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Finite Element Error Analysis and Solution Stability of Affine Optimal Control Problems *

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Abstract

We consider affine optimal control problems subject to semilinear elliptic PDEs. The results are two-fold; first, we continue the analysis of solution stability of control problems under perturbations appearing jointly in the objective functional and the PDE. In regard to this, we prove that a coercivity-type property, that appears in the context of optimal control problems where the optimal control is of bang-bang structure, is sufficient for solution stability estimates for the optimal controls. The second result is concerned with the obtainment of error estimates for the numerical approximation for a finite element and a variational discretization scheme. The error estimates for the optimal controls and states are obtained under several conditions of different strengths, that appeared recently in the context of solution stability. The approaches used for the proofs are motivated by the structure of the assumptions and enable an improvement of the error estimates for the finite element scheme for the optimal controls and states under a Hölder-type growth condition.

1 Introduction

To obtain error estimates for a finite element approximation scheme and to study solution stability of PDE-constrained optimal control problems where the Tikhonov regularization term is absent in the objective functional, one must take into consideration additional difficulties. This is due to the fact, that in this case, one cannot expect a coercivity-type growth of the second variation of the objective functional with respect to the L^2-distance of the controls. In this paper, we address these difficulties by considering several coercivity-type conditions on the joint growth of the first and second variation of the objective functional. In regard to solution stability of optimal controls and states under

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perturbations appearing in the objective functional and the state equation simultaneously, the result of this paper is a continuation of the investigation in [4, 11, 12]. In this paper, we consider a growth condition that is similar to the one in [11], but weaker and that still allows for a Lipschitz-type estimate for the optimal controls with respect to the perturbations. The main part of this paper, deals with error estimates for the numerical approximation for problems where the control appears at most in an affine way in the PDE and builds upon the papers [7, 8, 10, 13]. In comparison to the results therein, we obtain error estimates for the optimal controls without assuming the so-called structural assumption on the adjoint state together and a second-order sufficient condition assuming the second variation to be strict positive on a certain cone, see [6, 12] for a discussion of various assumptions on the growth of the objective functional. Instead, we work with the unified conditions established in [4, 11, 12]. This is an improvement since to assume the structural assumption on the adjoint state and a second order sufficient condition as in [7, **?**] seems quite strong. Especially since by assuming the structural assumption one can obtain already stability results as long as the second variation is not too negative, as shown in this paper. To be precise, we consider among others the condition: Given a reference optimal control $\bar{u}$ and a number $\gamma \in (0, 1]$, there exist positive constants α and c such that

$$J(u) - J(\bar{u}) \geq c\|u - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{1.1}$$

for all feasible controls u with $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$ and $(u - \bar{u}) \in D_{\bar{u}}^{\tau}$.
Here, $D_{\bar{u}}^{\tau}$ denotes a cone, that extends the cone of critical directions commonly used in optimal control of PDEs and will be specified later on. See also [3], where it was first introduced.

Conditions of type (1.1) arise naturally in the characterization of strict bang-bang optimal controls, appearing as a consequence of sufficient second-order optimality conditions and the structural assumption on the adjoint state, see [10]. A slightly stronger assumption that implies (1.1) was first considered in [18] for affine ODE optimal control problems and [11] for PDE optimal control problems. Recently (1.1) appeared in [16, 17] in the context of Eigenvalue optimization problems. There it was shown that for a certain type of Eigenvalue optimization problem, condition (1.1) is implied by a growth of the second variation. To relate (1.1) with the classical assumptions used in affine PDE-constrained optimal control problems we refer to Theorem 4 in Section 4. We remark that to apply condition (1.1) for solution stability, we need that the controls corresponding to the perturbed problems are minimizers. This is not the case under the slightly stronger condition in [11], where it is sufficient that the controls corresponding to the perturbed problem satisfy a first-order optimality condition.
Finally, we do not consider a sparsity-promoting term appearing in the objective functional, but the proofs in this paper can be easily adapted to include such a term. One can consider a semilinear elliptic non-monotone and non-coercive state equation as in [9] without any changes of the results in the section on solution stability. The sections on error estimates can be adapted to the case

of a non-monotone and non-coercive state equation using the results in [5]. To the author's best knowledge, the assumptions considered in this paper are the weakest so far that still allow error estimates for the numerical approximation for problems where the control appears at most in an affine way in the objective functional.

We shortly list the novelties in the paper. Under conditions similar to the one introduced in [4] in the context of solution stability and conditions (1.1), we derive error estimates for a finite element scheme in Theorem 6.5. For this, in the proof of Theorem 6.5, we argue similar as in the first arguments in [7, Theorem 7], but in contrast to the proof therein, we then use the approximation property of the linearized state to conclude the proof. This allows to obtain error estimates for bang-bang optimal controls similar as in [7, Theorem 9] and also allows to improve the error estimates for the optmial controls for $\gamma \in (0,1]$, from $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \leq ch^{\gamma^2}$ to $\|\bar{u}_y - \bar{u}\|_{L^1(\Omega)} \leq ch^{\gamma}$. Then, using the assumptions of [11, 4], we prove error estimates for a variational discretization in Theorem 6.9. In regard to solution stability, we prove that condition (1.1) is sufficient for solution stability under quite general perturbations for a distributed control problems in Theorem 5.2. Until now, to the author's best knowledge, it was an open question if condition (1.1) by itself allows for these results. We believe this approach to be feasible also for the obtainment of error estimates for the numerical approximation for a 2-dimensional Neumann boundary control problem, but postpone this analysis to future work.

The paper is structured as follows: In the remainder of this section, we state the main assumptions that hold throughout the paper and state some additional remarks on the notation. In Section 2, we collect results on the involved PDEs, and in Section 3 the optimal control problem is discussed. In Section 4, we investigate the sufficient conditions for local optimality. Section 5 is concerned with solution stability. In Section 6 we define the discretization schemes and prove error estimates. Let $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, be a bounded domain with Lipschitz boundary.

We investigate the following optimal control problem. Given functions $u_a, u_b \in L^\infty(\Omega)$ such that $u_a < u_b$ a.e in Ω, define the set of feasible controls by

$$\mathcal{U} := \{u \in L^\infty(\Omega) | \ u_a \leq u \leq u_b \ \text{ for a.a. } x \in \Omega\} \tag{1.2}$$

and consider the problem

$$\text{(P)} \quad \min_{u\in\mathcal{U}} \left\{ J(u) := \int_\Omega L(x, y(x), u(x)) \, \mathrm{d}x \right\}, \tag{1.3}$$

subject to

$$\begin{cases} \mathcal{A}y + f(\cdot, y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma. \end{cases} \tag{1.4}$$

Denote by y_u, the unique solution of the state equation that corresponds to the control u. The objective integrands L appearing in (1.3) satisfies additional smoothness conditions, given below in Assumption 2.

1.1 Main assumptions and notation

The following assumptions, close to those in [4, 6, 7, 10], are standing in all of the paper.

Assumption 1 *The following statements are fulfilled.*

(i) The operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$, is given by

$$\mathcal{A}y := -\sum_{i,j=1}^{n} \partial_{x_j}(a_{i,j}(x)\partial_{x_i} y),$$

where $a_{i,j} \in L^\infty(\Omega)$. In Section 6 we additionally assume that $a_{i,j} \in C^{0,1}(\bar{\Omega})$. Further, the $a_{i,j}$ satisfy the uniform ellipticity condition

$$\exists \lambda_{\mathcal{A}} > 0 : \ \lambda_{\mathcal{A}}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \quad \textit{for all } \xi \in \mathbb{R}^n \ \textit{ and a.a. } x \in \Omega.$$

(ii) We assume that $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function of class C^2 with respect to the second variable satisfying:

$$\begin{cases} f(\cdot,0) \in L^\infty(\Omega) \ \textit{and} \ \frac{\partial f}{\partial y}(x,y) \geq 0 \ \forall y \in \mathbb{R}, \\ \forall M > 0 \ \exists C_{f,M} > 0 \ \textit{s. t.} \ \left|\dfrac{\partial f}{\partial y}(x,y)\right| + \left|\dfrac{\partial^2 f}{\partial y^2}(x,y)\right| \leq C_{f,M} \ \forall |y| \leq M, \\ \forall \rho > 0 \ \textit{and} \ \forall M > 0 \ \exists \ \varepsilon > 0 \ \textit{such that} \\ \left|\dfrac{\partial^2 f}{\partial y^2}(x,y_2) - \dfrac{\partial^2 f}{\partial y^2}(x,y_1)\right| < \rho \ \forall |y_1|, |y_2| \leq M \ \textit{with} \ |y_2 - y_1| \leq \varepsilon, \end{cases}$$

for almost every $x \in \Omega$.

