DIPLOMARBEIT

Hamiltonian surface charges in 3d Einstein gravity

Differences in metric and Chern-Simons formulation with selected examples

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Abstract

Hamiltonian surface charges are introduced in terms of the covariant phase space formalism, then they are computed for Einstein gravity in three spacetime dimensions. In general this gives a different result depending on whether one uses the second order (metric) or first order (Cartan or Chern-Simons) formulation. This is studied explicitly for three different spacetime examples, only in one of which the difference between formalisms becomes apparent.

Contents

Introduction

The concept of Hamiltonian surface charges in the context of general relativity and holography has received a lot of attention in recent years. The foundations, however, go back to 1986, when Brown and Henneaux released their seminal paper [1] showing that the Hamiltonian charges obtained in the covariant phase space formalism correspond to generators of asymptotic symmetries and that their Dirac bracket algebra is isomorphic to the Lie algebra of the corresponding diffeomorphism generators, up to additional central charges. Finite and nonzero Hamiltonian charges are related to symmetry transformations between distinct physical states, all obeying some kind of boundary conditions that need to be specified. As a specific example Brown and Henneaux considered asymptotically Anti-de Sitter spacetimes in three dimensions (AdS_3) and found that the charges obey the centrally extended Witt algebra, also known as Virasoro algebra, familiar from conformal field theory in two dimensions (CFT_2). This precursor of the AdS/CFT correspondence was formulated more than a decade before Maldacena's famous paper [2].

Three is the lowest number of dimensions where Einstein gravity exists, but it is also particularly simple as there are no bulk degrees of freedom. Instead the theory features "topological" or "surface" degrees of freedom at the (possibly asymptotic) boundary. For asymptotically AdS_3 spacetimes Bañados, Teitelboim and Zanelli found a black hole solution, now referred to as BTZ black hole [3]. This provided researchers with a simplified black hole model to tackle the information loss problem and gain insights about quantum gravity. The BTZ black hole was later found to be a special representative of a class of asymptotically AdS_3 spacetimes called the Bañados geometries [4]. They feature two state-dependent functions and as many towers of charges. In later years more general AdS_3 boundary conditions were found, featuring four and six state-dependent functions [5, 6, 7]. One can, instead of the asymptotic region, also consider the symmetries and their associated charges at the (outer) horizon of the BTZ black hole [8, 9]. This leads to a different algebra that can be used to define and count microstates of the BTZ black hole [10]. In all of these cases the Hamiltonian surface charges allow for a systematic treatment of symmetries that are compatible with certain boundary conditions.

On a technical level, Einstein gravity in three spacetime dimensions can be treated in different mathematical formalisms. The metric formalism features the metric as the only field content of the theory (assuming no coupling to matter). In the Cartan formalism the metric is replaced by the vielbein and spin connection. If both are treated as individual fields, there are no second derivatives in the action, hence this formalism is also referred to as the first order formulation of Einstein gravity. In contrast, the metric formalism is referred to as second order formalism. The methods of the covariant phase space formalism allow to extract expressions for the Hamiltonian surface charges in both formalisms, however the results do, in general, not agree. This was recently given formal treatment in four spacetime dimensions [11]. The first part of the following text is concerned with repeating the main results of the covariant phase space formalism followed by an extensive analysis of the differences between the metric and Cartan formalism in three spacetime dimensions. In the specific case of three spacetime dimensions there is also another formulation of Einstein gravity in terms of a gauge theory called Chern-Simons theory [12]. As will be shown in the following, the Hamiltonian charges from the Chern-Simons formalism agree with those in the Cartan formalism. Thus, the differences between the metric formalism and the Cartan formalism are the same as between the metric formalism and the Chern-Simons formalism. However, these differences do not always occur. To highlight this fact three different examples will be given. For the Bañados geometries and the BTZ near horizon boundary conditions there is no difference in the Hamiltonian surface charges between formalisms. As a third example, a general locally AdS_3 metric in Gaussian null coordinates will be studied, which exhibits significant differences in the Hamiltonian surface charges, depending on the formalism.

Chapter 1

Conserved Hamiltonian surface charges in 3d Einstein gravity

The treatment of invariant surface charges in the covariant phase space formalism makes heavy use of the exterior calculus of differential forms. A p-form $\boldsymbol{\alpha}$ in d spacetime dimensions is given either as

$$\boldsymbol{\alpha} = \frac{1}{p!}\alpha_{\mu_1\dots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_p} \tag{1.1}$$

or, alternatively, as

$$\boldsymbol{\alpha} = \alpha^{\mu_1\dots\mu_{d-p}}\,(\mathrm{d}^p x)_{\mu_1\dots\mu_{d-p}}\,, \tag{1.2}$$

where

$$(\mathrm{d}^p x)_{\mu_1\dots\mu_{d-p}} = \frac{1}{(d-p)!}\frac{1}{p!}\epsilon_{\mu_1\dots\mu_{d-p}\nu_1\dots\nu_p}\,\mathrm{d}x^{\nu_1}\wedge\dots\wedge\mathrm{d}x^{\nu_p}\,. \tag{1.3}$$

Differential forms as abstract objects with all basis elements included are written as bold-faced letters. The only exceptions to this rule are the dreibein, the spin connection and the coordinate differentials.

The symbol $\epsilon_{\mu_1\dots\mu_{d-p}\nu_1\dots\nu_p}$ refers to the usual permutation symbol without any additional factors of $\sqrt{-g}$, i.e. ϵ is a tensor density, not a tensor. Raising the indices of the permutation symbol with some metric one picks up a factor of the inverse determinant of that metric, i.e.

$$\epsilon^{\mu_1\dots\mu_d} = \epsilon_{\nu_1\dots\nu_d}g^{\mu_1\nu_1}\dots g^{\mu_d\nu_d} = \det\left(g^{-1}\right)\epsilon_{\mu_1\dots\mu_d} = \det(g)^{-1}\epsilon_{\mu_1\dots\mu_d}\,. \tag{1.4}$$

This then leads to the identity

$$\epsilon^{\mu_1\dots\mu_p\nu_1\dots\nu_{d-p}}\epsilon_{\mu_1\dots\mu_p\rho_1\dots\rho_{d-p}} = p!\,\det(g)^{-1}\delta^{\nu_1\dots\nu_{d-p}}_{\rho_1\dots\rho_{d-p}} \tag{1.5}$$

with the antisymmetrizer $\delta^{\nu_1\dots\nu_{d-p}}_{\rho_1\dots\rho_{d-p}}$. The same is true for permutation symbols with anholonomic ("flat") indices with the metric replaced with the Minkowski metric η_{ab}, which has determinant -1.

In the following the key relations of the covariant phase space formalism will be sketched before it will be applied to Einstein gravity in three different contexts, the second order (metric) formulation, the first order (Cartan/dreibein) formulation and the Chern-Simons formulation. An accessible, but more elaborate introduction to the covariant phase space formalism is given by Compère and Fiorucci in [13], although they do not consider state-dependent diffeomorphisms. This special case is picked up e.g. in [11] and the appendix of [14].

1.1 Covariant phase space formalism

1.1.1 Presymplectic potential and form

The usual starting point for a typical field theory on a d-dimensional (pseudo-)Riemannian spacetime manifold $\mathcal{M}$ is the Lagrangian. If the volume element of spacetime is included in the definition, the Lagrangian is a top form given as

$$\mathbf{L}(\Phi) = \mathcal{L}(\Phi)\sqrt{-g}\,\mathrm{d}^d x \tag{1.6}$$

with some scalar function $\mathcal{L}(\Phi)$. The symbol Φ stands for the field content of the theory, including the metric and any possible combination of matter fields. The action of the theory is then given as

$$S = \int_{\mathcal{M}} \mathbf{L} \tag{1.7}$$

and the equations of motion can be found by computing the variation of S with respect to all fields Φ, which will be enumerated by an index i. This usually leads to terms where derivative operators act on $\delta\Phi_i$. One then performs partial integration on these terms, which yields total derivatives, i.e. surface terms. Thus,

$$\delta S = \int_{\mathcal{M}} \delta\mathbf{L} = \int_{\mathcal{M}} \left(\frac{\delta\mathbf{L}}{\delta\Phi^i}\delta\Phi^i + \mathrm{d}\boldsymbol{\Theta}\left(\delta\Phi;\Phi\right)\right). \tag{1.8}$$

Here $\frac{\delta\mathbf{L}}{\delta\Phi^i}$, in a slight abuse of notation, means the equations of motion (EOM). $\boldsymbol{\Theta}$ is a $d-1$-form called the presymplectic potential. Note that it is defined by (1.8) only up to an additional closed (and thus locally exact) form. It was assumed that the spacetime manifold $\mathcal{M}$ is at least $\mathcal{C}^2$ everywhere (and not only piecewise as in some applications), i.e. $\partial\partial\mathcal{M} = 0$. This avoids any corner contributions to the surface terms [15].

The key ingredient to the Hamiltonian surface charge is the presymplectic form $\boldsymbol{\omega}$, which is defined as

$$\boldsymbol{\omega}(\delta_1\Phi, \delta_2\Phi;\Phi) = \delta_1\boldsymbol{\Theta}(\delta_2\Phi;\Phi) - \delta_2\boldsymbol{\Theta}(\delta_1\Phi;\Phi) - \boldsymbol{\Theta}([\delta_1,\delta_2]\Phi;\Phi). \tag{1.9}$$

The term with the commutator serves to reproduce the presymplectic form defined by Lee and Wald, which is bilinear in the variations [14, 16]. It differs from the notion of an invariant presymplectic form defined by Barnich and Compère [17], which will not be discussed here. The term "presymplectic form" in the following will always refer to the Lee-Wald presymplectic form given by (1.9). Note that if one were to add an arbitrary exact form to the presymplectic potential, the presymplectic form would also change by a total derivative. It is important to keep this in mind as it will end up mattering in the following. Another ambiguity could come from the fact that only the bulk part of the action was considered and any surface terms that are added to preserve the variational principle were ignored. However, this is legitimate since an additional term $\mathrm{d}\mathbf{M}$ in the Lagrangian enters the presymplectic potential as an additional term $\delta\mathbf{M}$, which drops out in the presymplectic form.

1.1.2 Noether current and Noether-Wald surface charge

Under some symmetry transformation δ_ϵ with parameter ϵ the Lagrangian stays invariant, possibly up to a surface term. Thus,

$$\delta_\epsilon\mathbf{L} = \frac{\delta\mathbf{L}}{\delta\Phi^i}\delta_\epsilon\Phi^i + \mathrm{d}\boldsymbol{\Theta}\left(\delta_\epsilon\Phi;\Phi\right) = \mathrm{d}\mathbf{Y}_\epsilon\,. \tag{1.10}$$

From this one can define the Noether current of the symmetry as

$$\mathbf{J}_\epsilon := \mathbf{\Theta}(\delta_\epsilon \Phi; \Phi) - \mathbf{Y}_\epsilon \tag{1.11}$$

It is clear from (1.10) that

$$\mathrm{d}\mathbf{J}_\epsilon \approx 0, \tag{1.12}$$

where the symbol $\approx$ means the relation holds on-shell. So the on-shell Noether current must be a closed (and locally exact) form, i.e.

$$\mathbf{J}_\epsilon \approx \mathrm{d}\mathbf{Q}_\epsilon\,. \tag{1.13}$$

The $(d-2)$-form $\mathbf{Q}_\epsilon$ is called the Noether-Wald surface charge [13].

1.1.3 Conserved Hamiltonian surface charges

The variation of the conserved charge H_ξ (also called Hamiltonian surface charge) associated with the symmetry generated by ϵ is

$$\not\delta H_\epsilon = \int_\Sigma \boldsymbol{\omega}(\delta\Phi, \delta_\epsilon\Phi; \Phi). \tag{1.14}$$

Σ is a spacelike hypersurface in $\mathcal{M}$. The symbol $\not\delta$ is used as it is not clear at this point whether or not (1.14) is integrable in field space. If it turns out to be integrable, one obtains H_ξ via an integration in field space as

$$H_\epsilon(\Phi) = \int_{\overline{\Phi}}^{\Phi} \delta H_\epsilon + N_\epsilon(\overline{\Phi}). \tag{1.15}$$

Here Φ means the target field configuration at which the charge is evaluated and $\overline{\Phi}$ is some reference field configuration. $N_\epsilon(\overline{\Phi})$ is a charge associated with that reference field configuration. Integrability means that the result of the integral is independent of the path chosen between $\overline{\Phi}$ and Φ.

As mentioned previously, there is an ambiguity in the definition of $\mathbf{\Theta}$ that allows for an additional total derivative in $\mathbf{\Theta}$ that would also enter the presymplectic form and consequently the surface charge. Thus, the result (1.14) implicitly contains a convention of how to resolve this ambiguity. Also, the presymplectic form $\boldsymbol{\omega}$ in (1.14) is specifically the Lee-Wald presymplectic form (1.9). To keep track of these conventions the charge defined by (1.14) is called "Iyer-Wald surface charge" [18]. This is to distinguish it from the "Barnich-Brandt surface charge" [19], which makes use of the invariant presymplectic form built from the equations of motion instead of the Lagrangian. In the following the phrase "Hamiltonian surface charge" always refers to the Iyer-Wald surfac charge. However, for Killing symmetries of Einstein gravity the Iyer-Wald and Barnich-Brandt charges agree. Only in the case of symplectic symmetries, which will not be considered in the following, there is a difference [13, 14].

Gauge transformations

Depending on the type of symmetry the expression (1.14) can be further simplified. For (infinitesimal) gauge transformations with parameter $\epsilon = \lambda$ it may be the case that the Lagrangian is exactly invariant under the transformation and thus $\mathbf{Y}_\lambda = 0$. If then the presymplectic potential is also gauge invariant, i.e. $\delta_\lambda\mathbf{\Theta}(\delta\Phi;\Phi) = 0$, the variation of the surface charge is given by

$$\begin{aligned}\not\delta H_\lambda &= \int_\Sigma \left[\delta\boldsymbol{\Theta}(\delta_\lambda\Phi;\Phi) - \boldsymbol{\Theta}([\delta,\delta_\lambda]\Phi;\Phi)\right] \\ &\approx \int_\Sigma \mathrm{d}\left[\delta\mathbf{Q}_\lambda - \mathbf{Q}_{\delta\lambda}\right] \\ &\approx \oint_{\partial\Sigma}\left[\delta\mathbf{Q}_\lambda - \mathbf{Q}_{\delta\lambda}\right] = \oint_{\partial\Sigma}\mathbf{k}_\lambda(\delta\Phi;\Phi),\end{aligned} \tag{1.16}$$

where the form

$$\mathbf{k}_\lambda(\delta\Phi;\Phi) := \delta\mathbf{Q}_\lambda - \mathbf{Q}_{\delta\lambda} \tag{1.17}$$

was introduced for convenience. Note that in the last line of (1.16) the integral is over the boundary of Σ. If the gauge parameter λ is not field-dependent, then the Hamiltonian surface charge is integrable and coincides with the Noether-Wald charge of the symmetry integrated over $\partial\Sigma$.

