{S}(\mathbb{R}^d) \to \mathbb{R} \\ \phi \mapsto \|f\|_{\dot{H}^s(\mathbb{R}^d)} \end{cases},$$

which is a semi-norm on $\mathcal{S}(\mathbb{R}^d)$. Indeed as the distributional equation is linear in its argument, we get for $r \in \mathbb{R}$, $\phi \in \mathcal{S}(\mathbb{R}^d)$ and its solution $f \in \dot{H}_0^s(D)$ that $\|r\phi\|_{\dot{H}^{-s}(D)} = \|rf\|_{\dot{H}^s(\mathbb{R}^d)} = |r| \cdot \|f\|_{\dot{H}^s(\mathbb{R}^d)} = |r| \cdot \|\phi\|_{\dot{H}^{-s}(D)}$. The triangle inequality for $\|\cdot\|_{\dot{H}^{-s}(D)}$ follows from the triangle inequality of $\|\cdot\|_{\dot{H}^s(\mathbb{R}^d)}$.

We consider for $s \geqslant 0$ the functional

$$\Phi_D^s : \begin{cases} \mathcal{S}(\mathbb{R}^d) \to \mathbb{R} \\ \phi \mapsto \exp\left(-\frac{1}{2}\|\phi\|^2_{\dot{H}^{-s}(D)}\right) \end{cases}.$$

Here we want to point out that the semi-norm $\|\cdot\|_{\dot{H}^{-s}(D)}$ is well defined for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ and not only for $\phi \in \mathcal{S}_H(\mathbb{R}^d)$ due to the allowability of the domain D. Thus, we can define the fractional Gaussian field on D as an element on $\mathcal{S}'(\mathbb{R}^d)$ and not only on $\mathcal{S}'_H(\mathbb{R}^d)$. We first show that the assumptions of Bochner-Minlos are given: Clearly, it holds that $\Phi_D^s(0) = 1$. The continuity follows from the continuity of the map $\mathcal{S}(\mathbb{R}^d), \phi \mapsto \|\phi\|_{\dot{H}^{-s}(D)}$, that is discussed in Lemma 2.8.5 in the next section. Next we show that Φ_D^s is positive semi-definite to be able to use the Bochner-Minlos theorem and define a random element on $\mathcal{S}(\mathbb{R}^d)$. From the uniqueness of the Riesz representation theorem, we get the linearity of the map that gives us the solution of the distributional equation 2.7 for given $\phi \in \mathcal{S}(\mathbb{R}^d)$. Therefore we get the semi-definiteness of Φ_D^s by Lemma 2.4.13. Now we can define:

Definition 2.7.7 (The $FGF_s(D)$ with zero boundary conditions)**.** *For $s \geqslant 0$ and an allowable domain D we define h_D to be the unique random element of $\dot{H}^s(\mathbb{R}^d)'$, restricted to $\mathcal{S}'(\mathbb{R}^d)$ with characteristic function Φ_D^s, given by the Bochner-Minlos theorem. Then h_D is called the fractional Gaussian field with parameter s on D with zero boundary conditions.*

Remark 2.7.8. Again for all $\phi \in \mathcal{S}(\mathbb{R}^d)$ we get that

$$(h_D, \phi) \sim N\left(0, \|\phi\|^2_{\dot{H}^{-s}(D)}\right).$$

For every $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $supp(\phi) \subseteq \mathbb{R}^d \backslash D$ we have that $(\phi, g)_{L^2(\mathbb{R}^d)} = 0$ for all $g \in \dot{H}^{-s}(D)$ and therefore the solution of 2.7 is the zero function in $\dot{H}_0^s(D)$. Therefore we have $\|\phi\|_{\dot{H}^{-s}(D)} = 0$ and $(h_D, \phi) = 0$ in distribution. Thus the fractional Gaussian field is supported on the closure of D. Furthermore, as the topology on $\mathcal{S}(\mathbb{R}^d)$ is finer as the one in $\dot{H}^s(\mathbb{R}^d)$, we get that h_D is a tempered distribution on $\mathcal{S}(\mathbb{R}^d)$. Moreover, there is again a way to extend the domain of the $FGF_s(D)$ by constructing a Gaussian Hilbert space in a similar way as we did for the $FGF_s(\mathbb{R}^d)$. The domain, we can expand the fractional Gaussian field on, is the same as the completion of $C_c^\infty(D)$ under the topology induced by $\|\cdot\|_{\dot{H}^{-s}(D)}$, which induces a metric on $\mathcal{S}(D)$ by the map $d(\phi, \psi) := \|\phi - \psi\|_{\dot{H}^{-s}(D)}$.

Example 2.7.9. The explicit form of the covariance kernel of the fractional Gaussian field on an allowable domain D is highly dependent of the structure of D and in many cases it is very hard or not yet possible to derive an explicit form. We want to give one example without proof here and consider the unit ball $B_1(0) := \{x \in \mathbb{R}^d : \|x\|_2 \leqslant 1\}$. In that case we can find an explicit way of expressing the solution f of the distributional equation 2.7 for given $\phi \in \mathcal{S}(\mathbb{R}^d)$. It holds that

$$f(x) = \int_{B_1(0)} G_B^s(x,y)\phi(y)dxdy, \qquad x \in B_1(0).$$

First we consider the case $s \in (0,1)$. In that case (see equation (2.65) in [GGS10]) the covariance kernel is of the form

$$G_B^s(x,y) = \frac{\Gamma(1+\frac{d}{2})}{d\pi^{d/2}4^{d-1}((s-1)!)^2}|x-y|^{2H}\int_1^{\frac{\left||x|y-\frac{x}{|x|}\right|}{|x-y|}}(z^2-1)^{s-1}z^{1-d}dz, \qquad x,y \in B_1(0).$$

In the case of s being an positive integer (see Corollary 4 in [BGR61]) we get

$$G_B^s(x,y) = \frac{\Gamma(\frac{d}{2})}{4^s\pi^{d/2}\Gamma(s)^2}|x-y|^{2H}\int_0^{\frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}}(z+1)^{-d/2}z^{s-1}dz, \qquad x,y \in B_1(0).$$

In our last case (see Corollary 4 in [BGR61]) let $s > 1$ and not being a positive integer. We decompose $(-\Delta)^s = (-\Delta)^{s-\lfloor s\rfloor}(-\Delta)^{\lfloor s\rfloor}$. Using the first two cases we can conclude

$$\begin{aligned}\phi(x) &= (-\Delta)^{\lfloor s\rfloor}\int_{B_1(0)} G_B^{\lfloor s\rfloor}(x,y)\phi(y)dy\\ &= (-\Delta)^s \underbrace{\int_{B_1(0)}\int_{B_1(0)} G_B^{s-\lfloor s\rfloor}(x,y)G_B^{\lfloor s\rfloor}(y,z)\phi(z)dzdy}_{=:g(x)}.\end{aligned}$$

Therefore g is a solution of the distributional equation 2.7 with parameter s. From the uniqueness of that problem we get that

$$G_B^s(x,y) = \int_{B_1(0)} G^{\lfloor s\rfloor}(x,z)G^{s-\lfloor s\rfloor}(z,y)dz.$$

The function $(x,y) \mapsto G_B^s(x,y)$ is already our covariance kernel of the $FGF_s(B_1(0))$. Indeed if we take $\phi \in C_c^\infty(D)$ we have

$$\begin{aligned}\mathbb{E}[(h_D,\phi)^2] &= \|\phi\|^2_{\dot{H}^{-s}(D)} = \|f\|^2_{\dot{H}^s(\mathbb{R}^d)}\\ &= (f,f)_{\dot{H}^s(\mathbb{R}^d)} = (\hat{f}|\xi|^s, \hat{f}|\xi|^s)_{L^2(\mathbb{R}^d)}\\ &= ((-\Delta)^s f, f)_{L^2(\mathbb{R}^d)} = (\phi, f)_{L^2(\mathbb{R}^d)}\\ &= \int_{\mathbb{R}d} f(x)\phi(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_B^s(x,y)\phi(x)\phi(y)dxdy.\end{aligned}$$

As this consideration was independent of $B_1(0)$, we conclude, given that f is in an integral form with a kernel, that this very kernel is already the covariance kernel of the fractional Gaussian field.

2.8 The Markov property

In this section we want to generalize the idea of the Markov property for processes with a time axis onto a multidimensional level. The main idea is the following: We split our domain $\mathbb{R}^d$ into two parts D and $\mathbb{R}^d \backslash D$. Then we want to find out what the fractional Gaussian field looks like when we condition it to be fixed in $\mathbb{R}^d \backslash D$. The approach is clearly inspired by studying the Brownian motion considering the sets $[0, r)$ and $[r, \infty)$ and then get the Markov property. For the fractional Gaussian field we still get interesting results. Here we do that in an very arbitrary setting, later in the special case of the Gaussian free field, we look at this idea more closely and get more results. This section follows section 5 in [LSSW16].

We first introduce the term of a s-harmonic function.

Definition 2.8.1 (s-harmonic function and $Har_s(D)$)**.** *Let $s > 0$ and $D \subseteq \mathbb{R}^d$. Then we call a function $f : \mathbb{R}^d \to \mathbb{R}$ s-harmonic on D if $((-\Delta)^s f)|_D = 0$. Furthermore, we define the space*

$$Har_s(D) := \left\{ f \in \dot{H}^s(\mathbb{R}^d) : ((-\Delta)^s f)|_D = 0 \right\}$$

of all in D s-harmonic functions of $\dot{H}^s(\mathbb{R}^d)$.

Definition 2.8.2 (s-harmonic extension)**.** *Let $s \in \mathbb{R}$, $D \subseteq \mathbb{R}^d$ a domain and $f : \mathbb{R}^d \backslash D \to \mathbb{R}$ a function. If a function $g : \mathbb{R}^d \to \mathbb{R}$ satisfies the two conditions*

$$f|_{\mathbb{R}^d \backslash D} = g|_{\mathbb{R}^d \backslash D}, \qquad ((-\Delta)^s g)|_D = 0$$

we call g the s-harmonic extension of f on D.

These terms can be extended point-wise onto random functions f, g and, because of Remark 2.3.6, also onto topological dual spaces. We want to split the space $\dot{H}^s(\mathbb{R}^d)$ into a direct sum of subspaces.

Lemma 2.8.3. *For an allowable domain $D \subseteq \mathbb{R}^d$ we have*

$$\dot{H}^s(\mathbb{R}^d) = Har_s(D) \oplus \dot{H}^s_0(D).$$

Proof. Let $f \in Har_s(D)$ and $g \in \dot{H}^s_0(D)$. Then we have

$$(f, g)_{\dot{H}^s(\mathbb{R}^d)} = (\hat{f}|\xi|^s, \hat{g}|\xi|^s)_{L^2(\mathbb{R}^d)} = ((-\Delta)^s f, g)_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (-\Delta)^s f(\xi) g(\xi) d\xi = 0$$

as $(-\Delta)^s f|_D = 0$ and $g = 0$ outside D. So $Har_s(D)$ and $\dot{H}_0^s(D)$ are orthogonal. Now let $f \in \dot{H}^s(\mathbb{R}^d)$ and consider the linear functional $\dot{H}_0^s(D) \to 0, g \mapsto (f, g)_{\dot{H}_0^s(D)} = ((-\Delta)^s f, g)_{L^2(\mathbb{R}^d)}$. From the allowability of D it follows that this functional is also continuous. With Riesz' representation theorem we get that there exists $f_D \in \dot{H}_0^s(D)$ such that $(f, g)_{\dot{H}_0^s(D)} = (f_D, g)_{\dot{H}_0^s(D)}$. Thus $((-\Delta)^s(f - f_D), g)_{L^2(\mathbb{R}^d)} = 0$ for all $g \in \dot{H}_0^s(D)$ so $(-\Delta)^s(f - f_D) = 0$ on D and $f - f_D \in Har_s(D)$. In particular we can write $f = f - f_D + f_D$ with $f - f_D \in Har_s(D)$ and $f_D \in \dot{H}_0^s(D)$. ■

Remark 2.8.4. Since $\dot{H}^s(\mathbb{R}^d)$ is the direct sum of $Har_s(D)$ and $\dot{H}_0^s(D)$ there exist orthogonal projections $P_D^{Har} : \dot{H}^s(\mathbb{R}^d) \to Har_s(D), f \mapsto f - f_D$ and $P_D : \dot{H}^s(\mathbb{R}^d) \to \dot{H}_0^s(D), f \mapsto f_D$. As every orthogonal projection is a contraction, they are continuous. Furthermore, the two subspaces are closed.

Lemma 2.8.5. *The semi-norm $\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}, \phi \mapsto \|\phi\|_{\dot{H}^{-s}(D)}$ defined in Definition 2.7.6 is continuous.*

Proof. With the projection P_D we can now solve the equation 2.7 explicitly. For $\phi \in \mathcal{S}(\mathbb{R}^d)$ we define $f := P_D(-\Delta)^{-s}\phi \in \dot{H}_0^s(D)$. Indeed for $g \in \dot{H}_0^s(D)$ we have

$$\begin{aligned}
(f, g)_{\dot{H}^s(\mathbb{R}^d)} &= (P_D(-\Delta)^{-s}\phi, g)_{\dot{H}^s(\mathbb{R}^d)} \\
&= ((-\Delta)^{-s}\phi, g)_{\dot{H}^s(\mathbb{R}^d)} \\
&= (\widehat{(-\Delta)^{-s}\phi}|\xi|^s, \hat{g}|\xi|^s)_{L^2(\mathbb{R}^d)} \\
&= (\hat{\phi}|\xi|^{s-2s}, \hat{g}|\xi|^s)_{L^2(\mathbb{R}^d)} \\
&= (\hat{\phi}, \hat{g})_{L^2(\mathbb{R}^d)} = (\phi, g)_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

As P_D is a projection, and further with Remark 2.3.6 we get

$$\|\phi\|_{\dot{H}^{-s}(D)} = \|P_D(-\Delta)^{-s}\phi\|_{\dot{H}^s(\mathbb{R}^d)} \leqslant \|(-\Delta)^{-s}\phi\|_{\dot{H}^s(\mathbb{R}^d)} = \|\phi\|_{\dot{H}^{-s}(\mathbb{R}^d)}$$

and therefore the continuity of the map $\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}, \phi \mapsto \|\phi\|_{\dot{H}^{-s}(D)}$. ■

Theorem 2.8.6 (The Markov property of $FGF_s(\mathbb{R}^d)$)**.** *Let $r \geqslant 0$ and $D \subseteq \mathbb{R}^d$ be an allowable domain. Then there exists a coupling of random elements (h, h_D, h_D^{har}) such that*

(i) $h \sim FGF_s(\mathbb{R}^d)$

(ii) $h_D \sim FGF_s(D)$

(iii) h_D^{Har} is a random element on $Har_s(D)$ independent of h_D.

(iv) $h = h_D + h_D^{Har}$ *almost surely.*

Furthermore h_D^{Har} *and* h_D *are both determined by* h.

Proof. First we define the three elements via the Bochner-Minlos theorem. Consider the functionals

$$\Phi_D^s : \begin{cases} \mathcal{S}(\mathbb{R}^d) \to \mathbb{R} \\ \phi \mapsto \exp\left(-\frac{1}{2}\|\phi\|^2_{\dot{H}^{-s}(D)}\right) \end{cases}, \quad \Phi_{D,har}^s \begin{cases} \mathcal{S}_H(\mathbb{R}^d) \to \mathbb{R} \\ \phi \mapsto \exp\left(-\frac{1}{2}\left\|P_D^{Har}(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)}\right) \end{cases}.$$

Clearly, both are linear and with Lemma 2.8.5 we get that Φ_D^s is continuous. With a similar argument we see that $\Phi_{D,Har}^s$ is also continuous. Lemma 2.4.13 shows that Φ_D^s and $\Phi_{D,Har}^s$ are positive semi-definite. Thus, with the Bochner-Minlos theorem we get two random elements h_D and h_D^{Har} respectively, that are clearly independent. As $P_D^{Har}(\phi) = 0$, on $\dot{H}_0^s(D)$ we get that h_D^{Har} is a random element on $Har_s(D)$. According to Definition 2.7.7 h_D is a fractional Gaussian field with zero boundary conditions on D. Furthermore we have

$$\begin{aligned}
\Phi_D^s(\phi)\Phi_{D,Har}^s(\phi) &= \exp\left(-\frac{1}{2}\|\phi\|^2_{\dot{H}^s(D)} - \frac{1}{2}\left\|P_D^{Har}(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)}\right) \\
&= \exp\left(-\frac{1}{2}\left(\left\|P_D(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)} + \left\|P_D^{Har}(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)}\right)\right) \\
&= \exp\Bigg(-\frac{1}{2}\Bigg(\left\|P_D(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)} \\
&\quad + \underbrace{2\left(P_D(-\Delta)^{-s}\phi, P_D^{Har}(-\Delta)^{-s}\phi\right)_{\dot{H}^s(\mathbb{R}^d)}}_{=0} + \left\|P_D^{Har}(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)}\Bigg)\Bigg) \\
&= \exp\left(-\frac{1}{2}\left\|P_D(-\Delta)^{-s}\phi + P_D^{Har}(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)}\right) \\
&= \exp\left(-\frac{1}{2}\left\|(-\Delta)^{-s}\phi\right\|^2_{\dot{H}^s(\mathbb{R}^d)}\right) \\
&= \exp\left(-\frac{1}{2}\|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}\right).
\end{aligned}$$

As Φ_s, defined in Lemma 2.5.5, coinsides with $\phi \mapsto \exp(-\frac{1}{2}\|\phi\|^2_{\dot{H}^{-s}(\mathbb{R}^d)})$ on $\mathcal{S}'_H(\mathbb{R}^d)$, we have $\Phi_D^s\Phi_{D,Har}^s = \Phi_s$ for $\phi \in \mathcal{S}'_H(\mathbb{R}^d)$. Hence, due to the uniqueness part of Bochner-Minlos and the independence of h_D and h_D^{Har}, we get

$$\begin{aligned}
\mathbb{E}\left[e^{i(h_D + h_D^{Har},\phi)}\right] &= \mathbb{E}\left[e^{i(h_D,\phi)}\right]\mathbb{E}\left[e^{i(h_D^{Har},\phi)}\right] \\
&= \left(\int_{\mathcal{S}_H(\mathbb{R}^d)} e^{i(f,\phi)}d\mu_D(f)\right)\left(\int_{\mathcal{S}_H(\mathbb{R}^d)} e^{i(f,\phi)}d\mu_D^{Har}(f)\right) \\
&= \Phi_D^s(\phi)\Phi_{D,Har}^s(\phi)
\end{aligned}$$

$$
\begin{aligned}
&= \Phi_s(\phi) \\
&= \int_{\mathcal{S}_H(\mathbb{R}^d)} e^{i(f,\phi)} d\mu(f) = \mathbb{E}\big[e^{i(h,\phi)}\big].
\end{aligned}
$$

As the characteristic function of a random variable determines it almost surely uniquely, it follows that $h_D + h_D^{Har} := h \sim FGF_s(\mathbb{R})$ defines a fractional Gaussian field on $\mathbb{R}^d$. Furthermore, since $h_D^{Har}(\phi) = 0$ for $\phi \in \dot{H}s_0(D)$, we get $h = h_D$ on $\dot{H}s_0(D)$. As h_D only lives on D, h_D is determined by h, and thus, also $h_D^{Har} = h - h_D$. ∎

Definition 2.8.7. *For $s \geqslant 0$, an allowable domain $D \subseteq \mathbb{R}^d$ and $h \sim FGF_s(\mathbb{R}^d)$ we call the uniquely determined random element h_D^{Har} given in Theorem 2.8.6 the harmonic extension of h given its values on $\mathbb{R}^d \backslash D$.*

Let $D \subseteq \mathbb{R}^d$ be an allowable domain and let us denote $P_D h := h_D$ and $P_D^{Har} h := h_D^{Har}$. Consider another allowable domain $O \subseteq D$. As the projections $P_D, P_D^{Har}, P_O, P_O^{Har}$ all commute, we can split a fractional Gaussian field $h \sim FGF_s(\mathbb{R})$ up into the following parts

$$h = h_D + h_D^{Har} = P_O h_D + P_O h_D^{Har} + P_O^{Har} h_D + P_O^{Har} h_D^{Har}.$$

Now as $P_O P_D = P_O$, $P_O P_D^{Har} = 0$ and $P_O^{Har} P_D^{Har} = P_D^{Har}$ we get

$$h_D = h_O + P_O^{Har} h_D,$$

with $h_{O,D}^{Har} := P_O^{Har} h_D$ and h_O being independent. In an analogue way we get the following Corollary.

Corollary 2.8.8. *Let $s \geqslant 0$ and $O \subseteq D \subseteq \mathbb{R}^d$ be two allowable domains. Then there exists a coupling of random elements $(h_D, h_{O,D}^{Har}, h_O)$ such that*

(i) $h_D \sim FGF_s(D)$

(ii) $h_O \sim FGF_s(O)$

(iii) $h_D = h_{O,D}^{Har} + h_O$ *almost surely. Furthermore,* $h_{O,D}^{Har}$ *and* h_O *are both determined by* h_D. $h_{O,D}^{Har}$ *is called the harmonic extension of* h_D *given its values on* $D \backslash O$.

2.9 The fractional Brownian field and continuity properties of the $FGF_s(\mathbb{R}^d)$

In this section we introduce and prove the existence of the fractional Brownian field introduced by A. Yaglom in 1957 (see pages 292-338 in [Yag57]). The one dimensional case was discussed earlier by Paul Levy in 1953 and is called the fractional Brownian motion. We define the process over its covariance function, prove the existence with Kolmogorov's extension theorem and finally show that the fractional Brownian field can be interpreted as a fractional Gaussian field.