Assumption 2 *The function $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is Carathéodory and of class C^2 with respect to the second variable. In addition, we assume that*

$$\begin{cases} L(x,y,u) = L_a(x,y) + L_b(x,y)u \quad \textit{with} \quad L_a(\cdot,0), L_b(\cdot,0) \in L^1(\Omega), \\ \forall M > 0 \ \exists C_{L,M} > 0 \ \textit{such that} \\ \left|\dfrac{\partial L}{\partial y}(x,y,u)\right| + \left|\dfrac{\partial^2 L}{\partial y^2}(x,y,u)\right| \leq C_{L,M} \ \forall |y|, |u| \leq M, \\ \forall \rho > 0 \ \textit{and} \ M > 0 \ \exists \varepsilon > 0 \ \textit{such that} \\ \left|\dfrac{\partial^2 L}{\partial y^2}(x,y_2,u) - \dfrac{\partial^2 L}{\partial y^2}(x,y_1,u)\right| < \rho \ |y_1|, |y_2| \leq M \ \textit{with} \ |y_2 - y_1| \leq \varepsilon, \end{cases}$$

for almost every $x \in \Omega$.

2 Auxillary results for the state equation

We collect properties of solutions to linear and semilinear elliptic PDEs. The results in this section are standard, we refer to [4, 7]. Let $\alpha \in L^\infty(\Omega)$ be a nonnegative function. We consider the properties of solutions to the linear equation

$$\mathcal{A}y + \alpha y = h. \tag{2.1}$$

Theorem 2.1 *[4, Lemma 2.2] Let $h \in L^r(\Omega)$ with $r > n/2$. Then the linear equation (2.1) has a unique solution $y_h \in H_0^1(\Omega) \cap C(\bar{\Omega})$. Further there exists a positive constant C_r independent of α and h such that*

$$\|h_u\|_{H_0^1(\Omega)} + \|h_u\|_{C(\bar{\Omega})} \le C_r \|u\|_{L^r(\Omega)}. \tag{2.2}$$

Lemma 2.2 *[4, Lemma 2.3] Assume that $s \in [1, \frac{n}{n-2})$, s' is its conjugate, and let $\alpha \in L^\infty(\Omega)$ be a nonnegative function. Then, there exists a constant $C_{s'}$ independent of a such that*

$$\begin{cases} \|y_h\|_{L^s(\Omega)} \le C_{s'} \|h\|_{L^1(\Omega)}, \\ \|\varphi_h\|_{L^s(\Omega)} \le C_{s'} \|h\|_{L^1(\Omega)}, \end{cases} \quad \forall h \in H^{-1}(\Omega) \cap L^1(\Omega), \tag{2.3}$$

where y_h and φ_h satisfy the equations (2.1) and $\mathcal{A}^*\varphi_h + \alpha\varphi_h = h$, respectively, and $C_{s'}$ is given by (2.2) with $r = s'$.

For the semilinear state equation, we cite the following regularity result.

Theorem 2.3 *[7, Theorem 1] For every $u \in L^r(\Omega)$ with $r > n/2$ there exists a unique $y_u \in Y := H_0^1(\Omega) \cap C(\bar{\Omega})$ solution of (1.4). Moreover, there exists a constant $T_r > 0$ independent of u such that*

$$\|y_u\|_{H_0^1(\Omega)} + \|y_u\|_{C(\bar{\Omega})} \le T_r(\|u\|_{L^r(\Omega)} + \|f(\cdot, 0)\|_{L^\infty(\Omega)}).$$

If $u_k \rightharpoonup u$ weakly in $L^p(\Omega)$, then the strong convergence

$$\|y_{u_k} - y_u\|_{H_0^1(\Omega)} + \|y_{u_k} - y_u\|_{C(\bar{\Omega})} \to 0$$

holds. If further $u \in L^\infty(\Omega)$ and $\{a_{i,j}\} \in C^{0,1}(\bar{\Omega})$ we have $y_u \in W^{2,r}(\Omega)$ for all $r < \infty$ and

$$\|y_u\|_{W^{2,r}(\Omega)} \le M_0 r \Big(\|u\|_{L^\infty(\Omega)} + \|f(\cdot, 0)\|_{L^\infty(\Omega)} \Big)$$

for a positive constant M_0 independent of u and r.

For each $r > n/2$, we define the map $G_r : L^r(\Omega) \to H_0^1(\Omega) \cap C(\bar{\Omega})$ by $G_r(u) = y_u$.

Theorem 2.4 *[4, Theorem 2.6] Let Assumption 1 hold. For every $r > \frac{n}{2}$ the map G_r is of class C^2, and the first and second derivatives at $u \in L^r(\Omega)$ in the directions $v, v_1, v_2 \in L^r(\Omega)$, denoted by $z_{u,v} = G_r'(u)v$ and $z_{u,v_1,v_2} = G_r''(u)(v_1, v_2)$, are the solutions of the equations*

$$\mathcal{A}z + \frac{\partial f}{\partial y}(x, y_u)z = v, \tag{2.4}$$

$$\mathcal{A}z + \frac{\partial f}{\partial y}(x, y_u)z = -\frac{\partial^2 f}{\partial y^2}(x, y_u)z_{u,v_1}z_{u,v_2}, \tag{2.5}$$

respectively.

Lemma 2.5 *[4, Lemma 2.7] The following statements are fulfilled.*

(i) Suppose that $r > \frac{n}{2}$ and $s \in [1, \frac{n}{n-2})$. Then, there exist constants K_r depending on r and M_s depending on s such that for every $u, \bar{u} \in \mathcal{U}$

$$\|y_u - y_{\bar{u}} - z_{\bar{u},u-\bar{u}}\|_{L^s(\Omega)} \leq M_s \|y_u - y_{\bar{u}}\|^2_{L^2(\Omega)}. \tag{2.6}$$

(ii) Taking $C_X = K_2\sqrt{|\Omega|}$ if $X = C(\bar{\Omega})$ and $C_X = M_2$ if $X = L^2(\Omega)$, the following inequality holds

$$\|z_{u,v} - z_{\bar{u},v}\|_X \leq C_X \|y_u - y_{\bar{u}}\|_X \|z_{\bar{u},v}\|_X \ \forall u, \bar{u} \in \mathcal{U} \text{ and } \forall v \in L^2(\Omega). \tag{2.7}$$

(iii) Let be X as in (ii). There exists $\varepsilon > 0$ such that for all $\bar{u}, u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{C(\bar{\Omega})} \leq \varepsilon$ the following inequalities are satisfied

$$\frac{1}{2}\|y_u - y_{\bar{u}}\|_X \leq \|z_{\bar{u},u-\bar{u}}\|_X \leq \frac{3}{2}\|y_u - y_{\bar{u}}\|_X, \tag{2.8}$$

$$\frac{1}{2}\|z_{\bar{u},v}\|_X \leq \|z_{u,v}\|_X \leq \frac{3}{2}\|z_{\bar{u},v}\|_X \quad \forall v \in L^2(\Omega). \tag{2.9}$$

3 The optimal control problem

The optimal control problem (1.3)-(1.2) is well posed under Assumptions 1 and 2. By the direct method of calculus of variations one can easily prove that there exists at least one global minimizer, see [20, Theorem 5.7]. In this section, we discuss the structure of the optimal control problem.

Definition 1 *We say that $\bar{u} \in \mathcal{U}$ is an $L^r(\Omega)$-weak local minimum of problem (1.3)-(1.2), if there exists some positive ε such that*

$$J(\bar{u}) \leq J(u) \quad \forall u \in \mathcal{U} \text{ with } \|u - \bar{u}\|_{L^r(\Omega)} \leq \varepsilon.$$

We say that $\bar{u} \in \mathcal{U}$ is a strict weak local minimum if the above inequalities are strict for $u \neq \bar{u}$.

Theorem 3.1 *For every $r > \frac{n}{2}$, the functional $J : L^r(\Omega) \longrightarrow \mathbb{R}$ is of class C^2. Moreover, given $u, v, v_1, v_2 \in L^r(\Omega)$ we have*

$$\begin{aligned} J'(u)v &= \int_\Omega \Big[\frac{\partial L}{\partial y}(x, y_u, u)\Big] z_{u,v} + \Big[\frac{\partial L}{\partial u}(x, y_u, u)\Big] v \,\mathrm{d}x \\ &= \int_\Omega \Big[p_u + \frac{\partial L}{du}(x, y_u, u)\Big] v \,\mathrm{d}x, \\ J''(u)(v_1, v_2) &= \int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y_u, u) - p_u \frac{\partial^2 f}{\partial y^2}(x, y_u)\Big] z_{u,v_1} z_{u,v_2} \,\mathrm{d}x \\ &\quad + \int_\Omega \Big[\frac{\partial^2 L}{\partial u \partial y}(x, y_u, u)\Big](z_{u,v_1} v_2 + z_{u,v_2} v_1) \,\mathrm{d}x. \end{aligned}$$

Here, $p_u \in H_0^1(\Omega) \cap C(\bar{\Omega})$ is the unique solution of the adjoint equation

$$\left\{ \begin{array}{l} \mathcal{A}^* p + \dfrac{\partial f}{\partial y}(x, y_u) p = \dfrac{\partial L}{\partial y}(x, y_u, u) \text{ in } \Omega, \\ p = 0 \text{ on } \partial\Omega. \end{array} \right. \tag{3.1}$$

We introduce the Hamiltonian $\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ni (x, y, p, u) \mapsto H(x, y, p, u) \in \mathbb{R}$ in the usual way:

$$H(x, y, p, u) := L(x, y, u) + p(u - f(x, y)). \tag{3.2}$$

The local form of the Pontryagin type necessary optimality conditions for problem (1.3)-(1.2) in the next theorem is well known (see e.g. [2, 6, 20]).