This simplification is only valid if the Lagrangian and presymplectic potential are exactly invariant under the gauge transformation. With respect to the following examples this is the case for local Lorentz transformations of the Einstein-Hilbert-Palatini action (1.34), but not for gauge transformations of the Chern-Simons action (1.93).

Diffeomorphisms

For diffeomorphisms generated by an infinitesimal vector ξ the Lagrangian transforms as

$$\delta_\xi\mathbf{L} = \mathcal{L}_\xi\mathbf{L} = \mathrm{d}(i_\xi\mathbf{L}) + i_\xi(\mathrm{d}\mathbf{L}) = \mathrm{d}(i_\xi\mathbf{L}) \tag{1.18}$$

and thus $\mathbf{Y}_\xi = i_\xi\mathbf{L}$. Then

$$\begin{aligned}\not\delta H_\xi &= \int_\Sigma \left[\delta\boldsymbol{\Theta}(\delta_\xi\Phi;\Phi) - \delta_\xi\boldsymbol{\Theta}(\delta\Phi;\Phi) - \boldsymbol{\Theta}([\delta,\delta_\xi]\Phi;\Phi)\right] \\ &= \int_\Sigma \left[\delta\mathbf{J}_\xi + \delta\mathbf{Y}_\xi - \mathrm{d}\left(i_\xi\boldsymbol{\Theta}(\delta\Phi;\Phi)\right) - i_\xi\left(\mathrm{d}\boldsymbol{\Theta}(\delta\Phi;\Phi)\right) - \mathbf{J}_{\delta\xi} - \mathbf{Y}_{\delta\xi}\right] \\ &= \int_\Sigma \left[\delta\mathbf{J}_\xi + \delta\left(i_\xi\mathbf{L}\right) - \mathrm{d}\left(i_\xi\boldsymbol{\Theta}(\delta\Phi;\Phi)\right) - i_\xi\left(\mathrm{d}\boldsymbol{\Theta}(\delta\Phi;\Phi)\right) - \mathbf{J}_{\delta\xi} - \mathbf{Y}_{\delta\xi}\right] \\ &= \int_\Sigma \left[\delta\mathbf{J}_\xi + i_{\delta\xi}\mathbf{L} + i_\xi\delta\mathbf{L} - \mathrm{d}\left(i_\xi\boldsymbol{\Theta}(\delta\Phi;\Phi)\right) - i_\xi\left(\mathrm{d}\boldsymbol{\Theta}(\delta\Phi;\Phi)\right) - \mathbf{J}_{\delta\xi} - \mathbf{Y}_{\delta\xi}\right] \\ &\approx \int_\Sigma \left[\mathrm{d}\delta\mathbf{Q}_\xi - \mathrm{d}\left(i_\xi\boldsymbol{\Theta}(\delta\Phi;\Phi)\right) - \mathrm{d}\mathbf{Q}_{\delta\xi}\right] \\ &\approx \oint_{\partial\Sigma}\left[\delta\mathbf{Q}_\xi - i_\xi\boldsymbol{\Theta}(\delta\Phi;\Phi) - \mathbf{Q}_{\delta\xi}\right] = \oint_{\partial\Sigma}\mathbf{k}_\xi(\delta\Phi;\Phi)\end{aligned} \tag{1.19}$$

with

$$\mathbf{k}_\xi(\delta\Phi;\Phi) := \delta\mathbf{Q}_\xi - i_\xi\boldsymbol{\Theta}(\delta\Phi;\Phi) - \mathbf{Q}_{\delta\xi} \tag{1.20}$$

Again, the conserved Hamiltonian surface charge can be computed as an integral over the boundary of Σ.

Conservation criterion

Assuming integrability the difference of the Hamiltonian surface charge evaluated at two different surfaces $\partial\Sigma_1$ and $\partial\Sigma_2$ is

$$\int_{\overline{\Phi}}^{\Phi}\oint_{\partial\Sigma_1}\mathbf{k}_\epsilon(\delta\Phi;\Phi) - \int_{\overline{\Phi}}^{\Phi}\oint_{\partial\Sigma_2}\mathbf{k}_\epsilon(\delta\Phi;\Phi) = \int_{\overline{\Phi}}^{\Phi}\int_{\mathcal{S}}\mathrm{d}\mathbf{k}_\epsilon(\delta\Phi;\Phi)\,. \tag{1.21}$$

Here $\mathcal{S}$ means the codimension one hypersurface that is bounded by $\partial\Sigma_1$ and $\partial\Sigma_2$. As can be seen the Hamiltonian surface charge is conserved between $\partial\Sigma_1$ and $\partial\Sigma_2$ if there exists a surface $\mathcal{S}$ such that

$$\mathrm{d}\mathbf{k}_\epsilon(\delta\Phi;\Phi) = \boldsymbol{\omega}(\delta\Phi, \delta_\epsilon\Phi;\Phi) \approx 0 \tag{1.22}$$

everywhere on $\mathcal{S}$. This is, in general, not the case, but there are two special and interesting cases in the context of general relativity. For diffeomorphisms that obey the Killing equation the presymplectic form will be zero identically and the associated charge will be conserved everywhere. In the case of asymptotic Killing symmetries the charge will be conserved in the asymptotic region where the symmetry holds.

1.2 Metric formalism

Starting from the Einstein-Hilbert Lagrangian

$$\mathbf{L}_g = \frac{1}{16\pi G}\sqrt{-g}\,(R - 2\Lambda)\,\mathrm{d}^d x \tag{1.23}$$

computing the variation yields

$$\delta\mathbf{L}_g = -\frac{\sqrt{-g}}{16\pi G}(G^{\mu\nu} + \Lambda g^{\mu\nu})\delta g_{\mu\nu}\,\mathrm{d}^d x + \mathrm{d}\boldsymbol{\Theta}(\delta g; g) \tag{1.24}$$

with

$$\boldsymbol{\Theta}(\delta g; g) = \frac{\sqrt{-g}}{8\pi G} g^{\mu[\nu}\delta\Gamma^{\rho]}_{\nu\mu}\left(\mathrm{d}^{d-1}x\right)_\rho = \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\sigma\delta g^{\sigma\rho} - g^{\sigma\rho}\nabla_\sigma\delta g^\mu_\mu\right)\left(\mathrm{d}^{d-1}x\right)_\rho . \tag{1.25}$$

One has to be precise here what these variations mean. $\delta g^{\mu\nu}$ in this context means the variation of $g_{\mu\nu}$ with both indices raised by the metric. This is not the same as the variation of $g^{\mu\nu}$, which would have a different overall sign. The symbol δg^μ_μ means the variation of $g_{\mu\nu}$ with one index raised and contracted. For a diffeomorphism ξ

$$\begin{aligned}
\boldsymbol{\Theta}(\delta_\xi g; g) &= \frac{\sqrt{-g}}{16\pi G}\left(g^{\sigma\mu}g^{\rho\nu}\nabla_\sigma\mathcal{L}_\xi g_{\mu\nu} - g^{\sigma\rho}g^{\mu\nu}\nabla_\sigma\mathcal{L}_\xi g_{\mu\nu}\right)\left(\mathrm{d}^{d-1}x\right)_\rho \\
&= \frac{\sqrt{-g}}{16\pi G}\left(g^{\sigma\mu}g^{\rho\nu}\nabla_\sigma(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu) - g^{\sigma\rho}g^{\mu\nu}\nabla_\sigma(\nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu)\right)\left(\mathrm{d}^{d-1}x\right)_\rho \\
&= \frac{\sqrt{-g}}{16\pi G}\left(\nabla^\mu\nabla_\mu\xi^\rho + \nabla^\mu\nabla^\rho\xi_\mu - 2\nabla^\rho\nabla^\mu\xi_\mu\right)\left(\mathrm{d}^{d-1}x\right)_\rho .
\end{aligned} \tag{1.26}$$

The Noether current for a diffeomorphism is

$$\begin{aligned}
\mathbf{J}_\xi &= \boldsymbol{\Theta}(\delta_\xi g; g) - i_\xi\mathbf{L}_g \\
&= \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\sigma\nabla^\sigma\xi^\mu + \nabla_\sigma\nabla^\mu\xi^\sigma - 2\nabla^\mu\nabla_\sigma\xi^\sigma\right)\left(\mathrm{d}^{d-1}x\right)_\mu - \frac{\sqrt{-g}}{16\pi G}\xi^\mu(R - 2\Lambda)\left(\mathrm{d}^{d-1}x\right)_\mu \\
&= \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\sigma\nabla^\sigma\xi^\mu + 2[\nabla_\sigma,\nabla^\mu]\xi^\sigma - \nabla_\sigma\nabla^\mu\xi^\sigma - \xi^\mu(R - 2\Lambda)\right)\left(\mathrm{d}^{d-1}x\right)_\mu \\
&= \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\sigma\nabla^\sigma\xi^\mu + 2R^\mu{}_\sigma\xi^\sigma - \nabla_\sigma\nabla^\mu\xi^\sigma - \xi^\mu(R - 2\Lambda)\right)\left(\mathrm{d}^{d-1}x\right)_\mu \\
&\approx \frac{\sqrt{-g}}{16\pi G}\left(\nabla_\sigma\nabla^\sigma\xi^\mu - \nabla_\sigma\nabla^\mu\xi^\sigma\right)\left(\mathrm{d}^{d-1}x\right)_\mu \\
&\approx \mathrm{d}\mathbf{Q}_\xi
\end{aligned} \tag{1.27}$$

with the Noether-Wald charge

$$\mathbf{Q}_\xi = \frac{\sqrt{-g}}{8\pi G}\nabla^\sigma\xi^\mu \left(\mathrm{d}^{d-2}x\right)_{\mu\sigma} = -\frac{\sqrt{-g}}{8\pi G}\nabla^\mu\xi^\nu \left(\mathrm{d}^{d-2}x\right)_{\mu\nu}. \tag{1.28}$$

From this (keeping in mind the aforementioned sign convention for $\delta g^{\mu\nu}$)

$$\begin{aligned}\delta\mathbf{Q}_\xi &= -\frac{1}{8\pi G}\left[\delta\sqrt{-g}\nabla^\mu\xi^\nu + \sqrt{-g}\,\delta(\nabla^\mu\xi^\nu)\right]\left(\mathrm{d}^{d-2}x\right)_{\mu\nu}\\ &= -\frac{1}{8\pi G}\left[\frac{1}{2}\sqrt{-g}\,g^{\rho\sigma}\delta g_{\rho\sigma}\nabla^\mu\xi^\nu + \sqrt{-g}\,\delta(g^{\mu\rho}\nabla_\rho\xi^\nu)\right]\left(\mathrm{d}^{d-2}x\right)_{\mu\nu}\\ &= -\frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}g_{\rho\sigma}\delta g^{\rho\sigma}\nabla^\mu\xi^\nu - \delta g^{\mu\sigma}\nabla_\sigma\xi^\nu + g^{\mu\rho}\delta\Gamma^\nu{}_{\sigma\rho}\xi^\sigma + g^{\mu\rho}\nabla_\rho\delta\xi^\nu\right]\left(\mathrm{d}^{d-2}x\right)_{\mu\nu}\end{aligned} \tag{1.29}$$

and since

$$\begin{aligned}g^{\mu\rho}\delta\Gamma^\nu{}_{\sigma\rho}\xi^\sigma &= g^{\mu\rho}\frac{1}{2}g^{\nu\lambda}\left(\nabla_\sigma\delta g_{\rho\lambda} + \nabla_\rho\delta g_{\sigma\lambda} - \nabla_\lambda\delta g_{\sigma\rho}\right)\xi^\sigma\\ &= \frac{1}{2}\left(\nabla_\sigma\delta g^{\mu\nu} + g_{\sigma\kappa}\nabla^\mu\delta g^{\kappa\nu} - g_{\sigma\kappa}\nabla^\nu\delta g^{\kappa\mu}\right)\xi^\sigma\\ &= \frac{1}{2}\xi^\sigma\nabla_\sigma\delta g^{\mu\nu} - \xi_\sigma\nabla^{[\nu}\delta g^{\mu]\sigma}\end{aligned} \tag{1.30}$$

the variation of the Noether-Wald charge is

$$\delta\mathbf{Q}_\xi = -\frac{\sqrt{-g}}{8\pi G}\left[\frac{1}{2}\delta g^\sigma_\sigma\nabla^\mu\xi^\nu - \delta g^{\mu\rho}\nabla_\rho\xi^\nu - \xi_\sigma\nabla^\nu\delta g^{\mu\sigma} + \nabla^\mu\delta\xi^\nu\right]\left(\mathrm{d}^{d-2}x\right)_{\mu\nu}. \tag{1.31}$$

Finally, the form $\mathbf{k}^g_\xi$ can be computed as

$$\begin{aligned}\mathbf{k}^g_\xi &= \delta\mathbf{Q}_\xi - i_\xi\mathbf{\Theta}(\delta g; g) - \mathbf{Q}_{\delta\xi}\\ &= \frac{\sqrt{-g}}{8\pi G}\left(-\frac{1}{2}\delta g^\sigma_\sigma\nabla^\mu\xi^\nu + \delta g^{\mu\sigma}\nabla_\sigma\xi^\nu + \xi_\sigma\nabla^\nu\delta g^{\mu\sigma} - \xi^\nu\nabla_\sigma\delta g^{\sigma\mu} + \xi^\nu\nabla^\mu\delta g^\rho_\rho\right)\left(\mathrm{d}^{d-2}x\right)_{\mu\nu}.\end{aligned} \tag{1.32}$$

The superscript g stresses that $\mathbf{k}^g_\xi$ was obtained in the metric formalism. The (variation of the) Hamiltonian surface charge can then be found by integrating

$$\not{\delta} H_\xi \approx \oint_{\partial\Sigma}\mathbf{k}^g_\xi. \tag{1.33}$$

Also note that this result is valid in an arbitrary number of spacetime dimensions. This will not be true for the following sections, which are restricted to 3d Einstein gravity.

