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Asymptotic symmetries of ultracold Kerr/dS-black holes

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Kurzfassung

In dieser Arbeit wird zuerst ein Überblick über die wichtigsten Grundlagen zu erhaltenen Ladungen und Symmetrien, der Thermodynamik Schwarzer Löcher, Konformer Feldtheorie sowie Konformer Feldtheorie für verzerrte Raumzeiten als auch zur Kerr/CFT-Korrespondenz gegeben.
Anschließend betrachten wir Erhaltungsgrößen sowie die zentrale Ladung des ultrakalten Grenzfalles der Kerr/dS-Lösung. Nach einem Uplift des metrischen Tensors in Analogie zu [15] finden wir Randbedingungen mit Ähnlichkeiten zu jenen, welche für zwei-dimensionale Schwarze Löcher in [1] gefunden worden sind. Die zugehörige Algebra ist die gewarpede Wittalgebra mit endlichen, wohldefinierten Ladungen sowie einer verschwindenden Zentralladung.

Abstract

This thesis will first review fundamentals on conserved charges and symmetries, (black hole) thermodynamics, (warped) conformal field theory as well as the Kerr/CFT-correspondence. Afterwards, we study the conserved charges and central charge of the ultracold limit of the Kerr/dS-solution. After uplifting the metric in analogy to what was done in [15], we find boundary conditions similar to the two-dimensional ones studied in [1]. Their algebra turns out to be the warped Witt-algebra and we find associated finite conserved charges as well as a vanishing central charge.

Contents

Introduction

Motivation

The universe we live in contains black holes that are solutions to Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 0, \tag{0.0.1}$$

with a positive cosmological constant Λ, the metric $g_{\mu\nu}$, the curvature tensor $R_{\mu\nu}$ and its contraction R.[18] The solution closest to the black holes found in our reality is the Kerr/dS-solution,[4]

$$ds^2 = -\frac{\Delta_r}{\rho^2}(dt - \frac{a}{\Theta}\sin^2\theta d\phi)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta}{\rho^2}\sin^2\theta(adt - \frac{r^2+a^2}{\Theta}d\phi)^2 \tag{0.0.2}$$

with parameters

$$\Delta_r = (r^2 + a^2)(1 - \frac{r^2}{l^2}) - 2Mr, \quad \Delta_\theta = 1 + a^2\frac{\cos^2\theta}{l^2} \tag{0.0.3}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Theta = 1 + \frac{a^2}{l^2}. \tag{0.0.4}$$

This Kerr/dS spacetime contains a class of rotating black holes, such as the Kerr solution or the rotating Nariai solution (0.0.12), as well as some special cases, such as black holes whose horizons are in thermal equilibrium (lukewarm black holes) or the so-called cold black hole solutions that additionally have degenerate horizons, see [14] and chapter 7 for more details.
The limit of (7.1.1) that has vanishing temperature at the horizon r_C,

$$ds^2 = \tilde{\Gamma}(\theta)\Big(-d\tilde{t}^2 + d\tilde{r}^2 + \tilde{\alpha}(\theta)d\theta^2\Big) + \gamma(\theta)(d\phi + \tilde{k}\tilde{r}d\tilde{t})^2, \tag{0.0.5}$$

for which all three horizons (inner, outer and cosmological horizon) coincide, is therefore of great interest. We call it an "ultracold" black hole. In the line element (0.0.5), $\tilde{t}$ acts as the time coordinate, $\tilde{r}$ as the radial coordinate and θ and ϕ are angular coordinates. All other expressions are functions given by

$$\rho_c^2 = r_c^2 + a^2\cos\theta^2, \quad \tilde{\Gamma}(\theta) = \frac{\rho_c^2 r_c}{(a^2+r_c^2)}, \tag{0.0.6}$$

$$\tilde{\alpha}(\theta) = \frac{(r_c^2+a^2)}{r_c\Delta_\theta}, \quad \gamma(\theta) = \frac{\Delta_\theta(r_c^2+a^2)^2\sin\theta^2}{\rho_c^2\Theta^2}, \tag{0.0.7}$$

$$\tilde{k} = -\frac{2ar_c^2\Theta}{(a^2+r_c^2)^2}. \tag{0.0.8}$$

A summary of the metric's (0.0.5) derivation is given further down below starting out from equation (0.0.12).

This special black hole can be studied by e.g. utilizing symmetries. They can be used to naturally constrain our theory to a few (possibly unique) solutions. In the past, such a strategy has been successful a number of times already using asymptotic symmetries, which constrain the phase space and preserve the metric only in a select region, e.g. at infinity. Examples of such successes would e.g. be the AdS_3 or BTZ-black holes/solutions or also the famous Kerr/CFT-correspondence, see chapter 6 for details, which connects quantum theory and gravitational theory by finding that the entropy obtained from the both of them matches. The latter involved a quantity called the central charge. It is an additional term in the charge algebra up to which the charges' algebra matches the diffeomorphisms' algebra appearing after introducing a macroscopic length scale, which can be done by e.g. using certain boundary conditions.[9, 13] To be precise, given a certain diffeomorphism ζ that is allowed by a set of boundary conditions h conserving the given metric g, one can calculate a corresponding conserved charge [4]

$$Q_\zeta(h,g) = \int_S k_\zeta[h;g] \tag{0.0.9}$$

over a surface S. Here, a quantity preserved on-shell called the surface charge [4]

$$k_\zeta[h,g] = -\frac{1}{4}\varepsilon_{\alpha\beta\mu\nu}[\zeta^\nu D^\mu h - \zeta^\nu D_\sigma h^{\mu\sigma} + \zeta_\sigma D^\nu h^{\mu\sigma} + \frac{1}{2}hD^\nu\zeta\mu - h^{\nu\sigma}D_\sigma\zeta^\mu + \frac{1}{2}h^{\sigma\nu}(D^\mu\zeta_\sigma + D_\sigma\zeta^\mu)]dx^\alpha \wedge dx^\beta \tag{0.0.10}$$

is used. The charges' algebra can be calculated to be [9]

$$\{Q_{\zeta_m}, Q_{\zeta_n}\} = Q_{[\zeta_m,\zeta_n]} + \mathcal{C}_{\zeta_m,\zeta_n}[\bar{\Phi}], \tag{0.0.11}$$

with beforementioned central charge $\mathcal{C}_{\zeta_m,\zeta_n}[\bar{\Phi}]$. A more detailed explanation and derivation is given in chapter 1.3.2.

For the reasons mentioned above, we attempt to find answers to the following questions:

- What are the asymptotic symmetries preserving the ultracold solution (0.0.5)?

- Do finite and well-defined expressions for both conserved charges and central charge exist for a given set of boundary conditions?
- If yes, what are those expressions and what is the corresponding algebra?

Strategy and results

At first, we start out from the rotating Nariai-metric in static coordinates

$$ds^2 = \Gamma(\theta)\Big(-(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + \alpha(\theta)d\theta^2\Big) + \gamma(\theta)(d\phi + krdt)^2 \tag{0.0.12}$$

with

$$\rho_c^2 = r_c^2 + a^2\cos^2\theta, \quad \Gamma(\theta) = \frac{\rho_c^2 r_c}{b(a^2+r_c^2)}, \tag{0.0.13}$$

$$\alpha(\theta) = \frac{b(r_c^2+a^2)}{r_c\Delta_\theta}, \quad \gamma(\theta) = \frac{\Delta_\theta(r_c^2+a^2)^2\sin^2\theta}{\rho_c^2\Theta^2}, \tag{0.0.14}$$

$$k = -\frac{2ar_c^2\Theta}{b(a^2+r_c^2)^2}. \tag{0.0.15}$$

This is an extremal limit (meaning that at least some horizons fall together) of the Kerr/dS-solution for which outer and cosmological horizon coincide and we use it to rederive the ultracold solution (0.0.5). To do that, we first reparametrize radial and time coordinates [4]

$$r = \tilde{r}\sqrt{b}, \quad t = \tilde{t}\sqrt{b} \tag{0.0.16}$$

as well as some functions,

$$\tilde{k} = b\cdot k, \quad \tilde{\Gamma}(\theta) = b\cdot\Gamma(\theta), \quad \tilde{\alpha} = \frac{\alpha}{b} \tag{0.0.17}$$

appearing in the metric. Afterwards, the parameter $b \propto (r_c - r_-)$ is taken to 0, where r_c denotes the cosmological horizon and r_- the inner one.

We ensure that the equations of motions are still fulfilled by this metric for any value of the rotational parameter a and horizon r_c. Additionally, a special value for the ultracold solution's horizon,

$$r_{UC} = l\sqrt{\frac{-3+2\cdot\sqrt{3}}{3}}, \tag{0.0.18}$$

can be found by setting a to its extremal value $a^2 = \frac{r_c^2(1-3\frac{r_c^2}{l^2})}{1+\frac{r_c^2}{l^2}}$, which was found for the Kerr/dS-metric by employing the extremality conditions $\Delta_r\mid_{horizon} = 0$ and $\partial_r\Delta_r\mid_{horizon} = 0$[4].

We then attempt utilizing the generators,[4]

$$\zeta_\varepsilon = \varepsilon(\phi)\partial_\phi - r\varepsilon'(\phi)\partial_r, \quad \bar{\zeta} = \partial_\tau \tag{0.0.19}$$

in order to find expressions for the conserved charges, their algebra and central extension. After expanding $\varepsilon_n = -e^{-in\phi}$, one finds that their Lie-bracket generates a copy of the Witt algebra

$$i[\zeta_n, \zeta_m] = (n-m)\zeta_{n+m}. \tag{0.0.20}$$

The algebra of the corresponding conserved (quantized) charges L_m matches this algebra up to a central extension c:

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{0.0.21}$$

This strategy has worked for finding the Nariai-limit's as well as NHEK's (near horizon extremal Kerr) [19] conserved charges and central extension, which is why we use it as a first attempt. However, the resulting central charge for the ultracold black hole diverges as $\mathcal{O}(r^2)$. As the Nariai-metric's central charge

$$c = 3 \mid k \mid \int_0^\pi d\theta \sqrt{\Gamma(\theta)\alpha(\theta)\gamma(\theta)} = \frac{12ar_c^2}{b(a^2 + r_c^2)}. \tag{0.0.22}$$

is proportional to $1/b$, the divergence was expected. This is because we had arrived at the ultracold black hole by taking the limit $b \to 0$ for the Nariai metric(0.0.12).

As the ultracold solution (0.0.5) is a "fibered product of two-dimensional Minkowski space and the two-sphere"[4],we take inspiration from Godet and Marteau's paper on boundary conditions for $AdS_2/Mink^{(1,1)}$ [15] and change our strategy to uplifting the metric after a coordinate change to Eddington-Finkelstein coordinates.
For this, we replace the flat part in (0.0.5), $-d\tilde{t}^2 + d\tilde{r}^2$, by $(P(u)r + T(u))du^2 - 2dudr$,with two functions dependend on the retarded time u, $T(u)$ and $P(u)$. This is called the "uplift". The metric then reads

$$ds^2 = \tilde{\Gamma}(\theta)((P(u)r + T(u))du^2 - 2dudr + \tilde{\alpha(\theta)}d\theta^2) + \gamma(\theta)(d\phi + \tilde{k}rdu)^2. \tag{0.0.23}$$

From this, we solve $\mathcal{L}_\xi g_{\mu\nu} = \mathcal{O}(\delta g_{\mu\nu})$ to find the diffeomorphisms

$$\xi^u = \varepsilon(u)\partial_u, \quad \xi^r = -(r\varepsilon'(u) - \eta'(u))\partial_r + \mathcal{O}(\frac{1}{r}), \quad \xi^\phi = -\eta(u)\partial_\phi \tag{0.0.24}$$

conserving the uplifted metric (0.0.23) asymptotically under the conditions $\mathcal{L}_\xi g_{ur} = \mathcal{L}_\xi g_{rr} = 0$ and $\mathcal{L}_\xi g_{uu} = 2\tilde{\Gamma}(\theta)\delta_\xi P(u)r + \delta_\xi T(u)$, with functions $\varepsilon(u)$ and $\eta(u)$ periodic in the time coordinate u and the variation of the metric at the boundary $\delta g_{\mu\nu}$. These conditions have also already successfully been used for just the flat part $(P(u)r + T(u))d^2u - 2dudr$ in [3]. The so found diffeomorphisms are similar to the ones found for two-dimensional flat Rindler-type metrics described in [3]. The diffeomorphism algebra is the warped Witt-algebra

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [l_m, d_n] = -nd_{m+n}, \tag{0.0.25}$$

with generators l_m, d_m. The substructure is similar to the two-dimensional counterpart whose diffeomorphisms had admitted to the BMS$_2$-algebra [15]

$$[(\varepsilon_1, \eta_1), (\varepsilon_2, \eta_2)] = (\varepsilon_1\varepsilon_2' - \varepsilon_2\varepsilon_1', (\varepsilon_2\eta_2 - \varepsilon_2\eta_1)'). \tag{0.0.26}$$

Furthermore, we find conserved charges

$$Q_{UC} = -\int_0^{2\pi L} du\Big(\frac{(a^2+r_c^2)\eta(u)P(u)}{4\Theta\pi} + \frac{ar_c^2 T(u)\varepsilon(u)}{2\pi(a^2+r_c^2)}\Big). \tag{0.0.27}$$

After defining

$$L_m = Q_{UC}\,|_{\eta=0}, \qquad P_m = Q_{UC}\,|_{\varepsilon=0}, \tag{0.0.28}$$

we find that the charge algebra coincides with the generator algebra (0.0.25) and we therefore have a vanishing central charge.

Chapter 1

Symmetries and conserved charges

In this section, we will give a brief overview over symmetries, conserved charges and the most important aspects of Noether's theorem. It states that to each continuous symmetry of the action corresponds a conserved quantity and vice-versa. In gravitational theory, symmetries are local and therefore an adequate formulation of Noether's theorem will be needed that we will also introduce. We will then see that this leads to conserved charges that will be of utmost importance in describing a black hole's nature and inner workings as well as its thermodynamics and also, in later sections, to a correspondence between quantum mechanics and gravity theory.

1.1 Hamilton action principle and Noether theorem for classical field theory

The Hamilton action principle that the action is extremal, i.e. that for any well-defined action S that depends on general coordinates q_i with a Lagrangian L we find

$$\delta S[q_i] = \int dt \delta L[q_i, \dot{q}_i] = 0. \tag{1.1.1}$$

Applying partial integration to (1.1.1) yields the well known Euler-Lagrange equations in 0+1 dimensions

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{1.1.2}$$

We can generalize this to a field theory in D+1 dimensions by making the transition $t \to x^\mu, q_i \to \Phi_i(x^\mu), \dot{q}_i \to \partial_\nu \Phi_i(x^\mu), L \to \mathcal{L}(\Phi, \partial_\mu \Phi)$:

$$\frac{\partial \mathcal{L}}{\partial \Phi_i} - \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \Phi_i} = 0. \tag{1.1.3}$$

Here, the Lagrangian L has become a Lagrange density $\mathcal{L}$ and the generalized coordinates q_i have become continuous scalar fields Φ_i.

Now assuming a symmetry of this Lagragian, meaning that it is invariant under said symmetry transformation up to a partial derivative, $\delta \mathcal{L} = \partial_\mu K^\mu$, using the Lagragian's variation in combination with the Euler-Lagrange equations (1.1.3), yields the expression

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_i}\delta\Phi_i + \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\partial_\mu\delta\Phi_i = \partial_\mu(\frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\delta\Phi_i). \tag{1.1.4}$$

We can use this to define a conserved current, called Noether current

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\Phi_i}\delta\Phi_i - K_\mu, \tag{1.1.5}$$

by substracting (1.1.4) and $\delta\mathcal{L}$. This Noether current also obeys the continuity equation

$$\partial_\mu J^\mu = 0, \tag{1.1.6}$$

making it conserved indeed [17]. Additionally, we may use it to define a Noether charge, [22]

$$Q = \int d^{D-1}x_0 J^0(x), \tag{1.1.7}$$

which we can verify to be conserved as well by applying its derivative [9]

$$\partial_t Q = \int_\Sigma d\Sigma\partial_0 J^0 = -\int_{\partial\Sigma} J\cdot dS = 0. \tag{1.1.8}$$

Examples of conserved charges in e.g. Minkowski space would be [9]

- the energy of the system $P^0 = \int_\Sigma d\Sigma T^{00}$, resulting from homogenity and connected to time translations,
- the system's momentum $P^i = \int_\Sigma d\Sigma T^{i0}$, connected to translations in space and
- Lorenz transformations $M^{\mu\nu} = \int_{\partial\Sigma} d\Sigma(x^\mu T^{\nu 0} - x^\nu T^{\mu 0})$, resulting from the isotropy of Minkowski space and boost invariance.

