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# Vertices in higher-dimensional spherically reduced quantum gravity

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# Abstract

Following previous results the quantization of D-dimensional spherically reduced Einstein gravity with minimally coupled scalar matter is performed. The gravitational variables can be treated exactly using Eddington–Finkelstein gauge in the path integral leading to a non-local effective action for the matter sector. At tree-level it is known to generate two sorts of non-local four-point vertices which lead to finite S-matrix elements if the original spacetime has $D = 4$. Here, the stability of the finiteness of this amplitude is investigated for higher dimensions. After a long computation a first result is reached: For $D = 6$ the divergent terms of the highest transcendentality cancel.

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# Contents

# 1. Introduction

*Whoever makes the universe hides messages in transcendental numbers...*

—Carl Sagan

The long-sought goal of finding a consistent theory of quantum gravity, free of problems like UV-divergences, non-renormalizability and non-unitarity [1] still has not been reached. It even remains an open question whether one of the two most popular candidates today, namely string theory or loop quantum gravity will be the right one to fill this gap. Despite their strengths in many points both of them are beset with conceptual or technical difficulties: In string theory one has to choose a fixed background geometry which the strings are propagating on being in contradiction with the assumptions of general relativity where no such preferred background should exist. Loop quantum gravity on the other hand, although manifestly background independent, still poses problems in defining physical observables such as an S-matrix and moreover obscures how Einstein gravity emerges in a semiclassical limit. On top of that, both of them are mathematically rather complex making it especially hard to obtain explicit results.

It is therefore interesting to have a simplified theory which does allow analytic computations and to use it as a toy model for gaining intuition about how a general theory of quantum gravity might behave. This simplification is achieved by taking the Einstein-massless-Klein–Gordon model (EMKG) which is a classical model of Einstein gravity in D dimensions minimally coupled to a massless scalar field and imposing spherical symmetry. The situation is thereby effectively reduced to two dimensions and is described by a certain model of dilaton gravity non-minimally coupled to matter [2]. A minimally-coupled version was successfully quantized in a series of papers [3, 4, 5] using the path integral formalism where it turned out that the geometric sector can be integrated exactly. Conceptually this is an important point as it avoids a perturbative treatment of Einstein gravity which is known to be ill-defined because of its non-renormalizability [6]. Afterwards, perturbation theory was applied in the matter sector and a certain S-matrix element was computed. In this setting it describes the interaction of ingoing matter fields mediated by gravity where full quantum backreaction is taken into account. In [7, 8] the scattering amplitude was computed at tree-level in the matter loops with reinstated non-minimal coupling in the reduced model as opposed to the previous work and the dimension of the original spacetime was restricted to four. It turned out that the scattering amplitude at that level of perturbation theory consists of two diagrams, each of them being rather complicated and even divergent. Only in the end, when the two diagrams were summed a finite and moreover intriguingly simple result was found, a situation frequently encountered in other gauge theories such as Yang–Mills theory. The structure of the problem depends crucially on the chosen gauge and it might well be that a different gauge leads to a much simpler way of obtaining the final sum. Because of its practicality when it comes to solving the path integral one however usually chooses an Eddington–Finkelstein-like gauge for the geometric sector as was done in the above references and likewise will be done in this work.

The main goal here is to investigate the stability of the finiteness of this scattering amplitude when the dimension of the original non-reduced spacetime is higher than four. It will turn out that there are differences between odd and even dimensions and for computational convenience we will restrict to the even case. Also, as the length of the expressions becomes a lot harder to handle the higher the dimension gets the emphasis will be on the first non-trivial case which is $D = 6$. The structure of the thesis consists of two parts: Chapter 2 will at first be concerned with the path integral quantization of the model in the sum over histories approach [9] which is just a summary of the previous results. It is followed by a brief look at the effective geometry as predicted by the quantum equations of motion with certain boundary conditions which turns out to be a virtual Schwarzschild–Tangherlini black hole [10]. Finally, the tree-level vertex functions for the matter field are extracted. All these considerations work without having to restrict the spacetime dimension. In chapter 3 the finiteness of the scattering amplitude for $D = 6$ is investigated with some computational methods adopted from [7]. At first everything will be written out rather explicitly but after a certain point the use of *Mathematica* became inevitable making it impractical to show anything more than the main results because of the sheer size of the expressions. The final result of this thesis is the cancellation of the divergent terms at the highest level of transcendentality which is a first step towards showing the full stability of the scattering amplitude.

# 2. Spherically reduced EMKG model

Consider a spacetime manifold $\mathcal{M}$ in $D > 3$ dimensions with pseudo-Riemannian metric $\bar{g}$, the usual Levi–Civita connection and a minimally coupled massless scalar field $\phi$. Let the dynamics of this system be governed by the action

$$\Gamma_{EMKG}[\bar{g}, \phi] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{\bar{g}} \left( \bar{R} + \kappa \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + f(\phi) \right) \tag{2.1}$$

which is the Einstein-massless-Klein–Gordon (EMKG) model. $\bar{R}$ is the Ricci scalar in D dimensions corresponding to the chosen connection and $f(\phi)$ represents a possible self-interaction of the scalar field which will be set to zero in the following. The units are chosen such that $(\kappa)^{-1} = 8\pi G = 1 = c$ and we will work in east coast signature $(-, +, +, ..., +)$. Assume now that the group $SO(D-1)$ acts on $\mathcal{M}$ via isometries with the corresponding orbits being round $D-2$-dimensional spheres. In that case the line element can be decomposed as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + \Phi^2(x_0, x_1) d\Omega^2_{S^{D-2}} \tag{2.2}$$

where $\alpha, \beta = 0, 1$ and $d\Omega^2_{S^{D-2}}$ is the metric of the round $S^{D-2}$. Here, a new field $\Phi$ was introduced which has the geometric interpretation of an area radius. The possible solutions of the matter field are now restricted to the s-wave sector, meaning that no dependence on angular coordinates can occur and one can already suppose that the dynamics of the system is described by an effectively two-dimensional theory with the dynamical variables $(g, \Phi, \phi)$. Indeed, if one performs a spherical reduction, inserting (2.2) into (2.1) and integrating out the angular dependencies[1] after several lines one arrives at

$$\Gamma_{EMKG}[g, X, \phi] = \frac{1}{2} \int_{\mathcal{M}_2} d^2 x \sqrt{g} \left( X \frac{R}{2} - \frac{1}{2} U(X) (\partial X)^2 + V(X) + \frac{X}{2} (\partial \phi)^2 \right) \tag{2.3}$$

where the functions

$$U(X) = -\frac{D-3}{D-2} \frac{1}{X} \qquad\qquad V(X) = \frac{D-3}{2(D-2)} \lambda^2 X^{\frac{D-4}{D-2}} \tag{2.4}$$

[1] For an extensive treatment of this procedure see appendix C of [7].

were introduced and the area radius was redefined in terms of the *dilaton-field* $X$ as

$$\Phi =: \frac{D-2}{\lambda} X^{\frac{1}{D-2}}. \tag{2.5}$$

The constant $\lambda$ has inverse length dimension 1 and renders the dilaton dimensionless. From a physical perspective the area radius should be restricted to positive values which is inherited by the dilaton.

The matterless part of (2.3) is an example of a two-dimensional dilaton gravity model [2] described by the functions $U$ and $V$. It arises on general grounds when performing a spherical reduction of Einstein gravity and together with the matter part is the model of interest in this thesis. It should be noted, that the matter field was minimally coupled in the original action but is now non-minimally coupled to the dilaton. This has as a consequence that the matter stress energy tensor is not conserved anymore, it can be shown instead that $\nabla^\mu T_{\mu\nu} \propto (\partial_\nu X)$. Although this seems problematic at first sight, one has to keep in mind that this non-minimal coupling only is an artefact of the two-dimensional description meaning that the true physical interpretations have to take place in the non-reduced realm where the stress energy tensor is indeed covariantly conserved.
Regarding the following path integral quantization it is practical to use the first order formulation of dilaton gravity which promotes the connection to an independent dynamical variable. For this one introduces the vielbein $e^a_\mu$ by $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and chooses lightcone gauge $\eta_{\pm\mp} = 1$, $\eta_{\pm\pm} = 0$ for the Lorentz indices. The connection is represented by a tensor-valued two-form

$$\omega^a{}_b = \epsilon^a{}_b \omega \tag{2.6}$$

which has to be antisymmetric in its vielbein-indices and therefore is proportional to the Levi–Civita tensor in two dimensions. We can now define the gauge-covariant derivative $D$ and subsequently the torsion two-form

$$T^a := D \wedge e^a = d \wedge e^a + \epsilon^a{}_b \omega \wedge e^b \tag{2.7}$$

together with the curvature two-form

$$R^a{}_b = D \wedge \omega^a{}_b = \epsilon^a{}_b d\omega. \tag{2.8}$$

It can be seen that $R^a{}_b$ has only one independent component, which after elimination of the torsion part of the spin connection can be taken to be dual to the Ricci scalar of the corresponding second-order action (2.3). This elimination is implemented by introducing Lagrange multipliers $X^\pm$ which from the perspective of the first-order action are independent dynamical fields. It can be shown [7, 2] that

$$\Gamma_{1^{st}}[e^a,\omega,X,X^+,X^-] = \tag{2.9a}$$

$$\frac{1}{2}\int_{\mathcal{M}_2}\Big[X^+D\wedge e^- + X^-D\wedge e^+ + Xd\wedge\omega - \mathcal{V}(X,X^\pm)e^-\wedge e^+ \tag{2.9b}$$

$$+ F(X)d\phi\wedge *d\phi\Big] \tag{2.9c}$$

indeed reproduces (2.3) upon eliminating the fields $X^\pm$ and $\omega$ by their equations of motion. To be consistent with the literature, the coupling function $F(X) = -\frac{X}{2}$ was introduced and the dilaton gravity model dependence is now fully encoded in the potential function

$$\mathcal{V} := V(X) + X^+X^-U(X). \tag{2.10}$$

Switching into a coordinate patch on the world-sheet manifold $\mathcal{M}_2$ we can express the first-order action in a convenient way for starting a Hamiltonian analysis. This yields

$$\Gamma_{1^{st}} = \frac{1}{2}\int_{\mathcal{M}_2}\mathcal{L}_g + \mathcal{L}_m\, d^2x \tag{2.11}$$

where

$$\mathcal{L}_g = \tilde{\epsilon}^{\mu\nu}\Big[X^+(\partial_\mu - \omega_\mu)e^-_\nu + X^-(\partial_\mu + \omega_\mu)e^+_\nu + X\partial_\mu\omega_\nu\Big] - |e|\mathcal{V} \tag{2.12}$$

$$\mathcal{L}_m = \frac{F(X)}{|e|}\Big(\tilde{\epsilon}^{\mu\nu}e^+_\nu\partial_\mu\phi\Big)\Big(\tilde{\epsilon}^{\kappa\lambda}e^-_\kappa\partial_\lambda\phi\Big) \tag{2.13}$$

with $\tilde{\epsilon}^{\mu\nu}$ being the Levi–Civita symbol[2] and $|e| = \sqrt{|g|} = e^-_0e^+_1 - e^-_1e^+_0$ the determinant of the Zweibein.

