Dissertation

Cardinal Characteristics on Large Cardinals

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Univ.Prof. Dipl.–Ing. Dr.techn. Martin Goldstern
Institut für Diskrete Mathematik und Geometrie
Wiedner Hauptstraße 8–10, 1040 Wien
martin.goldstern@tuwien.ac.at

eingereicht an der Fakultät für Mathematik und Geoinformation der
Technischen Universität Wien von

Dipl.–Ing. Johannes Philipp Schürz, BSc
jschuerz@gmail.com

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Abstract

The study of cardinal characteristics on regular uncountable cardinals has significantly gained in popularity during the last decade. The generalizations of the Cantor and Baire space to regular uncountable cardinals κ naturally induce generalizations of the related cardinal characteristics. While for an arbitrary regular uncountable cardinal κ the picture can be quite different from the classical case, it turns out that if one requires κ to be a large cardinal, then many classical results generalize.

In this thesis we aim to further investigate cardinal characteristics related to the ideal of strong measure zero sets on inaccessible κ, define stationary variants of several combinatorial cardinal characteristics, present a new method to iterate forcing notions, which seems to be a very promising tool to separate cardinal characteristics in the higher Cichoń diagram, and use the recently developed technique of capturing to investigate the interaction between determinacy and forcing.

Kurzfassung

Das Studium von Kardinalzahlcharakteristiken auf regulären überabzählbaren Kardinalzahlen hat in den letzten zehn Jahren erheblich an Popularität gewonnen. Die Verallgemeinerungen des Cantor- und Baire-Raums auf reguläre überabzählbare Kardinalzahlen κ induzieren auf natürliche Weise Verallgemeinerungen der zugehörigen Kardinalzahlcharakteristiken. Während für eine beliebige reguläre überabzählbare Kardinalzahl κ das Bild ganz anders als im klassischen Fall aussehen kann, stellt sich heraus, dass sich viele klassische Ergebnisse verallgemeinern lassen, wenn man voraussetzt, dass κ eine große Kardinalzahl ist.

In dieser Arbeit wollen wir Kardinalzahlcharakteristiken im Zusammenhang mit dem Ideal der starken Nullmengen auf unerreichbaren κ weiter untersuchen, stationäre Varianten einiger kombinatorischer Kardinalzahlcharakteristiken definieren, eine neue Methode zur Iteration von Forcings vorstellen, die ein sehr vielversprechendes Werkzeug zu sein scheint, um Kardinalzahlcharakteristiken im höheren Cichoń-Diagramm zu trennen, und die vor kurzem entwickelte Technik des Capturings verwenden, um das Zusammenspiel zwischen Determiniertheit und Forcing zu untersuchen.

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Contents

Introduction

The foundations of modern Set Theory go back to Georg Cantor who introduced the notion of 'Menge' (German for 'set') around the end of the 19^{th} century. He postulated the existence of certain sets by a list of heuristic axioms, e.g. the set of all natural numbers, which we will denote by ω. He also compared these sets in size and showed that two sets, one properly containing the other, (e.g. the set of natural numbers and the set of algebraic numbers) can still have the same size. On the other hand, he proved that there are different sizes of infinity, which he called cardinalities.

Soon after the discovery of the Lebesgue measure by Henri Lebesgue in 1902, the investigation of cardinal characteristics on ω, in particular those of the Cantor space 2^ω, i.e. the space of 0-1 sequences equipped with the Tychonoff topology, started to attract more attention:
The union of how many Lebesgue measure zero sets is not Lebesgue measure zero? What is the smallest size of a set which is not Lebesgue measure zero? The union of how many meager sets covers the whole space? What is the smallest size of a family of meager sets such that every meager set is covered by a set of the family?
It was these questions that underlined the importance of Set Theory in other areas of mathematics such as Measure Theory and Topology.

In the 1920's Emile Borel introduced the notion of strong measure zero, a strengthening of Lebesgue measure zero, and conjectured that every strong measure zero set had to be countable. This statement is nowadays known as the Borel Conjecture.
A few years later Wacław Sierpiński discovered that assuming CH, i.e. the Continuum Hypothesis whose consistency relative to ZFC was not known at that time, the Borel Conjecture fails. In 1940, Kurt Gödel established the relative consistency of the Continuum Hypothesis and the Axiom of Choice by showing that both statements hold in the constructible universe.

In 1963, Paul Cohen (see [Coh63] and [Coh64]) used his revolutionizing technique of forcing to show that also the failure of the Continuum Hypothesis is consistent relative to ZFC. He also showed that the failure of the Axiom of Choice is consistent relative to ZF. Paul Cohen's results together with those of Kurt Gödel show that the Continuum Hypothesis is independent of ZFC and the Axiom of Choice is independent of ZF.
In 1976, Richard Laver (see [Lav76]) invented Laver forcing to show the relative consistency of the Borel Conjecture. His result together with Sierpiński's show that the Borel Conjecture is independent of ZFC.

In 1984, David Fremlin (see [Fre84]) picked up on some of the original questions about measure and category, and summarized the known inequalities between 12 cardinal characteristics (the additivity number, the covering number, the uniformity number and the cofinality number for measure and category, respectively, as well as the bounding number, the dominating number, $\aleph_1$ the first uncountable cardinal and $2^{\aleph_0}$ the size of the continuum) in a single diagram, which he called Cichoń's diagram:

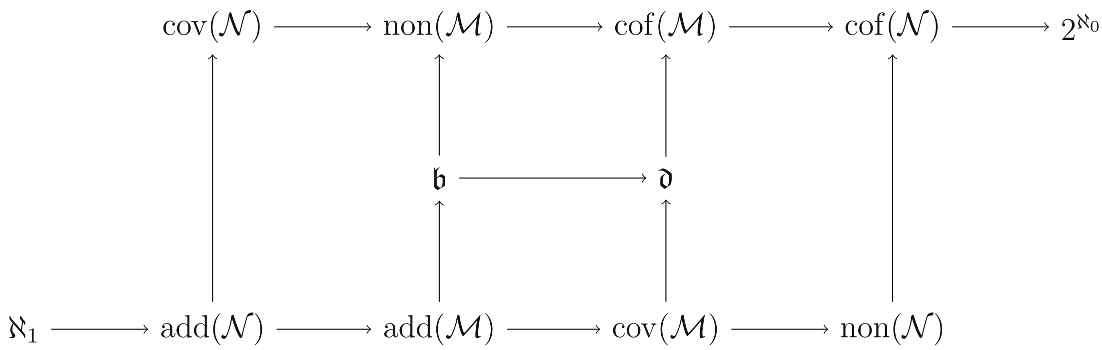

Figure 1: Cichoń's diagram

Here, an arrow from κ_1 to κ_2 denotes $\kappa_1 \leq \kappa_2$. Furthermore, the equalities $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$ hold true.
And indeed the pairwise inequalities represented in Cichoń's diagram are all that are provable in ZFC: Any assignment of the cardinals $\aleph_1$ and $\aleph_2$ to the 12 cardinal characteristics not contradicting the inequalities in the diagram is consistent relative to ZFC (see Chapter 7 in [BJ95]).

Since then the study of cardinal characteristics has only become more popular. Very recently, Martin Goldstern, Jakob Kellner, Diego Mejía and Saharon Shelah (see [GKS19] and [GKMS20]) showed that consistently all independent entries in Cichoń's diagram can be different. It remains open, whether all configurations of strict inequalities between the 12 cardinal characteristics, not contradicting the diagram, are consistent.
Also cardinal characteristics with a more combinatorial flavor such as the almost-disjointness number, the pseudo-intersection number, the reaping number, the splitting number, the tower number and the ultrafilter number have been studied extensively. Maybe most notably is the much celebrated result by Maryanthe Malliaris and S. Shelah (see [MS13]) that the pseudo-intersection number equals the tower number.

But what happens if one replaces ω by a regular uncountable cardinal κ and studies cardinal characteristics on κ, in particular those of the higher Cantor space 2^κ, which consists of all 0-1 sequences of length κ and carries the $<\kappa$-box topology (a topology canonically generalizing the Tychonoff topology on ω)? It turns out that this question is particularly interesting if κ is a large cardinal (e.g. inaccessible, weakly compact, strongly unfoldable, measurable, supercompact), since, in this case, κ is (similar to ω) also very large compared to all of its predecessors.

The first results in this field appeared in the 90's when Toshio Suzuki (see [Suz93]) and Jindřich Zapletal (see [Zap97]) showed that the statement 'the splitting number on κ is larger than κ' implies the existence of large cardinals. Meanwhile, James Cummings and S. Shelah (see [CS95]) were the first to investigate the bounding and dominating number on κ, and Aapo Halko (see [Hal96] and [HS01]) was the first to generalize the notion of strong measure zero to κ.

However, it was only in the last decade that the field of higher cardinal characteristics really gained in popularity: Sy-David Friedman and Giorgio Laguzzi (see [FL17]) introduced a notion of null ideal on 2^κ using a $\diamondsuit$-sequence on κ^+, and Jörg Brendle, Andrew Brooke-Taylor, S.-D. Friedman and Diana Montoya (see [BBTFM18]) established a version of a higher Cichoń diagram.
Meanwhile, S. Shelah (see [She17]) developed a different notion of null ideal on 2^κ for κ weakly compact, which is related to a generalization of random forcing, and is also compatible with $|2^\kappa|$ being larger than κ^+. Thomas Baumhauer, M. Goldstern and S. Shelah (see [BGS21]) used this version of a null ideal to define their own higher Cichoń diagram. It is this version of null ideal and higher Cichoń's diagram that we want to further investigate here.

Finally, let us note that while some results on ω easily generalize to κ, there are others which are only consistently true (see [LMRS16]), and some fail completely (e.g. see [FHK14] and [FKK16]). Furthermore, results from Omer Ben-Neria and Moti Gitik (see [BNG15]) and from Shimon Garti (see [BNG20]) suggest that the study of higher cardinal characteristics is very much related to the study of the large cardinal properties themselves.

Overview

This thesis is structured into six chapters:

We start the 1st chapter by giving basic definitions about large cardinals, ideals and forcing in general. We present Shelah's generalization of random forcing and show how it induces an ideal on 2^κ giving a generalization of the null ideal. We conclude the chapter by recalling the basic definitions about sharps, the projective hierarchy and determinacy.

In the 2nd chapter we investigate the ideal of strong measure zero sets on κ inaccessible. We check that Sacks and Silver forcing on κ satisfy an appropriate version of Axiom A (+ bounding), and give two different constructions showing the consistency of the statement '$|2^\kappa|$ has size κ^{++} and $\forall X \subseteq 2^\kappa$: X is strong measure zero iff X has size $\leq\kappa^+$'. We also investigate the notion of stationary strong measure zero (see [Sch19]).

In the 3rd chapter we approximate the ideal of strong measure zero sets on κ inaccessible with the help of 'generalized Yorioka' ideals, and use them to characterize its cofinality. We show that this cardinal characteristic can consistently be smaller, equal or even larger than $|2^\kappa|$, and conclude this chapter by showing that the additivity of the meager ideal can consistently be larger than the covering number of the strong measure zero ideal (see [Sch20]).

In the 4th chapter we define variants of the classical cardinal characteristics modulo the non-stationary ideal for κ regular uncountable. While some of them turn out to be trivial, we provide forcing constructions separating the non-trivial ones. However, many interesting questions remain open in this new field of study (see [Sch21]).

The 5th chapter mainly consists of a write-up of Shelah's Corrected Iteration (see [She19]). This kind of iteration seems to be a very promising new tool to separate cardinals in the higher Cichoń diagram. Indeed, we planned to use it to separate the bounding number on κ and the covering number of the higher null ideal. However, there are issues when actually applying the Corrected Iteration. We present the technical problems, and assuming they can be fixed, we sketch how to achieve the desired consistency result.

In the 6th chapter we show using the technique of capturing that $\mathbf{\Pi}^1_1$-determinacy is preserved under any countable support iteration of 'simply' definable, proper forcing notions. We also investigate connected components of symmetric $\mathbf{\Delta}^1_3$-relations on the reals, and conclude the chapter by showing that even without the existence of large cardinals, capturing can still be used to preserve certain regularity properties (see [SSS21]).

1 Preliminaries

We will start with several basic definitions most of which can be found in [Jec03]:

Definition 1.0.1. Let κ be an infinite cardinal. We say that:

- κ is regular iff $\mathrm{cf}(\kappa) = \kappa$ where $\mathrm{cf}(\alpha) := \min\{\mathrm{otp}(E) \colon E \subseteq \alpha \text{ is cofinal in } \alpha\}$ is defined for any limit ordinal α and otp denotes the order type.
- κ is inaccessible iff κ is regular and $\lambda^\lambda < \kappa$ for every $\lambda < \kappa$.
- κ is weakly compact iff κ is inaccessible and every $<\kappa$-splitting tree $T \subseteq \kappa^{<\kappa}$ has a branch of size κ.
- κ is measurable iff κ carries a $<\kappa$-complete ultrafilter.
- κ is supercompact iff for every $\theta > \kappa$ there exists an elementary embedding $j \colon V \to M$ with critical point κ such that $j(\kappa) > \theta$ and $M^\theta \subseteq M$.

Definition 1.0.2. We call $cl \subseteq \kappa$ a club iff cl is closed and unbounded in κ. Let $Cl := \{x \subseteq \kappa \colon \exists cl \subseteq x \; cl \text{ is club}\}$ denote the club filter and let $NS := \{x \subseteq \kappa \colon \exists cl \in Cl \; x \cap cl = \emptyset\}$ denote the non-stationary ideal.

Let us define the higher Cantor space:

Definition 1.0.3. We call $2^\kappa := \{f \colon f : \kappa \to 2 \text{ is a function}\}$ the higher Cantor space and equip it with the following topology:
For $\eta \in 2^{<\kappa}$ we define $[\eta] := \{x \in 2^\kappa \colon \eta \triangleleft x\}$. Let $\mathcal{B} := \{[\eta] \colon \eta \in 2^{<\kappa}\}$ and define the $<\kappa$-box topology to be the closure of $\mathcal{B}$ under arbitrary unions. [1]

Using this topology we can now generalize the notion of meagerness:

Definition 1.0.4. We call $D \subseteq 2^\kappa$ open dense iff D is open in the $<\kappa$-box topology and for every $\eta \in 2^{<\kappa}$ there exists $\nu \in 2^{<\kappa}$ such that $\eta \triangleleft \nu$ and $[\nu] \subseteq D$.
We call $X \subseteq 2^\kappa$ closed nowhere dense iff $2^\kappa \setminus X$ is open dense. We call $X \subseteq 2^\kappa$ meager iff $X \subseteq \bigcup_{i<\kappa} Y_i$ for some closed nowhere dense $(Y_i)_{i<\kappa}$.
We define $\mathcal{M}_\kappa := \{X \subseteq 2^\kappa \colon X \text{ is meager}\}$ the κ-Borel ideal of all meager sets of 2^κ.

Definition 1.0.5. Let $\mathcal{I}$ be an ideal. We say that $\mathcal{I}$ is $\leq\kappa$-complete iff for every $(Y_i)_{i<\kappa} \subseteq \mathcal{I}$ we have $\bigcup_{i<\kappa} Y_i \in \mathcal{I}$.

[1] Note that for $\eta \in 2^{<\kappa}$ the set $[\eta]$ is clopen in the $<\kappa$-box topology. Furthermore, if $(O_i)_{i<\lambda}$ is a family of open sets of size $\lambda < \kappa$ then $\bigcap_{i<\lambda} O_i$ is also open.

Definition 1.0.6. If $\mathcal{I} \subseteq \mathfrak{P}(2^\kappa)$ is a proper ideal containing all singletons, we can define the following cardinal characteristics:

- $\mathrm{add}(\mathcal{I}) := \min\{|\mathcal{F}|\colon \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} \notin \mathcal{I}\}$
- $\mathrm{cov}(\mathcal{I}) := \min\{|\mathcal{F}|\colon \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} = 2^\kappa\}$
- $\mathrm{non}(\mathcal{I}) := \min\{|X|\colon X \subseteq 2^\kappa \wedge X \notin \mathcal{I}\}$
- $\mathrm{cof}(\mathcal{I}) := \min\{|\mathcal{F}|\colon \mathcal{F} \subseteq \mathcal{I} \wedge \forall X \in \mathcal{I}\ \exists Y \in \mathcal{F}\ X \subseteq Y\}$

The following is an easy fact:

Fact 1.0.7. We have $\mathrm{add}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$ and $\mathrm{add}(\mathcal{I}) \leq \mathrm{non}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$.

Fact 1.0.8. $\mathcal{M}_\kappa$ is a $\leq\kappa$-complete, proper ideal containing all singletons, hence $\mathrm{add}(\mathcal{M}_\kappa)$, $\mathrm{cov}(\mathcal{M}_\kappa)$, $\mathrm{non}(\mathcal{M}_\kappa)$ and $\mathrm{cof}(\mathcal{M}_\kappa)$ are all defined. In particular $\mathrm{add}(\mathcal{M}_\kappa) \geq \kappa^+$.

Definition 1.0.9. We say that GCH at κ holds iff $\mathfrak{c}_\kappa := |2^\kappa| = \kappa^+$.

1.1 Forcing

For a detailed presentation of the theory of forcing see Chapter 4 in [Kun11]. If $\mathcal{P}$ is a forcing notion, we will use the convention to force downwards, i.e. $q \leq_\mathcal{P} p$ means that q is a stronger condition than p.

Definition 1.1.1. We say that a forcing notion $\mathcal{P}$ is $<\kappa$-closed iff for every $\lambda < \kappa$ every decreasing sequence $(p_i)_{i<\lambda} \subseteq \mathcal{P}$ has a lower bound in $\mathcal{P}$.

Definition 1.1.2. We say that a forcing notion $\mathcal{P}$ is $\leq\kappa$-strategically closed iff for every condition $p^* \in P$ Player I has a winning strategy in the following game of length κ:

- Player I starts the game, and always plays first in limit stages.
- Player I and II alternate playing conditions $p, q \in \mathcal{P}$ below p^*.
- If p_i denotes Player I's choice at stage $i < \kappa$ and q_i Player II's, then we require:
  - for every $i < \kappa$ and every $j < i$ we have $p_i \leq_\mathcal{P} q_j$.
  - for every $i < \kappa$ we have $q_i \leq_\mathcal{P} p_i$.

Player II wins the game iff at some stage $i^* < \kappa$ Player I has no legal move.

Fact 1.1.3. If $\mathcal{P}$ is a $\leq\kappa$-strategically closed forcing notion, then Π^1_1-absoluteness holds between V and $V^\mathcal{P}$, i.e. $(V^V_{\kappa+1}, V^V_\kappa, \in) \prec_{\Pi_1} (V^{V^\mathcal{P}}_{\kappa+1}, V^V_\kappa, \in)$ (see [FKK16]).

Definition 1.1.4. We say that a forcing notion $\mathcal{P}$ is κ-linked iff there exists $(P_i)_{i<\kappa}$ such that:

- $\mathcal{P} = \bigcup_{i<\kappa} P_i$

- $\forall i < \kappa \colon P_i$ is linked, i.e. $\forall p_1, p_2 \in P_i \colon p_1 \parallel p_2$, where $\parallel$ means compatible.

Definition 1.1.5. We say that a forcing notion $\mathcal{P}$ is κ-centered$_{<\kappa}$ iff there exists $(P_i)_{i<\kappa}$ such that:

- $\mathcal{P} = \bigcup_{i<\kappa} P_i$
- $\forall i < \kappa \colon P_i$ is centered$_{<\kappa}$, i.e. $\forall Q \in [P_i]^{<\kappa}\ \exists q \in \mathcal{P} \colon q$ is a lower bound of Q.

Definition 1.1.6. We say that a forcing notion $\mathcal{P}$ is $<\kappa$-directed closed iff

$$\forall Q \in [\mathcal{P}]^{<\kappa} \colon \big(\forall p_1, p_2 \in Q\ \exists q \in Q\ q \leq_{\mathcal{P}} p_1, p_2\big) \Rightarrow \exists r \in \mathcal{P}\ r \text{ is a lower bound of } Q$$

The following theorem is due to Laver (see [Lav78]):

Theorem 1.1.7. Let κ be supercompact. Then V can be prepared using a κ-c.c. forcing notion $\mathbb{P}$ preserving the supercompactness of κ such that $V^{\mathbb{P}} \models$ 'the supercompactness of κ is indestructible by $<\kappa$-directed closed forcing notions'.

Definition 1.1.8. We say that a forcing notion $\mathcal{P}$ is κ^κ-bounding iff $\Vdash_{\mathcal{P}} \forall f \in \kappa^\kappa\ \exists g \in \kappa^\kappa \cap V \colon f \leq^*_\kappa g$ (see Definition 1.2.4).

Definition 1.1.9. Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta \colon \alpha \leq \gamma, \beta < \gamma \rangle$ be an iteration. We say that $\mathcal{P}_\gamma$ has:

- $<\kappa$-support iff $\mathcal{P}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}_\beta$ for $\mathrm{cf}(\alpha) \geq \kappa$ and $\mathcal{P}_\alpha$ is the inverse limit of $(\mathcal{P}_\beta)_{\beta<\alpha}$ for $\mathrm{cf}(\alpha) < \kappa$.
- $\leq\kappa$-support iff $\mathcal{P}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}_\beta$ for $\mathrm{cf}(\alpha) > \kappa$ and $\mathcal{P}_\alpha$ is the inverse limit of $(\mathcal{P}_\beta)_{\beta<\alpha}$ for $\mathrm{cf}(\alpha) \leq \kappa$.

Let us now recall two very important forcing notions:

Definition 1.1.10. We denote the κ-Cohen forcing by $\mathbb{C}_\kappa$. We have $p \in \mathbb{C}_\kappa$ iff $p \in 2^{<\kappa}$ and define $q \leq_{\mathbb{C}_\kappa} p$ iff $p \triangleleft q$.

Definition 1.1.11. We denote the κ-Hechler forcing by $\mathbb{H}_\kappa$. We have $p \in \mathbb{H}_\kappa$ iff $p = (\rho_p, f_p)$ such that $\rho_p \in \kappa^{<\kappa}$, $f_p \in \kappa^\kappa$ and $\rho_p \triangleleft f_p$, and define $q \leq_{\mathbb{H}_\kappa} p$ iff $\rho_p \triangleleft \rho_q$ and $f_p \leq f_q$.

1.2 The generalized null ideal

In [She17] Shelah presents a generalization of random forcing for κ inaccessible and uses it to generalize the ideal of measure zero sets. We need the following definitions:

Definition 1.2.1. We define

- $S^\kappa_{\text{inc}} := \{\lambda < \kappa \colon \lambda \text{ is inaccessible}\}$
- $S \subseteq S^\kappa_{\text{inc}}$ is nowhere stationary iff for every regular uncountable $\delta \leq \kappa$ the set $S \cap \delta$ is a non-stationary subset of δ.

We will now define by induction on $\delta \in S^\kappa_{\text{inc}} \cup \{\kappa\}$

- a forcing notion $\mathbb{R}_\delta$ (whose definition will use the ideals $\text{id}(\mathbb{R}_\lambda)$ for $\lambda \in S^\kappa_{\text{inc}} \cap \delta$)
- ideals $\text{wid}(\mathbb{R}_\delta)$ and $\text{id}(\mathbb{R}_\delta)$ on 2^δ.

Definition 1.2.2. We have $p \in \mathbb{R}_\delta$ iff there exists $(\tau_p, S_p, (I^p_\lambda)_{\lambda\in S_p})$ such that

- $p \subseteq 2^{<\delta}$ is a tree, i.e. downwards closed.
- $\tau_p \in 2^{<\delta}$ is the trunk of p, i.e. the smallest node in p that has two successors.
- Above τ_p the tree p is fully branching, i.e. $\tau_p \lhd \eta \in p \Rightarrow \eta^\frown 0, \eta^\frown 1 \in p$.
- $S_p \subseteq S^\kappa_{\text{inc}} \cap \delta$ is nowhere stationary.
- For every $\lambda \in S_p$ we have $I^p_\lambda \in \text{id}(\mathbb{R}_\lambda)$
- If $\lambda \notin S_p$, λ is a limit ordinal and $\eta \in 2^\lambda$, then $\eta \in p$ iff $\forall i < \lambda\colon \eta \restriction i \in p$.
- If $\lambda \in S_p$ and $\eta \in 2^\lambda$, then $\eta \in p$ iff
  - $\forall i < \lambda\colon \eta \restriction i \in p$ and
  - $\eta \notin I^p_\lambda$

and we define $q \leq_{\mathbb{R}_\delta} p$ iff $q \subseteq p$. If G is $(V, \mathbb{R}_\kappa)$-generic, we define $r_G := \bigcup_{p\in G} \tau_p$. Furthermore, we define

$$\text{wid}(\mathbb{R}_\delta) := \{I \subseteq 2^\delta \colon \exists \mathcal{A} \subseteq \mathbb{R}_\delta\ \mathcal{A} \text{ is a maximal antichain } \wedge I \subseteq 2^\delta \setminus \bigcup_{p\in\mathcal{A}} [p]\}$$

where $[p] := \{x \in 2^\delta \colon \forall i < \delta\ x \restriction i \in p\}$. Define $\text{id}(\mathbb{R}_\delta)$ to be the $\leq\delta$-closure of $\text{wid}(\mathbb{R}_\delta)$.

The following are the most important properties of $\mathbb{R}_\kappa$, $\text{wid}(\mathbb{R}_\kappa)$ and $\text{id}(\mathbb{R}_\kappa)$:

Lemma 1.2.3. We have:

- $\mathbb{R}_\kappa$ is $\leq\kappa$-strategically closed and κ-linked. In particular, $\mathbb{R}_\kappa$ satisfies the κ^+-c.c.
- $\text{wid}(\mathbb{R}_\kappa)$ and $\text{id}(\mathbb{R}_\kappa)$ are $<\kappa$-complete, proper ideals with a κ-Borel basis and contain all singletons. Furthermore, $\text{id}(\mathbb{R}_\kappa)$ is even $\leq\kappa$-complete.
- G is $(V, \mathbb{R}_\kappa)$-generic iff $r_G \notin I$ for every κ-Borel set $I \in \text{wid}(\mathbb{R}_\kappa) \cap V$.
- For κ weakly compact, we have
$$\text{wid}(\mathbb{R}_\kappa) = \text{id}(\mathbb{R}_\kappa) = \{I \subseteq 2^\kappa \colon \exists S \in [S^\kappa_{\text{inc}}]^\kappa \text{ nowhere stationary } \exists (I_\lambda)_{\lambda\in S}$$
$$\big(\forall \lambda \in S\ I_\lambda \in \text{id}(\mathbb{R}_\lambda)\big) \wedge \big(\forall x \in 2^\kappa\ \ x \in I \Leftrightarrow \exists^\infty \lambda \in S\ x \restriction \lambda \in I_\lambda\big)\}.$$
  Here $\exists^\infty \lambda \in S$ means 'there exist unboundedly many $\lambda \in S$ with the desired property'.

In particular, the cardinal characteristics $\mathrm{add}(\mathrm{id}(\mathbb{R}_\kappa)), \mathrm{cov}(\mathrm{id}(\mathbb{R}_\kappa)), \mathrm{non}(\mathrm{id}(\mathbb{R}_\kappa))$ and $\mathrm{cof}(\mathrm{id}(\mathbb{R}_\kappa))$ are all defined.

We define two more cardinal characteristics:

Definition 1.2.4. Let $f, g \in \kappa^\kappa$. We say that $f \leq^*_\kappa g$ iff $|\{i < \kappa \colon f(i) > g(i)\}| < \kappa$ and define:

- $\mathfrak{b}_\kappa := \min\{|B| \colon B \subseteq \kappa^\kappa \wedge \forall f \in \kappa^\kappa\ \exists g \in B\ g \not\leq^*_\kappa f\}$ the bounding number
- $\mathfrak{d}_\kappa := \min\{|D| \colon D \subseteq \kappa^\kappa \wedge \forall f \in \kappa^\kappa\ \exists g \in D\ f \leq^*_\kappa g\}$ the dominating number

Theorem 1.2.5. In [BGS21] the authors proved the following relations between the various cardinal characteristics for κ inaccessible:

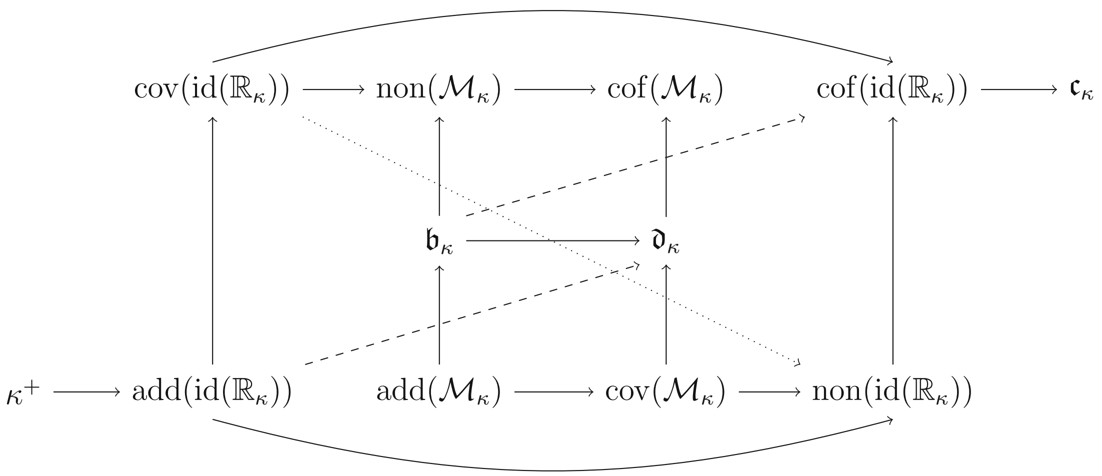

Figure 2: The higher Cichoń diagram

where an arrow from θ_1 to θ_2 denotes $\theta_1 \leq \theta_2$. Furthermore, the equalities $\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}_\kappa, \mathrm{cov}(\mathcal{M}_\kappa)\}$ and $\mathrm{cof}(\mathcal{M}_\kappa) = \max\{\mathfrak{d}_\kappa, \mathrm{non}(\mathcal{M}_\kappa)\}$ hold true.

It remains open whether $\mathrm{add}(\mathcal{M}_\kappa) < \mathrm{add}(\mathrm{id}(\mathbb{R}_\kappa))$ or $\mathrm{cof}(\mathrm{id}(\mathbb{R}_\kappa)) < \mathrm{cof}(\mathcal{M}_\kappa)$ are consistent.

1.3 Sharps

The following presentation of $x^\sharp$ for a real $x \in \omega^\omega$ can be found in Chapter 2.9 of [Kan03], Chapter 18 of [Jec03] and Chapter 10.2 of [Sch14a]. We will start with several definitions:

Definition 1.3.1. Let $\mathcal{L} := \{\in, P\}$ denote the language of Set Theory together with an additional unary predicate P, and define $\mathcal{L}^* := \mathcal{L} \cup \{c_n \colon n < \omega\}$ where c_n is a constant

symbol for every $n < \omega$. Let Form denote the set of formulas in the language $\mathcal{L}$ and Form* the set of formulas in the language $\mathcal{L}^*$, respectively.
Let $\mathcal{M}$ be an $\mathcal{L}^*$-structure. We define $T^{\mathcal{M}} := \{\sigma \in \text{Form}^* \colon \sigma \text{ is a sentence} \wedge \mathcal{M} \models \sigma\}$ the theory of $\mathcal{M}$.

Definition 1.3.2. Working in the theory $\text{ZFC}^* + V = L[P]$, where ZFC* denotes some large enough fragment of ZFC, let $<_{L[P]}$ denote the canonical well-order of $L[P](= V)$. For a formula $\varphi(x_0, ..., x_n) \in$ Form we define the Skolem function $h_\varphi \colon V^n \to V$ such that

$$h_\varphi(\bar{y}) := \begin{cases} \min_{<_{L[P]}}\{z \colon \varphi(z, \bar{y})\} & \text{if } \exists x_0 \colon \varphi(x_0, \bar{y}) \\ \emptyset & \text{else} \end{cases}$$

Definition 1.3.3. Let $\mathcal{M} = (M, E, A)$ be an $\mathcal{L}$-structure. Let $(X, <_X)$ be a linear order such that $X \subseteq M$. [2] We call $(X, <_X)$ a set of order indiscernibles for $\mathcal{M}$ iff for every formula $\varphi(x_0, ..., x_n) \in$ Form and every $y_0 <_X ... <_X y_n$ and $z_0 <_X ... <_X z_n$ we have

$$\mathcal{M} \models \varphi(y_0, ..., y_n) \leftrightarrow \varphi(z_0, ..., z_n)$$

For the rest of this section fix a real $x \in \omega^\omega$.

Definition 1.3.4. We call $T \subseteq \text{Form}^*$ an EM blueprint [3] for x iff $T = T^{(L_\delta[x], \in, x, (y_n)_{n<\omega})}$ such that δ is limit with $\omega < \delta < \omega_1$, $(L_\delta[x], \in, x, (y_n)_{n<\omega}) \models$ ‘ c_n is an ordinal $\wedge$ $c_n \in c_{n+1}$ ’ for every $n < \omega$, and $(\{y_n \colon n < \omega\}, \in)$ is a set of order indiscernibles for $(L_\delta[x], \in, x)$.

Lemma 1.3.5. Let T be an EM blueprint for x and let $\alpha < \omega_1$ be an infinite ordinal. Then there exists an $\mathcal{L}^*$-structure $\mathcal{M} = (M, E, A, (y_n)_{n<\omega})$ [4] and a set $X \subseteq \text{Ord}^{\mathcal{M}}$ with $\{y_n \colon n < \omega\} \subseteq X$ [5] such that $T = T^{\mathcal{M}}$, (X, E) is a set of order indiscernibles for (M, E, A) and $(X, E) \approx (\alpha, \in)$. Furthermore, we can require that $M = \bigcup_{\varphi \in \text{Form}} h_\varphi^{\mathcal{M}}[X^{<\omega}]$. In this case $(\mathcal{M}, X)$ is unique up to isomorphism.

If $\mathcal{M} = (M, E, A)$ is an $\mathcal{L}$-structure and $X \subseteq \text{Ord}^{\mathcal{M}}$ such that (X, E) is a set of order indiscernibles for $\mathcal{M}$ and $M = \bigcup_{\varphi \in \text{Form}} h_\varphi^{\mathcal{M}}[X^{<\omega}]$, it easily follows that:

- $\forall y \in (M \setminus X) \cap \text{Ord}^{\mathcal{M}} \colon (X \cup \{y\}, E)$ is not a set of order indiscernibles for $\mathcal{M}$
- $\forall y \in X \colon y \notin \bigcup_{\varphi \in \text{Form}} h_\varphi^{\mathcal{M}}[(X \setminus \{y\})^{<\omega}]$

Definition 1.3.6. Let T be an EM blueprint for x and let $\alpha < \omega_1$ be infinite. By the (T, α)-model we denote the (up to isomorphism) uniquely defined pair $(\mathcal{M}, X)$ such that $\mathcal{M} = (M, E, A, (y_n)_{n<\omega})$ is a model of T, $X \subseteq \text{Ord}^{\mathcal{M}}$ with $(y_n)_{n<\omega} \subseteq X$ and (X, E) is a set of order indiscernible of order type α, and $M = \bigcup_{\varphi \in \text{Form}} h_\varphi^{\mathcal{M}}[X^{<\omega}]$.

Definition 1.3.7. Let T be an EM blueprint for x . We call T well-founded iff $\forall \alpha < \omega_1 \colon$ the (T, α)-model is well-founded.

[2]Note that neither X nor $<_X$ must be an element of M.
[3]EM stands for Ehrenfeucht-Mostowski.
[4]In particular, for every $n < \omega$ we have $n \in x$ iff $n^{\mathcal{M}} \in A$.
[5]W.l.o.g. we assume that $\{y_n \colon n < \omega\}$ is an initial segment of X.

Definition 1.3.8. Let T be an EM blueprint for x. We call T unbounded iff for every formula $\varphi(x_0, ..., x_n) \in$ Form we have $h_\varphi(c_0, ..., c_n) \in \text{Ord} \to h_\varphi(c_0, ..., c_n) < c_{n+1} \in T$.

The following lemma motivates the notion 'unbounded':

Lemma 1.3.9. Let T be an unbounded EM blueprint for x and let $\alpha < \omega_1$. Let $(\mathcal{M}, X)$ be the (T, α)-model. Then X is unbounded in $\text{Ord}^{\mathcal{M}}$, i.e. $\forall x' \in \text{Ord}^{\mathcal{M}}\ \exists y \in X : x' \mathrel{E} y$.

Definition 1.3.10. Let T be an EM blueprint for x. We call T remarkable iff T is unbounded and for every formula $\varphi(x_0, ..., x_{m+n}) \in$ Form we have

$$h_\varphi(c_0, ..., c_{m+n}) < c_m \to h_\varphi(c_0, ..., c_{m+n}) = h_\varphi(c_0, ..., c_{m-1}, c_{m+n+1}, ..., c_{m+2n+1}) \in T.$$

The following lemma motivates the notion 'remarkable':

Lemma 1.3.11. Let T be a remarkable EM blueprint for x and $\alpha < \omega_1$. Let $(\mathcal{M}, X)$ be the (T, α)-model and we define $y_\omega := \min X \backslash \{y_n : n < \omega\}$ [6]. Then $\forall x' \in \text{Ord}^{\mathcal{M}} : x' \mathrel{E} y_\omega \Rightarrow x' \in \bigcup_{\varphi \in \text{Form}} h_\varphi^{\mathcal{M}}[(\{y_n : n < \omega\})^{<\omega}]$. In particular, X is closed in $\text{Ord}^{\mathcal{M}}$.

Lemma 1.3.12. If there exists a well-founded EM blueprint for x, then there exists a unique well-founded, remarkable EM blueprint for x.

Definition 1.3.13. We say $x^\sharp$ exists iff there exists a well-founded, remarkable EM blueprint for x. In this case, we denote by $x^\sharp$ the unique well-founded, remarkable EM blueprint for x, and identify it with a subset of ω.

Theorem 1.3.14. The following are equivalent:

- $x^\sharp$ exists.
- There exists a closed unbounded class $I_x \subseteq \text{Ord}$ [7] containing all uncountable cardinals such that for all cardinals $\kappa \in I_x$:
  - $|I_x \cap \kappa| = \kappa$
  - $(I_x \cap \kappa, \in)$ is a set of order indiscernibles for $(L_\kappa[x], \in, x)$.
  - $L_\kappa[x] = \bigcup_{\varphi \in \text{Form}} h_\varphi^{(L_\kappa[x], \in, x)}[(I_x \cap \kappa)^{<\omega}]$.

  In particular, $x^\sharp = \{\sigma \in \text{Form}^* : \sigma \text{ is a sentence} \land (L_{\aleph_\omega}[x], \in, x, (\aleph_n)_{n<\omega}) \vDash \sigma\}$.
- There exists a non-trivial, elementary embedding $j : L[x] \to L[x]$.
- There exists a countable structure $(L_\alpha[x], \in, U)$ such that
  - $(L_\alpha[x], \in)$ is a model of ZFC^- with a largest cardinal κ
  - $(L_\alpha[x], \in, U)$ is a model of Σ_0-separation
  - $(L_\alpha[x], \in, U) \vDash U$ is a $<\kappa$-complete ultrafilter
  - all iterated ultrapowers of $(L_\alpha[x], \in, U)$ are well-founded

Definition 1.3.15. We say that the reals are $\sharp$-closed iff $\forall x' \in \omega^\omega : x'^\sharp$ exists.

[6] Since w.l.o.g. $\{y_n : n < \omega\}$ is an initial segment of X, we have $y_n \mathrel{E} y_\omega$ for every $n < \omega$.
[7] The elements of I_x are called Silver indiscernibles.

1.4 Descriptive set theory and determinacy

The following can be found in Chapter 25 and 33 of [Jec03]. We start by defining the projective hierarchy of $(\omega^\omega)^k$ simultaneously for every $k < \omega$:

Definition 1.4.1. We call $A \subseteq (\omega^\omega)^k$ analytic or a $\mathbf{\Sigma}^1_1$ set iff A has a $\mathbf{\Sigma}^1_1$-definition, i.e. there exists a tree $T \subseteq (\omega^{<\omega})^k \times \omega^{<\omega}$ such that

$$A = \{\bar{x} \in (\omega^\omega)^k \colon \exists y \in \omega^\omega \ \forall n' < \omega \ (x_0 \restriction n', ..., x_{k-1} \restriction n', y \restriction n') \in T\}.$$

We call $A \subseteq (\omega^\omega)^k$ coanalytic or a $\mathbf{\Pi}^1_1$ set iff A has a $\mathbf{\Pi}^1_1$-definition: $A = (\omega^\omega)^k \setminus B$ for some $\mathbf{\Sigma}^1_1$ set $B \subseteq (\omega^\omega)^k$, and call $A \subseteq (\omega^\omega)^k$ a $\mathbf{\Delta}^1_1$ set iff A is both a $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ set.
Inductively, we call $A \subseteq (\omega^\omega)^k$ a $\mathbf{\Sigma}^1_{n+1}$ set iff A has a $\mathbf{\Sigma}^1_{n+1}$-definition:

$$A = \{\bar{x} \in (\omega^\omega)^k \colon \exists y \in \omega^\omega \ (\bar{x}, y) \in B\}$$

for some $\mathbf{\Pi}^1_n$ set $B \subseteq (\omega^\omega)^k \times \omega^\omega$. Similarly, we call $A \subseteq (\omega^\omega)^k$ a $\mathbf{\Pi}^1_{n+1}$ set iff A has a $\mathbf{\Pi}^1_{n+1}$-definition: $A = (\omega^\omega)^k \setminus B$ for some $\mathbf{\Sigma}^1_{n+1}$ set $B \subseteq (\omega^\omega)^k$, and call $A \subseteq (\omega^\omega)^k$ a $\mathbf{\Delta}^1_{n+1}$ set iff A both is a $\mathbf{\Sigma}^1_{n+1}$ and $\mathbf{\Pi}^1_{n+1}$ set.

The projective hierarchy of $2^\omega \times (\omega^\omega)^k$ for $k < \omega$ is defined analogously.

Next we state two absoluteness results:

Theorem 1.4.2. Let N be a countable transitive model satisfying ZFC*, i.e. a large enough fragment of ZFC. Then $\mathbf{\Sigma}^1_1$-absoluteness holds between N and V.

Theorem 1.4.3. Let M be transitive (proper class) model satisfying ZFC* with $\omega_1^V \in M$. Then $\mathbf{\Sigma}^1_2$-absoluteness holds between M and V.

Let us now turn to determinacy:

Definition 1.4.4. For $A \subseteq \omega^\omega$ we define the two player game G_A as follows:

- Player I starts the game.
- Player I and II alternate playing $a_n, b_n \in \omega$.

Player I wins the game iff $\langle a_0, b_0, a_1, b_1, ...\rangle \in A$.

Definition 1.4.5. Let $\sigma \colon \omega^{<\omega} \to \omega$ be a strategy. We call σ a winning strategy for Player I in the game G_A iff for every $b \in \omega^\omega$ we have $\langle \sigma(\emptyset), b_0, \sigma(\langle b_0\rangle), b_1, ...\rangle \in A$. Conversely, we call σ a winning strategy for Player II in the game G_A iff for every $a \in \omega^\omega$ we have $\langle a_0, \sigma(\langle a_0\rangle), a_1, \sigma(\langle a_0, a_1\rangle), ...\rangle \in \omega^\omega \setminus A$.
We call A determined iff one player has a winning strategy in the game G_A.

Definition 1.4.6. For a collection Γ of subsets of ω^ω we say that Γ-determinacy holds iff every set A in Γ is determined.

The following theorem is by Harrington (see [Har78]) and Martin (see [Mar70]):

Theorem 1.4.7. The following two statements are equivalent:

- $\mathbf{\Pi}^1_1$-determinacy holds.
- The reals are $\sharp$-closed. (see Definition 1.3.15)

2 Strong measure zero sets on 2^κ for κ inaccessible

In this chapter we will investigate the notion of strong measure zero sets on the higher Cantor space 2^κ for κ at least inaccessible as defined by Halko [Hal96]:

Definition 2.0.1. Let $X \subseteq 2^\kappa$. We call X strong measure zero iff

$$\forall f \in \kappa^\kappa \ \exists (\eta_i)_{i<\kappa} \colon \big(\forall i < \kappa \ \eta_i \in 2^{f(i)} \big) \wedge X \subseteq \bigcup_{i<\kappa} [\eta_i].$$

Let $\mathcal{SN} := \{X \subseteq 2^\kappa \colon X \text{ is strong measure zero}\}$ denote the collection of all strong measure zero sets.

The following is an easy fact:

Fact 2.0.2. $\mathcal{SN}$ is a $\leq\kappa$-complete, proper ideal on 2^κ which contains all singletons. Furthermore, $\mathcal{SN} \subseteq \mathrm{id}(\mathbb{R}_\kappa)$ (see Definition 1.2.2).

We shall give two different proofs showing the relative consistency of:

$$\text{ZFC} + \mathfrak{c}_\kappa = \kappa^{++} \wedge \mathcal{SN} = [2^\kappa]^{\leq\kappa^+}.$$

The first proof follows Goldstern, Judah and Shelah [GJS93] and we require κ to be strongly unfoldable (see Definition 2.3.1). In the second, somewhat better proof we follow Corazza [Cor89] and only require κ to be inaccessible.
Finally, we show that in the Corazza model every $X \in \mathcal{SN}$ is even stationary strong measure zero (see Definition 2.5.1). On the other hand, assuming GCH at κ, we show that there exists $X \in \mathcal{SN}$ such that X is not stationary strong measure zero.

Strong measure zero sets for κ regular uncountable have also been studied in [HS01], where the authors show that the Borel Conjecture at κ, i.e. the statement that 'all strong measure zero sets have size at most κ', is false for κ successor with $\kappa^{<\kappa} = \kappa$. The question, whether the Borel Conjecture for κ inaccessible is consistent, remains open [1] as is also stated in [KLLS16].

[1] This is related to the problem of how to add dominating reals without adding Cohen reals on κ (see [KKLW20]).

2.1 The Forcing

Let us assume that κ is an inaccessible cardinal, in particular $\lambda^\lambda < \kappa$ for $\lambda < \kappa$.

For $f \in \kappa^\kappa$, $f(0) > 1$ and strictly increasing, we define the 'f-perfect tree' forcing PT_f as follows:

Definition 2.1.1. Let $p \in PT_f$ iff

P1 $p \subseteq \kappa^{<\kappa}$, $p \neq \emptyset$ and p is a tree

P2 $\forall \eta \in p\ \forall i \in \mathrm{dom}(\eta)\colon \eta(i) < f(i)$

P3 $\forall \eta \in p\colon |\mathrm{succ}_p(\eta)| = 1 \vee \mathrm{succ}_p(\eta) = \{\eta^\frown\alpha\colon \alpha < f(\mathrm{dom}\,\eta)\}$,
where $\mathrm{succ}_p(\eta)$ denotes the successors of η in p.

P4 $\forall \eta \in p\ \exists \nu \in p\colon \eta \lhd \nu \wedge |\mathrm{succ}_p(\nu)| > 1$

P5 If $\lambda < \kappa$ is a limit, then $\forall \eta \in \kappa^\lambda\colon \eta \in p \Leftrightarrow \forall i < \lambda\ \ \eta \restriction i \in p$

P6 If $\lambda < \kappa$ is a limit, then
$\forall \eta \in \kappa^\lambda\colon \big(\eta \in p \wedge \{\nu \trianglelefteq \eta\colon |\mathrm{succ}_p(\nu)| > 1\}$ is unbounded in $\eta\big) \Rightarrow |\mathrm{succ}_p(\eta)| > 1$

We define $q \leq_{PT_f} p$ iff $q \subseteq p$.

If G is a (V, PT_f)-generic filter, we define $g \in \kappa^\kappa$ to be the unique real contained in $\bigcap_{p\in G}[p]$, where $[p] := \{x \in \kappa^\kappa\colon \forall i < \kappa\ x \restriction i \in p\}$.

Definition 2.1.2. Furthermore, we define:

- $\mathrm{split}_p(\eta)$ iff $|\mathrm{succ}_p(\eta)| > 1$
- $\mathrm{ht}_p(\eta) := \mathrm{otp}\,\{\nu \trianglelefteq \eta\colon \mathrm{split}_p(\nu)\}$, where otp denotes the order type
- For $i < \kappa$: $\mathrm{split}_i(p) := \{\eta \in p\colon \mathrm{split}_p(\eta) \wedge \mathrm{ht}_p(\eta) = i\}$

Lemma 2.1.3. *PT_f is $<\kappa$-closed.*

Proof. If $(p_j)_{j<\lambda}$ with $\lambda < \kappa$ is a decreasing sequence, it is easy to see that $p := \bigcap_{j<\lambda} p_j$ is a condition. □

Definition 2.1.4. For $i < \kappa$, we define $q \leq_i p$ iff $q \leq_{PT_f} p \wedge \mathrm{split}_i(p) \subseteq q$.

Fact 2.1.5. The following holds true:

- $q \leq_i p \Leftrightarrow q \leq_{PT_f} p \wedge \forall j < i\ \mathrm{split}_j(q) = \mathrm{split}_j(p)$
- $\forall b \in \kappa^\kappa\ \forall i < \kappa\colon b \in [p] \Rightarrow b \cap \mathrm{split}_i(p) \neq \emptyset$, i.e. $\mathrm{split}_i(p)$ is a front in p

Definition 2.1.6. We call a forcing notion $\mathcal{P}$ strongly κ^κ-bounding iff there is a sequence $(\leq_i)_{i<\kappa}$ of reflexive and transitive, binary relations on $\mathcal{P}$ such that:

- $(\mathcal{P}, \leq_{\mathcal{P}})$ is $<\kappa$-closed
- $\leq_i \subseteq \leq_{\mathcal{P}}$
- $\forall j < i\colon\ \leq_i \subseteq \leq_j$
- If $(p_j)_{j<\delta}$ is a fusion sequence of length $\delta \leq \kappa$, i.e. $\forall j < \delta\colon p_{j+1} \leq_j p_j$ and $\forall \lambda < \delta\ \forall j < \lambda\colon \lambda$ is limit $\Rightarrow p_\lambda \leq_j p_j$, then there exists a q_δ such that $\forall j < \delta\colon q_\delta \leq_j p_j$.
- If A is a maximal antichain, $p \in \mathcal{P}$ and $i < \kappa$, then there exists $q \leq_i p$ such that $A \upharpoonright q := \{r \in A\colon r \parallel q\}$ has size $< \kappa$, where $\parallel$ means compatible.

Fact 2.1.7. Obviously, strongly κ^κ-bounding implies κ^κ-bounding.

Lemma 2.1.8. Let $(p_j)_{j<\delta}$ be a fusion sequence in PT_f of length $\delta \leq \kappa$. Then there exists q_δ such that $\forall j < \delta\colon q_\delta \leq_j p_j$.

Proof. Define $q_\delta := \bigcap_{j<\delta} p_j$. We need to show that q_δ is a condition. Only P4 is non-trivial. Let $\eta \in q_\delta$ be arbitrary and set $j^* := \mathrm{ht}_{p_0}(\eta)$. Consider p_{j^*+1} and note that $\mathrm{ht}_{p_{j^*+1}}(\eta) \leq \mathrm{ht}_{p_0}(\eta)$. Find $\nu \in p_{j^*+1}$ with $\eta \lhd \nu$, $\mathrm{split}_{p_{j^*+1}}(\nu)$ and with minimal domain. For every $\rho \in \mathrm{succ}_{p_{j^*+1}}(\nu)$ we have $\mathrm{ht}_{p_{j^*+1}}(\rho) \leq j^* + 1$, hence we can deduce $\forall j < \delta\colon \rho \in p_j$. Thus $\rho \in q_\delta$ and $\mathrm{split}_{q_\delta}(\nu)$ follows. □

Definition 2.1.9. If $p \in PT_f$ is a condition and $\eta \in p$, let $p^{[\eta]} := \{\nu \in p\colon \nu \lhd \eta \vee \eta \lhd \nu\}$. [2]

Lemma 2.1.10. Let p be a forcing condition and $i < \kappa$. Then $|\mathrm{split}_i(p)| < \kappa$.

Proof. We will prove the lemma by induction on $i < \kappa$:

- $i = 0$ is trivial.
- $i \to i+1$: As $|\mathrm{split}_i(p)| < \kappa$ and p is always $<\kappa$-splitting, it follows that $|\mathrm{split}_{i+1}(p)| = |\bigcup_{\eta \in \mathrm{split}_i(p)} \mathrm{succ}_p(\eta)| < \kappa$.
- λ is a limit: As κ is inaccessible, it follows that $|\mathrm{split}_\lambda(p)| \leq |\prod_{i<\lambda} \mathrm{split}_i(p)| < \kappa$.

This finishes the proof. □

Theorem 2.1.11. PT_f is strongly κ^κ-bounding.

Proof. Only the antichain condition remains to be shown. Let A be a maximal antichain, $p \in PT_f$ and $i < \kappa$ be arbitrary [3]. Enumerate $\mathrm{split}_i(p)$ as $\{\eta_j\colon j < \delta\}$ with $\delta < \kappa$. For every η_j find $q_{\eta_j} \leq p^{[\eta_j]}$ such that q_{η_j} is compatible with a unique element from the antichain. Now set $q := \bigcup_{j<\delta} q_{\eta_j}$ which is a condition. Obviously $q \leq_i p$.
Now let $r \in A$ be compatible with q. Let $s \leq_{PT_f} q, r$. W.l.o.g. let s be such that $|\mathrm{split}_i(p) \cap s| = 1$, this is possible if the stem is simply long enough: Pick $b \in \kappa^\kappa$ with $b \in [s] \subseteq [p]$ and note that $b \cap \mathrm{split}_i(p) \neq \emptyset$, hence we have $\exists!\, j\colon \eta_j \in \mathrm{split}_i(p) \cap b \cap s$. Therefore $s \leq_{PT_f} p^{[\eta_j]}$, and since $s \leq_{PT_f} q$ we have $s \leq_{PT_f} q_{\eta_j}$. It follows that if $r \in A$ is compatible with q, then it is also compatible with some $q_{\eta_{j_r}}$. Hence, there exists a function from $A \upharpoonright q$ to δ which has to be injective, since any q_{η_j} is compatible with a unique element from the antichain. Now $|A \upharpoonright q| < \kappa$ follows easily. □

[2] Obviously, $p^{[\eta]}$ is a condition stronger than p.
[3] Work with $i + 1$, if i is a limit.

2.2 The Iteration

Assume that $V \vDash |2^\kappa| = \kappa^+$. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ be an iteration of length κ^{++} with $\leq\kappa$-support such that $\Vdash_\alpha \dot{\mathbb{Q}}_\alpha = P\dot{T}_{f_\alpha}$. The family $(f_\alpha)_{\alpha<\kappa^{++}}$ is in the ground model V and we require that every $f \in \kappa^\kappa \cap V$ appears cofinally often. We set $\mathbb{P} := \mathbb{P}_{\kappa^{++}}$.

Definition 2.2.1. We say that a forcing notion $\mathcal{P}$ is κ-proper iff for every regular and sufficiently large cardinal θ (e.g. $\theta > |\mathfrak{P}(\mathcal{P})|$), every elementary submodel $M \prec H(\theta)$ containing $\mathcal{P}$ such that $|M| = \kappa$ and ${}^{<\kappa}M \subseteq M$, and every $p \in \mathcal{P} \cap M$, there exists $q \leq_{\mathcal{P}} p$ such that q is $(\mathcal{P}, M)$-generic.

Note that there cannot exist a general preservation theorem for κ-properness by Rosłanowski [Ros18]. Therefore, we will have to work to ensure κ-properness.

Definition 2.2.2. The following generalizes the notion of strongly κ^κ-bounding:

- Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta \colon \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ be an iteration of strongly κ^κ-bounding forcing notions, i.e. $\forall \alpha < \kappa^{++} \colon \Vdash_\alpha$ '$\dot{\mathcal{Q}}_\alpha$ is strongly κ^κ-bounding'. Let $F \in [\kappa^{++}]^{<\kappa}$ and $i < \kappa$. We define $q \leq_{F,i} p$ iff $q \leq_{\mathcal{P}_{\kappa^{++}}} p$ and $\forall \alpha \in F \colon q \restriction \alpha \Vdash_\alpha q(\alpha) \leq_i^{\dot{\mathcal{Q}}_\alpha} p(\alpha)$.
- A sequence $\langle (p_i, F_i) \colon i < \delta \rangle$ of length $\delta \leq \kappa$ is called a fusion sequence iff:
  - $\forall j < \delta \colon p_{j+1} \leq_{F_j, j} p_j$
  - $\forall \lambda < \delta \; \forall j < \lambda \colon \lambda$ is limit $\Rightarrow p_\lambda \leq_{F_j, j} p_j$
  - F_j increasing and, if $\delta = \kappa$, then $\bigcup_{j<\delta} \operatorname{supp}(p_j) \subseteq \bigcup_{j<\delta} F_j$.
- We say that $\mathcal{P}_{\kappa^{++}}$ satisfies Axiom B iff for every fusion sequence of length $\delta \leq \kappa$ there exists a q_δ such that $\forall j < \delta \colon q_\delta \leq_{F_j, j} p_j$ and, in addition, for every maximal antichain $A \subseteq \mathcal{P}_{\kappa^{++}}$, every $F \in [\kappa^{++}]^{<\kappa}$, every $i < \kappa$ and every $p \in \mathcal{P}_{\kappa^{++}}$ there exists a $q \leq_{F,i} p$ such that $|A \restriction q| < \kappa$.

Note that this is similar to fusion with countable support.

Fact 2.2.3. Axiom B implies κ-properness and κ^κ-bounding.

Lemma 2.2.4. For every fusion sequence of length $\delta \leq \kappa$ in $\mathbb{P}$, there exists a $q_\delta \in \mathbb{P}$ such that $\forall j < \delta \colon q_\delta \leq_{F_j, j} p_j$.

Proof. We will only consider the case $\delta = \kappa$. For $\alpha \in \bigcup_{j<\delta} \operatorname{supp}(p_j)$ choose j_α minimal such that $\alpha \in F_{j_\alpha}$. Set $q_\delta(\alpha) := \bigcap_{j \geq j_\alpha} p_j(\alpha)$. Otherwise let $q_\delta(\alpha) = \mathbb{1}_{PT_{f_\alpha}}$. By induction on $\alpha \leq \kappa^{++}$ show that $q_\delta \restriction \alpha \in \mathbb{P}_\alpha$ (using Lemma 2.1.8), $q_\delta \restriction \alpha \Vdash_\alpha$ '$(p_j(\alpha))_{j \geq j_\alpha}$ is a fusion sequence' and use $\operatorname{supp}(q_\delta) = \bigcup_{j<\delta} \operatorname{supp}(p_j)$ for limit steps. Obviously $\forall j < \delta \colon q_\delta \leq_{F_j, j} p_j$. $\square$

Next we want to show the following theorem:

Theorem 2.2.5. $\mathbb{P}$ satisfies Axiom B.

In [BGS21], the authors describe a 'Fusion game', which could be used to prove Axiom B. However, we use different methods, which shall come in handy later.

In order to prove this theorem, we will need some lemmas. Until the proof of Theorem 2.2.5 fix $p \in \mathbb{P}$, $F \in [\kappa^{++}]^{<\kappa}$ and $i < \kappa$.

Lemma 2.2.6. For $\alpha \leq \kappa^{++}$ define the set

$$D_\alpha := \{s \in \mathbb{P}_\alpha : \forall \beta \in F \cap \alpha \; \exists x_\beta, \, y_\beta \in \kappa^{<\kappa}$$

$$s \restriction \beta \Vdash_\beta s(\beta) \cap \operatorname{split}_i(p(\beta)) = \{x_\beta\}\check{} \wedge \operatorname{succ}_{s(\beta)}(x_\beta)\check{} = \{y_\beta\}\check{}\}.$$

Then D_α is dense below $p \restriction \alpha$.

Proof. Fix some $p' \in \mathbb{P}_\alpha$ with $p' \leq_{\mathbb{P}_\alpha} p \restriction \alpha$. Since $|F| < \kappa$, $\mathbb{P}_\alpha$ is $<\kappa$-closed, $p' \restriction \beta \Vdash_\beta p'(\beta) \subseteq p(\beta)$ and $\Vdash_\beta$ '$\operatorname{split}_i(p(\beta))$ is a front in $p(\beta)$', we can inductively construct some $s \leq_{\mathbb{P}_\alpha} p'$ such that $s \in D_\alpha$. □

If $s \in D_\alpha$ we shall write x^s_β, y^s_β for the corresponding x_β, y_β.

In the next lemmas we shall slightly abuse notation: If $s \in \mathbb{P}_\alpha$ with $s \leq_{\mathbb{P}_\alpha} p' \restriction \alpha$ we shall identify s with $s ^\frown p' \restriction [\alpha, \kappa^{++})$, e.g. saying $s \leq_{\mathbb{P}} p'$. For such s and p' the next lemma defines a new condition $p'^{[s]}$:

Lemma 2.2.7. Let $p' \leq_{F,i+1} p$ and $s \in D_\alpha$ with $s \leq_{\mathbb{P}} p'$. Then there exists $p'^{[s]} \leq_{F,i+1} p'$ such that

(a) $\forall \alpha' \geq \alpha \colon p'^{[s]}(\alpha') = p'(\alpha')$,

(b) $s \leq_{\mathbb{P}} p'^{[s]}$ and

(c) $\forall s' \in D_\alpha \colon \big(s' \leq_{\mathbb{P}} p'^{[s]} \wedge \forall \beta \in F \cap \alpha \; y^{s'}_\beta = y^s_\beta \big) \Rightarrow s' \leq_{\mathbb{P}} s$.

Proof. We will only consider the case $\alpha = \kappa^{++}$. Note that $\forall \beta \in F \colon s \restriction \beta \Vdash_\beta \operatorname{split}_i(p(\beta)) = \operatorname{split}_i(p'(\beta))$. Construct a sequence $(r_\beta)_{\beta \leq \kappa^{++}}$ of extending conditions by induction: Assume that $r_\beta \in \mathbb{P}_\beta$ has been constructed, $r_\beta \leq_{F \cap \beta, i+1} p'$ and $s \restriction \beta \leq_{\mathbb{P}} r_\beta$. Now there are two cases:

- $\beta \notin F$: Define $r_{\beta+1}(\beta) := \begin{cases} s(\beta) & \text{if } s \restriction \beta \in \dot{G}_\beta \\ p'(\beta) & \text{else} \end{cases}$

- $\beta \in F$: Define $r_{\beta+1}(\beta) := \begin{cases} s(\beta) \cup (p'(\beta) \setminus p'(\beta)^{[y^s_\beta]}) & \text{if } s \restriction \beta \in \dot{G}_\beta \\ p'(\beta) & \text{else} \end{cases}$

Obviously, $r_{\beta+1} \leq_{F \cap (\beta+1), i+1} p'$ and $s \restriction (\beta + 1) \leq_{\mathbb{P}} r_{\beta+1}$. If γ is a limit, define r_γ to be the union of $\{r_\beta \colon \beta < \gamma\}$. Set $p'^{[s]} := r_{\kappa^{++}}$ and note that $\operatorname{supp}(p'^{[s]}) \subseteq \operatorname{supp}(p') \cup \operatorname{supp}(s)$.

We check property (c). Let $s' \in D_{\kappa^{++}}$ with $s' \leq_{\mathbb{P}} p'^{[s]}$ and $\forall \beta \in F\colon y^{s'}_\beta = y^s_\beta$. Assume that $s' \upharpoonright \beta \leq_{\mathbb{P}} s \upharpoonright \beta$. If $\beta \in F$, we have that $s' \upharpoonright \beta \Vdash_\beta s'(\beta) \leq_{\dot{PT}_{f_\beta}} p'^{[s]}(\beta) = s(\beta) \cup p'(\beta) \setminus p'(\beta)^{[y^s_\beta]}$. Since $y^{s'}_\beta = y^s_\beta$ it follows that $s' \upharpoonright \beta \Vdash_\beta s'(\beta) \leq_{\dot{PT}_{f_\beta}} s(\beta)$. The case $\beta \notin F$ is trivial. Hence $s' \upharpoonright (\beta+1) \leq_{\mathbb{P}} s \upharpoonright (\beta+1)$. □

The next lemma will be used for the successor step in the proof of Theorem 2.2.5.

Lemma 2.2.8. Assume that $\mathbb{P}_\alpha$ satisfies Axiom B. Let $p' \leq_{F,i+1} p$. Then there exists $q \leq_{F,i+1} p'$ such that $\exists \mu_q < \kappa \; \forall \beta \in F \cap (\alpha+1)\colon q \upharpoonright \beta \Vdash_\beta \varphi(\mu_q, q(\beta), i)$ where $\varphi(\mu, s, j)$ is the formula $|\mathrm{split}_j(s)| \leq \mu \wedge \mathrm{split}_j(s) \subseteq \mu^{\leq \mu}$.

Proof. Note that as κ remains inaccessible in $V^{\mathbb{P}_\alpha}$, it follows that $\Vdash_\alpha \exists \mu < \kappa \; \forall \beta \in F \cap (\alpha+1)\colon \varphi(\mu, p'(\beta), i)$. Therefore, since $\mathbb{P}_\alpha$ satisfies Axiom B, there exists $q \leq_{F,i+1} p'$ and $\mu_q < \kappa$ such that $\forall \beta \in F \cap (\alpha+1)\colon q \upharpoonright \beta \Vdash_\beta \varphi(\mu_q, p'(\beta), i)$. Since $\forall \beta \in F\colon q \upharpoonright \beta \Vdash_\beta \mathrm{split}_i(p'(\beta)) = \mathrm{split}_i(q(\beta))$, it follows that $\forall \beta \in F \cap (\alpha+1)\colon q \upharpoonright \beta \Vdash_\beta \varphi(\mu_q, q(\beta), i)$. □

Proof of Theorem 2.2.5. By Lemma 2.2.4 it remains to be shown that for every maximal antichain $A \subseteq \mathbb{P}$, every $F \in [\kappa^{++}]^{<\kappa}$, every $i < \kappa$ and every $p \in \mathbb{P}$ there exists a $q \leq_{F,i} p$ such that $|A \upharpoonright q| < \kappa$. We shall prove the theorem for $\mathbb{P}_\alpha$ by induction on $\alpha \leq \kappa^{++}$:

- $\alpha = 1$: This follows from Theorem 2.1.11.
- $\alpha \to \alpha+1$: Let $A \subseteq \mathbb{P}_{\alpha+1}$ be a maximal antichain, $p \in \mathbb{P}_{\alpha+1}$ a condition, $F \in [\alpha+1]^{<\kappa}$ a set and $i < \kappa$ an ordinal. Let q and μ_q be as in Lemma 2.2.8. Now consider the set:

  $$C = \{g \in \prod_{\beta \in F} f_\beta(\mu_q)^{\leq(\mu_q+1)}\colon \exists s \in D_{\alpha+1} \; s \leq_{\mathbb{P}_\alpha} q \wedge |A \upharpoonright s| = 1 \wedge \forall \beta \in F \; y^s_\beta = g(\beta)\}.$$

  Enumerate C as $(g_{j+1})_{j<\delta}$ with $\delta < \kappa$. Now construct a $\leq_{F,i+1}$-decreasing sequence $(t_j)_{j<\delta}$ by induction:

  - Set $t_0 := q$.
  - $j \to j+1$: If for g_{j+1} there still exists an $s \in D_{\alpha+1}$ with $s \leq_{\mathbb{P}_\alpha} t_j$ witnessing $g_{j+1} \in C$, pick such an s, call it s_{j+1}, and set $t_{j+1} := t_j^{[s_{j+1}]}$. Otherwise, set $t_{j+1} := t_j$. Obviously, we have $t_{j+1} \leq_{F,i+1} t_j$.
  - $\lambda < \delta$ is a limit: Set $t_\lambda := \bigcap_{j<\lambda} t_j$, i.e. $\forall \alpha' < \alpha+1\colon t_\lambda(\alpha') := \bigcap_{j<\lambda} t_j(\alpha')$. Then we have $t_\lambda \leq_{F,i+1} t_j$ for all $j < \lambda$.