Theorem 3.2 *If $\bar{u}$ is a weak local minimizer for problem (1.3)-(1.2), then there exist unique elements $\bar{y}, \bar{p} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that*

$$\left\{ \begin{array}{l} \mathcal{A}\bar{y} + f(x, \bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \partial\Omega. \end{array} \right. \tag{3.3}$$

$$\left\{ \begin{array}{l} \mathcal{A}^* \bar{p} = \dfrac{\partial H}{\partial y}(x, \bar{y}, \bar{p}, \bar{u}) \text{ in } \Omega, \\ \bar{p} = 0 \text{ on } \partial\Omega. \end{array} \right. \tag{3.4}$$

$$\int_\Omega \frac{\partial H}{\partial u}(x, \bar{y}, \bar{p}, \bar{u})(u - \bar{u}) \,\mathrm{d}x \geq 0 \quad \forall u \in \mathcal{U}. \tag{3.5}$$

4 The optimality conditions

We consider conditions on the objective functional that are sufficient for local optimality and solution stability under perturbations appearing jointly in the objective functional and the equation.

Assumption 3 *Let $\bar{u} \in \mathcal{U}$, $\gamma \in (n/(2+n), 1]$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants c and α such that*

$$J'(\bar{u})(u-\bar{u}) + \beta J''(\bar{u})(u-\bar{u})^2 \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{4.1}$$

for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$.

In this paper, we prove for the first time, that Assumption 3 for $\beta = 1/2$, is sufficient for solution stability if the control corresponding to the perturbed problem is a minimizer. Assumption 3 with $\beta = 1$ was considered in [11], it is the stronger assumption, in the sense, that if $\bar{u}$ satisfies the first-order optimality condtion, Assumption 3($\beta = 1$) implies Assumption 3($\beta = 1/2$). On the other hand, if Assumption 3($\beta = 1$) is satisfied, to obtain a solution stability result, the control corresponding to a perturbed problem needs to satisfy only the first-order necessary optimality condition. Under Assumption 3($\beta = 1/2$) we need that the controls corresponding to the pertrubed problems are minimizers. For the achievement of error estimates for the numerical approximation, this is not a constraint, here, we consider minimizers of the discrete problem. We state some technical results that are needed later on

Lemma 4.1 *[4, Proposition 5.3] Let $\bar{u} \in \mathcal{U}$ be given. It is equivalent:*

1. *There exist positive constants c and α such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{4.2}$$

   for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$.

2. *There exist positive constants c and α such that (4.2) holds for all $u \in \mathcal{U}$ with $\|y_u - y_{\bar{u}}\|_{L^\infty(\Omega)} < \alpha$.*

The next lemma is crucial for the estimations later on. It is well known for objective functionals with varying generality and was proven in several publications for case $\gamma = 1$. The proof for $\gamma < 1$ follows by exactly the same arguments.

Lemma 4.2 *Given $\gamma \in (n/(2+n), 1]$ and $\bar{u}, u \in \mathcal{U}$. Define $u_\theta := \bar{u} + \theta(u-\bar{u})$ for some $\theta \in [0, 1]$. For all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\left| J''(\bar{u})(u-\bar{u})^2 - J''(u_\theta)(u-\bar{u})^2 \right| \leq \epsilon\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$$

for all $\|u-\bar{u}\|_{L^1(\Omega)} < \delta$.

Further, we have the following result.

Theorem 4.3 *Let $\bar{u} \in \mathcal{U}$ be given. It is equivalent:*

1. *The control $\bar{u}$ satisfies Assumption 3($\beta = 1$).*

2. *There exist positive constants c and α such that*

$$J'(\bar{u})(u-\bar{u}) + J''(\bar{u}+\theta(u-\bar{u}))(u-\bar{u})^2 \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$$

for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$ and $\theta \in [0,1]$.

3. *There exist positive constants μ, β, such that*

$$J'(u)(u-\bar{u}) \geq \mu\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{4.3}$$

for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \beta$.

The direction from 1 to 3 was proven in [11, Lemma 12]. The direction from 3 to 1 follows by using Taylor's theorem and Lemma 4.2: Define $u_\theta : \bar{u} + \theta(u-\bar{u})$ for some $\theta \in [0,1]$. By Talyor's theorem, there exists θ such that

$$J'(u)(u-\bar{u}) - J'(\bar{u})(u-\bar{u}) = J''(u_\theta)(u-\bar{u})^2.$$

By Lemma 4.2 we obtain:

$$\begin{aligned}
&J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \\
&= J'(u)(u-\bar{u}) + J'(\bar{u})(u-\bar{u}) - J'(u)(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \\
&\geq \mu\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} - \Big|J''(\bar{u})(u-\bar{u})^2 - J''(u_\theta)(u-\bar{u})^2\Big|.
\end{aligned}$$

Select $\alpha < \min\{\delta, \beta\}$, such that $\mu > \varepsilon$. Then defining $c := \mu - \varepsilon$ the claim follows for controls $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$. The case 2 to 1 is trivial and the argument from 1 to 2 follows again by using Taylor's theorem and Lemma 4.2. The reformulation of Assumption 3($\beta = 1$) to (4.3) is useful to provide short proofs for the error estimates for the variational discretization later on. It appeared first in [14] in the context of ODE optimal control.

Theorem 4.4 *Assumptions 3 (with $\beta = 1$) together with (3.5) and Assumption 3 ($\beta = 1/2$) both imply strict weak local optimality.*

The claim for Assumption 3 with $\beta = 1$ was proven in [11]. We provide alternative proof. We prove the statement for $\beta = 1/2$. But this follows easily by Taylor's theorem and Lemma 4.2

$$J(u) - J(\bar{u}) = J'(\bar{u})(u-\bar{u}) + \frac{1}{2}J''(u_\theta)(u-\bar{u})^2 \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$$

for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$. This proves the claim for $\beta = 1/2$. If $\bar{u}$ satisfies (3.5), then Assumption 3 with $\beta = 1$ implies Assumption 3 with $\beta = 1/2$.

Theorem 4.5 *Let $\bar{u} \in \mathcal{U}$ be given. It is equivalent:*

1. *Assumption 3($\beta = 1/2$) holds.*

2. *There exist positive constants c and α such that*

$$J'(\bar{u})(u-\bar{u}) + \frac{1}{2}J''(\bar{u}+\theta(u-\bar{u}))(u-\bar{u})^2 \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}$$

for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)}$ and $\theta \in [0,1]$.

3. *There exists a positive constants c and α such that*

$$J(u) - J(\bar{u}) \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}, \tag{4.4}$$

for all $u \in \mathcal{U}$ with $\|u-\bar{u}\| < \alpha$.

The proof follows by the same arguments as in Theorem 4.3. To further relate the cases $\beta \in \{1/2, 1\}$ of Assumption 3 we state the following.

Theorem 4.6 *Let Assumption 3($\beta = 1/2$) be satisfied. Let there exist μ such that $c > \mu$ and*

$$J''(\bar{u}) \geq -\mu\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{4.5}$$

for all $u \in \mathcal{U}$ with $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$. Then Assumption 3($\beta = 1$) is satisfied with constant $c := \gamma - \mu/2$.

We consider the set

$$\Big\{v \in L^2(\Omega)\Big| v \geq 0 \text{ a.e. on } [\bar{u} = u_a] \text{ and } v \leq 0 \text{ a.e. on } [\bar{u} = u_b]\Big\} \tag{4.6}$$

and define for some $\tau > 0$

$$\begin{aligned}
D_{\bar{u}}^{\tau} &:= \Big\{v \in L^2(\Omega)\Big| v \text{ satisfies (4.6) and } v(x) = 0 \text{ if } \Big|\frac{\partial \bar{H}}{\partial u}(x)\Big| > \tau\Big\},\\
G_{\bar{u}}^{\tau} &:= \Big\{v \in L^2(\Omega)\Big| v \text{ satisfies (4.6) and } J'(\bar{u})(v) \leq \tau\|z_{\bar{u},v}\|_{L^1(\Omega)}\Big\},\\
C_{\bar{u}}^{\tau} &:= D_{\bar{u}}^{\tau} \cap G_{\bar{u}}^{\tau}.
\end{aligned}$$

Here, $\bar{H}$ denotes the Hamiltonian (3.2) corresponding to the reference control $\bar{u}$. If the control does not appear explicitly in the objective functional, $\frac{\partial \bar{H}}{\partial u}$ is given by the adjoint $\bar{p}$ corresponding to $\bar{u}$. Assumption 3 can be considered as acting only on the cone $D_{\bar{u}}^{\tau}$. For a prove see [11, Proposition 6.2] or [4, Coroallary 14]. Assumption 3 is equivalent to the assumption: Let $\bar{u} \in \mathcal{U}$ be given. There exist positive constants c and α, such that

$$J'(\bar{u})(u-\bar{u}) + \beta J''(\bar{u})(u-\bar{u})^2 \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \tag{4.7}$$

for all $u \in \mathcal{U}$ with $(u-\bar{u}) \in D_{\bar{u}}^{\tau}$ and $\|u-\bar{u}\|_{L^1(\Omega)} < \alpha$.

