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Multidimensionale Stochastische Integration

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Univ.-Prof. Dipl.-Math. Dr. rer. nat. habil. Uwe Schmock

Durch

Felix Franz Sadowski, BSc
Matrikelnummer 11806458

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Under the supervision of

Univ.-Prof. Dipl.-Math. Dr. rer. nat. habil. Uwe Schmock

By

Felix Franz Sadowski, BSc
Matriculation number 11806458

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(Signature Author) (Signature Supervisor)

Zusammenfassung

Die vorliegende Diplomarbeit zielt darauf ab, das stochastische Integral bezüglich mehrdimensionaler stetiger lokaler Martingale und in weiterer Folge stetiger Semimartingale einzuführen und dieses zu diskutieren. Da in der aktuariellen Praxis Vermögenswerte oft als stetige Semimartingale modelliert werden, ist ein möglichst allgemeiner Integralbegriff sehr wichtig, da das stochastische Integral genau den Gewinn oder Verlust einer Handelsstrategie bezüglich dieses Vermögenswertes darstellt. Des Weiteren werden viele Eigenschaften des bekannten Lebesgue-Stieltjes-Integrals auch für das neu eingeführt stochastische Integral nachgewiesen.

Kapitel 1 gibt eine kurze Übersicht über die stochastische Integration bezüglich eindimensionaler stetiger lokaler Martingale oder Semimartingale, gemäß [Sch23, Chapter 5]. Das darauffolgende Kapitel 2 basiert zum Teil auf [SC02, Chapter 3] und [CE15, Section 12.5] und liefert zuallererst für jedes d-dimensionale stetige lokale Martingal eine Darstellung seines Kovariationsprozesses als pfadweises Lebesgue-Stieltjes-Integral eines vorhersehbaren matrixwertigen Prozesses bezüglich der Spur des Kovariationsprozesses. Diese Darstellung wird verwendet, um für jedes $p \geq 1$ den normierten Vektorraum $L^p(M)$ zu definieren.

Darauf aufbauend wird in Kapitel 3, Abschnitt 3.1, das stochastische Integral bezüglich eines multidimensionalen stetigen lokalen Martingals definiert. Im Anschluss werden einige Eigenschaften, wie zum Beispiel die Linearität im Integranden sowie im Integrator, des soeben definierten stochastischen Integrals behauptet und gezeigt. Danach, in Abschnitt 3.2, wird das stochastische Integral bezüglich multidimensionaler adaptierter und stetiger Prozesse von lokalendlicher Variation definiert. Dies hat die Definition des stochastischen Integrals bezüglich d-dimensionaler stetiger Semimartingale zur Folge.

Um die vorhin erwähnte Darstellung von $[M]$ als pfadweises Lebesgue-Stieltjes-Integral in Theorem 2.7 konstruieren zu können, wird eine Verallgemeinerung des bekannten Satzes von Radon-Nikodým benötigt. Dieser Satz sowie die besagte Verallgemeinerung werden in Kapitel 4 bewiesen. Während der gesamten Diplomarbeit werden viele mehr oder weniger bekannte Resultate aus den unterschiedlichsten Teilbereichen der Mathematik verwendet, die mehrheitlich im Appendix (Kapitel 5) gesammelt zu finden sind.

Interessierte Lesende sind gerne eingeladen, sich einige Eigenschaften des stochastischen Integrals bezüglich eindimensionaler stetiger lokaler Martingale oder stetiger Semimartingale in Erinnerung zu rufen und zu versuchen, diese auf den in dieser Arbeit eingeführten Integralbegriff zu verallgemeinern. Alternativ könnten auch nicht notwendigerweise stetige Integratoren betrachtet werden. Die vorliegende Diplomarbeit stellt nicht den Anspruch dieses spannende Thema im Bereich der Stochastischen Analysis vollständig beleuchtet zu haben, sondern soll eine Einführung darstellen, auf der aufgebaut werden kann.

Stichwörter: Stochastische Integration, stetiges lokales Martingal, stetiges Semimartingale, Übergangs- oder Markovkern, Satz von Radon-Nikodým, signiertes or komplexes Maß auf einem δ-Ring.

Abstract

The thesis at hand aims to introduce and discuss stochastic integration with respect to multi-dimensional continuous local martingales and even continuous semimartingales. As assets are often modeled as continuous semimartingales and the stochastic integral corresponds to the profit or loss of a trading strategy w.r.t. this asset, this is a very relevant topic in the professional life of many practical mathematicians. Furthermore, many useful properties of the well-known Lebesgue–Stieltjes integral are being extended to our stochastic integral processes.

Chapter 1 provides a quick overview of stochastic integration with respect to continuous semimartingales, introduced in [Sch23, Chapter 5]. The following Chapter 2 relies in parts on [SC02, Chapter 3] and [CE15, Section 12.5] and first and foremost provides the covariation process of each d-dimensional continuous local martingale M with a representation as a pathwise Lebesgue–Stieltjes integral of a predictable matrix-valued process w.r.t. the trace of the covariation process. This representation will then be used to introduce the normed vector spaces $L^p(M)$ for $p \geq 1$.

Building on those findings, in Chapter 3 (Section 3.1) the stochastic integral w.r.t. M will be defined. Afterwards, some properties of this newly introduced stochastic integral, for example linearity in the integrand as well as the integrator, will be stated and proven. Furthermore, in Section 3.2, the stochastic integral w.r.t. multi-dimensional adapted and continuous process of locally finite variation is being defined as a pathwise Lebesgue–Stieljes integral. Those results then lead to the definition of the stochastic integral w.r.t. $\mathbb{K}^d$-valued continuous semimartingale $X = A + M$, being composed of a process of locally finite variation and a continuous local martingale in Section 3.4.

In order to construct the predictable integrand in the aforementioned representation of $[M]$ as a pathwise Lebesgue–Stieltjes integral in Theorem 2.7, an extension of the famous Radon–Nikodým theorem will be used. To prove the Radon–Nikodým theorem as well as many different generalizations of it is the duty of Chapter 4. Throughout this thesis, a lot of more or less well-known results of many different fields of mathematics are being used, which can be found in the appendix, Chapter 5.

Any reader interested in the topic of this thesis is welcome to consider properties of the stochastic integral w.r.t. one-dimensional continuous local martingales or semimartingales and try to generalize them for the stochastic integral introduced in this thesis. Another possible extension of this work would be to consider not necessarily continuous integrators. The thesis at hand is by no means to be considered as a conclusion, but much more an introduction to this fascinating topic in the field of Stochastic Analysis.

Keywords: Stochastic integration, continuous local martingale, continuous semimartingale, transition kernel, Radon–Nikodým theorem, signed or complex measure on a δ-ring.

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Last but certainly not least, a big thanks belongs to my family and close friends, who always supported me in stressful time and provided me with much needed distractions and balance.

Thank you all!

Felix Sadowski

Statement of originality

I hereby declare that I have authored the present master thesis independently and did not use any sources other than those specified. I have not yet submitted the work to any other examining authority in the same or comparable form. It has not been published yet.

Hornstein, 21.09.2023

Felix Sadowski

Contents

1 Introduction and the one-dimensional case

1.1 Introduction and motivation

As insurance companies hold quite large amounts of capital, investing this capital in different assets is a big part of the day-to-day work of many actuaries. Those assets are often modeled via continuous semimartingales and the profit or loss of a trading strategies can then be denoted by a stochastic integral of the trading strategy with respect to this continuous semimartingale. As those assets may often times be highly correlated, a multi-dimensional model can provide better results when considering a portfolio of assets instead of simply looking at each one separately. For instance, exchange-traded funds, when related to some stock index, for example ATX or DAX, are a separate asset consisting of a linear combination of other stocks in this index. Consequently, the exchange-traded fund and this linear combination are almost perfectly correlated. Intuitively, the stochastic integral of a d-dimensional integrable trading strategy w.r.t. a d-dimensional continuous semimartingale could then be defined by taking the sum of the componentwise stochastic integrals, see for example [Sch23, Definition 5.109(d)]. The following two examples, which can be found in [CE15, p. 284f], show, however that this definition –while easy to introduce – has some shortcomings. The example below corresponds to a trading strategy, which for example shorts an exchange-traded fund and holds a long position of the linear combination of the stocks in this exchange-traded fund.

Example 1.1. Take any $\mathbb{K}$-valued continuous semimartingale X and a $\mathbb{K}$-valued process H, which is integrable w.r.t. X. Intuitively, the profit or loss of the two-dimensional trading strategy $(H, H)^\top$ w.r.t. the two-dimensional continuous semimartingale $(X, -X)^\top$ is zero, as

$$\int_0^{\cdot} (H_t, H_t)^\top \,\mathrm{d}(X_t, -X_t)^\top = \int_0^{\cdot} H_t \,\mathrm{d}X_t - \int_0^{\cdot} H_t \,\mathrm{d}X_t = 0.$$

One would expect the same result for a trading strategy H, which is not integrable w.r.t. X. In this definition of the multi-dimensional stochastic integral, such processes cannot be considered and would not yield a result.

This thesis, however, introduces another definition of the stochastic integral with respect to multi-dimensional continuous semimartingales, where the integral in the display above is well-defined and equal to zero, even if $H \notin L(X)$, as Example 3.30 below shows. In order to provide another, much less trivial, example, the following metric on the space $\mathcal{S}$ of all $\mathbb{K}$-valued semimartingales will be introduced.

Definition 1.2 (Émery distance)**.** The *Émery distance* of two $\mathbb{K}$-valued continuous semimartingales X and Y is defined as

$$\rho(X,Y) = \sup_{\substack{H\in\mathcal{P}^1\\ \|H\|_\infty\le 1}} \left\{ \sum_{n=1}^{\infty} 2^{-n}\,\mathbb{E}\Big[\sup_{t\in[0,n]}\Big|\int_0^t H_s\,\mathrm{d}(X_s-Y_s)\Big|\wedge 1\Big]\right\},$$

where P^1 denotes the space of all one-dimensional predictable processes according to Definition 2.1 below and $\|\cdot\|_\infty$ denotes the uniform norm on the space of all bounded $\mathbb{K}$-valued processes, i.e.

$$\|H\|_\infty = \sup\{|H_t(\omega)| : (t,\omega)\in\mathbb{R}_+\times\Omega\}.$$

This metric was defined by Michel Émery in [Éme06, p. 264ff]. Furthermore, [Éme06, Théorème 1] states that $\mathcal{S}$ is complete w.r.t. ρ, when identifying processes that are up to indistinguishability equal. In the following example assume $\mathbb{K}=\mathbb{R}$.

Example 1.3 (Failing completeness w.r.t. the Émery distance)**.** For two independent real-valued standard Brownian motions B^1 and B^2 define the deterministic process $H_t = t$ for $t\in\mathbb{R}_+$ and the $\mathbb{R}$-valued continuous local martingales $X^1 = B^1$ and $X^2 = (1-H)\bullet B^1 + H\bullet B^2$. Note at this point that the processes X^1 and X^2 in this example are continuous local martingales. Furthermore, define the vector space

$$\mathcal{L}(X^1,X^2) = \{K^1\bullet X^1 + K^2\bullet X^2 : K^1\in L(X^1),\ K^2\in L(X^2)\}. \tag{1.1}$$

For more information about the stochastic integral w.r.t. one-dimensional continuous local martingales, denoted by the operator $\bullet$ in the display above, as well as the vector space of integrable process w.r.t. X^1 and X^2, i.e. $L(X^1)$ and $L(X^2)$ respectively, see [Sch23, Section 5.7]. Throughout this example, the integrands are real-valued, deterministic and continuously differentiable on $(0,\infty)$. Consequently, the stochastic integrals below can be calculated with the help of [Sch23, Corollary 5.62] and [Sch23, Example 5.63]. Keep in mind that for each $x\in\mathbb{R}_+$ follows $|1-x|^2\le 1\vee x^2$. Now consider for each $\epsilon>0$ and $t\in\mathbb{R}_+$ the pathwise integrals

$$\int_0^t \left|1-\frac{1}{H_s(\omega)+\epsilon}\right|^2 \mathrm{d}[X^1]_s(\omega) \le \int_0^t \max\left(1,\frac{1}{(s+\epsilon)^2}\right)\mathrm{d}s \le \int_0^t \max\left(1,\frac{1}{\epsilon^2}\right)\mathrm{d}s$$
$$= t\max\left(1,\frac{1}{\epsilon^2}\right) < \infty$$

and

$$\int_0^t \left|\frac{1}{H_s(\omega)+\epsilon}\right|^2 \mathrm{d}[X^2]_s(\omega) = \int_0^t \left|\frac{1}{s+\epsilon}\right|^2 \mathrm{d}[X^2]_s(\omega) \le \int_0^t \frac{1}{\epsilon^2}\,\mathrm{d}[X^2]_s(\omega) = \frac{[X^2]_t(\omega)}{\epsilon^2} < \infty.$$

Consequently, $1-(H+\epsilon)^{-1}\in L(X^1)$ as well as $(H+\epsilon)^{-1}\in L(X^2)$, whereby the sum of two stochastic integrals

$$
\begin{aligned}
Y^\epsilon &:= (1-(H+\epsilon)^{-1}) \bullet X^1 + (H+\epsilon)^{-1} \bullet X^2 \\
&= B^1 - \frac{1}{H+\epsilon} \bullet B^1 + \frac{1-H}{H+\epsilon} \bullet B^1 + \frac{H}{H+\epsilon} \bullet B^2 \\
&= \left(\frac{H+\epsilon}{H+\epsilon} - \frac{1}{H+\epsilon} + \frac{1-H}{H+\epsilon}\right) \bullet B^1 + \frac{H+\epsilon-\epsilon}{H+\epsilon} \bullet B^2 \\
&= \frac{\epsilon}{H+\epsilon} \bullet B^1 - \frac{\epsilon}{H+\epsilon} \bullet B^2 + B^2 \\
&= \frac{\epsilon}{H+\epsilon} \bullet (B^1 - B^2) + B^2
\end{aligned}
$$

is well-defined. Note that in the calculations above linearity of the stochastic integral in the integrand as well as the integrator and the chain rule for stochastic integrals have been applied. This in turn leads to

$$
\begin{aligned}
[Y^\epsilon - B^2]_t = \left[\frac{\epsilon}{H+\epsilon} \bullet (B^1 - B^2)\right]_t &= \int_0^t \left(\frac{\epsilon}{H_s+\epsilon}\right)^2 \mathrm{d}[B^1 - B^2]_s = 2\int_0^t \left(\frac{\epsilon}{s+\epsilon}\right)^2 \mathrm{d}s \\
= 2\int_\epsilon^{t+\epsilon} \frac{\epsilon^2}{u^2}\,\mathrm{d}u &= 2\left(-\frac{\epsilon^2}{t+\epsilon} + \frac{\epsilon^2}{\epsilon}\right) = 2\frac{\epsilon(t+\epsilon)-\epsilon^2}{t+\epsilon} = \frac{2\epsilon t}{t+\epsilon},
\end{aligned}
$$

which converges for each $t \in (0,\infty)$ to zero for $\epsilon \searrow 0$. The sequence

$$
\frac{\frac{2t}{n}}{t+\frac{1}{n}}, \qquad n \in \mathbb{N},
$$

converges even uniformly to zero, as for fixed $\epsilon > 0$ it follows that

$$
\frac{\frac{2t}{n}}{t+\frac{1}{n}} \leq \frac{\frac{2t}{n}}{t} = \frac{2}{n} \leq \epsilon, \qquad t \in (0,\infty),
$$

for each natural number $n \geq \frac{2}{\epsilon}$. Consequently, $(Y^{1/n})_{n\in\mathbb{N}}$ converges in $\mathcal{H}_0^2$ to B^2, see Definition 3.4 and Lemma 3.7 below, because for each $\epsilon > 0$ one may define

$$
n_\epsilon = \inf\{n \in \mathbb{N} : n \geq 2\epsilon^{-2}\},
$$

which results in

$$
\|Y^{1/n} - B^2\|_{\mathcal{H}_0^2} = \mathbb{E}\big[[Y^{1/n} - B^2]_\infty\big]^{1/2} \leq \mathbb{E}\big[[2/n]_\infty\big]^{1/2} = \sqrt{\frac{2}{n}} \leq \epsilon, \qquad n \geq n_\epsilon.
$$

Therefore, [Éme06, Theoreme 2(b)] implies the convergence of $(Y^{1/n})_{n\in\mathbb{N}}$ to B^2 in $\mathcal{S}$.

However, it will now be shown that $B^2 \notin \mathcal{L}(X^1, X^2)$. For proof by contradiction assume the existence of processes $K \in L(X^1)$ and $\tilde{K} \in L(X^2)$, such that

$$
\begin{aligned}
B^2 &= K \bullet X^1 + \tilde{K} \bullet X^2 = K \bullet B^1 + \tilde{K} \bullet \big((1-H) \bullet B^1 + H \bullet B^2\big) \\
&= \big(K + \tilde{K}(1-H)\big) \bullet B^1 + (\tilde{K}H) \bullet B^2.
\end{aligned}
$$

This then implies

$$\begin{aligned} t = [B^2]_t &= \big[\big(K + \tilde{K}(1-H)\big) \bullet B^1 + (\tilde{K}H) \bullet B^2, B^2\big]_t \\ &= \big[\big(K + \tilde{K}(1-H)\big) \bullet B^1, B^2\big]_t + \big[(\tilde{K}H) \bullet B^2, B^2\big]_t \\ &= \int_0^t K_s + \tilde{K}_s(1-H_s)\,\mathrm{d}[B^1,B^2]_s + \int_0^t \tilde{K}_s H_s\,\mathrm{d}[B^2]_s = \int_0^t \tilde{K}_s\, s\,\mathrm{d}s \end{aligned}$$

up to indistinguishability for each $t \in (0,\infty)$. Consequently, there exists a $(\lambda \otimes \mathbb{P})$-null set in $N \subset \mathbb{R}_+ \times \Omega$ such that $\tilde{K}_t(\omega) = 1/t$ for each pair $(t,\omega) \in N^{\mathrm{c}}$, where λ denotes the Lebesgue-measure on $\mathbb{R}_+$. As the example at hand should only be an easy demonstration of the usefulness of the theory of stochastic integration that will be introduced and discussed throughout this thesis, and the following is pretty standard procedure in the field of stochastic analysis, the next few sentences will only outline the proof of the statement above. In the later parts of this thesis, such proofs will be provided in much more detail. Fix such an ω that the display above holds for each $t \in \mathbb{R}_+$. Then the functions $s \mapsto \int_0^s 1\,\mathrm{d}u = s$ and $s \mapsto \int_0^s \tilde{K}_u(\omega)\,u\,\mathrm{d}u$ induce a induce a finite and a signed measure on $[0,n]$ for each $n \in \mathbb{N}$, respectively. By the display above, those measures agree on the set $\{[a,b) : a,b \in [0,n] \text{ with } a \leq b\}$, which is intersection stable and generates the Borel-σ-algebra $\mathcal{B}_{[0,n]}$. As furthermore $\int_0^n 1\,\mathrm{d}s = n = \int_0^n \tilde{K}_s(\omega)\,s\,\mathrm{d}s$, Lemma 5.14 in the appendix implies $\int_A 1\,\mathrm{d}s = \int_A \tilde{K}_s(\omega)\,s\,\mathrm{d}s$ for each $A \in \mathcal{B}_{[0,n]}$ for fixed $t \in \mathbb{R}_+$. Consequently, as the constant function 1 as well as $\tilde{K}_s(\omega)\,s$ are $\mathcal{B}_{[0,n]}$-measurable for each $n \in \mathbb{N}$, follows $1 = \tilde{K}_s(\omega)\,s$ or equivalently $\tilde{K}_s(\omega) = 1/s$ on $[0,n]$ outside of a set N_n satisfying $\lambda(N_n) = 0$ for each $n \in \mathbb{N}$. Therefore, $\tilde{K}_s(\omega)$ induces a finite measure (and not only a signed measure) on $\mathcal{B}_{[0,n]}$ for each $n \in \mathbb{N}$ and as such a σ-finite measure $\mathcal{B}_{\mathbb{R}_+}$. Consequently, $\tilde{K}_s(\omega) = 1/s$ for each $s \in \mathbb{R}_+$ outside of the λ-null set $\bigcup_{n\in\mathbb{N}} N_n$ and $\mathbb{P}$-almost all $\omega \in \Omega$.

Almost analogously one obtains up to indistinguishability

$$\begin{aligned} 0 = [B^1,B^2]_t &= \big[B^1, \big(K + \tilde{K}(1-H)\big) \bullet B^1 + (\tilde{K}H) \bullet B^2\big]_t \\ &= \big[B^1, \big(K + \tilde{K}(1-H)\big) \bullet B^1\big]_t + \big[B^1, (\tilde{K}H) \bullet B^2\big]_t \\ &= \int_0^t K_s + \tilde{K}_s(1-H_s)\,\mathrm{d}[B^1]_s + \int_0^t \tilde{K}_s H_s\,\mathrm{d}[B^1,B^2]_s \\ &= \int_0^t K_s\,\mathrm{d}s + \int_0^t \tilde{K}_s(1-s)\,\mathrm{d}s, \end{aligned}$$

for each $t \in (0,\infty)$ and in turn for each $(t,\omega) \in \mathbb{R}_+ \times \Omega$ outside of a $(\lambda \times \mathbb{P})$-null set

$$K_t(\omega) = -\tilde{K}_t(\omega)(1-t) = -\frac{1-t}{t} = 1 - \frac{1}{t}.$$

However, $1 - t^{-1} \notin L(B^1)$, because

$$\begin{aligned} &\int_0^1 \Big(1 - \frac{1}{s}\Big)^2 \mathrm{d}[B^1]_s = \int_0^1 \Big(1 - \frac{2}{s} + \frac{1}{s^2}\Big)\,\mathrm{d}s = \lim_{s\searrow 0}\Big((1-0) - 2\big(\ln(1) - \ln(s)\big) - (1 - 1/s)\Big) \\ &= \lim_{s\searrow 0}\big(2 + 2\ln(s) + 1/s\big) = \infty. \end{aligned}$$

Thus $(Y^{1/n})_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}(X^1, X^2)$, which has been defined in display (1.1), whose limit w.r.t. ρ, i.e. B^2, is not in $\mathcal{L}(X^1, X^2)$, which is a rather unpleasing result. Speaking in financial terms, it is not possible to hedge options, which are only depending on B^2 w.r.t. the financial market (X^1, X^2), even though $Y^{1/n}$ can be hedged for each $n \in \mathbb{N}$.

Consequently, one would like to consider another possibility of extending the idea of stochastic integration w.r.t. local martingales or even semimartingales to multi-dimensional processes. To avoid such negative examples as given above, the integral should provide the possibility for positive and negative terms in the integrand to cancel out. Furthermore, one would like for the vector space $\{H \bullet X : H \in L(X)\}$ to be complete w.r.t. the Émery distance for each semimartingale X. This thesis will provide such a notion of stochastic integration, which will be stated for $\mathbb{R}^d$-valued processes H and X in Theorem 3.32. However, it only discusses the theory for continuous integrators.

1.2 Underlying assumptions

Throughout this thesis, unless stated otherwise, the underlying filtered probability space will be denoted by $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, where the filtration $\mathbb{F}$ is right-continuous, i.e. $\mathcal{F}_t = \bigcap_{u\in(t,\infty)} \mathcal{F}_u$ for all $t \in \mathbb{R}_+$. Furthermore, it is assumed that all $\mathbb{P}$-null sets of $\mathcal{F}_\infty := \sigma\big(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}_t\big)$ are already elements of $\mathcal{F}_0$. Note that by [Sch23, Remark 5.55] one can always add all null sets of $\mathcal{F}_\infty$ to a given filtration and this enlargement inherits the right-continuity of $\mathbb{F}$ and does not change the martingale or independence properties of given processes. Finally, unless specified otherwise, all equalities and inequalities between stochastic processes are understood to hold up to indistinguishability.

For two stochastic processes X and Y the integral process of X w.r.t. Y, if it exists, may be denoted by

$$X \bullet Y = \int_0^\cdot X_s \,\mathrm{d}Y_s,$$

or more precisely for $(t, \omega) \in \mathbb{R}_+ \times \Omega$

$$(X \bullet Y)_t(\omega) = \int_0^t X_s(\omega) \,\mathrm{d}Y_s(\omega).$$

This stochastic integral may not be calculated for each $\omega \in \Omega$ but can always be considered as a stochastic process apart from a $\mathbb{P}$-null set. Depending on the nature of Y, this more easily readable convention will be used to represent stochastic integrals as well as pathwise Lebesgue–Stieltjes ones.

1.3 Stochastic integrals w.r.t. one-dimensional continuous semimartingales

Although this thesis focuses on the multivariate case, we firstly consider a $\mathbb{K}$-valued continuous local martingale M, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. As this introduction should not distract from the main points of the thesis, proofs are omitted in this section and the reader is referred to [Sch23, Chapter 5] for said proofs as well as a more detailed understanding of the one-dimensional case.

Definition 1.4 ($L(M)$)**.** A progressive process $V : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{K}$ is integrable with respect to a $\mathbb{K}$-valued continuous local martingale M, i.e. $V \in L(M)$, if and only if

$$\int_0^t |V_s|^2 \, \mathrm{d}[M]_s < \infty, \qquad t \in \mathbb{R}_+,$$

where $[M]$ denotes the *covariation process* of M defined below. The set $L(M)$ is a vector space.

The integral in the definition above is a pathwise Lebesgue–Stieltjes integral, as the covariation process of any continuous local martingale is of locally finite variation. As the covariation process is an integral building block in stochastic integration, a proper definition as well as some basic properties will be given in the following.

Definition 1.5 (Covariation process)**.** For two continuous local martingales M and N taking values in $\mathbb{K}^m$ and $\mathbb{K}^n$, respectively, the covariation process of M and N (i.e. $[M, N]$) is defined as a process of locally finite variation, for which $[M, N]_0 = 0$ holds and

$$MN^T - [M, N]$$

is again a continuous local martingale. Considering just one continuous local martingale M, the covariation process of M is defined as

$$[M] := [M, \overline{M}].$$

Theorem 1.6. *For any two continuous local martingales M and N the covariation process exists uniquely, is $\mathbb{K}$-bilinear and compatible with stopping for any $\mathbb{F}$-stopping time, where all equalities are to be understood up to indistinguishability.*

Those properties will be useful throughout this thesis, one instance being the following abstract definition of the stochastic integral with respect to continuous local martingales.

Definition 1.7 (One-dim. stochastic integral for continuous local martingales)**.** Let M be a $\mathbb{K}$-valued continuous local martingale and $V \in L(M)$. Then there exists an up to indistinguishability unique $\mathbb{K}$-valued continuous local martingale, denoted by $V \bullet M$, such that $(V \bullet M)_0 = 0$ as well as

$$[V \bullet M, N] = \int_0^{\cdot} V_s \, \mathrm{d}[M, N]_s$$

holds for every $\mathbb{K}$-valued continuous local martingale N. Then $V \bullet M$ is said to be the stochastic integral of V with respect to M and can also be denoted by

$$\int_0^\cdot V_s \, \mathrm{d}M_s.$$

This thesis will consider not only continuous local martingales, but an even bigger set of processes, namely *continuous semimartingales*.

Definition 1.8 (Continuous semimartingales)**.** Let X be a $\mathbb{K}^d$-valued process. If X can be represented as the sum $X = A + M$ consisting of an adapted continuous process A of locally finite variation starting at 0 and a continuous local martingale M, then X is called a *continuous semimartingale* and $A + M$ its *canonical decomposition*.

Note that the canonical decomposition of a continuous semimartingale is unique up to indistinguishability. In the definition above, A as well as M need to match the dimensions of X, in order for the sum to be well defined and the equality to hold. For a $\mathbb{K}$-valued adapted and continuous process of locally finite variation A starting at 0 the $\mathbb{K}$-vector space $\tilde{L}(A)$ is defined as the set of all progressive processes V, such that the integral process $\int_0^\cdot V_s \, \mathrm{d}A_s$ exists as a pathwise Lebesgue–Stieltjes integral. Keeping that in mind, the following definition comes quite naturally.

Definition 1.9 (One-dim. stochastic integral for continuous semimartingales)**.** Let X be a $\mathbb{K}$-valued continuous semimartingale, $X = A + M$ its canonical decomposition and the progressive process $V \in \tilde{L}(X) := \tilde{L}(A) \cap L(M)$, therefore the integrals below are well defined. The *stochastic integral process* of V w.r.t. X is then defined as

$$V \bullet X = \int_0^\cdot V_s \, \mathrm{d}X_s = \int_0^\cdot V_s \, \mathrm{d}A_s + \int_0^\cdot V_s \, \mathrm{d}M_s.$$

This definition implies that the process $V \bullet X$ is a $\mathbb{K}$-valued continuous semimartingale, too, with canonical decomposition $V \bullet X = \int_0^\cdot V_s \, \mathrm{d}A_s + \int_0^\cdot V_s \, \mathrm{d}M_s$, as long as $V \in \tilde{L}(X)$.

2 Integrable processes w.r.t. multi-dimensional continuous local martingales

After those preliminary statements, the main topic, the multi-dimensional case, will now begin and proofs as well as more details on the treated subjects will be provided. The following approach to defining the stochastic integral process w.r.t. multi-dimensional continuous local martingales is quite different to the one-dimensional one. However, it will be shown that those two approaches are still equivalent for $\mathbb{K}$-valued continuous local martingales and in both senses integrable processes. Most of the following two chapters rely on [CE15, Section 12.5] and [SC02, Chapter 3].

2.1 Preliminary definitions

Definition 2.1 (Predictable σ-algebra and predictable processes)**.** On the space $\mathbb{R}_+ \times \Omega$ the predictable σ-algebra Σ_p is defined as the sub-σ-algebra of $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ generated by the set of all adapted and left-continuous processes. A stochastic process is then said to be predictable, if it is measurable w.r.t. Σ_p. Set $\mathcal{P}^d$ to be the vector space of all $\mathbb{K}^d$-valued predictable processes.

Definition 2.2 (Predictable step processes)**.** For every $\overline{\mathbb{R}}_+$-valued pointwise increasing finite sequence of stopping times $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_{m+1}$ one may define a *predictable step process* by

$$H_t = \varphi_0 \mathbb{1}_{\{0\}}(t) + \sum_{n=1}^{m} \varphi_n \mathbb{1}_{(\tau_n, \tau_{n+1}]}(t), \qquad t \in \mathbb{R}_+,$$

where φ_0 is a bounded $\mathcal{F}_0$-measurable and each φ_n is a bounded $\mathcal{F}_{\tau_n}$-measurable $\mathbb{K}^d$-valued random vector for $n = 1, \dots, m$. Clearly H is left-continuous and its adaptedness will be shown in the lemma below, which then proves predictability. Note also that for each pair $(t, \omega) \in \mathbb{R}_+ \times \Omega$ at most one term in the sum on the right-hand side adds a value not equal to zero.

Lemma 2.3. *Every predictable step process H according to Definition 2.2 is adapted.*

Proof. Note at first that for each $t \in \mathbb{R}_+$ and stopping time τ the set $\{\omega \in \Omega : \tau(\omega) < t\}$ $(=: \{\tau < t\})$ is an element of $\mathcal{F}_t$. This can be seen by taking a pointwise strictly increasing sequence of positive real numbers $(t_n)_{n \in \mathbb{N}}$ converging to t and observing that

$$\{\tau < t\} = \bigcup_{n \in \mathbb{N}} \{\tau \leq t_n\} \in \mathcal{F}_t,$$

where for each $n \in \mathbb{N}$ the set $\{\tau \le t_n\} \in \mathcal{F}_{t_n} \subseteq \mathcal{F}_t$. Obviously this also implies $\{\tau < t\}^\mathsf{c} = \{\tau \ge t\} \in \mathcal{F}_t$. Similarly, one can consider for each $F \in \mathcal{F}_\tau$ the set

$$F \cap \{\tau < t\} = F \cap \Big(\bigcup_{n\in\mathbb{N}} \{\tau \le t_n\}\Big) = \bigcup_{n\in\mathbb{N}} \big(F \cap \{\tau \le t_n\}\big) \in \mathcal{F}_t,$$

as $F \cap \{\tau \le t_n\} \in \mathcal{F}_{t_n} \subseteq \mathcal{F}_t$ for each $n \in \mathbb{N}$.

Note at this point that

$$H_0^{-1}(A) = \varphi_0^{-1}(A) \in \mathcal{F}_0, \qquad A \in \mathcal{B}_{\mathbb{K}^d}.$$

Now define $\psi : \Omega \to \mathbb{K}^d$ to be the constant zero function and fix $A \in \mathcal{B}_{\mathbb{K}^d}$ and $t \in \mathbb{R}_+ \setminus \{0\}$. Then, due to the $\mathcal{F}_{\tau_n}$-measurability of φ_n, the set $\varphi_n^{-1}(A)$ belongs to $\mathcal{F}_{\tau_n}$ for each $n = 1, \dots, m$ and thus

$$H_t^{-1}(A) = \Big(\underbrace{\psi^{-1}(A)}_{\in\{\varnothing,\Omega\}\subseteq\mathcal{F}_t} \cap \big(\underbrace{\{\tau_1 \ge t\}}_{\in\mathcal{F}_t}\big) \cup \underbrace{\{\tau_{m+1} < t\}}_{\in\mathcal{F}_t}\big)\Big) \cup \Big(\bigcup_{n=1}^{m} \big(\underbrace{\varphi_n^{-1}(A) \cap \{\tau_n < t\}}_{\in\mathcal{F}_t} \cap \underbrace{\{\tau_{n+1} \ge t\}}_{\in\mathcal{F}_t}\big)\Big)$$

is an element of $\mathcal{F}_t$, which proves the adaptedness of H. □

Definition 2.4 (Processes of locally finite variation)**.** Let $\mathcal{V}^d$ denote the set of all $\mathbb{K}^d$-valued continuous adapted processes A that are of locally finite variation, meaning that for almost all $\omega \in \Omega$ the total variation of the continuous function $A_\cdot(\omega) : \mathbb{R}_+ \ni s \mapsto A_s(\omega) \in \mathbb{K}^d$ is finite on the interval $[0, t]$ for all $t \in \mathbb{R}_+$. Furthermore, define $\mathcal{V}_0^d = \{A \in \mathcal{V}^d : A_0 = 0\}$.

Note that some authors omit the word *locally* in the definition above and call such processes *finite variation processes*. Additionally, the set $\mathcal{V}^d$ is a vector space by [Sch23, Remark 5.47]. Consequently, the same holds for $\mathcal{V}_0^d$.

Definition 2.5 (The set $\mathcal{V}_0^+$)**.** Throughout this thesis, let $\mathcal{V}_0^+$ denote the set of all adapted, continuous, real valued and non-decreasing processes starting at 0.

Keep in mind that, because those processes are non-decreasing, their total variation coincides with the process itself and it is therefore also of locally finite variation.

Definition 2.6 (Positive semidefinite processes)**.** The term *positive semidefinite process* will in the following refer to $\mathbb{K}^{d\times d}$-valued processes, whose realizations at any given time t are positive semidefinite Hermitian matrices, i.e. $\pi_t(\omega)^\mathsf{H} = \bar{\pi}_t(\omega)^\mathsf{T} = \pi_t(\omega)$ and $\langle x, \pi_t(\omega)x\rangle \ge 0$ for all $x \in \mathbb{K}^d$, $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian form on $\mathbb{K}^d$ (which is linear in the first and semilinear in the second argument), namely

$$\langle x, y\rangle := y^\mathsf{H} x, \qquad x, y \in \mathbb{K}^d.$$

Note that for each $x \in \mathbb{K}^d$ one can always also consider the complex conjugate vector $\bar{x}$ and thus obtain

$$0 \le \langle \bar{x}, \pi_t(\omega)\bar{x}\rangle = \bar{x}^\mathsf{H}\pi_t^\mathsf{H}(\omega)\bar{x} = x^\mathsf{T}\pi_t(\omega)\bar{x}, \qquad x \in \mathbb{K}^d,$$

which implies that $\pi_t(\omega)$ is a positive semidefinite Hermitian matrix, if and only if $\pi_t(\omega)^\mathsf{H} = \pi_t(\omega)$ and $x^\mathsf{T}\pi_t(\omega)\bar{x} \ge 0$ for all $x \in \mathbb{K}^d$.

2.2 Integral representation of the covariation process of continuous local martingales

This section is devoted to the proof of the theorem below, which is one of the most essential theorems of this thesis.

Theorem 2.7. *For every $\mathbb{K}^d$-valued continuous local martingale $M = (M^1, \ldots, M^d)^\intercal$ one can define the process $C^{(M)} = \operatorname{tr}([M]) = \sum_{j=1}^{d}[M^j] \in \mathcal{V}_0^+$, which will most of the time simply be denoted by C, if the underlying continuous local martingale is obvious. Furthermore, there exists a $\mathbb{K}^{d\times d}$-valued predictable positive semidefinite process $\pi^{(M)}$, or simply π for readability, such that up to indistinguishability*

$$[M^i, \overline{M}^j] = \pi^{ij} \bullet C, \qquad (i,j) \in \{1,\ldots,d\}^2 \tag{2.1}$$

or equivalently, when viewed as a matrix-equality, $[M] = \pi \bullet C$. This process π is unique apart from some subset of $\mathbb{R}_+ \times \Omega$ of measure 0 with regards to $C \otimes \mathbb{P}$, where C denotes with a slight abuse of notation also the σ-finite transition kernel induced by the process C.

Proof. As $[M^j]$ is $\mathbb{R}_+$-valued, adapted, continuous, non-decreasing and starting at zero for each $j = 1, \ldots, d$, the same holds for C causing $C \in \mathcal{V}_0^+$. For each pair $(i,j) \in \{1,\ldots,d\}^2$ the covariation process $[M^j, \overline{M}^j]$ is a $\mathbb{K}$-valued, continuous and adapted process of locally finite variation, i.e. an element of $\mathcal{V}_0^1$. According to Lemma 5.28 in the appendix, $[M^j, \overline{M}^j]$ may be seen as a signed or complex transition kernel from Ω to $\mathbb{R}_+$ on the δ-ring $\mathcal{R} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]}$. Consequently, Definition 5.25 and Lemma 5.26 result in the signed or complex measure

$$([M^j, \overline{M}^j] \otimes \mathbb{P})(A) := \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)[M^j, \overline{M}^j](\mathrm{d}s, \omega) \Big) \mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{R} \otimes \mathcal{F},$$

on the product δ-ring $\mathcal{R} \otimes \mathcal{F}$ for each $(i,j) \in \{1,\ldots,d\}^2$. Similarly, the total variation process $\mathbb{V}_{[M^i,\overline{M}^j]}$ and C may be viewed as σ-finite transition kernels from Ω to $\mathbb{R}_+$, due to Lemma 5.27, and the functions

$$(\mathbb{V}_{[M^i,\overline{M}^j]} \otimes \mathbb{P})(A) := \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\mathbb{V}_{[M^i,\overline{M}^j]}(\mathrm{d}s, \omega) \Big) \mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},$$

as well as

$$(C \otimes \mathbb{P})(A) := \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)C(\mathrm{d}s, \omega) \Big) \mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},$$

are two measures on $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ by Lemma 5.24 for each pair $(i,j) \in \{1,\ldots,d\}^2$. Due to Lemma 5.30 in the appendix one can see that outside of a $\mathbb{P}$-null set holds

$$\mathbb{V}_{[M^i,\overline{M}^j]}(A,\omega) \le \sqrt{[M^i](A,\omega)}\sqrt{[\overline{M}^j](A,\omega)} \le \sqrt{C(A,\omega)}\sqrt{C(A,\omega)} = C(A,\omega), \qquad A \in \mathbb{B}_{\mathbb{R}_+}.$$

Consider now a set $A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ satisfying $(C \otimes \mathbb{P})(A) = 0$. Thus one may use Lemma 4.14 below to obtain for each predictable set A the result

$$\begin{aligned}
|[M^i, \overline{M}^j] \otimes \mathbb{P}|(A) &\le 2\,(\mathbb{V}_{[M^i,\overline{M}^j]} \otimes \mathbb{P})(A) = 2 \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\mathbb{V}_{[M^i,\overline{M}^j]}(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&\le 2 \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)C(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) = 2\,(C \otimes \mathbb{P})(A) = 0,
\end{aligned}$$

and thus

$$[M^i, \overline{M}^j] \otimes \mathbb{P} \ll C \otimes \mathbb{P}$$

on $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ and consequently also on the sub-σ-algebra Σ_p for each pair $(i,j) \in \{1,\dots,d\}^2$. Consequently, Theorem 4.15 below states the existence of a $(C \otimes \mathbb{P})$-almost everywhere unique predictable process f^{ij} satisfying

$$[M^i, \overline{M}^j] = \int_0^{\cdot} f^{ij}\,\mathrm{d}C$$

up to indistinguishability.

Intuitively define the $\mathbb{K}^{d\times d}$-valued stochastic matrix $f = (f^{ij})_{(i,j)\in\{1,\dots,d\}^2}$. Therefore, one obtains up to indistinguishability

$$f^{ij} = \frac{\mathrm{d}[M^i, \overline{M}^j]}{\mathrm{d}C} = \frac{\mathrm{d}[\overline{M}^j, M^i]}{\mathrm{d}C} = \frac{\mathrm{d}\overline{[M^j, \overline{M}^i]}}{\mathrm{d}C} = \frac{\overline{\mathrm{d}[M^j, \overline{M}^i]}}{\mathrm{d}C} = \bar{f}^{ji} \qquad i \leq j,$$

i.e. $f = f^{\mathsf{H}}$.

Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a countable dense subset in $\mathbb{K}^d$ and $D_k := \{(t,\omega) \in \mathbb{R}_+ \times \Omega : \lambda_k^{\mathsf{T}} f(t,\omega)\bar{\lambda}_k \geq 0\}$, which inherits the predictability of f. When taking the intersection of those sets, it follows by the continuity of the vector multiplication that

$$\bigcap_{k\in\mathbb{N}} D_k = \{(t,\omega) : \lambda^{\mathsf{T}} f(t,\omega)\bar{\lambda} \geq 0 \quad \forall \lambda \in \mathbb{K}^d\} =: D.$$

As $\lambda_k^{\mathsf{T}} M$ is again a $\mathbb{K}$-valued local martingale, the inequality

$$0 \leq [\lambda_k^{\mathsf{T}} M] = \left[\sum_{i=1}^d \lambda^i M^i, \sum_{j=1}^d \overline{M}^j \bar{\lambda}_k^j\right] = \sum_{i,j=1}^d \lambda_k^i [M^i, \overline{M}^j]\bar{\lambda}_k^j = (\lambda_k^{\mathsf{T}} f \bar{\lambda}_k) \bullet C$$

follows and implies that the complement of each of the sets D_k, and therefore also $D^{\mathsf{c}} = \bigcup_{k\in\mathbb{N}} D_k^{\mathsf{c}}$, are $C(\cdot,\omega)$-evanescent for $\mathbb{P}$-almost all $\omega \in \Omega$. Therefore one may define $\pi = f\mathbb{1}_D$. □

This theorem showed that the covariation process of a continuous local martingale M can be represented as a matrix-valued stochastic integral and in the following the setting as well as the symbols of Theorem 2.7 will remain the same, i.e. M denoting a $\mathbb{K}^d$-valued continuous local martingale with $[M] = \pi \bullet C$ up to indistinguishability.

Example 2.8. As in Example 1.3, let B^1 and B^2 be two independent, real valued standard Brownian motion and define $X^1 = B^1$ as well as $X^2 = (1 - H) \bullet B^1 + H \bullet B^2$, where $H_t = t$ for each $t \in \mathbb{R}_+$. Therefore one can consider Theorem 2.7 for the $\mathbb{R}^2$-valued continuous local martingale $M = (X^1, X^2)^{\mathsf{T}}$. Fix $t \in \mathbb{R}_+$ and note that the equalities in this example are understood to hold up to indistinguishability. Consequently, [Sch23, 5.102] combined with [Sch23, Example 5.70] lead to

$$\begin{aligned} C_t &= [X^1]_t + [X^2]_t = [B^1]_t + [(1-H)\bullet B^1 + H\bullet B^2]_t \\ &= t + [(1-H)\bullet B^1]_t + 2\,[(1-H)\bullet B^1, H\bullet B^2]_t + [H\bullet B^2]_t \\ &= t + \int_0^t (1-s)^2\,\mathrm{d}s + 2\int_0^t (1-s)s\;\mathrm{d}\underbrace{[B^1,B^2]}_{=0} + \int_0^t s^2\,\mathrm{d}s \\ &= t + (t - t^2 + \frac{t^3}{3}) + \frac{t^3}{3} = \frac{2t^3}{3} - t^2 + 2t. \end{aligned}$$

for each $t \in \mathbb{R}_+$. Furthermore, one may use Lemma 4.12(v) below in the fourth step to obtain

$$\pi_t^{11} = \frac{\mathrm{d}[X^1]_t}{\mathrm{d}C_t} = \frac{\mathrm{d}[B^1]_t}{\mathrm{d}C_t} = \frac{\mathrm{d}t}{\mathrm{d}C_t} = \left(\frac{\mathrm{d}C_t}{\mathrm{d}t}\right)^{-1} = \frac{1}{2t^2-2t+2}, = \frac{1}{2}\frac{1}{t^2-t+1} \qquad t \in \mathbb{R}_+.$$

Note at this point that due to

$$t_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}$$

the polynomial $t^2 - t + 1$ has no real roots and thus π_t^{11} is well defined for each $t \in \mathbb{R}_+$. Additionally,

$$[X^1, X^2]_t = [B^1, (1-H)\bullet B^1 + H\bullet B^2]_t = \int_0^t (1-s)\,\mathrm{d}s = t - \frac{t^2}{2}$$

for each $t \in \mathbb{R}_+$ implies

$$\pi_t^{12} = \pi_t^{21} = \frac{\mathrm{d}[X^1,X^2]_t}{\mathrm{d}C_t} = \frac{\mathrm{d}(t - t^2/2)}{\mathrm{d}C_t} = \frac{\mathrm{d}t}{\mathrm{d}C_t} - \frac{1}{2}\frac{\mathrm{d}t^2}{\mathrm{d}C_t} = \pi_t^{11} - \frac{1}{2}\underbrace{\frac{\mathrm{d}t^2}{\mathrm{d}t}}_{=2t}\frac{\mathrm{d}t}{\mathrm{d}C_t} = (1-t)\,\pi_t^{11},$$

where parts (i) and (i) of Lemma 4.12 have been used. Finally,

$$\pi_t^{22} = \frac{\mathrm{d}[X^2]_t}{\mathrm{d}C_t} = \frac{\mathrm{d}\,(2t^3/3 - t^2 + t)}{\mathrm{d}C_t} = \frac{\mathrm{d}\,(2t^3/3 - t^2 + 2t)}{\mathrm{d}C_t} - \frac{\mathrm{d}t}{\mathrm{d}C_t} = 1 - \pi_t^{11}, \qquad t \in \mathbb{R}_+.$$

As stated in Theorem 2.7, π_t is a positive semidefinite Hermitian matrix for each $t \in \mathbb{R}_+$. It will now be checked that this holds also in the simple example at hand. At first note that $\pi_t^{\mathsf{H}} = \pi$ is apparent. Furthermore, $\pi_t^{11} = (2t^2 - 2t + 2)^{-1} > 0$ for each $t \in \mathbb{R}_+$. Thus one may now calculate

$$\begin{aligned} \det(\pi_t) &= \pi_t^{11}\pi_t^{22} - \pi_t^{12}\pi_t^{21} = \pi_t^{11}(1-\pi_t^{11}) - \big((1-t)\pi_t^{11}\big)^2 \\ &= \frac{1}{2t^2-2t+2} - \frac{1}{(2t^2-2t+2)^2} - \frac{(1-t)^2}{(2t^2-2t+2)^2} \\ &= \frac{2t^2-2t+2-1-1+2t-t^2}{(2t^2-2t+2)^2} = \frac{t^2}{(2t^2-2t+2)^2} > 0 \end{aligned}$$

for each $t \in (0,\infty)$. Consequently, π_t is positive definite for each $t \in (0,\infty)$, due to *Sylvester's criterion* or in German better known as *Hauptminorenkriterium*, see [Hav12, Satz 9.10.13].

2.3 Definition of the normed vector spaces $L^p(M)$

The existence and uniqueness of the aforementioned matrix-valued integral representation of the covariation process for each $\mathbb{K}^d$-valued continuous local martingale allows the definition of the following normed vector spaces.

Definition 2.9 (The norms $\|\cdot\|_{L^p(M)}$ and the spaces $L^p(M)$)**.** Let M be a $\mathbb{K}^d$-valued continuous local martingale, whose covariation process can be represented as $[M] = \pi \bullet C$ according to Theorem 2.7, H a $\mathbb{K}^d$-valued predictable process, see Definition 2.1, and $p \in [1, \infty)$. One can then define the function

$$\|H\|_{L^p(M)} = \mathbb{E}\big[\big((H^\mathsf{T}\pi\overline{H}) \bullet C\big)_\infty^{p/2}\big]^{1/p},$$

and the corresponding set $L^p(M) := \{H \in \mathcal{P}^d : \|H\|_{L^p(M)} < \infty\}$. Two processes $H, H' \in L^p(M)$ are said to be equivalent if and only if $\|H - H'\|_{L^p(M)} = 0$ and from now on the equivalence class of a process and the process itself are to be thought of as the same. For more details, one may turn to [Sch23, Remark 13.5]. The fact that $\|\cdot\|_{L^p(M)} : L^p(M) \to \mathbb{R}_+$ is a norm and $L^p(M)$ is a vector space will be shown in Lemma 2.10 below.

At this point it is useful to be reminded that, due to the positive semidefiniteness of π, the integral process $(H^\mathsf{T}\pi\overline{H}) \bullet C$ is non-decreasing and $\overline{\mathbb{R}}_+$-valued. Consequently, for each $H \in L^p(M)$ and $t \in \mathbb{R}_+$ the integral $\big((H^\mathsf{T}\pi\overline{H}) \bullet C\big)_t$ is $\mathbb{P}$-almost surely finite for all $t \in \overline{\mathbb{R}}_+$. Furthermore, for all $H \in \mathcal{P}^d$ the process $H^\mathsf{T}\pi\overline{H}$ is also predictable and thus progressive. Consequently, as C is continuous for all continuous local martingales M, the integral process $(H^\mathsf{T}\pi\overline{H}) \bullet C$ is $\mathbb{P}$-almost surely well-defined, continuous and adapted per [Sch23, Lemma 5.49(c)] and thus predictable for all $H \in L^p(M)$.

At first glance it might not be obvious that the function $\|\cdot\|_{L^p(M)}$ indeed yields a norm on $L^p(M)$, so it will be proven in the following.

Lemma 2.10. *For $p \in [1, \infty)$ and a $\mathbb{K}^d$-valued continuous local martingale M the function $\|\cdot\|_{L^p(M)} : L^p(M) \to \mathbb{R}_+$ is a norm on the vector space $L^p(M)$.*

Proof. Due to the positive semidefiniteness of π, one may view $\|\cdot\|_{L^p(M)} : L^p(M) \to \mathbb{R}_+$ as a $\mathbb{R}_+$-valued function. In the first and most important step it will be shown that this function satisfies the triangle inequality. Fix therefore $p \in [1, \infty)$ and take two processes $H, K \in L^p(M)$. Keep in mind that by its introduction in Theorem 2.7 $\pi_t(\omega)$ is a positive semidefinite Hermitian matrix for each $t \in \mathbb{R}_+$ and $\omega \in \Omega$. Thus one can use Lemma 5.20 in the appendix to see that

$$(H_t + K_t)^\mathsf{T}\pi_t\overline{(H_t + K_t)} \leq \left(\sqrt{H_t^\mathsf{T}\pi_t\overline{H}_t} + \sqrt{K_t^\mathsf{T}\pi_t\overline{K}_t}\right)^2$$

holds for each pair $(t, \omega) \in \mathbb{R}_+ \times \Omega$. As stated above, for each $H \in L^p(M)$ the integral $\big((H^\mathsf{T}\pi\overline{H}) \bullet C\big)_\infty$ is finite for all ω outside of a $\mathbb{P}$-null set N_H, which may depend on H. This means that for such a $\omega \in (N_H \cup N_K)^\mathsf{c}$ the functions $\sqrt{H^\mathsf{T}\pi\overline{H}}$ and $\sqrt{K^\mathsf{T}\pi\overline{K}}$ are elements of $\mathcal{L}^2\big(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, C_\cdot(\omega)\big)$. In the following, let $\|\cdot\|_2$ denote the L^2-norm on the space

$L^2\big(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, C_\cdot(\omega)\big)$ being the quotient space of the aforementioned $\mathcal{L}^2\big(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}, C_\cdot(\omega)\big)$ w.r.t. $\|\cdot\|_2$. Thus one can use the *Minkowski inequality* (see for example [Gri18, Satz 8.3]), i.e. the triangle inequality for L^p-norms, to obtain

$$\begin{aligned}
&\left(\int_0^\infty (H_t+K_t)^\mathsf{T}\pi_t\overline{(H_t+K_t)}\,\mathrm{d}C_t\right)^{1/2} \le \left(\int_0^\infty \left(\sqrt{H_t^\mathsf{T}\pi_t\bar{H}_t}+\sqrt{K_t^\mathsf{T}\pi_t\bar{K}_t}\right)^2\,\mathrm{d}C_t\right)^{1/2}\\
&= \left\|\sqrt{H_t^\mathsf{T}\pi_t\bar{H}_t}+\sqrt{K_t^\mathsf{T}\pi_t\bar{K}_t}\right\|_2 \le \left\|\sqrt{H_t^\mathsf{T}\pi_t\bar{H}_t}\right\|_2+\left\|\sqrt{K_t^\mathsf{T}\pi_t\bar{K}_t}\right\|_2\\
&= \left(\int_0^\infty H_t^\mathsf{T}\pi_t\bar{H}_t\,\mathrm{d}C_t\right)^{1/2}+\left(\int_0^\infty K_t^\mathsf{T}\pi_t\bar{K}_t\,\mathrm{d}C_t\right)^{1/2}.
\end{aligned}$$

Now let $\|\cdot\|_p$ denote the L^p-norm on the space $L^p(\Omega,\mathcal{F},\mathbb{P})$ and note that $H\in L^p(M) \iff \sqrt{\big((H^\mathsf{T}\pi\bar{H})\bullet C\big)_\infty}\in\mathcal{L}^p(\Omega,\mathcal{F},\mathbb{P})$ and

$$\|H\|_{L^p(M)} = \left\|\sqrt{\big((H^\mathsf{T}\pi\bar{H})\bullet C\big)_\infty}\right\|_p$$

are a direct consequence of Definition 2.9. Therefore the triangle inequality

$$\begin{aligned}
\|H+K\|_{L^p(M)} &= \mathbb{E}\Big[\Big(\big((H+K)^\mathsf{T}\pi\overline{(H+K)}\big)\bullet C\Big)_\infty^{p/2}\Big]^{1/p}\\
&\le \mathbb{E}\Big[\Big(\sqrt{\big((H^\mathsf{T}\pi\bar{H})\bullet C\big)_\infty}+\sqrt{\big((K^\mathsf{T}\pi\bar{K})\bullet C\big)_\infty}\Big)^p\Big]^{1/p}\\
&= \left\|\sqrt{\big((H^\mathsf{T}\pi\bar{H})\bullet C\big)_\infty}+\sqrt{\big((K^\mathsf{T}\pi\bar{K})\bullet C\big)_\infty}\right\|_p\\
&\le \left\|\sqrt{\big((H^\mathsf{T}\pi\bar{H})\bullet C\big)_\infty}\right\|_p+\left\|\sqrt{\big((K^\mathsf{T}\pi\bar{K})\bullet C\big)_\infty}\right\|_p\\
&= \|H\|_{L^p(M)}+\|K\|_{L^p(M)}
\end{aligned}$$

follows by again using the Minkowski inequality in the second-to-last step.