  Set $t := \bigcap_{j<\delta} t_j$. Then $t \leq_{F,i} p$.

  We claim that $|A \upharpoonright t| < \kappa$. Let $s' \in D_{\alpha+1}$ with $s' \leq_{\mathbb{P}_\alpha} t$ be compatible with a unique element from the antichain. Hence, there exists an $g_{j+1} \in C$ such that $\forall \beta \in F\colon y^{s'}_\beta = g_{j+1}(\beta)$. Now as $y^{s'}_\beta = y^{s_{j+1}}_\beta$ holds and $|A \upharpoonright s_{j+1}| = 1$, it follows from Lemma 2.2.7 and $t_{j+1} = t_j^{[s_{j+1}]}$ that $s' \leq_{\mathbb{P}_\alpha} s_{j+1}$ and hence $A \upharpoonright s' = A \upharpoonright s_{j+1}$. Thus $A \upharpoonright t \subseteq \{r \in A\colon \exists j < \delta \; s_{j+1} \parallel r\}$ and $|A \upharpoonright t| < \kappa$ follows.

- $\gamma \leq \kappa^{++}$ is a limit: Let $A \subseteq \mathbb{P}_\gamma$ be an antichain, $p \in \mathbb{P}_\gamma$ a condition, $F \in [\gamma]^{<\kappa}$ a set and $i < \kappa$ an ordinal. Using the induction hypothesis and the fact that $\mathbb{P}_\gamma$ is $<\kappa$-closed, we can easily construct a decreasing sequence $(q_\beta)_{\beta \in F}$ with the following properties:
  - $\forall \beta \in F\colon q_\beta \leq_{F,i+1} p$
  - $\forall \beta \in F\ \forall \beta' \in F \cap \beta\colon q_\beta \leq_{F,i+1} q_{\beta'}$
  - $\forall \beta \in F\ \exists \mu_{q_\beta} < \kappa\ \forall \beta' \in F \cap (\beta+1)\colon q_\beta \restriction \beta' \Vdash_{\beta'} \varphi(\mu_{q_\beta}, q_\beta(\beta'), i)$

  Set $q := \bigcap_{\beta \in F} q_\beta$ and $\mu_q := \sup\{\mu_\beta \colon \beta \in F\}$. Then $q \leq_{F,i} p$ and satisfies $\forall \beta \in F\colon q \restriction \beta \Vdash_\beta \varphi(\mu_q, q(\beta), i)$. Now proceed as in the successor step.

This finishes the proof of Theorem 2.2.5. □

Finally, we want to show some antichain results:

Theorem 2.2.9. $\mathbb{P}$ has the κ^{++}-c.c.

The proof will easily follow from the following lemmas and noting that the set $\{\alpha < \kappa^{++}\colon \mathrm{cf}(\alpha) = \kappa^+\}$ is stationary in κ^{++}:

Lemma 2.2.10. Let $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta \colon \alpha \leq \gamma, \beta < \gamma \rangle$ be an iteration such that $\forall \alpha < \gamma\colon P_\alpha$ has the θ-c.c., where θ is regular uncountable, and P_γ is a direct limit. If either $\mathrm{cf}(\gamma) \neq \theta$ or the set $\{\alpha < \gamma\colon P_\alpha \text{ is a direct limit}\}$ is stationary, then P_γ satisfies the θ-c.c.

For the proof of the above lemma see Chapter 16 in [Jec03].

Lemma 2.2.11. $\forall \alpha < \kappa^{++} : \mathbb{P}_\alpha$ has a dense subset of size κ^+. Hence $\mathbb{P}_\alpha$ satisfies the κ^{++}-c.c.

In [BGS21] the authors use 'hereditary κ^+-names' to find a dense subset of size κ^+.

For the proof we will need the following definition by Baumgartner and Laver [BL79]:

Definition 2.2.12. Let $p \in \mathbb{P}$, $F \in [\kappa^{++}]^{<\kappa}$ and $i < \kappa$. We say that p is (F,i)-determined iff for every $(g,h) \in \prod_{\beta \in F} \kappa^{<\kappa} \times \prod_{\beta \in F} \kappa^{<\kappa}$ such that $\forall \beta \in F\colon h(\beta) \in \mathrm{succ}(g(\beta))$:

- either $\forall \beta \in F\colon p^{[h]} \restriction \beta \Vdash_\beta g(\beta) \in \mathrm{split}_i(p(\beta))$
- or $\exists \beta' \in F\colon \forall \beta < \beta'\ p^{[h]} \restriction \beta \Vdash_\beta g(\beta) \in \mathrm{split}_i(p(\beta)) \wedge p^{[h]} \restriction \beta' \Vdash_{\beta'} g(\beta') \notin \mathrm{split}_i(p(\beta'))$

where $p^{[h]}$ is defined inductively such that $p^{[h]}(\beta) := p^{[h(\beta)]}(\beta)$ if $\beta \in F \cap \beta'$ and $p^{[h]}(\beta) := p(\beta)$ else (see Definition 2.1.9).

Proof of Lemma 2.2.11. Let $\alpha < \kappa^{++}$ be arbitrary. We will show that the set

$$E_\alpha := \{p \in \mathbb{P}_\alpha \colon \forall \beta \in \mathrm{supp}(p)\ \forall i < \kappa\ \exists j \geq i\ \exists F \in [\alpha]^{<\kappa}\ \beta \in F \wedge p \text{ is } (F,j)\text{-determined}\}$$

is dense and has size κ^+. Hence $\mathbb{P}_\alpha$ will have the κ^{++}-c.c.

We will first show density. Let $p \in \mathbb{P}_\alpha$ be arbitrary. By induction construct a fusion sequence $(q_j, F_j)_{j<\kappa}$ below p such that $\forall j < \kappa\ \forall \beta \in F_j\colon \Vdash_\beta \operatorname{split}_j(q_{j+1}(\beta)) = \operatorname{split}_j(q_j(\beta))$ and $\forall j < \kappa\colon q_{j+1}$ is (F_j, j)-determined. Use a bookkeeping argument to construct the F_j's. Then $q_\kappa \in E_\alpha$, where q_κ denotes the fusion limit:

In the successor step do the following: Assume that q_j and F_j are defined. Using Lemma 2.2.8 find $q'_j \leq_{F_j, j+1} q_j$ such that $\exists \mu_{q'_j} < \kappa\ \forall \beta \in F_j\colon q'_j \restriction \beta \Vdash_\beta \varphi(\mu_{q'_j}, q'_j(\beta), j)$. Now if we want to make sure that q_{j+1} is (F_j, j)-determined, we only need to check $(g, h) \in \prod_{\beta \in F_j} \mu_{q'_j}{}^{\leq \mu_{q'_j}} \times \prod_{\beta \in F_j} f_\beta(\mu_{q'_j})^{\leq(\mu_{q'_j}+1)}$.
This product is of size $< \kappa$, so enumerate the relevant (g, h) as $((g_{k+1}, h_{k+1}))_{k<\delta}$ with $\delta < \kappa$. Similarly to the proof of Theorem 2.2.5 we construct q_{j+1} by induction on $k < \delta$:

- Set $q_j^0 := q'_j$
- $k \to k+1$: Assume that q_j^k is defined. Define the condition $s_{h_{k+1}}$ as follows:

  $$s_{h_{k+1}}(\beta) := \begin{cases} \mathbb{1}_{PT_{f_\beta}}^{[h_{k+1}(\beta)]} & \text{if } \beta \in F_j \\ \mathbb{1}_{PT_{f_\beta}} & \text{else} \end{cases}$$

  Since $\operatorname{supp}(s_{h_{k+1}})$ has size $< \kappa$, we can distinguish two cases:
  - Case 1: $\exists s \leq_{\mathbb{P}_\alpha} q_j^k, s_{h_{k+1}}$ such that $\forall \beta \in F_j\colon s \restriction \beta \Vdash_\beta g_{k+1}(\beta) \in \operatorname{split}_j(q_j^k(\beta))$. Then set $q_j^{k+1} := q_j^{k\,[s]}$.
  - Case 2: Else there exists $s \leq_{\mathbb{P}_\alpha} q_j^k$ and $\beta' \in F_j$ such that $s \restriction \beta' \leq_{\mathbb{P}_\alpha} s_{h_{k+1}} \restriction \beta'$, $\forall \beta < \beta'\ s \restriction \beta \Vdash_\beta g_{k+1}(\beta) \in \operatorname{split}_j(q_j^k(\beta))$ and $s \restriction \beta' \Vdash_{\beta'} g_{k+1}(\beta') \notin \operatorname{split}_j(q_j^k(\beta'))$. In this case set $q_j^{k+1} := q_j^{k\,[s \restriction \beta']}$.

  This follows because if case 2 does not occur, then, by noting that $g_{k+1}(\beta) \in \operatorname{split}_j(q_j^k(\beta))$ implies $h_{k+1}(\beta) \in q_j^k(\beta)$, an s satisfying case 1 can be constructed.
- ι is a limit: Set $q_j^\iota := \bigcap_{k<\iota} q_j^k$.

Define $q_{j+1} := \bigcap_{k<\delta} q_j^k$. Clearly, q_{j+1} is (F_j, j)-determined.

In a limit step λ set $q_\lambda := \bigcap_{j<\lambda} q_j$. Clearly the fusion limit q_κ has the required properties. This shows that E_α is dense.

Now we will show by induction on $\alpha < \kappa^{++}$ that $|E_\alpha| = \kappa^+$:

- $\alpha = 1$: Then $E_\alpha = \mathbb{P}_1$ which has size $|2^\kappa| = \kappa^+$.
- γ is a limit: If $p \in E_\gamma$ then $p \restriction \alpha \in E_\alpha$ for every $\alpha < \gamma$, hence $|E_\gamma| \leq |\bigcup_{H \in [\gamma]^{\leq\kappa}} \prod_{\beta \in H} \kappa^+| = \kappa^+$.

- $\alpha \to \alpha+1$: Let $p \in E_{\alpha+1}$. Then p is completely determined by $p \restriction \alpha \in E_\alpha$, $(F_i, j_i)_{i<\kappa}$ such that $\alpha \in F_i$, and $(b_i)_{i<\kappa} \in \prod_{i<\kappa} \mathfrak{P}(\prod_{\beta \in F_i} \kappa^{<\kappa} \times \prod_{\beta \in F_i} \kappa^{<\kappa})$. The F_i's are increasing such that $\operatorname{supp}(p) \subseteq \bigcup_{i<\kappa} F_i$ and $\forall i < \kappa$: p is (F_i, j_i)-determined. The b_i's consist of those $(g, h) \in \prod_{\beta \in F_i} \kappa^{<\kappa} \times \prod_{\beta \in F_i} \kappa^{<\kappa}$ for which we have $\forall \beta \in F_i \colon p^{[h]} \restriction \beta \Vdash_\beta g(\beta) \in \operatorname{split}_{j_i}(p(\beta))$. Therefore, the mapping

$$E_{\alpha+1} \ni p \mapsto (p \restriction \alpha, (F_i, j_i, b_i)_{i<\kappa})$$

is injective. Hence, $E_{\alpha+1}$ has size κ^+. □

2.3 The Model

Recall that $\mathbb{P}$ will be an iteration of forcings of the form PT_{f_α}.

In what follows we shall always refer to the pointwise (not just eventually) dominating relation. This does not make any difference, since if D is an eventually dominating family, there exists D' of the same cardinality, such that D' is pointwise dominating. This easily follows from $\kappa^{<\kappa} = \kappa$. Note that since $\mathbb{P}$ is κ^κ-bounding, $\kappa^\kappa \cap V$ will be a dominating family in $V^{\mathbb{P}}$.

Furthermore, we require that κ is strongly unfoldable:

Definition 2.3.1. We call a cardinal κ strongly unfoldable iff κ is inaccessible and for every cardinal θ and every $x \subseteq \kappa$ there exists a transitive model M, such that $x \in M$ and $M \vDash \mathrm{ZFC}$, and an elementary embedding $j \colon M \to N$ with critical point κ, such that $j(\kappa) \geq \theta$ and $V_\theta \subseteq N$.

Note that strong unfoldability is downward absolute to L (see [Vil98]).

The first step in our iteration is a 'Johnstone preparation' to make the strong unfoldability of κ indestructible by $<\kappa$-closed, κ-proper forcing notions (see [Joh08]). We then collapse $\mathfrak{c}_\kappa$ to κ^+ using a $<\kappa^+$-closed forcing notion.

So w.l.o.g. $V \vDash$ '$|2^\kappa| = \kappa^+$ and the strong unfoldability is indestructible under $<\kappa$-closed, κ-proper forcing extensions'.

For the rest of this section we will be concerned with the main theorem:

Theorem 2.3.2. $V^{\mathbb{P}} \vDash \mathcal{SN} = [2^\kappa]^{\leq \kappa^+}$.

We will need several lemmas for the proof:

Lemma 2.3.3. Let $\dot{x}$ be a $\mathbb{P}$-name for a real in 2^κ, $p \in \mathbb{P}$ a condition, $F \in [\kappa^{++}]^{<\kappa}$ a set and $i < \kappa$ an ordinal, and assume $p \Vdash_{\mathbb{P}} \dot{x} \notin V$. Since κ is weakly compact, there exists $\delta < \kappa$ such that $\forall s \in 2^\delta\ \exists q \leq_{F,i} p \colon q \Vdash_{\mathbb{P}} s \not\subseteq \dot{x}$. We will write $\delta_{p,F,i}$ for the least such δ.

Proof. Assume towards a contradiction that the statement is false, i.e. for some $\dot{x}, p, F, i$ we have:

$$\forall \delta < \kappa\ \exists s_\delta \in 2^\delta \colon \neg \big(\exists q \leq_{F,i} p \colon q \Vdash_{\mathbb{P}} s_\delta \not\subseteq \dot{x} \big).$$

Set $T := \{s_\delta \restriction j \colon j \leq \delta \wedge \delta < \kappa\}$. T is a $<\kappa$-branching tree of height κ and as κ is weakly compact, T must have an infinite branch x^*. Since $x^* \in V$ but $p \Vdash_{\mathbb{P}} \dot{x} \notin V$ there exists a $\mathbb{P}$-name $\dot{j}$ for an ordinal less than κ such that $p \Vdash_{\mathbb{P}} \dot{x} \restriction \dot{j} \neq x^* \restriction \dot{j}$. As $\mathbb{P}$ satisfies Axiom B there exists a $q \leq_{F,i} p$ such that $q \Vdash_{\mathbb{P}} \dot{j} \leq j^*$ for some $j^* < \kappa$.
We claim that for some $\delta < \kappa$ we have $q \Vdash_{\mathbb{P}} s_\delta \not\subseteq \dot{x}$. We have that $q \Vdash_{\mathbb{P}} \dot{x} \restriction j^* \neq x^* \restriction j^*$, and since $x^* \restriction j^* \in T$, there exists $\delta \geq j^*$ such that $x^* \restriction j^* = s_\delta \restriction j^*$. Hence $q \Vdash_{\mathbb{P}} \dot{x} \restriction j^* \neq s_\delta \restriction j^*$ and therefore $q \Vdash_{\mathbb{P}} s_\delta \not\subseteq \dot{x}$. But this is a contradiction. □

Definition 2.3.4. Let D be a dominating family. We say that H has index D iff $H = \{h_f \colon f \in D\}$ and $\forall i < \kappa \colon h_f(i) \in 2^{f(i)}$.

Fact 2.3.5.

$$X \in \mathcal{SN} \Leftrightarrow \exists D \text{ dominating } \exists H \text{ with index } D \colon X \subseteq \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)]$$

$$\Leftrightarrow \forall D \text{ dominating } \exists H \text{ with index } D \colon X \subseteq \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)]$$

If $\alpha < \kappa^{++}$ and G_α is a $(V, \mathbb{P}_\alpha)$-generic filter, then in $V^{\mathbb{P}_\alpha}$ we define $\mathbb{P}^{\alpha,\kappa^{++}} := \mathbb{R}_{\kappa^{++}}$, where $\langle \mathbb{R}_\varepsilon, \dot{\mathbb{Q}}_\zeta \colon \varepsilon \leq \kappa^{++}, \zeta < \kappa^{++} \rangle$ is an iteration of length κ^{++} with $\leq\kappa$-support such that $\Vdash_\varepsilon \dot{\mathbb{Q}}_\varepsilon = \dot{PT}_{f_{\alpha+\varepsilon}}$. It follows from standard proper forcing arguments that in V the forcing $\mathbb{P} \approx \mathbb{P}_\alpha \star \mathbb{P}/\dot{G}_\alpha$ is dense in $\mathbb{P}_\alpha \star \dot{\mathbb{P}}^{\alpha,\kappa^{++}}$.

Lemma 2.3.6. Let $D \in V$ be a dominating family, $\alpha < \kappa^{++}$ and $H \in V^{\mathbb{P}_\alpha}$ has index D. Then we have

$$\Vdash_{\mathbb{P}^{\alpha,\kappa^{++}}} \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)] \subseteq 2^\kappa \cap V^{\mathbb{P}_\alpha}$$

Proof. Assume that for some condition p and some $\mathbb{P}^{\alpha,\kappa^{++}}$-name $\dot{x}$ we have

$$p \Vdash_{\mathbb{P}^{\alpha,\kappa^{++}}} \dot{x} \notin V^{\mathbb{P}_\alpha} \wedge \dot{x} \in \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)].$$

Working in $V^{\mathbb{P}_\alpha}$ we will define a tree of conditions such that along every branch we have a fusion sequence. Furthermore, we will define an increasing sequence $(\delta_i)_{i<\kappa}$ of ordinals less than κ and an increasing sequence $(F_i)_{i<\kappa}$ such that $F_i \in [\kappa^{++}]^{<\kappa}$.

For every $i < \kappa$ and every $g \in \prod_{j \leq i} 2^{\delta_j}$ we shall construct a condition $p(g)$ below p satisfying :

- $\forall i < \kappa\ \forall g \in \prod_{j \leq i} 2^{\delta_j} \colon p(g) \Vdash_{\mathbb{P}^{\alpha,\kappa^{++}}} g(i) \not\subseteq \dot{x}$
- $\forall i < \kappa\ \forall g \in \prod_{j \leq i} 2^{\delta_j}\ \forall s_{i+1} \in 2^{\delta_{i+1}} \colon p(g^\frown s_{i+1}) \leq_{F_i,i} p(g)$

- $\forall \lambda < \kappa$: λ is limit $\Rightarrow \forall g \in \prod_{j \leq \lambda} 2^{\delta_j}$ $\forall j < \lambda$ $p(g) \leq_{F_j, j} p(g \restriction (j+1))$
- $\forall i < \kappa$ $\forall g \in \prod_{j \leq i} 2^{\delta_j}$: $\mathrm{supp}(p(g)) \subseteq \bigcup_{j<\kappa} F_j$

$i = 0$: Since κ remains weakly compact in $V^{\mathbb{P}_\alpha}$, we can use Lemma 2.3.3 below p to find $\delta_0 < \kappa$ and $p(s_0)$ for every $s_0 \in 2^{\delta_0}$ such that $p(g) \Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} s_0 \not\subseteq \dot{x}$. Set $F_0 := \{0\}$.

$i \to i+1$: Assume that $p(g)$ is defined for every $g \in \prod_{j \leq i} 2^{\delta_j}$. Again using Lemma 2.3.3 we set $\delta_{i+1} := \sup\{\delta_{p(g), F_i, i} : g \in \prod_{j \leq i} 2^{\delta_j}\}$ and find $p(g^\frown s_{i+1})$ for every $g \in \prod_{j \leq i} 2^{\delta_j}$ and $s_{i+1} \in 2^{\delta_{i+1}}$ with the required properties. Finally, we use a bookkeeping argument to define F_{i+1}.

λ is a limit: Every $h \in \prod_{i<\lambda} 2^{\delta_i}$ defines a fusion sequence $(p(h \restriction (i+1)))_{i<\lambda}$. Set $p(h) := \bigcap_{i<\lambda} p(h \restriction (i+1))$ and $F_\lambda := \bigcup_{i<\lambda} F_i$. Next define $\delta_\lambda := \sup\{\delta_{p(h), F_\lambda, \lambda} : h \in \prod_{i<\lambda} 2^{\delta_i}\}$ and for every $g \in \prod_{j \leq \lambda} 2^{\delta_j}$ and $s_\lambda \in 2^{\delta_\lambda}$ find $p(h^\frown s_\lambda)$ again using Lemma 2.3.3. Note that $p(h^\frown s_\lambda)$ is still a fusion limit of $(p(h \restriction (i+1)))_{i<\lambda}$.

Let $f \in D$ dominate the function $(\delta_i)_{i<\kappa}$. Set $s_i := h_f(i) \restriction \delta_i$. Now $(p(\langle s_0, \ldots, s_j \rangle))_{j<\kappa}$ is a fusion sequence and has a lower bound p_κ. It follows that $p_\kappa \Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} s_i \not\subseteq \dot{x}$ for every $i < \kappa$. Thus $p_\kappa \Vdash_{\mathbb{P}_{\alpha,\kappa^{++}}} \dot{x} \notin \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)]$. A contradiction. □

We will also need the following lemma:

Lemma 2.3.7. If for every bounded family $B \subseteq \kappa^\kappa$ of size $< \theta$ there exists a $g \in \kappa^\kappa$ such that g diagonalizes B, i.e. $\forall h \in B$ $\exists^\infty i < \kappa$: $g(i) = h(i)$ [4], then $\mathrm{non}(\mathcal{SN}) \geq \theta$.

Proof. Let $X \subseteq 2^\kappa$ be of size $< \theta$ and let $f \in \kappa^\kappa$. For $x \in X$ let $h_x(i) := x \restriction f(i)$. The family $\{h_x : x \in X\}$ can be coded as a family $B \subseteq \kappa^\kappa$ bounded by $(|2^{f(i)}|)_{i<\kappa}$. Now if g diagonalizes B, then g defines a covering for X with respect to f. □

Proof of Theorem 2.3.2. Since for every $\alpha < \kappa^{++}$ the forcing $\mathbb{P}_\alpha$ has dense subset of size κ^+ by Lemma 2.2.11 and is κ-proper, there are essentially only $|(\kappa^+)^\kappa| = \kappa^+$ many $\mathbb{P}_\alpha$-names for reals. Hence $V^{\mathbb{P}_\alpha} \models |2^\kappa| = \kappa^+$. As $\mathbb{P}$ satisfies the κ^{++}-c.c. and is also κ-proper, we see that $V^{\mathbb{P}} \models$ '$|2^\kappa| = \kappa^{++}$ and no cardinals are collapsed'.

Let us first show that $\mathcal{SN} \subseteq [2^\kappa]^{\leq\kappa^+}$. Let $X \subseteq 2^\kappa$ be of size κ^{++}, and let D be a dominating family in $V^{\mathbb{P}}$ which lies in V. We will show that there exists no H in $V^{\mathbb{P}}$ with index D such that $X \subseteq \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)]$. To this end let $H \in V^{\mathbb{P}}$ be such that H has index D, and note that H must be of size κ^+ as V satisfies GCH at κ. Since $\mathbb{P}$ satisfies the κ^{++}-c.c., H must already appear in some $V^{\mathbb{P}_\alpha}$. Now there must be an $x \in X$ such that $x \notin V^{\mathbb{P}_\alpha}$. Hence, it follows by Lemma 2.3.6 that $x \notin \bigcap_{f \in D} \bigcup_{i<\kappa} [h_f(i)]$. Therefore, X is not strong measure zero.
In order to show that $[2^\kappa]^{\leq\kappa^+} \subseteq \mathcal{SN}$ we use Lemma 2.3.7: Let $B \subseteq \kappa^\kappa$ be a bounded family of size $< \kappa^{++}$, hence B appears in some intermediate model. Find some large enough

[4] Here $\exists^\infty i < \kappa$ means 'there exist unboundedly many $i < \kappa$ with the desired property'.

α such that f_α dominates B. We will show that $\exists g \in \kappa^\kappa \cap V^{\mathbb{P}_{\alpha+1}}\ \forall h \in B\ \exists^\infty i\colon g(i) = h(i)$, hence $V^{\mathbb{P}} \vDash \text{non}(\mathcal{SN}) \geq \kappa^{++}$.
Let $h \in B$ and $j < \kappa$ be arbitrary. Define the set $D_{h,j} := \{p \in PT_{f_\alpha}^{V^{\mathbb{P}_\alpha}} : \exists i \geq j\ p \Vdash_{PT_{f_\alpha}} \dot{g}_\alpha(i) = h(i)\check{}\}$. By extending the stem of a condition $q \in PT_{f_\alpha}^{V^{\mathbb{P}_\alpha}}$, we can show that $D_{h,j}$ is dense in $PT_{f_\alpha}^{V^{\mathbb{P}_\alpha}}$. Therefore, g_α will diagonalize every $h \in B$. □

2.4 A model where every $X \subsetneq 2^\kappa$ of size $\mathfrak{c}_\kappa$ can uniformly continuously be mapped onto 2^κ

Again, assume $V \vDash |2^\kappa| = \kappa^+$, but now κ is only inaccessible.

First we will define two forcing notions:

We define $\mathbb{S}_\kappa$, the generalized Sacks forcing, which is due to Kanamori [Kan80], as follows:

Definition 2.4.1. Let $p \in \mathbb{S}_\kappa$ iff:

- $p \subseteq 2^{<\kappa}$, $p \neq \emptyset$
- $\forall \eta \in p\ \exists \nu \in p\colon \eta \triangleleft \nu \wedge \text{split}_p(\nu)$
- If λ is a limit, then $\forall \eta \in 2^\lambda\colon \eta \in p \Leftrightarrow \forall i < \lambda\ \ \eta \restriction i \in p$
- If λ is a limit, then
  $\forall \eta \in 2^\lambda\colon \big(\eta \in p \wedge \{\nu \subsetneq \eta\colon \text{split}_p(\nu)\} \text{ is unbounded in } \eta\big) \Rightarrow \text{split}_p(\eta)$

Define $q \leq_{\mathbb{S}_\kappa} p$ iff $q \subseteq p$. Set $q \leq_i p$ iff $q \leq_{\mathbb{S}_\kappa} p \wedge \text{split}_i(p) \subseteq q$.

If G is a $(V, \mathbb{S}_\kappa)$-generic filter we define $s_G \in 2^\kappa$ to be the unique real contained in $\bigcap_{p \in G}[p]$, where $[p] := \{x \in 2^\kappa\colon \forall i < \kappa\ x \restriction i \in p\}$.

And for $f \in \kappa^\kappa \cap V$ we define $\mathbb{I}_f$ the infinitely equal forcing as follows:

Definition 2.4.2. Let $p \in \mathbb{I}_f$ iff:

- $\text{dom}(p) \subseteq \kappa$
- $|\kappa \setminus \text{dom}(p)| = \kappa$
- $\kappa \setminus \text{dom}(p)$ is closed
- $\forall i \in \text{dom}(p)\colon p(i) \in 2^{f(i)}$

Define $q \leq_{\mathbb{I}_f} p$ iff $p \subseteq q$. Set $q \leq_i p$ iff $q \leq_{\mathbb{I}_f} p \wedge (\exists j \geq i \colon (\kappa \setminus \mathrm{dom}(q)) \cap j = (\kappa \setminus \mathrm{dom}(p)) \cap j \wedge \mathrm{otp}\,(\kappa \setminus \mathrm{dom}(p) \cap j) = i\,)$.

If G is a $(V, \mathbb{I}_f)$-generic filter we define $g \in (2^{<\kappa})^\kappa$ as $\bigcup_{p \in G} p$.

Note that $\mathbb{I}_f$ is also a 'tree forcing', hence Definition 2.1.2 can be used analogously. However, the conditions are Silver-like trees, therefore we need to modify some proofs.

Lemma 2.4.3. $\mathbb{S}_\kappa$ and $\mathbb{I}_f$ are strongly κ^κ-bounding.

Proof. We will only consider the forcing $\mathbb{I}_f$. Let A be a maximal antichain, $p \in \mathbb{I}_f$ and $i < \kappa$ be arbitrary [5]. Enumerate $\mathrm{split}_i(p)$ as $\{\eta_{j+1} \colon j < \delta\}$ with $\delta < \kappa$. Inductively define a sequence $(q_j)_{j<\delta}$ such that $q_0 = p$ and $q_{j+1} \leq_{\mathbb{I}_f} (q_j \setminus \eta_j) \cup \eta_{j+1}$ is compatible with a unique element from the antichain. If λ is a limit, define $q_\lambda := \bigcup_{j<\delta}(q_{j+1} \setminus \eta_{j+1})$. Now set $q := p \cup \bigcup_{j<\delta}(q_{j+1} \setminus \eta_{j+1})$. The rest follows easily. □

Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ be an iteration of length κ^{++} with $\leq\kappa$-support such that:

- if $\mathrm{cf}(\alpha) = \kappa^+$ or $\alpha = 0$ then $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \dot{\mathbb{S}}_\kappa$
- otherwise $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \dot{\mathbb{I}}_{f_\alpha}$ such that every $f \in \kappa^\kappa \cap V$ appears cofinally often

We set $\mathbb{P} := \mathbb{P}_{\kappa^{++}}$.

We will also need to modify Lemma 2.2.7. Let $p \in \mathbb{P}$, $F \in [\kappa^{++}]^{<\kappa}$ and $i < \kappa$.

Lemma 2.4.4. Let D_α be as in Lemma 2.2.6. Let $p' \leq_{F,i+1} p$ and $s \in D_\alpha$ with $s \leq_{\mathbb{P}} p'$. Then there exists $p'^{[s]} \leq_{F,i+1} p'$ such that

(a) $\forall \alpha' \geq \alpha \colon p'^{[s]}(\alpha') = p'(\alpha')$,

(b) $s \leq_{\mathbb{P}} p'^{[s]}$

(c) $\forall s' \in D_\alpha \colon \big(s' \leq_{\mathbb{P}} p'^{[s]} \wedge \forall \beta \in F \cap \alpha\ y^{s'}_\beta = y^s_\beta \big) \Rightarrow s' \leq_{\mathbb{P}} s$.

Proof. We will only consider the case $\alpha = \kappa^{++}$. Again, construct a sequence $(r_\beta)_{\beta \leq \kappa^{++}}$ by induction: Assume that $r_\beta \in \mathbb{P}_\beta$ has been constructed, $r_\beta \leq_{F\cap\beta,i+1} p'$ and $s \restriction \beta \leq_{\mathbb{P}} r_\beta$. Now there are 3 cases:

- $\beta \notin F$: Define $r_{\beta+1}(\beta) := \begin{cases} s(\beta) & \text{if } s \restriction \beta \in \dot{G}_\beta \\ p'(\beta) & \text{else} \end{cases}$

- $\beta \in F \wedge \dot{\mathbb{Q}}_\beta = \dot{\mathbb{S}}_\kappa$: Define $r_{\beta+1}(\beta) := \begin{cases} s(\beta) \cup (p'(\beta) \setminus p'(\beta)^{[y^s_\beta]}) & \text{if } s \restriction \beta \in \dot{G}_\beta \\ p'(\beta) & \text{else} \end{cases}$

[5] Again, work with $i + 1$, if i is a limit.

- $\beta \in F \wedge \dot{\mathbb{Q}}_\beta = \dot{\mathbb{I}}_{f_\beta}$: Define $r_{\beta+1}(\beta) := \begin{cases} p'(\beta) \cup (s(\beta) \setminus y^s_\beta) & \text{if } s \restriction \beta \in \dot{G}_\beta \\ p'(\beta) & \text{else} \end{cases}$

Obviously, $r_{\beta+1} \leq_{F\cap(\beta+1),i+1} p'$ and $s \restriction (\beta+1) \leq_{\mathbb{P}} r_{\beta+1}$. If γ is a limit, define r_γ to be the union of $\{r_\beta \colon \beta < \gamma\}$. Set $p'^{[s]} := r_{\kappa^{++}}$ and note that $\text{supp}(p'^{[s]}) \subseteq \text{supp}(p') \cup \text{supp}(s)$.

We check property (c). Let $s' \in D_{\kappa^{++}}$ with $s' \leq_{\mathbb{P}} p'^{[s]}$ and $\forall \beta \in F \colon y^{s'}_\beta = y^s_\beta$. Assume that $s' \restriction \beta \leq_{\mathbb{P}} s \restriction \beta$. If $\beta \in F \wedge \dot{\mathbb{Q}}_\beta = \dot{\mathbb{I}}_{f_\beta}$, we have that $s' \restriction \beta \Vdash_\beta s'(\beta) \leq_{\mathbb{P}} p'^{[s]}(\beta) = p'(\beta) \cup (s(\beta) \setminus y^s_\beta)$. As $y^{s'}_\beta = y^s_\beta$ and $s' \restriction \beta \Vdash_\beta s(\beta) \leq_{\dot{\mathbb{I}}_{f_\beta}} p'(\beta)$, it follows that $s' \restriction \beta \Vdash_\beta s'(\beta) \leq_{\dot{\mathbb{I}}_{f_\beta}} s(\beta)$. The other 2 cases are similar. Hence $s' \restriction (\beta+1) \leq_{\mathbb{P}} s \restriction (\beta+1)$. □

Lemma 2.4.5. $\mathbb{P}$ satisfies Axiom B.

Proof. See the proof of Theorem 2.2.5. □

Lemma 2.4.6. $\forall \alpha < \kappa^{++} : \mathbb{P}_\alpha$ has a dense subset of size κ^+, and $\mathbb{P}$ satisfies the κ^{++}-c.c.

Proof. See the proof of Theorem 2.2.9 and Lemma 2.2.11. □

Again, our goal is to show the following theorem:

Theorem 2.4.7. $V^{\mathbb{P}} \vDash \mathcal{SN} = [2^\kappa]^{\leq \kappa^+}$.

One direction is the following lemma:

Lemma 2.4.8. $V^{\mathbb{P}} \vDash [2^\kappa]^{\leq \kappa^+} \subseteq \mathcal{SN}$.

Proof. Let $f \in \kappa^\kappa \cap V$ and $X \in [2^\kappa]^{\leq \kappa^+}$. Since $\mathbb{P}$ satisfies the κ^{++}-c.c., there exists $\alpha < \kappa^{++}$ such that $X \in V^{\mathbb{P}_\alpha}$. Find $\alpha' > \alpha$ such that $\dot{\mathbb{Q}}_{\alpha'} = \dot{\mathbb{I}}_{f_{\alpha'}}$ and $f_{\alpha'} = f$. For $x \in X$ define the set $D_x := \{p \in \mathbb{I}^{V^{\mathbb{P}_{\alpha'}}}_{f_{\alpha'}} : \exists i < \kappa \ p \Vdash_{\mathbb{I}_{f_{\alpha'}}} \dot{g}_{\alpha'}(i) = x \restriction f(i)\check{}\}$. Obviously, D_x is dense in $\mathbb{I}^{V^{\mathbb{P}_{\alpha'}}}_{f_{\alpha'}}$, hence $(g_{\alpha'}(i))_{i\in\kappa}$ will be the required covering. □

The next lemma will be crucial for the proof of Theorem 2.4.7

Lemma 2.4.9. Let $\dot{\tau}$ be a $\mathbb{P}$-name for an element of 2^κ, $p \in \mathbb{P}$ and $p \Vdash_{\mathbb{P}} \dot{\tau} \notin V$. Then there exists $q \leq_{\mathbb{P}} p$ and $(A_\eta)_{\eta \in \text{split}(q(0))}$ such that $A_\eta \subseteq 2^\kappa$ are non-empty, clopen and:

- if $\eta_1 \perp \eta_2$ then $A_{\eta_1} \cap A_{\eta_2} = \emptyset$
- if $\eta_1 \lhd \eta_2$ then $A_{\eta_2} \subseteq A_{\eta_1}$
- $q^{[\eta]} \Vdash_{\mathbb{P}} \dot{\tau} \in A_\eta$

where $q^{[\eta]}(\beta) := q^{[\eta]}(0)$ if $\beta = 0$ and $q(\beta)$ otherwise.

Proof. We shall construct a fusion sequence $(q_i, F_i)_{i<\kappa}$ such that $q_{i+1} \leq_{F_i,i+1} q_i$. The condition q_{i+1} will have the required properties for $(A_\eta)_{\eta \in \text{split}_i(q_i(0))}$. Also recall the definition of $D^i_{\kappa^{++}}$:

$$D^i_{\kappa^{++}} = \{s \in \mathbb{P} \colon \forall \beta \in F_i \ \exists x_\beta, y_\beta \in \kappa^{<\kappa}$$
$$s \restriction \beta \Vdash_\beta s(\beta) \cap \text{split}_i(p(\beta)) = \{x_\beta\}\check{} \wedge \text{succ}_{s(\beta)}(x_\beta)\check{} = \{y_\beta\}\check{}\}.$$

- $i=0$: Pick F_0 with $0 \in F_0$. Set $q_0 := p$.
- λ is a limit: Set $q_\lambda := \bigcap_{i<\lambda} q_i$ and $F_\lambda := \bigcup_{i<\lambda} F_i$.
- $i \to i+1$: Pick $q'_i \leq_{F_i,i+1} q_i$ such that there exists a $\mu_{q'_i} < \kappa$ with $\forall \beta \in F_i \colon q'_i \restriction \beta \Vdash_\beta \varphi(\mu_{q'_i}, q'_i(\beta), i)$ (see Lemma 2.2.8). Enumerate $\operatorname{split}_i(q'_i(0))$ as $(\eta^i_{j+1})_{j<\delta_i}$ with $\delta_i < \kappa$.

  Now take care of the η^i_j's inductively. Simultaneously, define an increasing sequence $(X^i_j)_{j<\delta_i}$ with $X^i_j \subseteq 2^\kappa$ and $|X^i_j| < \kappa$. The X^i_j's will contain interpretations of $\dot\tau$.

  - $j=0$: Set $X^i_0 = \tilde X^i_0 := \emptyset$ and $l^i_0 = \tilde l^i_0 := 0$.
  - $j \to j+1$: Now work below $q_i'^{[\eta^i_{j+1}]}$. Find $q_i'^{[\eta^i_{j+1}]'} \leq_{F_i,i+1} q_i'^{[\eta^i_{j+1}]}$ and l^i_{j+1} such that $q_i'^{[\eta^i_{j+1}]'} \Vdash_{\mathbb{P}} \forall x \in X^i_j \colon \dot\tau \restriction l^i_{j+1} \neq x \restriction l^i_{j+1}$. This is possible, because $p \Vdash_{\mathbb{P}} \dot\tau \notin V$, $|X^i_j| < \kappa$, $\mathbb{P}$ is $<\kappa$-closed and satisfies Axiom B. In more detail: There exists a $\mathbb{P}$-name $\dot l$ for an ordinal $< \kappa$ such that the set $\{s \in \mathbb{P} \colon \exists l_s < \kappa \; s \Vdash_{\mathbb{P}} \forall x \in \check X^i_j \colon \dot\tau \restriction \dot l \neq x \restriction \dot l \wedge \dot l = l_s\}$ is dense. Now use Axiom B to find an upper bound for $\dot l$.

    By induction we now define decreasing sequences $({}^{j+1}q^k_{i+1})_{k<\kappa}$, such that ${}^{j+1}q^k_{i+1} \leq_{F_i \setminus \{0\},i+1} q_i$, and $(C^i_k)_{k<\kappa}$ with $C^i_k \subseteq \prod_{\beta\in F_i} f_\beta(\mu_{q'_i})^{\leq(\mu_{q'_i}+1)}$ [6], and set $q^k := {}^{j+1}q^k_{i+1}$.

    * $k=0$: Set $q^0 := q_i'^{[\eta^i_{j+1}]'}$. Define

    $$C^i_0 := \{g \in \prod_{\beta\in F_i} f_\beta(\mu_{q'_i})^{\leq(\mu_{q'_i}+1)} \colon$$
    $$\exists s \in D^i_{\kappa^{++}} \; s \leq_{\mathbb{P}} q^0 \wedge \forall \beta \in F_i \; y^s_\beta = g(\beta)\}.$$

    * $k \to k+1$: Similarly to the proof of Theorem 2.2.5 we take care of all $g \in C^i_k$ and witnesses ${}^i s^k_g$, and construct q^{k+1} using Lemma 2.4.4. Define:

    $$C^i_{k+1} := \{g \in \prod_{\beta\in F_i} f_\beta(\mu_{q'_i})^{\leq(\mu_{q'_i}+1)} \colon$$
    $$\exists s \in D^i_{\kappa^{++}} \; s \leq_{\mathbb{P}} q^{k+1} \wedge s \text{ decides } \dot\tau \restriction (k+1) \wedge \forall \beta \in F_i \; y^s_\beta = g(\beta)\}.$$

    * ξ is a limit: Set $q^\xi := \bigcap_{k<\xi} q^k$ and define:

    $$C^i_\xi := \{g \in \prod_{\beta\in F_i} f_\beta(\mu_{q'_i})^{\leq(\mu_{q'_i}+1)} \colon$$
    $$\exists s \in D^i_{\kappa^{++}} \; s \leq_{\mathbb{P}} q^\xi \wedge s \text{ decides } \dot\tau \restriction \xi \wedge \forall \beta \in F_i \; y^s_\beta = g(\beta)\}.$$

[6] If β has cofinality κ^+, then set f_β to be the identity.

As $(C^i_k)_{k<\kappa}$ is a decreasing sequence of length κ and $|C^i_k| < \kappa$ the sequence must eventually be constant. Denote this index by $\tilde{l}^i_{j+1}$. Note that C^i_k is non-empty by a density argument.

Now define X^i_{j+1}. Set

$$\tilde{X}^i_{j+1} := \{x \in 2^\kappa : \exists g \in C^i_{\tilde{l}^i_{j+1}} \ \forall k < \kappa \ {}^i s^k_g \Vdash_{\mathbb{P}} \dot{\tau} \restriction k = x \restriction k\}$$

and note that $({}^i s^k_g)_{k<\kappa}$ is necessarily a decreasing sequence by Lemma 2.4.4. Furthermore, note that $\tilde{X}^i_{j+1}$ is disjoint from X^i_j and set $X^i_{j+1} := X^i_j \cup \tilde{X}^i_{j+1}$.

– ι is a limit: Set $X^i_\iota := \bigcup_{j<\iota} X^i_j$ and $\tilde{X}^i_\iota = \emptyset$. Set $l^i_\iota = \tilde{l}^i_\iota := 0$.

Define $l^i := \max\big\{\sup\{l^i_j : j < \delta_i\}, \sup\{\tilde{l}^i_j : j < \delta_i\}\big\}$. Recall that $\tilde{X}^i_{j+1} = X^i_{j+1} \setminus X^i_j$. Define $A_{\eta^i_{j+1}} := \bigcup_{x \in \tilde{X}^i_{j+1}} [x \restriction l^i]$. W.l.o.g. let l^i be large enough such that the $A_{\eta^i_j}$'s are disjoint. We can assume this, since the $\tilde{X}^i_j$'s are disjoint and of size $< \kappa$. Define $q_{i+1} \in \mathbb{P}$ such that $q_{i+1}(0) := \bigcup_{j<\delta_i} {}^{j+1}q^{l^i+1}_{i+1}(0)$ and $q_{i+1}(\beta) := {}^{j+1}q^{l^i+1}_{i+1}(\beta)$ if ${}^{j+1}q^{l^i+1}_{i+1} \restriction \beta \in \dot{G}_\beta$ and $q_{i+1}(\beta) := \mathbb{1}_{\dot{\mathbb{Q}}_\beta}$ else for $\beta > 0$. Note that $q_{i+1} \leq_{F_i, i+1} q_i$ and $q^{[\eta^i_{j+1}]}_{i+1} = {}^{j+1}q^{l^i+1}_{i+1}$. Define F_{i+1} using a bookkeeping argument.

We claim that the fusion limit q_κ has the required properties. Let $i < \kappa$ and $\eta \in \operatorname{split}_i(q_\kappa(0))$ be arbitrary. We must show that $q^{[\eta]}_\kappa \Vdash_{\mathbb{P}} \dot{\tau} \in A_\eta$.
It follows that $\eta \in \operatorname{split}_i(q_{i+1}(0))$, and since $q_{i+1} \leq_{F_i, i+1} q_i$, we deduce $\eta \in \operatorname{split}_i(q_i(0))$. Therefore $\eta = \eta^i_{j+1}$ for some $j < \delta_i$. We will show that $q^{[\eta^i_{j+1}]}_{i+1} \Vdash_{\mathbb{P}} \dot{\tau} \in A_{\eta^i_{j+1}}$. Let $s \leq_{\mathbb{P}} q^{[\eta^i_{j+1}]}_{i+1}$ with $s \in D^i_{\kappa^{++}}$ and s decides $\dot{\tau} \restriction l^i$. As $q^{[\eta^i_{j+1}]}_{i+1} = {}^{j+1}q^{l^i+1}_{i+1}$ we have $s \leq_{\mathbb{P}} {}^{j+1}q^{l^i+1}_{i+1}$. Therefore $s \leq_{\mathbb{P}} {}^i s^{l^i}_g$ for some $g \in C^i_{l^i}$ and hence $s \Vdash_{\mathbb{P}} \dot{\tau} \restriction l^i = x \restriction l^i$ for some $x \in \tilde{X}^i_{j+1}$. $\square$

In particular, $\dot{\tau}$ can continuously be mapped onto the first Sacks real $\dot{s}_0$ by a function from V. Note that for $<\kappa$-closed forcing extensions it is clear how to evaluate the image of a new real $\dot{x}$ under a ground model κ-Borel function f: In the ground model f is completely determined by $(B_s)_{s\in\kappa^{<\kappa}}$ where $B_s := f^{-1}([s])$, and the following is a Π^1_1-statement:

$$\forall x \in 2^\kappa \ \forall i < \kappa \ \exists! s \in 2^i : x \in B_s \wedge \forall s, t \in \kappa^{<\kappa} : s \triangleleft t \Rightarrow B_t \subseteq B_s.$$

Note that the mapping $s \mapsto B_s$ is κ-Borel, since $|\kappa^{<\kappa}| = \kappa$. By Π^1_1-absoluteness for $<\kappa$-closed forcing extensions (see Fact 1.1.3), it follows that $f(\dot{x}^G) = \bigcup\{s \in \kappa^{<\kappa} : \dot{x}^G \in B_s\}$.

Lemma 2.4.10. Let $p \in \mathbb{S}_\kappa$ be a Sacks condition. Then there exists a homeomorphism $g\colon [p] \to 2^\kappa \times 2^\kappa$ such that $\forall x \in 2^\kappa : \{\eta \in 2^{<\kappa} : \exists y \in g^{-1}(\{x\} \times 2^\kappa) \ \eta \triangleleft y\}$ is a Sacks condition stronger than p.

Proof. First we define $e\colon p \to 2^{<\kappa} \times 2^{<\kappa}$ as follows:

- e is monotonous

- e is continuous
- $e(\emptyset) = (\emptyset, \emptyset)$
- $\eta \notin \text{split}(p) \Rightarrow e(\eta^\frown i) := e(\eta)$
- $\eta \in \text{split}_j(p)$
  - if j is a successor, then $e(\eta^\frown i) := (e_1(\eta)^\frown i, e_2(\eta))$
  - if j is a limit, then $e(\eta^\frown i) := (e_1(\eta), e_2(\eta)^\frown i)$

For $u \in [p]$ set $g(u) := v$ iff $\{v\} = \bigcap_{i<\kappa}[e(u \restriction i)]$. Then g is a homeomorphism, since the topologies on $[p]$ and $2^\kappa \times 2^\kappa$ both have a clopen basis.

Now we must show that $q_x := \{\eta \in 2^{<\kappa} \colon \exists y \in g^{-1}(\{x\} \times 2^\kappa)\ \ \eta \lhd y\}$ is a condition:

- Let $(\eta_j)_{j<\delta}$ with $\eta_j \in q_x$ be a strictly increasing sequence of length $< \kappa$. Set $\eta := \bigcup_{j<\delta} \eta_j$. It easily follows that $\nu \in q_x \Leftrightarrow x \in [e_1(\nu)]$. As $e(\eta) = \bigcup_{j<\delta} e(\eta_j)$ we see that $x \in [e_1(\eta)]$. Hence $\eta \in q_x$.
- It easily follows that $g^{-1}(\{x\} \times 2^\kappa)$ is a perfect set. It remains to be shown that splitting is continuous: Let $(\eta_j)_{j<\delta}$ be a strictly increasing sequence of length $< \kappa$ such that $\eta_j \in \text{split}(q_x)$. Again, set $\eta := \bigcup_{j<\delta} \eta_j$. It follows that $\eta_j \in \text{split}(p)$, hence $\eta \in \text{split}_\lambda(p)$ for some limit λ. But as $x \in [e_1(\eta)]$ and $[e_1(\eta)] = [e_1(\eta^\frown i)]$, it follows that $\eta^\frown i \in q_x$ for $i = 1, 2$, hence $\eta \in \text{split}(q_x)$.

Finally, note that q_x is obviously stronger than p. □

The following is an easy observation:

Lemma 2.4.11. Let $Y, Z \subseteq 2^\kappa$ be closed and $f \colon Y \to Z$ uniformly continuous, i.e. $\forall i < \kappa\ \exists j < \kappa\ \forall x \in 2^\kappa \colon f''([x \restriction j] \cap Y) \subseteq [f(x) \restriction i]$. Then f can be extended to a uniformly continuous, total function f^* with $f^{*\,\prime\prime}\, 2^\kappa \subseteq Z$.

Theorem 2.4.12. In $V^{\mathbb{P}}$ the following holds true: Every $X \subseteq 2^\kappa$ of size $\mathfrak{c}_\kappa$ can uniformly continuously be mapped onto 2^κ.

Proof. Again, as every $\mathbb{P}_\alpha$ with $\alpha < \kappa^{++}$ has a dense subset of size κ^+ by Lemma 2.4.6 and is κ-proper, there are essentially only $|(\kappa^+)^\kappa| = \kappa^+$ many $\mathbb{P}_\alpha$-names for reals. Hence $V^{\mathbb{P}_\alpha} \vDash |2^\kappa| = \kappa^+$. As $\mathbb{P}$ satisfies the κ^{++}-c.c. and is also κ-proper, we see that $V^{\mathbb{P}} \vDash$ '$|2^\kappa| = \kappa^{++}$ and no cardinals are collapsed'.

In $V^{\mathbb{P}}$ assume that $X \subseteq 2^\kappa$ and for every uniformly continuous function f there exists a $y \in 2^\kappa$ such that $y \notin f''X$. Pick such a y and denote it by $F(f)$.
Since $\mathbb{P}$ satisfies the κ^{++}-c.c. and is κ-proper, hence no new κ-Borel functions appear in limit steps of cofinality κ^+, we can now find $\beta < \kappa^{++}$ with $\text{cf}(\beta) = \kappa^+$ such that for every uniformly continuous function $f \in V^{\mathbb{P}_\beta}$ we have $F(f) \in V^{\mathbb{P}_\beta}$ and $\Vdash_{\mathbb{P}\beta,\kappa^{++}} F(f) \notin f''\, 2^\kappa$. We will show that $X \subseteq V^{\mathbb{P}_\beta}$, hence $|X| \leq \kappa^+$.

Working in $V^{\mathbb{P}_\beta}$ we assume that $p \Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} \dot\tau \in 2^\kappa \wedge \dot\tau \notin V^{\mathbb{P}_\beta}$. As $p(0) \in \mathbb{S}_\kappa^{V^{\mathbb{P}_\beta}}$ we can use Lemma 2.4.9 to find $q \leq_{\mathbb{P}_{\beta,\kappa^{++}}} p$ and $(A_\eta)_{\eta \in \mathrm{split}(q(0))}$ clopen sets. We define $Y := \bigcap_{i<\kappa} \bigcup_{\eta \in \mathrm{split}_i(q)} A_\eta$, and note that Y is closed and non-empty by Π^1_1-absoluteness. Define $f\colon Y \to [q(0)]$ as follows: $f(x) := \bigcup\{\eta \in 2^{<\kappa} \colon x \in A_\eta\}$, and note that $q \Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} f(\dot\tau) = \dot{s_0}$.

We shall show that $f\colon Y \to [q(0)]$ is uniformly continuous: Let $i < \kappa$ be arbitrary and consider $\eta \in \mathrm{split}_i(q)$. We know that if $x \in Y \cap A_\eta$, then $f(x) \in [\eta]$. We also know that $A_\eta = \bigcup_{x \in \tilde{X}^i_{j+1}} [x \restriction l^i]$ for some $j < \delta_i$ (see Lemma 2.4.9). Let $x \in Y$ be arbitrary. Then there exists $\eta \in \mathrm{split}_i(q)$ such that $[x \restriction l^i] \subseteq A_\eta$, hence $f''([x \restriction l^i] \cap Y) \subseteq [\eta] \subseteq [f(x) \restriction i]$ as $i \subseteq \mathrm{dom}(\eta)$. Therefore, f is uniformly continuous.

By Lemma 2.4.11 f can be extended to a uniformly continuous, total function f^* with $f^{*\,\prime\prime}\, 2^\kappa \subseteq [q(0)]$. Define $h := \pi_1 \circ g \circ f^*$ with g from Lemma 2.4.10 and π_1 the projection onto the first coordinate. Similarly to above, it follows that also g is uniformly continuous. Hence, h is a uniformly continuous function in $V^{\mathbb{P}_\beta}$.

Now let $x \in 2^\kappa \cap V^{\mathbb{P}_\beta}$ be arbitrary. Then $q_x \Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} h(\dot\tau) = x$. This follows, because $q \Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} f^*(\dot\tau) = \dot{s_0}$ and $q_x \Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} \dot{s_0} \in g^{-1}(\{x\} \times 2^\kappa)$. If we set $x := F(h)$ then we can conclude that $q_x \Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} \dot\tau \notin \dot X$, where $\dot X$ is a $\mathbb{P}^{\beta,\kappa^{++}}$-name for X. As $\dot\tau$ and p were arbitrary, it follows that $\Vdash_{\mathbb{P}_{\beta,\kappa^{++}}} \dot X \subseteq V^{\mathbb{P}_\beta}$. □

Proof of Theorem 2.4.7. We have already seen one inclusion. Now assume that $X \subseteq 2^\kappa$ is of size κ^{++}. By the above theorem we can conclude that X can uniformly continuously be mapped onto 2^κ. It can easily be seen that the image of a strong measure zero set under a uniformly continuous function is again strong measure zero. Hence $X \notin \mathcal{SN}$. □

2.5 Strong measure zero vs. stationary strong measure zero

Finally, we take a look at the following definition by Halko [Hal96]:

Definition 2.5.1. Let $X \subseteq 2^\kappa$. We call X stationary strong measure zero iff

$$\forall f \in \kappa^\kappa\ \exists (\eta_i)_{i<\kappa} \colon \big(\forall i < \kappa\ \eta_i \in 2^{f(i)}\big) \wedge X \subseteq \bigcap_{cl \in Cl} \bigcup_{i \in cl} [\eta_i].$$

So for every $x \in X$ the set $\{i < \kappa \colon x \in [\eta_i]\}$ is stationary.

The following lemma shows, why stationary strong measure zero is a natural generalization.

Lemma 2.5.2.

$$X \in \mathcal{SN} \Leftrightarrow \forall f \in \kappa^\kappa\ \exists (\eta_j)_{j<\kappa}\colon \left(\forall j<\kappa\ \eta_j \in 2^{f(j)}\right) \wedge X \subseteq \bigcap_{i<\kappa}\bigcup_{j\geq i}[\eta_j].$$

So for every $x \in X$ the set $\{j<\kappa\colon x \in [\eta_j]\}$ is unbounded.

Proof. Only the $\Rightarrow$ direction is non-trivial. Let $X \in \mathcal{SN}$ and $f \in \kappa^\kappa$ be a challenge. Partition κ into $(S_i)_{i<\kappa}$ such that $|S_i| = \kappa$. Now for every $i<\kappa$ find a covering $(\eta^i_j)_{j\in S_i}$ of X for the challenge $(f(j))_{j\in S_i}$. But then $(\eta_j)_{j<\kappa} := (\eta^i_j)_{j\in S_i, i<\kappa}$ has the required properties. □

The next theorem shows that the two notions coincide in the Corazza model.

Theorem 2.5.3. $V^{\mathbb{P}} \vDash \forall X \in \mathcal{SN}\colon$ *X is stationary strong measure zero*

Proof. Working in $V^{\mathbb{P}}$ let $X \in \mathcal{SN}$ be arbitrary. Find α such that $X \in V^{\mathbb{P}_\alpha}$. Let $f \in \kappa^\kappa \cap V$ and a club $cl \in V^{\mathbb{P}}$ be arbitrary. Since $\mathbb{P}$ satisfies Axiom B, we can assume w.l.o.g. that $cl \in V$. Find $\beta > \alpha$ such that $f = f_\beta$ and $\dot{\mathbb{Q}}_\beta = \dot{\mathbb{I}}_{f_\beta}$. Let $p \in \mathbb{I}^{V^{\mathbb{P}_\beta}}_{f_\beta}$ be arbitrary. Find $q \leq_{\mathbb{I}^{V^{\mathbb{P}_\beta}}_{f_\beta}} p$ such that $\kappa \setminus \mathrm{dom}(q) = \kappa \setminus \mathrm{dom}(p) \cap cl$. Then $q \Vdash_{\mathbb{I}^{V^{\mathbb{P}_\beta}}_{f_\beta}} X \subseteq \bigcup_{i\in cl}[\dot{g}_\beta(i)]$. Hence, the set $\{p \in \mathbb{I}^{V^{\mathbb{P}_\beta}}_{f_\beta}\colon p \Vdash_{\mathbb{I}^{V^{\mathbb{P}_\beta}}_{f_\beta}} X \subseteq \bigcup_{i\in cl}[\dot{g}_\beta(i)]\}$ is dense in $\mathbb{I}^{V^{\mathbb{P}_\beta}}_{f_\beta}$. Since cl was arbitrary, we see that $X \subseteq \bigcap_{cl\in Cl}\bigcup_{i\in cl}[g_\beta(i)]$. □

On the other hand, assuming $|2^\kappa| = \kappa^+$ we can prove the following theorem:

Theorem 2.5.4. Under GCH at κ there exists a set $X \in \mathcal{SN}$ which is not stationary strong measure zero.

Proof. We shall construct X by induction. First enumerate all $f \in \kappa^\kappa$, such that f is strictly increasing, as $(f_\alpha)_{\alpha<\kappa^+}$. Furthermore, define the set $\mathcal{S} := \{\sigma \in (2^{<\kappa})^\kappa\colon \forall i < \kappa\ \mathrm{dom}(\sigma(i)) = i+1\}$ and also enumerate it as $(\sigma_\alpha)_{\alpha<\kappa^+}$.
If $\alpha = 0$ define $x_0(i) := 1 - \sigma_0(i)\,(i)$ so that $x_0 \notin \bigcup_{i<\kappa}[\sigma_0(i)]$. Then choose $\tau_0 \in (2^{<\kappa})^\kappa$ such that $\forall i<\kappa\colon \mathrm{dom}(\tau_0(i)) = f_0(i)$, $\forall i<\kappa\colon \bigcup_{j\geq i}[\tau_0(j)]$ is open dense and $x_0 \in \bigcup_{i<\kappa}[\tau_0(i)]$.
Assume that $(x_\beta)_{\beta<\alpha}$ and $(\tau_\beta)_{\beta<\alpha}$ have already been constructed. Enumerate $(x_\beta)_{\beta<\alpha}$ and $(\tau_\beta)_{\beta<\alpha}$ as $(x'_{i+1})_{i<\kappa}$ and $(\tau'_{i+1})_{i<\kappa}$. Inductively, we will now construct x_α and a club cl:

- Set $cl_0 := 0$ and set $t_0 := \langle 1 - \sigma_\alpha(0)\,(0)\rangle$.
- If $i = i'+1$ and $t_{i'}$ as well as $cl_{i'}$ have already been defined, find $j > cl_{i'}$ such that $t_{i'} \lhd \tau'_i(j)$ and $\tau'_i(j) \ntrianglelefteq x'_i$. Set $cl_i := \mathrm{dom}(\tau'_i(j))$ and $t_i := \tau'_i(j)^\frown(1 - \sigma_\alpha(cl_i)\,(cl_i))$.
- If λ is a limit set $cl_\lambda := \sup\{cl_j\colon j<\lambda\}$ and set $t_\lambda := (\bigcup_{j<\lambda} t_j)^\frown(1 - \sigma_\alpha(cl_\lambda)\,(cl_\lambda))$.

Set $x_\alpha := \bigcup_{i<\kappa} t_i$ and $cl := \{cl_i : i < \kappa\}$. By construction it follows that $\forall \beta < \alpha : x_\alpha \in \bigcup_{i<\kappa} [\tau_\beta(i)]$, x_α is distinct from x_β for every $\beta < \alpha$ and $x_\alpha \notin \bigcup_{i \in cl} [\sigma_\alpha(i)]$. Finally, find τ_α such that $\forall i < \kappa : \mathrm{dom}(\tau_\alpha(i)) = f_\alpha(i)$, $\forall i < \kappa : \bigcup_{j \geq i} [\tau_\alpha(j)]$ is open dense and $\{x_\beta : \beta \leq \alpha\} \subseteq \bigcup_{i<\kappa} [\tau_\alpha(i)]$.
Set $X := \{x_\alpha : \alpha < \kappa^+\}$. Then $\forall \alpha < \kappa^+ : X \subseteq \bigcup_{i<\kappa} [\tau_\alpha(i)]$, because $\{x_\beta : \beta \leq \alpha\} \subseteq \bigcup_{i<\kappa} [\tau_\alpha(i)]$ by the construction of τ_α and $x_\beta \in \bigcup_{i<\kappa} [\tau_\alpha(i)]$ for $\beta > \alpha$ by the construction of x_β. Hence, X is strong measure zero. However, $\forall \sigma \in \mathcal{S} \, \exists x \in X \, \exists cl \in Cl : x \notin \bigcup_{i \in cl} [\sigma(i)]$. Therefore, X cannot be stationary strong measure zero. □

3 The cofinality of the strong measure zero ideal for κ inaccessible

In this chapter we continue to investigate the notion of strong measure zero on 2^κ for κ at least inaccessible.

Fact 3.0.1. Since $\mathcal{SN}$ is a $\leq\kappa$-complete, proper ideal on 2^κ which contains all singletons, the cardinal characteristics $\operatorname{add}(\mathcal{SN})$, $\operatorname{cov}(\mathcal{SN})$, $\operatorname{non}(\mathcal{SN})$ and $\operatorname{cof}(\mathcal{SN})$ are all defined.

In [Yor02], Yorioka introduced the so-called Yorioka ideals approximating the ideal of strong measure zero sets on 2^ω. We will generalize this notion to κ and use it to investigate $\operatorname{cof}(\mathcal{SN})$. Our aim is to show that $\operatorname{cof}(\mathcal{SN}) < \mathfrak{c}_\kappa$, $\operatorname{cof}(\mathcal{SN}) = \mathfrak{c}_\kappa$ as well as $\operatorname{cof}(\mathcal{SN}) > \mathfrak{c}_\kappa$ are all consistent relative to ZFC.
We also generalize the Galvin–Mycielski–Solovay theorem (see Chapter 8.1 in [BJ95]) to κ inaccessible. This result was originally proven by Wohofsky (see [Woh]). Finally, we follow Pawlikowski [Paw90] and show the relative consistency of $\operatorname{cov}(\mathcal{SN}) < \operatorname{add}(\mathcal{M}_\kappa)$ for κ strongly unfoldable (see Definition 2.3.1).

3.1 Prerequisites

We start with several definitions:

Definition 3.1.1. Let $f, g \in \kappa^\kappa$ and f strictly increasing.

- We define the partial order $\ll$ on κ^κ as follows: $f \ll g$ iff $\forall \delta < \kappa\ \exists \mu < \kappa\ \forall i \geq \mu\colon g(i) \geq f(i^\delta)$. Here i^δ is defined using ordinal arithmetic.
- For $\sigma \in (2^{<\kappa})^\kappa$ define $g_\sigma \in \kappa^\kappa$ as follows: $g_\sigma(i) := \operatorname{dom}(\sigma(i))$.
- For $\sigma \in (2^{<\kappa})^\kappa$ define $Y(\sigma) \subseteq 2^\kappa$ as follows: $Y(\sigma) := \bigcap_{i<\kappa} \bigcup_{j \geq i} [\sigma(j)]$.
- Define $\mathcal{S}(f) \subseteq (2^{<\kappa})^\kappa$ as follows: $\mathcal{S}(f) := \{\sigma \in (2^{<\kappa})^\kappa \colon f \ll g_\sigma\}$.
- Define $A \subseteq 2^\kappa$ to be f-small iff there exists $\sigma \in \mathcal{S}(f)$ such that $A \subseteq Y(\sigma)$.
- Define $\mathcal{I}(f) := \{A \subseteq 2^\kappa \colon A \text{ is } f\text{-small}\}$.

Definition 3.1.2. Let $f \in \kappa^\kappa$ be strictly increasing and let $\sigma \in \mathcal{S}(f)$. For every $\delta < \kappa$ we define M_σ^δ to be the minimal ordinal ≥ 2 such that $\forall i \geq M_\sigma^\delta \colon g_\sigma(i) \geq f(i^\delta)$.

Lemma 3.1.3. $\mathcal{I}(f)$ forms a $\leq\kappa$-complete ideal.

Proof. $\mathcal{I}(f)$ is obviously closed under subsets. Hence, we must show that it is also closed under κ-unions.

Let $(A_k)_{k<\kappa}$ be a family of f-small sets and $(\sigma_k)_{k<\kappa} \subseteq \mathcal{S}(f)$ such that $A_k \subseteq Y(\sigma_k)$. We shall find a $\tau \in \mathcal{S}(f)$ such that $\bigcup_{k<\kappa} Y(\sigma_k) \subseteq Y(\tau)$. For every $k < \kappa$ let $g_k := g_{\sigma_k}$ and let $M_k^\delta := M_{\sigma_k}^\delta$ for ever $\delta < \kappa$. Define $m_k := \sup\{M_j^{3\cdot\delta} \colon j, \delta \leq k\}$.

We need the following definitions for $i \geq m_0$:

- Define $c(i) > 0$ such that $m_0 + \sum_{j<c(i)} j \cdot m_j \leq i < m_0 + \sum_{j\leq c(i)} j \cdot m_j$.
- Define $d(i) := m_0 + \sum_{j<c(i)} j \cdot m_j$.
- Define $a(i)$ and $b(i)$ such that $i - d(i) = c(i) \cdot a(i) + b(i)$ with $b(i) < c(i)$.
- Define $e(i) := \sum_{j<c(i)} m_j$.