First we describe the usual way of constructing a Gaussian process via Kolmogorov's extension theorem.

Definition 2.9.1 (Projective family of probability measures). *Let $\mathcal{I} \neq \varnothing$ be an index set such that for all $i \in \mathcal{I}$ we have a measurable space $(\mathcal{S}_i, \mathcal{A}_i)$. Furthermore, let $(\mu_F)_{F \in \mathcal{F}}$ be a family of probability measures on $(\mathcal{S}_F, \mathcal{A}_F)$ where $\mathcal{S}_F := \bigtimes_{i \in F} \mathcal{S}_i$, $\mathcal{A}_F := \bigotimes_{i \in F} \mathcal{A}_i$ and $\mathcal{F}$ denotes the set of all nonempty finite subsets of $\mathcal{I}$. In addition, for $F, G \in \mathcal{F}$ with $F \subseteq G$, we consider the projection $\prod_{G,F} \mathcal{S}_G \to \mathcal{S}_F, (x_i)_{i \in G} \mapsto (x_i)_{i \in F}$. Then the family $(\mu_F)_{F \in \mathcal{F}}$ is called projective or Kolmogorov consistent if for all $F, G \in \mathcal{F}$ with $F \subseteq G$ we have*

$$\mu_F = \mu_G \textstyle\prod_{G,F}^{-1}.$$

Theorem 2.9.2 (Kolmogorov's extension theorem). *Let $I \neq \varnothing$ be an index set such that for all $i \in \mathcal{I}$ $(\mathcal{S}_i, \mathcal{B}_i)$ is a separable and complete metric space equipped with the Borel sigma algebra and $(\mu_F)_{f \in \mathcal{F}}$ a Kolmogorov consistent family of probability measures for $\mathcal{I}$. Then there exists a unique probability measure μ on $(\Omega, \mathcal{B})$ with $\Omega := \bigtimes_{i \in \mathcal{I}}$ and $\mathcal{B} := \bigotimes_{i \in \mathcal{I}}$ such that for the induced stochastic process $X_i : \Omega \to \mathcal{S}_i$ the marginal distributions on the finite subsets of $\mathcal{I}$ coincide with the projective family $(\mu_F)_{F \in \mathcal{F}}$, i.e. for all $f \in \mathcal{F}$ and $B \in \mathcal{B}_F$, where $\mathcal{S}_F$ and $\mathcal{B}_F$ are constructed like in Definition 2.9.1, it holds that $\mu([(X_i)i \in F \in B]) = \mu_F(B)$.*

Proof. For a rigorous proof see section 15.6 in [Sch21]. ■

Definition 2.9.3. *Let $\mathcal{I} \neq \varnothing$ and $C : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ be a function. Then C is called a covariance function if C is symmetric in its two arguments and positive semi-definite, i.e. for $n \in \mathbb{N}$, $c_1, ..., c_n \in \mathbb{R}$ and $i_1, ..., i_n \in \mathcal{I}$, it holds that*

$$\sum_{j,k=1}^{n} c_j C(i_j, i_k) c_k \geqslant 0.$$

Proposition 2.9.4 (Construction of Gaussian processes). *Let $\mathcal{I} \neq \varnothing$ be an index set and $C : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ a covariance function. For $F \in \mathcal{F}$ we define the probability measures $\mu_F := N(0, C(i_j, i_k)_{j,k \in F})$ on $\mathcal{S}_F := \bigtimes_{i \in F} \mathbb{R}_i$. Then the family $(\mu_F)_{F \in \mathcal{F}}$ is a Kolmogorov consistent family of probability measures and the assumptions of Kolmogorov's extension theorem are fulfilled. Thus, there exists a Gaussian process with covariance function given by C.*

Proof. See Exercise 2.86 in [Sch21]. ■

First we introduce the following function that we wish the covariance function of the fractional Brwonian field to be. We define for Hurst parameter $H > 0$

$$C_{FBF} : \begin{cases} \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \\ (x, y) \mapsto \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x - y|^{2H}\right) \end{cases}. \tag{2.8}$$

In the following Lemma we ensure that C_{FBF} is indeed a covariance function. The proof follows Remark 2.111 of [Sch21].

Lemma 2.9.5. *For $H \in (0,1)$ the function $C_{FBF} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a covariance function.*

Proof. We need to show symmetry and positive semi-definiteness. From its definition it clearly holds that $C_{FBF}(x,y) = C_{FBF}(y,x)$. The positive semi-definiteness requires a longer argument. Let us consider the following integral for $z > 0$ solved by the transformation $v = zu$

$$\int_0^\infty \frac{1-e^{-z^2u^2}}{u^{1+2H}}du \overset{v=zu}{=} \int_0^\infty \frac{1-e^{-v^2}}{\frac{v^{1+2H}}{z^{1+2H}}} z dv = z^{2H} \underbrace{\int_0^\infty \frac{1-e^{-v^2}}{v^{1+2H}}dv}_{=:C_H}.$$

Thus, we get

$$z^{2H} = \frac{1}{C_H}\int_0^\infty \frac{1-e^{-z^2u^2}}{u^{1+2H}}du.$$

For $z = 0$ this equation remains also true. Furthermore, by the rule of de L'Hospital the integral is well defined and finite. Furthermore, we can write the exponential term into a power series and can exchange the series with the integral due to the monotone convergence theorem

$$\begin{aligned}
&\int_0^\infty \frac{e^{-|x|^2u^2}(e^{2(x,y)u^2}-1)e^{-|y|^2u^2}}{u^{1+2H}}du \\
&= \int_0^\infty \frac{e^{-|x|^2u^2}\left(\sum_{k=1}^\infty \frac{(2(x\cdot y)u^2)^k}{k!}\right)e^{-|y|^2u^2}}{u^{1+2H}}du \\
&= \sum_{k=1}^\infty \frac{2^k}{k!}\int_0^\infty \frac{e^{-|x|^2u^2}(x\cdot y)^k e^{-|y|^2u^2}}{u^{1-2k+2H}}du.
\end{aligned}$$

Using $|x-y|^2 = |x|^2 + |y|^2 - 2x\cdot y$ and the series representation of the exponential function

$$e^{2(x\cdot y)^2u^2} - 1 = \sum_{k=0}^\infty \frac{2^k(x\cdot y)^{2k}u^{2k}}{k!} - 1 = \sum_{k=1}^\infty \frac{2^k(x\cdot y)^{2k}}{u^{-2k}k!},$$

we get for C_{FBF}

$$\begin{aligned}
C_{FBF}(x,y) &= \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right) \\
&= \frac{1}{2C_H}\int_0^\infty \frac{(1-e^{-|x|^2u^2}) + (1-e^{|y|^2u^2}) - (1-e^{-|x-y|^{2H}u^2})}{u^{1+2H}}du \\
&= \frac{1}{2C_H}\int_0^\infty \frac{(1-e^{-|x|^2u^2})(1-e^{-|y|^2u^2})}{u^{1+2H}}du + \frac{1}{2C_H}\int_0^\infty \frac{e^{-|x-y|^2u^2} - e^{-(|x|^2+|y|^2)u^2}}{u^{1+2H}}du
\end{aligned}$$

$$= \frac{1}{2C_H}\int_0^\infty \frac{(1-e^{-|x|^2u^2})(1-e^{-|y|^2u^2})}{u^{1+2H}}du + \frac{1}{2C_H}\int_0^\infty \frac{e^{-|x|^2u^2}(e^{2(x\cdot y)^2u^2}-1)e^{-|y|^2u^2}}{u^{1+2H}}du$$
$$= \frac{1}{2C_H}\int_0^\infty \frac{(1-e^{-|x|^2u^2})(1-e^{-|y|^2u^2})}{u^{1+2H}}du + \frac{1}{2C_H}\sum_{k=1}^\infty \frac{2^k}{k!}\int_0^\infty \frac{e^{-|x|^2u^2}(x\cdot y)^{2k}e^{-|y|^2u^2}}{u^{1-2k+2H}}du.$$

Finally, for $n \in \mathbb{N}$, $x_1, ..., x_n \in \mathbb{R}^d$ and $v_1, ..., v_n \in \mathbb{R}$ we get

$$\sum_{i,j=1}^\infty v_i C_{FBF}(x_i, x_j) v_j =$$
$$\frac{1}{2C_H}\Bigg(\int_0^\infty \frac{\sum_{i,j=1}^n v_i(1-e^{-|x_i|^2u^2})(1-e^{-|x_j|^2u^2})v_j}{u^{1+2H}}du$$
$$+\sum_{k=1}^\infty \frac{2^k}{k!}\int_0^\infty \frac{\sum_{i,j=1}^n v_i e^{-|x_i|^2u^2}(x_i\cdot x_j)^k e^{-|x_j|^2u^2}v_j}{u^{1-2k+2H}}du\Bigg)$$
$$= \frac{1}{2C_H}\left(\int_0^\infty \frac{\left(\sum_{i=1}^n v_i(1-e^{-|x_i|^2u^2})\right)^2}{u^{1+2H}}du + \sum_{k=1}^\infty \frac{2^k}{k!}\int_0^\infty \frac{\left(\sum_{i=1}^n v_i e^{-|x_i|^2u^2/k}x_i\right)^{2k}}{u^{1-2k+2H}}du\right) \geqslant 0$$

and therefore the positive semi-definiteness of C_{FBF}. All together C_{FBF} is a covariance function. ∎

Now we finally get the existence of the fractional Brownian field for Hurst parameter $H \in (0, 1)$.

Corollary 2.9.6 (Existence of fractional Brownian field)**.** *For the covariance function C_{FBF} given in equation 2.8, there exists a unique Gaussian process $B_H : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that*

$$\mathbb{E}[B_H(x)B_H(y)] = C_{FBF}(x, y) \text{ for all } x, y \in \mathbb{R}^d.$$

Proof. For the proof combine Lemma 2.9.5 and Proposition 2.9.4. ∎

Remark 2.9.7. We call the special case $d = 1$ of the fractional Brownian field the fractional Brownian motion. This makes sense as for Hurst parameter $H = \frac{1}{2}$ we get for $s, t \in \mathbb{R}$

$$C_{FBF}(s, t) = \tfrac{1}{2}\big(s + t - |s - t|\big),$$

which is the covariance function of a two-sided 1-dimensional standard Brownian motion. From the uniqueness part of Kolmogorov's extension theorem it then follows that these processes coincide, so the standard Brownian motion can be seen as a special case of fractional Brownian motion.

We can even go one step further and show that the fractional Brownian motion can be interpreted as a special case of the fractional Gaussian field. The following Lemma shows the connection between these two objects.

Lemma 2.9.8. *Let $s \in (d/2, d/2+1)$, i.e. for the corresponding Hurst parameter $H \in (0,1)$, and $h \sim FGF_s(\mathbb{R}^d)$. Then the process $B_H : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $B_H(x) := \frac{1}{\sqrt{2|C(s,d)|}}(h, \delta_x - \delta_0)$ is well defined and a fractional Brownian field with Hurst parameter $H \in (0,1)$, whereby δ_x denotes the Dirac measure at the point $x \in \mathbb{R}^d$.*

Proof. We need to show that for $x \in \mathbb{R}^d$ the distribution $\delta_x - \delta_0$ is in the domain of the fractional Gaussian field. This is clear as we know from Theorem 2.6.4 that the covariance kernel of h is of the form

$$G_s : \begin{cases} \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \\ (x,y) \mapsto C(s,d)|x-y|^{2H} \end{cases}.$$

Observe that in this case the constant $C(s,d)$ is negative. Therefore, for all $x, y \in \mathbb{R}^d$, it holds for B_H that

$$\begin{aligned} \mathbb{E}[B_H(x)B_H(y)] &= \mathbb{E}[\frac{1}{\sqrt{2|C(s,d)|}}(h, \delta_x - \delta_0)\frac{1}{\sqrt{2|C(s,d)|}}(h, \delta_y - \delta_0)] \\ &= -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\delta_x(u) - \delta_0(u))|u-v|^{2H}(\delta_y(v) - \delta_0(v))dudv \\ &= \frac{1}{2}\big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big) = C_{FBF}(x,y). \end{aligned}$$

From the uniqueness part of Kolmogorov's extension theorem we get that B_H is indeed a fractional Brownian field. ■

With theorem 2.9.2 and Proposition 2.9.4 we have seen a very common way of defining an Gaussian process. A further standard tool that is often used is the Kolmogorov–Chentsov continuity criterion for finding a modification of the Gaussian process that is locally Hölder continuous.

Theorem 2.9.9 (Kolmogorov-Chentsov continuity criterion)**.** *Let $(X_t)_{t \in \mathbb{R}^d}$ be a process with values in a complete and separable metric space $(\mathcal{S}, d)$, equipped with the Borel sigma algebra $\mathcal{B}$. Assume that there exist constants $a, \epsilon > 0$ such that for every compact set $K \subseteq \mathbb{R}^d$ there exists a constant $C_K > 0$ satisfying for all $s, t \in K$*

$$\mathbb{E}[d(X_s, X_t)^a] \leqslant C_K|s-t|^{d+\epsilon}.$$

Then there exists a modification $(Y_t)_{t \in \mathbb{R}^d}$ of $(X_t)_{t \in \mathbb{R}d}$ such that all its paths are locally Hölder continuous for all Hölder exponents $b \in (0, \frac{\epsilon}{a})$.

Proof. See theorem 2.102 in [Sch21]. ■

There also exists a version of that Theorem as a special case of Gaussian processes that is sometimes easier to apply. We will use it later in in the next subsection.

Corollary 2.9.10 (Kolmogorov's criterion for Gausssian processes)**.** *Let $(X_a)_{a \in A}$ be a centered Gaussian process with index set $A \subseteq \mathbb{R}^d$. If there exist $\epsilon, C > 0$ such that for all $a, a' \in A$ it holds that*

$$\mathbb{E}\big[(X_a - X_{a'})^2\big] \leqslant C|a - a'|^\epsilon,$$

then there exists a modification of $(X_a)_{a \in A}$ that is almost surely continuous.

Proof. For the proof see Lemma 3.19 in [WP21]. ■

Our goal now is to show that the fractional Brownian motion has indeed locally Hölder continuous paths.

Lemma 2.9.11. *The fractional Brownian motion with Hurst parameter $H \in (0,1)$ has a modification with locally Hölder continuous paths for all Hölder exponents in $(0, H)$.*

Proof. Let $x, y \in \mathbb{R}$. Then it follows with the covariance function of the fractional Brownian motion 2.8 that

$$\begin{aligned}\mathbb{E}[(B_H(x) - B_H(y))^2] &= \mathbb{E}[B_H(x)^2] - 2\mathbb{E}[B_H(x)B_H(y)] + \mathbb{E}[B_H(y)] \\ &= |x|^{2H} - \big(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\big) + |y|^{2H} \\ &= |x-y|^{2H}\end{aligned}$$

and thus $B_H(x) - B_H(y) \sim N(0, |x-y|^{2H})$. As for all positive integers n the n-th centered moment of a standard Gaussian random variable exists and is finite (see Exercise 2.35 in [Sch21]), there exists a constant C_n and for the fractional Brownian motion we get

$$\mathbb{E}[|B_H(x) - B_H(y)|^{2n}] \leqslant C_n|x-y|^{2Hn} = C_n|x-y|^{1+(2Hn-1)}.$$

With the Kolmogorov-Chentsov continuity criterion we now get for all integers n a modification of the fractional Brownian motion with locally Hölder continuous paths with Hölder exponent less than $\frac{2Hn-1}{2n}$. In conclusion, we arrive at the result that there exists a modification with paths locally Hölder continuous for all Hölder exponents less than $\sup_{n \in \mathbb{N}} \frac{2Hn-1}{2n} = H$. ■

At the very end of this chapter we present two further properties of the fractional Gaussian field. The first shows that the higher the Hurst parameter H gets, the more differentiable the fractional Gaussian field is.

Theorem 2.9.12. *Let $s > 0$, $h \sim FGF_s(\mathbb{R}^d)$ and H be the corresponding Hurst parameter and define $k := \lceil H \rceil$-1. Then, $h \in C^{k,\alpha}(\mathbb{R}^d)$ almost surely for all multi-indices α with $0 < |\alpha| < H - \lceil H \rceil$.*

Proof. See Proposition 6.2 in [LSSW16]. ■

The second theorem shows the existence of a big coupling of fractional Gaussian fields. It is very interesting, especially for generating and plotting fractional Gaussian fields. Indeed one can start with an white noise or a Brownian motion and get all the other fractional Gaussian fields by applying a suitable fractional Laplacian operator.

Theorem 2.9.13. *There exists a coupling of random fields $(h_s)_{s\in\mathbb{R}}$ with $h_s \sim FGF_s(\mathbb{R})$ for all $s \in \mathbb{R}$ and $h_s = (-\Delta)^{\frac{t-s}{2}} h_t$ for all $s, t \in \mathbb{R}$.*

Proof. See Proposition 6.3 in [LSSW16]. ∎

2.10 The Gaussian free field

One further special case of the fractional Gaussian field is the Gaussian free field, or short GFF, that corresponds to $FGF_1(\mathbb{R}^d)$ or $FGF_1(D)$. It can be interpreted as a natural generalization of the Brownian motion. For this case, there is much more literature than for the general case. There are many ways to approximate the Gaussian free field by a discrete version. Furthermore, there is a larger theory of the Markov property and so called local sets. In addition, the two dimensional case is particularly interesting due to its connections to complex analysis and the Schramm-Loewner evolution, discovered in 2000 by Oded Schramm. In this section we want to give an overview of some interesting facts about the Gaussian free field. We start analysing the covariance kernel, also called Green's function. Then we cite a version of the Markov property and define local sets. Finally, we show that the Gaussian free field can be represented as a random Fourier series. This section follows the lecture notes of Wendelin Werner [WP21].

In this entire section we assume a domain $D \subseteq \mathbb{R}^d$ to satisfy certain conditions. We want it to be a bounded, connected and open subset of $\mathbb{R}^d$, such that all its boundary points are regular. Here a boundary point $z \in \partial D$ is said to be regular, if for all d-dimensional Brownian motions $(B_t)_{t\in[0,\infty)}$ starting in z, we have that $\inf\{t \in [0,\infty) : B_t \notin D\} = 0$ almost surely. We first consider the Green's function of the fractional Gaussian field i.e. the covariance kernel, that we denote as $G_D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ in this section. For $y \in \mathbb{R}^d$, we define the following function

$$H_y : \mathbb{R}^d \to \mathbb{R}, x \mapsto \begin{cases} \dfrac{1}{2\pi} \log \dfrac{1}{|x-y|} & \text{for } d = 2 \\ \dfrac{1}{a_d |x-y|^{d-2}} & \text{for } d \geqslant 3 \end{cases},$$

where a_d denotes the surface of the d-dimensional unit ball. In the case of $D = \mathbb{R}^d$, we already get the Green's function $G : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, by defining $G(x,y) := H_y(x)$. It readily follows that this function is, up to a multiplicative constant, the unique harmonic function on $\mathbb{R}^d \backslash \{y\}$ that tends to zero for $|x| \to \infty$. Now for $D \neq \mathbb{R}^d$, we define the function

$$h_{y,D} : \begin{cases} D\backslash\{x\} \to \mathbb{R} \\ x \mapsto \mathbb{E}_x\big[H_y(B_\tau)\big] \end{cases},$$

whereby $\mathbb{E}_x$ denotes the expectation under which B is a Brownian motion that starts at x and τ is the stopping time $\tau := \inf\{t \in [0, \infty) : B_\tau \in \partial D\}$. In other words, we start a Brownian motion in the point $x \in D$ and stop it in the first moment, where it hits the boundary of D and consider the value of H_y at this boundary point. Then we take the expected value of this random value. This is well defined, as D is suitable for that problem. The function $h_{y,D}$ is then the unique solution of the Dirichlet problem with boundary conditions H_y, which means that it is the unique function on D that is harmonic and has the values of H_y on ∂D. The Green's function then can be represented as

$$G_D(x, y) := H_y(x) - h_{y,D}(x).$$

The following Proposition sums up the last thoughts.

Proposition 2.10.1. *Let $D \neq \mathbb{R}^d$ be a domain and $y \in D$. Then $x \mapsto G_D(x, y)$ is the unique continuous function on $\bar{D}\backslash\{y\}$, such that the following three properties holds true.*

i) It vanishes at ∂D, i.e. $G_D(x, y) = 0$ for $y \in \partial D$.

ii) It is harmonic in $D\backslash\{y\}$.

iii) The function $x \mapsto G_D(x, y) - H_y(x)$ stays bounded in a neighbourhood of y.