Furthermore, for any $\alpha\in\mathbb{K}$ and $H\in L^p(M)$ the equality

$$\begin{aligned}
\|\alpha H\|_{L^p(M)} &= \mathbb{E}\big[\big((\alpha H^\mathsf{T}\pi\overline{\alpha H})\bullet C\big)_\infty^{p/2}\big]^{1/p} = \mathbb{E}\big[\big(|\alpha|^2(H^\mathsf{T}\pi\bar{H})\bullet C\big)_\infty^{p/2}\big]^{1/p}\\
&= |\alpha|\,\mathbb{E}\big[\big((H^\mathsf{T}\pi\bar{H})\bullet C\big)_\infty^{p/2}\big]^{1/p} = |\alpha|\|H\|_{L^p(M)}
\end{aligned}$$

holds and obviously also $\|0\|_{L^p(M)}=0$. Therefore it is clear to see that for $H,K\in L^p(M)$ and $\alpha\in\mathbb{K}$, also $\alpha H+K\in L^p(M)$. Thus $L^p(M)$ is a vector space and $\|\cdot\|_{L^p(M)}$ is indeed a norm on $L^p(M)$. Note at this point that in order to really obtain a norm and not only a seminorm it was stated before that throughout this thesis one may not distinguish between a process and its equivalence class in $L^p(M)$. □

2.4 Integrable processes w.r.t. multi-dimensional continuous local martingales

In the following, this thesis fixes $p = 2$ and examines the normed vector space $L^2(M)$ further. Note that some authors, for example Cohen and Elliott in [CE15] as well as Shyraev and Cherny in [SC02], consider $p = 1$.
The space $L^2(M)$ can often be too restrictive and one may integrate predictable processes, which are only locally in $L^2(M)$. Such processes are defined below.

Definition 2.11 (The space $L^2_{\text{loc}}(M)$)**.** A process $H \in \mathcal{P}^d$ is in $L^2_{\text{loc}}(M)$ for a $\mathbb{K}^d$-valued continuous local martingale M, if and only if there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \tau_n = \infty$ almost surely, such that

$$\mathbb{E}\big[\big((H^\top \pi \overline{H}) \bullet C\big)_{\tau_n}\big] < \infty, \qquad n \in \mathbb{N}.$$

A process H is said to be integrable w.r.t. a $\mathbb{K}^d$-valued continuous local martingale M, if and only if $H \in L^2_{\text{loc}}(M)$.

By keeping in mind that the integral process $\big((H^\top \pi \overline{H}) \bullet C\big)_{\tau_n}$ stopped at a stopping time τ_n is the same as stopping the integrator C, it is apparent that

$$\mathbb{E}\big[\big((H^\top \pi \overline{H}) \bullet C^{\tau_n}\big)_\infty\big] = \mathbb{E}\big[\big((H^\top \pi \overline{H}) \bullet C\big)_{\tau_n}\big]$$

holds. Furthermore, as $C^{(M)} = \sum_{j=1}^d [M^j]$ by Theorem 2.7 the compatibility with stopping of the covariation process (see [Sch23, Theorem 5.65]) implies that $(C^{(M)})^{\tau_n} = C^{(M^{\tau_n})}$ up to indistinguishability as well as $\pi^{(M)} = \pi^{(M^{\tau_n})}$ on the stochastic integral $[0, \tau_n] := \{(t, \omega) \in \mathbb{R}_+ \times \Omega : t \leq \tau_n(\omega)\}$ for each $n \in \mathbb{N}$. Consequently, $H \in L^2_{\text{loc}}(M)$ if and only if there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \tau_n = \infty$ almost surely, such that for all $n \in \mathbb{N}$ follows $H \in L^2(M^{\tau_n})$, i.e.

$$\|H\|_{L^2(M^{\tau_n})} = \mathbb{E}\big[\big((H^\top \pi^{(M^{\tau_n})} \overline{H}) \bullet C^{(M^{\tau_n})}\big)_\infty\big]^{1/2} = \mathbb{E}\big[\big((H^\top \pi \overline{H}) \bullet C\big)_{\tau_n}\big]^{1/2} < \infty, \qquad n \in \mathbb{N}.$$

This definition implies that for each $t \in \mathbb{R}_+$ and $\mathbb{P}$-almost all $\omega \in \Omega$ exists a $n \in \mathbb{N}$, such that $t \leq \tau_n(\omega)$. Thus one obtains almost surely pathwise for fixed $t \in \mathbb{R}_+$ the upper bound $\big((H^\top \pi \overline{H}) \bullet C\big)_t \leq \big((H^\top \pi \overline{H}) \bullet C\big)_{\tau_n}$, which is $\mathbb{P}$-almost surely finite. Therefore, in the same way as for $H \in L^p(M)$ one can see that the integral process $(H^\top \pi \overline{H}) \bullet C$ is $\mathbb{P}$-almost surely well-defined, continuous, adapted and thus predictable for each $H \in L^2_{\text{loc}}(M)$.

Let M be some $\mathbb{K}^d$-valued continuous local martingale and $H \in L^2_{\text{loc}}(M)$. Then set $(\tilde{\tau}_n)_{n\in\mathbb{N}}$ to be the increasing sequence of stopping times discussed in Definition 2.11 and may $(\sigma)_{n\in\mathbb{N}}$ denote the localizing sequence for M, i.e. it is also increasing as well as $\lim_{n\to\infty} \sigma_n = \infty$ and, additionally, $M^{\sigma_n} - M_0$ is a martingale. Consider now for each $n \in \mathbb{N}$ the stopping time $\tau_n := \tilde{\tau}_n \wedge \sigma_n$. Obviously, $(\tau_n)_{n\in\mathbb{N}}$ is still increasing and tending almost surely to infinity for $n \to \infty$. By [Sch23, Lemma 4.135(a)], $M^{\tau_n} - M_0$ is a martingale for each $n \in \mathbb{N}$ and thus $(\tau_n)_{n\in\mathbb{N}}$ is localizing sequence for M. Furthermore,

$$\mathbb{E}\big[\big((H^\top \pi \overline{H}) \bullet C\big)_{\tau_n}\big] \leq \mathbb{E}\big[\big((H^\top \pi \overline{H}) \bullet C\big)_{\tilde{\tau}_n}\big] < \infty, \qquad n \in \mathbb{N},$$

holds and therefore one can always think of a single sequence of stopping times meeting the criteria of Definition 2.11 as well as being a localizing sequence for M.

Lemma 2.12. *The above defined set $L^2_{\mathrm{loc}}(M)$ is indeed a vector space.*

Proof. In a similar way as in Lemma 2.10 one can see that for $H, K \in L^2_{\mathrm{loc}}(M)$ and $\alpha \in \mathbb{K}$ also $\alpha H + K \in L^2_{\mathrm{loc}}(M)$ and therefore $L^2_{\mathrm{loc}}(M)$ is also a vector space. Namely, by Definition 2.11, there exist two increasing sequences $(\tau_n^H)_{n\in\mathbb{N}}$ and $(\tau_n^K)_{n\in\mathbb{N}}$, whose limits for $n \to \infty$ are both almost surely infinite, satisfying

$$H \in L^2(M^{\tau_n^H}) \quad \text{and} \quad K \in L^2(M^{\tau_n^K}), \qquad n \in \mathbb{N}.$$

By now defining for all $n \in \mathbb{N}$ the stopping time $\tau_n = \tau_n^H \wedge \tau_n^K$ one obtains an again increasing sequence of stopping times with the almost sure limit $\lim_{n\to\infty} \tau_n = \infty$ satisfying $H, K \in L^2(M^{\tau_n})$ for each $n \in \mathbb{N}$, as

$$\|H\|_{L^2(M^{\tau_n})} = \mathbb{E}\Big[\Big((H^\mathsf{T}\pi\bar{H}) \bullet C\Big)_{\tau_n}\Big]^{1/2} \le \mathbb{E}\Big[\Big((H^\mathsf{T}\pi\bar{H}) \bullet C\Big)_{\tau_n^H}\Big]^{1/2} < \infty, \qquad n \in \mathbb{N},$$

and analogously

$$\|K\|_{L^2(M^{\tau_n})} = \mathbb{E}\Big[\Big((K^\mathsf{T}\pi\bar{K}) \bullet C\Big)_{\tau_n}\Big]^{1/2} \le \mathbb{E}\Big[\Big((K^\mathsf{T}\pi\bar{K}) \bullet C\Big)_{\tau_n^K}\Big]^{1/2} < \infty, \qquad n \in \mathbb{N}.$$

As $L^2(M^{\tau_n})$ is a vector space by Lemma 2.10, also $\alpha H + K \in L^2(M^{\tau_n})$ for each $n \in \mathbb{N}$, which leads to $\alpha H + K \in L^2_{\mathrm{loc}}(M)$ and thus concludes the proof. □

Furthermore, the function $\|\cdot\|_{L^2(M)} : L^2_{\mathrm{loc}}(M) \to \bar{\mathbb{R}}_+$ defines a *pseudonorm* on $L^2_{\mathrm{loc}}(M)$ in the sense that it fulfills all criteria of a norm, except that it might be infinite. Additionally, $L^2(M) \subseteq L^2_{\mathrm{loc}}(M)$ and $L^2_{\mathrm{loc}}(M) \setminus L^2(M)$ is exactly the set of all $H \in L^2_{\mathrm{loc}}(M)$ satisfying $\|H\|_{L^2(M)} = \infty$.

The main reason for introducing this set is that the constant processes are not necessarily elements of $L^2(M)$ for each continuous local martingale M. For example, take a $\mathbb{R}^d$-valued Brownian motion B. Then one may use $[B]_t = tI_d$ for each $t \in \mathbb{R}_+$, where I_d denotes the $(d \times d)$-dimensional identity matrix (see [Sch23, Example 5.70(a)]) to see that $C^{(B)} = dt$ holds and thus the constant process $H \equiv (1, \dots, 1)^\mathsf{T} \in \mathbb{R}^d$ is not in $L^2(B)$, as

$$\begin{aligned}\mathbb{E}\big[((H_t^\mathsf{T}\pi_t^{(B)}\bar{H}_t) \bullet C_t^{(B)})_\infty\big] &= \mathbb{E}\Bigg[\Bigg(\sum_{i,j=1}^{d} \frac{\mathrm{d}[B^i, B^j]_t}{\mathrm{d}(dt)} \bullet (dt)\Bigg)_\infty\Bigg] \\ &= \mathbb{E}\Bigg[\Bigg(\sum_{j=1}^{d} \frac{\mathrm{d}t}{\mathrm{d}(dt)} \bullet (dt)\Bigg)_\infty\Bigg] = d\,\mathbb{E}\Big[\int_0^\infty \mathrm{d}t\Big] = \infty.\end{aligned}$$

However, H is an element of $L^2_{\mathrm{loc}}(B)$, as the lemma below the following definition shows.

Definition 2.13 (Locally bounded process)**.** A $\mathbb{K}^d$-valued process H is called *locally bounded*, if and only if there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ tending almost surely to infinity, such that for each $n \in \mathbb{N}$ there exists some $U_n \in \mathbb{R}_+$ satisfying $\|H_t(\omega)\mathbb{1}_{[0,\tau_n(\omega)]}(t)\|_p \le U_n$ for all $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and some $p \in [1,\infty]$, where

$$\|\cdot\|_p : \mathbb{K}^d \ni x \mapsto \Big(\sum_{j=1}^d |x^j|^p\Big)^{1/p} \in \mathbb{R}_+, \qquad p \in [1,\infty),$$

denotes the p-norm on $\mathbb{K}^d$ and

$$\|\cdot\|_\infty : \mathbb{K}^d \ni x \mapsto \max\{|x^j| : j = 1,\dots,d\} \in \mathbb{R}_+$$

the maximum norm on $\mathbb{K}^d$.

Note that by [Sch23, Remark 13.19(b)] all norms on $\mathbb{K}^d$ are equivalent, thus the (locally) boundedness property of a stochastic process does not depend on the choice of $p \in [1,\infty]$.

Lemma 2.14. *For each locally bounded process $H \in \mathcal{P}^d$ according to Definition 2.13 and $\mathbb{K}^d$-valued continuous local martingale M follows that H is an element of $L^2_{\mathrm{loc}}(M)$.*

Proof. In the same way as in the first step of the proof of Theorem 4.15 below define for each $n \in \mathbb{N}$ the stopping time

$$\sigma_n = \inf\{t \in \mathbb{R}_+ : C_t = n\}.$$

As always, the convention $\inf \varnothing = \infty$ is used. Therefore, $(\sigma_n)_{n\in\mathbb{N}}$ is increasing and tending almost surely towards infinity as $n \to \infty$, because C is increasing and continuous. Additionally,

$$\mathbb{E}\Big[\int_0^{\sigma_n} \sum_{j=1}^d \pi_t^{jj}\,\mathrm{d}C_t\Big] = \mathbb{E}\Big[\sum_{j=1}^d \int_0^{\sigma_n} \frac{\mathrm{d}[M^j]_t}{\mathrm{d}C_t}\,\mathrm{d}C_t\Big] = \mathbb{E}\Big[\sum_{j=1}^d \int_0^{\sigma_n} \mathrm{d}[M^j]_t\Big] = \mathbb{E}\Big[\sum_{j=1}^d [M^j]_{\sigma_n}\Big]$$
$$= \mathbb{E}\big[C_{\sigma_n}\big] \le \mathbb{E}[n] = n < \infty$$

holds for all $n \in \mathbb{N}$. Fix now $x \in \mathbb{K}^d$. Note at first that, due to Lemma 5.21 in the fourth step,

$$\begin{aligned}
|x^{\mathsf{T}}\pi_t(\omega)\bar{x}| &= \Big|\sum_{i,j=1}^d x^i\pi_t^{ij}(\omega)\bar{x}^j\Big| \le \sum_{i,j=1}^d |x^i\pi_t^{ij}(\omega)\bar{x}^j| \le \|x\|_\infty^2 \sum_{i,j=1}^d |\pi_t^{ij}(\omega)| \\
&\le \|x\|_\infty^2 \sum_{i,j=1}^d \sqrt{\pi_t^{ii}(\omega)}\sqrt{\pi_t^{jj}(\omega)} \le \|x\|_\infty^2 \sum_{i,j=1}^d \frac{\pi_t^{ii}(\omega)+\pi_t^{jj}(\omega)}{2} \\
&= \frac{\|x\|_\infty^2}{2}\sum_{i=1}^d\sum_{j=1}^d\big(\pi_t^{ii}(\omega)+\pi_t^{jj}(\omega)\big) = \frac{\|x\|_\infty^2}{2}\sum_{i=1}^d\big(d\pi_t^{ii}(\omega)+\mathrm{tr}(\pi_t(\omega))\big) \\
&= \frac{\|x\|_\infty^2}{2}\Big(d\,\mathrm{tr}(\pi_t(\omega))+d\,\mathrm{tr}(\pi_t(\omega))\Big) = d\|x\|_\infty^2\mathrm{tr}(\pi_t(\omega))
\end{aligned} \tag{2.2}$$

holds for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. In the fifth step above the inequality

$$ab \leq \frac{a^2+b^2}{2} \iff 0 \leq a^2+b^2-2ab=(a-b)^2, \qquad a,b \in \mathbb{R}_+,$$

has been used. One may now define for each $n \in \mathbb{N}$ the stopping time $\tilde{\tau}_n = \tau_n \wedge \sigma_n$, whereby $(\tilde{\tau}_n)_{n\in\mathbb{N}}$ is also increasing and tending almost surely towards infinity as $n \to \infty$. Thus the process H is in $L^2_{\text{loc}}(M)$, due to

$$\mathbb{E}\big[((H^\mathsf{T}\pi\overline{H})\bullet C)_{\tilde{\tau}_n}\big] \leq d\,\mathbb{E}\Big[\Big((\|H\|^2_\infty \sum_{j=1}^d \pi^{jj})\bullet C\Big)_{\tilde{\tau}_n}\Big] = d\,\mathbb{E}\Big[\Big((\|H\mathbb{1}_{[0,\tau_n]}\|^2_\infty \sum_{j=1}^d \pi^{jj})\bullet C\Big)_{\tilde{\tau}_n}\Big]$$

$$\leq dU_n^2\,\mathbb{E}\Big[\Big((\sum_{j=1}^d \pi^{jj})\bullet C\Big)_{\tilde{\tau}_n}\Big] \leq dU_n^2\,\mathbb{E}\Big[\Big((\sum_{j=1}^d \pi^{jj})\bullet C\Big)_{\sigma_n}\Big] \leq dU_n^2 n < \infty$$

for each $n \in \mathbb{N}$. □

The last lemma also implies that each $\mathbb{K}^d$-valued predictable step process is in $L^2_{\text{loc}}(M)$ for all $\mathbb{K}^d$-valued continuous local martingales M, as they are bounded by Definition 2.2.

2.5 Properties of the space $L^2(M)$

As $L^2(M)$ is a normed vector space, one may also consider convergence of stochastic process in $L^2(M)$. For starters, a relatively simple, but often times useful case will be examined in the lemma below.

Lemma 2.15. *As always, may M denote a $\mathbb{K}^d$-valued continuous local martingale and let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times converging $\mathbb{P}$-almost surely to ∞ as $n \to \infty$. Then for every predictable process $H \in L^2(M)$ the sequence of processes $(H_n)_{n\in\mathbb{N}}$ defined by $H_{n,t} = H_t \mathbb{1}_{[0,\tau_n]}(t)$ for each $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$ converges in $L^2(M)$ to H as $n \to \infty$ in the sense that*

$$\lim_{n\to\infty} \|H_n - H\|_{L^2(M)} = 0.$$

Proof. Note at first that due to the left-continuousness and adaptedness of $\mathbb{1}_{[0,\tau_n]}$ the processes H_n are predictable and thus also in $L^2(M)$. With the *dominated convergence theorem* in mind, the proof is quite straightforward. At first consider

$$\begin{aligned}\|H_n - H\|^2_{L^2(M)} &= \mathbb{E}\Big[\int_0^\infty (H_{n,t}-H_t)^\mathsf{T}\pi_t\overline{(H_{n,t}-H_t)}\,\mathrm{d}C_t\Big]\\ &= \mathbb{E}\Big[\int_0^\infty (H_t\mathbb{1}_{[0,\tau_n]}(t)-H_t)^\mathsf{T}\pi_t(\overline{H}_t\mathbb{1}_{[0,\tau_n]}(t)-\overline{H}_t)\,\mathrm{d}C_t\Big]\\ &= \mathbb{E}\Big[\int_{\tau_n}^\infty H_t^\mathsf{T}\pi_t\overline{H}_t\,\mathrm{d}C_t\Big]\end{aligned}$$

for each $n \in \mathbb{N}$. The sequence of $\mathcal{F}$-measurable and $\mathbb{P}$-integrable functions $(f_n)_{n\in\mathbb{N}}$ defined as

$$\Omega \ni \omega \mapsto f_n(\omega) = \int_{\tau_n(\omega)}^\infty H_t^\mathsf{T}(\omega)\pi_t(\omega)\overline{H}_t(\omega)C(\mathrm{d}t,\omega) \in \mathbb{R}_+$$

for each $n \in \mathbb{N}$ converges for $\mathbb{P}$-almost all $\omega \in \Omega$ to zero as $\tau_n(\omega) \to \infty$ for $n \to \infty$. Furthermore,

$$|f_n(\omega)| = f_n(\omega) \leq g(\omega) := \int_0^\infty H_t^\mathsf{T}(\omega)\pi_t(\omega)\overline{H}_t(\omega)C(\mathrm{d}t,\omega), \qquad n \in \mathbb{N},$$

where $g : \Omega \to \mathbb{R}_+$ is measurable and satisfies $\mathbb{E}[g] = \|H\|^2_{L^2(M)} < \infty$. Thus one can use the dominated convergence theorem, see Theorem 5.37 in the appendix, to obtain

$$\lim_{n\to\infty} \|H_n - H\|^2_{L^2(M)} = \lim_{n\to\infty} \mathbb{E}\left[\int_{\tau_n}^\infty H_t^\mathsf{T}\pi_t\overline{H}_t\,\mathrm{d}C_t\right] = \lim_{n\to\infty} \mathbb{E}[f_n] = \lim_{n\to\infty} \mathbb{E}[|f_n - 0|] = 0,$$

which concludes the proof. □

The proof of the following lemma is very similar to the one above.

Lemma 2.16. *Let again $H \in L^2(M)$ for a $\mathbb{K}^d$-valued continuous local martingale M and set $H_{n,t}(\omega) := H_t(\omega)\mathbb{1}_{\|H_t(\omega)\|_2 \leq n}$ for each $n \in \mathbb{N}$ and $(t,\omega) \in \mathbb{R}_+ \times \Omega$. Then $H_n \to H$ in $L^2(M)$ for $n \to \infty$.*

Proof. As in Lemma 2.15 one has to show that $\lim_{n\to\infty} \|H_n - H\|_{L^2(M)} = 0$. Similarly to above it is clear that for each $n \in \mathbb{N}$ the equality

$$\begin{aligned}
\|H_n - H\|^2_{L^2(M)} &= \mathbb{E}\left[\int_0^\infty (H_{n,t} - H_t)^\mathsf{T}\pi_t\overline{(H_{n,t} - H_t)}\,\mathrm{d}C_t\right] \\
&= \mathbb{E}\left[\int_0^\infty (H_t\mathbb{1}_{\|H_t\|_2\leq n} - H_t)^\mathsf{T}\pi_t(\overline{H}_t\mathbb{1}_{\|H_t\|_2\leq n} - \overline{H}_t)\,\mathrm{d}C_t\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\|H_t\|_2>n}H_t^\mathsf{T}\pi_t\overline{H}_t\,\mathrm{d}C_t\right]
\end{aligned}$$

holds. Thus one may define for each $n \in \mathbb{N}$ pathwise

$$\tilde{f}_n(t) = \mathbb{1}_{\|H_t\|_2>n}H_t^\mathsf{T}\pi_t\overline{H}_t,$$

which is $\mathcal{B}_{\mathbb{R}_+}$-measurable by Lemma 5.6 and $\mathbb{P}$-almost surely C-integrable. Additionally,

$$|\tilde{f}_n(t)| = \tilde{f}_n(t) \leq \tilde{g}(t) := H_t^\mathsf{T}\pi_t\overline{H}_t, \qquad n \in \mathbb{N},$$

where $\tilde{g}$ is also $\mathcal{B}_{\mathbb{R}_+}$-measurable and $\mathbb{P}$-almost surely C-integrable. Furthermore, as H is $\mathbb{K}^d$-valued, the Euclidean norm of its realization at any point $(t,\omega) \in \mathbb{R}_+ \times \Omega$ is finite and thus there exists an $n_0(t,\omega) \in \mathbb{N}$, such that $\|H_t(\omega)\|_2 \leq n$ for all $n \geq n_0(t,\omega)$, which implies the pointwise convergence to zero of $(\tilde{f}_n)_{n\in\mathbb{N}}$ for $n \to \infty$. Thus one can use the dominated convergence theorem, i.e. Theorem 5.37 in the appendix, to obtain

$$0 = \lim_{n\to\infty} \int_0^\infty |\tilde{f}_n(t) - 0|\,\mathrm{d}C_t = \lim_{n\to\infty} \int_0^\infty \tilde{f}_n(t)\,\mathrm{d}C_t = \lim_{n\to\infty} \int_0^\infty \mathbb{1}_{\|H_t\|_2>n}H_t^\mathsf{T}\pi_t\overline{H}_t\,\mathrm{d}C_t.$$

For each $n \in \mathbb{N}$ the function

$$f_n(\omega) = \int_0^\infty \mathbb{1}_{\|H_t(\omega)\|_2>n}H_t^\mathsf{T}(\omega)\pi_t(\omega)\overline{H}_t(\omega)C(\mathrm{d}t,\omega)$$

is $\mathcal{F}$-measurable and $\mathbb{P}$-integrable satisfying

$$|f_n(\omega)| = f_n(\omega) \le g(\omega),$$

which was defined in the proof of Lemma 2.15 above. By the previous findings of this proof one can see that $(f_n)_{n\in\mathbb{N}}$ also converges $\mathbb{P}$-almost surely to zero for $n \to \infty$. Therefore the dominated convergence theorem leads to

$$\begin{aligned}\lim_{n\to\infty} \|H_n - H\|^2_{L^2(M)} &= \lim_{n\to\infty} \mathbb{E}\Big[\int_0^\infty \mathbb{1}_{\|H_t\|_2 > n} H_t^\mathsf{T} \pi_t \bar{H}_t \, \mathrm{d}C_t\Big] = \lim_{n\to\infty} \mathbb{E}[f_n] \\ &= \lim_{n\to\infty} \mathbb{E}[|f_n - 0|] = 0,\end{aligned}$$

whereby $\lim_{n\to\infty} \|H_n - H\|_{L^2(M)} = 0$ follows. □

In remaining parts of this thesis one may often approximate some process in the normed vector space $L^2(M)$ by predictable step processes. Therefore it is essential to show that those processes are dense in the aforementioned space, which will be done below. The proof of the following theorem relies on [SC02, p. 16f].

Theorem 2.17. *For each $\mathbb{K}^d$-valued continuous local martingale M the predictable step processes in $L^2(M)$ are dense in $L^2(M)$.*

Proof. Assume at first that M is a $\mathbb{K}^d$-valued continuous local martingale, such that

$$\mathbb{E}\big[\big(\mathrm{tr}(\pi_t) \bullet C\big)_\infty\big]^{1/2} =: U < \infty.$$

Take now any $x \in \mathbb{K}^d$ and predictable set $A \in \Sigma_p$. Thus the $\mathbb{K}^d$-valued predictable process $x\mathbb{1}_A$ is an element of $L^2(M)$. To be precise, one may use inequality (2.2), which has already been introduced in the proof of Lemma 2.14, to see that

$$\begin{aligned}\|x\mathbb{1}_A\|_{L^2(M)} &= \mathbb{E}\big[\big((x^\mathsf{T}\mathbb{1}_A \pi \bar{x} \mathbb{1}_A) \bullet C\big)_\infty\big]^{1/2} \le \mathbb{E}\big[\big((x^\mathsf{T} \pi \bar{x}) \bullet C\big)_\infty\big]^{1/2} \\ &\le d\|x\|^2_\infty \, \mathbb{E}\big[\big(\mathrm{tr}(\pi_t) \bullet C\big)_\infty\big]^{1/2} < \infty,\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{K}^d$, due to the assumption above. Now define $\mathfrak{M}$ as the set of all predictable sets A, such that for each $x \in \mathbb{K}^d$ exists a sequence of predictable step processes, see Definition 2.2, in $L^2(M)$ that converges to $x\mathbb{1}_A$ w.r.t. the norm $\|\cdot\|_{L^2(M)}$. In the next step it is shown that $\mathfrak{M}$ is a *monotone class* per Definition 5.15.

(i) $\varnothing$ and Ω are trivially in $\mathfrak{M}$, as the constant processes 0 and x are predictable step processes.

(ii) Suppose $A, B \in \mathfrak{M}$ with $A \subseteq B$. Then there exist two sequences of predictable step processes $(H_n^A)_{n\in\mathbb{N}}$ and $(H_n^B)_{n\in\mathbb{N}}$ converging to $x\mathbb{1}_A$ and $x\mathbb{1}_B$, respectively. Thus one can see that $(H_n^B - H_n^A)_{n\in\mathbb{N}}$, which is again a sequence of predictable step processes per Lemma 3.2 in the next chapter, approximates $x\mathbb{1}_{B\setminus A}$ and therefore $B \setminus A \in \mathfrak{M}$, due to

$$\begin{aligned}&\lim_{n\to\infty} \|(H_n^B - H_n^A) - x\mathbb{1}_{B\setminus A}\|_{L^2(M)} = \lim_{n\to\infty} \|(H_n^B - H_n^A) - (x\mathbb{1}_B - x\mathbb{1}_A)\|_{L^2(M)} \\ &= \lim_{n\to\infty} \|(H_n^B - x\mathbb{1}_B) - (H_n^A - x\mathbb{1}_A)\|_{L^2(M)} \\ &\le \lim_{n\to\infty} \|H_n^B - x\mathbb{1}_B\|_{L^2(M)} + \lim_{n\to\infty} \|H_n^A - x\mathbb{1}_A\|_{L^2(M)} = 0.\end{aligned}$$

(iii) Now assume $A, B \in \mathfrak{M}$ with $A \cap B = \varnothing$. With the notation from *(ii)* is is apparent that $(H_n^A + H_n^B)_{n\in\mathbb{N}}$ converges to $x\mathbb{1}_{A\cup B}$ and thus $A \cup B \in \mathfrak{M}$, as

$$\lim_{n\to\infty} \|(H_n^A + H_n^B) - x\mathbb{1}_{A\cup B}\|_{L^2(M)} = \lim_{n\to\infty} \|(H_n^A + H_n^B) - (x\mathbb{1}_A + x\mathbb{1}_B)\|_{L^2(M)}$$
$$= \lim_{n\to\infty} \|(H_n^A - x\mathbb{1}_A) + (H_n^B - x\mathbb{1}_B)\|_{L^2(M)}$$
$$\leq \lim_{n\to\infty} \|H_n^A - x\mathbb{1}_A\|_{L^2(M)} + \lim_{n\to\infty} \|H_n^B - x\mathbb{1}_B\|_{L^2(M)} = 0.$$

(iv) Fix an increasing sequence $(A_n)_{n\in\mathbb{N}} \in \mathfrak{M}$, which implies that $\mathbb{1}_{A_n}$ converges pointwise to $x\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k}$ for $n \to \infty$. Then for each $n \in \mathbb{N}$ exists a sequences of predictable step processes $(H_{n,k})_{k\in\mathbb{N}}$ in $L^2(M)$ converging to $x\mathbb{1}_{A_n}$ in $L^2(M)$. In other words, for each $n \in \mathbb{N}$ and $\epsilon > 0$ there exist a natural number $k_n(\epsilon)$, such that

$$\|H_{n,k} - x\mathbb{1}_{A_n}\|_{L^2(M)} \leq \epsilon, \qquad k \geq k_n(\epsilon).$$

Similarly as in the proof of Lemma 2.16 on can now define for fixed $\omega \in \Omega$ and each $n \in \mathbb{N}$ the, due to Lemma 5.6 in the appendix, $\mathcal{B}_{\mathbb{R}_+}$-measurable function

$$\mathbb{R}_+ \ni t \mapsto f_n(t,\omega) = \mathbb{1}_{A_n}(t,\omega)\, x^\mathsf{T}\pi_t(\omega)\bar{x},$$

which is pointwise bounded by the $\mathbb{P}$-almost surely $C_\cdot(\omega)$-integrable function

$$\mathbb{R}_+ \ni t \mapsto g(t,\omega) = x^\mathsf{T}\pi_t(\omega)\bar{x}.$$

Consequently, the dominated convergence theorem, see Theorem 5.37 in the appendix, leads for $\mathbb{P}$-almost all $\omega \in \Omega$ to

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} \mathbb{1}_{A_n}(t,\omega)\, x^\mathsf{T}\pi_t(\omega)\bar{x}\, C(\mathrm{d}t,\omega) = \lim_{n\to\infty} \int_{\mathbb{R}_+} f_n(t,\omega)\, C(\mathrm{d}t,\omega)$$
$$= \int_{\mathbb{R}_+} f(t,\omega)\, C(\mathrm{d}t,\omega) = \int_{\mathbb{R}_+} \mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k}(t,\omega)\, x^\mathsf{T}\pi_t(\omega)\bar{x}\, C(\mathrm{d}t,\omega) =: \tilde{f}(\omega) < \infty.$$

Furthermore, one may now define

$$\Omega \ni \tilde{f}_n(\omega) = \int_{\mathbb{R}_+} \mathbb{1}_{A_n}(t,\omega)\, x^\mathsf{T}\pi_t(\omega)\bar{x}\, C(\mathrm{d}t,\omega), \qquad n \in \mathbb{N},$$

which is for each $n \in \mathbb{N}$ a $\mathcal{F}$-measurable function and pointwise less or equal to

$$\Omega \ni \tilde{g}(\omega) = \int_{\mathbb{R}_+} x^\mathsf{T}\pi_t(\omega)\bar{x}\, C(\mathrm{d}t,\omega).$$

Note at this point that as stated in the beginning of this proof

$$\mathbb{E}[g] = \mathbb{E}\bigg[\int_{\mathbb{R}_+} x^\mathsf{T}\pi_t(\omega)\bar{x}\, C(\mathrm{d}t,\omega)\bigg] \leq d\|x\|_1^2\, \mathbb{E}\bigg[\int_{\mathbb{R}_+} \sum_{j=1}^{d} \pi_t^{jj}(\omega)\, C(\mathrm{d}t,\omega)\bigg] < \infty.$$

Thus the dominated convergence theorem is again applicable, leading to

$$\lim_{n\to\infty} \|x\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k} - x\mathbb{1}_{A_n}\|^2_{L^2(M)} = \lim_{n\to\infty} \mathbb{E}\Big[\Big(\big((\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k} - \mathbb{1}_{A_n})\, x^\mathsf{T}\pi\bar{x}\big) \bullet C\Big)_\infty\Big]$$
$$= \lim_{n\to\infty} \mathbb{E}[\tilde{f}(\omega) - \tilde{f}_n(\omega)] = 0.$$

Thus $x\mathbb{1}_{A_n}$ converges in $L^2(M)$ to $x\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k}$, i.e. for each $\epsilon > 0$ exists a $n(\epsilon) \in \mathbb{N}$, such that

$$\|x\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k} - x\mathbb{1}_{A_n}\|_{L^2(M)} < \epsilon, \qquad n \geq n(\epsilon).$$

Consequently, the sequence of predictable step processes $\big(H_{n,k_n(2^{-n})}\big)_{n\in\mathbb{N}}$ converges in $L^2(M)$ to $x\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k}$, as for each $\epsilon > 0$ the triangle inequality implies that

$$\begin{aligned} &\|H_{n,k_n(2^{-n})} - x\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k}\|_{L^2(M)} \\ &\leq \|H_{n,k_n(2^{-n})} - x\mathbb{1}_{A_n}\|_{L^2(M)} + \|x\mathbb{1}_{A_n} - x\mathbb{1}_{\bigcup_{k\in\mathbb{N}} A_k}\|_{L^2(M)} \\ &\leq \frac{1}{2^n} + \frac{\epsilon}{2} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for all $n \geq \tilde{n}(\epsilon) := \min\{n \in \mathbb{N} : 2^{-n} \leq \epsilon/2\} \vee n(\epsilon/2)$. Therefore, $\bigcup_{k\in\mathbb{N}} A_k \in \mathfrak{M}$, making $\mathfrak{M}$ a monotone class.

By [JS13, Theorem 2.2(ii)] the set of all sets of the form $\{0\} \times B$ for $B \in \mathcal{F}_0$ and $(s, r] \times B$ for $B \in \mathcal{F}_s$ and $s < r$ being two non-negative real numbers, which will in this proof be denoted by $\mathfrak{G}$, generates the σ-algebra Σ_p. Each of those sets is also in $\mathfrak{M}$, as $x\mathbb{1}_{\{0\}\times B}(t, \omega) = x\mathbb{1}_{t=0}\mathbb{1}_B(\omega)$ is itself a predictable step process, because $x\mathbb{1}_B(\omega)$ is per assumption a bounded and $\mathcal{F}_0$-measurable random variable. Analogously, $x\mathbb{1}_{(s,r]\times B}(t, \omega) = x\mathbb{1}_B(\omega)\mathbb{1}_{(s,r]}(t)$ is again a rather simple predictable step process according to Definition 2.2.

Note at this point that $\mathfrak{G}$ is intersection stable, as for $(s_1, r_1] \times B_1$ and $(s_2, r_2] \times B_2$ the intersect is either $\varnothing$ or $(s_1 \vee s_2, r_1 \wedge r_2] \times B_1 \cap B_2$, which is again in $\mathfrak{G}$, because $B_1 \cap B_2$ is $\mathcal{F}_{s_1\vee s_2}$-measurable. Additionally, $\big(\{0\} \times B_0\big) \cap \big((s, r] \times B\big) = \varnothing$ for all $B_0 \in \mathcal{F}_0$, two non-negative real numbers $s < r$ and $B \in \mathcal{F}_s$. Lastly $\big(\{0\} \times B_1\big) \cap \big(\{0\} \times B_2\big) = \{0\} \times (B_1 \cap B_2) \in \mathfrak{G}$, as $B_1 \cap B_2 \in \mathcal{F}_0$ for all $B_1, B_2 \in \mathcal{F}_0$. Therefore one may now use the *Monotone class lemma*, Lemma 5.16 in the appendix, to obtain

$$\Sigma_p = \sigma(\mathfrak{G}) = \mathfrak{M}(\mathfrak{G}) \subseteq \mathfrak{M},$$

where $\mathfrak{M}(\mathfrak{G})$ denotes the minimal monotone class that is a superset of $\mathfrak{G}$. By the definition of $\mathfrak{M}$ it is clear that $\mathfrak{M} \subseteq \Sigma_p$ and therefore $\mathfrak{M} = \Sigma_p$. In other words: For all processes of the form $x\mathbb{1}_A$, where $x \in \mathbb{K}^d$, $A \in \Sigma_p$, exists a sequence of predictable step processes in $L^2(M)$ converging to it.

Let now $H \in L^2(M)$ be bounded. As H is Σ_p-measurable, one can use Lemma 5.35 in the appendix to obtain a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, where $f_n = \sum_{j=1}^{m_n} x_{n,j}\mathbb{1}_{A_{n,j}}$, converging uniformly to H. For each $n \in \mathbb{N}$ and $j \in \{1, \dots, m_n\}$ it follows by the aforementioned lemma that $x_{n,j} \in \mathbb{K}^d$ and $A_{n,j} \in \Sigma_p$. As seen in the previous steps of this proof, there exist sequences of predictable step process $(H_{n,j,k})_{k\in\mathbb{N}}$ in $L^2(M)$ converging to each $x_{n,j}\mathbb{1}_{A_{n,j}}$. As such for each pair $(n,j)\in\mathbb{N}\times\{1,\dots,m_n\}$ and each $\epsilon>0$ exists a $k_{n,j}(\epsilon)\in\mathbb{N}$ such that

$$\|H_{n,j,k}-x_{n,j}\mathbb{1}_{A_{n,j}}\|_{L^2(M)}\le\frac{\epsilon}{m_n},\qquad k\ge k_{n,j}(\epsilon).$$

Therefore the sequence $(H_{n,k})_{k\in\mathbb{N}}$ defined for each $(n,k)\in\mathbb{N}^2$ as $H_{n,k}=\sum_{j=1}^{m_n}H_{n,j,k}$, which are again predictable step processes by Lemma 3.2 in the next chapter, converges for all $n\in\mathbb{N}$ in $L^2(M)$ to f_n for $k\to\infty$, which can be seen by

$$\begin{aligned}\|H_{n,k}-f_n\|_{L^2(M)}&=\Big\|\Big(\sum_{j=1}^{m_n}H_{n,j,k}\Big)-\Big(\sum_{j=1}^{m_n}x_{n,j}\mathbb{1}_{A_{n,j}}\Big)\Big\|_{L^2(M)}\\&=\|\sum_{j=1}^{m_n}\big(H_{n,j,k}-x_{n,j}\mathbb{1}_{A_{n,j}}\big)\|_{L^2(M)}\le\sum_{j=1}^{m_n}\|H_{n,j,k}-x_{n,j}\mathbb{1}_{A_{n,j}}\|_{L^2(M)}\\&\le\sum_{j=1}^{m_n}\frac{\epsilon}{m_n}=m_n\frac{\epsilon}{m_n}=\epsilon\end{aligned}$$

for fixed $\epsilon>0$ and all $k\ge k_n(\epsilon):=\max\{k_{n,j}(\epsilon):j=1,\dots,m_n\}$. As $f_n\to H$ uniformly for $n\to\infty$, it follows that for each $\epsilon>0$ exists an $\tilde{n}(\epsilon)\in\mathbb{N}$, such that

$$\|f_n(t,\omega)-H_t(\omega)\|_1\le\frac{\epsilon}{dU},\qquad n\ge\tilde{n}(\epsilon)$$

holds simultaneously for all pairs $(t,\omega)\in\mathbb{R}_+\times\Omega$. Consequently, one may use inequality (2.2) in the second step to obtain

$$\begin{aligned}\|f_n-H\|_{L^2(M)}&=\mathbb{E}\Big[\int_0^\infty\big(f_n(t,\omega)-H_t(\omega)\big)^\mathsf{T}\pi_t(\omega)\overline{\big(f_n(t,\omega)-H_t(\omega)\big)}\,\mathrm{d}C_t(\omega)\Big]^{1/2}\\&\le d\,\mathbb{E}\Big[\int_0^\infty\Big(\|f_n(t,\omega)-H_t(\omega)\|_1^2\sum_{j=1}^d\pi_t^{jj}(\omega)\Big)\,\mathrm{d}C_t(\omega)\Big]^{1/2}\\&\le d\frac{\epsilon}{dU}\,\mathbb{E}\Big[\int_0^\infty\Big(\sum_{j=1}^d\pi_t^{jj}(\omega)\Big)\,\mathrm{d}C_t(\omega)\Big]^{1/2}\le d\frac{\epsilon}{dU}U\le\epsilon\end{aligned}$$

for each $n\ge\tilde{n}(\epsilon)$.

Consider now the sequence of predictable step functions $(H_{n,k_n(2^{-n})})_{n\in\mathbb{N}}$. This sequence approximates H in $L^2(M)$, because for each $\epsilon>0$ exists an

$$n(\epsilon):=\max\Big(\min\{n\in\mathbb{N}:2^{-n}\le\epsilon/2\},\tilde{n}(\epsilon/2)\Big),$$

such that

$$\|H_{n,k_n(2^{-n})}-H\|_{L^2(M)}\le\|H_{n,k_n(2^{-n})}-f_n\|_{L^2(M)}+\|f_n-H\|_{L^2(M)}\le\frac{1}{2^n}+\frac{\epsilon}{2}\le\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

holds for all $n\ge n(\epsilon)$.

Fix in the next step a general, not necessarily bounded, $H \in L^2(M)$ and define for each $n \in \mathbb{N}$ the predictable process $H_n = H\mathbb{1}_{\|H\|_2 \le n}$, which is obviously bounded by n. Thus for each $n \in \mathbb{N}$ there exists a sequence of predictable step processes $(H_{n,k})_{k\in\mathbb{N}}$ converging to H_n in $L^2(M)$ for $k \to \infty$. In other words, for each $\epsilon > 0$ there exists a $k_n(\epsilon) \in \mathbb{N}$, such that $\|H_{n,k} - H_n\|_{L^2(M)} \le \epsilon$ for all $k \ge k_n(\epsilon)$. Furthermore, the sequence $(H_n)_{n\in\mathbb{N}}$ converges in $L^2(M)$ to H as $n \to \infty$, due to Lemma 2.16, i.e. for all $\epsilon > 0$ exists some $\tilde{n}(\epsilon) \in \mathbb{N}$, such that $\|H_n - H\|_{L^2(M)} \le \epsilon$ for all $n \ge \tilde{n}(\epsilon)$. Similarly to above, consider now the sequence of predictable step functions $(H_{n,k_n(2^{-n})})_{n\in\mathbb{N}}$ and fix $\epsilon > 0$. By those preliminary findings, it is now apparent that $(H_{n,k_n(2^{-n})})_{n\in\mathbb{N}}$ converges to H in $L^2(M)$, as

$$\|H_{n,k_n(2^{-n})} - H\|_{L^2(M)} \le \|H_{n,k_n(2^{-n})} - H_n\|_{L^2(M)} + \|H_n - H\|_{L^2(M)} \le \frac{1}{2^n} + \frac{\epsilon}{2} \le \epsilon$$

holds for each $n \ge n(\epsilon) := \max\bigl(\min\{n \in \mathbb{N} : 2^{-n} \le \epsilon/2\}, \tilde{n}(\epsilon/2)\bigr)$.

In the last step, let M denote a general $\mathbb{K}^d$-valued continuous local martingale. As already seen in the proof of Lemma 2.14, there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ satisfying $\tau_n \le \tau_{n+1}$, $\lim_{n\to\infty} \tau_n = \infty$ almost surely as well as

$$\mathbb{E}\Bigl[\int_0^{\tau_n} \Bigl(\sum_{j=1}^d (\pi^{(M)})_t^{jj}\Bigr)\, \mathrm{d}C_t^{(M)}\Bigr]^{1/2} < \infty, \qquad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$ one may now define the process $H_n = H\mathbb{1}_{[0,\tau_n]}$. Fix now some $n \in \mathbb{N}$. The stopped process M^{τ_n} fulfills the assumption from the beginning of this proof, as

$$\mathbb{E}\Bigl[\int_0^{\infty} \Bigl(\sum_{j=1}^d (\pi^{(M^{\tau_n})})_t^{jj}\Bigr)\, \mathrm{d}C_t^{(M^{\tau_n})}\Bigr]^{1/2} = \mathbb{E}\Bigl[\int_0^{\tau_n} \Bigl(\sum_{j=1}^d (\pi^{(M)})_t^{jj}\Bigr)\, \mathrm{d}C_t^{(M)}\Bigr]^{1/2} < \infty$$

shows. Thus there exists a sequence of predictable step processes $(H_{n,k})_{k\in\mathbb{N}}$ converging to H in $L^2(M^{\tau_n})$ for $k \to \infty$. As

$$\lim_{k\to\infty} \|H_{n,k}\mathbb{1}_{[0,\tau_n]} - H\|_{L^2(M^{\tau_n})} = \lim_{k\to\infty} \|H_{n,k} - H\|_{L^2(M^{\tau_n})} = 0,$$

one may assume $H_{n,k}(t,\omega) = 0$ for $t > \tau_n(\omega)$ for each $k \in \mathbb{N}$ without loss of generality. For any process $K \in L^2(M)$ Definition 2.9 implies

$$\begin{aligned}
\|K\mathbb{1}_{[0,\tau_n]}\|_{L^2(M)} &= \mathbb{E}\bigl[\bigl((K^\intercal \mathbb{1}_{[0,\tau_n]}\pi^{(M)}\overline{K}\mathbb{1}_{[0,\tau_n]} \bullet C^{(M)}\bigr)_\infty\bigr]^{1/2} \\
&= \mathbb{E}\bigl[\bigl((K^\intercal \pi^{(M)}\overline{K} \bullet C^{(M)}\bigr)_{\tau_n}\bigr]^{1/2} \\
&= \mathbb{E}\bigl[\bigl((K^\intercal \pi^{(M)}\overline{K} \bullet (C^{(M)})^{\tau_n}\bigr)_\infty\bigr]^{1/2} \\
&= \mathbb{E}\bigl[\bigl((K^\intercal \pi^{(M^{\tau_n})}\overline{K} \bullet C^{(M^{\tau_n})}\bigr)_\infty\bigr]^{1/2} \\
&= \|K\|_{L^2(M^{\tau_n})}.
\end{aligned}$$

Thus for each $\epsilon > 0$ exists a $k_n(\epsilon) \in \mathbb{N}$, such that

$$\|H_{n,k} - H_n\|_{L^2(M)} = \|(H_{n,k} - H)\mathbb{1}_{[0,\tau_n]}\|_{L^2(M)} = \|H_{n,k} - H\|_{L^2(M^{\tau_n})} \le \epsilon, \qquad k \ge k_n(\epsilon).$$

Furthermore, Lemma 2.15 implies for each $\epsilon > 0$ the existence of an $\tilde{n}(\epsilon) \in \mathbb{N}$, such that

$$\|H_n - H\|_{L^2(M)} \leq \epsilon, \qquad n \geq \tilde{n}(\epsilon).$$

Thus one may use the same trick as above one more time, i.e. consider the sequence of predictable step processes $(H_{n,k_n(2^{-n})})_{n\in\mathbb{N}}$ and define for each $\epsilon > 0$ the natural number $n(\epsilon) := \max\big(\min\{n \in \mathbb{N} : 2^{-n} \leq \epsilon/2\}, \tilde{n}(\epsilon/2)\big)$. Thus one can easily see that $H_{n,k_n(2^{-n})} \to H$ in $L^2(M)$, due to

$$\|H_{n,k_n(2^{-n})} - H\|_{L^2(M)} \leq \|H_{n,k_n(2^{-n})} - H_n\|_{L^2(M)} + \|H_n - H\|_{L^2(M)} \leq \frac{1}{2^n} + \frac{\epsilon}{2} \leq \epsilon$$

for $n \geq n(\epsilon)$, which concludes the proof. □

3 The stochastic integral

3.1 The stochastic integral w.r.t. multi-dimensional continuous local martingales

At first one may only consider simple integrands, i.e. predictable step processes.

Definition 3.1 (Stochastic integral for predictable step processes)**.** Let $H_t = \varphi_0 \mathbb{1}_{\{0\}}(t) + \sum_{n=1}^m \varphi_n \mathbb{1}_{(\tau_n, \tau_{n+1}]}(t)$ be a $\mathbb{K}^d$-valued predictable step process (see Definition 2.2) and M a $\mathbb{K}^d$-valued continuous local martingale. Then the $\mathbb{K}$-valued stochastic integral of H w.r.t. M is defined pathwise as

$$H \bullet M = \int_0^\cdot H_t \, \mathrm{d}M_t = \sum_{n=1}^m \varphi_n^\top \big(M^{\tau_{n+1}} - M^{\tau_n}\big).$$

By [Sch23, Lemma 5.13(b)] for each $n \in \mathbb{N}$ the process $\varphi_n^\top \big(M^{\tau_{n+1}} - M^{\tau_n}\big)$ is a $\mathbb{K}$-valued continuous local martingale. Thus the integral process $H \bullet M$ is also a $\mathbb{K}$-valued continuous local martingale and as such also predictable. In fact, the integral process is even a martingale, which will be shown a bit further below. This definition also implies that $(H \bullet M)_0 = 0$ for all predictable step processes $H \in L^2(M)$.

For a $\mathbb{K}^d$-valued predictable step process $H = (H_1, \dots, H_d)^\top$ and $\mathbb{K}^d$-valued continuous local martingale $M = (M_1, \dots, M_d)^\top$ it can be easily seen that

$$\begin{aligned}(H \bullet M)_t &= \sum_{n=1}^m \varphi_n^\top \big(M_t^{\tau_{n+1}} - M_t^{\tau_n}\big) = \sum_{n=1}^m \sum_{j=1}^d \varphi_n^j \big(M^j_{\tau_{n+1} \wedge t} - M^j_{\tau_n \wedge t}\big) \\ &= \sum_{j=1}^d \sum_{n=1}^m \varphi_n^j \big(M^j_{\tau_{n+1} \wedge t} - M^j_{\tau_n \wedge t}\big) = \sum_{j=1}^d (H^j \bullet M^j)_t\end{aligned}$$

holds for all $t \in \mathbb{R}_+$. Furthermore, for each stopping time τ holds $(H \bullet M)^\tau = H \bullet M^\tau$. Additionally, the equality $(H \bullet M)^\tau = H^\tau \bullet M^\tau$ follows by [Sch23, p. 152].