The following are immediate consequences for $i \geq m_0$:

- $i \geq m_0 + \sum_{j<c'} j \cdot m_j \Rightarrow c(i) \geq c'$
- $a(i) < m_{c(i)}$
- $0 \leq b(i) < c(i) \leq e(i)$.
- $d(i) \leq c(i) \cdot e(i)$ (show by induction)
- $i = d(i) + c(i) \cdot a(i) + b(i) < c(i) \cdot e(i) + c(i) \cdot a(i) + c(i) = c(i) \cdot (e(i) + a(i) + 1) \leq ((e(i) + a(i)) \cdot (e(i) + a(i) + 1) \leq ((e(i) + a(i))^3$
- $\forall k < \kappa\ \forall^\infty l < \kappa\ \exists i < \kappa \colon e(i) + a(i) = l \wedge b(i) = k$

The last statement can be deduced as follows: Given $k < \kappa$ let $l \geq \sum_{j\leq k} m_j$ be arbitrary. Hence, there exists a $\tilde{c} > k$ such that $\sum_{j<\tilde{c}} m_j \leq l < \sum_{j\leq\tilde{c}} m_j$. Define $i := m_0 + \sum_{j<\tilde{c}} j \cdot m_j + \tilde{c} \cdot (l - \sum_{j<\tilde{c}} m_j) + k$. Then $c(i) = \tilde{c}$ follows, because $m_{\tilde{c}} \geq (l - \sum_{j<\tilde{c}} m_j) + 1$ and therefore $\tilde{c} \cdot m_{\tilde{c}} > \tilde{c} \cdot (l - \sum_{j<\tilde{c}} m_j) + k$. Hence $d(i) = m_0 + \sum_{j<\tilde{c}} j \cdot m_j$, $a(i) = l - \sum_{j<\tilde{c}} m_j$, $b(i) = k$ and $e(i) = \sum_{j<\tilde{c}} m_j$. Now $e(i) + a(i) = l$ follows.

We are ready to define τ: If $i \geq m_0$ set $\tau(i) := \sigma_{b(i)}(e(i) + a(i))$. Else set $\tau(i) := \langle\,\rangle$. We must show that τ has the required properties.
First let us show that $\bigcup_{k<\kappa} Y(\sigma_k) \subseteq Y(\tau)$. Let $x \in Y(\sigma_k)$ for some $k < \kappa$ be arbitrary. For every $l < \kappa$ large enough, there exists $i < \kappa$ such that $\tau(i) = \sigma_k(l)$. Hence $x \in Y(\tau)$.
Now we must show that $\tau \in \mathcal{S}(f)$. Let $\delta < \kappa$ be arbitrary. If $i \geq m_0 + \sum_{j\leq\delta} j \cdot m_j$ (hence $c(i) > \delta$), then the following (in-)equalities hold: $\tau(i) = \sigma_{b(i)}(e(i) + a(i))$ by definition, $g_{b(i)}(e(i) + a(i)) \geq f((e(i) + a(i))^{3\cdot\delta})$ since $e(i) + a(i) \geq M_{b(i)}^{3\cdot\delta}$ and $b(i), \delta < c(i)$, and $f((e(i) + a(i))^{3\cdot\delta}) \geq f(i^\delta)$ since f is strictly increasing and $i \leq (e(i) + a(i))^3$. Hence $g_\tau(i) \geq f(i^\delta)$. □

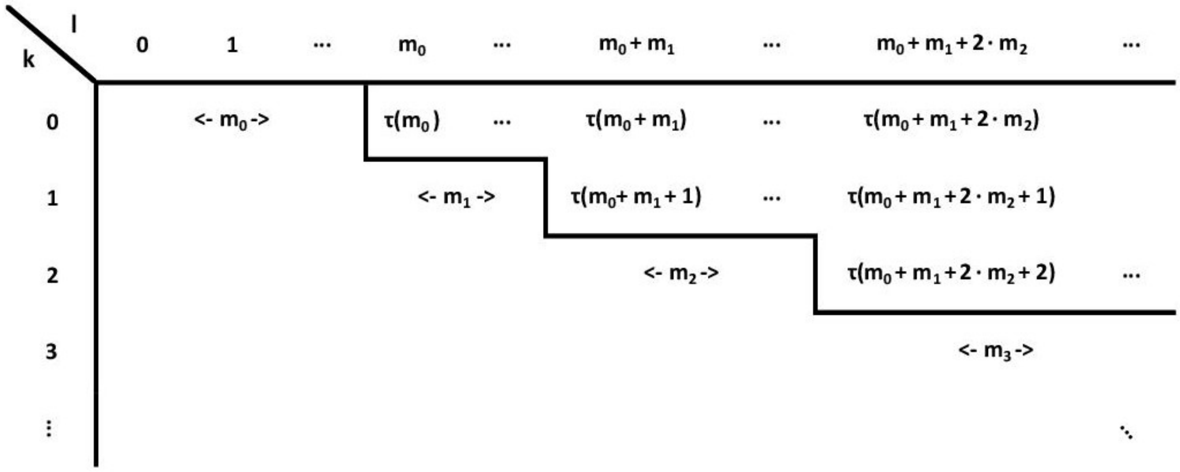

Figure 3: Definition of τ

The following fact is straightforward:

Fact 3.1.4. $\mathcal{SN} = \bigcap_{f \in \kappa^\kappa} \mathcal{I}(f)$.

The next lemma implicitly shows that comeager sets cannot be strong measure zero:

Lemma 3.1.5. Let $A \subseteq 2^\kappa$ be comeager. Then there exists $f \in \kappa^\kappa$ such that for every f-small set B the set $A \setminus B$ is non-empty.

Proof. Let A be comeager. We shall show that A contains a perfect set, which implies that there exists $f \in \kappa^\kappa$ such that A is not f-small: If $P \subseteq A$ is a perfect set, let $T \subseteq 2^{<\kappa}$ be a perfect tree such that $[T] = P$. Pick a function $f \in \kappa^\kappa$ such that $f(\delta) > \sup\{\mathrm{dom}(t) \colon t \text{ is in the } \delta\text{th splitting level of } T\}$. Let $\sigma \in \mathcal{S}(f)$ be arbitrary. Then an $x \in [T] \setminus Y(\sigma)$ can be constructed by induction.

Therefore, let us assume that $A = \bigcap_{i<\kappa} D_{i+1}$, where D_{i+1} are open dense and decreasing, and we inductively construct a perfect tree $T \subseteq 2^{<\kappa}$ such that no branches die out and $[T] \subseteq A$:

- Set $T_0 := \{t_{\langle\rangle}\}$ where $t_{\langle\rangle} := \langle\rangle$.
- If $i = i' + 1$ assume inductively that $T_{i'} = \{t_\eta \colon \eta \in 2^{i'}\}$ has already been defined and for every $t_\eta \in T_{i'}$ we have $[t_\eta] \subseteq \bigcap_{j<i'} D_{j+1}$. For every $t_\eta \in T_{i'}$ find $t'_\eta \triangleright t_\eta$ such that $[t'_\eta] \subseteq D_i$. Set $t_{\eta^\frown\langle i\rangle} := t'^\frown_\eta\langle i\rangle$ and $T_i := \{t_{\eta'} \colon \eta' \in 2^i\}$.
- If λ is a limit and $\eta \in 2^\lambda$ define $t_\eta := \bigcup_{\eta' \triangleleft \eta} t_{\eta'}$ and set $T_\lambda := \{t_\eta \colon \eta \in 2^\lambda\}$. Then we can deduce that $[t_\eta] \subseteq \bigcap_{j<\lambda} D_{j+1}$ for every $t_\eta \in T_\lambda$.

It follows from the construction that (the downward closure of) $T := \bigcup_{i<\kappa} T_i$ is a perfect tree and that $[T] \subseteq A$. $\square$

Conversely, the following fact holds true:

Fact 3.1.6. For every $f \in \kappa^\kappa$ strictly increasing there exists a comeager set $A \subseteq 2^\kappa$ such that $A \in \mathcal{I}(f)$.

Lemma 3.1.7. Assume GCH at κ and let $(f_\alpha)_{\alpha<\kappa^+}$ be a κ-scale such that f_α is strictly increasing. Then there exists a matrix $(A_\alpha^\beta)_{\alpha<\kappa^+}^{\beta<\kappa^+}$ with the following properties:

- $\forall \alpha, \beta < \kappa^+ \colon A_\alpha^\beta \subseteq 2^\kappa$ is comeager and f_α-small.
- $\forall \alpha, \beta, \beta' < \kappa^+ \colon \beta \leq \beta' \Rightarrow A_\alpha^\beta \subseteq A_\alpha^{\beta'}$.
- $\forall \alpha < \kappa^+ \ \forall f_\alpha\text{-small}\, B \subseteq 2^\kappa \ \exists \beta < \kappa^+ \colon B \subseteq A_\alpha^\beta$.
- $\forall \alpha < \kappa^+ \ \forall f_\alpha\text{-small}\, B \subseteq 2^\kappa \colon \alpha > 0 \Rightarrow \left(\bigcap_{\gamma<\alpha} A_\gamma^0 \right) \setminus B \neq \emptyset$. This means that for every $\alpha < \kappa^+$ the set $\bigcap_{\gamma<\alpha} A_\gamma^0$ is not f_α-small.

Proof. We shall construct A_α^β by a lexicographic induction on $(\alpha, \beta) \in \kappa^+ \times \kappa^+$: Assume that $(A_\gamma^\beta)_{\gamma<\alpha}^{\beta<\kappa^+}$ have already been defined. Since $\bigcap_{\gamma<\alpha} A_\gamma^0$ is comeager, there exists f such that $\bigcap_{\gamma<\alpha} A_\gamma^0$ is not f-small (see Lemma 3.1.5). W.l.o.g. let $f = f_\alpha$. Choose some $\tau_0 \in \mathcal{S}(f_\alpha)$ such that $Y(\tau_0)$ is comeager and set $A_\alpha^0 := Y(\tau_0)$. Next enumerate $\mathcal{S}(f_\alpha)$ as $(\sigma_\beta)_{\beta<\kappa^+}$ such that $\sigma_0 = \tau_0$. Finally choose $\tau_\beta \in \mathcal{S}(f_\alpha)$ inductively such that $\bigcup_{\gamma<\beta} Y(\tau_\gamma) \cup Y(\sigma_\beta) \subseteq Y(\tau_\beta)$. This is possible since $\mathcal{I}(f_\alpha)$ is $\leq\kappa$-complete. Set $A_\alpha^\beta := Y(\tau_\beta)$. □

Fact 3.1.8. If $(f_\alpha)_{\alpha<\kappa^+}$ and $(A_\alpha^\beta)_{\alpha<\kappa^+}^{\beta<\kappa^+}$ are as above, then for every $g \in \left(\kappa^+\right)^{\kappa^+}$ we have $\bigcap_{\alpha<\kappa^+} A_\alpha^{g(\alpha)} \in \mathcal{SN}$.

We are ready to prove the following theorem:

Theorem 3.1.9. Assume GCH at κ. Then $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_{\kappa^+}$.

Proof. First we prove $\mathrm{cof}(\mathcal{SN}) \leq \mathfrak{d}_{\kappa^+}$. Let $\mathcal{D}$ be a pointwise dominating family in $\left(\kappa^+\right)^{\kappa^+}$ of size $\mathfrak{d}_{\kappa^+}$. This family exists, because there is an eventually dominating family of size $\mathfrak{d}_{\kappa^+}$ and $\left(\kappa^+\right)^{\kappa} = \kappa^+$, since we have $|2^\kappa| = \kappa^+$. Let $(f_\alpha)_{\alpha<\kappa^+}$ be a κ-scale (which exists by GCH at κ) and let $(A_\alpha^\beta)_{\alpha<\kappa^+}^{\beta<\kappa^+}$ be the matrix from Lemma 3.1.7. Define $\mathcal{B} := \{X \subseteq 2^\kappa \colon \exists g \in \mathcal{D}\ X = \bigcap_{\alpha<\kappa^+} A_\alpha^{g(\alpha)}\}$. $\mathcal{B}$ has size $\leq \mathfrak{d}_{\kappa^+}$. We must show that $\mathcal{B}$ is cofinal in $\mathcal{SN}$.
First, we obviously have $\mathcal{B} \subseteq \mathcal{SN}$. Now let $Y \in \mathcal{SN}$. Then there exists an $h \in \left(\kappa^+\right)^{\kappa^+}$ such that $Y \subseteq \bigcap_{\alpha<\kappa^+} A_\alpha^{h(\alpha)}$. Hence, there exists $g \in \mathcal{D}$ such that g dominates h pointwise. In particular, $Y \subseteq \bigcap_{\alpha<\kappa^+} A_\alpha^{g(\alpha)} \in \mathcal{B}$.

Now let us show that $\mathrm{cof}(\mathcal{SN}) \geq \mathfrak{d}_{\kappa^+}$. Towards a contradiction we assume the opposite, i.e. there exists a $\mathcal{C}$ cofinal in $\mathcal{SN}$ of size $< \mathfrak{d}_{\kappa^+}$. Hence, for every $X \in \mathcal{C}$ there exists $g_X \in \mathcal{D}$ such that $X \subseteq \bigcap_{\alpha<\kappa^+} A_\alpha^{g_X(\alpha)}$. Let us define $\mathcal{D}' := \{g_X \colon X \in \mathcal{C}\} \subseteq \mathcal{D}$. Then $|\mathcal{D}'| < \mathfrak{d}_{\kappa^+}$. Hence, there exists h such that no $g \in \mathcal{D}'$ dominates it.
Inductively we will now construct $h' \in \left(\kappa^+\right)^{\kappa^+}$ and $\{x_\gamma \colon \gamma < \kappa^+\}$ such that:

- $\forall \alpha < \kappa^+ \colon h(\alpha) \leq h'(\alpha)$
- $\forall \alpha < \kappa^+ \colon \{x_\gamma \colon \gamma < \alpha\} \subseteq A_\alpha^{h'(\alpha)}$
- $x_\alpha \in \big(\bigcap_{\gamma \leq \alpha} A_\gamma^{h'(\gamma)} \big) \setminus A_\alpha^{h(\alpha)}$

Assume that $h' \restriction \alpha$ and $\{x_\gamma \colon \gamma < \alpha\}$ have already been defined. Simply by choosing $h'(\alpha)$ large enough we can ensure that $\{x_\gamma \colon \gamma < \alpha\} \subseteq A_\alpha^{h'(\alpha)}$. Also since $\bigcap_{\gamma<\alpha} A_\gamma^0 \subseteq \bigcap_{\gamma<\alpha} A_\gamma^{h'(\gamma)}$, it follows by Lemma 3.1.7 that $\big(\bigcap_{\gamma<\alpha} A_\gamma^{h'(\gamma)} \big) \setminus A_\alpha^{h(\alpha)} \neq \emptyset$. Again, if we choose $h'(\alpha)$ large enough, then also $\big(\bigcap_{\gamma\leq\alpha} A_\gamma^{h'(\gamma)} \big) \setminus A_\alpha^{h(\alpha)} \neq \emptyset$. Therefore, choose $h'(\alpha) \geq h(\alpha)$ large enough, and pick $x_\alpha \in \big(\bigcap_{\gamma\leq\alpha} A_\gamma^{h'(\gamma)} \big) \setminus A_\alpha^{h(\alpha)}$.
First, $\{x_\gamma \colon \gamma < \kappa^+\} \in \mathcal{SN}$, because $\{x_\gamma \colon \gamma < \kappa^+\} \subseteq \bigcap_{\gamma<\kappa^+} A_\gamma^{h'(\gamma)}$. Finally we show that no $X \in \mathcal{C}$ covers $\{x_\gamma \colon \gamma < \kappa^+\}$. It suffices to show that for every $g_X \in \mathcal{D}'$ we have $\{x_\gamma \colon \gamma < \kappa^+\} \nsubseteq \bigcap_{\alpha<\kappa^+} A_\alpha^{g_X(\alpha)}$. Let $g_X \in \mathcal{D}'$ be arbitrary. Find $\alpha < \kappa$ such that $g_X(\alpha) \leq h(\alpha)$. But then $x_\alpha \notin A_\alpha^{g_X(\alpha)}$. □

We can generalize Theorem 3.1.9 as follows:

Theorem 3.1.10. Assume that $\operatorname{add}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa$ and there exists a dominating family $\{f_\alpha \in \kappa^\kappa \colon \alpha < \mathfrak{d}_\kappa\}$ such that $\operatorname{add}(\mathcal{I}(f_\alpha)) = \operatorname{cof}(\mathcal{I}(f_\alpha)) = \mathfrak{d}_\kappa$. Then $\operatorname{cof}(\mathcal{SN}) = \mathfrak{d}_{\mathfrak{d}_\kappa}$.

Proof. First note that also $\mathfrak{b}_\kappa = \mathfrak{d}_\kappa$. Therefore, for every $\beta < \mathfrak{d}_\kappa$, the family $\{f_\alpha \colon \beta \leq \alpha < \mathfrak{d}_\kappa\}$ is also dominating. So w.l.o.g. we can assume that $\{f_\alpha \colon \alpha < \mathfrak{d}_\kappa\}$ is a κ-scale. Next, construct a matrix $(A_\alpha^\beta)_{\alpha<\mathfrak{d}_\kappa}^{\beta<\mathfrak{d}_\kappa}$ similar to Lemma 3.1.7 using $\operatorname{add}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa$.
Let $\mathcal{D}$ be a dominating a family in $\mathfrak{d}_\kappa^{\mathfrak{d}_\kappa}$ of size $\mathfrak{d}_{\mathfrak{d}_\kappa}$. Following the proof of Theorem 3.1.9 we define $\mathcal{B} := \{X \subseteq 2^\kappa \colon \exists g \in \mathcal{D}\ \exists \beta < \mathfrak{d}_\kappa\ X = \bigcap_{\alpha\geq\beta} A_\alpha^{g(\alpha)}\}$. Note that $|\mathcal{B}| \leq \mathfrak{d}_{\mathfrak{d}_\kappa}$ and $\mathcal{B} \subseteq \mathcal{SN}$, since $\{f_\alpha \colon \alpha < \mathfrak{d}_\kappa\}$ is a κ-scale. We will show that $\mathcal{B}$ is cofinal in $\mathcal{SN}$:
Let $Y \in \mathcal{SN}$, hence there exists $h \in \mathfrak{d}_\kappa^{\mathfrak{d}_\kappa}$ such that $Y \subseteq \bigcap_{\alpha<\mathfrak{d}_\kappa} A_\alpha^{h(\alpha)}$. But then there is $g \in \mathcal{D}$ and $\beta < \mathfrak{d}_\kappa$ such that $\forall \alpha \geq \beta \colon g(\alpha) \geq h(\alpha)$. Therefore $Y \subseteq \bigcap_{\alpha\geq\beta} A_\alpha^{g(\alpha)} \in \mathcal{B}$.
To prove $\mathfrak{d}_{\mathfrak{d}_\kappa} \leq \operatorname{cof}(\mathcal{SN})$ we proceed as in the proof of Theorem 3.1.9. □

3.2 Separating $\operatorname{cof}(\mathcal{SN})$ and $\mathfrak{c}_\kappa$

In this section we want to force $\operatorname{add}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa$ and there exists a dominating family $\{f_\alpha \colon \alpha < \mathfrak{d}_\kappa\}$ such that $\operatorname{add}(\mathcal{I}(f_\alpha)) = \operatorname{cof}(\mathcal{I}(f_\alpha)) = \mathfrak{d}_\kappa$ for every $\alpha < \mathfrak{d}_\kappa$. Then $\operatorname{cof}(\mathcal{SN}) = \mathfrak{d}_{\mathfrak{d}_\kappa}$ holds by Theorem 3.1.10. Using this we can separate $\operatorname{cof}(\mathcal{SN})$ and $\mathfrak{c}_\kappa$.

Definition 3.2.1. For $f \in \kappa^\kappa$ strictly increasing define the forcing notion $\mathbb{O}_f$:
Let $p \in \mathbb{O}_f$ iff $p = (\sigma_p, \delta_p, l_p, F_p)$ such that:

P1 $\sigma_p \in (2^{<\kappa})^{<\kappa}$, $\delta_p, l_p < \kappa$ and $F_p \subseteq \mathcal{S}(f)$ is infinite and of size $< \kappa$

P2 $|F_p| \cdot l_p \geq \operatorname{dom}(\sigma_p) \geq l_p \geq \sup\{M_\tau^{3\cdot\mu} \colon \tau \in F_p,\ \mu < \delta_p\} \cup |F_p|$

If $p = (\sigma_p, \delta_p, l_p, F_p)$ and $q = (\sigma_q, \delta_q, l_q, F_q)$ are conditions in $\mathbb{O}_f$ we define $q \leq_{\mathbb{O}_f} p$, i.e. q is stronger than p, iff:

Q1 $\sigma_p \subseteq \sigma_q$, $\delta_p \leq \delta_q$, $l_p \leq l_q$ and $F_p \subseteq F_q$

Q2 $\forall \mu < \delta_p \; \forall i \in \mathrm{dom}(\sigma_q) \setminus \mathrm{dom}(\sigma_p)\colon g_{\sigma_q}(i) \geq f(i^\mu)$

Q3 $\forall \tau \in F_p \; \forall i \in l_q \setminus l_p \; \exists j \in \mathrm{dom}(\sigma_q)\colon \sigma_q(j) = \tau(i)$

The following lemma will be crucial for many density arguments.

Lemma 3.2.2. Let $p = (\sigma_p, \delta_p, l_p, F_p) \in \mathbb{O}_f$. Let $\iota \geq \mathrm{dom}(\sigma_p)$, $\delta' \geq \delta_p$, $l' \geq l_p$ and $F' \supseteq F_p$. Then there exists an extension $q = (\sigma_q, \delta_q, l_q, F_q)$ such that $\mathrm{dom}(\sigma_q) \geq \iota$, $\delta_q = \delta'$, $l_q \geq l'$ and $F_q = F'$.

Proof. Let $\{\tau_k\colon k < |F_p|\}$ enumerate F_p and set

$$\tilde{l} := \max\big\{\sup\{M_\tau^{3\cdot\mu}\colon \tau \in F',\ \mu < \delta'\}, |F'|, l', \iota\big\}.$$

Set $l_q := l_p + \tilde{l}$. Hence $l_q \geq l'$, and we define σ_q as follows:

- $\sigma_q \restriction \mathrm{dom}(\sigma_p) = \sigma_p$
- For $i \in (\mathrm{dom}(\sigma_p) + |F_p| \cdot \tilde{l}) \setminus \mathrm{dom}(\sigma_p)$ such that $i = \mathrm{dom}(\sigma_p) + |F_p| \cdot a + b$, where $a < \tilde{l}$ and $b < |F_p|$, we set $\sigma_q(i) := \tau_b(l_p + a)$.

Then $\mathrm{dom}(\sigma_q) \geq \iota$. Set $\delta_q := \delta'$ and $F_q := F'$, and set $q := (\sigma_q, \delta_q, l_q, F_q)$. Now we must check that $q \in \mathbb{O}_f$ and $q \leq_{\mathbb{O}_f} p$.

Let us first check that $q \in \mathbb{O}_f$. The following inequalities hold:

$$|F_q| \cdot l_q \geq \mathrm{dom}(\sigma_q) = \mathrm{dom}(\sigma_p) + |F_p| \cdot \tilde{l} \geq l_q = l + \tilde{l} \geq \sup\{M_\tau^{3\cdot\mu}\colon \tau \in F_q,\ \mu < \delta_q\} \cup |F_q|$$

Therefore $q \in \mathbb{O}_f$.

Now let us check that $q \leq_{\mathbb{O}_f} p$:

(Q1) $\sigma_p \subseteq \sigma_q$, $\delta_p \leq \delta_q$, $l_p \leq l_q$ and $F_p \subseteq F_q$.

(Q2) We need to show that $\forall \mu < \delta_p \; \forall i \in \mathrm{dom}(\sigma_q) \setminus \mathrm{dom}(\sigma_p)\colon g_{\sigma_q}(i) \geq f(i^\mu)$. Let $\mu < \delta_p$ and $i \in \mathrm{dom}(\sigma_q) \setminus \mathrm{dom}(\sigma_p)$ be arbitrary such that $i - \mathrm{dom}(\sigma_p) = |F_p| \cdot a + b$. We note that $l_p + a \geq M_{\tau_b}^{3\cdot\mu}$ by the definition of $\mathbb{O}_f$, and the following inequalities hold:

$$(l_p + a)^3 \geq (l_p + a) \cdot (l_p + a + 1) \geq |F_p| \cdot (l_p + a + 1) \geq |F_p| \cdot l_p + |F_p| \cdot a + b \geq i$$

Therefore $g_{\sigma_q}(i) = g_{\tau_b}(l_p + a) \geq f((l_p + a)^{3\cdot\mu}) \geq f(i^\mu)$.

(Q3) Obviously, $\forall \tau \in F_p \; \forall i \in l_q \setminus l_p \; \exists j \in \mathrm{dom}(\sigma_q)\colon \sigma_q(j) = \tau(i)$ by the definition of σ_q and $l_q - l_p = \tilde{l}$. □

Lemma 3.2.3. $\mathbb{O}_f$ is $<\kappa$-closed.

Proof. Let $(p_k)_{k<\lambda}$ be a decreasing sequence of length $\lambda < \kappa$. Define $q := (\sigma_q, \delta_q, l_q, F_q)$, where $\sigma_q := \bigcup_{k<\lambda} \sigma_{p_k}$, $\delta_q := \bigcup_{k<\lambda} \delta_{p_k}$, $l_q := \bigcup_{k<\lambda} l_{p_k}$ and $F_q := \bigcup_{k<\lambda} F_{p_k}$.
Let us first check that $q \in \mathbb{O}_f$. Since $|F_q| \cdot l_q \geq |F_{p_k}| \cdot l_{p_k}$ for every $k < \lambda$, the following inequalities hold:

$$|F_q| \cdot l_q \geq \mathrm{dom}(\sigma_q) \geq l_q \geq \sup\{M_\tau^{3\cdot\mu} \colon \tau \in F_q,\, \mu < \delta_q\} \cup |F_q|$$

and therefore, q is indeed a condition.
Next let us check that q is a lower bound of $(p_k)_{k<\lambda}$. Fix p_k and we shall show that $q \leq_{\mathbb{O}_f} p_k$. (Q1) is trivially satisfied. For (Q2) fix $\mu < \delta_{p_k}$ and $i \in \mathrm{dom}(\sigma_q) \setminus \mathrm{dom}(\sigma_{p_k})$. Choose $k' > k$ such that $i \in \mathrm{dom}(\sigma_{p_{k'}})$. But then $g_{\sigma_{p_{k'}}}(i) \geq f(i^\mu)$. (Q3) can be shown similarly. □

Lemma 3.2.4. $\mathbb{O}_f$ is κ-linked [1].

Proof. For $\sigma \in (2^{<\kappa})^{<\kappa}$, $\delta < \kappa$ and $l < \kappa$ define the set $P_{(\sigma,\delta,l)} := \{p \in \mathbb{O}_f \colon \exists F_p \subseteq \mathcal{S}(f)\ p = (\sigma, \delta, l, F_p)\}$. We will show that $P_{(\sigma,\delta,l)}$ is linked. Then $\mathbb{O}_f$ will be κ-linked, because $\mathbb{O}_f = \bigcup_{\sigma\in(2^{<\kappa})^{<\kappa}} \bigcup_{\delta<\kappa} \bigcup_{l<\kappa} P_{(\sigma,\delta,l)}$.
Fix (σ, δ, l) and let $p_1, p_2 \in P_{(\sigma,\delta,l)}$. Set $F_q := F_{p_1} \cup F_{p_2}$ and note that $|F_q| = \max\{|F_{p_1}|, |F_{p_2}|\}$. Hence $l \geq |F_q|$ and therefore, $q := (\sigma, \delta, l, F_q)$ is a lower bound of p_1 and p_2. □

Assume that $V \models$ 'κ is inaccessible' and let $f \in \kappa^\kappa \cap V$ be strictly increasing. Furthermore, let G be a $(V, \mathbb{O}_f)$-generic filter. Define $\tau_G := \bigcup\{\sigma \in (2^{<\kappa})^{<\kappa} \colon \exists p \in G\ p = (\sigma, \delta_p, l_p, F_p)\}$. Then the following lemma is an easy observation:

Lemma 3.2.5. The following holds in $V^{\mathbb{O}_f}$:

1. $\tau_G \in (2^{<\kappa})^\kappa$
2. $g_{\tau_G} \gg f$, in particular $\tau_G \in \mathcal{S}(f)$
3. $\forall \tau \in \mathcal{S}(f) \cap V \colon Y(\tau) \subseteq Y(\tau_G)$

Hence, τ_G codes an f-small set which covers all ground model f-small sets.

Proof. ad 1.) By Lemma 3.2.2 the set $\{p \in \mathbb{O}_f \colon \mathrm{dom}(\sigma_p) \geq \iota\}$ is dense for every $\iota < \kappa$. Hence $\tau_G \in (2^{<\kappa})^\kappa$.

ad 2.) Let $\delta < \kappa$ be arbitrary. By a density argument there exists $p \in G$ such that $\delta_p \geq \delta + 1$. But then $g_{\tau_G}(i) \geq f(i^\delta)$ for all $i \geq \mathrm{dom}(\sigma_p)$.

ad 3.) Let $\tau \in \mathcal{S}(f) \cap V$ be arbitrary and fix $x \in Y(\tau)$, in particular the set $\{i < \kappa \colon x \in [\tau(i)]\}$ has size κ. By a density argument there exists $p, q \in G$ such that $\tau \in F_p$, l_q is arbitrarily large and $q \leq_{\mathbb{O}_f} p$. Hence, the set $\{i \geq \mathrm{dom}(\sigma_p) \colon x \in [\tau_G(i)]\}$ will also be of size κ. □

[1]Note that $\mathbb{O}_f$ is not κ-centered$_{<\kappa}$.

Definition 3.2.6. Let $\lambda > \kappa$ be a regular cardinal. Define B^λ to be a bijection between λ and $\lambda \times \lambda$ such that:

- If $B^\lambda(\alpha) = (\beta, \gamma)$ then $\beta \leq \alpha$
- If $B^\lambda(\alpha) = (\beta, \gamma)$, $B^\lambda(\alpha') = (\beta, \gamma')$ and $\alpha < \alpha'$, then $\gamma < \gamma'$

Furthermore, define $B^\lambda_0(\alpha)$ and $B^\lambda_1(\alpha)$ to be the projection of $B^\lambda(\alpha)$ onto the first and second coordinate, respectively.

Now we are ready to define the iteration:

Definition 3.2.7. Let $\langle \mathbb{P}_\epsilon, \dot{\mathbb{Q}}_\zeta : \epsilon \leq \lambda, \zeta < \lambda \rangle$ be a $<\kappa$-support iteration such that $\forall \epsilon < \lambda$: $\Vdash_\epsilon \dot{\mathbb{Q}}_\epsilon = \dot{\mathbb{H}}_\kappa \star \dot{\mathbb{O}}_{\dot{d}_{B^\lambda_0(\epsilon)}}$, where $\mathbb{H}_\kappa$ denotes κ-Hechler forcing and $\dot{d}_{B^\lambda_0(\epsilon)}$ is the generic κ-Hechler added by the first half of $\dot{\mathbb{Q}}_{B^\lambda_0(\epsilon)}$.

Note that $\mathbb{H}_\kappa$ is κ-centered$_{<\kappa}$.

The following lemma should be a straightforward consequence of Lemma 3.2.4:

Lemma 3.2.8. $\mathbb{P}_\lambda$ satisfies the κ^+-c.c. Furthermore, if $|2^\kappa| < \lambda$ and $\lambda^\kappa = \lambda$ or $|2^\kappa| \geq \lambda$, then there exists a dense set $D \subseteq \mathbb{P}_\lambda$ of size $\max\{|2^\kappa|, \lambda\}$.

Proof. The set

$$D := \{p \in \mathbb{P}_\lambda : \forall \epsilon < \lambda\ \exists \rho \in \kappa^{<\kappa}\ \exists \sigma \in (2^{<\kappa})^{<\kappa}\ \exists \delta < \kappa\ \exists l < \kappa\ \exists \dot{f}\ \exists (\dot{g}_k)_{k<\iota}$$

$$p \restriction \epsilon \Vdash_\epsilon \dot{p(\epsilon)} = ((\rho, \dot{f}), (\sigma, \delta, l, (\dot{g}_k)_{k<\iota}))\}$$

is dense in $\mathbb{P}_\lambda$. Let $Q \subseteq D$ be of size κ^+ and use a Δ-system argument to find $Q' \subseteq Q$ of size κ^+ such that Q' is linked. Hence $\mathbb{P}_\lambda$ satisfies the κ^+-c.c.
Show by induction on $\epsilon < \lambda$ that $|D \cap \mathbb{P}_\epsilon| \leq \max\{|2^\kappa|, \lambda\}$ using the κ^+-c.c. of $\mathbb{P}_\epsilon$ and the fact that every $\mathbb{P}_\epsilon$-name for an element of κ^κ is completely determined by a family of maximal antichains of size κ. $\square$

We are ready to state and prove the following theorem:

Theorem 3.2.9. Let $\lambda > \kappa$ be regular. Let $V \vDash$ '$|2^\kappa| \geq \lambda$ or $|2^\kappa| < \lambda \wedge \lambda^\kappa = \lambda$'. Then in $V^{\mathbb{P}_\lambda}$ the following holds true: $\mathrm{add}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa = \lambda$ and there exists a dominating family $\{f_\alpha \in \kappa^\kappa : \alpha < \lambda\}$ such that $\mathrm{add}(\mathcal{I}(f_\alpha)) = \mathrm{cof}(\mathcal{I}(f_\alpha)) = \lambda$. Hence $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_{\mathfrak{d}_\kappa} = \mathfrak{d}_\lambda$. Furthermore, $|2^\kappa| = \max\{|2^\kappa \cap V|, \lambda\}$, $\mathfrak{d}_\lambda = \mathfrak{d}^V_\lambda$ and $\mathrm{add}(\mathcal{SN}) = \mathrm{cov}(\mathcal{SN}) = \mathrm{non}(\mathcal{SN}) = \lambda$.

Proof. Using the κ^+-c.c., the following should be straightforward: The κ-Hechler reals $(\dot{d}_\epsilon)_{\epsilon<\lambda}$ form a κ-scale, hence $\mathfrak{d}_\kappa = \mathfrak{b}_\kappa = \lambda$. A $<\kappa$-support iteration adds κ-Cohen reals, which implies $\mathrm{cov}(\mathcal{M}_\kappa) = \lambda$, and since $\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}_\kappa, \mathrm{cov}(\mathcal{M}_\kappa)\}$ by Theorem 1.2.5, it follows that $\mathrm{add}(\mathcal{M}_\kappa) = \mathfrak{d}_\kappa = \lambda$. The family $(\dot{\tau}_{\epsilon'})_{\epsilon' \in B^{\lambda^{-1}}(\{\epsilon\} \times \lambda)}$ witnesses $\mathrm{add}(\mathcal{I}(\dot{d}_\epsilon)) = \mathrm{cof}(\mathcal{I}(\dot{d}_\epsilon)) = \lambda$, since $\mathcal{I}(\dot{d}_\epsilon))$ is a κ-Borel ideal. Hence $\mathrm{cof}(\mathcal{SN}) = \mathfrak{d}_{\mathfrak{d}_\kappa}$ by Theorem 3.1.10.

Since $\mathbb{P}_\lambda$ has a dense subset of size $\max\{|2^\kappa \cap V|, \lambda\}$, it satisfies the κ^+-c.c. and either $|2^\kappa \cap V| \geq \lambda$ or $\lambda^\kappa = \lambda$, it follows that $|2^\kappa| \leq \max\{|2^\kappa \cap V|, \lambda\}$. Again using the κ^+-c.c., it follows that $\mathbb{P}_\lambda$ is λ^λ-bounding, hence $\mathfrak{d}_\lambda = \mathfrak{d}_\lambda^V$. Using $\operatorname{add}(\mathcal{I}(\dot{d}_\epsilon)) = \lambda$ for all $\epsilon < \lambda$, we can deduce $\lambda \leq \operatorname{add}(\mathcal{SN})$. For $\operatorname{cov}(\mathcal{SN}) \leq \lambda$ we note that $2^\kappa \cap V^{\mathbb{P}_\epsilon} \in \mathcal{SN}$ for all $\epsilon < \lambda$ as being witnessed by $(\dot{\tau}_{\epsilon'})_{\epsilon' > \epsilon}$. For $\operatorname{non}(\mathcal{SN}) \leq \lambda$ we pick for every $\dot{\tau}_\epsilon$ possibly not distinct x_ϵ's such that $x_\epsilon \in 2^\kappa \setminus Y(\dot{\tau}_\epsilon)$ and set $X := \{x_\epsilon \colon \epsilon < \lambda\}$. It follows that $|X| \leq \lambda$ and $X \notin \mathcal{SN}$, since no $\dot{\tau}_\epsilon$ can cover X. □

Theorem 3.2.10. $\mathfrak{c}_\kappa < \operatorname{cof}(\mathcal{SN})$, $\mathfrak{c}_\kappa = \operatorname{cof}(\mathcal{SN})$ and $\mathfrak{c}_\kappa > \operatorname{cof}(\mathcal{SN})$ are all consistent relative to ZFC.

Proof. $\mathfrak{c}_\kappa < \operatorname{cof}(\mathcal{SN})$ holds under GCH at κ (see Theorem 3.1.9). For $\mathfrak{c}_\kappa = \operatorname{cof}(\mathcal{SN})$ assume that $V \vDash |2^\kappa| = \kappa^{++} \wedge \mathfrak{d}_{\kappa^+} = \kappa^{++}$ (e.g. by forcing over GCH with a κ^{++}-product of κ-Cohen forcing) and force with $\mathbb{P}_{\kappa^+}$. For $\mathfrak{c}_\kappa > \operatorname{cof}(\mathcal{SN})$ assume $V \vDash |2^\kappa| = \kappa^{+++} \wedge \mathfrak{d}_{\kappa^+} = \kappa^{++}$ (e.g. by forcing over GCH with a κ^{+++}-product of κ-Cohen forcing) and again force with $\mathbb{P}_{\kappa^+}$. □

3.3 A model for $\operatorname{cov}(\mathcal{SN}) < \operatorname{add}(\mathcal{M}_\kappa)$

In this section we first generalize the Galvin–Mycielski–Solovay theorem to κ inaccessible. Then we shall assume that κ is strongly unfoldable, and want to construct a model for $\operatorname{cov}(\mathcal{SN}) < \operatorname{add}(\mathcal{M}_\kappa)$. Indeed this will hold in the κ-Hechler model.

Definition 3.3.1. Let $X \subseteq 2^\kappa$. We call X meager-shiftable iff for every comeager $D \subseteq 2^\kappa$ there exists $y \in 2^\kappa$ such that $X + y \subseteq D$.

The following result is well-known in the classical case:

Theorem 3.3.2. (GMS) $X \in \mathcal{SN}$ iff X is meager-shiftable.

Proof. First we shall show that meager-shiftable implies strong measure zero: Let $X \subseteq 2^\kappa$ be meager-shiftable and let $f \in \kappa^\kappa$. Choose $(s_i)_{i<\kappa}$ such that $s_i \in 2^{f(i)}$ and $D := \bigcup_{i<\kappa}[s_i]$ is open dense. Since X is meager-shiftable, there exists $y \in 2^\kappa$ such that $X + y \subseteq D$. Define $t_i := s_i + y \restriction f(i)$. But then $X \subseteq \bigcup_{i<\kappa}[t_i]$. Hence X is strong measure zero.

We shall now show that strong measure zero implies meager-shiftable: Let $X \subseteq 2^\kappa$ be strong measure zero and let $D = \bigcap_{i<\kappa} D_i$ be an intersection of arbitrary dense open sets. W.l.o.g. let the D_i's be decreasing. Now construct a normal sequence $(c_i)_{i<\kappa}$, $c_i \in \kappa$, such that for every $i < \kappa$ and every $s \in 2^{c_i}$ there exists $t \in 2^{c_{i+1}}$ with $s \lhd t$ such that $[t] \subseteq D_i$. Define $f(i) := c_{i+1}$ and find $(s_i)_{i<\kappa}$ such that $s_i \in 2^{f(i)}$ and $X \subseteq \bigcap_{j<\kappa}\bigcup_{i\geq j}[s_i]$. We shall now inductively construct $y \in 2^\kappa$ such that $\forall i < \kappa\colon [s_i + y \restriction c_{i+1}] \subseteq D_i$:

- Choose $t_0 \in 2^{c_1}$ such that $[t_0] \subseteq D_0$ and set $y \restriction c_1 := s_0 + t_0$. Hence, $s_0 + y \restriction c_1 = t_0$ and so $[s_0 + y \restriction c_1] \subseteq D_0$.

- $i = i'+1$: Assume that $y \restriction c_{i'+1}$ has already be constructed. Find $t_i \rhd s_i \restriction c_i + y \restriction c_i$, $t_i \in 2^{c_{i+1}}$, such that $[t_i] \subseteq D_i$. Set $y \restriction c_{i+1} := s_i + t_i$. Then $s_i + y \restriction c_{i+1} = t_i$ and $y \restriction c_{i+1} \rhd y \restriction c_i$.

- λ is a limit: Set $y \restriction c_\lambda := \bigcup_{j<\lambda} y \restriction c_j$ and proceed as in the successor step to construct $y \restriction c_{\lambda+1}$.

Now we will show that $\forall i < \kappa\colon X + y \subseteq D_i$. To this end let $i < \kappa$ and $x \in X$ be arbitrary. We can now find $i' \geq i$ such that $x \in [s_{i'}]$. It follows that $x + y \in [s_{i'} + y \restriction c_{i'+1}] \subseteq D_{i'} \subseteq D_i$. This finishes the proof. □

Before we can construct the model, we will need some definitions:

Definition 3.3.3. We say that a forcing notion $\mathcal{P}$ has precaliber κ^+ iff for every $P \in [\mathcal{P}]^{\kappa^+}$ there exists $Q \in [P]^{\kappa^+}$ such that Q is centered$_{<\kappa}$.

While the previous definition is about forcing in general, the next definition is concerned with $\mathrm{cov}(\mathcal{SN})$:

Definition 3.3.4. Let $\{\mathcal{D}_\alpha \colon \alpha < \kappa^+\}$ be a sequence of families of open subsets of 2^κ. We call the family $\{\mathcal{D}_\alpha \colon \alpha < \kappa^+\}$ good iff $\forall E \in [\kappa^+]^{\kappa^+}\colon \bigcup_{\alpha \in E} \bigcap \mathcal{D}_\alpha = 2^\kappa$.

The motivation behind Definition 3.3.4 is that for each $\alpha < \kappa^+$ the set $\bigcap \mathcal{D}_\alpha$ could be a strong measure zero set. Then a good family corresponds to a family of strong measure zero sets of size κ^+ such that every subfamily of size κ^+ covers 2^κ.

Lemma 3.3.5. Suppose that $\{\mathcal{D}_\alpha \colon \alpha < \kappa^+\}$ is a good family in V, $V \vDash$ 'κ is weakly compact' and $\mathcal{P}$ is a $<\kappa$-closed forcing notion, which has precaliber κ^+. Then $\{\mathcal{D}_\alpha \colon \alpha < \kappa^+\}$ is also good in $V^{\mathcal{P}}$.

Proof. Towards a contradiction assume that there are names $\dot{x}$, $\dot{E}$ and a condition p such that $p \Vdash_{\mathcal{P}} \dot{x} \in 2^\kappa \wedge \dot{E} \in [\kappa^+]^{\kappa^+} \wedge \dot{x} \notin \bigcup_{\alpha \in \dot{E}} \bigcap \mathcal{D}_\alpha$. Working in V, for every $\alpha < \kappa^+$ find $\varepsilon_\alpha > \alpha$, $D_\alpha \in \mathcal{D}_{\varepsilon_\alpha}$ and conditions $p_\alpha \leq_{\mathcal{P}} p$ such that $p_\alpha \Vdash_{\mathcal{P}} \varepsilon_\alpha \in \dot{E} \wedge \dot{x} \notin D_\alpha$. Since $\mathcal{P}$ has precaliber κ^+, we can find $E^* \in [\kappa^+]^{\kappa^+}$ such that $\{p_\alpha \colon \alpha \in E^*\}$ is centered$_{<\kappa}$. Let $F \subseteq E^*$ be of size $< \kappa$ and let q_F be a lower bound of $\{p_\alpha \colon \alpha \in F\}$. Then $q_F \Vdash_{\mathcal{P}} \bigcup_{\alpha \in F} D_\alpha \neq 2^\kappa$. By Π^1_1-absoluteness [2] for $<\kappa$-closed forcing extensions (see Fact 1.1.3) $\bigcup_{\alpha \in F} D_\alpha \neq 2^\kappa$ must hold in V. Since κ is weakly compact it follows that $\bigcup_{\alpha \in E^*} D_\alpha \neq 2^\kappa$. Hence $\bigcup_{\beta \in \{\varepsilon_\alpha \colon \alpha \in E^*\}} \bigcap \mathcal{D}_\beta \neq 2^\kappa$. However, this is a contradiction to $\{\mathcal{D}_\alpha \colon \alpha < \kappa^+\}$ being a good family in V. □

We are now ready to prove the main theorem of this section:

Theorem 3.3.6. Let V satisfy $|2^\kappa| = \kappa^+$ and the strong unfoldability of κ is indestructible by $<\kappa$-closed, κ^+-c.c. forcing notions (see [Joh08]). Define $\mathbb{P}$ to be a $<\kappa$-support iteration of κ-Hechler forcing of length κ^{++}. Then $V^{\mathbb{P}} \vDash \mathrm{cov}(\mathcal{SN}) = \kappa^+ < \mathrm{add}(\mathcal{M}_\kappa) = \kappa^{++}$.

[2] This also guarantees that if $\mathbf{B}_1, \mathbf{B}_2 \in V$ are κ-Borel codes, then $V \vDash$ '$\mathbf{B}_1 = \mathbf{B}_2$' iff $V^{\mathcal{P}} \vDash$ '$\mathbf{B}_1 = \mathbf{B}_2$'.

Proof. Since $\mathrm{add}(\mathcal{M}_\kappa) = \min\{\mathfrak{b}_\kappa, \mathrm{cov}(\mathcal{M}_\kappa)\}$, it easily follows that $\mathrm{add}(\mathcal{M}_\kappa) = \kappa^{++}$ in $V^{\mathbb{P}}$. It remains to be shown that $V^{\mathbb{P}} \vDash \mathrm{cov}(\mathcal{SN}) = \kappa^+$: Working in $V^{\mathbb{P}}$ we define for $\alpha < \kappa^{++}$ the set

$$\mathcal{D}_\alpha := \{D\colon D \,\text{is dense open with}\ \kappa\text{-Borel code in}\, V^{\mathbb{P}_{\alpha+1}} \wedge 2^\kappa \cap V^{\mathbb{P}_\alpha} \subseteq D\}.$$

For $\tilde{E} \subseteq \kappa^{++}$ define $X_{\tilde{E}} := \bigcap_{\alpha\in\tilde{E}} \bigcap \mathcal{D}_\alpha$. If $\tilde{E}$ is cofinal in κ^{++}, then $X_{\tilde{E}}$ is smz: By GMS (see Theorem 3.3.2) it is enough to show that $X_{\tilde{E}}$ is meager-shiftable. To this end let D be a comeager set in $V^{\mathbb{P}}$ and find $\alpha' \in \tilde{E}$ such that D is coded in $V^{\mathbb{P}_{\alpha'}}$. But then $2^\kappa \cap V^{\mathbb{P}_{\alpha'}} \subseteq D + c_{\alpha'}$, where $c_{\alpha'}$ is some κ-Cohen real over $V^{\mathbb{P}_{\alpha'}}$ added by the next iterand of κ-Hechler forcing. Hence $D + c_{\alpha'} \in \mathcal{D}_{\alpha'}$, and therefore $X_{\tilde{E}} \subseteq \bigcap \mathcal{D}_{\alpha'} \subseteq D + c_{\alpha'}$.

We claim that $\forall x \in 2^\kappa\colon |\{\alpha < \kappa^{++}\colon x \notin \bigcap \mathcal{D}_\alpha\}| < \kappa^+$ holds in $V^{\mathbb{P}}$. Towards a contradiction assume that $\alpha^* < \kappa^{++}$ is the minimal ordinal such that there exists $E \in [\alpha^*]^{\kappa^+}$ with $\bigcup_{\alpha\in E} \bigcap \mathcal{D}_\alpha \neq 2^\kappa$. This observation means that the family $\{\mathcal{D}_\alpha\colon \alpha \in \alpha^*\}$ (note that $|\alpha^*| = \kappa^+$) is not good in $V^{\mathbb{P}}$. By the minimality of α^* it follows that E is cofinal in α^* and $\mathrm{otp}(E) = \kappa^+$, hence $\mathrm{cf}(\alpha^*) = \kappa^+$. Since the family $\{\mathcal{D}_\alpha\colon \alpha \in \alpha^*\}$ is in $V^{\mathbb{P}_{\alpha^*}}$, κ remains weakly compact in $V^{\mathbb{P}_{\alpha^*}}$, and the quotient forcing $\mathbb{P}/G_{\alpha^*}$ is $<\kappa$-closed and has precaliber κ^+, it follows by the previous lemma that $\{\mathcal{D}_\alpha\colon \alpha \in \alpha^*\}$ is also not good in $V^{\mathbb{P}_{\alpha^*}}$. Now working in $V^{\mathbb{P}_{\alpha^*}}$, for any $E' \in [\alpha^*]^{\kappa^+} \cap V^{\mathbb{P}_{\alpha^*}}$ if E' is not cofinal in α^*, then $2^\kappa \cap V^{\mathbb{P}_{\alpha^*}} \subseteq \bigcup_{\alpha\in E'} \bigcap \mathcal{D}_\alpha$ must hold by the minimality of α^*. If E' is cofinal in α^*, then we have $2^\kappa \cap V^{\mathbb{P}_{\alpha^*}} = \bigcup_{\alpha\in E'} 2^\kappa \cap V^{\mathbb{P}_\alpha} \subseteq \bigcup_{\alpha\in E'} \bigcap \mathcal{D}_\alpha$. But this is a contradiction to $\{\mathcal{D}_\alpha\colon \alpha \in \alpha^*\}$ not being good in $V^{\mathbb{P}_{\alpha^*}}$.

Again working in $V^{\mathbb{P}}$ let $\{\tilde{E}_\xi\colon \xi < \kappa^+\}$ be a partition of κ^{++} into cofinal subsets. It follows from the above claim that $\bigcup_{\xi<\kappa^+} X_{\tilde{E}_\xi} = 2^\kappa$, because for every $x \in 2^\kappa$ there must exist $\xi < \kappa^+$ such that for every $\alpha \in \tilde{E}_\xi$ we have $x \in \bigcap \mathcal{D}_\alpha$. Hence $\mathrm{cov}(\mathcal{SN}) = \kappa^+$. □

4 Cardinal characteristics on κ modulo non-stationary

Cardinal characteristics of $\mathfrak{P}(\kappa)$ for κ at least inaccessible have been studied extensively in [BTFFM17], [FMSS19], [FS18b], [RS17] and [RS19]. Similar to the classical case on ω, these cardinal characteristics are usually defined modulo the bounded ideal:

Definition 4.0.1. Let $x, y \in \mathfrak{P}(\kappa)$. We define:

- y splits x iff $|x \cap y| = \kappa$ and $|x \setminus y| = \kappa$.
  $\mathfrak{s}_\kappa := \min\{|\mathcal{S}| \colon \mathcal{S} \subseteq \mathfrak{P}(\kappa) \wedge \forall x \in \mathfrak{P}(\kappa)\ \exists y \in \mathcal{S}\ y \text{ splits } x\}$ the splitting number
  $\mathfrak{r}_\kappa := \min\{|\mathcal{R}| \colon \mathcal{R} \subseteq \mathfrak{P}(\kappa) \wedge \forall x \in \mathfrak{P}(\kappa)\ \exists y \in \mathcal{R}\ \neg\big(x \text{ splits } y\big)\}$ the reaping number

- $x \subseteq^* y$ iff $|x \setminus y| < \kappa$.
  $\mathcal{F} \subseteq \mathfrak{P}(\kappa)$ has the $<\kappa$-intersection property iff for every $\mathcal{F}' \subseteq \mathcal{F}$ of size $<\kappa$ we have that $|\bigcap_{x \in \mathcal{F}'} x| = \kappa$.
  $\mathfrak{p}_\kappa := \min\{|\mathcal{P}| \colon \mathcal{P} \subseteq \mathfrak{P}(\kappa) \wedge \mathcal{P}$ has the $<\kappa$-intersection property $\wedge \neg\big(\exists x \in \mathfrak{P}(\kappa)\ \forall y \in \mathcal{P}\ x \subseteq^* y\big)\}$ the pseudo intersection number
  $\mathfrak{t}_\kappa := \min\{|\mathcal{T}| \colon \mathcal{T} \subseteq \mathfrak{P}(\kappa) \wedge \mathcal{T}$ has the $<\kappa$-intersection property $\wedge \mathcal{T}$ is well-ordered by ${}^*\supseteq \wedge \neg\big(\exists x \in \mathfrak{P}(\kappa)\ \forall y \in \mathcal{T}\ x \subseteq^* y\big)\}$ the tower number

- x is almost disjoint from y iff $|x \cap y| < \kappa$.
  $\mathfrak{a}_\kappa := \min\{|\mathcal{A}| \colon \mathcal{A}$ is a maximal almost disjoint family $\wedge |\mathcal{A}| \geq \kappa\}$ the almost disjointness number

- $\mathcal{B} \subseteq \mathfrak{P}(\kappa)$ is a base for an ultrafilter $\mathcal{U}$ iff $\mathcal{U} = \{x \in \mathfrak{P}(\kappa) \colon \exists y \in \mathcal{B}\ y \subseteq x\}$.
  $\mathfrak{u}_\kappa := \min\{|\mathcal{B}| \colon \mathcal{B} \subseteq [\kappa]^\kappa \wedge \exists \mathcal{U} \subseteq \mathfrak{P}(\kappa)\ \mathcal{U}$ is an ultrafilter $\wedge \mathcal{B}$ is a base for $\mathcal{U}\}$ the ultrafilter number

In this chapter we intend to define variants of these cardinal characteristics modulo the non-stationary ideal for κ regular uncountable. To this end we recall the club filter $Cl = \{x \subseteq \kappa \colon \exists cl \subseteq x\ cl \text{ is club}\}$, the non-stationary ideal $NS = \{x \subseteq \kappa \colon \exists cl \in Cl\ x \cap cl = \emptyset\}$ and define the set of stationary sets $St := \mathfrak{P}(\kappa) \setminus NS$. Note that while the property $x \in Cl$ is upward absolute for models with the same cofinalities, the properties $x \in NS$ and $x \in St$ are in general not.

We will now define several relations on $St \times St$ modulo the non-stationary ideal and use them to define cardinal characteristics on St:

Definition 4.0.2. Let $x, y \in St$. We define:

- y stationarily splits x iff $x \cap y \in St$ and $x \setminus y \in St$.
  $\mathfrak{s}_\kappa^{cl} := \min\{|\mathcal{S}|: \mathcal{S} \subseteq St \wedge \forall x \in St\ \exists y \in \mathcal{S}\ y$ stationarily splits $x\}$ the stationary splitting number and
  $\mathfrak{r}_\kappa^{cl} := \min\{|\mathcal{R}|: \mathcal{R} \subseteq St \wedge \forall x \in St\ \exists y \in \mathcal{R}\ \neg\big(x$ stationarily splits $y\big)\}$ the stationary reaping number

- $x \subseteq^*_{cl} y$ iff $x \setminus y \in NS$.
  $\mathcal{F} \subseteq St$ has the $<\kappa$-stationary intersection property iff for every $\mathcal{F}' \subseteq \mathcal{F}$ of size $<\kappa$ we have that $\bigcap_{x \in \mathcal{F}'} x \in St$.
  $\mathfrak{p}_\kappa^{cl} := \min\{|\mathcal{P}|: \mathcal{P} \subseteq St \wedge \mathcal{P}$ has the $<\kappa$-stationary intersection property $\wedge \neg\big(\exists x \in St\ \forall y \in \mathcal{P}\ x \subseteq^*_{cl} y\big)\}$ the stationary pseudo intersection number
  $\mathfrak{t}_\kappa^{cl} := \min\{|\mathcal{T}|: \mathcal{T} \subseteq St \wedge \mathcal{T}$ has the $<\kappa$-stationary intersection property $\wedge \mathcal{T}$ is well-ordered by ${}^*_{cl}\supseteq \wedge \neg\big(\exists x \in St\ \forall y \in \mathcal{T}\ x \subseteq^*_{cl} y\big)\}$ the stationary tower number [1]

- x is stationary almost disjoint from y iff $x \cap y \in NS$.
  $\mathfrak{a}_\kappa^{cl} := \min\{|\mathcal{A}|: \mathcal{A}$ is a maximal stationary almost disjoint family $\wedge |\mathcal{A}| \geq \kappa\}$ the stationary almost disjointness number

- $\mathfrak{u}_\kappa^{cl} := \min\{|\mathcal{B}|: \mathcal{B} \subseteq St \wedge \exists \mathcal{U} \subseteq \mathfrak{P}(\kappa)\ \mathcal{U}$ is an ultrafilter $\wedge \mathcal{B}$ is a base for $\mathcal{U}\}$ [2] the stationary ultrafilter number
  $\mathfrak{u}_\kappa^{cl*} := \min\{|\mathcal{B}|: \mathcal{B} \subseteq St \wedge \exists \mathcal{U} \subseteq \mathfrak{P}(\kappa)\ \mathcal{U}$ is an ultrafilter $\wedge \mathcal{B} \cup Cl$ is a subbase for $\mathcal{U}\}$ [3] the stationary* ultrafilter number
  $\mathfrak{u}_\kappa^{me} := \min\{|\mathcal{B}|: \mathcal{B} \subseteq St \wedge \exists \mathcal{U} \subseteq \mathfrak{P}(\kappa)\ \mathcal{U}$ is a measure $\wedge \mathcal{B}$ is a base for $\mathcal{U}\}$ [4] the measure ultrafilter number
  $\mathfrak{u}_\kappa^{nm} := \min\{|\mathcal{B}|: \mathcal{B} \subseteq St \wedge \exists \mathcal{U} \subseteq \mathfrak{P}(\kappa)\ \mathcal{U}$ is a normal measure $\wedge \mathcal{B}$ is a base for $\mathcal{U}\}$ the normal measure ultrafilter number
  $\mathfrak{u}_\kappa^{nm*} := \min\{|\mathcal{B}|: \mathcal{B} \subseteq St \wedge \exists \mathcal{U} \subseteq \mathfrak{P}(\kappa)\ \mathcal{U}$ is a normal measure $\wedge \mathcal{B} \cup Cl$ is a subbase for $\mathcal{U}\}$ the normal measure* ultrafilter number

- Let $f, g \in \kappa^\kappa$ and define $f \leq^*_{cl} g$ iff $\{\alpha < \kappa: g(\alpha) < f(\alpha)\} \in NS$.
  $\mathfrak{b}_\kappa^{cl} := \min\{|B|: B \subseteq \kappa^\kappa \wedge \forall f \in \kappa^\kappa\ \exists g \in B\ g \not\leq^*_{cl} f\}$ the club bounding number
  $\mathfrak{d}_\kappa^{cl} := \min\{|D|: D \subseteq \kappa^\kappa \wedge \forall f \in \kappa^\kappa\ \exists g \in D\ f \leq^*_{cl} g\}$ the club dominating number

We will aim to establish some relations between these cardinal characteristics and also show some consistency results.

[1] Note that the notions of $\mathfrak{p}_\kappa^{cl}$ and $\mathfrak{t}_\kappa^{cl}$ introduced here are different to the ones defined in [FMSS19].
[2] In particular, $Cl \subseteq \mathcal{U}$.
[3] i.e. $\{y \in St: \exists x \in \mathcal{B}\ \exists cl \in Cl\ y = x \cap cl\}$ is a base for $\mathcal{U}$, since w.l.o.g. $\mathcal{B}$ is closed under intersections.
[4] i.e. $\mathcal{U}$ is a $<\kappa$-complete ultrafilter.

4.1 Results / Questions

The notions of the club bounding and dominating number have already been investigated by Cummings and Shelah (see [CS95]). In particular they showed the following theorem:

Theorem 4.1.1. Let κ be regular uncountable. Then $\mathfrak{b}_\kappa = \mathfrak{b}_\kappa^{cl}$. If $\kappa \geq \beth_\omega$ then $\mathfrak{d}_\kappa = \mathfrak{d}_\kappa^{cl}$.

The stationary almost disjointness number $\mathfrak{a}_\kappa^{cl}$ is trivial:

Lemma 4.1.2. Let κ be regular uncountable. Then $\mathfrak{a}_\kappa^{cl} = \kappa$.

Proof. Partition κ into κ many stationary sets $(x_i)_{i<\kappa}$. Define $y_i := \kappa \setminus \bigcup_{j \leq i} x_j$ and set $x_\kappa := \triangle_{i<\kappa} y_i$. Note that $x_i \cap x_\kappa \in NS$ for every $i < \kappa$. Now we have to distinguish two cases:

- If $x_\kappa \in St$, then we claim that the family $(x_i)_{i \leq \kappa}$ is maximal stationary almost disjoint. Towards a contradiction assume that $x^* \in St$ is stationary almost disjoint from x_i for every $i \leq \kappa$. We define a function $f \colon x^* \to \kappa$ such that $f(k)$ is the unique $i < \kappa$ such that $k \in x_i$. Equivalently $f(k) := \min\{i < \kappa \colon k \notin y_i\}$. If the set $\{k \in x^* \colon f(k) < k\}$ is stationary, then by Fodor's lemma (see Chapter 8 in [Jec03]) the set $\{k \in x^* \colon f(k) = \delta\}$ is stationary for some $\delta < \kappa$. But this implies that $x^* \cap x_\delta \in St$. Hence, the set $\{k \in x^* \colon f(k) \geq k\}$ is stationary, and therefore $x^* \cap x_\kappa \in St$. But this also leads to a contradiction, hence $(x_i)_{i \leq \kappa}$ is a maximal stationary almost disjoint family.

- If $x_\kappa \in NS$, then we proceed similarly and claim that $(x_i)_{i<\kappa}$ is maximal stationary almost disjoint. We define $f \colon x^* \to \kappa$ as above, and note that $\{k \in x^* \colon f(k) \geq k\}$ cannot be stationary. Hence, there exists $\delta < \kappa$ such that $x^* \cap x_\delta \in St$. □

Let us say a few words about the spectrum of stationary almost disjointness:

Definition 4.1.3. We define $\text{Spec}_{\text{sad}} := \{\gamma \geq \kappa \colon \exists \mathcal{A}\ \mathcal{A}$ is a maximal stationary almost disjoint family $\wedge\ |\mathcal{A}| = \gamma\}$.

Definition 4.1.4. Let $x \in St$. We say that $NS \restriction x$ is γ-saturated iff for every stationary almost disjoint family $\mathcal{A} \subseteq \mathfrak{P}(x)$ we have $|\mathcal{A}| < \gamma$.

Obviously, this definition agrees with the usual definition of saturation (see Chapter 22 in [Jec03]).

The next lemma will summarize some properties of Spec_{sad}:

Lemma 4.1.5. The following holds true for κ regular uncountable:

1. By Lemma 4.1.2 we have $\kappa \in \text{Spec}_{\text{sad}}$.
2. By [GS97] we have NS is not κ^+-saturated for $\kappa \geq \omega_2$, hence $\{\kappa\} \subsetneq \text{Spec}_{\text{sad}}$.
3. If $\diamondsuit_\kappa(\kappa)$ holds (see Definition 4.1.9), then $\mathfrak{c}_\kappa \in \text{Spec}_{\text{sad}}$.

4. By [Git86] it is consistent that κ is inaccessible and there exists $x \in St$ such that $x \cap \{i < \kappa \colon \mathrm{cf}(i) = j\} \in St$ for all cardinals $j < \kappa$ and $NS \restriction x$ is κ^+-saturated. By [JW85] it is consistent that κ is Mahlo and $NS \restriction \mathrm{Reg}$ is κ^+-saturated.

Question 4.1.6. Is it consistent that NS is $\mathfrak{c}_\kappa$-saturated for κ inaccessible? Is it even consistent that NS is κ^{++}-saturated and $\mathfrak{c}_\kappa$ is very large?
In [GS97] the authors ask whether the following is consistent for κ inaccessible: $\forall x \in St\ \exists y \in St \colon y \subseteq x \wedge NS \restriction y$ is κ^+-saturated.

Also the stationary pseudo intersection number $\mathfrak{p}^{cl}_\kappa$ and the stationary tower number $\mathfrak{t}^{cl}_\kappa$ are trivial:

Lemma 4.1.7. Let κ be regular uncountable. Then $\mathfrak{p}^{cl}_\kappa = \mathfrak{t}^{cl}_\kappa = \kappa$.

Proof. It will suffice to show that there exists a decreasing sequence $(x_i)_{i<\kappa}$ of stationary sets such that $\triangle_{i<\kappa} x_i = \{0\}$: Assume that x^* is a stationary pseudo intersection of $(x_i)_{i<\kappa}$. Again define $f \colon x^* \to \kappa$ such that $f(j) := \min\{i < \kappa \colon j \notin x_i\}$ and again we note that $\{j \in x^* \colon f(j) < j\} \in NS$. Hence, $x^* \subseteq^*_{cl} \triangle_{i<\kappa} x_i$ must hold, which leads to a contradiction.
Therefore, let us show that there exists such a sequence $(x_i)_{i<\kappa}$. Let $E^\kappa_\omega := \{i < \kappa \colon \mathrm{cf}(i) = \omega\}$ and for every $k \in E^\kappa_\omega$ let $(j^k_n)_{n<\omega}$ be a cofinal sequence in k. We claim that there exists $n^* < \omega$ such that for every $i < \kappa$ the set $x_i := \{k < \kappa \colon j^k_{n^*} \geq i\}$ is stationary. Assume towards a contradiction that for every $n < \omega$ there exist $i_n < \kappa$ such that $x_{i_n} \in NS$ and let cl_n be a club disjoint from x_{i_n}. We define $i^* := \sup_{n<\omega} i_n$ and $cl^* := \bigcap_{n<\omega} cl_n$. Let $k^* \in E^\kappa_\omega \cap cl^*$ with $k^* > i^*$. Then it follows that $j^{k^*}_n < i^*$ for every $n < \omega$. But this contradicts the assumption that $(j^{k^*}_n)_{n<\omega}$ is cofinal in k^*.
Hence, let n^* and $(x_i)_{i<\kappa}$ be as defined above. It remains to be shown that $\triangle_{i<\kappa} x_i = \{0\}$. Assume towards a contradiction that there exists $k > 0$ such that $k \in \triangle_{i<\kappa} x_i$. This means that $j^k_{n^*} \geq i$ for every $i < k$. But this is a contradiction. □

Next, we investigate the stationary reaping number $\mathfrak{r}^{cl}_\kappa$:

Theorem 4.1.8. $\mathfrak{r}^{cl}_\kappa \geq \kappa$ for κ inaccessible.

Proof. Let $(x_i)_{i<\lambda}$ with $\lambda < \kappa$ be a family of stationary sets and w.l.o.g. assume that $\kappa \subseteq^*_{cl} \bigcup_{i<\lambda} x_i$. Assume that $(x_{i,j})_{j<\lambda}$ is a partition of x_i into λ many stationary sets and define $x_{i,\lambda} := \kappa \setminus x_i$ for every $i < \lambda$. We will find a common refinement of the partitions $(x_{i,j})_{j\leq\lambda}$.
For every $s \in (\lambda+1)^\lambda$ define $y_s := \bigcap_{i<\lambda} x_{i,s(i)}$. Clearly, if $s_1, s_2 \in (\lambda+1)^\lambda$ with $s_1 \neq s_2$ then $y_{s_1} \cap y_{s_2} = \emptyset$. Now set $S := \{s \in (\lambda+1)^\lambda \colon y_s \in St\}$ and note that since $(\lambda+1)^\lambda < \kappa$ and every $x_{i,j} = \bigcup_{s\in(\lambda+1)^\lambda,\, s(i)=j} y_s$, we clearly have that $\kappa \subseteq^*_{cl} \bigcup_{s\in S} y_s$ and $(y_s)_{s\in S}$ refines every partition $(x_{i,j})_{j\leq\lambda}$.
Since the y_s are pairwise disjoint, one can now easily construct a set $y^* \in St$ which stationarily splits y_s for every $s \in S$, and hence stationarily splits x_i for every $i < \lambda$. □

We will later see that $\mathfrak{r}^{cl}_\kappa > \kappa$ can be forced.

