A CHARACTERIZATION FOR SOLUTIONS TO AUTONOMOUS OBSTACLE PROBLEMS WITH GENERAL GROWTH

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Oberseminar Analysis Eichstätt (27th June 2023)

SFB PDE Taming Complexity in Partial Differential Systems

TECHNISCHE UNIVERSITÄT WIEN

Motivation

S. Riccò, A. Torricelli: "A characterization for solutions to autonomous obstacle problems with general growth", arXiv (2023)

Standard growth conditions	Non-standard growth conditions
$\Downarrow$	$\Downarrow$ (?)
Minimizers are extremals	Minimizers are extremals
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Regularity of the solution(s)	Regularity of the solution(s)

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The Standard Obstacle Problem

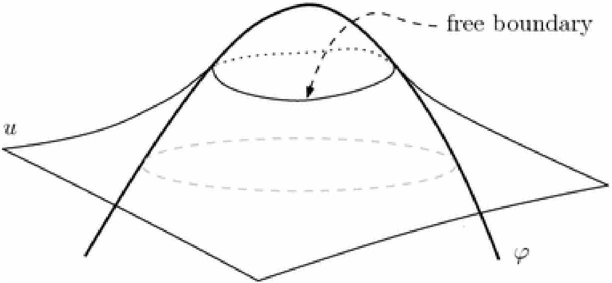

Figure: The obstacle φ, the solution u, and the free boundary $\partial\{u = \varphi\}$.

G. Fichera (1963) - elastostatics
G. Stampacchia (1964) - electrostatics
J.L. Lions, G. Stampacchia (1967) - concept of variational inequality

Figure by X. Ros-Oton (2018).

The Standard Obstacle Problem - II

Given a domain $\Omega \subset \mathbb{R}^n$, we seek to minimize the Dirichlet energy

$$J(v) := \int_\Omega |\nabla v|^2 \, dx,$$

considering functions v in the class

$$\mathcal{K} := \left\{ v \in W^{1,2}(\Omega) \quad \text{s.t.} \quad v\big|_{\partial\Omega} = f, \ v \geq \varphi \right\}.$$

We also need to assume a Compatibility Condition, in particular

$$\varphi \leq f \quad \text{on} \ \partial\Omega.$$

Since $J(v)$ is continuous and strictly convex on the convex set $\mathcal{K}$, the existence and uniqueness of a minimizer are guaranteed. We can also characterize this solution by means of a Variational Inequality, i.e.

$$\int_\Omega \nabla u \cdot \nabla (v - u) \, dx \geq 0 \qquad \forall \, v \in \mathcal{K}.$$

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Regularity of the Solutions

Remark 1

We may not get $u \in \mathcal{C}^2$ regardless how regular the set Ω and the obstacle φ are.

As a Counterexample, let $n = 2$ and consider the obstacle problem

$$\min_{v \geq \varphi} \int_\Omega |\nabla v|^2 \, dx \, dy,$$

where

$$\begin{cases} \Omega = \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < 4\}, \\ \varphi(x,y) = 1 - x^2 - y^2. \end{cases}$$

Define

$$S := \{(x,y) \in \mathbb{R}^2 \ : \ v(x,y) = \varphi(x,y)\} \quad \text{and} \quad \Lambda := \Omega \setminus S.$$

Then

- in S we have $\Delta v = \Delta \varphi = -4$,
- in Λ we have that double-sided variations are allowed, so $\Delta v = 0$.

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Regularity of the Solutions - II

Remark 2

The regularity of solutions to obstacle problems is influenced by the one of the obstacle.

- "Smooth" obstacle: $\varphi \in \mathcal{C}^2 \implies v \in W^{2,\infty}$,
- Lipschitz obstacle: $\varphi \in W^{1,\infty} \implies v \in W^{1,\infty}$,
- Hölder continuous obstacle: $\varphi \in \mathcal{C}^{0,\beta} \implies v \in \mathcal{C}^{0,\alpha}$, with $\alpha = \alpha(\beta)$,
- obstacle with Hölder continuous gradient: $\nabla\varphi \in \mathcal{C}^{0,\beta} \implies \nabla v \in \mathcal{C}^{0,\alpha}$,
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State of the Art

M. Carozza, J. Kristensen, A. Passarelli di Napoli -
Ann. Sc. Norm. Super. Pisa Cl. Sci. (2014)

Theorem (Carozza, Kristensen, Passarelli di Napoli - 2015)