Further, we have the following theorem that relates Assumption 3 to the assumptions made in [7, Theorem 9]. Let $\frac{\partial L_b}{\partial y} = 0$ and let $\bar{u}$ satisfy the following conditions: There exist positive constants c, k and α with $k < c$ such that

$$J'(\bar{u})(u-\bar{u}) \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \text{ for all } u \in \mathcal{U} \tag{4.8}$$

and

$$J''(\bar{u})(u-\bar{u}) \geq -k\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \text{ for all } (u-\bar{u}) \in C_{\bar{u}}^{\tau} \text{ with } \|u-\bar{u}\|_{L^1(\Omega)} < \alpha.$$

Then Assumption 3($\beta = 1$) holds for some appropriate constants. By Proposition 3 it is sufficient to prove the statement for the Assumption 3 on the cone $D_{\bar{u}}^{\tau}$. Thus we only need to consider the case $(u-\bar{u}) \notin G_{\bar{u}}^{\tau}$. But by definition of $(u-\bar{u}) \notin G_{\bar{u}}^{\tau}$, $J'(\bar{u})(u-\bar{u}) > \tau\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)}$. We estimate for some constant d independent of u

$$\left|J''(\bar{u})(u-\bar{u})^2\right| \leq d\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(\Omega)}\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)}.$$

By the assumption of the theorem, it also holds

$$J'(\bar{u})(u-\bar{u}) \geq c\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}.$$

Thus combining the estimates we obtain for $\|u-\bar{u}\|_{L^1(\Omega)}$ sufficiently small

$$\begin{aligned}
&J'(\bar{u})(u-\bar{u}) + J''(\bar{u})(u-\bar{u})^2 \\
&\geq \frac{1}{2}J'(\bar{u})(u-\bar{u}) + (\frac{1}{2}\tau - d\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(\Omega)})\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)} \\
&\geq c/2\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} + (\frac{1}{2}\tau - d\|z_{\bar{u},u-\bar{u}}\|_{L^\infty(\Omega)})\|z_{\bar{u},u-\bar{u}}\|_{L^1(\Omega)} \geq c/2\|u-\bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}}.
\end{aligned}$$

Remark 4.7 *If the of the Hamiltonian satisfies $C^1(\bar{\Omega})$ regularity, a sufficient condition is given in [13].*

5 Solution stability

We consider stability under perturbations appearing in the objective functional and the PDE simultaneously. This was also considered for instance in [4, 11, 19] for different conditions. In this section we additionally assume the second derivatives of L and f to be Lipschitz with respect to the y variable. We fix a positive constant M and define the set of feasible perturbations

$$\Gamma := \left\{\zeta := (\xi,\eta,\gamma) \in L^2(\Omega)\times L^2(\Omega)\times L^\infty(\Omega) | \|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)} \leq M\right\}.$$

The perturbed problem is given by

$$\min_{u\in\mathcal{U}}\left\{J_\zeta(u) := \int_\Omega L(x,y_u,u) + \rho u + \eta y_u \,\mathrm{d}x\right\} \tag{5.1}$$

subject to (1.2) and

$$\left\{\begin{array}{lll} \mathcal{A}y + f(\cdot,y) & = u+\xi & \text{in } \Omega, \\ y & = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{5.2}$$

The existence of a globally optimal solution to (5.1)-(5.2) is guaranteed by the assumptions on the optimal control problem and the direct method in the calculus of variations. We define

$$\bar{C} := \max_{u\in\mathcal{U}}\{\|y_u\|_{L^\infty(\Omega)}, \|p_u\|_{L^\infty(\Omega)}\}. \tag{5.3}$$

We need the next technical lemma, for a proof we refer to [4, Theorem 4.1].

Lemma 5.1 *Given $\xi \in L^2(\Omega)$, $u \in \mathcal{U}$ and $v \in L^2(\Omega)$, it holds*

$$\begin{aligned}\|y_u^\xi - y_u\|_{L^2(\Omega)} &\le C_2\|\xi\|_{L^2(\Omega)},\\ \|z_{u,v}^\xi - z_{u,v}\|_{L^s(\Omega)} &\le \bar{C}C_2^2\|\xi\|_{L^2(\Omega)}\|v\|_{L^1(\Omega)}.\end{aligned}$$

Theorem 5.2 *Let $\bar{u}$ satisfy Assumption 3 for some $\gamma \in (n/(n+2), 1]$. There exist positive constants c and α such that*

$$\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} \le c(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|\rho\|_{L^\infty(\Omega)})^\gamma,$$

for any minimizer $(\bar{y}^\zeta, \bar{p}^\zeta, \bar{u}^\zeta)$ of (5.1)-(5.2) with $\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} < \alpha$.

Denote by y_u^ξ the solution to 5.2 corresponding to the perturbation ξ and data u and define

$$J_\xi(u) := \int_\Omega L(x, y_u^\xi, u)\,\mathrm{d}x.$$

Let $\bar{u}^\zeta$ be a minimizer of the perturbed problem with perturbation $\zeta = (\xi, \eta, \gamma)$ and denote by $\bar{y}^\zeta$ the corresponding state and $\bar{p}^\zeta$ the corresponding adjoint state. We notice that $y_{\bar{u}^\zeta}^\zeta = y_{\bar{u}^\zeta}^\xi$ if ξ is as in ζ. In this case, it holds

$$J_\xi(\bar{u}^\zeta) + \int_\Omega \rho\bar{u}^\zeta + \eta y_{\bar{u}^\zeta}^\xi\,\mathrm{d}x \le J_\xi(\bar{u}) + \int_\Omega \rho\bar{u} + \eta y_{\bar{u}}^\xi\,\mathrm{d}x.$$

Therefore,

$$\begin{aligned}J_\xi(\bar{u}^\zeta) - J_\xi(\bar{u}) &\le \int_\Omega \rho(\bar{u} - \bar{u}^\zeta) + \eta(y_{\bar{u}}^\xi - y_{\bar{u}^\zeta}^\xi)\,\mathrm{d}x\\ &\le (\|\rho\|_{L^\infty(\Omega)} + C_2\|\eta\|_{L^2(\Omega)})\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)}.\end{aligned}$$

We estimate the term on the left side, define $u_\theta := \bar{u} + \theta(\bar{u}^\zeta - \bar{u})$ for some $\theta \in [0,1]$ and denote by y_{u_θ} and $p_{u\theta}$ the corresponding state and adjoint state.

$$\begin{aligned}&J_\xi(\bar{u}^\zeta) - J_\xi(\bar{u}) = J_\xi'(\bar{u})(\bar{u}^\zeta - \bar{u}) + \frac{1}{2}J_\xi''(u_\theta)(\bar{u}^\zeta - \bar{u})^2\\ &= \Big[J_\xi'(\bar{u})(\bar{u}^\zeta - \bar{u}) - J'(\bar{u})(\bar{u}^\zeta - \bar{u})\Big] + \Big[\frac{1}{2}(J_\xi''(u_\theta)(\bar{u}^\zeta - \bar{u})^2 - J''(u_\theta)(\bar{u}^\zeta - \bar{u})^2)\Big]\\ &+ \Big[J'(\bar{u})(\bar{u}^\zeta - \bar{u}) + \frac{1}{2}J''(u_\theta)(\bar{u}^\zeta - \bar{u})^2\Big] = I_1 + I_2 + I_3.\end{aligned}$$

We estimate the terms I_1, I_2, and I_3. By Theorem 4.5 there exist positive constants c and α such that $I_3 \geq c\|u-\bar u\|_{L^1(\Omega)}^{1+1/\gamma}$ for all $u \in \mathcal{U}$ with $\|u-\bar u\|_{L^1(\Omega)} < \alpha$. By assumption, this is satisfied by $\bar u^\zeta$. We continue with the term I_1:

$$\begin{aligned}|I_1| &\leq \Big| J'_\xi(\bar u)(\bar u^\zeta - \bar u) - J'(\bar u)(\bar u^\zeta - \bar u)\Big| \\ &\leq \Big|\int_\Omega \frac{\partial L}{\partial y}(x, y^\xi_{\bar u}, \bar u) z^\xi_{\bar u, \bar u^\zeta - \bar u} - \frac{\partial L}{\partial y}(x, y_{\bar u}, \bar u) z_{\bar u, \bar u^\zeta - \bar u}\,\mathrm{d}x\Big| \\ &\quad + \Big|\int_\Omega \Big[\frac{\partial L}{\partial u}(x, y^\xi_{\bar u}, \bar u) - \frac{\partial L}{\partial u}(x, y_{\bar u}, \bar u)\Big](\bar u^\zeta - \bar u)\,\mathrm{d}x\Big| = J_1 + J_2.\end{aligned}$$