Definition 4.1.9. Let $x \subseteq \kappa$ be stationary. We say that $\diamondsuit_\kappa(x)$ holds iff there exists a sequence $(s_i)_{i \in x}$ with $s_i \subseteq i$ such that for every $y \subseteq \kappa$ the set $\{i \in x \colon y \restriction i = s_i\}$ is stationary (see Chapter 27 in [Jec03]).

Question 4.1.10. Is $\mathfrak{r}^{cl}_\kappa = \kappa$ consistent? Does $\forall x \in St\colon \diamondsuit_\kappa(x)$ imply $\mathfrak{r}^{cl}_\kappa > \kappa$? How does $\mathfrak{r}^{cl}_\kappa$ relate to $\mathfrak{r}_\kappa$?

Concerning the various definitions of ultrafilter numbers:

Lemma 4.1.11. For κ measurable we have:

1. $\kappa^+ \leq \mathfrak{r}_\kappa \leq \mathfrak{u}_\kappa \leq \mathfrak{u}^{cl}_\kappa \leq \mathfrak{u}^{nm}_\kappa$
2. $\kappa \leq \mathfrak{r}^{cl}_\kappa \leq \mathfrak{u}^{cl^*}_\kappa \leq \mathfrak{u}^{nm^*}_\kappa$, $\mathfrak{u}^{cl^*}_\kappa \leq \mathfrak{u}^{cl}_\kappa$ and $\mathfrak{u}^{nm^*}_\kappa \leq \mathfrak{u}^{nm}_\kappa$
3. $\mathfrak{u}^{me}_\kappa = \mathfrak{u}^{nm}_\kappa$ and $\kappa^+ \leq \mathfrak{u}^{nm^*}_\kappa$

Proof. 1.) and 2.) should be obvious (using Theorem 4.1.8). Hence let us prove 3.): We clearly have $\mathfrak{u}^{me}_\kappa \leq \mathfrak{u}^{nm}_\kappa$. On the other hand let $\mathcal{U}$ be a measure such that there exists a base $\mathcal{B}$ of $\mathcal{U}$ with $|\mathcal{B}| = \mathfrak{u}^{me}_\kappa$. Let $V^\kappa/\mathcal{U}$ denote the ultrapower of V modulo $\mathcal{U}$, let $M := \text{mos}(V^\kappa/\mathcal{U})$ be the transitive collapse and $j\colon V \to M$ the elementary embedding. Pick $f\colon \kappa \to \kappa$ such that $\kappa = \text{mos}([f]_\mathcal{U})$. Then $\mathcal{V} := \{x \subseteq \kappa\colon \kappa \in j(x)\}$ is a normal measure and it easily follows that $\mathcal{V} = \{x \subseteq \kappa\colon \exists y \in \mathcal{U}\ f[y] \subseteq x\}$. Hence, $f[\mathcal{B}]$ is a base of $\mathcal{V}$ and $\mathfrak{u}^{nm}_\kappa \leq \mathfrak{u}^{me}_\kappa$ follows.
To show that $\kappa^+ \leq \mathfrak{u}^{nm^*}_\kappa$ we assume towards a contradiction that $\mathcal{U}$ is a normal measure and there exists $\mathcal{B} \subseteq \mathcal{U}$ with $|\mathcal{B}| = \kappa$ such that $\{y \in St\colon \exists x \in \mathcal{B}\ \exists cl \in Cl\ y = x \cap cl\}$ is a base of $\mathcal{U}$. If we enumerate $\mathcal{B}$ as $(x_i)_{i<\kappa}$ then we see that $\triangle_{i<\kappa} x_i \in \mathcal{U}$. But for every $x \in \mathcal{B}$ we have $x \not\subseteq^*_{cl} \triangle_{i<\kappa} x_i$ which leads to a contradiction. $\square$

Lemma 4.1.12. By [BTFFM17] the following is consistent: $\kappa^+ < \mathfrak{r}_\kappa = \mathfrak{u}^{nm}_\kappa < \mathfrak{c}_\kappa$.

Question 4.1.13. Are there any other provable relations between the various ultrafilter numbers? Are $\mathfrak{u}^{cl^*}_\kappa < \mathfrak{u}^{cl}_\kappa$ or $\mathfrak{u}^{nm^*}_\kappa < \mathfrak{u}^{nm}_\kappa$ consistent? Is even $\mathfrak{u}^{cl^*}_\kappa = \kappa$ consistent?

Let us now investigate the stationary splitting number $\mathfrak{s}^{cl}_\kappa$:

Theorem 4.1.14. For κ regular uncountable we have $\mathfrak{s}^{cl}_\kappa \geq \kappa$ iff κ is inaccessible.

Proof. We follow the proof of [Suz93]. First assume that κ is not inaccessible, hence there exists a minimal $\lambda < \kappa$ such that $|2^\lambda| \geq \kappa$. Let $f\colon \kappa \to 2^\lambda$ be injective and for every $s \in 2^{<\lambda}$ define $x_s := \{i < \kappa\colon s \triangleleft f(i)\}$. We set $X := \{x_s\colon s \in 2^{<\lambda} \wedge x_s \in St\}$ which is of size $|2^{<\lambda}| < \kappa$, and claim that X is a stationary splitting family. Towards a contradiction assume that $y \in St$ is not stationarily split by X. It follows that the set $S := \{s \in 2^{<\lambda}\colon y \subseteq^*_{cl} x_s\}$ is linearly ordered by $\triangleleft$, because for incompatible $s_1, s_2 \in 2^{<\lambda}$ we have that x_{s_1} and x_{s_2} are disjoint. Let us define $t := \bigcup S$ and note that $t \in 2^\lambda$. Now we can deduce that $y \subseteq f^{-1}(\{t\}) \cup \bigcup_{s \in 2^{<\lambda} \setminus S} (x_s \cap y)$. However, this leads to a contradiction, because y would be covered by a union of $< \kappa$ many non-stationary sets. On the other hand assume that κ is inaccessible and let $X \subseteq St$ be of size $\lambda < \kappa$.

Let $\theta > \kappa$ be a sufficiently large, regular cardinal, and choose an elementary submodel $M \prec H(\theta)$ with $\kappa, X \in M$, $X, 2^\lambda \subseteq M$ and $|M| < \kappa$. Now pick $i^* > \sup(M \cap \kappa)$ such that $i^* \in \bigcap_{cl \in Cl \cap M} cl$. The ordinal i^* induces a partition Y_0, Y_1 of X: set $Y_0 := \{x \in X : i^* \notin x\}$ and $Y_1 := \{x \in X : i^* \in x\}$. Since $2^\lambda \subseteq M$ we can deduce that also $Y_0, Y_1 \in M$, and hence $y := \bigcap Y_1 \setminus \bigcup Y_0 \in M$. If we can show that $y \in St$, this will imply that X is not a stationary splitting family. To this end let $cl \in Cl \cap M$ be arbitrary, and we obviously have $H(\theta) \models i^* \in y \cap cl$. By elementarity it follows that $M \models y \cap cl \neq \emptyset$, and since cl was arbitrary, we can deduce that $M \models y \in St$. Again by elementarity we have $y \in St$. □

The following definition already appeared in [HS18]:

Definition 4.1.15. Let $F \subseteq \mathfrak{P}(\kappa)$ be a uniform filter [5], i.e. for every $x \in F$ we have $|x| = \kappa$. We define:

- F is $<\kappa$-complete* iff for every $\lambda < \kappa$ and every $(x_i)_{i<\lambda}$ with $x_i \in F$ we have $|\bigcap_{i<\lambda} x_i| = \kappa$. [6]
- F is normal* iff for every $(x_i)_{i<\kappa}$ with $x_i \in F$ we have that $\triangle_{i<\kappa} x_i$ is stationary.
- F measures a set $X \subseteq \mathfrak{P}(\kappa)$ iff for every $x \in X$ either $x \in F$ or $\kappa \setminus x \in F$ holds true.

Note that we explicitly do not require that the (diagonal) intersection is again an element of F. Clearly, if F is normal*, then it is also $<\kappa$-complete*.

Definition 4.1.16. We say that κ has the normal* filter property iff for every $X \subseteq \mathfrak{P}(\kappa)$ of size $\leq\kappa$ there exists a normal* filter F measuring X.

The following notion clearly strengthens weak compactness and is downward absolute to L (see [JK69]):

Definition 4.1.17. Recall that κ is ineffable iff for every partition $f : [\kappa]^2 \to \{0, 1\}$ there exists a stationary homogeneous set $x \subseteq \kappa$.

The following theorem was proven in [DPZ80]:

Theorem 4.1.18. Let κ be regular uncountable. Then κ has the normal* filter property iff κ is ineffable.

Theorem 4.1.19. For κ regular uncountable we have $\mathfrak{s}_\kappa^{cl} > \kappa$ iff κ is ineffable.

Proof. We will show that $\mathfrak{s}_\kappa^{cl} > \kappa$ iff κ has the normal* filter property. Then this theorem follows by the previous theorem.
Let us first assume that $\mathfrak{s}_\kappa^{cl} > \kappa$ and let $X \subseteq \mathfrak{P}(\kappa)$ be of size $\leq\kappa$. We will show that there exists a normal* filter F measuring X. W.l.o.g. X is closed under compliments. Since

[5] In particular we can assume that F contains the co-bounded filter.
[6] Note that any $<\kappa$-complete* filter F can be extended to a $<\kappa$-complete filter $\tilde{F}$.

$\mathfrak{s}_\kappa^{cl} > \kappa$ there exists $y^* \in St$ such that X does not stationarily split y^*. Now we define $F := \{x \in X : y^* \subseteq^*_{cl} x\}$ and note that F is obviously an ultrafilter on X. We claim that F is normal*. Let $(x_i)_{i<\kappa}$ with $x_i \in F$ be arbitrary and $cl_i \in Cl$ with $y^* \cap cl_i \subseteq x_i$. Then $\triangle_{i<\kappa} x_i \supseteq \triangle_{i<\kappa} y^* \cap cl_i = y^* \cap \triangle_{i<\kappa} cl_i$ which is clearly stationary.
On the other hand assume that κ has the normal* filter property and let $X \subseteq St$ be of size κ. Then there exists a normal* filter F measuring X, and enumerate X as $(x_i)_{i<\kappa}$. Define $y_i := x_i$ if $x_i \in F$ and $y_i := \kappa \setminus x_i$ else. Since F is normal*, we can deduce that $y^* := \triangle_{i<\kappa} y_i \in St$. But no $x_i \in X$ can stationarily split y^*, hence $\mathfrak{s}_\kappa^{cl} > \kappa$. □

Before we can state the next theorem, we need the following definition:

Definition 4.1.20. Let α be a measurable cardinal and let $\mathcal{U}_0$, $\mathcal{U}_1$ and $\mathcal{U}$ be normal measures on α. We recall (see Chapter 19 in [Jec03]):

- the Mitchell order: $\mathcal{U}_0 \lhd \mathcal{U}_1$ iff $\mathcal{U}_0 \in V^\kappa/\mathcal{U}_1$, i.e. $\mathcal{U}_0$ is contained in the ultrapower of V modulo $\mathcal{U}_1$
- $o(\mathcal{U}) := \sup\{o(\mathcal{U}') + 1 : \mathcal{U}' \lhd \mathcal{U}\}$ the order of $\mathcal{U}$
- $o(\alpha) := \sup\{o(\mathcal{U}') : \mathcal{U}'$ is normal measure on $\alpha\}$ the order of α

It was proven by Zapletal (see [Zap97]) that $\mathfrak{s}_\kappa > \kappa^+$ has large consistency strength, and indeed the same proof shows:

Theorem 4.1.21. Let $\mathfrak{s}_\kappa^{cl} > \kappa^+$. Then there exists an inner model with a measurable cardinal α of order α^{++}. [7]

Let us now show some consistency results regarding $\mathfrak{s}_\kappa^{cl}$, $\mathfrak{b}_\kappa$, $\mathfrak{d}_\kappa$ and $\mathfrak{r}_\kappa^{cl}$. First we state a helpful tool:

Lemma 4.1.22. Let $V \vDash x \in St$ and let $\mathcal{P}$ be a $<\kappa$-closed forcing. Then $V^{\mathcal{P}} \vDash x \in St$.

Proof. Since being stationary is a Π^1_1-statement, the lemma follows by Π^1_1-absoluteness for $<\kappa$-closed forcing extensions (see Fact 1.1.3). □

Definition 4.1.23. Let $\mathcal{U}$ be a $<\kappa$-complete, normal ultrafilter on κ. We define $\mathbb{M}_\mathcal{U}$, the generalized Mathias forcing with respect to $\mathcal{U}$, as follows:

- A condition p is of the form (s^p, A^p) where $s^p \in [\kappa]^{<\kappa}$, $A^p \in \mathcal{U}$ and $\sup s^p \leq \min A^p$.
- Let $p = (s^p, A^p)$ and $q = (t^q, B^q)$ be conditions in $\mathbb{M}_\mathcal{U}$. We define $q \leq_{\mathbb{M}_\mathcal{U}} p$, in words q is stronger than p, if $s^p \subseteq t^q$, $B^q \subseteq A^p$ and $t^q \setminus s^p \subseteq A^p$.

If G is a $(V, \mathbb{M}_\mathcal{U})$-generic filter, we define $m_G := \bigcup_{p \in G} s^p$.

The next lemma follows immediately.

[7] This is equivalent to $\exists \mathcal{F} : L[\mathcal{F}] \vDash \exists \alpha : \alpha$ is measurable with order α^{++} (see [Mit83]) .

Lemma 4.1.24. Let $\mathcal{U}$ be a $<\kappa$-complete, normal ultrafilter. Then the forcing $\mathbb{M}_{\mathcal{U}}$ has the following properties:

- $\mathbb{M}_{\mathcal{U}}$ is κ-centered$_{<\kappa}$. In particular it satisfies the κ^+-c.c.
- $\mathbb{M}_{\mathcal{U}}$ is $<\kappa$-directed closed.

Lemma 4.1.25. Let $\mathcal{U}$ be a $<\kappa$-complete, normal ultrafilter on κ and let $V \vDash x \in St$. Then $\Vdash_{\mathbb{M}_{\mathcal{U}}} \dot{m}_G \in St \wedge (\dot{m}_G \subseteq^*_{cl} x \vee \dot{m}_G \cap x \in NS)$.

Proof. If $x \in \mathcal{U}$ then clearly $\Vdash_{\mathbb{M}_{\mathcal{U}}} \dot{m}_G \subseteq^* x$. On the other hand, if $x \notin \mathcal{U}$ then $\Vdash_{\mathbb{M}_{\mathcal{U}}} \dot{m}_G \cap x$ is bounded. Hence, it remains to be shown that $\Vdash_{\mathbb{M}_{\mathcal{U}}} \dot{m}_G \in St$. To this end let $p \in \mathbb{M}_{\mathcal{U}}$ and $\dot{cl}$ be a $\mathbb{M}_{\mathcal{U}}$-name for a club. Let $(p_i)_{i<\kappa}$ be a decreasing sequence of conditions below p interpreting $\dot{cl}$ as $cl^* \in V$, and w.l.o.g assume that $p_\lambda = \inf_{i<\lambda} p_i$ for every limit $\lambda < \kappa$. Let $A^* := \triangle_{i<\kappa} A^{p_i}$ denote the diagonal intersection of the A^{p_i}, and since $\mathcal{U}$ is a normal measure, we have that $A^* \in \mathcal{U}$. Hence, $A^* \cap \mathrm{Lim}(cl^*) \neq \emptyset$ where $\mathrm{Lim}(cl^*)$ is the club consisting only of the limit points of cl^*, and pick $i^* \in A^* \cap cl^*$. It follows that $i^* \in A^{p_{i^*}}$ and $p_{i^*} \Vdash_{\mathbb{M}_{\mathcal{U}}} i^* \in \dot{cl}$. If we define a condition $q := (s^{p_{i^*}} \cup \{i^*\}, A^{p_{i^*}} \setminus \{i^*\})$ then trivially $q \leq_{\mathbb{M}_{\mathcal{U}}} p_{i^*}$ and $q \Vdash_{\mathbb{M}_{\mathcal{U}}} i^* \in \dot{m}_G \cap \dot{cl}$. Hence $\Vdash_{\mathbb{M}_{\mathcal{U}}} \dot{m}_G \in St$. □

Theorem 4.1.26. Let κ be supercompact and indestructible by $<\kappa$-directed closed forcing notions (see Theorem 1.1.7). Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \kappa^{++}, \beta < \kappa^{++} \rangle$ be a $<\kappa$-support iteration such that $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha = \mathbb{M}_{\dot{\mathcal{U}}_\alpha}$ where $\dot{\mathcal{U}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a $<\kappa$-complete, normal ultrafilter, and set $\mathbb{P} := \mathbb{P}_{\kappa^{++}}$. Furthermore, assume that $V \vDash |2^\kappa| = \kappa^+$. Then $V^{\mathbb{P}} \vDash \mathfrak{s}^{cl}_\kappa = \mathfrak{b}_\kappa = \mathfrak{d}_\kappa = \mathfrak{r}^{cl}_\kappa = \mathfrak{c}_\kappa = \kappa^{++}$.

Proof. Since $\mathbb{P}$ satisfies the κ^+-c.c. and for every $\alpha < \kappa^{++}$ the forcing $\mathbb{P}_\alpha$ has a dense subset of size κ^+, we can deduce that $V^{\mathbb{P}} \vDash |2^\kappa| = \kappa^{++}$. It is easy to see that $V^{\mathbb{P}} \vDash \mathfrak{b}_\kappa = \mathfrak{d}_\kappa = \kappa^{++}$. Since $\mathbb{P}$ has $<\kappa$-support, it follows that $\mathbb{P}$ adds κ-Cohen reals, hence $V^{\mathbb{P}} \vDash \mathfrak{r}^{cl}_\kappa = \kappa^{++}$ (see Lemma 4.1.27). Now if $V^{\mathbb{P}} \vDash$ '$X \subseteq St$ is a set of size $\leq \kappa^+$', then by the κ^+-c.c. there exists $\alpha < \kappa^{++}$ such that $X \in V^{\mathbb{P}_\alpha}$ and by Π^1_1-downward absoluteness $V^{\mathbb{P}_\alpha} \vDash X \subseteq St$. By Lemma 4.1.25 X is not a stationary splitting family in $V^{\mathbb{P}_{\alpha+1}}$, hence by Lemma 4.1.22 X cannot be a stationary splitting family in $V^{\mathbb{P}}$. □

Lemma 4.1.27. Let G be a $(V, \mathbb{C}_\kappa)$-generic filter and let $c_G \subseteq \kappa$ denote the κ-Cohen real added by G. Let $x \subseteq \kappa$ be arbitrary. If $V \vDash x \in St$, then $V^{\mathbb{C}_\kappa} \vDash c_G$ stationarily splits x. Furthermore, if $\mathbb{P} := \prod_{\alpha<\kappa^+} \mathbb{C}_\kappa$ denotes the $<\kappa$-support product of κ-Cohen forcing, then $V^{\mathbb{P}} \vDash (c_\alpha)_{\alpha<\kappa^+}$ is a stationary splitting family.

Proof. We proceed similarly to the proof of 4.1.25: Let $p \in \mathbb{C}_\kappa$ and $\dot{cl}$ be a $\mathbb{C}_\kappa$-name for a club. Let $(p_i)_{i<\kappa}$ be a decreasing sequence below p interpreting $\dot{cl}$ as $cl^* \in Cl \cap V$. Again, w.l.o.g. assume that $p_\lambda = \inf_{i<\lambda} p_i$ for every limit $\lambda < \kappa$. Since x is stationary in V, we can find $i^* \in x \cap \mathrm{Lim}(cl^*)$ where $\mathrm{Lim}(cl^*)$ is again the club consisting only of the limit points of cl^*. Hence, there are $q_0, q_1 \in \mathbb{C}_\kappa$ below p_{i^*} such that $q_0 \Vdash_{\mathbb{C}_\kappa} i^* \in (x \setminus c_{\dot{G}}) \cap \dot{cl}$ and $q_1 \Vdash_{\mathbb{C}_\kappa} i^* \in x \cap c_{\dot{G}} \cap \dot{cl}$.
Let $\dot{x}$ be a $\mathbb{P}$-name for a stationary set in $V^{\mathbb{P}}$. By the κ^+-c.c. of $\mathbb{P}$ it follows that there exists $\alpha < \kappa^+$ such that $\dot{x}$ is a $\mathbb{P}_\alpha$-name, where $\mathbb{P}_\alpha := \prod_{\beta<\alpha} \mathbb{C}_\kappa$. By the above $V^{\mathbb{P}_{\alpha+1}} \vDash c_\alpha$ stationarily splits x. By Lemma 4.1.22 we have $V^{\mathbb{P}} \vDash c_\alpha$ stationarily splits x. □

The following proof already appeared in a similar version in [She84]:

Theorem 4.1.28. Let κ be supercompact and indestructible by $<\kappa$-directed closed forcing notions. Let $V \vDash |2^\kappa| = \kappa^+$ and define $\mathbb{R} := \mathbb{P} \star \dot{\mathbb{Q}}$ where $\mathbb{P} := \prod_{\alpha<\kappa^+} \mathbb{C}_\kappa$ and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a κ^{++} iteration of κ-Hechler forcing $\mathbb{H}_\kappa$ with $<\kappa$-support. Then $V^{\mathbb{R}} \vDash \mathfrak{s}^{cl}_\kappa = \kappa^+ \wedge \mathfrak{b}_\kappa = \mathfrak{d}_\kappa = \mathfrak{r}^{cl}_\kappa = \mathfrak{c}_\kappa = \kappa^{++}$.

Proof. Obviously, $\mathfrak{b}_\kappa = \mathfrak{d}_\kappa = \kappa^{++}$. Since $\mathbb{H}_\kappa$ adds κ-Cohen reals, we can deduce by 4.1.27 that $\mathfrak{r}^{cl}_\kappa = \kappa^{++}$. Since κ remains ineffable in $V^{\mathbb{R}}$ it follows that $\mathfrak{s}^{cl}_\kappa \geq \kappa^+$. It remains to be shown that $\mathfrak{s}^{cl}_\kappa \leq \kappa^+$. To this end we will show that $(c_\alpha)_{\alpha<\kappa^+}$ remains a stationary splitting family in $V^{\mathbb{R}}$ where the $(c_\alpha)_{\alpha<\kappa^+}$ are the generic κ-Cohen reals added by $\mathbb{P}$. Towards a contradiction assume that $\dot{x}$ is a $\mathbb{R}$-name and $(p, \dot{q})$ a condition in $\mathbb{R}$ such that $(p, \dot{q}) \Vdash_{\mathbb{R}} \dot{x} \in St \wedge (\forall \alpha < \kappa^+ \colon \dot{x} \subseteq^*_{cl} \dot{c}_\alpha \vee \dot{x} \cap \dot{c}_\alpha \in NS)$. Since $\mathbb{R}$ satisfies the κ^+-c.c. we can find $\alpha^* < \kappa^+$ such that the $\mathbb{R}$-name $\dot{x}$ does not depend on $\dot{c}_{\alpha^*}$. Since $\mathbb{P}$ is $<\kappa$-closed and $\Vdash_{\mathbb{P}}$ '$\dot{\mathbb{Q}}$ has $<\kappa$-support and is $<\kappa$-closed', we obviously have

$$\Vdash_{\mathbb{P}} \{q \in \dot{\mathbb{Q}} \colon \mathrm{dom}(q) \in \check{V} \wedge \exists \bar{\rho} \in (\kappa^{<\kappa})^{\mathrm{dom}(q)} \cap \check{V}$$

$$\forall \alpha \in \mathrm{dom}(q)\ \exists \dot{f}\ \ \Vdash_{\dot{\mathbb{Q}}} \dot{q}(\alpha) = (\bar{\rho}(\alpha), \dot{f})\} \text{ is dense in } \dot{\mathbb{Q}}$$

Hence, we can pick a condition $(p', \dot{q}') \leq_{\mathbb{R}} (p, \dot{q})$ such that all trunks of $(p', \dot{q}')$ are ground model objects, and $(p', \dot{q}')$ decides whether $\dot{x} \subseteq^*_{cl} c_{\alpha^*}$ or $\dot{x} \cap c_{\alpha^*} \in NS$, w.l.o.g. assume that $(p', \dot{q}') \Vdash_{\mathbb{R}} \dot{x} \subseteq^*_{cl} c_{\alpha^*}$. Now we define an automorphism π of $\mathbb{P}$ which fixes $\prod_{\alpha \in \kappa \setminus \{\alpha^*\}} \mathbb{C}_\kappa$ and $\Vdash_{\mathbb{P}} \dot{c}_{\alpha^*} \cap \pi(\dot{c}_{\alpha^*}) \subseteq \mathrm{dom}(p'(\alpha^*))$, in particular $p' = \pi(p')$. Now π induces an automorphism $\tilde{\pi}$ of $\mathbb{R}$, and since all trunks of $(p', \dot{q}')$ are ground model objects, we can deduce that $p' \Vdash_{\mathbb{P}} \dot{q}'$ and $\tilde{\pi}(\dot{q}')$ are compatible in $\dot{\mathbb{Q}}$. Hence there exists a condition $(p', \dot{r}) \leq_{\mathbb{R}} (p', \dot{q}'), (p', \tilde{\pi}(\dot{q}'))$, and since $\Vdash_{\mathbb{R}} \dot{x} = \tilde{\pi}(\dot{x})$ we can deduce that $(p', \dot{r}) \Vdash_{\mathbb{R}} \dot{x} \subseteq^*_{cl} c_{\alpha^*} \wedge \dot{x} \subseteq^*_{cl} \tilde{\pi}(c_{\alpha^*})$. But this immediately leads to a contradiction. $\square$

Lemma 4.1.29. Let κ be supercompact and indestructible by $<\kappa$-directed closed forcing notions. Let $V \vDash |2^\kappa| = \kappa^+$ and define $\mathbb{P} := \prod_{\alpha<\kappa^{++}} \mathbb{C}_\kappa$. Then $V^{\mathbb{P}} \vDash \mathfrak{s}^{cl}_\kappa = \mathfrak{b}_\kappa = \kappa^+ \wedge \mathfrak{d}_\kappa = \mathfrak{r}^{cl}_\kappa = \mathfrak{c}_\kappa = \kappa^{++}$.

Proof. The lemma immediately follows from the proof of Theorem 4.1.28. $\square$

Lemma 4.1.30. Let κ be supercompact and indestructible by $<\kappa$ directed-closed forcing notions. Let $V \vDash |2^\kappa| = \kappa^+$ and define $\mathbb{P} := \prod_{\alpha<\kappa^{++}} \mathbb{S}_\kappa$, i.e. a κ^{++}-product of κ-Sacks forcing with $\leq\kappa$-support. Then $V^{\mathbb{P}} \vDash \mathfrak{b}_\kappa = \mathfrak{d}_\kappa = \kappa^+ \wedge \mathfrak{r}^{cl}_\kappa = \mathfrak{c}_\kappa = \kappa^{++}$.

Proof. Since $\mathbb{P}$ is κ^κ-bounding (see Lemma 2.4.5), we have $V^{\mathbb{P}} \vDash \mathfrak{b}_\kappa = \mathfrak{d}_\kappa = \kappa^+$. It also follows that $V^{\mathbb{P}} \vDash$ '$Cl \cap V$ is cofinal in Cl', and therefore, it is easy to see that $V^{\mathbb{P}} \vDash$ '$\forall \alpha < \kappa^{++} \colon s_\alpha$ stationarily splits $St \cap V^{\mathbb{P}_\alpha}$', where $\mathbb{P}_\alpha := \prod_{\beta<\alpha} \mathbb{S}_\kappa$. Hence $V^{\mathbb{P}} \vDash \mathfrak{r}^{cl}_\kappa = \mathfrak{c}_\kappa = \kappa^{++}$. $\square$

It seems very reasonable to conjecture that $V^{\mathbb{P}} \vDash \mathfrak{s}^{cl}_\kappa = \kappa^+$.

Question 4.1.31. Is $\mathfrak{b}_\kappa < \mathfrak{s}^{cl}_\kappa$ consistent? Is even $\mathfrak{d}_\kappa < \mathfrak{s}^{cl}_\kappa$ consistent? How does $\mathfrak{s}^{cl}_\kappa$ relate to $\mathfrak{s}_\kappa$?

5 The Corrected Iteration

In this chapter we want to give a more transparent presentation of Shelah's Corrected Iteration (see [She19]), which seems to be a very promising tool to show further consistency results in the higher Cichoń diagram. For reasons of notational simplicity we will show how to construct a Corrected Iteration for κ-Hechler forcing.

Actually, we planned to use the Corrected Iteration to iterate the higher random forcing $\mathbb{R}_\kappa$ without adding dominating reals. This would yield the consistency of $\kappa^+ = \mathfrak{b}_\kappa < \operatorname{cov}(\operatorname{id}(\mathbb{R}_\kappa)) = \kappa^{++}$. Unfortunately, there seems to be a general problem when actually applying the Corrected Iteration.
We will address the issue in the last section, where we first show how to modify the Corrected Iteration to iterate $\mathbb{R}_\kappa$ and, assuming the issue can be fixed, sketch how to prove that the Corrected Iteration of higher random forcing does not add dominating reals.

Fix κ to be at least inaccessible. Let M be some well-founded partial order along which we want to iterate. We are looking for a definition of a forcing notion $\mathbb{Q}_M$ with the following properties:

- $\mathbb{Q}_M$ is a κ^+-c.c. forcing notion which does not add short sequences, i.e. $V^{<\kappa} \cap V^{\mathbb{Q}_M} = V^{<\kappa} \cap V$.
- For $s \in M$ we define $M_{<s} := \{t \in M \colon t < s\}$. Similarly, we define $M_{\leq s}$. We require that $\mathbb{Q}_{M_{<s}} \lessdot \mathbb{Q}_M$ as well as $\mathbb{Q}_{M_{\leq s}} \lessdot \mathbb{Q}_M$ for every $s \in M$. Hence, $\mathbb{Q}_M$ is an iteration.
- There exists a sequence $(\dot\eta_s)_{s\in M}$ such that for every $s \in M$ we have $\Vdash_{\mathbb{Q}_{M_{\leq s}}} \dot\eta_s \in \kappa^\kappa \wedge \dot\eta_s$ dominates $\kappa^\kappa \cap V^{\mathbb{Q}_{M_{<s}}}$.
- $\mathbb{Q}_M$ has '$<\kappa$-support', i.e. for every $\varphi \in \mathbb{Q}_M$ the set $\{s \in M \colon \exists i, j < \kappa\ \varphi \Vdash_{\mathbb{Q}_{M_{\leq s}}} \dot\eta_s(i) \neq j\}$ has size $< \kappa$.
- Let G be a $(V, \mathbb{Q}_M)$-generic filter. We require that $V[G] = V[(\dot\eta_s^G)_{s\in M}]$. Hence, G is completely determined by $(\dot\eta_s^G)_{s\in M}$.

Any ordinary iteration of κ-Hechler forcing along a well-order satisfies those requirements. However, the next requirement is crucial:

- Let G be a $(V, \mathbb{Q}_M)$-generic filter and let $f \colon M \to M$ be a strictly increasing function such that $f \in V$. Then the sequence $(\dot\eta_{f(s)}^G)_{s\in M}$ naturally defines a filter $G' \subseteq \mathbb{Q}_M$, which is also $(V, \mathbb{Q}_M)$-generic.

In the classical case, Judah and Shelah showed in [IHJS88] that for a finite support iteration of Suslin-c.c.c. forcing notions a similar claim is true.

Roughly, the construction will go as follows:

1. For any well-founded partial order $L \supseteq M$ define $\mathbb{P}^L_M$ to be an iteration of κ-Hechler forcing along L with 'partial memory'.

2. It turns out that we can find a 'sufficiently saturated' $L^* \supseteq M$ such that for any $L \supseteq L^*$ we have $\mathbb{P}^{L^*}_M \lessdot \mathbb{P}^L_M$. Roughly this will work, because, if L^* is 'sufficiently saturated', many automorphism arguments will go through.

3. Define $\mathbb{Q}_M$ to be the complete subforcing generated by the $(\dot{\eta}_s)_{s\in M}$ in $\mathbb{P}^{L^*}_M$. Note that this definition does not depend on L^*, because different $L^\star_1$ and $L^\star_2$ satisfying (2.) can be amalgamated to become an $L^\dagger \supseteq M$, and hence $\mathbb{P}^{L^\star_1}_M \lessdot \mathbb{P}^{L^\dagger}_M$ as well as $\mathbb{P}^{L^\star_2}_M \lessdot \mathbb{P}^{L^\dagger}_M$ hold. Therefore, $\mathbb{Q}_M$ is a definition for an iteration only depending on M.

4. Let $N \subseteq M$ with $N \in V$ be arbitrary, and similarly to $\mathbb{Q}_M$ define $\mathbb{Q}_N$. It turns out that there exists an L^{**} satisfying (2.) such that $\mathbb{P}^{L^{**}}_M = \mathbb{P}^{L^{**}}_N$. [1] Hence, $\mathbb{Q}_N$ is not only the complete subforcing of $\mathbb{P}^{L^{**}}_N$ generated by the $(\dot{\eta}_s)_{s\in N}$, but also the complete subforcing generated by the $(\dot{\eta}_s)_{s\in N}$ within $\mathbb{Q}_M$.

5. Now let $f\colon M \to M$ be a strictly increasing function such that $f \in V$, and set $N := \{f(s)\colon s \in M\}$. Let $(\eta_s)_{s\in M}$ be $(V, \mathbb{Q}_M)$-generic. By (4.) it follows that $(\eta_s)_{s\in N}$ is $(V, \mathbb{Q}_N)$-generic. Since, however, M and N are isomorphic, we can deduce that $\mathbb{Q}_M$ and $\mathbb{Q}_N$ are isomorphic as well, and hence $(\eta_s)_{s\in N}$ is also $(V, \mathbb{Q}_M)$-generic.

Of course, the above is only a very rough sketch and many subtleties and details need to be checked.

5.1 Prerequisites

Fix a well-founded partial order M (in a typical case $M = \kappa^{++}$) for which we want to construct the Corrected Iteration. Fix $\lambda_1 \geq |M|$ such that $\lambda_1^\kappa = \lambda_1$, and fix $\lambda_2 \geq \beth_2(\lambda_1)$ with $\lambda_2^\kappa = \lambda_2$. Pedantically, all notations should have the parameter $(M, \lambda_1, \lambda_2)$.

Definition 5.1.1. For a well-founded partial order L and $t \in L$ define:

- $L_{<t} := \{s \in L\colon s <_L t\}$
- $L_{\leq t} := \{s \in L\colon s \leq_L t\}$
- $\mathrm{dp}_L(t) := \bigcup\{\mathrm{dp}_L(s) + 1\colon s <_L t\}$ by induction

[1] Note that $\mathbb{P}^L_M$ will heavily depend on M, so it is a priori not clear that $\mathbb{P}^L_M = \mathbb{P}^L_N$ can hold for any L.

- $\infty_L := \bigcup\{\mathrm{dp}_L(t)+1 \colon t \in L\}$
- $L_\alpha := \{t \in L \colon \mathrm{dp}_L(t) < \alpha\}$ for $\alpha \leq \infty_L$

We will now define how an iteration parameter $\mathbf{m}$ looks like:

Definition 5.1.2. An iteration parameter $\mathbf{m}$ consists of:

(a) a well-founded partial order L such that $M \subseteq L$ as partial orders.

(b) sequences $\bar{u} = \langle u_t \colon t \in L\rangle$ and $\bar{\mathcal{P}} = \langle \mathcal{P}_t \colon t \in L\rangle$ such that $u_t \subseteq L_{<t}$ and $\mathcal{P}_t \subseteq \mathfrak{P}(u_t)$.

(c) an equivalence relation E on $L \setminus M$; by t/E we denote the equivalence class of t modulo E.

with the following restrictions:

(α) If $t_1, t_2 \in L \setminus M$ are not E-equivalent, then $t_1 <_L t_2 \Leftrightarrow \exists s \in M \colon t_1 <_L s <_L t_2$.

(β) If $t \in L \setminus M$ then $u_t \subseteq t/E \cup M$.

(γ) If $t \in L \setminus M$ then $|t/E| \leq \lambda_2$.

(δ) For every $t \in L$ the set $\mathcal{P}_t$ is closed under subsets.

(ε) For every $t \in L$ if $u \in \mathcal{P}_t$ then $\exists t' \in L \setminus M \colon u \subseteq t'/E \cup M$.

(ζ) If $t \in L \setminus M$ then $|\mathcal{P}_t| \leq \lambda_2$ and, for simplicity, $\mathcal{P}_t \subseteq [u_t]^{\leq\kappa}$.

(η) Within M we have 'full memory': $L_{<t} \cap M \subseteq u_t$ and $[L_{<t} \cap M]^{\leq\kappa} \subseteq \mathcal{P}_t$ for every $t \in L$.

We shall use the following notation: $\mathbf{m} = (L^{\mathbf{m}}, \langle u^{\mathbf{m}}_t \colon t \in L^{\mathbf{m}}\rangle, \langle \mathcal{P}^{\mathbf{m}}_t \colon t \in L^{\mathbf{m}}\rangle, E^{\mathbf{m}})$.

We shall refer to $s \in M$ as 'real' coordinates and to $t \in L \setminus M$ as 'fake' coordinates. M is the skeleton of the iteration parameter. Fake coordinates from different equivalence classes can only interact via M. The supports u can only reach into one equivalence class and M.

Definition 5.1.3. We define $\mathbf{M} := \{\mathbf{m} \colon \mathbf{m}$ is an iteration parameter$\}$, $\mathbf{M}_{\leq\theta} := \{\mathbf{m} \in \mathbf{M} \colon |L^{\mathbf{m}}| \leq \theta\}$ and $\mathbf{M}_{oc} := \{\mathbf{m} \in \mathbf{M} \colon \forall t_1, t_2 \in L^{\mathbf{m}} \setminus M \; t_1 \, E^{\mathbf{m}} \, t_2\}$. Here 'oc' stands for 'one (equivalence) class'.

For $\mathbf{m} \in \mathbf{M}$ we will now define the corresponding iteration of κ-Hechler forcing $\mathbb{P}^{\mathbf{m}}$:

Definition 5.1.4. By induction on $\alpha \leq \infty_{L^{\mathbf{m}}}$ we want to define the forcing notion $\mathbb{P}^{\mathbf{m}}_\alpha$:

- Define $\mathbb{P}^{\mathbf{m}}_1$ to be the set of functions p such that $\mathrm{dom}(p) \subseteq L^{\mathbf{m}}_1$, $|\mathrm{dom}(p)| < \kappa$ and for every $t \in \mathrm{dom}(p)$ we have $p(t) = (\rho, f)$ such that $\rho \in \kappa^{<\kappa}$, $f \in \kappa^\kappa$ and $\rho \lhd f$.
- If γ is a limit, we have two cases:

- $\mathrm{cf}(\gamma) \geq \kappa$: Set $\mathbb{P}^{\mathbf{m}}_{\gamma} := \bigcup_{\alpha<\gamma} \mathbb{P}^{\mathbf{m}}_{\alpha}$.
- $\mathrm{cf}(\gamma) < \kappa$: Define $\mathbb{P}^{\mathbf{m}}_{\gamma}$ to be the set of functions p such that $\mathrm{dom}(p) \subseteq L^{\mathbf{m}}_{\gamma}$, $|\mathrm{dom}(p)| < \kappa$ and for every $\alpha < \gamma$ we have $p \restriction L^{\mathbf{m}}_{\alpha} \in \mathbb{P}^{\mathbf{m}}_{\alpha}$.

- $\alpha \to \alpha + 1$: Define $\mathbb{P}^{\mathbf{m}}_{\alpha+1}$ to be the set of functions p such that $\mathrm{dom}(p) \subseteq L^{\mathbf{m}}_{\alpha+1}$, $|\mathrm{dom}(p)| < \kappa$, $p \restriction L^{\mathbf{m}}_{\alpha} \in \mathbb{P}^{\mathbf{m}}_{\alpha}$ and for every $t \in \mathrm{dom}(p)$ with $\mathrm{dp}_{L^{\mathbf{m}}}(t) = \alpha$ we have that $p(t) = (\rho, \sup_{j<\delta} B_j((\dot{\eta}_{t'})_{t' \in u_j}))$ where:
  - $\rho \in \kappa^{<\kappa}$
  - $\delta < \kappa$ and $(B_j)_{j<\delta}$ is a sequence of κ-Borel functions [2] $B_j : (\kappa^{\kappa})^{u_j} \to \kappa^{\kappa}$ in V, where $u_j \in \mathcal{P}_t$ for every $j < \delta$.
  - $\forall j < \delta \; \forall \bar{x} \in (\kappa^{\kappa})^{u_j} : \rho \triangleleft B_j(\bar{x})$ [3]
  - $(\dot{\eta}_{t'})_{t' \in u_j}$ is a subsequence of the generic sequence $(\dot{\eta}_t)_{t \in L^{\mathbf{m}}_{\alpha}}$ added by $\mathbb{P}^{\mathbf{m}}_{\alpha}$.

  We will use the notation $p(t) = (\rho^{p(t)}, \dot{B}^{p(t)}) = (\rho^{p(t)}, \sup_{j<\delta} B^{p(t)}_j((\dot{\eta}_{t'})_{t' \in u_j}))$.

We define $q \leq_{\mathbb{P}^{\mathbf{m}}} p$, in words q is stronger than p, inductively:

- $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$
- for every $t \in \mathrm{dom}(p)$ we have:
  $\rho^{p(t)} \triangleleft \rho^{q(t)}$ and $q \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{B}^{p(t)} \leq \dot{B}^{q(t)}$

Define $\dot{\eta}_t := \bigcup \{\rho \in \kappa^{<\kappa} : \exists p \in \dot{G} \; p(t) = (\rho, \dot{B}^{p(t)})\}$. Set $\mathbb{P}^{\mathbf{m}} := \mathbb{P}^{\mathbf{m}}_{\infty_L}$.

Notice the $\sup_{j<\delta}$ which we use in the definition for the successor step. This is crucial if s is a real coordinate: This way conditions can reach into different equivalence classes, and therefore different fake coordinates interplay at real coordinates.

Definition 5.1.5. For $p \in \mathbb{P}^{\mathbf{m}}$ we define:

- the full support $\mathrm{fsupp}(p) :=$

$$\{t \in L^{\mathbf{m}} : \exists \tilde{t} \in \mathrm{dom}(p) \; p(\tilde{t}) = (\rho^{p(\tilde{t})}, \sup_{j<\delta} B^{p(\tilde{t})}_j((\dot{\eta}_{t'})_{t' \in u_j})) \wedge \exists j < \delta \; t \in u_j\}$$

- the wide support $\mathrm{wsupp}(p) := \bigcup \{t/E : t \in \mathrm{fsupp}(p) \setminus M\} \cup M$

Lemma 5.1.6. The following facts hold true:

1. $\mathbb{P}^{\mathbf{m}}$ is a $<\kappa$-closed, κ^+-c.c. forcing notion.
2. $\Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{\leq t}} \dot{\eta}_t \in \kappa^{\kappa}$.
3. $\Vdash_{\mathbb{P}^{\mathbf{m}}} V[\dot{G}] = V[(\dot{\eta}_t)_{t \in L_{\mathbf{m}}}]$.

[2] Note that by Π^1_1-absoluteness (see Fact 1.1.3) it is clear how to evaluate the image of a new real $\dot{\eta}$ under a ground model κ-Borel function B for $<\kappa$-closed forcing extensions.

[3] Note that this statement is also absolute for $<\kappa$-closed forcing extension.

4. For any initial segment (i.e. downwards closed) $L \subseteq L^{\mathbf{m}}$ we have $\mathbb{P}^{\mathbf{m}} \restriction L \lessdot \mathbb{P}^{\mathbf{m}}$.

5. For any $\mathbb{P}^{\mathbf{m}}$-name $\dot{f}$ for an element of κ^κ there exists $u \subseteq L^{\mathbf{m}}$ of size $\leq\kappa$ and a ground model κ-Borel function $B\colon (\kappa^\kappa)^u \to \kappa^\kappa$ such that $\Vdash_{\mathbb{P}^{\mathbf{m}}} \dot{f} = B((\dot{\eta}_t)_{t\in u})$.

Proof. ad 1.) It should be obvious that $\mathbb{P}^{\mathbf{m}}$ is $<\kappa$-closed. For the κ^+-c.c. use a Δ-system argument and note that $\mathbb{H}_\kappa$ is κ-linked and $|\mathrm{dom}(p)| < \kappa$ for every $p \in \mathbb{P}^{\mathbf{m}}$.
ad 2., 3. and 4.) Trivial.
ad 5.) There are maximal antichains $(A_i)_{i<\kappa}$ which are all w.l.o.g. of size κ, such that for every $i < \kappa$ each $p \in A_i$ decides the value of $\dot{f}(i)$. Enumerate each A_i as $(p_{i,j})_{j<\kappa}$ and define $k_{i,j} \in \kappa$ such that $p_{i,j} \Vdash_{\mathbb{P}^{\mathbf{m}}} \dot{f}(i) = k_{i,j}$. Set $u := \bigcup_{i<\kappa} \bigcup_{j<\kappa} \mathrm{fsupp}(p_{i,j})$. Obviously, u is of size $\leq\kappa$. For $\bar{x} \in (\kappa^\kappa)^u$ and $i < \kappa$ define $B(\bar{x})(i) := k_{i,j^*}$ where $j^* := \min\{j < \kappa\colon p_{i,j} \in \mathbf{G}_{\bar{x}}\}$ if the minimum is well defined, else set $j^* := 0$. Here $\mathbf{G}_{\bar{x}}$ is the following set:

$$\{p \in \mathbb{P}^{\mathbf{m}} \cap \bigcup_{i<\kappa} A_i \colon \forall t \in \mathrm{dom}(p)\ (\, \rho^{p(t)} \lessdot x_t \wedge B^{p(t)}(\bar{x}) \leq x_t\,)\}.$$

Since $\bigcup_{i<\kappa} A_i$ is of size κ, B is obviously a κ-Borel function and for every $i, j < \kappa$ we have that $p_{i,j} \Vdash_{\mathbb{P}^{\mathbf{m}}} B((\dot{\eta}_t)_{t\in u})(i) = k_{i,j}$, hence $\Vdash_{\mathbb{P}^{\mathbf{m}}} \dot{f} = B((\dot{\eta}_t)_{t\in u})$. ◻

Lemma 5.1.7. Let $\mathbf{m} \in \mathbf{M}$ and define $\dot{Q}^{\mathbf{m}}_t$ to be the $(\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t})$-name for the quotient forcing $(\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{\leq t}) \,/\, (\dot{G} \restriction L^{\mathbf{m}}_{<t})$ for every $t \in L^{\mathbf{m}}$. Then the following holds true:

1. $\Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{Q}^{\mathbf{m}}_t \subseteq \dot{\mathbb{H}}_\kappa$ as partial orders.

2. Let $p, q \in \mathbb{P}^{\mathbf{m}}$ and $t \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. Then for every $r \in \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}$ we have $r \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} p(t)$ and $q(t)$ are incompatible in $\dot{Q}^{\mathbf{m}}_t$ iff one of the following conditions is satisfied:
   - $\rho^{p(t)}$ and $\rho^{q(t)}$ are incompatible
   - $\rho^{p(t)} \lessdot \rho^{q(t)}$ and $r \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{B}^{p(t)} \restriction \mathrm{dom}(\rho^{q(t)}) \nleq \rho^{q(t)}$
   - $\rho^{q(t)} \lessdot \rho^{p(t)}$ and $r \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{B}^{q(t)} \restriction \mathrm{dom}(\rho^{p(t)}) \nleq \rho^{p(t)}$

   In particular we have:
   $\Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} p(t)$ and $q(t)$ are compatible in $\dot{Q}^{\mathbf{m}}_t \Leftrightarrow p(t)$ and $q(t)$ are compatible in $\dot{\mathbb{H}}_\kappa$

3. Let $p, q \in \mathbb{P}^{\mathbf{m}}$ such that $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ but $q \nleq_{\mathbb{P}^{\mathbf{m}}} p$. Then there exists $t \in \mathrm{dom}(p)$ and $r \in \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{\leq t}$ such that $r \restriction L^{\mathbf{m}}_{\leq t} \leq_{\mathbb{P}^{\mathbf{m}}} q \restriction L^{\mathbf{m}}_{\leq t}$ and $r \restriction L^{\mathbf{m}}_{<t} \leq_{\mathbb{P}^{\mathbf{m}}} p \restriction L^{\mathbf{m}}_{<t}$, but $r \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} r(t)$ and $p(t)$ are incompatible in $\dot{Q}^{\mathbf{m}}_t$.

Proof. ad 1.) Trivial
ad 2.) Let $p, q \in \mathbb{P}^{\mathbf{m}}$, $t \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ and $r \in \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}$ be arbitrary. Assume w.l.o.g. that $\rho^{p(t)} \lessdot \rho^{q(t)}$ and $r \nVdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{B}^{p(t)} \restriction \mathrm{dom}(\rho^{q(t)}) \nleq \rho^{q(t)}$. Hence, there exist $r' \in \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}$ with $r' \leq_{\mathbb{P}^{\mathbf{m}}} r$ such that $r' \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{B}^{p(t)} \restriction \mathrm{dom}(\rho^{q(t)}) \leq \rho^{q(t)}$. Define a $(\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t})$-name $\dot{f}$ such that $\Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{f} \restriction \mathrm{dom}(\rho^{q(t)}) = \rho^{q(t)} \wedge \forall i \in \kappa \setminus \mathrm{dom}(\rho^{q(t)})\colon \dot{f}(i) =$

$\max\{\dot{B}^{p(t)}(i), \dot{B}^{q(t)}(i)\}$. If we set $\bar{r} := (\rho^{q(t)}, \dot{f})$, we can deduce that $\Vdash_{\mathbb{P}^{\mathbf{m}}\restriction L^{\mathbf{m}}_{<t}} \bar{r} \in \dot{Q}^{\mathbf{m}}_t$ and $r' \Vdash_{\mathbb{P}^{\mathbf{m}}\restriction L^{\mathbf{m}}_{<t}} \bar{r}$ is a common lower bound of $p(t)$ and $q(t)$ in $\dot{Q}^{\mathbf{m}}_t$.
ad 3.) It easily follows that there is $t \in \mathrm{dom}(p)$ such that $q \restriction L^{\mathbf{m}}_{<t} \leq_{\mathbb{P}^{\mathbf{m}}} p \restriction L^{\mathbf{m}}_{<t}$ but $q \restriction L^{\mathbf{m}}_{<t} \nVdash_{\mathbb{P}^{\mathbf{m}}\restriction L^{\mathbf{m}}_{<t}} q(t) \leq_{Q^{\mathbf{m}}_t} p(t)$. Hence, there exists $q' \in \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}$ such that $q' \leq_{\mathbb{P}^{\mathbf{m}}} q \restriction L^{\mathbf{m}}_{<t}$ and $q' \Vdash_{\mathbb{P}^{\mathbf{m}}\restriction L^{\mathbf{m}}_{<t}} q(t) \not\leq_{Q^{\mathbf{m}}_t} p(t)$. W.l.o.g assume that $\rho^{p(t)} \lhd \rho^{q(t)}$ and there exists $i < \kappa$ such that q' decides $\dot{B}^{p(t)} \restriction (i+1)$ and $\dot{B}^{q(t)} \restriction (i+1)$ and $q' \Vdash_{\mathbb{P}^{\mathbf{m}}\restriction L^{\mathbf{m}}_{<t}} \dot{B}^{p(t)}(i) \not\leq \dot{B}^{q(t)}(i)$. Now we can deduce that there exists $r \in \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{\leq t}$ such that $r \restriction L^{\mathbf{m}}_{<t} = q'$, $r \leq_{\mathbb{P}^{\mathbf{m}}} q \restriction L^{\mathbf{m}}_{\leq t}$ and $r \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}}\restriction L^{\mathbf{m}}_{<t}} r(t)$ and $p(t)$ are incompatible in $\dot{Q}^{\mathbf{m}}_t$. □

We will now have a look at how we can compare different iteration parameters:

Definition 5.1.8. Let $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$. Define $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ iff:

- $L^{\mathbf{m}_1} \subseteq L^{\mathbf{m}_2}$ as partial orders
- If $t \in L^{\mathbf{m}_1} \setminus M$ then $u^{\mathbf{m}_1}_t = u^{\mathbf{m}_2}_t$ and $\mathcal{P}^{\mathbf{m}_1}_t = \mathcal{P}^{\mathbf{m}_2}_t$
- If $s \in M$ then $u^{\mathbf{m}_1}_s = u^{\mathbf{m}_2}_s \cap L^{\mathbf{m}_1}$ and $\mathcal{P}^{\mathbf{m}_1}_s = \mathcal{P}^{\mathbf{m}_2}_s \cap [L^{\mathbf{m}_1}]^{\leq\kappa}$ [4]
- $E^{\mathbf{m}_1} = E^{\mathbf{m}_2} \restriction L^{\mathbf{m}_1} \times L^{\mathbf{m}_1}$

It can easily be seen that $(\mathbf{M}, \leq_{\mathbf{M}})$ is a partial order.

The next lemma shows some properties of $(\mathbf{M}, \leq_{\mathbf{M}})$, in particular it has amalgamation and is <Ord-closed.

Lemma 5.1.9. The following holds true:

1. Let $\mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_1$ and $\mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_2$ such that $L^{\mathbf{m}_1} \cap L^{\mathbf{m}_2} = L^{\mathbf{m}_0}$. Then there exists $\mathbf{m}_3 \in \mathbf{M}$ with $L^{\mathbf{m}_3} = L^{\mathbf{m}_1} \cup L^{\mathbf{m}_2}$ such that $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_3$ and $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}_3$.
2. Let $(\mathbf{m}_\alpha)_{\alpha<\gamma}$ be an $\leq_{\mathbf{M}}$-increasing sequence. Then there exists $\mathbf{m} \in \mathbf{M}$ which is an upper bound.

Proof. ad 1.)

- Define $L^{\mathbf{m}_3} := L^{\mathbf{m}_1} \cup L^{\mathbf{m}_2}$. In particular $t_1 \leq_{L^{\mathbf{m}_3}} t_2$ iff $t_1 \leq_{L^{\mathbf{m}_1}} t_2$ or $t_1 \leq_{L^{\mathbf{m}_2}} t_2$ or there exists $s \in M$ such that either $t_1 \leq_{L^{\mathbf{m}_1}} s \leq_{L^{\mathbf{m}_2}} t_2$ or $t_1 \leq_{L^{\mathbf{m}_2}} s \leq_{L^{\mathbf{m}_1}} t_2$ holds.
- If $t \in L^{\mathbf{m}_0} \setminus M$ set $u^{\mathbf{m}_3}_t := u^{\mathbf{m}_0}_t$ and $\mathcal{P}^{\mathbf{m}_3}_t := \mathcal{P}^{\mathbf{m}_0}_t$.
- If $t \in L^{\mathbf{m}_i} \setminus L^{\mathbf{m}_0}$ set $u^{\mathbf{m}_3}_t := u^{\mathbf{m}_i}_t$ and $\mathcal{P}^{\mathbf{m}_3}_t := \mathcal{P}^{\mathbf{m}_i}_t$.
- If $s \in M$ set $u^{\mathbf{m}_3}_s := u^{\mathbf{m}_1}_s \cup u^{\mathbf{m}_2}_s$ and $\mathcal{P}^{\mathbf{m}_3}_s := \mathcal{P}^{\mathbf{m}_1}_s \cup \mathcal{P}^{\mathbf{m}_2}_s$.
- Define $E^{\mathbf{m}_3} := E^{\mathbf{m}_1} \cup E^{\mathbf{m}_2}$.

[4] In particular, it follows that for every $u \in \mathcal{P}^{\mathbf{m}_2}_s \setminus \mathcal{P}^{\mathbf{m}_1}_s$ there exists $t \in L^{\mathbf{m}_2} \setminus L^{\mathbf{m}_1}$ such that $u \subseteq t/E^{\mathbf{m}_2} \cup M$.

It can easily be seen that $\mathbf{m}_3 \in \mathbf{M}$ and $\mathbf{m}_i \leq_{\mathbf{M}} \mathbf{m}_3$.

ad 2.) We define $\mathbf{m}$ similarly to above. In particular, we set $L^{\mathbf{m}} := \bigcup_{\alpha<\gamma} L^{\mathbf{m}_\alpha}$ as partial order. We must show that $L^{\mathbf{m}}$ is still well-founded. Let $A \subseteq L^{\mathbf{m}}$ be non-empty. We distinguish two cases:

- If $\forall t \in A \; \exists s \in A \cap M : s \leq_{L^{\mathbf{m}}} t$ then any minimal $s \in A \cap M$ is also minimal in A.
- If $\exists t \in A \; \forall s \in A \cap M : s \not\leq_{L^{\mathbf{m}}} t$ choose such a t and $\alpha < \gamma$ with $t \in L^{\mathbf{m}_\alpha}$. It follows by our construction that any minimal $t' \in A \cap t/E^{\mathbf{m}_\alpha}$ must also be minimal in A.

Again, all the other conditions are obviously satisfied. Hence, $\mathbf{m}$ is an upper bound. □

Note that if $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ then $\mathbb{P}^{\mathbf{m}_1} \subseteq \mathbb{P}^{\mathbf{m}_2}$ as sets, but, in general, not as partial orders. Furthermore, for $t \in L^{\mathbf{m}_1}$ we have that $\mathrm{dp}_{L^{\mathbf{m}_1}}(t) \leq \mathrm{dp}_{L^{\mathbf{m}_2}}(t)$, but, in general, not equal.

Now we define how to restrict iteration parameters:

Definition 5.1.10. Let $\mathbf{m} \in \mathbf{M}$ and let $L \subseteq L^{\mathbf{m}}$ such that $M \subseteq L$. We define $\mathbf{m} \restriction L := (L, \langle u_t^{\mathbf{m}} \cap L : t \in L \rangle, \langle \mathcal{P}_t^{\mathbf{m}} \cap [L]^{\leq\kappa} : t \in L \rangle, E^{\mathbf{m}} \cap L \times L)$.

The following lemma is straightforward:

Lemma 5.1.11. Let $\mathbf{m} \in \mathbf{M}$ and let $L \subseteq L^{\mathbf{m}}$ such that $M \subseteq L$. Then also $\mathbf{m} \restriction L \in \mathbf{M}$. Furthermore, if $\forall t \in L \setminus M : t/E^{\mathbf{m}} \subseteq L$ then even $\mathbf{m} \restriction L \leq_{\mathbf{M}} \mathbf{m}$.

Now we define a very important subclass of $\mathbf{M}$:

Definition 5.1.12. Set $\mathbf{M}_{ec} := \{\mathbf{m} \in \mathbf{M} : \forall \, \mathbf{m}_1, \mathbf{m}_2 \geq_{\mathbf{M}} \mathbf{m} \; \big(\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2 \Rightarrow \mathbb{P}^{\mathbf{m}_1} \lessdot \mathbb{P}^{\mathbf{m}_2}\big)\}$. Here 'ec' stands for existentially closed.

The point of the above definition is that the procedure of adding more and more fake coordinates stabilizes at $\mathbf{m} \in \mathbf{M}_{ec}$.

We are ready to state the first crucial theorem:

Theorem 5.1.13. For any $\mathbf{m} \in \mathbf{M}$ there exists $\mathbf{m}^* \in \mathbf{M}_{ec}$ such that $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}^*$. Hence, $\mathbf{M}_{ec}$ is not only upwards closed but also cofinal in $(\mathbf{M}, \leq_{\mathbf{M}})$.

In order to prove the above theorem, we will define an equivalence relation on $\mathbf{M}$:

Definition 5.1.14. We say that $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$ are equivalent iff there exists a function f such that:

- $f : L^{\mathbf{m}_1} \to L^{\mathbf{m}_2}$ is bijective
- $\forall s \in M : f(s) = s$
- $\forall t_1, t_2 \in L^{\mathbf{m}_1} : t_1 \leq_{L^{\mathbf{m}_1}} t_2 \Leftrightarrow f(t_1) \leq_{L^{\mathbf{m}_2}} f(t_2)$

- $\forall t_1, t_2 \in L^{\mathbf{m}_1} \setminus M \colon t_1 \, E^{\mathbf{m}_1} \, t_2 \Leftrightarrow f(t_1) \, E^{\mathbf{m}_2} \, f(t_2)$
- $\forall t \in L^{\mathbf{m}_1} \colon f[u_t^{\mathbf{m}_1}] = u_{f(t)}^{\mathbf{m}_2}$
- $\forall t \in L^{\mathbf{m}_1} \; \forall u \in [u_t^{\mathbf{m}_1}]^{\leq\kappa} \colon u \in \mathcal{P}_t^{\mathbf{m}_1} \Leftrightarrow f[u] \in \mathcal{P}_{f(t)}^{\mathbf{m}_2}$

We will call such an f an isomorphism. We will denote the equivalence by $\mathbf{m}_1 \approx_{\mathbf{M}} \mathbf{m}_2$.

Lemma 5.1.15. Let $\mathbf{m}_1, \mathbf{m}_2 \in M$ be iteration parameters and let $f \colon L^{\mathbf{m}_1} \to L^{\mathbf{m}_2}$ be an isomorphism. Then f canonically induces an isomorphism $\hat{f} \colon \mathbb{P}^{\mathbf{m}_1} \to \mathbb{P}^{\mathbf{m}_2}$.

Proof. Canonically define $\hat{f} \colon \mathbb{P}^{\mathbf{m}_1} \to \mathbb{P}^{\mathbf{m}_2}$, i.e. for $p \in \mathbb{P}^{\mathbf{m}_1}$ and $t \in \mathrm{dom}(p)$ define $\hat{f}(p)\,(f(t)) := (\rho^{p(t)}, \sup_{j<\delta} B_j^{p(t)}((\dot{\eta}_{t'})_{t' \in f[u]}))$. [5] Obviously $\hat{f} \colon \mathbb{P}^{\mathbf{m}_1} \to \mathbb{P}^{\mathbf{m}_2}$ is bijective. One can easily show by induction on $\mathrm{dp}_{L^{\mathbf{m}_1}}$ that $\forall p, q \in \mathbb{P}^{\mathbf{m}_1} \colon q \leq_{\mathbb{P}^{\mathbf{m}_1}} p \Leftrightarrow \hat{f}(q) \leq_{\mathbb{P}^{\mathbf{m}_2}} \hat{f}(p)$. $\square$

The next definition will be crucial:

Definition 5.1.16. Let $\mathbf{m} \in \mathbf{M}$ be an iteration parameter. We call $\mathbf{m}$ wide iff for every $\mathbf{m}' \in \mathbf{M}_{oc}$ there exist $(t_i)_{i<\lambda_2} \subseteq L^{\mathbf{m}} \setminus M$ such that $\forall i, j < \lambda_2 \colon i \neq j \Rightarrow \neg\, t_i \, E^{\mathbf{m}} \, t_j$ and $\mathbf{m} \restriction (t_i/E^{\mathbf{m}} \cup M) \approx_{\mathbf{M}} \mathbf{m}'$ for every $i < \lambda_2$.

The next lemma combined with Lemma 5.1.15 shows that $\mathbb{P}^{\mathbf{m}}$ has many automorphisms if $\mathbf{m}$ is wide:

Lemma 5.1.17. Let $\mathbf{m} \in \mathbf{M}$ be wide. Let $(t_i^1)_{i<i^*}$, $(t_i^2)_{i<i^*}$ and $(f_i)_{i<i^*}$ with $i^* < \lambda_2$ be such that:

- $\forall k \in \{1,2\} \; \forall i < i^* \colon t_i^k \in L^{\mathbf{m}} \setminus M$
- $\forall k \in \{1,2\} \; \forall i, j < i^* \colon i \neq j \Rightarrow \neg\, t_i^k \, E^{\mathbf{m}} \, t_j^k$
- f_i witnesses that $\mathbf{m} \restriction (t_i^1/E^{\mathbf{m}} \cup M) \approx_{\mathbf{M}} \mathbf{m} \restriction (t_i^2/E^{\mathbf{m}} \cup M)$ for every $i < i^*$

Then there exists an isomorphism $f \colon L^{\mathbf{m}} \to L^{\mathbf{m}}$ extending every f_i.

Proof. First we check that $f' := \bigcup_{i<i^*} f_i$ is a partial isomorphism, i.e. f' witnesses that $\mathbf{m} \restriction (\bigcup_{i<i^*} t_i^1/E^{\mathbf{m}} \cup M) \approx_{\mathbf{M}} \mathbf{m} \restriction (\bigcup_{i<i^*} t_i^2/E^{\mathbf{m}} \cup M)$. This holds because different equivalence classes only interact via M. Next we extend f' to a partial isomorphism f'' such that $\mathrm{dom}(f'') = \mathrm{ran}(f'')$. We can do this inductively using that $\mathbf{m}$ is wide. Now extend f'' to a total isomorphism $f \colon L^{\mathbf{m}} \to L^{\mathbf{m}}$ by defining f to be the identity on $L^{\mathbf{m}} \setminus \mathrm{dom}(f'')$. $\square$

We are now ready to prove Theorem 5.1.13:

[5] Here we set $B((x_t)_{t \in f[u]}) := B((x_{f(t)})_{t \in u})$ for a κ-Borel function $B \colon (\kappa^\kappa)^u \to \kappa^\kappa$.

Proof of Theorem 5.1.13. Let $\mathbf{m} \in \mathbf{M}$ be arbitrary. For every equivalence class $[\mathbf{m}'] \in \mathbf{M}_{oc}/_{\approx_\mathbf{M}}$ we want to add λ_2 many disjoint, $\approx_\mathbf{M}$-equivalent copies of $\mathbf{m}'$ to $\mathbf{m}$. Since $\mathbf{M}_{oc}/_{\approx_\mathbf{M}}$ contains only 2^{λ_2} many equivalence classes, this can be done inductively by Lemma 5.1.9. Call the resulting iteration parameter $\mathbf{m}^*$. Obviously, $\mathbf{m}^*$ is wide.

We must show that $\mathbf{m}^* \in \mathbf{M}_{ec}$. Let $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$ such that $\mathbf{m}^* \leq_\mathbf{M} \mathbf{m}_1 \leq_\mathbf{M} \mathbf{m}_2$. Obviously, $\mathbb{P}^{\mathbf{m}_1} \subseteq \mathbb{P}^{\mathbf{m}_2}$ as sets. We must show that $\leq_{\mathbb{P}^{\mathbf{m}_2}} \restriction \mathbb{P}^{\mathbf{m}_1} \times \mathbb{P}^{\mathbf{m}_1} = \leq_{\mathbb{P}^{\mathbf{m}_1}}$ and furthermore $\mathbb{P}^{\mathbf{m}_1} \lessdot \mathbb{P}^{\mathbf{m}_2}$. We will show by induction on $\alpha \leq \infty_{\mathbf{m}_1}$ that $\mathbb{P}^{\mathbf{m}_1}_\alpha \lessdot \mathbb{P}^{\mathbf{m}_2}$:

- $\alpha = 1$: This case is easy, as $\mathbb{P}^{\mathbf{m}_1}_1$ is a side by side product of κ-Hechler forcings, all of which also appear in $\mathbb{P}^{\mathbf{m}_2}_1$. Hence $\mathbb{P}^{\mathbf{m}_1}_1 \lessdot \mathbb{P}^{\mathbf{m}_2}_1 \lessdot \mathbb{P}^{\mathbf{m}_2}$.