1.2 Noether theorem for general relativity

In general relativity, usually the Einstein-Hilbert-action

$$S[g_{\mu\nu}] = -\frac{1}{2\kappa}\int d^D x\sqrt{-g}[R - 2\Lambda] \tag{1.2.1}$$

is used, where κ is inverse proportional to Newton's constant G, $g_{\mu\nu}$ is the metric, $R = R_{\mu\nu}g^{\mu\nu}$ the Ricci-scalar and Λ the cosmological constant. By applying the variational principle to (1.2.1), we find

$$\delta S = \int d^D x\sqrt{-g}[(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda)\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}] = 0 \tag{1.2.2}$$

and therefore the vacuum Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 0 \tag{1.2.3}$$

after neglecting the boundary term $\delta R_{\mu\nu}$.
The inhomogenous Einstein equations may be found from (1.2.3) by adding matter to our theory

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu}. \tag{1.2.4}$$

Conservation of this energy-momentum tensor $T_{\mu\nu}$ can be proven by taking the covariant derivative of both sides, leading to

$$\partial_\mu T^{\mu\nu} = 0 \tag{1.2.5}$$

due to metric compatibility $\partial^\mu g_{\mu\nu} = 0$ and conservation of the Einstein tensor $\partial_\mu G^{\mu\nu} = \partial_\mu (R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R) = 0$ [18].
This conserved tensor $T_{\mu\nu}$ is connected to the conserved Noether current (1.1.5) and thus to the Noether charge by the metric's $g_{\mu\nu}$ Killing vectors ξ^μ

$$J^\mu = T^\mu_\nu \xi^\nu. \tag{1.2.6}$$

However, for a theory not using Eucledian- or Minkowski-signature as it was used in the previous section, it is important to note that a term previously neglected (due to being equal to 1) will now appear in the expression for the charge and we therefore arrive at

$$Q(t) = \int_{\partial\Sigma} d^{D-1}x \sqrt{-g} J^0(x^\mu) = \int_{\partial\Sigma} d^{D-1}x \sqrt{-g} T^0_\mu \xi^\mu. \tag{1.2.7}$$

For a Killing-vector $\xi^0 = \xi^t$, which describes symmetries corresponding to time-translations, this charge will be the system's total energy [5].

1.2.1 Generalization

If we now not only look at global symmetries, but also consider gauge symmetries and transformations $x^\mu \to x^\mu + \zeta^\mu$, where ζ^μ is some diffeomorphism, we can say that two global symmetries of the Lagrangian are equivalent up to said gauge transformation and an additional symmetry that has a vanishing generator on shell. In addition, any two conserved currents are to be seen as equivalent if their only difference is a trivial current

$$J^\mu - \tilde{J}^\mu = \partial_\nu k^{[\mu\nu]} + t^\mu, \quad t^\mu \approx 0. \tag{1.2.8}$$

Here, should the equations of motion be fulfilled, the skew (2,0) tensor $k^{[\mu\nu]} = \frac{1}{2}(k^{\mu\nu} - k^{\nu\mu})$ can be used to arrive at an alternative expression for the conserved charges when integrating over a Cauchy-slice Σ

$$Q = \int_\Sigma (d^{D-1}x)_\mu J^\mu \approx \int_{\partial\Sigma} (d^{D-2}x)_{\mu\nu} k^{[\mu\nu]} \tag{1.2.9}$$

after assuming $\partial_\mu J^\mu \approx \partial_\mu \tilde{J}^\mu$. However, here, we have not invoked any constraints on $k^{[\mu\nu]}$, leaving the charge Q arbitary and leading to a vanishing current $J^\mu = 0$ for any diffeomorphism as the stress-energy tensor becomes trivial because $J^\mu = T^{\mu\nu}\xi_\nu \approx 0$. Despite that, constraining the skew tensor $k^{[\mu\nu]}$ to $(n-2)$-forms that uniquely vanish on shell, $dk = 0$, allows for the definition of a non-arbitrary charge conserved across surfaces S, $Q = \int_S k$, as well as a generalization of Noether's theorem. In analogy to mapping (global) continuous symmetries of the action to conserved quantities or currents, we therefore now have a bijection between gauge parameters $\lambda(x^\mu)$ that cause the fields' variations to vanish on-shell and the $(n-2)$-forms k [9].

1.2.2 Noether's second theorem and fundamental theorem of the phase space formalism

Similarly to (1.1.4), for a $(n-1)$-form S_ζ proportional to the equations of motion, the relation

$$dS_\zeta[\frac{\delta L}{\delta \Phi}, \Phi] = \frac{\delta L}{\delta \Phi}\delta_\zeta \Phi, \tag{1.2.10}$$

holds, where the Lagrangian L is an n-form and ζ^μ an infitesimal diffeomorphism. This is called "Noether's second theorem" and it can be proven for e.g. Einstein gravity with $\Phi = g_{\mu\nu}$ by explicit computation.

We now further define the variation $\delta = \delta\Phi^i \frac{\partial}{\partial \Phi^i} + \delta\Phi^i_\mu \frac{\partial}{\partial \Phi^i_\mu} + ...$ to be a one form anticommuting with the exterior derivative $d = dx^\mu \partial_\mu$, where $\partial_\mu = \frac{\partial}{\partial x^\mu} + \Phi^i_\mu \frac{\partial}{\partial \Phi^i} + \Phi^i_{\mu\nu} \frac{\partial}{\partial \Phi^i_\nu} + ...$ This together with using the total derivative of a presymplectic potential $\Theta(\delta\Phi_i, \Phi_i) = \partial_\mu \delta\Phi^i \frac{\delta \mathcal{L}}{\delta \partial_\mu \Phi^i}$, which is a boundary term in the Lagrangian's variation, lets us write

$$\delta\mathcal{L} = \frac{\delta \mathcal{L}}{\delta \Phi_i}\delta\Phi_i - d\Theta, \tag{1.2.11}$$

labeling $\delta\Theta = \omega$ the presymplectic form. Note that here, the presymplectic potential Θ is a $(n-1, 1)$-form and therefore making the presymplectic form ω a $(n-1, 2)$-form. The form degrees are given in the way that (spacetime form degree, covariant phase space form degree).
In a next step, we can find an expression for a conserved current by using (1.2.11) to find the Lagrange-density's variation along a diffeomorphism ζ^μ as well as Noether's second theorem (1.2.10) in order to obtain the expression $d(\zeta^\mu \partial_\mu \mathcal{L}) = dS_\zeta + d\Theta$ and therefore allowing us to find

$$J_\zeta = \zeta^\mu \partial_\mu \mathcal{L} - \Theta, \quad dJ_\zeta = dS_\zeta \approx 0. \tag{1.2.12}$$

This conserved current's derivative is locally exact due to a "fundamental property of thr covariant phase space"[9] and we may therefore also express it as $J_\zeta = S_\zeta + dQ_\zeta$, by virtue of which we find that the charge now reads

$$Q_\zeta = I_\zeta(J_\zeta - S_\zeta), \tag{1.2.13}$$

with the operator $I_\zeta = \frac{1}{n-k}\zeta^\alpha \frac{\partial}{\partial\partial_\mu\zeta^\alpha}\frac{\partial}{\partial dx^\mu}$. k denotes the form-degree of what I_ζ is acting on.
This expression for the charge (1.2.13) can be further simplified to the Noether-Wald surface charge

$$Q_\zeta[\Phi] = -I_\zeta\Theta \tag{1.2.14}$$

seeing as how neither S_ζ nor $\zeta^\mu\partial_\mu\mathcal{L}$ contain derivatives in the diffeomorphism ζ^μ and therefore drop out.
We can further connect the $(n-2,1)$-form k_ζ to the presymplectic form

$$\omega \approx dk_\zeta. \tag{1.2.15}$$

It can further (up to a total derivative) be expressed as

$$k_\zeta = -\delta Q_\zeta + \zeta^\mu\partial_\mu\Theta \tag{1.2.16}$$

using (1.2.14). We therefore find the theorem of the covariant phase space formalism that states that if a unique (up to a total derivative) infitesimal surface charge k_ζ exists that satisfies the relation (1.2.15), it is given in terms of the Noether-Wald charge (1.2.14) and presymplectic potential as given above in equation (1.2.16).

1.3 Surface charges and central extension

1.3.1 Asymptotic symmetries and asymptotic symmetry group

In general, diffeomorphisms ζ^μ preserving the metric can be found by solving the so called Killing's equation

$$\mathcal{L}_\zeta g_{\mu\nu} = 0, \tag{1.3.1}$$

where $\mathcal{L}_\zeta$ denotes the Lie-derivative along some vector field ζ. If this equation is only satisfied in an asymptotic region, e.g. for $r \to \infty$ with r being some radial coordinate, the diffeomorphism preserving the metric $g_{\mu\nu}$ merely asymptotically is called an asymptotic symmetry or asymptotic Killing-vector.
These symmetries may be constrained by a set of adequate boundary conditions that are obeyed by some set of field configurations ϕ, yielding a set of allowed diffeomorphisms $\{\zeta_a^\mu\}$ that contains vectors tangetial to ϕ and whose elements form a Lie-algebra $[\zeta_a, \zeta_b]^\mu = C_{ab}^c\zeta_c^\mu$.
Merely asymptotic symmetries whose associated conserved charge Q_ζ is non-zero have an effect on the actual physics of a system, all others are merely gauge transformations generating coordinate changes. Therefore, the asymptotic symmetry group can be defined as the quotient of all allowed diffeomorphisms over all trivial gauge transformations.

1.3.2 Surface charges and charge algebra

We next define a charge Q_ζ that is conserved if the presymplectic form vanishes on-shell $\omega \approx 0$ up to a total derivative term and exists if the surface charge k_ζ is integrable under the condition

$$\delta_1 \oint_S k_\zeta[\delta_2\Phi, \Phi] - \delta_2 \oint_S k_\zeta[\delta_1\Phi, \Phi] = 0. \tag{1.3.2}$$

Along some curve γ, we can then define

$$Q_\zeta = N_\zeta[\bar{\Phi}] + \int_\gamma \oint_S k_\zeta \tag{1.3.3}$$

using some reference charge N_ζ.

We can use the reference field $\bar{\Phi}$ to derive the charge algebra for two different asymptotic diffeomorphisms after defining the Lie-bracket for two infitesimal diffeomorphisms to be

$$\{Q_{\zeta_m}, Q_{\zeta_n}\} = \delta_{\zeta_n} Q_{\zeta_m} = \oint_S k_{\zeta_m}[\delta_{\zeta_n}\Phi; \Phi] \tag{1.3.4}$$

and thus

$$\{Q_{\zeta_m}, Q_{\zeta_n}\} = \delta_{\zeta_n} Q_{\zeta_m} = \oint_S k_{\zeta_m}[\delta_{\zeta_n}\Phi, \Phi] = (\oint_S k_{\zeta_m}[\delta_{\zeta_n}\Phi, \Phi] - \oint_S k_{\zeta_n}[\delta_{\zeta_m}\Phi, \Phi]) \tag{1.3.5}$$

$$+ \oint_S k_{\zeta_n}[\delta_{\zeta_m}\Phi, \Phi] = \int_\gamma \oint_S k_{\zeta_m}[\delta_{\zeta_n}\Phi, \Phi] + \oint_S k_{\zeta_m}[\delta_{\zeta_n}\bar{\Phi}, \bar{\Phi}]. \tag{1.3.6}$$

Using the integrability condition to see that $\int_\gamma \oint_S k_{\zeta_m}[\delta_{\zeta_n}\Phi, \Phi] = \int_\gamma \oint_S k_{[\zeta_m,\zeta_n]}$ as well as defining the central extension $\mathcal{C}_{\zeta_m,\zeta_n}[\bar{\Phi}] = \oint_S k_{\zeta_m}[\delta_{\zeta_n}\bar{\Phi}, \bar{\Phi}] - N_\zeta$, we then obtain the charge algebra as [9]

$$\{Q_{\zeta_m}, Q_{\zeta_n}\} = \int_\gamma \oint_S k_{[\zeta_m,\zeta_n]}[\delta_{\zeta_n}\Phi, \Phi] + \oint_S k_{\zeta_m}[\delta_{\zeta_n}\bar{\Phi}, \bar{\Phi}] = \tag{1.3.7}$$

$$Q_{[\zeta_m,\zeta_n]} + \mathcal{C}_{\zeta_m,\zeta_n}[\bar{\Phi}]. \tag{1.3.8}$$

It matches the diffeomorphism-algebra up to a central extension $\mathcal{C}_{\zeta_m,\zeta_n}$ that commutes with any surface charge Q_{ζ_m} and is antisymmetric under the exchange of the diffeomorphisms $\zeta_{m,n}$. It additionally preserves (anti)symmetry properties and is said to be non-trivial if it "cannot be absorbed into a normalization of the charges" and cannot be removed by a change of basis.[9]

We can also obtain an expression for the conserved charges in terms of the metric by linearizing the theory around some background $\bar{g}_{\mu\nu}$ with the metric's variation $h_{\mu\nu} = \delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$, defining the conserved charge as

$$Q_\zeta[g, \bar{g}] = \int_{\bar{g}}^{g} \oint_S k_\zeta[dg'; g'] \tag{1.3.9}$$

and its variation as

$$\delta_\zeta Q_\zeta[g,\bar{g}] = \int_\Sigma k_\zeta[dg';g'] \tag{1.3.10}$$

for all asymptotic symmetries ζ that preserve $dk_\zeta = 0$ on-shell. Afterwards, one can derive an expression for the surface charge and also the conserved charges for any conformal theory, which is a theory whose metric is invariant under conformal transformations up to a scale factor, see chapter 3 for details.