1.3 Cartan formalism

In the first order formulation of Einstein gravity the Einstein-Hilbert action is replaced by the Einstein-Hilbert-Palatini action, which in three spacetime dimensions is given as

$$S_{\text{EHP}} = \int_{\mathcal{M}}\mathbf{L}_e = \frac{\sigma}{16\pi G}\int_{\mathcal{M}}\left[\epsilon_{abc}R^{ab}\wedge e^c - \frac{\Lambda}{3}\epsilon_{abc}e^a\wedge e^b\wedge e^c\right]. \tag{1.34}$$

The 1-form $e^a = e^a_\mu\,\mathrm{d}x^\mu$ is called triad or dreibein[1]. Its components fulfill

$$g_{\mu\nu} = e^a_\mu e^b_\nu\eta_{ab} \tag{1.35}$$

[1] As "drei" means "three" in German, this nomenclature is, just like "triad", specific to the number of spacetime dimensions. The general term that does not specify the number of dimensions is "vielbein".

and taking the determinant of this expression establishes the relation

$$\sqrt{-g} = \sigma \ \mathrm{det}(e), \tag{1.36}$$

where σ is the sign of $\mathrm{det}(e)$ and depends on the relative orientation of the triad. While $\mathrm{det}(e)$ is a continuous function of spacetime, its sign will be either positive or negative everywhere. If at some point $\mathrm{det}(e)$ would change sign, the triad would degenerate at that point, which is unphysical. Hence, σ introduces a global sign that is needed to relate back to the metric formulation.

The Riemann curvature 2-form $R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ is given in terms of the spin connection. It is not to be confused with the presymplectic form, although both use the letter omega. This action is equivalent to the Einstein-Hilbert action from before. Introducing the dualized spin connection $\omega^a := \frac{1}{2}\epsilon^{abc}\omega_{bc}$ yields the Lagrangian

$$\mathbf{L}_e = \frac{\sigma}{8\pi G}\left[\mathrm{d}\omega_a \wedge e^a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c \wedge e^a - \frac{\Lambda}{6}\epsilon_{abc}e^a \wedge e^b \wedge e^c\right], \tag{1.37}$$

that has to be varied with respect to the dreibein and dualized spin connection independently, yielding

$$\begin{aligned}\delta\mathbf{L}_e = \frac{\sigma}{8\pi G}\Bigg[&\left(\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c - \frac{\Lambda}{2}\epsilon_{abc}e^b \wedge e^c\right) \wedge \delta e^a \\ &+ \left(\mathrm{d}e_a + \epsilon_{abc}\omega^b \wedge e^c\right) \wedge \delta\omega^a + \mathrm{d}\left(\delta\omega_a \wedge e^a\right)\Bigg].\end{aligned} \tag{1.38}$$

The EOM

$$\mathrm{d}\omega_a + \frac{1}{2}\epsilon_{abc}\omega^b \wedge \omega^c - \frac{\Lambda}{2}\epsilon_{abc}e^b \wedge e^c = 0, \tag{1.39}$$

$$\mathrm{d}e_a + \epsilon_{abc}\omega^b \wedge e^c = 0 \tag{1.40}$$

consist of the familiar Einstein equations, now in Cartan notation, and of the torsion constraint, that sets the torsion to zero. This was an implicit assumption in the second order formalism. The presymplectic potential is

$$\mathbf{\Theta}(\delta\Phi;\Phi) = \frac{\sigma}{8\pi G}\delta\omega_a \wedge e^a. \tag{1.41}$$

The letter Φ now stands for the dreibein and the spin connection, which are the independent fields of the theory.

A diffeomorphism ξ acts on the 1-form ω^a_μ as

$$\delta_\xi \omega^a_\mu = \mathcal{L}_\xi \omega^a_\mu = \xi^\nu \partial_\nu \omega^a_\mu + \partial_\mu \xi^\nu \omega^a_\nu \tag{1.42}$$

and analogously on e^a_μ. The Noether current associated with this diffeomorphism is then

$$\begin{aligned}
\mathbf{J}_\xi &= \mathbf{\Theta}(\delta_\xi\Phi;\Phi) - i_\xi\mathbf{L}_e \\
&= \frac{\sigma}{8\pi G}\Big[\xi^\mu\partial_\mu\omega_{a\nu}e^a_\rho + \partial_\nu\xi^\mu\omega_{a\mu}e^a_\rho - \xi^\mu\partial_\mu\omega_{a\nu}e^a_\rho - \xi^\mu\partial_\nu\omega_{a\rho}e^a_\mu - \xi^\mu\partial_\rho\omega_{a\mu}e^a_\nu \\
&\qquad\qquad -\frac{1}{2}\epsilon_{abc}\xi^\mu\omega^b_\mu\omega^c_\nu e^a_\rho - \frac{1}{2}\xi^\mu\epsilon_{abc}\omega^b_\nu\omega^c_\rho e^a_\mu - \frac{1}{2}\epsilon_{abc}\xi^\mu\omega^b_\rho\omega^c_\mu e^a_\nu \\
&\qquad\qquad +\frac{\Lambda}{2}\epsilon_{abc}\xi^\mu e^a_\mu e^b_\nu e^c_\rho\Big]\,\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho \\
&\approx \frac{\sigma}{8\pi G}\Big[\partial_\nu\xi^\mu\omega_{a\mu}e^a_\rho - \xi^\mu\partial_\rho\omega_{a\mu}e^a_\nu - \frac{1}{2}\epsilon_{abc}\xi^\mu\omega^b_\mu\omega^c_\nu e^a_\rho - \frac{1}{2}\epsilon_{abc}\xi^\mu\omega^b_\rho\omega^c_\mu e^a_\nu\Big]\,\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho \\
&\approx \frac{\sigma}{8\pi G}\left[\partial_\nu\left(\xi^\mu\omega_{a\mu}e^a_\rho\right) - \xi^\mu\omega_{a\mu}\partial_\nu e^a_\rho + \epsilon_{abc}\omega^a_\mu\omega^c_\nu e^b_\rho\xi^\mu\right]\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho \\
&\approx \frac{\sigma}{8\pi G}\partial_\nu\left(\xi^\mu\omega_{a\mu}e^a_\rho\right)\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho \\
&\approx \mathrm{d}\mathbf{Q}_\xi\,,
\end{aligned} \tag{1.43}$$

so

$$\mathbf{Q}_\xi = \frac{\sigma}{8\pi G}(i_\xi\omega_a)e^a. \tag{1.44}$$

The form $\mathbf{k}^e_\xi$ can then be worked out as

$$\begin{aligned}
\mathbf{k}^e_\xi &= \delta\mathbf{Q}_\xi - i_\xi\mathbf{\Theta}(\delta\Phi;\Phi) - \mathbf{Q}_{\delta\xi} \\
&= \frac{\sigma}{8\pi G}\left[(i_{\delta\xi}\omega_a)e^a + (i_\xi\delta\omega_a)e^a + (i_\xi\omega_a)\delta e^a - (i_\xi\delta\omega_a)e^a + \delta\omega_a(i_\xi e^a) - (i_{\delta\xi}\omega_a)e^a\right] \\
&= \frac{\sigma}{8\pi G}\left[(i_\xi\omega_a)\delta e^a + \delta\omega_a(i_\xi e^a)\right].
\end{aligned} \tag{1.45}$$

In the Cartan formulation of GR there is an additional symmetry that is not present in the second order formulation, the symmetry under local Lorentz transformations. The generator of such a transformation is an infinitesimal[2] antisymmetric matrix λ_{ab}. The dreibein transforms as

$$\delta_\lambda e^a_\mu = \lambda^a{}_b e^b_\mu, \tag{1.46}$$

while the Minkowski metric is of course invariant under local Lorentz transformations, i.e.

$$\delta_\lambda\eta_{ab} = 0. \tag{1.47}$$

The spin connection transforms as

$$\begin{aligned}
\omega^a{}_b \to (\delta^a_c + \lambda^a{}_c)\,\mathrm{d}(\delta^c_b - \lambda^c{}_b) + (\delta^a_c + \lambda^a{}_c)\omega^c{}_d(\delta^d_b - \lambda^d{}_b) + \mathcal{O}(\lambda^2) \\
= \omega^a{}_b - \mathrm{d}\lambda^a{}_b + \lambda^a{}_c\omega^c{}_b - \omega^a{}_c\lambda^c{}_b + \mathcal{O}(\lambda^2),
\end{aligned} \tag{1.48}$$

so

$$\delta_\lambda\omega^a{}_b = -\,\mathrm{d}\lambda^a{}_b + \lambda^a{}_c\omega^c{}_b - \omega^a{}_c\lambda^c{}_b. \tag{1.49}$$

[2]Considering only infinitesimal Lorentz transformations is, strictly speaking, a restriction to the component connected with the unit element instead of the full Lorentz group O(1, 2).

If the torsion constraint is fulfilled, the Einstein-Hilbert-Palatini Lagrangian is exactly invariant under local Lorentz transformations und thus the corresponding Noether current is given by

$$\begin{aligned}
\mathbf{J}_\lambda &= \boldsymbol{\Theta}(\delta_\lambda \Phi; \Phi) \\
&= \frac{\sigma}{8\pi G} \delta_\lambda \omega_a \wedge e^a \\
&= \frac{\sigma}{16\pi G} \epsilon_{abc} \left(-\mathrm{d}\lambda^{bc} + \lambda^b{}_d \omega^{dc} - \omega^b{}_d \lambda^{dc} \right) \wedge e^a \\
&= \frac{\sigma}{16\pi G} \epsilon_{abc} \left(-\mathrm{d}\lambda^{bc} - 2\omega^b{}_d \lambda^{dc} \right) \wedge e^a \\
&= -\frac{\sigma}{16\pi G} \epsilon_{abc} \left(\mathrm{d}(e^a \lambda^{bc}) - \lambda^{bc}\, \mathrm{d}e^a + 2\lambda^{dc} \omega^b{}_d \wedge e^a \right) \\
&= -\frac{\sigma}{16\pi G} \epsilon_{abc} \left(\mathrm{d}(e^a \lambda^{bc}) - \lambda^{bc}\, \mathrm{d}e^a - \lambda^{bc} \omega^a{}_d \wedge e^d \right) \\
&\approx -\frac{\sigma}{16\pi G} \epsilon_{abc}\, \mathrm{d}(e^a \lambda^{bc}) \\
&\approx \mathrm{d}\mathbf{Q}_\lambda \,.
\end{aligned} \tag{1.50}$$

Here it was used that

$$2\epsilon_{abc} \lambda^{dc} \omega^b{}_d \wedge e^a = -\epsilon_{abc} \lambda^{bc} \omega^a{}_d \wedge e^d, \tag{1.51}$$

which can be seen by explicitly performing the summation. Then

$$\begin{aligned}
2\epsilon_{abc} \lambda^{dc} \omega^b{}_d \wedge e^a = 2 \big(&\lambda^{13} \omega^2{}_1 \wedge e^1 + \lambda^{21} \omega^3{}_2 \wedge e^2 + \lambda^{32} \omega^1{}_3 \wedge e^3 \\
&-\lambda^{12} \omega^3{}_1 \wedge e^1 - \lambda^{23} \omega^1{}_2 \wedge e^2 - \lambda^{31} \omega^2{}_3 \wedge e^3 \big)
\end{aligned} \tag{1.52}$$

is the same as

$$\begin{aligned}
&-\epsilon_{abc} \lambda^{bc} \omega^a{}_d \wedge e^d \\
&\quad = -\big(\lambda^{23} \omega^1{}_d \wedge e^d + \lambda^{31} \omega^2{}_d \wedge e^d + \lambda^{12} \omega^3{}_d \wedge e^d \\
&\qquad -\lambda^{32} \omega^1{}_d \wedge e^d - \lambda^{13} \omega^2{}_d \wedge e^d - \lambda^{21} \omega^3{}_d \wedge e^d \big) \\
&\quad = -2 \big(\lambda^{23} \omega^1{}_2 \wedge e^2 + \lambda^{23} \omega^1{}_3 \wedge e^3 + \lambda^{31} \omega^2{}_1 \wedge e^1 \\
&\qquad +\lambda^{31} \omega^2{}_3 \wedge e^3 + \lambda^{12} \omega^3{}_1 \wedge e^1 + \lambda^{12} \omega^3{}_2 \wedge e^2 \big) \,.
\end{aligned} \tag{1.53}$$

Now,

$$\mathbf{Q}_\lambda \approx -\frac{\sigma}{16\pi G} \epsilon_{abc} e^a \lambda^{bc} \tag{1.54}$$

and thus the variation of the Hamiltonian charge corresponding to local Lorentz transformations is obtained by integrating the form

$$\mathbf{k}^e_\lambda = \delta \mathbf{Q}_\lambda - \mathbf{Q}_{\delta\lambda} = -\frac{\sigma}{16\pi G} \epsilon_{abc} \delta e^a \lambda^{bc}. \tag{1.55}$$

1.4 Difference in presymplectic potentials and dressing form

Naively computing Hamiltonian surface charges for the same spacetime in the first and second order formalism will not necessarily lead to the same results. This is already clear at the level of the presymplectic potentials, which don't have to agree between the two formulations. Although they come from equivalent actions that lead to equivalent equations of motion, they can still differ by an additional exact form. If they differ by such an exact form (and it will turn out that they do), this also makes a difference in the Hamiltonian surface charges and needs to be taken into account when comparing results form both formalisms.