- $\alpha \to \alpha+1$: First we show that $\leq_{\mathbb{P}^{\mathbf{m}_2}_{\alpha+1}} \restriction \mathbb{P}^{\mathbf{m}_1}_{\alpha+1} \times \mathbb{P}^{\mathbf{m}_1}_{\alpha+1} = \leq_{\mathbb{P}^{\mathbf{m}_1}_{\alpha+1}}$. Assume that $\mathbb{P}^{\mathbf{m}_1}_\alpha \lessdot \mathbb{P}^{\mathbf{m}_2}$. Let $p, q \in \mathbb{P}^{\mathbf{m}_1}_{\alpha+1}$ such that $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$. We have:
  - $q \leq_{\mathbb{P}^{\mathbf{m}_1}} p$ iff
  - $q \restriction L^{\mathbf{m}_1}_\alpha \leq_{\mathbb{P}^{\mathbf{m}_1}_\alpha} p \restriction L^{\mathbf{m}_1}_\alpha$ and for every $t \in \operatorname{dom}(p)$ with $\operatorname{dp}_{L^{\mathbf{m}_1}}(t) = \alpha$ we have that $q \restriction L^{\mathbf{m}_1}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{m}_1}_{<t}} q(t) \leq_\mathbb{H} p(t)$ iff
  - $q \restriction L^{\mathbf{m}_2}_\alpha \leq_{\mathbb{P}^{\mathbf{m}_2}_\alpha} p \restriction L^{\mathbf{m}_2}_\alpha$ and for every $t \in \operatorname{dom}(p)$ with $\operatorname{dp}_{L^{\mathbf{m}_1}}(t) = \alpha$ we have that $q \restriction L^{\mathbf{m}_2}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}_2} \restriction L^{\mathbf{m}_2}_{<t}} q(t) \leq_\mathbb{H} p(t)$ iff
  - $q \leq_{\mathbb{P}^{\mathbf{m}_2}} p$

  This holds because $\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{m}_1}_{<t} \lessdot \mathbb{P}^{\mathbf{m}_2} \restriction L^{\mathbf{m}_2}_{<t}$, the statement '$q(t) \leq_\mathbb{H} p(t)$' is arithmetical and $B(\bar{x}) = y$ is absolute between $V^{\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{m}_1}_{<t}}$ and $V^{\mathbb{P}^{\mathbf{m}_2} \restriction L^{\mathbf{m}_2}_{<t}}$. Hence $q \leq_{\mathbb{P}^{\mathbf{m}_1}} p$ iff $q \leq_{\mathbb{P}^{\mathbf{m}_2}} p$.

  Next we will show that if $p, q \in \mathbb{P}^{\mathbf{m}_1}_{\alpha+1}$ are incompatible in $\mathbb{P}^{\mathbf{m}_1}$, then they are also incompatible in $\mathbb{P}^{\mathbf{m}_2}$: Assume that $r \leq_{\mathbb{P}^{\mathbf{m}_2}} p$ and $r \leq_{\mathbb{P}^{\mathbf{m}_2}} q$ with $r \in \mathbb{P}^{\mathbf{m}_2}$. Furthermore, assume for every $t \in \operatorname{dom}(r)$ there exists $t' \in \operatorname{dom}(p) \cup \operatorname{dom}(q)$ such that $t \leq_{L^{\mathbf{m}_2}} t'$. We can assume this, because for any initial $L' \subseteq L^{\mathbf{m}_2}$ we have $\mathbb{P}^{\mathbf{m}_2} \restriction L' \lessdot \mathbb{P}^{\mathbf{m}_2}$ by Lemma 5.1.6.
  Enumerate $\operatorname{wsupp}(r) \setminus (\operatorname{wsupp}(p) \cup \operatorname{wsupp}(q))$ modulo $E^{\mathbf{m}_2}$ as $(t_i)_{i<\delta}$ for some $\delta < \kappa$, and for every $i < \delta$ find $t'_i \in L^{\mathbf{m}^*} \setminus (\operatorname{wsupp}(p) \cup \operatorname{wsupp}(q))$ (in particular $t'_i \in L^{\mathbf{m}_1}$) such that $\forall i, j < \delta \colon i \neq j \Rightarrow \neg\, t_i \, E^{\mathbf{m}^*} \, t_j$ and $\mathbf{m}_2 \restriction (t_i/E^{\mathbf{m}_2} \cup M) \approx_\mathbf{M} \mathbf{m}^* \restriction (t'_i/E^{\mathbf{m}^*} \cup M)$. This is possible, since $\mathbf{m}^*$ is wide.
  Define an isomorphism $f \colon L^{\mathbf{m}_2} \to L^{\mathbf{m}_2}$ which is the identity on $\operatorname{wsupp}(p) \cup \operatorname{wsupp}(q)$ and maps $t_i/E^{\mathbf{m}_2}$ onto $t'_i/E^{\mathbf{m}_2}$ for every $i < \delta$. This can be done using Lemma 5.1.17. By Lemma 5.1.15 the isomorphism f induces an automorphism $\hat{f}$ of $\mathbb{P}^{\mathbf{m}_2}$, and we can deduce that $\hat{f}(r) \leq_{\mathbb{P}^{\mathbf{m}_2}} p$ and $\hat{f}(r) \leq_{\mathbb{P}^{\mathbf{m}_2}} q$. Furthermore, we have $\hat{f}(r) \in \mathbb{P}^{\mathbf{m}_1}$. Since $\operatorname{dom}(p) \cup \operatorname{dom}(q)$ is cofinal in $\operatorname{dom}(\hat{f}(r))$, we can deduce that $\hat{f}(r) \in \mathbb{P}^{\mathbf{m}_1}_{\alpha+1}$. As we already know that $\leq_{\mathbb{P}^{\mathbf{m}_2}_{\alpha+1}} \restriction \mathbb{P}^{\mathbf{m}_1}_{\alpha+1} \times \mathbb{P}^{\mathbf{m}_1}_{\alpha+1} = \leq_{\mathbb{P}^{\mathbf{m}_1}_{\alpha+1}}$, it follows that also $\hat{f}(r) \leq_{\mathbb{P}^{\mathbf{m}_1}} p$ and $\hat{f}(r) \leq_{\mathbb{P}^{\mathbf{m}_1}} q$. Hence, p and q are also compatible in $\mathbb{P}^{\mathbf{m}_1}$.

  Finally we show that $\mathbb{P}^{\mathbf{m}_1}_{\alpha+1} \lessdot \mathbb{P}^{\mathbf{m}_2}$. Let $A \subseteq \mathbb{P}^{\mathbf{m}_1}_{\alpha+1}$ be a maximal antichain. It follows that $A \subseteq \mathbb{P}^{\mathbf{m}_2}$ is also an antichain. Towards a contradiction assume that A is not maximal in $\mathbb{P}^{\mathbf{m}_2}$ and let $q \in \mathbb{P}^{\mathbf{m}_2}$ be incompatible with every $p \in A$. Again, we can assume that for every $t \in \text{dom}(q)$ there exists $t' \in \bigcup_{p \in A} \text{dom}(p)$ such that $t \leq_{L^{\mathbf{m}_2}} t'$. Similar to above, define an isomorphism $f\colon L^{\mathbf{m}_2} \to L^{\mathbf{m}_2}$ fixing $\bigcup_{p \in A} \text{wsupp}(p)$ pointwise and mapping $\text{wsupp}(q) \setminus \bigcup_{p \in A} \text{wsupp}(p)$ into $L^{\mathbf{m}^*}$. Again by Lemma 5.1.15 it follows that f induces an automorphism $\hat{f}$ of $\mathbb{P}^{\mathbf{m}_2}$. We can deduce that $\hat{f}(q) \in \mathbb{P}^{\mathbf{m}_1}_{\alpha+1}$ and hence is compatible with some $p' \in A$. Let $r \in \mathbb{P}^{\mathbf{m}_1}_{\alpha+1}$ be a common lower bound. But this immediately leads to a contradiction, since $\hat{f}^{-1}(r)$ would be a lower bound of p' and q in $\mathbb{P}^{\mathbf{m}_2}$.

- γ is a limit ordinal: Assume inductively that for every $\alpha < \gamma$ we have $\mathbb{P}^{\mathbf{m}_1}_\alpha \lessdot \mathbb{P}^{\mathbf{m}_2}$. For $p, q \in \mathbb{P}^{\mathbf{m}_1}_\gamma$ we have:
  - $q \leq_{\mathbb{P}^{\mathbf{m}_1}} p$ iff
  - for every $\alpha < \gamma$ we have $q \restriction L^{\mathbf{m}_1}_\alpha \leq_{\mathbb{P}^{\mathbf{m}_1}} p \restriction L^{\mathbf{m}_1}_\alpha$ iff
  - for every $\alpha < \gamma$ we have $q \restriction L^{\mathbf{m}_2}_\alpha \leq_{\mathbb{P}^{\mathbf{m}_2}} p \restriction L^{\mathbf{m}_2}_\alpha$ iff
  - $q \leq_{\mathbb{P}^{\mathbf{m}_2}} p$

  Similar to the successor step prove that for $p, q \in \mathbb{P}^{\mathbf{m}_1}_\gamma$ we have p and q are compatible in $\mathbb{P}^{\mathbf{m}_2}$ iff they are compatible in $\mathbb{P}^{\mathbf{m}_1}$.

  Similar to the successor step prove that if $A \subseteq \mathbb{P}^{\mathbf{m}_1}$ is a maximal antichain, then A is also maximal in $\mathbb{P}^{\mathbf{m}_2}$.

Hence $\mathbb{P}^{\mathbf{m}_1} \lessdot \mathbb{P}^{\mathbf{m}_2}$ which finishes the proof. □

In particular, $\mathbf{M}_{ec}$ is non-empty.

5.2 The Corrected Iteration

In this section we want to properly define the Corrected Iteration and show some of its basic properties.

Definition 5.2.1. Let Var be a set of variables. We define $\mathcal{L}_{\kappa^+}(\text{Var})$ (the κ^+-propositional logic) inductively:

- $\text{Var} \subseteq \mathcal{L}_{\kappa^+}(\text{Var})$
- If $\varphi \in \mathcal{L}_{\kappa^+}(\text{Var})$ then $\neg\varphi \in \mathcal{L}_{\kappa^+}(\text{Var})$
- If $\alpha < \kappa^+$ and $\{\varphi_i \colon i < \alpha\} \subseteq \mathcal{L}_{\kappa^+}(\text{Var})$ then $\bigwedge_{i<\alpha} \varphi_i \in \mathcal{L}_{\kappa^+}(\text{Var})$

For an assignment $b\colon \text{Var} \to 2$ one inductively defines $\varphi[b] \in \{0, 1\}$ for $\varphi \in \mathcal{L}_{\kappa^+}(\text{Var})$ in the natural way.

The next lemma follows immediately:

Lemma 5.2.2. For every $\varphi \in \mathcal{L}_{\kappa^+}(\mathrm{Var})$ the mapping $B_\varphi\colon 2^{\mathrm{Var}} \ni b \mapsto \varphi[b] \in \{0,1\}$ is κ-Borel. [6] In particular, there exists $u_\varphi \subseteq \mathrm{Var}$ of size $\leq \kappa$ such that for every $b_1, b_2 \in 2^{\mathrm{Var}}$ if $b_1 \restriction u_\varphi = b_2 \restriction u_\varphi$ then $B_\varphi(b_1) = B_\varphi(b_2)$.

Definition 5.2.3. Let $\mathbf{m} \in \mathbf{M}$. Let $\mathrm{Var} := \{p_{t,i,j}\colon t \in L^{\mathbf{m}} \wedge i,j < \kappa\}$. For $p \in \mathbb{P}^{\mathbf{m}}$ and $\varphi \in \mathcal{L}_{\kappa^+}(\mathrm{Var})$ we define $p \Vdash_{\mathbb{P}^{\mathbf{m}}} \varphi$ is true iff $p \Vdash_{\mathbb{P}^{\mathbf{m}}} \varphi[b_{\dot G}] = 1$, where $b_{\dot G}$ is a $\mathbb{P}^{\mathbf{m}}$-name for an element of 2^{Var} such that for every $t \in L^{\mathbf{m}}$ and every $i,j < \kappa$ we have $\Vdash_{\mathbb{P}^{\mathbf{m}}} b_{\dot G}(p_{t,i,j}) = 1 \Leftrightarrow \dot\eta_t(i) = j$. Similarly we define $p \Vdash_{\mathbb{P}^{\mathbf{m}}} \varphi$ is false.
Let $L \subseteq L^{\mathbf{m}}$ be arbitrary. Set $\mathrm{Var} \restriction L := \{p_{t,i,j} \in \mathrm{Var}\colon t \in L\}$. We define $\mathbb{P}^{\mathbf{m}}[L] := \{\varphi \in \mathcal{L}_{\kappa^+}(\mathrm{Var} \restriction L)\colon \exists p \in \mathbb{P}^{\mathbf{m}}\ p \Vdash_{\mathbb{P}^{\mathbf{m}}} \varphi \text{ is true}\}$ and set $\psi \leq_{\mathbb{P}^{\mathbf{m}}[L]} \varphi$ iff $\Vdash_{\mathbb{P}^{\mathbf{m}}} \neg(\psi \wedge \neg\varphi)$ is true.[7]

The following facts are obvious:

Lemma 5.2.4. Let $\mathbf{m} \in \mathbf{M}$ and let $L \subseteq L^{\mathbf{m}}$ be arbitrary. The following is true:

- $\mathbb{P}^{\mathbf{m}}[L] \lessdot \mathbb{B}(\mathbb{P}^{\mathbf{m}})$, where $\mathbb{B}(\mathcal{P})$ denotes the Boolean completion of a forcing notion $\mathcal{P}$.
- $\mathbb{P}^{\mathbf{m}}[L^{\mathbf{m}}] = \mathbb{B}(\mathbb{P}^{\mathbf{m}})$.
- If $L \subseteq L^{\mathbf{m}}$ is an initial segment, then $\mathbb{P}^{\mathbf{m}} \restriction L$ is dense in $\mathbb{P}^{\mathbf{m}}[L]$.
- For every condition $p \in \mathbb{P}^{\mathbf{m}}$ there exist a unique condition $\pi(p) \in \mathbb{P}^{\mathbf{m}}[M]$, called the projection of p, such that for every $\varphi \in \mathbb{P}^{\mathbf{m}}[M]$ we have: p and φ are compatible iff $\pi(p)$ and φ are compatible.
- In particular, $\pi(p)$ is a reduct of p, i.e. for every $\varphi \in \mathbb{P}^{\mathbf{m}}[M]$ with $\varphi \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}^*})} \pi(p)$ we have p and φ are compatible.

Now we are ready to define the Corrected Iteration $\mathbb{Q}_M$:

Definition 5.2.5. Fix $\mathbf{m}^* \in \mathbf{M}_{ec}$. We define $\mathbb{Q}_M$ as the complete Boolean algebra generated by $(\dot\eta_s)_{s \in M}$ within $\mathbb{P}^{\mathbf{m}^*}$: $\mathbb{Q}_M := \mathbb{P}^{\mathbf{m}^*}[M]$. Furthermore, we define $\mathbb{Q}_M \restriction N := \mathbb{P}^{\mathbf{m}^*}[N]$ for $N \subseteq M$.

Next we show that our definition is well defined:

Lemma 5.2.6. $\mathbb{Q}_M$ does not depend on the choice of $\mathbf{m}^* \in \mathbf{M}_{ec}$.

Proof. Let $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}_{ec}$ and w.l.o.g. assume that $L^{\mathbf{m}_1} \cap L^{\mathbf{m}_2} = M$. By Lemma 5.1.9 there exists $\mathbf{m}_3$ such that $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_3$ and $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}_3$. Since both $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}_{ec}$ it follows that $\mathbb{P}^{\mathbf{m}_1} \lessdot \mathbb{P}^{\mathbf{m}_3}$ and $\mathbb{P}^{\mathbf{m}_2} \lessdot \mathbb{P}^{\mathbf{m}_3}$. Hence, it does not matter whether we define $\mathbb{Q}_M$ in $\mathbb{P}^{\mathbf{m}_3}$, in $\mathbb{P}^{\mathbf{m}_1}$ or in $\mathbb{P}^{\mathbf{m}_2}$. □

The following lemma summarizes the most important properties of the Corrected Iteration:

[6] Note that the κ-Borel algebra on 2^{Var} is the family $\mathcal{B} \subseteq \mathfrak{P}(2^{\mathrm{Var}})$ which is generated by the basic clopen sets only.

[7] If we factorize $\mathbb{P}^{\mathbf{m}}[L]$ modulo $\psi \leq_{\mathbb{P}^{\mathbf{m}}[L]} \varphi \wedge \varphi \leq_{\mathbb{P}^{\mathbf{m}}[L]} \psi$, we get a complete Boolean algebra.

Lemma 5.2.7. The following facts hold true:

1. $\mathbb{Q}_M$ is κ^+-c.c. and is $\leq\kappa$-strategically closed [8].
2. For every $s, t \in M$ we have $\mathbb{Q}_M \restriction M_{<s} \lessdot \mathbb{Q}_M \restriction M_{<t}$ if $s \leq_M t$. Hence, $(\mathbb{Q}_M \restriction M_{<s})_{s\in M}$ is indeed an iteration.
3. $\Vdash_{\mathbb{Q}_M} V[\dot{G}] = V[(\dot{\eta}_s)_{s\in M}]$.
4. $\Vdash_{\mathbb{Q}_M\restriction M_{\leq s}} \dot{\eta}_s$ dominates $V^{\mathbb{Q}_M\restriction M_{<s}}$
5. If $\mathrm{cf}(\alpha) \geq \kappa$ then $\bigcup_{\beta<\alpha} \mathbb{Q}_M \restriction M_\beta$ is dense in $\mathbb{Q}_M \restriction M_\alpha$. Hence, the iteration has '$<\kappa$-support'.

Proof. ad 1.) First we will show that $\mathbb{Q}_M$ satisfies the κ^+-c.c. Towards a contradiction assume that $(\varphi_i)_{i<\kappa^+}$ is a family of pairwise incompatible conditions in $\mathbb{Q}_M$. Let $(p_i)_{i<\kappa^+}$ be a sequence of conditions in $\mathbb{P}^{\mathbf{m}^*}$ such that $p_i \Vdash_{\mathbb{P}^{\mathbf{m}^*}} \varphi_i$ is true for every $i < \kappa^+$. But then the p_i's must also be pairwise incompatible, which contradicts the κ^+-c.c. of $\mathbb{P}^{\mathbf{m}^*}$. Now let us show that $\mathbb{Q}_M$ is $\leq\kappa$-strategically closed. Let $\varphi^* \in \mathbb{Q}_M$ be arbitrary and denote Player I's choice in stage $i < \kappa$ by φ_i and Player II's by ψ_i, which are all below φ^*. Player I's winning strategy is to inductively pick decreasing $p_i \in \mathbb{P}^{\mathbf{m}^*}$ and to set $\varphi_i := \pi(p_i)$. Hence, p_i will be compatible with ψ_i and there exists a common lower bound p_{i+1}. In a limit stage $\lambda < \kappa$ the condition $p_\lambda = \inf_{j<\lambda} p_j$ will ensure that $\bigwedge_{i<\lambda} \psi_i \in \mathbb{Q}_M$.

ad 2.) Trivial.

ad 3.) We will show that $\Vdash_{\mathbb{Q}_M} \varphi \in \dot{G} \Leftrightarrow B_\varphi(b^M_{\dot{G}}) = 1$ for every $\varphi \in \mathbb{Q}_M$, where $b^M_{\dot{G}}$ is a $\mathbb{Q}_M$-name for an element of $2^{\mathrm{Var}\restriction M}$ such that for every $s \in M$ and every $i, j < \kappa$ we have $\Vdash_{\mathbb{Q}_M} b_{\dot{G}^M}(p_{s,i,j}) = 1$ iff $\dot{\eta}_s(i) = j$ (see Lemma 5.2.2 and Definition 5.2.3). Then $\Vdash_{\mathbb{Q}_M} \dot{G} = \{\varphi \in \mathbb{Q}_M \colon B_\varphi(b^M_{\dot{G}}) = 1\}$.
Let $\varphi \in \mathbb{Q}_M$ be arbitrary. Let $\psi \in \mathbb{Q}_M$ be such that $\psi \Vdash_{\mathbb{Q}_M} \varphi \in \dot{G}$. Let $\psi' \in \mathbb{Q}_M$ be a common lower bound of φ and ψ, and let $p \in \mathbb{P}^{\mathbf{m}^*}$ be such that $p \Vdash_{\mathbb{P}^{\mathbf{m}^*}} B_{\psi'}(b^M_{\dot{G}}) = 1$. Then $\pi(p) \leq_{\mathbb{Q}_M} \psi$ and $\pi(p) \Vdash_{\mathbb{Q}_M} B_\varphi(b^M_{\dot{G}}) = 1$.
On the other hand, let $\psi \in \mathbb{Q}_M$ be such that $\psi \Vdash_{\mathbb{Q}_M} B_\varphi(b^M_{\dot{G}}) = 1$. Then $\psi \Vdash_{\mathbb{Q}_M} B_{\neg\varphi}(b^M_{\dot{G}}) = 0$. Hence, ψ and $\neg\varphi$ are incompatible in $\mathbb{Q}_M$. As $\{\varphi, \neg\varphi\} \subseteq \mathbb{Q}_M$ is a maximal antichain, we can deduce that $\psi \Vdash_{\mathbb{Q}_M} \varphi \in \dot{G}$.

ad 4.) Let $\dot{f}$ be a $\mathbb{Q}_M \restriction M_{<s}$-name for an element of κ^κ. Since $\Vdash_{\mathbb{Q}_M} \varphi \in \dot{G} \Leftrightarrow B_\varphi(b^M_{\dot{G}}) = 1$ for every $\varphi \in \mathbb{Q}_M$, we can find $u \in [M_{<s}]^{\leq\kappa}$ and a κ-Borel function $B \colon (\kappa^\kappa)^u \to \kappa^\kappa$ such that $\Vdash_{\mathbb{Q}_M} \dot{f} = B((\dot{\eta}_s)_{s\in u})$. Hence $\Vdash_{\mathbb{P}^{\mathbf{m}^*}\restriction L^{\mathbf{m}^*}_{\leq s}} \dot{\eta}_s$ eventually dominates $\dot{f}$. But $\mathbb{Q}_M \restriction M_{\leq s} \lessdot \mathbb{P}^{\mathbf{m}^*}[L^{\mathbf{m}^*}_{\leq s}]$, hence $\Vdash_{\mathbb{Q}_M\restriction M_{\leq s}} \dot{\eta}_s$ eventually dominates $\dot{f}$.

ad 5.) Let $\mathrm{cf}(\alpha) \geq \kappa$ and let $\varphi \in \mathbb{Q}_M \restriction M_\alpha$. Pick $p \in \mathbb{P}^{\mathbf{m}^*}_\alpha$ with $p \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}^*})} \varphi$. Since

[8] As a complete Boolean algebra $\mathbb{Q}_M$ cannot be $<\kappa$-closed.

$|\mathrm{dom}(p)| < \kappa$, it follows that there exists a $\beta < \alpha$ such that $p \in \mathbb{P}^{\mathbf{m}^*}_\beta$. We claim that $\pi(p) \in \mathbb{Q}_M \restriction M_\beta$. To this end let $\pi^{\mathbb{Q}_M \restriction M_\beta}(p) \in \mathbb{Q}_M \restriction M_\beta$ be the projection of p onto $\mathbb{Q}_M \restriction M_\beta$, and we will show that $\pi(p) = \pi^{\mathbb{Q}_M \restriction M_\beta}(p)$:
Since obviously $\pi(p) \leq_{\mathbb{Q}_M} \pi^{\mathbb{Q}_M \restriction M_\beta}(p)$, it follows for any $\varphi' \in \mathbb{Q}_M$ that $\pi^{\mathbb{Q}_M \restriction M_\beta}(p)$ and φ' are compatible if p and φ' are compatible. On the other hand assume towards a contradiction that there exists $\varphi' \in \mathbb{Q}_M$ such that $\pi^{\mathbb{Q}_M \restriction M_\beta}(p)$ and φ' are compatible, but p and φ' are incompatible. Let $\psi \in \mathbb{Q}_M$ be a common lower bound of $\pi^{\mathbb{Q}_M \restriction M_\beta}(p)$ and φ such that there exists $q \in \mathbb{P}^{\mathbf{m}^*}$ with $\psi = \pi(q)$. W.l.o.g. we can assume that $\mathrm{wsupp}(p) \cap \mathrm{wsupp}(q) = M$. By setting $\psi' := \pi^{\mathbb{Q}_M \restriction M_\beta}(q)$ and noting that $\pi^{\mathbb{Q}_M \restriction M_\beta}(q) \leq_{\mathbb{Q}_M} \pi^{\mathbb{Q}_M \restriction M_\beta}(p)$ we can show following the proof of Lemma 5.3.8 that p and $q \restriction L^{\mathbf{m}^*}_\beta$ are compatible. But then p and q must be compatible as well, which leads to a contradiction, since p and φ' were assumed to be incompatible. Hence $\pi(p) = \pi^{\mathbb{Q}_M \restriction M_\beta}(p)$. It follows that $\pi(p) \leq_{\mathbb{Q}_M} \varphi$ and therefore, $\bigcup_{\beta<\alpha} \mathbb{Q}_M \restriction M_\beta$ is dense in $\mathbb{Q}_M \restriction M_\alpha$. □

5.3 Different Iteration Parameters

In this section we want to show that if $N \subseteq M$ then $\mathbb{Q}_N \lessdot \mathbb{Q}_M$. Indeed this is not even trivial if N is an initial segment of M. In particular it is not obvious if $\mathbb{Q}_{M_\alpha} = \mathbb{Q}_M \restriction M_\alpha$. As a motivational example we will show this in the next lemma.
We will use the following notation: if we want to refer to a definition using N as the well-founded partial for which we want to construct the Corrected Iteration, we will denote this by a superscript N (e.g. $\mathbf{M}^N$, $\mathbf{M}^N_{ec}$, $\leq_{\mathbf{M}^N}$ etc.). Furthermore, elements of $\mathbf{M}^N$ we will denote by $\mathbf{n}$, $\mathbf{n}_i$, $\mathbf{n}^*$ etc.

Lemma 5.3.1. Let $N \subseteq M$ be an initial segment. Then $\mathbb{Q}_N \lessdot \mathbb{Q}_M$ and $\mathbb{Q}_N = \mathbb{Q}_M \restriction N$ hold true.

Proof. Let $\mathbf{m}^* \in \mathbf{M}_{ec}$. We define $\mathbf{n}^* := \mathbf{m}^* \restriction \{t \in L^{\mathbf{m}} \colon \exists s \in N\ t \leq_{L^{\mathbf{m}}} s\}$, i.e. $L^{\mathbf{n}^*} := \{t \in L^{\mathbf{m}^*} \colon \exists s \in N\ t \leq_{L^{\mathbf{m}}} s\}$, $u^{\mathbf{n}^*}_t := u^{\mathbf{m}^*}_t$, $\mathcal{P}^{\mathbf{n}^*}_t := \mathcal{P}^{\mathbf{m}^*}_t$ and $E^{\mathbf{n}^*} := E^{\mathbf{m}^*} \cap L^{\mathbf{n}^*} \times L^{\mathbf{n}^*}$. Since N is an initial segment of M, it can easily be checked that $\mathbf{n}^* \in \mathbf{M}^N$. The crucial point is that real coordinates in $\mathbf{m}^*$ remain such in $\mathbf{n}^*$.
We will now show that even $\mathbf{n}^* \in \mathbf{M}^N_{ec}$. Let $\mathbf{n}_1, \mathbf{n}_2 \in \mathbf{M}^N$ with $\mathbf{n}^* \leq_{\mathbf{M}^N} \mathbf{n}_1 \leq_{\mathbf{M}^N} \mathbf{n}_2$. W.l.o.g. we can assume that $L^{\mathbf{n}_2} \cap L^{\mathbf{m}^*} = L^{\mathbf{n}^*}$. Similar to Lemma 5.1.9 we can now 'amalgamate' $\mathbf{n}_i$ and $\mathbf{m}^*$ over $\mathbf{n}^*$:

- Define $L^{\mathbf{m}_i} := L^{\mathbf{n}_i} \cup L^{\mathbf{m}^*}$. In particular $t_1 \leq_{L^{\mathbf{m}_i}} t_2$ iff $t_1 \leq_{L^{\mathbf{n}_i}} t_2$ or $t_1 \leq_{L^{\mathbf{m}^*}} t_2$ or there exists $s \in N$ such that $t_1 \leq_{L^{\mathbf{n}_i}} s \leq_{L^{\mathbf{m}^*}} t_2$ holds.
- If $t \in L^{\mathbf{n}^*} \setminus M$ set $u^{\mathbf{m}_i}_t := u^{\mathbf{n}^*}_t$ and $\mathcal{P}^{\mathbf{m}_i}_t := \mathcal{P}^{\mathbf{n}^*}_t$.
- If $t \in L^{\mathbf{n}_i} \setminus L^{\mathbf{n}^*}$ set $u^{\mathbf{m}_i}_t := u^{\mathbf{n}_i}_t$ and $\mathcal{P}^{\mathbf{m}_i}_t := \mathcal{P}^{\mathbf{n}_i}_t$.
- If $t \in L^{\mathbf{m}^*} \setminus (L^{\mathbf{n}^*} \cup M)$ set $u^{\mathbf{m}_i}_t := u^{\mathbf{m}^*}_t$ and $\mathcal{P}^{\mathbf{m}_i}_t := \mathcal{P}^{\mathbf{m}^*}_t$.
- If $s \in N$ set $u^{\mathbf{m}_i}_s := u^{\mathbf{n}_i}_s$ and $\mathcal{P}^{\mathbf{m}_i}_s := \mathcal{P}^{\mathbf{n}_i}_s$.

- If $s \in M \setminus N$ set $u_s^{\mathbf{m}_i} := u_s^{\mathbf{m}^*}$ and $\mathcal{P}_s^{\mathbf{m}_i} := \mathcal{P}_s^{\mathbf{m}^*}$.
- Define $E^{\mathbf{m}_i} := E^{\mathbf{n}_i} \cup E^{\mathbf{m}^*}$.

One can easily check that $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$ and $\mathbf{m}^* \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$. Hence $\mathbb{P}^{\mathbf{m}^*} \lessdot \mathbb{P}^{\mathbf{m}_1} \lessdot \mathbb{P}^{\mathbf{m}_2}$. Using the fact that $L^{\mathbf{n}_i}$ is an initial segment of $L^{\mathbf{m}_i}$, we can show by induction on $\alpha \leq \infty_{L^{\mathbf{n}_i}}$ that $\mathbb{P}_\alpha^{\mathbf{n}_i} = \mathbb{P}^{\mathbf{m}_i} \restriction L_\alpha^{\mathbf{n}_i}$. But $\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{n}_1} \lessdot \mathbb{P}^{\mathbf{m}_2}$ and therefore $\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{n}_1} \lessdot \mathbb{P}^{\mathbf{m}_2} \restriction L^{\mathbf{n}_2}$. Hence $\mathbf{n}^* \in \mathbf{M}_{ec}^N$.
Since obviously $\mathbb{P}^{\mathbf{n}^*} \lessdot \mathbb{P}^{\mathbf{m}^*}$ holds, it follows that $\mathbb{Q}_N \lessdot \mathbb{Q}_M$ and furthermore $\mathbb{Q}_N = \mathbb{Q}_M \restriction N$. □

The remainder of this paper will deal with generalizing the previous lemma to any $N \subseteq M$. If N is not an initial segment of M the previous proof fails, because a $t \in M \setminus N$ might reach into more than λ_2 equivalence classes in $\mathbf{m}^*$. Indeed for the $\mathbf{m}^*$ we constructed in Theorem 5.1.13 this will definitely be the case.

It will suffice if we construct $\mathbf{m}^* \in \mathbf{M}_{ec} \cap \mathbf{M}_{\leq \lambda_2}$:

Theorem 5.3.2. Let $\mathbf{m}^* \in \mathbf{M}_{ec} \cap \mathbf{M}_{\leq \lambda_2}$. Let $N \subseteq M$ be arbitrary. Then there exists $\mathbf{n}^* \in \mathbf{M}_{ec}^N$ such that $\mathbb{P}^{\mathbf{n}^*} = \mathbb{P}^{\mathbf{m}^*}$. Hence $\mathbb{Q}_N \lessdot \mathbb{Q}_M$ and $\mathbb{Q}_N = \mathbb{Q}_M \restriction N$.
Furthermore, if G is a $(V, \mathbb{Q}_M)$-generic filter and $f\colon M \to M$ is a strictly increasing function such that $f \in V$, then the sequence $(\dot{\eta}_{f(s)}^G)_{s \in M}$ naturally defines a filter $G' \subseteq \mathbb{Q}_M$, which is also $(V, \mathbb{Q}_M)$-generic.

Proof. We define $\mathbf{n}^*$ as follows:

- Set $L^{\mathbf{n}^*} := L^{\mathbf{m}^*}$ as partial orders.
- Define $E^{\mathbf{n}^*} := L^{\mathbf{n}^*} \setminus N \times L^{\mathbf{n}^*} \setminus N$.
- If $t \in L^{\mathbf{n}^*} \setminus N$ set $u_t^{\mathbf{n}^*} := u_t^{\mathbf{m}^*}$ and $\mathcal{P}_t^{\mathbf{n}^*} := \mathcal{P}_t^{\mathbf{m}^*}$.
- If $s \in N$ set $u_s^{\mathbf{n}^*} := u_s^{\mathbf{m}^*}$ and $\mathcal{P}_s^{\mathbf{n}^*} := \mathcal{P}_s^{\mathbf{m}^*}$.

It is crucial that $L^{\mathbf{m}^*}$ is of size $\leq \lambda_2$, hence we can treat $L^{\mathbf{n}^*} \setminus N$ as one equivalence class, and therefore, we can set $u_t^{\mathbf{n}^*} := u_t^{\mathbf{m}^*}$ and $\mathcal{P}_t^{\mathbf{n}^*} := \mathcal{P}_t^{\mathbf{m}^*}$ even for $t \in M \setminus N$. It follows that $\mathbf{n}^* \in \mathbf{M}^N$.
Now let us show that $\mathbf{n}^* \in \mathbf{M}_{ec}^N$: Let $\mathbf{n}_1, \mathbf{n}_2 \in \mathbf{M}^N$ such that $\mathbf{n}^* \leq_{\mathbf{M}^N} \mathbf{n}_1 \leq_{\mathbf{M}^N} \mathbf{n}_2$. Similar to the previous proof we define $\mathbf{m}_i$ as follows:

- Set $L^{\mathbf{m}_i} := L^{\mathbf{n}_i}$ as partial orders.
- Define $E^{\mathbf{m}_i} := E^{\mathbf{m}^*} \cup E^{\mathbf{n}_i} \restriction \big((L^{\mathbf{n}_i} \setminus L^{\mathbf{m}^*}) \times (L^{\mathbf{n}_i} \setminus L^{\mathbf{m}^*})\big)$.
- If $t \in L^{\mathbf{n}_i} \setminus M$ set $u_t^{\mathbf{m}_i} := u_t^{\mathbf{n}_i}$ and $\mathcal{P}_t^{\mathbf{m}_i} := \mathcal{P}_t^{\mathbf{n}_i}$.
- If $s \in M$ set $u_s^{\mathbf{m}_i} := u_s^{\mathbf{n}_i}$ and $\mathcal{P}_s^{\mathbf{m}_i} := \mathcal{P}_s^{\mathbf{n}_i}$.

One can easily check that $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$ and $\mathbf{m}^* \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$. Hence $\mathbb{P}^{\mathbf{m}^*} \lessdot \mathbb{P}^{\mathbf{m}_1} \lessdot \mathbb{P}^{\mathbf{m}_2}$. By induction on $\alpha \leq \infty_{L^{\mathbf{n}_i}}$ show that $\mathbb{P}^{\mathbf{n}_i}_\alpha = \mathbb{P}^{\mathbf{m}_i}_\alpha$, recalling that $\mathbb{P}^{\mathbf{m}_i}$ does not depend on $E^{\mathbf{m}_i}$. Hence $\mathbf{n}^* \in \mathbf{M}^N_{ec}$. Since, in particular, $\mathbb{P}^{\mathbf{n}^*} = \mathbb{P}^{\mathbf{m}^*}$, it follows that $\mathbb{Q}_N \lessdot \mathbb{Q}_M$ and $\mathbb{Q}_N = \mathbb{Q}_M \restriction N$.
Let G be a $(V, \mathbb{Q}_M)$-generic filter and let $f\colon M \to M$ be a strictly increasing function in V. Set $N := f[M]$. The sequence $(\dot{\eta}^G_{f(s)})_{s \in M}$ obviously induces a $(V, \mathbb{Q}_M \restriction N)$-generic filter: Set $G' := \{\varphi \in \mathbb{Q}_M \restriction N \colon B_\varphi(b^M_G \restriction N) = 1\}$, where $b^M_G \restriction N(p_{f(s),i,j}) = 1$ iff $\dot{\eta}^G_{f(s)}(i) = j$ for every $i, j < \kappa$ and $s \in M$ (see Lemma 5.2.2 and Definition 5.2.3). Since $\mathbb{Q}_N = \mathbb{Q}_M \restriction N$, we can deduce that G' is a $(V, \mathbb{Q}_N)$-generic filter. But since $\mathbb{Q}_M$ and $\mathbb{Q}_N$ are obviously isomorphic, it follows that G' is $(V, \mathbb{Q}_M)$-generic. □

We are left with showing that $\mathbf{M}_{ec} \cap \mathbf{M}_{\leq\lambda_2} \neq \emptyset$. This however will require some work. If we look back at how we constructed the $\mathbf{m}^* \in \mathbf{M}_{ec}$ in Theorem 5.1.13, we notice two things: first, of course, that $|L^{\mathbf{m}^*}| = 2^{\lambda_2}$, and secondly that $\mathbf{m}^*$ is 'very saturated', i.e. every 'reasonable type' is satisfied λ_2-many times in $\mathbf{m}^*$. However, conditions and even antichains in $\mathbb{P}^{\mathbf{m}^*}$ only use $\leq\kappa$-many coordinates. Indeed it turns out that we do not need global automorphisms, but local ones will suffice. In particular, we do not need that every 'reasonable type' is satisfied in $\mathbf{m}^*$, but only those of 'size $\leq\kappa$'. This will be the reason why we can find $\mathbf{m}^* \in \mathbf{M}_{ec} \cap \mathbf{M}_{\leq\lambda_2}$.

We will need several definitions:

Definition 5.3.3. Let $\mathbf{m} \in \mathbf{M}$ and let $L \subseteq L^{\mathbf{m}}$ be arbitrary, in particular, not necessarily an initial segment. We define $\mathbb{P}^{\mathbf{m}} \restriction L := \{p \in \mathbb{P}^{\mathbf{m}} \colon \mathrm{fsupp}(p) \subseteq L\}$ and endow it with the partial ordering $\leq_{\mathbb{P}^{\mathbf{m}} \restriction L} := \leq_{\mathbb{P}^{\mathbf{m}}} \restriction (\mathbb{P}^{\mathbf{m}} \restriction L \times \mathbb{P}^{\mathbf{m}} \restriction L)$.

Definition 5.3.4. Let $\mathbf{m} \in \mathbf{M}$ and let $L \subseteq L^{\mathbf{m}}$ be arbitrary. For a condition $p \in \mathbb{P}^{\mathbf{m}}$ we define $p \restriction L$ as follows:

- Set $\mathrm{dom}(p \restriction L) := \mathrm{dom}(p) \cap L$.
- If $t \in \mathrm{dom}(p \restriction L)$ define $(p \restriction L)(t) := (\rho^{p(t)}, \sup_{j \in I_L} B^{p(t)}_j((\dot{\eta}_{t'})_{t' \in u_j}))$, where $p(t) = (\rho^{p(t)}, \sup_{j<\delta} B^{p(t)}_j((\dot{\eta}_{t'})_{t' \in u_j}))$ and $I_L := \{j < \delta \colon u_j \subseteq L\}$.

Note that if $M \subseteq L$, then $\mathbb{P}^{\mathbf{m} \restriction L} = \mathbb{P}^{\mathbf{m}} \restriction L$ as sets, but, in general, not as partial orders.

Lemma 5.3.5. Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L^{\mathbf{m}}$ such that $M \subseteq L$. Let $p \in \mathbb{P}^{\mathbf{m}}$ be a condition. Then the following facts hold true:

1. $p \restriction L \in \mathbb{P}^{\mathbf{m}} \restriction L$
2. $\mathbb{P}^{\mathbf{m}} \restriction L \subseteq \mathbb{P}^{\mathbf{m}}[L]$ as partial orders
3. $\Vdash_{\mathbb{P}^{\mathbf{m}}} p \in \dot{G} \Leftrightarrow \{p \restriction (t/E^{\mathbf{m}} \cup M) \colon t \in \mathrm{wsupp}(p) \setminus M\} \subseteq \dot{G}$

Proof. ad 1., 2.) Trivial.
ad 3.) Obviously, for any $q \in \mathbb{P}^{\mathbf{m}}$ if $q \Vdash_{\mathbb{P}^{\mathbf{m}}} p \in \dot{G}$, then also $q \Vdash_{\mathbb{P}^{\mathbf{m}}} \{p \restriction (t/E^{\mathbf{m}} \cup M) \colon t \in \mathrm{wsupp}(p) \setminus M\} \subseteq \dot{G}$. Hence $\Vdash_{\mathbb{P}^{\mathbf{m}}} p \in \dot{G} \Rightarrow \{p \restriction (t/E^{\mathbf{m}} \cup M) \colon t \in \mathrm{wsupp}(p) \setminus M\} \subseteq \dot{G}$.
On the other hand let $q \in \mathbb{P}^{\mathbf{m}}$ be such that $q \Vdash_{\mathbb{P}^{\mathbf{m}}} p \notin \dot{G}$. By Lemma 5.1.7 we can assume w.l.o.g that there exists $t \in \mathrm{dom}(p)$ such that $q \restriction L^{\mathbf{m}}_{<t} \leq_{\mathbb{P}^{\mathbf{m}}} p \restriction L^{\mathbf{m}}_{<t}$ and $q \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} p(t)$ and $q(t)$ are incompatible in $\dot{Q}^{\mathbf{m}}_t$. Again by Lemma 5.1.7, there are now three cases:

- $\rho^{p(t)}$ and $\rho^{q(t)}$ are incompatible. Find $t' \in \mathrm{wsupp}(p) \setminus M$ such that $t \in t'/E^{\mathbf{m}} \cup M$. But then $q \Vdash_{\mathbb{P}^{\mathbf{m}}} p \restriction (t'/E^{\mathbf{m}} \cup M) \notin \dot{G}$.

- $\rho^{q(t)} \triangleleft \rho^{p(t)}$ and there exists $i^* \in \mathrm{dom}(\rho^{p(t)})$ such that $q \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \rho^{p(t)}(i^*) < \dot{B}^{q(t)}(i^*)$. Again there exists $t' \in \mathrm{wsupp}(p) \setminus M$ with $t \in t'/E^{\mathbf{m}} \cup M$. It follows that $q \Vdash_{\mathbb{P}^{\mathbf{m}}} p \restriction (t'/E^{\mathbf{m}} \cup M) \notin \dot{G}$.

- $\rho^{p(t)} \triangleleft \rho^{q(t)}$ and there exists $j < \delta$ and $i^* \in \mathrm{dom}(\rho^{q(t)})$ such that $q \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \rho^{q(t)}(i^*) < B^{p(t)}_j((\dot{\eta}_{t'})_{t' \in u_j})(i^*)$. Note that this follows because $\Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \dot{B}^{p(t)} = \sup_{j<\delta} B^{p(t)}_j((\dot{\eta}_{t'})_{t' \in u})$. Now there exists $t' \in \mathrm{wsupp}(p) \setminus M$ such that $u_j \subseteq t'/E^{\mathbf{m}} \cup M$. Hence $q \Vdash_{\mathbb{P}^{\mathbf{m}}} p \restriction (t'/E^{\mathbf{m}} \cup M) \notin \dot{G}$.

In any case it follows that $q \Vdash_{\mathbb{P}^{\mathbf{m}}} \{p \restriction (t/E^{\mathbf{m}} \cup M) \colon t \in \mathrm{wsupp}(p) \setminus M\} \not\subseteq \dot{G}$. Hence $\Vdash_{\mathbb{P}^{\mathbf{m}}} \{p \restriction (t/E^{\mathbf{m}} \cup M) \colon t \in \mathrm{wsupp}(p) \setminus M\} \subseteq \dot{G} \Rightarrow p \in \dot{G}$. □

Definition 5.3.6. Let $\mathbf{m} \in \mathbf{M}$. We define $\mathcal{Y}_{\mathbf{m}} := \{(t, \bar{s}) \colon t \in L^{\mathbf{m}} \setminus M \wedge \bar{s} \in (t/E^{\mathbf{m}})^{<\kappa^+}$ is injective$\}$. Furthermore, we define an equivalence relation on $\mathcal{Y}_{\mathbf{m}}$. We say that $(t_1, \bar{s}_1)$ and $(t_2, \bar{s}_2)$ are 0-equivalent over $\mathbf{m}$ iff there exists a bijective function[9] $f \colon \bar{s}_1 \cup M \to \bar{s}_2 \cup M$, which we will call a weak isomorphism, such that:

- $f(\bar{s}_1(i)) = \bar{s}_2(i)$ for every $i < \mathrm{dom}(\bar{s}_1)$ [10]
- $\forall s \in M \colon f(s) = s$
- $\forall t'_1, t'_2 \in \bar{s}_1 \cup M \colon t'_1 \leq_{L^{\mathbf{m}}} t'_2 \Leftrightarrow f(t'_1) \leq_{L^{\mathbf{m}}} f(t'_2)$
- $\forall t \in \bar{s}_1 \cup M \colon f[u^{\mathbf{m}}_t \cap (\bar{s}_1 \cup M)] = u^{\mathbf{m}}_{f(t)} \cap (\bar{s}_2 \cup M)$
- $\forall t \in \bar{s}_1 \cup M \; \forall u \in [u^{\mathbf{m}}_t \cap (\bar{s}_1 \cup M)]^{\leq\kappa} \colon u \in \mathcal{P}^{\mathbf{m}}_t \cap [\bar{s}_1 \cup M]^{\leq\kappa} \Leftrightarrow f[u] \in \mathcal{P}^{\mathbf{m}}_{f(t)} \cap [\bar{s}_2 \cup M]^{\leq\kappa}$

Clearly f induces a bijective function $\hat{f} \colon \mathbb{P}^{\mathbf{m}} \restriction (\bar{s}_1 \cup M) \to \mathbb{P}^{\mathbf{m}} \restriction (\bar{s}_2 \cup M)$, but in general $\hat{f}$ does neither preserve $\leq_{\mathbb{P}^{\mathbf{m}}}$ nor compatibility in $\mathbb{P}^{\mathbf{m}}$. Hence, we further require:

- $\hat{f}$ is an isomorphism from $\mathbb{P}^{\mathbf{m}} \restriction (\bar{s}_1 \cup M)$ to $\mathbb{P}^{\mathbf{m}} \restriction (\bar{s}_2 \cup M)$
- $\hat{f}$ extends to an isomorphism $\hat{\hat{f}}$ from $\mathbb{P}^{\mathbf{m}}[\bar{s}_1 \cup M]$ to $\mathbb{P}^{\mathbf{m}}[\bar{s}_2 \cup M]$

[9] We will occasionally identify $\bar{s}$ with $\{\bar{s}(i) \colon i < \mathrm{dom}(\bar{s})\}$.
[10] This guarantees that there exists at most one such a function.

We say that $(t_1, \bar{s}_1)$ and $(t_2, \bar{s}_2)$ are 1-equivalent over $\mathbf{m}$ iff:

- $(t_1, \bar{s}_1)$ and $(t_2, \bar{s}_2)$ are 0-equivalent over $\mathbf{m}$
- for every $i \in \{1, 2\}$ and every $\bar{z}_i \in (t_i/E^{\mathbf{m}})^{<\kappa^+}$ there exists $\bar{z}_{3-i} \in (t_{3-i}/E^{\mathbf{m}})^{<\kappa^+}$ such that $(t_1, \bar{s}_1^\frown \bar{z}_1)$ and $(t_2, \bar{s}_2^\frown \bar{z}_2)$ are 0-equivalent over $\mathbf{m}$

Lemma 5.3.7. Let $\mathbf{m} \in \mathbf{M}$. The following holds true:

- The number of 0-equivalence classes over $\mathbf{m}$ is at most $\beth_1(\lambda_1)$.
- The number of 1-equivalence classes over $\mathbf{m}$ is at most $\beth_2(\lambda_1)$.

Proof. There are at most 2^{λ_1} many possibilities for $\mathbf{m} \restriction (\bar{s} \cup M)$ modulo isomorphism. $\mathbb{P}^{\mathbf{m}} \restriction (\bar{s} \cup M)$ as a set is uniquely determined by $\mathbf{m} \restriction (\bar{s} \cup M)$ and is at most of size $\lambda_1^\kappa = \lambda_1$. Hence, there are at most 2^{λ_1} many possibilities for $\mathbb{P}^{\mathbf{m}} \restriction (\bar{s} \cup M)$ as a partial order. $\mathbb{P}^{\mathbf{m}}[\bar{s} \cup M]$ is not completely determined by $\mathbf{m} \restriction (\bar{s} \cup M)$, but there are at most λ_1 many possibilities for $\mathbb{P}^{\mathbf{m}}[\bar{s} \cup M]$ as a set, all of which are of size $\leq \lambda_1$. Hence, there are at most 2^{λ_1} many possibilities for $\mathbb{P}^{\mathbf{m}}[\bar{s} \cup M]$ as a partial order. It follows that there are at most $\beth_1(\lambda_1)$ many 0-equivalence classes over $\mathbf{m}$.

For $(t, \bar{s}) \in \mathcal{Y}_{\mathbf{m}}$ we define a function $F_{(t,\bar{s})} \colon \mathcal{Y}_{\mathbf{m}} /_{\text{0-equ.}} \to 2$ as follows: $F_{(t,\bar{s})}(\,[(t', \bar{s}')]_{\text{0-equ.}}\,) = 1$ iff there exists $\bar{z} \in (t/E^{\mathbf{m}})^{<\kappa^+}$ such that $(t, \bar{s}^\frown \bar{z}) \in [(t', \bar{s}')]_{\text{0-equ.}}$. Obviously $(t_1, \bar{s}_1)$ and $(t_2, \bar{s}_2)$ are 1-equivalent over $\mathbf{m}$ iff they are 0-equivalent over $\mathbf{m}$ and $F_{(t_1,\bar{s}_1)} = F_{(t_2,\bar{s}_2)}$. Hence, there are at most $\beth_2(\lambda_1)$ many 1-equivalence classes over $\mathbf{m}$. □

Lemma 5.3.8. Let $\mathbf{m} \in \mathbf{M}$ be wide. Let $p, q \in \mathbb{P}^{\mathbf{m}}$ and $\psi \in \mathbb{P}^{\mathbf{m}}[M]$ such that:

- $\text{wsupp}(p) \cap \text{wsupp}(q) = M$
- ψ is a reduct of p
- q and ψ are compatible in $\mathbb{B}(\mathbb{P}^{\mathbf{m}})$

Then p and q are compatible.

Proof. Simultaneously we shall construct by induction decreasing sequences $(p_n)_{n<\omega}$, $(q_n)_{n<\omega}$ and $(\psi_n)_{n<\omega}$ such that for any $n < \omega$ we have $\text{dom}(p_n) \cap M \subseteq \text{dom}(q_{n+1})$ and $\text{dom}(q_n) \cap M \subseteq \text{dom}(p_{n+1})$, $\text{wsupp}(p_n) \cap \text{wsupp}(q_n) = M$, ψ_n is a reduct of p_n, and q_n and ψ_n are compatible (ψ_n will even be a reduct of q_n for $n \geq 1$).

Set $p_0 := p$, $q_0 := q$ and $\psi_0 := \psi$. They satisfy the requirements by assumption. Assume that p_n, q_n, ψ_n have been defined. Since q_n and ψ_n are compatible, we can find a lower bound q_n'. Obviously we can assume that $\text{dom}(p_n) \cap M \subseteq \text{dom}(q_n')$. However, it is not necessarily the case that $\text{wsupp}(p_n) \cap \text{wsupp}(q_n') = M$. Therefore, we define an automorphism $\hat{f}$ of $\mathbb{P}^{\mathbf{m}}$ fixing $\text{wsupp}(q_n)$ pointwise and moving $\text{wsupp}(q_n') \setminus \text{wsupp}(q_n)$ away from $\text{wsupp}(p_n)$. Then $\hat{f}(q_n')$ is also a lower bound of q_n and ψ_n, and $\text{wsupp}(p_n) \cap \text{wsupp}(\hat{f}(q_n')) = M$. Set $q_{n+1} := \hat{f}(q_n')$, and define $\psi_n' := \pi(q_{n+1})$. It easily follows that

$\psi'_n \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \psi_n$, hence p_n and ψ'_n are compatible.
Similar to above find p_{n+1} with the required properties. Set $\psi_{n+1} := \pi(p_{n+1})$ and notice that $\psi_{n+1} \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \psi'_n$. Hence, q_{n+1} and ψ_{n+1} are compatible (ψ_{n+1} is even a reduct of q_{n+1}).

Since $(p_n)_{n<\omega}$, $(q_n)_{n<\omega}$ and $(\psi_n)_{n<\omega}$ are decreasing sequences, there exist greatest lower bounds p^*, q^* and ψ^*. It follows that $\mathrm{dom}(p^*) \cap M = \mathrm{dom}(q^*) \cap M$ and $\mathrm{wsupp}(p^*) \cap \mathrm{wsupp}(q^*) = M$. Furthermore, since for every $s \in \mathrm{dom}(p^*) \cap M$ the condition ψ^* decides $\rho^{p^*(s)}$ and $\rho^{q^*(s)}$, it follows that $\rho^{p^*(s)} = \rho^{q*(s)}$ for every $s \in \mathrm{dom}(p^*) \cap M$. Hence, p^* and q^* are compatible, which proves that p and q are compatible. □

Lemma 5.3.9. Let $\mathbf{m} \in \mathbf{M}$ be wide. Let $\{\chi_i : i < i^*\} \subseteq \mathbb{B}(\mathbb{P}^{\mathbf{m}})$ with $i^* < \kappa$ be a family of conditions such that:

- There exists $\psi \in \mathbb{P}^{\mathbf{m}}[M]$ such that ψ is a reduct of χ_i for every $i < i^*$.
- There exists $(L_i)_{i<i^*}$ with $L_i \subseteq L^{\mathbf{m}}$ such that:
  - $\forall i, j < i^* : i \neq j \Rightarrow L_i \cap L_j = M$
  - $\forall i < i^* \; \forall t \in L_i : t/E^{\mathbf{m}} \subseteq L_i$
  - $\forall i < i^* : \chi_i \in \mathbb{P}^{\mathbf{m}}[L_i]$

Then there exists $p \in \mathbb{P}^{\mathbf{m}}$ which is a lower bound of $\{\chi_i : i < i^*\}$.

Proof. By induction we will construct decreasing sequences $(p_i)_{i<i^*}$ and $(\psi_i)_{i<i^*}$ such that p_i is a lower bound of $\{\chi_j : j < i\}$, $\forall j \geq i$: $\mathrm{wsupp}(p_i) \cap L_j = M$ and $\psi_i \in \mathbb{P}^{\mathbf{m}}[M]$ is a reduct of p_i.

Set $p_0 \in \mathbb{P}^{\mathbf{m}}$ to be any condition below ψ such that $\forall i < i^*$: $\mathrm{wsupp}(p_0) \cap L_i = M$. One can achieve this by using an automorphism argument. Set $\psi_0 := \pi(p_0)$.
Assume inductively that p_i and ψ_i have already been defined and satisfy the required properties. Since $\psi_i \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \psi$, it follows that χ_i and ψ_i are compatible. Pick a common lower bound $p'_i \in \mathbb{P}^{\mathbf{m}}$ and use an automorphism argument to make sure that $\mathrm{wsupp}(p'_i) \cap \mathrm{wsupp}(p_i) = M$. This is possible since $\chi_i \in \mathbb{P}^{\mathbf{m}}[L_i]$ and $\mathrm{wsupp}(p_i) \cap L_i = M$. If we set $\psi'_i := \pi(p'_i)$ and note that $\psi'_i \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \psi_i$, we can use the previous lemma to show that there exist a lower bound p''_i of p_i and p'_i. Again use an automorphism argument to make $\mathrm{wsupp}(p''_i) \setminus M$ disjoint from $\bigcup_{j>i} L_j$. Define $p_{i+1} := \hat{f}(p''_i)$ where $\hat{f}$ is the corresponding automorphism of $\mathbb{P}^{\mathbf{m}}$. Set $\psi_{i+1} := \pi(p_{i+1})$.
In the limit step define p_λ to be the greatest lower bound of $\{p_i : i < \lambda\}$ and set $\psi_\lambda := \pi(p_\lambda)$. Obviously ψ_λ is a lower bound of $\{\psi_i : i < \lambda\}$ and $\mathrm{wsupp}(p_\lambda) \cap \bigcup_{j \geq \lambda} L_j = M$.

Define p_{i^*} to be the greatest lower bound of $\{p_i : i < i^*\}$. Then p_{i^*} is obviously a lower bound of $\{\chi_i : i < i^*\}$. □

Lemma 5.3.10. Let $\mathbf{m} \in \mathbf{M}$ be wide. Let $((t_1^i, \bar{s}_1^i))_{i<i^*}$, $((t_2^i, \bar{s}_2^i))_{i<i^*}$ and $(f_i)_{i<i^*}$ be such that:

- $\forall k \in \{1,2\}\ \forall i < i^*\colon (t^i_k, \bar{s}^i_k) \in \mathcal{Y}_{\mathbf{m}}$
- $\forall k \in \{1,2\}\ \forall i,j < i^*\colon i \neq j \Rightarrow \neg\, t^i_k\, E^{\mathbf{m}}\, t^j_k$
- f_i witnesses that $(t^i_1, \bar{s}^i_1)$ and $(t^i_2, \bar{s}^i_2)$ are 0-equivalent over $\mathbf{m}$

Then $f := \bigcup_{i<i^*} f_i$ induces an isomorphism $\hat{f}\colon \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}^i_1 \cup M) \to \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}^i_2 \cup M)$.

Proof. Abbreviate $\bigcup_{i<i^*} \bar{s}^i_k \cup M$ by L_k. Canonically define $\hat{f}\colon \mathbb{P}^{\mathbf{m}} \restriction L_1 \to \mathbb{P}^{\mathbf{m}} \restriction L_2$, i.e. for $p \in \mathbb{P}^{\mathbf{m}} \restriction L_1$ and $t \in \operatorname{dom}(p)$ define $\hat{f}(p)\,(f(t)) := (\rho^{p(t)}, \sup_{j<\delta} B^{p(t)}_j((\dot{\eta}_{t'})_{t' \in f[u]}))$. Obviously $\hat{f}\colon \mathbb{P}^{\mathbf{m}} \restriction L_1 \to \mathbb{P}^{\mathbf{m}} \restriction L_2$ is bijective and extends every $\hat{f}_i$. We need to show that $\forall p,q \in \mathbb{P}^{\mathbf{m}} \restriction L_1\colon q \leq_{\mathbb{P}^{\mathbf{m}}} p \Leftrightarrow \hat{f}(q) \leq_{\mathbb{P}^{\mathbf{m}}} \hat{f}(p)$.

In order to show this, we will first verify the following claim: The following two statements are equivalent for $p,q \in \mathbb{P}^{\mathbf{m}}$ satisfying $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$.

(a)$_k$ $p,q \in \mathbb{P}^{\mathbf{m}} \restriction L_k \wedge q \leq_{\mathbb{P}^{\mathbf{m}}} p$

(b)$_k$ $p,q \in \mathbb{P}^{\mathbf{m}} \restriction L_k \wedge \forall i < i^*\colon q \restriction (\bar{s}^i_k \cup M) \wedge \pi(q) \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} p \restriction (\bar{s}^i_k \cup M) \wedge \pi(p)$

(a)$_k$ $\Rightarrow$ (b)$_k$: Let $p,q \in \mathbb{P}^{\mathbf{m}} \restriction L_k$ and $i < i^*$ be arbitrary, and assume towards a contradiction that there exists $q' \in \mathbb{P}^{\mathbf{m}}$ such that

$$q' \Vdash_{\mathbb{P}^{\mathbf{m}}} \big(\, q \restriction (\bar{s}^i_k \cup M) \wedge \pi(q) \text{ is true} \big) \ \wedge\ \big(\, p \restriction (\bar{s}^i_k \cup M) \wedge \pi(p) \text{ is false} \big).$$

Using an automorphism argument we can assume that $\operatorname{wsupp}(q') \cap \operatorname{wsupp}(q) \subseteq t_i/E^{\mathbf{m}} \cup M$. Now we use the previous lemma with $\chi_{i'} := q \restriction (\bar{s}^{i'}_k \cup M)$ for $i' \neq i$, $\chi_i := q'$ and $\psi := \pi(q')$ noting that $\pi(q') \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \pi(q)$ and $\{i' < i^*\colon q \restriction (\bar{s}^{i'}_k \cup M) \neq q \restriction M\}$ is of size $< \kappa$, hence $\{q \restriction (\bar{s}^{i'}_k \cup M)\colon i' < i^*\} \cup \{q'\}$ is of size $< \kappa$. Therefore, we get a condition q'' which is a lower bound of q' and by Lemma 5.3.5 $q'' \Vdash_{\mathbb{P}^{\mathbf{m}}} q \in \dot{G}$. Since $q \leq_{\mathbb{P}^{\mathbf{m}}} p$, it follows that $q'' \Vdash_{\mathbb{P}^{\mathbf{m}}} p \in \dot{G}$. But this immediately leads to a contradiction, since q' and p must be incompatible.

(b)$_k$ $\Rightarrow$ (a)$_k$: Let $p,q \in \mathbb{P}^{\mathbf{m}} \restriction L_k$ be arbitrary and assume towards a contradiction that $q \not\leq_{\mathbb{P}^{\mathbf{m}}} p$. Since $\operatorname{dom}(p) \subseteq \operatorname{dom}(q)$, it follows by Lemma 5.1.7 that there exists $q' \leq_{\mathbb{P}^{\mathbf{m}}} q$ such that $q' \Vdash_{\mathbb{P}^{\mathbf{m}}} p \notin \dot{G}$. However, $q' \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} q \restriction (\bar{s}^i_k \cup M) \wedge \pi(q)$ for every $i < i^*$, hence $q' \Vdash_{\mathbb{P}^{\mathbf{m}}} \{p \restriction (\bar{s}^i_k \cup M)\colon i < i^*\} \subseteq \dot{G}$. By Lemma 5.3.5 it follows that $q' \Vdash_{\mathbb{P}^{\mathbf{m}}} p \in \dot{G}$, which is a contradiction.

Next we show that for every $p \in \mathbb{P}^{\mathbf{m}} \restriction L_1$ we have $\pi(p) = \pi(\hat{f}(p))$, i.e. for every $\varphi \in \mathbb{P}^{\mathbf{m}}[M]$ we have $\pi(p)$ is compatible with φ iff $\hat{f}(p)$ is compatible with φ:
Let $\pi(p)$ be compatible with φ and let $\psi \in \mathbb{P}^{\mathbf{m}}[M]$ be a common lower bound. Enumerate $\operatorname{wsupp}(p) \setminus M$ modulo $E^{\mathbf{m}}$ as $(t^j_1)_{j<j^*}$ with $j^* < \kappa$. Since the condition $\pi(p)$ is a reduct of $p \restriction (\bar{s}^j_1 \cup M)$, we can deduce that ψ is a reduct of $p \restriction (\bar{s}^j_1 \cup M) \wedge \psi$ for every $j < j^*$. Furthermore, since $\hat{\hat{f}}_j\colon \mathbb{P}^{\mathbf{m}}[\bar{s}^j_1 \cup M] \to \mathbb{P}^{\mathbf{m}}[\bar{s}^j_2 \cup M]$ is an isomorphism, it follows that ψ is also a reduct of $\hat{f}(p) \restriction (\bar{s}^j_2 \cup M) \wedge \psi$. By Lemma 5.3.9 and Lemma 5.3.5 we

can deduce that $\hat{f}(p)$ and ψ are compatible. Hence, $\hat{f}(p)$ and φ are compatible.
On the other hand, assume that $\hat{f}(p)$ and φ are compatible and let $q \in \mathbb{P}^{\mathbf{m}}$ be a common lower bound. It follows that $\pi(q) \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \varphi$ and it is a reduct of $\hat{f}(p) \restriction (\bar{s}_2^j \cup M) \wedge \pi(q)$ for every $j < j^*$. Furthermore, since $\hat{\hat{f}}_j^{-1} \colon \mathbb{P}^{\mathbf{m}}[\bar{s}_2^j \cup M] \to \mathbb{P}^{\mathbf{m}}[\bar{s}_1^j \cup M]$ is an isomorphism, we can deduce that $\pi(q)$ is also a reduct of $p \restriction (\bar{s}_1^j \cup M) \wedge \pi(q)$. Again by Lemma 5.3.9 and Lemma 5.3.5 it follows that p and φ are compatible. Hence, $\pi(p)$ and φ are compatible.

Now let $p, q \in \mathbb{P}^{\mathbf{m}} \restriction L_1$ such that $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$. We can conclude that the following are equivalent:

- $q \leq_{\mathbb{P}^{\mathbf{m}}} p$
- $\forall i < i^* \colon q \restriction (\bar{s}_1^i \cup M) \wedge \pi(q) \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} p \restriction (\bar{s}_1^i \cup M) \wedge \pi(p)$ by $(\mathrm{a})_1 \Leftrightarrow (\mathrm{b})_1$
- $\forall i < i^* \colon \hat{f}_i(q \restriction (\bar{s}_1^i \cup M)) \wedge \pi(q) \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \hat{f}_i(p \restriction (\bar{s}_1^i \cup M)) \wedge \pi(p)$ since $\hat{\hat{f}}_i \colon \mathbb{P}^{\mathbf{m}}[\bar{s}_1^i \cup M] \to \mathbb{P}^{\mathbf{m}}[\bar{s}_2^i \cup M]$ is an isomorphism, $\hat{f} = \hat{\hat{f}} \restriction \mathbb{P}^{\mathbf{m}}(\bar{s}_1^i \cup M)$ and $\hat{\hat{f}}_i \restriction \mathbb{P}^{\mathbf{m}}[M]$ is the identity for every $i < i^*$
- $\forall i < i^* \colon \hat{f}_i(q \restriction (\bar{s}_1^i \cup M)) \wedge \pi(\hat{f}(q)) \leq_{\mathbb{B}(\mathbb{P}^{\mathbf{m}})} \hat{f}_i(p \restriction (\bar{s}_1^i \cup M)) \wedge \pi(\hat{f}(p))$ since $\pi(r) = \pi(\hat{f}(r))$ for every $r \in \mathbb{P}^{\mathbf{m}} \restriction L_1$
- $\hat{f}(q) \leq_{\mathbb{P}^{\mathbf{m}}} \hat{f}(p)$ by $(\mathrm{b})_2 \Leftrightarrow (\mathrm{a})_2$ and $\hat{f}_i(r \restriction (\bar{s}_1^i \cup M)) = \hat{f}(r) \restriction (\bar{s}_2^i \cup M)$ for every $r \in \mathbb{P}^{\mathbf{m}} \restriction L_1$ and every $i < i^*$

Hence $\hat{f} \colon \mathbb{P}^{\mathbf{m}} \restriction L_1 \to \mathbb{P}^{\mathbf{m}} \restriction L_2$ is an isomorphism. □

Lemma 5.3.11. Let $\mathbf{m} \in \mathbf{M}$. Assume that $L_0, L_1, L_2 \subseteq L^{\mathbf{m}}$ such that:

- L_2 is an initial segment of $L^{\mathbf{m}}$.
- $L_0 = L_1 \cap L_2$
- $\mathbb{P}^{\mathbf{m}} \restriction L_0 \lessdot \mathbb{P}^{\mathbf{m}} \restriction L_2$
- $L_1 \setminus L_0$ is disjoint from M
- if $t \in L_1 \setminus L_0$ then $(t/E^{\mathbf{m}} \cup M) \cap L^{\mathbf{m}}_{<t} \subseteq L_1$

Then $\mathbb{P}^{\mathbf{m}} \restriction L_1 \lessdot \mathbb{P}^{\mathbf{m}}$.