## 2.1. Hamiltonian analysis

Before turning to the path integral it is important to analyse the dynamical structure of this system. We already anticipate that there will be some gauge redundancies present as there are still two diffeomorphisms and one local Lorentz boost to be performed with arbitrary transformation parameters. In fact, eventually the only physical local degree of freedom left will be the scalar field which goes along

[2] We use the convention $\tilde{\epsilon}^{01} = 1$ together with $\epsilon^{-+} = \tilde{\epsilon}^{-+} = -1$. This is consistent with the usual definitions $\epsilon_{\mu\nu} = \sqrt{|g|}\tilde{\epsilon}_{\mu\nu}$ and $\epsilon_{ab} = \tilde{\epsilon}_{ab}$ if $\epsilon_{\mu\nu} = -e^a_\mu e^b_\nu\epsilon_{ab}$.

with the well-known fact that the matterless model of two-dimensional dilaton gravity is a field theory of topological type [11, 12]. These gauge redundancies have to be tackled if one wants to perform the path integral so that gauge equivalent field configurations are not integrated over. There are several ways this can be done, one of them being the BRST formalism which needs as a starting point a Hamiltonian analysis. In the spirit of [5, 7] we define the canonical coordinates $\phi$ and

$$q_i = (\omega_1, e_1^-, e_1^+) \qquad \bar{q}_i = (\omega_0, e_0^-, e_0^+) \tag{2.14}$$

together with their conjugate momenta

$$p^i = (X, X^+, X^-) \qquad \bar{p}^i = 0 \qquad \pi = \frac{\partial \mathcal{L}_m}{\partial \partial_0 \phi} \tag{2.15}$$

and perform the Legendre transformation to the canonical Hamiltonian density

$$\mathcal{H} = \pi\dot{\phi} + p^i \dot{q}_i - \mathcal{L}, \tag{2.16}$$

where $\dot{q} = \partial_0 q$ and $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$. It can be seen that there are three primary constraints $\bar{p}^i \approx 0$ appearing. Following [13] we therefore need the consistency conditions[3]

$$\dot{\bar{p}}^i = \{\bar{p}^i, \mathcal{H}'\} \approx 0 \tag{2.17}$$

which can be evaluated using the fundamental Poisson brackets

$$\{q_i, p^{j\prime}\} = \delta_i^j \delta(x^1 - x^{1\prime}) \tag{2.18a}$$
$$\{\bar{q}_i, \bar{p}^{j\prime}\} = \delta_i^j \delta(x^1 - x^{1\prime}) \tag{2.18b}$$
$$\{\phi, \pi'\} = \delta(x^1 - x^{1\prime}) \tag{2.18c}$$

and lead to three secondary constraints

$$G_1 = \partial_1 p^1 + p^3 q_3 - p^2 q_2 \tag{2.19a}$$
$$G_2 = \partial_1 p^2 + p^2 q_1 - q_3 \mathcal{V} + \frac{F(p^1)}{4q_2}\left[\partial_1\phi - \frac{\pi}{F(p^1)}\right]^2 \tag{2.19b}$$
$$G_3 = \partial_1 p^3 - p^3 q_1 + q_2 \mathcal{V} - \frac{F(p^1)}{4q_2}\left[\partial_1\phi + \frac{\pi}{F(p^1)}\right]^2 . \tag{2.19c}$$

A short calculation yields that the canonical Hamiltonian density can be expressed in terms of these secondary constraints as

$$\mathcal{H} = -\bar{q}_i G_i \tag{2.20}$$

[3] $\mathcal{H}'$ has to be read as $\mathcal{H}(x')$.

which shows that $\mathcal{H}$ vanishes weakly, a well-known feature of a diffeomorphism invariant theory [13]. Here the algorithm stops as we have

$$\{G_i, \mathcal{H}'\} \approx 0 \ . \tag{2.21}$$

When computing the bracket relations between the constraints one finds

$$\{\bar{p}^i, G_j'\} = 0 \qquad\qquad \forall i, j \tag{2.22}$$

$$\{G_i, G_j'\} = C_{ij}{}^k G_k \delta(x^1 - x^{1\prime}) \tag{2.23}$$

where the non-vanishing components of the structure functions $C_{ij}{}^k = -C_{ji}{}^k$ read

$$\begin{aligned} C_{12}{}^2 &= -1 \qquad & C_{23}{}^1 &= -\frac{\partial \mathcal{V}}{\partial p_1} + \frac{F'(p_1)}{|e|F(p_1)}\mathcal{L}_m \\ C_{13}{}^3 &= 1 \qquad & C_{23}{}^2 &= -\frac{\partial \mathcal{V}}{\partial p_2} \\ & & C_{23}{}^3 &= -\frac{\partial \mathcal{V}}{\partial p_3} . \end{aligned} \tag{2.24}$$

As is apparent from (2.22)-(2.23), all six constraints are first class and therefore generate gauge transformations. The initially 14 phase space degrees of freedom are therefore reduced by twice the number of first class constraints which yields two phase space degrees of freedom or respectively one local physical degree of freedom corresponding to the matter field, just as expected.

One can observe from the form of $\mathcal{H}$ in (2.20) that the $\bar{q}_i$ in fact act like Lagrange multipliers[4] enforcing the constraints $G_i$. On the other hand they are canonical variables whose conjugate momenta $\bar{p}_i$ have to vanish on the constraint surface. From a BRST-perspective it is therefore possible to regard them as part of the non-minimal sector and effectively reduce the physical phase space to $(q^i, \phi)$ together with their momenta. This of course does not change the number of first class constraints but permits to write the extended Hamiltonian action as

$$\Gamma_E = \int \dot{q}_i p^i + \dot{\bar{q}}^i \bar{p}_i + \dot{\phi}\pi + \bar{q}^i G_i \ . \tag{2.25}$$

We now associate to each constraint a ghost/antighost pair

$$G_i \longrightarrow (c_i, p_c^i) \tag{2.26}$$

$$\bar{p}_i \longrightarrow (b_i, p_b^i) \tag{2.27}$$

[4]To make this explicit in the notation we from now on switch the index of $\bar{q}_i \mapsto \bar{q}^i$ and $\bar{p}^i \mapsto \bar{p}_i$.

with the ghosts $c_i, b_i$ being real odd fields and the antighosts $p_c^i, p_b^i$ being imaginary odd fields so that the graded Poisson bracket relations

$$\{c^i, p_j^{c\prime}\} = -\delta_j^i \delta(x^1 - x^{1\prime}) \tag{2.28}$$

$$\{b^i, p_j^{b\prime}\} = -\delta_j^i \delta(x^1 - x^{1\prime}) \tag{2.29}$$

hold. Having defined these the BRST charge $\Omega$ can be determined by homological methods found in [13] and reads

$$\Omega = \Omega_{min} + \Omega_{nonmin} \tag{2.30}$$

$$\Omega_{min} = c^i G_i + \frac{1}{2} c^i c^j C_{ij}{}^k p_k^c \tag{2.31}$$

$$\Omega_{nonmin} = b^i \bar{p}_i \ . \tag{2.32}$$

It should be pointed out that although the structure functions are field-dependent the homological perturbation series stops after one iteration which is normally only found to hold in ordinary Yang–Mills theory and is thus a non-trivial feature in this case [7]. As the canonical Hamiltonian in the minimal sector vanishes, its BRST-invariant extension is exact and can be written in the form

$$H_{BRST} = \{\Psi, \Omega\} \tag{2.33}$$

with the gauge fixing fermion $\Psi$. In the context of dilaton gravity it became customary to choose temporal gauge for the geometric variables [5] meaning a restriction of the components $\bar{q}_i$ to some constants. This can be implemented into the dynamics of the extended phase space by a gauge fixing fermion of the form $\Psi = p_i^b \mathcal{X}^i + p_i^c \bar{q}^i$ with appropriate functions $\mathcal{X}^i$. However in [7] there was found a shortcut for evaluating the path integral leading to the same outcome. This choice reading

$$\Psi = p_2^c \tag{2.34}$$

shall be made here as well.