Lemma 3.2. *Let G, H be two $\mathbb{K}^d$-valued predictable step processes, $\alpha \in \mathbb{K}$ and M a $\mathbb{K}^d$-valued continuous local martingale. Then the process $\alpha G + H$ is again a $\mathbb{K}^d$-valued predictable step process, which leads to the set of all $\mathbb{K}^d$-valued predictable step processes being a vector space, and the above defined stochastic integral is linear in the integrand, i.e.*

$$(\alpha G + H) \bullet M = \alpha (G \bullet M) + H \bullet M.$$

Proof.

Step 1 (Vector space property). For notational convenience let

$$G_t = \varphi_0^G \mathbb{1}_{\{0\}}(t) + \sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{(\tau_n^G, \tau_{n+1}^G]}(t), \qquad t \in \mathbb{R}_+$$

and

$$H_t = \varphi_0^H \mathbb{1}_{\{0\}}(t) + \sum_{n=1}^{m^H} \varphi_n^H \mathbb{1}_{(\tau_n^H, \tau_{n+1}^H]}(t), \qquad t \in \mathbb{R}_+.$$

Without loss of generality assume $m^G \leq m^H$ and define $\varphi_n^G = 0$ as well as $\tau_{n+1}^G = \tau_{m^G+1}^G$ for all $n = m^G + 1, \dots, m^H$. Similarly to [Sch23, p. 152] one may define

$$\tau_n = \max_{\substack{I \subseteq \{G,H\} \times \{1,\dots,n\} \\ |I| = n+1}} \min_{(i,j) \in I} \tau_j^i, \qquad n \in \{1, \dots, m^H + 1\},$$

and

$$\tau_n = \max_{\substack{I \subseteq \{G,H\} \times \{1,\dots,m^H+1\} \\ |I| = 2m^H+3-n}} \min_{(i,j) \in I} \tau_j^i, \qquad n \in \{m^H + 2, \dots, 2m^H + 2\},$$

where $|I|$ denotes the cardinality, i.e. the number of elements, of I. Fix now $n \in \{1, \dots 2m^H + 2\}$ and note that each element of I is itself a pair consisting of a capital letter (either G or M) and a natural number smaller than or equal to n. By [Sch23, Lemma 3.12(b)] follows that τ_n is a stopping time, as $\min_{(i,j) \in I} \tau_j^i$ is a stopping time, due to it being the minimum of finitely many stopping times for each set $I \subseteq \{G, H\} \times \{1, \dots, n \wedge (m^H + 1)\}$. Thus τ_n is the maximum of finitely many stopping times and consequently again a stopping time, by using [Sch23, Lemma 3.12(b)] once more.

By definition, $\tau_1 = \tau_1^G \wedge \tau_1^H = \min\{\tau_j^i : i \in \{G, H\},\ j \in \{1, \dots, m^H + 1\}\}$, as both sequences $(\tau_n^i)_{n \in \{1,\dots,m^H\}}$ for $i \in \{G, H\}$ are increasing. Thus one may now use induction and assume that

$$\tau_k = \min\Big(\{\tau_j^i : i \in \{G, H\},\ j \in \{1, \dots, m^H + 1\}\} \setminus \{\tau_1, \dots, \tau_{k-1}\}\Big)$$

for all $k = 1, \dots, n-1$ and fixed $n \in \{1, \dots, m^H + 1\}$, which also implies $\tau_1 \leq \dots \leq \tau_{n-1}$. Note that there exists exactly one set $\tilde{I} \subseteq \{G, H\} \times \{1, \dots, n\}$ satisfying $|\tilde{I}| = n + 1$, such that $\tilde{I} \cap \{\tau_j : j = 1, \dots, n-1\} = \varnothing$, as the cardinality of $\{G, H\} \times \{1, \dots, n\} = 2n$. Thus each subset of it with cardinality $n + 1$ leaves out exactly $2n - (n + 1) = n - 1$ elements. As both sequences $(\tau_n^i)_{n \in \{1,\dots,m^i\}}$ for $i \in \{G, H\}$ are increasing,

$$\min_{(i,j) \in I} \tau_j^i \leq \tau_{n-1} \leq \min_{(i,j) \in \tilde{I}} \tau_j^i, \qquad I \subseteq \{G, H\} \times \{1, \dots, n\} \text{ satisfying } |I| = n + 1,\ I \neq \tilde{I},$$

holds, which implies

$$\tau_n = \min_{(i,j) \in \tilde{I}} \tau_j^i = \min\Big(\{\tau_j^i : i \in \{G, H\},\ j \in \{1, \dots, m^H + 1\}\} \setminus \{\tau_1, \dots, \tau_{n-1}\}\Big),$$

for each $n \in \{1, \dots, m^H+1\}$. Similarly, for fixed (m^H+1+n), where again $n \in \{1, \dots, m^H+1\}$, there exists exactly one subset $\tilde{I} \subseteq \{G, H\} \times \{1, \dots, m^H+1\}$ satisfying

$$|\tilde{I}| = 2m^H + 3 - (m^H + 1 + n) = m^H - n + 2,$$

such that $\tilde{I} \cap \{\tau_j : j = 1, \dots, (m^H + 1 + n) - 1\} = \varnothing$, as the cardinality of $\{G, H\} \times \{1, \dots, m^H + 1\} = 2m^H + 2$. Thus each subset of it with cardinality $m^H - n + 2$ leaves out exactly $2m^H + 2 - (m^H - n + 2) = m^H + n$ elements. Consequently,

$$\min_{(i,j)\in I} \tau_j^i \leq \tau_{m^H+n} \leq \min_{(i,j)\in \tilde{I}} \tau_j^i, \quad I \subseteq \{G, H\} \times \{1, \dots, m^H+1\} \text{ with } |I| = 2m^H+3-n,\ I \neq \tilde{I},$$

holds for all $n \in \{1, \dots, m^H + 1\}$ in the same way as above, leading to

$$\tau_n = \min_{(i,j)\in \tilde{I}} \tau_j^i = \min\Big(\big\{\tau_j^i : i \in \{G, H\},\ j \in \{1, \dots, m^H + 1\}\big\} \setminus \{\tau_1, \dots, \tau_{n-1}\}\Big)$$

for each $n \in \{1, \dots, 2m^H + 2\}$. Thus the sequence $(\tau_n)_{n\in\{1,\dots,2m^H+2\}}$ is an increasing finite sequence of stopping times satisfying

$$\big\{\tau_n(\omega) : n \in \{1, \dots, 2m^H + 2\}\big\} = \big\{\tau_n^i(\omega) : i \in \{G, H\},\ n \in \{1, \dots, m^i + 1\}\big\}$$

for each $\omega \in \Omega$.

Furthermore,

$$\begin{aligned}
\sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{(\tau_n^G, \tau_{n+1}^G]}(t) &= \sum_{k=1}^{2m^H+1} \mathbb{1}_{(\tau_k,\tau_{k+1}]}(t) \Big(\sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{(\tau_n^G, \tau_{n+1}^G]}(t)\Big) \\
&= \sum_{k=1}^{2m^H+1} \sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{(\tau_n^G, \tau_{n+1}^G]}(t) \mathbb{1}_{(\tau_k,\tau_{k+1}]}(t) \\
&= \sum_{k=1}^{2m^H+1} \Big(\sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{\{(\tau_k,\tau_{k+1}] \subseteq (\tau_n^G, \tau_{n+1}^G]\}}\Big) \mathbb{1}_{(\tau_k,\tau_{k+1}]}(t) \\
&= \sum_{k=1}^{2m^H+1} \Big(\sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{\{(\tau_k,\tau_{k+1}] \subseteq (\tau_n^G, \tau_{n+1}^G]\}}\, \mathbb{1}_{\{\tau_k < \tau_{k+1}\}}\Big) \mathbb{1}_{(\tau_k,\tau_{k+1}]}(t)
\end{aligned}$$

holds for each $t \in \mathbb{R}_+$. Note that for each pair $(\omega, k) \in \Omega \times \{1, \dots 2m^H + 1\}$ exists at most one $n \in \{1, \dots, m^G\}$, such that

$$\mathbb{1}_{\{(\tau_k,\tau_{k+1}] \subseteq (\tau_n^G, \tau_{n+1}^G]\}}(\omega)\, \mathbb{1}_{\{\tau_k < \tau_{k+1}\}}(\omega) = 1$$

and define $\Omega \ni \omega \mapsto \psi(\omega) \equiv 0 \in \mathbb{K}^d$. Additionally, for each two stopping times σ and τ and a set $F \in \mathcal{F}_\sigma$ follows

$$F \cap \{\sigma < \tau\} = F \cap \big(\{\sigma \leq \tau\} \setminus \{\sigma = \tau\}\big) = \underbrace{\big(F \cap \{\sigma \leq \tau\}\big)}_{\in \mathcal{F}_{\sigma\wedge\tau}} \setminus \underbrace{\{\sigma = \tau\}}_{\in \mathcal{F}_{\sigma\wedge\tau}} \in \mathcal{F}_{\sigma\wedge\tau},$$

due to [Sch23, Lemma 3.12(g) and (h)]. Consequently, for fixed $k \in \{1, \dots, 2m^H + 1\}$ the random vector given by

$$\begin{aligned}\Omega \ni \omega \mapsto \tilde{\varphi}_k^G &= \sum_{n=1}^{m^G} \varphi_n^G(\omega) \mathbb{1}_{\{(\tau_k, \tau_{k+1}] \subseteq (\tau_n^G, \tau_{n+1}^G]\}}(\omega)\, \mathbb{1}_{\{\tau_k < \tau_{k+1}\}}(\omega) \\ &= \mathbb{1}_{\{\tau_k < \tau_{k+1}\}}(\omega) \sum_{n=1}^{m^G} \varphi_n^G(\omega) \mathbb{1}_{\{\tau_n^G \le \tau_k\}}(\omega) \mathbb{1}_{\{\tau_{n+1}^G \ge \tau_{k+1}\}}(\omega) \\ &= \mathbb{1}_{\{\tau_k < \tau_{k+1}\}}(\omega) \sum_{n=1}^{m^G} \varphi_n^G(\omega) \mathbb{1}_{\{\tau_n^G \le \tau_k < \tau_{n+1}^G\}}(\omega)\end{aligned}$$

is bounded, $\mathbb{K}^d$-valued and $\mathcal{F}_{\tau_k}$-measurable, as one can use [Sch23, Lemma 3.12(d), (g) and (h)] to obtain

$$\begin{aligned}&\left(\mathbb{1}_{\{\tau_k < \tau_{k+1}\}} \sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{\{\tau_n^G \le \tau_k < \tau_{n+1}^G\}} \right)^{-1}(A) \\ &= \Big(\underbrace{\psi^{-1}(A)}_{\{\varnothing,\Omega\} \subseteq \mathcal{F}_{\tau_k}} \cap \Big(\underbrace{\{\tau_k < \tau_1^G\}}_{\in \mathcal{F}_{\tau_k \wedge \tau_1^G} \subseteq \mathcal{F}_{\tau_k}} \cup \underbrace{\{\tau_k \ge \tau_{m^G+1}^G\}}_{\in \mathcal{F}_{\tau_k}} \cup \underbrace{\{\tau_k = \tau_{k+1}\}}_{\in \mathcal{F}_{\tau_k}} \Big)\Big) \\ &\cup \left(\bigcup_{n=1}^{m^G} \underbrace{\underbrace{(\varphi_n^G)^{-1}(A)}_{\in \mathcal{F}_{\tau_n^G}} \cap \{\tau_n^G \le \tau_k\} \cap}_{\in \mathcal{F}_{\tau_n^G \wedge \tau_k} \subseteq \mathcal{F}_{\tau_k}} \underbrace{\{\tau_k < \tau_{n+1}^G\}}_{\in \mathcal{F}_{\tau_k \wedge \tau_{n+1}^G} \subseteq \mathcal{F}_{\tau_k}} \cap \underbrace{\{\tau_k < \tau_{k+1}\}}_{\in \mathcal{F}_{\tau_k \wedge \tau_{k+1}} \subseteq \mathcal{F}_{\tau_k}} \right) \in \mathcal{F}_{\tau_k}\end{aligned}$$

for all $A \in \mathcal{B}_{\mathbb{K}^d}$. Analogously one can see that for each $t \in \mathbb{R}_+$ holds

$$\sum_{n=1}^{m^H} \varphi_n^H \mathbb{1}_{(\tau_n^H, \tau_{n+1}^H]}(t) = \sum_{k=1}^{2m^H+1} \left(\sum_{n=1}^{m^H} \varphi_n^H \mathbb{1}_{\{(\tau_k, \tau_{k+1}] \subseteq (\tau_n^H, \tau_{n+1}^H]\}}\, \mathbb{1}_{\{\tau_k < \tau_{k+1}\}} \right) \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t)$$

and the random vector defined as

$$\Omega \ni \omega \mapsto \tilde{\varphi}_k^H = \sum_{n=1}^{m^H} \varphi_n^H(\omega) \mathbb{1}_{\{(\tau_k, \tau_{k+1}] \subseteq (\tau_n^H, \tau_{n+1}^H]\}}(\omega)\, \mathbb{1}_{\{\tau_k < \tau_{k+1}\}}(\omega)$$

is also bounded, $\mathbb{K}^d$-valued and $\mathcal{F}_{\tau_k}$-measurable for each $k \in \{1, \dots, 2m^H + 1\}$.

Consequently,

$$\begin{aligned}\alpha G_t + H_t &= (\alpha \varphi_0^G + \varphi_0^H) \mathbb{1}_{\{0\}}(t) + \alpha \sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{(\tau_n^G, \tau_{n+1}^G]}(t) + \sum_{n=1}^{m^H} \varphi_n^H \mathbb{1}_{(\tau_n^H, \tau_{n+1}^H]}(t) \\ &= (\alpha \varphi_0^G + \varphi_0^H) \mathbb{1}_{\{0\}}(t) + \alpha \sum_{k=1}^{2m^H+1} \tilde{\varphi}_k^G \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t) + \sum_{k=1}^{2m^H+1} \tilde{\varphi}_k^H \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t) \\ &= (\alpha \varphi_0^G + \varphi_0^H) \mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{2m^H+1} (\alpha \tilde{\varphi}_k^G + \tilde{\varphi}_k^H) \mathbb{1}_{(\tau_k, \tau_{k+1}]}(t)\end{aligned}$$

for each $t \in \mathbb{R}_+$, where $\alpha\varphi_0^G + \varphi_0^H$ is a bounded $\mathcal{F}_0$-measurable random vector and $\alpha\tilde{\varphi}_k^G + \tilde{\varphi}_k^H$ is a bounded $\mathcal{F}_{\tau_k}$-measurable random vector for each $k \in \{1, \dots, 2m^H + 1\}$. Thus $\alpha G + H$ is a $\mathbb{K}^d$-valued predictable step process by Definition 2.2 and the set of all $\mathbb{K}^d$-valued predictable step processes is consequently a vector space.

Step 2 (Linearity of the integral). As $\alpha G + H$ is a $\mathbb{K}^d$-valued predictable step process, the stochastic integral $(\alpha G + H) \bullet M$ exists according to Definition 3.1. Thus one may use the previous findings of this proof to obtain

$$\begin{aligned}(\alpha G + H) \bullet M &= \Big((\alpha\varphi_0^G + \varphi_0^H)\mathbb{1}_{\{0\}} + \sum_{k=1}^{2m^H+1} (\alpha\tilde{\varphi}_k^G + \tilde{\varphi}_k^H)\mathbb{1}_{(\tau_k,\tau_{k+1}]} \Big) \bullet M \\ &= \sum_{k=1}^{2m^H+1} (\alpha\tilde{\varphi}_k^G + \tilde{\varphi}_k^H)^\mathsf{T}(M^{\tau_{k+1}} - M^{\tau_k}) \\ &= \alpha \sum_{k=1}^{2m^H+1} (\tilde{\varphi}_k^G)^\mathsf{T}(M^{\tau_{k+1}} - M^{\tau_k}) + \sum_{k=1}^{2m^H+1} (\tilde{\varphi}_k^H)^\mathsf{T}(M^{\tau_{k+1}} - M^{\tau_k}).\end{aligned}$$

Define now for each $n \in \{1, \dots, m^G\}$ and $\omega \in \Omega$ the natural number

$$k_n^G(\omega) = \inf\{k \in \{1, \dots, 2m^H + 2\} : \tau_n^G(\omega) = \tau_k(\omega)\}.$$

Consequently, for each pair $(t, \omega) \in \mathbb{R}_+ \times \Omega$ follows

$$\begin{aligned}&\sum_{k=1}^{2m^H+1} (\tilde{\varphi}_k^G)^\mathsf{T}(M_t^{\tau_{k+1}} - M_t^{\tau_k}) = \sum_{k=1}^{2m^H+1} \Big(\mathbb{1}_{\{\tau_k < \tau_{k+1}\}} \sum_{n=1}^{m^G} \varphi_n^G \mathbb{1}_{\{\tau_n^G \le \tau_k < \tau_{n+1}^G\}} \Big)^\mathsf{T} (M_t^{\tau_{k+1}} - M_t^{\tau_k}) \\ &= \sum_{k=1}^{2m^H+1} \sum_{n=1}^{m^G} (\varphi_n^G)^\mathsf{T} \mathbb{1}_{\{\tau_n^G \le \tau_k < \tau_{n+1}^G\}} (M_t^{\tau_{k+1}} - M_t^{\tau_k}) \\ &= \sum_{n=1}^{m^G} (\varphi_n^G)^\mathsf{T} \sum_{k=1}^{2m^H+1} \mathbb{1}_{\{\tau_n^G \le \tau_k < \tau_{n+1}^G\}} (M_t^{\tau_{k+1}} - M_t^{\tau_k}) = \sum_{n=1}^{m^G} (\varphi_n^G)^\mathsf{T} \sum_{k=k_n(\omega)}^{k_{n+1}(\omega)-1} M_t^{\tau_{k+1}} - M_t^{\tau_k} \\ &= \sum_{n=1}^{m^G} (\varphi_n^G)^\mathsf{T} (M_t^{\tau_{k_{n+1}}} - M_t^{\tau_{k_n}}) = \sum_{n=1}^{m^G} (\varphi_n^G)^\mathsf{T} (M_t^{\tau_{n+1}^G} - M_t^{\tau_n^G}) = (G \bullet M)_t\end{aligned}$$

and analogously

$$\sum_{k=1}^{2m^H+1} (\tilde{\varphi}_k^H)^\mathsf{T}(M^{\tau_{k+1}} - M^{\tau_k}) = H \bullet M.$$

Thus one may now combine those results to obtain

$$\begin{aligned}(\alpha G + H) \bullet M &= \alpha \sum_{k=1}^{2m^H+1} (\tilde{\varphi}_k^G)^\mathsf{T}(M^{\tau_{k+1}} - M^{\tau_k}) + \sum_{k=1}^{2m^H+1} (\tilde{\varphi}_k^H)^\mathsf{T}(M^{\tau_{k+1}} - M^{\tau_k}) \\ &= \alpha(G \bullet M) + H \bullet M,\end{aligned}$$

which completes the proof. □

Another very important property of the above defined stochastic integral for $\mathbb{K}^d$-valued predictable step processes w.r.t. $\mathbb{K}^d$-valued continuous local martingales will be shown in the following lemma.

Lemma 3.3. *Let H be a predictable step process and M a continuous local martingale, which are both $\mathbb{K}^d$-valued. Then the quadratic variation process of the stochastic integral*

$$[H \bullet M] = \int_0^{\cdot} \left(\sum_{i,j=1}^{d} H_s^i \pi_s^{ij} \overline{H}_s^j \right) \mathrm{d}C_s = (H^\mathsf{T} \pi \overline{H}) \bullet C$$

up to indistinguishability.

Proof. Let $H = (H_1, \dots, H_d)^\mathsf{T}$, where $H^j = \varphi_0^j \mathbb{1}_{\{0\}}(t) + \sum_{n=1}^m \varphi_n^j \mathbb{1}_{(\tau_n, \tau_{n+1}]}(t)$ for each $j = 1, \dots, d$. Thus for all $t \in \mathbb{R}_+$ one obtains

$$\begin{aligned}
[H \bullet M]_t &= \left[\sum_{j=1}^{d} H^j \bullet M^j \right]_t = \sum_{j,k=1}^{d} [H^j \bullet M^j, \overline{H}^k \bullet \overline{M}^k]_t \\
&= \sum_{j,k=1}^{d} \left[\sum_{n=1}^{m} \varphi_n^j (M^j_{\tau_{n+1}\wedge s} - M^j_{\tau_n \wedge s}), \sum_{l=1}^{m} \bar{\varphi}_l^k (\overline{M}^k_{\tau_{l+1}\wedge s} - \overline{M}^k_{\tau_l \wedge s}) \right]_t \\
&= \sum_{j,k=1}^{d} \sum_{n,l=1}^{m} \left[\varphi_n^j (M^j_{\tau_{n+1}\wedge s} - M^j_{\tau_n \wedge s}), \bar{\varphi}_l^k (\overline{M}^k_{\tau_{l+1}\wedge s} - \overline{M}^k_{\tau_l \wedge s}) \right]_t .
\end{aligned}$$

Considering each $n = 1, \dots, m$ separately, for each $\omega \in \Omega$ the processes $\varphi_n^j (M^j_{\tau_{n+1}\wedge s} - M^j_{\tau_n \wedge s})$ are zero regardless of φ_n^j up to $\tau_n(\omega)$ for each $j = 1, \dots, d$. Thus also the covariation process is zero. Therefore, the linearity of the covariation process extends in this case not only to $\mathcal{F}_0$, but to all $\mathcal{F}_{\tau_n}$-measurable random variables, particularly φ_n^j. An analogous conclusion can be drawn for the processes $\bar{\varphi}_l^k (\overline{M}^k_{\tau_{l+1}\wedge s} - \overline{M}^k_{\tau_l \wedge s})$, where $l = 1, \dots, m$ and $k = 1, \dots, d$. Furthermore, those processes are constant, except when $\tau_n(\omega) < s \le \tau_{n+1}(\omega)$, which leads to the covariation process $\left[\varphi_n^j (M^j_{\tau_{n+1}\wedge s} - M^j_{\tau_n \wedge s}), \bar{\varphi}_l^k (\overline{M}^k_{\tau_{l+1}\wedge s} - \overline{M}^k_{\tau_l \wedge s}) \right]$ being zero, whenever $n \neq l$, because then at least one of the two continuous local martingales in the covariation bracket would be constant on every interval in $\mathbb{R}_+$. Consequently,

$$[H \bullet M]_t = \sum_{j,k=1}^{d} \sum_{n=1}^{m} \left[\varphi_n^j (M^j_{\tau_{n+1}\wedge s} - M^j_{\tau_n \wedge s}), \bar{\varphi}_n^k (\overline{M}^k_{\tau_{n+1}\wedge s} - \overline{M}^k_{\tau_n \wedge s}) \right]_t, \qquad t \in \mathbb{R}_+.$$

By carefully using the above mentioned extended linearity argument twice, one obtains

$$\begin{aligned}
[H \bullet M]_t &= \sum_{j,k=1}^{d} \sum_{n=1}^{m} \varphi_n^j \bar{\varphi}_n^k \left[(M^j_{\tau_{n+1}\wedge s} - M^j_{\tau_n \wedge s}), (\overline{M}^k_{\tau_{n+1}\wedge s} - \overline{M}^k_{\tau_n \wedge s}) \right]_t \\
&= \sum_{j,k=1}^{d} \sum_{n=1}^{m} \varphi_n^j \bar{\varphi}_n^k \Big([M^j_{\tau_{n+1}\wedge s}, \overline{M}^k_{\tau_{n+1}\wedge s}]_t - [M^j_{\tau_{n+1}\wedge s}, \overline{M}^k_{\tau_n \wedge s}]_t - [M^j_{\tau_n \wedge s}, \overline{M}^k_{\tau_{n+1}\wedge s}]_t \\
&\quad + [M^j_{\tau_n \wedge s}, \overline{M}^k_{\tau_n \wedge s}]_t \Big)
\end{aligned}$$

for each $t \in \mathbb{R}_+$, which then leads to

$$\begin{aligned}[H \bullet M]_t &= \sum_{j,k=1}^{d} \sum_{n=1}^{m} \varphi_n^j \bar{\varphi}_n^k \big([M^j, \overline{M}^k]_t^{\tau_{n+1}} - [M^j, \overline{M}^k]_t^{\tau_n} \big) \\ &= \sum_{j,k=1}^{d} \sum_{n=1}^{m} \varphi_n^j \bar{\varphi}_n^k \int_{\tau_n \wedge t}^{\tau_{n+1} \wedge t} \pi_s^{jk} \, \mathrm{d}C_s = \big((H^\top \pi \overline{H}) \bullet C \big)_t, \end{aligned}$$

where in the second-to-last equality Theorem 2.7 is used. □

Up to this point only one given continuous local martingale was being considered. In the definitions below two Banach spaces, whose elements are continuous (local) martingales, will be introduced. In the following let $\mathcal{M}$ and $\mathcal{M}_{\text{loc}}$ denote the vector space of all $\mathbb{K}$-valued continuous martingales and continuous local martingales, respectively.

Definition 3.4 (The Banach space $\mathcal{H}^2$ and the norm $\|\cdot\|_{\mathcal{H}^2}$)**.** On the above introduced space $\mathcal{M}$ set the function $\|\cdot\|_{\mathcal{H}^2} : \mathcal{M} \to \overline{\mathbb{R}}_+$ to be

$$\|M\|_{\mathcal{H}^2} = \mathbb{E}\Big[\sup_{t \in \mathbb{R}_+} |M_t|^2 \Big]^{1/2}$$

and define the space $\mathcal{H}^2 = \{ M \in \mathcal{M} : \|M\|_{\mathcal{H}^2} < \infty \}$.

From this point on, no distinction will be made between a process in $\mathcal{H}^2$ and its equivalence class of all processes in $\mathcal{H}^2$ that are equal up to indistinguishability, in order for $\|\cdot\|_{\mathcal{H}^2}$ to define a norm on $\mathcal{H}^2$.
In the following let $\tilde{\mathcal{H}}^2$ denote the set of all aforementioned equivalence classes of $\mathbb{K}$-valued *càdlàg* martingales M satisfying $\|M\|_{\mathcal{H}^2} < \infty$, as opposed to $\mathcal{H}^2$ including only the continuous ones. Note that càdlàg is an acronym of the French phrase *continue à droite, limite à gauche*, which means that for each $\omega \in \Omega$ the path $M_{\cdot}(\omega)$ is right-continuous and the left-hand limit exists for each $t \in (0, \infty)$. Consequently, $\mathcal{H}^2 \subseteq \tilde{\mathcal{H}}^2$. Then $(\tilde{\mathcal{H}}^2, \|\cdot\|_{\mathcal{H}^2})$ is a Banach space, as shown in [CE15, Lemma 10.1.5]. Furthermore, as stated by [CE15, Remark 10.1.11], the limit in $\tilde{\mathcal{H}}^2$ of a sequence of $\mathbb{K}$-valued continuous martingales in $\mathcal{H}^2$ is then itself again a $\mathbb{K}$-valued continuous martingale and consequently also an element of $\mathcal{H}^2$. Thus the above defined set $\mathcal{H}^2$ equipped with the norm $\|\cdot\|_{\mathcal{H}^2}$ is indeed a Banach space, as for $\mathbb{K}$-valued continuous martingales M^1 and M^2 as well as $\alpha \in \mathbb{K}$ also $\alpha M^1 + M^2$ is a $\mathbb{K}$-valued continuous martingale. Similarly, one can also consider continuous *local* martingales M satisfying $\|M\|_{\mathcal{H}^2} < \infty$. However, the following lemma shows that this does not yield a generalization of the space $\mathcal{H}^2$.

Lemma 3.5. *If for $M \in \mathcal{M}_{\text{loc}}$ the value $\|M\|_{\mathcal{H}^2}$ is finite, then M is a continuous martingale and thus also in $\mathcal{H}^2$.*

Proof. Fix $M \in \mathcal{M}_{\text{loc}}$, such that $\|M\|_{\mathcal{H}^2} < \infty$, which directly shows that the random variable $\sup_{s \in \mathbb{R}_+} |M_s|$ is in $\in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, *Jensen's inequality* [Gri18, Satz 8.1] applied to the convex function $\mathbb{R}_+ \ni x \mapsto x^2$ implies that $\sup_{s \in \mathbb{R}_+} |M_s|$ is also an element of $\in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Additionally, let $(\tau_n)_{n \in \mathbb{N}}$ be a localizing sequence of M. Then for each

$t \in \mathbb{R}_+$ also $M_t \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ as well as $M_{t\wedge\tau_n} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for each $n \in \mathbb{N}$, as for all $\omega \in \Omega$ it is evident that

$$|M_{t\wedge\tau_n(\omega)}(\omega)| \leq \sup_{s\in\mathbb{R}_+} |M_s(\omega)|, \qquad (t,n) \in \mathbb{R}_+ \times \mathbb{N}$$

holds. Furthermore, as the pointwise limit $\lim_{n\to\infty} \tau_n = \infty$ one may use the pathwise continuity of M to obtain $\lim_{n\to\infty} M_{t\wedge\tau_n} = M_t$. Consequently, for two non-negative real numbers $s \leq t$ the *conditional dominated convergence theorem* [Sch23, Theorem 17.15(j)] leads almost surely to

$$\mathbb{E}[M_t|\mathcal{F}_s] = \lim_{n\to\infty} \mathbb{E}[M_{t\wedge\tau_n}|\mathcal{F}_s] = \lim_{n\to\infty} \mathbb{E}[M_{t\wedge\tau_n} - M_0|\mathcal{F}_s] + M_0 = \lim_{n\to\infty} M_{s\wedge\tau_n} - M_0 + M_0 = M_s,$$

where the martingale property of $M^{\tau_n} - M_0$ for each $n \in \mathbb{N}$ has been used in the third step. Therefore, M is a martingale. □

Further examination of the space $\mathcal{H}^2$ yields the result that there exists an equivalent norm on a subspace of it, which is induced by a scalar product. This subspace $\mathcal{H}^2_0$ is defined as the set of all $M \in \mathcal{H}^2$ starting at zero. It is apparent that $(\mathcal{H}^2_0, \|\cdot\|_{\mathcal{H}^2})$ is again a Banach space, when identifying up to indistinguishability equal processes.

Lemma 3.6. *The function $\mathcal{H}^2_0 \times \mathcal{H}^2_0 \ni (M,N) \mapsto \langle M,N\rangle_{\mathcal{H}^2_0} := \mathbb{E}\big[[M,\bar{N}]_\infty\big] \in \mathbb{K}$ defines a scalar product on $\mathcal{H}^2_0$.*

Proof. For readablility set $\langle M,N\rangle = \langle M,N\rangle_{\mathcal{H}^2_0}$ throughout this proof.

The *Burkholder–Davis–Gundy inequalities* [Sch23, Theorem 5.84] combined with the monotonicity of the root function on $\mathbb{R}_+$ imply

$$\mathbb{E}\big[[M]_\infty\big]^{1/2} \leq \|M\|_{\mathcal{H}^2} \leq 2\,\mathbb{E}\big[[M]_\infty\big]^{1/2} \tag{3.1}$$

for all $M \in \mathcal{H}^2_0$. Thus this function takes indeed only finite values, as for $M, N \in \mathcal{H}^2_0$ the bounds

$$\begin{aligned}
|\langle M,N\rangle| &\leq \mathbb{E}\big[|[M,\bar{N}]_\infty(\omega)|\big] \leq \mathbb{E}\big[\mathbb{V}_{[M,\bar{N}]}(\mathbb{R}_+,\omega)\big] \leq \mathbb{E}\Big[\sqrt{[M](\mathbb{R}_+,\omega)}\sqrt{[N](\mathbb{R}_+,\omega)}\Big] \\
&\leq \sqrt{\mathbb{E}\big[[M](\mathbb{R}_+,\omega)\big]}\sqrt{\mathbb{E}\big[[N](\mathbb{R}_+,\omega)\big]} = \mathbb{E}\big[[M]_\infty\big]^{1/2}\,\mathbb{E}\big[[N]_\infty\big]^{1/2} \\
&\leq \|M\|_{\mathcal{H}^2}\,\|N\|_{\mathcal{H}^2} < \infty
\end{aligned}$$

hold, where in the third step equation (5.30) in the appendix and in the fourth one the Hölder inequality [Gri18, Satz 8.2] have been used.

Now fix M_1, M_2 and $M_3 \in \mathcal{H}^2_0$ and $\alpha \in \mathbb{K}$ and see that the conditions in Definition 5.17 below, namely

(i)

$$\begin{aligned}
\langle \alpha M_1 + M_2, M_3\rangle &= \mathbb{E}\big[[\alpha M_1 + M_2, \bar{M}_3]_\infty\big] = \alpha\mathbb{E}\big[[M_1,\bar{M}_3]_\infty\big] + \mathbb{E}\big[[M_2,\bar{M}_3]_\infty\big] \\
&= \alpha\langle M_1, M_3\rangle + \langle M_2, M_3\rangle,
\end{aligned}$$

(ii)

$$\langle M_1, \alpha M_2 + M_3 \rangle = \mathbb{E}\big[[M_1, \overline{\alpha M_2 + M_3}]_\infty\big] = \bar{\alpha}\mathbb{E}\big[[M_1, \bar{M}_2]_\infty\big] + \mathbb{E}\big[[M_1, \bar{M}_3]_\infty\big]$$
$$= \bar{\alpha}\langle M_1, M_2 \rangle + \langle M_1, M_3 \rangle, \qquad \text{and}$$

(iii)

$$\langle M_1, M_2 \rangle = \mathbb{E}\big[[M_1, \bar{M}_2]_\infty\big] = \mathbb{E}\big[[\bar{M}_2, M_1]_\infty\big] = \overline{\mathbb{E}\big[[M_2, \bar{M}_1]_\infty\big]} = \overline{\langle M_2, M_1 \rangle}$$

follow directly from the linearity of the expectation as well as [Sch23, Theorem 5.65]. Thus $\langle \cdot, \cdot \rangle$ is a symmetric bilinear or Hermitian sesquilinear form, corresponding to $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Furthermore, by [Sch23, Theorem 5.76(d)], for each $M \in \mathcal{H}_0^2$ the inequality $\langle M, M \rangle \geq 0$ holds, as $[M]$ is up to indistinguishability non-negative. Additionally, the aforementioned Burkholder–Davis–Gundy inequalities imply that

$$\langle M, M \rangle = \mathbb{E}\big[[M]_\infty\big] = 0 \iff \|M\|_{\mathcal{H}^2} = 0 \iff M \overset{\text{uti}}{=} 0$$

holds, proving the positive definiteness of $\langle \cdot, \cdot \rangle$ and consequently concluding the proof. □

Therefore, the following lemma is a direct result of the Burkholder–Davis–Gundy inequalities (3.1) mentioned in the proof above.

Lemma 3.7. *The norm on $\mathcal{H}_0^2$ that is induced by the above defined scalar product, i.e.*

$$\|M\|_{\mathcal{H}_0^2} := \sqrt{\langle M, M \rangle_{\mathcal{H}_0^2}} = \mathbb{E}\big[[M]_\infty\big]^{1/2},$$

is equivalent to $\|\cdot\|_{\mathcal{H}^2}$.

Consequently $(\mathcal{H}_0^2, \|\cdot\|_{\mathcal{H}_0^2})$ inherits the completeness of $(\mathcal{H}_0^2, \|\cdot\|_{\mathcal{H}^2})$ and is therefore also a Banach space. This will become very useful in the following, as Lemma 3.3 directly leads to

$$\|H \bullet M\|_{\mathcal{H}_0^2} = \mathbb{E}\big[[H \bullet M]_\infty\big]^{1/2} = \mathbb{E}\big[\big((H^\top \pi \bar{H}) \bullet C\big)_\infty\big]^{1/2} = \|H\|_{L^2(M)} \tag{3.2}$$

for all predictable step processes H in $L^2(M)$. Therefore, for each $\mathbb{K}^d$-valued continuous local martingale M and $H \in L^2(M)$ being a predictable step process the stochastic integral $H \bullet M$ is a continuous martingale starting at zero, not only a local martingale.

As stated in Lemma 2.17, each $H \in L^2(M)$ can be approximated by a sequence $(H_n)_{n\in\mathbb{N}}$ of predictable step processes in $L^2(M)$ and as such, $(H_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(M)$. Therefore, for each $\epsilon > 0$ exists an $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0$ the norm

$$\|(H_n \bullet M) - (H_m \bullet M)\|_{\mathcal{H}_0^2} = \|(H_n - H_m) \bullet M\|_{\mathcal{H}_0^2} = \|H_n - H_m\|_{L^2(M)} \leq \epsilon,$$

which makes the sequence of stochastic integrals $(H_n \bullet M)_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathcal{H}_0^2$. Note that in the first step of the display above Lemma 3.2 has been used. Due to the completeness of this space, there exists a unique limit of $(H_n \bullet M)_{n\in\mathbb{N}}$, which will in the following simply be denoted by $H \bullet M \in \mathcal{H}_0^2$.

Definition 3.8. Let M be a $\mathbb{K}^d$-valued continuous local martingale and $H \in L^2(M)$, according to Definition 2.9. Then the stochastic integral of H with respect to M

$$H \bullet M = \int_0^\cdot H_t \, \mathrm{d}M_t$$

is defined as the continuous martingale in $\mathcal{H}_0^2$ that is the $\mathcal{H}_0^2$-limit of the sequence $(H_n \bullet M)_{n\in\mathbb{N}}$, where the sequence of predictable step processes $(H_n)_{n\in\mathbb{N}}$ converges to H in $L^2(M)$.

Lemma 3.9. *In the setting of the definition above, the stochastic integral process $H \bullet M$ exists uniquely up to indistinguishability in $\mathcal{H}_0^2$, i.e. it does not depend on the sequence $(H_n)_{n\in\mathbb{N}}$ approximating H in $L^2(M)$.*

Proof. As in Definition 3.8 let $(H_n)_{n\in\mathbb{N}}$ denote a sequence of predictable step processes converging to H in $L^2(M)$ and $H \bullet M$ the $\mathcal{H}_0^2$-limit of $(H_n \bullet M)_{n\in\mathbb{N}}$. Furthermore, let $(\tilde{H}_n)_{n\in\mathbb{N}}$ be another sequence of predictable step processes converging to H in $L^2(M)$ and $\tilde{Y}$ the $\mathcal{H}_0^2$-limit of $(\tilde{H}_n \bullet M)_{n\in\mathbb{N}}$. Thus one can use the linearity of the stochastic integral for predictable step processes as well as equality (3.2) to see that

$$\begin{aligned} \|\tilde{Y} - (H \bullet M)\|_{\mathcal{H}_0^2} &= \mathbb{E}\big[[\tilde{Y} - (H \bullet M)]_\infty\big]^{1/2} = \lim_{n\to\infty} \mathbb{E}\big[[(\tilde{H}_n \bullet M) - (H_n \bullet M)]_\infty\big]^{1/2} \\ &= \lim_{n\to\infty} \mathbb{E}\big[[(\tilde{H}_n - H_n) \bullet M]_\infty\big]^{1/2} = \lim_{n\to\infty} \|\tilde{H}_n - H_n\|_{L^2(M)} = 0 \end{aligned}$$

holds, which implies the equality up to indistinguishability of $\tilde{Y}$ and $H \bullet M$. □

Lemma 3.10. *The isometry (3.2) holds for all $H \in L^2(M)$.*

Proof. Let again $(H_n)_{n\in\mathbb{N}}$ denote a sequence of predictable step processes converging to H in $L^2(M)$. Then by Definition 3.8 follows

$$\|H \bullet M\|_{\mathcal{H}_0^2} = \lim_{n\to\infty} \|H_n \bullet M\|_{\mathcal{H}_0^2} = \lim_{n\to\infty} \|H_n\|_{L^2(M)} = \|H\|_{L^2(M)},$$

which proves the lemma. □

The lemma below will be used on multiple occasions throughout the remainder of this chapter. Note that the linearity of the stochastic integral proven in the first part will be extended to processes $H \in L^2_{\text{loc}}(M)$ in Lemma 3.23.

Lemma 3.11. *For each $\mathbb{K}^d$-valued continuous local martingale M and two processes $H, G \in L^2(M)$ follow the two statements below.*

(i) Let $\alpha \in \mathbb{K}$. Then $(\alpha H + G) \in L^2(M)$ and

$$(\alpha H + G) \bullet M = \alpha(H \bullet M) + G \bullet M.$$

(ii) For each $n \in \mathbb{N}$ define $H_n = H \mathbb{1}_{\|H\|_2 \le n}$, which leads to

$$H_n \bullet M \to H \bullet M$$

in $\mathcal{H}_0^2$ as $n \to \infty$.

Proof. The linearity follows from Lemma 3.2 and the second part is a consequence of the first part.

(*i*) Let $(H_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ denote two sequences of predictable step process in $L^2(M)$ converging to H and G in $L^2(M)$, respectively. As always those sequences exist due to Lemma 2.17 in the last chapter. Then Lemma 2.10 implies $(\alpha H+G)\in L^2(M)$. Define now for each $n\in\mathbb{N}$ the process $K_n=\alpha H_n+G_n$, which is also a predictable step process by Lemma 3.2 as well as an element of $L^2(M)$. Consequently, the sequence $(K_n)_{n\in\mathbb{N}}$ converges in $L^2(M)$ to the predictable process $\alpha H+G$ for $n\to\infty$, because

$$\begin{aligned}&\lim_{n\to\infty}\|K_n-\alpha H+G\|_{L^2(M)}=\lim_{n\to\infty}\|\alpha H_n+G_n-\alpha H+G\|_{L^2(M)}\\&=\lim_{n\to\infty}\|\alpha(H_n-H)+(G_n-G)\|_{L^2(M)}\\&\le\alpha\lim_{n\to\infty}\|H_n-H\|_{L^2(M)}+\lim_{n\to\infty}\|G_n-G\|_{L^2(M)}=0.\end{aligned}$$

Due to Definition 3.8 and Lemma 3.9, the stochastic integral $(\alpha H+G)\bullet M$ is the H_0^2-limit of the sequence $(K_n\bullet M)_{n\in\mathbb{N}}$ and thus one can use the linearity of the stochastic integral of predictable step processes w.r.t. continuous local martingales in Lemma 3.2 to obtain

$$\begin{aligned}(\alpha H+G)\bullet M&=\lim_{n\to\infty}(K_n\bullet M)=\lim_{n\to\infty}\big((\alpha H_n+G_n)\bullet M\big)\\&=\lim_{n\to\infty}\big(\alpha(H_n\bullet M)+(G_n\bullet M)\big)=\alpha\lim_{n\to\infty}(H_n\bullet M)+\lim_{n\to\infty}(G_n\bullet M)\\&=\alpha(H\bullet M)+G\bullet M.\end{aligned}$$

(*ii*) Lemma 2.16 implies $H_n\in L^2(M)$ for each $n\in\mathbb{N}$ as well as $H_n\to H$ as $n\to\infty$ in $L^2(M)$ and thus one can use Lemma 3.10 and the linearity of the stochastic integral proven in part (i) to get

$$\lim_{n\to\infty}\|(H_n\bullet M)-(H\bullet M)\|_{\mathcal{H}_0^2}=\lim_{n\to\infty}\|(H_n-H)\bullet M\|_{\mathcal{H}_0^2}=\lim_{n\to\infty}\|H_n-H\|_{L^2(M)}=0.$$

□

Fix now a stopping time τ as well as again $H\in L^2(M)$ and $(H_n)_{n\in\mathbb{N}}$, a sequence of predictable step processes converging to H in $L^2(M)$. Then the sequence of processes $(H_n\mathbb{1}_{[0,\tau]})_{n\in\mathbb{N}}$ converges to $H\mathbb{1}_{[0,\tau]}$ in $L^2(M)$, as

$$\begin{aligned}&\lim_{n\to\infty}\|H_n\mathbb{1}_{[0,\tau]}-H\mathbb{1}_{[0,\tau]}\|_{L^2(M)}=\lim_{n\to\infty}\|(H_n-H)\mathbb{1}_{[0,\tau]}\|_{L^2(M)}\\&\le\lim_{n\to\infty}\|H_n-H\|_{L^2(M)}=0\end{aligned}\tag{3.3}$$

shows.

Consider some process $H\in L^2_{\text{loc}}(M)$. By Definition 2.11 there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty}\tau_n=\infty$ almost surely, such that

$$\mathbb{E}[\big((\mathbb{1}_{[0,\tau_n]}H^\intercal\pi\bar{H})\bullet C\big)_\infty]=\mathbb{E}[\big((H^\intercal\pi\bar{H})\bullet C\big)_{\tau_n}]<\infty,\qquad n\in\mathbb{N}.$$

This implies that the process $H\mathbb{1}_{[0,\tau_n]}$ is an element of $L^2(M)$ for each $n \in \mathbb{N}$ and as such there exists the stochastic integral $(H\mathbb{1}_{[0,\tau_n]}) \bullet M$. Note that for each $m \geq n$ the equality

$$\big((H\mathbb{1}_{[0,\tau_m]}) \bullet M\big)^{\tau_n} = (H\mathbb{1}_{[0,\tau_m]}) \bullet M^{\tau_n} = (H\mathbb{1}_{[0,\tau_m]}\mathbb{1}_{[0,\tau_n]}) \bullet M = (H\mathbb{1}_{[0,\tau_n]}) \bullet M$$

holds. Fix now some $n, m \in \mathbb{N}$ with $n \leq m$ and a sequence of predictable step processes $(H_{m,k})_{k\in\mathbb{N}}$ converging in $L^2(M)$ to $H\mathbb{1}_{[0,\tau_m]}$. Thus the sequence $(H_{m,k}\mathbb{1}_{[0,\tau_n]})_{k\in\mathbb{N}}$ converges to $H\mathbb{1}_{[0,\tau_n]}$ in $L^2(M)$ by equation (3.3). Due to Lemma 3.9, the sequence $\big((H_{m,k}\mathbb{1}_{[0,\tau_n]})\bullet M\big)_{k\in\mathbb{N}}$ then approximates $(H\mathbb{1}_{[0,\tau_n]}) \bullet M$ in $\mathcal{H}_0^2$ for all $m \geq n$.

Definition 3.12 (Stochastic integral w.r.t. $\mathbb{K}^d$-valued continuous local martingales)**.** Let M be a $\mathbb{K}^d$-valued continuous local martingale and $H \in L^2_{\text{loc}}(M)$. Then the stochastic integral of H w.r.t. M is a process $H \bullet M$ satisfying

$$(H \bullet M)^{\tau_n} = (H\mathbb{1}_{[0,\tau_n]}) \bullet M, \qquad n \in \mathbb{N} \tag{3.4}$$

up to indistinguishability, where $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times satisfying all conditions of Definition 2.11.

Lemma 3.13. *The above defined stochastic integral exists uniquely up to indistinguishability and is a $\mathbb{K}$-valued continuous local martingale starting at zero.*

Proof. Note at first that equation (3.4) implies for some fixed $n \in \mathbb{N}$ almost surely

$$(H \bullet M)_0 = (H \bullet M)_0^{\tau_n} = \big((H\mathbb{1}_{[0,\tau_n]}) \bullet M\big)_0 = 0.$$

It is apparent that $\mathbb{P}$-almost surely for each pair $(t, \omega) \in \mathbb{R}_+ \times \Omega$ there exists an $n_0 \in \mathbb{N}$, such that $t \leq \tau_n(\omega)$ for all $n \geq n_0$. Consequently, the almost sure limit $\lim_{n\to\infty} H\mathbb{1}_{[0,\tau_n]} = H$. Therefore the almost sure limit $\lim_{m\to\infty}(H\mathbb{1}_{[0,\tau_m]}) \bullet M$ satisfies

$$\big(\lim_{m\to\infty}(H\mathbb{1}_{[0,\tau_m]}) \bullet M\big)^{\tau_n} = \lim_{m\to\infty}\big((H\mathbb{1}_{[0,\tau_m]}) \bullet M\big)^{\tau_n} = \lim_{m\to\infty}(H\mathbb{1}_{[0,\tau_n]}) \bullet M = (H\mathbb{1}_{[0,\tau_n]}) \bullet M$$

for each $n \in \mathbb{N}$. Thus one can set $H \bullet M = \lim_{m\to\infty}(H\mathbb{1}_{[0,\tau_m]}) \bullet M$.

Let now Y denote another process satisfying (3.4). Thus for each $t \in \mathbb{R}_+$ and $\mathbb{P}$-almost all $\omega \in \Omega$ exists a $n \in \mathbb{N}$, such that $t \leq \tau_n(\omega)$, leading to

$$\begin{aligned} Y_t(\omega) = Y_t^{\tau_n(\omega)}(\omega) &= \big((H_s(\omega)\mathbb{1}_{[0,\tau_n(\omega)]}(s)) \bullet M_s(\omega)\big)_t = (H_s(\omega) \bullet M_s(\omega))_t^{\tau_n(\omega)} \\ &= (H_s(\omega) \bullet M_s(\omega))_t, \end{aligned}$$

which shows that the two processes Y and $H \bullet M$ agree up to indistinguishability.

Suppose now $(\tilde{\tau}_n)_{n\in\mathbb{N}}$ to be another sequence of stopping times meeting all criteria of Definition 2.11 and let $\tilde{Y}$ be the up to indistinguishability unique integral process of $H \bullet M$ satisfying

$$\tilde{Y}^{\tilde{\tau}_n} = (H\mathbb{1}_{[0,\tilde{\tau}_n]}) \bullet M, \qquad n \in \mathbb{N}.$$

Similarly to the last step there exists for each $t \in \mathbb{R}_+$ and $\mathbb{P}$-almost all $\omega \in \Omega$ a $n \in \mathbb{N}$, such that $t \leq \tau_n(\omega)$ as well as $t \leq \tilde{\tau}_n(\omega)$. Consequently,

$$
\begin{aligned}
\tilde{Y}_t(\omega) &= \tilde{Y}_t^{\tilde{\tau}_n(\omega)}(\omega) = \big((H_s(\omega)\mathbb{1}_{[0,\tilde{\tau}_n(\omega)]}(s)) \bullet M_s(\omega)\big)_t \\
&= \big((H_s(\omega)\mathbb{1}_{[0,\tau_n(\omega)]}(s)) \bullet M_s(\omega)\big)_t = (H_s(\omega) \bullet M_s(\omega))_t^{\tau_n(\omega)} = (H_s(\omega) \bullet M_s(\omega))_t
\end{aligned}
$$

and as such $\tilde{Y} = H \bullet M$ up to indistinguishability.

As the pointwise limit of adapted processes this stochastic integral is also adapted and the continuousness follows directly from equation (3.4), thus $H \bullet M$ is also predictable. By definition 3.8 for $H \in L^2(M)$ the stochastic integral $H \bullet M \in \mathcal{H}_0^2$ is again a continuous martingale.

As mentioned earlier, the sequence $(\tau_n)_{n\in\mathbb{N}}$ can be assumed to not only meet all criteria of Definition 2.11, but to also be a localizing sequence for the continuous local martingale M. Consider now for each $n \in \mathbb{N}$ the stopped integral process

$$
(H \bullet M)^{\tau_n} = H \bullet M^{\tau_n} = (H\mathbb{1}_{[0,\tau_n]}) \bullet M,
$$

where the right-hand side is a martingale, as $H\mathbb{1}_{[0,\tau_n]} \in L^2(M)$. Consequently, the integral process $H \bullet M$ is again a continuous local martingale with localizing sequence $(\tau_n)_{n\in\mathbb{N}}$, which concludes the proof. □

3.2 The stochastic integral w.r.t. multi-dimensional continuous processes of locally finite variation

In order to define stochastic integrals w.r.t. continuous semimartingales, one needs to at first consider integrators of locally finite variation.

Lemma 3.14. *For each $A \in \mathcal{V}_0^d$ define the process*

$$
V_t(\omega) = \sum_{j=1}^{d} \mathbb{V}_{A^j(\omega)}([0,t]), \qquad (t,\omega) \in \mathbb{R}_+ \times \Omega,
$$

where $\mathbb{V}_B([0,t])$ denotes the pathwise total variation of the process B on the interval $[0,t]$. Then there exists a $(V \otimes \mathbb{P})$-almost everywhere unique $\mathbb{K}^d$-valued predictable process v, satisfying $A^j = v^j \bullet V$ up to indistinguishability for each $j \in \{1,\dots,d\}$.