Proof. See Lemma 3.7 in [WP21] ■

Furthermore, one can consider the Green's function to be the inverse of the Laplacian operator. Let $f \in C_c(D)$, i.e. f is continuous with compact support in D, then we can look at the following construction. We define

$$G_D(f) : \begin{cases} \overline{D} \to \mathbb{R} \\ x \mapsto \int_D f(y) G_D(x, y) dy \end{cases}.$$

Then it follows that $G_D(f)$ is a continuous function on $\overline{D}$ that is twice continuously differentiable in D and vanishes on ∂D. Furthermore, we have $-\Delta G_D(f) = f$. We can use that to show that for all Schwartz functions the Green's function is indeed a positive definite integral kernel. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then we have for $h \sim GFF$

$$\begin{aligned}
\mathbb{E}\big[(h, \phi)^2\big] &= \int_D \int_D \phi(x) G_D(x, y) \phi(y) dx dy \\
&= \int_D G_D(\phi)(x) \phi(x) dx \\
&= \int_D G_D(\phi)(x)(-\Delta) G_D(\phi)(x) dx \\
&= \int_D \Big|\nabla G_D(x)\Big|^2 dx \geqslant 0.
\end{aligned}$$

We will add one last comment about the Green's function. As D is bounded, we can find an orthonormal basis $(\phi_n)_{n\in\mathbb{N}}$ of $L^2(D)$, such that all ϕ_n vanish on $\partial(D)$ and are eigenfunctions of $-\Delta$, i.e. $-\Delta\phi_n = \lambda_n$ for some $\lambda_n \geqslant 0$. As $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal basis, we have for all $f \in L^2(D)$

$$f = \sum_{n=1}^{\infty} f_n \phi_n, \text{ where } f_n := \int_D f(x)\phi_n(x)dx.$$

If we apply $-\Delta$ to f, we get

$$-\Delta f = \sum_{n=1}^{\infty} f_n(-\Delta)\phi_n = \sum_{n=1}^{\infty} f_n \lambda_n \phi_n.$$

Thus, we can find an explicit way to represent the above mentioned function $G_D(\cdot)$, seen as the inverse operator of $-\Delta$ on $L^2(D)$. For $f \in L^2(D)$ we have

$$G_D(f)(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} f_n \phi_n(x). \tag{2.9}$$

We use that to get a representation of the Green's function in terms of the orthonormal basis $(\phi_n)_{n\in\mathbb{N}}$. Using $(\phi_n, \phi_m)_{L^2(D)} = \delta_{\{n=m\}}$, we have

$$\begin{aligned} G_D(x,y) &= \sum_{n=1}^{\infty} \int_D \phi_n(z) G_D(x,z)dz \phi_n(y) \\ &= \sum_{n=1}^{\infty} G_D(\phi_n)(x)\phi_n(y) \\ &\overset{2.9}{=} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(\int_D \phi_m(z)\phi_n(z)dz\right) \frac{1}{\lambda_m}\phi_m(x)\phi_n(y) \\ &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n}\phi_n(x)\phi_m(y), \end{aligned}$$

as $(\phi_n)_{n\in\mathbb{N}}$ is an orthonormal basis.

Let us now continue with the second topic. Clearly we cannot evaluate the Gaussian free field point-wise and thus, it makes even less sense to talk about continuity. But we can find a subset of the domain of the Gaussian free field on which we will try to find a continuous modification. Therefore, we need to notice that the Gaussian free field can be defined on a larger domain as the Schwartz functions in D. In particular we can extend the process on all functions f that fulfill

$$\int_D \int_D f(x) G_D(x,y) f(y) dxdy < \infty. \tag{2.10}$$

In [WP21] the Gaussian free field is directly defined on all these functions. A special subclass of functions that satisfy 2.10 is the set of the so called cycle averages that we will introduce now.

Definition 2.10.2 (cycle averages)**.** *Let $z \in \mathbb{R}^d$ and $r > 0$. Then $\lambda_{z,r}$ is defined to be the density of the uniform measure on the surface of the ball $B_r(z)$ with center z and radius r. It is easy to see that for all domains $D \subseteq \mathbb{R}^d$, such that $B_r(z) \subseteq D$, the density $\lambda_{z,r}$ can be integrated over the Green's function as in 2.10 and thus, $\Gamma(\lambda_{z,r})$ is well defined. We denote*

$$\gamma(z,r) := \Gamma(\lambda_{z,r}).$$

An interesting fact is that one can construct a standard Brownian motion out of a Gaussian free field.

Proposition 2.10.3. *For $\overline{B_{r_0}(z)} \subseteq D$, we define the process $(B_t^{z,r_0}) \in [0,\infty)$ by*

$$B_t^{z,r_0} := \begin{cases} (\gamma(z, r_0 e^{-t}) - \gamma(z,r_0) & \text{for } d = 2 \\ \gamma(z,(t + r_0^{2-d})^{1/(2-d)}) - \gamma(z,r_0) & \text{for } d \geqslant 3 \end{cases}.$$

Then $(B_t^{z,r_0})_{t\in[0,\infty)}$ is a standard Brownian motion. Furthermore, if one considers countably many disjoint balls $B_{r_n}(z_n), n \in \mathbb{N}$, contained in D, then the corresponding processes $(B_t^{z_j,r_j})_{t\in[0,\infty)}$ are independent Brownian motions.

Proof. Let Γ be a Gaussian free field on an domain D and f a function that agrees with condition 2.10 and is supported in $D\backslash\overline{B_{r_0}(z)}$. As the Green's function is, according to Proposition 2.10.1, harmonic in both its arguments except the diagonal, it follows for all $0 < r < r_0$

$$\begin{aligned} \mathbb{E}\big[\gamma(z,r)\Gamma(f)\big] &= \int_D \int_D f(x) G_D(x,y)\lambda_{z,r}(y)dxdy \\ &= \int_D f(x) \int_D G_D(x,y)\lambda_{z,r}(y)dydx \\ &= \int_D f(x)G_D(z)dx \end{aligned}$$

and thus, for all $0 < r < r_0$, we get

$$\mathbb{E}\big[(\gamma(z,r) - \gamma(z,r_0))\Gamma(f)\big] = 0.$$

Since Γ is Gaussian distributed, we get that the process $(\gamma(z,r) - \gamma(z,r_0))_{r\in(0,r_0]}$ is independent of all $\Gamma(f)$, where f is supported in $D\backslash\overline{B_{r_0}(z)}$. Now, we want to determine the covariance structure of this process. Using again the harmonicity and the symmetry of the Green's function, we get for $0 < r < r' < r_0$

$$\begin{aligned} \mathbb{E}\big[(\gamma(z,r) - \gamma(z,r'))^2\big] &= \int_D \int_D \Big(\lambda_{z,r}(x) - \lambda_{z,r'}(x)\Big) G_D(x,y) \Big(\lambda_{z,r}(y) - \lambda_{z,r'}(y)\Big) dy \\ &= \int_D \lambda_{z,r}(x) \Bigg(\underbrace{\int_D G_D(x,y)\lambda_{z,r}(y)dy}_{=G_D(x,z)}\Bigg) dx - \int_D \Bigg(\underbrace{\int_D \lambda_{z,r}(x) G_D(x,y)dx}_{=G_D(z,y)}\Bigg) \lambda_{z,r'}(y)dy \\ &\quad - \int_D \lambda_{z,r'}(x) \Bigg(\underbrace{\int_D G_D(x,y)\lambda_{z,r}(y)dy}_{=G_D(z,y)}\Bigg) dx + \int_D \Bigg(\underbrace{\int_D \lambda_{z,r'}(x) G_D(x,y)dx}_{=G_D(z,y)}\Bigg) \lambda_{z,r'}(y)dy \\ &= \int_D \lambda_{z,r}(x) G_D(x,z)dx - \int_D \lambda_{z,r'}(x)G_D(x,z)dx. \end{aligned}$$

With Proposition 2.10.1, it follows that this is equal to

$$\int_D \lambda_{z,r}(x) G_{B_{r'}(z)}(x,z)dx = \begin{cases} \log(\frac{r'}{r}) & \text{if } d = 2, \\ r^{2-d} - (r')^{2-d} & \text{if } d \geqslant 3 \end{cases}, \tag{2.11}$$

where $G_{B_{r'}(z)}$ denotes the Green's function of the domain $B_{r'}(z)$. Thus, the first result follows. As we have seen that these processes are independent of all $\Gamma(f)$ such that the support of f is contained in $D\backslash B_{r_0}(z)$, the second result follows. ■

Proposition 2.10.4. *There exists a modification of the Gaussian free field on D such that the process $\gamma : \{(z,r) : z \in D, r \in (0, d(z, \partial D))\} \to \mathbb{R}, (z,r) \mapsto \gamma(z,r)$ is continuous.*

Idea of the proof. We want to apply Corollary 2.9.10. For a change in the radius we get an estimate with 2.11. For a change of the center it is easy to find a similar one of the form

$$\mathbb{E}\big[(\gamma(z,r) - \gamma(z',r))^2\big] \leqslant C(r_0)|z - z'|$$

for all $d(z,z') < r_0$, $d(z,\partial D), d(z',\partial D) > r_0$. Thus, we get the existence of a constant $C(r_0, D) > 0$ such for all $r, r' > r_0$ and $z, z' \in D$ with $d(z,z') \leqslant \frac{r_0}{2}$ and $d(z,\partial D) > r_0$, it holds that

$$\mathbb{E}\big[(\gamma(z,r) - \gamma(z',r'))^2\big] \leqslant C(r_0, D)\big(|z - z'| + |r - r'|\big).$$

Using Corollary 2.9.10, we can conclude that there exists a modification of the Gaussian free field such that for all (z,r) with $d(z,\partial D) > r_0$ the process $(z,r) \mapsto \gamma(z,r)$ is continuous. As we can do this whole procedure for a sequence of $r_{0,n} \to 0$, we see that there exists a modification such that $(z,r) \mapsto \gamma(z,r)$ is continuous on the whole set $\{(z,r) : z \in D, r \in (0, d(z,\partial D))\}$. ■

Another point regarding the Gaussian free field, that one has to think about, are scaling properties, analogue to Proposition 2.5.2. In the case of $d \geqslant 3$, we can deduce directly from the covariance structure of the Gaussian free field that for a domain $D \subseteq \mathbb{R}^d$, $r > 0$ and $x, y \in D$ we have for the Green's function

$$G_{rD}(rx, ry) = r^{2-d} G_D(x,y),$$

where rD denotes the scaled domain. Thus, we get in law for a Gaussian free field Γ_D on D the following property

$$\Gamma_{rD} \overset{d}{=} r^{d/2-1}\Gamma_D.$$

In the case $d = 2$, we obtain a very special property. With Proposition 2.5.2 it follows, plugging in $s = 1$, that the Gaussian free field is scaling invariant. However, that is not the only property. It is even conformal invariant, i.e. if we have an angle preserving bijection $\Phi : D \to \tilde{D}$, Gaussian free fields $\Gamma_D, \Gamma_{\tilde{D}}$ with domains D and $\tilde{D}$ respectively, and any $\tilde{f}$ on $\tilde{D}$ that agrees with 2.10 it holds that

$$\Gamma_{\tilde{D}}(\tilde{f}) = \Gamma_D(f),$$

where f is the push forward map under Φ, defined by

$$f : \begin{cases} D \to \mathbb{R} \\ x \mapsto \tilde{f}(\Phi^{-1}(x))\left|(\Phi^{-1})'(x))\right|^2 \end{cases}.$$

Let us conclude this subsection with one more topic. We want to describe the Markov property in an analogue way as in Corollary 2.8.8. As we are now in a more specific setting, we are able to get a stronger version of the Markov property. We state the version in the lecture notes of Wendelin Werner (see Proposition 4.3 in [WP21]).

Theorem 2.10.5 (weak Markov property of the GFF). *Let $D \subseteq \mathbb{R}^d$ be a domain and $A \subseteq D$ a compact set such that the boundary of $D \backslash A$ is regular and Γ a Gaussian free field in D. Then there exist two generalized random functions Γ^A, Γ_A on D such that the following holds*

i) Γ^A and Γ_A are independent Gaussian processes.

ii) $\Gamma^A|_{D\backslash A}$ is a Gaussian Free field in $D \backslash A$.

iii) There exists a version of Γ_A such that $\Gamma_A|_{D\backslash A}$ is almost surely equal to a harmonic function in $D\backslash A$.

Proof. See section 4.1 in [WP21]. ■

Remark 2.10.6. The idea of the weak Markov property for the Gaussian free field is the following. One chooses a suitable compact set $A \subseteq D$ where we know what happens. By Theorem 2.10.5, we can now restrict Γ to the set A where we know it and extend it on the complement by the unique harmonic function, that has zero boundary on ∂D and the values of Γ on ∂A. Here one has to be careful, as Γ is not a function and therefore this step requires more care. Once we have found that harmonic function, we can sample an independent Gaussian free field on $D\backslash A$ with zero boundary conditions. If we now sum them up, we just end up with a Gaussian free field in D. This is particularly interesting for sampling a Gaussian free field. It would be analogue to receive a simulation of a Brownian motion by sampling Gaussian random variables step by step.

Remark 2.10.7. Analogue to chapter 2.8, we get a result for splitting a Gaussian free field into two parts twice. Let $A \subseteq B \subseteq D$ be two compact subsets of D that satisfy the assumptions of Theorem 2.10.5, then it follows in an analogue way that

$$\Gamma^B = (\Gamma^A)^B \text{ and } \Gamma_B = \Gamma_A + (\Gamma^A)_B.$$

Remark 2.10.8. Here we define a function f being harmonic by $\Delta f = 0$. If f is continuous, this is equivalent to the mean value property, i.e. for all balls, completely included in the domain of f, the value of f at the center is the same as the average of f on the surface of the ball. We will use this fact later to show the equivalence between the strong Markov property and a set A being a local set.

Here we call this decomposition the weak Markov property of the Gaussian free field, based on the weak Markov property of the Brownian motion. As in the case of the Brownian motion we can even find an analogue of the strong version of the Markov property, namely that the set A is not deterministic any more.

Definition 2.10.9 (Strong Markov property of the GFF)**.** *Let Γ be a Gaussian free field on D and A a random compact subset of $\overline{D}$ such that $D \backslash A$ has a regular boundary. We say A satisfies the strong Markov property for Γ, if there exist two random generalized functions Γ_A, Γ^A such that*

i) $\Gamma = \Gamma_A + \Gamma^A$

ii) Γ_A *is linear in its argument and there exists a random function h_A in the complement of A that is harmonic almost surely and such that $\Gamma_A(\phi) = h_A(\phi)$ on the event that the support of ϕ is contained in $D \backslash A$.*

iii) Conditionally on (A, Γ_A), Γ^A is a Gaussian free field in $D \backslash A$ with zero boundary conditions.

Here the notion $h_A(\phi)$ denotes the integral

$$h_A(\phi) = \int_D h_A(x)\phi(x)dx.$$

Remark 2.10.10. As every harmonic function is determined by its values on the boundary, Γ_A is measurable with respect to the sigma-algebra generated by Γ restricted to A. As conditionally on (A, Γ_A) the process Γ^A is a Gaussian free field in $D \backslash A$ and Γ_A can be explicitly described in a measurable way by A, it is, conditionally on A, independent of Γ^A. Thus, if A is deterministic, we get the weak Markov property. Therefore, every deterministic set satisfies the strong Markov property.

By Remark 2.10.10, we have seen the Definition 2.10.10 makes sense as an extension of the weak Markov property. Nevertheless, it is not an easy definition to work with. For example, if one wants to prove that, under additional assumptions, the union of two sets, that satisfy the strong Markov property, satisfies again the strong Markov property, the proof gets very long and complicated. Therefore, one can derive an equivalent notion, the so called local sets, that is easier to work with.

Definition 2.10.11 (dyadic approximation)**.** *Let $n \in \mathbb{N}$ and $A \subseteq \overline{D}$ compact, further for $i \in \mathbb{Z}^d$ let*

$$Q_i := \left[\frac{i_1}{2^n}, \frac{i_1+1}{2^n}\right] \times \cdots \times \left[\frac{i_d}{2^n}, \frac{i_d+1}{2^n}\right].$$

Then, we define the dyadic approximation of the set A as follows

$$A_n := \bigcup_{i \in \mathbb{Z}^d, Q_i \cap A \neq \emptyset} Q_i.$$

Remark 2.10.12. We call sets that can be represented as a finite union of cubes Q_i of length 2^{-n} intersected with D 2^{-n}-dyadic sets. In the case of $A \subseteq D$, it follows by the boundedness of D that A_n is a finite union of cubes $Q_i \cap D$ and therefore a 2^{-n}-dyadic set. In addition, it holds that $A_n \supseteq A$ and

$$\bigcap_{n \in \mathbb{N}} A_n = A.$$

Furthermore, one can define the dyadic approximation of a random set in an analogue way. Here it is important that the dyadic approximation of a random set is a deterministic and therefore measurable function of the random set.

Definition 2.10.13 (dyadic local sets)**.** *Let Γ be a Gaussian free field on D and $n \in \mathbb{N}$. Then we call a random set A an 2^{-n}-dyadic local set of Γ, if it is a random 2^{-n}-dyadic set and for all deterministic 2^{-n}-dyadic sets B the process Γ^B, defined by Theorem 2.10.5, is independent of the sigma-algebra generated by (Γ_B, B).*

A direct consequence of this definition is the following Lemma.

Lemma 2.10.14. *For all $n \in \mathbb{N}$, every 2^{-n}-dyadic local set satisfies the strong Markov property.*

Proof. Let A be a 2^{-n}-dyadic local set. It readily follows that A is a compact random set such that $D \backslash A$ has regular boundary. Now for every deterministic 2^{-n}-dyadic set B, by Theorem 2.10.5, there exist Γ^B and Γ_B that split Γ up with corresponding harmonic functions h_B on $D \backslash B$. As D is bounded, there are only finitely many 2^{-n}-dyadic sets in D. Thus, the following random processes are well defined

$$\Gamma_A := \sum_B \mathbb{1}_{\{A=B\}} \Gamma_B, \; \Gamma^A := \sum_B \mathbb{1}_{\{A=B\}} \Gamma^B, \; h_A := \sum_B \mathbb{1}_{\{A=B\}} h_B.$$

Furthermore, it follows that $\Gamma_A + \Gamma^A = \Gamma$ almost surely, h_A is almost surely harmonic on $D \backslash A$ and $\Gamma_A(\phi) = h_A(\phi)$ on the event that the support of ϕ is contained in $D \backslash A$. Finally, conditionally on A and therefore also conditionally on (A, Γ_A), Γ^A is a Gaussian free field in $D \backslash A$. ∎

Remark 2.10.15. On the other hand it is easy to see that every random 2^{-n}-dyadic set, that satisfies the strong Markov property, is a 2^{-n}-dyadic local set.

Now we are ready to define local sets.

Definition 2.10.16 (local set)**.** *Let Γ be a Gaussian free field on D and A a random compact set such that $D \backslash A$ has a regular boundary. Then A is a local set of Γ, if every 2^{-n}-dyadic approximation is a 2^{-n}-dyadic local set of Γ.*

Let us show that the two notions, A satisfying the strong Markov property and A being a local set, coincide.

Theorem 2.10.17. *Let Γ be a Gaussian free field on D and $A \subseteq \overline{D}$ a random compact set. Then A satisfies the strong Markov property for Γ if and only if it is a local set for Γ.*

Idea of the proof. We start assuming that A satisfies the strong Markov property, thus there exist Γ_A and Γ^A. Let $n \in \mathbb{N}$. By Lemma 2.10.14, conditionally on (Γ_A, A), we can split the process Γ^A, that is a Gaussian free field in $D \backslash A$, further up. We get

$$\Gamma^A = (\Gamma^A)^{A_n} + (\Gamma^A)_{A_n}.$$

It follows that, conditionally on (Γ_A, A) and $(\Gamma^A)_{A_n}$, the process $(\Gamma^A)^{A_n}$ is a Gaussian free field in $D \backslash A_n$ and $\Gamma_A + (\Gamma^A)_{A_n}$ is restricted to $D \backslash A_n$, as a sum of harmonic functions, again a harmonic function. We define $\Gamma_{A_n} := \Gamma_A + (\Gamma^A)_{A_n}$ and $\Gamma^{A_n} := \Gamma - \Gamma_{A_n}$. As A_n is a deterministic and therefore measurable function of A and the information of A, we have split up Γ into two parts such that one is conditioned on (A_n, Γ_{A_n}) a Gaussian free field in $D \backslash A_n$ and the other is a harmonic function in $D \backslash A_n$. As already mentioned in Remark 2.10.15, it follows that A_n is a 2^{-n}-dyadic local set.