## 2.2. Path integral quantization

Having introduced the ghost fields and $H_{BRST}$ we can write down the gauge fixed path integral

$$Z[J_i, j_i, \sigma] = \int (\mathcal{D}Q)\, (\mathcal{D}P)\, (\mathcal{D}c^i)\, (\mathcal{D}p_i^c)\, (\mathcal{D}b^i)\, (\mathcal{D}p_i^b) \times \exp\Big[i \int \mathcal{L}_{ext} + J_i p^i + j_i q_i + \sigma\phi\Big] \tag{2.35}$$

with a Lagrangian density on the extended phase space

$$\mathcal{L}_{ext} = P\dot{Q} + p_i^b \dot{b}^i + p_i^c \dot{c}^i - \{p_2^c, \Omega\} \tag{2.36}$$

and the triplets $Q = (q_i, \bar{q}^i, \phi)$, $P = (p^i, \bar{p}_i, \pi)$ together with sources $(J_i, j_i, \sigma)$ for the minimal sector. In contrast to the usual choice of sources for the canonical coordinates there are also introduced sources for the momenta $p^i$ which turn out to be useful later on when integrating them [3, 5]. The first few integrations over the non-minimal sector and the ghosts are rather trivial and are done in the following order:

$$\bar{p}_i \to \bar{q}^i \to p_i^b, b^i \to p_i^c, c^i. \tag{2.37}$$

As an important intermediate step we chose three integration constants for the $\bar{q}^i$-integrals so that

$$\bar{q}^i = (0, 1, 0)^i \tag{2.38}$$

which fixes the beforementioned temporal gauge. The path integral now reads

$$\begin{aligned} Z[J_i, j_i, \sigma] = \int (\mathcal{D}q_i)\ (\mathcal{D}p^i)\ (\mathcal{D}\phi)\ (\mathcal{D}\pi)\ \mathrm{Det}\Big(\partial_0^2(\partial_0 + p_2 U)\Big) \\ \times \exp\Big[\frac{i}{\hbar}\int p^i \dot{q}_i + \pi\dot{\phi} + G_2 + \mathcal{L}_{(s)}\Big] \end{aligned} \tag{2.39}$$

where the functional determinants arise from integrating out the ghost fields and $\mathcal{L}_{(s)}$ contains the source terms. There was also reinstated a factor $\hbar$ to to keep track of the loop order later on. As a next step one performs the Gaussian integral over the scalar field momentum $\pi$ leading to

$$\begin{aligned} Z[J_i, j_i, \sigma] = \int (\mathcal{D}q_i)\ (\mathcal{D}p^i)\ (\mathcal{D}\phi)\ \sqrt{4q_2 F}\ \mathrm{Det}\Big(\partial_0^2(\partial_0 + p^2 U)\Big) \\ \times \exp\Big[\frac{i}{\hbar}\int p^i \dot{q}_i + q_1 p^2 - q_3\mathcal{V} + F\big(\partial_0\phi\partial_1\phi - q_2(\partial_0\phi)^2\big) + \mathcal{L}_{(s)}\Big] \end{aligned} \tag{2.40}$$

with another new factor in the measure. It was now argued in [5, 2] that a diffeomorphism covariant path integral of a scalar field $\varphi$ is given by

$$\int (\mathcal{D}\varphi \sqrt[4]{|g|})\ \exp\Big[i\int d^2x\ \big(\sqrt[4]{|g|}\varphi\big) M \big(\sqrt[4]{|g|}\varphi\big)\Big] = (\mathrm{Det}\ M)^{-1/2} \tag{2.41}$$

where $M$ is some operator. Comparing this with the path integral in our case one finds that the factor $\sqrt{q_2}$ in the measure should be replaced with $\sqrt{q_3}$ by hand as in the gauge (2.38) we have $\sqrt{|g|} = q_3$.

In order to perform the $q_i$-integrals next we rewrite this exchanged factor as

$$\sqrt{q_3} = \int (\mathcal{D}\mathcal{X})(\mathcal{D}u)(\mathcal{D}\bar{u}) \exp\Big[i \int (\mathcal{X}^2 + u\bar{u})q_3\Big] \tag{2.42}$$

$$:= \int (\mathcal{D}\mathcal{X})(\mathcal{D}u)(\mathcal{D}\bar{u}) \exp\Big[i \int hq_3\Big] \tag{2.43}$$

with the auxiliary fields $\mathcal{X}$ having even parity and $u, \bar{u}$ having odd parity. Now one can see the strength of the chosen gauge: All the $q_i$ appear linearly in the exponent and can thus be integrated exactly yielding three delta functions

$$\delta^{(1)} := \delta\Big(\partial_0 p^1 - p^2 - j_1\Big) \tag{2.44a}$$

$$\delta^{(2)} := \delta\Big(\partial_0 p^2 - j_2 + F(p^1)(\partial_0\phi)^2\Big) \tag{2.44b}$$

$$\delta^{(3)} := \delta\Big((\partial_0 + p^2 U)p^3 + V - \hbar h - j_3\Big) \tag{2.44c}$$

which in turn can be used to integrate the $p^i$ and leave only the scalar field $\phi$ in the path integral. With this integration also the Faddeev–Popov determinant in (2.40) cancels. Thus, the whole geometric sector is treated non-perturbatively meaning that quantum gravitational effects are accounted for to full extent within the range of the specified boundary conditions for $p^i$ and $q_i$ [5]. Following the last reference these boundary conditions are chosen such that global quantum fluctuations are excluded from the field space integrated over. This is achieved by choosing the integration constants properly when solving the differential equations in (2.44a)-(2.44c) which amounts to a full fixing of the residual gauge freedom [7]. It should be noted that the differential equations in the delta functions are just the classical equations of motion for the $p^i$ as determined from the gauge fixed action (2.40). We therefore have

$$Z[J_i, j_i, \sigma] = \int (\mathcal{D}\phi)\ (\mathcal{D}\mathcal{X})(\mathcal{D}u)(\mathcal{D}\bar{u})\ \sqrt{F(p^1_{cl})} \times \exp\Big[\frac{i}{\hbar} \int F(p^1_{cl})\partial_0\phi\partial_1\phi + \mathcal{L}_{amb} + \mathcal{L}_{(s)}\Big] \tag{2.45}$$

where an additional term

$$\mathcal{L}_{amb} = e^{Q(p^1_{cl})}\left[j_3 + \hbar h - V(p^1_{cl})\right] \tag{2.46}$$

with

$$Q(p^1) = \int^{p^1} U(y)\, dy \tag{2.47}$$

was introduced arising from an ambiguity in the source terms $J_i p^i_{cl}$. Its necessity is discussed at length in [7, 5], essentially it assures a non-trivial action of the scalar field when $J_i = j_i = 0$ which is of course important for investigating scattering processes involving it. Moreover the integral over the auxiliary field $h = \mathcal{X} + u\bar{u}$ would only pull a factor proportional to $J_3$ in front of the exponential which would make the whole path integral vanish for vanishing sources. Including $\mathcal{L}_{amb}$ avoids this and we get after evaluation of the auxiliary integrals

$$\begin{aligned} Z[\sigma] = \int (\mathcal{D}\phi) \sqrt{F(p^1_{cl}) e^{Q(p^1_{cl})}} \\ \times \exp\left[\frac{i}{\hbar} \int F(p^1_{cl}) \partial_0\phi \partial_1\phi - V(p^1_{cl}) e^{Q(p^1_{cl})} + \sigma\phi\right] \end{aligned} \tag{2.48}$$

where $J_i = j_i = 0$. The additional factor $e^{Q(p^1_{cl})}$ in the measure is recognized as the solution $q^{cl}_3$ of the classical equation of motion (2.58) determined from (2.40). As it implicitly depends on the matter field through $p^i_{cl}$ it accounts for the full backreaction of the scalar field on the geometry [5]. In fact it can be shown that the other components of $q_i$ also enter only classically and we generally have

$$\langle q_i \rangle \equiv q^{cl}_i \qquad \langle p^i \rangle \equiv p^i_{cl} \tag{2.49}$$

meaning that the geometric sector of our model fulfills the classical equations (2.58) of motion as determined from the action in (2.40). In fact, a stronger version of this arises for pure dilaton gravity where computing the quantum effective action for the geometric variables leads back to the classical action meaning the theory is locally quantum trivial [3]. The non-trivial loop corrections in the matter case arise from the effective action of the scalar field and in order to compute the different vertices one has to find the solutions $p^i_{cl}, q^{cl}_i$ and insert them into (2.48). For generic matter contributions this task is quite non-trivial and the equations of motion can only be solved in a weak matter approximation[8, 7, 5] meaning that the energy of the scalar field is a lot smaller than the Planck mass,

$$(\partial_0\phi)^2 \ll m_{Planck}. \tag{2.50}$$

Thus, at this point we have to start working perturbatively in the matter sector and can employ ordinary QFT methods such as isolating the quadratic terms in $\phi$ for the propagator and pulling all higher polynomials as interactions in front of the path integral with $\phi \to \frac{\hbar}{i}\frac{\delta}{\delta\sigma}$. The most work however still lies in the integration of the equations of motion as there are multiple integrals which have to be done in a certain order to respect boundary conditions implemented by certain regularizations of the Green functions [5]. As we are only interested in the tree level vertices of the scalar field we can use a certain trick to extract them which was first introduced in that last reference.

## 2.3. Lowest order tree-graphs

As mentioned before, the expectation values of the geometric variables fulfill the classical equations of motion with sources. We can therefore go back to (2.40) and switch to a saddle-point approximation for the geometric sector which is exact in this case

$$Z[\langle q_i\rangle_j, \langle p^i\rangle_J, \sigma] = \int (\mathcal{D}\phi)\; e^{\frac{i}{\hbar}W[\langle q_i\rangle_j, \langle p^i\rangle_J, \sigma]} \tag{2.51}$$

where the inverse Legendre transformation of the quantum effective action

$$\begin{aligned} W[\langle q_i\rangle_j, \langle p^i\rangle_J, \sigma] = \int \langle p^i\rangle\langle \dot q_i\rangle + \langle q_1\rangle\langle p^2\rangle - \langle q_3\rangle\mathcal{V}(\langle p^1\rangle) & \quad (2.52a)\\ + 2F(\langle p^1\rangle)\big(\Phi_1 - \langle q_2\rangle\Phi_0\big) & \quad (2.52b)\\ + j_i\langle q^i\rangle + J_i\langle p_i\rangle + \sigma\phi & \quad (2.52c) \end{aligned}$$

and the combinations

$$\Phi_0 := \frac{1}{2}\big(\partial_0\phi\big)^2 \qquad\qquad \Phi_1 := \frac{1}{2}\big(\partial_0\phi\partial_1\phi\big) \tag{2.53}$$

were introduced. The factor in the measure of (2.40) was left out as by a similar trick as in (2.43) it would contribute terms of $\mathcal{O}(\hbar)$ to $W$ which are of higher loop order and thus can be neglected. Note, that $\langle q_i\rangle$ and $\langle p^i\rangle$ depend functionally on the scalar field and the sources which is denoted by the subscripts in the arguments of the above generating functional. As we are only interested in vertices for the scalar field we can now set $j_i = J_i = 0$. At tree level the generating functional for $\phi$-vertex functions is just the classical action in $\phi$ [14] and we can therefore use

$$\tilde W[\phi] = \int \langle p^i\rangle\langle \dot q_i\rangle + \langle q_1\rangle\langle p^2\rangle - \langle q_3\rangle\mathcal{V}(\langle p^1\rangle) + 2F(\langle p^1\rangle)\big(\Phi_1 - \langle q_2\rangle\Phi_0\big) \tag{2.54}$$

to generate two sorts[5] of 4-point vertices by functional differentiation [2, 15, 5]

$$V_a^{(4)} = \int dx^2 dy^2\; v_a^{(4)}(x,y)\Phi_0(x)\Phi_0(y), \qquad v_a^{(4)}(x,y) = \frac{\delta^2\tilde W}{\delta\Phi_0(x)\delta\Phi_0(y)} \tag{2.55}$$

$$V_b^{(4)} = \int dx^2 dy^2\; v_b^{(4)}(x,y)\Phi_0(x)\Phi_1(y), \qquad v_b^{(4)}(x,y) = \frac{\delta^2\tilde W}{\delta\Phi_0(x)\delta\Phi_1(y)}\;. \tag{2.56}$$