1.3.3 Conserved charges in Einstein-gravity

For the Lagrangian $\mathcal{L} = \frac{1}{16\pi G}\sqrt{-g}R$, where the only field is the metric $g_{\mu\nu}$, the presymplectic potential previously mentioned is

$$\Theta^\mu[h,g] = \frac{\sqrt{-g}}{16\pi G}(\nabla_\nu h^{\mu\nu} - \nabla^\mu h^\nu_\nu). \tag{1.3.11}$$

For $h = \delta_\zeta g = \mathcal{L}_\zeta g$, we find that

$$\Theta^\mu \approx \frac{\sqrt{-g}}{16\pi G}\nabla_\nu(\nabla^\nu\zeta^\mu - \nabla^\mu\zeta^\nu) \tag{1.3.12}$$

and

$$Q_\zeta = -I_\zeta\Theta = \frac{\sqrt{-g}}{8\pi G}\nabla^\mu\zeta^\nu(d^{n-2}x)_{\mu\nu}, \tag{1.3.13}$$

leading to the surface charge formula [9]

$$k_\zeta[h,g] = \frac{\sqrt{-g}}{8\pi G}(d^{n-2}x)_{\mu\nu}(\zeta^\mu\nabla_\sigma h^{\nu\sigma} - \zeta^\mu\nabla^\nu h + \zeta_\sigma\nabla^\nu h^{\mu\sigma} + \frac{1}{2}h\nabla^\nu\zeta^\mu - h^{\rho\nu}\nabla_\rho\zeta^\mu). \tag{1.3.14}$$

The surface charge that we will need to use is called the Barnich-Brandt charge, which differs from (1.3.14) by just an additional boundary term and for Einstein gravity reads [9]

$$k_\zeta[h,g] = \frac{\sqrt{-g}}{8\pi G}(d^{n-2}x)_{\mu\nu}(\zeta^\mu\nabla_\sigma h^{\nu\sigma} - \zeta^\mu\nabla^\nu h + \zeta_\sigma\nabla^\nu h^{\mu\sigma} + \frac{1}{2}h\nabla^\nu\zeta^\mu - \frac{1}{2}h^{\rho\nu}\nabla_\rho\zeta^\mu + \frac{1}{2}h^\nu_\sigma\nabla^\mu\zeta^\sigma) \tag{1.3.15}$$

or rather [19]

$$k_\zeta[h,g] = -\frac{1}{4}\varepsilon_{\alpha\beta\mu\nu}[\zeta^\nu\nabla^\mu h - \zeta^\nu\nabla_\sigma h^{\mu\sigma} + \zeta_\sigma\nabla^\nu h^{\mu\sigma} + \frac{1}{2}h\nabla^\nu\zeta\mu - h^{\nu\sigma}\nabla_\sigma\zeta^\mu + \frac{1}{2}h^{\sigma\nu}(\nabla^\mu\zeta_\sigma + \nabla_\sigma\zeta^\mu)]dx^\alpha\wedge dx^\beta. \tag{1.3.16}$$

1.3.4 Charges in Topologically Massive Gravity

A big class of black holes can be described using the theory of Topologically Massive Gravity (TMG), where higher order curvature corrections are introduced to the action in the form of an additional Chern-Simons term, see below. The action reads [10]

$$S_{TMG} = \frac{1}{16\pi}\int d^3x\sqrt{-g}(R+2) - \frac{1}{96\pi\nu}\int d^3x\sqrt{-g}\varepsilon^{\lambda\mu\nu}\Gamma^r_{\lambda\sigma}(\partial_\mu\Gamma^\sigma_{r\nu} + \frac{2}{3}\Gamma^\sigma_{\mu\tau}\Gamma^r_{\nu r}), \tag{1.3.17}$$

the surface charge will have a contribution from the gravitational side denoted by the Barnich-Brandt charge (1.3.15), but also a contribution from the gravitational Chern-Simons term [10]

$$\begin{aligned} k^{\mu\nu}_{gCS}[\zeta, h, g] = \\ \frac{1}{3\nu}k^{\mu\nu}_{Barnich}[\eta, h, g] - \frac{1}{6\nu}\zeta_\lambda(2\varepsilon^{\mu\nu\rho}\delta(G^{\lambda}{}_{\rho}) - \varepsilon^{\mu\nu\lambda}\delta G) + \\ \frac{1}{6\nu}\varepsilon^{\mu\nu\rho}[\zeta_\rho h^{\lambda\sigma}G_{\sigma\lambda} + \frac{1}{2}h(\zeta_\sigma G^\sigma_\rho + \frac{1}{2}\zeta_\rho R)], \end{aligned} \tag{1.3.18}$$

with $\eta^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho}\nabla_\nu\zeta_\rho$, the totally antisymmetric tensor $\varepsilon^{\mu\nu\rho}$ and the Christoffel symbol Γ.

Due to this additional term in the action (1.3.17), the vacuum Einstein-equations now obtain an additional term and change to the third order differential equations [16]

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} + \frac{1}{\mu}C_{\alpha\beta} = 0, \tag{1.3.19}$$

where $C_{\alpha\beta} = \varepsilon_\alpha{}^{\mu\nu}\nabla_\mu(R_{\nu\beta} - \frac{1}{4}g_{\nu\beta}R)$ is the Cotton-tensor and μ the mass of the graviton. All "normal" solutions of Einstein-gravity are also solutions of these equations, but there will also be a new set of solutions appearing that are of great interest to us, namely warped spacetimes, see chapter 4.3 for more details.

Chapter 2

Black Hole thermodynamics

This section will give a brief summary of the thermodynamics of black holes by presenting brief derivations and/or proofs of the 4 laws of thermodynamics of black holes that stand in analogy to the laws of thermodynamics from classical theory.

2.1 Zeroth law

We start by defining a special quantity called the surface gravity κ as a "non-affine parameter of the null geodesic" [18]

$$\xi^\mu \nabla_\mu \xi^\nu \mid_{\mathcal{H}} = \kappa \xi^\nu, \tag{2.1.1}$$

where ξ denotes a Killing vector that is normal to the black hole horizon $\mathcal{H}$, making it a Killing horizon. After using the orthogonality of ξ on the horizon $\mathcal{H}$ on (2.1.1), we get $\xi_{[\mu}\nabla_\nu \xi_{\rho]}$, which after expansion and utilizing Killing's equation, $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, becomes

$$\xi_\rho \nabla_\mu \xi_\nu = -2\xi_{[\mu}\nabla_{\nu]}\xi_\rho. \tag{2.1.2}$$

After multiplying this by $\nabla^\mu \xi^\nu$ and using (2.1.1) as well as Killing's equation yields an explicit expression for the surface gravity

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu) \mid_{\mathcal{H}} . \tag{2.1.3}$$

Acting on this with $\xi^\mu \nabla_\mu$ yields the relations

$$\xi^{mu} \nabla_\mu \kappa^2 \mid_{\mathcal{H}} = -\frac{1}{2}\xi^\mu \nabla_\mu [(\nabla^\alpha \xi^\nu)(\nabla_\alpha \xi_\nu)] \mid_{\mathcal{H}} = -\xi^\mu (\nabla^\alpha \xi^\nu) \nabla_\mu \nabla_\alpha \xi_\nu \mid_{\mathcal{H}} . \tag{2.1.4}$$

Now, we introduce the curvature tensor, the Riemann-tensor defined as $[\nabla_\mu, \nabla_\nu]\xi^\rho = R^\rho{}_{\alpha\mu\nu}\xi^\alpha$. For a Killing vector, there exists the Killing vector lemma

$$\nabla_\mu \nabla_\nu \xi^\rho = R^\rho{}_{\nu\mu\alpha}\xi^\alpha. \tag{2.1.5}$$

Using this as well as the fact that the Riemann-tensor is antisymmetric in its last two indeces from (2.1.4) we find

$$\xi^\mu \nabla_\mu \kappa^2 \mid_{\mathcal{H}} = 0. \tag{2.1.6}$$

This proves that the surface gravity κ is constant along orbits of the Killing vector ξ generating the horizon.
Now, imagine a $(n-2)$-dimensional spacelike hypersurface S on which $\xi^\mu \mid_S = 0$, making the (null) hypersurfaces that generate the Killing horizon intersect. S then denotaes a bifurcation two-sphere. Taking a tangent vector field s^μ to this bifurcation two-sphere and acting on (2.1.3) with $s^\mu \nabla_\mu$ as well as using that $\xi^\mu \mid_S = 0$ on S, we find

$$s^\mu \nabla_\mu \kappa^2 \mid_S = 0. \tag{2.1.7}$$

Therefore, the surface gravity κ is constant along the bifurcation sphere S.
With this we proved the zeroth law of black hole thermodynamics, namely that the surface gravity κ is constant on a bifurcate Killing horizon $\mathcal{H}$.[18]

2.2 First law

The derivation of the first law works using conserved charges $\delta Q_\zeta = \int_{S^\infty} k_\zeta$. Because of the uniqueness condition $dk_\zeta = 0$ and in order to have integrability, this integral must be the same as the one over the horizon H and therefore for a generator of H of the form $\zeta = \partial_t + \Omega \partial_\phi$, with the temporal coordinate t and angular coordinate ϕ, we have

$$\int_{S^\infty} k_\zeta = \int_{S^\infty} k_{\partial_t} + \Omega_H \int_{S^\infty} k_{\partial_\phi} = \delta M - \Omega_H \delta J = \oint_H k_\zeta[g_{\mu\nu}, \delta g_{\mu\nu}]. \tag{2.2.1}$$

The last term consists of the two seperate terms

$$-\delta \oint_H K_\zeta[g_{\mu\nu}] = -\delta(\frac{\kappa A}{8\pi G}) \tag{2.2.2}$$

and

$$\oint_H K_{\delta\zeta}[g_{\mu\nu}] - \oint_H \zeta\Theta[g_{\mu\nu}, \delta g_{\mu\nu}] = \oint_H \frac{dA}{8\pi G}\delta\kappa, \tag{2.2.3}$$

where

$$K_\zeta[g] = \frac{\sqrt{-g}}{16\pi G}(\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu)(d^{n-2}x)_{\mu\nu}, \tag{2.2.4}$$

$$\Theta[h, g] = \frac{\sqrt{-g}}{16\pi G}(\nabla_\nu \delta g^{\mu\nu} - \nabla^\mu \delta g)(d^{n-1}x)_\mu \tag{2.2.5}$$

and the intergation measure for $(n-2)$-forms

$$\sqrt{-g}(d^2x)_{\mu\nu} = \frac{1}{2}(\xi_\mu n_\nu - n_\mu \xi_\nu)dA \tag{2.2.6}$$

were used. Putting all together, we finally arrive at

$$\oint_H k_\zeta[g_{\mu\nu}, \delta g_{\mu\nu}] = \frac{\kappa}{8\pi G}\delta A \tag{2.2.7}$$

and therefore the first law of black hole thermodynamics [11]

$$\delta M - \Omega_H \delta J = \frac{\kappa}{8\pi G}\delta A. \tag{2.2.8}$$

2.3 Second law

The second law states that if the null energy condition, $R_{\alpha\beta}\xi^\alpha\xi^\beta \geq 0$, is satisfied for any future directed null vector field ξ, a black hole's area A will never decrease

$$\delta A \geq 0. \tag{2.3.1}$$

This can be seen to be true using the null, twist free version of Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{d-1}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - R_{\alpha\beta}\xi^\alpha\xi^\beta, \tag{2.3.2}$$

where d is the dimension of the spacetime. Because this law only holds if the null energy condition is fulfilled, the inequality

$$\frac{d\theta}{d\lambda} \leq -\frac{1}{d-1}\theta^2, \tag{2.3.3}$$

implying

$$\frac{1}{\theta(\lambda)} \geq \frac{1}{\theta_0} + \frac{\lambda}{d-1}, \tag{2.3.4}$$

must hold true. Now assuming that the congruence is at first converging, i.e. that θ_0 is negative, with decreasing λ, $\theta \to -\infty$ at some point in time, implying a singularity of the congruence. However, an event horizon's generators can never run into such a caustic if the horizon is generated by "null geodesics without future end points"[11] and therefore $\theta \geq 0$, meaning that the area of a black hole cannot decrease indeed. [11]

2.4 Summary

To summarize, we also state the laws of classical thermodynamics as an analogy to black hole thermodynamics: [23]

Law	Black hole dynamics	Thermodynamics
0	$\kappa = const$ on the Killing horizon H	temperature $T = const.$ in a body that is in thermal equilibrium
1	$\delta M - \Omega_H \delta J = \frac{\kappa}{8\pi G}\delta A$	$dE = TdS$+work terms
2	$\delta A \geq 0$	$\delta S \geq 0$

Chapter 3

Conformal Field Theory

This section will provide a summary of the basic and most important aspects of conformal field theories (CFT). A CFT is a theory whose metric is invariant under conformal transformations up to a scale factor.

There are three such transformations:

- translations in time
- scaling transformations
- special conformal transformations.

We can derive the conformal version of the Killing equation by starting out with some metric $g_{\mu\nu}$ that is supposed to remain invariant under a conformal transformation and making the ansatz

$$g_{\alpha\beta} \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} = \lambda(x) g_{\mu\nu}, \tag{3.0.1}$$

with some scale factor $\lambda(x)$. Now we insert a finite translation $x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x^{\nu})$ to find

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = \lambda(x) g_{\mu\nu}. \tag{3.0.2}$$

Taking the trace yields the remaining unknown scaling factor $\lambda(x) = \frac{2}{d}\partial\varepsilon$ and thus the conformal Killing equation

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = \frac{2}{d}(\partial\varepsilon) g_{\mu\nu}. \tag{3.0.3}$$

If we let ∂_{μ} act on this and then add three different index permutations up, we arrive at

$$2\partial_{\mu}\partial_{\nu}\varepsilon_{\rho} = \frac{2}{d}(-g_{\mu\nu}\partial_{\rho} + g_{\rho\mu}\partial_{\nu} + g_{\nu\rho}\partial_{\nu})(\partial\varepsilon). \tag{3.0.4}$$

We can use this in order to obtain the CFT's infitesimal generators by inserting different ansaetze for ε_{μ} into it:

- after taking inspiration from Poincare-transformations, expanding and inserting $\varepsilon_{\mu} = a_{\mu} + b_{\mu\nu}x^{\nu} + C_{\mu\nu\rho}x^{\nu}x^{\rho}$, where a_{μ} is some constant associated with translations in spacetime, yields

$$P_{\mu} = -i\partial_{\mu}. \tag{3.0.5}$$

We also obtain $C_{\mu\nu\rho} = C_{\mu\rho\nu}$ as a symmetric tensor in its last two indeces.

- When only using the middle term of beforementioned ansatz $\varepsilon_\mu = b_{\mu\nu}x^\nu$ and inserting it into the conformal Killing equation (3.0.3), we obtain $b_{\mu\nu} = m_{\mu\nu} + \alpha g_{\mu\nu}$, where $m_{\mu\nu}$ is some antisymmetric tensor that generates Lorenz transformations

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{3.0.6}$$

and the symmetric part of $b_{\mu\nu}$ generates dilatations

$$D = -ix^\mu \partial_\mu. \tag{3.0.7}$$

- Inserting $\varepsilon_\mu = \varepsilon_{\mu\nu\rho}x^\nu x^\rho$ into (3.0.4) yields the tensor $C_{\mu\nu\rho} = g_{\mu\rho}b_\nu + g_{\mu\nu}b_\rho - g_{\nu\rho}b_\mu$ with $b_\mu = \frac{1}{d}C^\nu_{\nu\mu}$. Its non-trivial part generates special conformal transformations

$$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu). \tag{3.0.8}$$

These generator's Lie brackets provide us with the conformal algebra for $d \geq 3$

$$[D, P_\mu] = iP_\mu \quad [D, K_\mu] = -iK_\mu \tag{3.0.9}$$
$$[K_\mu, P_\nu] = 2i(g_{\nu\mu}D - L_{\mu\nu}) \quad [K_\rho, L_{\mu\nu}] = i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) \tag{3.0.10}$$
$$[P_\rho, L_{\mu\nu}] = i(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu) \tag{3.0.11}$$
$$[L_{\mu\nu}, L_{\rho\sigma}] = i(g_{\nu\rho}L_{\mu\sigma} + ...). \tag{3.0.12}$$

Together they make up the conformal group, which for d-dimensional Minkowski space is $SO(d, 2)$ and $SO(d + 1, 1)$ for d-dimentional Euclidean space. [21]

3.1 Conformal group and representations

The conformal group (3.0.12) is made up of $\frac{(d+1)(d+2)}{2}$ generators in d dimensions, which coincides with the number of generators of the $SO(d + 2)$-type algebra. We therefore redefine the generators as

$$J_{\mu\nu} = L_{\mu\nu} \tag{3.1.1}$$
$$J_{-1\mu} = \frac{1}{2}(P_\mu - K_\mu) \tag{3.1.2}$$
$$J_{0\mu} = \frac{1}{2}(P_\mu + K_\mu) \tag{3.1.3}$$
$$J_{0-1} = D, \tag{3.1.4}$$

yielding the Lorenz-type algebra

$$[J_{mn}, J_{pq}] = i(g_{ma}J_{na} + ...), \tag{3.1.5}$$

where $g_{ma} = diag(-1, -1, 1, 1....1)$.

Representations: Let us now consider a field Φ on which we use a scaling transformation

$$\Phi(\lambda x) = \lambda^{-\Delta}\Phi(x), \tag{3.1.6}$$

where the scaling dimension Δ is an eigenvalue of the dilatation operator. Using the Jacobian $\mid \frac{\partial x'}{\partial x} \mid = \Lambda(x)^{-d/2}$ and committing the field to a coordinate transformation, we obtain

$$\Phi'(x') = \mid \frac{\partial x'}{\partial x} \mid^{-\frac{\Delta}{2}} \Phi(x). \tag{3.1.7}$$

A field that transforms in this way is called a quasi-primary. [21]

3.2 Witt and Virasoro algebra

In order to obtain the famous Witt algebra, we first start out with a holomorphic transformation

$$z' = f(z) = z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n(-z^{n+1}), \tag{3.2.1}$$

where the last equality is just the general Laurent-expansion for an arbitary holomorphic function. The infitesimal generators for individual Laurent-modes in 2 dimensions are

$$l_m = -z^{m+1}\partial_z, \tag{3.2.2}$$

and their Lie algebra provides us with the Witt-algebra

$$[l_m, l_n] = (n - m)l_{n+m}, \tag{3.2.3}$$

which is infinite dimensional.