Let us now assume that A is a local set for Γ. Therefore, we have, for all $n \in \mathbb{N}$, a splitting $\Gamma^{A_n}, \Gamma_{A_n}$. We want to construct the two processes Γ^A and Γ_A. As A_n is a measurable function of A and $\bigcap_{n \geqslant m} A_n = A$ for all $m \in \mathbb{N}$, the sigma-algebra generated by A is the same as the sigma-algebra generated by $(A_n, n \geqslant m)$ for all $m \in \mathbb{N}$. Furthermore, as A_n is already determined by A_{n+1} for all n, Γ_{A_m} is measurable with respect to the Γ_{A_n} for all $m \geqslant n$. Let us now define the sigma-algebras

$$\begin{aligned} \mathcal{G}_n :&= \sigma(A, \Gamma_{A_n}) = \sigma(\Gamma_{A_n}, A_n, A_{n+1}, A_{n+2}, ...) \\ &= \sigma(\Gamma_{A_n}, \Gamma_{A_{n+1}}, \Gamma_{A_{n+2}}, ..., A_n, A_{n+1}, A_{n+2}, ...), \end{aligned}$$

that is a decreasing sequence of sigma-algebras. Clearly Γ_{A_n} is measurable with respect to $\mathcal{G}_n$. We have, using Remark 2.10.7,

$$\Gamma_{A_n} = \Gamma_{A_{n+1}} + (\Gamma^{A_{n+1}})_{A_n},$$

where $(\Gamma^{A_{n+1}})_{A_n}$ is independent of $(A_{n+1}, \Gamma_{A_n+1})$, and therefore of $\mathcal{G}_{n+1}$, and centered Gaussian. Thus, we get

$$\Gamma_{A_{n+1}} = \mathbb{E}\big[\Gamma_{A_n} | \mathcal{G}_{n+1}\big].$$

In conclusion, $(\Gamma_{A_n}(\phi))_{n \in \mathbb{N}}$ is an inverse martingale with respect to the inverse filtration $(\mathcal{G}_n)_{n \in \mathbb{N}}$ for all $\phi \in \mathcal{S}(D)$ and therefore converges almost surely and in L^p for all $p > 1$ to some limit, we call $\Gamma_A(\phi)$. Further, we define

$$\Gamma^A := \Gamma - \Gamma_A.$$

Now, we need to show that this splitting agrees with the definition of the strong Markov property in 2.10.9. We start showing that Γ_A restricted to $D\backslash A$ is a harmonic function. Remember that D is open, thus, for every $z \in D$ we have $d(z, \partial D) > 0$. We define for $z \in D$, $0 < r < d(z, \partial D)$ and $n \in \mathbb{N}$ the random variable $\gamma_n(z, r)$ as the average of Γ_{A_n} on the surface of the ball $B_r(z) \subseteq D$. As also the process $\gamma_n(z, r)$ is a reverse martingale, it converges almost surely and in L^p for all $p > 1$ to the spherical average of Γ_A, we denote by $\gamma_\infty(z, r)$. By Remark 2.10.8, this process needs to coincide almost surely with the harmonic function h_{A_n} on $D\backslash A_n$ and is therefore independent of the chosen $r < d(z, \partial D)$. Now, we define the event $E(z, r)$ of the ball $B_r(z)$ being contained in $D\backslash A$ and the sequence of the harmonic functions $\mathbb{1}_{E(z,r)} h_{A_n}(z)$ converges almost surely and in all L^p for $p > 1$ to some limit. As again this process is independent of $r < d(z, \partial D)$ and therefore can be defined on the whole event $D\backslash A$, we can define

$$h_A(z) := \mathbb{1}_{D\backslash A}(z) h_{A_n}(z).$$

We notice that for all suitable f, we have $\Gamma_A(f) = h_A(f)$ on the event that the support of f is contained in $D\backslash A$. Thus, on this event we have

$$\Gamma_A(f) = \lim_{n\to\infty} \Gamma_{A_n}(f) = \lim_{n\to\infty} h_{A_n}(f) = h_A(f).$$

To use Remark 2.10.8, we need to show that there exists a continuous version of that process that has the mean value property. We want to apply the Kolmogorov-Chentsov continuity criterion 2.9.9. Let $z, z' \in D$ with $d(z, \partial D), d(z', \partial D) > 2r$. Using the L^p convergence, conditional Jensen inequality and equation 2.11, we conclude

$$\begin{aligned}
\mathbb{E}\Big[(\gamma_\infty(z,r) - \gamma_\infty(z',r))^{2d+2}\Big] &= \lim_{n\to\infty} \mathbb{E}\Big[(\gamma_n(z,r) - \gamma_n(z',r))^{2d+2}\Big] \\
&= \lim_{n\to\infty} \mathbb{E}\Big[\mathbb{E}\big[\gamma(z,r) - \gamma(z',r) | \mathcal{G}_n\big]^{2d+2}\Big] \\
&\leqslant \mathbb{E}\Big[(\gamma(z,r) - \gamma(z',r))^{2d+2}\Big] \\
&= \mathbb{E}\Big[(\gamma(z,r) - \gamma(z',r)^2\Big]^{d+1} (2d+1)! \leqslant C(r)|z - z'|^{d+1}.
\end{aligned}$$

Note that for the equality in the last line, we used that for $X \sim N(0, \sigma^2)$, we have $\mathbb{E}[X^{2n}] = (2n-1)!\sigma^n$. Using the Kolmogorov-Chentsov continuity criterion, we get that there exists a continuous modification of Γ_A on the event $D\backslash A$. We need to show that this contiunuous version also has the mean value property. We already know that for $z \in D$ and $3r > d(z, \partial D)$, it holds that

$$h_A(z) = \gamma_\infty(z, r) = \lim_{n\to\infty} \gamma_n(z, r).$$

Furthermore, the averages $\gamma_n(z, r)$ converge almost surely. Thus, to see that $h_A(z)$ is equal to its average on $\partial B_r(z)$, which we denote by $\gamma_A(z, r)$, we need to show that the averages $\gamma_n(z, r)$ converge to the average of h_A. This is, because of the uniqueness of the limit, almost surely true. Therefore, it suffices to show

$$\mathbb{1}_{E(z,r)} h_{A_n}(z) \to \mathbb{1}_{E(z,r)} \gamma_A(z, r).$$

This follows readily as it is bounded in L^2. For the details see Proposition in [WP21]. Finally, we need to show that Γ^A is conditioned on (A, Γ_A) a Gaussian free field in $D\backslash A$. As Γ^{A_n} is a Gaussian free field in $D\backslash A_n$ for all $n \in \mathbb{R}$, it is independent of $\mathcal{G}_n \supseteq \mathcal{G}_\infty \supseteq \sigma(A, \Gamma_A)$. Furthermore, Γ^{A_n} is independent of A_m for $m \geqslant n$ as with Remark 2.10.7 we get $\Gamma^{A_n} = (\Gamma^{A_m})^{A_n}$. In conclusion Γ^A is independent of Γ_A. ∎

Finally, we show, using the equality of the two notions, the following lemma.

Lemma 2.10.18. *Let Γ be a Gaussian free field in D and A, A' two local sets of γ that are, conditioned on Γ, independent. Then also $A \cup A'$ is a local set.*

Proof. We first show the result for two 2^{-n}-dyadic local sets. Let B be a deterministic 2^{-n}-dyadic set, $f_1, ..., f_m, g_1, ..., g_m$ suitable functions in the domain of the Gaussian free field and $U_1, ..., U_m, V_1, ..., V_m$ open sets in $\mathbb{R}$. We define the following, with respect to $\sigma(\Gamma^B)$ and $\sigma(\Gamma_B)$, measurable events

$$U^B := \Big\{\Gamma^B(f_j) \in U_j : j \in \{1, ..., m\}\Big\} \text{ and } V_B := \Big\{\Gamma_B(g_j) \in V_j : j \in \{1, ..., m\}\Big\}.$$

Then, the set of all events of the form of U^B is stable under intersection and generates $\sigma(\Gamma^B)$, analogue to that, the set of all events of the form V_B is intersection stable and generates $\sigma(\Gamma_B)$. Furthermore, for a 2^{-n}-dyadic set $B' \subseteq B$, the set of events of the form $V_B, V_B \cap \{A = B'\}$ generates $\sigma(\Gamma_B, \{A = B'\})$ and analogue for A'. Now, for all 2^{-n}-dyadic sets B_1, B_2 with $B_1 \cup B_2 = B$ we get

$$\begin{aligned}\mathbb{P}\big[U^B, V_B, A = B_1, A' = B_2\big] &= \mathbb{E}\Big[\mathbb{P}\big[U^B, V_B, A = B_1, A' = B_2 | \Gamma\big]\Big] \\ &= \mathbb{E}\Big[\mathbb{1}_{U^B \cap V_B}\mathbb{P}\big[A = B_1, A' = B_2|\Gamma\big]\Big] \\ &= \mathbb{E}\Big[\mathbb{1}_{U^B}\mathbb{1}_{V_B}\mathbb{P}\big[A = B_1|\Gamma\big]\mathbb{P}\big[A' = B_2|\Gamma\big]\Big].\end{aligned}$$

As Γ^B is independent of $\sigma(\Gamma_B, \{A = B_1\})$, it follows that $\mathbb{P}[A = B_1|\Gamma] = \mathbb{P}[A = B_1|\Gamma_B]$ and also that $\mathbb{P}[A' = B_2|\Gamma] = \mathbb{P}[A' = B_2|\Gamma_B]$. Using that Γ^B and Γ_B are independent, we have

$$\begin{aligned}\mathbb{P}\Big[U^B, V_B, A = B_1, A' = B_2\Big] &= \mathbb{P}\big[U^B\big]\mathbb{E}\Big[\mathbb{1}_{V_B}\mathbb{P}\big[A = B_1|\Gamma\big]\mathbb{P}\big[A' = B_2|\Gamma\big]\Big] \\ &= \mathbb{P}\big[U^B\big]\mathbb{P}\Big[V_B, A = B_1, A' = B_2\Big].\end{aligned}$$

Now summing over all B_1, B_2 satisfying $B_1 \cup B_2 = B$, we get

$$\mathbb{P}\Big[U^B, V_B, A \cup A' = B\Big] = \mathbb{P}\big[U^B\big]\mathbb{P}\Big[V_B, A \cup A' = B\Big].$$

Thus, we get that $\Gamma^{A \cup A'}$ is independent of $(\Gamma_{A \cup A'}, A \cup A')$ and $A \cup A'$ is a 2^{-n}-dyadic local set. With Lemma 2.10.14 the result follows. ∎

3 Convergence of the Gaussian multiplicative chaos associated to the fractional Brownian field

In this chapter we define the Gaussian multiplicative chaos of a fractional Brownian field and want to describe its limit in an useful way when the Hurst parameter converges to 0. Let B_H be a fractional Brownian motion, then the Gaussian multiplicative chaos associated to that fractional Brownian field B_H is a random measure, that is formally given by

$$M_\gamma^H(dx) := \exp\left(\gamma B_H(x) - \frac{\gamma^2}{2}\mathbb{E}\big[B_H(x)^2\big]\right)dx,$$

where $\gamma > 0$ is a constant. Often the Gaussian multiplicative chaos is shortly denoted by GMC. We are interested in the case when the Hurst parameter H tends to 0. The first approach to this topic was the definition of a Gaussian multiplicative chaos associated to a log-correlated Gaussian field that was introduced by J. Kahane, see [Kah85]. In the first three sections we want to present and prove a result of P. Hager and E. Neuman (see Theorem 2.4 of [HN20]) that shows convergence in probability. In the second part we discuss normalizations of fractional Brownian fields that fit the assumptions of the convergence result.

3.1 The statement of convergence of the Gaussian multiplicative chaos

In the following, D denotes always a bounded domain. This section follows section 2.1 of [HN20].

Definition 3.1.1 (family of normalized fractional Brownian fields)**.** *Let D be a bounded domain and $H_0 \in (0, \frac{1}{2})$. Then we call $(X_H)_{H\in(0,H_0)}$ a family of normalized fractional Brownian fields if for every $H \in (0, H_0)$, X_H is a random element of tempered distributions on D. Furthermore, we have the following covariance structure*

$$\mathbb{E}\big[X_H(x)X_h(y)\big] = C(H,h)\left(\frac{1 - \|x-y\|^{H+h}}{H+h} + g_{H,h}(x,y)\right) \tag{3.1}$$

for $x, y \in D$ and $H, h \in (0, H_0)$, whereby $C(H,h) > 0$ is a constant that depends only on H and h and $g_{H,h} : D \times D \to \mathbb{R}$ is a bounded function for every $h, H \in (0, H_0)$.

Definition 3.1.2 (Gaussian multiplicative chaos). *Let $(X_H)_{H\in(0,H_0)}$ be a family of normalized fractional Brownian fields and $\gamma > 0$. Then we call the random measure M_γ^H on D with density*

$$M_\gamma^H(dx) := \exp\Big(\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}\big[X_H(x)^2\big]\Big)\,dx,$$

the multiplicative chaos associated to $(X_H)_{H\in(0,H_0)}$.

The interesting case of the Gaussian multiplicative chaos is when H tends to 0. Is there a useful way of normalizing to get a non trivial limit? This is, in the general setting, a very hard question to answer and for many cases still unknown. Here we present a result of P. Hager and E. Neuman from 2020 (see Theorem 2.4 in [HN20]).

Theorem 3.1.3. *Let $H_0 \in (0, \frac{1}{2})$, $(X_H)_{H\in(0,H_0)}$ be a family of normalized fractional Brownian fields on a domain D, $\gamma > 0$ and M_γ^H the associated Gaussian multiplicative chaos. Let the covariance function 3.1 of $(X_H)_{H\in(0,H_0)}$ fulfill the following two properties:*

i) The function $C : (0, H_0)^2 \to \mathbb{R}_+$ that describes the constant of 3.1 is uniformly continuous and it holds that

$$\lim_{\bar{H}\to 0} \sup_{0<h,H<\bar{H}} |C(H, h) - 1| = 0. \tag{3.2}$$

ii) The function $g : (0, H_0)^2 \to \mathbb{R}, (h, H) \to g_{H,h}(x, y)$ that describes the bounded functions of 3.1 is uniformly continuous, uniformly in $x, y \in D$, and there exists a bounded function $g : D \times D \to \mathbb{R}$ such that

$$\lim_{(H,h)\to 0} \sup_{x,y\in D} \big|g_{H,h}(x, y) - g(x, y)\big| = 0. \tag{3.3}$$

Then there exists a constant $\gamma^* > \sqrt{\frac{7d}{4}}$ that depends only on the dimension, such that for all $\gamma \leqslant \gamma^*$ the sequence of random measures $(M_\gamma^H)_{H\in(0,H_0)}$ converges as $H \to 0$ to a Borel measure M_γ on D in probability with respect to the weak topology of measures.

Remark 3.1.4. The constant γ^* can be calculated explicitly. For ρ defined in 3.16 we have

$$\gamma^*(d) = \sqrt{\frac{d}{1-\rho}} > \sqrt{\frac{7d}{4}},$$

where $\rho \approx 0.42872$.

Remark 3.1.5. From the two assumptions 3.2 and 3.3 of Theorem 3.1.3 and the covariance structure of the normalized fraction Brownian field, given in 3.1, the point-wise convergence of the covariance function for all $x, y \in D$, $x \neq y$

$$\lim_{H\to 0} \mathbb{E}\big[X_H(x)X_H(y)\big] = \log \frac{1}{\|x-y\|} + g(x,y),$$

where g is the function defined in 3.3, readily follows.

Remark 3.1.6. The proof of Theorem 3.1.3 is very long and extensive. First, we prove the uniform integrability of the random measures $(M_\gamma^H)_{H\in(0,H_0)}$. This is done in section 3.2. Then we prove convergence in section 3.3. The main idea of the proof of uniform integrability is the principle of good points. This approach is based on the work of N. Berestycki on the construction of the Gaussian multiplicative chaos (see [Ber17]). Theorem 3.1.3 also generalizes Corollary 2.2 in [NR18] to a higher dimensional setting.

3.2 The Uniformly Integrability of the Gaussian multiplicative chaos

In this section we show that for all $\gamma < \gamma^*$ the family of random measures $(M_\gamma^H)_{H\in(0,H_0)}$ from Theorem 3.1.3 is uniformly integrable. This takes, in that setting, a bigger amount of effort. We start recalling the definition of a family of random measures being uniformly integrable.

Definition 3.2.1 (uniformly integrable family of random measures)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{I} \neq \varnothing$ an index set. Then a family $(\mu_i)_{i\in\mathcal{I}}$ of random measures on $(\Omega, \mathcal{F})$ is uniformly integrable if for all $A \in \mathcal{F}$ the family of random variables $(\mu_i(A))_{iin\mathcal{I}}$ is uniformly integrable in the usual sense.*

First, we introduce the so called good points where the mass of our limiting measure should be concentrated on.

Definition 3.2.2 (good points)**.** *Let $\bar{H} \in (0, H_0)$ and $H \in (0, \frac{\bar{H}}{2})$. Define the set*

$$J_{H,\bar{H}} := \left\{ H + \frac{1}{n} : n \in \mathbb{N} \text{ and } \frac{1}{\bar{H}-H} < n \leqslant \frac{1}{H} \right\}.$$

Let $x \in D$ and $\alpha > 0$. Then we call the subset of Ω

$$G_\alpha^{H,\bar{H}}(X) := \left\{ \omega \in \Omega : X^h(x)(\omega) \leqslant \frac{\alpha}{h+H} \text{ for all } h \in J_{H,\bar{H}} \right\}$$

the event of x being a good point of order α.

The following Lemma is the first step of our proof of uniformly integrability. It shows that the limiting measure, unless it does not exist, will be concentrated on the good points.

Lemma 3.2.3. *For all $\bar{H} \in (0, H_0)$ and orders $\alpha > 0$ there exists a positive constant $p_\alpha^{\bar{H}} > 0$ that only depends on $\bar{H}$ and α, such that we get the uniform bound from below*

$$\mathbb{P}\big[G_\alpha^{H,\bar{H}}(x)\big] \geqslant 1 - p_\alpha^{\bar{H}} \text{ for all } x \in D \text{ and } 0 < H < \frac{\bar{H}}{2}.$$

Furthermore it holds that $p_\alpha^{\bar{H}} \to 0$ if $\bar{H} \to 0$.

Before we prove Lemma 3.2.3, we shortly show an easy but useful estimate.

Proposition 3.2.4. *Let $Z \sim N(0, \sigma^2)$. Then, for all $x > 0$ the following inequality holds*

$$\mathbb{P}\big[Z > x\big] \leqslant \exp\Big(-\frac{x^2}{2\sigma^2}\Big).$$

Proof. Define $f(x) := \exp(-\frac{x^2}{2\sigma^2}) - \mathbb{P}[Z > x]$, that is smooth. We want to show that for all $x > 0$ we have $f(x) \geqslant 0$. Calculating the derivative, we get that f has a maximum at $x^* := \frac{\sigma}{\sqrt{2\pi}}$. It follows $f(x^*) \geqslant f(0) = 1 - \frac{1}{2} = \frac{1}{2} > 0$. Furthermore, it readily follows that f is monotone increasing in $[0, x^*]$ and monotone decreasing in $[x^*, \infty)$. Therefore, $f \geqslant 0$ in the first interval. Finally we get for $x \geqslant x^*$

$$f(x) \geqslant \lim_{y\to\infty} f(y) = \lim_{y\to\infty} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) - \mathbb{P}\big[Z > y\big] = 0$$

and the result follows. ∎

Proof. of Lemma 3.2.3 Using the covariance function 3.1 of the family of fractional Brownian fields $(X_H)_{H\in(0,H_0)}$ and the two assumptions 3.2 and 3.3 of Theorem 3.1.3 we get that there exist two constants $c_1, c_2 > 0$ such that for all $h \in (0, H_0)$ it holds that

$$0 \leqslant \mathbb{E}\big[X_H(x)^2\big] = C(h,h)\Big(\frac{1}{2h} + g_{h,h}(x,x)\Big) \leqslant (1+c_1)\Big(\frac{1}{2h} + c_2\Big). \tag{3.4}$$

Here it is important that the functions C and g are uniformly continuous to be able to extend them in $\mathbb{R}$ (for more details see Theorem 10.45 in [Cla14]). Now, we start estimating the complementary probability of a point $x \in D$ being a good point of order $\alpha > 0$ with a rough but good enough bound

$$\mathbb{P}[\Omega \backslash G_\alpha^{H,\bar{H}}(x)] = \mathbb{P}\Big[\exists h \in J_{H,\bar{H}} : X_h(x) > \frac{\alpha}{h+H}\Big] \leqslant \sum_{h\in J_{H,\bar{H}}} \mathbb{P}\Big[X_h(x) > \frac{\alpha}{h+H}\Big]. \tag{3.5}$$

As X_h is normally distributed we get with Proposition 3.2.4, equation 3.4 and setting $h = H + \frac{1}{n}$ as $h \in J_{H,\bar{H}}$

$$\mathbb{P}\Big[X_h(x) > \frac{\alpha}{h+H}\Big] \leqslant \exp\Big(-\frac{\alpha^2(2H+\frac{1}{n})^{-2}}{2\mathbb{E}[X_{H+\frac{1}{n}}(x)^2]}\Big) \leqslant \exp\Big(-\frac{\alpha^2}{2(c_1+1)}\underbrace{\frac{(2H+\frac{1}{n})^{-2}}{(H+1n)^{-1}+c_2}}_{\beta(H,n)}\Big).$$

We further estimate using $H + \frac{1}{n} \leqslant 1$ and setting $m := \lfloor \frac{1}{H} \rfloor$

$$\begin{aligned}\beta(H,n) &= \frac{1}{(2H+\frac{1}{n})^2}\frac{H+\frac{1}{n}}{1+c_2(H+\frac{1}{n})}\\ &= \frac{1}{2(2H+\frac{1}{n})^2}\frac{2H+\frac{2}{n}}{1+c_2(H+\frac{1}{n})}\\ &\geqslant \frac{1}{2(2H+\frac{1}{n})}\frac{1}{1+c_2(H+\frac{1}{n})}\\ &\geqslant \frac{1}{4(H+\frac{1}{n})(1+c_2)} \geqslant \frac{1}{4(\frac{1}{m}+\frac{1}{n})(1+c_2)}.\end{aligned}$$

Putting the last two calculations together we get

$$\mathbb{P}\Big[X_h(x) > \frac{\alpha}{h+H}\Big] \leqslant \exp\Big(-\underbrace{\frac{\alpha^2}{8(c_1+1)(c_2+1)}}_{=:\kappa>0}\frac{1}{\frac{1}{m}+\frac{1}{n}}\Big).$$

From the definition of good points 3.2.2 it follows that the sum in equation 3.5 starts at $\lfloor \frac{1}{\bar{H}-H} \rfloor - 1 \geqslant \lfloor \frac{1}{\bar{H}} \rfloor - 1$ and continues up to m. Putting all together, we arrive at

$$\mathbb{P}[\Omega \backslash G_\alpha^{H,\bar{H}}(x)] \leqslant \sum_{n=\lfloor \frac{1}{\bar{H}} \rfloor -1}^{m} \exp\Big(-\frac{\kappa}{\frac{1}{n}+\frac{1}{m}}\Big) \leqslant \exp\Big(-\frac{m\kappa}{2}\Big) + \sum_{n=\lfloor \frac{1}{\bar{H}} \rfloor -1}^{m-1} \exp\Big(-\frac{\kappa}{\frac{1}{n}+\frac{1}{m-1}}\Big).$$

Using the last inequality iteratively, we get

$$\mathbb{P}[\Omega \backslash G_\alpha^{H,\bar{H}}(x)] \leqslant \sum_{n=\lfloor \frac{1}{\bar{H}} \rfloor -1}^{m} \exp\Big(-\frac{n\kappa}{2}\Big) \leqslant \sum_{n=\lfloor \frac{1}{\bar{H}} \rfloor -1}^{\infty} \exp\Big(-\frac{n\kappa}{2}\Big) =: p_\alpha^{\bar{H}}.$$

As the series converges, it follows that $p_\alpha^{\bar{H}} \to 0$ if $\bar{H} \to 0$. ∎

Lemma 3.2.5. *Let $\alpha > \gamma$. Then there exists for every $\epsilon \in (0, \frac{\alpha}{\gamma} - 1)$ a sufficiently small $\bar{H} > 0$ such that for $p_{\alpha-\gamma(1+\epsilon)}^{\bar{H}}$ defined as in Lemma 3.2.3 and for all $x \in D$ and $0 < H \leqslant \frac{\bar{H}}{2}$ it holds that*

$$\mathbb{E}\big[e^{\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]} \mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)\}}\big] \geqslant 1 - p_{\alpha-\gamma(1+\epsilon)}^{\bar{H}},$$

where $\mathbb{1}_A$ denotes the characteristic function of the subset A.