Proof. The crucial point of the proof will be the fact that $L_2 \cup (L_1 \setminus L_0)$ is an initial segment of $L^{\mathbf{m}}$. To see this let $t_1 \in L_1 \setminus L_0$ and $t_2 \leq_{\mathbf{m}} t_1$. If $t_1 \, E^{\mathbf{m}} \, t_2$ then $t_2 \in L_1$, hence $t_2 \in L_2 \cup (L_1 \setminus L_0)$. If $\neg\, t_1 \, E^{\mathbf{m}} \, t_2$ then there exists by definition an $s \in M$ with $t_2 \leq_{\mathbf{m}} s \leq_{\mathbf{m}} t_1$. It follows that $s \in L_2$ and hence $t_2 \in L_2$.

Furthermore, since $(t/E^{\mathbf{m}} \cup M) \cap L^{\mathbf{m}}_{<t} \subseteq L_1$ for every $t \in L_1 \setminus L_0$, we have:

(†) For every $p \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup (L_1 \setminus L_0))$ and every $t \in \mathrm{dom}(p) \cap L_1 \setminus L_0$ it follows that $\Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} p(t) = (p \restriction L_1)(t)$.

For $p, q \in \mathbb{P}^{\mathbf{m}} \restriction L_1$ we have by definition $q \leq_{\mathbb{P}^{\mathbf{m}} \restriction L_1} p$ iff $q \leq_{\mathbb{P}^{\mathbf{m}}} p$. Set $L' := L_1 \setminus L_0$. We will show by induction on $\alpha \leq \infty_{L'}$ that $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha) \lessdot \mathbb{P}^{\mathbf{m}}$:

- If $\alpha = 0$ then $L'_\alpha = \emptyset$ and $\mathbb{P}^{\mathbf{m}} \restriction L_0 \lessdot \mathbb{P}^{\mathbf{m}}$ holds by assumption.
- $\alpha \to \alpha + 1$: We assume inductively that $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha) \lessdot \mathbb{P}^{\mathbf{m}}$.

  Assume that $p, q \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ are compatible in $\mathbb{P}^{\mathbf{m}}$ and w.l.o.g. let $r \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_{\alpha+1})$ be a common lower bound. We can assume this, because $L_2 \cup L'_{\alpha+1}$ is an initial segment of $L^{\mathbf{m}}$. Since $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha) \lessdot \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_\alpha)$, there exists $r' \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha)$ which is a reduct of $r \restriction (L_2 \cup L'_\alpha)$. Define a condition $\bar{r} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ such that $\bar{r}(t) := r'(t)$ if $t \in L_0 \cup L'_\alpha$ and $\bar{r}(t) := r(t)$ if $\mathrm{dp}_{L'}(t) = \alpha$. By (†) we can deduce $\bar{r} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$.
  Towards a contradiction assume that $\bar{r} \not\leq_{\mathbb{P}^{\mathbf{m}}} p$. W.l.o.g. we have $\mathrm{dom}(p) \subseteq \mathrm{dom}(\bar{r})$ and hence, by Lemma 5.1.7, there exists $\bar{r}' \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_{\alpha+1})$ with $\bar{r}' \leq_{\mathbb{P}^{\mathbf{m}}} \bar{r}$ and $t \in \mathrm{dom}(p)$ such that $\bar{r}' \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \bar{r}'(t)$ and $p(t)$ are incompatible in $\dot{Q}^{\mathbf{m}}_t$.
  There are now 2 cases: If $\mathrm{dp}_{L'}(t) < \alpha$ pick a condition $\bar{r}'' \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha)$ which is a reduct of $\bar{r}' \restriction (L_2 \cup L'_\alpha)$. Then $\bar{r}'' \leq_{\mathbb{P}^{\mathbf{m}}} \bar{r} \restriction (L_0 \cup L'_\alpha)$ and $\bar{r}''$ is incompatible with $p \restriction (L_0 \cup L'_\alpha)$. Since $r' = \bar{r} \restriction (L_0 \cup L'_\alpha)$, the conditions r and $\bar{r}''$ are compatible. But this leads to a contradiction, because $r \leq_{\mathbb{P}^{\mathbf{m}}} p$.
  If $\mathrm{dp}_{L'}(t) = \alpha$ then $\bar{r}(t)$ and $p(t)$ are $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha)$-names by definition. Since $r' \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha)$ is a reduct of $r \restriction (L_2 \cup L'_\alpha)$ it follows that $\bar{r} \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \bar{r}(t) = r(t) \leq_{\mathbb{H}} p(t)$. But this leads to a contradiction, because $\bar{r}' \restriction (L_2 \cup L'_\alpha) \leq_{\mathbb{P}^{\mathbf{m}}} \bar{r} \restriction (L_0 \cup L'_\alpha)$ and $\bar{r}' \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \bar{r}'(t) \leq_{\mathbb{H}} \bar{r}(t)$.
  Similarly show that $\bar{r} \leq_{\mathbb{P}^{\mathbf{m}}} q$. Hence, p and q are also compatible in $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$.

  Now let $A \subseteq \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ be a maximal antichain, and assume that A is not maximal in $\mathbb{P}^{\mathbf{m}}$. W.l.o.g. let $p \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_{\alpha+1})$ be incompatible with every element from A. Again, there exists $p' \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha)$ which is a reduct of $p \restriction (L_2 \cup L'_\alpha)$. Define a condition $\bar{p} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ such that $\bar{p}(t) := p'(t)$ if $t \in L_0 \cup L'_\alpha$ and $\bar{p}(t) := p(t)$ if $\mathrm{dp}_L(t) = \alpha$. Again, we can deduce $\bar{p} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ by (†).
  Since A is maximal in $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ there exists $q \in A$ such that $\bar{p}$ and q are compatible in $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$. Let $r \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ be a common lower bound of $\bar{p}$ and q. Since $r \restriction (L_0 \cup L'_\alpha) \leq_{\mathbb{P}^{\mathbf{m}}} \bar{p} \restriction (L_0 \cup L'_\alpha)$, it follows that $r \restriction (L_0 \cup L'_\alpha)$ and $p \restriction (L_2 \cup L'_\alpha)$ are compatible in $\mathbb{P}^{\mathbf{m}} \restriction L_2$ and let $r' \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_\alpha)$ be a common lower bound. Define a condition $\bar{r} \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_{\alpha+1})$ such that $\bar{r}(t) := r'(t)$ if $t \in L_2 \cup L'_\alpha$ and $\bar{r}(t) := r(t)$ if $\mathrm{dp}_L(t) = \alpha$. It follows that $\bar{r}$ is a common lower bound of r and p. This however immediately leads to a contradiction, since r and p must be incompatible. Hence, A is also maximal in $\mathbb{P}^{\mathbf{m}}$.

- If γ is a limit we assume inductively that $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha) \lessdot \mathbb{P}^{\mathbf{m}}$ for every $\alpha < \gamma$.

  Assume that $p, q \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$ are compatible in $\mathbb{P}^{\mathbf{m}}$ and let $r \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_\gamma)$ be a common lower bound. There exists $r' \in \mathbb{P}^{\mathbf{m}} \restriction L_0$ which is a reduct of $r \restriction L_2$. Define a condition $\bar{r} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$ such that $\bar{r}(t) := r'(t)$ if $t \in L_0$ and $\bar{r}(t) := r(t)$ if $t \in L'_\gamma$. Again, we can deduce $\bar{r} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_{\alpha+1})$ by (†).
  It follows that for every $\alpha < \gamma$ the condition $\bar{r} \restriction (L_0 \cup L'_\alpha)$ is a reduct of $r \restriction (L_2 \cup L'_\alpha)$: For $r'' \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha)$ such that $r'' \leq_{\mathbb{P}^{\mathbf{m}}} \bar{r} \restriction (L_0 \cup L'_\alpha)$ one can inductively construct a lower bound of $r \restriction (L_2 \cup L'_\alpha)$ and r'' in $\mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_\alpha)$.
  Towards a contradiction assume that $\bar{r} \not\leq_{\mathbb{P}^{\mathbf{m}}} p$. Again by Lemma 5.1.7 there exists $\bar{r}' \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_\gamma)$ with $\bar{r}' \leq_{\mathbb{P}^{\mathbf{m}}} \bar{r}$ and $t \in \mathrm{dom}(p)$ such that $\bar{r}' \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} \bar{r}'(t)$ and $p(t)$ are incompatible in $\dot{Q}^{\mathbf{m}}_t$. Fix $\alpha < \gamma$ such that $t \in L'_\alpha$. Pick a condition $\bar{r}'' \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\alpha)$ which is a reduct of $\bar{r}' \restriction (L_2 \cup L'_\alpha)$. Then $\bar{r}'' \leq_{\mathbb{P}^{\mathbf{m}}} \bar{r} \restriction (L_0 \cup L'_\alpha)$ and $\bar{r}''$ is incompatible with $p \restriction (L_0 \cup L'_\alpha)$. But then r and $\bar{r}''$ are compatible, which leads to a contradiction, because $r \restriction (L_2 \cup L'_\alpha) \leq_{\mathbb{P}^{\mathbf{m}}} p \restriction (L_0 \cup L'_\alpha)$. Similarly show that $\bar{r} \leq_{\mathbb{P}^{\mathbf{m}}} q$. Hence, p and q are also compatible in $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$.

  Now let $A \subseteq \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$ be a maximal antichain, and assume that A is not maximal in $\mathbb{P}^{\mathbf{m}}$. Let $p \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_\gamma)$ be incompatible with every element from A. Again, there exists $p' \in \mathbb{P}^{\mathbf{m}} \restriction L_0$ which is a reduct of $p \restriction L_2$. Define a condition $\bar{p} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$ such that $\bar{p}(t) := p'(t)$ if $t \in L_0$ and $\bar{p}(t) := p(t)$ if $t \in L'_\gamma$. Again, we can deduce $\bar{p} \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$ by (†).
  Since A is maximal in $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$ there exists $q \in A$ such that $\bar{p}$ and q are compatible in $\mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$. Let $r \in \mathbb{P}^{\mathbf{m}} \restriction (L_0 \cup L'_\gamma)$ be a common lower bound of $\bar{p}$ and q. Since $r \restriction L_0 \leq_{\mathbb{P}^{\mathbf{m}}} \bar{p} \restriction L_0$, it follows that $r \restriction L_0$ and $p \restriction L_2$ are compatible in $\mathbb{P}^{\mathbf{m}} \restriction L_2$ and let $r' \in \mathbb{P}^{\mathbf{m}} \restriction L_2$ be a common lower bound. Define a condition $\bar{r} \in \mathbb{P}^{\mathbf{m}} \restriction (L_2 \cup L'_\gamma)$ such that $\bar{r}(t) := r'(t)$ if $t \in L_2$ and $\bar{r}(t) := r(t)$ if $t \in L_1 \setminus L_0$. It follows by induction on $\mathrm{dom}(\bar{r})$ that $\bar{r}$ is a common lower bound of r and p. This however immediately leads to a contradiction, since r and p must be incompatible. Hence, A is also maximal in $\mathbb{P}^{\mathbf{m}}$. □

Lemma 5.3.12. Let $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$ such that $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$. Assume that $L_0, L_1, L_2 \subseteq L^{\mathbf{m}_2}$ such that:

- L_2 is an initial segment of $L^{\mathbf{m}_2}$.
- L_1 is an initial segment of $L^{\mathbf{m}_1}$.
- $L_0 = L_1 \cap L_2$
- $\mathbb{P}^{\mathbf{m}_1} \restriction L_0 \lessdot \mathbb{P}^{\mathbf{m}_2} \restriction L_2$
- $L_1 \setminus L_0$ is disjoint from M

Then $\mathbb{P}^{\mathbf{m}_1} \restriction L_1 \lessdot \mathbb{P}^{\mathbf{m}_2}$.

Proof. Set $L' := L_1 \setminus L_0$. We will show by induction on $\alpha \leq \infty_{L'}$ that $\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\alpha) \lessdot \mathbb{P}^{\mathbf{m}_2}$:

- If $\alpha = 0$ then $L'_\alpha = \emptyset$ and we know that $\mathbb{P}^{\mathbf{m}_1} \restriction L_0 \lessdot \mathbb{P}^{\mathbf{m}_2}$ holds by assumption.

- $\alpha \to \alpha + 1$: We assume inductively that $\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\alpha) \lessdot \mathbb{P}^{\mathbf{m}_2}$. Let $p, q \in \mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_{\alpha+1})$ such that $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$. We have:
  - $q \leq_{\mathbb{P}^{\mathbf{m}_1}} p$ iff
  - $q \restriction (L_0 \cup L'_\alpha) \leq_{\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\alpha)} p \restriction (L_0 \cup L'_\alpha)$ and for every $t \in \mathrm{dom}(p)$ with $\mathrm{dp}_{L'}(t) = \alpha$ we have that $q \restriction L^{\mathbf{m}_1}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{m}_1}_{<t}} q(t) \leq_{\mathbb{H}} p(t)$ iff
  - $q \restriction (L_2 \cup L'_\alpha) \leq_{\mathbb{P}^{\mathbf{m}_2} \restriction (L_2 \cup L'_\alpha)} p \restriction (L_2 \cup L'_\alpha)$ and for every $t \in \mathrm{dom}(p)$ with $\mathrm{dp}_{L'}(t) = \alpha$ we have that $q \restriction L^{\mathbf{m}_2}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}_2} \restriction L^{\mathbf{m}_2}_{<t}} q(t) \leq_{\mathbb{H}} p(t)$ iff
  - $q \leq_{\mathbb{P}^{\mathbf{m}_2}} p$

  This holds because $L_2 \cup L'_\alpha$ is an initial segment of $L^{\mathbf{m}_2}$, $\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{m}_1}_{<t} \lessdot \mathbb{P}^{\mathbf{m}_2} \restriction L^{\mathbf{m}_2}_{<t}$, the statement '$q(t) \leq_{\mathbb{H}} p(t)$' is arithmetical and $B(\bar{x}) = y$ is absolute between $V^{\mathbb{P}^{\mathbf{m}_1} \restriction L^{\mathbf{m}_1}_{<t}}$ and $V^{\mathbb{P}^{\mathbf{m}_2} \restriction L^{\mathbf{m}_2}_{<t}}$. Hence $\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_{\alpha+1}) = \mathbb{P}^{\mathbf{m}_2} \restriction (L_0 \cup L'_{\alpha+1})$ as partial orders.
  By Lemma 5.3.11 with $\mathbf{m} := \mathbf{m}_2$ it follows that $\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_{\alpha+1}) \lessdot \mathbb{P}^{\mathbf{m}_2}$.

- If γ is a limit we assume that for every $\alpha < \gamma$ we have $\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\alpha) \lessdot \mathbb{P}^{\mathbf{m}_2}$. Let $p, q \in \mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\gamma)$ such that $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$. We have:
  - $q \leq_{\mathbb{P}^{\mathbf{m}_1}} p$ iff
  - $q \restriction (L_0 \cup L'_\alpha) \leq_{\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\alpha)} p \restriction (L_0 \cup L'_\alpha)$ for every $\alpha < \gamma$ iff
  - $q \restriction (L_2 \cup L'_\alpha) \leq_{\mathbb{P}^{\mathbf{m}_2} \restriction (L_2 \cup L'_\alpha)} p \restriction (L_2 \cup L'_\alpha)$ for every $\alpha < \gamma$ iff
  - $q \leq_{\mathbb{P}^{\mathbf{m}_2}} p$

  Hence $\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\gamma) = \mathbb{P}^{\mathbf{m}_2} \restriction (L_0 \cup L'_\gamma)$ as partial orders.
  Again, by Lemma 5.3.11 with $\mathbf{m} := \mathbf{m}_2$ it follows that $\mathbb{P}^{\mathbf{m}_1} \restriction (L_0 \cup L'_\gamma) \lessdot \mathbb{P}^{\mathbf{m}_2}$. □

From this point on we denote by $\mathbf{m}^*$ the iteration parameter in $\mathbf{M}_{ec}$ which we constructed in Theorem 5.1.13.

Lemma 5.3.13. Let $\mathbf{m} \in \mathbf{M}$ be wide. Then the following holds true:

1. Let $(t_1, \bar{s}_1) \in \mathcal{Y}_{\mathbf{m}}$. Let $t_2 \in L^{\mathbf{m}} \setminus M$ be such that $\mathbf{m} \restriction (t_1/E^{\mathbf{m}} \cup M) \approx_{\mathbf{M}} \mathbf{m} \restriction (t_2/E^{\mathbf{m}} \cup M)$. Then there exists $\bar{s}_2 \in (t_2/E^{\mathbf{m}})^{<\kappa^+}$ such that $(t_1, \bar{s}_1)$ and $(t_2, \bar{s}_2)$ are 1-equivalent over $\mathbf{m}$.

2. Let $\mathbf{m}^* \leq_{\mathbf{M}} \mathbf{m}$ and $(t_1, \bar{s}_1), (t_2, \bar{s}_2) \in \mathcal{Y}_{\mathbf{m}^*}$. Then $(t_1, \bar{s}_1)$ and $(t_2, \bar{s}_2)$ are k-equivalent over $\mathbf{m}^*$ iff they are k-equivalent over $\mathbf{m}$ for $k \in \{0, 1\}$.

Proof. ad 1) Let $f \colon L^{\mathbf{m}} \to L^{\mathbf{m}}$ be an isomorphism mapping $t_1/E^{\mathbf{m}}$ onto $t_2/E^{\mathbf{m}}$, which exists by Lemma 5.1.17. If we set $\bar{s}_2(i) := f(\bar{s}_1(i))$ for every $i < \mathrm{dom}(\bar{s}_1)$, then it easily follows that $(t_1, \bar{s}_1)$ and $(t_2, \bar{s}_2)$ are 1-equivalent over $\mathbf{m}$.
ad 2) This is should be straightforward using the fact that $\mathbb{P}^{\mathbf{m}^*} \lessdot \mathbb{P}^{\mathbf{m}}$. □

We are ready to prove the crucial theorem:

Theorem 5.3.14. $\mathbf{M}_{ec} \cap \mathbf{M}_{\leq \lambda_2}$ is non-empty.

Proof. We will construct $\mathbf{n}^* \in \mathbf{M}_{ec} \cap \mathbf{M}_{\leq \lambda_2}$ from within $\mathbf{m}^*$: For every $[(t, \bar{s})]_{\text{1-equ.}} \in \mathcal{Y}_{\mathbf{m}^*}/_{\text{1-equ.}}$ we want to add λ_2 many disjoint, $\approx_{\mathbf{M}}$-equivalent copies of $\mathbf{m}^* \restriction (t/E^{\mathbf{m}^*} \cup M)$ to $\mathbf{m}^* \restriction M$. This can be done by Lemma 5.1.9 using that $\mathbf{m}^*$ is wide. Since $|\mathcal{Y}_{\mathbf{m}^*}/_{\text{1-equ.}}| = \lambda_2$ by Lemma 5.3.7, it follows that also $|L^{\mathbf{n}^*}| = \lambda_2$. Obviously $\mathbf{n}^* \leq_{\mathbf{M}} \mathbf{m}^*$.

It remains to be shown that $\mathbf{n}^* \in \mathbf{M}_{ec}$. Let $\mathbf{n}_1, \mathbf{n}_2 \in \mathbf{M}_{ec}$ such that $\mathbf{n}^* \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$. W.l.o.g. we can assume that $L^{\mathbf{n}_i} \cap L^{\mathbf{m}^*} = L^{\mathbf{n}^*}$ for $i \in \{1, 2\}$, hence we can amalgamate $\mathbf{n}_i$ and $\mathbf{m}^*$ over $\mathbf{n}^*$ to get $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$ with $\mathbf{m}^* \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$. Since $\mathbf{m}^* \in \mathbf{M}_{ec}$ we have $\mathbb{P}^{\mathbf{m}^*} \lessdot \mathbb{P}^{\mathbf{m}_1} \lessdot \mathbb{P}^{\mathbf{m}_2}$. It will suffice to show that $\mathbb{P}^{\mathbf{n}_i} \lessdot \mathbb{P}^{\mathbf{m}_i}$ for $i \in \{1, 2\}$, because then $\mathbb{P}^{\mathbf{n}_i} \lessdot \mathbb{P}^{\mathbf{m}_2}$ for $i \in \{1, 2\}$, and hence $\mathbb{P}^{\mathbf{n}_1} \lessdot \mathbb{P}^{\mathbf{n}_2}$ follows.

Let $\mathbf{n}, \mathbf{m} \in \mathbf{M}$ be such that $\mathbf{n}^* \leq_{\mathbf{M}} \mathbf{n}$, $\mathbf{m}^* \leq_{\mathbf{M}} \mathbf{m}$ and $\mathbf{n} \leq_{\mathbf{M}} \mathbf{m}$. For $\alpha \leq \infty_M$ we define $L^{\mathbf{m}}_{<_M \alpha} := \{t \in L^{\mathbf{m}} \colon \exists s \in M\ t \leq_{L^{\mathbf{m}}} s \wedge t \neq s \wedge \mathrm{dp}_M(s) \leq \alpha\}$ and $L^{\mathbf{m}}_{\leq_M \alpha} := \{t \in L^{\mathbf{m}} \colon \exists s \in M\ t \leq_{L^{\mathbf{m}}} s \wedge \mathrm{dp}_M(s) \leq \alpha\}$. We will show by induction on $\alpha \leq \infty_M$ that $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M \alpha} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$ as well as $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M \alpha} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$. This will then yield $\mathbb{P}^{\mathbf{n}} \lessdot \mathbb{P}^{\mathbf{m}}$. If we set $\mathbf{n} := \mathbf{n}_i$ and $\mathbf{m} := \mathbf{m}_i$, then we get $\mathbb{P}^{\mathbf{n}_i} \lessdot \mathbb{P}^{\mathbf{m}_i}$.

- $\alpha = 0$: We obviously have $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M 0} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$, because $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M 0} \cap L^{\mathbf{n}})$ is a side by side product of iterations of κ-Hechler forcing all of which also appear in $\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<_M 0}$. This follows in particular from the fact that for $t_1, t_2 \in L^{\mathbf{m}}_{<_M 0}$ with $\neg\, t_1\, E^{\mathbf{m}}\, t_2$ neither $t_1 \leq_{L^{\mathbf{m}}} t_2$ nor $t_2 \leq_{L^{\mathbf{m}}} t_1$ can hold.

  Since $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M 0} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$ we can deduce for $p, q \in \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ that $q \leq_{\mathbb{P}^{\mathbf{n}}} p$ iff $q \leq_{\mathbb{P}^{\mathbf{m}}} p$.

  Assume that $p, q \in \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ are compatible in $\mathbb{P}^{\mathbf{m}}$ and let $r \in \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{\leq_M 0}$ be a common lower bound. Enumerate $(\mathrm{wsupp}(p) \cup \mathrm{wsupp}(q) \cup \mathrm{wsupp}(r)) \setminus M$ modulo $E^{\mathbf{m}}$ as $(t_i)_{i<i^*}$ and for every $i < i^*$ enumerate $t_i/E^{\mathbf{m}} \cap (\mathrm{fsupp}(p) \cup \mathrm{fsupp}(q) \cup \mathrm{fsupp}(r)\,)$ as $\bar{s}_i$. Obviously, $\bar{s}_i \in (t_i/E^{\mathbf{m}})^{<\kappa^+}$. Since $\mathbf{m}^*$ is wide, it follows that for every $i < i^*$ there are λ_2 many disjoint, $\approx_{\mathbf{M}}$-equivalent copies of $\mathbf{m} \restriction (t_i/E^{\mathbf{m}} \cup M)$ in $\mathbf{m}^*$. By Lemma 5.3.13 it follows that for every $i < i^*$ there exists $(t'_i, \bar{s}'_i) \in \mathcal{Y}_{\mathbf{m}^*}$ which is 0-equivalent over $\mathbf{m}$ to $(t_i, \bar{s}_i)$. By the construction of $\mathbf{n}^*$ it follows that for every $i < i^*$ there exist λ_2 many $E^{\mathbf{n}^*}$-disjoint $(t''_i, \bar{s}''_i) \in \mathcal{Y}_{\mathbf{n}^*}$ which are 0-equivalent over $\mathbf{m}^*$ to $(t'_i, \bar{s}'_i)$. Again by Lemma 5.3.13, $(t''_i, \bar{s}''_i)$ and $(t'_i, \bar{s}'_i)$ are also 0-equivalent over $\mathbf{m}$ for every $i < i^*$, and hence $(t''_i, \bar{s}''_i)$ and $(t_i, \bar{s}_i)$ are 0-equivalent over $\mathbf{m}$ for every $i < i^*$.
  For $t_i \in \mathrm{wsupp}(r) \setminus (\mathrm{wsupp}(p) \cup \mathrm{wsupp}(q))$ choose 0-equivalent over $\mathbf{m}$, mutually $E^{\mathbf{n}^*}$-disjoint $(t''_i, \bar{s}''_i) \in \mathcal{Y}_{\mathbf{n}^*}$ such that they are also $E^{\mathbf{n}^*}$-disjoint from $\mathrm{wsupp}(p) \cup \mathrm{wsupp}(q)$, and let f_i be the corresponding weak isomorphisms. For $t_i \in \mathrm{wsupp}(p) \cup \mathrm{wsupp}(q)$ set $(t''_i, \bar{s}''_i) := (t_i, \bar{s}_i)$ and define f_i to be the identity.
  Using Lemma 5.3.10 it follows that $f := \bigcup_{i<i^*} f_i$ induces an isomorphism $\hat{f} \colon \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}_i \cup M) \to \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}''_i \cup M)$ such that $\hat{f}(p) = p$ and $\hat{f}(q) = q$. Since

$r \in \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}_i \cup M)$, it follows that $\hat{f}(r) \in \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}''_i \cup M) \subseteq \mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{n}}$ and it is a lower bound of p and q in $\mathbb{P}^{\mathbf{m}}$. Since $f(s) = s$ for every $s \in M$, we can deduce that $\hat{f}(r) \in \mathbb{P}^{\mathbf{m}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ [11]. Since $\mathbb{P}^{\mathbf{m}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}}) = \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ as partial orders, it follows that $\hat{f}(r)$ is also a lower bound of p and q in $\mathbb{P}^{\mathbf{n}}$. Hence, p and q are also compatible in $\mathbb{P}^{\mathbf{n}}$.

Now let $A \subseteq \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ be a maximal antichain. It follows that A is also an antichain in $\mathbb{P}^{\mathbf{m}}$, and assume towards a contradiction that A is not maximal in $\mathbb{P}^{\mathbf{m}}$. Hence, there exists $q \in \mathbb{P}^{\mathbf{m}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ such that q is incompatible with every $p \in A$. Again, enumerate $(\bigcup_{p\in A} \operatorname{wsupp}(p) \cup \operatorname{wsupp}(q)) \setminus M$ modulo $E^{\mathbf{m}}$ as $(t_i)_{i<i^*}$ and for every $i < i^*$ enumerate $t_i/E^{\mathbf{m}} \cap (\bigcup_{p\in A} \operatorname{fsupp}(p) \cup \operatorname{fsupp}(q)\,)$ as $\bar{s}_i$. Obviously, $\bar{s}_i \in (t_i/E^{\mathbf{m}})^{<\kappa^+}$. Again, for every $i < i^*$ we can find λ_2 many $E^{\mathbf{n}^*}$-disjoint $(t''_i, \bar{s}''_i) \in \mathcal{Y}_{\mathbf{n}^*}$ which are 1-equivalent over $\mathbf{m}$ using Lemma 5.3.13 and noting how $\mathbf{n}^*$ was constructed. For $t_i \in \operatorname{wsupp}(q) \setminus \bigcup_{p\in A} \operatorname{wsupp}(p)$ choose 1-equivalent over $\mathbf{m}$, mutually $E^{\mathbf{n}^*}$-disjoint $(t''_i, \bar{s}''_i) \in \mathcal{Y}_{\mathbf{n}^*}$ such that they are also $E^{\mathbf{n}^*}$-disjoint from $\bigcup_{p\in A} \operatorname{wsupp}(p)$, and let f_i be the corresponding weak isomorphisms. For $t_i \in \bigcup_{p\in A} \operatorname{wsupp}(p)$ set $(t''_i, \bar{s}''_i) := (t_i, \bar{s}_i)$ and define f_i to be the identity.
Using Lemma 5.3.10 it follows that $f := \bigcup_{i<i} f_i$ induces an isomorphism $\hat{f}\colon \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}_i \cup M) \to \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<i^*} \bar{s}''_i \cup M)$ such that $\hat{f}(p) = p$ for every $p \in A$. Again $\hat{f}(q) \in \mathbb{P}^{\mathbf{m}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$, and since $\mathbb{P}^{\mathbf{m}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}}) = \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ as partial orders, there exists $p' \in A$ such that $\hat{f}(q)$ and p' are compatible in $\mathbb{P}^{\mathbf{n}}$. Hence, let $r \in \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M 0} \cap L^{\mathbf{n}})$ be a common lower bound. Obviously, r is also a lower bound of $\hat{f}(q)$ and p' in $\mathbb{P}^{\mathbf{m}}$.
We will now aim to extend $\hat{f}^{-1}$ to an isomorphism $\hat{g}\colon \mathbb{P}^{\mathbf{m}} \restriction L_1 \to \mathbb{P}^{\mathbf{m}} \restriction L_2$ such that $\bigcup_{i<i^*} \bar{s}''_i \cup M \subseteq L_1 \subseteq L^{\mathbf{n}}$, $\bigcup_{i<i^*} \bar{s}_i \cup M \subseteq L_2 \subseteq L^{\mathbf{m}}$ and $r \in \mathbb{P}^{\mathbf{m}} \restriction L_1$. Then $\hat{g}(r) \in \mathbb{P}^{\mathbf{m}}$ will witness that $\hat{g}(p') = p'$ and $\hat{g}(\hat{f}(q)) = q$ are compatible in $\mathbb{P}^{\mathbf{m}}$, which is a contradiction. To this end enumerate $(\bigcup_{p\in A} \operatorname{wsupp}(p) \cup \operatorname{wsupp}(r)) \setminus M$ modulo $E^{\mathbf{m}}$ as $(t''_i)_{i<j^*}$ extending $(t''_i)_{i<i^*}$, and enumerate $t''_i/E^{\mathbf{m}} \cap (\bigcup_{p\in A} \operatorname{fsupp}(p) \cup \operatorname{fsupp}(r))$ as $\bar{z}''_i$ extending $\bar{s}''_i$ for every $i < i^*$. [12] Since $(t''_i, \bar{s}''_i)$ and $(t_i, \bar{s}_i)$ are 1-equivalent over $\mathbf{m}$ for every $i < i^*$, we can find $\bar{z}_i \in (t_i/E^{\mathbf{m}})^{<\kappa^+}$ extending $\bar{s}$ such that $(t''_i, \bar{z}''_i)$ and $(t_i, \bar{z}_i)$ are 0-equivalent over $\mathbf{m}$, and let g_i be the corresponding weak isomorphisms. For $i \in j^* \setminus i^*$ enumerate $t''_i/E^{\mathbf{m}} \cap \operatorname{fsupp}(r)$ as $\bar{z}''_i$, set $(t_i, \bar{z}_i) := (t''_i, \bar{z}''_i)$ and define g_i to be the identity.
Since for every $i < i^*$ the isomorphism $\hat{g}_i$ obviously extends $\hat{f}_i^{-1}$, it follows that the isomorphism $\hat{g}\colon \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<j^*} \bar{z}''_i \cup M) \to \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<j^*} \bar{z}_i \cup M)$, which extends $\hat{g}_i$ for every $i < j^*$ and exists by Lemma 5.3.10, necessarily extends $\hat{f}^{-1}$. Clearly, we have $r \in \mathbb{P}^{\mathbf{m}} \restriction (\bigcup_{i<j^*} \bar{z}''_i \cup M)$.

[11] Note that weak isomorphisms do not necessarily preserve $\operatorname{dp}_{L^{\mathbf{m}}}$, but they preserve dp_M. This is the reason why we do induction along $L^{\mathbf{m}}_{<_M \alpha}$ and $L^{\mathbf{m}}_{\leq_M \alpha}$ respectively.

[12] W.l.o.g. let $\operatorname{wsupp}(\hat{f}(q)) \subseteq \operatorname{wsupp}(r)$ and $\operatorname{fsupp}(\hat{f}(q)) \subseteq \operatorname{fsupp}(r)$.

- $\alpha \to \alpha+1$: We assume inductively that $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M \alpha} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$. By Lemma 5.3.12 we can deduce $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M \alpha+1} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$.

  Since $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M \alpha+1} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$ we can deduce for $p, q \in \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M \alpha+1} \cap L^{\mathbf{n}})$ that $q \leq_{\mathbb{P}^{\mathbf{n}}} p$ iff $q \leq_{\mathbb{P}^{\mathbf{m}}} p$.
  Show that $p, q \in \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M \alpha+1} \cap L^{\mathbf{n}})$ are compatible in $\mathbb{P}^{\mathbf{m}}$ iff they are compatible in $\mathbb{P}^{\mathbf{n}}$ similar to the base case.
  Let $A \subseteq \mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M \alpha+1} \cap L^{\mathbf{n}})$ be a maximal antichain. Show that A is also maximal in $\mathbb{P}^{\mathbf{m}}$ similar to the base case.

- If γ is a limit we assume inductively that $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M \alpha} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$ for every $\alpha < \gamma$. Set $L := \bigcup_{\alpha<\gamma} L^{\mathbf{m}}_{<_M \alpha}$. We must first show that $\mathbb{P}^{\mathbf{n}} \restriction (L \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$.
  Obviously, $\mathbb{P}^{\mathbf{n}} \restriction (L \cap L^{\mathbf{n}}) = \mathbb{P}^{\mathbf{m}} \restriction (L \cap L^{\mathbf{n}})$ as partial orders. Use a weak automorphism argument to show that $p, q \in \mathbb{P}^{\mathbf{n}} \restriction (L \cap L^{\mathbf{n}})$ are compatible in $\mathbb{P}^{\mathbf{m}}$ iff they are compatible in $\mathbb{P}^{\mathbf{n}}$. Again using a weak automorphism argument show that if $A \subseteq \mathbb{P}^{\mathbf{n}} \restriction (L \cap L^{\mathbf{n}})$ is a maximal antichain, then it is also maximal in $\mathbb{P}^{\mathbf{m}}$.
  Next use Lemma 5.3.12 to show that $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M \gamma} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$.

  Now that we know that $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{<_M \gamma} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$, we can show $\mathbb{P}^{\mathbf{n}} \restriction (L^{\mathbf{m}}_{\leq_M \gamma} \cap L^{\mathbf{n}}) \lessdot \mathbb{P}^{\mathbf{m}}$ similar to the base case.

This shows that $\mathbf{n}^* \in \mathbf{M}_{ec} \cap \mathbf{M}_{\leq \lambda_2}$. □

5.4 Iterating $\mathbb{R}_\kappa$ without adding dominating reals

Last but not least we want to discuss how to iterate Shelah's higher random forcing for κ supercompact using a Corrected Iteration, and show that this way no dominating reals on κ^κ are being added. Unfortunately, there seems to be a general problem when actually applying the Corrected Iteration.
We shall first elaborate on the problem and explain the difficulties concerning it. Then, assuming this problem can be fixed, we sketch how to achieve the consistency of $\kappa^+ = \mathfrak{b}_\kappa < \mathrm{cov}(\mathrm{id}(\mathbb{R}_\kappa)) = \kappa^{++}$.

Let κ be supercompact and let $\mathbb{Q}_M$ denote the Corrected Iteration along a well-founded partial order M (of a $\leq\kappa$-strategically closed, κ^+-c.c. 'simply definable' forcing notion $\mathcal{Q}$ preserving supercompactness) which we constructed in the previous sections. Since $\mathbb{Q}_M$ is a complete Boolean algebra, it obviously cannot be $<\kappa$-directed closed. However, we would like that $\mathbb{Q}_M$ satisfies a slightly weaker property:

Conjecture 5.4.1. Let $\theta > \kappa$ be a regular and sufficiently large cardinal. Let $N \prec H(\theta)$ be of size $<\kappa$ such that $\mathbb{Q}_M \in N$, $\kappa_N := \sup(\kappa \cap N)$ is inaccessible and $N^{<\kappa_N} \subseteq N$. Then every $(N, \mathbb{Q}_M)$-generic filter G_N has a lower bound in $\mathbb{Q}_M$.

Since $\mathbb{Q}_M$ is a complete subforcing of $\mathbb{P}^{\mathbf{m}^*}$, which is a standard iteration (along a well-founded partial order) of $\mathcal{Q}$, this conjecture seems plausible. Indeed it appears as an

'obvious' claim in ([She20]) for a Corrected Iteration of the respective forcing considered there.
However, a more careful analysis of the conjecture leads to the following problem: If one wants to construct a lower bound $\varphi \in \mathbb{Q}_M$ of G_N, one needs to construct a witness $p_\varphi \in \mathbb{P}^{\mathbf{m}^*}$ at the same time. If one tries to do this inductively, the discontinuity of the projection $\pi \colon \mathbb{P}^{\mathbf{m}^*} \to \mathbb{Q}_M$ could possibly ruin the construction in limit steps.
And indeed this issue is non-trivial: by a result of Kunen (see [Kun78]) κ-Cohen forcing has a complete subforcing which adds a κ-Aronszajn tree, and hence this complete subforcing destroys the weak compactness of κ. But since the conjecture in particular implies that if V is appropriately prepared, supercompactness is preserved (see [Koe06]), we see that even κ-Cohen forcing has a complete subforcing for which the respective conjecture cannot hold. Therefore, the only hope to prove this conjecture (for a Corrected iteration of suitable $\mathcal{Q}$) is to use the fact that $\mathbb{Q}_M$ is still 'quite similar' to a standard iteration of $\mathcal{Q}$.

We aim to establish the consistency of $\kappa^+ = \mathfrak{b}_\kappa < \mathfrak{d}_\kappa = \mathrm{cov}(\mathrm{id}(\mathbb{R}_\kappa)) = \mathrm{non}(\mathrm{id}(\mathbb{R}_\kappa)) = \mathrm{cof}(\mathrm{id}(\mathbb{R}_\kappa)) = \kappa^{++}$. To this end we will define similar to κ-Hechler forcing a Corrected Iteration for Shelah's higher random forcing $\mathbb{R}_\kappa$. For our purpose $M := \kappa^{++}$ will suffice.

Definition 5.4.2. Let $\mathbf{m} \in \mathbf{M}$ be an iteration parameter (see Definition 5.1.2). Similar to Definition 5.1.4 we will now define by induction on $\alpha \leq \infty_{L^\mathbf{m}}$ the forcing notion $\mathbb{P}^\mathbf{m}_\alpha$:

- Define $\mathbb{P}^\mathbf{m}_1$ to be the set of functions p such that $\mathrm{dom}(p) \subseteq L^\mathbf{m}_1$, $|\mathrm{dom}(p)| < \kappa$ and for every $t \in \mathrm{dom}(p)$ we have $p(t) = (\tau, S, Cl, (I_\lambda)_{\lambda \in S^\kappa_{\mathrm{inc}}})$ such that $\tau \in 2^{<\kappa}$, $S \subseteq S^\kappa_{\mathrm{inc}}$ nowhere stationary, Cl is a club disjoint from S and $I_\lambda \in \mathrm{id}(\mathbb{R}_\lambda)$ for every $\lambda \in S^\kappa_{\mathrm{inc}}$.
- If γ is a limit, we have two cases:
  - $\mathrm{cf}(\gamma) \geq \kappa$: Set $\mathbb{P}^\mathbf{m}_\gamma := \bigcup_{\alpha<\gamma} \mathbb{P}^\mathbf{m}_\alpha$.
  - $\mathrm{cf}(\gamma) < \kappa$: Define $\mathbb{P}^\mathbf{m}_\gamma$ to be the set of functions p such that $\mathrm{dom}(p) \subseteq L^\mathbf{m}_\gamma$, $|\mathrm{dom}(p)| < \kappa$ and for every $\alpha < \gamma$ we have $p \restriction L^\mathbf{m}_\alpha \in \mathbb{P}^\mathbf{m}_\alpha$.
- $\alpha \to \alpha + 1$: Define $\mathbb{P}^\mathbf{m}_{\alpha+1}$ to be the set of functions p such that $\mathrm{dom}(p) \subseteq L^\mathbf{m}_{\alpha+1}$, $|\mathrm{dom}(p)| < \kappa$, $p \restriction L^\mathbf{m}_\alpha \in \mathbb{P}^\mathbf{m}_\alpha$ and for every $t \in \mathrm{dom}(p)$ with $\mathrm{dp}_{L^\mathbf{m}}(t) = \alpha$ we have
  $$p(t) = \big(\tau, \bigcup_{j<\delta} B^1_j((\dot{r}_{t'})_{t' \in u_j}), \bigcap_{j<\delta} B^2_j((\dot{r}_{t'})_{t' \in u_j}), (\bigcup_{j<\delta} B^3_j((\dot{r}_{t'})_{t' \in u_j})(\lambda)\,)_{\lambda \in (S^\kappa_{\mathrm{inc}} \setminus \delta)}\big)$$
  where:
  - $\tau \in 2^{<\kappa}$
  - $\delta \leq \mathrm{dom}(\tau)$, $(B^i_j)_{j<\delta}$ is a sequence of ground model κ-Borel functions for every $i \in \{1, 2, 3\}$, and $u_j \in \mathcal{P}_t$ for every $j < \delta$ such that
    * $B^1_j \colon (2^\kappa)^{u_j} \to \mathfrak{P}(S^\kappa_{\mathrm{inc}})$ and $\forall \bar{x} \in (2^\kappa)^{u_j} \colon B^1_j(\bar{x})$ nowhere stationary below κ
    * $B^2_j \colon (2^\kappa)^{u_j} \to \mathfrak{P}(\kappa)$ and $\forall \bar{x} \in (2^\kappa)^{u_j} \colon B^2_j(\bar{x})$ is club $\wedge\, B^1_j(\bar{x}) \cap B^2_j(\bar{x}) = \emptyset$

- $B^3_j \colon (2^\kappa)^{u_j} \to (\mathfrak{P}(2^{<\kappa}))^{S^\kappa_{\text{inc}}}$ and $\forall \bar{x} \in (2^\kappa)^{u_j}\ \forall \lambda \in S^\kappa_{\text{inc}} \colon B^3_j(\bar{x})(\lambda) \in \text{id}(\mathbb{R}_\lambda)$ [13]

– $(\dot{r}_{t'})_{t' \in u_j}$ is a subsequence of the generic sequence $(\dot{r}_t)_{t \in L^{\mathbf{m}}_\alpha}$ added by $\mathbb{P}^{\mathbf{m}}_\alpha$.

We will use the notation $p(t) = (\tau_{p(t)}, \dot{B}^1_{p(t)}, \dot{B}^2_{p(t)}, \dot{B}^3_{p(t)})$. Clearly $\Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} p(t) = (\tau_{p(t)}, \dot{B}^1_{p(t)}, \dot{B}^2_{p(t)}, \dot{B}^3_{p(t)}) \in \mathbb{R}_\kappa$. [14]

We define $q \leq_{\mathbb{P}^{\mathbf{m}}} p$ inductively:

- $\text{dom}(p) \subseteq \text{dom}(q)$
- for every $t \in \text{dom}(p)$ we have $q \restriction L^{\mathbf{m}}_{<t} \Vdash_{\mathbb{P}^{\mathbf{m}} \restriction L^{\mathbf{m}}_{<t}} q(t) \leq_{\mathbb{R}_\kappa} p(t)$

Define $\dot{r}_t := \bigcup\{\tau \in \kappa^{<\kappa} \colon \exists p \in \dot{G}\ p(t) = (\tau, \dot{B}^1_{p(t)}, \dot{B}^2_{p(t)}, \dot{B}^3_{p(t)})\}$. Set $\mathbb{P}^{\mathbf{m}} := \mathbb{P}^{\mathbf{m}}_{\infty_L}$.

Since $\mathbb{R}_\kappa$ is not $<\kappa$-closed, we can not expect that $\mathbb{P}^{\mathbf{m}}$ is. However, we have the following lemma:

Lemma 5.4.3. $\mathbb{P}^{\mathbf{m}}$ is $\leq\kappa$-strategically closed. Furthermore, it satisfies all other properties of Lemma 5.1.6.

Following Definition 5.2.5 we can now define the Corrected Iteration of $\mathbb{R}_\kappa$:

Definition / Assumption 5.4.4. Fix $\mathbf{m}^* \in \mathbf{M}_{ec}$. We define $\mathbb{Q}_{\kappa^{++}} := \mathbb{P}^{\mathbf{m}^*}[\kappa^{++}]$. We shall assume that Conjecture 5.4.1 holds for the Corrected Iteration $\mathbb{Q}_{\kappa^{++}}$ of higher random forcing.

Since we know that V can be appropriately prepared such that this assumption in particular implies that supercompactness and therefore, weak compactness is preserved in $V^{\mathbb{Q}_{\kappa^{++}}}$, we have:

Lemma 5.4.5. Let $\dot{I}$ be a $\mathbb{Q}_{\kappa^{++}}$-name for an element of $\text{id}(\mathbb{R}_\kappa)$. Then there exist κ-Borel functions $(B^i)_{i \in \{1,2,3\}}$ in V and $u \subseteq \kappa^{++}$ of size $\leq \kappa$ such that:

$$\Vdash_{\mathbb{Q}_{\kappa^{++}}} B^1((\dot{r}_s)_{s \in u}) \subseteq S^\kappa_{\text{inc}} \text{ is nowhere stationary below } \kappa$$
$$\wedge\ B^2((\dot{r}_s)_{s \in u}) \subseteq \kappa \text{ is club} \wedge B^1((\dot{r}_s)_{s \in u}) \cap B^2((\dot{r}_s)_{s \in u}) = \emptyset$$
$$\wedge\ B^3((\dot{r}_s)_{s \in u}) \in (\mathfrak{P}(2^{<\kappa}))^{S^\kappa_{\text{inc}}} \wedge \forall \lambda \in S^\kappa_{\text{inc}} \colon B^3((\dot{r}_s)_{s \in u})(\lambda) \in \text{id}(\mathbb{R}_\lambda)$$
$$\wedge\ \dot{I} \subseteq \{x \in 2^\kappa \colon \exists^\infty \lambda \in B^1((\dot{r}_s)_{s \in u})\ x \restriction \lambda \in B^3((\dot{r}_s)_{s \in u})(\lambda)\}$$

Now this easily implies that $\Vdash_{\mathbb{Q}_{\kappa^{++}}} \text{cov}(\text{id}(\mathbb{R}_\kappa)) = \kappa^{++}$. Since $\mathbb{Q}_{\kappa^{++}}$ adds κ-Cohen reals, we also have $\Vdash_{\mathbb{Q}_{\kappa^{++}}} \mathfrak{d}_\kappa = \text{non}(\text{id}(\mathbb{R}_\kappa)) = \text{cof}(\text{id}(\mathbb{R}_\kappa)) = \kappa^{++}$.

But why does $\mathfrak{b}_\kappa$ stay small in $V^{\mathbb{Q}_{\kappa^{++}}}$?

To show this we will need the following definition:

[13] Hence $\forall \bar{x} \in (2^\kappa)^u\ \forall \lambda \in S^\kappa_{\text{inc}} \setminus \delta \colon \bigcup_{j<\delta} B^3_j(\bar{x})(\lambda) \in \text{id}(\mathbb{R}_\lambda)$ where $u := \bigcup_{j<\delta} u_j$.

[14] Pedantically, $(\tau_{p(t)}, \dot{B}^1_{p(t)}, \dot{B}^3_{p(t)} \restriction \dot{B}^1_{p(t)})$ is a witness for a condition $p(t)$ in $\mathbb{R}_\kappa$ and $\dot{B}^2_{p(t)}$ witnesses that $\dot{B}^1_{p(t)}$ is non-stationary.

Definition 5.4.6. We say that $\boxdot_\kappa$ holds iff for any sufficiently large, regular $\theta > \kappa$ and any forcing notion $\mathcal{Q} \in H(\theta)$ which is $\leq\kappa$-strategically closed, the set $\mathcal{S}_{\theta,\mathcal{Q}}$ consisting of $N \in [H(\theta)]^{<\kappa}$ with the following properties, is a stationary subset of $[H(\theta)]^{<\kappa}$:

- $N \prec H(\theta)$ and $\mathcal{Q} \in N$.
- The Mostowski collapse of N is $\mathbb{A}_N$ with $\mathrm{mos}_N \colon N \to \mathbb{A}_N$.
- $\kappa_N := \sup(\kappa \cap N)$ is inaccessible and $N^{<\kappa_N} \subseteq N$.
- $\exists \theta_N < \kappa \colon \mathbb{A}_N \subseteq H(\theta_N) \wedge \mathbb{A}_N \cap \mathrm{Ord} = \theta_N$.
- $\exists G_N \subseteq \mathrm{mos}_N(\mathcal{Q}) \colon G_N$ is an $(\mathbb{A}_N, \mathrm{mos}_N(\mathcal{Q}))$-generic filter.
- $H(\theta_N) = \mathbb{A}_N[G_N]$.

It turns out that we can force $\boxdot_\kappa$:

Lemma 5.4.7. After some preliminary forcing we have that $\boxdot_\kappa$ holds in the extension and κ is still supercompact. Hence, we can assume that $V \vDash \boxdot_\kappa$.
Under Assumption 5.4.4 we can even assume that $V \vDash$ '$\mathcal{S}'_{\theta,\mathbb{Q}_{\kappa^{++}}} := \{N \in \mathcal{S}_{\theta,\mathbb{Q}_{\kappa^{++}}} \colon \kappa_N$ is weakly compact$\}$ is stationary in $[H(\theta)]^{<\kappa}$' for any $\theta > \kappa$ regular and sufficiently large.

Now assume towards a contradiction that there exists a $\mathbb{Q}_{\kappa^{++}}$-name $\dot{f}$ and a condition $\varphi \in \mathbb{Q}_{\kappa^{++}}$ such that $\varphi \Vdash_{\mathbb{Q}_{\kappa^{++}}} \dot{f}$ dominates $\kappa^\kappa \cap V$. Since $\mathbb{Q}_{\kappa^{++}} \lessdot \mathbb{P}^{\mathbf{m}^*}$ we can work with a $p_\varphi \in \mathbb{P}^{\mathbf{m}^*}$ below φ.
Let $\theta > \kappa$ be a regular and sufficiently large cardinal. Let us pick a sequence $(N_i)_{i<\kappa}$ such that $N_i \in \mathcal{S}'_{\theta,\mathbb{Q}_{\kappa^{++}}}$, $i, \dot{f}, p_\varphi \in N_i$ and $(\kappa_{N_i})_{i<\kappa}$ is unbounded in κ. We can do this, because by $\boxdot_\kappa$ and Assumption 5.4.4 the set $\mathcal{S}'_{\theta,\mathbb{Q}_{\kappa^{++}}}$ is stationary in $[H(\theta)]^{<\kappa}$.
Let $f^* \in \kappa^\kappa$ be a function such that $\forall i < \kappa \colon f^*(i) > \kappa_{N_i}$ and let $q \in \mathbb{P}^{\mathbf{m}^*}$ with $q \leq_{\mathbb{P}^{\mathbf{m}^*}} p_\varphi$ and $i^* < \kappa$ be such that $q \Vdash_{\mathbb{P}^{\mathbf{m}^*}} \forall i \geq i^* \colon \dot{f}(i) \geq f^*(i)$. W.l.o.g. let $i^* > |\mathrm{dom}(q)| \cup \bigcup_{t \in \mathrm{dom}(q)} \mathrm{dom}(\tau_{q(t)})$.
Note that by elementarity the forcing notion $\mathrm{mos}_{N_{i^*}}(\mathbb{Q}_{\kappa^{++}})$ is just the Corrected Iteration of $\mathbb{R}_{\kappa_{N_{i^*}}}$ of length $(\kappa_{N_{i^*}})^{++}$ in $\mathbb{A}_{N_{i^*}}$.

Theorem 5.4.8. Let $(\dot{r}_\alpha^{G_{N_{i^*}}})_{\alpha<(\kappa_{N_{i^*}})^{++}}$ be the generic sequence of $\kappa_{N_{i^*}}$-reals added by $G_{N_{i^*}} \subseteq \mathrm{mos}_{N_{i^*}}(\mathbb{Q}_{\kappa^{++}})$ over $\mathbb{A}_{N_{i^*}}$. Under Assumption 5.4.4 we can now find $F \colon (\kappa_{N_{i^*}})^{++} \to (\kappa_{N_{i^*}})^{++}$ strictly increasing with $F \in \mathbb{A}_{N_{i^*}}$ such that:

- The sequence $(\dot{r}_{F(\alpha)}^{G_{N_{i^*}}})_{\alpha<(\kappa_{N_{i^*}})^{++}}$ induces a $(\mathbb{A}_{N_{i^*}}, \mathrm{mos}_{N_{i^*}}(\mathbb{Q}_{\kappa^{++}}))$-generic filter $G'_{N_{i^*}}$. (Here we use Theorem 5.3.2, i.e. the crucial property of the Corrected Iteration.)
- There exists a lower bound $r \in \mathbb{P}^{\mathbf{m}^*}$ of $\mathrm{mos}^{-1}_{N_{i^*}}[G'_{N_{i^*}}] \subseteq \mathbb{Q}_{\kappa^{++}}$ such that $r \leq_{\mathbb{P}^{\mathbf{m}^*}} q$.

Since $\mathrm{mos}^{-1}_{N_{i^*}}[G'_{N_{i^*}}]$ is a $(N_{i^*}, \mathbb{Q}_{\kappa^{++}})$-generic filter and $i^* \in N_{i^*}$, we have that $r \Vdash_{\mathbb{P}^{\mathbf{m}^*}} \dot{f}(i^*) \in N_{i^*} \cap \kappa$. But this is a contradiction to $q \Vdash_{\mathbb{P}^{\mathbf{m}^*}} \dot{f}(i^*) \geq f^*(i^*) > \kappa_{N_{i^*}}$. Hence, $\mathbb{Q}_{\kappa^{++}}$ does not add any dominating reals on κ^κ.

6 Preserving $\mathbf{\Pi}^1_1$-determinacy

In this chapter [1] we study the preservation of $\mathbf{\Pi}^1_1$-determinacy under 'simply' definable, proper forcing notions and their iterations. One of the earliest results on preservation of large cardinals [2] by forcing is the Levy-Solovay theorem (see Chapter 21 in [Jec03]) which shows that measurability is preserved by small forcing notions via lifting of elementary embeddings. Similar preservation results have been proven for many other large cardinal notions such as weakly compact, Ramsey, supercompact, huge, strong or Woodin cardinals (see [Hjo95]).

Besides small forcings, there are several other classes of forcing notions which preserve large cardinals under certain conditions: Laver showed in [Lav78] that the supercompactness of κ becomes indestructible under $<\kappa$-directed closed forcing notions after some preliminary preparation, and Johnstone showed in [Joh08] a similar result for κ strongly unfoldable.

We aim to prove the preservation of $\mathbf{\Pi}^1_1$-determinacy under any countable support iteration of 'simply' definable, proper forcing notions using the technique of capturing. While such a preservation result (for an iteration of length ω_2) easily follows from the existence of a measurable cardinal using Levy-Solovay's theorem, we do not want to make any additional assumptions on the existence of stronger large cardinals.

Surprisingly, rather little was known beforehand: Woodin showed in [Woo82] that Cohen and random forcing preserve $\mathbf{\Delta}^1_2$-determinacy. Schlicht proved in [Sch14b] that $\mathbf{\Sigma}^1_2$- absolutely c.c.c. forcing notions preserve $\mathbf{\Pi}^1_{n+1}$-determinacy, and, together with Castiblanco, extended this to Sacks, Silver, Miller, Mathias and Laver forcing (see [CS21]).

We also aim to investigate connected components of symmetric $\mathbf{\Delta}^1_3$-relations on the reals and show preservation of regularity properties such as the $\mathbf{\Delta}^1_2$- or $\mathbf{\Sigma}^1_2$-Baire property.

6.1 Prerequisites

We start with several important definitions which we will use throughout this chapter.

First, we define a class of 'simply' definable, proper forcing notions:

[1] This chapter was a joint project with Jonathan Schilhan and Philipp Schlicht.
[2] Note that $\mathbf{\Pi}^1_1$-determinacy is a large cardinal property by Theorem 1.4.7.

Definition 6.1.1. Let $\mathbb{P} = (\mathrm{dom}(\mathbb{P}), \leq_{\mathbb{P}})$ be a forcing notion such that $\mathrm{dom}(\mathbb{P}) \subseteq \omega^\omega$. Following the notation of [GJ92] we say that $\mathbb{P}$ is Suslin iff $\mathrm{dom}(\mathbb{P})$ and $\leq_{\mathbb{P}}$ have $\mathbf{\Sigma}^1_1$-definitions. We say that $\mathbb{P}$ is strongly Suslin iff additionally the incompatibility relation $\perp_{\mathbb{P}}$ also has a $\mathbf{\Sigma}^1_1$-definition.

Although most classical forcings are not literally defined on ω^ω, they are usually defined on some Polish space.

Definition 6.1.2. Let $\mathbb{P}$ be a Suslin forcing. Then a countable transitive model N satisfying ZFC*, i.e. a large enough fragment of ZFC, and containing all the parameters for the definition of $\mathbb{P}$ is called a candidate for $\mathbb{P}$. We say that $\mathbb{P}$ is proper-for-candidates iff for every candidate N for $\mathbb{P}$ and every $p \in \mathbb{P}^N$ [3] there exists $q \leq_{\mathbb{P}} p$ such that q is $(N, \mathbb{P})$-generic.

The definition of proper-for-candidates for strongly Suslin forcing is similar and includes the parameter for the definition of $\perp_{\mathbb{P}}$. Note that for a strongly Suslin forcing $\mathbb{P}$ it follows that '$\mathbb{P}$ is proper-for-candidates' is a $\mathbf{\Pi}^1_3$-property. Furthermore, note that all the classical tree forcings such as Sacks forcing $\mathbb{S}$, Silver forcing $\mathbb{SI}$, Miller forcing $\mathbb{MI}$, Mathias forcing $\mathbb{M}$ or Laver forcing $\mathbb{L}$ are Suslin, proper-for-candidates forcing notions.

Let us now turn to the notion of capturing. This technique was originally introduced in [CS21]:

Definition 6.1.3. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions such that $\mathbb{Q}$ is definable. We say that $\mathbb{Q}$ captures $\mathbb{P}$ iff for every $p \in \mathbb{P}$, every $\mathbb{P}$-name $\dot{\tau}$ for a real and every $y \in \omega^\omega$, there exists some $z \in \omega^\omega$ and some $q \leq_{\mathbb{P}} p$ such that:

$$q \Vdash_{\mathbb{P}} \exists G_{\mathbb{Q}} \colon G_{\mathbb{Q}} \text{ is } (L[y,z], \mathbb{Q}^{L[y,z]})\text{-generic} \wedge \dot{\tau} \in L[y,z][G_{\mathbb{Q}}].$$

We define a weaker version where the forcing $\mathbb{Q}$ may depend on $\dot{\tau}$:

Definition 6.1.4. Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ is captured (by forcing notions with property φ) iff for every $p \in \mathbb{P}$, every $\mathbb{P}$-name $\dot{\tau}$ for a real and every $y \in \omega^\omega$, there exists some $z \in \omega^\omega$, some $\mathbb{Q} \in L[y,z]$ (with $L[y,z] \vDash \varphi(\mathbb{Q})$) and some $q \leq_{\mathbb{P}} p$ such that:

$$q \Vdash_{\mathbb{P}} \exists G_{\mathbb{Q}} \colon G_{\mathbb{Q}} \text{ is } (L[y,z], \mathbb{Q})\text{-generic} \wedge \dot{\tau} \in L[y,z][G_{\mathbb{Q}}].$$

And we define a stronger version that provides a uniform $\mathbb{P}$-name for the relevant $\mathbb{Q}$-generic filter.

Definition 6.1.5. Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions such that $\mathbb{Q}$ is definable. We say that $\mathbb{Q}$ uniformly captures $\mathbb{P}$ iff for every $p \in \mathbb{P}$ and every $\mathbb{P}$-name $\dot{\tau}$ for a real, there exists some $z \in \omega^\omega$ and some $\mathbb{P}$-name $\dot{G}_{\mathbb{Q}}$ such that for every $y \in \omega^\omega$, there is some $q \leq_{\mathbb{P}} p$ such that:

$$q \Vdash_{\mathbb{P}} \dot{G}_{\mathbb{Q}} \text{ is } (L[z,y], \mathbb{Q}^{L[y,z]})\text{-generic} \wedge \dot{\tau} \in L[z][\dot{G}_{\mathbb{Q}}].$$

[3] Note that by $\mathbf{\Sigma}^1_1$-absoluteness we have $\mathbb{P}^N = \mathbb{P} \cap N$

Finally, let us define the following large cardinal property:

Definition 6.1.6. We say that ω_1 is inaccessible to the reals iff $\omega_1^{L[x]} < \omega_1$ for every real $x \in \omega^\omega$.

Note that this large cardinal property can be viewed as a regularity property: By a result of Solovay (see Chapter 14 in [Kan03]) it is equivalent to the statement that every $\mathbf{\Sigma}^1_2$ set has the perfect set property, and by a result of Brendle and Löwe (see [BL99]), it is equivalent to the statement that every $\mathbf{\Sigma}^1_2$ set has the property of Baire in the dominating topology.

6.2 Preserving 'The reals are $\sharp$-closed'

Let us show how capturing can be used to preserve $\mathbf{\Pi}^1_1$-determinacy. By Theorem 1.4.7 it will be enough to preserve that the reals are $\sharp$-closed:

Theorem 6.2.1. Assume that $V \models$ 'The reals are $\sharp$-closed' and let $\mathbb{P}$ be a forcing notion that is captured. Then also $V^{\mathbb{P}} \models$ 'The reals are $\sharp$-closed'.

Proof. Working in $V^{\mathbb{P}}$ let $x \in \omega^\omega$ be arbitrary. Since $\mathbb{P}$ is captured, there exist $z \in \omega^\omega \cap V$, a forcing notion $\mathbb{Q} \in L[z]$ and an $(L[z], \mathbb{Q})$-generic filter $G_{\mathbb{Q}}$ such that $x \in L[z][G_{\mathbb{Q}}]$. Since $z^\sharp$ exists, there is a non-trivial, elementary embedding $j : L[z] \to L[z]$ with $\mathrm{crit}(j) > |\mathbb{Q}|^{L[z]}$. Using a Levy-Solovay argument, j can be lifted to $j^* : L[z][G_{\mathbb{Q}}] \to L[z][G_{\mathbb{Q}}]$, and we can conclude that $j^* \restriction L[x] \colon L[x] \to L[x]$ is a non-trivial, elementary embedding. Hence $x^\sharp$ exists. □

We can also preserve that ω_1 is inaccessible to the reals:

Theorem 6.2.2. Assume that $V \models$ 'ω_1 is inaccessible to the reals' and let $\mathbb{P}$ be a forcing notion that is captured by forcing notions not collapsing ω_1. Then also $V^{\mathbb{P}} \models$ 'ω_1 is inaccessible to the reals'.

Proof. Working in $V^{\mathbb{P}}$ let $x \in \omega^\omega$ be arbitrary. Since $\mathbb{P}$ is captured by forcing notions not collapsing ω_1, there exists $z \in \omega^\omega \cap V$, a forcing notion $\mathbb{Q} \in L[z]$ such that $L[z] \models$ '$\mathbb{Q}$ does not collapse ω_1' and an $(L[z], \mathbb{Q})$-generic filter $G_{\mathbb{Q}}$ such that $x \in L[z][G_{\mathbb{Q}}]$. Hence, we can deduce $\omega_1^{L[x]} \leq \omega_1^{L[z][G_{\mathbb{Q}}]} = \omega_1^{L[z]} < \omega_1$. □

We will also need the following technical lemma:

Lemma 6.2.3. Assume that $V \models$ 'The reals are $\sharp$-closed' and let $\mathbb{P}$ and $\mathbb{P}'$ be forcing notions that are captured. Furthermore, let $\theta > \omega$ be a regular and sufficiently large cardinal, let $M \prec H(\theta)$ be a countable, elementary submodel with $\mathbb{P}' \in M$, let $\mathrm{mos} \colon M \to N$ denote be Mostowski collapse, and let $g \in V^{\mathbb{P}}$ be an $(N, \mathrm{mos}(\mathbb{P}'))$-generic filter. Then $V^{\mathbb{P}} \models \forall x \in \omega^\omega \cap N[g] \colon x^\sharp \text{ exists} \wedge x^\sharp \in \omega^\omega \cap N[g]$.

Proof. Working in $V^{\mathbb{P}}$ let $x \in \omega^\omega \cap N[g]$ be arbitrary. By Theorem 6.2.1 we know that $x^\sharp$ exists in $V^{\mathbb{P}}$, hence it remains to be shown that $x^\sharp \in \omega^\omega \cap N[g]$. Since $N \vDash$ '$\mathrm{mos}(\mathbb{P}')$ is captured', there exist $z \in \omega^\omega \cap N$, a forcing notion $\mathbb{Q} \in L[z] \cap N$ and an $(L[z] \cap N, \mathbb{Q})$-generic filter $G_{\mathbb{Q}} \in N[g]$ such that $x \in (L[z] \cap N)[G_{\mathbb{Q}}]$. Since indiscernibles for well-founded, remarkable EM blueprints are absolute for transitive models satisfying ZFC*, i.e. a large enough fragment of ZFC, and $z^\sharp \in N$, we can deduce that $(I_z)^N = I_z \cap N$, where I_z is the class of Silver indiscernibles for $L[z]$. In particular, this implies that $G_{\mathbb{Q}}$ is also $(L[z], \mathbb{Q})$-generic. Now pick an increasing sequence $(\alpha_n)_{n<\omega} \subseteq I_z \cap N$ in N with α_0 large enough and let $\alpha^* := \sup\{\alpha_n \colon n < \omega\} \in I_z \cap N$. Since $x \in L_{\alpha^*}[z][G_{\mathbb{Q}}] \subseteq L_{\aleph_\omega}[z][G_{\mathbb{Q}}]$ and $\{\alpha_n \colon n < \omega\} \cup \{\aleph_n \colon n < \omega\} \cup \{\alpha^*, \aleph_\omega\} \subseteq I_z$, we can conclude that $T^{(L_{\alpha^*}[x], \in, x, (\alpha)_{n<\omega})} = T^{(L_{\aleph_\omega}[x], \in, x, (\aleph_n)_{n<\omega})}$ (see Definition 1.3.1). By Theorem 1.3.14 this implies that $x^\sharp \in \omega^\omega \cap N[g]$. □

6.3 Capturing iterations of 'simply' definable, proper forcing notions

In [CS21] the authors showed the following lemma:

Lemma 6.3.1. If ω_1 is inaccessible to the reals, then Cohen forcing uniformly captures Sacks and Silver forcing, and Mathias forcing uniformly captures Miller, Mathias and Laver forcing.