If we now consider a central extension $Z(n, m)$ of the algebra

$$[L_n, L_m] = (n - m)L_{n+m} + Z(n, m), \tag{3.2.4}$$

we will want to check whether it is trivial. To do this, we can make use of the Jacobi identities to obtain conditions on Z. Note that for the Virasoro algebra we are intending to derive with this, the central extension will be non-trivial and directly proportional to a central charge c, which was described in chapter 1.3.2.

First of all, we will commit to a change of basis $L_n \to L_n + A(n), Z \to Z - (n-m)A(n+m)$ which helps us find $Z(n,0) = Z(0,m) = Z(1,-1) = 0$ as well as $Z(n,m) = -Z(m,n)$ by choosing $A(n) = \frac{Z(n,0)}{n}$ and $A(0) = \frac{Z(1,-1)}{2}$. We can freely choose such constraints on A since it does not change the fundamental physics. The central extension's antisymmetric property is a result of the antisymmetry of the commutator.
By now computing the generators' algebra again, we find the global subalgebra

$$[L_{\pm 1}, L_0] = \pm L_{\pm 1}, \quad [L_1, L_{-1}] = 2L_0, \tag{3.2.5}$$

but no central extension. However, the Jacobi identity still has to hold, so we can use it to find

- For $k = 0$, $n + m \neq 0$: $(m+n)Z(m,n) = 0$ and therefore $m = -n$ and $Z(n,m) = Z(n)\delta_{n+m,0}$.
- For $k = 1$: $Z(n) = \frac{Z(2)}{6}(n^3 - n)$, which is a unique central extension of the Witt algebra to the Virasoro algebra.

Defining $Z(2) = \frac{c}{2}$, we therefore find the Virasoro algebra [21]

$$[L_n, L_m] = (n-m)L_{m+n} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}. \tag{3.2.6}$$

3.3 (Quasi-)Primary fields and OPEs

3.3.1 Primaries and quasi-primaries

We again consider a field $\phi(z,\overline{z})$ under a scaling transformation $z \to \lambda z$:

$$\phi'(z,\overline{z}) = \lambda^h \overline{\lambda}^{\overline{h}} \phi(\lambda z, \overline{\lambda}\overline{z}) \tag{3.3.1}$$

with the (anti-)holomorphic scaling dimensions $h, \overline{h}$. If ϕ transforms under the local conformal transformation $z \to f(z)$ as

$$\phi' = (\frac{\partial f}{\partial z})^h (\frac{\partial f}{\partial \overline{z}})^{\overline{h}} \phi(\lambda z, \overline{\lambda}\overline{z}), \tag{3.3.2}$$

and this holds for all $f, \overline{f}$, then ϕ is called a primary field. Note that while not all quasi-primary fields are primaries, all primary fields are trivially quasi-primaries.
Example: For $f(z) = z + \varepsilon(z)$, we have $(\frac{\partial f}{\partial z})^h = 1 + h\partial_z \varepsilon(z) + \mathcal{O}(\varepsilon^2)$ and therefore the transformation[21]

$$\delta_\varepsilon \phi = (h\partial_z \varepsilon + \varepsilon \partial_z + \overline{h}\partial_{\overline{z}}\overline{\varepsilon} + \overline{\varepsilon}\partial_{\overline{z}})\phi. \tag{3.3.3}$$

3.3.2 Operator product expansions and Ward identity

Ward identities: The ward identities describe how the invariance of the action of a theory under a continuous symmetry constraints the correlation functions' forms.
Given the general form of a path integral as $Z = \int \mathcal{D}\phi e^{-S(\phi)}$, one may write correlation functions as

$$\langle O_1(x_1)...O_n(x_n)\rangle = \frac{1}{2}\int \mathcal{D}\phi e^{-S(\phi)}\Pi_i O_i(x_i). \tag{3.3.4}$$

Now doing an infitesimal transformation $\phi' = \phi + \Sigma(x^\mu)\delta\phi$, with some coordinate dependend function Σ, the path integral changes to

$$\int \mathcal{D}\phi' e^{-S(\phi')} = \int \mathcal{D}\phi e^{-S(\phi - \int J^\alpha \partial_\alpha \Sigma dx)} \approx \int \mathcal{D}\phi e^{-S(\phi)}(1 - \int J^\alpha \partial_\alpha \Sigma dx) \tag{3.3.5}$$

with a current J^α. Integration by parts yields the quantum mechanic analogue of the continuity equation (and we now also have a quantum version of Noether's theorem)

$$\langle \partial_\alpha J^\alpha \rangle = 0. \tag{3.3.6}$$

Now, adding an operator transformed under the transformation done before $O'_n = O_n + \Sigma(x^\alpha)\delta O_n$, with Σ chosen so that only for $x = x_n$, $\Sigma(x_n) = 0$ (meaning that support is away from the operator insertion x_n), will lead to $O'_n = O_n$ after inserting $x^\alpha = x^n$. The operator's expectation value will then become

$$\int \mathcal{D}\phi e^{-S(\phi)} O_n(x_n) \approx \int \mathcal{D}\phi e^{-S(\phi)}(1 - \int J^\alpha \partial_\alpha \Sigma dx) O_n(x_n) \tag{3.3.7}$$

We therefore find $\langle \partial_\alpha J^\alpha(x) O_n(x_n)\rangle = 0 \quad \forall x \neq x_n$, which can be generalized to

$$\langle \partial_\alpha J^\alpha(x) \Pi_i O_i(x_i)\rangle = 0 \quad \forall x \neq x_i. \tag{3.3.8}$$

If $\Sigma = const$ in a support including $x = x_i$ and 0 otherwise, again the same procedure will in the end yield the Ward identities

$$\partial_\mu \langle J^\mu(y) \langle O_1(x_1)...O_n(x_n)\rangle = \sum_{i=1}^{n} \delta(y - x_i)\langle O_1(x_1)..O_i(x_i)..O_n(x_n)\rangle. \tag{3.3.9}$$

OPEs: An operator product expansion or OPE describes "what is happening when two local operators approach each other", meaning that if we insert two local operators at points close to each other they can be approximated by a "sting" of operators at one of these two points. This is valid within correlation functions. The OPE shows a singular behaviour as the two points approach each other, which is also its only non-trivial part. This can be seen to be true when considering the Ward identities (3.3.9): writing the correlation function on the left hand side out as an integral and applying the residue formula at the singular point will yield a non-vanishing expression for that point only.

The OPE is generally given by

$$O_1(z, \overline{z}) O_2(\omega, \overline{\omega}) = \sum_j C_{12j}(z - \omega, \overline{z} - \overline{\omega}) O_j(y). \tag{3.3.10}$$

For the OPE between an operator O and the stress tensor $T(z)$, its singular part describes how the operator acts under a conformal transformation and we therefore are interested in calculating it.[11]

To do so, let us start out with a 2d CFT that contains symmetries of the form $x^\mu \to x^\mu + \varepsilon^\mu(x)$ and therefore currents $j^\nu = T_{\mu\nu}\varepsilon^\nu$. For consistency, the energy-stress tensor $T_{\mu\nu}$ has to be trace-free and fulfill the conservation equation, which can be expressed in complex coordinates as

$$T_{z\overline{z}} = 0, \quad \partial_{\overline{z}} T_{zz} = 0 = \partial_z T_{\overline{z}\overline{z}}, \tag{3.3.11}$$

from which also follows that the tensor is holomorphic $T_{zz} = T_{zz}(z) = T(z)$. Since we have symmetries, we are also provided with an infinite set of conserved charges, one corresponding to each appearing symmetry,

$$Q = \int dx j_0 = \frac{1}{2\pi i}\oint_C [dz\varepsilon(z)T(z) + h.c.], \tag{3.3.12}$$

with the conserved current $j_0 = T^{\mu 0}\varepsilon_0$, which generates symmetry transformations of the form

$$\delta O = [Q, O] \tag{3.3.13}$$

for a given operator O. Similarly we therefore find that an infitesimal transformation for the fields

$$\delta_\varepsilon \phi(m, \overline{n}) = \frac{1}{2\pi i}\oint_C dz[\varepsilon(z)T(z), \phi(\omega, \overline{\omega})] + h.c.. \tag{3.3.14}$$

Now reinterprating the commutator appearing here as a radial ordering instead of a time ordering of the contour integral $\oint_{C(\omega)} R(A(z)B(\omega))$ in order to end up with a radial quantization, see subsection 3.4 for details,

$$R(A(z)B(\omega)) = A(z)B(\omega) \quad \forall \mid z \mid\rangle \mid \omega \mid \quad \vee \quad B(\omega)A(z) \quad \forall \mid z \mid \langle\mid \omega \mid \tag{3.3.15}$$

leads to the transformation

$$\delta_\varepsilon \phi(\omega, \overline{\omega}) = \frac{1}{2\pi i}\oint_{C(\omega)} dz\varepsilon(z)R[T(z)\phi(\omega, \overline{\omega})] + anti. - hol.. \tag{3.3.16}$$

Comparing this to the infitesimal transformation (3.3.3) leads to an example of an OPE valid for any primary field

$$R(T(z)\phi(\omega, \overline{\omega})) = \frac{h}{(z-\omega)^2}\phi(\omega, \overline{\omega}) + \frac{\partial_\omega \phi}{z - \omega} + regularization. \tag{3.3.17}$$

Next, let us expand

$$T(z) = \sum_n z^{-n-2} L_n, \tag{3.3.18}$$

with

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z), \tag{3.3.19}$$

which leads us to the mode algebra for the stress-energy tensor $T_{\mu\nu}$, the Virasoro algebra (3.2.6). Let us further expand $\varepsilon(z) = -\varepsilon_n z^{n+1}$ to its n-th Laurent mode. This yields the charges [21]

$$Q_n = \oint \frac{dz}{2\pi i} T(z)\varepsilon(z) = -\varepsilon_n L_n. \tag{3.3.20}$$

3.3.3 Sugawara construction

Let us define the field $\phi(z, \overline{z})$ as a field describing a free boson with currents $j(z) = i\partial_\phi$ and $\overline{j} = i\overline{\partial}_\phi$. Expanding these currents to

$$j(z) = \sum_n z^{-n-1} J_n \tag{3.3.21}$$

and using their Heisenberg commutator yields an affine (Kac-Moody) algebra

$$[J_n, J_m] = kn\delta_{n+m,0}, \tag{3.3.22}$$

with some normalization constant k, where we demand that the vacuum condition $J_n \mid 0\rangle = 0$ is fulfilled. The Sugawara stress tensor's modes are bilinear in the currents J_n and given by

$$L_n = \frac{1}{2k} : \sum_p J_{n+m-p} J_p : . \tag{3.3.23}$$

This form is obtained by seeking an energy-momentum tensor that has the classical form $\frac{1}{2k}\sum_a J^a J^a$, but with a normal ordering, meaning that all creation operators are left to the annihilation operators. [13] These modes' and currents' commutator is

$$[L_n, J_m] = -mJ_{n+m}. \tag{3.3.24}$$

It is now also straight forward to calculate the modes' bracket algebra (where we are skipping a few steps)

$$[L_n, L_m] = \frac{n-m}{2k}(\sum_{p\geq 0} J_{n+m-p} J_p + \sum_{p\langle 0} J_p J_{n+m-p}) + \tag{3.3.25}$$

$$\frac{1}{2\kappa}\sum_{p=0}^{n-1}(n-p)[J_p, J_{n+m-p}] = \tag{3.3.26}$$

$$(n-m)L_{n+m} + \frac{1}{12}(n^3 - n)\delta_{n+m,0}, \tag{3.3.27}$$

which is just the Virasoro algebra (3.2.6) for $c = 1$.

Since therefore $T(z)$ obeys the Virasoro algebra, the OPE [13]

$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2T\omega}{(z-\omega)^2} + \frac{\partial_\omega T(\omega)}{z-\omega}. \tag{3.3.28}$$

is implied, since this is uniquely fixing the constant appearing in (3.3.23) to be $1/(2k)$, just as we had chosen, since otherwise we would not end up with a proper energy-momentum tensor.[13] In addition, from this we see that $T(z)$ is not a primary after comparison to (3.3.17), but rather a quasi-primary, as explained below.

Since we are now equipped with an OPE between $T(\omega)$ and $T(z)$, we have all the tools needed to calculate $T(\omega)$'s transformation behaviour under a conformal transformation $z \to z + \varepsilon(z)$:

$$\begin{aligned} \delta_\varepsilon T = -\frac{1}{2\pi i}\oint_{C_z} dz\varepsilon(z)T(z)T(\omega) = \\ -\frac{c}{12}\varepsilon'''(\omega) - 2\varepsilon'(\omega)T(\omega) - \varepsilon(\omega)\partial T(\omega) \end{aligned} \tag{3.3.29}$$

for infitesimal and

$$T'(\omega) = (\frac{d\omega}{dz})^2[T(z) - \frac{c}{12}\{\omega, z\}] \tag{3.3.30}$$

for finite transformations, where $\{\omega, z\}$ is the Schwarzian derivative [11]

$$\{\omega, z\} = \frac{\omega'''}{\omega'} - \frac{3}{2}\Big(\frac{\omega''}{\omega'}\Big). \tag{3.3.31}$$

Here, ω' denotes the derivative of ω with respect to z. From $T's$ transformation behaviour under the global part of conformal transformations, we also see that it is in fact a quasi-primary.

3.4 Radial quantization

Considering a 2 dimensional Euclidean cylinder with parametrization $\omega = Z + i\phi$, where ϕ is 2π-periodic, and the conformal transformation

$$z = e^{\frac{2\pi\omega}{L}}, \tag{3.4.1}$$

we can now conformally map constant Z slices on the cylinder to constant $\mid z \mid$ slices on the planar circle, giving a correspondence between the time and radial evolution of those two respectively. In addition, by having such a conformal map, we are enabled to understand the theory on the plane/cylinder by understanding the theory on the cylinder/plane.

The stress tensor's transformation behaviour under this map is given by (3.3.30) and results in

$$T_{cylinder}(\omega) = (\frac{2\pi}{L})^2 z^2 [T_{plane}(z) - \frac{c}{24z^2}]. \quad (3.4.2)$$

Since the only scale invariant function depending on z is $f(z) = z^{-2}$, the ground state energy on the plane vanishes. However, due to not being translation invariant, the ground state on the circle has a Casimir energy

$$\langle T_{cyl} \rangle = -\frac{c}{24}(\frac{2\pi}{L})^2. \quad (3.4.3)$$

Now expanding the stress tensor on the plane as in (3.3.18) and the one on the cylinder as $T_{cyl} = \sum_{m=-\infty}^{\infty} L_m^{cyl} e^{-m\omega}$ with the modes given by (3.3.19), the energy on the cylinder will be

$$T_{cyl}(\omega) = \sum_m L_m^{plane} z^{-m} - \frac{c}{24} = \sum_m L_m^{cyl} e^{-m\omega}, \quad (3.4.4)$$

leading to

$$L_0^{cyl} = L_0^{plane} - \frac{c}{24}. \quad (3.4.5)$$

Since time translations are given by translations of Z, the system's energy is

$$H = \int_0^{2\pi} d\phi T_{ZZ} = \int d\phi (T(\omega) + \overline{T}(\overline{\omega})) = L_0^{cyl} + \overline{L}_0^{cyl}, \quad (3.4.6)$$

and analogously, the angular momentum is given by

$$J = i(L_0^{cyl} - \overline{L}_0^{cyl}). \quad (3.4.7)$$

On the plane the vacuum groundstates will be trivial and on the cylinder they are

$$L_0^{vac} = -\frac{c}{24}, \quad \overline{L}_0^{vac} = -\frac{\overline{c}}{24}. \quad (3.4.8)$$

From that the vacuum ground state energy and angular momentum can be easily obtained by insertion.[11]

Chapter 4

Warped conformal field theory

This section very closely follows [10] and summarizes its most important and basic aspects.