Proof. As $X_H(x)$ is a centered Gaussian random variable, we get

$$\mathbb{E}\big[e^{\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]}\big] = 1.$$

We therefore can define a equivalent probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ via

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := e^{\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]}.$$

With Girsanov's theorem we see that the Gaussian process $(X_h)_{h\in(0,1)}$ has the same variance as before but a shifted mean under the new measure $\mathbb{Q}$. Let $\mathbb{E}_{\mathbb{Q}}[\cdot]$ denote the expectation under the measure $\mathbb{Q}$. We want to estimate the mean of $(X_h)_{h\in(0,1)}$ under the new measure $\mathbb{Q}$. Using the covariance function 3.1 and the second assumption 3.3 of Theorem 3.1.3, we can bound the functions $g_{H,h}$ by a constant $c>0$, such that for all $x \in D$, it holds that

$$\mathbb{E}_{\mathbb{Q}}\big[X_h(x)\big] = \gamma\mathbb{E}\big[X_h(x)X_H(x)\big] \leqslant C(H,h)\gamma\left(\frac{1}{H+h}+c\right).$$

Furthermore, from the definition of $J_{H,\bar{H}}$ it follows that all $h \in J_{H,\bar{H}}$ are bounded by $2H \leqslant h \leqslant \bar{H}$. Now let $0<\epsilon<\frac{\alpha}{\gamma}-1$. As due to 3.3, $sup_{0<h,H<\bar{H}}|C_{h,H}-1| \to 0$ when $\bar{H}\to 0$, we get for $\bar{H}$ sufficiently small

$$\mathbb{E}\big[X_h(x)X_H(x)\big] \leqslant \gamma\frac{1+\epsilon}{h+H} \text{ for all } x\in D \text{ and } h \in J_{H,\bar{H}}.$$

All together, it follows that

$$\begin{aligned}
&\mathbb{E}\big[e^{\gamma X_H(x)-\frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]}\mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)\}}\big] = \mathbb{E}_{\mathbb{Q}}\big[\mathbb{1}_{G_\alpha^{H,\bar{H}}(x)}\big] = \mathbb{Q}(G_\alpha^{H,\bar{H}}(x))\\
&= \mathbb{Q}\left(X^h(x) \leqslant \frac{\alpha}{h+H} \text{ for all } h \in J_{H,\bar{H}}\right)\\
&= \mathbb{P}\left(X_h(x) \leqslant \frac{\alpha}{h+H} - \gamma\mathbb{E}\big[X_h(x)X_H(x)\big] \text{ for all } h \in J_{H,\bar{H}}\right)\\
&\geqslant \mathbb{P}\left(X_h(x) \leqslant \frac{\alpha}{h+H} - \gamma\frac{1+\epsilon}{h+H} \text{ for all } h \in J_{H,\bar{H}}\right)\\
&= \mathbb{P}\left(X_h(x) \leqslant (\alpha-\gamma(1+\epsilon))\frac{1}{h+H} \text{ for all } h \in J_{H,\bar{H}}\right)\\
&= \mathbb{P}\big[G_{\alpha-\gamma(1+\epsilon)}^{H,\bar{H}}(x)\big].
\end{aligned}$$

Now, with Lemma 3.2.3 we get that

$$\mathbb{E}\big[e^{\gamma X_H(x)-\frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]}\mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)}\big] \geqslant 1-p_{\alpha-\gamma(1+\epsilon)}^{\bar{H}},$$

which concludes the proof. ∎

Our goal is now to define a good approximation of our random measure M_γ^H using the concept of good points. Let $\mathcal{B}(D)$ denote the sigma algebra of measurable sets in D. For all $0<\bar{H}<H_0$, $H\in(0,\frac{\bar{H}}{2})$, $\alpha>\gamma$ and $A\in\mathcal{B}(D)$, we define the approximation as

$$I_{\alpha,\gamma}^{H,\bar{H}}(A) := \int_A e^{\gamma X_H(x)-\frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]}\mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)\}}dx, \tag{3.6}$$

which defines a random measure on D. In fact, $I_{\alpha,\gamma}^{H,\bar{H}}$ is M_γ^H restricted to the event of good points. In Lemma 3.2.5 we showed that $I_{\alpha,\gamma}^{H,\bar{H}}$ almost defines a probability measure. Now, we want to show that $I_{\alpha,\gamma}^{H,\bar{H}}$ is square integrable.

Theorem 3.2.6. *For all $\gamma < \gamma^*(d)$ and all $\alpha > \gamma$ close enough to γ, there exists a sufficiently small $\bar{H} > 0$ such that*

$$\sup_{0<H\leqslant\bar{H}/2}\ \sup_{A\in\mathcal{B}(D)} \mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big] < \infty.$$

The proof of this theorem takes a lot of effort. We therefore show the main idea of it and skip one step, that needs a very long calculation. For the proof in all details see Proposition 3.4 in [HN20]. First, we shortly prove an inequality that is needed in the proof of Theorem 3.2.6.

Proposition 3.2.7. *Let $z \in (0,1]$ and $\alpha \in (0,1)$. Then, it holds that*

$$\frac{1-z^\alpha}{\alpha} \leqslant -\log(z).$$

Proof. The inequality is equivalent to $z^\alpha - \log(z)\alpha \geqslant 1$. For $z = 1$ we get $1^\alpha - \log(1)\alpha = 1 - 0 \geqslant 1$, so the inequality is true. Now, we calculate the derivative for $z \in (0,1]$

$$\tfrac{d}{dz}\big(z^\alpha - \log(z)\alpha\big) = \alpha z^{\alpha-1} - \frac{\alpha}{z} = \alpha\big(z^{\alpha-1} - \frac{1}{z}\big) \leqslant \alpha\big(1 - \frac{1}{z}\big) \leqslant 0.$$

As for $z = 1$ the inequality is true and for $z \leqslant 1$ the derivative of the left hand side is less than $\frac{d}{dz}1 = 0$, we get, using the fundamental theorem of calculus, that the inequality holds for all $z \in (0,1]$. ■

Proof of 3.2.6 As the integrand in 3.6 is positive, it immediately follows that for all $A \in \mathcal{B}(D)$ we have

$$\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big] \leqslant \mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(D)^2\big].$$

Thus, it suffices to show that

$$\sup_{0<H<\bar{H}/2} \mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(D)^2\big] < \infty,$$

for small enough $\bar{H} > 0$. For $x, y \in D$ and $H \in (0, H_0)$, let $Z_H(x,y) := \gamma X_H(x) + \gamma X_H(y) - \frac{\gamma^2}{2}\mathbb{E}[(X_H(x) + X_H(y))^2]$. Then it follows

$$\gamma X_H(x) - \frac{\gamma^2}{2} X_H(x)^2 + \gamma X_(y) - \frac{\gamma^2}{2} X_(y)^2 = Z_H(x,y) + \gamma^2 \mathbb{E}\big[X_H(x)X_(y)\big].$$

Note that the term $\gamma^2\mathbb{E}\big[X_H(x)X_(y)\big]$ is deterministic and with 3.1 completely known to us. We define an equivalent probability measure $\mathbb{Q}_{x,y}$ by

$$\frac{d\mathbb{Q}_{x,y}}{d\mathbb{P}} = e^{Z_H(x,y)}. \tag{3.7}$$

As $\gamma X_H(x) + \gamma X_H(y)$ is centered Gaussian, $\mathbb{Q}_{x,y}$ is well defined. We want to use that probability measure to find an estimate for $\mathbb{E}[I_{\alpha,\gamma}^{H,\bar{H}}(D)^2]$. With Fubini's theorem, the covariance function 3.1, our considerations about the measure $\mathbb{Q}_{x,y}$ and a uniform bound $C > 0$ for all the functions $g_{h,H}$, we get

$$\begin{aligned}
&\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(D)^2\big] \\
&= \mathbb{E}\left[\left(\int_D e^{\gamma X_H(x)-\frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]}\mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)\}}dx\right)^2\right] \\
&= \mathbb{E}\left[\int_D\int_D e^{\gamma X_H(x)-\frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]+\gamma X_H(y)-\frac{\gamma^2}{2}\mathbb{E}[X_H(y)^2]}\mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)\cap G_\alpha^{H,\bar{H}}(y)\}}dxdy\right] \\
&= \int_D\int_D \mathbb{E}\Big[e^{\gamma X_H(x)-\frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]+\gamma X_H(y)-\frac{\gamma^2}{2}\mathbb{E}[X_H(y)^2]}\mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)\cap G_\alpha^{H,\bar{H}}(y)\}}\Big]dxdy \\
&= \int_D\int_D \mathbb{E}\Big[e^{Z_H(x,y)+\gamma^2\mathbb{E}\left[X_H(x)X_(y)\right]}\mathbb{1}_{\{G_\alpha^{H,\bar{H}}(x)\cap G_\alpha^{H,\bar{H}}(y)\}}\Big]dxdy \\
&= \int_D\int_D e^{\gamma^2\mathbb{E}[X_H(x)X_H(y)]}\mathbb{Q}_{x,y}\big[G_\alpha^{H,\bar{H}}(x)\cap G_\alpha^{H,\bar{H}}(y)\big]dxdy \\
&= \int_D\int_D \exp\left(C(H,H)\gamma^2\frac{1-\|x-y\|^{2H}}{2H}+g_{H,H}(x,y)\right)\mathbb{Q}_{x,y}\big[G_\alpha^{H,\bar{H}}(x)\cap G_\alpha^{H,\bar{H}}(y)\big]dxdy \\
&\leqslant e^C\int_D\int_D \exp\left(C(H,H)\gamma^2\frac{1-\|x-y\|^{2H}}{2H}\right)\mathbb{Q}_{x,y}\big[G_\alpha^{H,\bar{H}}(x)\cap G_\alpha^{H,\bar{H}}(y)\big]dxdy. \qquad (3.8)
\end{aligned}$$

We need to bound the last term of this estimate uniformly in H. For that purpose, we split the integral into four regions and bound the term region by region. For three of them, this is an easy procedure. One of them requires a long argument. First we define

$$\kappa := \max_{z\in[1,2]}\frac{2(1-e^{-z})^2}{z(2-e^{-2z})}, \tag{3.9}$$

which is well defined and finite as $[1,2]$ is compact and we take a maximum over a continuous function. In fact, we have $\kappa \in (1,2)$. We choose the regions that split $D \times D$ in four parts, in the following way

$$\begin{aligned}
R_1 &:= \left\{(x,y)\in D\times D : \|x-y\| < e^{-\kappa/H}\right\} \\
R_2 &:= \left\{(x,y)\in D\times D : e^{-\kappa/H} \leqslant \|x-y\| < e^{-2/\bar{H}}\right\} \\
R_3 &:= \left\{(x,y)\in D\times D : e^{-2/\bar{H}} \leqslant \|x-y\| < 1\right\} \\
R_4 &:= \left\{(x,y)\in D\times D : 1 \leqslant \|x-y\|\right\}.
\end{aligned} \tag{3.10}$$

As $\frac{H}{\bar{H}} \in (0,\frac{1}{2}]$, it readily follows that the four regions are pairwise disjoint, nonempty and fill up $D \times D$. In addition as D is bounded, also all R_j are bounded and so they have finite measure. We denote the four integrals resulting from our partition as

$$I_j^{H,\bar{H}}(D) := \int\int_{R_i} \exp\left(C(H,H)\gamma^2\frac{1-\|x-y\|^{2H}}{2H}\right)\mathbb{Q}_{x,y}\big[G_\alpha^{H,\bar{H}}(x)\cap G_\alpha^{H,\bar{H}}(y)\big]dxdy,$$

where $j \in \{1,2,3,4\}$. $I_j^{H,\bar{H}}$ also depends on γ and α, but for readability we do not put them into the notation. In total, we get with $K := e^C$,

$$\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(D)^2\big] \leqslant K \sum_{j=1}^{4} I_j^{H,\bar{H}}(D).$$

Now we want to bound each $I_j^{H,\bar{H}}$. We start with $I_1^{H,\bar{H}}$. Let $\epsilon > 0$, again by assumption 3.2, we see that we can bound $C(H,H)$ by $1+\epsilon$ for $H \in (0, \frac{\bar{H}}{2})$, $\bar{H}$ sufficiently small. As J_1 has finite measure, we get

$$\begin{aligned}
I_1^{H,\bar{H}}(D) &\leqslant \int\int_{R_1} \exp\left(C(H,H)\gamma^2 \frac{1-\|x-y\|^{2H}}{2H}\right) dxdy \\
&\leqslant \int\int_{R_1} \exp\left((1+\epsilon)\gamma^2 \frac{1-\|x-y\|^{2H}}{2H}\right) dxdy \\
&\leqslant \int\int_{R_1} \exp\left((1+\epsilon)\gamma^2 \frac{1}{2H}\right) dxdy \\
&\leqslant \exp\left(1+\epsilon)\gamma^2 \frac{1}{2H}\right) |R_1| \\
&\leqslant \exp\left((1+\epsilon)\gamma^2 \frac{1}{2H} - \frac{\kappa d}{H}\right),
\end{aligned} \tag{3.11}$$

where C arises from the volume of the unit ball in $\mathbb{R}^d$. Now for the last term to be finite we need $(1+\epsilon)\gamma^2 \frac{1}{2H} - \frac{\kappa d}{H}$ to stay finite for all $H \in (0, \frac{\bar{H}}{2})$. As ϵ is arbitrarily small and $\kappa > 1$, we have

$$(1+\epsilon)\gamma^2 \frac{1}{2H} - \frac{\kappa d}{H} \leqslant \frac{\gamma^2}{2H} - \frac{d}{H}.$$

For $\gamma^2 < 2d$ it follows that the first integral is finite, so it holds that

$$\sup_{H \in (0,\bar{H})} I_1^{H,\bar{H}}(D) < \infty.$$

For the second bound we use the inequality in 3.2.7. For $H \in (0, \frac{1}{2})$ and for $x, y \in \|x-y\| < 1$ we get

$$\frac{1-\|x-y\|^{2H}}{2H} \leqslant -\log\|x-y\|.$$

Plugging in this inequality and using that for all $(x,y) \in R_3$, we have $\|x-y\| \geqslant \exp(\frac{2}{H})$, so we arrive at

$$\begin{aligned}
I_3^{H,\bar{H}}(D) &\leqslant \int\int_{R_3} \exp\left(\gamma^2 \frac{1-\|x-y\|^{2H}}{2H}\right) dxdy \\
&\leqslant \int\int_{R_3} \exp\Big(\gamma^2(-\log\|x-y\|)\Big) dxdy
\end{aligned}$$

$$\leqslant \int\int_{R_3} \exp\left(\gamma^2 \frac{2}{\bar{H}}\right) dxdy$$
$$\leqslant \exp\left(\gamma^2 \frac{2}{\bar{H}}\right) |D|^2.$$

Therefore, it follows that

$$\sup_{0<H\leqslant \bar{H}/2} I_3^{H,\bar{H}}(D) < \infty.$$

The bound for $I_4^{H,\bar{H}}(D)$ is easy to get. As in R_4, $1 - \|x-y\| \leqslant 0$ and $C(H,H) > 0$, it follows for all $H \in (0, \frac{\bar{H}}{2})$ that

$$\begin{aligned} I_4^{H,\bar{H}}(D) &\leqslant \int\int_{R_4} \exp\left(C(H,H)\gamma^2 \frac{1-\|x-y\|^{2H}}{2H}\right) dxdy \\ &\leqslant \int\int_{R_4} \exp(0) dxdy \\ &= |R_4| \leqslant |D| < \infty. \end{aligned}$$

Finally, we need a bound for $I_2^{H,\bar{H}}(D)$, which one is not so easy to obtain and takes a lot of effort. Therefore, we skip that step here and refer the reader to Proposition 3.5 in [HN20] for a rigorous proof. The upper bound γ^* is determined in that prove. Now, having all four bounds we eventually conclude that

$$\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(D)^2\big] \leqslant K \sum_{j=1}^{4} I_j^{H,\bar{H}}(D) < \infty,$$

and the square integrability of $I_{\alpha,\gamma}^{H,\bar{H}}(A)$ follows for all $A \in \mathcal{B}(D)$. ∎

Analogue to $I_{\alpha,\gamma}^{H,\bar{H}}$, we can define for $\alpha > \gamma$, $\bar{H} \in (0, H_0)$, $H \in (0, \frac{\bar{H}}{2})$ and $A \in \mathcal{B}(D)$

$$L_{\alpha,\gamma}^{H,\bar{H}}(A) := \int_A e^{\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]} \mathbb{1}_{\{\Omega \setminus G_\alpha^{H,\bar{H}}(x)\}} dx.$$

Then clearly it holds that $M_\gamma^H(A) = I_{\alpha,\gamma}^{H,\bar{H}}(A) + L_{\alpha,\gamma}^{H,\bar{H}}(A)$. With Lemma 3.2.5 we immediately get the following result for $L_{\alpha,\gamma}^{H,\bar{H}}$.

Corollary 3.2.8. *For all $\alpha > \gamma$ and $\epsilon > 0$ there exists a $\bar{H} > 0$ such that*

$$\sup_{0>H<\bar{H}/2} \sup_{A \in \mathcal{B}(D)} \mathbb{E}\big[L_{\alpha,\gamma}^{H,\bar{H}}(A)\big] \leqslant \epsilon.$$

Proof. Let $\alpha > \gamma$ and $\epsilon > 0$, we choose a $\delta \in (0, \frac{\alpha}{\gamma} - 1)$. Recall that D is bounded and therefore $|D| < \infty$. With Lemma 3.2.5 and Fubini's theorem we get a $\bar{H} > 0$ sufficiently small such that

$$\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(D)\big] = \mathbb{E}\left[\int_D e^{\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]} \mathbb{1}_{\{\Omega \setminus G_\alpha^{H,\bar{H}}(x)\}} dx\right]$$

$$\geqslant \int_D \big(1 - p^{\bar{H}}_{\alpha-\gamma(1+\delta)}\big)dx = |D|\big(1 - p^{\bar{H}}_{\alpha-\gamma(1+\delta)}\big).$$

As $\mathbb{E}[M^H_\gamma(x)] = 1$ for all $x \in D$ and $H \in (0, H_0)$, we have, with Fubini's theorem, $|D| = \mathbb{E}[M^H_\gamma(D)] = \mathbb{E}[I^{H,\bar{H}}_{\alpha,\gamma}(D)] + \mathbb{E}[L^{H,\bar{H}}_{\alpha,\gamma}(D)]$. Thus, we get

$$\begin{aligned}\sup_{0<H<\bar{H}/2}\sup_{A\in\mathcal{B}(D)} \mathbb{E}\big[L^{H,\bar{H}}_{\alpha,\gamma}(A)\big] &\leqslant \sup_{0<H<\bar{H}/2} \mathbb{E}\big[L^{H,\bar{H}}_{\alpha,\gamma}(D)\big] \\ &= \sup_{0<H<\bar{H}/2}\Big(|D| - \mathbb{E}\big[I^{H,\bar{H}}_{\alpha,\gamma}(D)\big]\Big) \leqslant |D|p^{\bar{H}}_{\alpha-\gamma(1+\delta)}.\end{aligned}$$

Finally, by Lemma 3.2.5 we can choose $\bar{H} > 0$ small enough to get

$$\sup_{0<H<\bar{H}/2}\sup_{A\in\mathcal{B}(D)} \mathbb{E}\big[L^{H,\bar{H}}_{\alpha,\gamma}(A)\big] \leqslant |D|p^{\bar{H}}_{\alpha-\gamma(1+\delta)} \leqslant \epsilon$$

and the result follows. ■

The result of Corollary 3.2.8 shows the interesting fact that the random measure M^H_γ is more and more supported in the good points the smaller H gets. This we want to use to prove the uniform integrability of $(M^H_\gamma)_{H\in(0,H_0)}$. We now have all the ingredients for that.