The term J_1 is estimated using the mean value theorem and Lemma 5.1

$$\begin{aligned}|J_1| &\leq \Big|\int_\Omega \Big[\frac{\partial L}{\partial y}(x, y^\xi_{\bar u}, \bar u) - \frac{\partial L}{\partial y}(x, y_{\bar u}, \bar u)\Big] z^\xi_{\bar u, \bar u^\zeta - \bar u}\,\mathrm{d}x\Big| \\ &\quad + \Big|\int_\Omega \frac{\partial L}{\partial y}(x, y_{\bar u}, \bar u)\Big[z^\xi_{\bar u, \bar u^\zeta - \bar u} - z_{\bar u, \bar u^\zeta - \bar u}\Big]\,\mathrm{d}x\Big| \\ &\leq C_{L,M} C_2 \|\xi\|_{L^2(\Omega)} \|z^\xi_{\bar u, u^\zeta - \bar u}\|_{L^2(\Omega)} + \|\psi_M\|_{L^2(\Omega)} \|z^\xi_{\bar u, \bar u^\zeta - \bar u} - z_{\bar u, \bar u^\zeta - \bar u}\|_{L^2(\Omega)} \\ &\leq C_2(C_{L,M} + C_2 C_{f,M}\|\psi_M\|_{L^2(\Omega)})\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}.\end{aligned}$$

The term J_2 is estimated by using again the mean value theorem and Lemma 5.1

$$\begin{aligned}|J_2| &\leq \Big\|\frac{\partial L}{\partial u}(y^\xi_{\bar u}, \bar u) - \frac{\partial L}{\partial u}(y_{\bar u}, \bar u)\Big\|_{L^\infty(\Omega)} \|\bar u^\zeta - \bar u\|_{L^1(\Omega)} \\ &\leq C_2 C_{L,M}\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|_{L^1(\Omega)}.\end{aligned}$$

To estimate I_2 we write

$$\begin{aligned}I_2 &= \int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y^\xi_{u_\theta}, u_\theta) - \frac{\partial^2 L}{\partial y^2}(x, y_{u_\theta}, u_\theta)\Big](z^\xi_{u_\theta, \bar u^\zeta - \bar u})^2\,\mathrm{d}x \\ &\quad + \int_\Omega \Big[p_{u_\theta}\frac{\partial^2 f}{\partial y^2}(x, y_{u_\theta}) - p^\xi_{u_\theta}\frac{\partial^2 f}{\partial y^2}(x, y^\xi_{u_\theta})\Big](z^\xi_{u_\theta, \bar u^\zeta - \bar u})^2\,\mathrm{d}x \\ &\quad + \int_\Omega \Big[\frac{\partial^2 L}{\partial y^2}(x, y_{u_\theta}, u_\theta) + p_{u_\theta}\frac{\partial^2 f}{\partial y^2}(x, y_{u_\theta})\Big]\Big[(z^\xi_{u_\theta, \bar u^\zeta - \bar u})^2 - z^2_{u_\theta, \bar u^\zeta - \bar u}\Big]\,\mathrm{d}x \\ &\quad + 2\int_\Omega \Big[\frac{\partial L}{\partial u}(x, y^\xi_{u_\theta}, u_\theta) - \frac{\partial L}{\partial u}(x, y_{u_\theta}, u_\theta)\Big] z^\xi_{u_\theta, \bar u^\zeta - \bar u}(\bar u^\zeta - \bar u)\,\mathrm{d}x \\ &\quad + 2\int_\Omega \frac{\partial L}{\partial u}(x, y_{u_\theta}, u_\theta)\Big[z^\xi_{u_\theta, \bar u^\zeta - \bar u} - z_{u_\theta, \bar u^\zeta - \bar u}\Big](\bar u^\zeta - \bar u)\,\mathrm{d}x = \sum_{i=1}^5 K_i.\end{aligned}$$

For the first term, we find by Theorem 2.1, Lemma 2.2 and Lemma 5.1

$$|K_1| \leq \mathrm{Lip}_{L,M} C_2 C_\infty \|u_b\|_{L^\infty(\Omega)}\|\xi\|_{L^2(\Omega)}\|\bar u^\zeta - \bar u\|^2_{L^1(\Omega)}.$$

The estimate for the second and third terms follows by Theorem 2.1, Lemma 2.2, Lemma 5.1 and (5.3)

$$|K_2| \leq C_2\Big(C_{f,M}C_\infty + \mathrm{Lip}_{f,M}\bar{C}\Big)\|\xi\|_{L^2(\Omega)}\|\bar{u}^\zeta - \bar{u}\|^2_{L^1(\Omega)},$$
$$|K_3| \leq 2\Big(C_{L,M} + \bar{C}C_{f,M}\Big)\|\xi\|_{L^2(\Omega)}\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)}.$$

For the fourth and fifth terms using the same arguments, we find

$$|K_4| \leq C_{L,M}C_\infty^2\bar{C}\|\xi\|_{L^2(\Omega)}\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)},$$
$$|K_5| \leq 2\bar{C}C_2^2\|u_b\|_{L^\infty(\Omega)}\|L_b(\cdot, y_{u_\theta}(\cdot))\|_{L^\infty(\Omega)}\|\xi\|_{L^2(\Omega)}\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)}.$$

Summarizing, we conclude the existence of a positive constant c such that

$$|I_2| \leq c\|\xi\|_{L^2(\Omega)}\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)}.$$

Further, it holds $I_3 - |I_1| - |I_2| \leq (\|\rho\|_{L^\infty(\Omega)} + \|\xi\|_{L^2(\Omega)})\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)}$. By the estimates on terms I_1, I_2 and I_3 we conclude the existence of a positive constant c with

$$\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \leq c(\|\rho\|_{L^\infty(\Omega)} + \|\xi\|_{L^2(\Omega)})\|\bar{u} - \bar{u}^\zeta\|_{L^1(\Omega)},$$

for all $\|\bar{u}^\zeta - \bar{u}\|_{L^1(\Omega)} < \alpha$.

6 Discrete model and error estimates

We come to the main part of this manuscript. The goal is to prove error estimates for the numerical approximation under mainly Assumptions 3 for $\gamma \in (n/(n+2), 1]$. Additionally, we consider assumptions introduced in [4]. In a remark at the end of the next subsection, we address assumptions that allow us to admit $\gamma \in (0, 1]$.

6.0.1 The finite element scheme

The finite element scheme we consider, is close to the one in [7], we also refer to [1] for an overview of the finite elements method. In this section, we consider Ω to be convex and let $\{\tau_h\}_{h>0}$ be a quasi-uniform family of triangulations of $\bar{\Omega}$. Denote $\bar{\Omega}_h = \cup_{T\in\tau_h} T$ and assume that every boundary node of Ω_h is a point of Γ. Further, suppose that there exists a constant $C_\Gamma > 0$ independent of h such that the distance d_Γ satisfies $d_\Gamma(x) < C_\Gamma h^2$ for every $x \in \Gamma_h = \partial\Omega_h$. Then we can infer that the existence of a constant $C_\Omega > 0$ independent of h such that

$$|\Omega \setminus \Omega_h| \leq C_\Omega h^2, \tag{6.1}$$

where $|\cdot|$ denotes the Lebesgue measure. We define the finite-dimensional space

$$Y_h = \{z_h \in C(\bar{\Omega}) : z_{h|T} \in P_1(T)\ \forall T \in \tau_h \text{ and } z_h \equiv 0 \text{ on } \Omega \setminus \Omega_h\},$$

where $P_i(T)$ denotes the polynomials in T of degree at most i.
For $u \in L^2(\Omega)$, the associated discrete state is the unique element $y_h(u) \in Y_h$ that solves

$$a(y_h, z_h) + \int_{\Omega_h} f(x, y_h) z_h \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega_h} u z_h \, \mathrm{d}x \quad \forall z_h \in Y_h, \tag{6.2}$$

where

$$a(y, z) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij} \partial_{x_i} y \partial_{x_j} z \, \mathrm{d}x \quad \forall y, z \in H^1(\Omega).$$

The proof of the existence and uniqueness of a solution for (6.2) is standard.