Let $F : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ *be a* $\mathcal{C}^1$ *convex integrand satisfying*

$$F(\xi) \geq \theta(|\xi|)$$

for all $\xi \in \mathbb{R}^{N\times n}$*, where* $\theta : [0,+\infty) \longrightarrow [0,+\infty)$ *is an increasing, convex and superlinear function. Let* $g \in W^{1,1}(\Omega, \mathbb{R}^N)$ *with* $F(sDg) \in L^1(\Omega)$ *for some number* $s > 1$*. Then minimizers* u *in* $W^{1,1}_g(\Omega, \mathbb{R}^N)$ *of the functional*

$$\mathcal{F}(v,\Omega) := \int_\Omega F(Dv(x))\,dx$$

are characterized by the conditions

$$F^*(F'(Du)) \in L^1(\Omega), \qquad \langle F'(Du), Du\rangle \in L^1(\Omega)$$

and by $\operatorname{div} F'(Du) = 0$ *in the distributional sense in* Ω*.*

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M. Eleuteri, A. Passarelli di Napoli - Ann.Acad.Sci. Fenn.Math. (2022)

$$\min\left\{\int_\Omega F(Dz) \quad \text{s.t.} \quad z\in\mathbb{K}^F_\psi(\Omega)\right\}, \tag{P1}$$

- $\Omega\subset\mathbb{R}^n$ is open and bounded with $n\geq 2$,
- $F:\mathbb{R}^n\longrightarrow\mathbb{R}$ is a function of class $\mathcal{C}^1$ and satisfies the hypotheses

$$l|\xi|^p\leq F(\xi)\leq L\left(1+|\xi|^q\right), \tag{A1}$$

$$\nu|V_p(\xi)-V_p(\eta)|^2\leq F(\xi)-F(\eta)-\langle F'(\eta),\xi-\eta\rangle, \tag{A2}$$

$$F(\lambda\xi)\leq C(\lambda)F(\xi), \tag{A3}$$

for all $\xi,\eta\in\mathbb{R}^n$ and $\lambda>1$, where $0<l<L$, $\nu>0$, $1<p\leq q<\infty$ and

$$V_p(\xi):=\left(1+|\xi|^2\right)^{\frac{p-2}{4}}\xi.$$

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$$V_p(\xi) := \left(1 + |\xi|^2\right)^{\frac{p-2}{4}} \xi.$$

State of the Art - III

The function $\psi \longrightarrow [-\infty, +\infty)$ is the obstacle and it is such that

$$\psi \text{ is } \mathrm{Cap}_q\text{–quasi–continuous,} \tag{O1}$$

where q is the exponent in (A1). Moreover, given $u_0 \in W^{1,q}(\Omega)$ a fixed boundary value, there exists a function $g \in u_0 + W_0^{1,q}(\Omega)$ with

$$\psi \leq \widetilde{g} \qquad \mathrm{Cap}_q\text{–a.e. on } \Omega. \tag{O2}$$

where $\widetilde{g}$ denotes the precise representative of the function g. The class of admissible functions is defined as

$$\mathbb{K}_\psi^F(\Omega) := \left\{ z \in u_0 + W_0^{1,p}(\Omega) \ : \ \widetilde{z} \geq \psi \ \mathrm{Cap}_q\text{–a.e. on } \Omega, \ F(Dz) \in L^1(\Omega) \right\}.$$

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State of the Art - IV

Theorem (Eleuteri, Passarelli di Napoli - 2022)

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a $\mathcal{C}^1$ function satisfying (A1), (A2) *and* (A3)*. Assume moreover that* (O1) *and* (O2) *hold true. If $u \in \mathbb{K}^F_\psi(\Omega)$ is the solution to the obstacle problem* (P1)*, then*

$$F^*(F'(Du)) \in L^1(\Omega), \qquad \langle F'(Du), Du\rangle \in L^1(\Omega)$$

and

$$\operatorname{div} F'(Du) \leq 0 \qquad \text{in the distributional sense.}$$

Moreover, it holds the following

$$\int_\Omega F(Du)\, dx = [\![F'(Du), \psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega F^*(F'(Du))\, dx.$$

- (A3) can be exchanged for $F(cDu_0) \in L^1(\Omega)$, with $c > 1$,
- (O2) has the role of a compatibility condition and implies that $\mathbb{K}_\psi(\Omega)$ is not empty.