Theorem 3.2.9. *For all $\gamma < \gamma^*(d)$ the family of random measures $(M^H_\gamma)_{H\in(0,H_0)}$ is uniformly integrable.*

Proof. As already mentioned, we have for all $A \in \mathcal{B}(D)$, $\bar{H} \in (0, H_0)$, $H \in (0, \frac{\bar{H}}{2})$ and $\alpha > \gamma$

$$M^H_\gamma(A) = I^{H,\bar{H}}_{\alpha,\gamma}(A) + L^{H,\bar{H}}_{\alpha,\gamma}.$$

First we have a look at $I^{H,\bar{H}}_{\alpha,\gamma}$. We see that for all $A \in \mathcal{B}(D)$, we get for α sufficiently close to γ and $\bar{H} > 0$ small enough,

$$\sup_{0<H\leqslant\bar{H}/2}\sup_{A\in\mathcal{B}(D)} \mathbb{E}\big[I^{H,\bar{H}}_{\alpha,\gamma}(A)^2\big] < \infty.$$

With Hölder's inequality we get for $n \in \mathbb{N}$ and $A \in \mathcal{B}(D)$

$$\sup_{0<H\leqslant\bar{H}/2} \mathbb{E}\big[I^{H,\bar{H}}_{\alpha,\gamma}(A)\mathbb{1}_{\{I^{H,\bar{H}}_{\alpha,\gamma}(A)>n\}}\big] \leqslant \sup_{0<H\leqslant\bar{H}/2} \mathbb{E}\big[I^{H,\bar{H}}_{\alpha,\gamma}(A)^2\big]^{1/2}\mathbb{P}\big[I^{H,\bar{H}}_{\alpha,\gamma}(A) > n\big]^{1/2}.$$

As $\mathbb{P}[I^{H,\bar{H}}_{\alpha,\gamma}(A) > n]^{1/2} \to 0$ when $n \to \infty$, the uniform integrability follows. For $L^{H,\bar{H}}_{\alpha,\gamma}$ the uniform integrability follows directly from Corollary 3.2.8. Now let $\epsilon > 0$. As $I^{H,\bar{H}}_{\alpha,\gamma}(A)$ is uniformly integrable, there exists a $\delta > 0$ such that for all $B \in \mathcal{B}(D)$ with $\mathbb{P}[B] < \delta$ it holds that

$$\sup_{0<H<\bar{H}/2} \mathbb{E}\big[I^{H,\bar{H}}_{\alpha,\gamma}(A)\mathbb{1}_B\big] < \frac{\epsilon}{2}.$$

Furthermore, with Corollary 3.2.8 it follows for $\bar{H}$ sufficiently small that

$$\sup_{0<H<\bar{H}2} \mathbb{E}\big[L_{\alpha,\gamma}^{H,\bar{H}}(A)\big] < \frac{\epsilon}{2}.$$

Combining these results, it follows that

$$\sup_{0<H<\bar{H}/2} \mathbb{E}\big[M_\gamma^H(A)\mathbb{1}_B\big] \leqslant \sup_{0<H<\bar{H}/2} \mathbb{E}\big[L_{\alpha,\gamma}^{H,\bar{H}}(A)\mathbb{1}_B\big] + \sup_{0<H<\bar{H}/2} \mathbb{E}\big[L_{\alpha,\gamma}^{H,\bar{H}}(A)\big] < \epsilon,$$

and thus, the uniform integrability of $(M_\gamma^H)_{H\in(0,\bar{H}/2)}$. Therefore, for all $(M_\gamma^H)_{H\in(0,H_0)}$ to be uniformly integrable, it suffices to show that $(M_\gamma^H)_{H\in[\bar{H},H_0)}$ is bounded in $L^2(\Omega,\mathcal{F},\mathbb{P})$. As already done, we can, due to the assumptions 3.2 and 3.3, estimate $|C(h,H)| \leqslant C_1$ and $|g_{h,H}(x,y)| \leqslant C_2$ for all $0 < h, H < H_0$ and $x, y \in D$, where $C_1, C_2 > 0$ are constants. With the covariance function 3.1 we get, similar as in the proof of Theorem 3.2.6, for all $H \in [\bar{H}, H_0)$ and $A \in \mathcal{B}(D)$

$$\begin{aligned}
\mathbb{E}\big[M_\gamma^H(A)^2\big] &\leqslant \int\int_A \exp\Big(C(H,H)\gamma^2\frac{1-\|x-y\|^{2H}}{2H} + g_{H,H}(x,y)\Big)dxdy \\
&\leqslant \int\int_A \exp\Big(C_1\gamma^2\frac{1-\|x-y\|^{2H}}{2H} + C_2\Big)dxdy \\
&\leqslant \int\int_{\|x-y\|<1} \exp\Big(C_1\gamma^2\frac{1-\|x-y\|^{2H}}{2H} + C_2\Big)dxdy \\
&\quad + \int\int_{\|x-y\|\geqslant 1} \exp\Big(C_1\gamma^2\frac{1-\|x-y\|^{2H}}{2H} + C_2\Big)dxdy.
\end{aligned}$$

As $\|x-y\|^{2H} < 1$ if $\|x-y\| < 1$, we can estimate the first integral by

$$\begin{aligned}
&\int\int_{\|x-y\|<1} \exp\Big(C_1\gamma^2\frac{1-\|x-y\|^{2H}}{2H} + C_2\Big)dxdy \\
&\leqslant \int\int_{\|x-y\|<1} \exp\Big(C_1\gamma^2\frac{1}{2H} + C_2\Big)dxdy \\
&\leqslant \int\int_{\|x-y\|<1} \exp\Big(C_1\gamma^2\frac{1}{2\bar{H}} + C_2\Big)dxdy \\
&\leqslant C\exp\Big(C_1\gamma^2\frac{1}{2\bar{H}} + C_2\Big) < \infty.
\end{aligned}$$

From $|D| < \infty$ it follows that also the second integral is finite, since

$$\begin{aligned}
&\int\int_{\|x-y\|\geqslant 1} \exp\Big(C_1\gamma^2\frac{1-\|x-y\|^{2H}}{2H} + C_2\Big)dxdy \\
&\leqslant \int\int_{\|x-y\|\geqslant 1} \exp(C_2)dxdy \\
&\leqslant |D|^2\exp(C_2) < \infty.
\end{aligned}$$

All together, we have for all $A \in \mathcal{B}(D)$ that

$$\sup_{H \in [\bar{H}, H_0)} \mathbb{E}\big[M_\gamma^H(A)^2\big] < \infty$$

and the uniform integrability of $(M_\gamma^H)_{H\in(0,H_0)}$ follows. ■

3.3 Convergence of the Gaussian multiplicative chaos

In this section we want to prove the convergence of $(M_\gamma^H)_{H\in(0,H_0)}$ as $H \to 0$. First, we show that $(I_{\alpha,\gamma}^{H,\bar{H}}(A))_{H\in(0,\bar{H})}$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $A \in \mathcal{B}(D)$. This will take more effort. The proofs of these two results follow chapter 6 in [HN20]. Second, we conclude that for all $A \in \mathcal{B}(D)$, $(M_\gamma^H(A))_{H\in(0,H_0)}$ converges in $L^1(\Omega, \mathcal{F}, \mathbb{P})$, that is, using the last section, a quicker result. Finally, we derive from those two results the convergence of the random measures $(M_\gamma^H)_{H\in(0,H_0)}$ with respect to the weak topology of measures on D. Here we follow chapter 6 in [HN20] and chapter 6 in [Ber17].

First we show a direct consequence of the assumptions 3.2 and 3.3 of Theorem 3.1.3. We will need it later for the proof that $(I_{\alpha,\gamma}^{H,\bar{H}}(A))_{H\in(0,H_0)}$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proposition 3.3.1. *Let $H_1 \in (0, \frac{\bar{H}}{2})$. Then for all $x, y \in D$ the limits*

$$C(h,0) := \lim_{H\to 0} C(h,H) \text{ and } g_h(x,y) := \lim_{H\to 0} g_{h,H}(x,y)$$

exist and the following two statements are true.

i) For all $A \in \mathcal{B}(D)$ it holds that

$$\lim_{H\to 0} \sup_{x\in A, h\geqslant H_1} \left| \mathbb{E}\big[X_h(x)X_H(x)\big] - C(h,0)\left(\frac{1}{h} + g_h(x,x)\right)\right| = 0.$$

ii) For $\beta \in (0, e^{-2/\bar{H}})$ it holds that

$$\lim_{H\to 0} \sup_{\|x-y\|\geqslant\beta, h\geqslant H_1} \left| \mathbb{E}\big[X_h(x)X_H(y)\big] - C(h,0)\left(\frac{1-\|x-y\|^h}{h} + g_h(x,y)\right)\right| = 0.$$

Proof. The existence of the two limits is a direct result of 3.2 and 3.3, as every uniformly continuous function can be extended continuously (for more details see Theorem 10.45 in [Cla14]). With 3.1 we get for some constants $C_1, C_2 > 0$

$$\begin{aligned}
&\lim_{H\to 0} \sup_{x\in A, h\geqslant H_1} \left| \mathbb{E}\big[X_h(x)X_H(x)\big] - C(h,0)\left(\frac{1}{h} + g_h(x,x)\right)\right| \\
&= \lim_{H\to 0} \sup_{x\in A, h\geqslant H_1} \left| C(h,H)\left(\frac{1}{H+h} + g_{H,h}(x,x)\right) - C(h,0)\left(\frac{1}{h} + g_h(x,x)\right)\right| \\
&\leqslant \lim_{H\to 0} \sup_{x\in A, h\geqslant H_1} \Bigg(\big|C(h,H) - C(h,0)\big| \left|\frac{1}{H+h} + g_{H,h}(x,x)\right|
\end{aligned}$$

$$+ C(h,0)\left(\left|\frac{1}{H+h} - \frac{1}{h}\right| + \left|g_{h,H}(x,x) - g_h(x,x)\right|\right)\right)$$

$$\leqslant \lim_{H\to 0} \sup_{x\in A, h\geqslant H_1} \left(\left|C(h,H) - C(h,0)\right| C_1 + C_2\left(\left|\frac{1}{H+h} - \frac{1}{h}\right| + \left|g_{h,H}(x,x) - g_h(x,x)\right|\right)\right) = 0.$$

The second limit works the same way. We have

$$\lim_{H\to 0} \sup_{\|x-y\|\geqslant \beta, h\geqslant H_1} \left|\mathbb{E}\big[X_h(x)X_H(y)\big] - C(h,0)\left(\frac{1-\|x-y\|^h}{h} + g_h(x,y)\right)\right|$$

$$= \lim_{H\to 0} \sup_{\|x-y\|\geqslant \beta, h\geqslant H_1} \left|C(h,H)\left(\frac{1-\|x-y\|^{H+h}}{H+h} + g_{H,h}(x,y)\right) - C(h,0)\left(\frac{1-\|x-y\|^h}{h} + g_h(x,y)\right)\right|$$

$$\leqslant \lim_{H\to 0} \sup_{\|x-y\|\geqslant \beta, h\geqslant H_1} \left(|C(h,H) - C(h,0)|C_1 + C_2\left|\frac{1-\|x-y\|^{H+h}}{H+h} - \frac{1-\|x-y\|^h}{h}\right|\right.$$

$$\left. + C_2|g_{h,H}(x,y) - g_h(x,y)|\right) = 0.$$

■

Now, we need to focus on showing the convergence of $(I_{\alpha,\gamma}^{H,\bar{H}}(A))_{H\in(0,\bar{H})}$ in $L^2(\Omega,\mathcal{F},\mathbb{P})$. Let $A \in \mathcal{B}(D)$ and $h, H \in (0,\bar{H})$. Then it holds that

$$\mathbb{E}\big[(I_{\alpha,\gamma}^{h,\bar{H}}(A) - I_{\alpha,\gamma}^{H,\bar{H}}(A))^2\big] = \mathbb{E}\big[I_{\alpha,\gamma}^{h,\bar{H}}(A)^2\big] - 2\mathbb{E}\big[I_{\alpha,\gamma}^{h,\bar{H}}(A)I_{\alpha,\gamma}^{H,\bar{H}}(A)\big] + \mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big].$$

In order to make the expression on the left hand side small, we try to find a sharp upper bound and a sharp lower bound for

$$\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big] \text{ and } \mathbb{E}\big[I_{\alpha,\gamma}^{h,\bar{H}}(A)I_{\alpha,\gamma}^{H,\bar{H}}(A)\big]$$

respectively. The following two Lemmas will show that such bounds exist.

Lemma 3.3.2. *Let $A \in \mathcal{B}(D)$, $\bar{H} \in (0,H_0)$ and $\alpha > \gamma$ sufficiently close to γ. Then there exists a nonnegative function g_α depending only on $\alpha, \bar{H}$ and γ such that*

$$\limsup_{H\to 0} \mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big] \leqslant \int_A\int_A e^{\gamma^2 g(x,y)} \frac{g_\alpha(x,y)}{\|x-y\|^{\gamma^2}} dxdy,$$

where $g(x,y)$ is the limiting function in 3.3.

Idea of the proof. We can use a lot of work that we have already done in the last section. Let us choose $\beta \in (0, e^{-2/\bar{H}})$. As 3.8 in the proof of Theorem 3.2.6, we can estimate $\mathbb{E}[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2]$ in the following way

$$\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big]$$

$$\leqslant K\int_A\int_A \mathbb{1}_{\{\|x-y\|<\beta\}} \exp\left(C(H,H)\gamma^2\frac{1-\|x-y\|^{2H}}{2H}\right)\mathbb{Q}_{x,y}\big[G_{\alpha,\gamma}^{H,\bar{H}}(x) \cap G_{\alpha,\gamma}^{H,\bar{H}}(y)\big]dxdy$$

$$+ K \int_A \int_A \mathbb{1}_{\{\|x-y\|\geqslant\beta\}} \exp\Big(C(H,H)\gamma^2 \frac{1-\|x-y\|^{2H}}{2H} \Big) \mathbb{Q}_{x,y}\big[G_{\alpha,\gamma}^{H,\bar{H}}(x) \cap G_{\alpha,\gamma}^{H,\bar{H}}(y)\big] dxdy$$
$$=: KJ_1(\beta,H) + KJ_2(\beta,H).$$

We want to estimate $J_1(\beta,H)$ and $J_2(\beta,H)$. Recall the four regions in 3.10 we split $D\times D$ with. As $\beta < e^{-2/\bar{H}}$, the support of the integrand of $J_1(\beta,H)$ is contained in $R_1 \cup R_2$. Thus, we get

$$\begin{aligned}
&J_1(\beta,H)\\
&\leqslant \int\int_{R_1} \mathbb{1}_{\{\|x-y\|<\beta\}} \exp\Big(C(H,H)\gamma^2 \frac{1-\|x-y\|^{2H}}{2H} \Big) \mathbb{Q}_{x,y}\big[G_{\alpha,\gamma}^{H,\bar{H}}(x) \cap G_{\alpha,\gamma}^{H,\bar{H}}(y)\big] dxdy\\
&+ \int\int_{R_2} \mathbb{1}_{\{\|x-y\|<\beta\}} \exp\Big(C(H,H)\gamma^2 \frac{1-\|x-y\|^{2H}}{2H} \Big) \mathbb{Q}_{x,y}\big[G_{\alpha,\gamma}^{H,\bar{H}}(x) \cap G_{\alpha,\gamma}^{H,\bar{H}}(y)\big] dxdy\\
&=: J_{1,1}(\beta,H) + J_{1,2}(\beta,H).
\end{aligned}$$

Using the estimate in 3.11, we get

$$J_{1,1}(\beta,H) \leqslant C \int\int_{\{\|x-y\|<\beta\}\cap R_1} \exp\Big((1+\epsilon)\frac{\gamma^2}{2H} \Big) dxdy < \infty.$$

In an similar way we get an estimate for R_2. We cannot show the details here, as we skipped that part in the proof of theorem 3.2.6. For the details see section 4 in [HN20]. We have

$$J_{1,2}(\beta,H) \leqslant C \int\int_{\|x-y\|<\beta} \|x-y\|^{-(1-\eta)\gamma^2} dxdy < \infty,$$

where $\eta > 0$ can be chosen such that the right hand side is finite. Summing up what we did, we can find a function $g_{\alpha,1}(\beta)$ such that $g_{\alpha,1}(\beta) \to 0$ if $\beta \to 0$ and

$$\sup_{0<H\leqslant\bar{H}} J_1(\beta,H) \leqslant g_{\alpha,1}(\beta). \tag{3.12}$$

Next we want to bound $J_2(\beta,H)$. We want to use the equivalent probability measure $\mathbb{Q}_{x,y}$, as defined in 3.7 again. With Girsanov's theorem, it follows that under $\mathbb{Q}_{x,y}$, the vector $(X_h(x),X_h(y))_{h\in(0,\bar{H}]}$ is again Gaussian distributed with the same variance but a different mean, given by

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}_{x,y}}\big[X_h(x)\big] &= \gamma\mathbb{E}\big[X_h(x)(X_H(x)+X_H(y))\big]\\
&= \gamma C(h,H)\Big(\frac{1}{H+h} + g_{H,h}(x,x) + \frac{1-\|x-y\|^{2H}}{H+h} + g_{H,h}(x,y) \Big),\\
\mathbb{E}_{\mathbb{Q}_{x,y}}\big[X_h(y)\big] &= \gamma\mathbb{E}\big[X_h(y)(X_H(x)+X_H(y))\big]\\
&= \gamma C(h,H)\Big(\frac{1-\|x-y\|^{2H}}{H+h} + g_{H,h}(y,x) + \frac{1}{H+h} + g_{H,h}(y,y) \Big).
\end{aligned}$$

With Proposition 3.3.1 it follows that on the event $\|x-y\| \geqslant \beta$, the joint law of $(X_h(x), X_{(}y))_{h\in(0,\bar{H}]}$ converges weakly and uniformly on compact sets $K \subseteq (0,\bar{H}]$ under the measure $\mathbb{Q}_{x,y}$ to a joint distribution $(Y_h(x), Y_h(y))_{h\in(0,\bar{H}]}$ for $H \to 0$. Furthermore, it has the same covariance structure as $(X_h(x), X_h(y))_{h\in(0,\bar{H}]}$ but shifted mean, i.e.

$$\mathbb{E}\big[Y_h(x)\big] = \gamma C(h,0)\left(\frac{2-\|x-y\|^h}{h} + g_h(x,x) + g_h(x,y)\right),$$
$$\mathbb{E}\big[Y_h(y)\big] = \gamma C(h,0)\left(\frac{2-\|x-y\|^h}{h} + g_h(y,y) + g_h(x,y)\right).$$

Let us define, for $\bar{H} \in (0, H_0)$, $\alpha > \gamma$ sufficiently close to γ and $x \in D$, the event

$$\tilde{G}_\alpha^{\bar{H}}(x) := \Bigg\{\omega \in \Omega : Y_h(x)(\omega) \leqslant \frac{\alpha}{h} + \gamma C(h,0)\left(\frac{2-\|x-y\|^h}{h} + g_h(x,x) + g_h(x,y)\right) \text{ for all } h \in (0,\bar{H}]\Bigg\}.$$

Comparing with the definition of good points in 3.2.2, we immediately see that the event $\tilde{G}_\alpha^{\bar{H}}(x)$ under $\mathbb{P}$ has the same probability as the event $G_\alpha^{H,\bar{H}}(x)$ under the measure $\mathbb{Q}_{x,y}$. It may be easily shown that the probability of the event $\tilde{G}_\alpha^{H_1}(x) \cap \tilde{G}_\alpha^{H_1}(y)$ converges uniformly to 1 as $H_1 \to 0$ on the event $\|x-y\| \geqslant \beta$. Furthermore, it follows that

$$\lim_{H\to 0} \mathbb{Q}_{x,y}\big[G_\alpha^{H,\bar{H}}(x) \cap G_\alpha^{H,\bar{H}}(y)\big] = \mathbb{P}\big[\tilde{G}_\alpha^{\bar{H}}(x) \cap \tilde{G}_\alpha^{\bar{H}}(y)\big] =: g_\alpha(x,y). \tag{3.13}$$

Using 3.1 and 3.2, we eventually get, uniformly in $\|x-y\| \geqslant \beta$, that

$$\lim_{H\to 0} \mathbb{E}\big[X_H(x)X_H(y)\big] = -\log\|x-y\| + g(x,y).$$

Since g is bounded, it follows with dominated convergence that

$$\begin{aligned}
&\lim_{H\to 0} J_2(\beta, H) \qquad (3.14)\\
&= \lim_{H\to 0}\int_A\int_A \mathbb{1}_{\{\|x-y\|\geqslant\beta\}} \exp\left(C(H,H)\gamma^2\frac{1-\|x-y\|^{2H}}{2H}\right)\mathbb{Q}_{x,y}\big[G_{\alpha,\gamma}^{H,\bar{H}}(x) \cap G_{\alpha,\gamma}^{H,\bar{H}}(y)\big]dxdy\\
&\leqslant C\lim_{H\to 0}\int_A\int_A \mathbb{1}_{\{\|x-y\|\geqslant\beta\}} e^{\gamma^2\mathbb{E}[X_H(x)X_H(y)]}\mathbb{Q}_{x,y}\big[G_\alpha^{H,\bar{H}} \cap G_\alpha^{H,\bar{H}}(y)\big]dxdy\\
&\leqslant \int_A\int_A \mathbb{1}_{\{\|x-y\|\geqslant\beta\}} e^{\gamma^2 g(x,y)}\frac{g_\alpha(x,y)}{\|x-y\|^{\gamma^2}}dxdy. \qquad (3.15)
\end{aligned}$$

We want to show that this estimate is finite, in order to be able to use dominated convergence. This part we again have to shortcut as we did not go through the whole proof of 3.2.6. For more details see Lemma 4.2 in [HN20]. In their paper P. Hager and E. Neuman showed that for $\delta > 0$ there exists $\bar{H} > 0$ small enough satisfying

$$\lim_{H\to 0} \mathbb{Q}_{x,y}\big[G_\alpha^{H,\bar{H}}(x) \cap G_\alpha^{H,\bar{H}}(y)\big] \leqslant \frac{C}{\gamma^2}\exp\left(\gamma^2\rho\log\|x-y\|\right),$$

where ρ is defined as

$$\rho := \max_{z\in[0,2]} \frac{2(1-e^{-z})^2}{z(2-e^{-2z})} \approx 0.42872. \tag{3.16}$$

Using Lemma 4.2 in [HN20] and 3.13, we conclude for $\gamma < \gamma^*$ that

$$\begin{aligned}
&\sup_{\beta \in (0, e^{-2/\bar{H}})} \int_A \int_A \mathbb{1}_{\{\|x-y\| \geqslant \beta\}} e^{\gamma^2 g(x,y)} \frac{g_\alpha(x,y)}{\|x-y\|^{\gamma^2}} dxdy \\
&\leqslant \sup_{\beta \in (0, e^{-2/\bar{H}})} C(\gamma) \int_A \int_A \mathbb{1}_{\{\|x-y\| \geqslant \beta\}} \frac{\exp(\gamma^2 \rho \log \|x-y\|)}{\|x-y\|^{\gamma^2}} dxdy \\
&\leqslant C(\gamma) \int_A \int_A \frac{1}{\|x-y\|^{\gamma^2(1-\rho)}} dxdy < \infty.
\end{aligned}$$