Lemma 6.1 *[7, Lemma 3] There exists a constant $c > 0$, which depends on the data of the problem but is independent of the discretization parameter h, such that for every $u \in \mathcal{U}$*

$$\|y_h(u) - y_u\|_{L^2(\Omega)} \le ch^2, \tag{6.3}$$

$$\|y_h(u) - y_u\|_{L^\infty(\Omega)} \le ch^2 |\log h|^2. \tag{6.4}$$

The set of feasible controls for the discrete problem is given by

$$U_h := \{u_h \in L^\infty(\Omega_h) : u_{h|T} \in P_0(T) \ \forall T \in \tau_h\}.$$

By Π_h we denote the linear projection onto U_h in the $L^2(\Omega_h)$ by

$$(\Pi_h u)_{|T} = \frac{1}{|T|} \int_T u \, \mathrm{d}x, \quad \forall T \in \tau_h.$$

By $u_h \rightharpoonup u$ weak* in $L^\infty(\Omega)$ we mean, as in [7], the following

$$\int_{\Omega_h} u_h v \, \mathrm{d}x \to \int_{\Omega} u v \, \mathrm{d}x \quad \forall \ v \in L^1(\Omega).$$

Lemma 6.2 *[7, Lemma 4] Given $1 < p < \infty$ there exists a positive constant K_p that depends on p and Ω but is independent of h such that*

$$\|u - \Pi_h u\|_{W^{-1,p}(\Omega_h)} \le K_p h \|u\|_{L^p(\Omega)} \quad \forall \ u \in L^p(\Omega).$$

We define

$$J_h(u) := \int_{\Omega_h} L(x, y_h(u), u) \, \mathrm{d}x.$$

The discrete problem is given by

$$\min_{u_h \in \mathcal{U}_h} J_h(u_h), \tag{6.5}$$

where $\mathcal{U}_h := U_h \cap \mathcal{U}$. This set is compact and nonempty and the existence of a global solution of (6.5) follows from the continuity of J_h in $\mathcal{U}_h$. For $u \in L^2(\Omega)$, the discrete adjoint state $p_h(u) \in Y_h$ is the unique solution of

$$a(z_h, p_h) + \int_{\Omega_h} \frac{\partial f}{\partial y}(x, y_h(u)) p_h z_h \, \mathrm{d}x = \int_{\Omega_h} \frac{\partial L}{\partial y}(x, y_h(u)) z_h \, \mathrm{d}x \quad \forall z_h \in Y_h. \tag{6.6}$$

One can calculate that

$$J_h'(u)(v) = \int_{\Omega_h} p_h(u)v \, \mathrm{d}x.$$

A local solution of (6.5) satisfies the variational inequality

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) \geq 0 \quad \forall u_h \in \mathcal{U}_h.$$

We consider the next two assumptions on the optimal control problem that were first introduced in [4]. They present a weakening of Assumption 3 and are applicable to obtain state stability for possibly non-bang-bang optimal controls.

Assumption 4 *Let $\bar{u} \in \mathcal{U}$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants c and α with*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)}$$

for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$.

Assumption 5 *Let $\bar{u} \in \mathcal{U}$ and $\beta \in \{1/2, 1\}$ be given. There exist positive constants c and α with*

$$J'(\bar{u})(u - \bar{u}) + \beta J''(\bar{u})(u - \bar{u})^2 \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}\|u - \bar{u}\|_{L^1(\Omega)}$$

for all $u \in \mathcal{U}$ with $\|u - \bar{u}\|_{L^1(\Omega)} < \alpha$.

6.1 Discretization with piecewise constant controls

We prove that under Assumption 3, the estimates in Theorem 6.5 hold true. Additionally, it is possible to derive error estimates under Assumptions 4 and 5.

Lemma 6.3 *Consider $\bar{u} \in \mathcal{U}$ satisfying the first order optimality condition and define $\bar{\sigma} := p_{\bar{u}} + L_b(x, y_{\bar{u}})$. Assume that $\bar{\sigma}$ is Lipschitz on Ω, then*

$$J'(\bar{u})(\Pi_h\bar{u} - \bar{u}) \leq \mathrm{Lip}_{\bar{\sigma}} h\|\bar{u} - \Pi_h\bar{u}\|_{L^1(\Omega)}.$$

We argue as in [7, Lemma 7]. The next lemma is crucial to obtain error estimates for the numerical approximation without assuming the structural assumption.

Lemma 6.4 *We assume that $p_{\bar{u}} + L_b(x, y_{\bar{u}})$, corresponding to the reference solution $\bar{u}$, is Lipschitz continuous on Ω and we select $\beta = 1/2$ in the assumptions referenced below.*

1. *Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 3. Then there exists a positive constant c independent of h, such that for h sufficiently small*

$$\|\bar{u} - \Pi_h\bar{u}\|_{L^1(\Omega_h)} \leq ch. \tag{6.7}$$

2. *If $\bar{u} \in \mathcal{U}$ satisfies Assumption 5, there exists a positive constant c independent of h such that for h sufficiently small*

$$\|y_{\bar{u}} - y_{\Pi_h \bar{u}}\|_{L^2(\Omega)} \leq ch. \tag{6.8}$$

3. *Finally, if $\bar{u} \in \mathcal{U}$ satisfies Assumption 4, there exists a positive constant c independent of h such that for h sufficiently small*

$$\|y_{\bar{u}} - y_{\Pi_h \bar{u}}\|_{L^2(\Omega)} \leq ch^{\frac{1}{2}}. \tag{6.9}$$

The assumption in Lemma 6.3, that $p_{\bar{u}} + L_b(x, y_{\bar{u}})$ is Lipschitz, is not a big constraint. By the assumptions in this section, the adjoint state is already Lipschitz. We begin proving (6.7). Let $\bar{u} \in \mathcal{U}$ satisfy Assumption 3, then there exist positive constants c and α such that for h with $\|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega_h)} < \alpha$

$$J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) + \beta J''(\bar{u})(\Pi_h \bar{u} - \bar{u})^2 \geq c\|\Pi_h \bar{u} - \bar{u}\|^2_{L^1(\Omega_h)}.$$

We remark that there exists h_0 such that condition $\|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega_h)} < \alpha$ is satisfied for all $h < h_0$. This follows since $\|\Pi_h \bar{u} - \bar{u}\|_{L^2(\Omega_h)} \to 0$. By Lemma 6.3, we obtain

$$J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) = \int_\Omega (p_{\bar{u}} + L_b(x, y_{\bar{u}}))(\Pi_h \bar{u} - \bar{u})\,\mathrm{d}x \leq \mathrm{Lip} h\|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega)}.$$

The right-hand side is further estimated employing the Peter-Paul inequality for some $\varepsilon < c$, where c is the constant appearing in condition (3)

$$\mathrm{Lip} h\|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega)} \leq \frac{(\mathrm{Lip} h)^2}{2\varepsilon} + \frac{\varepsilon\|\Pi_h \bar{u} - \bar{u}\|^2_{L^1(\Omega)}}{2}. \tag{6.10}$$

Given $n < p$, using [7, Lemma 1, Lemma 4], the second variation is estimated by

$$\begin{aligned} J''(\bar{u})(\Pi_h \bar{u} - \bar{u})^2 &\leq \left\|\frac{\partial^2 L}{\partial^2 y}(\cdot, y_{\bar{u}}) - p_{\bar{u}}\frac{\partial^2 f}{\partial^2 y}(\cdot, y_{\bar{u}})\right\|_{L^\infty(\Omega)} \|z_{\bar{u}, \Pi_h \bar{u} - \bar{u}}\|^2_{L^2(\Omega)} \\ &\leq C_p(\bar{C}C_{f,M} + C_{L,M})\|\Pi_h \bar{u} - \bar{u}\|^2_{W^{-1,p}} \leq C_p K_p^2(\bar{C}C_{f,M} + C_{L,M})h^2. \end{aligned}$$

Now, the first claim follows by absorbing the second term of (6.10). The proofs of the second and third claims follow by the same arguments from the estimate

$$\begin{aligned} &J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) + J''(\bar{u})(\Pi_h \bar{u} - \bar{u})^2 \\ &\leq \left(\mathrm{Lip}_{\bar{\sigma}} h + \left\|\frac{\partial^2 L}{\partial^2 y}(\cdot, y_{\bar{u}}) - p_{\bar{u}}\frac{\partial^2 f}{\partial^2 y}(\cdot, y_{\bar{u}})\right\|_{L^\infty(\Omega)} \|\Pi_h \bar{u} - \bar{u}\|_{W^{-1,p}(\Omega)}\right)\|\Pi_h \bar{u} - \bar{u}\|_{L^1(\Omega)}. \end{aligned}$$

Theorem 6.5 *Let $\bar{u}$ be a local solution of* (P). *Consider the constant α corresponding to the Assumptions 3, 4 or 5. Consider discrete controls $\bar{u}_h \in \mathcal{U}_h$ that satisfy $\|\bar{u}_h - \Pi_h \bar{u}\|_{L^1(\Omega_h)} < \alpha$ and*

$$ch^2 \geq J_h(\bar{u}_h) - J_h(\Pi_h \bar{u}). \tag{6.11}$$

for some positive constant c. We recall that $\bar{y}$ is the solution of (1.4) *and $y(\bar{u}_h)$ denotes the solution of* (6.6) *for $\bar{u}_h$.*

1. *Let* $L_b = 0$ *in the objective functional and let* $\bar{u}$ *satisfy Assumption 4. Then, there exists a positive constant* c *independent of* h *such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq c\sqrt{h}. \tag{6.12}$$