It will be seen in the following section that they both are non-local, however with each pair of outer legs attached to a single point. Also, $V_a^{(4)}$ is symmetric under

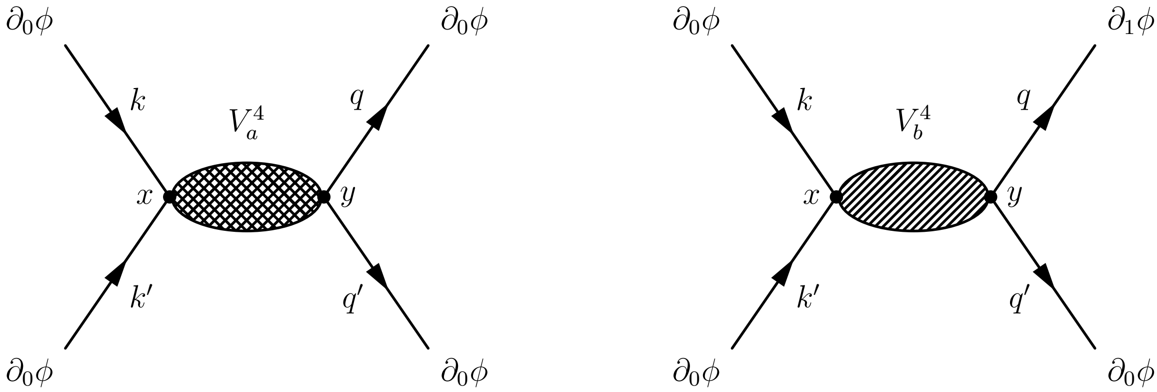


Figure 2.1.: The two lowest-order contributions $V_a^{(4)}$ and $V_b^{(4)}$ to the 4-point vertex. Outer legs with ingoing momenta $k, k'$ and outgoing momenta $q, q'$ are attached.

exchanging the leg pairs whereas $V_b^{(4)}$ is not which can be seen in the diagrams of figure 2.1.

The trick in [5] consists now in circumventing the functional differentiations which would need first the solutions $\langle q_i \rangle$ and $\langle p^i \rangle$ for generic $\phi$ but instead assume the two matter contributions to be localized at a single point, i.e.

$$\Phi_0(x) = c_0\delta(x-y) \qquad\qquad \Phi_1(x) = c_1\delta(x-y) \tag{2.57}$$

where two constants $c_0, c_1$ were introduced. The equations of motion[6]

$$\begin{aligned} \partial_0 p_1 &= p_2 & \partial_0 q_1 &= q_3\frac{\partial\mathcal{V}}{\partial p_1} - 2F'(p_1)\big(\Phi_1 - q_2\Phi_0\big) \\ \partial_0 p_2 &= -2F(p_1)\Phi_0 & \partial_0 q_2 &= -q_1 + q_3p_3U(p_1) \\ \partial_0 p_3 &= -\mathcal{V} & \partial_0 q_3 &= q_3p_2U(p^1) \end{aligned} \tag{2.58}$$

are then solved for this matter distribution which can be done exactly provided certain boundary conditions are imposed to fix the homogeneous solutions. The symmetric and asymmetric vertex functions $v_a^{(4)}$ and $v_b^{(4)}$ can then be extracted by plugging the solutions into (2.54) and expanding to linear order in $c_0$ or $c_1$. Moreover, it turns out that it is sufficient to only take the last part of the effective action $2F(p^1)\big(\Phi_1 - q_2\Phi_0\big)$ for this expansion as all other terms lead to vertices with at least 6 outer legs or higher loop diagrams [7].

[5]Note, that the third possible combination $\Phi_1\Phi_1$ does not appear as the $\langle p^i \rangle$ by (2.44a)-(2.44b) only depend on $\Phi_0$.

[6]For notational convenience, from now on we switch to only lower indices of the canonical variables.

## 2.3.1. Effective fields

Solving (2.58) with (2.57) can be done separately for $x^0 < y^0$ and $x^0 > y^0$ where the matter parts vanish. As we have six first order equations we will at first get six undetermined homogeneous solutions for each of the patches. At the overlap $x^0 = y^0$ there have to be imposed matching conditions which can be found by observing the form of (2.58): $p_1, p_3, q_2, q_3$ have to be continuous and $p_2, q_1$ jump at the overlap. The size of the jump is determined by the matter dependent terms in the respective equations. With this one gets six matching conditions and consequently the number of undetermined homogeneous solutions is reduced by half. We can now choose freely which solutions we want to keep and take the outer ones thinking of them as asymptotic solutions describing a classical vacuum background. Conceptually, this is not in interference with the background independence of the path integral quantization carried out before as it still holds in the bulk. By the asymptotic conditions we rather fix the notion of an "asymptotic observer" which is a useful ingredient for the scattering problem [16].

In fact, there are still the constraints of section 2.1 between the geometric variables which were not listed in (2.58) and come out as Ward identities in the path integral formalism [7, 15]. They reduce the number of independent homogeneous solutions to three. By residual gauge transformations we can fix two of them leaving one physical asymptotic constant which we give the suggestive name $\mathcal{C}_\infty$. Provided we fix the residual gauge freedom by

$$p_1\big|_{x^0\to\infty} = x^0 \tag{2.59a}$$

$$q_3\big|_{x^0\to\infty} = 1 \cdot e^{Q(p_1)} \tag{2.59b}$$

it can be shown that $\mathcal{C}_\infty$ corresponds to the integration constant of $p_3$ and is on the other hand given by

$$\mathcal{C}^{(g)}\big|_{x^0\to\infty} = \left(e^{Q(p_1)} p_2 p_3 + w(p_1)\right)\Big|_{x^0\to\infty} = \mathcal{C}_\infty \tag{2.60}$$

which indeed is the asymptotic value of the conserved Casimir[7] featuring prominently in pure dilaton gravity models [2, 17]. (2.59a) can be seen to fix the asymptotic behaviour of the dilaton. Thinking back to its initial interpretation as essentially an area radius we see that the "time" coordinate $x^0$ with respect to which our Hamiltonian evolves as well should be thought of as a radius. Additionally (2.59a) fixes $p_2|_{x^0\to\infty} = 1$ and the Ward identities determine the last two homogeneous

[7]The function $w(p_1) = \int^{p_1} dz\, e^{Q(z)} V(z)$ is defined in the usual way [2].

solutions. In summary we then get

$$\begin{aligned}
p_1 &= x^0 + \big(x^0 - y^0\big) 2F(y^0) h_0 && (2.61)\\
p_2 &= 1 - 2F(y^0) h_0 && (2.62)\\
p_3 &= e^{-Q(p_1)} \Big[ \mathcal{C}_\infty - 2F(y^0) w(y^0) h_0 - \frac{w(p_1)}{p_2} \Big] && (2.63)\\
q_2 &= \mathcal{C}_\infty - w(p_1) - 2(x^0 - y^0) F'(y^0) h_1 && (2.64)\\
&\quad + \Big[ 4F(y^0)\big(w(x^0) - w(y^0)\big) - 2(x^0 - y^0)\Big(F'(y^0) w(y^0) + F(y^0) w'(y^0)\Big)\Big]\\
q_3 &= e^{Q(p_1)} && (2.65)
\end{aligned}$$

where

$$h_i := c_i \delta(x^1 - y^1) \Theta(y^0 - x^0) \tag{2.66}$$

and the solution for $q_1$ was left out as it is not explicitly needed. It is interesting to see what geometry this effective solution corresponds to. The effective line element takes the form

$$ds^2 = 2q_3 dx^0 \big(dx^1 + q_2 dx^0\big) = 2dr du + K(r,u) du^2 \tag{2.67}$$

where we introduced Eddington–Finkelstein coordinates

$$dr = b\, q_3 dx^0 \qquad\qquad du = \frac{dx^1}{b} \tag{2.68}$$

which are adapted to the chosen temporal gauge. When inserting the potentials for the EMKG-model the scale factor $b$ can be chosen to any convenient value which we set to $b^{-1} = (D-2)\sqrt{2}$. The functions appearing in the solutions (2.61)-(2.65) are

$$\begin{aligned}
U(r) &= \Omega\big(\sqrt{2}r\big)^{2-D} & V(r) &= \frac{\lambda^2}{2}\Omega\big(\sqrt{2}r\big)^{D-4}\\
e^{Q(r)} &= \big(\sqrt{2}r\big)^{3-D} & w(r) &= \frac{\lambda^2}{2} e^{-Q(r)} && (2.69)\\
F(r) &= -\frac{1}{2}\big(\sqrt{2}r\big)^{D-2}
\end{aligned}$$

where we defined the constant

$$\Omega := \frac{D-3}{D-2} \,. \tag{2.70}$$

We can now integrate (2.68) and find for the asymptotic region ($p_1 = x^0$)

$$x^0 = \big(\sqrt{2}r\big)^{D-2} \qquad\qquad x^1 = \frac{u}{\sqrt{2}(D-2)} \,. \tag{2.71}$$

Employing the correspondences $(x^0, x^1) \leftrightarrow (r, u)$ and $(y^0, y^1) \leftrightarrow (r', u')$ it can be shown to linear order in $c_0$ and $c_1$ that the asymptotic Killing norm reads

$$K\big|_{x^0\to\infty} = -1 + \mathcal{O}(\mathcal{C}_\infty) \tag{2.72}$$

where the $\mathcal{O}(\mathcal{C}_\infty)$ term still contains spacetime dependence. We now fix

$$\mathcal{C}_\infty = 0 \tag{2.73}$$

which yields an asymptotic Minkowski background. For setting up the scattering problem later on this is highly advantageous as we can just use the expansions of free matter fields on flat space for our asymptotic states. In the region $x^0 < y^0$ we get

$$K\big|_{x^0<y^0} = -1 + \frac{2m}{r^{D-3}} + ar - d \tag{2.74}$$

with

$$m = \frac{2^{\frac{1-D}{2}}}{(D-2)^2}\,\delta(x^1 - y^1)\Big[(D-2)(y^0)^{\frac{2D-5}{D-2}}c_0 - y^0 c_1\Big] \tag{2.75}$$

$$a = \frac{\sqrt{2}}{(D-2)^2}\,\delta(x^1 - y^1)\Big[(2D-5)(D-2)(y^0)^{\frac{D-3}{D-2}}c_0 + c_1\Big] \tag{2.76}$$

$$d = 2y^0 c_0\,\delta(x^1 - y^1) \tag{2.77}$$

and $x = x(r, u)$, $y = y(r', u')$. It is seen that this region is described by a Schwarzschild–Tangherlini black hole with an additional Rindler term which generalizes the four dimensional case investigated in [7, 2]. From a scattering perspective at our level of perturbation theory it mediates the interaction between two incoming "particles". It is clear that this solution with (2.57) is not fulfilling the equations of motion of the matter field which coined the name virtual black hole for this intermediate scattering state [16, 18] in analogy to virtual particles in ordinary QFT.