Now, we have shown that the estimate in 3.14 is finite and can use dominated convergence. We arrive at

$$\lim_{\beta \to 0} \lim_{H \to 0} J_2(\beta, H) \leqslant \int_A \int_A e^{\gamma^2 g(x,y)} \frac{g_\alpha(x,y)}{\|x-y\|^{\gamma^2}} dxdy.$$

Together with 3.12 we conclude that

$$\mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big] < \infty.$$

■

Lemma 3.3.3. *For $A \in \mathcal{B}(D)$, $\alpha > \gamma$ sufficiently close to γ, it holds that*

$$\liminf_{h,H \to 0} \mathbb{E}\big[I_{\alpha,\gamma}^{h,\bar{H}}(A) I_{\alpha,\gamma}^{H,\bar{H}}(A)\big] \geqslant \int_A \int_A e^{\gamma^2 g(x,y)} \frac{g_\alpha(x,y)}{\|x-y\|^{\gamma^2}} dxdy.$$

Proof. The proof is very similar to that one of Lemma 3.3.2. We will not show the details here. For a full version see Lemma 6.4 in [HN20]. ■

Corollary 3.3.4. *For $A \in \mathcal{B}(D)$, $\alpha > \gamma$ sufficiently close to γ, $\bar{H} \in (0, H_0)$ and $\gamma < \gamma^*$, $(I_{\alpha,\gamma}^{H,\bar{H}}(A))_{H \in (0,\bar{H})}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and therefore converges to some limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. Let $A \in \mathcal{B}(D)$, $\alpha > \gamma$ sufficiently close to γ, $\bar{H} \in (0, H_0)$ and $\gamma < \gamma^*$. Then we get

$$\mathbb{E}\big[(I_{\alpha,\gamma}^{h,\bar{H}}(A) - I_{\alpha,\gamma}^{H,\bar{H}}(A))^2\big] = \mathbb{E}\big[I_{\alpha,\gamma}^{h,\bar{H}}(A)^2\big] - 2\mathbb{E}\big[I_{\alpha,\gamma}^{h,\bar{H}}(A) I_{\alpha,\gamma}^{H,\bar{H}}(A)\big] + \mathbb{E}\big[I_{\alpha,\gamma}^{H,\bar{H}}(A)^2\big].$$

Due to Lemma 3.3.2 and Lemma 3.3.3 we can estimate

$$\begin{aligned}
&0 \leqslant \limsup_{h,H \to 0} \mathbb{E}\big[(I_{\alpha,\gamma}^{h,\bar{H}}(A) - I_{\alpha,\gamma}^{H,\bar{H}}(A))^2\big] \\
&\leqslant 2 \int_A \int_A e^{\gamma^2 g(x,y)} \frac{g_\alpha(x,y)}{\|x-y\|^{\gamma^2}} dxdy - 2 \int_A \int_A e^{\gamma^2 g(x,y)} \frac{g_\alpha(x,y)}{\|x-y\|^{\gamma^2}} dxdy = 0.
\end{aligned}$$

Therefore $\mathbb{E}\big[(I_{\alpha,\gamma}^{h,\bar{H}}(A) - I_{\alpha,\gamma}^{H,\bar{H}}(A))^2\big]$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. As $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is complete, the sequence converges to some limit. ■

Corollary 3.3.5. *For all $A \in \mathcal{B}(D)$ the sequence of random variables $(M_\gamma^H(A))_{H\in(0,H_0)}$ converges in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.*

Proof. Analogue as in the proof of Theorem 3.2.9, for all $A \in \mathcal{B}(D)$, $\bar{H} \in (0, H_0)$ and $H \in (0, \frac{\bar{H}}{2})$, we can split up our random measure

$$M_\gamma^H(A) = I_{\alpha,\gamma}^{H,\bar{H}}(A) + L_{\alpha,\gamma}^{H,\bar{H}}(A).$$

Now let $\epsilon > 0$ and $A \in \mathcal{B}(D)$. By Corollary 3.2.8 we can make the second term small. Thus, we can choose $\bar{H} > 0$ sufficiently close to 0 such that

$$\sup_{0<H\leqslant\bar{H}/2} \mathbb{E}\big[L_{\alpha,\gamma}^{H,\bar{H}}(A)\big] \leqslant \frac{\epsilon}{4}.$$

As we have shown in Corollary 3.3.4, $(I_{\alpha,\gamma}^{H,\bar{H}}(A))_{H\in(0,H_0)}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Therefore, we can choose a $H_1 \in (0, \bar{H})$ such that for all $h, H \in (0, H_1)$ we get with Jensen's inequality

$$\mathbb{E}\big[|I_{\alpha,\gamma}^{h,\bar{H}}(A) - I_{\alpha,\gamma}^{H,\bar{H}}(A)|\big]^2 \leqslant \mathbb{E}\big[(I_{\alpha,\gamma}^{h,\bar{H}}(A) - I_{\alpha,\gamma}^{H,\bar{H}}(A))^2\big] \leqslant \frac{\epsilon^2}{4}.$$

Summing up we get for all $h, H \in (0, H_1)$

$$\begin{aligned}\mathbb{E}\big[|M_\gamma^h(A) - M_\gamma^H|\big] &\leqslant \mathbb{E}\big[|I_{\alpha,\gamma}^{h,\bar{H}}(A) - I_{\alpha,\gamma}^{H,\bar{H}}(A)|\big] + \mathbb{E}\big[|L_{\alpha,\gamma}^{h,\bar{H}}(A) - L_{\alpha,\gamma}^{H,\bar{H}}(A)|\big] \\ &\leqslant \frac{\epsilon}{2} + \mathbb{E}\big[|L_{\alpha,\gamma}^{h,\bar{H}}(A)|\big] + \mathbb{E}\big[|L_{\alpha,\gamma}^{H,\bar{H}}(A)|\big] \\ &\leqslant \frac{\epsilon}{2} + 2\frac{\epsilon}{4} = \epsilon.\end{aligned}$$

Thus $(M_\gamma^H)_{H\in(0,H_0)}$ is a Cauchy sequence in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. As $L^1(\Omega, \mathcal{B}(D), \mathbb{P})$ is complete, the convergence result follows. ∎

Proof of Theorem 3.1.3 Let $A \in \mathcal{B}(D)$. By Corollary 3.3.5, we know that $(M_\gamma^H(A))_{H\in(0,H_0)}$ converges in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. As convergence in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ implies convergence in probability for $\gamma < \gamma^*$, we define

$$\mathcal{A} := \Big\{A = [x_1, y_1) \times ... \times [x_d, y_d) : x_i, y_i \in \mathbb{Q}, x_i \leqslant y_i \text{ for all } i \in \{1, ..., n\} \text{ and } \overline{A} \subseteq D\Big\}.$$

Then $\mathcal{A}$ is non empty, countable, intersection stable and generates the Borel sets $\mathcal{B}(D)$ on D. As for every sequence $(H_n)_{n\in\mathbb{N}}$ with $H_n \to 0$ and all $A \in \mathcal{A}$ we have that $(M_\gamma^H(A))$ converges in probability, we can find a subsequence $(H_{n_k})_{k\in\mathbb{N}}$ such that $(M_\gamma^{H_{n_k}}(A))_{k\in\mathbb{N}}$ converges almost surely to some limit. Thus, by the countability of $\mathcal{A}$, we find a subsequence $(H_{n_l})_{l\in\mathbb{N}}$ such that for all $A \in \mathcal{A}$ and D at the same time, the sequence $(M_\gamma^{H_{n_l}}(A))_{l\in\mathbb{N}}$ converges almost surely to some limit. We denote that limit by $M_\gamma(A)$ for $A \in \mathcal{A}$. As $\mathcal{A}$ is intersection stable, generated the Borel sets and $M_\gamma^{H_{n_l}}(D) \to M_\gamma$ almost surely, there exists a random measure on $\mathcal{B}(D)$ that extends M_γ such that $(M_\gamma^{H_{n_l}})_{l\in\mathbb{N}}$ converges almost surely in the sense of weak convergence to M_γ. We want to show uniqueness of the limit. Let $A \in \mathcal{A}$. We want to have

$$M_\gamma(A) = \sup_{B \in \mathcal{A}, B \subseteq A} M_\gamma(B). \tag{3.17}$$

It immediately follows that the right-hand side is less or equal then the left-hand side. For the other inequality we need to make an argument. With Fubini's theorem we get for $A \in \mathcal{A}$ and $H \in (0, H_0)$

$$\begin{aligned}
\mathbb{E}\big[M_\gamma^H(A)\big] &= \mathbb{E}\Big[\int_A e^{\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]} dx\Big] \\
&= \int_A \mathbb{E}\big[e^{\gamma X_H(x) - \frac{\gamma^2}{2}\mathbb{E}[X_H(x)^2]}\big] dx \\
&= \int_A dx = |A|.
\end{aligned}$$

As $M_\gamma^H(A)$ converges in $L^1(\Omega, \mathcal{F}, \mathbb{P})$, we get that

$$\mathbb{E}\big[M_\gamma(A)\big] = \lim_{H \to 0} \mathbb{E}\big[M_\gamma^H(A)\big] = |A|.$$

Now since all $M_\gamma^H(A)$ and $M_\gamma(A)$ are non negative random variables it follows that in 3.17 equality holds. Likewise, there also holds equality in

$$M_\gamma(A) = \inf_{B \in \mathcal{A}, B \supseteq A} M_\gamma(B) \tag{3.18}$$

for all $A \in \mathcal{A}$. Using that $M_\gamma^{H_{n_l}}(D) \to M_\gamma(D)$ almost surely, we get that the random measures $\{M_\gamma^{H_{n_l}}, M_\gamma\}$ are tight in the space of Borel measures on D, equipped with the topology of weak convergence. Let $\tilde{M}_\gamma$ be another limit in probability with respect to the weak convergence. Using Portmanteau's theorem and the equation 3.17 and 3.18, it follows that $\tilde{M}_\gamma(A) = M_\gamma(A)$ for all $A \in \mathcal{A}$ and therefore, by the uniqueness of the measures, that $\tilde{M}_\gamma = M_\gamma$. This implies the weak convergence in probability of $(M_\gamma^H)_{H \in (0, H_0)}$. ■

3.4 Normalization of fractional Brownian fields, statements

In this section we want to show some examples of normalized fractional Brownian fields that agree with the assumptions of Theorem 3.1.3. For that purpose we introduce two ways of generating families of fractional Brownian fields, together with a class of normalizing kernels. The main part will be to prove that all such normalized fractional Brownian fields indeed agree with the assumptions of Theorem 3.1.3. This will be done in the next section. We start giving some examples of fractional Brownian fields. This section follows section 2.2 of [HN20].

Example 3.4.1 (Mandelbrot-van-Ness representation). Let $d = 1$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$ a filtered probability space and $(W_t)_{t\in\mathbb{R}}$ be a two-sided Brownian motion. Then for $H \in (0,1)\backslash\{\frac{1}{2}\}$ we consider

$$\tilde{B}_H(t) := C(H)\int_{\mathbb{R}} \big((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\big)dW_s,\ t\in\mathbb{R} \tag{3.19}$$

where $C(H) > 0$ is a constant only depending on H. Mandelbrot and Van-Ness showed in [MVN68] that $(\tilde{B}_H(t))_{t\in\mathbb{R}}$ defines a fractional Brownian motion with Hurst parameter H. Furthermore V. Dobrićand and F. Ojeda calculated in [DO06] the covariance structure of the process $(\tilde{B}_H(t))_{t\in\mathbb{R}}$ that is given by

$$\mathbb{E}\big[\tilde{B}_H(t)\tilde{B}_h(s)\big] = C_1(h,H)\big(|s|^{h+H} + |t|^{h+H} - |t-s|^{h+H}\big) - C_2(h,H)f_{h,H}(s,t) \tag{3.20}$$

whereby $s,t\in\mathbb{R}$ and $h,H\in(0,1)$ with $H+h\neq 1$, and $f_{h,H}:\mathbb{R}^2\to\mathbb{R}$ given by

$$f_{h,H}(s,t) := \operatorname{sgn}(s)|s|^{h+H} + \operatorname{sgn}|t|h + H - \operatorname{sgn}(t-s)|t-s|^{h+H}.$$

For this family of fractional Brownian motions there is no useful extension to higher dimensions.

Example 3.4.2. Another example of a construction of a family of fractional Brownian fields is the following. Let again $d = 1$, $H\in(0,1)\backslash\{\frac{1}{2}\}$. We define

$$B_H(t) := C_3(1,H)\int_{\mathbb{R}} \big(|t-s|^{H-\frac{1}{2}} - |s|^{H-\frac{1}{2}}\big)dW_s,\ t\in\mathbb{R}, \tag{3.21}$$

where $H\in(0,1)\backslash\{\frac{1}{2}\}$ and $C_3(1,H) > 0$ is a constant only depending on H. In [DO06] the covariance structure of this process was calculated and is given by

$$\mathbb{E}\big[B_H(t)B_h(s)\big] = c_3^1(H,h)\big(|s|^{h+H} + |t|^{h+H} - |t-s|^{h+H}\big),\ s,t\in\mathbb{R}, \tag{3.22}$$

where $c_3^1(h,H) > 0$ is a constant depending on h and H. This process can be extended in a multidimensional setting in the following way. For $d \geqslant 1$ we define

$$B_H(t) := C_3(d,H)\int_{\mathbb{R}^d} \big(\|x-y\|^{H-\frac{1}{2}} - \|y\|^{H-\frac{1}{2}}\big)W(dy),\ x\in\mathbb{R}^d, \tag{3.23}$$

where W is a white noise in $\mathbb{R}^d$ and again $H\in(0,1)\backslash\{\frac{1}{2}\}$. In Lemma 2.8 of [HN20] it was shown that the covariance structure of this process is given by

$$\mathbb{E}\big[B_H(x)B_h(y)\big] = c_3^d(H,h)\big(\|x\|^{H+h} + \|y\|^{H+h} - \|x-y\|^{H+h}\big) \tag{3.24}$$

where $x,y\in\mathbb{R}^d$ and $c_3^d(H,h) > 0$ is a constant only depending on d, H and h.

Next, we define a class of normalizing kernels for which we will make use of generating families of normalized fractional Brownian fields agreeing with Definition 3.1.

Definition 3.4.3 (Normalizing kernels)**.** *Let $d \geqslant 1$, $H_0 \in (0, \frac{1}{2})$ and $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function. We say that ψ is a normalizing kernel of order H_0 if the following conditions hold true:*

i) For all $y \in D$ the map $x \mapsto \psi(x, y)$ is almost everywhere continuous and it holds that

$$\int_{\mathbb{R}^d} \psi(x, y) dx = 1. \tag{3.25}$$

ii) ψ is bounded in the following ways:

$$\sup_{y \in D} \int_{\mathbb{R}^d} \|x\|^{2H_0} \psi(x, y) dx < \infty \tag{3.26}$$

$$\sup_{y \in D} \int_{\mathbb{R}^d} \min\{\log \|x - y\|, 0\}^2 \psi(x, y) dx < \infty \tag{3.27}$$

$$\sup_{y,v} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \min\{\log \|x - u\|, 0\}^2 \psi(x, y) \psi(u, v) dx du < \infty. \tag{3.28}$$

The set of normalizing kernels of order H_0 is denoted by $\mathcal{N}_{H_0}(D)$.

The result of generating families of fractional Broenian motions by P. Hager and E. Neunman is the following.

Theorem 3.4.4. *Let $(B_H)_{H \in (0, H_0)}$ be a family of fractional Brownian fields constructed either by 3.21 or 3.23 and $\psi \in \mathcal{N}_{H_0}(D)$. We define*

$$X_H(x) := \Gamma(H)^{\frac{1}{2}} \left(B_H(x) - \int_{\mathbb{R}^d} B_H(y) \psi(y, x) dy \right), \; x \in D, H \in (0, H_0), \tag{3.29}$$

where $\Gamma(\cdot)$ denotes the gamma function and $H_0 = 1$ as in the case of Example 3.4.1 and $H_0 = \frac{1}{2}$ as in Example 3.4.2. Then $(X_H)_{H \in (0, H_o)}$ is a family of normalized fractional Brownian fields according to Definition 3.1.1 which agrees with the assumptions of Theorem 3.1.3, i.e. the associated Gaussian multiplicative chaos converges in probability as $H \to 0$ with respect to the weak topology.

The proof of this theorem takes some effort and is shown in the next section. Using this result and Levy's continuity theorem on the space of tempered distributions, one gets a very strong convergence result. First we state Levy's theorem.

Theorem 3.4.5 (Levy's continuity theorem on the space of tempered distributions)**.** *Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of generalized random fields on $\mathcal{S}(\mathbb{R}^d)$ and $(\Phi_n)_{n \in \mathbb{N}}$ their characteristic functions given by 2.2, where all h_n are seen as probability measures on $\mathcal{S}'(\mathbb{R}^d)$. If (Φ_n) converges point-wise to a functional $\Phi : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ that is continuous at the origin, then there exists a generalized random field h on $\mathcal{S}(\mathbb{R}^d)$ such that Φ is its characteristic function and h_n converges in distribution to h in probability with respect to the strong topology.*

Proof. See Theorem 2.3 in [BDW17]: ■

A very interesting consequence of this theorem is the following.

Corollary 3.4.6. *Let $(h_n)_{n\in\mathbb{N}}$ and h be generalized random fields on $\mathcal{S}(\mathbb{R}^d)$. Then the following statements are equivalent:*

i) $(h_n)_{n\in\mathbb{N}}$ *converges in distribution with respect to the strong topology to* h.

ii) $(h_n)_{n\in\mathbb{N}}$ *converges in distribution with respect to the weak topology to* h.

iii) The corresponding characteristic functions $(\Phi_n)_{n\in\mathbb{N}}$ *converge point-wise on* $\mathcal{S}(\mathbb{R}^d)$.

iv) $(h_n(\phi)_{n\in\mathbb{N}}$ *converges in distribution for all* $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Proof. See Corollary 2.4 in [BDW17]. ■

Remark 3.4.7. Let $(X_H)_{H\in(0,H)}$ be a family of random fields constructed as in Theorem 3.4.4. Then the theorem states that the covariance structure is of a form as in 3.1 and in addition fulfills the assumptions 3.2 and 3.3 of Theorem 3.1.3. Therefore, by Remark 3.1.5 it holds that the covariance kernels converge point-wise to a log correlated covariance kernel i.e.

$$\lim_{H\to 0} \mathbb{E}\big[X_H(x)X_H(y)\big] = \log\frac{1}{\|x-y\|} + g(x,y),$$

where g is a bounded function. If one showed now that

$$\lim_{H\to 0} \mathbb{E}\big[(X_H,\phi)(X_H,\phi)\big] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d)}\left(\log\frac{1}{\|x-y\|} + g(x,y)\right)dxdy, \text{ for } \phi,\psi \in \mathcal{S}(\mathbb{R}^d),$$

one would get, with Corollary 3.4.6, the weak convergence in probability of $(X_H)_{H\in(0,H_0)}$ to a log correlated field X. A special case of that result was proven as Theorem 2.1 of [NR18]. Nevertheless, the convergence of the Gaussian multiplicative chaos associated to $(X_H)_{H\in(0,H_0)}$ follows from Theorem 3.1.3.

We give a famous example of a process that can be represented such that it fulfills the requirements of Theorem 3.4.4.

Example 3.4.8 (fractional Ornstein-Uhlenbeck process)*.* Let $m\in\mathbb{R}$ be a mean and parameter $\alpha,\gamma > 0$ given. We define the fractional Ornstein-Uhlenbeck process via the fractional Brownian motion $(B_H)_{H\in(0,\frac{1}{2})}$ in the following way

$$dZ_H(t) = \gamma dB_H(t) - \alpha(Z_H(t)-m)dt, t\in\mathbb{R}.$$

In section 1 in [CKM03] it was shown that the Ornstein-Uhlenbeck process has the following representation.

$$Z_H(t) = m + \gamma B_H(t) - \gamma \int_{-\infty}^{t} \alpha e^{-\alpha(t-s)} B_H(s) ds.$$

If one subtracts the mean and scales the process by $\sqrt{\Gamma(H)}$, it is possible to find a representation of the process in the form given in Theorem 3.4.4. It readily follows that the normalizing kernel needed is given by

$$\psi : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x, y) \mapsto \mathbb{1}_{\{x \leqslant y\}} \alpha e^{-\alpha(y-x)} \end{cases}.$$

Furthermore, ψ is indeed a normalizing kernel of order H_0 for all $H_0 \in (0, \frac{1}{2})$ and every bounded domain D.

3.5 Normalization of fractional Brownian fields, proof

In this section we present the proof of Theorem 3.4.4. It follows section 8 in [HN20]. First we denote the integral in 3.29 as

$$I_H(x) := \int_{\mathbb{R}^d} B_H(y) \psi(y, x) dy, \; x \in \mathbb{R}^d.$$

We start showing simple properties of $I_H(x)$.