Proof. The proof is very similar to the first steps in the proof of Theorem 2.7. By assumption, $A^j \in \mathcal{V}_0^1$ for each $j \in \{1,\dots,d\}$. Furthermore, as $\mathbb{V}_{A^j}$ is $\mathbb{R}_+$-valued, adapted, continuous, non-decreasing and starting at zero for each $j \in \{1,\dots,d\}$ follows $V \in \mathcal{V}_0^+$. Fix now $j \in \{1,\dots,d\}$. According to Lemma 5.28 in the appendix, A^j may be seen as a signed or complex transition kernel from Ω to $\mathbb{R}_+$ on the δ-ring $\mathcal{R} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]}$. Consequently, Definition 5.25 and Lemma 5.26 result in the signed or complex measure

$$
(A^j \otimes \mathbb{P})(B) := \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_B(s,\omega) A^j(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega), \qquad B \in \mathcal{R} \otimes \mathcal{F},
$$

on the product δ-ring $\mathcal{R} \otimes \mathcal{F}$. Similarly, the processes $\mathbb{V}_{A^j}$ and V may be viewed as two σ-finite transition kernels from Ω to $\mathbb{R}_+$, due to Lemma 5.27, and the functions

$$(\mathbb{V}_{A^j} \otimes \mathbb{P})(B) := \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_B(s,\omega) \mathbb{V}_{A^j}(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega), \qquad B \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},$$

as well as

$$(V \otimes \mathbb{P})(B) := \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_B(s,\omega) V(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega), \qquad B \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},$$

are two measures on $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ by Lemma 5.24. Note that the linearity of the Lebesgue–Stieltjes integral in the integrand as well as the integrator implies

$$\begin{aligned}
(V \otimes \mathbb{P})(B) &= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_B(s,\omega) V(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_B(s,\omega) \Big(\sum_{j=1}^d \mathbb{V}_{A^j}(\mathrm{d}s,\omega) \Big) \Big) \mathbb{P}(\mathrm{d}\omega) = \int_\Omega \Big(\sum_{j=1}^d \int_{\mathbb{R}_+} \mathbb{1}_B(s,\omega) \mathbb{V}_{A^j}(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&= \sum_{j=1}^d \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_B(s,\omega) \mathbb{V}_{A^j}(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) = \sum_{j=1}^d (\mathbb{V}_{A^j} \otimes \mathbb{P})(B)
\end{aligned}$$

for each $B \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$. Thus it is apparent that $\mathbb{V}_{A^j} \otimes \mathbb{P} \ll V \otimes \mathbb{P}$ holds on $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ for each $j \in \{1, \dots, d\}$. Consider now a set $B \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ satisfying $(V \otimes \mathbb{P})(B) = 0$. Thus one may use Lemma 4.14 below to obtain

$$|A^j \otimes \mathbb{P}|(B) \le 2\, (\mathbb{V}_{A^j} \otimes \mathbb{P})(B) = 0$$

and thus

$$A^j \otimes \mathbb{P} \ll V \otimes \mathbb{P}$$

on $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ and thus also on the sub-σ-algebra Σ_p for each $j \in \{1, \dots, d\}$. Consequently, Theorem 4.15 proves the lemma. □

Definition 3.15 (Stochastic integral w.r.t. $\mathbb{K}^d$-valued continuous processes of locally finite variation)**.** Let $A \in \mathcal{V}_0^d$ be a process of locally finite variation starting at 0 let V be defined as in Lemma 3.14. Then there exists a $(V \otimes \mathbb{P})$-almost everywhere unique predictable process $v = (v^1, \dots, v^d)^\mathsf{T}$, satisfying $A^j = v^j \bullet V$, as stated by the lemma above. If there is some degree of ambiguity regarding the underlying process A, the processes v and V will be denoted by $v^{(A)}$ and $V^{(A)}$, respectively. The space $L(A)$ is then defined to include all $\mathbb{K}^d$-valued predictable processes H, such that almost surely $(|H^\mathsf{T} v| \bullet V)_t < \infty$ holds for each $t \in \mathbb{R}_+$. For such a process H the integral process is defined as the pathwise Lebesgue–Stieltjes integral

$$H \bullet A = (H^\mathsf{T} v) \bullet V,$$

which is again adapted, continuous, starting at zero and of locally finite variation, as stated by [Sch23, Lemma 5.49(c)] and thus in $\mathcal{V}_0^1$.

Note that for $\mathbb{K}$-valued processes $A \in \mathcal{V}_0^1$ and $H \in L(A)$ the pathwise Lebesgue–Stieltjes integral $H \bullet A$ does already exist, which could cause problems, if the above defined integral does not agree with $H \bullet A$ in the sense of a regular pathwise Lebesgue–Stieltjes integral. Luckily, this is not the case, as for each $t \in \mathbb{R}_+$ the polar decomposition of A in [Sch23, Theorem 15.128(c)] implies

$$(|H| \bullet \mathbb{V}_A)_t = \big((|H|\,|v^{(A)}|) \bullet V^{(A)}\big)_t = (|H^\mathsf{T} v^{(A)}| \bullet V^{(A)})_t < \infty,$$

whereby Lemma 4.12(iii) is applicable, which leads to

$$\int_0^t H^\mathsf{T} v^{(A)}\,\mathrm{d}V^{(A)} = \int_0^t H \frac{\mathrm{d}A}{\mathrm{d}\mathbb{V}_A}\,\mathrm{d}\mathbb{V}_A = \int_0^t H\,\mathrm{d}A$$

for each $t \in \mathbb{R}_+$ and the above defined integral does not cause ambiguity.

This newly defined integral is linear in the integrand as well as the integrator, which will be shown in the following lemma.

Lemma 3.16. *Let A, B be two processes in $\mathcal{V}_0^d$. Then the statements below follow.*

(i) $L(A)$ is a vector space.

(ii) Let $H, K \in L(A)$ and $\alpha \in \mathbb{K}$, then due to part (i) holds $(\alpha H + K) \in L(A)$ and

$$(\alpha H + K) \bullet A = \alpha(H \bullet A) + K \bullet A.$$

(iii) Let $H \in L(A) \cap L(B)$ and again $\alpha \in \mathbb{K}$, then $H \in L(\alpha A + B)$ and

$$H \bullet (\alpha A + B) = \alpha(H \bullet A) + H \bullet B.$$

Proof. (i) Fix $H, K \in L(A)$ as well as $\alpha \in \mathbb{K}$ and let v and V be as in Lemma 3.14. Thus by using the linearity of the Lebesgue–Stieltjes integral and the triangle inequality fulfilled by $|\cdot| : \mathbb{K} \to \mathbb{R}_+$ it follows that

$$\int_0^t |(\alpha H_s + K_s)^\mathsf{T} v_s|\,\mathrm{d}V_s = \int_0^t |\alpha H_s^\mathsf{T} v_s + K_s^\mathsf{T} v_s|\,\mathrm{d}V_s \le \int_0^t |\alpha H_s^\mathsf{T} v_s| + |K_s^\mathsf{T} v_s|\,\mathrm{d}V_s$$
$$= |\alpha| \int_0^t |H_s^\mathsf{T} v_s|\,\mathrm{d}V_s + \int_0^t |K_s^\mathsf{T} v_s|\,\mathrm{d}V_s < \infty$$

holds for all $t \in \mathbb{R}_+$, whereby $L(A)$ is a vector space.

(ii) Similarly to above, one can again use the linearity of the Lebesgue–Stieltjes integral to obtain

$$\big((\alpha H + K)^\mathsf{T} v\big) \bullet V = \big(\alpha H^\mathsf{T} v + K^\mathsf{T} v\big) \bullet V = \alpha\big((H^\mathsf{T} v) \bullet V\big) + (K^\mathsf{T} v) \bullet V.$$

(iii) At first consider $B \equiv 0$ and without loss of generality $\alpha \neq 0$. By the definition of the total variation measure, Definition 4.10 in the next chapter, follows $V^{(\alpha A)} = |\alpha| V^{(A)}$. Thus again the linearity of the Lebesgue–Stieltjes integral implies pathwise

$$(v^{(\alpha A)})^j \bullet V^{(\alpha A)} = \alpha A^j = \mathrm{e}^{\mathrm{i}\varphi}\, |\alpha| A^j = \mathrm{e}^{\mathrm{i}\varphi}\, |\alpha| \big((v^{(A)})^j \bullet V^{(A)}\big) = \mathrm{e}^{\mathrm{i}\varphi} (v^{(A)})^j \bullet V^{(\alpha A)}$$

for all $j = 1, \ldots, d$, where $\alpha = |\alpha|\, \mathrm{e}^{\mathrm{i}\varphi}$ for some $\varphi \in [0, 2\pi)$. Consequently, $v^{(\alpha A)} = \mathrm{e}^{\mathrm{i}\varphi}\, v^{(A)}$ must hold $(V^{(\alpha A)} \otimes \mathbb{P})$-almost everywhere by the uniqueness in Lemma 3.14. Thus $H \in L(\alpha A)$, as

$$\big(|H^\mathsf{T} v^{(\alpha A)}| \bullet V^{(\alpha A)}\big)_t = \big(|H^\mathsf{T}\, \mathrm{e}^{\mathrm{i}\varphi}\, v^{(A)}| \bullet (|\alpha| V^{(A)})\big)_t = |\alpha| \big(|H^\mathsf{T} v^{(A)}| \bullet V^{(A)}\big)_t < \infty$$

holds almost surely for each $t \in \mathbb{R}_+$. Note that $|\,\mathrm{e}^{\mathrm{i}\varphi}\,| = 1$ has been used in the second step above. Furthermore,

$$\begin{aligned} H \bullet (\alpha A) &= (H^\mathsf{T} v^{(\alpha A)}) \bullet V^{(\alpha A)} = (H^\mathsf{T}\, \mathrm{e}^{\mathrm{i}\varphi}\, v^{(A)}) \bullet (|\alpha| V^{(A)}) = |\alpha|\, \mathrm{e}^{\mathrm{i}\varphi} \big((H^\mathsf{T} v^{(A)}) \bullet V^{(A)}\big) \\ &= \alpha \big((H^\mathsf{T} v^{(A)}) \bullet V^{(A)}\big) = \alpha (H \bullet A) \end{aligned}$$

holds, as the Lebesgue–Stieltjes integral is linear in the integrand as well as the integrator.

Consequently, one may consider now $\alpha = 1$ and drop the assumption $B \equiv 0$. As $H \in L(A) \cap L(B)$ the sets

$$N_A := \{\omega \in \Omega : \big(|H(\omega)^\mathsf{T} v^{(A)}(\omega)| \bullet V^{(A)}(\omega)\big)_t = \infty, \text{ for some } t \in \mathbb{R}_+\}$$

and

$$N_B := \{\omega \in \Omega : \big(|H(\omega)^\mathsf{T} v^{(B)}(\omega)| \bullet V^{(B)}(\omega)\big)_t = \infty, \text{ for some } t \in \mathbb{R}_+\}$$

are $\mathbb{P}$-null sets. Thus the same holds for $N_A \cup N_B$. Fix now some $\omega \in (N_A \cup N_B)^\mathsf{c}$ and $t \in \mathbb{R}_+$. Note at this point that the process $V^{(\tilde{A})}$ induces pathwise a σ-finite measure on $\mathcal{B}_{\mathbb{R}_+}$ for each process $\tilde{A} \in \mathcal{V}_0^d$. Furthermore, $A_\cdot(\omega)$ as well as $B_\cdot(\omega)$ induce signed or complex measures on the δ-ring $\mathcal{R} := \bigcup_{n \in \mathbb{N}} \mathcal{B}_{[0,n]}$ of all relatively compact Borel sets of $\mathbb{R}_+$, see the proof of Lemma 3.14. Therefore one can use Lemma 4.12 to obtain

$$\begin{aligned} \infty > \big(|H^\mathsf{T} v^{(A)}| \bullet V^{(A)}\big)_t &= \left(\left| H^\mathsf{T} \frac{\mathrm{d}A}{\mathrm{d}V^{(A)}} \right| \bullet V^{(A)}\right)_t \\ &= \left(\left| H^\mathsf{T} \frac{\mathrm{d}A}{\mathrm{d}V^{(A)}} \right| \frac{\mathrm{d}V^{(A)}}{\mathrm{d}(V^{(A)} + V^{(B)})} \bullet (V^{(A)} + V^{(B)})\right)_t \\ &= \left(\left| H^\mathsf{T} \frac{\mathrm{d}A}{\mathrm{d}V^{(A)}} \frac{\mathrm{d}V^{(A)}}{\mathrm{d}(V^{(A)} + V^{(B)})} \right| \bullet (V^{(A)} + V^{(B)})\right)_t \\ &= \left(\left| H^\mathsf{T} \frac{\mathrm{d}A}{\mathrm{d}(V^{(A)} + V^{(B)})} \right| \bullet (V^{(A)} + V^{(B)})\right)_t \end{aligned}$$

and, analogously,

$$\big(|H^\top v^{(B)}| \bullet V^{(B)}\big)_t = \left(\left|H^\top \frac{\mathrm{d}B}{\mathrm{d}(V^{(A)}+V^{(B)})}\right| \bullet (V^{(A)}+V^{(B)})\right)_t$$

as well as

$$\big(|H^\top v^{(A+B)}| \bullet V^{(A+B)}\big)_t = \left(\left|H^\top \frac{\mathrm{d}(A+B)}{\mathrm{d}(V^{(A)}+V^{(B)})}\right| \bullet (V^{(A)}+V^{(B)})\right)_t$$

for each $t \in \mathbb{R}_+$. Consequently, $H \in L(A+B)$ holds, as for each $t \in \mathbb{R}_+$ follows

$$\begin{aligned}
\big(|H^\top v^{(A+B)}| \bullet V^{(A+B)}\big)_t &= \left(\left|H^\top \frac{\mathrm{d}(A+B)}{\mathrm{d}(V^{(A)}+V^{(B)})}\right| \bullet (V^{(A)}+V^{(B)})\right)_t \\
&= \left(\left|H^\top \frac{\mathrm{d}A}{\mathrm{d}(V^{(A)}+V^{(B)})} + H^\top \frac{\mathrm{d}B}{\mathrm{d}(V^{(A)}+V^{(B)})}\right| \bullet (V^{(A)}+V^{(B)})\right)_t \\
&\leq \left(\left|H^\top \frac{\mathrm{d}A}{\mathrm{d}(V^{(A)}+V^{(B)})}\right| \bullet (V^{(A)}+V^{(B)})\right)_t \\
&\quad + \left(\left|H^\top \frac{\mathrm{d}B}{\mathrm{d}(V^{(A)}+V^{(B)})}\right| \bullet (V^{(A)}+V^{(B)})\right)_t \\
&= \big(|H^\top v^{(A)}| \bullet V^{(A)}\big)_t + \big(|H^\top v^{(B)}| \bullet V^{(B)}\big)_t < \infty.
\end{aligned}$$

By taking the same steps as above again and simply omitting the absolute value (which leads to $\leq$ becoming $=$ in the fourth step of the display above) one indeed obtains

$$H \bullet (A+B) = (H^\top v^{(A+B)}) \bullet V^{(A+B)} = (H^\top v^{(A)}) \bullet V^{(A)} + (H^\top v^{(B)}) \bullet V^{(B)} = H \bullet A + H \bullet B,$$

which concludes the proof. □

3.3 Properties of the stochastic integral w.r.t. multi-dimensional continuous local martingales

One may now consider sequences of continuous martingales and their limits and use the Lemma 5.22 in the appendix to prove the following two statements.

Lemma 3.17. *Let $(M_n)_{n\in\mathbb{N}}$ and $(N_n)_{n\in\mathbb{N}}$ be two sequences of $\mathbb{K}$-valued continuous martingales in $\mathcal{H}_0^2$, converging in $\mathcal{H}_0^2$ to M and N, respectively, then*

$$\mathbb{E}\big[[M,N]_t\big] = \lim_{n\to\infty} \mathbb{E}\big[[M_n,N_n]_t\big], \qquad t \in \mathbb{R}_+.$$

Proof. An analogue result as in Lemma 3.6 also holds, when not considering the entire positive real half-axis $\mathbb{R}_+$ but only the interval $[0,t]$ for each $t \in \mathbb{R}_+$. To be precise, for each $t \in \mathbb{R}_+$ the space $\mathcal{H}_0^2(t) :=$ {continuous martingales $M = (M_s)_{s\in[0,t]}$ starting at zero and satisfying $\|M\|_{\mathcal{H}^2(t)} := \mathbb{E}[\sup_{s\in[0,t]} |M_s|^2]^{1/2} < \infty$} is a Banach space on which $\langle M,N\rangle_t := \mathbb{E}\big[[M,\bar{N}]_t\big]$ induces the equivalent norm $\mathbb{E}\big[[\cdot]_t\big]^{1/2}$, making $(\mathcal{H}_0^2(t), \mathbb{E}\big[[\cdot]_t\big]^{1/2})$ a Hilbert-space.

Fix now $t \in \mathbb{R}_+$. Then, by viewing M and for each $n \in \mathbb{N}$ also M_n as continuous martingales on the interval $[0,t]$, they are also elements of the aforementioned Hilbert space $(\mathcal{H}_0^2(t), \mathbb{E}\big[[\cdot]_t\big]^{1/2})$. The same holds for N and $(N_n)_{n\in\mathbb{N}}$. Consequently,

$$\mathbb{E}\big[[M,N]_t\big] = \lim_{n\to\infty} \mathbb{E}\big[[M_n,N_n]_t\big], \qquad t \in \mathbb{R}_+$$

is a direct consequence of Lemma 5.22 below. □

Lemma 3.18. *As in the last lemma, let $(M_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{H}_0^2$ converging in $\mathcal{H}_0^2$ to M and, analogously, let $(N_n)_{n\in\mathbb{N}}$ converge in $\mathcal{H}_0^2$ to N. Then there exist two subsequences $(M_{k_n})_{n\in\mathbb{N}}$ and $(N_{l_n})_{n\in\mathbb{N}}$, such that*

$$[M,N]_t(\omega) = \lim_{n\to\infty} [M_{k_n}, N_{l_n}]_t(\omega)$$

for $\mathbb{P}$-almost all $\omega \in \Omega$ and all $t \in \mathbb{R}_+$.

Proof. Fix a pair $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and define the set

$$\mathbb{M}(t,\omega) := \{M_\cdot(\omega) : M = (M_s)_{s\in[0,t]} \text{ is a } \mathbb{K}\text{-valued continuous martingale}\}.$$

Obviously, $\mathbb{M}(t,\omega) \subseteq C([0,t],\mathbb{K})$, where $C([0,t],\mathbb{K})$ denotes the space of all continuous functions from the interval $[0,t]$ into $\mathbb{K}$. Furthermore, it is apparent that $\mathbb{M}(t,\omega)$ is a vector space. In the same way as in Lemma 3.6 it follows that the function

$$\langle M_\cdot(\omega), N_\cdot(\omega)\rangle := [M_\cdot(\omega), \bar{N}_\cdot(\omega)]_t(\omega) = [M,\bar{N}]_t(\omega)$$

is a scalar product on $\mathbb{M}(t,\omega)$, making $\big(\mathbb{M}(t,\omega), [\cdot]_t(\omega)^{1/2}\big)$ a Pre-Hilbert space. As per assumption $(M_n)_{n\in\mathbb{N}}$ converges to M in $\mathcal{H}_0^2$ for $n \to \infty$, i.e.

$$0 = \lim_{n\to\infty} \|M_n - M\|_{\mathcal{H}_0^2} = \lim_{n\to\infty} \mathbb{E}\big[[M_n - M]_\infty\big]^{1/2},$$

follows that the sequence of $\mathbb{P}$-integrable random variables $([M_n - M]_\infty)_{n\in\mathbb{N}}$, or to be more precise their equivalence classes w.r.t. the L^1-norm, converges in $L^1(\Omega,\mathcal{F},\mathbb{P})$ to zero. By [Sch09, Satz 8.4.8] this sequence also converges in measure to zero, which in turn implies the existence of a subsequence $([M_{k_n} - M]_\infty)_{n\in\mathbb{N}}$ that converges $\mathbb{P}$-almost surely to zero, as stated in [Gri18, p. 64]. Consequently, for $\mathbb{P}$-almost all $\omega \in \Omega$ and each $t \in \mathbb{R}_+$ follows

$$\lim_{n\to\infty} [M_{k_n} - M]_t(\omega) \le \lim_{n\to\infty} [M_{k_n} - M]_\infty(\omega) = 0,$$

which means that $\big((M_{k_n})_\cdot(\omega)\big)_{n\in\mathbb{N}}$ converges to $M_\cdot(\omega)$ in $\mathbb{M}(t,\omega)$. Analogously, there exists a subsequence $(N_{l_n})_{n\in\mathbb{N}}$, such that the same holds almost surely for $\big((N_{l_n})_\cdot(\omega)\big)_{n\in\mathbb{N}}$ and $N_\cdot(\omega)$ for each $t \in \mathbb{R}_+$. Thus one can use Lemma 5.22 in the appendix to obtain for all $\omega \in \Omega$ outside of a $\mathbb{P}$-null set

$$[M,N]_t(\omega) = \lim_{n\to\infty} [M_{k_n}, N_{l_n}]_t(\omega), \qquad t \in \mathbb{R}_+,$$

which concludes the proof. □

This lemma will be used in the proof of the following theorem.

Theorem 3.19. *Let M be again a $\mathbb{K}^d$-valued continuous local martingale and $H \in L^2_{\mathrm{loc}}(M)$. Then the stochastic integral process $H \bullet M$ is the up to indistinguishability unique $\mathbb{K}$-valued continuous local martingale starting at zero that satisfies up to indistinguishability*

$$[H \bullet M, N] = H \bullet [M, N] \tag{3.5}$$

for each $\mathbb{K}$-valued continuous local martingale N, where $H \in L([M,N])$ and the right-hand side is the stochastic integral w.r.t. $[M,N] \in \mathcal{V}_0^d$ according to Definition 3.15.

Proof.
Step 1 (Predictable step process H). For readability, set $v = v^{([M,N])}$ and $V = V^{([M,N])}$ throughout this proof. Let $H = (H^1, \dots, H^d)^\mathsf{T}$ at first be a predictable step process. Note at this point that Lemma 4.12(iv) in the next chapter implies the $\mathbb{P}$-almost sure integrability of $v^j(\cdot,\omega)$ w.r.t. $V_\cdot(\omega)$ on the interval $[0,t]$ for each $t \in \mathbb{R}_+$ and $j = 1, \dots, d$, because

$$\int_{[0,t]} |v_s^j| \, \mathrm{d}V_s = \int_{[0,t]} \left| \frac{\mathrm{d}[M^j,N]_s}{\mathrm{d}V_s} \right| \mathrm{d}V_s \leq 2 \int_{[0,t]} \frac{\mathrm{d}\mathbb{V}_{[M^j,N]}([0,s])}{\mathrm{d}V_s} \, \mathrm{d}V_s = 2\mathbb{V}_{[M^j,N]}([0,t]) < \infty.$$

Therefore it is apparent that by the boundedness of H there exists a $c > 0$, such that for each $t \in \mathbb{R}_+$ follows

$$(|H^\mathsf{T} v| \bullet V)_t \leq c \Big(\Big| \sum_{j=1}^d v^j \Big| \bullet V \Big)_t \leq c \sum_{j=1}^d \Big(\Big| \frac{\mathrm{d}[M^j,N]}{\mathrm{d}V_t} \Big| \bullet V \Big)_t \leq 2c \sum_{j=1}^d \mathbb{V}_{[M^j,N]}([0,t] < \infty.$$

As stated above, $H \bullet M = \sum_{j=1}^d H^j \bullet M^j$ and consequently the right-hand side actually fulfills (3.5), as for every $t \in \mathbb{R}_+$

$$\begin{aligned}
[H \bullet M, N]_t &= \Big[\sum_{j=1}^d H^j \bullet M^j, N \Big]_t = \sum_{j=1}^d [H^j \bullet M^j, N]_t \\
&= \sum_{j=1}^d \Big[\sum_{n=1}^m \varphi_n^j (M^j_{\tau_{n+1} \wedge s} - M^j_{\tau_n \wedge s}), N_s \Big]_t = \sum_{j=1}^d \sum_{n=1}^m \Big[\varphi_n^j (M^j_{\tau_{n+1} \wedge s} - M^j_{\tau_n \wedge s}), N_s \Big]_t
\end{aligned}$$

holds. As already stated in the proof of lemma 3.3, by viewing the right-hand side for each $n = 1, \dots, m$ separately the linearity of the covariation process extends in this case to all $\mathcal{F}_{\tau_n}$-measurable random variables, particularly φ_n^j for all $j = 1, \dots d$, which leads to

$$\begin{aligned}
[H \bullet M, N]_t &= \sum_{j=1}^d \sum_{n=1}^m \varphi_n^j \Big[(M^j_{\tau_{n+1} \wedge s} - M^j_{\tau_n \wedge s}), N_s \Big]_t \\
&= \sum_{j=1}^d \sum_{n=1}^m \varphi_n^j \big([M^j_{\tau_{n+1} \wedge s}, N_s]_t - [M^j_{\tau_n \wedge s}, N_s]_t \big) = \sum_{j=1}^d \sum_{n=1}^m \varphi_n^j \big([M^j, N]_t^{\tau_{n+1}} - [M^j, N]_t^{\tau_n} \big) \\
&= \sum_{j=1}^d \sum_{n=1}^m \varphi_n^j \int_{\tau_n \wedge t}^{\tau_{n+1} \wedge t} v_s^j \, \mathrm{d}V_s = \sum_{j=1}^d \sum_{n=1}^m \int_{\tau_n \wedge t}^{\tau_{n+1} \wedge t} H_s^j v_s^j \, \mathrm{d}V_s = \sum_{j=1}^d \int_0^t H_s^j v_s^j \, \mathrm{d}V_s \\
&= \int_0^t \sum_{j=1}^d H_s^j v_s^j \, \mathrm{d}V_s = \int_0^t H_s^\mathsf{T} v_s \, \mathrm{d}V_s = (H \bullet [M,N])_t
\end{aligned}$$

for each $t \in \mathbb{R}_+$ and almost all $\omega \in \Omega$.

Step 2 (Bounded $H \in L^2(M)$). In the following let $\|\cdot\|_2$ denote the Euclidean norm on $\mathbb{K}^d$ and fix a bounded process $H \in L^2(M)$, i.e. there exists a $c > 0$ satisfying $\|H_t(\omega)\|_2 \le c$ for each pair $(t,\omega) \in \mathbb{R}_+ \times \Omega$, and let as always $(H_n)_{n\in\mathbb{N}}$ denote a sequence of predictable step processes converging in $L^2(M)$ to H. Fix now $n \in \mathbb{N}$ and consider

$$H_n(t,\omega) = \varphi_{n,0}\mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{m_n} \varphi_{n,k}\mathbb{1}_{(\tau_{n,k},\tau_{n,k+1}]}(t,\omega), \qquad (t,\omega) \in \mathbb{R}_+ \times \Omega,$$

according to Definition 2.2. Thus one can define

$$\tilde{\varphi}_{n,k} = \varphi_{n,k}\Big(\mathbb{1}_{\|\varphi_{n,k}\|_2 \le c} + \frac{c}{\|\varphi_{n,k}\|_2}\mathbb{1}_{\|\varphi_{n,k}\|_2 > c}\Big), \qquad k = 0,\ldots,m_n,$$

which is for each $k = 0,\ldots,m_n$ again a bounded $\mathbb{K}^d$-valued random vector and inherits the respective measurability of $\varphi_{n,k}$. Therefore, $(\tilde{H}_n)_{n\in\mathbb{N}}$ defined for each $n \in \mathbb{N}$ as

$$\tilde{H}_n(t,\omega) = \tilde{\varphi}_{n,0}\mathbb{1}_{\{0\}}(t) + \sum_{k=1}^{m_n} \tilde{\varphi}_{n,k}\mathbb{1}_{(\tau_{n,k},\tau_{n,k+1}]}(t,\omega), \qquad (t,\omega) \in \mathbb{R}_+ \times \Omega,$$

is also a sequence of predictable step processes, which is uniformly bounded by c. Furthermore, this sequence converges in $L^2(M)$ to H, due to $\|\tilde{H}_n - H\|_{L^2(M)} \le \|H_n - H\|_{L^2(M)}$. Consequently, as each predictable step process $\tilde{H}_n$ is bounded by c, $|H_n^j(t,\omega)v^j(t,\omega)| \le c|v^j(t,\omega)|$ for each pair $(t,\omega) \in \mathbb{R}_+ \times \Omega$ and $n \in \mathbb{N}$. Thus the dominated convergence theorem, see Theorem 5.37 in the appendix, is applicable in the fifth step of the display below. By Definition 3.8, $(H_n \bullet M)_{n\in\mathbb{N}}$ converges in $\mathcal{H}_0^2$ to $H \bullet M$. By Lemma 3.18 one obtains the existence of a subsequence $(H_{k_n} \bullet M)_{n\in\mathbb{N}}$, such that

$$\begin{aligned}
[H \bullet M, N]_t &= \Big[\lim_{n\to\infty}(H_{k_n} \bullet M), N\Big]_t = \lim_{n\to\infty}[H_{k_n} \bullet M, N]_t = \lim_{n\to\infty}\big((H_{k_n}^\top v) \bullet V\big)_t \\
&= \sum_{j=1}^{d} \lim_{n\to\infty}\big((H_{k_n}^j v^j) \bullet V\big)_t = \sum_{j=1}^{d}\Big(\big(\lim_{n\to\infty} H_{k_n}^j v^j\big) \bullet V\Big)_t = \Big(\big(\lim_{n\to\infty} H_{k_n}^\top v\big) \bullet V\Big)_t \\
&= \big((H^\top v) \bullet V\big)_t = (H \bullet [M,N])_t
\end{aligned}$$

for each $t \in \mathbb{R}_+$ and $\mathbb{P}$-almost all $\omega \in \Omega$. Thus one can now consider the function $\mathbb{R}_+ \ni t \mapsto H(t,\omega)^\top v(t,\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$ as a Radon–Nikodým derivative of $[H \bullet M, N]_\cdot(\omega)$ w.r.t. $V_\cdot(\omega)$ according to Theorem 4.15 in the next chapter. By the same argumentation as above, also $\pi_\cdot^{ij}(\omega)$ is $\mathbb{P}$-almost surely $C_\cdot(\omega)$-integrable on the interval $[0,t]$ for each $t \in \mathbb{R}_+$ and $(i,j) \in \{1,\ldots,d\}^2$ and

$$|H_n^i(t,\omega)\pi_t^{ij}(\omega)\overline{H}_n^j(t,\omega)| \le c^2\,\pi_t^{ij}(\omega), \qquad (t,\omega) \in \mathbb{R}_+ \times \Omega,\ n \in \mathbb{N}.$$

Thus the dominated convergence theorem is again applicable and leads combined with Lemma 3.3 for each $t \in \mathbb{R}_+$ almost surely to

$$\begin{aligned}
[H \bullet M]_t &= \Big[\lim_{n\to\infty}(H_{k_n} \bullet M)\Big]_t = \lim_{n\to\infty}[H_{k_n} \bullet M]_t = \lim_{n\to\infty}\big(H_{k_n}^\top \pi \overline{H}_{k_n}) \bullet C\big)_t \\
&= \big(\lim_{n\to\infty} H_{k_n}^\top \pi \overline{H}_{k_n}) \bullet C\big)_t = \big(H^\top \pi \overline{H}) \bullet C\big)_t.
\end{aligned}$$

Step 3 ($H \in L^2(M)$). Consider now a (possibly unbounded) process $H \in L^2(M)$ and define for each $n \in \mathbb{N}$ the bounded process $H_n = H\mathbb{1}_{\|H\|_2 \le n}$ as in Lemma 2.16. Due to the positive semidefiniteness of π, the function $t \mapsto H_n(t,\omega)^\top \pi_t(\omega) H_n(t,\omega)$ is $\mathcal{B}_{\mathbb{R}_+}$-measurable, $\mathbb{R}_+$-valued and satisfies

$$H_n(t,\omega)^\top \pi_t(\omega) H_n(t,\omega) \le H_{n+1}(t,\omega)^\top \pi_t(\omega) H_{n+1}(t,\omega), \qquad n \in \mathbb{N}.$$

Furthermore, $H_n \in L^2(M)$ implies the $C_\cdot(\omega)$-integrability of this function for $\mathbb{P}$-almost all $\omega \in \Omega$. Furthermore, Lemma 3.11(ii) states that the sequence $(H_n \bullet M)_{n\in\mathbb{N}}$ converges in $\mathcal{H}_0^2$ to $H \bullet M$. Thus one may use the *monotone convergence theorem*, Theorem 5.38 in the appendix, and again Lemma 3.18 to get a subsequence $(H_{k_n} \bullet M)_{n\in\mathbb{N}}$ for which

$$\begin{aligned}[H \bullet M] &= \Big[\lim_{n\to\infty} (H_{k_n} \bullet M)\Big] = \lim_{n\to\infty} [H_{k_n} \bullet M] = \lim_{n\to\infty} \big(H_{k_n}^\top \pi \overline{H}_{k_n}) \bullet C\big) \\ &= \big(\lim_{n\to\infty} H_{k_n}^\top \pi \overline{H}_{k_n}\big) \bullet C = (H^\top \pi \overline{H}) \bullet C\end{aligned}$$

up to indistinguishability.

Similarly to above, for each $n \in \mathbb{N}$ the $\mathcal{B}_{\mathbb{R}_+}$-measurable and $\mathbb{R}_+$-valued function $\mathbb{R}_+ \ni t \mapsto f_n(t,\omega) := |H_n(t,\omega)^\top v(t,\omega)|$ is for $\mathbb{P}$-almost all $\omega \in \Omega$ integrable w.r.t. $V_\cdot(\omega)$ on $[0,t]$, as one can use Lemma 5.30 in the appendix in the fourth step to obtain

$$\begin{aligned}&\int_{[0,t]} |H_n(s,\omega)^\top v(s,\omega)|\, V(\mathrm{d}s,\omega) = \int_{[0,t]} \left|\frac{\mathrm{d}[H_n \bullet M, N]_s(\omega)}{V(\mathrm{d}s,\omega)}\right| V(\mathrm{d}s,\omega) \\ &\le 2\int_{[0,t]} \frac{\mathrm{d}\mathbb{V}_{[H_n\bullet M,N]}([0,s],\omega)}{V(\mathrm{d}s,\omega)} V(\mathrm{d}s,\omega) = 2\mathbb{V}_{[H_n\bullet M,N]}([0,t],\omega) \\ &\le 2\sqrt{[H_n \bullet M]([0,t],\omega)}\sqrt{[N]([0,t],\omega)} = 2\sqrt{\big(H_n^\top \pi \overline{H}_n) \bullet C\big)_t}\sqrt{[N]_t} \\ &\le 2\sqrt{\big(H^\top \pi \overline{H}) \bullet C\big)_t}\sqrt{[N]_t} = 2\sqrt{[H \bullet M]_t}\sqrt{[N]_t} < \infty\end{aligned}$$

for each $t \in \mathbb{R}_+$, due to the findings of the last step and Lemma 4.12(iv) below. Furthermore, for fixed $\omega \in \Omega$ and $n \in \mathbb{N}$ holds $f_n(t,\omega) \le f_{n+1}(t,\omega)$ for all $t \in \mathbb{R}_+$. As the last display also implies

$$\sup_{n\in\mathbb{N}} \int_{[0,t]} |H_n(s,\omega)^\top v(s,\omega)|\, V(\mathrm{d}s,\omega) \le \sup_{n\in\mathbb{N}} \big(2\sqrt{[H \bullet M]_t}\sqrt{[N]_t}\big) = 2\sqrt{[H \bullet M]_t}\sqrt{[N]_t} < \infty,$$

one may use again the monotone convergence theorem to obtain for $\mathbb{P}$-almost all $\omega \in \Omega$ the integrability of $\mathbb{R}_+ \ni s \mapsto |H(s,\omega)^\top v(s,\omega)| = \lim_{n\to\infty} f_n(s,\omega)$ w.r.t. $V_\cdot(\omega)$ on $[0,t]$ for each $t \in \mathbb{R}_+$, which implies $H \in L([M,N])$ per Definition 3.15. Obviously, this function is a pointwise upper bound of f_n for each $n \in \mathbb{N}$. Consequently,

$$\begin{aligned}[H \bullet M, N]_t &= \Big[\lim_{n\to\infty} (H_{k_n} \bullet M), N\Big]_t = \lim_{n\to\infty} [H_{k_n} \bullet M, N]_t = \lim_{n\to\infty} \big((H_{k_n}^\top v) \bullet V\big)_t \\ &= \big((\lim_{n\to\infty} H_{k_n}^\top v) \bullet V\big)_t = \big((H^\top v) \bullet V\big)_t = (H \bullet [M,N])_t\end{aligned}$$

for each $t \in \mathbb{R}_+$ and $\mathbb{P}$-almost all $\omega \in \Omega$ follows. Note here that in the second-to-last step of the display above the dominated convergence theorem has been used once more, which is applicable by the previous findings in this step of the proof.

Step 4 ($H \in L^2_{\text{loc}}(M)$). In the second-to-last step, let H denote a general process in $L^2_{\text{loc}}(M)$ and $(\tau_n)_{n\in\mathbb{N}}$ the corresponding increasing sequence of stopping times according to Definition 2.11. Then for $\mathbb{P}$-almost all $\omega \in \Omega$, each $t \in \mathbb{R}_+$ there exists a $n \in \mathbb{N}$, such that $t \leq \tau_n(\omega)$. Thus one can see that $H \in L([M, N])$, due to

$$(|H^\mathsf{T} v| \bullet V)_t = (|\mathbb{1}_{[0,\tau_n]} H^\mathsf{T} v| \bullet V)_t < \infty,$$

as well as

$$[H \bullet M, N]_t = [(H\mathbb{1}_{[0,\tau_n]}) \bullet M, N]_t = \big((H\mathbb{1}_{[0,\tau_n]}) \bullet [M, N]\big)_t = (H \bullet [M, N])_t.$$

Analogously, one obtains

$$[H \bullet M]_t = [(H\mathbb{1}_{[0,\tau_n]}) \bullet M]_t = \big(((H\mathbb{1}_{[0,\tau_n]})^\mathsf{T} \pi \overline{H} \mathbb{1}_{[0,\tau_n]}) \bullet C\big)_t = \big((H^\mathsf{T} \pi \overline{H}) \bullet C\big)_t, \qquad t \in \mathbb{R}_+.$$

Step 5 (Uniqueness). Suppose Y is a $\mathbb{K}$-valued continuous local martingale starting at zero as well as satisfying equation (3.5), i.e.

$$[Y, N] = H \bullet [M, N]$$

up to indistinguishability for each $\mathbb{K}$-valued continuous local martingale N. Therefore, the linearity of the covariation process implies

$$[H \bullet M - Y, N] = [H \bullet M, N] - [Y, N] = H \bullet [M, N] - H \bullet [M, N] = 0, \qquad N \in \mathcal{M}_{\text{loc}}.$$

By considering now $N = \overline{H \bullet M - Y}$ one obtains

$$[H \bullet M - Y] = [H \bullet M - Y, \overline{H \bullet M - Y}] = 0,$$

which implies, by using [Sch23, Corollary 5.85] that the continuous local martingale $H \bullet M - Y$ is up to indistinguishability equal to its starting point, which is zero. □

As already stated in Lemma 3.3, for predictable step processes H there exists a often times useful formula for the covariation of the integral process. This result also holds for general processes $H \in L^2_{\text{loc}}(M)$, which will be stated in the lemma below. Note that the proof of this equality has already been given during the proof of the last theorem.

Lemma 3.20. *Let M be a $\mathbb{K}^d$-valued continuous local martingale and $H \in L^2_{\text{loc}}(M)$. Then for the covariation of the integral process $H \bullet M$ holds up to indistinguishability*

$$[H \bullet M] = \int_0^\cdot \left(\sum_{i,j=1}^d H_s^i \pi_s^{ij} \overline{H}_s^j \right) \mathrm{d}C_s = (H^\mathsf{T} \pi \overline{H}) \bullet C.$$

Example 3.21 (One-dimensional case)**.** Consider now a $\mathbb{K}$-valued continuous local martingale M and $H \in L^2_{\text{loc}}(M) \cap L(M)$, where $L(M)$ has been introduced in Definition 1.4. Obviously, this new definition of the multi-dimensional stochastic integral should coincide for such a $\mathbb{K}$-valued process H with the one from Definition 1.7, otherwise the one-dimensional case would not be well defined. However, this is not the case, as equation (3.5) coincides with the equality in Definition 1.7. By revisiting Theorem 2.7 the one-dimensional case also implies $C = [M]$ and thus $\pi = 1$ up to indistinguishability.

Example 3.22 ($H^j \in L^2_{\mathrm{loc}}(M^j)$)**.** For a $\mathbb{K}^d$-valued continuous local martingale $M = (M^1, \dots, M^d)^\mathsf{T}$ and a predictable process $H = (H^1, \dots, H^d)^\mathsf{T}$ let $H^j \in L^2_{\mathrm{loc}}(M^j)$ for each $j = 1, \dots, d$, which implies that $\sum_{j=1}^d H^j \bullet M^j$ is well-defined. Thus there exists for each $j = 1, \dots, d$ a sequence of stopping times $(\tau_n^j)_{n\in\mathbb{N}}$ satisfying all criteria of Definition 2.11. Then one may define for each $n \in \mathbb{N}$ the function $\tau_n = \bigwedge_{j=1}^d \tau_n^j$, i.e. the pointwise minimum, which is again a stopping time by [Sch23, Lemma 3.12(b)]. Obviously, this sequence is also increasing and satisfying $\lim_{n\to\infty} \tau_n = \infty$ almost surely. As both H^j and M^j are one-dimensional for each $j = 1, \dots, d$,

$$\int_0^{\tau_n} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t \le \int_0^{\tau_n^j} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t = \int_0^{\tau_n^j} |H_t^j|^2 \frac{\mathrm{d}[M^j]_t}{\mathrm{d}C_t} \,\mathrm{d}C_t = \int_0^{\tau_n^j} |H_t^j|^2 \,\mathrm{d}[M^j]_t \tag{3.6}$$

follows, which is $\mathbb{P}$-almost surely finite for all $n \in \mathbb{N}$. Thus for almost all $\omega \in \Omega$ holds

$$H^j \sqrt{\pi^{jj}} \in \mathcal{L}^2\big([0, \tau_n(\omega)], \mathcal{B}_{[0,\tau_n(\omega)]}, C_\cdot(\omega)\big), \qquad (n, j) \in \mathbb{N} \times \{1, \dots, d\}.$$

One can now use Lemma 5.21 below in the second and the Hölder inequality [Gri18, Satz 8.2] in the fourth step to obtain almost surely

$$\begin{aligned}
\int_0^{\tau_n} H_t^\mathsf{T} \pi_t \overline{H}_t \,\mathrm{d}C_t &\le \int_0^{\tau_n} \sum_{i,j=1}^d |H_t^i|\, |\pi_t^{ij}|\, |\overline{H}_t^j| \,\mathrm{d}C_t \le \int_0^{\tau_n} \sum_{i,j=1}^d |H_t^i| \sqrt{\pi_t^{ii}} \sqrt{\pi_t^{jj}} |\overline{H}_t^j| \,\mathrm{d}C_t \\
&= \sum_{i,j=1}^d \int_0^{\tau_n} \Big(|H_t^i| \sqrt{\pi_t^{ii}}\Big)\Big(\sqrt{\pi_t^{jj}} |\overline{H}_t^j|\Big) \,\mathrm{d}C_t \\
&\le \sum_{i,j=1}^d \bigg(\int_0^{\tau_n} |H_t^i|^2 \pi_t^{ii} \,\mathrm{d}C_t\bigg)^{1/2} \bigg(\int_0^{\tau_n} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t\bigg)^{1/2}.
\end{aligned}$$

In the next step, one may take the expectation of both sides of inequality (3.6) to see

$$\mathbb{E}\Big[\int_0^{\tau_n} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t\Big] \le \mathbb{E}\Big[\int_0^{\tau_n^j} |H_t^j|^2 \,\mathrm{d}[M^j]_t\Big] < \infty, \qquad (n, j) \in \mathbb{N} \times \{1, \dots, d\},$$

whereby

$$\bigg(\int_0^{\tau_n} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t\bigg)^{1/2} \in \mathcal{L}^2\big(\Omega, \mathcal{F}, \mathbb{P}\big), \qquad (n, j) \in \mathbb{N} \times \{1, \dots, d\},$$

follows. Consequently, H is an element of $L^2_{\mathrm{loc}}(M)$, as

$$\begin{aligned}
\mathbb{E}\Big[\int_0^{\tau_n} H_t^\mathsf{T} \pi_t \overline{H}_t \,\mathrm{d}C_t\Big] &\le \mathbb{E}\bigg[\sum_{i,j=1}^d \bigg(\int_0^{\tau_n} |H_t^i|^2 \pi_t^{ii} \,\mathrm{d}C_t\bigg)^{1/2} \bigg(\int_0^{\tau_n} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t\bigg)^{1/2}\bigg] \\
&= \sum_{i,j=1}^d \mathbb{E}\bigg[\bigg(\int_0^{\tau_n} |H_t^i|^2 \pi_t^{ii} \,\mathrm{d}C_t\bigg)^{1/2} \bigg(\int_0^{\tau_n} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t\bigg)^{1/2}\bigg] \\
&\le \sum_{i,j=1}^d \mathbb{E}\bigg[\int_0^{\tau_n} |H_t^i|^2 \pi_t^{ii} \,\mathrm{d}C_t\bigg]^{1/2} \mathbb{E}\bigg[\int_0^{\tau_n} |H_t^j|^2 \pi_t^{jj} \,\mathrm{d}C_t\bigg]^{1/2} < \infty
\end{aligned}$$

follows for each $n \in \mathbb{N}$ by using the Hölder inequality again in the last step. Therefore, the stochastic integral $H \bullet M$ exists.

Fix now a $\mathbb{K}$-valued continuous local martingale N and let $v = (v^1, \dots, v^d)^\mathsf{T}$ be the predictable process introduced in Lemma 3.14 satisfying $[M^j, N] = v^j \bullet V^{([M,N])}$ for each $j \in \{1, \dots, d\}$. Furthermore, one may also consider for each $j = 1, \dots, d$ the stochastic integral $H^j \bullet M^j$, which satisfies $[H^j \bullet M^j, N] = H^j \bullet [M^j, N]$ by Theorem 3.19. Thus for each $t \in \mathbb{R}_+$ the equality

$$
\begin{aligned}
\Big[\sum_{j=1}^d H^j \bullet M^j, N\Big]_t &= \sum_{j=1}^d [H^j \bullet M^j, N]_t = \sum_{j=1}^d (H^j \bullet [M^j, N])_t \\
&= \sum_{j=1}^d \left(\left(H^j \frac{\mathrm{d}[M^j, N]}{\mathrm{d}\mathbb{V}_{[M^j,N]}}\right) \bullet \mathbb{V}_{[M^j,N]}\right)_t = \sum_{j=1}^d \left(\left(H^j \frac{\mathrm{d}[M^j, N]}{\mathrm{d}\mathbb{V}_{[M^j,N]}} \frac{\mathrm{d}\mathbb{V}_{[M^j,N]}}{\mathrm{d}V^{([M,N])}}\right) \bullet V^{([M,N])}\right)_t \\
&= \sum_{j=1}^d \left(\Bigg(H^j \underbrace{\frac{\mathrm{d}[M^j, N]}{\mathrm{d}V^{([M,N])}}}_{=v^j}\Bigg) \bullet V^{([M,N])}\right)_t = \left(\Big(\sum_{j=1}^d H^j v^j\Big) \bullet V^{([M,N])}\right)_t \\
&= \big((H^\mathsf{T} v) \bullet V^{([M,N])}\big)_t = (H \bullet [M, N])_t
\end{aligned}
$$

follows up to indistinguishability by the linearities of the covariation process and the Lebesgue–Stieltjes integral as well as Lemma 4.12 in the next chapter. Thus the uniqueness in Theorem 3.19 implies $H \bullet M = \sum_{j=1}^d H^j \bullet M^j$ up to indistinguishability.

Theorem 3.19 and Lemma 3.16 will now be used to proof the following important properties of the stochastic integral introduced in section 3.1.

Lemma 3.23. *Let M and $\tilde{M}$ be two $\mathbb{K}^d$-valued continuous local martingales, $\alpha \in \mathbb{K}$ and $H \in L^2_{\mathrm{loc}}(M)$. Then the following three statements hold, where the equalities are as always understood up to indistinguishability.*

(i) For each $G \in L^2_{\mathrm{loc}}(M)$ follows $(\alpha H + G) \in L^2_{\mathrm{loc}}(M)$ and

$$(\alpha H + G) \bullet M = \alpha(H \bullet M) + G \bullet M.$$

(ii) For each stopping time τ the equalities

$$(H \bullet M)^\tau = H \bullet M^\tau = (H \mathbb{1}_{[0,\tau]}) \bullet M$$

hold.

(iii) If additionally $H \in L^2_{\mathrm{loc}}(\tilde{M})$, then $H \in L^2_{\mathrm{loc}}(\alpha M + \tilde{M})$ as well as

$$H \bullet (\alpha M + \tilde{M}) = \alpha(H \bullet M) + H \bullet \tilde{M}.$$

Proof. (*i*) Note at first that $(\alpha H + G) \in L^2_{\text{loc}}(M)$, because $L^2_{\text{loc}}(M)$ is a vector space due to Lemma 2.12. Let N be some $\mathbb{K}$-valued continuous local martingale. Then one can use the linearity of the covariation process and Lemma 3.16(ii) in the last step to obtain

$$\begin{aligned}[\alpha(H \bullet M) + G \bullet M, N] &= \alpha[H \bullet M, N] + [G \bullet M, N] \\ &= \alpha(H \bullet [M, N]) + G \bullet [M, N] = (\alpha H + G) \bullet [M, N]\end{aligned}$$

up to indistinguishability and the uniqueness in Theorem 3.19 concludes the proof.

(*ii*) For any $t \in \overline{\mathbb{R}}_+$ it is apparent that both

$$(H \bullet M)^\tau_t = \int_0^{t\wedge\tau} H_s \,\mathrm{d}M_s = \int_0^t H_s \,\mathrm{d}M_{s\wedge\tau} = (H \bullet M^\tau)_t$$

and

$$(H \bullet M)^\tau_t = \int_0^{t\wedge\tau} H_s \,\mathrm{d}M_s = \int_0^t H_s \mathbb{1}_{[0,\tau]}(t) \,\mathrm{d}M_s = \big((H\mathbb{1}_{[0,\tau]}) \bullet M\big)_t$$

are true.