### 2.3.2. Extracting the vertex functions

Having the solutions for the geometric variables at hand we can now extract the two vertex functions as described before. We take the relevant term in the action which we call $\mathcal{L}_0$ evaluated on the solutions

$$\mathcal{L}_0\big|_{eom} = 2F(p^1)\big(\Phi_1 - q_2\Phi_0\big) = p_1 q_2 \Phi_0 - p_1\Phi_1 \tag{2.78}$$

and expand in a double Taylor series in the $c_i$. The symmetric vertex then has contributions from only the first term in (2.78) and reads

$$V_a^{(4)} = \int d^2x d^2y \Phi_0(x)\Phi_0(y)\Big(\frac{dq_2}{dc_0}p_1 + q_2\frac{dp_1}{dc_0}\Big)\Big|_{c_i=0} \tag{2.79}$$

whereas the asymmetric vertex has contributions from each of the terms

$$V_b^{(4)} = \int d^2x d^2y \Big(\Phi_0(x)\Phi_1(y)\frac{dq_2}{dc_1}p_1 - \Phi_1(x)\Phi_0(y)\frac{dp_1}{dc_0}\Big)\Big|_{c_i=0}. \tag{2.80}$$

We now insert the functions

$$w(X) = (D-2)^2 X^{\Omega} \qquad\qquad F(X) = -\frac{X}{2} \tag{2.81}$$

corresponding to our potentials and choose $\lambda = \sqrt{2}(D-2)$. This leads to the final form of the lowest order 4-point vertices adapted to our model

$$\begin{aligned} V_a^{(4)} = \int_x\int_y \Phi_0(x)\Phi_0(y)\, x^0 y^0 \Theta(y^0 - x^0)\delta(x^1 - y^1) \\ \times\Big[2(D-2)^2\Big((x^0)^{\Omega} - (y^0)^{\Omega}\Big) \\ - (x^0 - y^0)\big(2D^2 - 9D + 10\big)\Big((x^0)^{\frac{1}{2-D}} + (y^0)^{\frac{1}{2-D}}\Big)\Big] \end{aligned} \tag{2.82}$$

$$V_b^{(4)} = -\int_x\int_y \Phi_0(x)\Phi_1(y)\, x^0\, |x^0 - y^0|\delta(x^1 - y^1) \tag{2.83}$$

where (2.70) was used. They are indeed non-local in the coordinates $x^0, y^0$ and local in $x^1, y^1$.

# 3. Scattering amplitude

The main goal is to determine the scattering amplitude

$$T(q, q'; k, k') = \left\langle q, q' \middle| V_a^{(4)} + V_b^{(4)} \middle| k, k' \right\rangle \tag{3.1}$$

corresponding to the 4-point vertex determined in the last section. In general we would like to have an expression for any dimension $D > 3$ which however turned out to be too tedious a task with the chosen methods. Nevertheless the computation will be kept as general as possible until a certain point where time constraints have forced a condensation to the case $D = 6$. Providing an additional measurement point to $D = 4$ in [19] this is at least a step in the right direction. The final aim of this chapter is the partial cancellation process between several divergent terms leaving only the lowest order divergences together with finite terms in the amplitude. Showing the full cancellation of divergences and computing the finite result is left open for future work.

Before the actual calculation can start we should however make sense of the brackets in (3.1).

## 3.1. Asymptotics

We have seen that for our boundary conditions asymptotically no virtual black hole can act which means that we can build the asymptotic Fock space in the usual "QFT-way", i.e. on a Minkowski background. The viewpoint taken for the scattering process here is the one from D dimensions as it is considered the more natural one and also makes it easier to compare the situation for different values of $D$. This means however that there will appear explicit factors of surfaces of $(D-2)$ spheres $S$ here and there which we will mostly absorb in overall constants later on.

The equation of motion for the scalar field asymptotically is just the Klein–Gordon equation which can be solved in D dimensions (see appendix B) and yields under the assumption of a spherically symmetric matter configuration the expansion in modes

$$\phi(r,t) = \int_0^\infty dk \, \sqrt{\frac{k^{D-2}}{2}} r^{-\alpha} J_\alpha(kr) \left( a_k^+ e^{ikt} + a_k^- e^{-ikt} \right), \qquad \alpha = \frac{D-3}{2} \tag{3.2}$$

where for technical reasons involving convergence of the integrals following later only standing waves instead of propagating ones were allowed as they are regular at the origin [19, 8, 7]. $J_\alpha(kr)$ denote the Bessel functions of the first kind. The Hamiltonian reads

$$H = \frac{S}{2}\int_0^\infty dr\, r^{D-2}\Big(\dot{\phi}^2 + (\partial_r\phi)^2\Big) = S\int_0^\infty k^{D-2}dk\, k a_k^+ a_k^- \tag{3.3}$$

which was obtained using (A.1) and can be seen to have the correct volume element $Sk^{D-2}dk$. The fields $a_k^\pm$ were promoted to operators with

$$[a_k^-, a_{k'}^+] = \frac{\delta(k-k')}{k^{D-2}S} \qquad\qquad [a_k^-, a_{k'}^-] = 0 = [a_k^+, a_{k'}^+] \tag{3.4}$$

and by the chosen normalization factor are compatible with the equal time commutation relation

$$[\phi(t,r), \pi(t,r')] = \frac{i\delta(r-r')}{r^{D-2}S} \tag{3.5}$$

satisfied by $\phi$ and $\pi = \partial_t\phi$. Using the vacuum state defined by $a_k^- |0\rangle = 0$ we can now generate the asymptotic Fock space by the action of ladder operators and e.g. obtain normalized two particle states by

$$|k_1, k_2\rangle = \frac{1}{\sqrt{2!}} a_{k_1}^+ a_{k_2}^+ |0\rangle \ . \tag{3.6}$$

## 3.2. Wick contractions

When evaluating (3.1) a first step is to look at the operator valued part in the vertex integrals (2.82)-(2.83) and isolate the terms that contribute to the connected diagrams in figure 2.1, i.e. the part in normal order[1]. In this way all four outer legs are attached to the amputated vertex. From now on the procedure will be split into the part for the symmetric vertex (2.82) and one for the asymmetric vertex (2.83).

[1]Normal ordering is defined here in the sense that all $a_k^-$'s are always to the right of all $a_k^+$'s. So, e.g.

$$: a_q^- a_k^+ a_{q'}^+ a_{k'}^- : = a_k^+ a_{q'}^+ a_{k'}^- a_q^- \ . \tag{3.7}$$

### 3.2.1. Symmetric vertex

The expression

$$\Phi_0(x)\Phi_0(y) = \frac{1}{4}\Big(\partial_0\phi(x)\Big)^2\Big(\partial_0\phi(y)\Big)^2 \tag{3.8}$$

is at first transformed to asymptotic Eddington–Finkelstein coordinates (2.71) defined before

$$x^0 = (\sqrt{2}r)^{D-2} \qquad\qquad x^1 = \frac{u}{(D-2)\sqrt{2}} \tag{3.9}$$

and in order to insert (3.2) we use the relation

$$t = u + r \tag{3.10}$$

between retarded time and Minkowski time. In the following $x$ and $y$ will still be used as arguments but have to be understood as shorthand notation for $(u, r)$ and $(u', r')$. Now we can evaluate the derivative

$$\partial_0\phi(x) = \frac{1}{(D-2)2^{\frac{D-2}{2}}r^{D-3}}\partial_r\phi(u,r) =: \phi^-(x) + \phi^+(x) \tag{3.11}$$

where positive and negative energy modes

$$\phi^-(x) = \frac{1}{(D-2)2^{\frac{D-1}{2}}r^{2\alpha}}\int_0^\infty dk\,\sqrt{k^{2D-5}}\,\partial_r\Xi(k,r)a_k^+e^{iku} \tag{3.12a}$$

$$\phi^+(x) = \frac{1}{(D-2)2^{\frac{D-1}{2}}r^{2\alpha}}\int_0^\infty dk\,\sqrt{k^{2D-5}}\,\partial_r\mathcal{X}(k,r)a_k^-e^{-iku} \tag{3.12b}$$

with

$$\Xi(k,r) := k^{-\alpha}r^{-\alpha}J_\alpha(kr)e^{ikr} \tag{3.13}$$

$$\mathcal{X}(k,r) := k^{-\alpha}r^{-\alpha}J_\alpha(kr)e^{-ikr} \tag{3.14}$$

were defined. This is then inserted into

$$\langle q,q'|\,\Phi_0(x)\Phi_0(y)\,|k,k'\rangle = \langle q,q'|\,:\Phi_0(x)\Phi_0(y):+\,\mathit{contractions}\,|k,k'\rangle \tag{3.15}$$

where the right hand side was obtained using Wick's theorem. All the contractions among the fields themselves lead to disconnected diagrams which are neglected. The normal ordered part can be written in terms of $\phi^+$ and $\phi^-$ reading

