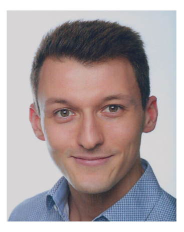

# Finite Element and Isogeometric Stabilized Methods for the Advection-Diffusion-Reaction Equation [1]

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## Background

**Demand:**

The linear, steady **advection-diffusion-reaction (ADR) equation** has proven to be a challenge for numerical simulations: e.g., the Galerkin finite element method may suffer from **instabilities** associated with reaction and advection as well as their interaction. State-of-the-art stabilization approaches [2] are not globally conservative, a desirable property of numerical methods. In addition, higher-order basis functions have not been considered to date.

**Contribution:**

The **GSC stabilization** approach yields finite element and isogeometric stabilized methods that are globally conservative. The proposed stabilization parameters are motivated by exact solutions of homogeneous problems in one dimension and linear basis functions. Concerning the generalization for multidimensional problems, **streamline** and **directional** GSC methods are distinguished. In addition, B-spline elements surpass Lagrange finite elements.

## Need for Stabilization

The steady ADR differential operator reads

$$\mathcal{L}u = ru + \mathbf{a} \cdot \nabla u - \nabla \cdot (\kappa \nabla u)$$

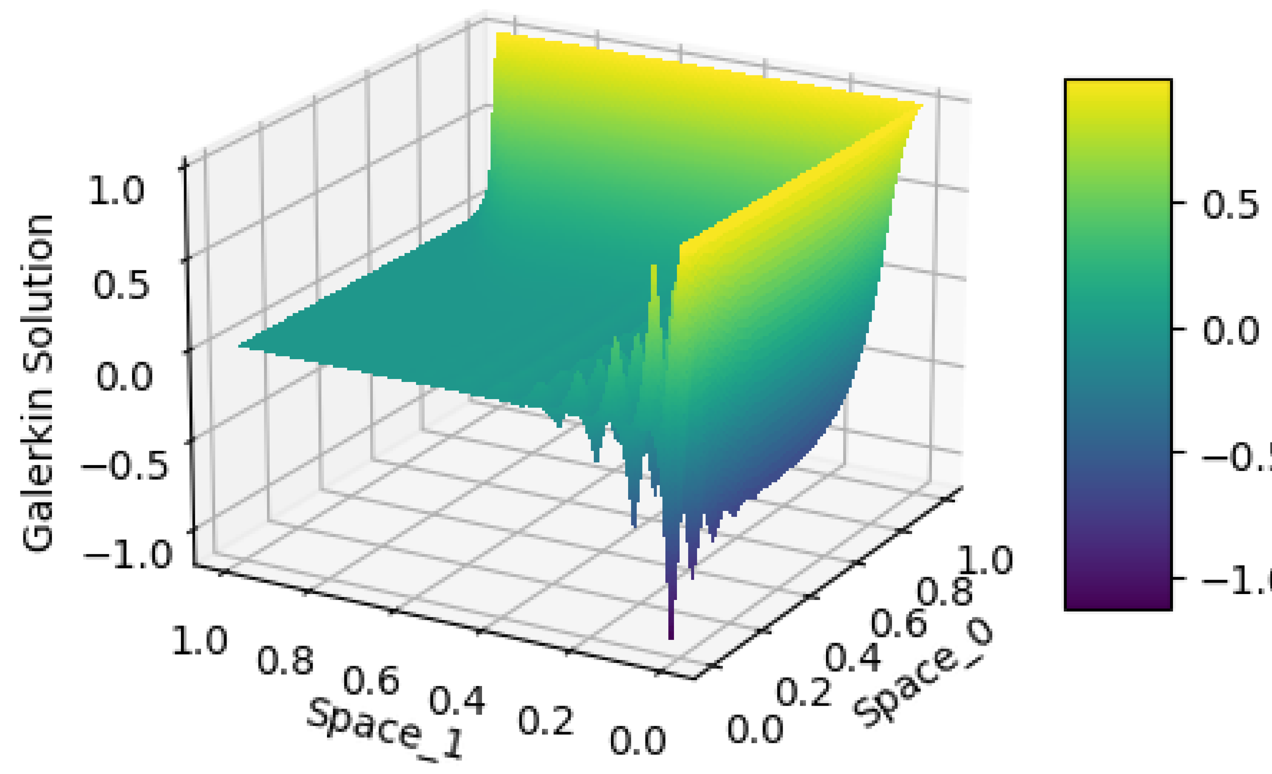
and is used to define nonhomogeneous ADR problems on domains $\Omega$ with boundaries $\Gamma$

$$\begin{aligned} RES(u) = \mathcal{L}u - f = 0 \quad &\text{in} \quad \Omega, \\ u = g \quad &\text{on} \quad \Gamma. \end{aligned}$$

ADR problems with, e.g., sharp boundary layers cannot always be reliably solved by Galerkin methods: unphysical oscillations may occur. Therefore, **two residual-based stabilization terms** are added in GSC methods:

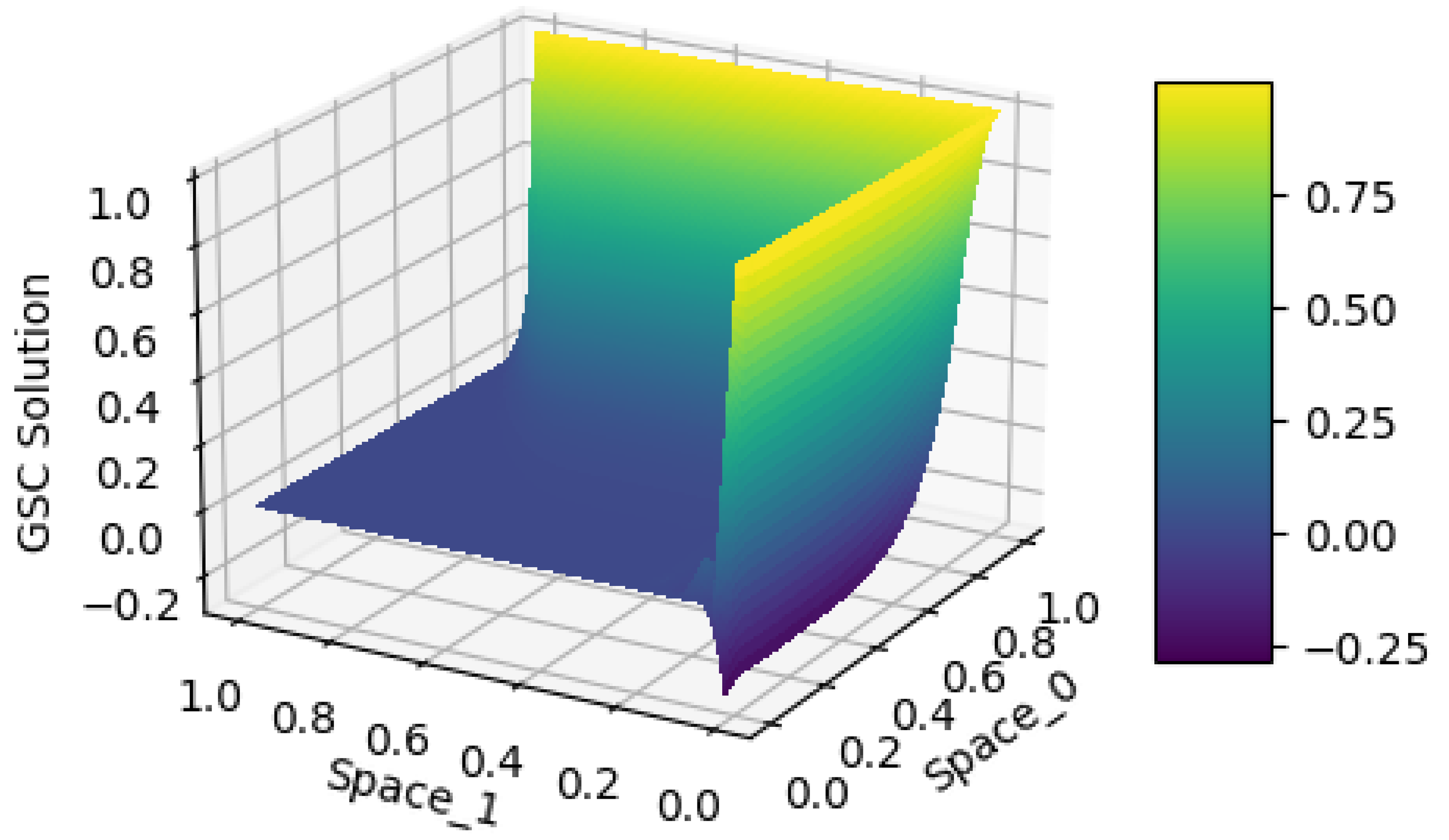
$$(\tau_{\mathbf{R}} \nabla(rw^h), \nabla RES(u^h))_{\tilde{\Omega}} \quad \text{and} \quad (\tau_{\mathbf{A}} \mathbf{a} \cdot \nabla w^h, RES(u^h))_{\tilde{\Omega}}.$$

Analytical solutions of homogeneous, one-dimensional ADR problems allow to find stabilization parameters yielding nodal exactness for linear basis functions.