(*iii*) Let $(\tau_n)_{n\in\mathbb{N}}$ and $(\tilde{\tau}_n)_{n\in\mathbb{N}}$ denote the two sequences of stopping times discussed in Definition 2.11, which exist due to $H \in L^2_{\text{loc}}(M)$ and $H \in L^2_{\text{loc}}(\tilde{M})$, respectively. As always, the sequence $(\sigma_n)_{n\in\mathbb{N}}$ defined for each $n \in \mathbb{N}$ as $\sigma_n = \tau_n \wedge \tilde{\tau}_n$ is also increasing and tending almost surely to infinity. Fix now $n \in \mathbb{N}$. Consequently, Lemma 3.20, Theorem 3.19 applied to $N := \overline{H \bullet (\alpha M + \tilde{M})}$ and Lemma 3.16(iii) lead to

$$\begin{aligned}&\mathbb{E}\big[\big((H^\intercal \pi^{(\alpha M+\tilde{M})}\overline{H}) \bullet C^{(\alpha M+\tilde{M})}\big)_{\sigma_n}\big] = \mathbb{E}\big[[H \bullet (\alpha M + \tilde{M})]_{\sigma_n}\big] \\ &= \mathbb{E}\big[\big(H \bullet [(\alpha M + \tilde{M}), N]\big)_{\sigma_n}\big] = \mathbb{E}\Big[\Big(H \bullet \big(\alpha[M,N] + [\tilde{M},N]\big)\Big)_{\sigma_n}\Big] \\ &= \mathbb{E}\big[\alpha\big(H \bullet [M,N]\big)_{\sigma_n} + \big(H \bullet [\tilde{M},N]\big)_{\sigma_n}\big] \\ &= \alpha\,\mathbb{E}\big[\big(H \bullet [M,N]\big)_{\sigma_n}\big] + \mathbb{E}\big[\big(H \bullet [\tilde{M},N]\big)_{\sigma_n}\big] \\ &\le \alpha\,\mathbb{E}\big[\big(H \bullet [M,N]\big)_{\sigma_n}\big] + \mathbb{E}\big[\big(H \bullet [\tilde{M},N]\big)_{\sigma_n}\big] \\ &= \alpha\,\mathbb{E}\big[[H \bullet M, N]_{\sigma_n}\big] + \mathbb{E}\big[[H \bullet \tilde{M}, N]_{\sigma_n}\big]\end{aligned}$$

Furthermore, one may use Lemma 5.30 in the appendix in the last step to obtain

$$\begin{aligned}&\big|[H \bullet (\alpha M + \tilde{M}), \overline{H \bullet M}]\big| = \big|H \bullet [\alpha M + \tilde{M}, \overline{H \bullet M}]\big| \\ &= \big|H \bullet \big(\alpha[M, \overline{H \bullet M}] + [\tilde{M}, \overline{H \bullet M}]\big)\big| \le \big|\alpha(H \bullet [M, \overline{H \bullet M}])\big| + \big|H \bullet [\tilde{M}, \overline{H \bullet M}]\big| \\ &= |\alpha|\,\big|[H \bullet M, \overline{H \bullet M}]\big| + \big|[H \bullet \tilde{M}, \overline{H \bullet M}]\big| \le |\alpha|\,[H \bullet M] + \mathbb{V}_{[H\bullet\tilde{M},\overline{H\bullet M}]} \\ &\le |\alpha|\,[H \bullet M] + \sqrt{[H \bullet \tilde{M}]}\sqrt{[H \bullet M]}\end{aligned}$$

and analogously

$$\big|[H \bullet (\alpha M + \tilde{M}), \overline{H \bullet \tilde{M}}]\big| \le [H \bullet \tilde{M}] + |\alpha|\,\sqrt{[H \bullet \tilde{M}]}\sqrt{[H \bullet M]}.$$

Combining the last three displays results in

$$
\begin{aligned}
&\mathbb{E}\big[\big((H^\top \pi^{(\alpha M+\tilde{M})}\overline{H}) \bullet C^{(\alpha M+\tilde{M})}\big)_{\sigma_n}\big] = \big|\mathbb{E}\big[\big((H^\top \pi^{(\alpha M+\tilde{M})}\overline{H}) \bullet C^{(\alpha M+\tilde{M})}\big)_{\sigma_n}\big]\big| \\
&\le |\alpha|\,\big|\mathbb{E}\big[[H \bullet M, N]_{\sigma_n}\big]\big| + \big|\mathbb{E}\big[[H \bullet \tilde{M}, N]_{\sigma_n}\big]\big| \\
&\le |\alpha|\,\mathbb{E}\big[\big|[H \bullet M, \overline{H \bullet (\alpha M + \tilde{M})}]_{\sigma_n}\big|\big] + \mathbb{E}\big[\big|[H \bullet \tilde{M}, \overline{H \bullet (\alpha M + \tilde{M})}]_{\sigma_n}\big|\big] \\
&= |\alpha|\,\mathbb{E}\big[\big|[H \bullet (\alpha M + \tilde{M}), \overline{H \bullet M}]_{\sigma_n}\big|\big] + \mathbb{E}\big[\big|[H \bullet (\alpha M + \tilde{M}), \overline{H \bullet \tilde{M}}]_{\sigma_n}\big|\big] \\
&\le |\alpha|\,\mathbb{E}\Big[|\alpha|\,[H \bullet M]_{\sigma_n} + \sqrt{[H \bullet \tilde{M}]_{\sigma_n}}\sqrt{[H \bullet M]_{\sigma_n}}\Big] \\
&+ \mathbb{E}\Big[[H \bullet \tilde{M}]_{\sigma_n} + |\alpha|\,\sqrt{[H \bullet \tilde{M}]_{\sigma_n}}\sqrt{[H \bullet M]_{\sigma_n}}\Big] \\
&= |\alpha|^2\,\mathbb{E}\big[[H \bullet M]_{\sigma_n}\big] + 2|\alpha|\,\mathbb{E}\Big[\sqrt{[H \bullet \tilde{M}]_{\sigma_n}}\sqrt{[H \bullet M]_{\sigma_n}}\Big] + \mathbb{E}\big[[H \bullet \tilde{M}]_{\sigma_n}\big].
\end{aligned}
$$

Due to

$$
\mathbb{E}\big[[H \bullet M]_{\sigma_n}\big] = \mathbb{E}\big[\big((H^\top \pi^{(M)}\overline{H}) \bullet C^{(M)}\big)_{\sigma_n}\big] \le \mathbb{E}\big[\big((H^\top \pi^{(M)}\overline{H}) \bullet C^{(M)}\big)_{\tau_n}\big] < \infty
$$

and

$$
\mathbb{E}\big[[H \bullet \tilde{M}]_{\sigma_n}\big] = \mathbb{E}\big[\big((H^\top \pi^{(\tilde{M})}\overline{H}) \bullet C^{(\tilde{M})}\big)_{\sigma_n}\big] \le \mathbb{E}\big[\big((H^\top \pi^{(\tilde{M})}\overline{H}) \bullet C^{(\tilde{M})}\big)_{\tilde{\tau}_n}\big] < \infty,
$$

the Hölder inequality [Gri18, Satz 8.2] is applicable and results in

$$
\mathbb{E}\Big[\sqrt{[H \bullet \tilde{M}]_{\sigma_n}}\sqrt{[H \bullet M]_{\sigma_n}}\Big] \le \mathbb{E}\big[[H \bullet \tilde{M}]_{\sigma_n}\big]^{1/2}\,\mathbb{E}\big[[H \bullet M]_{\sigma_n}\big]^{1/2}.
$$

Consequently, H is an element of $L^2_{\text{loc}}(\alpha M + \tilde{M})$, as for each $n \in \mathbb{N}$ holds

$$
\begin{aligned}
&\mathbb{E}\big[\big((H^\top \pi^{(\alpha M+\tilde{M})}\overline{H}) \bullet C^{(\alpha M+\tilde{M})}\big)_{\sigma_n}\big] \\
&\le |\alpha|^2\,\mathbb{E}\big[[H \bullet M]_{\sigma_n}\big] + 2|\alpha|\,\mathbb{E}\Big[\sqrt{[H \bullet \tilde{M}]_{\sigma_n}}\sqrt{[H \bullet M]_{\sigma_n}}\Big] + \mathbb{E}\big[[H \bullet \tilde{M}]_{\sigma_n}\big] \\
&\le |\alpha|^2\,\mathbb{E}\big[\big((H^\top \pi^{(M)}\overline{H}) \bullet C^{(M)}\big)_{\tau_n}\big] + \mathbb{E}\big[\big((H^\top \pi^{(\tilde{M})}\overline{H}) \bullet C^{(\tilde{M})}\big)_{\tilde{\tau}_n}\big] \\
&+ 2|\alpha|\,\mathbb{E}\big[[H \bullet \tilde{M}]_{\sigma_n}\big]^{1/2}\,\mathbb{E}\big[[H \bullet M]_{\sigma_n}\big]^{1/2} < \infty.
\end{aligned}
$$

The second claim of the last part of this lemma can be proven much quicker and simpler, as one may use again the linearity of the covariation process, Lemma 3.16(iii) and Theorem 3.19 to see that

$$
\begin{aligned}
&H \bullet [\alpha M + \tilde{M}, N] = H \bullet (\alpha[M, N] + [\tilde{M}, N]) = \alpha(H \bullet [M, N]) + H \bullet [\tilde{M}, N] \\
&= \alpha[H \bullet M, N] + [H \bullet \tilde{M}, N] = [\alpha(H \bullet M) + H \bullet \tilde{M}, N]
\end{aligned}
$$

up to indistinguishability for each $\mathbb{K}$-valued continuous local martingale N, whereby the uniqueness in Theorem 3.19 concludes the proof. □

Now one may again consider the processes X^1 and X^2 defined in Example 1.3 and further examined in Example 2.8.

Example 3.24. In the setting of Example 1.3 in Section 1.1 define for each $\epsilon > 0$ the deterministic $\mathbb{R}^2$-valued process

$$\tilde{H}^\epsilon = \big(1 - (H+\epsilon)^{-1}, (H+\epsilon)^{-1}\big)^\mathsf{T}$$

which is predictable, as stated by [Sch23, p. 286], as well as $M = (X^1, X^2)^\mathsf{T}$. Note at this point that

$$\|\tilde{H}_t^\epsilon\|_2^2 = \big|1-(H_t+\epsilon)^{-1}\big|^2 + \big|(H_t+\epsilon)^{-1}\big|^2 \leq \max\big\{1, (t+\epsilon)^{-2}\big\} + (t+\epsilon)^{-2} \leq \max\big\{1, \epsilon^{-2}\big\} + \epsilon^{-2},$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^2$. Consequently, as $\tilde{H}$ is bounded, it is also an element of $L^2_{\text{loc}}(M)$ by Lemma 2.14 and there exists an increasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ tending almost surely to infinity, such that $\tilde{H} \in L^2(M^{\tau_n})$ for each $n \in \mathbb{N}$. Consider now any real-valued continuous local martingale N and a two-dimensional predictable process K satisfying $[X^j, N] = K^j \bullet C$ for $j = 1, 2$, according to Theorem 3.19. Consequently,

$$\begin{aligned}
&[\tilde{H}^\epsilon \bullet M, N] = ((\tilde{H}^\epsilon)^\mathsf{T} K) \bullet C = \left(K^1 - \frac{K^1}{H+\epsilon} + \frac{K^2}{H+\epsilon}\right) \bullet C \\
&= [X^1, N] - [(H+\epsilon)^{-1} \bullet X^1, N] + [(H+\epsilon)^{-1} \bullet X^2, N] \\
&= [B^1, N] - (H+\epsilon)^{-1} \bullet [B^1, N] + (H+\epsilon)^{-1} \bullet [(1-H) \bullet B^1 + H \bullet B^2, N] \\
&= [B^1, N] - (H+\epsilon)^{-1} \bullet [B^1, N] \\
&+ (H+\epsilon)^{-1} \bullet \big((1-H) \bullet [B^1, N]\big) + (H+\epsilon)^{-1} \bullet \big(H \bullet [B^2, N]\big) \\
&= [B^1, N] - (H+\epsilon)^{-1} \bullet [B^1, N] + \frac{1-H}{H+\epsilon} \bullet [B^1, N] + \frac{H}{H+\epsilon} \bullet [B^2, N] \\
&= [B^1, N] - \frac{H}{H+\epsilon} \bullet [B^1, N] + \frac{H}{H+\epsilon} \bullet [B^2, N] \\
&= \frac{\epsilon}{H+\epsilon} \bullet [B^1, N] + \frac{H+\epsilon-\epsilon}{H+\epsilon} \bullet [B^2, N] \\
&= \frac{\epsilon}{H+\epsilon} \bullet [B^1 - B^2, N] + [B^2, N] = \left[\frac{\epsilon}{H+\epsilon} \bullet (B^1 - B^2) + B^2, N\right]
\end{aligned}$$

for each $\epsilon > 0$, by [Sch23, Corollary 5.102] and the linearity of the stochastic integral as well as the chain rule for stochastic integrals, see [Sch23, Lemma 5.116]. Consequently, the uniqueness in Theorem 3.19 leads for each $\epsilon > 0$ to

$$\tilde{H}^\epsilon \bullet M = \frac{\epsilon}{H+\epsilon} \bullet (B^1 - B^2) + B^2$$

up to indistinguishability, just the same as in Example 1.3. Therefore, the findings of this example in Section 1.1 can be applied to the case at hand, which leads to the convergence of the sequence $(\tilde{H}^{1/n} \bullet M)_{n\in\mathbb{N}}$ to B^2 with respect to the topology induced by ρ.

3.4 The stochastic integral w.r.t. multi-dimensional continuous semimartingales

After all this preliminary work, the following definition comes quite easily.

Definition 3.25 (Stochastic integral w.r.t. $\mathbb{K}^d$-valued continuous semimartingales)**.** Let $X = A + M$ be a $\mathbb{K}^d$-valued continuous semimartingale and its canonical decomposition. A $\mathbb{K}^d$-valued process H is defined to be in $L(X)$, if and only if it is integrable w.r.t. A as well as M, according to Definition 3.15 and Definition 2.11, respectively, i.e. $L(X) = L(A) \cap L^2_{\text{loc}}(M)$. For those processes, the multi-dimensional stochastic integral w.r.t. X is defined as

$$H \bullet X = H \bullet A + H \bullet M.$$

Note that the integral process $H \bullet X$ is again a continuous semimartingale and $H \bullet A + H \bullet M$ is its canonical decomposition. The following theorem provides some useful properties of the stochastic integral, whose proves are straightforward, when considering the aforementioned canonical decomposition of the integral process.

Theorem 3.26 (Linearity of the stochastic integral w.r.t. continuous semimartingales)**.** *Let X and Y be two $\mathbb{K}^d$-valued continuous semimartingales. Then the following three statements hold.*

(i) $L(X)$ defines a vector space.

(ii) Let $H, G \in L(X)$ and $\alpha \in \mathbb{K}$, then $(\alpha H + G) \in L(X)$ and

$$(\alpha H + G) \bullet X = \alpha(H \bullet X) + G \bullet X.$$

(iii) Let $H \in L(X) \cap L(Y)$ and again $\alpha \in \mathbb{K}$, then $H \in L(\alpha X + Y)$ and

$$H \bullet (\alpha X + Y) = \alpha(H \bullet X) + H \bullet Y.$$

Proof. (i) Combine Lemma 3.16(i) with Lemma 2.12.

(ii) Combine Lemma 3.16(ii) with Lemma 3.23(i).

(iii) Combine Lemma 3.16(iii) with Lemma 3.23(ii). □

Another useful property of the Lebesgue–Stieltjes integral, which can be translated to stochastic integrals, is the *chain rule*. The proof of this will be split into the following two lemmata.

Lemma 3.27 (Chain rule for continuous processes of locally finite variation)**.** *Consider the three processes $A \in \mathcal{V}^d_0$, $H \in L(A)$ and G, which is a $\mathbb{K}$-valued predictable process. Then $GH \in L(A)$ if and only if $G \in L(H \bullet A)$, which also implies*

$$(GH) \bullet A = G \bullet (H \bullet A).$$

Proof. To prove the first statement, assume at first $GH \in L(A)$. Due to the last display of [Sch23, Lemma 5.49] in the fourth and the chain rule for Lebesgue integrals [Sch09, Satz 9.2.2(1)] in the fifth step,

$$\begin{aligned}(|G^\top v^{(H\bullet A)}| \bullet V^{(H\bullet A)})_t &= \big((|G|\,|v^{(H\bullet A)}|) \bullet V^{(H\bullet A)}\big)_t = (|G| \bullet \mathbb{V}_{H\bullet A})_t \\ &= (|G| \bullet \mathbb{V}_{(H^\top v^{(A)})\bullet V^{(A)}})_t = \big(|G| \bullet (|H^\top v^{(A)}| \bullet V^{(A)})\big)_t = (|GH^\top v^{(A)}| \bullet V^{(A)})_t < \infty\end{aligned}$$

follows almost surely for each $t \in \mathbb{R}_+$ and thus $G \in L(H\bullet A)$. In the second step of the display above, the fact $|v^{(H\bullet A)}| = 1$ has been used, which is stated in [Sch23, Theorem 15.128(c)]. This theorem is applicable, as $H \bullet A$ is one-dimensional. Now suppose $G \in L(H \bullet A)$. In the same way as above, one can see

$$\big(|(GH)^\top v^{(A)}| \bullet V^{(A)}\big)_t = (|G^\top v^{(H\bullet A)}| \bullet V^{(H\bullet A)})_t < \infty, \qquad t \in \mathbb{R}_+,$$

almost surely, leading to $GH \in L(A)$. Furthermore, the chain rule for Lebesgue–Stieltjes integrals [Sch23, Lemma 16.6] in the second step leads to

$$G \bullet (H \bullet A) = G \bullet \big((H^\top v^{(A)}) \bullet V^{(A)}\big) = (GH^\top v^{(A)}) \bullet V^{(A)} = (GH) \bullet A,$$

which concludes the proof. □

Lemma 3.28 (Chain rule for continuous local martingales)**.** *Let M denote a $\mathbb{K}^d$-valued continuous local martingale, $H \in L^2_{\text{loc}}(M)$ and G be a $\mathbb{K}$-valued predictable process. Then $GH \in L^2_{\text{loc}}(M)$ if and only if $G \in L^2_{\text{loc}}(H \bullet M)$, which also implies*

$$(GH) \bullet M = G \bullet (H \bullet M).$$

Proof. As $H \bullet M$ is one-dimensional, Lemma 3.20, Example 3.21 and the chain rule for Lebesgue integrals [Sch09, Satz 9.2.2(1)] in the third step lead to

$$\begin{aligned}\int_0^\cdot G_t^\top \pi_t^{(H\bullet M)} \bar{G}_t \,\mathrm{d}C_t^{(H\bullet M)} &= \int_0^\cdot |G_t|^2 \,\mathrm{d}[H \bullet M]_t = \int_0^\cdot |G_t|^2 \,\mathrm{d}\Big(\int_0^t H_s^\top \pi_s^{(M)} \bar{H}_s \,\mathrm{d}C_s^{(M)}\Big) \\ &= \int_0^\cdot |G_t|^2 H_t^\top \pi_t^{(M)} \bar{H}_t \,\mathrm{d}C_t^{(M)} = \int_0^\cdot (GH)_t^\top \pi_t^{(M)} \overline{GH}_t \,\mathrm{d}C_t^{(M)}.\end{aligned}$$

Let now $(\tau_n)_{n\in\mathbb{N}}$ denote an increasing sequence of stopping times fulfilling the criteria of Definition 2.11 for $GH \in L^2_{\text{loc}}(M)$, which is assumed at this point. Thus the last display implies

$$\mathbb{E}\big[((G^\top \pi^{(H\bullet M)} \bar{G}) \bullet C^{(H\bullet M)})_{\tau_n}\big] = \mathbb{E}\Big[\Big(((GH)^\top \pi^{(M)} \overline{GH}) \bullet C^{(M)}\Big)_{\tau_n}\Big] < \infty, \qquad n \in \mathbb{N},$$

resulting in $G \in L^2_{\text{loc}}(H \bullet M)$. Conversely, assume now $G \in L^2_{\text{loc}}(H \bullet M)$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be an according sequence of stopping discussed in Definition 2.11. Then also $GH \in L^2_{\text{loc}}(M)$ holds, due to

$$\mathbb{E}\Big[\Big(((GH)^\top \pi^{(M)} \overline{GH}) \bullet C^{(M)}\Big)_{\sigma_n}\Big] = \mathbb{E}\big[((G^\top \pi^{(H\bullet M)} \bar{G}) \bullet C^{(H\bullet M)})_{\sigma_n}\big] < \infty, \qquad n \in \mathbb{N}.$$

Similarly to the proof of Lemma 3.23(iii) one may now use Lemma 3.27 and Theorem 3.19, which result up to indistinguishability in

$$(GH) \bullet [M,N] = G \bullet (H \bullet [M,N]) = G \bullet [H \bullet M, N] = [G \bullet (H \bullet M), N]$$

for each $\mathbb{K}$-valued continuous local martingale N. Therefore, Theorem 3.19 implies $G \bullet (H \bullet M) = (GH) \bullet M$ up to indistinguishability. □

Theorem 3.29 (Chain rule for continuous semimartingales)**.** *Let $X = A+M$ be a $\mathbb{K}^d$-valued continuous semimartingale, $H \in L(X)$ and furthermore G a $\mathbb{K}$-valued predictable process. Then $GH \in L(X)$ if and only if $G \in L(H \bullet X)$ and the equation*

$$(GH) \bullet X = G \bullet (H \bullet X),$$

which can be equivalently written as

$$\int_0^{\cdot} G_t H_t \,\mathrm{d}X_t = \int_0^{\cdot} G_t \,\mathrm{d}\Big(\int_0^t H_s \,\mathrm{d}X_s\Big),$$

follows.

Proof. Let at first $GH \in L(X)$, which is by Definition 3.25 equivalent to $GH \in L(A) \cap L^2_{\mathrm{loc}}(M)$. Therefore, $G \in L(H \bullet A) \cap L^2_{\mathrm{loc}}(H \bullet M)$, by Lemma 3.27 and Lemma 3.28, respectively. Thus Definition 3.25 leads to $G \in L(H \bullet X)$, as $H \bullet A + H \bullet M$ is the canonical decomposition of the $\mathbb{K}$-valued continuous semimartingale $H \bullet X$. Assume now $G \in L(H \bullet X)$, i.e. $G \in L(H \bullet A) \cap L^2_{\mathrm{loc}}(M)$. In the same way as before, one obtains $GH \in L(A) \cap L^2_{\mathrm{loc}}(M)$ and thus $GH \in L(X)$. Furthermore, Definition 3.25 in the first and last step implies

$$(GH) \bullet X = (GH) \bullet A + (GH) \bullet M = G \bullet (H \bullet A) + G \bullet (H \bullet M) = G \bullet X,$$

where Lemma 3.27 and Lemma 3.28 have been used in the second step. □

Example 3.30. Reconsider now Example 1.1 in Section 1.1 and fix a $\mathbb{K}$-valued continuous semimartingale $X = A + M$ and a predictable process H, which is not necessarily in $L(X)$. The trading strategy $(H,H)^\intercal$ associated with an portfolio of the two assets modeled by $(X,-X)^\intercal$ should intuitively result in zero profit or loss. This newly defined stochastic integral does indeed yield this result, as

$$C = 2[M], \qquad \pi^{11} = \pi^{22} = \frac{\mathrm{d}[M]}{\mathrm{d}C} = \frac{1}{2} \quad \text{and} \quad \pi^{12} = \bar{\pi}^{21} = \frac{\mathrm{d}[M, -\overline{M}]}{\mathrm{d}C} = -\frac{1}{2}$$

result in $(H,H)^\intercal \in L^2\big((M,-M)^\intercal\big)$, because

$$\|(H,H)^\intercal\|^2_{L^2\left((M,-M)^\intercal\right)} = \mathbb{E}\Big[\int_{\mathbb{R}_+} \underbrace{(H_t,H_t)\pi_t}_{=(0,0)}(H_t,H_t)^{\mathsf{H}} \,\mathrm{d}C_t\Big] = 0 < \infty.$$

Thus the strategy $(H,H)^\intercal$ is in the same equivalence class w.r.t. $\|\cdot\|_{L^2\left((M,-M)^\intercal\right)}$ as the process $(0,0)^\intercal$, which is a predictable step process per Definition 2.2. Consequently, the stochastic integral according to Definition 3.8 $(H,H)^\intercal \bullet (M,-M)^\intercal \equiv 0$, as the constant sequence $\big((0,0)^\intercal\big)_{n\in\mathbb{N}}$ converges trivially to $(H,H)^\intercal$ in $L^2\big((M,-M)^\intercal\big)$.

Similarly, for the two-dimensional process $(A,-A)^\intercal \in \mathcal{V}_0^2$ follows

$$V = 2\mathbb{V}_A, \qquad \text{and} \qquad v^1 = \frac{\mathrm{d}A}{\mathrm{d}V} = -\frac{\mathrm{d}(-A)}{\mathrm{d}V} = -v^2,$$

which implies $(H,H)^\intercal \in L\big((A,-A)^\intercal\big)$, and as such also $(H,H)^\intercal \in L\big((X,-X)^\intercal\big)$, because

$$\big(|(H,H)\,v| \bullet V\big)_t = \big(|Hv^1 + Hv^2| \bullet V\big)_t = \big(|Hv^1 - Hv^1| \bullet V\big)_t = 0 < \infty, \qquad t \in \mathbb{R}_+,$$

and, analogously,

$$(H,H)^\intercal \bullet (A,-A)^\intercal = \big((H,H)\,v\big) \bullet V = \big(Hv^1 + Hv^2\big) \bullet V = \big(Hv^1 - Hv^1\big) \bullet V = 0.$$

Finally, this yields the expected result via

$$(H,H)^\intercal \bullet (X,-X)^\intercal = (H,H)^\intercal \bullet (M,-M)^\intercal + (H,H)^\intercal \bullet (A,-A)^\intercal = 0.$$

Example 3.31. Fix the notation of Example 3.24, where it was shown that the sequence $(\tilde{H}^{1/2} \bullet M)_{n\in\mathbb{N}}$ to B^2 with respect to the topology induced by ρ. In the next step one would like to see that there exists a $\mathbb{R}^d$ valued predictable process $\tilde{H} \in L^2_{\mathrm{loc}}(M)$, such that $B^2 = \tilde{H} \bullet M$ up to indistinguishability. Consider now the deterministic process

$$\tilde{H}_t = \big(1 - H_t^{-1}, H_t^{-1}\big)^\intercal \mathbb{1}_{t>0} = \big(1 - t^{-1}, t^{-1}\big)^\intercal \mathbb{1}_{t>0}, \qquad t \in \mathbb{R}_+.$$

Then Example 2.8 leads for each $t \in \mathbb{R}_+$ to

$$\begin{aligned}
\big((\tilde{H}^\intercal \pi \tilde{H}) \bullet C\big)_t &= \int_{(0,t]} \Big((1 - H_s^{-1})^2 \pi_s^{11} + 2(1 - H_s^{-1}) H_s^{-1} \pi_s^{12} + H_s^{-2} \pi_s^{22}\Big)\,\mathrm{d}C_s \\
&= \int_{(0,t]} \bigg(\Big(1 - \frac{1}{s}\Big)^2 \frac{\mathrm{d}s}{\mathrm{d}C_s} + 2\Big(1 - \frac{1}{s}\Big)\frac{1}{s}\Big(\frac{\mathrm{d}s}{\mathrm{d}C_s} - \frac{1}{2}\frac{\mathrm{d}s^2}{\mathrm{d}C_s}\Big) + \frac{1}{s^2}\Big(\frac{2}{3}\frac{\mathrm{d}s^3}{\mathrm{d}C_s} - \frac{\mathrm{d}s^2}{\mathrm{d}C_s} + \frac{\mathrm{d}s}{\mathrm{d}C_s}\Big)\bigg)\,\mathrm{d}C_s \\
&= \int_{(0,t]} \bigg(\Big(1 - \frac{2}{s} + \frac{1}{s^2} + \frac{2}{s} - \frac{2}{s^2} + \frac{1}{s^2}\Big)\frac{\mathrm{d}s}{\mathrm{d}C_s} + \Big(-\frac{1}{s} + \frac{1}{s^2} - \frac{1}{s^2}\Big)\frac{\mathrm{d}s^2}{\mathrm{d}C_s} + \frac{2}{3s^2}\frac{\mathrm{d}s^3}{\mathrm{d}C_s}\bigg)\,\mathrm{d}C_s \\
&= \int_{(0,t]} 1\,\mathrm{d}s - \int_{(0,t]} \frac{1}{s}\,\mathrm{d}s^2 + \int_{(0,t]} \frac{2}{3s^2}\,\mathrm{d}s^3 \\
&= \int_{(0,t]} \Big(1 - \frac{1}{s}2s + \frac{2}{3s^2}3s^2\Big)\,\mathrm{d}s = \int_{(0,t]} (1 - 2 + 2)\,\mathrm{d}s = t < \infty,
\end{aligned}$$

where in the fifth step the chain rule and

$$s^2 = \int_0^s 2u\,\mathrm{d}u \qquad \text{and} \qquad s^2 = \int_0^s 3u^2\,\mathrm{d}u$$

has been used. Consequently, the increasing sequence of deterministic stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined for each $n \in \mathbb{N}$ as $\tau_n = n$ tends to infinity and satisfies

$$\mathbb{E}\big[\big((\tilde{H}^\intercal \pi \tilde{H}) \bullet C\big)_{\tau_n}\big] = \mathbb{E}\big[\big((\tilde{H}^\intercal \pi \tilde{H}) \bullet C\big)_n\big] = \mathbb{E}[n] = n < \infty, \qquad n \in \mathbb{N},$$

which implies $\tilde{H} \in L^2_{\text{loc}}(M)$ and thus the stochastic integral $\tilde{H} \bullet M$ is well defined.

Similarly as in Example 3.24, consider now a real-valued continuous local martingale N and a two-dimensional predictable process K satisfying $[X^j, N] = K^j \bullet C$ for $j = 1, 2$, according to Theorem 3.19. Therefore,

$$\begin{aligned}[\tilde{H} \bullet M, N] = (\tilde{H}^\intercal K) \bullet C &= \left(K^1 - \frac{K^1}{H} + \frac{K^2}{H}\right) \bullet C \\ &= [X^1, N] - [H^{-1} \bullet X^1, N] + [H^{-1} \bullet X^2, N] \\ &= [B^1, N] - H^{-1} \bullet [B^1, N] + H^{-1} \bullet [(1 - H) \bullet B^1 + H \bullet B^2, N] \\ &= [B^1, N] - \frac{1}{H} \bullet [B^1, N] + \frac{1 - H}{H} \bullet [B^1, N] + \frac{H}{H} \bullet [B^2, N] = [B^2, N],\end{aligned}$$

which implies $\tilde{H} \bullet M = B^2$ up to indistinguishability, as stated in Theorem 3.19.

In the setting of the stochastic integral introduced in this thesis, the process B^2 can be denoted by a stochastic integral with respect to M, which was not possible in Section 1.1, as seen in Example 1.3.

The following theorem also circles back to Section 1.1 and makes sure that no setting similar to the negative Example 1.3 can exist for the stochastic integral defined in this thesis, when considering only the in applications much more often used case $\mathbb{K} = \mathbb{R}$. For the proof of this theorem the reader is referred to [Mém80, Théorème V.4].

Theorem 3.32. *Let X denote a $\mathbb{R}^d$-valued continuous semimartingale. Then the set $\{H \bullet X : H \in L(X)\}$, which is a vector space by Theorem 3.26, is complete w.r.t. the Émery distance, see Definition 1.2.*

Assume now that the $\mathbb{R}^d$-valued continuous semimartingale models a financial market consisting of d underlying assets. Each $H \in L(X)$ can be seen as a trading strategy w.r.t. X and $H \bullet X$ is the profit or loss of this trading strategy. If any financial instrument can be represented by a well-defined stochastic integral $H \bullet X$ w.r.t. X, then it can be hedged. In this context, the theorem above states that if there exists a sequence of financial instruments, which can be hedged w.r.t. X, converging w.r.t. the topology induced by ρ, then the same holds for the limit.

4 The Radon–Nikodým theorem

4.1 The Radon–Nikodým theorem for σ-finite measures

Throughout this thesis, $\mathcal{L}^p(\Omega, \mathcal{F}, \nu)$ denotes the space of all $\mathcal{F}$-measurable functions satisfying

$$\|f\|_{L^p(\nu)} := \int_\Omega |f|^p \, \mathrm{d}\nu < \infty,$$

The function $\|\cdot\|_{L^p(\nu)} : \mathcal{L}^p(\Omega, \mathcal{F}, \nu) \to \mathbb{R}_+$ is a seminorm. Consequently, one may call two functions f_1 and f_2 in $\mathcal{L}^p(\Omega, \mathcal{F}, \nu)$ equivalent, if and only if $\|f_1 - f_2\|_{L^p(\nu)} = 0$, which means they agree ν-almost everywhere. Therefore, on the factor space $L^p(\Omega, \mathcal{F}, \nu)$, consisting of all equivalence classes in $\mathcal{L}^p(\Omega, \mathcal{F}, \nu)$, $\|\cdot\|_{L^p(\nu)}$ is an actual norm. This chapter is often concerned with ν-almost everywhere unique functions in $\mathcal{L}^p(\Omega, \mathcal{F}, \nu)$. Note that one may also regard such functions as unique equivalence classes in $L^p(\Omega, \mathcal{F}, \nu)$.

In the following section the very useful Radon–Nikodým theorem will be constructed and proven in a similar fashion as in [Wil91, Chapter 14.13].
Note that in this section, all measures and functions are $[0, \infty) = \mathbb{R}_+$-valued. At first, the separability of the σ-algebra $\mathcal{F}$ is assumed, i.e. the existence of a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of Ω, such that $\mathcal{F} = \sigma(A_n : n \in \mathbb{N})$.

Theorem 4.1 (The Radon–Nikodým theorem for finite measures)**.** *Let μ denote a finite measure on the probability triple $(\Omega, \mathcal{F}, \nu)$. If $\mu \ll \nu$ on $\mathcal{F}$, i.e. for all $A \in \mathcal{F}$ with $\nu(A) = 0$ also $\mu(A) = 0$ must hold (say μ is* absolutely continuous *with regards to ν on $\mathcal{F}$), then there exists a ν-almost surely uniquely defined non-negative function $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \nu)$, such that*

$$\mu(A) = \int_A \mathrm{d}\mu = \int_A f \, \mathrm{d}\nu, \qquad A \in \mathcal{F}. \tag{4.1}$$

Such a function f is then understood to be a Radon–Nikodým derivative of μ w.r.t. ν on $\mathcal{F}$ and may also be denoted as

$$f = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}.$$

In the proof of this theorem, the following lemma will be used in the first step.

Lemma 4.2. *Let $\mu \ll \nu$ on $\mathcal{F}$ be the same as in Theorem 4.1. Then for each $\epsilon > 0$ there exists a $\delta > 0$, such that for every $A \in \mathcal{F}$*

$$\nu(A) < \delta \Rightarrow \mu(A) < \epsilon$$

holds.

Proof. In this case proof by contradiction will be used. Let's assume the statement not to be true. Therefore there must exist some $\epsilon > 0$, such that there is also a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ satisfying simultaneously

$$\nu(A_n) < 2^{-n} \quad \textit{and} \quad \mu(A_n) \geq \epsilon.$$

For $B := \limsup_{n\in\mathbb{N}} A_n$ one can see that $\nu(B) = 0$ by the Generalized First Borel–Cantelli Lemma (see Lemma 5.7 in the appendix), as $\sum_{n=1}^{\infty} \nu(A_n) \leq \sum_{n=1}^{\infty} 2^{-n} < \infty$. On the other hand, the *Reverse Fatou Lemma* (see for example [Wil91, p. 27]) states $\mu(\limsup_{n\in\mathbb{N}} A_n) \geq \limsup_{n\in\mathbb{N}} \mu(A_n)$ for finite measures μ and $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$. This leads to

$$\mu(B) = \mu(\limsup_{n\in\mathbb{N}} A_n) \geq \limsup_{n\in\mathbb{N}} \mu(A_n) \geq \lim_{n\in\mathbb{N}} \mu(A_n) \geq \epsilon > 0,$$

which directly contradicts the assumption $\mu \ll \nu$ and therefore proves the lemma. □

Proof of Theorem 4.1. Throughout the proof of this theorem and the one of the following lemma all expectations are to be understood w.r.t. the underlying probability measure ν, i.e. $\mathbb{E}[f] = \int_\Omega f \,\mathrm{d}\nu$. As $\mathcal{F}$ is separable by assumption, one can represent it as $\sigma(A_n : n \in \mathbb{N})$ and define for each $n \in \mathbb{N}$

$$\mathcal{F}_n = \sigma(A_1, \ldots, A_n).$$

An *atom* A of a σ-algebra $\mathcal{F}$ is a set in $\mathcal{F}$, where the empty set and A itself are the only subsets of A that are again also in $\mathcal{F}$. Keeping this definition in mind, each $\mathcal{F}_n$ is therefore made up of the $2^{r(n)}$ possible unions of the atoms $A_{n,1}, \ldots A_{n,r(n)}$. In order to see this easily, one has to understand that each atom can be represented as

$$A_{n,k} = H_1 \cap H_2 \cap \cdots \cap H_n,$$

where $H_j = A_j$ or $H_j = A_j^{\mathsf{c}}$. Working with those atoms provides the opportunity to define the function below unambiguously, because for each $\omega \in \Omega$ there exists exactly one $k \in \{1, \ldots, r(n)\}$, such that $\omega \in A_{n,k}$. The existence is obvious, as $\Omega \in \mathcal{F}_n$. If there were to exist $k \neq \tilde{k} \in \{1, \ldots, r(n)\}$ satisfying $\omega \in A_{n,k}$ and $\omega \in A_{n,\tilde{k}}$, then $\omega \in A_{n,k} \cap A_{n,\tilde{k}}$. This however would mean that this intersect would be a nonempty strict subset of both $A_{n,k}$ and $A_{n,\tilde{k}}$ while also belonging to $\mathcal{F}$, contradicting the definition of an atom.
For each $n \in \mathbb{N}$ and $\omega \in \Omega$ with $\omega \in A_{n,k}$ now define $f_n : \Omega \to [0, \infty)$ as

$$f_n(\omega) := \begin{cases} 0 & \text{if } \nu(A_{n,k}) = 0, \\ \frac{\mu(A_{n,k})}{\nu(A_{n,k})} & \text{if } \nu(A_{n,k}) > 0. \end{cases}$$

Keep in mind that due to the definition of an atom for each $A \in \mathcal{F}_n$ and $k \in \{1, \ldots, r(n)\}$ the intersect $A \cap A_{n,k}$ is either $A_{n,k}$ or $\varnothing$. From this it is clear to see that each f_n is

$\mathcal{F}_n$-measurable and that for every $A \in \mathcal{F}_n$

$$\begin{aligned}\mathbb{E}[f_n \mathbb{1}_A] &= \int_A f_n(\omega)\,\mathrm{d}\nu(\omega) = \sum_{k=1}^{r(n)} \int_{A\cap A_{n,k}} f_n(\omega)\,\mathrm{d}\nu(\omega) = \sum_{\substack{k=1\\ A_{n,k}\subseteq A}}^{r(n)} \int_{A_{n,k}} f_n(\omega)\,\mathrm{d}\nu(\omega)\\ &= \sum_{\substack{k=1\\ A_{n,k}\subseteq A\\ \nu(A_{n,k})>0}}^{r(n)} \int_{A_{n,k}} \frac{\mu(A_{n,k})}{\nu(A_{n,k})}\,\mathrm{d}\nu(\omega) = \sum_{\substack{k=1\\ A_{n,k}\subseteq A\\ \nu(A_{n,k})>0}}^{r(n)} \frac{\mu(A_{n,k})}{\nu(A_{n,k})}\int_{A_{n,k}} \mathrm{d}\nu(\omega)\\ &= \sum_{\substack{k=1\\ A_{n,k}\subseteq A\\ \nu(A_{n,k})>0}}^{r(n)} \mu(A_{n,k}) = \sum_{\substack{k=1\\ A_{n,k}\subseteq A}}^{r(n)} \mu(A_{n,k}) = \mu(A)\end{aligned} \tag{4.2}$$

holds, where in the second-to-last equality the fact $\nu(A) = 0 \Rightarrow \mu(A) = 0$ for $A \in \mathcal{F}$ is used. Furthermore, by setting $A = \Omega$ in the previous equation, it is apparent that $f_n \in \mathcal{L}^1(\Omega, \mathcal{F}_n, \nu)$ due to the finiteness of μ. Therefore f_n can be seen as a Radon–Nikodým derivative of μ w.r.t. ν on the space $(\Omega, \mathcal{F}_n)$.

By the previous findings, $M := (f_n)_{n\in\mathbb{N}}$ is an integrable, time-discrete process adapted to the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Equation (4.2) suffices to see that $\mathbb{E}[f_{n+1}\mathbb{1}_A] = \mu(A) = \mathbb{E}[f_n\mathbb{1}_A]$ for every $A \in \mathcal{F}_n$ and therefore $\mathbb{E}[f_{n+1}|\mathcal{F}_n] = f_n$ for each $n \in \mathbb{N}$. Consequently, M is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \nu)$. Keeping in mind that each f_n is non-negative, i.e. $(f_n)^- = 0$, one can now use *Doob's almost sure convergence theorem* (see Theorem 5.8 in the appendix) to obtain almost surely a ν-integrable and $\mathcal{F}$-measurable random variable

$$f_\infty = \lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} f_n,$$

as $\mathcal{F}_\infty = \mathcal{F}$ per definition of $\mathcal{F}_n$.

Fix now $\epsilon > 0$. Lemma 4.2 necessitates the existence of a $\delta > 0$, such that

$$\nu(A) < \delta \Rightarrow \mu(A) < \epsilon.$$

holds for all $A \in \mathcal{F}$. Select a $C \in (0, \infty)$ that satisfies

$$C^{-1}\mu(\Omega) < \delta.$$

This causes the boundedness of $\nu(\{f_n > C\})$ by δ simultaneously for all $n \in \mathbb{N}$, as

$$\begin{aligned}\nu(\{f_n > C\}) &= \int_{\{f_n>C\}} \mathrm{d}\nu = C^{-1}\int_{\{f_n>C\}} C\,\mathrm{d}\nu < C^{-1}\int_{\{f_n>C\}} f_n\,\mathrm{d}\nu \le C^{-1}\int_\Omega f_n\,\mathrm{d}\nu\\ &= C^{-1}\mathbb{E}[f_n] = C^{-1}\mu(\Omega) < \delta.\end{aligned}$$

Therefore M is uniformly integrable (see Definition 5.9 in the appendix), because

$$\mathbb{E}[|f_n|\mathbb{1}_{\{|f_n|>C\}}] = \mathbb{E}[f_n\mathbb{1}_{\{f_n>C\}}] = \mu(\{f_n > C\}) < \epsilon, \qquad n \in \mathbb{N},$$

holds, where in the second step equation (4.2) is used again. This allows the usage of Theorem 5.12 in the appendix, which suffices to see that the martingale $M = (f_n)_{n\in\mathbb{N}}$ converges to f_∞ in $\mathcal{L}^1$. For every $A \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ there exists a $n_0 \in \mathbb{N}$, such that $A \in \mathcal{F}_n$ for all $n \geq n_0$. Consequently, for such a set A

$$\int_A f_\infty \, \mathrm{d}\nu = \mathbb{E}[f_\infty \mathbb{1}_A] = \mathbb{E}[\lim_{n\to\infty} f_n \mathbb{1}_A] = \lim_{n\to\infty} \mathbb{E}[f_n \mathbb{1}_A] = \lim_{n\to\infty} \mu(A) = \mu(A)$$

remains true. The set $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is *intersection-stable*, as for each pair $A, \tilde{A} \in \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ there exists a $n_0 \in \mathbb{N}$ in a way that both $A, \tilde{A} \in \mathcal{F}_{n_0}$ implying $A \cap \tilde{A} \in \mathcal{F}_{n_0}$, as $\mathcal{F}_{n_0}$ is a σ-algebra. Note that

$$\mathcal{F} = \sigma(A_n : n \in \mathbb{N}) = \sigma\big(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n\big).$$

Then, by Lemma 5.14 in the appendix, the previous equality also holds more general for all $A \in \mathcal{F}$. Therefore f_∞ is a Radon–Nikodým derivative of μ w.r.t. ν on $(\Omega, \mathcal{F})$.

In order to prove the uniqueness ν-a.s. let f and $\tilde{f}$ be two possible Radon–Nikodým derivatives satisfying equation (4.1). Then, due to the required $\mathcal{F}$-measurability, the set

$$A := \{f \neq \tilde{f}\} = \{f > \tilde{f}\} \cup \{f < \tilde{f}\}$$

is in $\mathcal{F}$ and thus must be a ν null set, as

$$\int_{\{f>\tilde{f}\}} f \, \mathrm{d}\nu = \mu(\{f > \tilde{f}\}) = \int_{\{f>\tilde{f}\}} \tilde{f} \, \mathrm{d}\nu$$

implies $\nu(\{f > \tilde{f}\})$ to be zero and the same can be done for $\{f < \tilde{f}\}$. □

The separability of $\mathcal{F}$, which was assumed in the beginning of this chapter is a rather unpleasant restriction and will therefore be dropped in the following.

Lemma 4.3. *Theorem 4.1 remains true, even if $\mathcal{F}$ may not be separable.*

Proof. To avoid confusion it should be stated here that the terms *sub-* and *superset* are not to be understood in the strict sense, unless explicitly stated otherwise. Furthermore, throughout this proof $\mathcal{F}$ is not allowed to be separable.

Define Sep as the set of all separable sub-σ-algebras of $\mathcal{F}$. Note at this point that each $\mathcal{G}$ in Sep is a strict sub-σ-algebra of $\mathcal{F}$ and thus there exists at least one set $A \in \mathcal{F}$ with $A \notin \mathcal{G}$. Therefore one may consider the separable σ-algebra $\sigma(\mathcal{G}, A)$ and see that each $\mathcal{G}$ in Sep has at least one strict super-σ-algebra in Sep.

Due to Theorem 4.1 there exists for each $\mathcal{G} \in Sep$ a Radon–Nikodým derivative $f_\mathcal{G} \in \mathcal{L}^1(\Omega, \mathcal{G}, \nu)$. In the first step of the proof consider the family $(f_\mathcal{G})_{\mathcal{G}\in Sep}$. This family will be proven to be a "Cauchy sequence", where one should understand this phrase as follows: For all $\epsilon > 0$ there exists a $\mathcal{K} \in Sep$ such that for each two supersets $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{K}$ in Sep the inequality $\|f_{\mathcal{G}_1} - f_{\mathcal{G}_2}\|_1 < \epsilon$ must hold, where $\|\cdot\|_1$ denotes the L^1-norm on $L^1(\Omega, \mathcal{F}, \nu)$. Let's assume the opposite, meaning the existence of an $\epsilon_0 > 0$, such that for every $\mathcal{K} \in Sep$

exist at least two supersets $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{K}$ in *Sep* with $\|f_{\mathcal{G}_1} - f_{\mathcal{G}_2}\|_1 \geq \epsilon_0$. Keep in mind that every $\mathcal{K}$ in *Sep* has at least two different super-σ-algebras in *Sep*, namely the above mentioned strict super-σ-algebra and itself. Now fix some $\mathcal{K}_1 \in Sep$ and the corresponding σ-algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ in *Sep* fulfilling the inequality above. Due to the triangle-inequality

$$\epsilon_0 \leq \|f_{\mathcal{G}_1} - f_{\mathcal{G}_2}\|_1 \leq \|f_{\mathcal{G}_1} - f_{\mathcal{K}_1}\|_1 + \|f_{\mathcal{K}_1} - f_{\mathcal{G}_2}\|_1$$

holds and therefore at least one of the summands on the right-hand side has to be bigger or equal to $\epsilon_0/2$. Define then $\mathcal{K}_2$ to be either $\mathcal{G}_1$ or $\mathcal{G}_2$, such that

$$\|f_{\mathcal{K}_1} - f_{\mathcal{K}_2}\|_1 \geq \frac{\epsilon_0}{2}$$

is satisfied. The same can then in turn be done with $\mathcal{K}_2$ leading to the definition of $\mathcal{K}_3 \supseteq \mathcal{K}_2 \supseteq \mathcal{K}_1$. This leads to the existence of a sequence $\mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \cdots$ in *Sep* satisfying

$$\|f_{\mathcal{K}_n} - f_{\mathcal{K}_{n+1}}\|_1 \geq \frac{\epsilon_0}{2}, \qquad n \in \mathbb{N}.$$

In a similar way as in the proof of Theorem 4.1 one obtains that $(f_{\mathcal{K}_n})_{n\in\mathbb{N}}$ is a uniformly integrable martingale w.r.t. the filtration $(\mathcal{K}_n)_{n\in\mathbb{N}}$ and therefore converges in $\mathcal{L}^1$, which makes the inequality above impossible.

Therefore one can find for each $n \in \mathbb{N}$ a $\mathcal{K}_n \in Sep$ satisfying that if $\mathcal{K}_n \subseteq \mathcal{G}_1 \cap \mathcal{G}_2$ for $\mathcal{G}_1$ and $\mathcal{G}_2$ both in *Sep*, $\|f_{\mathcal{G}_1} - f_{\mathcal{G}_2}\|_1 < 2^{-(n+1)}$ holds. Define now $\mathcal{H}_n = \sigma(\mathcal{K}_1 \cup \cdots \cup \mathcal{K}_n)$ for each $n \in \mathbb{N}$. From this point on, the Radon–Nikodým derivatives $f_{\mathcal{H}_n}$ will be considered as equivalence classes in the normed vector space $L^1(\Omega, \mathcal{H}_n, \nu)$. Then it follows from the completeness of this aforementioned space (for more details about the completeness of L^p-spaces the reader is referred to [Wil91, p. 65f]) that the limit

$$f := \lim_{n\to\infty} f_{\mathcal{H}_n}$$

exists almost surely as well as in L^1. Thus for each $n \in \mathbb{N}$ there exists some $n_0 \in \mathbb{N}$, such that for all $m \geq n_0$

$$\|f - f_{\mathcal{H}_m}\|_1 \leq 2^{-(n+1)}$$

holds. Therefore, for every $n \in \mathbb{N}$ there exists some $n_0 \in \mathbb{N}$ so that for $m := \max\{n, n_0\}$ and every $\mathcal{G} \in Sep$ with $\mathcal{H}_m \subseteq \mathcal{G}$

$$\|f - f_{\mathcal{G}}\|_1 \leq \|f - f_{\mathcal{H}_m}\|_1 + \|f_{\mathcal{H}_m} - f_{\mathcal{G}}\|_1 \leq 2^{-(n+1)} + 2^{-(m+1)} \leq 2^{-(n+1)} + 2^{-(n+1)} = 2^{-n}.$$

So in that sense, one can say $(f_{\mathcal{G}})_{\mathcal{G}\in Sep} \to f$ in L^1.

With slight abuse of notation, some function in the equivalence class f will now be fixed and in the following be denoted also by $f \in \mathcal{L}^1(\Omega, \mathcal{G}, \nu)$. It therefore now suffices to show equation (4.1) for such a function f. In order to do so fix some $A \in \mathcal{F}$ and let $\epsilon > 0$ and $\mathcal{K} \in Sep$ be such that for each $Sep \ni \mathcal{G} \supseteq \mathcal{K}$ the inequality $\|f - f_{\mathcal{G}}\|_1 < \epsilon$ holds, which is always possible due to the previous steps. Furthermore note that $\sigma(\mathcal{K}, A) \in Sep$ and thus

$$|\mathbb{E}[f\mathbb{1}_A] - \mu(A)| = |\mathbb{E}[(f - f_{\sigma(\mathcal{K},A)})\mathbb{1}_A]| \leq |\mathbb{E}[f - f_{\sigma(\mathcal{K},A)}]| \leq \|f - f_{\sigma(\mathcal{K},A)}\|_1 < \epsilon.$$

As this holds for each $\epsilon > 0$, the lemma is proven, as the uniqueness can be shown in the same way as in Theorem 4.1. □

Those results can be extended even further making the Radon–Nikodým theorem even more useful.

Theorem 4.4. *Let μ and ν denote one finite and one σ-finite measure on $(\Omega, \mathcal{F})$, respectively. If $\mu \ll \nu$ on $\mathcal{F}$, then there exists a ν-almost everywhere uniquely defined non-negative function $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \nu)$, such that*

$$\mu(A) = \int_A \mathrm{d}\mu = \int_A f \, \mathrm{d}\nu, \qquad A \in \mathcal{F}. \tag{4.3}$$

Such a function f is then understood to be a Radon–Nikodým derivative of μ w.r.t. ν on $\mathcal{F}$ and may also be denoted as

$$f = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}.$$

Proof. Note at this point that if $\nu(\Omega) = 0$ (and consequently $\mu(\Omega) = 0$) any non-negative function $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \nu)$ can be regarded as a Radon–Nikodým derivative of μ w.r.t. ν on $\mathcal{F}$ and the theorem is proven. Therefore in the remaining parts of the proof, this extreme case will no longer be considered and $\nu(\Omega) > 0$ is pre-conditioned.

At first assume ν to be finite. Then by defining $\tilde{\nu}(A) := \frac{\nu(A)}{\nu(\Omega)}$ for each $A \in \mathcal{F}$ one obtains a probability measure $\tilde{\nu}$, for which there exists the Radon–Nikodým derivative $\tilde{f}$. By the linearity of the integral it is clear to see that

$$\mu(A) = \int_A \tilde{f} \, \mathrm{d}\tilde{\nu} = \int_A \frac{\tilde{f}}{\nu(\Omega)} \, \mathrm{d}\nu, \qquad A \in \mathcal{F}$$

and therefore $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = \frac{\tilde{f}}{\nu(\Omega)} =: f$.

Now ν is assumed to no longer be finite but σ-finite. One can therefore find a partition consisting of $\mathcal{F}$-measurable nonempty sets $\Omega = \bigcup_{n \in \mathbb{N}} A_n$ with $A_n \cap A_m = \varnothing$ for $n \neq m$ in such a way that $\nu(A_n) < \infty$ for all $n \in \mathbb{N}$, i.e. $\nu \restriction_{\mathcal{F}_n}$ is a finite measure on $(A_n, \mathcal{F}_n)$, where $\mathcal{F}_n := \{A \cap A_n : A \in \mathcal{F}\}$ is the trace σ-algebra of $\mathcal{F}$ on A_n. Therefore there exists a Radon–Nikodým derivative $f_n \in \mathcal{L}^1(A_n, \mathcal{F}_n, \nu \restriction_{\mathcal{F}_n})$ for each $n \in \mathbb{N}$. Considering now a $A \in \mathcal{F}$ one obtains by using the *monotone convergence theorem* (see Theorem 5.38 in the appendix) in the last equality

$$\begin{aligned} \mu(A) = \mu(A \cap \bigcup_{n \in \mathbb{N}} A_n) &= \sum_{n=1}^{\infty} \mu(A \cap A_n) = \sum_{n=1}^{\infty} \int_{A \cap A_n} f_n \, \mathrm{d}\nu = \sum_{n=1}^{\infty} \int_A f_n \mathbb{1}_{A_n} \, \mathrm{d}\nu \\ &= \lim_{m \to \infty} \sum_{n=1}^{m} \int_A f_n \mathbb{1}_{A_n} \, \mathrm{d}\nu = \lim_{m \to \infty} \int_A \sum_{n=1}^{m} f_n \mathbb{1}_{A_n} \, \mathrm{d}\nu = \int_A \sum_{n=1}^{\infty} f_n \mathbb{1}_{A_n} \, \mathrm{d}\nu \end{aligned}$$

Defining then

$$f(\omega) = \sum_{n=1}^{\infty} f_n(\omega) \mathbb{1}_{A_n}(\omega)$$

gives a Radon–Nikodým derivative on of μ w.r.t. ν on $\mathcal{F}$, as it is obviously $\mathcal{F}$-measurable and due to the now proven equation (4.3) applied to $A = \Omega$ one can see that $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \nu)$ as long as μ is finite. □

At last, yet another extension of the Radon–Nikodým theorem will be provided. To be precise, one wants to use it even for two σ-finite measures. This however comes with a price tag, as it is then no longer possible to prove the integrability of the Radon–Nikodým derivative. By setting $A = \Omega$ in the first equality of the upcoming theorem it can be easily seen that for a measure μ that is indeed not finite but σ-finite, the Radon–Nikodým derivative is not integrable w.r.t. ν.