Our main theorem generalizes the above lemma:

Theorem 6.3.2. Assume that ω_1 is inaccessible to the reals and let $\langle \mathbb{P}_\alpha, \dot{P}_\beta \colon \alpha \leq \kappa, \beta < \kappa \rangle$ with $\mathbb{P} := \mathbb{P}_{\kappa^{++}}$ be a countable support iteration of length κ of Suslin forcing notions $\dot{P}_\alpha$ such that for every $\alpha < \kappa$,

$$\Vdash_{\mathbb{P}_\alpha} \dot{P}_\alpha \text{ is proper-for-candidates in } L[A] \text{ for every } A \in [\omega^\omega]^\omega.$$ [4] [5]

Then for every $p \in \mathbb{P}$, every $\mathbb{P}$-name $\dot{\tau}$ for a real and every $y \in \omega^\omega$ there exist $z \in \omega^\omega$ and $\mathbb{Q} \in L[y,z]$ such that $L[y,z] \vDash$ '$\mathbb{Q} = \langle \mathbb{Q}_\alpha, \dot{Q}_\beta \colon \alpha \leq \alpha^*, \beta < \alpha^* \rangle$ is a countable support iteration of length $\alpha^* < \omega_1$ of Suslin, proper-for-candidates forcing notions $\dot{Q}_\beta$' and there exists $p' \in \mathbb{P}$ with $p' \leq_{\mathbb{P}} p$ such that $p' \Vdash_{\mathbb{P}} \exists G_{\mathbb{Q}} \colon G_{\mathbb{Q}}$ is $(L[y,z], \mathbb{Q})$-generic $\wedge\ \dot{\tau} \in L[y,z][G_{\mathbb{Q}}]$. In particular, $\mathbb{P}$ is captured by forcing notions of size $< \omega_1^V$.

$\mathbb{Q}$ is obtained in a very concrete way from $\mathbb{P}$, essentially as an iteration of certain iterands of $\mathbb{P}$. For example, if $\mathbb{P}$ is an iteration of Sacks forcing, then $\mathbb{Q}$ is an iteration of Sacks forcing in $L[y,z]$.

[4] Of course, $L[A]$ must contain the real parameters for the definition of P_α, and note that A can be uncountable in $L[A]$. Also note that if $N \in V^{\mathbb{P}_\alpha}$ is a candidate for P_α, then the statement '$p' \in P_\alpha$ is (N, P_α)-generic' is a $\mathbf{\Pi}^1_2$-property. Hence $V^{\mathbb{P}_\alpha} \vDash P_\alpha$ is proper-for-candidates.

[5] This is a technical requirement which we will need for the proof. Note that if $\dot{P}_\alpha$ is provably proper-for-candidates this is trivially satisfied. Also, if $\dot{P}_\alpha$ is strongly Suslin and proper-for-candidates, then it is also proper-for candidates in every $L[A]$, since $\mathbf{\Pi}^1_3$-statements are downward absolute.

We will prove this theorem in several steps using techniques from [IHJS88]. We start with an arbitrary condition $p^* \in \mathbb{P}$, an arbitrary $\mathbb{P}$-name $\dot\tau$ for a real and an arbitrary $y \in \omega^\omega$. Let $\theta > \omega$ be a sufficiently large, regular cardinal and let $M \prec H(\theta)$ be a countable, elementary submodel such that $p^*, \mathbb{P}, \dot\tau, y, \kappa \in M$. Let $\mathrm{mos}\colon M \to N$ denote the Mostowski collapse and let us define $\bar{\mathbb{P}} := \mathrm{mos}(\mathbb{P})$, $\bar p^* := \mathrm{mos}(p^*)$, $\dot{\bar\tau} := \mathrm{mos}(\dot\tau)$ and $\alpha^* := \mathrm{mos}(\kappa)$. [6] Since $N \in H(\omega_1)$ we can code it as a real $z \in \omega^\omega$.

Now working in $L[z]$:

Definition / Lemma 6.3.3. By induction on $\alpha \leq \alpha^*$ we now want to define:

- the countable support iteration $\mathbb{Q}_\alpha$
- a function $i_\alpha\colon \bar{\mathbb{P}}_\alpha \to \mathbb{Q}_\alpha$
- the notion $q \in \mathbb{Q}_\alpha$ is $(N, \bar{\mathbb{P}}_\alpha)$-generic

and prove

- $\mathbb{Q}_\alpha$ is an iteration of Suslin forcings [7]
- i_α is an embedding, i.e. for every $\bar p_1, \bar p_2 \in \bar{\mathbb{P}}_\alpha$ we have $\bar p_2 \leq_{\bar{\mathbb{P}}_\alpha} \bar p_1$ iff $i_\alpha(\bar p_2) \leq_{\mathbb{Q}_\alpha} i_\alpha(\bar p_1)$

We set $\mathbb{Q} := \mathbb{Q}_{\alpha^*}$.

Proof. If $\alpha = 1$:

- We define $\mathbb{Q}_1 := Q_0 := \bar{\mathbb{P}}_1^{L[z]}$, i.e. $\bar{\mathbb{P}}_1$ has a Suslin definition in N which we can evaluate in $L[z]$.
- We set $i_1 := \mathrm{id}_{\bar{\mathbb{P}}_1}$. By $\mathbf{\Sigma}^1_1$-absoluteness we have $\bar{\mathbb{P}}_1 \subseteq \mathbb{Q}_1$ and i_1 is an embedding.
- We define $q \in \mathbb{Q}_1$ is $(N, \bar{\mathbb{P}}_1)$-generic iff $q \Vdash_{\mathbb{Q}_1} i_1^{-1}[\dot G_{\mathbb{Q}_1}]$ is an $(N, \bar{\mathbb{P}}_1)$-generic filter.[8]

$\alpha \to \alpha + 1$:

- We define a $\mathbb{Q}_\alpha$-name $\dot Q_\alpha$ for a forcing notion such that:
  - $q \Vdash_{\mathbb{Q}_\alpha} \dot Q_\alpha = (\dot{\bar P}_\alpha[i_\alpha^{-1}[\dot G_{\mathbb{Q}_\alpha}]])^{L[z][\dot G_{\mathbb{Q}_\alpha}]}$ iff $q \in \mathbb{Q}_\alpha$ is $(N, \bar{\mathbb{P}}_\alpha)$-generic.
  - $q \Vdash_{\mathbb{Q}_\alpha} \dot Q_\alpha = \{\mathbb{1}\}$ iff $q \in \mathbb{Q}_\alpha$ is $(N, \bar{\mathbb{P}}_\alpha)$-antigeneric. [9]

  If $G_{\mathbb{Q}_\alpha}$ is a $(L[z], \mathbb{Q}_\alpha)$-generic filter and $q \in G_{\mathbb{Q}_\alpha}$ is $(N, \bar{\mathbb{P}}_\alpha)$-generic, then by definition $G_{\bar{\mathbb{P}}_\alpha} := i_\alpha^{-1}[G_{\mathbb{Q}_\alpha}]$ is an $(N, \bar{\mathbb{P}}_\alpha)$-generic filter and $\dot{\bar P}_\alpha[G_{\bar{\mathbb{P}}_\alpha}]$ is a Suslin definition in $N[G_{\bar{\mathbb{P}}_\alpha}]$ which we can evaluate in $L[z][G_{\mathbb{Q}_\alpha}]$. Since $\Vdash_{\mathbb{Q}_\alpha} \dot Q_\alpha$ is Suslin, we have that $\mathbb{Q}_{\alpha+1} := \mathbb{Q}_\alpha \star \dot Q_\alpha$ is an iteration of Suslin forcings.

[6]The variable $\bar p$ will range over conditions in $\bar{\mathbb{P}}$, and will not denote $\mathrm{mos}(p)$ for some $p \in \mathbb{P}$.
[7]We will later see that $\mathbb{Q}_\alpha$ is even an iteration of Suslin, proper-for-candidates forcing notions.
[8]Obviously, this notion here coincides with the classical notion of $(N, \bar{\mathbb{P}}_1)$-genericity.
[9]We define $q \in \mathbb{Q}_\alpha$ to be $(N, \bar{\mathbb{P}}_\alpha)$-antigeneric iff $\forall q' \leq_{\mathbb{Q}_\alpha} q\colon q'$ is not $(N, \bar{\mathbb{P}}_\alpha)$-generic.

- We will define $i_{\alpha+1}\colon \bar{\mathbb{P}}_{\alpha+1} \to \mathbb{Q}_{\alpha+1}$ such that $i_{\alpha+1}$ extends i_α. For $\bar{p} \in \bar{\mathbb{P}}_{\alpha+1}$ we define $i_{\alpha+1}(\bar{p}) := i_\alpha(\bar{p} \restriction \alpha)^\frown \dot{q}_{\bar{p}}(\alpha)$, where $\dot{q}_{\bar{p}}(\alpha)$ is a $\mathbb{Q}_\alpha$-name such that:
  - $q \Vdash_{\mathbb{Q}_\alpha} \dot{q}_{\bar{p}}(\alpha) = \dot{\bar{p}}(\alpha)[i_\alpha^{-1}[\dot{G}_{\mathbb{Q}_\alpha}]]$ iff $q \in \mathbb{Q}_\alpha$ is $(N, \bar{\mathbb{P}}_\alpha)$-generic.
  - $q \Vdash_{\mathbb{Q}_\alpha} \dot{q}_{\bar{p}}(\alpha) = \dot{\mathbb{1}}$ iff $q \in \mathbb{Q}_\alpha$ is $(N, \bar{\mathbb{P}}_\alpha)$-antigeneric.

  Note that $\Vdash_{\mathbb{Q}_\alpha} \dot{q}_{\bar{p}}(\alpha) \in \dot{Q}_\alpha$. Using the induction hypothesis and $\mathbf{\Sigma}^1_1$-absoluteness, it easily follows that $i_{\alpha+1}$ is an embedding. [10]

- We define $q \in \mathbb{Q}_{\alpha+1}$ is $(N, \bar{\mathbb{P}}_{\alpha+1})$-generic iff $q \Vdash_{\mathbb{Q}_{\alpha+1}} i_{\alpha+1}^{-1}[\dot{G}_{\mathbb{Q}_{\alpha+1}}]$ is an $(N, \bar{\mathbb{P}}_{\alpha+1})$-generic filter

$\lambda \leq \alpha^*$ is a limit:

- We define $\mathbb{Q}_\lambda$ to be the countable support limit of $\langle \mathbb{Q}_\alpha, \dot{Q}_\beta \colon \alpha < \lambda, \beta < \lambda \rangle$. Using the induction hypothesis $\mathbb{Q}_\lambda$ is obviously an iteration of Suslin forcings.
- For $\bar{p} \in \bar{\mathbb{P}}_\lambda$ we define $i_\lambda(\bar{p})$ to be the union of $(i_\alpha(\bar{p} \restriction \alpha))_{\alpha<\lambda}$. Using the induction hypothesis we clearly have $i_\lambda(\bar{p}) \in \mathbb{Q}_\lambda$ and i_λ is an embedding.
- We define $q \in \mathbb{Q}_\lambda$ is $(N, \bar{\mathbb{P}}_\lambda)$-generic iff $q \Vdash_{\mathbb{Q}_\lambda} i_\lambda^{-1}[\dot{G}_{\mathbb{Q}_\lambda}]$ is an $(N, \bar{\mathbb{P}}_\lambda)$-generic filter. □

Now working in V: Let $\pi : N \to M$ denote the uncollapse and note that obviously $\pi(\alpha + 1) = \pi(\alpha) + 1$ for every $\alpha < \alpha^*$.

Definition / Lemma 6.3.4. By induction on $\alpha \leq \alpha^*$ we now want to define:

- a function $j_\alpha\colon \mathbb{Q}_\alpha \to \mathbb{P}_{\pi(\alpha)}$
- the notion $p \in \mathbb{P}_{\pi(\alpha)}$ is $(L[z], \mathbb{Q}_\alpha)$-generic

and prove

- j_α is an embedding, i.e. for every $q_1, q_2 \in \mathbb{Q}_\alpha$ we have $q_2 \leq_{\mathbb{Q}_\alpha} q_1$ iff $j_\alpha(q_2) \leq_{\mathbb{P}_{\pi(\alpha)}} j_\alpha(q_1)$
- if $p \in \mathbb{P}_{\pi(\alpha)}$ is $(L[z], \mathbb{Q}_\alpha)$-generic and $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \exists q \in j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]\colon q$ is $(N, \bar{\mathbb{P}}_\alpha)$-generic, then

$$(\dagger_\alpha) \quad \forall \bar{p} \in \bar{\mathbb{P}}_\alpha \colon p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \pi(\bar{p}) \in \dot{G}_{\mathbb{P}_{\pi(\alpha)}} \Leftrightarrow j_\alpha(i_\alpha(\bar{p})) \in \dot{G}_{\mathbb{P}_{\pi(\alpha)}}$$

Proof. If $\alpha = 1$:

- Since $\bar{\mathbb{P}}_1$, $\mathbb{P}_1$ and hence $\mathbb{Q}_1$ have the same Suslin definition, we can deduce by $\mathbf{\Sigma}^1_1$-absoluteness that $\mathbb{Q}_1 \subseteq \mathbb{P}_1$ and $j_1 := \mathrm{id}_{\mathbb{Q}_1}$ is an embedding.
- We define $p \in \mathbb{P}_1$ is $(L[z], \mathbb{Q}_1)$-generic iff $p \Vdash_{\mathbb{P}_1} j_1^{-1}[\dot{G}_{\mathbb{P}_1}]$ is an $(L[z], \mathbb{Q}_1)$-generic filter.

[10] Note that the embedding will not preserve incompatibility.

- Since $\pi(\bar{p}) = \bar{p} = j_\alpha(i_\alpha(\bar{p}))$ for every $\bar{p} \in \bar{\mathbb{P}}_1$, we have that $(\dagger_1)$ trivially follows.

$\alpha \to \alpha + 1$:

- We will define $j_{\alpha+1}\colon \mathbb{Q}_{\alpha+1} \to \mathbb{P}_{\pi(\alpha+1)}$ such that $j_{\alpha+1}$ extends j_α. For $q \in \mathbb{Q}_{\alpha+1}$ we define $j_{\alpha+1}(q) := j_\alpha(q \restriction \alpha)^\frown \dot{p}_q(\pi(\alpha))$, where $\dot{p}_q(\pi(\alpha))$ is a $\mathbb{P}_{\pi(\alpha)}$-name such that:
  - $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \dot{p}_q(\pi(\alpha)) = \dot{q}(\alpha)[j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]]$ iff $p \in \mathbb{P}_{\pi(\alpha)}$ is $(L[z], \mathbb{Q}_\alpha)$-generic.
  - $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \dot{p}_q(\pi(\alpha)) = \dot{\mathbb{1}}$ iff $p \in \mathbb{P}_{\pi(\alpha)}$ is $(L[z], \mathbb{Q}_\alpha)$-antigeneric. [11]

  Since $\mathbb{P}_{\pi(\alpha+1)} = \mathbb{P}_{\pi(\alpha)} \star \dot{P}_{\pi(\alpha)}$ it remains to be shown that $\Vdash_{\mathbb{P}_{\pi(\alpha)}} \dot{p}_q(\pi(\alpha)) \in \dot{P}_{\pi(\alpha)}$:
  - This is obvious if $p \in \mathbb{P}_{\pi(\alpha)}$ is $(L[z], \mathbb{Q}_\alpha)$-antigeneric.
  - If $p \in \mathbb{P}_{\pi(\alpha)}$ is $(L[z], \mathbb{Q}_\alpha)$-generic, but $p \Vdash_{\mathbb{P}_{\pi(\alpha)}}$ '$\exists q \in j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]\colon q$ is $(N, \bar{\mathbb{P}}_\alpha)$ -antigeneric', then $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \dot{Q}_\alpha[j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]] = \{\dot{\mathbb{1}}\}$, hence $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \dot{p}_q(\pi(\alpha)) = \dot{q}(\alpha)[j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]] = \dot{\mathbb{1}} \in \dot{P}_{\pi(\alpha)}$.
  - Now if $p \in \mathbb{P}_{\pi(\alpha)}$ is $(L[z], \mathbb{Q}_\alpha)$-generic and $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \exists q \in j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]\colon q$ is $(N, \bar{\mathbb{P}}_\alpha)$-generic, then we can use $(\dagger_\alpha)$ to deduce that

$$p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \text{`}\dot{\bar{P}}_\alpha[i_\alpha^{-1}[j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]]]\,, \dot{P}_{\pi(\alpha)} \text{ and hence } \dot{Q}_\alpha[j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]]$$

$$\text{have the same Suslin definition'}.$$

  Hence, by $\mathbf{\Sigma}^1_1$-absoluteness, it follows that $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \dot{p}_q(\pi(\alpha)) \in \dot{P}_{\pi(\alpha)}$.

  Again using $\mathbf{\Sigma}^1_1$-absoluteness and the induction hypothesis, we see that $j_{\alpha+1}$ is an embedding. [12]

- We define $p \in \mathbb{P}_{\pi(\alpha+1)}$ is $(L[z], \mathbb{Q}_{\alpha+1})$-generic iff $p \Vdash_{\mathbb{P}_{\pi(\alpha+1)}} j_{\alpha+1}^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha+1)}}]$ is an $(L[z], \mathbb{Q}_{\alpha+1})$-generic filter.

- Assume that $p \in \mathbb{P}_{\pi(\alpha+1)}$ satisfies the assumptions for $(\dagger_{\alpha+1})$. Using the induction hypothesis $(\dagger_\alpha)$, we can deduce that $p \restriction \pi(\alpha) \Vdash_{\mathbb{P}_{\pi(\alpha)}} \pi(\dot{\bar{p}})(\pi(\alpha)) = j_{\alpha+1}(i_{\alpha+1}(\dot{\bar{p}}))(\pi(\alpha))$ for every $\bar{p} \in \bar{\mathbb{P}}_{\alpha+1}$. Hence $p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \pi(\bar{p}) \in \dot{G}_{\mathbb{P}_{\pi(\alpha)}} \star \dot{G}_{\dot{P}_{\pi(\alpha)}} \Leftrightarrow j_{\alpha+1}(i_{\alpha+1}(\bar{p})) \in \dot{G}_{\mathbb{P}_{\pi(\alpha)}} \star \dot{G}_{\dot{P}_{\pi(\alpha)}}$ for every $\bar{p} \in \bar{\mathbb{P}}_{\alpha+1}$.

$\lambda \leq \alpha^*$ is a limit: Let $\lambda' := \sup(\pi(\lambda) \cap M)$.

- For $q \in \mathbb{Q}_\lambda$ let $j_\lambda(q) \restriction \lambda' \in \mathbb{P}_{\lambda'}$ be the union of $(j_\alpha(q \restriction \alpha))_{\alpha<\lambda}$. Extend $j_\lambda(q) \restriction \lambda'$ trivially to get $j_\lambda(q) \in \mathbb{P}_{\pi(\lambda)}$. Using the induction hypothesis we clearly have that j_λ is an embedding.

- We define $p \in \mathbb{P}_{\pi(\lambda)}$ is $(L[z], \mathbb{Q}_\lambda)$-generic iff $p \Vdash_{\mathbb{P}_{\pi(\lambda)}} j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\lambda)}}]$ is an $(L[z], \mathbb{Q}_\lambda)$-generic filter.

[11] Again, we define $p \in \mathbb{P}_{\pi(\alpha)}$ to be $(L[z], \mathbb{Q}_\alpha)$-antigeneric iff $\forall p' \leq_{\mathbb{P}_{\pi(\alpha)}} p\colon p'$ is not $(L[z], \mathbb{Q}_\alpha)$-generic.
[12] Note that again the embedding will not preserve incompatibility.

- Assume that $p \in \mathbb{P}_{\pi(\lambda)}$ satisfies the assumptions for $(\dagger_\lambda)$. Using the induction hypothesis $(\dagger_\alpha)$ for every $\alpha < \lambda$ and noting that $\Vdash_{\mathbb{P}_{\lambda'}} p' \in \dot{G}_{\mathbb{P}_{\lambda'}} \Leftrightarrow \forall n < \omega \colon p' \restriction \alpha_n \in \dot{G}_{\mathbb{P}_{\alpha_n}}$ holds for every $p' \in \mathbb{P}_{\lambda'}$ and every cofinal sequence $(\alpha_n)_{n<\omega} \subseteq \lambda' \cap M$, we can deduce that $p \Vdash_{\mathbb{P}_{\lambda'}} \pi(\bar{p}) \restriction \lambda' \in \dot{G}_{\mathbb{P}_{\lambda'}} \Leftrightarrow j_\lambda(i_\lambda(\bar{p})) \restriction \lambda' \in \dot{G}_{\mathbb{P}_{\lambda'}}$ for every $\bar{p} \in \bar{\mathbb{P}}_\lambda$. Since both $\pi(\bar{p})$ and $j_\lambda(i_\lambda(\bar{p}))$ are trivial extensions of $\pi(\bar{p}) \restriction \lambda'$ and $j_\lambda(i_\lambda(\bar{p})) \restriction \lambda'$, respectively, it easily follows that $p \Vdash_{\mathbb{P}_{\pi(\lambda)}} \pi(\bar{p}) \in \dot{G}_{\mathbb{P}_{\pi(\lambda)}} \Leftrightarrow j_\lambda(i_\lambda(\bar{p})) \in \dot{G}_{\mathbb{P}_{\pi(\lambda)}}$ for every $\bar{p} \in \bar{\mathbb{P}}_\lambda$. □

Lemma 6.3.5. For every $q \in \mathbb{Q}$ there exists $p \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} j_{\alpha^*}(q)$ and p is $(L[z], \mathbb{Q})$-generic.

Proof. By induction on $\alpha \leq \alpha^*$ we will show the following claim:

$$(\ddagger_\alpha) \quad \forall \beta < \alpha \; \forall p \in \mathbb{P}_{\pi(\beta)} \; \forall q \in \mathbb{Q}_\alpha \colon \big(p \leq_{\mathbb{P}_{\pi(\beta)}} j_\alpha(q) \restriction \pi(\beta) \wedge p \text{ is } (L[z], \mathbb{Q}_\beta)\text{-generic}\big) \Rightarrow$$

$$\big(\exists p' \in \mathbb{P}_{\pi(\alpha)} \colon p' \leq_{\mathbb{P}_{\pi(\alpha)}} j_\alpha(q) \wedge p' \restriction \pi(\beta) = p \wedge p' \text{ is } (L[z], \mathbb{Q}_\alpha)\text{-generic}\big)$$

Clearly, for $\alpha = \alpha^*$ the claim implies the lemma.

Since ω_1 is inaccessible to the reals and hence $L[z] \vDash$ 'ω_1^V is inaccessible', there exists $\gamma < \omega_1$ such that $L_\gamma[z] \vDash \mathrm{ZFC}^*$, i.e. a large enough fragment of ZFC, and $\mathfrak{P}^{L[z]}(\mathbb{Q}) \in L_\gamma[z]$.

- If $\alpha = 1$ we must simply show that for every $q \in \mathbb{Q}_1$ there exists $p \in \mathbb{P}_1$ such that $p \leq_{\mathbb{P}_1} q$ and p is $(L[z], \mathbb{Q}_1)$-generic). Since $\mathbb{P}_1$ is proper-for-candidates and $L_\gamma[z]$ is a candidate for $\mathbb{Q}_1$, such a condition p obviously exists.

- $\alpha \to \alpha + 1$: Let $q \in \mathbb{Q}_{\alpha+1}$ and $p \in \mathbb{P}_{\pi(\beta)}$ such that $p \leq_{\mathbb{P}_{\pi(\beta)}} j_{\alpha+1}(q) \restriction \pi(\beta)$ and p is $(L[z], \mathbb{Q}_\beta)$-generic be arbitrary. Using $(\ddagger_\alpha)$ we can assume w.l.o.g. that $\beta = \alpha$. Let $G_{\mathbb{P}_{\pi(\alpha)}}$ be a $(V, \mathbb{P}_{\pi(\alpha)})$-generic filter containing p. Hence $G_{\mathbb{Q}_\alpha} := j_\alpha^{-1}[G_{\mathbb{P}_{\pi(\alpha)}}]$ is an $(L_\gamma[z], \mathbb{Q}_\alpha)$-generic filter. There are now two cases:
  - If there is $q' \in G_{\mathbb{Q}_\alpha}$ such that q' is $(N, \bar{\mathbb{P}}_\alpha)$-antigeneric, then $\dot{Q}_\alpha[G_{\mathbb{Q}_\alpha}] = \{\mathbb{1}\}$. Hence $\mathbb{1} \in \dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}]$ is trivially $(L_\gamma[z][G_{\mathbb{Q}_\alpha}], \dot{Q}_\alpha[G_{\mathbb{Q}_\alpha}])$-generic.
  - If there is $q' \in G_{\mathbb{Q}_\alpha}$ such that q' is $(N, \bar{\mathbb{P}}_\alpha)$-generic, then $\dot{Q}_\alpha[G_{\mathbb{Q}_\alpha}]$ and $\dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}]$ have the same Suslin definition. Since $L_\gamma[z][G_{\mathbb{Q}_\alpha}] \in V[G_{\mathbb{P}_{\pi(\alpha)}}]$ is a candidate for $\dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}]$ and $\dot{q}(\alpha)[G_{\mathbb{Q}_\alpha}] \in \dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}]^{L_\gamma[z][G_{\mathbb{Q}_\alpha}]}$, there exists $p' \in \dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}]$ such that $p' \leq_{\dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}]} \dot{q}(\alpha)[G_{\mathbb{Q}_\alpha}]$ and p' is $(L_\gamma[z][G_{\mathbb{Q}_\alpha}], \dot{Q}_\alpha[G_{\mathbb{Q}_\alpha}])$-generic.

  In any case there exists a $\mathbb{P}_{\pi(\alpha)}$-name $\dot{p}_{p'}(\pi(\alpha))$ for an element of $\dot{P}_{\pi(\alpha)}$ such that

  $$p \Vdash_{\mathbb{P}_{\pi(\alpha)}} \dot{p}_{p'}(\pi(\alpha)) \leq_{\dot{P}_{\pi(\alpha)}} \dot{q}(\alpha)[j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]] = \dot{j_{\alpha+1}(q)(\pi(\alpha))} \wedge$$

  $$\dot{p}_{p'}(\pi(\alpha)) \text{ is } (L[z][j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]], \dot{Q}_\alpha[j_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]])\text{-generic.}$$

  Hence $p' := p^\frown \dot{p}_{p'}(\pi(\alpha)) \in \mathbb{P}_{\pi(\alpha+1)}$ has the required properties.

- $\lambda \leq \alpha^*$ is a limit: Let $q \in \mathbb{Q}_\lambda$, $\beta < \lambda$ and $p \in \mathbb{P}_{\pi(\beta)}$ be arbitrary such that $p \leq_{\mathbb{P}_{\pi(\beta)}} j_\lambda(q) \restriction \pi(\beta)$ and p is $(L[z], \mathbb{Q}_\beta)$-generic. Let $\lambda' := \sup(\pi(\lambda) \cap M)$. Let $(\alpha_n)_{n<\omega} \in L[z]$ with $\alpha_0 := \beta$ be a cofinal sequence in λ and enumerate every dense subset of $\mathbb{Q}_\lambda$ in $L_\gamma[z]$ as $(D_n)_{n<\omega}$ [13]. By induction on $n < \omega$ we will now construct:
  - $(q_n)_{n<\omega}$ decreasing such that $q_n \in \mathbb{Q}_\lambda$, $q_{n+1} \restriction \alpha_n = q_n \restriction \alpha_n$ and the set $E_n := \{q' \in \mathbb{Q}_{\alpha_n} : q'^\frown q_{n+1} \restriction [\alpha_n, \lambda) \in D_n\}$ is dense below $q_{n+1} \restriction \alpha_n$ for every $n < \omega$
  - $(p_n)_{n<\omega}$ decreasing such that $p_n \in \mathbb{P}_{\pi(\alpha_n)}$, $p_{n+1} \restriction \pi(\alpha_n) = p_n$, $p_n \leq_{\mathbb{P}_{\pi(\alpha_n)}} j_\lambda(q_n) \restriction \pi(\alpha_n)$ and p_n is $(L[z], \mathbb{Q}_{\alpha_n})$-generic for every $n < \omega$

  If $n = 0$ we set $q_0 := q$ and $p_0 := p$. Then p_0 and q_0 satisfy the requirements by assumption.
  $n \to n+1$: The existence of $q_{n+1} \in \mathbb{Q}_\lambda$ with the required properties follows from a standard 'iteration of proper forcing' argument. Now we use $(\ddagger_{\alpha_{n+1}})$ to extend p_n to $p_{n+1} \in \mathbb{P}_{\pi(\alpha_{n+1})}$ with the required properties.

  Let $p_\omega \in \mathbb{P}_{\lambda'}$ be the union of $(p_n)_{n<\omega}$ and extend p_ω trivially to get $p' \in \mathbb{P}_{\pi(\lambda)}$. Since $p' \leq_{\mathbb{P}_{\pi(\lambda)}} j_\lambda(q_n)$ and $p' \restriction \pi(\alpha_n)$ is $(L[z], \mathbb{Q}_{\alpha_n})$-generic for every $n < \omega$, we can deduce that $p' \Vdash_{\mathbb{P}_{\pi(\lambda)}} \forall n < \omega \colon E_n \cap j_{\alpha_n}^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha_n)}}] \neq \emptyset$ and therefore $p' \Vdash_{\mathbb{P}_{\pi(\lambda)}} \forall n < \omega \colon D_n \cap j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\lambda)}}] \neq \emptyset$. But why does $p' \Vdash_{\mathbb{P}_{\pi(\lambda)}} j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\lambda)}}]$ is a filter?
  To this end we define for every $q_1, q_2 \in \mathbb{Q}_\lambda$ such that q_1 and q_2 are incompatible in $\mathbb{Q}_\lambda$ the set $D_{q_1,q_2} := \{q' \in \mathbb{Q}_\lambda \colon \exists \alpha < \lambda\ q' \restriction \alpha \Vdash_{\mathbb{Q}_\alpha} \dot{q}'(\alpha) \perp_{\dot{Q}_\alpha} \dot{q}_1(\alpha) \vee \dot{q}'(\alpha) \perp_{\dot{Q}_\alpha} \dot{q}_2(\alpha)\}$. It can easily be seen that $D_{q_1,q_2} \in L[z]$ is dense. Let $p'' \leq_{\mathbb{P}_{\pi(\lambda)}} p'$ be arbitrary such that there exists $q' \in \mathbb{Q}_\lambda$ with $p'' \Vdash_{\mathbb{P}_{\pi(\lambda)}} q' \in D_{q_1,q_2} \cap j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\lambda)}}]$, and let $\alpha < \lambda$ be such that $q' \restriction \alpha \Vdash_{\mathbb{Q}_\alpha} \dot{q}'(\alpha) \perp_{\dot{Q}_\alpha} \dot{q}_1(\alpha) \vee \dot{q}'(\alpha) \perp_{\dot{Q}_\alpha} \dot{q}_2(\alpha)$. Since $p'' \restriction \pi(\alpha+1) \Vdash_{\mathbb{P}_{\pi(\alpha+1)}} j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha+1)}}]$ is a $(L[z], \mathbb{Q}_{\alpha+1})$-generic filter, it follows that $p'' \Vdash_{\mathbb{P}_{\pi(\lambda)}} q_1 \notin j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\lambda)}}] \vee q_2 \notin j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\lambda)}}]$.
  In particular, $p' \Vdash_{\mathbb{P}_{\pi(\lambda)}} j_\lambda^{-1}[\dot{G}_{\mathbb{P}_{\pi(\lambda)}}]$ is a filter and hence p' is $(L[z], \mathbb{Q}_\lambda)$-generic. □

Lemma 6.3.6. $L[z] \models$ '$\mathbb{Q}$ is an iteration of proper-for-candidates forcing notions', i.e. $L[z] \models \forall \alpha < \alpha^* \colon \Vdash_{\mathbb{Q}_\alpha} \dot{Q}_\alpha$ is proper-for-candidates.

Proof. Let $\alpha < \alpha^*$ and $q \in \mathbb{Q}_\alpha$ be arbitrary such that q is either $(N, \bar{\mathbb{P}}_\alpha)$-generic or $(N, \bar{\mathbb{P}}_\alpha)$-antigeneric. Hence, there are two cases:

- If q is $(N, \bar{\mathbb{P}}_\alpha)$-antigeneric, then $q \Vdash_{\mathbb{Q}_\alpha} \dot{Q}_\alpha = \{\mathbb{1}\}$ and hence $q \Vdash_{\mathbb{Q}_\alpha} \dot{Q}_\alpha$ is proper-for-candidates.
- If q is $(N, \bar{\mathbb{P}}_\alpha)$-generic, we use Lemma 6.3.5 to find $p \in \mathbb{P}_{\pi(\alpha)}$ such that $p \leq_{\mathbb{P}_{\pi(\alpha)}} j_\alpha(q)$ and p is $(L[z], \mathbb{Q}_\alpha)$-generic. Hence, p satisfies the assumptions for $(\dagger_\alpha)$. Let $G_{\mathbb{P}_{\pi(\alpha)}}$ be a $(V, \mathbb{P}_{\pi(\alpha)})$-generic filter with $p \in G_{\mathbb{P}_{\pi(\alpha)}}$ and set $G_{\mathbb{Q}_\alpha} := j_\alpha^{-1}[G_{\mathbb{P}_{\pi(\alpha)}}]$. Then $G_{\mathbb{Q}_\alpha}$ is a $(L[z], \mathbb{Q}_\alpha)$-generic filter with $q \in G_{\mathbb{Q}}$, and by $(\dagger_\alpha)$ we can deduce that $\dot{Q}_\alpha[G_{\mathbb{Q}_\alpha}]$ and $\dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}]$ have the same Suslin definition. Hence $\dot{Q}_\alpha[G_{\mathbb{Q}_\alpha}] =$

[13] This enumeration only exists in V, but this will suffice.

$(\dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}])^{L[z][G_{\mathbb{Q}_\alpha}]}$. Since $L[z] \models$ ' ω_1^V is inaccessible', we can deduce $|\omega^\omega \cap L[z][G_{\mathbb{Q}_\alpha}]| < \omega_1$. Since $L[z] \models$ ' α^* is countable', we can now find $A \in ([\omega^\omega]^\omega)^{V[G_{\mathbb{P}_{\pi(\alpha)}}]}$ such that $L[z][G_{\mathbb{Q}_\alpha}] = L[A]$. By

$$V[G_{\mathbb{P}_{\pi(\alpha)}}] \models \forall A \in [\omega^\omega]^\omega \colon L[A] \models \text{`}(\dot{P}_{\pi(\alpha)}[G_{\mathbb{P}_{\pi(\alpha)}}])^{L[A]} \text{ is proper-for-candidates'}$$

which is our technical requirement for Theorem 6.3.2, we can deduce that $L[z][G_{\mathbb{Q}_\alpha}] \models$ ' $\dot{Q}_\alpha[G_{\mathbb{Q}_\alpha}]$ is proper-for-candidates'. Since $q \in G_{\mathbb{Q}_\alpha}$, there exists $q' \in G_{\mathbb{Q}_\alpha}$ with $q' \leq_{\mathbb{Q}_\alpha} q$ and $q' \Vdash_{\mathbb{Q}_\alpha} \dot{Q}_\alpha$ is proper-for-candidates. □

Lemma 6.3.7. For every condition $\bar{p} \in \bar{\mathbb{P}}$ there exists $q \in \mathbb{Q}$ such that $q \leq_{\mathbb{Q}} i_{\alpha^*}(\bar{p})$ and q is $(N, \bar{\mathbb{P}})$-generic.

Proof. By Lemma 6.3.6 we know that $\mathbb{Q}$ is an iteration of Suslin, proper-for-candidates forcing notions in $L[z]$. Hence, the proof is just a simpler version of the proof of Lemma 6.3.5. □

Proof of Theorem 6.3.2. By Lemma 6.3.7 there exists $q' \in \mathbb{Q}$ such that $q' \leq_{\mathbb{Q}} i_{\alpha^*}(\bar{p}^*)$ and q' is $(N, \bar{\mathbb{P}})$-generic. By Lemma 6.3.5 there exists $p' \in \mathbb{P}$ such that $p' \leq_{\mathbb{P}} j_{\alpha^*}(q')$ and p' is $(L[z], \mathbb{Q})$-generic. Hence, p' satisfies the assumptions for $(\dagger_{\alpha^*})$ and we can deduce that $p' \Vdash_{\mathbb{P}} \dot{\bar{\tau}}[i_{\alpha^*}^{-1}[j_{\alpha^*}^{-1}[\dot{G}_{\mathbb{P}}]]] = \dot{\tau}[\dot{G}_{\mathbb{P}}]$. Furthermore, since $p' \Vdash_{\mathbb{P}} \bar{p}^* \in i_{\alpha^*}^{-1}[j_{\alpha^*}^{-1}[\dot{G}_{\mathbb{P}}]]$ it follows that $p' \Vdash_{\mathbb{P}} p^* \in \dot{G}_{\mathbb{P}}$. Therefore, we can assume w.l.o.g. that $p' \leq_{\mathbb{P}} p^*$, and it follows that

$$p' \Vdash_{\mathbb{P}} \exists G_{\mathbb{Q}} \colon G_{\mathbb{Q}} \text{ is } (L[z], \mathbb{Q})\text{-generic} \wedge \dot{\tau} \in L[z][G_{\mathbb{Q}}].$$

Since $L[z] \models$ ' ω_1^V is inaccessible', we can deduce that $L[z] \models |\mathbb{Q}| < \omega_1^V$. This finishes the proof. □

6.4 Capturing products of iterations of Sacks forcing

We recall the definition Sacks forcing:

Definition 6.4.1. Let $p \in \mathbb{S}$ iff:

- $p \subseteq 2^{<\omega}$, $p \neq \emptyset$ and p is a tree.
- $\forall \eta \in p\ \exists \nu \in p \colon \eta \lhd \nu \wedge |\mathrm{succ}_p(\nu)| > 1$, where $\mathrm{succ}_p(\eta)$ denotes the successors of η in p.

Define $q \leq_{\mathbb{S}} p$ iff $q \subseteq p$.

If G is a $(V, \mathbb{S})$-generic filter we define $s_G \in 2^\omega$ to be the unique real contained in $\bigcap_{p \in G} [p]$ where $[p] := \{x \in 2^\omega \colon \forall n < \omega\ x \restriction n \in p\}$.

Definition 6.4.2. Furthermore, we define:

- $\mathrm{split}_p(\eta)$ iff $|\mathrm{succ}_p(\eta)| > 1$

- $\mathrm{ht}_p(\eta) := |\{\nu \lhd \eta \colon \nu \neq \eta \wedge \mathrm{split}_p(\nu)\}|$
- For $n < \omega$: $\mathrm{split}_n(p) := \{\eta \in p \colon \mathrm{split}_p(\eta) \wedge \mathrm{ht}_p(\eta) = n\}$

Set $q \leq_n p$ iff $q \leq_{\mathbb{S}} p \wedge \mathrm{split}_n(p) \subseteq q$.

Fact 6.4.3. The following holds true:

- $q \leq_n p \Leftrightarrow q \leq_{\mathbb{S}} p \wedge \forall m < n\ \mathrm{split}_m(q) = \mathrm{split}_m(p)$
- $\forall x \in \omega^\omega\ \forall n < \omega \colon x \in [p] \Rightarrow x \cap \mathrm{split}_n(p) \neq \emptyset$, i.e. $\mathrm{split}_n(p)$ is a front in p

Definition 6.4.4. Let $p \in \mathbb{S}$ and let $\eta \in p$. We define the condition $p^{[\eta]} := \{\nu \in p \colon \nu \lhd \eta \vee \eta \lhd \nu\}$.

Lemma 6.4.5. If $(p_n)_{n<\omega} \subseteq \mathbb{S}$ is a fusion sequence, i.e. $\forall n < \omega \colon p_{n+1} \leq_n p_n$, then $q := \bigcap_{n<\omega} p_n \in \mathbb{S}$ and $\forall n < \omega \colon q \leq_n p_n$.

Let $\mathbb{P}$ denote a countable support iteration of Sacks forcing of length κ.

Definition 6.4.6. Let $p, q \in \mathbb{P}$. Let $F \in [\kappa]^{<\omega}$ and let $n < \omega$. We define:

- $\mathrm{supp}(p)$ the support of p.
- $q \leq_{F,n} p$ iff $\forall \beta \in F \colon q \restriction \beta \Vdash_\beta q(\beta) \leq_n p(\beta)$.

Lemma 6.4.7. Let $(p_n)_{n<\omega} \subseteq \mathbb{P}$ be a fusion sequence, i.e. there exists $(F_n)_{n<\omega} \subseteq [\kappa]^{<\omega}$ increasing such that $\bigcup_{n<\omega} \mathrm{supp}(p_n) \subseteq \bigcup_{n<\omega} F_n$ and $\forall n < \omega \colon p_{n+1} \leq_{F_n,n} p_n$, then there exists $q \in \mathbb{P}$ with $\forall n < \omega \colon q \leq_{F_n,n} p_n$.

Lemma 6.4.8. Let $A \subseteq \mathbb{P}$ be a maximal antichain, $F \in [\kappa]^{<\omega}$, $n < \omega$ and $p \in \mathbb{P}$. Then there exists $q \leq_{F,n} p$ such that $|A \restriction q| < \omega$ with $A \restriction q := \{r \in A \colon r \parallel q\}$, where $\parallel$ means compatible.

Lemma 6.4.9. For $F \in [\kappa]^{<\omega}$, $n < \omega$ and $p \in \mathbb{P}$ the set

$$D_{F,n}(p) := \{s \in \mathbb{P} \colon \forall \beta \in F\ \exists \eta_\beta,\ \nu_\beta \in 2^{<\omega}$$

$$s \restriction \beta \Vdash_\beta s(\beta) \cap \mathrm{split}_n(p(\beta)) = \{\eta_\beta\} \ \wedge\ \mathrm{succ}_{s(\beta)}(\eta_\beta) = \{\nu_\beta\}\}.$$

is open dense below p.

If $F \in [\kappa]^{<\omega}$, $n < \omega$, $p \in \mathbb{P}$ and $s \in D_{F,n}(p)$ are clear from the context, we write η^s_β and ν^s_β for the corresponding η_β and ν_β.

Lemma 6.4.10. Let $p \in \mathbb{P}$, $F \in [\kappa]^{<\omega}$, $n < \omega$ and $s \in D_{F,n}(p)$. Then there exists $p^{[s]} \leq_{F,n+1} p$ such that

- $\mathrm{supp}(p^{[s]}) = \mathrm{supp}(p) \cup \mathrm{supp}(s)$,
- $s \leq_{\mathbb{P}} p^{[s]}$ and

- $\forall s' \in D_{F,n}(p)\colon (s' \leq_{\mathbb{P}} p^{[s]} \wedge \forall \beta \in F\ \nu^{s'}_\beta = \nu^s_\beta) \Rightarrow s' \leq_{\mathbb{P}} s$.

Note that $p^{[s]}$ depends on p and s as well as on F and n. For notational simplicity, however, we will suppress F and n if they are clear from the context.

Let us now turn to the product $\mathbb{P}^2$. By $((\dot{s}^k_\beta)_{\beta<\kappa})_{k\in\{0,1\}}$ we denote the sequence of Sacks reals added by $\mathbb{P}^2$ and define:

Definition 6.4.11. Let $(p^0,p^1),(q^0,q^1) \in \mathbb{P}^2$, $F \in [\kappa]^{<\omega}$ and $n < \omega$. We define $(q^0,q^1) \leq_{F,n} (p^0,p^1)$ iff $q^k \leq_{F,n} p^k$ for $k \in \{0,1\}$.

We will aim to show the following theorem:

Theorem 6.4.12. Assume that ω_1 is inaccessible to the reals. Then $\mathbb{P}^2$ is captured.

We will need several lemmas for the proof.

Lemma 6.4.13. Let $(p^0,p^1) \in \mathbb{P}^2$ and $\dot{\tau}$ be a $\mathbb{P}^2$-name for a real. Then there exists $(q^0,q^1) \leq_{\mathbb{P}^2} (p^0,p^1)$, $(F_n)_{n<\omega} \subseteq [\kappa]^{<\omega}$ and for every $n<\omega$: $C^k_n \subseteq \prod_{\beta\in F_n} 2^{<\omega} \times \prod_{\beta\in F_n} 2^{<\omega}$ finite for $k \in \{0,1\}$ and $A_n\colon C^0_n \times C^1_n \to 2^n$ such that:

1. if $(\bar\eta,\bar\nu) \in C^k_n$ then for every $\beta \in F_n$ we have $\bar\nu(\beta) = \bar\eta(\beta)^\frown i_\beta$ for some $i_\beta \in \{0,1\}$.
2. if $(\bar\eta,\bar\nu) \in C^k_n$ then for every $\beta \in F_n$ we have $q^{k\,[\bar\nu]} \restriction \beta \Vdash_\beta \bar\eta(\beta) \in \mathrm{split}_n(q^k(\beta))$ where $q^{k\,[\bar\nu]}$ is defined inductively such that for every $\alpha < \kappa$ we have that $q^{k\,[\bar\nu]} \restriction \alpha \Vdash_\alpha q^{k\,[\bar\nu]}(\alpha) = q^k(\alpha)^{[\bar\nu(\alpha)]}$ if $\alpha \in F_n$ and $q^{k\,[\bar\nu]} \restriction \alpha \Vdash_\alpha q^{k\,[\bar\nu]}(\alpha) = q^k(\alpha)$ else.
3. if $s \in D_{F_n,n}(q^k)$ then $((\eta^s_\beta)_{\beta\in F_n}, (\nu^s_\beta)_{\beta\in F_n}) \in C^k_n$.
4. $A_n = (\rho_{(\bar\nu^0,\bar\nu^1)})_{(\bar\eta^0,\bar\nu^0)\in C^0_n,\,(\bar\eta^1,\bar\nu^1)\in C^1_n}$ such that for every $((\bar\eta^0,\bar\nu^0),(\bar\eta^1,\bar\nu^1)) \in C^0_n \times C^1_n$ we have $(q^{0\,[\bar\nu^0]}, q^{1\,[\bar\nu^1]}) \Vdash_{\mathbb{P}^2} \dot\tau \restriction n = \rho_{(\bar\nu^0,\bar\nu^1)}$.

Proof. By induction on $n < \omega$ we will construct a fusion sequence $((p^0_n,p^1_n))_{n<\omega}$ such that $\forall n<\omega\colon (p^0_{n+1},p^1_{n+1}) \leq_{F_n,n+1} (p^0_n,p^1_n)$, and the required sets C^k_n and A_n. The fusion limit (q^0,q^1) will have the required properties.

- $n = 0$: Set $(p^0_0,p^1_0) := (p^0,p^1)$ and $F_0 := \{0\}$.
- $n \to n+1$: Assume that (p^0_n,p^1_n), F_n, C^k_{n-1} and A_{n-1} have already been defined. Using Lemma 6.4.8 find $\tilde{p}^k_n \leq_{F_n,n+1} p^k_n$ and $l < \omega$ such that for every $\beta \in F_n$ we have $\tilde{p}^k_n \restriction \beta \Vdash_\beta \mathrm{split}_n(\tilde{p}^k_n(\beta)) \subseteq 2^{<l}$.
  Enumerate $\left(\prod_{\beta\in F_n} 2^{<l} \times \prod_{\beta\in F_n} 2^{\leq l}\right)^2$ as $((\bar\eta^0_m,\bar\nu^0_m),(\bar\eta^1_m,\bar\nu^1_m))_{m<\tilde{l}}$ for some $\tilde{l} < \omega$. By induction on $m < \tilde{l}$ construct a $\leq_{F_n,n+1}$-decreasing sequence $(({}^m p^0_n, {}^m p^1_n))_{m\leq\tilde{l}}$:
  - $m = 0$: Set $({}^0 p^0_n, {}^m p^1_n) := (\tilde{p}^0_n,\tilde{p}^1_n)$.
  - $m \to m+1$: Assume that $({}^m p^0_n, {}^m p^1_n)$ has already been defined. If there exists $s^k \in D_{F_n,n}({}^m p^k_n)$ such that $\forall \beta \in F_n\colon \bar\eta^k_m(\beta) = \eta^{s^k}_\beta \wedge \bar\nu^k_m(\beta) = \nu^{s^k}_\beta$ for $k \in \{0,1\}$, we can w.l.o.g. assume that also $(s^0,s^1) \Vdash_{\mathbb{P}^2} \dot\tau \restriction n = \rho$ for some $\rho \in 2^n$. In this case pick such (s^0,s^1) and ρ, call it $({}^m s^0_n, {}^m s^1_n)$ and $\rho_{(\bar\nu^0_m,\bar\nu^1_m)}$, respectively, and set $({}^{m+1}p^0_n, {}^{m+1}p^1_n) := ({}^m p^{0\,[{}^m s^0_n]}_n, {}^m p^{1\,[{}^m s^1_n]}_n)$. Else set $({}^{m+1}p^0_n, {}^{m+1}p^1_n) := ({}^m p^0_n, {}^m p^1_n)$ and $\rho_{(\bar\nu^0_m,\bar\nu^1_m)}$ is undefined.

Define $(p^0_{n+1}, p^1_{n+1}) := ({}^{\tilde{l}}p^0_n, {}^{\tilde{l}}p^1_n)$. Clearly we have $(p^0_{n+1}, p^1_{n+1}) \leq_{F_n,n+1} (p^0_n, p^1_n)$. We define

$$C^k_n := \{(\bar{\eta}, \bar{\nu}) \in \prod_{\beta \in F_n} 2^{<\omega} \times \prod_{\beta \in F_n} 2^{<\omega} \colon$$
$$\exists s \in D_{F_n,n}(p^k_{n+1})\ \forall \beta \in F_n\ \bar{\eta}(\beta) = \eta^s_\beta \wedge \bar{\nu}(\beta) = \nu^s_\beta\}$$

and for $((\bar{\eta}^0, \bar{\nu}^0), (\bar{\eta}^1, \bar{\nu}^1)) \in C^0_n \times C^1_n$ we set $A_n(((\bar{\eta}^0, \bar{\nu}^0), (\bar{\eta}^1, \bar{\nu}^1))) := \rho_{(\bar{\nu}^0_m, \bar{\nu}^1_m)}$ iff $((\bar{\eta}^0, \bar{\nu}^0), (\bar{\eta}^1, \bar{\nu}^1)) = ((\bar{\eta}^0_m, \bar{\nu}^0_m), (\bar{\eta}^1_m, \bar{\nu}^1_m))$ and $\rho_{(\bar{\nu}^0_m, \bar{\nu}^1_m)}$ is defined. Use a bookkeeping argument to define $F_{n+1} \supseteq F_n$.

We must show that the fusion limit (q^0, q^1) has the required properties:

- ad 1.) This is obviously satisfied.
- ad 3.) Let $n < \omega$ be arbitrary. Since $q^k \leq_{F_n,n+1} p^k_{n+1}$ and therefore $D_{F_n,n}(q^k) \subseteq D_{F_n,n}(p^k_{n+1})$, this is satisfied by the definition of C^k_n.
- ad 2. and ad 4.) Let $n < \omega$ and $(\bar{\eta}^k, \bar{\nu}^k) \in C^k_n$ be arbitrary. Let $s^k \in D_{F_n,n}(p^k_{n+1})$ for $k \in \{0,1\}$ witness $(\bar{\eta}^k, \bar{\nu}^k) \in C^k_n$ such that (s^0, s^1) decides $\dot{\tau} \restriction n$. Let $m < \tilde{l}$ such that $(\bar{\eta}^k, \bar{\nu}^k) = (\bar{\eta}^k_m, \bar{\nu}^k_m)$. Since we also have $D_{F_n,n}(p^k_{n+1}) \subseteq D_{F_n,n}({}^m p^k_n)$, it follows that there exists ${}^m s^k_n \in D_{F_n,n}({}^m p^k_n)$ with $((\eta^{{}^m s^k_n}_\beta)_{\beta \in F_n}, (\eta^{{}^m s^k_n}_\beta)_{\beta \in F_n}) = (\bar{\eta}^k_m, \bar{\nu}^k_m)$ and $\rho_{(\bar{\nu}^0_m, \bar{\nu}^1_m)} \in 2^n$ such that $({}^m s^0_n, {}^m s^1_n) \Vdash_{\mathbb{P}^2} \dot{\tau} \restriction n = \rho_{(\bar{\nu}^0_m, \bar{\nu}^1_m)}$. Furthermore, we have $({}^{m+1}p^0_n, {}^{m+1}p^1_n) := ({}^m p^0_n{}^{[{}^m s^0_n]}, {}^m p^1_n{}^{[{}^m s^1_n]})$ and $\rho_{(\bar{\nu}^0_m, \bar{\nu}^1_m)}$ is defined. Hence $A_n(((\bar{\eta}^0, \bar{\nu}^0), (\bar{\eta}^1, \bar{\nu}^1))) = \rho_{(\bar{\nu}^0_m, \bar{\nu}^1_m)}$.
  By induction on $\beta \in F_n$ we will now show that $q^{k\,[\bar{\nu}^k]} \restriction \beta \Vdash_\beta \bar{\eta}^k(\beta) \in \operatorname{split}_n(q^k(\beta))$:
  - $\beta = 0$: Since $\bar{\eta}^k(0) \in \operatorname{split}_n(p^k_{n+1}(0))$ and $\operatorname{split}_n(p^k_{n+1}(0)) = \operatorname{split}_n(q^k(0))$ we have that $\bar{\eta}^k(0) \in \operatorname{split}_n(q^k(0))$.
  - $\beta > 0$: Assume that for every $\beta' \in F \cap \beta$ we have $q^{k\,[\bar{\nu}^k]} \restriction \beta' \Vdash_{\beta'} \bar{\eta}^k(\beta') \in \operatorname{split}_n(q^k(\beta'))$. Therefore $q^{k\,[\bar{\nu}^k]} \restriction \beta$ is well defined. Since $q^k \leq_{\mathbb{P}} {}^{m+1}p^k_n$, $q^{k\,[\bar{\nu}^k]} \restriction \beta \in D_{F_n \cap \beta, n}({}^m p^k_n \restriction \beta)$ and therefore, by Lemma 6.4.10, $q^{k\,[\bar{\nu}^k]} \restriction \beta \leq_{\mathbb{P}} {}^m s^k_n \restriction \beta$, it follows that $q^{k\,[\bar{\nu}^k]} \restriction \beta \Vdash_\beta \bar{\eta}^k(\beta) \in \operatorname{split}_n({}^m p^k_n(\beta))$. Since $q^{k\,[\bar{\nu}^k]} \restriction \beta \Vdash_\beta \operatorname{split}_n(q^k(\beta)) = \operatorname{split}_n({}^m p^k_n(\beta))$ we have $q^{k\,[\bar{\nu}^k]} \restriction \beta \Vdash_\beta \bar{\eta}^k(\beta) \in \operatorname{split}_n(q^k(\beta))$.

  Since $q^{k\,[\bar{\nu}^k]} \leq_{\mathbb{P}} {}^m s^k_n$ for $k \in \{0,1\}$, we have $(q^{0\,[\bar{\nu}^0]}, q^{1\,[\bar{\nu}^1]}) \Vdash_{\mathbb{P}^2} \dot{\tau} \restriction n = \rho_{(\bar{\nu}^0, \bar{\nu}^1)}$. □

Definition 6.4.14. Let $\tilde{q}^k \leq_{\mathbb{P}} q^k$ and let $m < \omega$. We define $\tilde{q}^k$ to be m-okay iff for every $(\bar{\eta}, \bar{\nu}) \in C^k_m$ one of the following two cases applies:

(Case 1.) $\forall \beta \in F_m \colon \tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \operatorname{split}_{\tilde{q}^k(\beta)}(\bar{\eta}(\beta))$

(Case 2.) $\exists \beta \in F_m\ \forall \beta' \in F_m \cap \beta \colon \tilde{q}^{k\,[\bar{\nu}]} \restriction \beta' \Vdash_{\beta'} \operatorname{split}_{\tilde{q}^k(\beta')}(\bar{\eta}(\beta'))$ and $\tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \bar{\eta}(\beta) \notin \tilde{q}^k(\beta)$

and set $C^k_m(\tilde{q}^k) := \{(\bar{\eta}, \bar{\nu}) \in C^k_m \colon \forall \beta \in F_m \colon \tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \operatorname{split}_{\tilde{q}^k(\beta)}(\bar{\eta}(\beta))\}$.

We define $\tilde{q}^k$ to be m-good iff for every $n \geq m$ we have:

- $\tilde{q}^k$ is n-okay
- $\forall(\bar{\eta},\bar{\nu}) \in C^k_n(\tilde{q}^k)\ \forall\beta\in\kappa\colon \tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \tilde{q}^{k\,[\bar{\nu}]}(\beta) = q^{k\,[\bar{\nu}]}$.
- $\forall\beta\in\kappa\setminus F_m\colon \tilde{q}^k \restriction \beta \Vdash_\beta \tilde{q}^k(\beta) = q^k(\beta)$

Lemma 6.4.15. Let $\tilde{q}^k \le_{\mathbb{P}} q^k$ be m-okay and m^*-good for some $m^* \ge m$. Let $(\bar{\eta},\bar{\nu}) \in C^k_m(\tilde{q}^k)$ and let $(\bar{\bar{\eta}},\bar{\bar{\nu}}) \in C^k_{m'}(\tilde{q}^k)$ for some $m' > m^*$ such that $\forall\beta\in F_m\colon \bar{\nu}(\beta) \triangleleft \bar{\bar{\eta}}(\beta)$. Then there exists a condition $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle} \le_{\mathbb{P}} \tilde{q}^k$ such that:

1. $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}$ is $m'+1$-good.
2. $\forall\beta\in F_m\colon \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta) \cap \mathrm{split}_{m+1}(q^k(\beta)) = \tilde{q}^k(\beta) \cap \mathrm{split}_{m+1}(q^k(\beta))$
   In particular, $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}$ is m-okay with $C^k_m(\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}) = C^k_m(\tilde{q}^k)$.
3. $\forall\beta\in\kappa\colon \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]}(\beta) = \tilde{q}^{k\,[\bar{\bar{\nu}}]}(\beta)$
   In particular, $\forall\beta\in F_{m'}\colon \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \bar{\bar{\nu}} \triangleleft \dot{s}^k_\beta$.

Proof. We will define $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}$ by induction on $\beta < \kappa$ and simultaneously prove $(2.)_\beta$ and $(3.)_\beta$:

- $\beta = 0$: We set $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(0) := \left(\tilde{q}^k(0) \setminus \tilde{q}^k(0)^{[\bar{\nu}(0)]}\right) \cup \tilde{q}^k(0)^{[\bar{\bar{\nu}}(0)]}$ and see that $(2.)_0$ and $(3.)_0$ obviously hold true.
- $\beta > 0$: Here we distinguish three cases:
  - $\beta \notin F_{m'}$: Wet set $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta) := \tilde{q}^k(\beta)$ and see that $(3.)_\beta$ obviously holds true.
  - $\beta \in F_{m'} \setminus F_m$: We define $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta) := \begin{cases} \tilde{q}^k(\beta)^{[\bar{\bar{\nu}}(\beta)]} & \text{if } \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \in G_\beta \\ \tilde{q}^k(\beta) & \text{else} \end{cases}$
    Since $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta$ is well-defined by $(2.)_{\beta'}$ for $\beta' < \beta$, $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \le_{\mathbb{P}} \tilde{q}^{k\,[\bar{\nu}]} \restriction \beta$ by $(3.)_{\beta'}$ for $\beta' < \beta$ and $\tilde{q}^k$ is m'-okay, we have that $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \mathrm{split}_{\tilde{q}^k(\beta)}(\bar{\bar{\eta}}(\beta))$ and hence $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta)$ is well-defined. $(3.)_\beta$ follows immediately.
  - $\beta \in F_m$: Similarly we define
    $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta) := \begin{cases} \left(\tilde{q}^k(\beta) \setminus \tilde{q}^k(\beta)^{[\bar{\nu}(\beta)]}\right) \cup \tilde{q}^k(\beta)^{[\bar{\bar{\nu}}(\beta)]} & \text{if } \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \in G_\beta \\ \tilde{q}^k(\beta) & \text{else} \end{cases}$
    Again, since $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta$ is well-defined by $(2.)_{\beta'}$ for $\beta' < \beta$, $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \le_{\mathbb{P}} \tilde{q}^{k\,[\bar{\nu}]} \restriction \beta$ by $(3.)_{\beta'}$ for $\beta' < \beta$ and $\tilde{q}^k$ is m'-okay, we have that $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \mathrm{split}_{\tilde{q}^k(\beta)}(\bar{\bar{\eta}}(\beta))$ and hence $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta)$ is well-defined. $(2.)_\beta$ and $(3.)_\beta$ follow immediately.

It remains to be shown that $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}$ is $m'+1$-good: We clearly have $\forall\beta\in\kappa\setminus F_{m'}\colon \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta) = \tilde{q}^k(\beta) = q^k(\beta)$. Now let $n \ge m'+1$ and $(\bar{\bar{\eta}},\bar{\bar{\nu}}) \in C^k_n$ be arbitrary. We will simultaneously show by induction on $\beta \in F_n$ that either (Case 1.) or (Case 2.) applies and $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\bar{\nu}}]} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\bar{\nu}}]}(\beta) = q^{k\,[\bar{\bar{\nu}}]}(\beta)$ if $\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle\,[\bar{\bar{\nu}}]} \restriction \beta \Vdash_\beta \mathrm{split}_{\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(\beta)}(\bar{\bar{\eta}}(\beta))$:

- $\beta = 0$: We have four cases:
  - $\mathrm{split}_{\tilde{q}^k(0)}(\bar{\bar{\eta}}(0))$, $\bar{\nu}(0) \triangleleft \bar{\bar{\eta}}(0)$ and $\bar{\bar{\nu}}(0) \triangleleft \bar{\bar{\eta}}(0)$: Then clearly $\mathrm{split}_{\tilde{q}^{k\,\langle\bar{\bar{\nu}}\rangle}(0)}(\bar{\bar{\eta}}(0))$.

- $\text{split}_{\tilde{q}^k(0)}(\bar{\eta}(0))$ and $\bar{\nu}(0) \ntriangleleft \bar{\eta}(0)$: Again clearly $\text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(0)}(\bar{\eta}(0))$.

Clearly, in both of the above cases we have $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]}(0) = \tilde{q}^{k\,[\bar{\nu}]}(0) = q^{k\,[\bar{\nu}]}(0)$.

- $\text{split}_{\tilde{q}^k(0)}(\bar{\eta}(0))$, $\bar{\nu}(0) \triangleleft \bar{\eta}(0)$ and $\bar{\nu}(0) \ntriangleleft \bar{\eta}(0)$: Then $\bar{\eta}(0) \notin \tilde{q}^{k\,\langle\bar{\nu}\rangle}(0)$.
- $\bar{\eta}(0) \notin \tilde{q}^k(0)$: Then clearly $\bar{\eta}(0) \notin \tilde{q}^{k\,\langle\bar{\nu}\rangle}(0)$.

- $\beta \in F_n \setminus F_{m'}$: We assume inductively that $\forall \beta' \in F_n \cap \beta\colon \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta' \Vdash_{\beta'} \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta')}(\bar{\eta}(\beta'))$. Again, since $\tilde{q}^{k\,\langle\bar{\nu}\rangle} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta) = q^k(\beta)$ it follows that $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)}(\bar{\eta}(\beta))$ and $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]}(\beta) = q^{k\,[\bar{\nu}]}(\beta)$.

- $\beta \in F_{m'} \setminus F_m$: Again, we assume inductively that $\forall \beta' \in F_n \cap \beta\colon \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta' \Vdash_{\beta'} \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta')}(\bar{\eta}(\beta'))$. We have three cases:
  - $\forall \beta' \in F_{m'} \cap \beta\colon \bar{\nu}(\beta') \triangleleft \bar{\eta}(\beta')$ and $\bar{\nu}(\beta) \triangleleft \bar{\eta}(\beta)$: Then $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \leq_{\mathbb{P}} \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ and hence $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)}(\bar{\eta}(\beta))$.
  - $\exists \beta' \in F_{m'} \cap \beta\colon \bar{\nu}(\beta') \ntriangleleft \bar{\eta}(\beta')$: Let $\beta^* := \min\{\beta' \in F_{m'} \cap \beta\colon \bar{\nu}(\beta') \ntriangleleft \bar{\eta}(\beta')\}$. Since $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta^* \Vdash_{\beta^*} \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta^*)}(\bar{\eta}(\beta^*))$ and $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta^* \leq_{\mathbb{P}} \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta^*$, it follows that $\beta^* \in F_m$ and $\bar{\nu}(\beta^*) \ntriangleleft \bar{\eta}(\beta^*)$. Hence $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ and $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ are incompatible, and again $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)}(\bar{\eta}(\beta))$.

  Clearly, in both of the above cases we have $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]}(\beta) = \tilde{q}^{k\,[\bar{\nu}]}(\beta) = q^{k\,[\bar{\nu}]}(\beta)$.

  - $\forall \beta' \in F_{m'} \cap \beta\colon \bar{\nu}(\beta') \triangleleft \bar{\eta}(\beta')$ and $\bar{\nu}(\beta) \ntriangleleft \bar{\eta}(\beta)$: Then $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \leq_{\mathbb{P}} \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ and $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \bar{\eta} \notin \tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)$.

- $\beta \in F_m \setminus \{0\}$: Again, we assume inductively that $\forall \beta' \in F_n \cap \beta\colon \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta' \Vdash_{\beta'} \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta')}(\bar{\eta}(\beta'))$ and $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \leq_{\mathbb{P}} \tilde{q}^{k\,[\bar{\nu}]} \restriction \beta$. We have four cases:
  - $\tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^k(\beta)}(\bar{\eta}(\beta))$, $\forall \beta' \in F_{m'} \cap \beta\colon \bar{\nu}(\beta') \triangleleft \bar{\eta}(\beta')$ and $\bar{\nu}(\beta) \triangleleft \bar{\eta}(\beta)$: Then $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \leq_{\mathbb{P}} \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ and therefore $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)}(\bar{\eta}(\beta))$.
  - $\tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^k(\beta)}(\bar{\eta}(\beta))$ and $\exists \beta' \in F_{m'} \cap \beta\colon \bar{\nu}(\beta') \ntriangleleft \bar{\eta}(\beta')$: Similar to above it follows that $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ and $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ are incompatible and therefore $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)}(\bar{\eta}(\beta))$.

  Clearly, in both of the above cases we have $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]}(\beta) = \tilde{q}^{k\,[\bar{\nu}]}(\beta) = q^{k\,[\bar{\nu}]}(\beta)$.
  - $\tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \text{split}_{\tilde{q}^k(\beta)}(\bar{\eta}(\beta))$, $\forall \beta' \in F_{m'} \cap \beta\colon \bar{\nu}(\beta') \triangleleft \bar{\eta}(\beta')$ and $\bar{\nu}(\beta) \ntriangleleft \bar{\eta}(\beta)$: Then $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \leq_{\mathbb{P}} \tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta$ and therefore $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \bar{\eta} \notin \tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)$.
  - $\tilde{q}^{k\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \bar{\eta}(\beta) \notin \tilde{q}^k(\beta)$: Then clearly $\tilde{q}^{k\,\langle\bar{\nu}\rangle\,[\bar{\nu}]} \restriction \beta \Vdash_\beta \bar{\eta}(\beta) \notin \tilde{q}^{k\,\langle\bar{\nu}\rangle}(\beta)$. □

W.l.o.g. we can assume that $\text{supp}(q^0) = \text{supp}(q^1)$. Let $\text{mos} : \text{supp}(q^k) \to \alpha^*$ denote the transitive collapse of $\text{supp}(q^k)$ with $\alpha^* < \omega_1$, let $\pi\colon \alpha^* \to \text{supp}(q^k)$ denote the uncollapse, and let $y \in \omega^\omega$ be arbitrary. Code the 'transitive collapse' of $(F_n)_{n<\omega}$,

$((C^k_n)_{n<\omega})_{k\in\{0,1\}}$ and $(A_n)_{n<\omega}$ as a real $z \in \omega^\omega$. Let $((\dot{c}^k_\beta)_{\beta<\alpha^*})_{k\in\{0,1\}}$ be a sequence of $\mathbb{P}^2$-names for reals such that $\Vdash_{\mathbb{P}^2} \dot{c}^k_\beta(n) = i$ iff $\eta^\frown i \lhd \dot{s}^k_{\pi(\beta)}$ for some $\eta \in \mathrm{split}_n(q^k(\pi(\beta)))$.