In order to define a Warped Conformal Field Theory/WCFT, consider the symmetry structure of a 2 dimensional Lorentzian type of theory that is invariant under global $SL(2,R)_R \times U(1)_L$ transformations. In the "nontrivial minimal case" [10] of this, we talk about a WCFT. In general, warping corresponds to a deformation that results reduces the isometry group of the deformed metric to $SL(2,R)_R \times U(1)_L$ and changes its asymptotics.

4.1 WCFT algebra and transformations

We start out by defining the right moving energy momentum tensor

$$T_\zeta = -\frac{1}{2\pi}\int dx^- \zeta(x^-)T(x^-) \tag{4.1.1}$$

that generates infitesimal coordinate transformations in x^- and the right moving Kac-Moody current

$$P_\chi = -\frac{1}{2\pi}\int dx^- \chi(x^-)P(x^-) \tag{4.1.2}$$

that generates gauge transformations in x^+, where $x^\pm$ denotes left/right moving coordinates. We further demand the ground state's invariance under the global symmetries.
Applying the coordinate change $x^- = e^{i\phi}$, with test functions $\zeta_n = (x^-)^n$, as well as defining $L_n = iT_{\zeta_{n+1}}$ and $P_n = P_{\chi_n}$ leads to the commutator algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m} \tag{4.1.3}$$

$$[P_n, P_m] = \frac{k}{2}n\delta_{n+m} \tag{4.1.4}$$

$$[L_n, P_m] = -mP_{m+n}. \tag{4.1.5}$$

Defining $\delta_{\varepsilon+\gamma} = \delta_\varepsilon + \delta\gamma$, from the WCFT algebra one can guess that T and P's infitesimal transformations are

$$\delta_\varepsilon T(x^-) = -\varepsilon(x^-)\partial_- T(x^-) - 2\partial_\varepsilon(x^-)T(x^-) - \frac{c}{12}\partial_-^3\varepsilon \tag{4.1.6}$$

$$\delta_\gamma T(x^-) = -\partial_-\gamma(x^-)P(x^-) \tag{4.1.7}$$

$$\delta_\varepsilon P(x^-) = -\varepsilon(x^-)\partial_- P(x^-) - \partial_-\varepsilon(x^-)P(x^-). \tag{4.1.8}$$

The finite transformations are then

$$x^- = f(\omega^-) \qquad\qquad x^+ = \omega^+ + g(\omega^-), \tag{4.1.9}$$

and are denoted by arbitary functions $f(\omega^-)$ and $g(\omega^-)$. They will reduce to two quantities used here, $\varepsilon(\omega^-) = -\delta\omega^-$ and $\gamma(\omega^-) = -\delta\omega^+$, and in general the currents' (4.1.2) and stress-energy tensor's (4.1.1) finite transformation behaviour can be uniquely fixed by demanding the finite coordinate transformations'(4.1.9) reduction to the infitesimal version (4.1.8). In both transformation laws, there will be an anomaly k appearing due to the mixing of both generators. The energy-momentum tensor transforms as

$$T'(\omega^-) = \Big(\frac{\partial x^-}{\partial \omega^-}\Big)^2 \{T(x^-) - \frac{c}{12}\Big[\frac{\omega^{-'''}}{\omega^{-'}} - \frac{3}{2}(\frac{\omega^{-''}}{\omega^{-'}})^2\}\Big] + x^{-'}x^{+'}P(x^-) - \frac{k}{4}(x^{+'})^2 \tag{4.1.10}$$

and the current as

$$P'(\omega^-) = x^{-'}\Big(P(x^-) + \frac{k}{2}\omega^{+'}\Big), \tag{4.1.11}$$

where we denoted partial derivatives of $\omega^\pm$ along x^- and $x^\pm$ along ω^- with a $'$ respectively.

4.1.1 Map to the cylinder

We can now, analogously to an ordinary CFT, construct a map from the plane to the cylinder where we map $x^- \to \phi$ by adding a tilt α and choosing

$$f(\omega^-) = f(\phi) = e^{i\phi}, \quad \omega^+ = t, \quad g(\omega^-) = g(\phi) = 2\alpha\phi. \tag{4.1.12}$$

We will see that this change of coordinates will lead to additional anomalous terms as opposed to the usual transformation of partial derivatives.

Now, just as we did for an ordinary CFT, we can again define modes on the cylinder,

$$P_n^\alpha = -\frac{1}{2\pi}\int d\phi P^\alpha(\phi)e^{in\phi} \tag{4.1.13}$$

and

$$L_n^\alpha = -\frac{1}{2\pi}\int d\phi T^\alpha(\phi)e^{in\phi}, \tag{4.1.14}$$

that are connected to the modes used in the WCFT algebra (4.1.5) by

$$P_n^\alpha = P_n + k\alpha\delta_n, \quad L_n^\alpha = L_n + 2\alpha P_n + (k\alpha^2 - \frac{c}{24})\delta_n \tag{4.1.15}$$

and where

$$P^\alpha = ix^- P(x^-) - k\alpha, \tag{4.1.16}$$

$$T^\alpha = -\frac{1}{x^2}T(x^-) + \frac{c}{24} + 2i\alpha x^- P(x^-) - k\alpha^2. \tag{4.1.17}$$

The above expression is obtained by using the finite transformations (4.1.9) with the functions as defined in (4.1.12). One then obtains (4.1.15) by inserting the mode expansion for energy momentum tensor and the right-moving Kac-Moody current.

This additionally yields an expression for the generators' finite transformation behaviours

$$P'(\phi') = \frac{1}{\lambda}(P(\phi) - k\gamma) \tag{4.1.18}$$

and

$$T'(\phi') = \left(\frac{1}{\lambda}\right)^2 (T(\phi) + 2\gamma P(\phi) - k\gamma^2), \tag{4.1.19}$$

after changing the size as well as tilt parameter, which leads to the coordinate change $\phi = \frac{\phi'}{\lambda}$ and $t = t' + 2\frac{\gamma}{\lambda}\phi'$, and then inserting for the coordinates $x^\pm$ and the transformation rules (4.1.11) and (4.1.10). We then also find expressions for the conserved charges

$$Q_{\partial_t'} = Q_{\partial_t} + k\gamma, \quad Q_{\partial_{\phi'}} = \frac{1}{\lambda}(Q_{\partial_\phi} + 2\gamma Q_{\partial_t} + k\gamma^2). \tag{4.1.20}$$

4.2 Representations and vacuum energies

Now, in order to obtain unitary representations of the algebra (4.1.5), we will also demand hermicity in compatibility to unitarity. For hermicity, we demand $L_{-n} = L_n^\dagger$ and $P_{-n} = P_n^\dagger$. In order to also obtain unitarity, we first require that the primary states obey $P_n \mid p, h\rangle = 0, \quad L_n \mid p, h\rangle = 0$ for all $n > 0$ and define the ground states as

$$P_0 \mid p, h\rangle = p \mid p, h\rangle, \quad L_0 \mid p, h\rangle = h \mid p, h\rangle, \tag{4.2.1}$$

where the states' positivity puts constraints on the constants used: $c, k > 0, h \geq 0$. These constraints can be further specified by defining the modes

$$L_n' = L_n - \frac{1}{k}\sum_m : P_{n+m}P_{-m} :, \tag{4.2.2}$$

as it was similarly done for an ordinary CFT in section 3.3.3, and then calculating their norm to give

$$h \geq \frac{p^2}{k}, \quad c \geq 1. \tag{4.2.3}$$

This achieves unitarity in compatibility with hermicity. Furthermore, note that the modes (4.2.2) obey the Virasoro algebra

$$[L'_n, L'_m] = (n-m)L'_{n+m} + \frac{c-1}{12}n(n-1)(n+1)\delta_{n+m}. \tag{4.2.4}$$

We can further use (4.2.2) for $n = 0$ with $L_0 := -\frac{c}{24}$ or also the norms of the descendants of P_n and L_n to find the constraint

$$L_0 \geq \frac{P_0^2}{k} - \frac{c}{24} \tag{4.2.5}$$

necessary for unitarity. Using this in combination with $P_0^2 = k^2\alpha^2$, which we can obtain from (4.1.15) for $n = 0$ and by setting $P_0 = 0$, we find the vacuum states

$$P_0^{\alpha,vac} = k\alpha, \quad L_0^{\alpha,vac} = k\alpha^2 - \frac{c}{24}, \tag{4.2.6}$$

which also holds true in non-unitary theories, see [10] for details.

4.3 Thermodynamics of warped black holes in TMG

WCFTs also play a role in the theory of TMGs: Among the solutions of the TMG-Einstein equations (1.3.19) are also warped spacetimes with local $SL(2,R) \times U(1)$ symmetries, like Warped AdS, to which spacelike stretched black holes with $\nu > 1$ count. Globally, they are different from the global spacelike WAdS-metric

$$ds^2 = \frac{1}{\nu^2+3}[-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2], \tag{4.3.1}$$

as the isometries are broken to $U(1) \times U(1)$ by an identification, but locally they look like a spacelike stretched version of global WAdS. The metric for these warped black holes is

$$d^2s = d^2t + \frac{d^2r}{(\nu^2+3)(r-r_+)(r-r_-)} - \left(2\nu r - \sqrt{r_+r_-(\nu^2+3)}\right)dtd\phi + \tag{4.3.2}$$

$$\frac{r}{4}[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+r_-(\nu^2+3)}]d^2\phi, \tag{4.3.3}$$

where r_+ and r_- are the outer and inner Killing horizon respectively, ν is a parameter of the action (1.3.17) and the rest are coordinates. In the extremal case, $r_+ = r_- = 0$, this becomes an identification of

$$d^2s = d^2t + \frac{d^2r}{r^2(\nu^2+3)} - 2\nu r dt d\phi + \frac{3}{4}(\nu^2-1)r^2 d^2\phi, \tag{4.3.4}$$

which is Poincare spacelike WAdS, covering a patch of global WAdS.

In order to be able to derive the thermodynamics of this warped black hole, we will first need to find out what partition function is the right one to consider. We can use the coordinate identifications $(t,\phi) \sim (t+i\beta, \phi+i\beta\Omega)$, with the inverse Hawking temperature $\beta = -\frac{2\pi}{3k}(1+\frac{T_L}{T_R})$ and angular potential $\beta\Omega = \frac{1}{T_R}$ and parameters.

$$T_L = \frac{\nu^2+3}{8\pi}\left(r_+ + r_- - \frac{1}{\nu}\sqrt{r_+r_-(\nu^2+3)}\right) \tag{4.3.5}$$
$$T_R = \frac{\nu^2+3}{8\pi}(r_+ - r_-) \tag{4.3.6}$$

We can use them to make a coordinate change to Poincare WAdS (4.3.4) near the boundary

$$\phi' = -\frac{1}{2\pi T_R}e^{-2\pi T_R \phi} + \mathcal{O}(1/r^2) \tag{4.3.7}$$
$$t' = t + \frac{2}{k}\mathcal{M}\phi + \mathcal{O}(1/r) \tag{4.3.8}$$

in a compact form. With this coordinate change, the generators' transformation behaviour takes the form

$$P'(\phi) = -2\pi T_R \phi' P(\phi') - k\alpha \tag{4.3.9}$$

and

$$T'(\phi) = (2\pi T_R\phi')^2 T(\phi') - 4\pi T_R \alpha \phi' P(\phi') - k\alpha^2 - \frac{c}{6}\pi^2 T_R^2, \tag{4.3.10}$$

when choosing $\alpha = \frac{\mathcal{M}}{k}$. The zero modes then are

$$P_0 = \mathcal{M} = k\alpha, \quad L_0 = -\mathcal{L} = \frac{c}{6}\pi^2 T_R^2 + k\alpha^2. \tag{4.3.11}$$

These are the black hole's mass and angular momentum respectively that can also be calculated as described in chapter 1.3.4 using the boundary conditions with associated asymptotic symmetries as proposed in [8]. In the primed plane, the thermal identification is $(t', \phi') \sim (t' + i\beta_0, \phi')$, with $\beta_0 = -\frac{2\pi}{3k}$.
If we now also introduce an exponential map for the coordinate t similar to that of ϕ,

$$t' \to \frac{1}{2\pi T_L}e^{2\pi T_L(\frac{k}{2\mathcal{M}}t + \phi)}, \tag{4.3.12}$$

we arrive at a Minkowski vacuum in the primed plane. In order to get rid of the conserved charge contained within the coordinate transformation, we define the new deformed BTZ-coordinates

$$t_R = \phi, \quad t_L = \frac{k}{2\mathcal{M}}t + \phi, \tag{4.3.13}$$

and therefore the infitesimal charges

$$\delta Q_{\partial_L} = \frac{2\mathcal{M}}{k}\delta\mathcal{M}, \quad \delta Q_{\partial_R} = -\delta\mathcal{L} - \frac{2\mathcal{M}}{k}\delta\mathcal{M}. \tag{4.3.14}$$

The partition function would be given by

$$Z = Tr(e^{-\beta_R Q_R - \beta_L Q_L}), \tag{4.3.15}$$

as the charges just simply correspond to the conserved quantities mass and angular momentum that we usually see in it, and integrated the charges read

$$Q_L = \frac{P_0^2}{k}, \quad Q_R = L_0 - \frac{P_0^2}{k}. \tag{4.3.16}$$

We have therefore found a partition function that fits our black hole, where the inverse temperatures $\beta_{L,R}$ are given by the parameters $T_{L,R}$ we had introduced earlier.
As we have already used coordinates that can be defined on a deformed BTZ black hole (4.3.13), we will use the opportunity to make a coordinate change to the deformed BTZ metric

$$ds^2 = ds^2_{BTZ} + \frac{1}{48}(\nu^2 - 1)\xi_\mu \xi_\nu dx_b^\mu dx_b^\nu, \tag{4.3.17}$$

with the AdS_3 BTZ black hole metric with AdS-radius $l_b = \frac{2}{\sqrt{3+\nu^2}}$

$$ds^2_{BTZ} = (8M_{BTZ} - \frac{r_b^2}{l_b^2})d^2t_b + \frac{d^2 r_b}{-8M_{BTZ} + \frac{r_b^2}{l_b^2} + \frac{16J^2_{BTZ}}{r_b^2}} - 8J_{BTZ}dt_b d\phi_b + r_b^2 d\phi_b^2 \tag{4.3.18}$$

and the Killing vector responsible for the deformation $\xi = \frac{1}{\mathcal{M}}(l_b \partial_{t_b} + \partial_{\phi_b})$.
The needed coordinate change is

$$t_R = \phi = \phi_b - \frac{t_b}{l_b} \quad t_L = \phi_b + \frac{t_b}{l_b} \tag{4.3.19}$$

$$r_b^2 = 3\mathcal{M}(2r - \frac{1}{\nu}\sqrt{r_+ r_-(\nu^2 + 3)}) + 4l_b J_{BTZ} \tag{4.3.20}$$

and the conserved charges of the BTZ black hole and the warped black hole parameters are related by the expressions

$$\mathcal{M} = \frac{1}{6}\sqrt{8(M_{BTZ} - \frac{J_{BTZ}}{l_b})} \tag{4.3.21}$$

$$\mathcal{L} = -\frac{M_{BTZ}}{3\nu} - \frac{1+3\nu^2}{\nu(\nu^2+3)}\frac{J_{BTZ}}{l_b}. \tag{4.3.22}$$

To find the ground state, we try using deformed global AdS instead of (4.3.13), which can be achieved by inserting $M_{BTZ} = -\frac{1}{8}$, $J_{BTZ} = 0$ and yields

$$\mathcal{M}^{vac} = -\frac{i}{6}m \quad -\mathcal{L}^{vac} = -\frac{c}{24} - \frac{1}{36k}. \tag{4.3.23}$$

This matches with the form of the ground state defined in equation (4.2.6). Note that this is the ground state indeed as the deformed BTZ metric (4.3.17) minimizes Q_R and is smooth for our chosen values of parameters. One then finds the Wald-like entropy,

$$S = -\frac{2\pi}{3k}\mathcal{M} + 2\pi\sqrt{\frac{c}{6}(-\mathcal{L} - \frac{\mathcal{M}^2}{k})}, \tag{4.3.24}$$

see [11], which matches the entropy found in [11] by conventional means.[10] It can also be derived by the means of a WCFT using the Warped Cardy formula we will discuss in the next chapter, see 5.2.14.