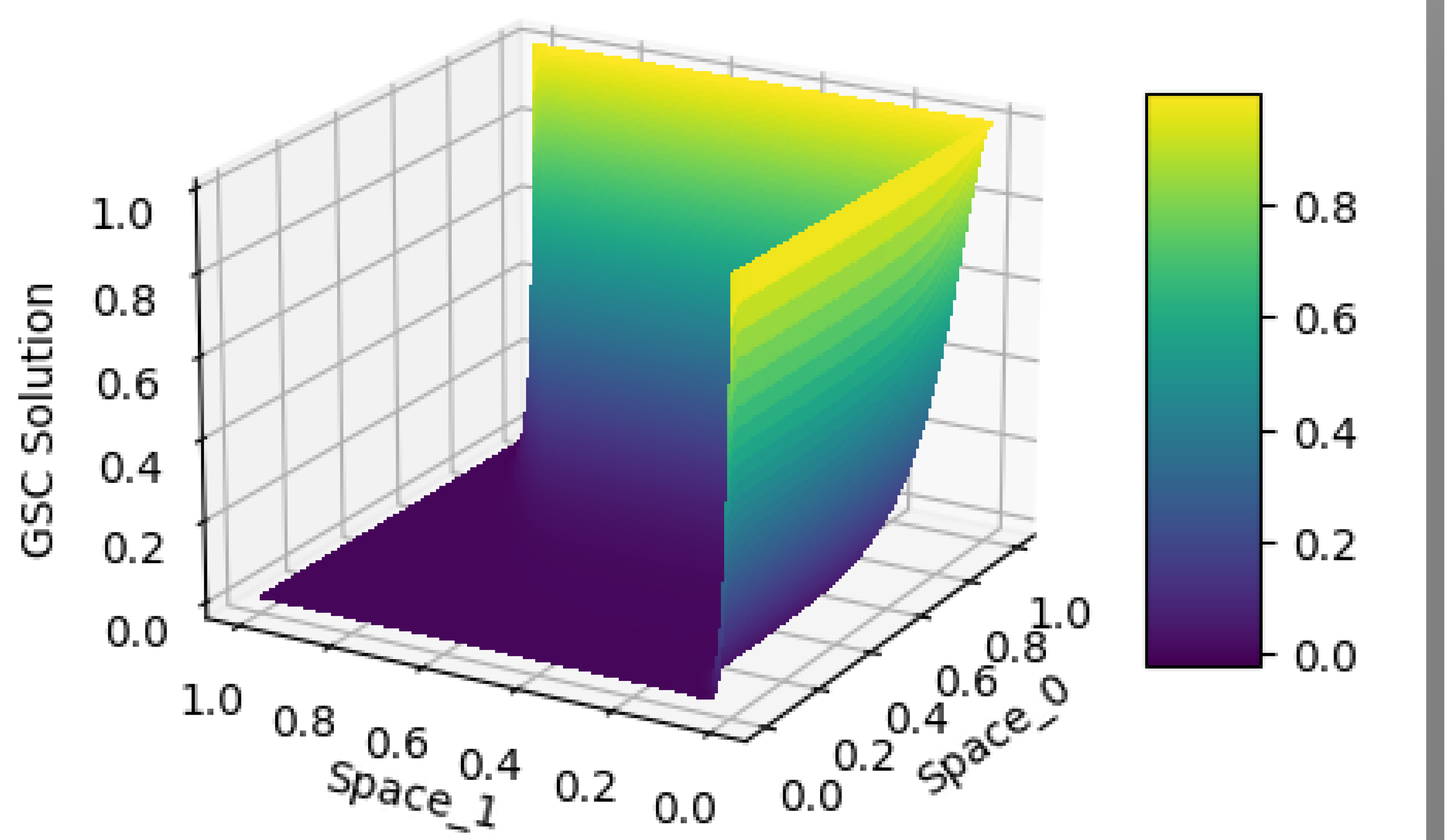



## Generalizations for Multidimensional Problems

In **streamline** GSC methods, $\tau_{\mathbf{R}}$ and $\tau_{\mathbf{A}}$ are generalized in the **classical** way: the velocity is represented by its Euclidean norm in the streamline operator.




In **directional** GSC methods, $\tau_{\mathbf{R}}$ and $\tau_{\mathbf{A}}$ are generalized in a **novel** way: diagonal matrices are transformed from element principal to Euclidean axes.




## Take-Home Messages

- The **GSC stabilization** approach yields globally-conservative methods for the **ADR** equation and its parameters are based on exact solutions.
- **Streamline** GSC methods are obtained by the classical multidimensional generalization, whereas **directional** GSC methods result from a novel one.
- B-splines consistently outperform Lagrange elements of the same order when the same number of unknowns is used in numerical studies.

## References

[1] K. Key, M.R.A. Abdelmalik, S. Elgeti, T.J.R. Hughes, and F.A. Baidoo, Finite Element and Isogeometric Stabilized Methods for the Advection-Diffusion-Reaction Equation, *Compututer Methods in Applied Mechanics and Engineering*, **116354** (2023)

[2] G. Hauke, G. Sangalli, and M.H. Doweidar, Combining Adjoint Stabilized Methods for the Advection-Diffusion-Reaction Problem, *Mathematical Models and Methods in Applied Sciences*, **17**(02):305–326 (2007)

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This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 33849990/GRK2379 (IRTG Modern Inverse Problems).