Lemma 6.4.16. Let $\mathbb{C}_{\alpha^*} := \prod_{\beta<\alpha^*} \mathbb{C}$ denote the finite support product of α^* many Cohen forcings. Then there exists $(r^0, r^1) \leq_{\mathbb{P}^2} (q^0, q^1)$ such that $(r^0, r^1) \Vdash_{\mathbb{P}^2} ((\dot{c}^k_\beta)_{\beta<\alpha^*})_{k\in\{0,1\}}$ is $(L[z], \mathbb{C}^2_{\alpha^*})$-generic $\land\ \dot{\tau} \in L[z][((\dot{c}^k_\beta)_{\beta<\alpha^*})_{k\in\{0,1\}}]$.

Proof. For notational simplicity let us assume that $\mathrm{supp}(q^k) = \alpha^*$, i.e. mos is the identity. Let $(D_m)_{m<\omega}$ enumerate all dense open subsets of $\mathbb{C}^2_{\alpha^*}$ contained in $L[z]$. Working in V we will now construct by induction on $m < \omega$ a decreasing sequence $((q^0_m, q^1_m))_{m<\omega}$ and an increasing sequence of natural numbers $(n_m)_{m<\omega}$ such that:

1. $\forall m < \omega$: $\mathrm{supp}(q^k_m) = \alpha^*$
2. $\forall m < \omega$: q^k_m is n_m-good.
3. $\forall m < \omega\ \forall m' < m\ \forall \beta \in F_{n_{m'}}$: $q^k_m \restriction \beta \Vdash_\beta q^k_m(\beta) \cap \mathrm{split}_{n_{m'}+1}(q^k(\beta)) = q^k_{m'}(\beta) \cap \mathrm{split}_{n_{m'}+1}(q^k(\beta))$
4. $\forall m < \omega$: $(q^0_{m+1}, q^1_{m+1}) \Vdash_{\mathbb{P}^2} ((\dot{c}^k_\beta \restriction n_{m+1})_{\beta\in F_{n_{m+1}}})_{k\in\{0,1\}} \in D_m$

Hence:

- If $m = 0$ we set $(q^0_0, q^1_0) := (q^0, q^1)$ and $n_0 := 0$. (1.) and (2.) are obviously satisfied. (3.) and (4.) are vacuously true.

- $m \to m+1$: Assume that (q^0_m, q^1_m) and n_m have already been defined and satisfy (1.) - (4.). Enumerate $C^0_{n_m}(q^0_m) \times C^1_{n_m}(q^1_m)$ as $(((\bar{\eta}^0_l, \bar{\nu}^0_l), (\bar{\eta}^1_l, \bar{\nu}^1_l)))_{l<\tilde{l}}$ for some $\tilde{l} < \omega$. By induction on $l < \tilde{l}$ we will now construct a decreasing sequence $((^l q^0_m, {}^l q^1_m))_{l\leq\tilde{l}}$ and an increasing sequence of natural numbers ${}^l n_m$ such that ${}^l q^k_m$ is n_m-okay with $C^k_{n_m}({}^l q^k_m) = C^k_{n_m}(q^k_m)$ and ${}^l n_m$-good:
  - $l = 0$: We set $({}^0 q^0_m, {}^0 q^1_m) := (q^0_m, q^1_m)$ and ${}^0 n_m := n_m$. Obviously, ${}^0 q^k_m$ is n_m-good by assumption.
  - $l \to l+1$: Assume that $({}^l q^0_m, {}^l q^1_m)$ and ${}^l n_m$ have already been defined. Since ${}^l q^k_m$ is n_m-okay with $C^k_{n_m}({}^l q^k_m) = C^k_{n_m}(q^k_m)$, we see that ${}^l q^k_m{}^{[\bar{\nu}^k_l]}$ is well-defined. Let $s^k \in D_{F_{{}^l n_m}, {}^l n_m}(q^k)$ with $s^k \leq_{\mathbb{P}} {}^l q^k_m{}^{[\bar{\nu}^k_l]}$. Since ${}^l q^k_m$ is ${}^l n_m$-good, this implies that $((\eta^{s^k}_\beta)_{\beta\in F_{{}^l n_m}}, (\nu^{s^k}_\beta)_{\beta\in F_{{}^l n_m}}) \in C^k_{{}^l n_m}({}^l q^k_m)$ and $\forall \beta \in F_{n_m} : \nu^k_l(\beta) \lhd \eta^{s^k}_\beta$. Furthermore, we have

  $$\forall \beta \in \kappa \colon\ {}^l q^k_m{}^{[(\nu^{s^k}_\beta)_{\beta\in F_{{}^l n_m}}]} \restriction \beta \Vdash_\beta {}^l q^k_m{}^{[(\nu^{s^k}_\beta)_{\beta\in F_{{}^l n_m}}]}(\beta) = q^{k\,[(\nu^{s^k}_\beta)_{\beta\in F_{{}^l n_m}}]}(\beta).$$

  We see that ${}^l q^k_m{}^{[(\nu^{s^k}_\beta)_{\beta\in F_{{}^l n_m}}]} \Vdash_{\mathbb{P}} (\dot{c}^k_\beta \restriction ({}^l n_m + 1))_{\beta\in F_{{}^l n_m}} = \tilde{p}^k$ for some condition $\tilde{p}^k \in \mathbb{C}_{\alpha^*}$. Let $({}^l\hat{p}^0, {}^l\hat{p}^1) \leq_{\mathbb{C}^2_{\alpha^*}} (\tilde{p}^0, \tilde{p}^1)$ such that $({}^l\hat{p}^0, {}^l\hat{p}^1) \in D_m$. W.l.o.g. we can assume that there exists ${}^l\tilde{n} > {}^l n_m$ such that $\forall k \in \{0,1\}$: $\mathrm{dom}({}^l\hat{p}^k) =$

$F_{l_{\tilde{n}}} \wedge \forall \beta \in F_{l_{\tilde{n}}} \operatorname{dom}({}^l\hat{p}^k(\beta)) = {}^l\tilde{n}+1$. Now we can easily find $s_l^k \in D_{F_{l_{\tilde{n}}}, {}^l\tilde{n}}(q^k)$ with $s_l^k \leq_{\mathbb{P}} {}^l q_m^k{}^{[(\nu_\beta^{s^k})_{\beta \in F_{l_{n_m}}}]}$ such that $s_l^k \Vdash_{\mathbb{P}} (\dot{c}_\beta^k \restriction ({}^l\tilde{n}+1))_{\beta \in F_{l_{\tilde{n}}}} = {}^l\hat{p}^k$. Note that $\forall \beta \in F_{n_m} \colon \bar{\nu}_l^k(\beta) \lhd \eta_\beta^{s_l^k}$.

We can now apply Lemma 6.4.15 with $\tilde{q}^k := {}^l q_m^k$, $m := n_m$, $m^* := {}^l n_m$, $m' := {}^l\tilde{n}$, $(\bar{\eta}, \bar{\nu}) := (\bar{\eta}_l^k, \bar{\nu}_l^k)$ and $(\bar{\tilde{\eta}}, \bar{\tilde{\nu}}) := ((\eta_\beta^{s_l^k})_{\beta \in F_{l_{\tilde{n}}}}, (\nu_\beta^{s_l^k})_{\beta \in F_{l_{\tilde{n}}}})$, and set ${}^{l+1}q_m^k := {}^l q_m^k{}^{\langle(\nu_\beta^{s_l^k})_{\beta \in F_{l_{\tilde{n}}}}\rangle}$ and ${}^{l+1}n_m := {}^l\tilde{n}+1$. By Lemma 6.4.15 we have that ${}^{l+1}q_m^k$ is n_m-okay with $C_{n_m}^k({}^{l+1}q_m^k) = C_{n_m}^k({}^l q_m^k) = C_{n_m}^k(q_m^k)$ and ${}^{l+1}n_m$-good.

We set $q_{m+1}^k := {}^{\tilde{l}}q_m^k$ and $n_{m+1} := {}^{\tilde{l}}n_m$. We immediately see that (1.) and (2.) are satisfied.

ad (3.): Using induction on $l < \tilde{l}$ and Lemma 6.4.15 it follows that

$$\forall \beta \in F_{n_m} \colon q_{m+1}^k \restriction \beta \Vdash_\beta q_{m+1}^k(\beta) \cap \operatorname{split}_{n_m+1}(q^k(\beta)) = q_m^k(\beta) \cap \operatorname{split}_{n_m+1}(q^k(\beta)).$$

But this implies

$$\forall \beta \in F_{n_{m'}} \colon q_{m+1}^k \restriction \beta \Vdash_\beta q_{m+1}^k(\beta) \cap \operatorname{split}_{n_{m'}+1}(q^k(\beta)) = q_m^k(\beta) \cap \operatorname{split}_{n_{m'}+1}(q^k(\beta))$$

for every $m' < m+1$. Using the induction hypothesis for q_m^k and noting that $q_{m+1}^k \leq_{\mathbb{P}} q_m^k$ we can deduce

$$\forall \beta \in F_{n_{m'}} \colon q_{m+1}^k \restriction \beta \Vdash_\beta q_{m+1}^k(\beta) \cap \operatorname{split}_{n_{m'}+1}(q^k(\beta)) = q_{m'}^k(\beta) \cap \operatorname{split}_{n_{m'}+1}(q^k(\beta))$$

for every $m' < m+1$.

ad (4.): Let $\tilde{s}^k \in D_{F_{n_m}, n_m}(q^k)$ with $\tilde{s}^k \leq_{\mathbb{P}} q_{m+1}^k$ be arbitrary. Hence, there exists $l < \tilde{l}$ with $(((\eta_\beta^{\tilde{s}^0})_{\beta \in F_{n_m}}, (\nu_\beta^{\tilde{s}^0})_{\beta \in F_{n_m}}), ((\eta_\beta^{\tilde{s}^1})_{\beta \in F_{n_m}}, (\nu_\beta^{\tilde{s}^1})_{\beta \in F_{n_m}})) = ((\bar{\eta}_l^0, \bar{\nu}_l^0), (\bar{\eta}_l^1, \bar{\nu}_l^1))$.

By Lemma 6.4.15 we see that ${}^{l+1}q_m^k{}^{[(\nu_\beta^{\tilde{s}^k})_{\beta \in F_{n_m}})]} \leq_{\mathbb{P}} {}^l q_m^k{}^{[(\nu_\beta^{s_l^k})_{\beta \in F_{l_{\tilde{n}}}}]}$ and hence $\tilde{s}^k \leq_{\mathbb{P}} {}^l q_m^k{}^{[(\nu_\beta^{s_l^k})_{\beta \in F_{l_{\tilde{n}}}}]}$. Since ${}^l q_m^k{}^{[(\nu_\beta^{s_l^k})_{\beta \in F_{l_{\tilde{n}}}}]}$ decides $(\dot{c}_\beta^k \restriction {}^{l+1}n_m)_{\beta \in F_{l_{\tilde{n}}}}$ and $s_l^k \leq_{\mathbb{P}} {}^l q_m^k{}^{[(\nu_\beta^{s_l^k})_{\beta \in F_{l_{\tilde{n}}}}]}$, we have that

$$({}^l q_m^0{}^{[(\nu_\beta^{s_l^0})_{\beta \in F_{l_{\tilde{n}}}}]}, {}^l q_m^1{}^{[(\nu_\beta^{s_l^1})_{\beta \in F_{l_{\tilde{n}}}}]}) \Vdash_{\mathbb{P}^2} ((\dot{c}_\beta^k \restriction {}^{l+1}n_m)_{\beta \in F_{l_{\tilde{n}}}})_{k \in \{0,1\}} = ({}^l\hat{p}^0, {}^l\hat{p}^1) \in D_m$$

and therefore $(\tilde{s}^0, \tilde{s}^1) \Vdash_{\mathbb{P}^2} ((\dot{c}_\beta^k \restriction n_{m+1})_{\beta \in F_{n_{m+1}}})_{k \in \{0,1\}} \in D_m$.

Now we define r^k such that for every $\beta < \alpha^*$ we have $r^k \restriction \beta \Vdash_\beta r^k(\beta) = \bigcap_{m<\omega} q_m^k(\beta)$ (and $r^k \restriction \beta \Vdash_\beta r^k(\beta) = \mathbb{1}_{\mathbb{S}}$ for $\beta \in \kappa \setminus \alpha^*$). We will show by induction on $\beta < \alpha^*$ that $r^k \restriction \beta \Vdash_\beta r^k(\beta) \in \mathbb{S}$:

- $\beta = 0$: Let $\eta \in r^k(0)$ be arbitrary and w.l.o.g. we can assume that $\eta \in \operatorname{split}_l(q^k(0))$ for some $l < \omega$. Let $m^* := \min\{m < \omega \colon l \leq n_m\}$. Let $\tilde{\eta} \in q_{m^*}^k(0) \cap \operatorname{split}_{n_{m^*}}(q^k(0))$ such that $\eta \lhd \tilde{\eta}$. Since $q_{m^*}^k$ is, in particular, n_{m^*}-okay, we have $\operatorname{split}_{q_{m^*}^k(0)}(\tilde{\eta})$. By (3.) it follows that $\operatorname{split}_{q_{m'}^k(0)}(\tilde{\eta})$ for every $m' > m^*$. Hence, we can deduce $\operatorname{split}_{r^k(0)}(\tilde{\eta})$ and therefore $r^k(0) \in \mathbb{S}$.

- $\beta > 0$: We assume that for every $\beta' < \beta$ we have $r^k \restriction \beta' \Vdash_{\beta'} r^k(\beta') \in \mathbb{S}$. Hence, $r^k \restriction \beta$ is a condition and $\forall m < \omega\colon r^k \restriction \beta \leq_{\mathbb{P}} q^k_m \restriction \beta$. Towards a contradiction assume that there exists $s \leq_{\mathbb{P}} r^k \restriction \beta$ and $\eta \in 2^{<\omega}$ such that $s \Vdash_\beta \eta \in r^k(\beta) \wedge \forall \eta' \in r^k(\beta)\colon \eta \lhd \eta' \Rightarrow \neg \operatorname{split}_{r^k(\beta)}(\eta')$. W.l.o.g. $s \Vdash_\beta \eta \in \operatorname{split}_l(q^k(\beta))$ for some $l < \omega$. Let $m^* := \{m < \omega\colon l \leq n_m \wedge \beta \in F_m\}$. Let $s' \in D_{F_{m^*} \cap \beta, n_{m^*}}(q^k \restriction \beta)$ with $s' \leq_{\mathbb{P}} s$ and $\tilde{\eta} \in 2^{<\omega}$ with $\eta \lhd \tilde{\eta}$ such that $s' \Vdash_\beta \tilde{\eta} \in q^k_{m^*}(\beta) \cap \operatorname{split}_{n_{m^*}}(q^k(\beta))$. Since $q^k_{m^*}$ is n_{m^*}-okay and $s' \leq_{\mathbb{P}} q^k_{m^*}{}^{[(\nu^{s'}_{\beta'})_{\beta' \in F_{m^*} \cap \beta}]}$, we have $q^k_{m^*}{}^{[(\nu^{s'}_{\beta'})_{\beta' \in F_{m^*} \cap \beta}]} \restriction \beta \Vdash_\beta \operatorname{split}_{q^k_{m^*}(\beta)}(\tilde{\eta})$. By (4.) we have $s' \Vdash_\beta \operatorname{split}_{q^k_{m'}(\beta)}(\tilde{\eta})$ for every $m' > m^*$. But this implies $s' \Vdash_\beta \operatorname{split}_{r^k(\beta)}(\tilde{\eta})$ which is a contradiction.

It remains to be shown that $(r^0, r^1) \Vdash_{\mathbb{P}^2} \dot{\tau} \in L[z][((\dot{c}^k_\beta)_{\beta<\alpha^*})_{k \in \{0,1\}}]$. But this immediately follows, because, given $((\dot{c}^k_\beta)_{\beta<\alpha^*})_{k\in\{0,1\}}$, one can inductively reconstruct $((\dot{s}^k_\beta)_{\beta<\alpha^*})_{k\in\{0,1\}}$ using $(F_n)_{n<\omega}$ and $((C^k_n)_{n<\omega})_{k\in\{0,1\}}$, and $\dot{\tau}$ can be reconstructed from $((\dot{s}^k_\beta)_{\beta<\alpha^*})_{k\in\{0,1\}}$ and $(A_n)_{n<\omega}$. □

Proof of Theorem 6.4.12. Let $(p^0, p^1) \in \mathbb{P}^2$, $\dot{\tau}$ a $\mathbb{P}^2$-name for a real and $y \in \omega^\omega$ be arbitrary. Using Lemma 6.4.13, Lemma 6.4.15 and Lemma 6.4.16 we can deduce that there exists $z \in \omega^\omega$, $\alpha^* < \omega_1^{L[y,z]}$ and $(q^0, q^1) \leq_{\mathbb{P}^2} (p^0, p^1)$ such that

$$(q^0, q^1) \Vdash_{\mathbb{P}^2} \exists H\colon H \text{ is } (L[y,z], \mathbb{C}^2_{\alpha^*})\text{-generic} \wedge \dot{\tau} \in L[y,z][H].$$

This shows that $\mathbb{P}^2$ is captured. □

Lemma 6.4.17. Let $\theta > \omega$ be a sufficiently large, regular cardinal and let $M \prec H(\theta)$ be a countable, elementary submodel such that $\mathbb{P} \in M$. Let $g \in V$ be an $(M, \mathbb{P})$-generic filter and $p \in \mathbb{P} \cap M$ be a condition. Then there exists $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}} g \times \dot{G}$ is $(M, \mathbb{P}^2)$-generic.

Proof. We proceed very similar to Lemma 6.4.13: Let $(D_n)_{n<\omega}$ enumerate all the dense open subsets of $\mathbb{P}^2$ contained M. Working in M we will construct by induction on $n < \omega$ a decreasing sequence $(p^0_n)_{n<\omega} \subseteq g$ and a fusion sequence $(p^1_n)_{n<\omega} \subseteq \mathbb{P} \cap M$ such that $\forall n < \omega\colon p^1_{n+1} \leq_{F_n, n+1} p^1_n$ and $p^1_{n+1} \Vdash_{\mathbb{P}} \exists p' \in \dot{G}\colon (p^0_{n+1}, p') \in D_n \cap M$. Then the fusion limit q will have the required property.

- If $n = 0$ let $p^0_0 \in g$ be some condition, and set $p^1_0 := p$ and $F_0 := \{0\}$.
- $n \to n+1$: Assume that p^0_n, p^1_n and F_n have already been defined. Using Lemma 6.4.8 we find $\tilde{p}^1_n \leq_{F_n, n+1} p^1_n$ and $l < \omega$ such that $\forall \beta \in F_n\colon \tilde{p}^1_n \restriction \beta \Vdash_\beta \operatorname{split}_{\underline{n}}(\tilde{p}^1_n(\beta)) \subseteq 2^{<l}$. Now enumerate $\prod_{\beta \in F_n} 2^{<l} \times \prod_{\beta \in F_n} 2^{\leq l}$ as $(\bar{\eta}_m, \bar{\nu}_m)_{m<\tilde{l}}$ for some $\tilde{l} < \omega$. By induction on $m < \tilde{l}$ construct a decreasing sequence $({}^m p^0_n)_{m \leq \tilde{l}} \subseteq g$ and a $\leq_{F_n, n+1}$-decreasing sequence $({}^m p^1_n))_{m \leq \tilde{l}} \subseteq M$:
  - $m = 0$: Set ${}^0 p^0_n := p^0_n$ and ${}^0 p^1_n := \tilde{p}^1_n$.
  - $m \to m+1$: Assume that ${}^m p^0_n$ and ${}^m p^1_n$ have already been defined. If there exists $s^1 \in D_{F_n, n}({}^m p^1_n)$ such that $\forall \beta \in F_n\colon \bar{\eta}_m(\beta) = \eta^{s^1}_\beta \wedge \bar{\nu}_m(\beta) = \nu^{s^1}_\beta$,

then there exist $s^0 \in \mathbb{P}$ with $s^0 \in g$, $s^0 \leq_{\mathbb{P}} {}^m p^0_n$ and $(s^0, s^1) \in D_n$. [14]
In this case pick such (s^0, s^1), call it $({}^m s^0_n, {}^m s^1_n)$, and set ${}^{m+1}p^0_n := s^0$ and ${}^{m+1}p^1_n := {}^m p^1_n{}^{[{}^m s^1_n]}$. Else set $({}^{m+1}p^0_n, {}^{m+1}p^1_n) := ({}^m p^0_n, {}^m p^1_n)$.
We clearly have ${}^{m+1}p^0_n \in g$, ${}^{m+1}p^1_n \in M$, ${}^{m+1}p^0_n \leq_{\mathbb{P}} {}^m p^0_n$ and ${}^{m+1}p^1_n \leq_{F_n,n} {}^m p^1_n$ by Lemma 6.4.10.

We set $p^0_{n+1} := {}^{\tilde{l}}p^0_n$ and $p^1_{n+1} := {}^{\tilde{l}}p^1_n$ and note that clearly $p^0_{n+1} \in g$, $p^1_{n+1} \in M$, $p^0_{n+1} \leq_{\mathbb{P}} p^0_n$ and $p^1_{n+1} \leq_{F_n,n} p^1_n$. Define $F_{n+1} \supseteq F_n$ using a bookkeeping argument.
It remains to be shown that p^1_{n+1} has the required properties. Now working in V let $s \in D_{F_n,n}(p^1_{n+1})$ be arbitrary. Hence, there exists $m < \tilde{l}$ such that $(\bar{\eta}_m, \bar{\nu}_m) = ((\eta^s_\beta)_{\beta \in F_n}, (\nu^s_\beta)_{\beta \in F_n})$. Since $D_{F_n,n}(p^1_{n+1}) \subseteq D_{F_n,n}({}^m p^1_n)$ and M is an elementary submodel, there exists $s' \in D_{F_n,n}({}^m p^1_n) \cap M$ with $(\bar{\eta}_m, \bar{\nu}_m) = ((\eta^{s'}_\beta)_{\beta \in F_n}, (\nu^{s'}_\beta)_{\beta \in F_n})$.
Therefore, we defined ${}^m s^0_n \in g$ and ${}^m s^1_n \in D_{F_n,n}({}^m p^1_n)$ with $((\eta_\beta^{{}^m s^1_n})_{\beta \in F_n}, (\nu_\beta^{{}^m s^1_n})_{\beta \in F_n}) = (\bar{\eta}_m, \bar{\nu}_m)$ and $({}^m s^0_n, {}^m s^1_n) \in D_n$ in the construction above. Since ${}^{m+1}p^1_n = {}^m p^1_n{}^{[{}^m s^1_n]}$ and $s \leq_{\mathbb{P}} p^1_{n+1} \leq_{\mathbb{P}} {}^{m+1}p^1_n$, we have $s \leq_{\mathbb{P}} {}^m s^1_n$ by Lemma 6.4.10. Hence $s \Vdash_{\mathbb{P}} {}^m s^1_n \in \dot{G} \wedge (p^0_{n+1}, {}^m s^1_n) \in D_n \cap M$. $\square$

6.5 Symmetric $\mathbf{\Delta}^1_3$-relations

We start with the following definition:

Definition 6.5.1. Let $\mathbb{P}$ be a forcing notion and let $E \subseteq \omega^\omega \times \omega^\omega$ be a symmetric relation. We call E a $\mathbb{P}$-absolute $\mathbf{\Delta}^1_3$-relation iff E both has a $\mathbf{\Sigma}^1_3$- and a $\mathbf{\Pi}^1_3$-definition, which remain equivalent in every $\mathbb{P}$-generic extension. We call E thin iff there is no perfect subset of pairwise E-incompatible reals.
Furthermore, we define $F_E \subseteq \omega^\omega \times \omega^\omega$ to be the smallest equivalence relation containing E. [15] We call the equivalence classes of F_E the connected components of E.

We will need the following lemma, which implies that symmetric, $\mathbb{P}$-absolute $\mathbf{\Delta}^1_3$-relations are absolute between V and $V^{\mathbb{P}}$.

Lemma 6.5.2. Assume that $V \vDash$ 'The reals are $\sharp$-closed' and let $\mathbb{P}$ be a forcing notion that is captured by forcing notions of size $< \omega_1^V$. Then $\mathbf{\Sigma}^1_3$-absoluteness holds between V and $V^{\mathbb{P}}$.

Proof. Let $\varphi(x)$ be a $\mathbf{\Sigma}^1_3$-definition and let $\psi(x,y)$ be a $\mathbf{\Pi}^1_2$-definition such that $\varphi(x)$ is equivalent to $\exists y \in \omega^\omega \colon \psi(x,y)$. Let $a \in \omega^\omega \cap V$ and assume that $V^{\mathbb{P}} \vDash \varphi(a)$. Hence, there exists $b \in \omega^\omega \cap V^{\mathbb{P}}$ with $V^{\mathbb{P}} \vDash \psi(a,b)$.
Since $\mathbb{P}$ is captured by forcing notions of size $< \omega_1^V$, there exist $z \in \omega^\omega \cap V$, $\mathbb{Q} \in L[a,z]$ with $|\mathbb{Q}| < \omega_1^V$ and $H \in V^{\mathbb{P}}$ which is $(L[a,z], \mathbb{Q})$-generic such that $b \in L[a,z][H]$. By $\mathbf{\Sigma}^1_2$-absoluteness we have $L[a,z][H] \vDash \psi(a,b)$ and therefore $L[a,z][H] \vDash \varphi(a)$. Hence,

[14] To see this, note that the set $E_n := \{p' \in \mathbb{P} \colon \exists s' \leq_{\mathbb{P}} s^1\ (p', s') \in D_n\}$ is open dense, hence there exists a $p' \in E_n \cap g$. Set $s^0 := p'$ and w.l.o.g. assume that $s^0 \leq_{\mathbb{P}} {}^m p^0_n$ and $(s^0, s^1) \in D_n$.
[15] Clearly, if E is a $\mathbf{\Sigma}^1_3$-relation, F_E is a $\mathbf{\Sigma}^1_3$-relation.

there exists $q \in H$ such that $q \Vdash_{\mathbb{Q}}^{L[a,z]} \varphi(a)$. Since $|\mathbb{Q}| < \omega_1^V$ and $\{a, z\}^\sharp$ exists, we can find an $(L[a, z], \mathbb{Q})$-generic filter H' containing q in V. Hence $L[a, z][H'] \models \varphi(a)$ and, by $\mathbf{\Sigma}^1_3$-upward absoluteness, we have $V \models \varphi(a)$.

□

The following arguments were inspired by [Hjo93] and can also be found in [CS21].

Lemma 6.5.3. Assume that $V \models$ 'The reals are $\sharp$-closed' and $\mathbb{P}$ is a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is captured. If E is a thin, symmetric $\mathbf{\Pi}^1_3$-relation, $p \in \mathbb{P}$ a condition and $\dot{\tau}$ a $\mathbb{P}$-name for a real such that $p \Vdash_{\mathbb{P}} \dot{\tau} \notin V$, then

$$D := \{p' \in \mathbb{P} : (p', p') \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{\tau}^{\dot{G}_1} \, E \, \dot{\tau}^{\dot{G}_2}\}$$

is dense below p, where $\dot{G}_1 \times \dot{G}_2$ denotes the $\mathbb{P} \times \mathbb{P}$-name for the $(V, \mathbb{P} \times \mathbb{P})$-generic filter.

Proof. Towards a contradiction, assume that D is not dense below p. Pick a condition $q \leq_{\mathbb{P}} p$ such that for any $r \leq_{\mathbb{P}} q$, there are $r_0, r_1 \leq_{\mathbb{P}} r$ with $(r_0, r_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg \dot{\tau}^{\dot{G}_1} \, E \, \dot{\tau}^{\dot{G}_2}$. Let $\theta > \omega$ be a regular and sufficiently large cardinal, let $M \prec H(\theta)$ be a countable, elementary submodel containing all the relevant parameters, and let $\mathrm{mos}\colon M \to N$ denote the Mostowski collapse. Let $\bar{\mathbb{P}} := \mathrm{mos}(\mathbb{P})$, $\bar{q} := q$ and $\dot{\bar{\tau}} := \mathrm{mos}(\dot{\tau})$. Working in V let $(D_n)_{n<\omega}$ enumerate all dense open subsets of $\bar{\mathbb{P}} \times \bar{\mathbb{P}}$ in N. We can now inductively construct a tree $(\bar{q}_s)_{s \in 2^{<\omega}}$ of conditions in $\bar{\mathbb{P}}$ such that:

1. $\bar{q}_\varnothing = \bar{q}$
2. $\bar{q}_{s^\frown i} \leq_{\bar{\mathbb{P}}} \bar{q}_s$ for $i \in \{0, 1\}$
3. $(\bar{q}_{s^\frown 0}, \bar{q}_{s^\frown 1}) \Vdash_{\bar{\mathbb{P}}\times\bar{\mathbb{P}}} \neg \dot{\bar{\tau}}^{\dot{g}_1} \, E \, \dot{\bar{\tau}}^{\dot{g}_2}$ [16]
4. $(\bar{q}_s, \bar{q}_t) \in D_0 \cap \cdots \cap D_{n-1}$ for $s, t \in 2^n$ with $s \neq t$

For $x \in 2^\omega$, we define $g_x := \{\bar{p}' \in \bar{\mathbb{P}} \colon \exists n < \omega \; \bar{q}_{x\restriction n} \leq_{\mathbb{P}} \bar{p}'\}$. Now it easily follows that for $x, y \in \omega^\omega$ with $x \neq y$ the filter $g_x \times g_y$ is $(N, \bar{\mathbb{P}} \times \bar{\mathbb{P}})$-generic with

$$N[g_x \times g_y] \models \neg \, \bar{\tau}^{g_x} \, E \, \bar{\tau}^{g_y}.$$

Since $\forall x' \in \omega^\omega \cap N[g_x \times g_y]\colon x'^\sharp \in \omega^\omega \cap N[g_x \times g_y]$ by Lemma 6.2.3, we have $\mathbf{\Sigma}^1_2$-absoluteness between $N[g_x \times g_y]$ and V, hence $V \models \neg \, \bar{\tau}^{g_x} \, E \, \bar{\tau}^{g_y}$ by $\mathbf{\Sigma}^1_3$-upward absoluteness. Since the map $2^\omega \ni x \mapsto \tau^{g_x}$ is continuous and injective (since $N \models$ '$\bar{q} \Vdash_{\bar{\mathbb{P}}} \dot{\bar{\tau}} \notin V$'), there exists a perfect set of pairwise E-incompatible reals. This, however, contradicts our assumption that E is thin. □

We can now prove the following theorem:

[16] Here, $\dot{g}_1 \times \dot{g}_2$ denotes the $\bar{\mathbb{P}} \times \bar{\mathbb{P}}$-name for the $(N, \bar{\mathbb{P}} \times \bar{\mathbb{P}})$-generic filter.

Theorem 6.5.4. Assume that $V \models$ 'The reals are $\sharp$-closed', let $\mathbb{P}$ denote a countable support iteration of Sacks forcing and let E be a thin, symmetric, $\mathbb{P}$-absolute $\mathbf{\Delta}^1_3$-relation. Then $V^{\mathbb{P}} \models \forall x \in \omega^\omega \setminus V\ \exists y \in \omega^\omega \cap V \colon x\, E\, y$.
In particular, $V^{\mathbb{P}} \models$ '$\forall x \in \omega^\omega\ \exists y \in \omega^\omega \cap V \colon x\, F_E\, y$', i.e. no new connected components of E appear in $V^{\mathbb{P}}$.

Proof. Let $p \in \mathbb{P}$ be a condition and $\dot{\tau}$ a $\mathbb{P}$-name for a real such that $p \Vdash_{\mathbb{P}} \dot{\tau} \notin V$. Since $\mathbb{P} \times \mathbb{P}$ is captured by Theorem 6.4.12, we can use Lemma 6.5.3 to find $q \leq_{\mathbb{P}} p$ such that $(q, q) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{\tau}^{\dot{G}_1}\, E\, \dot{\tau}^{\dot{G}_2}$.
Let $\theta > \omega$ be a regular and sufficiently large cardinal, let $M \prec H(\theta)$ be a countable, elementary submodel containing all the relevant parameters and let $g \in V$ be an $(M, \mathbb{P})$-generic filter containing q. By Lemma 6.4.17 we can now find $r \leq_{\mathbb{P}} q$ such that

$$r \Vdash_{\mathbb{P}} g \times (\dot{G} \cap M) \text{ is } (M, \mathbb{P} \times \mathbb{P})\text{-generic.}$$

Let G be a $(V, \mathbb{P})$-generic filter containing r, and let $\mathrm{mos}\colon M \to N$ denote the Mostowski collapse. We can now deduce that $\mathrm{mos}[g] \times \mathrm{mos}[G \cap M]$ is a $(N, \mathrm{mos}(\mathbb{P}) \times \mathrm{mos}(\mathbb{P}))$-generic filter with $N[\mathrm{mos}[g] \times \mathrm{mos}[G \cap M]] \models$ '$\dot{\tau}^g\, E\, \dot{\tau}^G$'. Working in $V^{\mathbb{P}}$ we see that $\forall x \in \omega^\omega \cap N[\mathrm{mos}[g] \times \mathrm{mos}[G \cap M]] \colon x^\sharp \in \omega^\omega \cap N[\mathrm{mos}[g] \times \mathrm{mos}[G \cap M]]$ by Lemma 6.2.3, hence we have $\mathbf{\Sigma}^1_2$-absoluteness between $N[\mathrm{mos}[g] \times \mathrm{mos}[G \cap M]]$ and $V^{\mathbb{P}}$. By $\mathbf{\Sigma}^1_3$-upward absoluteness we get $V^{\mathbb{P}} \models$ '$\dot{\tau}^g\, E\, \dot{\tau}^G$' with $\dot{\tau}^g \in V$.
Since $\mathbb{P}$ is captured by forcing notions of size $< \omega_1^V$ by Theorem 6.3.2, we have $\mathbf{\Sigma}^1_3$-absoluteness between V and $V^{\mathbb{P}}$ by Lemma 6.5.2, hence E and, therefore, F_E are absolute. Together with the first part, this finishes the proof. □

6.6 Regularity properties

In this section we will not assume the existence of large cardinals. We start with the following definition:

Definition 6.6.1. Recall that a set $X \subseteq 2^\omega$ has the Baire property iff there exists an open set $O \subseteq 2^\omega$ such that $X \triangle O = X \setminus O \cup O \setminus X$ is meager. Similarly, recall that X is Lebesgue measurable iff there exists a G_δ set $B \subseteq 2^\omega$ such that $X \triangle B$ is null.
For a collection Γ of subsets of 2^ω we write $\mathrm{BP}(\Gamma)$ to denote that every set in Γ has the Baire property, and $\mathrm{LM}(\Gamma)$ to denote that every set in Γ is Lebesgue measurable.

We will be particularly interested in the cases where Γ is the collection of $\mathbf{\Delta}^1_2$ or $\mathbf{\Sigma}^1_2$ subsets of 2^ω. The following characterizations are well known (see Chapter 9.2 and Chapter 9.3 in [BJ95]).

Theorem 6.6.2. $\mathrm{BP}(\mathbf{\Delta}^1_2)$ holds iff for every $x \in \omega^\omega$ there exists a Cohen real over $L[x]$, and $\mathrm{BP}(\mathbf{\Sigma}^1_2)$ holds iff for every $x \in \omega^\omega$ there is a comeager set of Cohen reals over $L[x]$. Analogously, $\mathrm{LM}(\mathbf{\Delta}^1_2)$ holds iff for every $x \in \omega^\omega$ there exists a random real over $L[x]$, and $\mathrm{LM}(\mathbf{\Sigma}^1_2)$ holds iff for every $x \in \omega^\omega$ there is a measure one set of random reals over $L[x]$.

The next theorem shows how $\mathrm{BP}(\boldsymbol{\Delta}^1_2)$ can be preserved:

Theorem 6.6.3. Let $\mathbb{P}$ be uniformly captured by Cohen forcing and assume that $V \vDash \mathrm{BP}(\boldsymbol{\Delta}^1_2)$. Then also $V^{\mathbb{P}} \vDash \mathrm{BP}(\boldsymbol{\Delta}^1_2)$.

Proof. By the above theorem, $\mathrm{BP}(\boldsymbol{\Delta}^1_2)$ holds iff there exists a Cohen real over $L[x]$ for every real $x \in \omega^\omega$. Moreover, note that if c is a Cohen real over $L[x, y]$, then it is also a Cohen real over $L[x]$. Now assume that $p \in \mathbb{P}$ and $\dot{\tau}$ is a $\mathbb{P}$-name for a real. By uniform capturing, there exist $z \in \omega^\omega$ and a $\mathbb{P}$-name $\dot{c}$ such that for every $y \in \omega^\omega$, there is a $q \leq_{\mathbb{P}} p$ with

$$q \Vdash_{\mathbb{P}} \dot{c} \text{ is a Cohen real over } L[z, y] \wedge \dot{\tau} \in L[z][\dot{c}].$$

Let c_0 be a Cohen real over $L[z]$. If we set $y := c_0$, then we find $q \leq_{\mathbb{P}} p$ such that

$$q \Vdash_{\mathbb{P}} \dot{c} \text{ is a Cohen real over } L[z, c_0] \wedge \dot{\tau} \in L[z][\dot{c}].$$

By mutual genericity, we have

$$q \Vdash_{\mathbb{P}} c_0 \text{ is a Cohen real over } L[z][\dot{c}] \supseteq L[\dot{\tau}].$$

This finishes the proof. □

Similarly, for $\mathrm{LM}(\boldsymbol{\Delta}^1_2)$:

Theorem 6.6.4. Let $\mathbb{P}$ be uniformly captured by random forcing and assume that $V \vDash \mathrm{LM}(\boldsymbol{\Delta}^1_2)$. Then also $V^{\mathbb{P}} \vDash \mathrm{LM}(\boldsymbol{\Delta}^1_2)$.

The next theorem shows how $\mathrm{BP}(\boldsymbol{\Sigma}^1_2)$ can be preserved:

Theorem 6.6.5. Let $\mathbb{P}$ be uniformly captured by Cohen forcing and assume that $V \vDash \mathrm{BP}(\boldsymbol{\Sigma}^1_2)$. Then also $V^{\mathbb{P}} \vDash \mathrm{BP}(\boldsymbol{\Sigma}^1_2)$.

Proof. By Theorem 6.6.2, $\mathrm{BP}(\boldsymbol{\Sigma}^1_2)$ holds iff there is a comeager set of Cohen reals over $L[x]$ for every real $x \in \omega^\omega$. Equivalently, $\bigcup \mathcal{B}$ is meager, where $\mathcal{B}$ is the collection of all Borel meager subsets of 2^ω coded in $L[x]$. Let $p \in \mathbb{P}$ and $\dot{\tau}$ be a $\mathbb{P}$-name for a real. Again, by uniform capturing there exist $z \in \omega^\omega$ and a $\mathbb{P}$-name $\dot{c}$ with the required properties. Consider the collection $\tilde{\mathcal{B}}$ of all Borel meager subsets of $2^\omega \times 2^\omega$ coded in $L[z]$. By our assumption, there exists a Borel meager set $B \subseteq 2^\omega \times 2^\omega$ coded in V such that $\bigcup \tilde{\mathcal{B}} \subseteq B$. Let B be coded by $y \in \omega^\omega$. Then there is $q \leq_{\mathbb{P}} p$ such that

$$q \Vdash_{\mathbb{P}} \dot{c} \text{ is a Cohen real over } L[z, y] \wedge \dot{\tau} \in L[z][\dot{c}].$$

Let G be a $(V, \mathbb{P})$-generic filter with $q \in G$. Working in $V[G]$, let $X := \{x \in 2^\omega : (\dot{c}^G, x) \in B\}$. We claim that X is meager and contains every Borel meager set coded in $L[\dot{\tau}^G]$. To see that X is meager, recall that by the Kuratowski-Ulam Theorem (see Chapter 15 in [Oxt80]) there exists a comeager set $Y \subseteq 2^\omega$ coded in $L[z, y]$, such that for every $u \in Y$ the set $\{x \in 2^\omega : (u, x) \in B\}$ is meager. Since $\dot{c}^G$ is a Cohen real over $L[z, y]$, we have $\dot{c}^G \in Y$ and hence X is indeed meager. Now assume that Z is a Borel meager set coded in $L[z][\dot{c}^G] \supseteq L[\dot{\tau}^G]$. Then there is a Borel meager set $B' \subseteq 2^\omega \times 2^\omega$ coded in $L[z]$ such that $Z = \{x \in 2^\omega : (\dot{c}^G, x) \in B'\}$. Since $B' \subseteq B$, we have $Z \subseteq X$. □

Similarly, for $\mathrm{LM}(\mathbf{\Delta}^1_2)$:

Theorem 6.6.6. Let $\mathbb{P}$ be uniformly captured by random forcing and assume that $V \vDash \mathrm{LM}(\mathbf{\Sigma}^1_2)$. Then also $V^{\mathbb{P}} \vDash \mathrm{BP}(\mathbf{\Sigma}^1_2)$.

Next, we give some examples of forcing notions that are captured also without the existence of large cardinals:

Example 6.6.7. Assuming $\mathrm{BP}(\mathbf{\Delta}^1_2)$, Sacks forcing $\mathbb{S}$ is uniformly captured by Cohen forcing.

Proof. Assume that $p \in \mathbb{S}$ and $\dot{\tau}$ is an $\mathbb{S}$-name for a real. Using continuous reading of names, there exist $p' \leq_{\mathbb{P}} p$ and $f\colon [p'] \to 2^\omega$ continuous such that $p' \Vdash_{\mathbb{S}} \dot{\tau} = f(\dot{x}_{\text{gen}})$, and let p' and f be coded by some real $z \in \omega^\omega$. Furthermore, let $\eta\colon [p'] \to 2^\omega$ be the canonical homeomorphism, let $\dot{c}$ be an $\mathbb{S}$-name for $\eta(\dot{x}_{\text{gen}})$ and let $y \in \omega^\omega$ be arbitrary. Consider the forcing $\mathbb{A}$ consisting of finite subtrees of p' ordered by end-extension. Then $\mathbb{A} \in L[z,y]$ is a countable forcing notion. Since there is a Cohen real over $L[z,y]$, there also exists an $(L[z,y]), \mathbb{A})$-generic filter G. It is easy to see that $q := \bigcup G \subseteq p$ is a perfect tree such that for every branch $x \in [q]$ we have that $\eta(x)$ is a Cohen real over $L[z,y]$. Since this statement is absolute, we have

$$q \Vdash_{\mathbb{S}} \dot{c} \text{ is a Cohen real over } L[z,y] \wedge \dot{\tau} \in L[z][\dot{c}].$$

This finishes the proof. □

Example 6.6.8. Assuming $\mathrm{BP}(\mathbf{\Delta}^1_2)$, any countable support product or iteration of Sacks forcing is uniformly captured by Cohen forcing.

Proof. Let us consider the product first: Let $\mathbb{P}$ be a countable support product of Sacks forcing, $\bar{p} \in \mathbb{P}$ and $\dot{\tau}$ a $\mathbb{P}$-name for a real. Using continuous reading of names, there exist $\bar{p}' \leq_{\mathbb{P}} \bar{p}$ and $f\colon \prod_{i \in \mathrm{supp}(\bar{p}')}[\bar{p}'(i)] \to \omega^\omega$ continuous such that $\bar{p}' \Vdash_{\mathbb{P}} \dot{\tau} = f(\dot{x}_{\text{gen}} \restriction \mathrm{supp}(\bar{p}'))$. W.l.o.g. we can assume that $\mathrm{supp}(\bar{p}')$ is a countable ordinal, and let $z \in \omega^\omega$ code $\bar{p}'$ and f. Furthermore, let $\eta\colon \prod_{i \in \mathrm{supp}(\bar{p}')}[\bar{p}'(i)] \to 2^\omega$ be a canonical homeomorphism, let $\dot{c}$ be a $\mathbb{P}$-name for $\eta(\dot{x}_{\text{gen}} \restriction \mathrm{supp}(\bar{p}'))$ and let $y \in \omega^\omega$ be arbitrary. For every $i \in \mathrm{supp}(\bar{p}')$, let $\mathbb{A}_i$ be the forcing notion consisting of finite subtrees of $\bar{p}'(i)$ ordered by end-extension and we define $\mathbb{A}$ to be the finite support product of $(\mathbb{A}_i)_{i \in \mathrm{supp}(\bar{p}')}$. Hence, $\mathbb{A}$ is a countable forcing notion in $L[z,y]$. Since we assume $\mathrm{BP}(\mathbf{\Delta}^1_2)$, there exists an $(L[z,y], \mathbb{A})$-generic filter adding a subtree $\bar{q}(i)$ of $\bar{p}'(i)$ for every $i \in \mathrm{supp}(\bar{p}')$. It is easy to see that for any $\bar{x} \in \prod_{i \in \mathrm{supp}(\bar{p}')} \bar{q}(i)$, we have that $\eta(\bar{x})$ is a Cohen real over $L[z,y]$. Since this statement is absolute, we have

$$\bar{q} \Vdash_{\mathbb{P}} \dot{c} \text{ is a Cohen real over} L[z,y] \wedge \dot{\tau} \in L[z][\dot{c}].$$

This finishes the proof for the product.

The proof for the iteration is essentially the same: Let $\mathbb{P}'$ denote a countable support iteration of Sacks forcing, $\bar{p} \in \mathbb{P}'$ and $\dot{\tau}$ a $\mathbb{P}'$-name for a real. It follows that there exist $\bar{p}' \leq_{\mathbb{P}'} \bar{p}$, a continuous function $f\colon (2^\omega)^{\mathrm{supp}(\bar{p}')} \to \omega^\omega$ and $\varphi\colon (2^\omega)^{\mathrm{supp}(\bar{p}')} \to (2^\omega)^{\mathrm{supp}(\bar{p}')}$ with the following properties (see [FS18a]):

1. $\bar{p}' \Vdash_{\mathbb{P}'} \dot{\tau} = f(\varphi(\dot{x}_{\text{gen}} \restriction \text{supp}(\bar{p}')))$.

2. For any condition $\bar{r}$ in the countable support product of $\mathbb{S}$ along $\text{supp}(\bar{p}')$, there exists $\bar{q} \leq_{\mathbb{P}'} \bar{p}'$ such that $\bar{q} \Vdash_{\mathbb{P}'} \varphi(\dot{x}_{\text{gen}} \restriction \text{supp}(\bar{p}')) \in \prod_{i \in \text{supp}(\bar{p}')}[\bar{r}(i)]$.

Again, w.l.o.g. we can assume that $\text{supp}(\bar{p}')$ is a countable ordinal, and let $z \in \omega^\omega$ code $\bar{p}'$, f and φ. Let $\eta\colon (2^\omega)^{\text{supp}(\bar{p}')} \to 2^\omega$ be a canonical homeomorphism, let $\dot{c}$ be a $\mathbb{P}'$-name for $\eta(\varphi(\dot{x}_{\text{gen}} \restriction \text{supp}(\bar{p}')))$, and let $y \in \omega^\omega$ be arbitrary. Similar to above, we use a Cohen real over $L[z,y]$ to find $\bar{r}$ in the countable support product of $\mathbb{S}$ along $\text{supp}(\bar{p}')$ such that for any $\bar{x} \in \prod_{i \in \text{supp}(\bar{p}')}[r(i)]$ we have that $\eta(\bar{x})$ is a Cohen real over $L[z,y]$. Since this statement is absolute, we can use (2.) to find $\bar{q} \leq_{\mathbb{P}'} \bar{p}'$ such that

$$\bar{q} \Vdash_{\mathbb{P}'} \dot{c} \text{ is a Cohen real over } L[z,y] \wedge \dot{\tau} \in L[z][\dot{c}].$$

This finishes the proof for the iteration $\square$

Example 6.6.9. Assuming $\text{BP}(\mathbf{\Delta}^1_2)$, Silver forcing $\mathbb{SI}$ is uniformly captured by Cohen forcing.

Proof. Let $p \in \mathbb{SI}$ and $\dot{\tau}$ be an $\mathbb{SI}$-name for a real. Using continuous reading of names, we find $p' \leq_{\mathbb{SI}} p$ and $f\colon 2^\omega \to 2^\omega$ continuous such that $p' \Vdash_{\mathbb{SI}} \dot{\tau} = f(\dot{x}_{\text{gen}})$, and let p' and f be coded by a real $z \in \omega^\omega$. Define $\eta : 2^\omega \to 2^\omega$ such that $\eta(x)(n) = i$ iff $x(m) = i$ where m is the n'th element of $\omega \backslash \text{dom}(p')$. Let $\dot{c}$ be a $\mathbb{SI}$-name for $\eta(\dot{x}_{\text{gen}})$ and let $y \in \omega^\omega$ be arbitrary. Consider the forcing $\mathbb{A} := \{s \in \text{Fin}(\omega, 2)\colon \exists n < \omega\ \text{dom}(s) \subseteq n \wedge p' \restriction n \subseteq s\}$ ordered by end-extension and note that $\mathbb{A} \in L[z,y]$. Since there is a Cohen real over $L[z,y]$, there also exists an $(L[z,y]), \mathbb{A})$-generic filter G. It is easy to see that $q := \bigcup G \in \mathbb{SI}$ with $q \leq_{\mathbb{SI}} p'$, and that $\eta(x)$ is a Cohen real over $L[z,y]$ for every $x \supseteq q$. Since this statement is absolute, we have

$$q \Vdash_{\mathbb{SI}} \dot{c} \text{ is a Cohen real over } L[z,y] \wedge \dot{\tau} \in L[z][\dot{c}].$$

This finishes the proof. $\square$

Example 6.6.10. Assuming $\text{BP}(\mathbf{\Sigma}^1_2)$, Miller forcing $\mathbb{MI}$ is uniformly captured by Cohen forcing.

Proof. Let $p \in \mathbb{MI}$ and $\dot{\tau}$ be an $\mathbb{MI}$-name for a real. Using continuous reading of names, there exist $p' \leq_{\mathbb{MI}} p$ and $f\colon [p'] \to 2^\omega$ continuous such that $p' \Vdash_{\mathbb{MI}} \dot{\tau} = f(\dot{x}_{\text{gen}})$, and let p' and f be coded by some real $z \in \omega^\omega$. Let $\eta\colon [p'] \to \omega^\omega$ be the canonical homeomorphism, let $\dot{c}$ be an $\mathbb{MI}$-name for $\eta(\dot{x}_{\text{gen}})$ and let $y \in 2^\omega$ be arbitrary. Since $\text{BP}(\mathbf{\Sigma}^1_2)$ holds, the set of Cohen reals $C \subseteq \omega^\omega$ over $L[z,y]$ is comeager. In particular, also $\eta^{-1}(C)$ is comeager in $[p']$ and therefore, contains the branches of a superperfect tree $q \leq_{\mathbb{MI}} p'$. It is easy to see that for every $x \in [q]$ we have that $\eta(x)$ is a Cohen real over $L[z,y]$. Since this statement is absolute, we have

$$q \Vdash_{\mathbb{MI}} \dot{c} \text{ is a Cohen real over } L[z,y] \wedge \dot{\tau} \in L[z][\dot{c}].$$

This finishes the proof. $\square$

Example 6.6.11. Assuming $\mathrm{LM}(\mathbf{\Sigma}^1_2)$, Sacks forcing $\mathbb{S}$ is uniformly captured by random forcing.

Proof. Assume that $p \in \mathbb{S}$ and $\dot{\tau}$ is an $\mathbb{S}$-name for a real. Using continuous reading of names, there exist $p' \leq_{\mathbb{P}} p$ and $f\colon [p'] \to 2^\omega$ continuous such that $p' \Vdash_{\mathbb{S}} \dot{\tau} = f(\dot{x}_{\mathrm{gen}})$, and let p' and f be coded by some real $z \in \omega^\omega$. Furthermore, let $\eta\colon [p'] \to 2^\omega$ be the canonical homeomorphism, let $\dot{r}$ be an $\mathbb{S}$-name for $\eta(\dot{x}_{\mathrm{gen}})$ and let $y \in \omega^\omega$ be arbitrary. Since $\mathrm{LM}(\mathbf{\Sigma}^1_2)$ holds, the set of random reals $R \subseteq 2^\omega$ over $L[z, y]$ has measure one. In particular, R contains a perfect set, hence $\eta^{-1}(R)$ contains the branches of a perfect tree $q \leq_{\mathbb{S}} p'$. It is easy to see that for every branch of $[q]$ we have that $\eta(x)$ is a random real over $L[z, y]$. Since this statement is absolute, we have

$$q \Vdash_{\mathbb{MI}} \dot{r} \text{ is a random real over } L[z, y] \wedge \dot{\tau} \in L[z][\dot{r}].$$

This finishes the proof. □

Since Miller forcing is captured by Cohen forcing under $\mathrm{BP}(\mathbf{\Sigma}^1_2)$, Theorem 6.6.5 implies that Miller forcing preserves $\mathrm{BP}(\mathbf{\Sigma}^1_2)$. This does not hold for $\mathrm{BP}(\mathbf{\Delta}^1_2)$:

Theorem 6.6.12. Let V be the forcing extension obtained by adding ω_1-many Cohen reals over L. Then $\mathrm{BP}(\mathbf{\Delta}^1_2)$ holds in V but not in $V^{\mathbb{MI}}$.

Proof. Working in V assume towards a contradiction that for some $p \in \mathbb{MI}$ and $\mathbb{MI}$-name $\dot{c}$ we have

$$p \Vdash_{\mathbb{MI}} \dot{c} \in \mathbb{Z}^\omega \text{ is a Cohen real over } L[\dot{x}_{\mathrm{gen}}].$$

Note that for technical reasons we will consider a Cohen real to be an element of $\mathbb{Z}^\omega$ which is $\mathbb{Z}^{<\omega}$-generic, where $\mathbb{Z}$ is the set of integers. Using continuous reading of names, we may assume w.l.o.g. that there exists $f\colon [p] \to \mathbb{Z}^\omega$ continuous such that $p \Vdash_{\mathbb{MI}} \dot{c} = f(\dot{x}_{\mathrm{gen}})$. Let p and f be coded by some real $z \in \omega^\omega$.

Claim 6.6.13. There exists $q \leq_{\mathbb{MI}} p$ such that for every $x \in [q]$ we have that $f(x)$ is a Cohen real over $L[x]$.

Proof. For every $\alpha < \omega_1$, the set B_α of $(x, u) \in \omega^\omega \times \mathbb{Z}^\omega$ such that u is in a closed nowhere dense subset of $\mathbb{Z}^\omega$ coded in $L_\alpha[x]$ is a $\Delta^1_1(y)$ set, where $y \in \omega^\omega$ is a real coding α.[17] In particular, B_α is coded in L for every $\alpha < \omega_1$, since $\omega_1^L = \omega_1$. Now note that for every $\alpha < \omega_1$ the set $\{x \in [p] : (x, f(x)) \in B_\alpha\}$ is bounded and coded in $L[z]$. Otherwise, it would contain the branches of a superperfect tree $r \leq_{\mathbb{MI}} p$ (see Chapter 21.F in [Kec95]). But then

$$r \Vdash_{\mathbb{MI}} f(\dot{x}_{\mathrm{gen}}) \text{ is not a Cohen real over } L[\dot{x}_{\mathrm{gen}}],$$

since the statement '$\,$for every branch $x \in [r]$ we have $((x, f(x)) \in B_\alpha)$' is a $\mathbf{\Pi}^1_1$-property and therefore absolute. Let $\eta\colon \omega^\omega \to [p]$ be the canonical homeomorphism and we see that $\eta^{-1}(\{x \in [p] : (x, f(x)) \in B_\alpha\})$ is bounded as well. The statement '$\,\eta^{-1}(\{x \in$

[17] To see this, note that once we fix a real y coding a well-ordering of ω of type α, we easily get a Borel function mapping x to a code for $L_\alpha[x]$.

$[p] : (x, f(x)) \in B_\alpha\})$ is bounded' is a $\Sigma^1_2(z)$-property and therefore absolute between $L[z]$ and V. Since there is a Cohen real over $L[z]$, there exists a real $c \in \omega^\omega$ which is unbounded over $L[z]$. In particular, c is unbounded over $\eta^{-1}(\{x \in [p] : (x, f(x)) \in B_\alpha\})$ for every $\alpha < \omega_1$. If we pick $q \leq_{\mathbb{MI}} p$ such that for every $x \in [q]$ we have $c \leq^* \eta^{-1}(x)$, this finishes the proof of the claim □

The set $\{f(x) : x \in [q]\}$ is analytic. If it is not contained in a σ-compact subset of $\mathbb{Z}^\omega$, it contains the branches of a superperfect tree T, hence $[T]$ is a superperfect set of Cohen reals over L.
On the other hand, if $\{f(x) : x \in [q]\}$ is contained in a σ-compact subset of $\mathbb{Z}^\omega$, then there exist $a, b \in \mathbb{Z}^\omega$ such that for every $x \in [q]$ we have $a \leq^* f(x) \leq^* b$. Now consider the analytic set $A := \{x + f(x) : x \in [q]\}$. It follows that A is a set of Cohen reals over L, since for every $x \in [q]$ the real $x + f(x)$ is the image of $f(x)$ under a homeomorphism of $\mathbb{Z}^\omega$ in $L[x]$, and hence also Cohen over $L[x]$. We show that A is not contained in a σ-compact set. To this end, let $d \in \omega^\omega$ be arbitrary. Since $[q]$ is unbounded, there exists $x \in [q]$ such that $x \not\leq^* d - a$. But then for infinitely many $n < \omega$ we have $x(n) + f(x)(n) > d(n) - a(n) + f(x)(n) \geq d(n) - a(n) + a(n) = d(n)$, and hence $x + f(x) \not\leq^* d$. In particular, A contains the branches of a superperfect tree T, hence $[T[$ is a superperfect set of Cohen reals over L.
We have shown that in any case there exists a superperfect set of Cohen reals over L in V. This, however, is impossible in the Cohen model by a result of Spinas (see [Spi95]). This finishes the proof Theorem 6.6.12. □

Pawlikowski showed in [Paw86] that if c is a Cohen real over V and r is a random real over $V[c]$, then there exists a Cohen real over $V[r]$ in $V[c][r]$. In particular, this shows that random forcing preserves $\mathrm{BP}(\mathbf{\Delta}^1_2)$. Using similar arguments, we generalize his result:

Theorem 6.6.14. Let $\mathbb{P}$ be captured by ω^ω-bounding forcing notions and assume that $V \vDash \mathrm{BP}(\mathbf{\Delta}^1_2)$. Then also $V^{\mathbb{P}} \vDash \mathrm{BP}(\mathbf{\Delta}^1_2)$.

Proof. Let $C(2^\omega)$ denote the space of continuous functions $f : 2^\omega \to 2^\omega$ equipped with the topology of uniform convergence, and note that $C(2^\omega)$ is a Polish space (see Chapter 4.E in [Kec95]). Its topology is generated by open sets of the form

$$\{g \in C(2^\omega) : \forall x \in 2^\omega (g(x) \restriction k = f(x) \restriction k)\}$$

for $f \in C(2^\omega)$ and $k < \omega$. We will need the following lemma (see Chapter 3.2 in [BJ95]):

Lemma 6.6.15. Let $F \subseteq 2^\omega \times 2^\omega$ be closed such that every vertical section of F is nowhere dense. Then the set $\tilde{F} := \{f \in C(2^\omega) : \exists x \in 2^\omega\ (x, f(x)) \in F\}$ is closed nowhere dense in $C(2^\omega)$.

Proof. Let $f \notin \tilde{F}$ be arbitrary. Hence, for every $x \in 2^\omega$ there exist $s_x, t_x \in 2^{<\omega}$ such that $s_x \lhd x$, $[s_x] \times [t_x] \cap F = \emptyset$ and $f''[s_x] \subseteq [t_x]$. By compactness there are finitely many $x_0, \dots, x_n$ such that $\bigcup_{i \leq n} [s_{x_i}] = 2^\omega$. Let $k^* := \max\{|t_{x_i}| : i \leq n\}$. Then we see that

$\{h \in C(2^\omega) : \forall x \in 2^\omega (h(x) \restriction k^* = f(x) \restriction k^*)\}$ is open, contains f and is contained in the complement of $\tilde{F}$. Since f was arbitrary, this shows that $C(2^\omega) \setminus \tilde{F}$ is open.

To show that $C(2^\omega) \setminus \tilde{F}$ is dense let $f \in C(2^\omega)$ and $k < \omega$ be arbitrary. For every $x \in 2^\omega$ we find $s_x \triangleleft x$ and t_x such that $f''[s_x] \subseteq [f(x) \restriction k]$, $f(x) \restriction k \triangleleft t_x$ and $[s_x] \times [t_x] \cap F = \emptyset$. By compactness there are finitely many $x_0, \ldots, x_n$ such that $\bigcup_{i \leq n} [s_{x_i}] = 2^\omega$. Now consider the function g mapping $x \in 2^\omega$ to $t_{x_i}{}^\frown 0^\frown 0^\frown \ldots$, where $i \leq n$ is least such that $s_{x_i} \triangleleft x$. Then g is continuous, $g \notin \tilde{F}$ and $g(x) \restriction k = f(x) \restriction k$ for every $x \in 2^\omega$.
Altogether we have shown that $C(2^\omega) \setminus \tilde{F}$ is open dense, hence $\tilde{F}$ is closed nowhere dense. This finishes the proof of the lemma. $\square$

Let $\langle s_n : n \in \omega \rangle$ be some canonical enumeration of $2^{<\omega}$, and we say that $x \in 2^\omega$ codes a closed nowhere dense subset of 2^ω iff $\{s_n : x(n) = 1\}$ is a nowhere dense subtree of $2^{<\omega}$. For such x let T_x denote the corresponding nowhere dense tree.

Now let $p \in \mathbb{P}$ and $\dot{\tau}$ a $\mathbb{P}$-name for an element of 2^ω be arbitrary. By the assumptions of the theorem, there exist $z \in \omega^\omega$, a $\mathbb{P}$-name $\dot{H}$, a forcing notion $\mathbb{Q} \in L[z]$ such that $L[z] \models$ ‘ $\mathbb{Q}$ is ω^ω-bounding ’ and $q \leq_{\mathbb{P}} p$ such that

$$q \Vdash_{\mathbb{P}} \dot{H} \text{ is } (L[z], \mathbb{Q})\text{-generic} \wedge \dot{\tau} \in L[z][\dot{H}].$$

Let $\dot{\sigma} \in L[z]$ be a $\mathbb{Q}$-name and $r \leq_{\mathbb{P}} q$ such that $r \Vdash_{\mathbb{P}} \dot{\sigma}^{\dot{H}} = \dot{\tau}$.

Since $\mathrm{BP}(\boldsymbol{\Delta}^1_2)$ holds, there exists a $g \in C(2^\omega)$ not contained in any closed nowhere dense subset of $C(2^\omega)$ coded in $L[z]$.

Claim 6.6.16. $r \Vdash_{\mathbb{P}} g(\dot{\tau})$ is Cohen over $L[\dot{\tau}]$.

Proof. Let G be a $(V, \mathbb{P})$-generic filter containing r and set $H := \dot{H}^G$. For every real $x \in L[z, \dot{\sigma}^H] \supseteq L[\dot{\tau}^G]$ there exists a Borel function $f : 2^\omega \to 2^\omega$ coded in $L[z]$ such that $f(\dot{\sigma}^H) = x$, since $\omega_1^{L[z,\dot{\sigma}^H]} = \omega_1^{L[z]}$ (otherwise $\mathbb{Q}$ would not be ω^ω-bounding in $L[z]$). In particular, if $x \in L[z, \dot{\sigma}^H]$ codes a closed nowhere dense subset of 2^ω, then there exists a Borel function $f \in L[z]$ such that $f(y)$ codes a closed nowhere dense subset of 2^ω for every real $y \in 2^\omega$, and $f(\dot{\sigma}^H) = x$. Now f is the projection onto the first two coordinates of a set $[T]$ for some tree $T \subseteq 2^{<\omega} \times 2^{<\omega} \times \omega^{<\omega}$. Therefore, there exists a $\mathbb{Q}$-name $\dot{w}$ in $L[z]$ for an element of ω^ω such that $(\sigma^H, f(\sigma^H), \dot{w}^H) \in [T]$. Since $L[z] \models$ ‘ $\mathbb{Q}$ is ω^ω-bounding ’, we may find a finitely branching subtree S of T in $L[z]$ such that $(\dot{\sigma}^H, f(\dot{\sigma}^H), \dot{w}^H) \in [S]$. The projection of $[S]$ onto the first two coordinates is then a compact subset of $2^\omega \times 2^\omega$, hence a partial, continuous function on some closed set $X \subseteq 2^\omega$ containing $\dot{\sigma}^H$. Consider the set $F := \{(x, y) : x \in X \wedge y \in T_{f(x)}\}$ and note that F is closed and each section of F is nowhere dense. By the above lemma, $\tilde{F}$ is a closed nowhere dense subset of $C(2^\omega)$, and it is obviously coded in $L[z]$. Hence, $g \notin \tilde{F}$ and it follows that for every $x \in 2^\omega$ we have $g(x) \notin T_{f(x)}$. In particular, we note that $g(\dot{\sigma}^H) \notin T_{f(\dot{\sigma}^H)}$. Hence, we have shown that $g(\dot{\tau}^G) = g(\dot{\sigma}^H)$ is not contained in any closed nowhere dense subset of 2^ω coded in $L[\tau^G]$, i.e. $g(\dot{\tau}^G)$ is a Cohen real over $L[\dot{\tau}^G]$. This finishes the proof of the claim. $\square$

By Theorem 6.6.2 this finishes the proof of Theorem 6.6.14. $\square$

We immediately get the following corollary:

Corollary 6.6.17. Let $\mathbb{P}$ be captured by ω^ω-bounding forcing notions and assume that $V \models \mathrm{BP}(\mathbf{\Sigma}^1_2)$. Then also $V^{\mathbb{P}} \models \mathrm{BP}(\mathbf{\Sigma}^1_2)$.

Proof. By Theorem 6.6.14, we have $V^{\mathbb{P}} \models \mathrm{BP}(\mathbf{\Delta}^1_2)$. Since $\mathrm{BP}(\mathbf{\Delta}^1_2)$ implies that $\mathrm{BP}(\mathbf{\Sigma}^1_2)$ is equivalents to the statement ' $L[x] \cap \omega^\omega$ is bounded for every real $x \in \omega^\omega$ ' (see Chapter 9.3 in [BJ95]), and by capturing we have that every $L[x]$ is contained in an ω^ω-bounding forcing extension of a model $L[z]$ for some $z \in \omega^\omega \cap V$, it follows that $V^{\mathbb{P}} \models \mathrm{BP}(\mathbf{\Sigma}^1_2)$. □

Example 6.6.18. Random forcing captures itself. In particular, it preserves $\mathrm{BP}(\mathbf{\Delta}^1_2)$ and $\mathrm{BP}(\mathbf{\Sigma}^1_2)$.

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