To compare the two presymplectic potentials, first note that

$$\omega_{bc\mu} = e_{b\sigma}\partial_\mu e^\sigma_c + e_{b\sigma}\Gamma^\sigma_{\mu\lambda}e^\lambda_c = e_{b\sigma}\nabla_\mu e^\sigma_c \tag{1.56}$$

and thus

$$\begin{aligned}\mathbf{\Theta}_e &= \frac{\sigma}{8\pi G}\delta\omega_a \wedge e^a = \frac{\sigma}{16\pi G}\delta\omega_{bc\mu}e_{a\nu}\epsilon^{abc}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\ &= \frac{\sigma}{16\pi G}\left(\delta e_{b\sigma}\nabla_\mu e^\sigma_c + e_{b\sigma}\nabla_\mu \delta e^\sigma_c + e_{b\sigma}\delta\Gamma^\sigma_{\mu\lambda}e^\lambda_c\right)e_{a\nu}\epsilon^{abc}\epsilon^{\mu\nu\alpha}\det(g)\left(\mathrm{d}^2x\right)_\alpha .\end{aligned} \tag{1.57}$$

The last term is

$$\begin{aligned}\frac{\sigma\det(g)}{16\pi G}e_{b\sigma}\delta\Gamma^\sigma_{\mu\lambda}e^\lambda_c e_{a\nu}\epsilon^{abc}\epsilon^{\mu\nu\alpha}\left(\mathrm{d}^2x\right)_\alpha &= \frac{\sigma\det(g)}{16\pi G}\delta\Gamma^\sigma_{\mu\lambda}g_{\kappa\sigma}e^\nu_a e^\kappa_b e^\lambda_c \epsilon^{abc}\epsilon^\mu{}_\nu{}^\alpha\left(\mathrm{d}^2x\right)_\alpha \\ &= \frac{\sigma\det(g)\det(e)}{16\pi G}\delta\Gamma^\sigma_{\mu\lambda}g_{\kappa\sigma}\epsilon^{\nu\kappa\lambda}\epsilon^\mu{}_\nu{}^\alpha\left(\mathrm{d}^2x\right)_\alpha \\ &= \frac{\sigma\det(g)\det(e)}{16\pi G}\delta\Gamma^\sigma_{\mu\lambda}g^{\lambda\rho}\epsilon_{\nu\sigma\rho}\epsilon^{\nu\alpha\mu}\left(\mathrm{d}^2x\right)_\alpha \\ &= \frac{\sigma\det(e)}{16\pi G}\delta^{\alpha\mu}_{\sigma\rho}g^{\lambda\rho}\delta\Gamma^\sigma_{\mu\lambda}\left(\mathrm{d}^2x\right)_\alpha \\ &= \frac{\sqrt{-g}}{8\pi G}g^{\lambda[\mu}\delta\Gamma^{\alpha]}_{\mu\lambda}\left(\mathrm{d}^2x\right)_\alpha \\ &= \mathbf{\Theta}_g .\end{aligned} \tag{1.58}$$

The first two terms in the second line of (1.57) then must be a total derivative. Using the identity

$$\begin{aligned}e_{b\sigma}\nabla_\mu\delta e^\sigma_c &= e^\rho_b\nabla_\mu\left(g_{\rho\sigma}\delta e^\sigma_c\right) \\ &= e^\rho_b\nabla_\mu\left(\delta e_{c\rho} - e^\sigma_c\delta\left(e^d_\rho e_{d\sigma}\right)\right) \\ &= e^\rho_b\nabla_\mu\left(\delta e_{c\rho} - e^\sigma_c\delta e^d_\rho e_{d\sigma} - e^\sigma_c e^d_\rho\delta e_{d\sigma}\right) \\ &= -e^\rho_b\nabla_\mu\left(e^\sigma_c e^d_\rho\delta e_{d\sigma}\right) \\ &= -\nabla_\mu e^\sigma_c\delta e_{b\sigma} - e^\rho_b e^\sigma_c\nabla_\mu\left(e^d_\rho\delta e_{d\sigma}\right)\end{aligned} \tag{1.59}$$

the first two terms of (1.57) are

$$\begin{aligned}&-\frac{\sigma\det(g)}{16\pi G}e_{a\nu}e^\rho_b e^\sigma_c\nabla_\mu\left(e^d_\rho\delta e_{d\sigma}\right)\epsilon^{abc}\epsilon^{\mu\nu\alpha}\left(\mathrm{d}^2x\right)_\alpha \\ =&\frac{\sigma\det(g)\det(e)}{16\pi G}\epsilon_\nu{}^{\rho\sigma}\epsilon^{\nu\mu\alpha}\nabla_\mu\left(e^d_\rho\delta e_{d\sigma}\right)\left(\mathrm{d}^2x\right)_\alpha \\ =&\frac{\det(g)\sqrt{-g}}{16\pi G}g^{\rho\beta}g^{\sigma\gamma}\epsilon_{\nu\beta\gamma}\epsilon^{\nu\mu\alpha}\nabla_\mu\left(e^d_\rho\delta e_{d\sigma}\right)\left(\mathrm{d}^2x\right)_\alpha \\ =&\frac{\sqrt{-g}}{8\pi G}g^{\rho[\mu}g^{\alpha]\sigma}\nabla_\mu\left(e^d_\rho\delta e_{d\sigma}\right)\left(\mathrm{d}^2x\right)_\alpha \\ =&\frac{\sqrt{-g}}{8\pi G}\nabla_\mu\left(e^{d[\mu}g^{\alpha]\sigma}\delta e_{d\sigma}\right)\left(\mathrm{d}^2x\right)_\alpha \\ =&-\frac{\sqrt{-g}}{8\pi G}\nabla_\mu\left(\delta e^{d[\mu}e^{\alpha]}_d\right)\left(\mathrm{d}^2x\right)_\alpha \\ =&-\partial_\mu\left(\frac{\sqrt{-g}}{8\pi G}\delta e^{d[\mu}e^{\alpha]}_d\right)\left(\mathrm{d}^2x\right)_\alpha\end{aligned} \tag{1.60}$$

$$
\begin{aligned}
&= -\,\mathrm{d}\left[\frac{\sqrt{-g}}{8\pi G}\delta e^{d[\mu}e_d^{\alpha]}\,(\mathrm{d}x)_{\alpha\mu}\right]\\
&= -\,\mathrm{d}\boldsymbol{\alpha}
\end{aligned}
$$

with

$$
\begin{aligned}
\boldsymbol{\alpha} := \frac{\sqrt{-g}}{8\pi G}\delta e^{d[\mu}e_d^{\alpha]}\,(\mathrm{d}x)_{\alpha\mu} &= \frac{\sigma\det(e)}{16\pi G}\delta e^{d\mu}e_d^{\alpha}\epsilon_{\alpha\mu\nu}\,\mathrm{d}x^{\nu}\\
&= \frac{\sigma}{16\pi G}\delta e^{d\mu}e_d^{\alpha}e_\alpha^a e_\mu^b e_\nu^c\epsilon_{abc}\,\mathrm{d}x^{\nu}\\
&= \frac{\sigma}{16\pi G}\epsilon_{abc}\delta e^{a\mu}e_\mu^b e^c
\end{aligned}
\tag{1.61}
$$

called the "dressing form" [11]. So indeed

$$
\boldsymbol{\Theta}_g = \boldsymbol{\Theta}_e + \mathrm{d}\boldsymbol{\alpha}\,. \tag{1.62}
$$

This means that, in general, the presymplectic forms of the second and first order formulation will not agree. The difference, however, is quantified by the form $\boldsymbol{\alpha}$ that is easily computed in the Cartan formalism and its exterior derivative can be added to the old "bare" presymplectic potential. Any results derived from this new "dressed" presymplectic potential will then reproduce the results form the metric formalism.

Due to the inclusion of the dressing form the presymplectic form gains an additional contribution

$$
\begin{aligned}
\boldsymbol{\omega}^{\alpha}(\delta\Phi,\delta_\epsilon\Phi;\Phi) &= \delta\,\mathrm{d}\boldsymbol{\alpha}(\delta_\epsilon\Phi;\Phi) - \delta_\epsilon\,\mathrm{d}\boldsymbol{\alpha}(\delta\Phi;\Phi) - \mathrm{d}\boldsymbol{\alpha}([\delta,\delta_\epsilon]\Phi;\Phi)\\
&= \frac{\sigma}{16\pi G}\epsilon_{abc}\left[\mathrm{d}\left(\delta(\delta_\epsilon e^{a\mu}e_\mu^b e^c)\right) - \delta_\epsilon\left(\mathrm{d}(\delta e^{a\mu}e_\mu^b e^c)\right) - \mathrm{d}\left([\delta,\delta_\epsilon]e^{a\mu}e_\mu^b e^c\right)\right].
\end{aligned}
\tag{1.63}
$$

Note that while the variation δ in field space is assumed to commute with the exterior derivative d, this is not necessarily the case for the symmetry transformation δ_ϵ. This will now be studied for gauge transformations and diffeomorphisms.

1.4.1 Gauge transformations

In the case of local Lorentz transformations (1.63) yields

$$
\begin{aligned}
\boldsymbol{\omega}^{\alpha}_{\lambda} &= \frac{\sigma}{16\pi G}\epsilon_{abc}\left[\mathrm{d}\left(\delta(\lambda^a{}_d e^{d\mu}e_\mu^b e^c)\right) - \delta_\lambda\left(\mathrm{d}(\delta e^{a\mu}e_\mu^b e^c)\right) - \mathrm{d}\left(\delta\lambda^a{}_d e^{d\mu}e_\mu^b e^c\right)\right]\\
&= \frac{\sigma}{16\pi G}\epsilon_{abc}\left[\mathrm{d}\left(\lambda^a{}_d\delta(e^{d\mu}e_\mu^b e^c)\right) - \delta_\lambda\left(\mathrm{d}(\delta e^{a\mu}e_\mu^b e^c)\right)\right]\\
&= \frac{\sigma}{16\pi G}\epsilon_{abc}\left[\mathrm{d}\left(\lambda^a{}_d\delta(e^{d\mu}e_\mu^b e^c)\right) - \mathrm{d}\left(\delta_\lambda(\delta e^{a\mu}e_\mu^b e^c)\right)\right.\\
&\qquad\left. + \mathrm{d}\lambda^a{}_d\,\delta e^{d\mu}e_\mu^b\wedge e^c + \mathrm{d}\lambda^b{}_d\,e^{a\mu}e_\mu^d\wedge e^c + \mathrm{d}\lambda^c{}_d\,\delta e^{a\mu}e_\mu^b\wedge e^d\right]\\
&= \frac{\sigma}{16\pi G}\epsilon_{abc}\left[\mathrm{d}\left(\lambda^a{}_d e^{d\mu}\delta e_\mu^b e^c + \lambda^a{}_d e^{d\mu}e_\mu^b\delta e^c - \delta e^{a\mu}\lambda^b{}_d e_\mu^d e^c - \delta e^{a\mu}e_\mu^b\lambda^c{}_d e^d\right)\right.\\
&\qquad\left. + \mathrm{d}\lambda^a{}_d\,\delta e^{d\mu}e_\mu^b\wedge e^c + \mathrm{d}\lambda^b{}_d\,\delta e^{a\mu}e_\mu^d\wedge e^c + \mathrm{d}\lambda^c{}_d\,\delta e^{a\mu}e_\mu^b\wedge e^d\right]
\end{aligned}
\tag{1.64}
$$

Note that the last line is

$$
\frac{\sigma}{16\pi G}\epsilon_{abc}\,\mathrm{d}\lambda^a{}_d\wedge\left[\delta e^{d\mu}e_\mu^b e^c + \delta e^{c\mu}e_\mu^d e^b + \delta e^{b\mu}e_\mu^c e^d\right]. \tag{1.65}
$$

The expression inside the brackets is invariant under cyclic permutations of b, c and d. The ϵ_{abc} prefactor makes it also antisymmetric in b and c. This means that the bracket is proportional to ϵ^{bcd}. Together with ϵ_{abc} this gives $-2\delta^d_a$, but since $\lambda^a{}_d$ is antisymmetric the whole expression (1.65) is zero.

Now using

$$\begin{aligned} e^{d\mu}\delta e^b_\mu &= e^d_\nu g^{\nu\mu}\delta e^b_\mu \\ &= e^d_\nu \delta e^{b\nu} - e^d_\nu \delta(e^\nu_a e^{a\mu}) e^b_\mu \\ &= e^d_\nu \delta e^{b\nu} - e^d_\nu \delta e^\nu_b - \delta e^{d\mu} e^b_\mu \\ &= -\delta e^{d\mu} e^b_\mu \end{aligned} \tag{1.66}$$

in (1.64) gives

$$\boldsymbol{\omega}^\alpha_\lambda = \frac{\sigma}{16\pi G}\epsilon_{abc}\,\mathrm{d}\left[\lambda^{ab}\delta e^c - \lambda^a{}_d\left(\delta e^{d\mu}e^b_\mu e^c + \delta e^{c\mu}e^d_\mu e^b + \delta e^{b\mu}e^c_\mu e^d\right)\right]. \tag{1.67}$$

The last three terms are, again, invariant under cyclic permutations of b, c and d. The same argument as above then leads to them being zero and the first term is the only contribution left. This establishes

$$\mathbf{k}^\alpha_\lambda = \frac{\sigma}{16\pi G}\epsilon_{abc}\lambda^{ab}\delta e^c. \tag{1.68}$$

This neatly cancels (1.55), so that in the Cartan formalism $\mathbf{k}^e_\lambda + \mathbf{k}^\alpha_\lambda = 0$. This should come at no surprise as it is equivalent to saying that the local Lorentz transformations have no charges in the metric formalism, which makes sense as the symmetry is not present there. It is, however, a nice consistency check. The more interesting symmetry that is present in both formalisms is that of diffeomorphisms.

1.4.2 Diffeomorphisms

Note that since $\delta_\xi \mathrm{d} = \mathrm{d}i_\xi \mathrm{d} + i_\xi \mathrm{d}^2 = \mathrm{d}i_\xi\mathrm{d} = \mathrm{d}^2 i_\xi + \mathrm{d}i_\xi\mathrm{d} = \mathrm{d}\delta_\xi$ the variation δ_ξ commutes with d. This means that for a diffeomorphism (1.63) yields

$$\boldsymbol{\omega}^\alpha_\xi = \frac{\sigma}{16\pi G}\epsilon_{abc}\,\mathrm{d}\left[\mathcal{L}_\xi e^{a\mu}\delta(e^b_\mu e^c_\nu) - \delta e^{a\mu}\mathcal{L}_\xi(e^b_\mu e^c_\nu)\right]\mathrm{d}x^\nu \tag{1.69}$$

and thus

$$\mathbf{k}^\alpha_\xi = \frac{\sigma}{16\pi G}\epsilon_{abc}\left[\mathcal{L}_\xi e^{a\mu}\delta(e^b_\mu e^c_\nu) - \delta e^{a\mu}\mathcal{L}_\xi(e^b_\mu e^c_\nu)\right]\mathrm{d}x^\nu\,. \tag{1.70}$$

By adding $\mathbf{k}^\alpha_\xi$ to $\mathbf{k}^e_\xi$ the result for the Hamiltonian charges from the metric formulation can be recovered, i.e.

$$\not\delta H^g_\xi = \oint_{\partial\Sigma}\mathbf{k}^g_\xi = \oint_{\partial\Sigma}\left[\mathbf{k}^e_\xi + \mathbf{k}^\alpha_\xi\right] \tag{1.71}$$

Note, however, that in general

$$\mathbf{k}^g_\xi \neq \mathbf{k}^e_\xi + \mathbf{k}^\alpha_\xi, \tag{1.72}$$

but rather only

$$\mathbf{k}^g_\xi = \mathbf{k}^e_\xi + \mathbf{k}^\alpha_\xi + \mathrm{d}f \tag{1.73}$$

with some arbitrary scalar function f, which does not change (1.71).

It should be stressed once more what this means. The Hamiltonian charges associated with the same diffeomorphisms of the exact same spacetime will, in general, not agree in the first and second order formulation of Einstein gravity. This is due to the additional symmetry of local Lorentz transformations that is only present in the Cartan formulation. However, there is a systematic way of compensating the difference in Hamiltonian charges. First, one can quantify the difference in the presymplectic potentials, which leads to the concept of the dressing form. From the dressing form the difference in the Hamiltonian charges can be calculated.