Theorem 4.5 (The Radon–Nikodým theorem for σ-finite measures)**.** *Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$. If $\mu \ll \nu$ on $\mathcal{F}$, then there exists a ν-almost everywhere uniquely defined $\mathcal{F}$-measurable function f, such that*

$$\mu(A) = \int_A \mathrm{d}\mu = \int_A f \,\mathrm{d}\nu, \qquad A \in \mathcal{F}.$$

Note that in contrast to the last theorems, f is still $\mathbb{R}_+$-valued, it however may not be in $\mathcal{L}^1(\Omega, \mathcal{F}, \nu)$. Such a function f is then understood to be a Radon–Nikodým derivative of μ w.r.t. ν on $\mathcal{F}$ and may also be denoted as

$$f = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}.$$

Proof. Let μ be not finite but σ-finite. Then, exactly like in the theorem above, one can find a partition consisting of $\mathcal{F}$-measurable nonempty sets $\Omega = \bigcup_{n\in\mathbb{N}} A_n$ with $A_n \cap A_m = \varnothing$ for $n \neq m$ such that $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$. The Radon-Nikodym derivative f is then defined the same as above, namely

$$f(\omega) = \sum_{n=1}^{\infty} f_n(\omega) \mathbb{1}_{A_n}(\omega)$$

and is therefore again $\mathcal{F}$-measurable. Note that for any given $\omega \in \Omega$ only on of the indicator functions on the right-hand side are nonzero, and therefore $f(\omega) < \infty$ holds for each $\omega \in \Omega$. □

In the course of this section, the conditions on the measures μ and ν have gotten weaker and weaker, starting at finite μ and a probability measure ν and ending with two σ-finite measures. For more general measures however, the conclusion of the above theorems may fail, which will be shown by the following examples.

Example 4.6 (Absolutely continuous measures without a density)**.** On the measurable space $(\{0\}, \{\varnothing, \{0\}\})$ set $\mu(\{0\}) = 1$ and $\nu(\{0\}) = \infty$. Therefore $\mu \ll \nu$ (even $\mu \sim \nu$ as also $\nu \ll \mu$, i.e. the two measures are equivalent), as the empty set is the only null set for both measures. However, there does not exist a function $f : \{0\} \to \mathbb{R}_+$ satisfying

$$1 = \mu(\{0\}) = \int_{\{0\}} f \,\mathrm{d}\nu = f(0)\nu(\{0\}),$$

as $\nu(\{0\}) = \infty$. Thus the σ-finiteness of ν in the Radon-Nikodym Theorems is essential.

Example 4.7 (Multiple different densities)**.** In the same setting as above, every function $f : \{0\} \to (0, \infty]$ can be considered as a derivative of the not σ-finite measure ν w.r.t. itself on $\{\varnothing, \{0\}\}$, so the uniqueness ν-almost everywhere of a density could fail as well as its existence, which was demonstrated in the example above [Sch23, p. 253f].

4.2 The Radon–Nikodým theorem for signed or complex measures on δ-rings

Definition 4.8 (Signed and complex measures on δ-rings)**.** Let $\mathcal{R}$ be a δ-ring (see [Sch23, Definition 15.106]) on a set S and $\mathcal{F} := \sigma(\mathcal{R})$ its generated σ-algebra. According to the Jordan decomposition, every signed (or complex) measure μ on $\mathcal{R}$ can be written as $\mu = \mu_+ - \mu_-$ (or $\mu = \mu_+^R - \mu_-^R + \mathrm{i}(\mu_+^I - \mu_-^I)$), where the right-hand side consists of $\mathbb{R}_+$-valued measures in $\mathcal{R}$.

Assumption 4.9. Throughout this section, the existence of a sequence $(A_n)_{n\in\mathbb{N}}$ of disjoint sets in $\mathcal{R}$ that satisfy $\bigcup_{n\in\mathbb{N}} A_n = S$, is assumed.

Note at this point that in the throughout this thesis most important case, i.e. $S = \mathbb{R}_+$ and $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]}$, this assumption is trivially fulfilled by defining $A_n = [n, n+1)$ for $n \in \mathbb{N} \cup \{0\}$.

Definition 4.10 (Total variation measure)**.** The *total variation measure* $|\mu|$ of a $\mathbb{K}$-valued measure μ on $\mathcal{R}$ is defined for every $A \in \mathcal{F}$ by

$$|\mu|(A) = \sup_{\pi\in\Pi} \sum_{B\in\pi} |\mu(B)|,$$

in which Π is the set of all countable collections of disjoint subsets of A in $\mathcal{R}$.

Note that for two $\mathbb{K}$-valued measures μ, ν on $\mathcal{R}$ and $a \in \mathbb{K}$, the total variation measure has the properties

$$|a\mu|(A) = \sup_{\pi\in\Pi} \sum_{B\in\pi} |a\mu(B)| = \sup_{\pi\in\Pi} \sum_{B\in\pi} |a|\,|\mu(B)| = |a| \sup_{\pi\in\Pi} \sum_{B\in\pi} |\mu(B)| = |a|\,|\mu|(A), \qquad A \in \mathcal{F}$$

and

$$\begin{aligned}
|\mu + \nu|(A) &= \sup_{\pi\in\Pi} \sum_{B\in\pi} |(\mu+\nu)(B)| \leq \sup_{\pi\in\Pi} \sum_{B\in\pi} (|\mu(B)| + |\nu(B)|) \\
&= \sup_{\pi\in\Pi} \Big(\big(\sum_{B\in\pi} |\mu(B)|\big) + \big(\sum_{B\in\pi} |\nu(B)|\big) \Big) \leq \Big(\sup_{\pi\in\Pi} \sum_{B\in\pi} |\mu(B)| \Big) + \Big(\sup_{\pi\in\Pi} \sum_{B\in\pi} |\nu(B)| \Big) \\
&= |\mu|(A) + |\nu|(A), \qquad A \in \mathcal{F}.
\end{aligned}$$

Each σ-finite $\mathbb{R}_+$-valued measure on $\mathcal{R}$ can be extended to an $\overline{\mathbb{R}}_+$-valued measures on $\mathcal{F}$. In the following, these minimal extensions will be denoted by the same symbols as the underlying measures on $\mathcal{R}$. Therefore, the *total variation measure* $|\mu|$ of a signed measure μ can be written as $|\mu| = \mu_+ + \mu_-$. For a complex measure μ it can be shown that $|\mathrm{Re}(\mu)| = \mu_+^R + \mu_-^R$, $|\mathrm{Im}(\mu)| = \mu_+^I + \mu_-^I$ and $|\mu| \leq |\mathrm{Re}(\mu)| + |\mathrm{Im}(\mu)|$. Additionally, Assumption 4.9 implies that $|\mu|$ is indeed a σ-finite $\overline{\mathbb{R}}_+$-valued measure on $\mathcal{F}$ [Sch23, p. 467ff].

Let μ and ν be two signed (or complex) measures on a δ-ring. One may call μ *absolutely continuous* w.r.t. ν, i.e. $\mu \ll \nu$ on $\mathcal{F}$, if and only if the total variation measure $|\mu|$ is absolutely continuous w.r.t. $|\nu|$ on $\mathcal{F}$.

Theorem 4.11 (The Radon–Nikodým theorem for signed measures on δ-rings)**.** *In the setting of Definition 4.8 let μ and ν be two signed measures on a δ-ring $\mathcal{R}$ on a set S and $\mathcal{F} := \sigma(\mathcal{R})$, satisfying $\mu \ll \nu$ on $\mathcal{F}$. Then there exists a Radon–Nikodým derivative of μ w.r.t. ν on $\mathcal{R}$, i.e. a $\mathcal{F}$-measurable function $f : S \to \mathbb{R}$ that satisfies*

$$\mu(A) = \int_A f \, \mathrm{d}\nu = \int_A f \, \mathrm{d}\nu_+ - \int_A f \, \mathrm{d}\nu_-$$

for all sets $A \in \mathcal{R}$. This function is $|\nu|$-almost everywhere unique and may be written as $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$.

Proof.

Step 1 (Existence). Due to Definition 4.8 and the assumption $\mu \ll \nu$, $|\mu|$ and $|\nu|$ are two σ-finite measures on the measure space $(S, \mathcal{F})$ satisfying $|\mu| \ll |\nu|$. Therefore, due to the Radon–Nikodým theorem for σ-finite measures Theorem 4.5, there exists $|\nu|$-almost everywhere uniquely a $\mathcal{F}$-measurable function $\tilde{f} : S \to \mathbb{R}_+$ satisfying

$$|\mu|(A) = \int_A \tilde{f} \, \mathrm{d}|\nu|, \qquad A \in \mathcal{F}.$$

Let $A \in \mathcal{R}$ be an arbitrary set and P_μ, N_μ and P_ν, N_ν be the Hahn decomposition on δ-rings (see [Sch23, Theorem 15.116 and Remark 15.117]) of S w.r.t. μ and ν, respectively, which exists, because by the assumption made just above the start of this theorem, there exists sequence $(A_n)_{n \in \mathbb{N}} \in \mathcal{R}$ of disjoint sets, whose union is S. Note that those four sets are elements of $\mathcal{F}$, not necessarily of $\mathcal{R}$ and that the Hahn decomposition of $A \in \mathcal{R}$ w.r.t. μ can be easily identified as $A^+_\mu = A \cap P_\mu$ and $A^-_\mu = A \cap N_\mu$. Therefore, one can define the $\mathcal{F}$-measurable function

$$f(a) = \begin{cases} \tilde{f}(a) & \text{for } a \in (P_\mu \cap P_\nu) \cup (N_\mu \cap N_\nu), \\ -\tilde{f}(a) & \text{for } a \in (P_\mu \cap N_\nu) \cup (N_\mu \cap P_\nu). \end{cases}$$

By using the equalities

$$\mu_+(A) = \mu(A^+_\mu), \qquad \mu_-(A) = -\mu(A^-_\mu), \qquad \mu_+(A^-_\mu) = \mu_-(A^+_\mu) = 0 \tag{4.4}$$

presented by the Jordan decomposition one can prove that the previously defined function f is indeed the Radon–Nikodým derivative we are looking for. By switching the signs of terms equal to zero, the third equality is justified and one gets

$$\begin{aligned} \mu(A) &= \mu_+(A) - \mu_-(A) = \mu_+(A^+_\mu) + \mu_+(A^-_\mu) - \mu_-(A^+_\mu) - \mu_-(A^-_\mu) \\ &= \mu_+(A^+_\mu) + \mu_-(A^+_\mu) - \mu_+(A^-_\mu) - \mu_-(A^-_\mu) = |\mu|(A^+_\mu) - |\mu|(A^-_\mu) \\ &= |\mu|(A \cap P_\mu) - |\mu|(A \cap N_\mu) = \int_{A \cap P_\mu} \tilde{f} \, \mathrm{d}|\nu| - \int_{A \cap N_\mu} \tilde{f} \, \mathrm{d}|\nu|, \end{aligned}$$

where the Radon–Nikodym theorem is used in the last equation. By using analogous equations as in (4.4) for ν the right-hand side can be further split into

$$\int_{A\cap P_\mu} \tilde f \,\mathrm{d}|\nu| - \int_{A\cap N_\mu} \tilde f \,\mathrm{d}|\nu| = \int_{A\cap P_\mu} \tilde f \,\mathrm{d}\nu_+ + \int_{A\cap P_\mu} \tilde f \,\mathrm{d}\nu_- - \int_{A\cap N_\mu} \tilde f \,\mathrm{d}\nu_+ - \int_{A\cap N_\mu} \tilde f \,\mathrm{d}\nu_-$$
$$= \int_{(A\cap P_\mu)\cap P_\nu} \tilde f \,\mathrm{d}\nu - \int_{(A\cap P_\mu)\cap N_\nu} \tilde f \,\mathrm{d}\nu - \int_{(A\cap N_\mu)\cap P_\nu} \tilde f \,\mathrm{d}\nu + \int_{(A\cap N_\mu)\cap N_\nu} \tilde f \,\mathrm{d}\nu$$
$$= \int_{A\cap(P_\mu\cap P_\nu)} \tilde f \,\mathrm{d}\nu + \int_{A\cap(N_\mu\cap N_\nu)} \tilde f \,\mathrm{d}\nu - \int_{A\cap(P_\mu\cap N_\nu)} \tilde f \,\mathrm{d}\nu - \int_{A\cap(N_\mu\cap P_\nu)} \tilde f \,\mathrm{d}\nu$$
$$= \int_{A\cap\big((P_\mu\cap P_\nu)\cup(N_\mu\cap N_\nu)\big)} \tilde f \,\mathrm{d}\nu - \int_{A\cap\big((P_\mu\cap N_\nu)\cup(N_\mu\cap P_\nu)\big)} \tilde f \,\mathrm{d}\nu$$
$$= \int_A \tilde f \mathbb{1}_{[(P_\mu\cap P_\nu)\cup(N_\mu\cap N_\nu)]} - \tilde f \mathbb{1}_{[(P_\mu\cap N_\nu)\cup(N_\mu\cap P_\nu)]} \,\mathrm{d}\nu = \int_A f \,\mathrm{d}\nu$$

and thus f may be regarded as a Radon–Nikodým derivative of μ w.r.t. ν on $\mathcal{R}$.

Step 2 (Uniqueness). To show the $|\nu|$-almost everywhere uniqueness consider two Radon–Nikodým derivatives f_1 and f_2 and consider the set $\{f_1 < f_2\}$, which is in $\mathcal{F}$, due to the $\mathcal{F}$-measurability of both f_1 and f_2. By the assumption made just above the start of this theorem, there exists sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{R}$ of disjoint sets satisfying $\bigcup_{n\in\mathbb{N}} A_n = S$. Define for each $n \in \mathbb{N}$ the trace-δ-ring $\mathcal{R}_n = \{A \cap A_n : A \in \mathcal{R}\}$, which is the same as the trace-σ-algebra $\{A \cap A_n : A \in \mathcal{F}\}$ by [Sch23, Lemma 15.110(a)] as well as a (not necessarily strict) subset of $\mathcal{R}$ by the definition of a δ-ring. Furthermore, as $P_\nu \in \mathcal{F}$, the same holds for $\{f_1 < f_2\} \cap P_\nu$. Thus the set $(\{f_1 < f_2\} \cap P_\nu) \cap A_n$ is an element of $\mathcal{R}_n \subseteq \mathcal{R}$ for all $n \in \mathbb{N}$ one may now see that

$$\int_{\{f_1<f_2\}} f_1 \,\mathrm{d}\nu_+ = \int_{\{f_1<f_2\}\cap P_\nu} f_1 \,\mathrm{d}\nu = \int_{(\{f_1<f_2\}\cap P_\nu)\cap(\bigcup_{n\in\mathbb{N}} A_n)} f_1 \,\mathrm{d}\nu$$
$$= \sum_{n\in\mathbb{N}} \int_{(\{f_1<f_2\}\cap P_\nu)\cap A_n} f_1 \,\mathrm{d}\nu = \sum_{n\in\mathbb{N}} \mu(\{f_1 < f_2\} \cap P_\nu \cap A_n) = \sum_{n\in\mathbb{N}} \int_{(\{f_1<f_2\}\cap P_\nu)\cap A_n} f_2 \,\mathrm{d}\nu$$
$$= \int_{(\{f_1<f_2\}\cap P_\nu)\cap(\bigcup_{n\in\mathbb{N}} A_n)} f_2 \,\mathrm{d}\nu = \int_{\{f_1<f_2\}\cap P_\nu} f_2 \,\mathrm{d}\nu = \int_{\{f_1<f_2\}} f_2 \,\mathrm{d}\nu_+.$$

An analogous result follows also for the measure ν_-, as $N_\nu \in \mathcal{F}$, and for the set $\{f_1 > f_2\} \in \mathcal{F}$ with respect to both ν_+ and ν_-. Consequently, $f_1 = f_2$ must hold $|\nu|$-almost everywhere, due to

$$\int_{\{f_1<f_2\}} f_1 \,\mathrm{d}|\nu| = \int_{\{f_1<f_2\}} f_1 \,\mathrm{d}\nu_+ + \int_{\{f_1<f_2\}} f_1 \,\mathrm{d}\nu_- = \int_{\{f_1<f_2\}} f_2 \,\mathrm{d}\nu_+ + \int_{\{f_1<f_2\}} f_2 \,\mathrm{d}\nu_-$$
$$= \int_{\{f_1<f_2\}} f_2 \,\mathrm{d}|\nu|$$

and analogously

$$\int_{\{f_1>f_2\}} f_1 \,\mathrm{d}|\nu| = \int_{\{f_1>f_2\}} f_2 \,\mathrm{d}|\nu|,$$

resulting in $|\nu|(\{f_1 \neq f_2\}) = |\nu|(\{f_1 < f_2\}) + |\nu|(\{f_1 > f_2\}) = 0$, as $\{f_1 \neq f_2\}$ is the union of the disjoint sets $\{f_1 < f_2\}$ and $\{f_1 > f_2\}$. □

Obviously, this theorem can be extended to a complex measure $\mu = \mu^R + \mathrm{i}\mu^I$ and a signed measure ν on $\mathcal{R}$, as $\mu \ll \nu$ implies that $\mu^R \ll \nu$ as well as $\mu^I \ll \nu$ hold. Then one could use Theorem 4.11 for the real and imaginary part separately, which results in

$$\mu(A) = \mu^R(A) + \mathrm{i}\mu^I(A) = \int_A f^R \,\mathrm{d}\nu + \mathrm{i}\int_A f^I \,\mathrm{d}\nu = \int_A f^R + \mathrm{i}f^I \,\mathrm{d}\nu, \qquad A \in \mathcal{R}$$

and the Radon–Nikodým derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = f^R + \mathrm{i}f^I$.

The Radon–Nikodým derivative has many useful properties, some of which will be proven in the following lemma.

Lemma 4.12. *As in Theorem 4.11 let μ and ν be two signed or complex measures on a δ-ring $\mathcal{R}$, $\mathcal{F} := \sigma(\mathcal{R})$ and $a \in \mathbb{K}$. Thus the following four statements hold.*

(i) The Radon–Nikodým derivative is linear, i.e. if λ is a signed measure on $\mathcal{R}$ with $\mu \ll \lambda$ and $\nu \ll \lambda$ on $\mathcal{F}$, then also $(a\mu + \nu) \ll \lambda$ on $\mathcal{F}$ and

$$\frac{\mathrm{d}(a\mu + \nu)}{\mathrm{d}\lambda} = a\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} + \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}, \qquad |\lambda|\text{-almost everywhere.}$$

(ii) If $\mu \ll \nu \ll \lambda$ on $\mathcal{F}$, where λ is a σ-finite measure on $\mathcal{F}$ and ν is restricted to be a signed measure, then

$$\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}, \qquad \lambda\text{-almost everywhere.}$$

(iii) Let again λ be a σ-finite measure on $\mathcal{F}$ and h be a $|\mu|$-integrable and $\mathcal{F}$-measurable function. Thus if $\mu \ll \lambda$, then

$$\int_A h \,\mathrm{d}\mu = \int_A h\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \,\mathrm{d}\lambda, \qquad A \in \mathcal{R}.$$

(iv) If λ denotes once more a σ-finite measure on $\mathcal{F}$ satisfying $\mu \ll \lambda$ and μ is a signed measure on $\mathcal{R}$, then

$$\frac{\mathrm{d}|\mu|}{\mathrm{d}\lambda} = \left|\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right|$$

follows on $\mathcal{R}$. For a complex measure μ on $\mathcal{R}$ this then implies

$$\left|\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right| = \left|\frac{\mathrm{d}(\mu^R + \mathrm{i}\mu^I)}{\mathrm{d}\lambda}\right| = \left|\frac{\mathrm{d}\mu^R}{\mathrm{d}\lambda} + \mathrm{i}\frac{\mathrm{d}\mu^I}{\mathrm{d}\lambda}\right| \leq \left|\frac{\mathrm{d}\mu^R}{\mathrm{d}\lambda}\right| + \left|\frac{\mathrm{d}\mu^I}{\mathrm{d}\lambda}\right| = \frac{\mathrm{d}|\mu^R|}{\mathrm{d}\lambda} + \frac{\mathrm{d}|\mu^I|}{\mathrm{d}\lambda} \leq 2\frac{\mathrm{d}|\mu|}{\mathrm{d}\lambda}.$$

(v) If μ and ν are two signed measures satisfying $\mu \ll \nu$ and $\nu \ll \mu$, i.e. they have the same null-sets, then

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} \neq 0, \quad \nu\text{-almost everywhere} \qquad \text{and} \qquad \frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \left(\frac{\mathrm{d}\mu}{\mathrm{d}\nu}\right)^{-1}, \quad \nu\text{-almost everywhere.}$$

Proof. (i) Fix some set $A \in \mathcal{F}$, such that $|\lambda|(A) = 0$. Therefore,

$$|a\mu + \nu|(A) \leq |a\mu|(A) + |\nu|(A) = |a|\,|\mu|(A) + |\nu|(A) = 0$$

holds, which follows directly from the properties of the total variation measure below Definition 4.10 and implies $(a\mu + \nu) \ll \lambda$ on $\mathcal{F}$. Take now any set $A \in \mathcal{R}$ and consider

$$(a\mu + \nu)(A) = a\mu(A) + \nu(A) = a \int_A \frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda + \int_A \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_A a\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} + \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda.$$

Thus the $|\lambda|$-almost everywhere uniqueness of the Radon–Nikodým derivative in Theorem 4.11 proves the first part of this lemma.

(ii) For each $A \in \mathcal{R}$ one can use the chain rule for Lebesgue integrals [Sch09, Satz 9.2.2(1)] twice in the third as well as part (i) in the fifth step to obtain

$$\begin{aligned}
\mu_+^R(A) &= \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\,\mathrm{d}\nu = \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\,\mathrm{d}\nu_+ - \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\,\mathrm{d}\nu_- \\
&= \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\frac{\mathrm{d}\nu_+}{\mathrm{d}\lambda}\,\mathrm{d}\lambda - \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\frac{\mathrm{d}\nu_-}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\Big(\frac{\mathrm{d}\nu_+}{\mathrm{d}\lambda} - \frac{\mathrm{d}\nu_-}{\mathrm{d}\lambda}\Big)\,\mathrm{d}\lambda \\
&= \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\frac{\mathrm{d}(\nu_+ - \nu_-)}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda
\end{aligned}$$

and, analogously,

$$\mu_-^R(A) = \int_A \frac{\mathrm{d}\mu_-^R}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda, \qquad \mu_+^I(A) = \int_A \frac{\mathrm{d}\mu_+^I}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda, \qquad \mu_-^I(A) = \int_A \frac{\mathrm{d}\mu_-^I}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda,$$

which combined leads to

$$\begin{aligned}
\mu(A) &= \mu_+^R(A) - \mu_-^R(A) + \mathrm{i}\big(\mu_+^I(A) - \mu_-^I(A)\big) \\
&= \int_A \frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda - \int_A \frac{\mathrm{d}\mu_-^R}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda + \mathrm{i}\int_A \frac{\mathrm{d}\mu_+^I}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda - \mathrm{i}\int_A \frac{\mathrm{d}\mu_-^I}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda \\
&= \int_A \Big(\frac{\mathrm{d}\mu_+^R}{\mathrm{d}\nu} - \frac{\mathrm{d}\mu_-^R}{\mathrm{d}\nu} + \mathrm{i}\frac{\mathrm{d}\mu_+^I}{\mathrm{d}\nu} - \mathrm{i}\frac{\mathrm{d}\mu_-^I}{\mathrm{d}\nu}\Big)\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_A \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda.
\end{aligned}$$

Thus one may use again the uniqueness in Theorem 4.11 conclude the proof of the second part.

(iii) Note that h is μ_+^R-, μ_-^R-, μ_+^I- and μ_-^I-integrable by assumption. Thus for each $A \in \mathcal{R}$ one can again use the chain rule for Lebesgue integrals [Sch09, Satz 9.2.2(2)] in the third step as well as the linearities of the Lebesgue integral and the Radon–Nikodým derivative to get

$$\begin{aligned}
\int_A h\,\mathrm{d}\mu &= \int_A h\,\mathrm{d}\mu_+^R - \int_A h\,\mathrm{d}\mu_-^R + \mathrm{i}\int_A h\,\mathrm{d}\mu_+^I - \mathrm{i}\int_A h\,\mathrm{d}\mu_-^I \\
&= \int_A h\frac{\mathrm{d}\mu_+^R}{\mathrm{d}\lambda}\,\mathrm{d}\lambda - \int_A h\frac{\mathrm{d}\mu_-^R}{\mathrm{d}\lambda}\,\mathrm{d}\lambda + \mathrm{i}\int_A h\frac{\mathrm{d}\mu_+^I}{\mathrm{d}\lambda}\,\mathrm{d}\lambda - \mathrm{i}\int_A h\frac{\mathrm{d}\mu_-^I}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \int_A h\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\,\mathrm{d}\lambda.
\end{aligned}$$

(iv) Fix $A \in \mathcal{R}$, revisit the construction of the Radon–Nikodým derivative in Theorem 4.11 and take the notation from there. As λ is a σ-finite measure on $\mathcal{F}$ it is apparent that $P_\lambda = S$ and $N_\lambda = \varnothing$. Consequently, $f(a) = \tilde{f}(a)$ for $a \in P_\mu$ and $f(a) = -\tilde{f}(a)$ for $a \in N_\mu$, which leads to $|f| = \tilde{f}$ and thus

$$
\begin{aligned}
|\mu|(A) = |\mu|(A \cap P_\mu) + |\mu|(A \cap N_\mu) &= \int_{A\cap P_\mu} \tilde{f}\,\mathrm{d}\lambda + \int_{A\cap N_\mu} \tilde{f}\,\mathrm{d}\lambda \\
&= \int_{A\cap P_\mu} |f|\,\mathrm{d}\lambda + \int_{A\cap N_\mu} |f|\,\mathrm{d}\lambda = \int_A |f|\,\mathrm{d}\lambda.
\end{aligned}
$$

(v) This follows directly from part (ii), as

$$
1 = \frac{\mathrm{d}\nu}{\mathrm{d}\nu} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\nu}, \qquad \nu\text{-almost everywhere.}
$$

□

The following theorem can be proven in exactly the same way as Theorem 4.11 by using Theorem 4.4 instead of Theorem 4.5 in the beginning of the proof and keeping the above mentioned extension to complex measures μ in mind.

Theorem 4.13 (The Radon–Nikodým theorem for signed or complex measures on σ-algebras)**.** *Let μ and ν denote one finite $\mathbb{K}$-valued and one signed or σ-finite measure on $(\Omega, \mathcal{F})$, respectively. If $\mu \ll \nu$ on $\mathcal{F}$, then there exists a $|\nu|$-almost surely uniquely defined $\mathbb{K}$-valued function $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \nu)$, such that*

$$
\mu(A) = \int_A \mathrm{d}\mu = \int_A f\,\mathrm{d}\nu, \qquad A \in \mathcal{F}.
$$

Such a function f is then understood to be a Radon–Nikodým derivative of μ w.r.t. ν on $\mathcal{F}$ and may also be denoted as

$$
f = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}.
$$

4.3 The Radon–Nikodým theorem for signed or complex transition kernels

The theorem below plays an essential role in the proofs of Theorem 2.7 and Lemma 3.14. At first though a Lemma is stated, which is helpful in its proof. Note at this point that $\mathcal{R} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]}$ is a δ-ring and consists of all relatively compact sets in $\mathbb{R}_+$, see [Sch23, Example 15.107(d)].

Lemma 4.14. *Let V denote a process in $\mathcal{V}_0^1$, which may be seen as a signed or complex transition kernel from Ω to $\mathbb{R}_+$ on $\mathcal{R}$, as stated in Lemma 5.28 in the appendix, satisfying pathwise $V = \mathbb{V}^{R,+} - \mathbb{V}^{R,-} + \mathrm{i}(\mathbb{V}^{I,+} - \mathbb{V}^{I,-})$. Consequently, the map*

$$
(V\otimes\mathbb{P})(A) := (\mathbb{V}^{R,+}\otimes\mathbb{P})(A) - (\mathbb{V}^{R,-}\otimes\mathbb{P})(A) + \mathrm{i}\big((\mathbb{V}^{I,+}\otimes\mathbb{P})(A) - (\mathbb{V}^{I,-}\otimes\mathbb{P})(A)\big), \quad A \in \mathcal{R}\otimes\mathcal{F},
$$

is a signed or complex measure on $\mathcal{R}\otimes\mathcal{F}$, due to Definition 5.25, and Lemma 5.26 implies

$$
(V\otimes\mathbb{P})(A) = \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) V(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{R}\otimes\mathcal{F}.
$$

Similarly, the total variation process $\mathbb{V}_V$ *can be viewed as a* σ*-finite transition kernel from* Ω *to* $\mathbb{R}_+$ *by Lemma 5.27 and induces the measure*

$$(\mathbb{V}_V \otimes \mathbb{P})(A) = \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) \mathbb{V}_V(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},$$

as stated in Lemma 5.24. Then for the total variation of $V \otimes \mathbb{P}$ *holds*

$$|V \otimes \mathbb{P}|(A) \leq 2(\mathbb{V}_V \otimes \mathbb{P})(A), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}.$$

In the case $\mathbb{K} = \mathbb{R}$*, one would get the even more convenient result*

$$|(V \otimes \mathbb{P})|(A) = (\mathbb{V}_V \otimes \mathbb{P})(A), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}.$$

Proof. By the properties of the total variation measure in [Sch23, Theorem 15.128] follows

$$\begin{aligned}
&|(V \otimes \mathbb{P})|(A) \\
&\leq |\mathrm{Re}(V \otimes \mathbb{P})|(A) + |\mathrm{Im}(V \otimes \mathbb{P})|(A) \\
&= |(\mathbb{V}^{R,+} \otimes \mathbb{P}) - (\mathbb{V}^{R,-} \otimes \mathbb{P})|(A) + |(\mathbb{V}^{I,+} \otimes \mathbb{P}) - (\mathbb{V}^{I,-} \otimes \mathbb{P})|(A) \\
&= (\mathbb{V}^{R,+} \otimes \mathbb{P})(A) + (\mathbb{V}^{R,-} \otimes \mathbb{P})(A) + (\mathbb{V}^{I,+} \otimes \mathbb{P}) + (\mathbb{V}^{I,-} \otimes \mathbb{P})(A) \\
&= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) \underbrace{\big(\mathbb{V}^{R,+} + \mathbb{V}^{R,-}\big)(\mathrm{d}s,\omega)}_{|\mathrm{Re}(V)|(\mathrm{d}s,\omega)} \Big) \mathbb{P}(\mathrm{d}\omega) \\
&\quad + \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) \underbrace{\big(\mathbb{V}^{I,+} + \mathbb{V}^{I,-}\big)(\mathrm{d}s,\omega)}_{|\mathrm{Im}(V)|(\mathrm{d}s,\omega)} \Big) \mathbb{P}(\mathrm{d}\omega) \\
&\leq 2 \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\, \mathbb{V}_V(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&= 2\, (\mathbb{V}_V \otimes \mathbb{P})(A)
\end{aligned}$$

for each $A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$, which simplifies to

$$\begin{aligned}
|(V \otimes \mathbb{P})|(A) = |(\mathbb{V}^{R,+} \otimes \mathbb{P}) - (\mathbb{V}^{R,-} \otimes \mathbb{P})|(A) &= (\mathbb{V}^{R,+} \otimes \mathbb{P})(A) + (\mathbb{V}^{R,-} \otimes \mathbb{P})(A) \\
&= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) \underbrace{\big(\mathbb{V}^{R,+} + \mathbb{V}^{R,-}\big)(\mathrm{d}s,\omega)}_{\mathbb{V}_V(\mathrm{d}s,\omega)} \Big) \mathbb{P}(\mathrm{d}\omega) = (\mathbb{V}_V \otimes \mathbb{P})(A),
\end{aligned}$$

whenever $\mathbb{K} = \mathbb{R}$. □

Theorem 4.15 (The Radon–Nikodým theorem for a signed or complex and a σ-finite transition kernel on a δ-ring)**.** *Let* $C \in \mathcal{V}_0^+$ *denote an adapted, continuous, real-valued and non-decreasing process starting at zero and* $V \in \mathcal{V}_0^1$*. When viewing* C *and* V *as a* σ*-finite transition kernel and a signed or complex transition kernel, respectively, assume that*

$$V \otimes \mathbb{P} \ll C \otimes \mathbb{P}$$

on Σ_p. Then there exists a $(C \otimes \mathbb{P})$-almost everywhere unique predictable process f satisfying

$$V = \int_0^{\cdot} f_s \, \mathrm{d}C_s$$

up to indistinguishability. Furthermore, $f(\cdot,\omega)$ may be $\mathbb{P}$-almost surely seen as the $C(\cdot,\omega)$-almost everywhere unique Radon–Nikodým derivative of $V(\cdot,\omega)$ w.r.t. $C(\cdot,\omega)$ on $\mathcal{R}$ and be denoted by $\frac{\mathrm{d}V(\cdot,\omega)}{\mathrm{d}C(\cdot,\omega)}$, according to Theorem 4.11.

Proof.
Step 1 (Construction of the density for a stopped process). As stated in Lemma 5.28, V can be viewed as a signed or complex transition kernel from Ω to $\mathbb{R}_+$ in $\mathcal{R}$ and by Lemma 5.26, $V \otimes \mathbb{P}$ is a signed or complex measure on the δ-ring $\mathcal{R} \otimes \mathcal{F}$. Simlarly, C may be seen as a σ-finite transition kernel from Ω to $\mathbb{R}_+$ and

$$(C \otimes \mathbb{P})(A) = \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) \, C(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},$$

is a measure on $(\mathbb{R}_+ \times \Omega, \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F})$, due to Lemma 5.27 and Lemma 5.24, respectively, because $\mathbb{P}$ is a (probability) measure on $(\Omega, \mathcal{F})$. Throughout this proof the notation of the processes V and C will be sightly abused, as the same symbol is used for the process itself as well as its induced transition kernel. Note at this point that for two sets $A \in \mathcal{B}_{\mathbb{R}_+}$ and $F \in \mathcal{F}$ the product $A \times F$ is an element of $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ and

$$\begin{aligned} (C \otimes \mathbb{P})(A \times F) &= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_{A \times F}(s,\omega) \, C(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\ &= \int_\Omega \mathbb{1}_F(\omega) \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s) \, C(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) = \int_F C(A,\omega) \, \mathbb{P}(\mathrm{d}\omega) \end{aligned} \tag{4.5}$$

holds.

Let's fix the convention $\inf \varnothing = \infty$ and define a series of functions as

$$\tau_n(\omega) = \inf\{t \in \mathbb{R}_+ : C_t(\omega) = n\}, \qquad n \in \mathbb{N},$$

which are stopping times due to [Sch23, Lemma 3.52(b)]. Obviously, C^{τ_n} is therefore pathwise bounded by n. Correspondingly, define the sets A_n as those subsets of the product space $\mathbb{R}_+ \times \Omega$, on which τ_n has not yet been called upon:

$$A_n = \{(t,\omega) \in \mathbb{R}_+ \times \Omega : \tau_n(\omega) > t\}, \qquad n \in \mathbb{N}.$$

Consider also the sets

$$\tilde{A}_n := \bigcup_{q \in \mathbb{Q}_+} \Big([0,q] \times \{\omega \in \Omega : \tau_n(\omega) > q\} \Big), \qquad n \in \mathbb{N},$$

which are in $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$, as $[0,q] \in \mathcal{B}_{\mathbb{R}_+}$ and $\{\omega \in \Omega : \tau_n(\omega) > q\} = \{\omega \in \Omega : \tau_n(\omega) \leq q\}^{\mathsf{c}} \in \mathcal{F}_q \subseteq \mathcal{F}$ for all $q \in \mathbb{Q}_+$ and $n \in \mathbb{N}$. Thus $\tilde{A}_n$ is simply the countable union of measurable sets and thus itself measurable. Fix now $n \in \mathbb{N}$ and some $(t, \omega) \in A_n$. As such $\tau_n(\omega) > t$ holds. Then there exists some $q \in \mathbb{Q}_+$ satisfying $\tau_n(\omega) > q > t$, which implies $(t, \omega) \in \tilde{A}_n$ and consequently $A_n \subseteq \tilde{A}_n$. For the next step consider $(t, \omega) \in \tilde{A}_n$, which implies the existence of a rational number q, such that $t \leq q$ and $\tau_n(\omega) > q \geq t$. Therefore the pair (t, ω) is also an element of A_n, leading to $\tilde{A}_n \subseteq A_n$ and consequently $A_n = \tilde{A}_n$ for all $n \in \mathbb{N}$. As such, all sets A_n are also elements of the product σ-algebra $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$. The definition of τ_n implies that $A_n \subseteq A_{n+1}$ for each $n \in \mathbb{N}$ and as C is continuous, and therefore pathwise bounded on compact intervals, also $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}_+ \times \Omega$ holds. In the following one would like to make use of the fact that for each $n \in \mathbb{N}$ the set A_n is predictable. Equivalently it will be shown that $\mathbb{1}_{A_n} : \mathbb{R}_+ \times \Omega \to \{0, 1\}$ is a Σ_p-measurable stochastic process. This indicator function can alternatively be written as $\mathbb{1}_{[0,\tau_n(\omega))}(t)$. Unfortunately this shows that the process is not left-continuous so it is at first glance not obvious that it is really predictable. To prove this one can define for each $n \in \mathbb{N}$ a series of stopping times $(\tau_{n,k})_{k\in\mathbb{N}}$ by

$$\tau_{n,k}(\omega) = \inf\{t \in \mathbb{R}_+ : C_t(\omega) = n - \frac{1}{k}\}.$$

Thus for each $n \in \mathbb{N}$ the sequence $(\tau_{n,k})_{k\in\mathbb{N}}$ is an *announcing sequence* for τ_n making it a *predictable stopping time* (see [Sch23, Definition 6.38]). Consequently for each pair $(t, \omega) \in \mathbb{R}_+ \times \Omega$

$$\lim_{k\to\infty} \mathbb{1}_{[0,\tau_{n,k}(\omega)]}(t) = \mathbb{1}_{[0,\tau_n(\omega))}(t) = \mathbb{1}_{A_n}(t, \omega)$$

and $\mathbb{1}_{A_n}$ is then predictable as the pointwise limit of the predictable processes $\mathbb{1}_{[0,\tau_{n,k}]}$ (see [Sch23, Remark 7.93]), which leads to $A_n \in \Sigma_p$ for all $n \in \mathbb{N}$. Consequently, the measure $C \otimes \mathbb{P}$ is σ-finite, as for each $n \in \mathbb{N}$ holds

$$\begin{aligned}
(C \otimes \mathbb{P})(A_n) &= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_{A_n}(s, \omega) C(\mathrm{d}s, \omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_{A_n}(s, \omega) C^{\tau_n}(\mathrm{d}s, \omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&\leq \int_\Omega \Big(\int_{\mathbb{R}_+} C^{\tau_n}(\mathrm{d}s, \omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&\leq \int_\Omega \underbrace{C^{\tau_n}(\mathbb{R}_+, \omega)}_{\leq n} \mathbb{P}(\mathrm{d}\omega) \leq n\mathbb{P}(\Omega) = n < \infty.
\end{aligned}$$

Analogously, one may define the stopping times

$$\sigma_n(\omega) = \inf\{t \in \mathbb{R}_+ : \mathbb{V}_t(\omega) = n\}, \qquad n \in \mathbb{N},$$

where $\mathbb{V}$ denotes the total variation process of V, and the sets

$$B_n = \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \sigma_n(\omega) > t\}, \qquad n \in \mathbb{N}.$$

By the same argumentation as above follows $B_n \subseteq B_{n+1}$, $B_n \in \Sigma_p$ and $\mathbb{V}^n := \mathbb{V}_{V^{\sigma_n}} \leq n$ for all $n \in \mathbb{N}$ as well as $\bigcup_{n\in\mathbb{N}} B_n = \mathbb{R}_+ \times \Omega$. As the total variation process of the stopped process V^{σ_n} is bounded by n for each $n \in \mathbb{N}$, the same holds for the positive and the negative variation of the real and imaginary part of V^{σ_n}. Thus the right-hand side of

$$V^{\sigma_n}(A,\omega) = \mathbb{V}^{R,+}_{V^{\sigma_n}}(A,\omega) - \mathbb{V}^{R,-}_{V^{\sigma_n}}(A,\omega) + \mathrm{i}\big(\mathbb{V}^{I,+}_{V^{\sigma_n}}(A,\omega) - \mathbb{V}^{I,-}_{V^{\sigma_n}}(A,\omega)\big)$$

consists of four $\mathbb{R}_+$-valued terms for each $A \in \mathcal{B}_{\mathbb{R}_+}$, $\omega \in \Omega$ and $n \in \mathbb{N}$. For readability, the subscript V^{σ_n} will be omitted in the following. Consequently, one may use Lemma 5.26 to see that

$$\begin{aligned}
&(V^{\sigma_n} \otimes \mathbb{P})(A) = \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) V^{\sigma_n(\omega)}(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&= (\mathbb{V}^{R,+} \otimes \mathbb{P})(A) - (\mathbb{V}^{R,-} \otimes \mathbb{P})(A) + \mathrm{i}\big((\mathbb{V}^{I,+} \otimes \mathbb{P})(A) - (\mathbb{V}^{I,-} \otimes \mathbb{P})(A)\big), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}
\end{aligned}$$

is a signed or complex measure on $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ and not only on the δ-ring $\mathcal{R} \otimes \mathcal{F}$, as

$$\begin{aligned}
(\mathbb{V}^j \otimes \mathbb{P})(\mathbb{R}_+ \times \Omega) &= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_{\mathbb{R}_+ \times \Omega}(s,\omega) \mathbb{V}^j(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{V}^j(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \\
&\le \int_\Omega \underbrace{\mathbb{V}^j(\mathbb{R}_+,\omega)}_{\le n} \mathbb{P}(\mathrm{d}\omega) \le n\mathbb{P}(\Omega) = n < \infty
\end{aligned}$$

for each $j \in \{(R,+),(R,-),(I,+),(I,-)\}$. Analogously it can be seen that $\mathbb{V}^n$ also induces a finite measure on $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$ by defining

$$(\mathbb{V}^n \otimes \mathbb{P})(A) = \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega) \mathbb{V}^n(\mathrm{d}s,\omega) \Big) \mathbb{P}(\mathrm{d}\omega) \le n < \infty, \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},\ n \in \mathbb{N}.$$

The total variation of $V^{\sigma_n} \otimes \mathbb{P}$ is bounded by

$$|(V^{\sigma_n} \otimes \mathbb{P})|(A) \le 2(\mathbb{V}^n \otimes \mathbb{P})(A), \qquad A \in \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F},$$

by Lemma 4.14.

Furthermore, by assumption holds

$$V^{\sigma_n} \otimes \mathbb{P} \ll V \otimes \mathbb{P} \ll C \otimes \mathbb{P}, \qquad n \in \mathbb{N},$$

on the predictable σ-algebra $\Sigma_p \subseteq \mathcal{B}_{\mathbb{R}_+} \times \mathcal{F}$. Therefore, Theorem 4.13 is applicable, which results for each $n \in \mathbb{N}$ in the existence of a $(C \otimes \mathbb{P})$-almost everywhere unique $\mathbb{K}$-valued function $f_n \in \mathcal{L}^1\big(\mathbb{R}_+ \times \Omega, \Sigma_p, C \otimes \mathbb{P}\big)$ satisfying

$$(V^{\sigma_n} \otimes \mathbb{P})(A) = \int_A f_n \,\mathrm{d}(C \otimes \mathbb{P}), \qquad A \in \Sigma_p.$$

Therefore, one may use 4.12(iv) to see that for $A \in \Sigma_p$ and $n \in \mathbb{N}$ the integral

$$\int_A |f_n| \,\mathrm{d}(C \otimes \mathbb{P}) \le 2|(V^{\sigma_n} \otimes \mathbb{P})|(A) \le 4(\mathbb{V}^n \otimes \mathbb{P})(A) < \infty$$

is bounded. Furthermore, for each $A \in \Sigma_p$ and $n \in \mathbb{N}$ follows

$$\begin{aligned}&\int_\Omega\Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)V^{\sigma_n(\omega)}(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) = (V^{\sigma_n}\otimes\mathbb{P})(A)\\ &= \int_A f_n(s,\omega)\,\mathrm{d}(C\otimes\mathbb{P})(s,\omega) = \int_{\mathbb{R}_+\times\Omega} \mathbb{1}_A(s,\omega)f_n(s,\omega)\,\mathrm{d}(C\otimes\mathbb{P})(s,\omega)\\ &= \int_\Omega\Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)f_n(s,\omega)C(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega),\end{aligned}$$

where in the last step Fubini's theorem for transition kernels [Gri18, Satz 9.5] has been used. Now consider sets of the form $A\times F \in \Sigma_p$, where $A \in \mathcal{B}_{\mathbb{R}_+}$ and $F \in \mathcal{F}$. By using the same arguments as in equation (4.5) the equality

$$\begin{aligned}&\int_F V^{\sigma_n(\omega)}(A,\omega)\mathbb{P}(\mathrm{d}\omega) = \int_\Omega\Big(\int_{\mathbb{R}_+} \mathbb{1}_{A\times F}(s,\omega)V^{\sigma_n(\omega)}(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega)\\ &= \int_\Omega\Big(\int_{\mathbb{R}_+} \mathbb{1}_{A\times F}(s,\omega)f_n(s,\omega)C(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega)\\ &= \int_F\Big(\int_A f_n(s,\omega)C(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega)\end{aligned}$$

follows by Lemma 4.12(iii). Note that the map $f_n(\cdot,\omega):\mathbb{R}_+\to\mathbb{K}$ for fixed $\omega\in\Omega$ is $\mathcal{B}_{\mathbb{R}_+}$-measurable, due to Lemma 5.6, as f_n is $\mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{F}$-measurable, due to $\Sigma_p \subseteq \mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{F}$. Furthermore, one may use again [Gri18, Satz 9.5] to see that $f_n(\cdot,\omega)$ is $\mathbb{P}$-almost surely $C(\cdot,\omega)$-integrable, as

$$\int_\Omega\Big(\int_{\mathbb{R}_+} |f_n(s,\omega)|C(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) = \int_{\mathbb{R}_+\times\Omega} |f_n(s,\omega)|\,\mathrm{d}(C\otimes\mathbb{P})(s,\omega) < \infty,$$

which leads to $f_n(\cdot,\omega)\in\mathcal{L}^1\big(\mathbb{R}_+,\mathcal{B}_{\mathbb{R}_+},C(\cdot,\omega)\big)$ for almost all $\omega\in\Omega$ and each $n\in\mathbb{N}$.

Fix now $n\in\mathbb{N}$, $s,t\in\mathbb{R}_+$ satisfying $s\le t$ and some set $F\in\mathcal{F}_s$. Thus the set $(s,t]\times F$ is an element of Σ_p by [JS13, Theorem 2.2(ii)] and one may use the second-to-last display to obtain

$$\int_F V^{\sigma_n(\omega)}\big((s,t],\omega\big)\mathbb{P}(\mathrm{d}\omega) = \int_F\Big(\int_s^t f_n(u,\omega)C(\mathrm{d}u,\omega)\Big)\mathbb{P}(\mathrm{d}\omega),$$

which then leads to

$$\begin{aligned}&\mathbb{E}\Big[\Big(\big(V_t^{\sigma_n(\omega)}(\omega)-V_s^{\sigma_n(\omega)}(\omega)\big)-\int_s^t f_n(u,\omega)C(\mathrm{d}u,\omega)\Big)\mathbb{1}_F(\omega)\Big]\\ &= \int_F V^{\sigma_n(\omega)}\big((s,t],\omega\big)\mathbb{P}(\mathrm{d}\omega) - \int_F\Big(\int_s^t f_n(u,\omega)C(\mathrm{d}u,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) = 0.\end{aligned}$$

Consequently,

$$\mathbb{E}\Big[\big(V_t^{\sigma_n(\omega)}(\omega)-V_s^{\sigma_n(\omega)}(\omega)\big)-\int_s^t f_n(u,\omega)C(\mathrm{d}u,\omega)\;\Big|\;\mathcal{F}_s\Big]=0,\qquad s\le t,$$

holds. Furthermore, the process $V^{\sigma_n} - \int_0^\cdot f_n \,\mathrm{d}C$ is adapted, continuous and starting at zero, because $V^{\sigma_n} \in \mathcal{V}_0^1$ and $\int_0^\cdot f_n \,\mathrm{d}C$ is adapted, continuous and starting at zero by [Sch23, Lemma 5.49(c)]. Additionally, this process is also integrable, as

$$\mathbb{E}\Big[\Big|V_t^{\sigma_n(\omega)}(\omega) - \int_0^t f_n(s,\omega)\, C(\mathrm{d}s,\omega)\Big|\Big] \le \mathbb{E}\big[|V_t^{\sigma_n(\omega)}(\omega)|\big] + \mathbb{E}\Big[\Big|\int_0^t f_n(s,\omega)\, C(\mathrm{d}s,\omega)\Big|\Big]$$
$$\le \mathbb{E}\big[\mathbb{V}_t^n(\omega)\big] + \mathbb{E}\Big[\int_0^t |f_n(s,\omega)|\, C(\mathrm{d}s,\omega)\Big] < \infty$$

holds for all $t \in \mathbb{R}_+$. Consequently, the process $V^{\sigma_n} - \int_0^\cdot f_n \,\mathrm{d}C$ is a continuous martingale, due to [Sch23, Remark 4.2(b)]. Furthermore, [Sch23, Lemma 5.49(c)] also implies that $\int_0^\cdot f_n \,\mathrm{d}C$ is of locally finite variation, and thus the same holds for $V^{\sigma_n} - \int_0^\cdot f_n \,\mathrm{d}C$. Therefore, one may use [Sch23, Lemma 5.51] to obtain that

$$V^{\sigma_n} - \int_0^\cdot f_n \,\mathrm{d}C = 0$$

and equivalently

$$V^{\sigma_n} = \int_0^\cdot f_n \,\mathrm{d}C$$

hold up to indistinguishability, i.e. the set

$$N_n := \bigcup_{t\in\mathbb{R}_+} \Big\{\omega \in \Omega : V_t^{\sigma_n(\omega)}(\omega) \ne \int_0^t f_n(s,\omega)\, C(\mathrm{d}s,\omega)\Big\}$$

is contained in a $\mathbb{P}$-null set, for each $n \in \mathbb{N}$.

Step 2 (The density for the unstopped process). To avoid ambiguousness, one can now define $\tilde{B}_n = B_n \setminus B_{n-1}$ for each $n \in \mathbb{N}$, where $B_0 := \varnothing$, and see that $\bigcup_{n\in\mathbb{N}} \tilde{B}_n = \bigcup_{n\in\mathbb{N}} B_n = \mathbb{R}_+ \times \Omega$. Note that the above argued predictability of each of the sets B_n implies that also $\tilde{B}_n$ is predictable for each $n \in \mathbb{N}$. Therefore, set

$$f(t,\omega) = f_n(t,\omega), \qquad \text{for} \quad (t,\omega) \in \tilde{B}_n.$$

As f may also be written as $\sum_{n=1}^\infty f_n \mathbb{1}_{\tilde{B}_n}$ and $\tilde{B}_n \in \Sigma_p$ it is also Σ_p-measurable. Note at this point that $f_n \restriction_{B_n} = f_m \restriction_{B_n}$ holds outside of a set $\tilde{N}$, which is contained in a $(C \otimes \mathbb{P})$-null set for all $n \le m$, due to the uniqueness in Theorem 4.13. As stated in the proof of [Gri18, Satz 9.1] for each $\omega \in \Omega$ and $n \in \mathbb{N}$ the set

$$\tilde{B}_n(\cdot,\omega) := \{t \in \mathbb{R}_+ : (t,\omega) \in \tilde{B}_n\} = \{t \in \mathbb{R}_+ : \sigma_{n-1}(\omega) \le t < \sigma_n(\omega)\} = \big[\sigma_{n-1}(\omega), \sigma_n(\omega)\big),$$

with $\sigma_0 := 0$, is an element of $\mathcal{B}_{\mathbb{R}_+}$, which leads to the $\mathcal{B}_{\mathbb{R}_+}$-measurability of the map $f(\cdot,\omega) : \mathbb{R}_+ \to \mathbb{C}$, and $\bigcup_{n\in\mathbb{N}} \tilde{B}_n(\cdot,\omega) = \mathbb{R}_+$ for almost all $\omega \in \Omega$.