2. *Let* $L_b = 0$ *in the objective functional and let* $\bar{u}$ *satisfy Assumption 5. Then, there exists a positive constant* c *independent of* h *such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch. \tag{6.13}$$

3. *Let* $\frac{\partial L_b}{\partial y} = 0$ *in the objective functional and let* $\bar{u}$ *satisfy Assumption 3*$(\beta = 1/2)$. *Then, there exists a positive constant* c *independent of* h *such that*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch^{\gamma}. \tag{6.14}$$

If $\frac{\partial L_b}{\partial y} \neq 0$ *we obtain the estimate*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} \leq ch^{\frac{(1+\min\{1/r,1/\gamma\})\gamma}{\gamma+1}}. \tag{6.15}$$

It is clear that the condition (6.11) is satisfied if the controls $\bar{u}_h$ are minimizers of the corresponding discrete problems. We notice that for a discrete control $\bar{u}_h$ that satisfies the assumptions of the theorem, we have for some positive constant c

$$\begin{aligned} J(\bar{u}_h) - J(\bar{u}) &= \Big[J(\bar{u}_h) - J_h(\bar{u}_h)\Big] + \Big[J_h(\bar{u}_h) - J_h(\Pi_h\bar{u})\Big] + \Big[J_h(\Pi_h\bar{u}) - J(\Pi_h\bar{u})\Big] \\ &\quad + \Big[J(\Pi_h\bar{u}) - J(\bar{u})\Big] = I_1 + I_2 + I_3 + I_4 \leq ch^{1+\gamma}. \end{aligned}$$

We give a short argument why this is true. For the second term, it follows from the fact that $I_2 \geq -ch^2$ by (6.11). For the term I_3 we use the estimates in Lemma 6.1 and (6.1), to obtain

$$\begin{aligned} I_3 &= \int_\Omega L(x, y_{\Pi_h\bar{u}}, \Pi_h\bar{u})\,\mathrm{d}x - \int_{\Omega_h} L(x, y(\Pi_h\bar{u}), \Pi_h\bar{u})\,\mathrm{d}x \\ &= \int_{\Omega\setminus\Omega_h} L(x, y_{\Pi_h\bar{u}}, \Pi_h\bar{u})\,\mathrm{d}x + \int_{\Omega_h} L(x, y_{\Pi_h\bar{u}}, \Pi_h\bar{u}) - L(x, y(\Pi_h\bar{u}), \Pi_h\bar{u})\,\mathrm{d}x \\ &\geq -h^2\Big(\Big\|\frac{\partial L_a}{\partial y}(x, y_\theta)\Big\|_{L^2(\Omega)} + \Big\|\frac{\partial L_b}{\partial y}(x, y_\vartheta)\Pi_h\bar{u}\Big\|_{L^2(\Omega_h)} + C_\Omega\|L(x, y_\vartheta, \Pi_h\bar{u})\|_{L^\infty(\Omega)}\Big). \end{aligned}$$

For the first term, similar arguments guarantee the estimate

$$\begin{aligned} I_1 &= -\int_{\Omega\setminus\Omega_h} L(x, y_{\bar{u}_h}, \bar{u}_h)\,\mathrm{d}x + \int_{\Omega_h} L(x, y(\bar{u}_h), \bar{u}_h) - L(x, y_{\bar{u}_h}, \bar{u}_h)\,\mathrm{d}x \\ &\geq -h^2\Big(\Big\|\frac{\partial L_a}{\partial y}(x, y_\theta)\Big\|_{L^2(\Omega)} + \Big\|\frac{\partial L_b}{\partial y}(x, y_\vartheta)\bar{u}_h\Big\|_{L^2(\Omega_h)} + C_\Omega\|L(x, y_{\bar{u}_h}, \bar{u}_h)\|_{L^\infty(\Omega)}\Big). \end{aligned}$$

The last term can be estimated using Lemma 6.4 and the fact that by Lemma 6.2, $\|y_{\Pi_h\bar u}-\bar y\|_{L^2(\Omega)}\le c\|\Pi_h\bar u-\bar u\|_{W^{-1,2}}\le K_2 h\|\bar u\|_{L^2(\Omega)}$. To shorten notation, we denote by $L_{a,y}$ and $L_{b,y}$ the derivatives of L_a and L_b by y. By Taylor's theorem

$$
\begin{aligned}
&J(\Pi_h(\bar u))-J(\bar u)=\int_\Omega L(x,y_{\Pi_h(\bar u)},\Pi_h(\bar u))-L(x,y_{\bar u},\bar u)\,\mathrm{d}x\\
&=\int_\Omega L_{a,y}(x,y_\theta)(y_{\Pi_h\bar u}-y_{\bar u})\,\mathrm{d}x\\
&+\int_\Omega L_{b,y}(x,y_\vartheta)(y_{\Pi_h\bar u}-y_{\bar u})\Pi_h\bar u\,\mathrm{d}x+\int_\Omega L_b(x,y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x\\
&=\int_\Omega L_{a,y}(x,y_{\bar u})(y_{\Pi_h\bar u}-y_{\bar u})\,\mathrm{d}x+\int_\Omega L_{b,y}(x,y_{\bar u})(y_{\Pi_h\bar u}-y_{\bar u})\bar u\,\mathrm{d}x\\
&+\int_\Omega L_b(x,y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x+\int_\Omega(L_{a,y}(x,y_\theta)-L_{a,y}(x,y_{\bar u}))(y_{\Pi_h\bar u}-y_{\bar u})\,\mathrm{d}x\\
&+\int_\Omega(L_{b,y}(x,y_\vartheta)-L_{b,y}(x,y_{\bar u}))(y_{\Pi_h\bar u}-y_{\bar u})\Pi_h\bar u\,\mathrm{d}x\\
&+\int_\Omega L_b(x,y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x+\int_\Omega L_{b,y}(x,y_{\bar u})(y_{\Pi_h\bar u}-y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&J(\Pi_h(\bar u))-J(\bar u)\\
&=\int_\Omega(L_{a,y}(x,y_{\bar u})+L_{b,y}(x,y_{\bar u})\bar u)z_{\bar u,\Pi_h\bar u-\bar u}\,\mathrm{d}x+\int_\Omega L_b(x,y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x\\
&+\int_\Omega(L_{a,y}(x,y_{\bar u})+L_{b,y}(x,y_{\bar u})\Pi_h\bar u)(y_{\Pi_h\bar u}-y_{\bar u}-z_{\bar u,\Pi_h\bar u-\bar u})\,\mathrm{d}x\\
&+\int_\Omega(L_{a,y}(x,y_\theta)-L_{a,y}(x,y_{\bar u}))(y_{\Pi_h\bar u}-y_{\bar u})\,\mathrm{d}x\\
&+\int_\Omega(L_{b,y}(x,y_\vartheta)-L_{b,y}(x,y_{\bar u}))(y_{\Pi_h\bar u}-y_{\bar u})\Pi_h\bar u\,\mathrm{d}x\\
&+\int_\Omega L_{b,y}(x,y_{\bar u})(y_{\Pi_h\bar u}-y_{\bar u})(\Pi_h\bar u-\bar u)\,\mathrm{d}x=\sum_{i=1}^{6}I_i.
\end{aligned}
$$

We provide the estimates for (6.14) under Assumption 3. Integration by parts, Lemma 6.3 and Lemma 6.4 guarantee the existence of a positive constant c with

$$|I_1+I_2|=\int_\Omega(p_{\bar u}+L_b(x,y_{\bar u}))(\Pi_h\bar u-\bar u)\,\mathrm{d}x\le ch^{1+\gamma}.$$

The term I_3 can be estimated using (2.6) and (6.7)

$$|I_3|\le\|\psi_M\|_{L^2(\Omega)}\|y_{\Pi_h\bar u}-y_{\bar u}\|^2_{L^2(\Omega)}\le ch^2.$$

For the terms I_4 and I_5, we use (6.7) and the local Lipschitz property of $L_{a,y}$ and $L_{b,y}$ to infer the existence of a constant c such that $|I_4|,|I_5|\le c\|y_{\Pi_h\bar u}-$

$y_{\bar{u}}\|^2_{L^2(\Omega)} \leq ch^2$. Finally, for some number $r > n/2$, we estimate $|I_6| \leq \|\Pi_h \bar{u} - \bar{u}\|^{1+1/r}_{L^1(\Omega)} \leq ch^{1+1/r}$. If the term I_6 is absent we obtain the better estimate (6.14), if not (6.15) holds. To continue the proof of (6.14), we conclude from the estimates of the terms I_i, and by Theorem 4.5 the existence of a positive constant k such that

$$kh^{1+\gamma} \geq J(\bar{u}_h) - J(\bar{u}) \geq c\|\bar{u}_h - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)}, \tag{6.16}$$

where in (6.16), the constant c is from Theorem 4.4. Now, this is equivalent to

$$(c/k)^{\frac{-\gamma}{\gamma+1}} h^\gamma \geq \|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}.$$

The proofs for the claims (6.12) and (6.13) follow by similar arguments.