4.4 WCFT and the BMS$_2$-algebra

WCFT can be connected to a special algebra first introduced by Afshar et al. in [2], which will be briefly summarized here and be of further importance in section 7.3.
For a metric of the general form

$$ds^2 = 2V(u,r)du^2 - 2dudr, \tag{4.4.1}$$

in Eddington-Finkelstein coordinates the asymptotic Killing-vectors preserving this metric are

$$\xi = \varepsilon(u)\partial_u - (\varepsilon'(u)r - \eta(u))\partial_r, \tag{4.4.2}$$

where $\eta(u)$ and $\varepsilon(u)$ are arbitary functions dependend on the retarded time u. They preserve the conditions that $\mathcal{L}_\xi g_{ur} = \mathcal{L}_\xi g_{rr} = 0$ [1], where for e.g. Rindler-type black holes one may also use the condition that additionally $\mathcal{L}_\xi g_{uu} = \delta_\xi P(u)r + \delta_\xi T(u)$, where $P(u)$ and $T(u)$ are conserved charges contained within $V(u,r)$[3]. They form the infinite-dimensional BMS$_2$-algebra with commutators [2]

$$[(\varepsilon_1, \eta_1), (\varepsilon_2, \eta_2)] = (\varepsilon_1\varepsilon_2' - \varepsilon_2\varepsilon_1', (\varepsilon_1\eta_2 - \varepsilon_2\eta_1)'). \tag{4.4.3}$$

When expanding into Laurent-modes $\varepsilon = -u^{n+1}, \eta = 0$, one obtains the Witt-algebra

$$[l_m, l_n] = (n-m)l_{n+m}, \tag{4.4.4}$$

with $l_m = \xi_m \mid_{\eta=0}$ and for $\varepsilon = 0, \eta = u^{n-1}$, the commutator vanishes and we therefore call $\chi_m = \xi_m \mid_{\varepsilon=0}$ a spin-0 supertranslation. Also, the mixed commutator of these two cases reads [2]

$$[l_m, l_n] = -(n+m)\chi_{n+m}. \tag{4.4.5}$$

Additionally, this algebra corresponds to finite coordinate transformations

$$u' = F(u), \quad r' = \frac{1}{F(u)'}(r + G'(u)), \tag{4.4.6}$$

with arbitary periodic functions $G(u)$.[15]
The conserved currents $P(u)$ and $T(u)$ then transform as

$$\delta_\xi P = \varepsilon P' + \varepsilon' P + \varepsilon'', \quad \delta_\xi T = \varepsilon T' + 2\varepsilon' T + \eta P - \eta'. \tag{4.4.7}$$

By defining modes $L_n = (-ie^{inu}, 0), J_n = (0, -ie^{inu})$, one obtains the BMS$_2$-algebra that can be centrally extended to [1]

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c}{12}n^3\delta_{n+m,0} \tag{4.4.8}$$

$$[\mathcal{L}_n, \mathcal{J}_m] = -(m+n)\mathcal{J}_{n+m} + (\lambda n - ik)\delta_{n+m,0} \tag{4.4.9}$$

$$[\mathcal{J}_n, \mathcal{J}_m] = 0. \tag{4.4.10}$$

For a WCFT with coordinates t and ϕ, after applying a coordinate change

$$t \to t + G(\phi), \quad \phi \to F(\phi), \tag{4.4.11}$$

one finds the same transformation behaviour as well as the same mode-algebra as just described.[15] Furthermore, one can use the metric (4.4.2) in its most general form with $V(u,r) = -\frac{r^2}{2} + P(u)r + T(u)$ to uplift metrics that contain a flat part but replacing said part with (4.4.2). When taking the flat limit, this reduces to $V(u,r) = P(u)r + T(u)$ [1].

In the case of a higher dimensional metric that is to be uplifted, one will have to additionally transform e.g. the azimuthal angle ϕ to [15]

$$\phi \to \phi - G(u), \tag{4.4.12}$$

which will also result in an additional Killing-vector $\xi^\phi = G(u)\partial_\phi$. When taking the retarded time u to be periodic $u \sim u + 2\pi L$, with some length scale L, one is able to obtain the mode algebra as the algebra (4.4.10) after expanding the functions $\eta(u)$ and $\varepsilon(u)$ in Fourier-modes similar to L_n and J_n instead.[3]

Chapter 5

Cardy formula

For this short section, we will start out by deriving the Cardy formula for the ordinary CFTs described in chapter 3 and afterwards derive the Warped Cardy formula for the theories described in the previous section 4.

5.1 Cardy formula for an ordinary CFT

For an ordinary CFT, we start by using a partition function

$$Z(q,\bar{q}) = Tr_{\mathcal{H}} q^{L_0} q^{\bar{L_0}}, \tag{5.1.1}$$

where $q = e^{2\pi i \tau}$. Here, $\tau = \frac{1}{2\pi}(\theta + i\beta)$ is a modular parameter, meaning that it is a parameter that can be used in order to specify the (complex) structure on our space. [20]
Using

$$H = L_0 + \bar{L_0} \tag{5.1.2}$$

$$J = L_0 - \bar{L_0}, \tag{5.1.3}$$

we can express the generators $L_0, \bar{L_0}$ in terms of the vacuum charges H and J. Therefore, the partition function becomes

$$Z = Tr_{\mathcal{H}} \exp(i\theta J - \beta H). \tag{5.1.4}$$

Now, assuming the theory is defined on a cylinder, we define a torus by identifying the thermal correlators' periodicity on the spatial and thermal circle as

$$(t,\phi) \sim (t, \phi + 2\pi) \sim (t + i\beta, \phi + \theta) \tag{5.1.5}$$

will lead to

$$Z(\tau,\bar{\tau}) = Tr_{\mathcal{H}} \exp(-2\pi J) = Z(\beta,\theta). \tag{5.1.6}$$

After making the coordinate transformation $x^{\pm} = \phi \pm it$, we find the relation

$$(t, \phi) \to (x^+, x^-) \sim (\phi + \theta + i(t+\beta), \phi + \theta - i(t+\beta)) = (x^+ + 2\pi\tau, x^- + 2\pi\bar{\tau}). \tag{5.1.7}$$

As we are in an ordinary CFT, we can use its symmetries to independently rescale the coordinates $x^\pm \to x'^\pm = \lambda^\pm x^\pm$. Therefore, (5.1.7) becomes

$$(x'^+, x'^-) \sim (x'^+ + 2\pi\lambda^+, x'^- + 2\pi\lambda^-) \sim (x'^+ + 2\pi\tau\lambda^+, x'^- + 2\pi\tau\lambda^-). \tag{5.1.8}$$

We can use this relation to exchange the spatial and thermal cycles by identifying $\beta' = -2\pi\tau i\beta\lambda^\pm$ from (5.1.7) as well as imposing $t \geq 0, \beta' > 0$, leading to the simplest choice of $\tau\lambda^\pm = -1$ and therefore

$$(x'^+, x'^-) \sim \Big(x'^+ - \frac{2\pi}{\tau}, x'^- - \frac{2\pi}{\tau}\Big) \tag{5.1.9}$$

as well as

$$Z(\tau, \bar{\tau}) = Z\Big(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\Big). \tag{5.1.10}$$

The latter equality is called the S-transformation of the theory's partition function.
Plugging $\tau' = -\frac{1}{\tau} = -\frac{2\pi(\theta - i\beta)}{\theta^2+\beta^2}$ and $\beta' = \frac{4\pi^2\beta}{\theta^2+\beta^2}$ into the partition function (5.1.4), after some simplification we find the entropy

$$S(\beta, \theta) = -\frac{8\pi^2}{\theta^2+\beta^2}(H_{vac}\beta + iJ_{vac}\theta), \tag{5.1.11}$$

where we have applied the limit $\beta \to 0$ as we wish to calculate it for the vacuum state.
If we now perform a Legendre-transformation to $S(H, J)$ and simplify again, we find the Cardy-formula for an ordinary CFT

$$S(L_0, \bar{L_0}) = 2\pi\Big(\sqrt{\frac{c}{6}L_0} + \sqrt{\frac{\bar{c}}{6}\bar{L_0}}\Big). \tag{5.1.12}$$

5.1.1 Entropy and central charge for a BTZ-Black Hole

We will briefly show an example of the application of the Cardy formula on a black hole referred to as BTZ-black hole with metric

$$ds^2 = (8GM - \frac{r^2}{l^2})dt^2 + \frac{dr^2}{(8GM - \frac{r^2}{l^2} + \frac{16G^2J^2)}{r^2}} + 8GJdtd\phi + r^2d\phi^2, \tag{5.1.13}$$

where l is the AdS-radius and J and M are the black holes' angular momentum and mass given by

$$M = Q_{\partial_t} = \frac{L_+^2 + L_-^2}{8Gl^2} \tag{5.1.14}$$

$$J = Q_{\partial_\phi} = \frac{-L_+ + L_-}{4Gl}, \tag{5.1.15}$$

where $L_\pm$ are constants. The black hole has horizons at the values

$$r_\pm = \sqrt{2Gl(lM+J)} \pm \sqrt{2Gl(lM-J)} \tag{5.1.16}$$

and its Bekenstein-Hawking entropy is given by

$$S = \frac{A}{4G} = \frac{\pi r_+}{2G}. \tag{5.1.17}$$

The central charge obtained using Brown-Hennaux boundary conditions is

$$c_+ = c_- = \frac{3l}{2G}. \tag{5.1.18}$$

By using this on the expression for the black hole's outer horizon r_+, (5.1.16), we find the entropy

$$S = \frac{2\pi}{2G}\left(\sqrt{GlL_0^+} + \sqrt{GlL_0^-}\right), \tag{5.1.19}$$

with $L_0^\pm = Q_{\partial_\pm} = \frac{lM \pm J}{2}$. We then also see that the Cardy-formula exactly reproduces the Bekenstein-Hawking entropy after insertion of (5.1.18). [11]

5.2 Cardy formula for a warped Conformal Field Theory

In order to derive the warped Cardy formula, we start out by looking at a WCFT with coordinates (t, ϕ) and thermal correlators periodic under a complex shift with $(t+i\beta, \phi+\theta)$ as well as symmetries $\varphi \to f(\phi), \quad t \to t - g(\phi)$. The shift θ is related to the shift used in section 4.3 by $\theta = i\beta\Omega$. We therefore have two conserved charges, namely the energy $P_0 = Q[\partial_t]$ and angular momentum $L_0 = Q[\partial_\phi]$. Putting this theory on a circle with $\phi \sim \phi + 2\pi$ and finite temperature as well as angular potential will result in the partition function

$$Z(\beta, \theta) = Tr(e^{-\beta P_0 + i\theta L_0}). \tag{5.2.1}$$

In order to exchange the thermal and angular cycle just as we did for an ordinary CFT in the previous section, we choose the ansatz

$$\phi' = \lambda\phi \quad t' = t - 2\gamma\phi, \tag{5.2.2}$$

which leads to the identifications

$$(t', \phi') = (t - 2\gamma\phi, \lambda\phi) \sim (t' + i\beta - 2\gamma\theta, \phi' + \lambda\theta). \tag{5.2.3}$$

As we are looking at the full circle, we set $\theta = 2\pi$, resulting in $(t', \phi') \sim (t' - 4\pi\gamma, \phi' + 2\pi\lambda)$, which can be used to identify $-4\pi\gamma = i\beta - 2\gamma\theta, \quad 2\gamma\theta = i\beta$, leading to

$$\gamma = \frac{i\beta}{2\theta}, \quad \lambda = \frac{2\pi}{\theta}. \tag{5.2.4}$$

After insertion we find

$$(t', \phi') \sim \Big(t' - 4\pi\frac{i\beta}{2\theta}, \phi' + 2\pi\frac{2\pi}{\theta}\Big) = \Big(t' - \frac{2\pi i\beta}{\theta}, \phi' + \frac{4\pi^2}{\theta}\Big) = \tag{5.2.5}$$

$$(t' + i\beta', \phi' + \theta') \tag{5.2.6}$$

and therefore the warped modular transformation

$$\beta' = -\frac{2\pi}{\theta}\beta, \quad \theta' = \frac{4\pi^2}{\theta}. \tag{5.2.7}$$

Using (4.1.18) we then find

$$P' = \frac{4\pi^2}{\theta^2}T'(\phi') + \frac{\theta' i\beta'}{4\pi^2}P'(\phi') + \frac{k\beta^2}{4\theta^2}, \tag{5.2.8}$$

and, after simplifying the middle term, the partition function

$$Z(\beta, \theta) = Tr\Big(\exp[-\beta P_0' + i\theta L_0']\Big) = Tr\Big(\exp[-\frac{2\pi\beta}{\theta}P_0 + \frac{k\beta^2}{4\theta^2}\theta i - \frac{4\pi^2 i\theta}{\theta^2}L_0]\Big). \tag{5.2.9}$$

For small θ we can replace $(L_0, P_0) \to (L_0^{vac}, P_0^{vac})$, leading to the entropy [19]

$$S = -\frac{2\pi\beta}{\theta}P_0^{vac} - \frac{8\pi^2 i}{\theta}L_0^{vac}. \tag{5.2.10}$$

Now using $\theta = i\beta\Omega$ and applying a Legendre transformation from (P_0, L_0) to (P_0^{vac}, L_0^{vac}), [12]

$$L_0 = -i\frac{\partial \ln Z}{\partial\theta} = -\frac{2\pi i\beta}{\theta^2}P_0^{vac} - \frac{k\beta^2}{4\theta^2} + \frac{4\pi^2}{\theta^2}L_0^{vac} = \tag{5.2.11}$$

$$-\frac{\beta}{4\theta^2}(8\pi i P_0^{vac} + k) + \frac{4\pi^2}{\theta^2}L_0^{vac}, \tag{5.2.12}$$

$$P_0 = -\frac{\partial \ln Z}{\partial\beta} = \frac{2\pi}{\theta}P_0^{vac} - \frac{k\beta}{2\theta}i, \tag{5.2.13}$$

we find the Cardy-formula for a WCFT

$$S = -\frac{4iP_0P_0^{vac}\pi}{k} + 4\pi\sqrt{-(L_0 - \frac{P_0^2}{k})(L_0^{vac} - \frac{P_0^{vac^2}}{k})} \tag{5.2.14}$$

after expressing β and θ in terms of $P_0, L_0, L_0^{vac}, P_0^{vac}$, which are given by (4.2.6). Plugging in for L_0^{vac} and P_0^{vac} from (4.2.6), we find that the Cardy-formula becomes

$$S = -4\pi i\alpha P_0 + 2\pi\sqrt{\frac{c}{6}\Big(L_0 - \frac{P_0^2}{k}\Big)}, \tag{5.2.15}$$

where $c \geq 1, k > 0$ and L_0 is assumed to be bounded from below. In the case of a unitary theory where P_0 is hermitian, the first term in (5.2.15) vanishes because $k\alpha \to 0$ and therefore, the entropy has no negative contributions. However, if the theory is not unitary, (5.2.15) is not an entropy as it does have a negative contribution.

Chapter 6

Kerr/CFT correspondence

This chapter serves as a small introduction to the Kerr/CFT correspondence, which finds a correspondence between gravity and quantum theory.

6.1 Central Charge for extremal Kerr

We start out with the Kerr-metric

$$d\hat{s}^2 = -\frac{\Delta}{\rho^2}(d\hat{t} - a\sin^2\theta d\hat{\phi})^2 + \frac{\sin^2\theta}{\rho^2}\big((\hat{r}^2+a^2)d\hat{\phi} - ad\hat{t}\big)^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \rho^2 d\theta^2, \tag{6.1.1}$$

where $\Delta = \hat{r}^2 - 2Mr + a^2$, $\rho^2 = \hat{r}^2 + a^2\cos^2\theta$. The metric corresponding to the region near the extreme horizon $\hat{r} = M$ can from this be obtained by defining the near-horizon coordinates

$$t = \frac{\lambda\hat{t}}{2M}, \quad y = \frac{\lambda M}{\hat{r} - M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M}, \tag{6.1.2}$$

and another change to global coordinates (r, τ, φ)

$$y = (\cos\tau\sqrt{1+r^2} + r)^{-1}, \quad t = y\sin\tau\sqrt{1+r^2}, \quad \phi = \varphi + \ln\frac{\cos\tau + r\sin\tau}{1+\sin\tau\sqrt{1+r^2}} \tag{6.1.3}$$

yields the global Near Horizon Extremal Kerr (NHEK)-metric

$$ds^2 = 2J\Omega^2\Big(-(1+r^2)d\tau^2 + \frac{dr^2}{1+r^2} + d\theta^2 + \Psi^2(d\varphi + rd\tau)^2\Big) \tag{6.1.4}$$

that covers the whole of the NHEK geometry. It contains the angular momentum $J = M^2$ as well as the functions $\Omega^2 = \frac{1+\cos^2\theta}{2}, \Psi = \frac{2\sin\theta}{1+\cos^2\theta}$.