Consider now the unstopped process V. Due to the previous findings of this proof holds

$$\begin{aligned} V_t(\omega) &= V_t(\omega)\mathbb{1}_{\bigcup_{n\in\mathbb{N}}\tilde{B}_n}(t,\omega) = \sum_{n\in\mathbb{N}} V_t(\omega)\mathbb{1}_{\tilde{B}_n}(t,\omega) = \sum_{n\in\mathbb{N}} V_t^{\sigma_n}(\omega)\mathbb{1}_{\tilde{B}_n}(t,\omega) \\ &= \sum_{n\in\mathbb{N}} \mathbb{1}_{\tilde{B}_n}(t,\omega)\int_0^t f_n(s,\omega)\,C(\mathrm{d}s,\omega) = \int_0^t \sum_{n\in\mathbb{N}} \mathbb{1}_{\tilde{B}_n}(t,\omega) f_n(s,\omega)\,C(\mathrm{d}s,\omega) \\ &= \int_0^t \sum_{n\in\mathbb{N}} \mathbb{1}_{\tilde{B}_n}(s,\omega) f_n(s,\omega)\,C(\mathrm{d}s,\omega) = \int_0^t f(s,\omega)\,C(\mathrm{d}s,\omega) \end{aligned}$$

for all $t \in \mathbb{R}_+$ and $\omega \in N^{\mathsf{c}}$, where $N := \bigcup_{n\in\mathbb{N}} N_n$ is contained in a $\mathbb{P}$-null set, such that $(t,\omega) \in \tilde{N}^{\mathsf{c}}$, i.e. $V = f \bullet C$ up to indistinguishability. Note that in the second-to-last step the above mentioned fact $f_n \restriction_{B_n} = f_m \restriction_{B_n}$ for all $n \le m$ has been used and the interchange of the formally infinite sum and the integral is valid, as at most one summand is greater than zero for the fixed pair $(t,\omega) \in \mathbb{R}_+ \times \Omega$.

Fix now $n \in \mathbb{N}$ and consider the interval $[0,n]$. Note that the set $\mathcal{E} := \{(a,b] : a,b \in [0,n],$ such that $a \le b\}$ generates the Borel-σ-algebra on $[0,n]$, denoted by $\mathcal{B}_{[0,n]}$. Furthermore, the set $\mathcal{E}$ is intersection stable, as for two sets $(a_1,b_1]$ and $(a_2,b_2]$ in $\mathcal{E}$ the intersect is either $\varnothing \in \mathcal{E}$ or the interval $(a_1 \vee a_2, b_1 \wedge b_2]$, which is also in $\mathcal{E}$. Fix now $\omega \in \Omega$ outside of a $\mathbb{P}$-null set, such that $V = f \bullet C$ and consider V as well as $f \bullet C$ pathwise as two signed or complex measures on $([0,n], \mathcal{B}_{[0,n]})$. Thus for each $t \in [0,n]$ holds

$$V\big((0,t],\omega\big) = V_t(\omega) - V_0(\omega) = V_t(\omega) = \int_0^t f(s,\omega)C(\mathrm{d}s,\omega) = (f \bullet C)\big((0,t],\omega\big),$$

which leads to

$$\begin{aligned} &V\big((a,b],\omega\big) = V\big((0,b],\omega\big) - V\big((0,a],\omega\big) = (f \bullet C)\big((0,b],\omega\big) - (f \bullet C)\big((0,a],\omega\big) \\ &\quad = (f \bullet C)\big((a,b],\omega\big) \end{aligned}$$

Thus the measures $V(\cdot,\omega)$ and $(f \bullet C)(\cdot,\omega)$ agree on $\mathcal{E}$. Furthermore, the continuity of V and $f \bullet C$ imply

$$V\big([0,n],\omega\big) = V\big((0,n],\omega\big) = (f \bullet C)\big((0,n],\omega\big) = (f \bullet C)\big([0,n],\omega\big).$$

Thus Lemma 5.14 in the appendix is applicable, resulting in $V(A,\omega) = (f \bullet C)(A,\omega)$ for all $A \in \sigma(\mathcal{E}) = \mathcal{B}_{[0,n]}$ for each $n \in \mathbb{N}$. Consider now a set $A \in \mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]}$. Then there exists an $n \in \mathbb{N}$, such that $A \in \mathcal{B}_{[0,n]}$ and thus $V(A,\omega) = (f \bullet C)(A,\omega)$ holds for each $A \in \mathcal{R}$. As $\mathcal{B}_{\mathbb{R}_+} = \sigma(\mathcal{R})$ and f is a $\mathcal{B}_{\mathbb{R}_+}$-measurable function satisfying $V(A,\omega) = (f \bullet C)(A,\omega)$ for all $A \in \mathcal{R}$, $f(\cdot,\omega)$ may be $\mathbb{P}$-almost surely seen as the $C(\cdot,\omega)$-almost everywhere unique Radon–Nikodým derivative of $V(\cdot,\omega)$ w.r.t. $C(\cdot,\omega)$ on $\mathcal{R}$ and be denoted by $\frac{\mathrm{d}V(\cdot,\omega)}{\mathrm{d}C(\cdot,\omega)}$, according to Theorem 4.11.

Step 3 (Uniqueness). To proof the $(C \otimes \mathbb{P})$-almost everywhere uniqueness define now for each $n \in \mathbb{N}$ the predictable trace-σ-algebra $\Sigma_p^n = \{A \cap B_n : A \in \Sigma_p\}$ on B_n and let f denote a predictable process satisfying $V = f \bullet C$ up to indistinguishability. As stated above,

$f = \frac{\mathrm{d}V(\cdot,\omega)}{\mathrm{d}C(\cdot,\omega)}$ holds for $\mathbb{P}$-almost all $\omega \in \Omega$ and $C(\cdot,\omega)$-almost all $t \in \mathbb{R}_+$. Consequently, $f(\cdot,\omega)$ is almost surely locally integrable w.r.t. $C(\cdot,\omega)$, as Lemma 4.12(iv) implies

$$\begin{aligned}\int_a^b |f(s,\omega)| C(\mathrm{d}s,\omega) = \int_a^b \left|\frac{\mathrm{d}V(s,\omega)}{\mathrm{d}C(s,\omega)}\right| C(\mathrm{d}s,\omega) &\leq 2\int_a^b \frac{\mathrm{d}\mathbb{V}_V(s,\omega)}{\mathrm{d}C(s,\omega)} C(\mathrm{d}s,\omega) \\ &= 2\mathbb{V}_V\big((a,b],\omega\big) \leq 2\mathbb{V}_V\big([0,b],\omega\big) < \infty.\end{aligned}$$

for each pair of non-negative real numbers satsisfying $a < b$. View now again V as well as its total variation $\mathbb{V}_V$ and C again as a signed or complex and two σ-finite transition kernels from Ω to $\mathbb{R}_+$ according to Lemma 5.28 and Lemma 5.27, respectively. The by $\mathbb{V}_V$ induced measure $\mathbb{V}_V \otimes \mathbb{P}$ on Σ_p^n is finite for each $n \in \mathbb{N}$, as it coincides on B_n with $\mathbb{V}^n \otimes \mathbb{P}$, which leads to

$$\begin{aligned}&\int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\,(|f| \bullet C)(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) \\ &\leq 2\int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\,\mathbb{V}_V(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) = 2(\mathbb{V}_V \otimes P)(A) < \infty\end{aligned}$$

for each $A \in \Sigma_p^n$, i.e. $\mathbb{1}_A(\cdot,\omega)$ is $\mathbb{P}$-almost surely integrable w.r.t. $(f \bullet C)(\cdot,\omega)$. Thus the chain rule for Lebesgue–Stieltjes integrals (see for example [Sch23, Lemma 16.6]) results via

$$\int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)|f(s,\omega)|\,C(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) = \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\,(|f| \bullet C)(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) < \infty$$

in the almost sure integrability of $\mathbb{1}_A(\cdot,\omega)f(\cdot,\omega)$ w.r.t. $C(\cdot,\omega)$ for each $A \in \Sigma_p^n$. This in turn leads to

$$\begin{aligned}(V \otimes P)(A) &= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\,V(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) \\ &= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)\,(f \bullet C)(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) \\ &= \int_\Omega \Big(\int_{\mathbb{R}_+} \mathbb{1}_A(s,\omega)f(s,\omega)\,C(\mathrm{d}s,\omega)\Big)\mathbb{P}(\mathrm{d}\omega) \\ &= \int_{\mathbb{R}_+ \times \Omega} \mathbb{1}_A(s,\omega)f(s,\omega)\;\mathrm{d}(C \otimes \mathbb{P})(s,\omega) \\ &= \int_A f(s,\omega)\;\mathrm{d}(C \otimes \mathbb{P})(s,\omega)\end{aligned}$$

due to the chain rule for Lebesgue–Stieltjes integrals in the third and [Gri18, Satz 9.5] in the sixth step. Consequently, as $f\restriction_{B_n}$ is also Σ_p^n-measurable, it can be viewed as a Radon–Nikodým derivative of $V \otimes P$ w.r.t. $C \otimes P$ on Σ_p^n, according to Theorem 4.13 and is as such $(C \otimes P)$-almost everywhere unique on B_n for all $n \in \mathbb{N}$. Therefore, as $\bigcup_{n\in\mathbb{N}} B_n = \mathbb{R}_+ \times \Omega$ the process f is also $(C \otimes P)$-almost everywhere unique on $\mathbb{R}_+ \times \Omega$, which concludes the proof. □

5 Appendices

5.1 Appendix on measurable functions and measure theory

As the following lemmata are used on more than one occasion throughout this thesis, the appendix starts with some theory of $\overline{\mathbb{R}}$-valued measurable functions on a measurable space $(\Omega, \mathcal{F})$.

Lemma 5.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a series of $\overline{\mathbb{R}}$-valued $\mathcal{F}$-measurable functions. Then the pointwise supremum $\sup_{n\in\mathbb{N}} f_n$ and pointwise infimum $\inf_{n\in\mathbb{N}} f_n$ are also $\mathcal{F}$-measurable functions.*

Proof. A function $f : \Omega \to \overline{\mathbb{R}}$ is defined to be $\mathcal{F}$-measurable, if and only if for all $A \in \mathcal{B}_{\overline{\mathbb{R}}}$ the set $f^{-1}(A) = \{\omega \in \Omega : f(\omega) \in A\}$ is in $\mathcal{F}$. By [Sch09, Satz 7.1.2] it suffices to show that the sets $\{\omega \in \Omega : f(\omega) \leq a\}$ are elements of $\mathcal{F}$ for all $a \in \mathbb{R}$ to prove the $\mathcal{F}$-measurability of f.

Fix now $a \in \mathbb{R}$. Thus

$$\{\omega \in \Omega : \sup_{n\in\mathbb{N}} f_n(\omega) \leq a\} = \{\omega \in \Omega : f_n(\omega) \leq a \text{ for all } n \in \mathbb{N}\} = \bigcap_{n\in\mathbb{N}} \{\omega \in \Omega : f_n(\omega) \leq a\}$$

is in $\mathcal{F}$, as $\{\omega \in \Omega : f_n(\omega) \leq a\} \in \mathcal{F}$ for each $n \in \mathbb{N}$ and a σ-algebra is closed under countable intersection. Thus $\sup_{n\in\mathbb{N}} f_n$ is $\mathcal{F}$-measurable.

Similarly, [Sch09, Satz 7.1.2] also states that a $\overline{\mathbb{R}}$-valued function is $\mathcal{F}$-measurable, if and only of $\{\omega \in \Omega : f(\omega) < a\} \in \mathcal{F}$ for each $a \in \mathbb{R}$. Fix now a real number a and see that

$$\{\omega \in \Omega : \inf_{n\in\mathbb{N}} f_n(\omega) < a\} = \{\omega \in \Omega : f_n(\omega) < a \text{ for some } n \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}} \{\omega \in \Omega : f_n(\omega) < a\}$$

is in $\mathcal{F}$, which concludes the proof. □

From this observation the next lemma follows directly, as for a series $(\alpha_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ the *limes inferior* and *limes superior* are defined as

$$\liminf_{n\to\infty} \alpha_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} \alpha_n \qquad \text{and} \qquad \limsup_{n\to\infty} \alpha_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} \alpha_n,$$

respectively.

Lemma 5.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a series of $\overline{\mathbb{R}}$-valued $\mathcal{F}$-measurable functions. Then the pointwise limes inferior $\liminf_{n\to\infty} f_n$ and pointwise limes superior $\limsup_{n\to\infty} f_n$ are also $\mathcal{F}$-measurable functions.*

Proof. For each $k \in \mathbb{N}$ define the function $g_k = \inf_{n \geq k} f_n$, which is $\mathcal{F}$-measurable as the pointwise infimum of the sequence of $\mathcal{F}$-measurable functions $(f_{n+k-1})_{n\in\mathbb{N}}$. Consequently, $\liminf_{n\to\infty} f_n = \sup_{k\in\mathbb{N}} g_k$ is also $\mathcal{F}$-measurable.

Analogously, one may define $\tilde{g}_k = \sup_{n\geq k} f_n$ for each $k \in \mathbb{N}$ and see that $\limsup_{n\to\infty} f_n = \inf_{k\in\mathbb{N}} \tilde{g}_k$ is $\mathcal{F}$-measurable. □

Ultimately, this leads to the measurability of the pointwise limit of measurable functions.

Lemma 5.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a series of $\overline{\mathbb{R}}$-valued $\mathcal{F}$-measurable functions converging pointwise for each $\omega \in \Omega$ to a $\overline{\mathbb{R}}$-valued function f. Then f is also $\mathcal{F}$-measurable.*

Proof. For a converging sequence of real numbers the limit coincides with the limes inferior as well as the limes superior and the statement follows from the lemma above. □

Often times one has to consider measurable functions on a product space. Thus one at first has to define a product σ-algebra.

Definition 5.4 (Product σ-algebra)**.** Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces and π_1 and π_2 be the canonical projections, i.e.

$$\pi_1 : \Omega_1 \times \Omega_2 \ni (\omega_1, \omega_2) \mapsto \omega_1 \in \Omega_1 \qquad \text{and} \qquad \pi_2 : \Omega_1 \times \Omega_2 \ni (\omega_1, \omega_2) \mapsto \omega_2 \in \Omega_2.$$

Then the product σ-algebra is being generated by the inverse images of sets in $\mathcal{F}_1$ and $\mathcal{F}_2$ under the corresponding canonical projection, i.e.

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \sigma\big(\{\pi_j^{-1}(A_j) : A_j \in \mathcal{F}_j,\ j = 1, 2\}\big).$$

This definition obviously implies that the canonical projections are $(\mathcal{F}_1 \otimes \mathcal{F}_2)$-$\mathcal{F}_j$-measurable for $j = 1, 2$. Analogously one could define the product δ-ring as

$$\mathcal{R}_1 \otimes \mathcal{R}_2 = \delta\big(\{\pi_j^{-1}(A_j) : A_j \in \mathcal{R}_j,\ j = 1, 2\}\big),$$

where $\mathcal{R}_j$ is a δ-ring on Ω_j for $j = 1, 2$.

Lemma 5.5 ($\mathcal{F}_1 \otimes \mathcal{F}_2$-measurable functions)**.** *Consider another measurable space $(S, \mathcal{S})$ and a map $f : S \to \Omega_1 \times \Omega_2$. Then f is $\mathcal{S}$-$(\mathcal{F}_1 \otimes \mathcal{F}_2)$-measurable, if and only if $\pi_j \circ f : S \to \Omega_j$ is $\mathcal{S}$-$\mathcal{F}_j$-measurable for $j = 1, 2$.*

Proof. As stated above, the canonical projections are $(\mathcal{F}_1 \otimes \mathcal{F}_2)$-$\mathcal{F}_j$-measurable for $j = 1, 2$. Consequently, the $\mathcal{S}$-$(\mathcal{F}_1 \otimes \mathcal{F}_2)$-measurablity of f implies the $\mathcal{S}$-$\mathcal{F}_j$-measurability of $\pi_j \circ f$ for $j = 1, 2$.

Assume now the $\mathcal{S}$-$\mathcal{F}_j$-measurability of $\pi_j \circ f$ for $j = 1, 2$. Then one may define $\mathfrak{G} = \{A \in \mathcal{F}_1 \otimes \mathcal{F}_2 : f^{-1}(A) \in \mathcal{S}\}$. This set is again a σ-algebra, as

(i) $f^{-1}(\Omega_1 \times \Omega_2) = S \in \mathcal{S}$ and thus $\Omega_1 \times \Omega_2 \in \mathfrak{G}$,

(ii) for each $A \in \mathfrak{G}$ the complement is also in $\mathfrak{G}$, because

$$f^{-1}(A^{\mathsf{c}}) = \{s \in S : f(s) \in A^{\mathsf{c}}\} = \{s \in S : f(s) \notin A\} = \{s \in S : f(s) \in A\}^{\mathsf{c}} = f^{-1}(A)^{\mathsf{c}},$$

(iii) and for a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{G}$ and its union

$$\begin{aligned} f^{-1}\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) &= \{s \in S : f(s) \in \bigcup_{n\in\mathbb{N}} A_n\} \\ &= \{s \in S : \text{ there exists a } n \in \mathbb{N} \text{ such that } f(s) \in A_n\} \\ &= \bigcup_{n\in\mathbb{N}} \{s \in S : f(s) \in A_n\} = \bigcup_{n\in\mathbb{N}} f^{-1}(A_n) \end{aligned}$$

follows, which is therefore again in $\mathcal{S}$ leading to $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{G}$.

Per assumption, for each $j = 1, 2$ and $A_j \in \mathcal{F}_j$ the inverse image $(\pi_j \circ f)^{-1}(A_j) = f^{-1}\big(\pi_j^{-1}(A_j)\big)$ is in $\mathcal{S}$. Consequently, as $\pi_j^{-1}(A_j) \in \mathcal{F}_1 \otimes \mathcal{F}_2$, the sets $\pi_j^{-1}(A_j)$ are in $\mathfrak{G}$ for all $j = 1, 2$ and $A_j \in \mathcal{F}_j$. This leads to $\mathcal{F}_1 \otimes \mathcal{F}_2 \subseteq \mathfrak{G}$, because those sets generate the product σ-algebra and as such it is the smallest one containing all those sets. By the definition of $\mathfrak{G}$ it is apparent that also $\mathfrak{G} \subseteq \mathcal{F}_1 \otimes \mathcal{F}_2$ and thus $\mathfrak{G} = \mathcal{F}_1 \otimes \mathcal{F}_2$. Therefore f is $\mathcal{S}$-$(\mathcal{F}_1 \otimes \mathcal{F}_2)$-measurable [Sch23, p. 435]. □

Lemma 5.6. *In the setting of Definition 5.4 let $f : \Omega_1 \times \Omega_2 \to \mathbb{K}$ be $(\mathcal{F}_1 \otimes \mathcal{F}_2)$-$\mathcal{B}_\mathbb{K}$-measurable. Then the functions $f(\cdot, \omega_2) : \Omega_1 \to \mathbb{K}$ for fixed $\omega_2 \in \Omega_2$ and $f(\omega_1, \cdot) : \Omega_2 \to \mathbb{K}$ for fixed $\omega_1 \in \Omega_1$ are $\mathcal{F}_1$-$\mathcal{B}_\mathbb{K}$- and $\mathcal{F}_2$-$\mathcal{B}_\mathbb{K}$-measurable, respectively.*

Proof. For fixed $\omega_2 \in \Omega_2$ consider at first $f(\cdot, \omega_2) : \Omega_1 \ni \omega_1 \mapsto f(\omega_1, \omega_2)$. Then one can define $g_{\omega_2} : \Omega_1 \ni \omega_1 \mapsto (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ and see that

$$\pi_1 \circ g_{\omega_2} : \Omega_1 \ni \omega_1 \mapsto \omega_1 \in \Omega_1 \qquad \text{and} \qquad \pi_2 \circ g_{\omega_2} : \Omega_1 \ni \omega_1 \mapsto \omega_2 \in \Omega_2,$$

are the identity and a constant function, respectively. As such they are $\mathcal{F}_1$-$\mathcal{F}_1$- and $\mathcal{F}_1$-$\mathcal{F}_2$-measurable, respectively, which implies the $\mathcal{F}_1$-$(\mathcal{F}_1 \otimes \mathcal{F}_2)$-measurabiliy of g_{ω_2} for each $\omega_2 \in \Omega_2$, due to Lemma 5.5. Consequently, $f(\cdot, \omega_2) = f \circ g_{\omega_2}$ is $\mathcal{F}_1$-$\mathcal{B}_\mathbb{K}$-measurable for all $\omega_2 \in \Omega_2$.

Analogously, one can define $g_{\omega_1} : \Omega_2 \ni \omega_2 \mapsto (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ for each $\omega_1 \in \Omega_1$. By the same steps as above one can show that this function is $\mathcal{F}_2$-$\mathcal{F}_1 \otimes \mathcal{F}_2$, which leads to the $\mathcal{F}_2$-$\mathcal{B}_\mathbb{K}$-measurability of $f(\omega, \cdot) = f \circ g_{\omega_1}$ for all $\omega_1 \in \Omega_1$. □

The First Borel–Cantelli Lemma is one of the most famous lemmata in probability theory. It can, however, be quite easily extended to more general measures, not only probability measures. Let $(\Omega, \mathcal{F}, \nu)$ be therefore a measure space, not necessarily a probability space. The more well-known, but also more restrictive Borel–Cantelli Lemma can for example be found in [Wil91, p. 27].

Lemma 5.7 (Generalized First Borel–Cantelli Lemma)**.** *For a sequence $(A_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{F}$ define $B = \limsup_{n\in\mathbb{N}} A_n$. Then the property $\sum_{n=1}^{\infty} \nu(A_n) < \infty$ implies that $\nu(B) = 0$.*

Proof. For each $m \in \mathbb{N}$ define $B_m = \bigcup_{n\geq m} A_n$. As $B = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} A_n$ per definition, one obtains

$$\nu(B) \leq \nu(B_m) \leq \sum_{n\geq m} \nu(A_n).$$

As this remains true for each $m \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \nu(A_n) < \infty$ the lemma is proven by taking the limit $m \to \infty$ on the right-hand side. □

5.2 Appendix on martingale convergence

Doob's almost sure convergence theorem bears the existence of a limit for $t \to \infty$ for some sub- or supermartingales. In this theory, however, we only need a special case of this theorem, where the underlying process is a martingale. The following theorem will therefore only be concerned with this special case. For the more general theorem as well as a proof the reader is referred to [Sch23, Theorem 7.57].

Theorem 5.8 (Doob's almost sure convergence theorem)**.** *Let $T \subseteq \mathbb{R}_+$ and $M = (M_t)_{t\in T}$ be a $\mathbb{K}^d$-valued martingale satisfying componentwise (in the case $\mathbb{K} = \mathbb{C}$ the real and imaginary part of each component have to fulfill this condition separately) either*

$$\sup_{t\in T} \mathbb{E}[(M_t^j)^+] < \infty \quad \text{or} \quad \sup_{t\in T} \mathbb{E}[(M_t^j)^-] < \infty,$$

where $(\cdot)^+$ and $(\cdot)^-$ denote the positive and negative parts of the process, respectively. If $t^* := \sup_{t\in T} \notin T$ there exists a limiting $\mathbb{K}^d$-valued, integrable and $\mathcal{F}_{t^*}$-measurable random vector M_{t^*} in the sense that

$$M_{t^*} = \lim_{\substack{t\in F \\ t\uparrow t^*}} M_t \quad a.s.$$

for each countable $F \subseteq T$ satisfying $\sup F = t^*$.

Unfortunately, without further conditions, the almost sure convergence of the above theorem cannot be extended to more useful convergence types, such as the $\mathcal{L}^1$-convergence. For this matter, a new kind of stochastic processes is needed, namely *uniformly integrable* processes, which will be defined below. For readability and because we do not need the general case in this thesis, only $\mathbb{R}$-valued random variables will be discussed. Just note that the extensions to $\mathbb{C}$ or even more general $\mathbb{C}^d$ can be done straightforwardly by viewing the real and imaginary parts separately and the multiple dimensions componentwise.

Definition 5.9 (Uniformly integrable families of random variables)**.** Let $X = (X_t)_{t\in T}$ be a family of $\mathbb{R}$-valued random variables on a set $T \subseteq \mathbb{R}_+$. X is then said to be uniformly integrable, if and only if for each $\epsilon > 0$ there exists $c_\epsilon \geq 0$, such that

$$\mathbb{E}[|X_t| \mathbb{1}_{\{|X_t|>c_\epsilon\}}] < \epsilon, \qquad t \in T.$$

Such families are bounded in $\mathcal{L}^1$, because

$$\mathbb{E}[|X_t|] = \mathbb{E}[|X_t| \mathbb{1}_{\{|X_t|>c_1\}}] + \mathbb{E}[|X_t| \mathbb{1}_{\{|X_t|\leq c_1\}}] \leq 1 + c_1 < \infty.$$

The following lemma shows the usefulness of uniformly integrability.

Lemma 5.10. *For a sequence $(X_n)_{n\in\mathbb{N}}$ and a random variable X, which are both real-valued and in $\mathcal{L}^1$ that satisfy $X_n \to X$ in probability and $(X_n)_{n\in\mathbb{N}}$ is uniformly integrable, the limit $X_n \to X$ also holds in $\mathcal{L}^1$, i.e. $\mathbb{E}[|X_n - X|] \to 0$ for $n \to \infty$.*

Proof. Let for $c \in [0,\infty)$ the function $\varphi_c : \mathbb{R} \to [-c,c]$ be defined as

$$\varphi_c(x) = \begin{cases} -c & \text{if } x < -c, \\ x & \text{if } -c < x < c, \\ c & \text{if } x > c. \end{cases}$$

Fix $\epsilon > 0$. Due to $X \in \mathcal{L}^1$ there exists $c \in [0,\infty)$, such that

$$\begin{aligned} &\mathbb{E}[|\varphi_c(X) - X|] \\ &= \mathbb{E}[|\varphi_c(X) - X|\mathbb{1}_{\{X>c\}}] + \mathbb{E}[|\varphi_c(X) - X|\mathbb{1}_{\{X<-c\}}] + \mathbb{E}[|\varphi_c(X) - X|\mathbb{1}_{\{|X|\le c\}}] \\ &= \mathbb{E}[|c - X|\mathbb{1}_{\{X>c\}}] + \mathbb{E}[|c + X|\mathbb{1}_{\{X<-c\}}] + 0 < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}. \end{aligned}$$

Similarly, due to the uniformly integrability of $(X_n)_{n\in\mathbb{N}}$, there exists also a $\tilde{c} \in [0,\infty)$, for which

$$\mathbb{E}[|\varphi_{\tilde{c}}(X_n) - X_n|] < \frac{\epsilon}{3}, \qquad n \in \mathbb{N}$$

holds. By defining $C := \max\{c, \tilde{c}\}$ both expectations above are bounded by $\frac{\epsilon}{3}$. Apparently, $|\varphi_C(x) - \varphi_C(y)| < |x - y|$ holds and therefore one knows that $\varphi_C(X_n) \to \varphi_C(X)$ in probability, as $n \to \infty$. By using the *bounded convergence theorem* [Wil91, Theorem 13.6] there exists an $n_0 \in \mathbb{N}$ such that

$$\mathbb{E}[|\varphi_C(X_n) - \varphi_C(X)|] < \frac{\epsilon}{3}$$

holds for all $n \ge n_0$. This leads to

$$\mathbb{E}[|X_n - X|] \le \mathbb{E}[|\varphi_C(X_n) - X_n|] + \mathbb{E}[|\varphi_C(X_n) - \varphi_C(X)|] + \mathbb{E}[|\varphi_C(X) - X|] < \epsilon$$

for all $n \ge n_0$ by the triangle inequality, which concludes the proof. □

Definition 5.11 (Uniformly integrable martingales)**.** Quite naturally, a martingale $M = (M_t)_{t\in T}$ is called a *uniformly integrable martingale*, if and only if the family $(M_t)_{t\in T}$ is uniformly integrable.

Theorem 5.12. *For every real-valued uniformly integrable martingale $M = (M_n)_{\in\mathbb{N}}$ the limit*

$$M_\infty := \lim_{n\to\infty} M_n$$

exists not only almost surely, but also in $\mathcal{L}^1$.

Proof. As stated above, the uniformly integrabiliy of M causes its boundedness is $\mathcal{L}^1$. Therefore, by *Doob's almost sure convergence theorem* Theorem 5.8, M_∞ exists almost surely. This implies the convergence of $(M_n)_{\in\mathbb{N}}$ to M_∞ in probability as $n \to \infty$. Lemma 5.10 therefore proves this theorem [Wil91, p. 127ff]. □

5.3 Appendix on set theory

For some set Ω define $\mathcal{P}(\Omega)$ as the set of all subsets of Ω, also known as the *power set*. The following Lemma is often times helpful when trying to proof the equivalence of two signed or even complex measures on a measure space, when one knows they agree on some subset of the σ-algebra. The proof relies heavily on *Dynkin's π-λ lemma* (see for example [Sch23, Theorem 15.66]) that states that for every *intersection-stable* subset $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ the generated *Dynkin system* $\mathcal{D}(\mathcal{E})$ agrees with the generated σ-algebra $\sigma(\mathcal{E})$. Therefore the definition of a *Dynkin system* is provided beforehand.

Definition 5.13 (Dynkin system)**.** A set $\mathcal{D} \subseteq \mathcal{P}(\Omega)$ is called a *Dynkin system*, if and only if the three conditions below are fulfilled:

(i) $\Omega \in \mathcal{D}$,

(ii) for every $D \in \mathcal{D}$ also its compliment D^{c} must be in $\mathcal{D}$, and

(iii) if a sequence $(D_n)_{n\in\mathbb{N}} \in \mathcal{D}$ satisfies $D_n \cap D_m = \varnothing$ for all $n \neq m$, then $\bigcup_{n\in\mathbb{N}} D_n$ has to belong to $\mathcal{D}$.

Lemma 5.14. *Let $(\Omega, \mathcal{F})$ be a measurable space, $\mu, \nu : \mathcal{F} \to \mathbb{K}$ be two signed or complex measures on it and $\mathcal{E} \subseteq \mathcal{F}$ be* intersection-stable*, meaning that the implication $E_1, E_2 \in \mathcal{E} \Rightarrow E_1 \cap E_2 \in \mathcal{E}$ holds. If then $\mu(E) = \nu(E)$ for all $E \in (\mathcal{E} \cup \Omega)$, then the same remains true for all sets in the generated σ-algebra $\sigma(\mathcal{E})$.*

Proof. Define $\mathcal{D} = \{D \in \sigma(\mathcal{E}) : \mu(D) = \nu(D)\}$ as the subset of $\sigma(\mathcal{E})$, on which the to be proven proposition holds. Naturally, $\mathcal{D} = \sigma(\mathcal{E})$ has to be shown. By definition $\mathcal{E} \subseteq \mathcal{D} \subseteq \sigma(\mathcal{E})$. It is also apparent that $\mathcal{D}$ is a *Dynkin system*, as

(i) $\Omega \in \mathcal{D}$, per assumption,

(ii) for $D \in \mathcal{D}$ $\mu(D^{\mathsf{c}}) = \mu(\Omega) - \mu(D) = \nu(\Omega) - \nu(D) = \nu(D^{\mathsf{c}})$ and therefore $D^{\mathsf{c}} \in \mathcal{D}$, and

(iii) if a sequence $(D_n)_{n\in\mathbb{N}} \in \mathcal{D}$ satisfies $D_n \cap D_m = \emptyset$ for all $n \neq m$, then $\mu(\bigcup_{n\in\mathbb{N}} D_n) = \sum_{n\in\mathbb{N}} \mu(D_n) = \sum_{n\in\mathbb{N}} \nu(D_n) = \nu(\bigcup_{n\in\mathbb{N}} D_n)$, due to the σ-additivity of signed or complex measures, leading to $\bigcup_{n\in\mathbb{N}} D_n \in \mathcal{D}$.

Consequently, as $\mathcal{D}$ is a *Dynkin system* and a superset of $\mathcal{E}$, $\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D} \subseteq \sigma(\mathcal{E})$. *Dynkin's π-λ lemma* states $\mathcal{D}(\mathcal{E}) = \sigma(\mathcal{E})$, which in turn leads to $\mathcal{D}(\mathcal{E}) = \mathcal{D} = \sigma(\mathcal{E})$ concluding the proof. □

Another special kind of subset of $\mathcal{P}(\Omega)$ are *monotone classes*, which will be defined and discussed in the following.

Definition 5.15 (Monotone class)**.** A set $\mathfrak{M} \subseteq \mathcal{P}(\Omega)$ is a monotone class, if and only if the following four criteria are met:

(i) $\varnothing$ as well as Ω are in $\mathfrak{M}$,

(ii) for each two sets $A \subseteq B$ both in $\mathfrak{M}$ also $B \setminus A \in \mathfrak{M}$,

(iii) for all $A, B \in \mathfrak{M}$ satisfying $A \cap B = \varnothing$ also $A \cup B \in \mathfrak{M}$, and

(iv) for an increasing sequence $(A_n)_{n\in\mathbb{N}} \in \mathfrak{M}$ also $\bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{M}$.

Lemma 5.16 (Monotone class lemma)**.** *Let $\mathcal{E} \subseteq \mathcal{P}(\Omega)$ be* intersection stable *as defined in Lemma 5.14. Thus it follows that*

$$\mathfrak{M}(\mathcal{E}) = \sigma(\mathcal{E}),$$

where $\mathfrak{M}(\mathcal{E})$ denotes the generated monotone class and $\sigma(\mathcal{E})$ the generated σ-algebra.

Proof. It is quite clear to see that every σ-algebra is also a monotone class and therefore $\sigma(\mathcal{E}) \supset \mathfrak{M}(\mathcal{E})$. Thus it remains to show that $\mathfrak{M}(\mathcal{E})$ is a σ-algebra. Therefore it will be proven in the following that $\mathfrak{M}(\mathcal{E})$ is intersection stable. To do so one may define for each set $A \in \mathfrak{M}(\mathcal{E})$

$$\mathfrak{M}_A = \{B \in \mathfrak{M}(\mathcal{E}) : A \cap B \in \mathfrak{M}(\mathcal{E})\}.$$

Take any set $A \in \mathcal{E}$. As per assumption, $A \cap B \in \mathcal{E}$ for each $B \in \mathcal{E}$ and therefore $\mathcal{E} \subseteq \mathfrak{M}_A$. Note that $\mathfrak{M}_A$ defines a monotone class, as

(i) $A \cap \varnothing = \varnothing \in \mathfrak{M}(\mathcal{E})$ and $A \cap \Omega = A \in \mathfrak{M}(\mathcal{E})$, therefore both $\varnothing$ and Ω are in $\mathfrak{M}_A$.

(ii) for each two sets $B_1 \subseteq B_2$ both in $\mathfrak{M}_A$ follows $A \cap (B_2 \setminus B_1) = (A \cap B_2) \setminus (A \cap B_1) \in \mathfrak{M}(\mathcal{E})$, because both $(A \cap B_1)$ and $(A \cap B_2)$ are elements of $\mathfrak{M}(\mathcal{E})$ and $(A \cap B_1) \subseteq (A \cap B_2)$ holds, which results in $B_2 \setminus B_1 \in \mathfrak{M}_A$.

(iii) for all $B_1, B_2 \in \mathfrak{M}_A$ satisfying $B_1 \cap B_2 = \varnothing$ follows $A \cap (B_1 \cup B_2) = (A \cap B_1) \cup (A \cap B_2) \in \mathfrak{M}(\mathcal{E})$, because both $(A \cap B_1)$ as well as $(A \cap B_2)$ are in $\mathfrak{M}(\mathcal{E})$ satisfying $(A \cap B_1) \cap (A \cap B_2) \subseteq B_1 \cap B_2 = \varnothing$. Therefore $A \cap (B_1 \cup B_2) \in \mathfrak{M}(\mathcal{E})$ and consequently $B_1 \cup B_2 \in \mathfrak{M}_A$.

(iv) for an increasing sequence $(B_n)_{n\in\mathbb{N}} \in \mathfrak{M}_A$ follows $A \cap \left(\bigcup_{n\in\mathbb{N}} B_n\right) = \bigcup_{n\in\mathbb{N}} (A \cap B_n) \in \mathfrak{M}(\mathcal{E})$, as $(A \cap B_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathfrak{M}(\mathcal{E})$, whereby $\bigcup_{n\in\mathbb{N}} B_n \in \mathfrak{M}_A$.

Because $\mathfrak{M}(\mathcal{E})$ the smallest monotone class containing $\mathcal{E}$ and as just now shown $\mathfrak{M}_A$ is also a monotone class containing $\mathcal{E}$, it is apparent that $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}_A$ holds. Thus for every $A \in \mathcal{E}, B \in \mathfrak{M}(\mathcal{E})$ it was shown that $A \cap B \in \mathfrak{M}(\mathcal{E})$.

Take now some $A \in \mathfrak{M}(\mathcal{E})$. By switching places of A and B in the last sentence one obtains again $\mathcal{E} \subseteq \mathfrak{M}_A$. One can now perform the same steps as before with this now more general set A and get the same result, namely that $\mathfrak{M}_A$ is a monotone class for general $A \in \mathfrak{M}(\mathcal{E})$ and thus $\mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}_A$. In other words: For each $A, B \in \mathfrak{M}(\mathcal{E})$ the intersect $A \cap B$ is also in $\mathfrak{M}(\mathcal{E})$, i.e. $\mathfrak{M}(\mathcal{E})$ is intersection stable for two and therefore also finitely many sets.

From that it follows that for $A, B \in \mathfrak{M}(\mathcal{E})$ the union $A \cup B = \left(A^{\mathsf{c}} \cap B^{\mathsf{c}}\right)^{\mathsf{c}} \in \mathfrak{M}(\mathcal{E})$, as monotone classes are by definition stable w.r.t. complementary sets, and $\mathfrak{M}(\mathcal{E})$ is thus stable under finite unions. For the last step fix some general sequence $(A_n)_{n\in\mathbb{N}} \in \mathfrak{M}(\mathcal{E})$ and define

$$B_n = \bigcup_{j=1}^{n} A_j, \qquad n \in \mathbb{N}.$$

By the previous findings of this proof, $B_n \in \mathfrak{M}(\mathcal{E})$ as a finite union and also $B_n \subseteq B_{n+1}$ for each $n \in \mathbb{N}$. Consequently, by the definition of a monotone class, $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n \in \mathfrak{M}(\mathcal{E})$. Thus $\mathfrak{M}(\mathcal{E})$ is not only a monotone class, but also a σ-algebra, which concludes the proof [LG16, p. 261f]. □

5.4 Appendix on scalar products and pre-Hilbert spaces

The appendix now switches topics and will in this section concern itself with some linear algebra and more precisely symmetric bilinear or *Hermitian sesquilinear forms* on vector spaces, depending on $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. In order to improve readability and to not take the focus away from the findings and conclusions of this section, only the vocabulary of the complex case will be stated in the following. When considering $\mathbb{K} = \mathbb{R}$, the reader is asked to switch to the correct names for the real case, for example think of a symmetric bilinear form when reading Hermitian sesquilinear form.

Definition 5.17 (Hermitian sesquilinear forms and scalar products)**.** A function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$, where V denotes a vector space over $\mathbb{K}$ is said to be a Hermitian sesquilinear form, if and only if it satisfies the three equalities below for all $x, y, z \in V$ and $\alpha \in \mathbb{K}$, namely

(i) $\langle \alpha x + y, z\rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$,

(ii) $\langle x, \alpha y + z\rangle = \bar{\alpha}\langle x, y\rangle + \langle x, z\rangle$ and

(iii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

One may call these conditions *linearity in the first argument*, *semilinearity in the second argument* and *Hermitian symmetry*. Depending on the subject the linearity and semilinearity may also be switched. A *scalar* product is defined as a positive definite Hermitian sesquilinear form and induces the norm $\|x\| := \sqrt{\langle x, x\rangle}$ for all $x \in V$.

Note at this point that one can use the conditions (i) and (iii) in of Definition 5.17 to obtain

$$\langle x, \alpha y + z\rangle = \overline{\langle \alpha y + z, x\rangle} = \bar{\alpha}\overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \bar{\alpha}\langle x, y\rangle + \langle z, y\rangle,$$

which is condition (ii). Thus if one wants to check, if a given function is a Hermitian sesquilinear form, it suffices to show (i) and (iii). However, throughout this thesis, all three conditions are being shown, as the proof of (ii) is typically not too hard.

The following theorem is one of the most well-known inequalities in the field of linear algebra and normed vector spaces. In most cases, the theorem is stated for general norms on a vector space or those generated by some scalar product. This thesis however extends the conclusion also for not necessarily positive definite symmetric or Hermitian sesquilinear forms but positive *semi*definite ones. The more restrictive case can be found as [Hav12, Satz 11.3.3] from where the proof is being extended.

Theorem 5.18 (Generalized Cauchy–Schwarz inequality)**.** *Let V be a vector space over $\mathbb{K}$ and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ be a positive semidefinite Hermitian sesquilinear form. Then the inequality*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

holds for all $x, y \in V$.

Proof. Note at first that due to the assumed positive semidefiniteness $\langle x, x\rangle \geq 0$, i.e. $\mathbb{R}_+$-valued, holds for all $x \in V$. Fix now some $\epsilon > 0$ and define

$$\alpha = \frac{\langle x, y\rangle}{\langle y, y\rangle + \epsilon} \in \mathbb{K}.$$

Positive semidefiniteness and the definition of a Hermitian sesquilinear form suffice to prove

$$\begin{aligned} 0 \leq \langle x - \alpha y, x - \alpha y\rangle &= \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha\langle y, x\rangle + |\alpha|^2\langle y, y\rangle \\ &= \langle x, x\rangle - \frac{\overline{\langle x, y\rangle}}{\langle y, y\rangle + \epsilon}\langle x, y\rangle - \frac{\langle x, y\rangle}{\langle y, y\rangle + \epsilon}\langle y, x\rangle + \frac{|\langle x, y\rangle|^2}{(\langle y, y\rangle + \epsilon)^2}\langle y, y\rangle \\ &= \langle x, x\rangle - \frac{2|\langle x, y\rangle|^2}{\langle y, y\rangle + \epsilon} + \frac{|\langle x, y\rangle|^2}{(\langle y, y\rangle + \epsilon)^2}\langle y, y\rangle. \end{aligned}$$

Equivalently, one may write

$$2|\langle x, y\rangle|^2 - |\langle x, y\rangle|^2\frac{\langle y, y\rangle}{\langle y, y\rangle + \epsilon} \leq \langle x, x\rangle(\langle y, y\rangle + \epsilon)$$

for all $\epsilon > 0$. Now there are two cases two be differentiated: In the first one $\langle y, y\rangle = 0$ holds. Therefore the fraction on the left-hand side is equal to zero for each $\epsilon > 0$ and can thus be disregarded. This leads to

$$2|\langle x, y\rangle|^2 \leq \langle x, x\rangle(\langle y, y\rangle + \epsilon)$$

and, by taking the limit $\epsilon \searrow 0$ on the right-hand side and dividing both sides by 2,

$$|\langle x, y\rangle|^2 = 0 \quad (= \langle x, x\rangle\langle y, y\rangle),$$

which trivially proves the inequality. Let's now consider the other case, i.e. $\langle y, y\rangle > 0$. Then one can again take the limit $\epsilon \searrow 0$ on both sides of the equation and obtain

$$2|\langle x, y\rangle|^2 - |\langle x, y\rangle|^2 = |\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle. \qquad \square$$

In order to use this theorem in the earlier parts of this thesis, the following lemmata prove useful.

Lemma 5.19. *Every $\mathbb{K}^{d\times d}$-valued Hermitian matrix π induces a Hermitian sesquilinear form $\langle\cdot,\cdot\rangle : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}$ by defining*

$$\langle x, y\rangle = x^{\mathsf{T}}\pi\bar{y}.$$

Furthermore, $\langle\cdot,\cdot\rangle$ inherits the positive or negative (semi)definiteness of π.

Proof. As in Definition 5.17 let $x, y, z \in \mathbb{K}^d$ and $\alpha \in \mathbb{K}$. Thus

(i) $\langle \alpha x + y, z\rangle = (\alpha x + y)^{\mathsf{T}}\pi\bar{z} = \alpha(x^{\mathsf{T}}\pi\bar{z}) + y^{\mathsf{T}}\pi\bar{z} = \alpha\langle x, z\rangle + \langle y, z\rangle$,

(ii) $\langle x, \alpha y + z\rangle = x^{\mathsf{T}}\pi(\overline{\alpha y + z}) = \bar{\alpha}(x^{\mathsf{T}}\pi\bar{y}) + x^{\mathsf{T}}\pi\bar{z} = \bar{\alpha}\langle x, y\rangle + \langle x, z\rangle$ as well as

(iii) $\langle x, y\rangle = x^{\mathsf{T}}\pi\bar{y} = (x^{\mathsf{T}}\pi\bar{y})^{\mathsf{T}} = \bar{y}^{\mathsf{T}}\pi^{\mathsf{T}}(x^{\mathsf{T}})^{\mathsf{T}} = \bar{y}^{\mathsf{T}}\bar{\pi}x = \overline{y^{\mathsf{T}}\pi\bar{x}} = \overline{\langle y, x\rangle}$

follow directly.

Assume π to be positive definite, the other cases can be shown analogously. Thus $x^{\mathsf{H}}\pi x > 0$ holds for all $x \in \mathbb{K}^d \setminus \{0\}$. Fix now $x \in \mathbb{K}^d \setminus \{0\}$ and consider its complex conjugate vector $\bar{x}$, which leads to

$$0 < \bar{x}^{\mathsf{H}}\pi\bar{x} = \overline{(\bar{x})}^{\mathsf{T}}\pi\bar{x} = x^{\mathsf{T}}\pi\bar{x} = \langle x, x\rangle,$$

which proves the positive definiteness of $\langle\cdot,\cdot\rangle$. □

Lemma 5.20. *Let π be a $\mathbb{K}^{d\times d}$-valued positive semidefinite Hermitian matrix and $\langle\cdot,\cdot\rangle : \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K}$ the corresponding positive semidefinite Hermitian sesquilinear form. Then for all $x, y \in \mathbb{K}^d$ the inequality*

$$(x + y)^{\mathsf{T}}\pi(\overline{x + y}) \le \left(\sqrt{x^{\mathsf{T}}\pi\bar{x}} + \sqrt{y^{\mathsf{T}}\pi\bar{y}}\right)^2$$

holds.

Proof. Let's at first consider the left-hand side of the inequality and obtain

$$(x + y)^{\mathsf{T}}\pi(\overline{x + y}) = x^{\mathsf{T}}\pi\bar{x} + x^{\mathsf{T}}\pi\bar{y} + y^{\mathsf{T}}\pi\bar{x} + y^{\mathsf{T}}\pi\bar{y} = x^{\mathsf{T}}\pi\bar{x} + 2\mathrm{Re}(x^{\mathsf{T}}\pi\bar{y}) + y^{\mathsf{T}}\pi\bar{y}, \quad (5.1)$$

because $y^{\mathsf{T}}\pi\bar{x} = \bar{x}^{\mathsf{T}}\bar{\pi}y = \overline{x^{\mathsf{T}}\pi\bar{y}}$. Now one can focus on the term $\mathrm{Re}(x^{\mathsf{T}}\pi\bar{y})$ and see that its square is bounded from above by

$$|\mathrm{Re}(x^{\mathsf{T}}\pi\bar{y})|^2 \le |x^{\mathsf{T}}\pi\bar{y}|^2 = |\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle = (x^{\mathsf{T}}\pi\bar{x})(y^{\mathsf{T}}\pi\bar{y}),$$

due to the Cauchy–Schwarz inequality in Theorem 5.18. As both sides are $\mathbb{R}_+$-valued, one my take the square-root on both sides and the inequality still remains true. Therefore by revisiting equation (5.1) one obtains

$$(x + y)^{\mathsf{T}}\pi(\overline{x + y}) \le x^{\mathsf{T}}\pi\bar{x} + 2\sqrt{(x^{\mathsf{T}}\pi\bar{x})}\sqrt{(y^{\mathsf{T}}\pi\bar{y})} + y^{\mathsf{T}}\pi\bar{y} = \left(\sqrt{x^{\mathsf{T}}\pi\bar{x}} + \sqrt{y^{\mathsf{T}}\pi\bar{y}}\right)^2,$$

which proves the lemma. □

The absolute values of positive semidefinite matrices are in some sense bounded by their diagonal elements, which may often times be helpful when trying to bound sesquilinear forms such as the one defined above.

Lemma 5.21. *Let $\pi \in \mathbb{K}^{d\times d}$ be a positive semidefinite Hermitian matrix. Then π^{jj} is a non-negative real number for each $j = 1, \dots, d$ and*

$$|\pi^{ij}| \leq \sqrt{\pi^{ii}}\sqrt{\pi^{jj}}, \qquad (i,j) \in \{1,\dots,d\}^2.$$

Proof. Consider for each $j = 1,\dots d$ the unit vector $e_j \in \{0,1\}^d$, which consists of all zeros except for a one in the j-th entry. Then the positive semidefiniteness of π implies

$$0 \leq (e^j)^{\mathsf{H}}\pi e^j = (e^j)^{\mathsf{T}}\pi e^j = \pi^{j\cdot}e^j = \pi^{jj},$$

where $\pi^{j\cdot}$ denotes the j-the row of π. Furthermore, the function $\langle\cdot,\cdot\rangle : \mathbb{K}^d \times \mathbb{K}^d \ni (x,y) \mapsto x^{\mathsf{T}}\pi\bar{y} \in \mathbb{K}$ is a positive semidefinite Hermitian sesquilinear form, as stated in Lemma 5.19. Thus one can use again the Cauchy–Schwarz inequality in Theorem 5.18 in the third step to obtain

$$|\pi^{ij}| = |e_i^{\mathsf{T}}\pi\bar{e}_j| = |\langle e_i, e_j\rangle| \leq \sqrt{\langle e_i, e_i\rangle}\sqrt{\langle e_j, e_j\rangle} = \sqrt{e_i^{\mathsf{T}}\pi\bar{e}_i}\sqrt{e_j^{\mathsf{T}}\pi\bar{e}_j} = \sqrt{\pi^{ii}}\sqrt{\pi^{jj}}$$

for each pair $(i,j) \in \{1,\dots d\}^2$. □

Keep in mind that a *pre-Hilbert space* $(H, \|\cdot\|_H)$ is normed vector space, where the norm is induced by a positive definite Hermitian sesquilinear form, i.e. $\|x\|_H = \sqrt{\langle x,x\rangle}$ for all $x \in H$. If H is complete w.r.t. $\|\cdot\|_H$ then it is called a *Hilbert space*. If $\langle\cdot,\cdot\rangle$ is only positive semidefinite, then $\|\cdot\|_H$ is a seminorm, as the Cauchy–Schwarz inequality in Theorem 5.18 holds for positive semidefinite Hermitian sesquilinear forms and not only for scalar products. Thus the triangle inequality holds for $\|\cdot\|_H$, due to

$$\begin{aligned}\|x+y\|_H^2 &= |\langle x+y, x+y\rangle| \leq \langle x,x\rangle + |\langle x,y\rangle| + |\langle y,x\rangle| + \langle y,y\rangle \\ &\leq \langle x,x\rangle + 2\sqrt{\langle x,x\rangle\langle y,y\rangle} + \langle y,y\rangle = \left(\sqrt{\langle x,x\rangle} + \sqrt{\langle y,y\rangle}\right)^2\end{aligned}$$

for each pair $(x,y) \in H^2$. Furthermore,

$$\|\alpha x\|_H^2 = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\bar{\alpha}\langle x,x\rangle} = \sqrt{|\alpha|^2\langle x,x\rangle} = |\alpha|\,\|x\|_H^2, \qquad \alpha \in \mathbb{K},\ x \in H,$$

and thus $\|\cdot\|_H$ meets all criteria of a seminorm.

Lemma 5.22. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two sequences in a $\mathbb{K}$-vector space H converging to x and y, respectively, with respect to the seminorm $\|\cdot\|_H$ induced by the positive semidefinite Hermitian sesquilinear form $\langle\cdot,\cdot\rangle$ on H. Then the sequence $(\langle x_n, y_n\rangle)_{n\in\mathbb{N}}$ converges to $\langle x,y\rangle$ in $(\mathbb{K}, |\cdot|)$.*

Proof. Note at first that $z_n \to z$ for $n \to \infty$ in H means for every $\epsilon > 0$ exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ follows $\|z_n - z\|_H \leq \epsilon$. Furthermore, a convergent sequence is bounded, i.e. there exists a $M > 0$, such that $\|z_n\|_H \leq M$ for all $n \in \mathbb{N}$ as well as $\|z\|_H \leq M$. Define M as such a bound holding simultaneously for both convergent sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$.

Fix $\epsilon > 0$ and take $n_0 \in \mathbb{N}$, such that

$$M\|x_n - x\|_H \leq \frac{\epsilon}{2} \quad \text{as well as} \quad M\|y_n - y\|_H \leq \frac{\epsilon}{2}, \qquad n \geq n_0$$

holds. Thus for such $n \geq n_0$ it follows that

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle| \\ &\leq |\langle x_n, y_n\rangle - \langle x_n, y\rangle| + |\langle x_n, y\rangle - \langle x, y\rangle| = |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|, \end{aligned}$$

where in the second step the triangle inequality satisfied by $|\cdot|$ and in the last one the linearity of the scalar product have been used. By the Cauchy–Schwarz inequality, Theorem 5.18, as well as the previously in this proof discussed upper bounds it follows that

$$\begin{aligned} |\langle x_n, y_n\rangle - \langle x, y\rangle| &\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle| \\ &\leq \sqrt{\langle x_n, x_n\rangle}\sqrt{\langle y_n - y, y_n - y\rangle} + \sqrt{\langle x_n - x, x_n - x\rangle}\sqrt{\langle y, y\rangle} \\ &= \|x_n\|_H \|y_n - y\|_H + \|x_n - x\|_H \|y\|_H \\ &\leq M\|y_n - y\|_H + M\|x_n - x\|_H \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which concludes the proof. □

5.5 Appendix on transition kernels

In the following, a new kind of "measure-like" functions will be introduced and briefly discussed. For more information on this topic as well as a proof of Lemma 5.24 the reader is referred to [Sch23, Section 15.8]. Note that throughout this section the existence of a sequence of sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{S}$ satisfying $\bigcup_{n\in\mathbb{N}} A_n = S$ is being assumed. The findings of this section have been used in the earlier parts of this thesis on the measure space $(S, \mathcal{S}) = (\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$, for which for example the sequence of intervals $([0, n])_{n\in\mathbb{N}}$ satisfy this assumption.