1.4.3 Kosmann derivative

For a spacetime isometry

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 0 \tag{1.74}$$

the metric is invariant under δ_ξ. For the same isometry in the Cartan formulation, the dreibein need not be invariant as long as the metric is invariant. Since

$$\delta_\lambda g_{\mu\nu} = \delta_\lambda \left(e^a_\mu e^b_\nu \eta_{ab}\right) = \delta_\lambda e^a_\mu e^b_\nu \eta_{ab} + e^a_\mu \delta_\lambda e^b_\nu \eta_{ab} + e^a_\mu e^b_\nu \delta_\lambda \eta_{ab} = 0, \tag{1.75}$$

where δ_λ denotes a local Lorentz transformation

$$\delta_\lambda e^a_\mu = \lambda^a{}_b e^b_\mu, \qquad \delta_\lambda \eta_{ab} = 0 \tag{1.76}$$

with the antisymmetric generator λ^{ab}, the triad is allowed to transform with a gauge transformation in addition to the diffeomorphism ξ. This freedom can be used to solve a particular issue with the generic Lie derivative. Note that the expression

$$\delta_\xi e^a_\mu = \mathcal{L}_\xi e^a_\mu \tag{1.77}$$

is, in general, not gauge covariant. This is solved by formulating a gauge covariant Lie derivative $L_\xi = \mathcal{L}_\xi + \delta_{i_\xi \omega}$, that contains an additional gauge transformation with gauge parameter $i_\xi \omega^a{}_b$. Replacing $\mathcal{L}_\xi \to L_\xi$ in (1.77) yields

$$L_\xi e^a_\mu = \mathcal{L}_\xi e^a_\mu + i_\xi \omega^a{}_b e^b_\mu = \xi^\nu \partial_\nu e^a_\mu + e^a_\nu \partial_\mu \xi^\nu + \xi^\nu \omega^a{}_{b\nu} e^b_\mu \tag{1.78}$$

and under an arbitrary infinitesimal gauge transformation

$$\begin{aligned} \delta_\lambda L_\xi e^a_\mu &= \xi^\nu \partial_\nu \lambda^a{}_b e^b_\mu + \lambda^a{}_b \xi^\nu \partial_\nu e^b_\mu + \lambda^a{}_b e^b_\nu \partial_\mu \xi^\nu - \xi^\nu \partial_\nu \lambda^a{}_b e^b_\mu + \xi^\nu \lambda^a{}_c \omega^c{}_{b\nu} e^b_\mu \\ &\quad - \xi^\nu \omega^a{}_{c\nu} \lambda^c{}_b e^b_\mu + \xi^\nu \omega^a{}_{b\nu} \lambda^b{}_c e^c_\mu \\ &= \lambda^a{}_b \left(\xi^\nu \partial_\nu e^b_\mu + e^b_\nu \partial_\mu \xi^\nu + \xi^\nu \omega^b{}_{c\nu} e^c_\mu\right) \\ &= \lambda^a{}_b L_\xi e^b_\mu. \end{aligned} \tag{1.79}$$

So, to recap, if $\mathcal{L}_\xi$ is a spacetime isometry, it will still be an isometry if an arbitrary gauge transformation is added to it. Thus, the ordinary Lie derivative $\mathcal{L}_\xi$ can be replaced by the gauge covariant Lie derivative L_ξ. But since this is a spacetime isometry it is still a transformation that leaves the metric invariant and as such can be written solely as a gauge transformation, i.e.

$$L_\xi e^a_\mu = \mathcal{L}_\xi e^a_\mu + i_\xi \omega^a{}_b e^b_\mu = \lambda(\xi)^a{}_b e^b_\mu \tag{1.80}$$

and

$$\begin{aligned} L_\xi \omega^a{}_{b\mu} &= \mathcal{L}_\xi \omega^a{}_{b\mu} - \partial_\mu(i_\xi \omega^a{}_b) + (i_\xi \omega^a{}_c)\omega^c{}_{b\mu} - \omega^a{}_{c\mu}(i_\xi \omega^c{}_b) \\ &= -\partial_\mu \lambda(\xi)^a{}_b + \lambda(\xi)^a{}_c \omega^c{}_{b\mu} - \omega^a{}_{c\mu} \lambda(\xi)^c{}_b. \end{aligned} \tag{1.81}$$

Both conditions are fulfilled by [11]

$$\lambda(\xi)^a{}_b = e^\mu_b \xi^\nu \partial_\nu e^a_\mu + e^a_\nu e^\mu_b \partial_\mu \xi^\nu + i_\xi \omega^a{}_b. \tag{1.82}$$

A gauge transformation with the gauge parameter of (1.82) reproduces the isometry generated by ξ. This can now be used to define the Kosmann derivative [11] acting on the dreibein as

$$\mathcal{K}^{(e)}_\xi e^a_\mu := L_\xi e^a_\mu - \delta_{\lambda(\xi)} e^a_\mu, \tag{1.83}$$

which, if ξ is a Killing vector (i.e. an isometry), fulfills

$$\mathcal{K}^{(e)}_\xi e^a_\mu = 0 \tag{1.84}$$

by construction. One can split the Kosmann derivative in a diffeomorphism $\mathcal{L}_\xi$ and a Lorentz transformation with parameter

$$\bar{\lambda}^a{}_b = i_\xi \omega^a{}_b - \lambda(\xi)^a{}_b = -e^\mu_b \xi^\nu \partial_\nu e^a_\mu - e^a_\nu e^\mu_b \partial_\mu \xi^\nu \tag{1.85}$$

Its total Hamiltonian charge in the Cartan formulation then reproduces the charge of the diffeomorphism ξ in the metric formulation. This can be seen by first computing

$$\begin{aligned}\boldsymbol{\Theta}^e(\delta_{\bar{\lambda}}\Phi;\Phi) &= \frac{\sigma}{16\pi G}\epsilon_{abc}\left(-\mathrm{d}\bar{\lambda}^{bc}\wedge e^a + 2\bar{\lambda}^b{}_d \omega^{dc}\wedge e^a\right)\\ &= \frac{\sigma}{16\pi G}\epsilon_{abc}\left(\mathrm{d}\left(\bar{\lambda}^{ba}\wedge e^c\right) + \bar{\lambda}^{bc}\,\mathrm{d}e^a + 2\bar{\lambda}^{bd}e^a\wedge\omega^c{}_d\right).\end{aligned} \tag{1.86}$$

Writing all the index summations of the last term explicitly (using the antisymmetry of ω^{ab} and λ^{ab}) gives

$$2\left(\bar{\lambda}^{21}e^1\wedge\omega^3{}_1 + \bar{\lambda}^{13}e^3\wedge\omega^2{}_3 + \bar{\lambda}^{32}e^2\wedge\omega^1{}_2 - \bar{\lambda}^{23}e^3\wedge\omega^1{}_3 - \bar{\lambda}^{31}e^1\wedge\omega^2{}_1 - \bar{\lambda}^{12}e^2\wedge\omega^3{}_2\right). \tag{1.87}$$

Doing the same for $-\lambda^{bc}e^d\wedge\omega^a{}_d$ yields

$$\begin{aligned}&-2\left(\bar{\lambda}^{12}e^d\wedge\omega^3{}_d + \bar{\lambda}^{23}e^d\wedge\omega^1{}_d + \bar{\lambda}^{31}e^d\wedge\omega^2{}_d\right)\\ &= 2\left(\bar{\lambda}^{21}e^1\wedge\omega^3{}_1 - \bar{\lambda}^{12}e^2\wedge\omega^3{}_2 + \bar{\lambda}^{32}e^2\wedge\omega^1{}_2\right.\\ &\qquad\left.-\bar{\lambda}^{23}e^3\wedge\omega^1{}_3 + \bar{\lambda}^{13}e^3\wedge\omega^2{}_3 - \bar{\lambda}^{31}e^1\wedge\omega^2{}_1\right).\end{aligned} \tag{1.88}$$

So,

$$2\epsilon_{abc}\bar{\lambda}^{bd}e^a\wedge\omega^c{}_d = -\epsilon_{abc}\lambda^{bc}e^d\wedge\omega^a{}_d = \epsilon_{abc}\lambda^{bc}\omega^a{}_d\wedge e^d. \tag{1.89}$$

This can be used in (1.86) to obtain

$$\begin{aligned}\boldsymbol{\Theta}^e(\delta_{\bar{\lambda}}\Phi;\Phi) &= \frac{\sigma}{16\pi G}\epsilon_{abc}\left(\mathrm{d}\left(\bar{\lambda}^{ba}e^c\right) + \bar{\lambda}^{bc}\,\mathrm{d}e^a + \lambda^{bc}\omega^a{}_d\wedge e^d\right)\\ &\approx \frac{\sigma}{16\pi G}\epsilon_{abc}\,\mathrm{d}\left(\bar{\lambda}^{ba}e^c\right).\end{aligned} \tag{1.90}$$

And since

$$\begin{aligned}\boldsymbol{\alpha}(\delta_\xi\Phi;\Phi) &= \frac{\sigma}{16\pi G}\epsilon_{abc}\delta_\xi e^{a\mu}e^b_\mu e^c = -\frac{\sigma}{16\pi G}\epsilon_{abc}e^{a\mu}\delta_\xi e^b_\mu e^c\\ &= -\frac{\sigma}{16\pi G}\epsilon_{abc}e^{a\mu}\left(\xi^\nu\partial_\nu e^b_\mu + \partial_\mu\xi^\nu e^b_\nu\right)e^c\\ &= \frac{\sigma}{16\pi G}\epsilon_{abc}\bar{\lambda}^{ba}e^c\end{aligned} \tag{1.91}$$

it holds that

$$\boldsymbol{\Theta}^e(\delta_{\bar{\lambda}}\Phi;\Phi) = \mathrm{d}\boldsymbol{\alpha}(\delta_\xi\Phi;\Phi). \tag{1.92}$$

This means that one can compute the Hamiltonian charge of the Kosmann derivative, a combination of diffeomorphism and gauge transformation, within the Cartan formulation and reproduce the result for the same diffemorphism in the metric formulation.

1.5 Chern-Simons formalism

Einstein gravity in three spacetime dimensions with negative cosmological constant can be described by the difference of two Chern-Simons (CS) actions, each of the form

$$S_{\mathrm{CS}}[\mathbf{A}] = \frac{k}{4\pi}\int_{\mathcal{M}}\left\langle \mathbf{A}\wedge \mathrm{d}\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}\right\rangle \tag{1.93}$$

with the Killing form $\langle\rangle$, the gauge connection form $\mathbf{A} = \mathbf{A}^a t_a = A_\mu\, \mathrm{d}x^\mu = A^a_\mu t_a\, \mathrm{d}x^\mu$ with gauge generators t_a and $k = 1/(4G)$. The Killing form introduces a metric

$$\kappa_{ab} := \langle t_a, t_b\rangle \tag{1.94}$$

on the Lie algebra. It is symmetric since the Killing form is invariant under cyclic permutations. This also means that

$$\langle t_a t_b t_c\rangle = \frac{1}{2}\langle t_a, [t_b, t_c]\rangle = \frac{1}{2}\kappa_{ad} f^d{}_{bc} = \frac{1}{2}\kappa_{bd} f^d{}_{ca} = \frac{1}{2}\kappa_{cd} f^d{}_{ab} \tag{1.95}$$

with the structure constants $f^a{}_{bc}$ of the Lie algebra. The CS action can then be written as

$$S_{\mathrm{CS}}[\mathbf{A}] = \frac{k}{4\pi}\int_{\mathcal{M}}\left(A^a_\mu \partial_\nu A^b_\rho + \frac{1}{3}A^a_\mu A^c_\nu A^d_\rho f^b{}_{cd}\right)\kappa_{ab}\, \mathrm{d}x^\mu\wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho\,. \tag{1.96}$$

The action that recovers the Cartan formulation of Einstein gravity in three spacetime dimensions consists of the difference of two CS actions and is given as

$$S_{\mathrm{AdS}_3} = \frac{\sigma k}{4\pi}\int_{\mathcal{M}}\left\langle \mathbf{A}^+\wedge \mathrm{d}\mathbf{A}^+ + \frac{2}{3}\mathbf{A}^+\wedge\mathbf{A}^+\wedge\mathbf{A}^+ - \mathbf{A}^-\wedge \mathrm{d}\mathbf{A}^- - \frac{2}{3}\mathbf{A}^-\wedge\mathbf{A}^-\wedge\mathbf{A}^-\right\rangle . \tag{1.97}$$

Varying the Lagrangian with respect to $\mathbf{A}^+$ and $\mathbf{A}^-$

$$\begin{aligned}\delta\mathbf{L}_{\mathrm{AdS}_3} = \frac{\sigma k}{4\pi}\langle 2\left(\mathrm{d}\mathbf{A}^+ + \mathbf{A}^+\wedge\mathbf{A}^+\right)\wedge\delta\mathbf{A}^+ - \mathrm{d}\left(\mathbf{A}^+\wedge\delta\mathbf{A}^+\right)\\ -2\left(\mathrm{d}\mathbf{A}^- + \mathbf{A}^-\wedge\mathbf{A}^-\right)\wedge\delta\mathbf{A}^- + \mathrm{d}\left(\mathbf{A}^-\wedge\delta\mathbf{A}^-\right)\rangle\end{aligned} \tag{1.98}$$

yields the equations of motion (EOM)

$$\mathbf{F}^\pm := \mathrm{d}\mathbf{A}^\pm + \mathbf{A}^\pm\wedge\mathbf{A}^\pm = 0, \tag{1.99}$$

or, in components,

$$\mathbf{F}^{a\pm}_{\mu\nu} = \left(\partial_\mu A^{a\pm}_\nu - \partial_\nu A^{a\pm}_\mu + A^b_\mu A^c_\nu f^a{}_{bc}\right) = 0. \tag{1.100}$$

These EOM are equivalent to the Einstein equations on 3-dimensional anti-de Sitter space (AdS_3) for the metric given the identifications

$$\mathbf{A}^\pm := (\omega^a{}_\mu \pm e^a_\mu/\ell) J_a\, \mathrm{d}x^\mu \tag{1.101}$$

with the $\mathfrak{so}(2,1)$ algebra elements J_a. Since

$$\langle J_a, J_b\rangle = \frac{\ell}{2}\eta_{ab} \tag{1.102}$$

the metric is then recovered by

$$g_{\mu\nu} = \frac{\ell}{2}\left\langle\left(A^+_\mu - A^-_\mu\right)\left(A^+_\nu - A^-_\nu\right)\right\rangle. \tag{1.103}$$