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Formulation of the Problem

We consider the obstacle problem

$$\min\left\{\int_\Omega F(Dv(x))\,dx \quad \text{s.t.} \quad v\in\mathbb{K}_\psi(\Omega)\right\}, \tag{P}$$

where

- $\Omega\subset\mathbb{R}^n$ is open and bounded,
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Hypotheses

We consider $\phi : \mathbb{R}^n \longrightarrow [0, +\infty)$ of class $\mathcal{C}^1$ and strictly convex such that

$$\phi(\xi) := \theta(|\xi|) \qquad \forall\, \xi \in \mathbb{R}^n,$$

where $\theta : [0, +\infty) \longrightarrow [0, +\infty)$ is a superlinear function at infinity.
We suppose that

$$F - \phi \qquad \text{is a convex function.} \tag{H1}$$

This implies that there exist $C \in \mathbb{R}$ and $M > 0$ such that

$$F(\xi) - \frac{1}{2}\,\phi(\xi) \geq C \qquad \forall\, \xi \in \mathbb{R}^n \text{ with } |\xi| \geq M,$$

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Main Theorem

Theorem (R., Torricelli - 2023)

Let F be a non-negative function of class $\mathcal{C}^1$, satisfying (H1) *with ϕ defined as before, and let $u_0 \in W^{1,1}(\Omega)$ be such that $F(Du_0), F(tDu_0) \in L^1(\Omega)$ for some $t > 1$. Then, the minimizer $u \in W^{1,1}_{u_0}(\Omega)$ of the minimization problem* (P) *is characterized by*

$$F^*(F'(Du)) \in L^1(\Omega), \qquad \langle F'(Du), Du\rangle \in L^1(\Omega) \tag{1}$$

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An Additional Definition

C. Scheven, T. Schmidt - J. Differential Equations (2016)
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Given $\sigma \in L^1(\Omega, \mathbb{R}^n)$ and $U \in W^{1,1}(\Omega)$, let us define the measure $[\![\sigma, U]\!]_{u_0}$ as

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$$\min_{v \in \mathbb{K}_\psi(\Omega)} \int_\Omega G(Dv)\, dx = \max_{\sigma \in L^1(\Omega, \mathbb{R}^n)} \left([\![\sigma, \psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega G^*(\sigma)\, dx \right).$$

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Outline of the Proof

1. Approximation

Constructing a sequence of approximating functions $(F_k)_k$.

2. Perturbed Problems

Considering sequences of approximating problems, since the one generated from $(F_k)_k$ lose the necessary properties.

3. Variational Inequality and Passage to the Limit

Proving a variational inequality for the perturbed problems and the convergence of the sequences we are considering.

4. Validity of the Characterization

Proving the validity of the theses, i.e. (1), (2) and (3).

Approximating Sequence

We can build the approximating sequence $(F_k)_k$ such that

- $F_k(\xi) \leq F_{k+1}(\xi)$ for all $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$,
- $F_k \nearrow F$ pointwise as $k \to \infty$,
- F_k are of class $\mathcal{C}^\infty$, Lipschitz continuous and convex for all $k \in \mathbb{N}$.

By construction though, the functions F_k are NOT superlinear at infinity for any $k \in \mathbb{N}$, which means that we cannot guarantee the existence of solutions of

$$\min \left\{ \int_\Omega F_k(Dv(x))\, dx \quad \text{s.t.} \quad v \in \mathbb{K}_\psi(\Omega) \right\}.$$

Auxiliary Result

Theorem (Ekeland Variational Principle for Metric Spaces)

Let (V, d) be a complete metric space and $J : V \to \mathbb{R}$ be a lower semicontinuous functional bounded from below. Given $\varepsilon > 0$ and $w \in V$ such that

$$J(w) \leq \inf_V J + \varepsilon,$$

then, for every $\lambda > 0$, there exists $v_\lambda \in V$ such that

i) $d(w, v_\lambda) \leq \lambda$,

ii) $J(v_\lambda) \leq J(w)$,

iii) v_λ *is the unique minimizer of the functional*

$$w \longmapsto J(w) + \frac{\varepsilon}{\lambda} d(w, v_\lambda).$$

- $V = \mathbb{K}_\psi(\Omega)$ is a complete metric space if endowed with the norm

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Right Functionals

$$I_k(v) := \int_\Omega F_k(Dv)\,dx \qquad \text{and} \qquad I(v) := \int_\Omega F(Dv)\,dx.$$

- We know that

$$\inf_{v\in\mathbb{K}_\psi(\Omega)} I(v) = I(u) = \int_\Omega F(Du)\,dx,$$

- since $F_k \nearrow F$ pointwise as $k \to \infty$, then $I_k \leq I$ pointwise for all $k \in \mathbb{N}$,
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$\Longrightarrow$ $v_k \longrightarrow u \in \mathbb{K}_\psi(\Omega)$ in $W^{1,1}(\Omega)$ (using generalized Young measures),