Definition 5.23 (Finite and σ-finite transition kernels)**.** Fix two measurable spaces $(S, \mathcal{S})$ and $(\Omega, \mathcal{F})$. Then a function $K : \mathcal{S} \times \Omega \to \overline{\mathbb{R}}_+$ is a *transition kernel* from Ω to S, if and only if the following two criteria are met:

(i) The function $K(\cdot, \omega) : \mathcal{S} \to \overline{\mathbb{R}}_+$ is a measure for each $\omega \in \Omega$.

(ii) The function $K(A, \cdot) : \Omega \to \overline{\mathbb{R}}_+$ is $\mathcal{F}$-measurable for each $A \in \mathcal{S}$.

If $K(\cdot, \omega)$ is a finite measure on $(S, \mathcal{S})$ for each $\omega \in \Omega$, i.e. $K(S, \omega) < \infty$, then K is called a *finite transition kernel*. Furthermore, K is said to be σ-finite, if one can find a sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{S}$ with $\bigcup_{n\in\mathbb{N}} A_n = S$ in such a way that $K(A_n, \omega) < \infty$ holds for all $\omega \in \Omega$ and $n \in \mathbb{N}$.

Lemma 5.24. *In the same setting as above, let $K : \mathcal{S} \times \Omega \to \overline{\mathbb{R}}_+$ be a σ-finite transition kernel and μ a measure on $(\Omega, \mathcal{F})$. Then the function*

$$(K \otimes \mu)(A) := \int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) K(\mathrm{d}s, \omega)\Big)\mu(\mathrm{d}\omega), \qquad A \in \mathcal{S} \otimes \mathcal{F},$$

is a measure on $(S \times \Omega, \mathcal{S} \otimes \mathcal{F})$.

Intuitively, those ideas can be extended to $\mathbb{K}$-valued transition kernels. As per [Sch23, Remark 15.131] the set

$$\mathcal{R}(K) := \{A \in \mathcal{S} : K(A, \omega) < \infty \text{ for all } \omega \in \Omega\}$$

defines a δ-ring for each transition kernel K.

Definition 5.25. Let again $(S, \mathcal{S})$ and $(\Omega, \mathcal{F})$ denote two measurable spaces and $K : \mathcal{S} \times \Omega \to \mathbb{C}$ be a function. If there exists a decomposition of $K = K_1 - K_2 + \mathrm{i}(K_3 - K_4)$, where for each $j = 1, \dots, 4$ the function $K_j : \mathcal{S} \times \Omega \to \mathbb{R}$ is a transition kernel according to definition 5.23, $K : \mathcal{R} \times \Omega \to \mathbb{C}$ is a *complex transition kernel*, where $\mathcal{R} \subseteq \mathcal{R}(K) := \mathcal{R}(K_1) \cap \cdots \cap \mathcal{R}(K_4)$ is a δ-ring. If $K_3 \equiv K_4 \equiv 0$, K is a *signed transition kernel*. Analogously to Lemma 5.24, for each measure μ on $(\Omega, \mathcal{F})$ one can define the signed or complex measure

$$(K \otimes \mu)(A) = (K_1 \otimes \mu)(A) - (K_2 \otimes \mu)(A) + \mathrm{i}\big((K_3 \otimes \mu)(A) - (K_4 \otimes \mu)(A)\big), \quad A \in \mathcal{R} \otimes \mathcal{F}.$$

Lemma 5.26. *The defining equality*

$$(K \otimes \mu)(A) = \int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) K(\mathrm{d}s, \omega) \Big) \mu(\mathrm{d}\omega), \qquad A \in \mathcal{R} \otimes \mathcal{F}$$

of Lemma 5.24 holds also for signed or complex transition kernels $K : \mathcal{R} \times \Omega \to \mathbb{K}$.

Proof. Due to the linearity of the Lebesgue integral in the integrator as well as the integrand, one may also write in the setting of the definition above

$$\begin{aligned}
&(K \otimes \mu)(A) = (K_1 \otimes \mu)(A) - (K_2 \otimes \mu)(A) + \mathrm{i}\big((K_3 \otimes \mu)(A) - (K_4 \otimes \mu)(A)\big) \\
&\int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) K_1(\mathrm{d}s, \omega) \Big) \mu(\mathrm{d}\omega) - \int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) K_2(\mathrm{d}s, \omega) \Big) \mu(\mathrm{d}\omega) \\
&+ \mathrm{i} \Big(\int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) K_3(\mathrm{d}s, \omega) \Big) \mu(\mathrm{d}\omega) - \int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) K_4(\mathrm{d}s, \omega) \Big) \mu(\mathrm{d}\omega) \Big) \\
&= \int_\Omega \Bigg(\int_S \mathbb{1}_A(s, \omega) K_1(\mathrm{d}s, \omega) - \int_S \mathbb{1}_A(s, \omega) K_2(\mathrm{d}s, \omega) \\
&+ \mathrm{i} \int_S \mathbb{1}_A(s, \omega) K_3(\mathrm{d}s, \omega) - \mathrm{i} \int_S \mathbb{1}_A(s, \omega) K_4(\mathrm{d}s, \omega) \Bigg) \mu(\mathrm{d}\omega) \\
&= \int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) \Big(K_1(\mathrm{d}s, \omega) - K_2(\mathrm{d}s, \omega) + \mathrm{i}\big(K_3(\mathrm{d}s, \omega) - K_4(\mathrm{d}s, \omega)\big) \Big) \Big) \mu(\mathrm{d}\omega) \\
&= \int_\Omega \Big(\int_S \mathbb{1}_A(s, \omega) K(\mathrm{d}s, \omega) \Big) \mu(\mathrm{d}\omega)
\end{aligned}$$

for each $A \in \mathcal{R} \otimes \mathcal{F} \subseteq \mathcal{S} \otimes \mathcal{F}$. ☐

Lemma 5.27. *On the two measurable spaces* $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$ *and* $(\Omega, \mathcal{F})$ *each process* $\mathbb{V} \in \mathcal{V}_0^+$ *can be viewed as a* σ*-finite transition kernel from* Ω *to* $\mathbb{R}_+$*.*

Proof. Note that by [Sch23, Lemma 5.49] the total variation process is $\mathbb{R}_+$-valued, increasing, continuous and adapted.

By Definition 2.4 the total variation process $\mathbb{V}$ is pathwise finite on intervals of the form $[0, t]$ (and thus also on $(0, t]$) for each $t \in \mathbb{R}_+$. Thus one can simply consider the sequence $\big(\{0\}, ((0, n])_{n\in\mathbb{N}}\big)$ of sets in $\mathcal{B}_{\mathbb{R}_+}$ and see that $\{0\} \cup \bigcup_{n\in\mathbb{N}}(0, n] = \{0\} \cup (0, \infty) = \mathbb{R}_+$ and $\mathbb{V}(\{0\}, \omega) = 0$ (which will be shown below) as well as $\mathbb{V}\big((0, n], \omega\big) < \infty$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$, proving the σ-finiteness of $\mathbb{V}$.

(i) As $\mathbb{V}$ is pathwise non-decreasing and continuous, it induces for all $\omega \in \Omega$ a unique measure $\mathbb{V}(\cdot, \omega) : \mathcal{B}_{\mathbb{R}_+} \to \overline{\mathbb{R}}_+$, due to [Gri18, Satz 3.1]. Furthermore, [Sch23, Definition 5.20] directly implies that the total variation of any function on the degenerate interval $\{t\} = [t]$ is zero for each $t \in \mathbb{R}_+$ and thus $\mathbb{V}(\{0\}, \omega) = 0$ for all $\omega \in \Omega$.

(ii) Now for each $A \in \mathcal{B}_{\mathbb{R}_+}$ take a closer look at the function $\mathbb{V}(A, \cdot) : \Omega \ni \omega \mapsto \mathbb{V}(A, \omega)$. As stated above, the funnction $\omega \mapsto \mathbb{V}(\{0\}, \omega) \equiv 0$ and as a constant function it is measurable for all σ-algebras on Ω. In order to show that the second condition of Definition 5.23 also holds for all $A \in \mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$, the *monotone class lemma* will be used. At first, define the sets

$$\mathcal{E} = \{(a, b] : a, b \in \mathbb{R}_+,\ a \leq b\}$$

and

$$\mathfrak{M}_n = \{A \in \mathcal{B}_{\mathbb{R}_+\setminus\{0\}} : \mathbb{V}(A \cap (0, n], \cdot) \text{ is } \mathcal{F}\text{-measurable}\}, \qquad n \in \mathbb{N}.$$

Note at this point that $\mathcal{E}$ generates the Borel-σ-algebra $\mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$ and $\mathcal{E} \subseteq \mathfrak{M}_n$ for all $n \in \mathbb{N}$, as for each $a, b \in \mathbb{R}_+$ with $a \leq b$

$$\mathbb{V}((a, b] \cap (0, n], \cdot) = \begin{cases} \mathbb{V}(\varnothing, \cdot) \equiv 0 & \text{if } n \leq a, \\ \mathbb{V}((a, n], \cdot) = \mathbb{V}_n(\cdot) - \mathbb{V}_a(\cdot) & \text{if } a < n < b, \\ \mathbb{V}((a, b], \cdot) = \mathbb{V}_b(\cdot) - \mathbb{V}_a(\cdot) & \text{if } b \leq n \end{cases}$$

holds, where the right-hand side is $\mathcal{F}_{\min(n,b)}$- and $\mathcal{F}_a$-measurable, respectively, due to the adaptedness of $\mathbb{V}$ and consequently $\mathcal{F}$-measurable. Furthermore, $\mathbb{V}(A \cap (0, n], \omega)$ is finite for all $n \in \mathbb{N}$, $\omega \in \Omega$ and $A \in \mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$. Now it will be demonstrated that the set $\mathfrak{M}_n$ is a monotone class for fixed $n \in \mathbb{N}$.

(1) The set $\varnothing \in \mathfrak{M}_n$, as $\mathbb{V}(\varnothing \cap (0, n], \cdot) = \mathbb{V}(\varnothing, \cdot) \equiv 0$. Furthermore, $\mathbb{V}\big((\mathbb{R}_+ \setminus \{0\}) \cap (0, n], \cdot\big) = \mathbb{V}((0, n], \cdot) = \mathbb{V}_n(\cdot)$ is $\mathcal{F}_n$ and thus also $\mathcal{F}$ measurable.

(2) Let $A, B \in \mathfrak{M}_n$ satisfying $A \subseteq B$. Then

$$\begin{aligned} \omega \mapsto \mathbb{V}\big((B \setminus A) \cap (0, n], \omega\big) &= \mathbb{V}\big((B \cap [0, n]) \setminus (A \cap (0, n]), \omega\big) \\ &= \mathbb{V}(B \cap (0, n], \omega) - \mathbb{V}(A \cap (0, n], \omega) \end{aligned}$$

implies the $\mathcal{F}$-measurability of $\mathbb{V}\big((B \setminus A) \cap (0, n], \cdot\big)$ due to [Gri18, Satz 4.4] and thus $B \setminus A \in \mathfrak{M}_n$. As already mentioned, all terms in the equation above are finite.

(3) Now consider $A, B \in \mathfrak{M}_n$, such that $A \cap B = \varnothing$. Thus one can use

$$\omega \mapsto \mathbb{V}\big((A \cup B) \cap (0,n], \omega\big) = \mathbb{V}(A \cap (0,n], \omega) + \mathbb{V}(B \cap (0,n], \omega)$$

and again [Gri18, Satz 4.4] to see that $A \cup B \in \mathfrak{M}_n$, as $\mathbb{V}\big((A \cup B) \cap (0,n], \cdot\big)$ is $\mathcal{F}$-measurable.

(4) Last but not least, fix an increasing sequence $(A_k)_{k\in\mathbb{N}} \in \mathfrak{M}_n$ and define for each $k \in \mathbb{N}$ the set $B_k = A_k \setminus A_{k-1}$, where $A_0 := \varnothing$, which is again in $\mathfrak{M}_n$, which was already proven in part (ii). Obviously, $\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k$ holds. Thus, due to the σ-additivity of the measures $\mathbb{V}(\cdot, \omega)$ for all $\omega \in \Omega$,

$$\omega \mapsto \mathbb{V}\Big(\big(\bigcup_{k=1}^{\infty} A_k\big) \cap (0,n], \omega\Big) = \mathbb{V}\Big(\big(\bigcup_{k=1}^{\infty} B_k\big) \cap (0,n], \omega\Big) = \mathbb{V}\Big(\bigcup_{k=1}^{\infty} (B_k \cap (0,n]), \omega\Big)$$
$$= \sum_{k=1}^{\infty} \mathbb{V}(B_k \cap (0,n], \omega) = \lim_{k\to\infty} \sum_{j=1}^{k} \mathbb{V}(B_j \cap (0,n], \omega)$$

holds, where the last term is the pointwise limit of the sequence of $\mathcal{F}$-measurable functions $\big(\sum_{j=1}^{k} \mathbb{V}(B_j \cap (0,n], \cdot)\big)_{k\in\mathbb{N}}$ and as such is itself $\mathcal{F}$-measurable, due to Lemma 5.3. Therefore $\bigcup_{k=1}^{\infty} A_k \in \mathfrak{M}_n$, which is sufficient to show that $\mathfrak{M}_n$ is a monotone class for all $n \in \mathbb{N}$.

For the next step fix two sets in $\mathcal{E}$. Then the two intervals are $(a_1, b_1]$ and $(a_2, b_2]$, for some $a_1 \le b_1$ and $a_2 \le b_2$, where a_1, b_1, a_2, b_2 are in $\mathbb{R}_+$, and the intersect is either $\varnothing$, or $\big(a_1 \vee a_2, b_1 \wedge b_2\big]$. Either way, the intersect is again an element of $\mathcal{E}$ making it *intersection stable*. By its definition, $\mathfrak{M}_n$ is a subset of $\mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$. One can now use the *monotone class lemma* Lemma 5.16 in the second equality to obtain

$$\mathcal{B}_{\mathbb{R}_+\setminus\{0\}} = \sigma(\mathcal{E}) = \mathfrak{M}(\mathcal{E}) \subseteq \mathfrak{M}_n,$$

from which $\mathfrak{M}_n = \mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$ follows for all $n \in \mathbb{N}$. This implies the $\mathcal{F}$-measurability of $\Omega \ni \omega \mapsto \mathbb{V}(A \cap (0,n], \omega)$ for all $n \in \mathbb{N}$ and $A \in \mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$. Furthermore, as $\big((0,n]\big)_{n\in\mathbb{N}}$ is an increasing sequence satisfying $\lim_{n\to\infty}(0,n] = \mathbb{R}_+ \setminus \{0\}$, the function

$$\Omega \ni \omega \mapsto \mathbb{V}(A, \omega) = \sup_{n\in\mathbb{N}} \mathbb{V}(A \cap (0,n], \omega)$$

is the pointwise supremum of $\mathcal{F}$-measurable functions and as such also $\mathcal{F}$-measurable, which was proven in Lemma 5.1, for all $A \in \mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$.
For a more general $A \in \mathcal{B}_{\mathbb{R}_+}$ it is clear that either $A \in \mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$ or there exists a $\tilde{A} \in \mathcal{B}_{\mathbb{R}_+\setminus\{0\}}$, such that $A = \tilde{A} \cup \{0\}$. In the first case, the $\mathcal{F}$-measurability of $\mathbb{V}(A, \cdot)$ has already been proven. In the second case, this function is also $\mathcal{F}$-measurable, because

$$\omega \mapsto \mathbb{V}(A, \omega) = \mathbb{V}(\tilde{A} \cup \{0\}, \omega) = \mathbb{V}(\tilde{A}, \omega) + \mathbb{V}(\{0\}, \omega) = \mathbb{V}(\tilde{A}, \omega),$$

which concludes the proof. □

Lemma 5.28. *Each $V \in \mathcal{V}_0^1$ can be viewed as a finite signed or complex transition kernel from Ω to $\mathbb{R}_+$ on the δ-ring $\mathcal{R} := \bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]} \subset \mathcal{B}_{\mathbb{R}_+}$.*

Proof. At first, the case $\mathbb{K} = \mathbb{R}$ is assumed. Therefore, due to the right-continuity and locally finite variation of V, it can be pathwise represented as the difference of two real, right-continuous and non-decreasing functions, namely $V_\cdot(\omega) = F_1(\cdot,\omega) - F_2(\cdot,\omega)$ for each $n \in \mathbb{N}$ [Gri18, Satz 6.5]. As per [Gri18, Satz 3.1] the function $F_k(\cdot,\omega) : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1,2$ can be viewed as the distribution function of a unique σ-finite measure $\tilde{\mu}_k(\cdot,\omega)$ on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$. For $a, b \in \mathbb{R}_+$ satisfying $a \le b$ one may then define

$$V((a,b],\omega) = F_1(b,\omega) - F_1(a,\omega) - F_2(b,\omega) + F_2(a,\omega) = \tilde{\mu}_1((a,b],\omega) - \tilde{\mu}_2((a,b],\omega),$$

where both terms on the right-hand side are finite. As in the lemma above, it is only natural to set $\tilde{\mu}_1(\{0\},\omega) = \tilde{\mu}_2(\{0\},\omega) = 0$ for all $\omega \in \Omega$.

As per [Sch23, Example 15.107(d)] the countable union $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_{[0,n]}$ is the set of all relatively compact Borel sets of $\mathbb{R}_+$ and also a δ-ring. For each $A \in \mathcal{R}$ there exists a $b \in \mathbb{R}_+$, such that $A \subseteq \big(\{0\} \cup (0,b]\big)$ and thus

$$\tilde{\mu}_k(A,\omega) \le \tilde{\mu}_k\big(\{0\} \cup (0,b],\omega\big) = \tilde{\mu}_k\big(\{0\},\omega\big) + \tilde{\mu}_k\big((0,b],\omega\big) = \tilde{\mu}_k\big((0,b],\omega\big) < \infty, \quad k = 1,2.$$

More generally, $V(\cdot,\omega) : \mathcal{R} \to \mathbb{R}$ can then be viewed as a finite signed measure on the δ-ring $\mathcal{R}$ via

$$V(A,\omega) = \tilde{\mu}_1(A,\omega) - \tilde{\mu}_2(A,\omega), \qquad A \in \mathcal{R},\ \omega \in \Omega.$$

As such, there exists a unique Jordan decomposition (see [Sch23, Theorem 15.119], such that $V(\cdot,\omega) = \mu_+^R(\cdot,\omega) - \mu_-^R(\cdot,\omega)$ into mutually singular $\mathbb{R}_+$-valued measures. As per [Sch23, p. 165f] for two real numbers $0 \le a \le b$ and all $\omega \in \Omega$ the measures $\mu_+^R\big((a,b],\omega\big)$ and $\mu_-^R\big((a,b],\omega\big)$ coincide with the positive and negative variation of $V_\cdot(\omega)$ on the interval $(a,b]$, namely $\mathbb{V}_V^{R,+}([a,b])$ and $\mathbb{V}_V^{R,-}([a,b])$, respectively, and thus are $\mathbb{R}_+$-valued, increasing, continuous, adapted and starting at zero, see [Sch23, Lemma 5.49(b)].

In the general case, where $\mathbb{K} = \mathbb{C}$, the real and imaginary part of the process V have to be considered separately. This leads pathwise to the complex measure $V(\cdot,\omega) = \mathbb{V}_V^{R,+}(\cdot,\omega) - \mathbb{V}_V^{R,-}(\cdot,\omega) + \mathrm{i}\big(\mathbb{V}_V^{I,+}(\cdot,\omega) - \mathbb{V}_V^{I,-}(\cdot,\omega)\big)$ being composed of four finite measures on the δ-ring $\mathcal{R}$ denoting the postive and negative variation of the real and imaginary part of the covariation process, respectively.

Consequently, the lemma is proven by showing that $\mathbb{V}(A,\cdot) : \Omega \to \mathbb{R}_+$ is a transition kernel for each $\mathbb{V} \in \{\mathbb{V}_V^{R,+}, \mathbb{V}_V^{R,-}, \mathbb{V}_V^{I,+}, \mathbb{V}_V^{I,-}\}$. As stated above, those positive and negative variation processes are continuous, adapted and starting at zero. Consequently, they are also in $\mathcal{V}_0^1$, as they are of locally finite variation. Thus the last lemma implies that they can be viewed as σ-finite transition kernels from Ω to $\mathbb{R}_+$. □

The following theorem provides a very useful upper bound of integrals with respect to the total variation of some covariation process. In this thesis, only the result is stated and for the proof the reader is referred to [Sch23, Theorem 5.92].

Theorem 5.29. *For two $\mathbb{K}$-valued continuous local martingales M and N let $\mathbb{V}_t := \mathbb{V}_{[M,N]}([0,t])$ denote the total variation of their covariation process. As always the notation of total and quadratic variation processes will be abused in the sense that the same symbols will also be used for their induced σ-finite transition kernels on $(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+})$, respectively. Then there exists a $\mathbb{P}$-null set, such that outside of it*

$$\int_A |\langle U_s, V_s\rangle| \,\mathrm{d}\mathbb{V}_s \leq \left(\int_A |U_s|^2 \,\mathrm{d}[M]_s\right)^{1/2} \left(\int_A |V_s|^2 \,\mathrm{d}[N]_s\right)^{1/2}, \qquad A \in \mathcal{B}_{\mathbb{R}_+} \tag{5.2}$$

holds simultaneously for all $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$-measurable processes $U, V : \mathbb{R}_+ \times \Omega \to \mathbb{K}^d$. Note that in the above inequality the convention $0 \cdot \infty = \infty \cdot 0 = 0$ is used, as the integrals may be infinite.

Lemma 5.30. *Let M, N be two $\mathbb{K}$-valued continuous local martingales, $\mathbb{V}$ denote the total variation of their covariation process and consider it as well as $[M]$ and $[N]$ as σ-finite transition kernels. Then*

$$\mathbb{V}(A,\omega) \leq \sqrt{[M](A,\omega)}\sqrt{[N](A,\omega)}$$

holds for every set $A \in \mathcal{B}_{\mathbb{R}_+}$ and all $\omega \in \Omega$ outside the aforementioned $\mathbb{P}$-null set.

Proof. Let A be an arbitrary set in $\mathcal{B}_{\mathbb{R}_+}$ and $\omega \in \Omega$ be not in the null set mentioned in the theorem above. By defining $U \equiv V \equiv 1$ in inequality (5.2), one obtains

$$\mathbb{V}(A,\omega) = \int_A \mathrm{d}\mathbb{V}_s(\omega) \leq \left(\int_A \mathrm{d}[M]_s(\omega)\right)^{1/2} \left(\int_A \mathrm{d}[N]_s(\omega)\right)^{1/2} = \sqrt{[M](A,\omega)}\sqrt{[N](A,\omega)},$$

which concludes the proof. □

Definition 5.31 (Absolutely continuous transition kernels)**.** In the setting of Definition 5.23 let K_1 and K_2 denote two σ-finite transition kernels and μ a measure on $(\Omega, \mathcal{F})$. Then K_1 is said to be *absolutely continuous* w.r.t. K_2 on the triple $(S \times \Omega, \mathcal{S} \otimes \mathcal{F}, \mu)$, which will be denoted by $K_1 \ll K_2$, if and only if the measure $K_1(\cdot, \omega)$ is absolutely continuous w.r.t. the measure $K_2(\cdot,\omega)$ on $\mathcal{S}$ for μ-almost all $\omega \in \Omega$.

Lemma 5.32. *For two σ-finite transition kernels K_1, K_2 and a measure μ on $(\Omega, \mathcal{F})$ holds*

$$K_1 \ll K_2 \text{ on } (S \times \Omega, \mathcal{S} \otimes \mathcal{F}, \mu) \Rightarrow K_1 \otimes \mu \ll K_2 \otimes \mu \text{ on } \mathcal{S} \otimes \mathcal{F}.$$

Proof. Assume $K_1 \ll K_2$ on $(S \times \Omega, \mathcal{S} \otimes \mathcal{F}, \mu)$ and fix a set $A \in \mathcal{S} \otimes \mathcal{F}$ satisfying

$$(K_2 \otimes \mu)(A) = \int_\Omega \Big(\int_S \mathbb{1}_A(s,\omega) K_2(\mathrm{d}s,\omega)\Big)\mu(\mathrm{d}\omega) = 0.$$

As for fixed $\omega \in \Omega$ the integral $\int_S \mathbb{1}_A(s,\omega)K_2(\mathrm{d}s,\omega)$ is non-negative, it must be μ-almost everywhere equal to zero, i.e. there exists a set $N \in \mathcal{F}$ with $\mu(N) = 0$ and $\int_S \mathbb{1}_A(s,\omega)K_2(\mathrm{d}s,\omega) = 0$ for all $\omega \in N^{\mathrm{c}}$. Fix now such a ω. Then the function

$$s \mapsto \int_S \mathbb{1}_A(s,\omega)K_2(\mathrm{d}s,\omega) = 0,$$

which is equivalent to $A(\cdot,\omega)$, which is an element of $\mathcal{S}$ by Lemma 5.6, being a $K_2(\cdot,\omega)$-null set. By assumption, there exists a μ-null set $\tilde{N} \in \mathcal{F}$, such that $K_1(\cdot,\omega) \ll K_2(\cdot,\omega)$ on $\mathcal{S}$ for each $\omega \in \tilde{N}^{\mathsf{c}}$, which then leads to

$$\begin{aligned}(K_1 \otimes \mu)(A) &= \int_\Omega \Big(\int_S \mathbb{1}_A(s,\omega) K_1(\mathrm{d}s,\omega) \Big) \mu(\mathrm{d}\omega) \\ &= \int_{(N \cup \tilde{N})^{\mathsf{c}}} \Big(\int_S \mathbb{1}_A(s,\omega) K_1(\mathrm{d}s,\omega) \Big) \mu(\mathrm{d}\omega) + \int_{N \cup \tilde{N}} \Big(\int_S \mathbb{1}_A(s,\omega) K_1(\mathrm{d}s,\omega) \Big) \mu(\mathrm{d}\omega) \\ &= \int_\Omega 0 \, \mu(\mathrm{d}\omega) + 0 = 0,\end{aligned}$$

as $\mu(N \cup \tilde{N}) \leq \mu(N) + \mu(\tilde{N}) = 0$, whereby the lemma is proven. □

5.6 Appendix on the uniform approximation of bounded measurable functions

As per [Gri18, Satz 4.6] each measurable function f on a measurable space $(\Omega, \mathcal{F})$ can be pointwise approximated by simple measurable functions. The following theorem modifies those findings to the uniform approximation of bounded functions.

Definition 5.33. Let $(V, \|\cdot\|)$ be a normed vector space. A sequence of bounded V-valued functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a bounded V-valued function f, if and only if for each $\epsilon > 0$ exists a $n_0 \in \mathbb{N}$, such that

$$\|f_n(\omega) - f(\omega)\| \leq \epsilon, \qquad n \geq n_0.$$

holds simultaneously for all $\omega \in \Omega$, see [Kal14, Definition 6.6.5].

Theorem 5.34. *For every bounded $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{R}$ there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, where $f_n = \sum_{j=1}^{m_n} x_{n,j} \mathbb{1}_{A_{n,j}}$, such that for each $n \in \mathbb{N}$ holds $x_{n,j} \in \mathbb{R}$ and $A_{n,j} \in \mathcal{F}$ for all $j = 1, \dots, m_n$ as well as $\bigcup_{j=1}^{m_n} A_{n,j} = \Omega$ and $A_{n,j} \cap A_{n,k} = \varnothing$ for $j \neq k$, which is converging uniformly to f.*

Proof. The above mentioned theorem [Gri18, Satz 4.6] is being proven by defining for each $n \in \mathbb{N}$ and $j = -4^n, \dots, 4^n - 1$ the sets

$$A_{n,j} = \{\omega \in \Omega : j/2^n \leq f_n(\omega) < (j+1)/2^n\}$$

as well as $A_{n,-4^n-1} = \{\omega \in \Omega : f_n(\omega) < -2^n\}$ and $A_{n,4^n} = \{\omega \in \Omega : 2^n \leq f_n(\omega)\}$. Furthermore set

$$x_{n,j} = \begin{cases} -2^n & \text{for } j = -4^n - 1, \\ j/2^n & \text{for } j = -4^n, \dots, 4^n - 1, \\ 2^n & \text{for } j = 4^n. \end{cases}$$

Therefore $f_n := \sum_{j=-4^n-1}^{4^n} x_{n,j} \mathbb{1}_{A_{n,j}}$ is a $\mathcal{F}$-measurable, as $A_{n,j} \in \mathcal{F}$ for all $j = -4^n - 1, \dots, 4^n$, which follows directly from the measurability of f, simple function (apart from an index-shift) satisfying

$$f_n(\omega) = \begin{cases} -2^n & \text{for } f(\omega) < -2^n, \\ j/2^n & \text{for } j/2^n \leq f_n(\omega) < (j+1)/2^n, \\ 2^n & \text{for } 2^n \leq f_n(\omega). \end{cases}$$

Now fix some $\omega \in \Omega$. Then there exists a $n \in \mathbb{N}$ such that $-2^n \leq f(\omega) < 2^n$ and as such a $j \in \{-4^n, \ldots, 4^n - 1\}$ satisfying $j/2^n \leq f_n(\omega) < (j+1)/2^n$. Consequently

$$|f_n(\omega) - f(\omega)| = |j/2^n - f(\omega)| < \frac{1}{2^n}$$

and thus $f_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Fix $\epsilon > 0$ and define $n_\epsilon \in \mathbb{N}$ in a way that $\frac{1}{2^n} < \epsilon$ for all $n \geq n_\epsilon$. As f is bounded, there exists an $n_f \in \mathbb{N}$, such that $\|f\|_\infty := \sup_{\omega\in\Omega} |f(\omega)| \leq 2^n$ for all $n \geq n_f$. Thus for all $\omega \in \Omega$

$$|f_n(\omega) - f(\omega)| < \frac{1}{2^n} < \epsilon, \qquad n \geq n_0 := \max(n_\epsilon, n_f),$$

holds, which concludes the proof. □

Lemma 5.35. *For every bounded $\mathcal{F}$-measurable function $f : \Omega \to \mathbb{K}^d$ there exists a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$, where $f_n = \sum_{j=1}^{m_n} x_{n,j} \mathbb{1}_{A_{n,j}}$, such that for each $n \in \mathbb{N}$ holds $x_{n,j} \in \mathbb{K}^d$ and $A_{n,j} \in \mathcal{F}$ for all $j = 1, \ldots, m_n$ as well as $\bigcup_{j=1}^{m_n} A_{n,j} = \Omega$ and $A_{n,j} \cap A_{n,k} = \varnothing$ for $j \neq k$, which is converging uniformly to f.*

Proof. Consider at first a bounded $\mathcal{F}$-measurable $\mathbb{C}$-valued function $f = f^R + \mathrm{i} f^I$. Thus by the theorem above there exist two sequences of simple $\mathbb{R}$-valued functions $(f_n^R)_{n\in\mathbb{N}}$ and $(f_n^I)_{n\in\mathbb{N}}$ converging uniformly to f^R and f^I, respectively, where

$$f_n^R = \sum_{j=1}^{m_n^R} x_{n,j}^R \mathbb{1}_{A_{n,j}^R} \qquad \text{and} \qquad f_n^I = \sum_{j=1}^{m_n^I} x_{n,j}^I \mathbb{1}_{A_{n,j}^I}.$$

For $j = 1, \ldots, m_n := m_n^R m_n^I$ and $k = 1, \ldots, m_n^I$ one can now define

$$A_{n,j} = A_{n,k}^R \cap A_{n,j-(k-1)m_n^R}^I, \qquad \text{if } (k-1)m_n^R < j \leq k m_n^R.$$

Thus for $j, k \in \{1, \ldots, m_n\}$ with $j \neq k$ the intersect $A_{n,j} \cap A_{n,k} = \varnothing$ and

$$\begin{aligned} \bigcup_{j=1}^{m_n} A_{n,j} &= \bigcup_{k=1}^{m_n^I} \bigcup_{j=(k-1)m_n^R+1}^{k m_n^R} A_{n,j} = \bigcup_{k=1}^{m_n^I} \bigcup_{j=(k-1)m_n^R+1}^{k m_n^R} \left(A_{n,k}^R \cap A_{n,j-(k-1)m_n^R}^I\right) \\ &= \bigcup_{k=1}^{m_n^I} A_{n,k}^R \cap \Bigg(\bigcup_{j=(k-1)m_n^R+1}^{k m_n^R} A_{n,j-(k-1)m_n^R}^I \Bigg) = \bigcup_{k=1}^{m_n^I} A_{n,k}^R \cap \Bigg(\bigcup_{j=1}^{m_n^R} A_{n,j}^I \Bigg) \\ &= \bigcup_{k=1}^{m_n^I} A_{n,k}^R \cap \Omega = \bigcup_{k=1}^{m_n^I} A_{n,k}^R = \Omega. \end{aligned}$$

Naturally, one can then define for each $n \in \mathbb{N}$, $j = 1, \ldots, m_n$ and $k = 1, \ldots, m_n^I$ the $\mathbb{C}$-valued coefficients

$$x_{n,j} = x_{n,k}^R + \mathrm{i}x_{n,j-(k-1)m_n^R}^I, \qquad \text{if } (k-1)m_n^R < j \leq km_n^R,$$

and the simple function $f_n = \sum_{j=1}^{m_n} x_{n,j} \mathbb{1}_{A_{n,j}}$. Consequently, for each $n \in \mathbb{N}$ and $\omega \in \Omega$ there exists exactly one $j \in \{1, \ldots, m_n\}$ such that $\omega \in A_{n,j}$ and thus $f_n(\omega) = x_{n,j}$.

Fix now some $\epsilon > 0$. As $f_n^R \to f^R$ uniformly for $n \to \infty$ there exist $n_0^R \in \mathbb{N}$ and such that

$$|f_n^R(\omega) - f^R(\omega)| \leq \frac{\epsilon}{2}, \qquad n \geq n_0^R,$$

holds simultaneously for all $\omega \in \Omega$. That implies for each $\omega \in \Omega$ and $n \geq n_0^R$ the existence of a unique $j_n^R(\omega) \in \{1, \ldots, m_n^R\}$ satisfying $\omega \in A_{n,j_n^R(\omega)}^R$ and consequently

$$|x_{n,j_n^R(\omega)}^R - f^R(\omega)| = |f_n^R(\omega) - f^R(\omega)| \leq \frac{\epsilon}{2}, \qquad n \geq n_0^R.$$

Analogously, there exists an n_0^I such that for each $\omega \in \Omega$ and $n \geq n_0^I$ there exists uniquely a $j_n^I(\omega) \in \{1, \ldots, m_n^I\}$ satisfying $\omega \in A_{n,j_n^I(\omega)}^I$ and consequently

$$|x_{n,j_n^I(\omega)}^I - f^I(\omega)| = |f_n^I(\omega) - f^I(\omega)| \leq \frac{\epsilon}{2}, \qquad n \geq n_0^I.$$

Then, as stated before, for each $\omega \in \Omega$ and $n \geq n_0 := \max(n_0^R, n_0^I)$ there exists exactly one $j_n(\omega) \in \{1, \ldots, m_n\}$ such that $\omega \in A_{n,j_n(\omega)}$. Because all three sets $\{A_{n,j} : j = 1, \ldots, m_n\}$, $\{A_{n,j}^R : j = 1, \ldots, m_n^R\}$ and $\{A_{n,j}^I : j = 1, \ldots, m_n^I\}$ consist of pairwise disjoint sets, the equality $A_{n,j_n(\omega)} = A_{n,j_n^R(\omega)}^R \cap A_{n,j_n^I(\omega)}^I$ must hold for each $\omega \in \Omega$ and $n \geq n_0$. Consequently for such a pair (ω, n) follows

$$\begin{aligned}
&|f_n(\omega) - f(\omega)| = |x_{n,j_n(\omega)} - f(\omega)| = |\big(x_{n,j_n^R(\omega)}^R + \mathrm{i}x_{n,j_n^I(\omega)}^I\big) - \big(f^R(\omega) + \mathrm{i}f^I(\omega)\big)| \\
&= |\big(x_{n,j_n^R(\omega)}^R - f^R(\omega)\big) + \mathrm{i}\big(x_{n,j_n^I(\omega)}^I - f^I(\omega)\big)| \leq |x_{n,j_n^R(\omega)}^R - f^R(\omega)| + |x_{n,j_n^I(\omega)}^I - f^I(\omega)| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

which suffices to show the uniform convergence of f_n to f as $n \to \infty$.

The multi-dimensional case can be shown analogously. For simplicity, assume that $\mathbb{K}^d$ is equipped with the norm $\|x\|_1 := \sum_{j=1}^d |x^j|$. $\square$

Obviously, for each $n \in \mathbb{N}$ a simple function $f_n = \sum_{j=1}^{m_n} x_{n,j} \mathbb{1}_{A_{n,j}}$ is bounded by

$$\|f_n\| = \Big\|\sum_{j=1}^{m_n} x_{n,j} \mathbb{1}_{A_{n,j}}\Big\| \leq \big(\max_{j \in \{1,\ldots,d\}} \|x_{n,j}\|\big) \sum_{j=1}^{m_n} \mathbb{1}_{A_{n,j}} \leq \max_{j \in \{1,\ldots,d\}} \|x_{n,j}\| < \infty,$$

where $\|\cdot\|$ denotes any norm on $\mathbb{K}^d$. Thus the following lemma is applicable for a sequence of simple functions converging uniformly.

Lemma 5.36. *Let E be some non-empty set, $(V\|\cdot\|)$ a normed vector space and $(f_n)_{n\in\mathbb{N}}$ a sequence of bounded and uniformly converging functions, where $E \ni x \mapsto f_n(x) \in V$ for each $n \in \mathbb{N}$. Then the sequence is uniformly bounded, i.e. there exists a constant $C > 0$, such that*

$$\|f_n(x)\| \leq C, \qquad x \in E,\ n \in \mathbb{N}. \tag{5.3}$$

Proof. As $(f_n)_{n\in\mathbb{N}}$ is converging uniformly, there exists an $n_0 \in \mathbb{N}$, such that for all $n, m \in \mathbb{N}$ satisfying $n \geq n_0$ and $m \geq n_0$ holds

$$\|f_n(x) - f_m(x)\| \leq 1, \qquad x \in E.$$

Furthermore, as each function f_n is bounded, there exists $C_n > 0$, such that

$$\|f_n(x)\| \leq C_n, \qquad x \in E,\ n \in \mathbb{N}.$$

Consequently,

$$\|f_n(x)\| = \|f_n(x) - f_{n_0}(x) + f_{n_0}(x)\| \leq \|f_n(x) - f_{n_0}(x)\| + \|f_{n_0}(x)\| \leq 1 + C_{n_0}, \qquad x \in E,$$

for all $n \geq n_0$. Thus one may now define $C = \max\{C_1, \dots, C_{n_0-1}, 1+C_{n_0}\}$, which is finite as the maximum of only finitely many real numbers and does indeed fulfill inequality (5.3). □

5.7 Appendix on integral convergence theorems

When working with integrals and limits, being able to interchange them is often very useful. In general, this is not possible. One needs to check, if some conditions are fulfilled. The two most common theorems, stating under which criteria the limit and integral may be exchanged, will be stated below. For the following two theorems, there will be assumed to be an underlying measure space $(\Omega, \mathcal{F}, \mu)$. For the proofs of those theorems, the reader is referred to [BR07, Section 2.8].

Theorem 5.37 (Dominated convergence theorem)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{F}$-measurable, μ-integrable and $\mathbb{K}$-valued functions, such that there exists a $\mathcal{F}$-measurable and μ-integrable $\mathbb{R}_+$-valued function g satisfying*

$$|f_n(\omega)| \leq g(\omega), \qquad n \in \mathbb{N},$$

for μ-almost all $\omega \in \Omega$. If the sequence $(f_n)_{n\in\mathbb{N}}$ converges μ-almost everywhere to a $\mathcal{F}$-measurable function f for $n \to \infty$, then f is μ-almost everywhere $\mathbb{K}$-valued and μ-integrable and

$$\lim_{n\to\infty} \int_\Omega f_n(\omega)\, \mathrm{d}\mu(\omega) = \int_\Omega f(\omega)\, \mathrm{d}\mu(\omega) \quad \text{as well as} \quad \lim_{n\to\infty} \int_\Omega |f_n(\omega) - f(\omega)|\, \mathrm{d}\mu(\omega) = 0.$$

If $(f_n)_{n\in\mathbb{N}}$ converges for all $\omega \in \Omega$ to a function f, then the $\mathcal{F}$-measurability of f follows from [Sch23, Lemma 15.29].

Theorem 5.38 (Monotone convergence theorem)**.** *Let now* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of* $\mathcal{F}$*-measurable and* μ*-integrable* $\mathbb{R}_+$*-valued functions, such that for each* $\omega \in \Omega$

$$f_n(\omega) \leq f_{n+1}(\omega), \qquad n \in \mathbb{N}.$$

Furthermore, assume $\sup_{n\in\mathbb{N}} \int_\Omega f_n(\omega)\,\mathrm{d}\mu(\omega) < \infty$*. Then the pointwise limit* $f(\omega) := \lim_{n\to\infty} f_n(\omega)$ *is again* $\mathcal{F}$*-measurable,* μ*-integrable and* μ*-almost everywhere* $\mathbb{R}_+$*-valued satisfying*

$$\lim_{n\to\infty} \int_\Omega f_n(\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega) \quad \text{as well as} \quad \lim_{n\to\infty} \int_\Omega |f_n(\omega) - f(\omega)|\,\mathrm{d}\mu(\omega) = 0.$$

If one does not require $\sup_{n\in\mathbb{N}} \int_\Omega f_n(\omega)\,\mathrm{d}\mu(\omega) < \infty$ in the last theorem, the pointwise limit f is still $\mathcal{F}$-measurable and

$$\lim_{n\to\infty} \int_\Omega f_n(\omega)\,\mathrm{d}\mu(\omega) = \int_\Omega f(\omega)\,\mathrm{d}\mu(\omega)$$

holds, the integral on the right-hand side as well as $f(\omega)$ may however be infinite [CE15, Theorem 1.3.29].

Notation and Symbols

Notation

The different notations used throughout this thesis below may be found in alphabetical order, where Greek letters are ordered in the way the are spelled in the Latin alphabet.

- A, a $\mathbb{K}^d$-valued process of locally finite variation, see Definition 2.4, or a specified set
- α, an element of $\mathbb{K}$, unless specified otherwise
- $\mathcal{B}_E$, Borel-σ-algebra of the set E, being most of the time a subset of $\mathbb{R}$
- $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$, the product σ-algebra on $\mathbb{R}_+ \times \Omega$
- $C := \sum_{j=1}^{d} [M^j]$, see Theorem 2.7
- $\mathbb{C}$, the field of complex numbers
- *càdlàg*, continue à droite, limite à gauche, meaning right-continuous with left-hand limits at each point of the domain except its smallest point
- δ-ring, see [Sch23, Definition 15.106]
- $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$, Radon–Nikodým derivative of μ w.r.t. ν, see Chapter 4
- e, Euler's number
- $\varnothing := \{\}$, empty set
- $\mathbb{E}[X]$, the expectation of a measurable random variable X, i.e. $\int_\Omega X \,\mathrm{d}\mathbb{P}$
- $e_j \in \{0,1\}^d$, j-th unit vector consisting of zeros except for a one in the j-th entry
- $\mathcal{F}$, σ-algebra (most of the time on Ω)
- $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, filtration of $\mathcal{F}$, which is per assumption right-continuous and contains all $\mathbb{P}$-null sets of $\mathcal{F}_\infty$
- H, a predictable process, unless stated otherwise
- $\mathcal{H}^2$, Banach space of all $\mathbb{K}$-valued continuous martingales M, for which $\|M\|_{\mathcal{H}^2} := \mathbb{E}[\sup_{t \in \mathbb{R}_+} |M_t|^2]^{1/2} < \infty$ holds, see Definition 3.4
- $\mathcal{H}^2_0$, Hilbert space of all processes in $\mathcal{H}^2$ starting at zero combined with the norm $\|M\|_{\mathcal{H}^2_0}$, see Lemma 3.6 and Lemma 3.7

- $\mathrm{i} := \sqrt{-1}$, imaginary unit
- $\mathrm{Im}(z) = y$, the imaginary part of the complex number $z = x + \mathrm{i}y$
- I_d, the $(d \times d)$-dimensional identity matrix
- *i.e.*, abbreviation of the Latin phrase *id est*, being used as *which means* or *in other words*
- $\mathbb{K}$: $\mathbb{R}$ or $\mathbb{C}$
- $L(A)$, vector space of predictable processes that are integrable with respect to a $\mathbb{K}^d$-valued continuous process of locally finite variation A, see Definition 3.15
- $\mathcal{L}^p(\Omega, \mathcal{F}, \nu)$, space of all $\mathcal{F}$-measurable functions satisfying $\|f\|_{L^p(\nu)} := \int_\Omega |f|^p \,\mathrm{d}\nu < \infty$
- $L^p(\Omega, \mathcal{F}, \nu)$, Banach space of all equivalence classes in $\mathcal{L}^p(\Omega, \mathcal{F}, \nu)$ w.r.t. the norm $\|\cdot\|_{L^p(\nu)}$
- $L^p(M)$, see Definition 2.9
- $L^2_{\mathrm{loc}}(M)$, vector space of predictable processes that are integrable with respect to a $\mathbb{K}^d$-valued continuous local martingale M, see Definition 2.11
- $L(X)$, vector space of predictable processes that are integrable with respect to a $\mathbb{K}^d$-valued continuous semimartingale X, see Definition 3.25
- $\liminf_{n\to\infty} \alpha_n := \sup_{k\in\mathbb{N}} \inf_{n\geq k} \alpha_n$, limes inferior of a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$
- $\limsup_{n\to\infty} \alpha_n := \inf_{k\in\mathbb{N}} \sup_{n\geq k} \alpha_n$, limes superior of a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$
- $\liminf_{n\to\infty} A_n := \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} A_n$, limes inferior of a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of Ω
- $\limsup_{n\to\infty} A_n := \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_n$, limes superior of a sequence $(A_n)_{n\in\mathbb{N}}$ of subsets of Ω
- M, a $\mathbb{K}^d$-valued continuous local martingale, unless specified otherwise, see [Sch23, Definition 4.132]
- $\mathcal{M}$, vector space of all $\mathbb{K}$-valued continuous martingales
- $\mathcal{M}_{\mathrm{loc}}$, vector space of all $\mathbb{K}$-valued continuous local martingales
- μ, (signed or complex) measure one a σ-algebra or δ-ring
- $\mathbb{N}$, set of natural numbers, i.e. $\{1, 2, 3, \dots\}$
- ν, (signed or complex) measure one a σ-algebra or δ-ring
- Ω, underlying sample space
- ω, an element of Ω

- $(\Omega, \mathcal{F}, \mathbb{P})$, probability space
- $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, filtered probability space
- $\mathbb{P}$, probability measure (most of the time on Ω)
- $\mathcal{P}^d$, vector space of predictable processes, see Definition 2.1
- $\mathcal{P}(\Omega)$, power set, i.e. set of all subsets of Ω, sometimes also denoted by 2^Ω
- π, a positive semidefinite process, according to Theorem 2.7
- *positive semidefinite process*, see Definition 2.6
- *predictable step process* $H = \varphi_0 \mathbb{1}_0 + \sum_{n=1}^{m} \varphi_n \mathbb{1}_{(\tau_n, \tau_{n+1}]}$, see Definition 2.2
- $\mathbb{Q}_+$, the set of non-negative rational numbers
- $\mathbb{R}$, the field of real numbers
- $\overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$, the extended real numbers
- $\mathbb{R}_+ = [0, \infty)$, the set of real numbers greater than or equal to zero
- $\overline{\mathbb{R}}_+ = [0, \infty] = \mathbb{R}_+ \cup \{\infty\}$
- $\mathcal{R}$, a δ-ring, see [Sch23, Definition 15.106]
- $\mathrm{Re}(z) = x$, the real part of the complex number $z = x + \mathrm{i}y$
- $\mathcal{S}$, vector space of all $\mathbb{K}$-valued semimartingales
- Σ_p, predictable σ-algebra, see Definition 2.1
- *simple function*, see Lemma 5.35
- *stopping time* $\tau : \Omega \to \overline{\mathbb{R}}_+$, such that $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \in \mathbb{R}_+$, see [Sch23, Definition 3.7]
- t, an element of $\mathbb{R}_+$, unless stated otherwise
- τ, a stopping time, unless specified otherwise
- $\mathrm{tr}(\pi) = \sum_{j=1}^{d} \pi^{jj}$, the trace of a $(d \times d)$-dimensional matrix π
- *transition kernel*, see Definition 5.23
- *up to indistinguishability*, two stochastic processes X and Y are equal up to indistinguishability, if and only if $\bigcup_{t\in\mathbb{R}_+} \{X_t \neq Y_t\}$ is a subset of some $\mathbb{P}$-null set, see [Sch23, Definition 2.96(b)]
- $\mathbb{V}_F$, the total variation of a function F, see [Sch23, Definition 5.20]

- $\mathcal{V}_0^+$: the space of all adapted, continuous, real-valued and non-decreasing processes, see Definition 2.5
- $\mathcal{V}^d$: the space of all $\mathbb{K}^d$-valued continuous adapted processes, of locally finite variation, see Definition 2.4
- $\mathcal{V}_0^d$: the space of all processes in $\mathcal{V}^d$ starting at zero, see Definition 2.4
- *w.r.t.*, abbreviation of *with respect to*
- X, a $\mathbb{K}^d$-valued continuous semimartigale with canonical decomposition $X = A + M$, unless specified otherwise, see [Sch23, Definition 5.105]
- $\mathbb{1}_A$, indicator function of the set A

Symbols

- $X \bullet Y$, integral process of X w.r.t. Y, if it exists
- $H \bullet M$, where M denotes a $\mathbb{K}^d$-valued predictable step process, is the stochastic integral of H w.r.t. M, according to Definition 3.1, Definition 3.8 and Definition 3.12, depending on if H is a $\mathbb{K}^d$-valued predictable step process, an element of $H^2(M)$ or an element of $H^2_{\text{loc}}(M)$, respectively
- $H \bullet A$, where $A \in \mathcal{V}_0^d$ is the stochastic integral of H w.r.t. A, according to Definition 3.15
- $[M, N]$, covariation process of the $\mathbb{K}^d$-valued continuous local martingales M and N, see Definition 1.5
- $[M] = [M, \bar{M}]$, covariation process of the $\mathbb{K}^d$-valued continuous local martingale M
- For a signed or complex measure μ let $|\mu|$ denote the total variation measure, which is not to be confuse with $|\mu(A)|$, which is the absolute value of $\mu(A)$ for some set A, see Definition 4.10
- $\ll$, absolute continuous
- A^{c}, the complement of a set A
- π^{T}, the transpose of a $\mathbb{K}^{n\times d}$-valued matrix π, i.e. $\pi^{ij} = (\pi^{\mathsf{T}})^{ji}$ for $(i,j) \in \{1,\dots,n\} \times \{1,\dots,d\}$
- π^{H}, the Hermitian adjoint of a $\mathbb{K}^{n\times d}$-valued matrix π, i.e. $\pi^{ij} = (\bar{\pi}^{\mathsf{T}})^{ji}$ for $(i,j) \in \{1,\dots,n\} \times \{1,\dots,d\}$
- $\|\cdot\|_{L^p(M)}$, see Definition 2.9
- $\|\cdot\|_p$, the p-norm on $\mathbb{K}^d$ or an L^p-norm, depending on the context and the input

- $\bar{z} := a - ib$, the complex conjugate of a complex number $z = a + ib$
- $|z| := \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$, the absolute value of a complex number $z = a + ib$
- $[0, \tau] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega : t \leq \tau(\omega)\}$, the stochastic interval for a stopping time τ.
- $\subseteq$, non-strict subset
- $\supseteq$, non-strict superset
- $\cup, \bigcup$, union of two or more sets
- $\cap, \bigcap$, intersection of two or more sets
- $\times$, Cartesian product of two sets
- $\otimes$, product of σ-algebra, δ-rings, (possibly signed or complex) measures or between a σ-finite transition kernel or a signed or transition kernel and a measure, according to Lemma 5.24 and Lemma 5.26, respectively
- $\wedge$, minimum of two real numbers or pointwise minimum of two stopping times
- $\|\cdot\|$, unspecified norm on some vector space
- $\langle \cdot, \cdot \rangle$, unspecified inner product in some vector space
- $\equiv$, when a function is constantly equal to some value, for example $f \equiv a \in \mathbb{R}$
- $f\restriction_A$, where f is a function and A a subset of its domain, is the restriction of f on A
- $(f \circ g)(\omega) := f\big(g(\omega)\big)$, composition of two functions f and g
- $\mapsto$, defines a function, where the left side is mapped to the right side

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