1.5.1 Gauge transformations and diffeomorphisms

Since the CS gauge connections $\mathbf{A}^{\pm}$ are one-forms in their spacetime content, they transform under a diffeomorphism ξ in the usual sense

$$\delta_\xi \mathbf{A}^{\pm} = \left(\xi^\nu \partial_\nu A^{\pm}_\mu + \partial_\mu \xi^\nu A^{\pm}_\nu\right) \mathrm{d}x^\mu \,. \tag{1.104}$$

On the other hand one can act on the gauge connections with an (infinitesimal) gauge transformation with parameters $\lambda^{\pm}$ to obtain

$$\delta_\lambda \mathbf{A}^{\pm} = \mathrm{d}\lambda^{\pm} + \left[\mathbf{A}^{\pm}, \lambda^{\pm}\right]. \tag{1.105}$$

A particularly interesting case is $\lambda^{\pm} = A^{\pm}_\mu \xi^\mu$. Then

$$\begin{aligned} \delta_\lambda \mathbf{A}^{\pm} &= \partial_\nu A^{\pm}_\mu \xi^\mu \,\mathrm{d}x^\nu + A^{\pm}_\mu \partial_\nu \xi^\mu \,\mathrm{d}x^\nu + \xi^\mu [A^{\pm}_\nu, A^{\pm}_\mu]\,\mathrm{d}x^\nu \\ &= \left(\partial_\nu A^{\pm}_\mu \xi^\mu + A^{\pm}_\mu \partial_\nu \xi^\mu + \xi^\mu \partial_\mu A^{\pm}_\nu - \xi^\mu \partial_\mu A^{\pm}_\nu + \xi^\mu [A^{\pm}_\nu, A^{\pm}_\mu]\right) \mathrm{d}x^\nu \\ &= \delta_\xi A^{\pm}_\nu \,\mathrm{d}x^\nu + \xi^\mu F^{\pm}_{\nu\mu} \,\mathrm{d}x^\nu \\ &\approx \delta_\xi \mathbf{A}^{\pm}, \end{aligned} \tag{1.106}$$

that is, gauge transformations with infinitesimal parameter $\lambda^{\pm} = A^{\pm}_\mu \xi^\mu$ are on-shell equivalent to diffeomorphisms of the CS form.

The Chern-Simons action is not exactly invariant under infinitesimal gauge transformations, but rather

$$\delta_\lambda \mathbf{L}_{\mathrm{AdS}_3} = \mathrm{d}\mathbf{Y}_\lambda \approx \frac{\sigma k}{4\pi} \,\mathrm{d}\left\langle \mathrm{d}\lambda^+ \wedge \mathbf{A}^+ - \mathrm{d}\lambda^- \wedge \mathbf{A}^- \right\rangle . \tag{1.107}$$

Note that this expression was already simplified using the equations of motion.

1.5.2 Conserved surface charges

As can be seen from (1.98), the presymplectic potential for 3d Einstein gravity in the CS formulation is

$$\boldsymbol{\Theta}(\delta \mathbf{A}^{\pm}; \mathbf{A}^{\pm}) = -\frac{\sigma k}{4\pi} \left\langle \mathbf{A}^+ \wedge \delta \mathbf{A}^+ - \mathbf{A}^- \wedge \delta \mathbf{A}^- \right\rangle . \tag{1.108}$$

It can now be used to find the Hamiltonian surface charges for gauge transformations and diffeomorphisms in the Chern-Simons formulation of 3d Einstein gravity.

Gauge transformations

To find the charges associated with gauge transformations with parameters $\lambda^{\pm}$ for the plus and minus sectors one has to keep in mind that these gauge transformations do not leave the Lagrangian invariant, but introduce a surface term $\mathrm{d}\mathbf{Y}_\lambda$. Furthermore, the presymplectic potential is not invariant under gauge transformations and thus the simplification (1.16) is not necessarily correct. Instead one may compute

$$\begin{aligned} \boldsymbol{\omega}(\delta \mathbf{A}^{\pm}, \delta_\lambda \mathbf{A}^{\pm}; \mathbf{A}^{\pm}) &= \delta \boldsymbol{\Theta}(\delta_\lambda \mathbf{A}^{\pm}; \mathbf{A}^{\pm}) - \delta_\lambda \boldsymbol{\Theta}(\delta \mathbf{A}^{\pm}; \mathbf{A}^{\pm}) - \boldsymbol{\Theta}([\delta, \delta_\lambda] \mathbf{A}^{\pm}; \mathbf{A}^{\pm}) \\ &= -\frac{\sigma k}{2\pi} \left\langle \delta \mathbf{A}^+ \wedge \delta_\lambda \mathbf{A}^+ - \delta \mathbf{A}^- \wedge \delta_\lambda \mathbf{A}^- \right\rangle \\ &= -\frac{\sigma k}{2\pi} \Big\langle \delta \mathbf{A}^+ \wedge \mathrm{d}\lambda^+ + \delta \mathbf{A}^+ \wedge \left[\mathbf{A}^+, \lambda^+\right] \\ &\qquad - \delta \mathbf{A}^- \wedge \mathrm{d}\lambda^- - \delta \mathbf{A}^- \wedge \left[\mathbf{A}^-, \lambda^-\right] \Big\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma k}{2\pi} \left\langle \mathrm{d}(\delta\mathbf{A}^+ \wedge \lambda^+) - \delta\,\mathrm{d}\mathbf{A}^+\,\lambda^+ - 2\delta\mathbf{A}^+ \wedge \mathbf{A}^+\lambda^+ \right. \\
&\qquad\qquad \left. - \,\mathrm{d}(\delta\mathbf{A}^- \wedge \lambda^-) + \delta\,\mathrm{d}\mathbf{A}^-\,\lambda^- + 2\delta\mathbf{A}^- \wedge \mathbf{A}^-\lambda^- \right\rangle \\
&= \frac{\sigma k}{2\pi} \left\langle \mathrm{d}(\delta\mathbf{A}^+ \wedge \lambda^+) - \delta\,\mathrm{d}\mathbf{A}^+\,\lambda^+ - \delta(\mathbf{A}^+ \wedge \mathbf{A}^+)\lambda^+ \right. \\
&\qquad\qquad \left. - \,\mathrm{d}(\delta\mathbf{A}^- \wedge \lambda^-) + \delta\,\mathrm{d}\mathbf{A}^-\,\lambda^- + \delta(\mathbf{A}^- \wedge \mathbf{A}^-)\lambda^- \right\rangle \\
&= \frac{\sigma k}{2\pi} \left\langle \mathrm{d}(\delta\mathbf{A}^+ \wedge \lambda^+) - \delta\mathbf{F}^+\lambda^+ - \mathrm{d}(\delta\mathbf{A}^- \wedge \lambda^-) + \delta\mathbf{F}^-\lambda^- \right\rangle \\
&\approx \frac{\sigma k}{2\pi} \mathrm{d}\left\langle \delta\mathbf{A}^+ \wedge \lambda^+ - \delta\mathbf{A}^- \wedge \lambda^- \right\rangle .
\end{aligned}
\tag{1.109}
$$

Then

$$
\not\delta H_\lambda^{\mathrm{CS}} = \oint_{\partial\Sigma} \mathbf{k}_\lambda^{\mathrm{CS}} \tag{1.110}
$$

with

$$
\mathbf{k}_\lambda^{\mathrm{CS}} = \frac{\sigma k}{2\pi} \left\langle \delta\mathbf{A}^+\lambda^+ - \delta\mathbf{A}^-\lambda^- \right\rangle . \tag{1.111}
$$

Diffeomorphisms

The Noether current associated with an infinitesimal diffeomorphism ξ is given as

$$
\begin{aligned}
\mathbf{J}_\xi &= \mathbf{\Theta}[\delta_\xi \mathbf{A}^\pm; \mathbf{A}^\pm] - i_\xi \mathbf{L}[\mathbf{A}^\pm] \\
&= \frac{\sigma k}{4\pi} \left\langle -A_\nu^+ \delta_\xi A_\rho^+ + A_\nu^- \delta_\xi A_\rho^- \right. \\
&\qquad \left. -\xi^\mu \left(3A_{[\mu}^+ \partial_\nu A_{\rho]}^+ + 2A_{[\mu}^+ A_\nu^+ A_{\rho]}^+ - 3A_{[\mu}^- \partial_\nu A_{\rho]}^- - 2A_{[\mu}^- A_\nu^- A_{\rho]}^- \right) \right\rangle \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\
&= \frac{\sigma k}{4\pi} \left\langle -A_\nu^+ \xi^\mu \partial_\mu A_\rho^+ - A_\nu^+ \partial_\rho \xi^\mu A_\mu^+ + A_\nu^- \xi^\mu \partial_\mu A_\rho^- + A_\nu^- \partial_\rho \xi^\mu A_\mu^- \right. \\
&\qquad - \xi^\mu \left(A_\mu^+ \partial_\nu A_\rho^+ + A_\nu^+ \partial_\rho A_\mu^+ + A_\rho^+ \partial_\mu A_\nu^+ + 2A_\mu^+ A_\nu^+ A_\rho^+ \right) \\
&\qquad \left. + \xi^\mu \left(A_\mu^- \partial_\nu A_\rho^- + A_\nu^- \partial_\rho A_\mu^- + A_\rho^- \partial_\mu A_\nu^- + 2A_\mu^- A_\nu^- A_\rho^- \right) \right\rangle \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\
&= \frac{\sigma k}{4\pi} \left\langle -\partial_\rho \left(A_\nu^+ \xi^\mu A_\mu^+ \right) - \xi^\mu A_\mu^+ F_{\nu\rho}^+ + \partial_\rho \left(A_\nu^- \xi^\mu A_\mu^- \right) + \xi^\mu A_\mu^- F_{\nu\rho}^- \right\rangle \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\
&\approx \frac{\sigma k}{4\pi} \mathrm{d}\left\langle \mathbf{A}^+ \xi^\mu A_\mu^+ - \mathbf{A}^- \xi^\mu A_\mu^- \right\rangle \\
&\approx \mathrm{d}\mathbf{Q}_\xi .
\end{aligned}
\tag{1.112}
$$

Thus

$$
\mathbf{Q}_\xi = \frac{\sigma k}{4\pi} \left\langle \mathbf{A}^+ \xi^\mu A_\mu^+ - \mathbf{A}^- \xi^\mu A_\mu^-(\xi) \right\rangle . \tag{1.113}
$$

Then

$$
\begin{aligned}
\mathbf{k}_\xi^{\mathrm{CS}} &= \delta\mathbf{Q}_\xi - i_\xi \mathbf{\Theta}[\delta\mathbf{A}^\pm; \mathbf{A}^\pm] - \mathbf{Q}_{\delta\xi} \\
&= \frac{\sigma k}{4\pi} \left\langle \delta\mathbf{A}^+ \xi^\mu A_\mu^+ + \mathbf{A}^+ \delta\xi^\mu A_\mu^+ + \mathbf{A}^+ \xi^\mu \delta A_\mu^+ - \delta\mathbf{A}^- \xi^\mu A_\mu^- \right. \\
&\qquad - \mathbf{A}^- \delta\xi^\mu A_\mu^- - \mathbf{A}^- \xi^\mu \delta A_\mu^- + \xi^\mu A_\mu^+ \delta\mathbf{A}^+ - \xi^\mu \delta A_\mu^+ \mathbf{A}^+ \\
&\qquad \left. - \xi^\mu A_\mu^- \delta\mathbf{A}^- + \xi^\mu \delta A_\mu^- \mathbf{A}^- - \mathbf{A}^+ \delta\xi^\mu A_\mu^+ + \mathbf{A}^- \delta\xi^\mu A_\mu^- \right\rangle \\
&= \frac{\sigma k}{2\pi} \left\langle \xi^\mu A_\mu^+ \delta\mathbf{A}^+ - \xi^\mu A_\mu^- \delta\mathbf{A}^- \right\rangle .
\end{aligned}
\tag{1.114}
$$

Note in particular that this agrees with the result (1.111) for the gauge parameters $\lambda^{\pm} = \xi^{\mu} A^{\pm}_{\mu}$, which is consistent with (1.106).

Also,

$$
\begin{aligned}
\mathbf{k}^{\mathrm{CS}}_{\xi} &= \frac{\sigma k}{2\pi} \left\langle \xi^{\mu} A^{+}_{\mu} \delta \mathbf{A}^{+} - \xi^{\mu} A^{-}_{\mu} \delta \mathbf{A}^{-} \right\rangle \\
&= \frac{\sigma}{8\pi G} \frac{\ell}{2} \eta_{ab} \xi^{\mu} \left(A^{a+}_{\mu} \delta \mathbf{A}^{b+} - A^{a-}_{\mu} \delta \mathbf{A}^{b-} \right) \\
&= \frac{\sigma \ell}{16\pi G} \eta_{ab} \xi^{\mu} \left((\omega^{a}{}_{\mu} + e^{a}_{\mu}/\ell)(\delta\omega^{b} + \delta e^{b}/\ell) - (\omega^{a}{}_{\mu} - e^{a}_{\mu})(\delta\omega^{a} - \delta e^{a}/\ell) \right) \\
&= \frac{\sigma}{8\pi G} \xi^{\mu} \left(\omega^{a}{}_{\mu} \delta e_{a} + e^{a}_{\mu} \delta\omega_{a} \right) \\
&= \mathbf{k}^{e}_{\xi}.
\end{aligned}
\tag{1.115}
$$

The Hamiltonian charge for a diffeomorphism in the Chern-Simons formulation will be the same as in the Cartan formulation. The implications are the same as already discussed for the metric and Cartan formulation. Computing Hamiltonian charges in the Chern-Simons formulation will not necessarily reproduce the results from the metric formulation unless one actively compensates that difference. This is accomplished by translating the form $\mathbf{k}^{\alpha}_{\xi}$ into CS variables. First, note that

$$
e^{a}_{\mu} = \frac{\ell}{2} \left(A^{+a}_{\mu} - A^{-a}_{\mu} \right) \tag{1.116}
$$

This can be inserted into (1.70) to obtain $\mathbf{k}^{\alpha}_{\xi}$ in terms of the gauge connections. Since all that matters is its on-shell value, all Lie derivatives can be replaced with gauge transformations.

This concludes the general discussion of Hamiltonian surface charges in 3d Einstein gravity. The following section is devoted to specific examples that highlight the differences in charges between different formalisms that sometimes, but not always appear.

Chapter 2

Examples of boundary conditions and charges

The analysis of the previous chapter will now be applied to certain boundary conditions for 3d Einstein gravity with negative cosmological constant.