$$\begin{aligned} :\Phi_0(x)\Phi_0(y): =& \frac{1}{4}\Big(\phi^-(x)\phi^-(x)\phi^+(y)\phi^+(y) + \phi^-(y)\phi^-(y)\phi^+(x)\phi^+(x) \\ &+ \phi^-(x)\phi^-(y)\phi^+(x)\phi^+(y) + \phi^-(y)\phi^-(x)\phi^+(x)\phi^+(y) \\ &+ \phi^-(x)\phi^-(y)\phi^+(y)\phi^+(x) + \phi^-(y)\phi^-(x)\phi^+(y)\phi^+(x)\Big) \\ &+ \ldots \end{aligned} \tag{3.16}$$

where the ellipsis denotes terms with unequal numbers of negative and positive modes which will be left out as well because they yield zero when evaluated on the asymptotic states. The terms in the brackets are now contracted with the outer legs leading to a symmetry factor $2 \cdot 2!$ in each case. It is convenient to first determine the single contractions

$$\phi^+(x)\,|k\rangle = \phi^+(x) a_k^+ \,|0\rangle \tag{3.17a}$$

$$= \frac{1}{(D-2) 2^{\frac{D-1}{2}} r^{2\alpha}} \int_0^\infty d\tilde{k}\, \sqrt{\tilde{k}^{2D-5}}\, \partial_r \mathcal{X}(\tilde{k}, r) e^{-i\tilde{k}u} \frac{\delta(k-\tilde{k})}{k^{D-2} S} \,|0\rangle \tag{3.17b}$$

$$= \frac{1}{(D-2) S 2^{\frac{D-1}{2}} r^{2\alpha} \sqrt{k}} \partial_r \mathcal{X}(k, r) e^{-iku} \,|0\rangle \tag{3.17c}$$

and

$$\langle q|\, \phi^-(x) = \langle 0|\, \frac{1}{(D-2) S 2^{\frac{D-1}{2}} r^{2\alpha} \sqrt{q}} \partial_r \Xi(q, r) e^{iqu} \,. \tag{3.18}$$

They are then used to compute

$$\langle q, q'|\, : \Phi_0(x)\Phi_0(y) :\, |k, k'\rangle = \frac{1}{2} \big\langle 0 \big| a_q^- a_{q'}^- : \Phi_0(x)\Phi_0(y) : a_k^+ a_{k'}^+ \big| 0 \big\rangle \tag{3.19a}$$

$$= \frac{1}{8} \Big( \frac{2^{\frac{1-D}{2}}}{(D-2)S} \Big)^4 \frac{e^{iu(q+q') - iu'(k+k')}}{(rr')^{4\alpha} \sqrt{kk'qq'}} \sum_{legs} I(q, q'; k, k') \tag{3.19b}$$

where

$$I(q, q'; k, k') := \Big(\partial_r \Xi(q, r)\Big) \Big(\partial_r \Xi(q', r)\Big) \Big(\partial_{r'} \mathcal{X}(k, r')\Big) \Big(\partial_{r'} \mathcal{X}(k', r')\Big) \tag{3.20}$$

and $\sum_{legs}$ denotes a sum over all 24 permutations of the outer momenta, assigning a minus sign to every outgoing $k, k'$ and every ingoing $q, q'$:

$$\begin{aligned} \sum_{legs} I(q, q'; k, k') = I(q, q'; k, k') + I(q', q; k, k') + ... \\ + I(-k, q'; -q, k') + I(q, -k'; k, -q')... \,. \end{aligned} \tag{3.21}$$

After having transformed the integral (2.82) to the new coordinates (2.71) we can plug in (3.19b) which after several lines of simplifications leads to

$$T_a = \big\langle q, q' \big| : V_a^{(4)} : \big| k, k' \big\rangle \tag{3.22}$$

$$= 2\pi \frac{2^{\frac{D-14}{2}} \delta(k + k' - q - q')}{(D-2)^2 S^4 \sqrt{kk'qq'}} \int_{r,r'=0}^\infty dr\, dr'\, \Theta(r' - r) \times \tag{3.23}$$

$$\times \Big[ r^{D-2} \Big( (5 - 2D) r + r' \Big) - r'^{D-2} \Big( r + (5 - 2D) r' \Big) \Big] \sum_{legs} I(q, q'; k, k')$$

The $u$ and $u'$ integrals were evaluated trivially leading to the momentum conserving delta function.

### 3.2.2. Asymmetric vertex

The same steps are done for the asymmetric vertex where we evaluate the expression

$$\begin{aligned}\langle q,q'| :\Phi_0(x)\Phi_1(y): |k,k'\rangle \\ = \frac{1}{4}\Big\langle q,q'\Big| :\Big(\partial_0\phi(x)\Big)^2\Big(\partial_0\phi(y)\partial_1\phi(y)\Big)^2 : \Big|k,k'\Big\rangle\ .\end{aligned} \tag{3.24}$$

Inserting the expansion for the scalar field (3.2) we can again split each term into positive and negative energy modes

$$\partial_0\phi(x) = \phi^+(x) + \phi^-(x) \tag{3.25}$$

$$\partial_1\phi(x) = \psi^+(x) + \psi^-(x) \tag{3.26}$$

with $\phi^\pm$ defined as in (3.12) and

$$\psi^-(x) = i(D-2)\int_0^\infty dk\ \sqrt{k^{2D-5}}\,k\,\Xi(k,r)a_k^+e^{iku} \tag{3.27a}$$

$$\psi^+(x) = -i(D-2)\int_0^\infty dk\ \sqrt{k^{2D-5}}\,k\,\mathcal{X}(k,r)a_k^-e^{-iku}\ . \tag{3.27b}$$

The product is easily brought into normal order leading to

$$\begin{aligned}:\Phi_0(x)\Phi_1(y): =&\frac{1}{4}\Big(\phi^-(x)\phi^-(x)\phi^+(y)\psi^+(y) + \phi^-(y)\psi^-(y)\phi^+(x)\phi^+(x)\\ &+ \phi^-(x)\phi^-(y)\phi^+(x)\psi^+(y) + \phi^-(x)\psi^-(y)\phi^+(x)\phi^+(y)\\ &+ \phi^-(x)\psi^-(y)\phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y)\phi^+(x)\psi^+(y)\Big)\\ &+ \dots\end{aligned} \tag{3.28}$$

where like before the ellipsis denotes irrelevant terms. With the single contractions (3.17)-(3.18) and

$$\psi^+(x)\,|k\rangle = -\frac{i(D-2)k}{S\sqrt{k}}\mathcal{X}(k,r)e^{-iku}\,|0\rangle \tag{3.29a}$$

$$\langle q|\,\psi^-(x) = \langle 0|\,\frac{i(D-2)q}{S\sqrt{q}}\Xi(q,r)e^{iqu} \tag{3.29b}$$

we can evaluate (3.24) yielding

$$\langle q,q'| : \Phi_0(x)\Phi_1(y) : |k,k'\rangle = \frac{1}{2}\langle 0\big|a^-_q a^-_{q'} : \Phi_0(x)\Phi_1(y) : a^+_k a^+_{k'}\big|0\rangle \tag{3.30}$$

$$= \frac{i(2-D)}{8S}\Big(\frac{2^{\frac{1-D}{2}}}{(D-2)S}\Big)^3 \frac{e^{iu(q+q')-iu'(k+k')}}{r^{4\alpha}r'^{2\alpha}\sqrt{kk'qq'}} \sum_{legs} \tilde{I}(q,q';k,k') \ . \tag{3.31}$$

Here the altered definition

$$\tilde{I}(q,q';k,k') := \Big(\partial_r\, \Xi(q,r)\Big)\Big(\partial_r \Xi(q',r)\Big)\Big(k\ \mathcal{X}(k,r')\Big)\Big(\partial_{r'}\mathcal{X}(k',r')\Big) \tag{3.32}$$

was introduced whereas the leg sum is defined like in the symmetric case. Inserting into (2.83) after transforming the integrals to the new coordinates (2.71) we get

$$T_b = \big\langle q,q'\big| : V_b^{(4)} : \big|k,k'\big\rangle \tag{3.33}$$

$$= 2\pi i \frac{2^{\frac{D-12}{2}}\delta(k+k'-q-q')}{(D-2)S^4\sqrt{kk'qq'}} \int_{r,r'=0}^{\infty} dr\, dr' \left|r'^{D-2} - r^{D-2}\right| r \sum_{legs} \tilde{I}(q,q';k,k')$$

and the $u, u'$ integrals were again evaluated trivially.

## 3.3. Integration over radial coordinates for $D = 6$

The following integrations over $r$ and $r'$ are highly non-trivial and were eventually only manageable for the special case $D = 6$. Some of the techniques used here can also be found in [7, 8].

### 3.3.1. Preparation

i) Both of the amplitudes $T_a$ and $T_b$ contain a leg sum in the radial integrals summing products $I, \tilde{I}$ of differentiated higher order Bessel functions. An important simplification is achieved by noting that

$$\partial_r \Xi(k,r) = \partial_r\big(k^{-\alpha}r^{-\alpha}J_\alpha(kr)e^{ikr}\big) \tag{3.34}$$

$$= \frac{k}{r}\partial_k \Xi(k,r) \tag{3.35}$$

and likewise

$$\partial_r \mathcal{X}(q,r) = \frac{q}{r}\partial_q \mathcal{X}(q,r) \ . \tag{3.36}$$

It is therefore possible to write

$$I(q,q';k,k') = \frac{qq'kk'}{r^2r'^2}\partial_q\partial_{q'}\partial_k\partial_{k'}\ \Xi(q,r)\Xi(q',r)\mathcal{X}(k,r')\mathcal{X}(k',r') \tag{3.37}$$

$$\tilde{I}(q,q';k,k') = \frac{qq'kk'}{r^2r'}\partial_q\partial_{q'}\partial_{k'}\ \Xi(q,r)\Xi(q',r)\mathcal{X}(k,r')\mathcal{X}(k',r') \tag{3.38}$$

allowing to pull all the momentum derivatives together with the leg sums out of the integrals.