Proposition 3.5.1. *Let $(B_H)_{H \in (0, H_0)}$ be a fractional Brownian field defined as in Example 3.4.1 or Example 3.4.2. Then the integral $I_H(x)$ is well defined for all $x \in \mathbb{R}^d$, almost surely finite and Gaussian.*

Proof. By Lemma 2.9.11, it holds that $(B_H)_{H \in (0, H_0)}$ has almost surely Hölder continuous paths. Therefore, it is also measurable. Furthermore, by Lemma 5 and Remark 5 in [KMM15] for all $\mathrm{R}\epsilon > 0$ there exists an almost surely finite random variable Y_ϵ such that

$$|B_H(x)| \overset{a.s.}{\leqslant} Y_\epsilon \big(1 + \|x\|^{H+\epsilon}\big) \text{ for all } x \in \mathbb{R}^d.$$

As $\psi \in \mathcal{N}_{H_0}(D)$, it follows, by choosing $\epsilon = H_0$ and equations 3.25 and 3.26, that

$$\begin{aligned} \int_{\mathbb{R}^d} |B_H(y)| \psi(y, x) dy &\leqslant \int_{\mathbb{R}^d} Y_\epsilon \big(1 + \|y\|^{H+\epsilon}\big) \psi(y, x) dx \\ &\leqslant Y_\epsilon \Bigg(\underbrace{\int_{\mathbb{R}^d} \psi(y, x) dx}_{=1} + \underbrace{\int_{\mathbb{R}^d} \|y\|^{2H_0} \psi(y, x) dx}_{<\infty} \Bigg) < \infty \end{aligned}$$

uniformly for all $x \in D$. Therefore the integral is well defined and almost surely finite. As the Riemann integral is defined over converging Riemann sums and those are, in our case, finite sums of centered Gaussian random variables, it follows that the Riemann integral is also centered Gaussian (see Theorem 2.60 in [Sch21]). ■

Next we want to calculate the covariance structure of the normalization X_H. There we have to distinguish between the two cases. The important part is that it agrees with 3.1.

Lemma 3.5.2. *The family of random fields $(X_H)_{H\in(0,H_0)}$, given in 3.29, is a family of normalized fractional Brownian motions according to Definition 3.1.*

Proof. First we treat the case, where $(B_H)_{H\in(0,H_0)}$ is given by 3.23. In that case we have

$$\mathbb{E}\big[B_H(x)B_h(y)\big] = c_3^d(H,h)\big(\|x\|^{H+h} + \|y\|^{H+h} - \|x-y\|^{H+h}\big),\ x,y\in\mathbb{R}^d,\ h,H\in(0,H_0).$$

We want to use Fubini's theorem in order to be able to exchange the integral with the expectation. Therefore we estimate using $|ab|\leqslant a^2+b^2$ for $a,b\in\mathbb{R}$ and 3.26

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbb{E}\big[|B_H(x)B_h(u)|\big]\psi(u,y)du &\leqslant \int_{\mathbb{R}^d}\frac{\mathbb{E}\big[B_H(x)^2+B_H(u)^2\big]}{2}\psi(u,y)du\\ &= \int_{\mathbb{R}^d}\Big(c_3^d(H,H)\|x\|^{2H} + c_3^d(h,h)\|u\|^{2h}\Big)\psi(u,y)du < \infty.\end{aligned}$$

In an analogue way, with 3.25 and 3.26, it follows that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{E}\big[|B_H(v)B_h(u)|\big]\psi(u,y)\psi(v,x)dudv\\ &\leqslant \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(c_3^d(H,H)\|u\|^{2H} + c_3^d(h,h)\|v\|^{2h}\Big)\psi(u,y)\psi(v,x)dudv\\ &\leqslant c_3^d(H,H)\int_{\mathbb{R}^d}\underbrace{\int_{\mathbb{R}^d}\psi(v,x)dv}_{=1}\|u\|^{2H}\psi(u,y)du + c_3^d(h,h)\int_{\mathbb{R}^d}\underbrace{\int_{\mathbb{R}^d}\psi(u,y)du}_{=1}\|v\|^{2h}\psi(v,x)dv\\ &\leqslant c_3^d(H,H)\int_{\mathbb{R}^d}\|u\|^{2H}\psi(u,y)du + c_3^d(h,h)\int_{\mathbb{R}^d}\|v\|^{2h}\psi(v,x)dv < \infty.\end{aligned}$$

Therefore, we are allowed to use Fubini's theorem and get

$$\begin{aligned}&\frac{1}{\Gamma(H)^{\frac12}\Gamma(h)^{\frac12}}\mathbb{E}\big[X_H(x)X_h(y)\big]\\ &= \mathbb{E}\Big[\Big(B_H(x) - \int_{\mathbb{R}^d}B_H(u)\psi(u,x)du\Big)\Big(B_H(y) - \int_{\mathbb{R}^d}B_H(v)\psi(v,y)dv\Big)\Big]\\ &= \mathbb{E}\big[B_H(x)B_h(y)\big] - \int_{\mathbb{R}^d}\mathbb{E}\big[B_H(x)B_h(u)\big]\psi(u,y)du\\ &\quad - \int_{\mathbb{R}^d}\mathbb{E}\big[B_H(y)B_h(v)\big]\psi(v,x)dv + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{E}\big[B_H(u)B_h(v)\big]\psi(u,y)\psi(v,x)dudv.\end{aligned}$$

First we consider, using the covariance structure of $(B_H)_{H\in(0,H_0)}$ given in 3.24,

$$\frac{1}{c_3^d(H,h)}\int_{\mathbb{R}^d}\mathbb{E}\big[B_H(x)B_h(u)\big]\psi(u,y)du$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \left(\|x\|^{H+h} + \|u\|^{H+h} - \|x-u\|^{H+h}\right)\psi(u,y)du \\
&= \|x\|^{H+h} \underbrace{\int_{\mathbb{R}^d} \psi(u,y)du}_{=1} + \int_{\mathbb{R}^d} \|u\|^{H+h}\psi(u,y)du - \int_{\mathbb{R}^d} \|x-u\|^{H+h}\psi(u,y)du.
\end{aligned}
$$

Furthermore, we get

$$
\begin{aligned}
&\frac{1}{c_3^d(H,h)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{E}\big[B_H(u)B_h(v)\big]\psi(u,y)\psi(v,x)dudv \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(\|u\|^{H+h} + \|v\|^{H+h} - \|u-v\|^{H+h}\right)\psi(u,y)\psi(v,x)dudv \\
&= \int_{\mathbb{R}^d} \|u\|^{H+h}\psi(u,y)du + \int_{\mathbb{R}^d} \|v\|^{H+h}\psi(v,x)dv - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|u-v\|^{H+h}\psi(u,y)\psi(v,x)dudv.
\end{aligned}
$$

Now, we define the constant $C(H,h)$ and the functions $g_{H,h}$

$$
\begin{aligned}
C(H,h) &:= c_3^d(H,h)\sqrt{\Gamma(H)\Gamma(h)}(H+h) \\
g_{H,h}(x,y) &:= -\int_{\mathbb{R}^d} \frac{1-\|x-u\|^{H+h}}{H+h}\psi(u,y)du - \int_{\mathbb{R}^d} \frac{1-\|y-v\|^{H+h}}{H+h}\psi(v,x)dv \\
&\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1-\|u-v\|^{H+h}}{H+h}\psi(u,y)\psi(v,x)dudv. \qquad (3.30)
\end{aligned}
$$

It clearly holds that $C(H,h)$ is finite. Furthermore, as $\psi \in \mathcal{N}_{H_0}(D)$ and with 3.26, it follows that $g_{H,h}$ is finite. Now, putting all together, we get

$$
\begin{aligned}
&\frac{1}{c_3^d(H,h)\sqrt{\Gamma(H)\Gamma(h)}}\mathbb{E}\big[X_H(x)X_h(y)\big] \\
&= \|x\|^{H+h} + \|y\|^{H+h} - \|x-y\|^{H+h} - \|x\|^{H+h} - \int_{\mathbb{R}^d} \|u\|^{H+h}\psi(u,y)du \\
&+ \int_{\mathbb{R}^d} \|x-u\|^{H+h}\psi(u,y)du - \|y\|^{H+h} - \int_{\mathbb{R}^d} \|v\|^{H+h}\psi(v,x)dv + \int_{\mathbb{R}^d} \|y-v\|^{H+h}\psi(v,x)dv \\
&+ \int_{\mathbb{R}^d} \|u\|^{H+h}\psi(u,y)du + \int_{\mathbb{R}^d} \|v\|^{H+h}\psi(v,x)dv - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|u-v\|^{H+h}\psi(u,y)\psi(v,x)dudv \\
&= -\|x-y\|^{H+h} + \int_{\mathbb{R}^d} \|x-u\|^{H+h}\psi(u,y)du + \int_{\mathbb{R}^d} \|y-v\|^{H+h}\psi(v,x)dv \\
&- \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \|u-v\|^{H+h}\psi(u,y)\psi(v,x)dudv \\
&\overset{3.25}{=} 1 - \|x-y\|^{H+h} - \int_{\mathbb{R}^d} \left(1-\|x-u\|^{H+h}\right)\psi(u,y)du - \int_{\mathbb{R}^d} \left(1-\|y-v\|^{H+h}\right)\psi(v,x)dv \\
&+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left(1-\|u-v\|^{H+h}\right)\psi(u,y)\psi(v,x)dudv.
\end{aligned}
$$

Using 3.5 we finally get

$$\mathbb{E}\big[X_H(x)X_h(y)\big] = C(H,h)\bigg(\frac{1-\|x-y\|^{H+h}}{H+h} + g_{H,h}(x,y)\bigg).$$

Eventually, it follows that $(X_H)_{H\in(0,H_0)}$ is a family of normalized fractional Brownian motions.

Now for $(B_H)_{H\in(0,H_0)}$, defined as in Example 3.4.2, the whole procedure works in a very analogue way. We will skip the details here and just write down the outcome. The constant is given by

$$\tilde{C}(H,h) := C_1(H,h)\sqrt{\Gamma(H)\Gamma(h)}(H+h). \tag{3.31}$$

Clearly, this is finite. Moreover, the family of functions $\tilde{g}_{H,h}$ is given by

$$\tilde{g}_{H,h}(x,y) := g_{H,h} + \frac{C_2(H,h)}{C_1(H,h)(H+h)}\bigg(\int_{\mathbb{R}} \operatorname{sgn}(u)|x-u|^{H+h}\psi(u,y)du \\ + \int_{\mathbb{R}} \operatorname{sgn}(v)|y-v|^{H+h}\psi(v,x)dv - \int_{\mathbb{R}}\int_{\mathbb{R}} sgn(v-u)|u-v|^{H+h}\psi(u,y)\psi(v,x)dudv\bigg). \tag{3.32}$$

Again, the boundedness of $\tilde{g}_{H,h}$ follows by 3.26. ■

Next, we want to prove that the maps $(H,h)\to C(H,h)$ and $(H,h)\to g_{H,h}$ agree with the assumptions 3.2 and 3.3 of Theorem 3.1.3. For that purpose, we will cite a proposition that we need for the proof.

Remark 3.5.3. For $a\in\mathbb{R}$ it holds that $e^a-1\geqslant a$. This is equivalent to $e^a-1-a\geqslant 0$. For $a=0$ we have $e^0-1-0=0\geqslant 0$. Furthermore the derivative is given by e^a-1 and therefore it is $\leqslant 0$ for $a\leqslant 0$ and $\geqslant 0$ for $a\geqslant 0$. With the fundamental theorem of calculus the inequality follows. Furthermore, for $b>0$ we get $b^a-1=e^{a\log(b)}-1\geqslant a\log(b)$.

Proposition 3.5.4. *For $H_0\in[\frac{1}{2},1]$ and $h\in(0,2H_0]$ it holds that*

$$0\leqslant \log\frac{1}{z}-\frac{1-z^h}{h}\leqslant\frac{3h}{2}\log^2(z),\ \text{for } z\in(0,1]$$
$$0\leqslant \log\frac{1}{z}-\frac{1-z^h}{h}\leqslant 4H_0^2h\big(z^{2H_0}-1-\log(z)\big),\ \text{for } z\in(1,\infty).$$

Proof. The proof is very elemental. For $z>0$ and $h\in[0,1]$ we define

$$\beta(h,z) := \log\frac{1}{z}-\frac{1-z^h}{h}. \tag{3.33}$$

First we discuss the lower bound $\beta(h,z)\geqslant 0$. This is equivalent to $h\beta(h,z)\geqslant 0$ and further

$$0\leqslant h\beta(h,z) = -h\log(z)-1+z^h = -h\log(z)-1+e^{h\log(z)} \\ \Leftrightarrow h\log(z)\leqslant e^{h\log(z)}-1,$$

which is indeed true by Remark 3.5.3. We apply Taylor's theorem to the function $h \mapsto 1-z^h$ at $h=0$ and get

$$1-z^h = \sum_{k=1}^{\infty} \frac{-\frac{\partial^k}{\partial h^k} z^h}{k!} h^k = \sum_{k=1}^{\infty} \frac{-\log(z)^k z^h}{k!} h^k.$$

It follows that

$$\lim_{h\to 0} \frac{1-z^h}{h} = -\log(z). \tag{3.34}$$

Thus, we can conclude

$$\begin{aligned}
\lim_{h\to 0} \frac{\beta(h,z)}{h} &= \lim_{h\to 0} \left(\frac{-\log(z)}{h} - \frac{1-z^h}{h^2} \right) \\
&= \lim_{h\to 0} \left(\frac{-\log(z)}{h} + \sum_{k=1}^{\infty} \frac{\log(z)^k z^h}{k!} h^{k-2} \right) \\
&= \lim_{h\to 0} \left(\frac{-\log(z)}{h} + \frac{\log(z) z^h}{h} + \frac{\log(z)^2 z^h}{2} \right) \\
&= \lim_{h\to 0} \left(\frac{\log(z)\left(z^h-1\right)}{h} + \frac{\log(z)^2 z^h}{2} \right) \\
&= -\log(z) \lim_{h\to 0} \frac{1-z^h}{h} + \frac{\log(z)^2}{2} \\
&\overset{3.34}{=} \log(z)\log(z) + \frac{\log(z)^2}{2} = \frac{3\log(z)^2}{2}.
\end{aligned}$$

Furthermore, we calculate

$$\begin{aligned}
\frac{\partial}{\partial h} \frac{\beta(h,z)}{h} &= \frac{\partial}{\partial h} \left(\log\frac{1}{z} - \frac{1-z^h}{h} \right) \\
&= \frac{\partial}{\partial h} \left(\frac{-h\log(z) - (1-z^h)}{h^2} \right) \\
&= \frac{\left(-\log(z) + \log(z) z^h\right) h^2 + \left(h\log(z) + (1-z^h)\right) 2h}{h^4} \\
&= \frac{h\log(z)(1+z^h) + 2(1-z^h)}{h^3}.
\end{aligned}$$

For $z \in (0,1]$ it follows with Remark 3.5.3

$$\frac{\partial}{\partial h} \frac{\beta(h,z)}{h} \leqslant \frac{2h\log(z) + 2(1-e^{h\log(z)})}{h^3} \leqslant \frac{2h\log(z) - 2h\log(z)}{h^3} = 0.$$

Thus, by the fundamental theorem of calculus it follows for $z \in (0,1]$

$$0 \leqslant \frac{\beta(h,z)}{h} \leqslant \lim_{h\to 0} \frac{\beta(h,z)}{h} = \frac{3\log(z)^2}{2}$$

and, by multiplying the inequality with $h>0$, the result follows. Now we treat the case $z \in (1,\infty)$. Again, with Remark 3.2.7, we can estimate

$$\frac{\partial}{\partial h}\frac{\beta(h,z)}{h} = \frac{h\log(z)(1+z^h)+2(1-z^h)}{h^3} \geqslant \frac{2h\log(z)-2h\log(z)}{h^3} = 0.$$

By the fundamental theorem of calculus and using $H_0 \geqslant \frac{1}{2}$ and $\log(z) > 0$, it follows that

$$=\leqslant \frac{\beta(h,z)}{h} \leqslant \frac{\beta(2H_0,z)}{2H_0} = -\frac{\log(z)}{2H_0} - \frac{1-z^{2H_0}}{4H_0^2} \leqslant \frac{1-z^{2H-0}-\log(z)}{4H_0^2}.$$

By multiplying the inequality with $h > 0$, the result follows. ■

Now we are finally able to proof Theorem 3.4.4.

Idea of the proof of Theorem 3.4.4. First we consider the map $(0,H_0)^2 \to \mathbb{R}_+, (H,h) \mapsto C(H,h)$ that was defined in 3.30. This map can be calculated explicitly (see section 8 in [HN20]). It is given by

$$C(H,h) = \sqrt{\Gamma(H)\Gamma(h)}\frac{\Gamma(\frac{H+h+1}{2})\sqrt{\Gamma(\frac{H+d}{2})H\Gamma(2H)\sin(H\pi)\Gamma(\frac{h+d}{2})h\Gamma(2h)\sin(h\pi)}}{\Gamma(\frac{H+h+d}{2})\Gamma(H+h)\sin(\frac{H+h}{2}\pi)\sqrt{\Gamma(H+\frac{1}{2})\Gamma(h+\frac{1}{2})}}.$$

For the limit $(H,h) \to 0$, we consider first

$$\lim_{z\to 0}\frac{sin(z)}{z} = 1$$

and the following property of the Gamma function for $z > 0$

$$z\Gamma(z) = \Gamma(z+1) \qquad \Rightarrow \qquad \lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0}\Gamma(z+1) = \Gamma(1) = 1.$$

Thus, it follows for $a, b > 0$ that

$$\lim_{z\to 0}\Gamma(az)\sin(bz) = \frac{b}{a}\lim_{z\to 0}\Gamma(az)az\frac{\sin(bz)}{bz} = \frac{b}{a}.$$

Furthermore, using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, it follows that

$$\lim_{(H,h)\to 0} C(H,h) = \lim_{(H,h)\to 0}\sqrt{\frac{1}{Hh}}\frac{\Gamma(\frac{H+h+1}{2})\sqrt{\Gamma(\frac{H+h+d}{2})H2H\frac{1}{H\pi}\Gamma(\frac{h+d}{2})h\frac{1}{2h}h\pi}}{\Gamma(\frac{H+h+d}{2})\frac{1}{H+h}\frac{H+h}{2}\pi\sqrt{\Gamma(H+\frac{1}{2})\Gamma(h+\frac{1}{2})}} = 1.$$

For $\tilde{C}(H,h)$, the result follows in an analogue way.

Now, we want to prove that assumption 3.3 holds for $g_{H,h}$. Again, we start with the case of Example 3.4.1. First we show the uniform continuity. In particular we show that it has a uniformly bounded derivative. With Proposition 3.2.7 for $z > 0$ and $h \in (0, 2H_0)$ we estimate

$$\left|\frac{\partial}{\partial z}\frac{1-z^h}{h}\right| = \left|\frac{-z^h\log(z)h-(1-z^h)}{h^2}\right|$$

$$
\begin{aligned}
&= \left| \frac{-z^h \log(z)h + \log(z)h + \log(\frac{1}{z})h - (1-z^h)}{h^2} \right| \\
&\leqslant \left| \frac{1-z^h}{h} \right| |\log(z)| + \left| \frac{\log(\frac{1}{z}) - \frac{1-z^h}{h}}{h} \right| \\
&\leqslant C\big(\log_-^2(z) + z^{2H_0}\big),
\end{aligned}
$$

where $C > 0$ is a constant. Now, plugging that into the definition of $g_{H,h}$ and interchanging the integral and differentiation, we get that the map $(H,h) \mapsto g_{H,h}(x,y)$ is indeed differentiable for all $x,y \in D$ and using 3.26, 3.27 and 3.28 it follows that the derivative is bounded for $(h,H) \in (0,H_0)^2$ uniformly in $x,y \in D$.

In the second case our function $\tilde{g}_{H,h}$ is given in 3.32. As we have shown the assumption 3.3 for $g_{H,h}$ already, we only need to consider the second part of $\tilde{g}_{H,h}$. Therefore, we consider the second part

$$
\begin{aligned}
f_{H,h}(x,y) &:= \tilde{g}_{H,h}(x,y) - g_{H,h}(x,y) \\
&= \frac{C_2(H,h)}{C_1(H,h)(H+h)} \Bigg(\int_{\mathbb{R}} \operatorname{sgn}(u)|x-u|^{H+h} \psi(u,y) du \\
&+ \int_{\mathbb{R}} \operatorname{sgn}(v)|y-v|^{H+h} \psi(v,x) dv - \int_{\mathbb{R}} \int_{\mathbb{R}} sgn(v-u)|u-v|^{H+h} \psi(u,y)\psi(v,x) du dv \Bigg).
\end{aligned}
$$

As we can estimate $|\operatorname{sgn}(\cdot)| \leqslant 1$ and using 3.26, it readily follows that

$$
\begin{aligned}
\sup_{x,y \in D} |f_{H,h}(x,y)| &\leqslant \left| \frac{C_2(H,h)}{C_1(H,h)(H+h)} \right| \Bigg(\underbrace{\int_{\mathbb{R}} |x-u|^{H+h} \psi(u,y) du}_{<\infty} \\
&+ \underbrace{\int_{\mathbb{R}} |y-v|^{H+h} \psi(v,x) dv}_{<\infty} - \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} |u-v|^{H+h} \psi(u,y)\psi(v,x) du dv}_{<\infty} \Bigg) \\
&\leqslant C \left| \frac{C_2(H,h)}{C_1(H,h)(H+h)} \right|
\end{aligned}
$$

for a constant $C > 0$. Finally, using $\lim_{z \to 0} \frac{sin(z)}{z} = 1$, we get

$$
\left| \frac{C_2(H,h)}{C_1(H,h)(H+h)} \right| = \left| \frac{\sin(\frac{\pi}{2}(h-H)) \sin(\frac{\pi}{2}(h+H))}{\cos(\frac{\pi}{2}(h-H)) \cos(\frac{\pi}{2}(h+H))(H+h)} \right| \leqslant C(H+h),
$$

where $C > 0$ is a constant. Thus $f_{H,h}$ converges to 0 uniformly in (h,H) uniformly for all $x,y \in D$. ■

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