2.1 Bañados geometries

Consider the Bañados geometries [4] with metric

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathcal{L}^+(x^+)(\mathrm{d}x^+)^2 + \mathcal{L}^-(x^-)(\mathrm{d}x^-)^2 + (\mathrm{e}^{2\rho/\ell} + \mathcal{L}^-(x^-)\mathcal{L}^+(x^+)\mathrm{e}^{-2\rho/\ell})\,\mathrm{d}x^-\,\mathrm{d}x^+ \quad (2.1)$$

($\mathcal{L}^\pm(x^\pm)$ are some state-dependent functions of the lightcone-coordinates $x^\pm$) and the asymptotic Killing vector

$$\begin{aligned}\xi = &-\frac{\ell}{2}\left(\epsilon^{+\prime}(x^+) + \epsilon^{-\prime}(x^-)\right)\partial_\rho \\ &+\left(\epsilon^+(x^+) - \frac{1}{2}\mathrm{e}^{-2\rho/\ell}\ell^2\epsilon^{-\prime\prime}(x^-) + \mathcal{O}(\mathrm{e}^{-4\rho/\ell})\right)\partial_+ \\ &+\left(\epsilon^-(x^-) - \frac{1}{2}\mathrm{e}^{-2\rho/\ell}\ell^2\epsilon^{+\prime\prime}(x^+) + \mathcal{O}(\mathrm{e}^{-4\rho/\ell})\right)\partial_-\end{aligned} \quad (2.2)$$

with arbitrary functions $\epsilon^\pm(x^\pm)$. The asymptotic Killing vector preserves the asymptotic structure of the metric at $\rho\to\infty$. The corresponding Hamiltonian surface charge is given by

$$\oint H_\xi = -\frac{1}{8\pi G}\oint_{\rho\to\infty}\mathrm{d}\phi\,\left(\epsilon^{+\prime}(x^+)\delta\mathcal{L}(x^+) - \epsilon^{-\prime}(x^-)\delta\mathcal{L}(x^-)\right) \quad (2.3)$$

where the angular coordinate $\phi = (x^+ + x^-)/(2\ell)$ runs from 0 to 2π. The charge is integrable if the functions $\epsilon^\pm(x^\pm)$ are not state-dependent.

The same calculation can be performed in the Cartan formalism with the triad

$$e_\rho = J_3, \quad (2.4a)$$

$$e_+ = \frac{1}{2}\left(\mathrm{e}^{\rho/\ell} - \mathcal{L}^+(x^+)\mathrm{e}^{-\rho/\ell}\right)J_1 - \frac{1}{2}\left(\mathrm{e}^{\rho/\ell} + \mathcal{L}^+(x^+)\mathrm{e}^{-\rho/\ell}\right)J_2, \quad (2.4b)$$

$$e_- = -\frac{1}{2}\left(\mathrm{e}^{\rho/\ell} - \mathcal{L}^-(x^-)\mathrm{e}^{-\rho/\ell}\right)J_1 - \frac{1}{2}\left(\mathrm{e}^{\rho/\ell} + \mathcal{L}^-(x^-)\mathrm{e}^{-\rho/\ell}\right)J_2. \quad (2.4c)$$

The gauge connection forms are then given as

$$\mathbf{A}^{\pm} = \frac{\pm 1}{\ell} J_3 \, \mathrm{d}\rho + \frac{\mathrm{d}x^{\pm}}{\ell} \left(\mathrm{e}^{\rho/\ell} \left(J_1 \mp J_2 \right) - \mathcal{L}^{\pm} \mathrm{e}^{-\rho/\ell} \left(J_1 \pm J_2 \right) \right) . \tag{2.5}$$

A brief calculation shows that $\boldsymbol{\alpha} \to 0$ as $\rho \to \infty$. Therefore, in the case of the Bañados geometries, the presymplectic potentials agree in the asymptotic region and the results for the Hamiltonian charges of the metric and Cartan formalism agree.

2.2 Near horizon symmetry algebra

One can formulate boundary conditions for BTZ geometries at the horizon instead of the asymptotic region [8, 9]. In the metric formalism and Gaussian normal coordinates such boundary conditions are given by

$$\begin{aligned} \mathrm{d}s^2 =& \mathrm{d}r^2 - \left((a^2\ell^2 - \Omega^2) \cosh^2(r/\ell) - a^2\ell^2 \right) \mathrm{d}t^2 + 2 \left(\gamma\Omega \cosh^2(r/\ell) + a\omega\ell^2 \sinh^2(r/\ell) \right) \mathrm{d}t \, \mathrm{d}\phi \\ &+ \left(\gamma^2 \cosh^2(r/\ell) - \omega^2\ell^2 \sinh^2(r/\ell) \right) \mathrm{d}\phi^2 . \end{aligned} \tag{2.6}$$

The symbols $a, \Omega, \omega, \gamma$ represent functions of t and ϕ and the equations of motion demand that $\partial_t \gamma = \partial_\phi \Omega$ and $\partial_t \omega = -\partial_\phi a$. The Killing vector that preserves the general form of this metric is given as

$$\xi = \frac{\eta^+ \mathcal{J}^- + \eta^- \mathcal{J}^+}{\zeta^+ \mathcal{J}^- + \zeta^- \mathcal{J}^+} \partial_t + \frac{\eta^+ \zeta^- - \eta^- \zeta^+}{\zeta^+ \mathcal{J}^- + \zeta^- \mathcal{J}^+} \partial_\phi . \tag{2.7}$$

The $\eta^{\pm}$ are arbitrary functions of ϕ while $\mathcal{J}^{\pm}$ and $\zeta^{\pm}$ are just reparametrizations of the functions $\gamma, \Omega, \omega, a$ in the metric related via

$$\mathcal{J}^{\pm} = \gamma \ell^{-1} \pm \omega \tag{2.8}$$

and

$$\zeta^{\pm} = -a \pm \Omega \ell^{-1} . \tag{2.9}$$

Only the functions $\mathcal{J}^{\pm}$ are allowed to vary while the chemical potentials $\zeta^{\pm}$ are considered to be fixed.

This allows to rewrite the metric (2.6) in terms of the functions $\mathcal{J}^{\pm}$ and $\zeta^{\pm}$. It is then straightforward to compute its variation (only allowing $\mathcal{J}^{\pm}$ to vary). The Hamiltonian surface charge is

$$\not\delta H_\xi = \frac{\ell}{16G\pi} \oint_{r=0} \mathrm{d}\phi \left(\eta^+(\phi) \delta\mathcal{J}^+(t,\phi) + \eta^-(\phi) \delta\mathcal{J}^-(t,\phi) \right) . \tag{2.10}$$

The integral is performed over $r = 0$, which corresponds to the horizon of the spacetime (2.6), but the radial coordinate actually drops out in (2.10).

In the Chern-Simons formalism the triad

$$e_t = \sqrt{\ell^2 a^2 \sinh^2(r/\ell) - \cosh^2(r/\ell) \Omega^2} \, J_1 , \tag{2.11a}$$

$$e_r = J_3 , \tag{2.11b}$$

$$e_\phi = - \frac{\ell^2 a \sinh^2(r/\ell)\omega + \cosh^2(r/\ell)\gamma\Omega}{\sqrt{\ell^2 a^2 \sinh^2(r/\ell) - \cosh^2(r/\ell)\Omega^2}} J_1 + \frac{\ell \sinh(2r/\ell)(a\gamma + \omega\Omega)}{\sqrt{4\ell^2 a^2 \sinh^2(r/\ell) - 4\cosh^2(r/\ell)\Omega^2}} J_2 , \tag{2.11c}$$

reproduces the metric (2.6) via (1.35). The Hamiltonian charge calculated in the Cartan formalism reproduces again the result from the metric formalism. Again, there is no need to compute the dressing form and worry about different results in different formalisms.

2.3 3d spacetime in Gaussian null coordinates

In this section a general 3d spacetime in Gaussian null coordinates will be discussed in detail. Consider the gauge choice

$$g_{rr} = g_{r\phi} = 0, \qquad g_{vr} = \eta(v,\phi). \tag{2.12}$$

Here v takes the role of an advanced time coordinate, r can be seen as radial and ϕ as angular spatial coordinates. The same gauge was used to expand the metric near a null hypersurface, which is additionally defined by $g^{rr}|_{r=0} = 0$. This led to three towers of charges [20]. In the following, however, the requirement for $r = 0$ to be a null hypersurface will be dropped. The most general line element compatible with this gauge choice can be written as

$$\mathrm{d}s^2 = -\mathcal{V}(v,r,\phi)\,\mathrm{d}v^2 + 2\eta(v,\phi)\,\mathrm{d}v\,\mathrm{d}r + h(v,r,\phi)\left(\mathrm{d}\phi + \mathcal{U}(v,r,\phi)\,\mathrm{d}v\right)^2. \tag{2.13}$$

The equations of motion

$$\mathcal{E}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{\ell^2}g_{\mu\nu} = 0 \tag{2.14}$$

constrain the functions $\mathcal{V}$, h and $\mathcal{U}$. $\mathcal{E}_{rr}$ fixes

$$h = \Omega^2 + rh_1 + r^2\frac{h_1^2}{4\Omega^2}. \tag{2.15}$$

Ω and h_1 appear as integration constants and are functions of v and ϕ. In order to guarantee a negative determinant of the metric, the function h_1 most be positive everywhere. $\mathcal{E}_{r\phi}$ gives

$$\mathcal{U} = \frac{4f_0\Omega^2 + rf_1\left(rh_1 + 4\Omega^2\right) - r^2h_1\partial_\phi\eta}{\left(rh_1 + 2\Omega^2\right)^2} \tag{2.16}$$

with the integration constants f_0 and f_1, which are again functions of v and ϕ. $\mathcal{E}_{\phi\phi}$ fixes

$$\begin{aligned}\mathcal{V} = &-\frac{1}{\left(rh_1 + 2\Omega^2\right)^2}\left(r^2F_0h_1^2 + r^3F_1h_1^2 + r^4h_1^2\eta^2/\ell^2 + 4f_0^2\Omega^2 + 4rF_0h_1\Omega^2\right.\\ &+ 4r^2F_1h_1\Omega^2 + 4r^3h_1\eta^2\Omega^2/\ell^2 + 4F_0\Omega^4 + 4rF_1\Omega^4 + 4r^2\eta^2\Omega^4/\ell^2\\ &+ r^2f_1^2\left(rh_1 + 3\Omega^2\right) + 2r^2f_1\Omega^2\partial_\phi\eta - r^3h_1\left(\partial_\phi\eta\right)^2 - r^2\Omega^2\left(\partial_\phi\eta\right)^2\\ &\left.+ 2rf_0\left(f_1\left(rh_1 + 4\Omega^2\right) - rh_1\partial_\phi\eta\right)\right)\end{aligned} \tag{2.17}$$

with integration constants F_0 and F_1 as functions of v and ϕ. Then $\mathcal{E}_{vr}$ allows to express F_1 in terms of the other integration constants and η

$$\begin{aligned}F_1 =&\frac{1}{\ell^2h_1\Omega}\Big(-f_1^2\Omega + 4\eta^2\Omega^3/\ell^2 + \Omega\left(\partial_\phi\eta\right)^2 - 2f_1\eta\partial_\phi\Omega\\ &+ 2\eta\left(\partial_\phi\eta\partial_\phi\Omega + \Omega\left(\partial_\phi f_1 - \partial_\phi^2\eta - \partial_v h_1\right) + h_1\partial_v\Omega\right)\Big).\end{aligned} \tag{2.18}$$

The solution is then parametrized by the six functions η, F_0, f_0, f_1, Ω and h_1 of v and ϕ. The last two equations constrain, but do not fully determine these functions, for example one can find expressions for $\partial_v F_0$ and $\partial_v f_1$.

Taylor expanding the metric around $r = 0$ yields

$$g_{\mu\nu} = \begin{pmatrix} -F_0(v,\phi) - rF_1(v,\phi) + \mathcal{O}(r^2) & \eta(v,\phi) & f_0(v,\phi) + rf_1(v,\phi) + \mathcal{O}(r^2) \\ \eta(v,\phi) & 0 & 0 \\ f_0(v,\phi) + rf_1(v,\phi) + \mathcal{O}(r^2) & 0 & \Omega(v,\phi)^2 + rh_1(v,\phi) + \mathcal{O}(r^2) \end{pmatrix}. \tag{2.19}$$

All the $\mathcal{O}(r^2)$ contributions are exactly of second order in r. There are no higher order contributions. The function $F_1(v,\phi)$ is not arbitrary, but given by (2.18).