ii) In general the integrals behave differently for even and odd dimensions which has to do with the fact that the Bessel functions $J_\alpha(kr)$ can only be expressed in terms of elementary functions if

$$\alpha = n + \frac{1}{2} \qquad\qquad n \in \mathbb{N}^*\ , \tag{3.39}$$

i.e. if $D = 2\alpha + 3$ is even. In that case it is possible to employ the correspondence (A.3) between $J_\alpha(kr)$ and the spherical Bessel function $j_n(kr)$ of order $n$ which in turn is related to the zeroth order spherical Bessel function

$$j_0(kr) = \frac{\sin(kr)}{kr} \tag{3.40}$$

by the Rayleigh-formula (A.4). This yields for the upper restriction of $\alpha$

$$J_{n+\frac{1}{2}}(kr) = (-1)^n\sqrt{\frac{2kr}{\pi}}\left(\frac{k}{r}\right)^n\left(\frac{1}{k}\frac{d}{dk}\right)^n j_0(kr) \tag{3.41}$$

and will be used in the following. As already mentioned we will further restrict to $n = 1$ from now on which is the simplest “non-trivial” case apart from [7]. The case of odd dimensions shall be put aside for now as it is unclear how to handle the appearing expressions when it comes to performing the integrals. It might be a possible topic for future research. Evaluating (3.23) and (3.33) for $D = 6$ gives

$$T_a = 2\pi \frac{\delta(k + k' - q - q')}{(4S)^4\sqrt{kk'qq'}} \int_{r,r'=0}^{\infty} dr\, dr'\, \Theta(r' - r)(r' - r)\times \tag{3.42}$$

$$\times \left[7r^4 + 6r^3r' + 6r^2r'^2 + 6rr'^3 + 7r'^4\right] \sum_{legs} I(q, q'; k, k')$$

$$= 2\pi \frac{\delta(k + k' - q - q')}{2\,(4S)^4\sqrt{kk'qq'}} \int_{r,r'=0}^{\infty} dr\, dr'\, \left|r' - r\right|\times \tag{3.43}$$

$$\times \left[7r^4 + 6r^3r' + 6r^2r'^2 + 6rr'^3 + 7r'^4\right] \sum_{legs} I(q, q'; k, k')$$

$$T_b = 2\pi i \frac{16\,\delta(k + k' - q - q')}{2(4S)^4\sqrt{kk'qq'}} \int_{r,r'=0}^{\infty} dr\, dr'\, \left|r'^4 - r^4\right| \sum_{legs} \tilde{I}(q, q'; k, k') \tag{3.44}$$

where we used the $r \leftrightarrow r'$ symmetry of $T_a$ to convert the step function into an absolute value.

iii) All step functions included in the absolute values will be rewritten as integrals by Fourier transformation (A.30). We have

$$\left|r' - r\right| = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{d\tau}{\tau + i\epsilon} + \frac{d\tau}{\tau - i\epsilon}\right] \left(\frac{\partial_\tau}{i}\right) e^{i\tau(r'-r)} \tag{3.45}$$

$$\left|r'^4 - r^4\right| = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{d\tau}{\tau + i\epsilon} + \frac{d\tau}{\tau - i\epsilon}\right] e^{i\tau(r'-r)}\, (r'^4 - r^4) \tag{3.46}$$

where in the first line we additionally generate the term $(r' - r)$ by a differentiation with respect to $\tau$. In order to shorten the notation we will abbreviate the integral measure for the $\tau$ integrals by

$$[d\tau] := \lim_{\epsilon\to 0} \left[\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\right]. \tag{3.47}$$

iv) The quantities $k, k', q, q', E_x, E_y$ will be treated as independent variables from now on which allows to differentiate after the energies without affecting the momenta. This makes it possible to generate the polynomial expressions in (3.43)-(3.44) by

$$(7r^4 + 6r^3r' + 6r^2r'^2 + 6rr'^3 + 7r'^4) = D_{E_xE_y} e^{iE_xr} e^{-iE_yr'}\Big|_{E_x,E_y=0} \tag{3.48}$$

$$(r^4 - r'^4) = \tilde{D}_{E_xE_y} e^{iE_xr} e^{-iE_yr'}\Big|_{E_x,E_y=0} \tag{3.49}$$

where the energy differential operators

$$D_{E_xE_y} := \Big(7\frac{\partial^4}{\partial E_x^4} - 6\frac{\partial^3}{\partial E_x^3}\frac{\partial}{\partial E_y} + 6\frac{\partial^2}{\partial E_x^2}\frac{\partial^2}{\partial E_y^2} - 6\frac{\partial}{\partial E_x}\frac{\partial^3}{\partial E_y^3} + 7\frac{\partial^4}{\partial E_y^4}\Big) \quad (3.50)$$

$$\tilde{D}_{E_xE_y} := \Big(\frac{\partial^4}{\partial E_x^4} - \frac{\partial^4}{\partial E_y^4}\Big) \quad (3.51)$$

were defined. In the following parts the evaluation $E_x = 0 = E_y$ after the action of these operators is implicitly understood and will not be written out.

### 3.3.2. Symmetric vertex

After applying the tools above as well as the definitions for the functions $\Xi, \mathcal{X}$ in (3.13)-(3.14) one arrives at the expression

$$T_a = cD_{E_xE_y} \sum_{legs} q^2 q'^2 k^2 k'^2 \partial_q \partial_{q'} \partial_k \partial_{k'} \frac{1}{qq'kk'} \int_\tau [d\tau]\, \partial_\tau \left(I_1 \cdot I_2\right) \quad (3.52)$$

where we abbreviated the prefactor as

$$c = -\frac{2\delta(k + k' - q - q')}{\pi^2 (4S)^4 (kk'qq')^{\frac{3}{2}}} \,. \quad (3.53)$$

The radial integrals were pulled into the last term and were split into a product of

$$I_1 = \lim_{\varepsilon \to 0} \int_0^\infty dr\; r^{-6} \left(\frac{d}{dq} j_0(qr)\right) \left(\frac{d}{dq'} j_0(q'r)\right) e^{i(q+q'+E_x-\tau+i\varepsilon)r} \quad (3.54)$$

$$I_2 = \lim_{\varepsilon \to 0} \int_0^\infty dr'\; r'^{-6} \left(\frac{d}{dk} j_0(kr')\right) \left(\frac{d}{dk'} j_0(k'r')\right) e^{-i(k+k'+E_y-\tau-i\varepsilon)r'} \quad (3.55)$$

with a regulator[2] $\varepsilon$. In analogy to [7, 8] we now define the operators

$$D_{qq'} := q^2 q'^2 \partial_q \partial_{q'} \frac{1}{qq'} = qq'\partial_q\partial_{q'} - q\partial_q - q'\partial_{q'} + 1 \quad (3.56)$$

$$D_{kk'} := k^2 k'^2 \partial_k \partial_{k'} \frac{1}{kk'} = kk'\partial_k\partial_{k'} - k\partial_k - k'\partial_{k'} + 1 \quad (3.57)$$

which have the useful property (A.6). The integrals $I_1, I_2$ are of rather similar form, so the following steps will be essentially the same for both of them. They shall be shown explicitly for $I_1$.

[2] In principle the two integrals have to be regulated independently with two parameters, say $\varepsilon$ and $\tilde{\varepsilon}$. It however turned out that it does not matter in which order they are taken to zero which justifies taking the same symbol for both of them.

At first the momentum derivatives are pulled out leading to

$$\begin{aligned} I_1 = \quad & \partial_q \partial_{q'} \lim_{\varepsilon\to 0} \int_0^\infty dr\; r^{-6}\, j_0(qr) j_0(q'r) e^{i(q+q'+E_x-\tau+i\varepsilon)r} \\ & -i\big(\partial_q + \partial_{q'}\big) \lim_{\varepsilon\to 0} \int_0^\infty dr\; r^{-5}\, j_0(qr) j_0(q'r) e^{i(q+q'+E_x-\tau+i\varepsilon)r} \\ & - \lim_{\varepsilon\to 0} \int_0^\infty dr\; r^{-4}\, j_0(qr) j_0(q'r) e^{i(q+q'+E_x-\tau+i\varepsilon)r} \end{aligned} \tag{3.58}$$

where the second and third line result from using the inverse product rule. With

$$j_0(qr) = \frac{e^{iqr} - e^{-iqr}}{2iqr} \tag{3.59}$$

we can further rewrite

$$\begin{aligned} & j_0(qr) j_0(q'r) e^{i(q+q'-\tau+i\varepsilon)r} \\ & = \frac{1}{4qq'r^2} \Big( - e^{i(2q+2q'+E_x-\tau+i\varepsilon)r} + e^{i(2q+E_x-\tau+i\varepsilon)r} \\ & \qquad + e^{i(2q'+E_x-\tau+i\varepsilon)r} - e^{i(E_x-\tau+i\varepsilon)r} \Big) \\ & = \frac{1}{4qq'r^2} \sum_{qq'} e^{i(Q-\tau-i\varepsilon)r} \end{aligned} \tag{3.60}$$

where we defined the sign sum

$$\begin{aligned} \sum_{qq'} f(\pm q, \pm q', x) := & - f(+q, +q', x) + f(+q, -q', x) \\ & + f(-q, +q', x) - f(-q, -q', x) \end{aligned} \tag{3.61}$$

and

$$Q := q + q' \pm q \pm q' + E_x\;. \tag{3.62}$$

Insertion into (3.58) yields

$$\begin{aligned} I_1 = \quad & \partial_q \partial_{q'} \frac{1}{4qq'} \lim_{\varepsilon\to 0} \sum_{qq'} \int_0^\infty dr\; r^{-8}\, e^{i(Q-\tau+i\varepsilon)r} \\ & -i\big(\partial_q + \partial_{q'}\big) \frac{1}{4qq'} \lim_{\varepsilon\to 0} \sum_{qq'} \int_0^\infty dr\; r^{-7}\, e^{i(Q-\tau+i\varepsilon)r} \\ & - \frac{1}{4qq'} \lim_{\varepsilon\to 0} \sum_{qq'} \int_0^\infty dr\; r^{-6}\, e^{i(Q-\tau+i\varepsilon)r} \end{aligned} \tag{3.63}$$

and it can be seen that the integrals are of the form (A.24). They are all evaluated in the same way using that formula with the singular limit (A.25) and each one yields a divergent as well as a finite part in the limit $\delta \to 0$. Thus, in total we get

$$I_1 = I_1^\circ + I_1^\infty \ . \tag{3.64}$$

Taking a closer look at the divergent part

$$\begin{aligned} I_1^\infty = \lim_{\delta,\varepsilon\to 0} \frac{i}{4\delta}\Big[ \quad & -\frac{1}{7!}\partial_q\partial_{q'}\frac{1}{qq'}\sum_{qq'}(Q-\tau+i\varepsilon)^7 \\ & +\frac{1}{6!}\left(\partial_q+\partial_{q'}\right)\frac{1}{qq'}\sum_{qq'}(Q-\tau+i\varepsilon)^6 \\ & -\frac{1}{5!}\frac{1}{qq'}\sum_{qq'}(Q-\tau+i\varepsilon)^5\Big] \end{aligned} \tag{3.65}$$

we can evaluate the sign sums together with the differentiations in *Mathematica* and find

$$I_1^\infty = \lim_{\delta,\varepsilon\to 0}\frac{iqq'}{9\delta}\left(q+q'-\tau+i\varepsilon\right) \ . \tag{3.66}$$