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Use of Ekeland Variational Principle

There exists $(\varepsilon_k)_k \subset \mathbb{R}$ such that $\varepsilon_k \to 0$ and, for every $k \in \mathbb{N}$,

$$\int_\Omega F(Du)\,dx = \varepsilon_k^2 + \inf_{v \in \mathbb{K}_\psi(\Omega)} \int_\Omega F_k(Dv)\,dx.$$

If we notice that

$$\inf_{v \in \mathbb{K}_\psi(\Omega)} \int_\Omega F_k(v)\,dx \leq \int_\Omega F_k(Du)\,dx \leq \int_\Omega F(Du)\,dx,$$

we can make use of Ekeland Variational Principle for every step $k \in \mathbb{N}$, getting that there exists a sequence $(u_k)_k \subset \mathbb{K}_\psi(\Omega)$ such that

i) $u_k \longrightarrow u$ in $W^{1,1}(\Omega)$, since $d(u, u_k) \leq \varepsilon_k$ for every $k \in \mathbb{N}$,

ii) for every $k \in \mathbb{N}$ it holds

$$\int_\Omega F_k(Du_k)\,dx \leq \int_\Omega F_k(Du)\,dx \leq \int_\Omega F(Du)\,dx,$$

iii) for every $k \in \mathbb{N}$, u_k is the unique minimizer of

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Variational Inequality and Convergence

We define, for every $k \in \mathbb{N}$,

$$\sigma_k := F_k'(Du_k) \qquad \text{and} \qquad \sigma^* := F'(Du),$$

and it holds, for every $k \in \mathbb{N}$, the following Variational Inequality, thanks to (iii):

$$\int_\Omega \langle \sigma_k, \, D\eta - Du_k \rangle \, dx \geq -\varepsilon_k \left(C + \int_\Omega |D\eta| \, dx \right) \qquad \forall \, \eta \in \mathbb{K}_\psi(\Omega).$$

Moreover, thanks to (i),

- $\sigma_k \longrightarrow \sigma^*$ locally uniformly and in measure on Ω as $k \to \infty$,
- the two following pointwise extremality relations hold

$$\langle \sigma_k, Du_k \rangle = F_k^*(\sigma_k) + F_k(Du_k), \qquad \forall \, k \in \mathbb{N}$$

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Characterization

$$F^*(F'(Du)) \in L^1(\Omega) \quad \Longrightarrow \quad \langle F'(Du), Du\rangle \in L^1(\Omega)$$

Integrating the first pointwise extremality relation over Ω and using the Variational Inequality with $\eta = u_0 \in \mathbb{K}_\psi(\Omega)$, it holds

$$\int_\Omega F_k^*(\sigma_k)\,dx \le \overline{C}\left(\int_\Omega F(tDu_0)\,dx - \int_\Omega F(Du_0)\,dx + \int_\Omega |Du_0|\,dx + C\right) < \infty,$$

where $t > 1$ and where we used the properties of the F_k and thesis (ii) of Ekeland Variational Principle. Since $F_k^* \searrow F^*$ as $k \to \infty$ then

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$$\int_\Omega \langle \sigma^*, D\varphi \rangle \, dx \geq 0 \qquad \forall \, \varphi \in \mathcal{C}_0^\infty(\Omega), \; \varphi \geq 0 \quad \Longrightarrow \quad \operatorname{div} \sigma^* \leq 0$$

Lemma

If $F : \mathbb{R}^n \to \mathbb{R}$ is convex, then its polar $F^* : \mathbb{R}^n \to \mathbb{R}$ is superlinear at infinity.

There exists $\overline{\theta} : [0, +\infty) \to [0, +\infty)$ increasing, convex and superlinear such that

$$\overline{\theta}(|\xi|) \leq F^*(\xi) \leq F_k^*(\xi) \qquad \forall \, \xi \in \mathbb{R}^n,$$

$$\Longrightarrow \quad \sup_{k \in \mathbb{N}} \int_\Omega \overline{\theta}(|\sigma_k|) \, dx \leq \int_\Omega F_k^*(\sigma_k) \, dx < \infty.$$

Using De La Vallè-Poussin Theorem, this yields that the $(\sigma_k)_k$ are equi-integrable. Thanks to Vitali Convergence Theorem and the convergence in measure of $(\sigma_k)_k$ to σ^*, we have that $(\sigma_k)_k$ converges to σ^* also in $L^1(\Omega)$.

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Possible Generalizations

- Removing the positivity assumption on F.
- Weakening the hypotheses on the obstacle ψ.

- Vectorial problem.

THANK YOU FOR YOUR ATTENTION!