To find the diffeomorphisms that preserve the form of (2.19) the Ansatz

$$\xi = \xi^v(v,r,\phi)\partial_v + \xi^r(v,r,\phi)\partial_r + \xi^\phi(v,r,\phi)\partial_\phi \tag{2.20}$$

can be inserted into the Killing equation

$$\mathcal{L}_\xi g_{\mu\nu} = \delta_\xi g_{\mu\nu}. \tag{2.21}$$

The equation $\mathcal{L}_\xi g_{rr} = 0$ gives

$$\xi^v(v,r,\phi) = T \tag{2.22}$$

with some arbitrary function $T(v,\phi)$. In a similar fashion $\mathcal{L}_\xi g_{r\phi}$ fixes

$$\xi^\phi = Y - \frac{2r\eta\partial_\phi T}{rh_1 + 2\Omega^2}, \tag{2.23}$$

with the integration constant $Y(v,\phi)$. In the equation $\mathcal{L}_\xi g_{vr} = \delta_\xi\eta(v,\phi)$ the right hand side must not be r-dependent. Integration then yields

$$\begin{aligned}\xi^r = \frac{1}{\eta\,(rh_1 + 2\Omega^2)}\big(&rh_1\left(\eta K - r\left(\partial_v T\eta + T\partial_v\eta + Y\partial_\phi\eta\right) + r\delta_\xi\eta\right) + 2\eta\Omega^2 K + 2r\partial_\phi T f_0\eta \\ &+ r^2\partial_\phi T f_1\eta + r^2\partial_\phi T\eta\partial_\phi\eta - 2r\partial_v T\eta\Omega^2 - 2rT\partial_v\eta\Omega^2 + 2r\delta_\xi\eta\Omega^2 - 2rY\partial_\phi\eta\Omega^2\big).\end{aligned} \tag{2.24}$$

This expression contains a v- and ϕ-dependent integration constant that includes the arbitrary function $K(v,\phi)$. Furthermore, $\delta_\xi\eta(v,\phi)$ can be chosen at will. It is convenient to choose

$$\delta_\xi\eta = -W\eta - \frac{f_0\eta\partial_\phi T}{\Omega^2} + Y\partial_\phi\eta + 2\eta\partial_v T + T\partial_v\eta \tag{2.25}$$

with the new arbitrary function $W(v,\phi)$. In total, the Killing vector

$$\begin{aligned}\xi^v &= T \\ \xi^r &= \frac{1}{rh_1\Omega^2 + 2\Omega^4}\big(rh_1\left(K\Omega^2 - r\left(W\Omega^2 + f_0\partial_f T - \Omega^2\partial_v T\right)\right) \\ &\qquad + \Omega^2\left(2K\Omega^2 + r\left(-2W\Omega^2 + rf_1\partial_\phi T + 2\Omega^2\partial_v T + r\partial_\phi T\partial_\phi\eta\right)\right)\big) \\ \xi^\phi &= Y - \frac{2r\eta\partial_\phi T}{rh_1 + 2\Omega^2}\end{aligned} \tag{2.26}$$

is parametrized by four arbitrary functions $T(v,\phi)$, $Y(v,\phi)$, $K(v,\phi)$ and $W(v,\phi)$. Since in the following the surface charges will be computed at the $r=0$ hypersurface and they only contain single derivatives of the Killing vector, it is only relevant up to linear order and a Taylor expansion yields

$$\begin{aligned}\xi^v &= T, \\ \xi^r &= K + (\partial_v T - W)\,r + \mathcal{O}(r^2), \\ \xi^\phi &= Y - \frac{\eta\partial_\phi T}{\Omega^2}r + \mathcal{O}(r^2).\end{aligned} \tag{2.27}$$

The transformation behavior of all state-dependent functions is given by comparing the leading and subleading order in a Taylor expansion of (2.21). Then the commutator algebra of the Killing vector can be computed making use of the adjusted bracket [21]

$$[\xi_1,\xi_2]_{\text{adj. bracket}} := [\xi_1,\xi_2] - \hat{\delta}_{\xi_1}\xi_2 + \hat{\delta}_{\xi_2}\xi_1. \tag{2.28}$$

The first term is the usual commutator of vector fields, equivalent to the Lie derivative, whereas $\hat{\delta}$ requires the components of the following vector to be viewed as functions of the state-dependent functions, which are then transformed under the subscript vector. This gives the Killing vector algebra

$$[\xi(K_1, T_1, W_1, Y_1), \xi(K_2, T_2, W_2, Y_2)]_{\text{adj. bracket}} = \xi(K_{12}, T_{12}, W_{12}, Y_{12}) \tag{2.29}$$

with

$$\begin{aligned} K_{12} &= Y_1\partial_\phi K_2 + T_1\partial_v K_2 + K_2\left(W_1 - \partial_v T_1\right) - (1 \leftrightarrow 2), \\ T_{12} &= Y_1\partial_\phi T_2 + T_1\partial_v T_2 - (1 \leftrightarrow 2), \\ W_{12} &= Y_1\partial_\phi W_2 + T_1\partial_v W_2 + \partial_\phi T_2\partial_v Y_1 + \frac{f_1K_2\partial_\phi T_1 + \eta\partial_\phi K_2\partial_\phi T_1 + K_2\partial_\phi T_1\partial_\phi\eta}{\Omega^2} \\ &\quad + \frac{f_0h_1K_1\partial_\phi T_2}{\Omega^4} - (1 \leftrightarrow 2), \\ Y_{12} &= Y_1\partial_\phi Y_2 + T_1\partial_v Y_2 + \frac{\eta K_2\partial_\phi T_1}{\Omega^2} - (1 \leftrightarrow 2)\,. \end{aligned} \tag{2.30}$$

Here $-(1 \leftrightarrow 2)$ refers to repeating all the previous terms with opposite sign and exchanging the labels 1 and 2.

Under the Killing vector (2.26) the state-dependent functions transform as

$$\begin{aligned} \delta_\xi F_0 &= 2F_0\partial_v T + T\partial_v F_0 - 2\eta\partial_v K - 2f_0\partial_v Y + Y\partial_\phi F_0 \\ &\quad + \frac{K(\partial_\phi\eta)^2 - f_1^2K - 2K\eta\partial_\phi^2\eta + 4K\eta^2\Omega^2/\ell^2}{h_1} \\ &\quad + \frac{2K\eta\partial_\phi\eta\partial_\phi\Omega}{h_1\Omega} + \frac{2K\eta\Omega}{h_1}\left[\partial_\phi\left(\frac{f_1}{\Omega}\right) - \partial_v\left(\frac{h_1}{\Omega}\right)\right], \end{aligned} \tag{2.31}$$

$$\delta_\xi\eta = Y\partial_\phi\eta - W\eta - \frac{f_0\eta\partial_\phi T}{\Omega^2} + 2\eta\partial_v T + T\partial_v\eta, \tag{2.32}$$

$$\delta_\xi f_0 = f_1K + \eta\partial_\phi K - F_0\partial_\phi T + \partial_\phi(f_0Y) + \partial_v(f_0T) + \Omega^2\partial_v Y, \tag{2.33}$$

$$\delta_\xi\Omega = \frac{h_1K}{2\Omega} + \frac{f_0\partial_\phi T}{\Omega} + \partial_\phi(\Omega Y) + T\partial_v\Omega, \tag{2.34}$$

$$\begin{aligned} \delta_\xi f_1 &= 2f_1\partial_v T + h_1\partial_v Y + \partial_\phi(Yf_1) - \eta\partial_\phi W - f_1W + T\partial_v f_1 - \partial_\phi T\partial_v\eta \\ &\quad + \partial_\phi T\left[\frac{(f_1^2 - 2\eta\partial_\phi f_1 - (\partial_\phi\eta)^2 + 2\eta\partial_\phi^2\eta + 2\eta\partial_v h_1 - 4\eta^2\Omega^2/\ell^2)}{h_1}\right. \\ &\quad \left. + \frac{2\eta\partial_\phi\Omega(f_1 - \partial_\phi\eta)}{h_1\Omega} - \frac{1}{\Omega}\partial_\phi\left(\frac{\eta f_0}{\Omega}\right)\right] + \frac{Kh_1(f_1 - \partial_\phi\eta) - 2f_0\eta\partial_\phi^2 T}{2\Omega^2}, \end{aligned} \tag{2.35}$$

$$\delta_\xi h_1 = \frac{h_1^2K}{2\Omega^2} + 2f_1\partial_\phi T - 2\Omega\partial_\phi\left(\frac{\eta\partial_\phi T}{\Omega}\right) + h_1\left(2\partial_\phi Y - W\right) + Y\partial_\phi h_1 + \partial_v(Th_1). \tag{2.36}$$

The variation of the Hamiltonian surface charge as computed in the metric formalism is given by[1]

$$\not{\delta} H^g_\xi = \frac{1}{16\pi G}\oint_{r=0} \mathrm{d}\phi\left(W\delta\Omega + Y\delta\Upsilon + T\not{\delta}\mathcal{A} + K\not{\delta}\mathcal{B}\right) \tag{2.37}$$

[1]In computing $\sqrt{-g}$ it was assumed that the product $\eta\Omega$ is positive. Otherwise, the Hamiltonian surface charge receives an overall negative sign.

where

$$\Upsilon := \frac{f_0 h_1}{\eta\Omega} - \frac{f_1\Omega}{\eta}. \tag{2.38}$$

The expressions $\not\delta\mathcal{A}$ and $\not\delta\mathcal{B}$ are no total variational derivatives and are defined as

$$\begin{aligned}\not\delta\mathcal{A} := &\frac{(F_1+\partial_v\eta)\,\delta\Omega}{\eta} - \delta\left(\frac{\sqrt{F_0}h_1}{\Omega\eta}\right)\sqrt{F_0} - \delta\left(\frac{f_1}{\eta\Omega}\right)f_0 + \delta\left(\partial_\phi\left(\frac{f_0}{\Omega}\right)\frac{1}{\eta}\right)\eta \\ &- 2\delta\left(\frac{\partial_v\Omega}{\sqrt{\eta}}\right)\sqrt{\eta} + \eta\partial_\phi\left(\frac{\delta\Omega f_0}{\Omega^2\eta}\right),\end{aligned} \tag{2.39}$$

$$\not\delta\mathcal{B} := \delta\left(\frac{h_1}{\Omega}\right) - \frac{h_1\delta\eta}{2\eta\Omega}. \tag{2.40}$$

Hence, the Hamiltonian surface charge is, for state-independent W, Y, T and K, not integrable. In general, integrability can be achieved by an invertible redefinition ("change of basis") of these functions and the addition of a "corner term" to the presymplectic potential [22, 23], i.e. a shift

$$\mathbf{\Theta} \to \mathbf{\Theta} + \mathrm{d}c\,. \tag{2.41}$$

To compute the charge in the Cartan formulation the triad

$$e_v = \frac{h\mathcal{U}^2 - \mathcal{V} - \eta^2}{2\eta}J_1 + \frac{h\mathcal{U}^2 - \mathcal{V} + \eta^2}{2\eta}J_2, \tag{2.42a}$$

$$e_r = J_1 + J_2, \tag{2.42b}$$

$$e_\phi = \frac{h\mathcal{U}}{\eta}J_1 + \frac{h\mathcal{U}}{\eta}J_2 + \sqrt{h}J_3 \tag{2.42c}$$

can be used. It leads to the Hamiltonian surface charge

$$\not\delta H^e_\xi = \frac{1}{16\pi G}\oint_{r=0}\mathrm{d}\phi\left(Y\delta\left(\Upsilon - \frac{\Omega\partial_\phi\eta}{\eta}\right) + T\not\delta\mathcal{C} + K\delta\left(\frac{h_1}{\Omega}\right)\right) \tag{2.43}$$

with

$$\not\delta\mathcal{C} := f_0\delta\left(\frac{\partial_\phi\eta}{\eta\Omega}\right) - \delta\left(\frac{F_0h_1 + f_0f_1}{\eta\Omega}\right) - 2\delta\left(\partial_v\Omega\right) + \frac{2\delta\eta\partial_v\Omega + F_1\delta\Omega}{\eta} + \frac{2f_1\delta f_0 + h_1\delta F_0}{2\eta\Omega}. \tag{2.44}$$

Note that the function W does not appear in (2.43). Assuming state-independent Y, T and K, the term containing K is now integrable. The coefficient of T is different, but still not integrable. The main point, however, is that the charges in the two formalisms do not agree. This is due to a nonzero dressing form leading to

$$\begin{aligned}\mathbf{k}^\alpha_\xi = \frac{1}{16\pi G}\Bigg(&W\delta\Omega - \frac{Kh_1\delta\eta}{2\eta\Omega} + \frac{Tf_0\delta\Omega\partial_\phi}{\Omega^2} - \frac{2T\delta f_0\partial_\phi}{\Omega} + \frac{Tf_0\delta\eta\partial_\phi}{\eta\Omega} \\ &+ \frac{Y\delta\Omega\partial_\phi\eta - Y\delta\eta\Omega\partial_\phi - Y\delta\eta\partial_\phi\Omega + T\delta\Omega\partial_v\eta - T\delta\eta\partial_v\Omega}{\eta}\Bigg).\end{aligned} \tag{2.45}$$

One can now check that

$$(\mathbf{k}^g_\xi)_\phi - (\mathbf{k}^e_\xi)_\phi - (\mathbf{k}^\alpha_\xi)_\phi = \frac{1}{16\pi G}\partial_\phi\left(\frac{T\delta f_0}{\Omega} + \frac{Y\delta\eta\Omega}{\eta} - \frac{Tf_0\delta\eta}{\eta\Omega}\right), \tag{2.46}$$

which vanishes when integrated over ϕ given that the functions in (2.46) are 2π-periodic in ϕ. Since ϕ is an angular coordinate with identification $\phi \sim \phi + 2\pi$ any single-valued function of ϕ must be periodic. In all previous sections it was assumed that functions of ϕ are single-valued, but this assumption could also be dropped. For example, one could allow a quasi-periodic function

$$T(v, \phi) = \sum_{n \in \mathbb{Z}} T_n(v) \mathrm{e}^{\mathrm{i} n \phi} + T_\phi(v) \phi \tag{2.47}$$

for which

$$T(v, \phi + 2\pi) = T(v, \phi) + 2\pi \, T_\phi(v) \tag{2.48}$$

As a consequence, the integral of (2.46) over ϕ from 0 to 2π would not necessarily vanish.

To conclude, the general 3d spacetime in Gaussian null coordinates serves as an example where the Hamilonian surface charge differs between the metric and first order Cartan (and therefore Chern-Simons) formalism. By introducing the form $\mathbf{k}_\xi^\alpha$ the difference between $\mathbf{k}_\xi^g$ and $\mathbf{k}_\xi^e$ can be compensated, but only up to an additional exact term. Allowing only single-valued functions, this term (of which in the previous example only the ϕ-component is of relevance) then vanishes upon integration.

Conclusion and outlook

A careful treatment of the Hamiltonian surface charge in 3d Einstein gravity using the covariant phase space formalism shows that there are, in general, differences depending on whether the charges are computed in second order metric or first order Cartan formalism. A calculation in the Chern-Simons formalism recovers the results from the first order Cartan formalism. The difference can be attributed to an additional symmetry that is present in the Cartan formulation of Einstein gravity, the symmetry under local Lorentz transformations. To resolve the difference in surface charges one can make use of the ambiguity in the presymplectic potential. Since the latter is only defined up to a closed and thus locally exact form, one can add a specific corner term, which has been called the dressing form, to the presymplectic potential. This procedure is well supported in the covariant phase space formalism as there is no a priori prescription how to resolve the ambiguity surrounding the presymplectic potential. The addition of corner terms to the presymplectic potential is also intimately related to the renormalization of charges and the integrability problem [22].

However, as the examples from chapter 2 highlight, ignoring the ambiguity regarding the presymplectic potential and strictly following the Iyer-Wald prescription can lead to contradicting results for the Hamiltonian surface charges, but this is not always the case. In the example of the Bañados geometries and the near horizon boundary conditions presented in chapter 2 there is no difference in the Hamiltonian surface charge between metric and first order Cartan formalism, even without adding a corner term. This is not the case, however, for the general 3d spacetime in Gaussian null coordinates, where different results for the (variation of the) Hamiltonian surface charge were found in the metric and first order Cartan formalism. Only by explicitly computing the contribution from the dressing form the contradiction can be resolved. It remains an open question why the first two examples exhibit vanishing dressing form while the third example does not.

There is another interesting question surrounding the results of section 2.3. Assuming the problem of integrability can be solved, there will be a Hamiltonian surface charge parametrized by four arbitrary functions. This is two less than what is maximally possible [7], but the only restriction on the spacetime of section 2.3 is the gauge choice. It appears that the gauge choice alone removes two towers of charges. Exactly why and how this happens could be a topic of further research.

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