Remark 6.6 *In the proof of Theorem 6.5, Assumption 3 is used to guarantee the existence of positive constants c and α such that*

$$J(\bar{u}_h) - J(\bar{u}) \geq c\|\bar{u}_h - \bar{u}\|^{1+\frac{1}{\gamma}}_{L^1(\Omega)}$$

for all $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} < \alpha$. This growth can be obtained under the stronger assumptions of Theorem 4 for all $\gamma \in (0,1]$. Thus, the constraint $\gamma \in (n/(2+n), 1]$ can be weakened to $\gamma \in (0,1]$ for the cost of considering stronger conditions.

For a numerical example supporting the theoretical error estimate achieved in this paper, especially for the case $\gamma < 1$, we refer to [7].

6.2 Variational discretization

We prove that Assumption 3 with $\beta = 1$ is sufficient for approximation error estimates for a variational discretization. We refer to the [15] for the idea and introduction of variational discretization. Additionally, we consider Assumptions 4 and 5 for $\beta = 1$. Although we consider weaker conditions in this paper, the estimates under the estimates given in Theorem 6.9 below agree with the estimates in [7, Remark 7] for the variational discretization. For these assumptions, we can formulate an analog to Theorem 4.3. For a proof we refer to [4, Lemma 4.5, Lemma 5.4].

Theorem 6.7 *We have the following equivalence.*

1. *Assumption 4 with $\beta = 1$ holds for $\bar{u} \in \mathcal{U}$.*

2. *There exist positive constants c and α such that*

$$J'(u)(u - \bar{u}) \geq c\|z_{\bar{u},u-\bar{u}}\|^2_{L^2(\Omega)}$$

for all $u \in \mathcal{U}$ with $\|\bar{u} - u\|_{L^1(\Omega)} < \alpha$.

Further, it is equivalent

1. *Assumption 5 for $\beta = 1$ holds for $\bar{u} \in \mathcal{U}$.*
2. *There exist positive constants c and α such that*

$$J'(u)(u - \bar{u}) \geq c\|z_{\bar{u},u-\bar{u}}\|_{L^2(\Omega)}\|\bar{u} - u\|_{L^1(\Omega)} \tag{6.17}$$

for all $u \in \mathcal{U}$ with $\|\bar{u} - u\|_{L^1(\Omega)} < \alpha$.

Theorem 6.8 *[7, Theorem 9] Let $\bar{u}_h$ denote a solution to (6.5). We denote by $y_{\bar{u}_h}$ and $p_{\bar{u}_h}$ the solution to the continuous state equation and to the corresponding adjoint equation with respect to $\bar{u}_h$. By $p(\bar{u}_h)$ we denote the discrete adjoint equation corresponding to $\bar{u}_h$ and $p^h_{\bar{u}_h}$ denotes the solution to the following equation*

$$\begin{cases} \mathcal{A}^* p + \frac{\partial f}{\partial y}(\cdot, \bar{y}_h)p & = \frac{\partial L}{\partial y}(\cdot, y(\bar{u}_h)) \quad \text{in } \Omega, \\ p & = 0 \quad \text{on } \Gamma. \end{cases}$$

Then there the following estimates hold

$$\|p_{\bar{u}_h} - p^h_{\bar{u}_h}\|_{L^\infty(\Omega)} \leq ch^2 \tag{6.18}$$

$$\|p(\bar{u}_h) - p^h_{\bar{u}_h}\|_{L^\infty(\Omega)} \leq ch^2|\log h|^2. \tag{6.19}$$

We come to the error estimates for the variational discretization.

Theorem 6.9 *Let $\bar{u}_h$ be a sequence of solutions to the first-order optimality condition of the discrete problems such that $\|\bar{u}_h - \bar{u}\|_{L^1(\Omega_h)} < \alpha$. Here, α is the constant appearing in Theorem 4.3 or Theorem 6.7 depending on the selected growth assumption.*

1. *Let Assumptions 4 be satisfied by $\bar{u} \in \mathcal{U}$. There exists a positive constant c such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^2(\Omega)} \leq ch. \tag{6.20}$$

2. *Let Assumptions 5 be satisfied by $\bar{u} \in \mathcal{U}$. There exists a positive constant c such that*

$$\|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^2(\Omega)} \leq c(h|\log h|)^2. \tag{6.21}$$

3. *Let Assumptions 3 be satisfied by $\bar{u} \in \mathcal{U}$ for some $\gamma \in (n/(2+n), 1]$. There exists a positive constant c such that*

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} + \|y(\bar{u}_h) - \bar{y}\|_{L^2(\Omega)} + \|p(\bar{u}_h) - \bar{p}\|_{L^2(\Omega)} \leq c(h|\log h|)^{2\gamma}. \tag{6.22}$$

We consider (6.22). Since $\bar{u}_h$ satisfies the first-order necessary optimality condition of the discrete problem, it holds

$$0 \geq J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}) = J'(\bar{u}_h)(\bar{u}_h - \bar{u}) + J_h'(\bar{u}_h)(\bar{u}_h - u) - J'(\bar{u}_h)(\bar{u}_h - \bar{u}),$$

and by Theorem 4.3, (4.3)

$$c\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}^{1+\frac{1}{\gamma}} \leq J'(\bar{u}_h)(\bar{u}_h - \bar{u}) - J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}).$$

We use that $\bar{u}_h = \bar{u}$ on $\Omega \setminus \Omega_h$ by definition and write

$$J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}) - J'(\bar{u}_h)(\bar{u}_h - \bar{u}) = \int_{\Omega_h} (p(\bar{u}_h) + L_b(x, y(\bar{u}_h)))(\bar{u}_h - \bar{u})\,\mathrm{d}x$$
$$- \int_{\Omega_h} (p_{\bar{u}_h} + L_b(x, y_{\bar{u}_h}))(\bar{u}_h - \bar{u})\,\mathrm{d}x = I.$$

To estimate the term I, we follow similar reasoning as in [7], using (6.4) (6.18), (6.19) and also using the local Lipschitz property of L_b with respect to y, to infer

$$\begin{aligned}
I &\leq C_{L,M}(\|\bar{p}(\bar{u}_h) - p_{\bar{u}_h}\|_{L^\infty(\Omega)} + \|y(\bar{u}_h) - y_{\bar{u}_h}\|_{L^\infty(\Omega)})\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \\
&\leq C_{L,M}(\|p(\bar{u}_h) - p^h_{\bar{u}_h}\|_{L^\infty(\Omega)} + \|p^h_{\bar{u}_h} - p_{\bar{u}_h}\|_{L^\infty(\Omega)})\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \\
&+ C_{L,M}\|y(\bar{u}_h) - y_{\bar{u}_h}\|_{L^\infty(\Omega)}\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \\
&\leq CC_{L,M}(h^2 + 2h^2|\log h|^2)\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)},
\end{aligned}$$

for some positive constant C. Altogether, we obtain

$$\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)} \leq CC_{L,M}(h^2 + 2h^2|\log h|^2)^\gamma. \tag{6.23}$$

Applying the estimates (6.3), (6.18) and (6.19) the claim (6.22) holds for the controls. For the states we use (6.4) to find

$$\begin{aligned}
\|y(\bar{u}_h) - y_{\bar{u}}\|_{L^2(\Omega)} &\leq \|y(\bar{u}_h) - y_{\bar{u}_h}\|_{L^2(\Omega)} + \|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^2(\Omega)} \\
&\leq ch^2 + \|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}
\end{aligned}$$

and the estimate follows from (6.23). For Assumption 5, by (6.17) of Theorem 6.7

$$c\|\bar{u}_h - \bar{u}\|_{L^1(\Omega)}\|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^1(\Omega)} \leq J'(\bar{u}_h)(\bar{u}_h - \bar{u}) - J_h'(\bar{u}_h)(\bar{u}_h - \bar{u}).$$

Estimating as before, we obtain the existence of a positive constant c that satisfies

$$\|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^2(\Omega)} \leq c(h^2 + h^2|\log h|^2).$$

By again (6.3), (6.18) and (6.19) the claim (6.21) holds. Finally, consider Assumption 4. To estimate the term I, we use (6.18)-(6.19) to find

$$\begin{aligned}
I &\leq \|p(\bar{u}_h) - p_{\bar{u}_h}\|_{L^2(\Omega)}\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \\
&\leq (\|p(\bar{u}_h) - p^h_{\bar{u}_h}\|_{L^2(\Omega)} + \|p^h_{\bar{u}_h} - p_{\bar{u}_h}\|_{L^2(\Omega)})\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \leq 2ch^2\|u_a - u_b\|_{L^\infty(\Omega)},
\end{aligned}$$

for some positive constant c. Taking the root, this leads to the estimate $\|y_{\bar{u}_h} - y_{\bar{u}}\|_{L^2(\Omega)} \leq ch$, and by (6.3), (6.18) and the claim (6.20) holds.

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