In order to obtain the boundary charges, we next choose the boundary conditions

$$h_{\mu\nu} \sim \mathcal{O}\begin{pmatrix} r^2 & 1 & \frac{1}{r} & \frac{1}{r^2} \\ & 1 & \frac{1}{r} & \frac{1}{r} \\ & & \frac{1}{r} & \frac{1}{r^2} \\ & & & \frac{1}{r^3}, \end{pmatrix} \tag{6.1.5}$$

with $h_{\mu\nu} = g_{\mu\nu} - \bar{g}$. The asymptotic symmetry group's bracket algebra can be obtained by variation of the charges and then reads

$$\{Q_{\zeta_m}, Q_{\zeta_n}\}_{D.B.} = Q_{[\zeta_m,\zeta_n]} + \frac{1}{8\pi G}\int_{\partial\Sigma} k_{\zeta_m}[L_{\zeta_n}\hat{g}, \bar{g}] \tag{6.1.6}$$

The diffeomorphisms allowed by the boundary conditions (6.1.5) obey the algebra

$$i[\zeta_m, \zeta_n] = (m-n)\zeta_{m+n}, \tag{6.1.7}$$

the first term is

$$Q_{[\zeta_m,\zeta_n]} = (m-n)Q_{\zeta_{m+n}}. \tag{6.1.8}$$

With the integral in (6.1.6) resulting in $-i(m^3+2m)\delta_{m+n}J$ and defining

$$L_n = Q_{\zeta_n} + \frac{3J}{2}\delta_n, \tag{6.1.9}$$

we see that the left term in (6.1.6) becomes $Q_{\zeta_{m+n}} = L_{m+n} - \frac{3J}{2}\delta_{m+n}$ and therefore find

$$[L_m, L_n] = [Q_{\zeta_m}, Q_{\zeta_n}] + \frac{3\beta}{2}([Q_{\zeta_m}, \delta_n] + [\delta_m, Q_{\zeta_n}]) = \tag{6.1.10}$$

$$(n-m)L_{m+n} + (m^3 - m)J\delta_{m+n}. \tag{6.1.11}$$

This is the Virasoro algebra (3.2.6) with the central charge

$$c_L = 12J. \tag{6.1.12}$$

6.2 Temperature and entropy

Expanding the quantum field Φ in eigenmodes of the asymptotic energy ω and angular momentum m results in

$$\Phi = \Sigma_{\omega,m,l}\phi_{\omega ml}e^{-i\omega\hat{t}+im\hat{\phi}}f_l(r,\theta). \tag{6.2.1}$$

When changing the coordinates to $t = \frac{\lambda\hat{t}}{2M}$ and $\phi = \hat{\phi} - \frac{\hat{t}}{2M}$, we find that the expansion's (6.2.1) Boltzmann-weighing factor can be expressed in these coordinates as

$$e^{-\frac{\omega-\Omega_H m}{T_H}} = e^{-\frac{\omega-\frac{am}{2Mr_+}}{r_+-M}4\pi Mr_+} = e^{-\frac{2\pi}{r_+-M}(2Mr_+\omega-am)} \tag{6.2.2}$$

after defining $n_R = \frac{2M\omega-m}{\lambda}, n_L = m$. Since $\alpha \propto -\frac{2\pi\lambda r_+}{r_+-M} \Rightarrow \frac{2\pi am\lambda r_+}{r_+-M}$ gets left over as well as $\beta = \frac{2\pi(r_+-a)}{r_+-M}$, we find the right- and left-moving temperatures

$$T_1 := -\frac{1}{\alpha}, \quad T_2 := \frac{1}{\beta} \tag{6.2.3}$$

$$\lim_{extr} T_1 \to 0, \quad T_2 \to \frac{1}{2\pi}, \tag{6.2.4}$$

because $M^2 \to J$ and $r_+ = M + \sqrt{M^2 - a^2} \to M + \sqrt{M^2 - \frac{J^2}{M^2}} = M + \sqrt{M^2 - \frac{M^4}{M^2}} \to M, a \to M$.
From the Cardy-formula we know that the entropy is given by

$$S = \frac{\pi^2}{\beta} c_L T_L = \frac{\pi^2}{3} \cdot \frac{12J}{2\pi} = 2\pi J, \tag{6.2.5}$$

which is the same as the Bekenstein-Hawking entropy [19]

$$S_{BH} = \frac{area}{4} = \frac{8\pi r_+ M}{4G} \to 2\pi M^2 = 2\pi J. \tag{6.2.6}$$

We therefore see that the entropy resulting from gravitational theory (6.2.6) matches the one obtained when using quantum theory (6.2.5), therefore establishing a correspondence between the two.

Chapter 7

Kerr/dS solution

In this section, we will first very briefly review a few important aspects about the Kerr/dS solution, including its thermodynamics, and then have a short look at the special case of Nariai, where the cosmological horizon coincides with the outer horizon, $r_c = r_+$. [4] Afterwards, we will study the ultracold solution of Kerr/dS, for which all three horizons coincide.

7.1 Basic aspects

The Kerr-dS metric is [4]

$$d^2s = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a}{\Theta}\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta_r}d^2r + \frac{\rho^2}{\Delta_\theta}d^2\theta + \frac{\Delta_\theta}{\rho^2}\sin^2\theta(adt - \frac{r^2+a^2}{\Theta}d\phi)^2 \tag{7.1.1}$$

with parameters

$$\Delta_r = (r^2+a^2)\left(1 - \frac{r^2}{l^2}\right) - 2Mr, \quad \Delta_\theta = 1 + a^2\frac{\cos^2\theta}{l^2} \tag{7.1.2}$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \Theta = 1 + \frac{a^2}{l^2} \tag{7.1.3}$$

and coordinates (t, r, θ, ϕ). The horizons can be calculated from the zeros of the Killing norm Δ_r, which gives the equation $r^4 - r^2(l^2 - a^2) + 2Ml^2r - l^2a^2 = 0$. The radius' extrema can be found when holding $a = a_{max} = \frac{M}{4}\sqrt{6\sqrt{3}+9}$ fixed, which yields new equations for the extremal radius

$$2r_H + m \mp \sqrt{m^2 + 8r_Hm} - 2\frac{r_H^3}{l^2} = 0. \tag{7.1.4}$$

From this, now fixing the dS radius $l = l_{max} = \frac{M}{4}(2\sqrt{3}+3)^{3/2}$, one finds the extremal radius

$$r_{max} = (\frac{3}{4} + \frac{\sqrt{3}}{2})M, \tag{7.1.5}$$

for which all three horizons coincide $r_+ = r_- = r_c = r_{max}$[14], as it is the case for the ultracold solution [4] we will tackle later.

However, using the original equation $r^4 - r^2(l^2 - a^2) + 2Ml^2 r - l^2 a^2 = 0$ and taking $l \to \infty$ will yield the usual Kerr event horizon

$$r_K = M + \sqrt{M^2 - a^2}, \tag{7.1.6}$$

and taking $a \to 0$ instead will result in an equation that yields the Schwarzschild-event horizon r_S and the cosmological horizon r_c

$$\frac{r^3}{l^2} - r + 2M = 0. \tag{7.1.7}$$

On the other hand, for large parameters a, one obtains pure dS.

By introducing the mass parameter $m_N = \frac{l}{3\sqrt{3}}$, we can also differentiate between a number of classes of black holes:

- $m_N > M > 0$ will result in a black hole in dS spacetime,
- coinciding parameter m_N and M will result in the Nariai-solution with coinciding event and cosmological horizon and
- a naked singularity is described when $M > m_N$. [14]

Furthermore, for $g_{tt} = 0$, one finds two ergoregions, namely one associated with each of the two largest roots of $\Delta_r = 0$, the cosmological horizon r_C and the outer horizon r_H. On the horizons, the stationary Killing field $\zeta^\mu = (1, 0, 0, 0)$ corresponding to the time coordinate t is spacelike since $g_{tt} > 0$ at that location. Therefore, for some points away from the horizons, ζ^μ must be null, and, because there is a region between the outer and cosmological horizons where ζ^μ is timelike, there must be two ergospheres.[6]

7.1.1 Thermodynamics

The horizon's angular velocity can be calculated to be

$$\Omega_H = -\frac{g_{t\phi}}{g_{\phi\phi}} |_{r=r_c} = \frac{a}{r_c^2 + a^2}. \tag{7.1.8}$$

For symmetries, we have the stationary ζ^μ Killing vector we have already used to discuss the metric's ergoregions and an axial Killing vector $\psi^\mu = (0, 0, 0, 1)$ [14], where the first one is associated with the conserved charge corresponding to the mass [4]

$$\mathcal{M} = Q_{\partial_t} = -\frac{M}{\Theta^2} \tag{7.1.9}$$

and the latter to the second conserved quantity, the angular momentum

$$J = Q_{\partial_\phi} = -\frac{aM}{\Theta^2}. \tag{7.1.10}$$

The expressions for the parameters a and M appearing in them can be obtained from the black hole's horizons using the extremality conditions $\Delta_r\mid_{horizon}=0, \partial_r\Delta_r\mid_{horizon}=0$ that yield a system of equations that can be solved to find [4]

$$a^2=\frac{r_c^2(1-3\frac{r_c^2}{l^2})}{1+\frac{r_c^2}{l^2}},\quad M=\frac{r_c(1-\frac{r_c^2}{l^2})^2}{1+\frac{r_c^2}{l^2}}. \tag{7.1.11}$$

The two Killing vectors combined give a Killing vector that is normal to the horizon r_c, $\chi^\mu=\zeta^\mu+\Omega_H\psi^\mu$ and can be used to calculated the surface gravity [14]

$$\kappa=\sqrt{-\frac{1}{2}(\nabla_\mu\chi_\nu)^2}=\frac{1}{2(r_c^2+a^2)\Theta}\frac{d\Delta_r}{dr}\mid_{r=r_c}. \tag{7.1.12}$$

The Bekenstein-Hawking entropy can be easily calculated from the black hole horizon's area $A=4\pi\frac{r_c^2+a^2}{\Theta}$ to give [4]

$$S=\pi\frac{r_c^2+a^2}{\Theta}. \tag{7.1.13}$$

This satisfies the first law of black hole thermodynamics (2.2.8) with

$$\delta\mathcal{M}=T_H\delta S+\Omega_H\delta J, \tag{7.1.14}$$

with the inverse Hawking temperature [14]

$$\beta_H=\frac{1}{T_H}=\mid\frac{2\pi}{\kappa}\mid=\frac{2\pi(r_c^2+a^2)(l^2+a^2)}{\mid 2r_c^3+r_c(a^2-l^2)+Ml^2\mid}. \tag{7.1.15}$$

7.2 Nariai limit

In the Nariai limit $r_c=r_+=\frac{3M+\sqrt{9M^2-8a^2(1-\frac{a^2}{l^2})}}{2(1-\frac{a^2}{l^2})}$, the conserved charges remain unchanged, but for non-rotating Nariai $a\to 0$, the Hawking temperature will approach [14]

$$T_H=\frac{r_c^3-Ml^2}{2\pi r_c^2 l^2}, \tag{7.2.1}$$

but is otherwise obtained by using the usual relation $T_H=\frac{\kappa}{2\pi}$. [18]
In this limit, the chemical potential associated to the angular momentum, the left moving temperature T_L, defined as

$$\frac{1}{T_L}=\frac{\delta S}{\delta J}, \tag{7.2.2}$$

after inserting the expressions (7.1.11), can be found to be [4]

$$T_L=\frac{(r_c^2+a^2)}{4\pi a r_c\Theta}\frac{(6\frac{r_c^2}{l^2}+3\frac{r_c^4}{l^4}-1)}{1+\frac{r_c^2}{l^2}}. \tag{7.2.3}$$

7.2.1 Rotating Nariai

The rotating Nariai metric can now be obtained by first defining a non-extremality parameter $\tau = \frac{r_c - r_+}{r_c}$, which can be taken to be small since $r_c = r_+$ for this class of solutions. [4] Because of its coinciding horizons, the Nariai black hole is also in a state of thermal equilibrium since the two horizons have the same temperature [14], which, with the help of τ, can be approximated to

$$T_H \approx \frac{b\tau}{4\pi}. \tag{7.2.4}$$

Introducing a parameter $b = \frac{r_c(r_c - r_-)(3r_c + r_-)}{l^2(a^2 + r_c^2)}$ together with the cosmological angular velocity $\Omega_c = \Omega_H \cdot \Theta$, we can now define the near horizon coordinates

$$t = b\lambda\hat{t}, \quad r = \frac{r_c - \hat{r}}{\lambda r_c}, \quad \phi = \hat{\phi} - \Omega_c \hat{t}. \tag{7.2.5}$$

Inserting these into the metric (7.1.1) and then taking the limits $(\lambda, \tau) \to 0$ with fixed $\frac{\tau}{\lambda}, t, r, \phi$ lets us arrive at the rotating Nariai metric

$$d^2s = \Gamma(\theta)\Big(r(r-\tau)d^2t - \frac{d^2r}{r(r-\tau)} + \alpha(\theta)d^2\theta\Big) + \gamma(\theta)(d\phi + krdt)^2. \tag{7.2.6}$$

The new functions used for simplification are

$$\rho_c^2 = r_c^2 + a^2 \cos\theta^2, \quad \Gamma(\theta) = \frac{\rho_c^2 r_c}{b(a^2 + r_c^2)}, \tag{7.2.7}$$

$$\alpha(\theta) = \frac{b(r_c^2 + a^2)}{r_c \Delta_\theta}, \quad \gamma(\theta) = \frac{\Delta_\theta (r_c^2 + a^2)^2 \sin\theta^2}{\rho_c^2 \Theta^2}, \tag{7.2.8}$$

$$k = -\frac{2ar_c^2\Theta}{b(a^2 + r_c^2)^2}. \tag{7.2.9}$$

An additional coordinate change

$$r \to \frac{\tau}{2}(r+1), \quad t \to \frac{2}{\tau}t, \quad \phi \to \phi - kt \tag{7.2.10}$$

will lead to $dt^2 \frac{2}{\tau} - (r+1)(\frac{\tau}{2}(r+1) - \tau) \widehat{=} dt^2(r^2 - 1)$ (and analoguely for the other coordinates) and therefore the rotating Nariai metric in static coordinates with isometry group $U(1) \times SL(2, R)$[4]

$$ds^2 = \Gamma(\theta)\Big(-(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + \alpha(\theta)d\theta^2\Big) + \gamma(\theta)(d\phi + krdt)^2. \tag{7.2.11}$$

The corresponding generators are

$$
\begin{aligned}
K_0 &= \partial_t, \\
\overline{K_0} &= \partial_\phi, \\
K_1 &= \frac{r\sinh t}{\sqrt{1-r^2}}\partial_t + \cosh t\sqrt{1-r^2}\partial_r - \frac{k\sinh t}{\sqrt{1-r^2}}\partial_\phi, \\
\overline{K_1} &= \frac{r\cosh t}{\sqrt{1-r^2}}\partial_t + \sinh t\sqrt{1-r^2}\partial_r - \frac{k\cosh t}{\sqrt{1-r^2}}\partial_\phi.
\end{aligned} \tag{7.2.12}
$$

7.2.2 Boundary conditions, asymptotic symmetries and charges

The boundary conditions chosen are analogous to the ones used for NHEK in chapter 6 and are

$$
h_{\mu\nu} \sim \mathcal{O}\begin{pmatrix} r^2 & 1 & \frac{1}{r} & \frac{1}{r^2} \\ & 1 & \frac{1}{r} & \frac{1}{r} \\ & & \frac{1}{r} & \frac{1}{r^2} \\ & & & \frac{1}{r^3}, \end{pmatrix} \tag{7.2.13}
$$

with the additional condition $Q_{\partial_t} = 0$ to ensure finite charges. These allow for the left-moving diffeomorphisms

$$
\zeta_\varepsilon = \varepsilon(\phi)\partial_\phi - r\varepsilon'(\phi)\partial_r, \quad \bar{\zeta} = \partial_t \tag{7.2.14}
$$

that generate a copy of the Witt algebra

$$
i[\zeta_n, \zeta_m] = (n-m)\zeta_{n+m} \tag{7.2.15}
$$

after expanding $\varepsilon_n = -e^{-in\phi}$. Using these, the conserved charges

$$
Q_\zeta(\mathcal{L}_{\zeta_m} g, g) = \int k_{\zeta_m}[\mathcal{L}_{\zeta_m} g, g] \tag{7.2.16}
$$

can again be found using the Barnich-Brandt surface charge (1.3.15) and the algebra of asymptotic symmetries is again given by (6.1.6). The Lie derivatives of the metric (7.2.11) are

$$
\mathcal{L}_\zeta g_{tt} = -2ie^{-im\phi} m r^2 \Big(\Gamma(\theta) + k^2\cdot\gamma(\theta)\Big) \tag{7.2.17}
$$

$$
\mathcal{L}_\zeta g_{rr} = -\frac{2ie^{-im\phi} m\Gamma(\theta)}{(1-r^2)} \cdot \Big(\frac{r^2}{(1-r^2)} + 1\Big) \tag{7.2.18}
$$

$$
\mathcal{L}_\zeta g_{\phi r} = -\frac{e^{-im\phi} m^2 r\Gamma(\theta)}{(1-r^2)} \tag{7.2.19}
$$

$$
\mathcal{L}_\zeta g_{\phi\phi} = 2ie^{-im\phi} m\gamma(\theta). \tag{7.2.20}
$$

Using (6.1.9) to obtain the modes L_m after calculating the charges' algebra and comparing to the Virasoro algebra (3.2.6) will result in the central charge

$$c = 3 \mid k \mid \int_0^{\pi} d\theta \sqrt{\Gamma(\theta)\alpha(\theta)\gamma(\theta)} = \frac{12ar_c^2}{b(a^2 + r_c^2)}. \tag{7.2.21}$$

After using the cardy formula and plugging in for the parameters b, a(7.1.11) and T_L(7.2.3), we see that the entropy is the same as the Bekenstein-Hawking entropy we had calculated before in equation (7.1.13).[4]

7.3 Ultracold solution

This section concerns itself solely with the ultracold solution, an extremal limit of the Kerr/dS-metric, for which $r_+ = r_- = r_c$. Note that for this solution, the left moving temperature (7.2.3) will vanish. Additionally, it belongs to the class of so called "cold" solutions, whose horizons have the same temperature and additionally, some of them coincide, as was also the case with the Nariai limit discussed in the last section 7.2. [7]
We can obtain from the Nariai metric in static coordinates (7.2.11) by rescaling the coordinates in order to avoid difficulties with the parameter b now vanishing: $r = \tilde{r}\sqrt{b}, t = \tilde{t}\sqrt{b}$. The new coordinates can then be expressed as

$$\tilde{r} = \frac{r_c - \hat{r}}{(a^2 + r_c^2)^2}, \quad \tilde{t} = \frac{b}{\sqrt{b}}\lambda\hat{t}. \tag{7.3.1}$$

Therefore, some of the functions (7.2.9) will also be rescaled:

$$\tilde{k} = b \cdot k = -\frac{2ar_c^2\Theta}{(a^2 + r_c^2)^2}, \quad \tilde{\Gamma}(\theta) = b \cdot \Gamma(\theta). \tag{7.3.2}$$

The first term in (7.2.11) will then be $\frac{\Gamma(\tilde{\theta})}{b}\Big(- (1 - \tilde{r}^2 b)d^2\tilde{t}b + \frac{d^2\tilde{r}b}{1-\tilde{r}^2 b} + \alpha(\theta)d^2\theta\Big)$, giving us

$$\tilde{\alpha} = \frac{\alpha}{b} \tag{7.3.3}$$

as well as, when disregarding terms of $\mathcal{O}(b)$ and after inserting for the other parts of (7.2.11), the ultracold solution's metric [4]

$$ds^2 = \tilde{\Gamma}(\theta)\Big(- d\tilde{t}^2 + d\tilde{r}^2 + \tilde{\alpha}(\theta)d\theta^2\Big) + \gamma(\theta)(d\phi + \tilde{k}\tilde{r}d\tilde{t})^2. \tag{7.3.4}$$

7.3.1 Thermodynamics

From the metric (7.3.4) we can again calculate the Bekenstein-Hawking entropy

$$S = \frac{(r_c^2 + a^2)\pi}{\Theta}. \tag{7.3.5}$$

The horizon's angular velocity is

$$\Omega_C = -2\tilde{k}\tilde{r}\mid_{r_c} \tag{7.3.6}$$

and with the Killing vector $\xi = \frac{1}{\Theta}(\partial_{\tilde{t}} - 2\tilde{k}\tilde{r}\partial_{\varphi})$, the surface gravity can be calculated to be

$$\kappa = \frac{1}{\Theta}(\tilde{k} + \tilde{r}d\tilde{k})\mid_{r_C}. \tag{7.3.7}$$

The Hawking temperature will vanish at the special value of the horizon (7.3.8), as described in the following subsection 7.3.2.

7.3.2 Equations of motions

The ultracold solution (7.3.4) fulfills Einstein's equations of motion with a positive cosmological constant $\Lambda = 3/l^2$. We should mention that since we took the limit $b \to 0$, for a fixed to what was given in (7.1.11), this is only true for

$$r_C = l\sqrt{\frac{-3 + 2 \cdot \sqrt{3}}{3}}. \tag{7.3.8}$$

This is a saddle point of the Killing-metric, which therefore makes it a horizon. This also results in the surface gravity (7.3.7) becoming

$$\kappa = \frac{1}{\Theta}\tilde{k}. \tag{7.3.9}$$

However, changing a also changes the value of r_C, so the metric (7.3.4) does in fact solve Einstein's equations for any value of r_C.

7.3.3 Central charge from the Witt generators

Since they have worked in determining the boundary charges for NHEK 6.2.6 and Nariai 7.2 before, we attempt using the generators (7.2.14). One obtains the Lie-derivatives

$$\mathcal{L}_\zeta g_{tt} = -2ie^{-im\phi} m r^2 \gamma(\theta) \tilde{k}^2 \tag{7.3.10}$$
$$\mathcal{L}_\zeta g_{rr} = -2ie^{-im\phi} m \tilde{\Gamma}(\theta) \tag{7.3.11}$$
$$\mathcal{L}_\zeta g_{\phi r} = -e^{-im\phi} m^2 r \tilde{\Gamma}(\theta) \tag{7.3.12}$$
$$\mathcal{L}_\zeta g_{\phi\phi} = 2ie^{-im\phi} m \gamma(\theta). \tag{7.3.13}$$

The central charge obtained from utilizing the variation of the conserved charges (1.3.8) and extracting the anomolous term would then turn out to be

$$c = \frac{12 a r_C^2}{a^2 + r_C^2} r^2 + \mathcal{O}(1/r) \tag{7.3.14}$$

and therefore diverge at the boundary, suggesting that we need a different set of boundary conditions. Finding this set will be the topic in the next subsection 7.3.4.

7.3.4 New boundary conditions from uplifting the metric

7.3.4.1 Uplifting the metric

In order to be able to find adequate new boundary conditions for the metric (7.3.4) in the hope of getting a finite central extension, we use a similar approach to the one Godet and Marteau used for the NHEK geometry in [15]. The basic procedure was described in setion 4.4. We are therefore looking for diffeomorphisms corresponding to a coordinate change of the form [15]

$$u \to F(u), \quad r \to \frac{1}{F'(u)}\Big(r + G'(u)\Big), \quad \phi \to \phi - G(u) \tag{7.3.15}$$

where $G(u)$ is some periodic function dependend on the retarded time u and $F(u)$ is a reparametrization of the circle.

As a first step, we do a coordinate change to Eddington-Finkelstein coordinates $u = t - r$, after which the metric (7.3.4) reads

$$ds^2 = \tilde{\Gamma}(\theta)(du^2 - 2dudr + \tilde{\alpha(\theta)}d\theta^2) + \gamma(\theta)(d\phi + \tilde{k}rdu)^2. \tag{7.3.16}$$

Now we replace the flat part $du^2 - 2dudr$ with $2\Big(P(u)r + T(u)\Big)du^2 - 2dudr$ [2] to receive the uplifted ultracold metric

$$ds^2 = \tilde{\Gamma}(\theta)\Big(2(P(u)r + T(u))du^2 - 2dudr + \tilde{\alpha(\theta)}d\theta^2\Big) + \gamma(\theta)(d\phi + \tilde{k}rdu)^2. \tag{7.3.17}$$

From (7.3.17), we can now calculate the diffeomorphisms conserving this metric by solving the asymptotic Killing equation $\mathcal{L}_\xi g_{\mu\nu} = \mathcal{O}(\delta g_{\mu\nu})$ under the conditions $\mathcal{L}_\xi g_{ur} = \mathcal{L}_\xi g_{rr} = 0$ and $\mathcal{L}_\xi g_{uu} = 2\tilde{\Gamma}(\theta)\delta_\xi P(u)r + \delta_\xi T(u)$, with the charges' variations given as in (4.4.7). From the ur-component, one obtains $\partial_u\xi^u = -\partial_r\xi^r$ and the $u\phi$ and uu-components together give $\xi^r = -(\partial_r g_{u\phi})^{-1}(\partial_u\xi^\phi + \partial_u\xi^u) - (\partial_r g_{uu})^{-1}(2g_{uu}\partial_u\xi^u + 2\xi^u\partial_u g_{uu} + 2\partial_u\xi^u) + (\partial_r g_{uu})^{-1}(2\tilde{\Gamma(\theta)}\delta_\xi P(u)r + \delta_\xi T(u))$. We therefore find

$$\xi^u = \varepsilon(u)\partial_u, \quad \xi^r = -(r\varepsilon'(u) - \eta'(u))\partial_r + \mathcal{O}(\frac{1}{r}), \quad \xi^\phi = -\eta(u)\partial_\phi \tag{7.3.18}$$

with some functions $\varepsilon(u)$ and $\eta(u)$. Since we found diffeomorphisms (7.3.18) corresponding to a WCFT-algebra, we can take u to be periodic[3, 15]

$$u \sim u + 2\pi L, \tag{7.3.19}$$

with some length scale L. This can be understood in terms of a WCFT, where the (boundary) time coordinate u is exchanged with the azimuthal angle ϕ, see section 4.4. We can therefore also take the functions $\varepsilon(u)$ and $\eta(u)$ to be periodic in u. Expanding either of these two functions in Fourier modes will lead to the same result as expanding in Laurent modes and taking u to be complex instead. [3] As we can see, these diffeomorphisms share similarities with (4.4.2) and (7.3.15). After defining a set of generators

$$l_m = \xi\mid_{\varepsilon(u)=-e^{-inu},\eta(u)=0}, \quad p_m = \xi\mid_{\varepsilon(u)=0,\eta(u)=-e^{-inu}}, \tag{7.3.20}$$

we find that their bracket-algebra is

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [l_m, p_n] = -np_{m+n}, \quad [p_n, p_m] = 0. \tag{7.3.21}$$

Therefore, for $\varepsilon = 0$, one finds that a spin-0 supertranslation, for $\eta = 0$ one finds the Witt-algebra. We also see that our algebra admits to the warped Witt-algebra.
Furthermore, the diffeomorphisms (7.3.18) also show similarities to the ones preserving the quasi-Rindler boundary conditions found in [3], that is dual to a warped CFT[3], namely (4.4.2) with the additional Killing-vector as described in section 4.4. The difference is that the algebra one obtains in two dimensions turns out to be the BMS_2-algebra instead. It is however interesting to note that we do obtain a similar substructure in four dimensions.

The metric's Lie-derivatives are given by

$$\mathcal{L}_\xi g_{uu} = -2\varepsilon(u)mri\tilde{k}\gamma(\theta) + 2i\varepsilon(u)m(1-r)(\tilde{\Gamma}(\theta)P(u) + \tilde{k}^2\gamma(\theta)r) + 2i\varepsilon(u)m \\ (r^2\tilde{k}^2\gamma(\theta) + 2\tilde{\Gamma}(\theta)(rP(u)+T(u))) - 2\tilde{\Gamma}(\theta)(-\varepsilon(u)m^2r + \eta''(u)) - 2\tilde{\Gamma}(\theta)(rP(u)+T(u)) \quad (7.3.22)$$

and

$$\mathcal{L}_\xi g_{u\phi} = \varepsilon(u)\gamma(\theta)\Big(im + \tilde{k}r(1-im) + \tilde{k}m\Big) \quad (7.3.23)$$

$$. \quad (7.3.24)$$

7.3.4.2 Conserved charges and central charge

We find finite charges

$$Q_{UC} = -\int_0^{2\pi L} du \left(\frac{(a^2+r_c^2)\eta(u)P(u)}{4\Theta\pi} + \frac{ar_c^2T(u)\varepsilon(u)}{2\pi(a^2+r_c^2)}\right) \quad (7.3.25)$$

by integration over a constant r, ϕ surface. The functions $P(u)$ and $T(u)$ used for the uplift (7.3.17) can now also be understood as the conserved currents of this spacetime.[15].
The central charge can now be calculated by comparing the charge algebra of the charges (7.3.25) with the generator algebra (7.3.21) and finding possible anomalous terms. For that, we define

$$L_m = Q_{UC}\,|_{\eta=0}, \qquad P_m = Q_{UC}\,|_{\varepsilon=0} \quad (7.3.26)$$

in analogy to the generators (7.3.20).The charge algebra then exactly coincides with the generator algebra (7.3.21) and therefore the central charge vanishes.

Conclusion and Outlook

In conclusion, starting from the Nariai metric in static coordinates (7.2.11), we rederive the ultracold limit (7.3.4) of the Kerr/dS-metric by taking the parameter b appearing in (7.2.11) to 0, after reparametrization of time and radial coordinates and associated functions. This is the extremal limit in which all three horizons of the Kerr/dS-solution coincide and we check that the equations of motions are fulfilled for any a and horizon r_C.
Using the value (7.1.11) found for the parameter a for the Kerr/dS-solution yields a special value for the single ultracold horizon

$$r_{UC} = l\sqrt{\frac{-3+2\cdot\sqrt{3}}{3}} \approx 0.39. \tag{7.3.27}$$

Furthermore, implementing the boundary conditions (7.2.14) used for the Nariai-case to obtain the central charge, leads to a divergence of the central extension quadratic in the radial coordinate r. This is why we changed our strategy to uplifting the metric after a coordinate change to Eddington-Finkelstein coordinates, as was done by Godet and Marteau for the NHEK geometry in [15], and calculate appropriate boundary conditions and associated charges from there. We find the diffeomorphisms (7.3.18) that are similar to the ones found for the flat two dimensional counterpart in [3], with an additional periodic term in the radial component and an additional angular component. Their algebra turns out to be the warped Witt-algebra

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [l_m, p_n] = -np_{m+n}, \quad [p_n, p_m] = 0 \tag{7.3.28}$$

with a similar substructure as was found in two dimensions albeit not the same overall structure.
Using these boundary conditions, we find finite, non-trivial, well defined charges

$$Q_{UC} = -\int_0^{2\pi L} du \left(\frac{(a^2+r_c^2)\eta(u)P(u)}{4\Theta\pi} + \frac{ar_c^2 T(u)\varepsilon(u)}{2\pi(a^2+r_c^2)}\right), \tag{7.3.29}$$

containing the finite conserved currents $T(u)$ and $P(u)$ that were used for uplifting the metric as well as periodic functions appearing in the generators, $\varepsilon(u)$ and $\eta(u)$. We also find that charge and generator algebra exactly match, meaning that the central charge vanishes and the charges therefore admit to a centerless WCFT-algebra.
The next step would have been to see whether it would be possible to find a matching type of Cardy formula, which would be complicated since usually that would require a non-vanishing central

charge. If it is possible, however, it would be interesting to see whether that generates a matching entropy to (7.3.5).

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