It can be readily seen that this is eliminated by the differential operator $D_{qq'}$ with property (A.6). We therefore only need to concentrate on the finite part. In several lines it can be shown to be of the form

$$\begin{aligned} I_1^\circ = \frac{i}{4}\Big[ \quad & \frac{1}{7!}\partial_{qq'}\frac{1}{qq'}\lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau+i\varepsilon)^7\ln(Q-\tau+i\varepsilon) \\ & -\frac{1}{6!}\left(\partial_q+\partial_{q'}\right)\frac{1}{qq'}\lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau+i\varepsilon)^6\ln(Q-\tau+i\varepsilon) \\ & +\frac{1}{5!}\frac{1}{qq'}\lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau+i\varepsilon)^5\ln(Q-\tau+i\varepsilon)\Big] \end{aligned} \tag{3.67}$$

which simplifies by setting the terms $i\varepsilon$ in the polynomial expressions to zero as they will only lead to terms vanishing in the limit $\varepsilon \to 0$ after the following $\tau$-integration. It is however important to keep them in the logarithms as they will provide a prescription for circling the logarithmic cuts in the complex $\tau$-plane. One last time we introduce new operators consisting of the prefactors in (3.67)

$$L_7^Q := \frac{1}{7!}\partial_{qq'}\frac{1}{qq'} \qquad L_6^Q := \frac{1}{6!}(\partial_q+\partial_{q'})\frac{1}{qq'} \qquad L_5^Q := \frac{1}{5!}\frac{1}{qq'} \tag{3.68}$$

giving us the final expression

$$I_1^\circ = \frac{i}{4}\Big[\quad L_7^Q \lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau)^7\ln(Q-\tau+i\varepsilon)$$
$$-L_6^Q \lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau)^6\ln(Q-\tau+i\varepsilon)$$
$$+L_5^Q \lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau)^5\ln(Q-\tau+i\varepsilon)\Big]\ . \tag{3.69}$$

The integral $I_2$ is done analogously leading to

$$I_2^\circ = \frac{i}{4}\Big[-L_7^K \lim_{\varepsilon\to 0}\sum_{kk'}(K-\tau)^7\ln(K-\tau-i\varepsilon)$$
$$+L_6^K \lim_{\varepsilon\to 0}\sum_{kk'}(K-\tau)^6\ln(K-\tau-i\varepsilon)$$
$$-L_5^K \lim_{\varepsilon\to 0}\sum_{kk'}(K-\tau)^5\ln(K-\tau-i\varepsilon)\Big] \tag{3.70}$$

where

$$K := k + k' \pm k \pm k' + E_y\ . \tag{3.71}$$

As opposed to $I_1$ there is one non-trivial intermediate step to be mentioned: After applying (A.24) we at first get logarithms of the form $\ln(-K+\tau+i\varepsilon)$. In the final result (3.70) a minus sign was pulled out leading to

$$\ln(-K+\tau+i\varepsilon) = \ln(K-\tau-i\varepsilon) + i\pi \tag{3.72}$$

with an additional term on the RHS. Doing this for each integral we get a purely polynomial contribution similar to the divergent part $I_2^\infty$ which just like (3.66) vanishes under the action of $D_{kk'}$. This justifies leaving out the terms $i\pi$.

### 3.3.3. Asymmetric vertex

The starting point is expression (3.33) and again all the tools of section 3.3.1 are applied. Going directly to the case $D = 6$ we get

$$T_b = 16c\,\tilde{D}_{E_xE_y}\sum_{legs} q^2 q'^2 k^2 k'^2 \partial_q\partial_{q'}\partial_{k'}\frac{1}{qq'kk'}\int_\tau [d\tau]\,\big(J_1\cdot J_2\big) \tag{3.73}$$

with

$$J_1 = \lim_{\varepsilon\to 0}\int_0^\infty dr\; r^{-5}\left(\frac{d}{dq}j_0(qr)\right)\left(\frac{d}{dq'}j_0(q'r)\right)e^{i(q+q'+E_x-\tau+i\varepsilon)r} \tag{3.74}$$

$$J_2 = \lim_{\varepsilon\to 0}\int_0^\infty dr'\; r'^{-5}\left(\frac{d}{dk}j_0(kr')\right)\left(\frac{d}{dk'}j_0(k'r')\right)e^{-i(k+k'+E_y-\tau-i\varepsilon)r'}\;. \tag{3.75}$$

We can again define momentum operators

$$D_{qq'} := q^2 q'^2 \partial_q \partial_{q'} \frac{1}{qq'} = qq'\partial_q\partial_{q'} - q\partial_q - q'\partial_{q'} + 1 \tag{3.76}$$

$$\tilde{D}_{kk'} := k^2 k'^2 \partial_{k'} \frac{1}{kk'} = k\big(k'\partial_{k'} - 1\big) \tag{3.77}$$

and again pull the remaining derivatives out of the integral. Looking at the case $J_1$ more explicitly we get

$$\begin{aligned} J_1 = \quad & \partial_q\partial_{q'} \lim_{\varepsilon\to 0}\int_0^\infty dr\; r^{-5}\, j_0(qr) j_0(q'r) e^{i(q+q'+E_x-\tau+i\varepsilon)r} \\ & -i\big(\partial_q + \partial_{q'}\big) \lim_{\varepsilon\to 0}\int_0^\infty dr\; r^{-4}\, j_0(qr) j_0(q'r) e^{i(q+q'+E_x-\tau+i\varepsilon)r} \\ & - \lim_{\varepsilon\to 0}\int_0^\infty dr\; r^{-3}\, j_0(qr) j_0(q'r) e^{i(q+q'+E_x-\tau+i\varepsilon)r} \end{aligned} \tag{3.78}$$

where again the definition of the sign sum (3.61) was used to rewrite the spherical Bessel functions like in (3.60). This leaves us with

$$\begin{aligned} J_1 = \quad & \partial_q\partial_{q'} \frac{1}{4qq'} \lim_{\varepsilon\to 0} \sum_{qq'} \int_0^\infty dr\; r^{-7}\, e^{i(Q-\tau+i\varepsilon)r} \\ & -i\big(\partial_q + \partial_{q'}\big) \frac{1}{4qq'} \lim_{\varepsilon\to 0} \sum_{qq'} \int_0^\infty dr\; r^{-6}\, e^{i(Q-\tau+i\varepsilon)r} \\ & - \frac{1}{4qq'} \lim_{\varepsilon\to 0} \sum_{qq'} \int_0^\infty dr\; r^{-5}\, e^{i(Q-\tau+i\varepsilon)r} \end{aligned} \tag{3.79}$$

which is evaluated with (A.24). There is again a divergent and a finite part

$$J_1 = J_1^\circ + J_1^\infty \tag{3.80}$$

with the divergent part reading

$$J_1^\infty = \lim_{\delta,\varepsilon\to 0}\frac{1}{4\delta}\Big[ \quad -\frac{1}{6!}\partial_q\partial_{q'}\frac{1}{qq'}\sum_{qq'}(Q-\tau+i\varepsilon)^6 \\ +\frac{1}{5!}(\partial_q+\partial_{q'})\frac{1}{qq'}\sum_{qq'}(Q-\tau+i\varepsilon)^5 \\ -\frac{1}{4!}\frac{1}{qq'}\sum_{qq'}(Q-\tau+i\varepsilon)^4\Big] \tag{3.81}$$

and evaluating to

$$J_1^\infty = \lim_{\delta,\varepsilon\to 0}\frac{qq'}{9\delta}\ . \tag{3.82}$$

This is cancelled by $D_{qq'}$ rendering the integral $J_1$ finite. Note, that the main difference to the symmetric vertex lies in the powers in $r$ in the integrals (3.74)-(3.75). The fact that they are one order lower than in the symmetric case leads to the divergent parts $J_1^\infty, J_2^\infty$ being bilinear in the momenta. Therefore even though the new differential operator in (3.77) is only of first order, the integral $J_2$ is also finite as $\tilde{D}_{kk'}$ manages to kill the divergent terms.

We focus on the finite parts $J_1^\circ, J_2^\circ$ and find

$$J_1^\circ = \frac{1}{4}\Big[ \quad M_6^Q\lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau)^6\ln(Q-\tau+i\varepsilon) \\ -M_5^Q\lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau)^5\ln(Q-\tau+i\varepsilon) \\ +M_4^Q\lim_{\varepsilon\to 0}\sum_{qq'}(Q-\tau)^4\ln(Q-\tau+i\varepsilon)\Big] \tag{3.83}$$

$$J_2^\circ = \frac{1}{4}\Big[ \quad M_6^K\lim_{\varepsilon\to 0}\sum_{kk'}(K-\tau)^6\ln(K-\tau-i\varepsilon) \\ -M_5^K\lim_{\varepsilon\to 0}\sum_{kk'}(K-\tau)^5\ln(K-\tau-i\varepsilon) \\ +M_4^K\lim_{\varepsilon\to 0}\sum_{kk'}(K-\tau)^4\ln(K-\tau-i\varepsilon)\Big] \tag{3.84}$$

where we left out all terms $i\varepsilon$ in the polynomial expressions and introduced the operators

$$M_6^Q := \frac{1}{6!}\partial_{qq'}\frac{1}{qq'} \qquad M_5^Q := \frac{1}{5!}(\partial_q+\partial_{q'})\frac{1}{qq'} \qquad M_4^Q := \frac{1}{4!}\frac{1}{qq'} \tag{3.85}$$

with analogous versions for $K$.

## 3.4. Integration over $\tau$ for $D = 6$

In the last section we have managed to reduce the radial integrals to sign sums with certain differential operators in front of them making the only remaining integration the one over $\tau$. At this point the two amplitudes read

$$T_a = c\, D_{E_xE_y} \sum_{legs} D_{qq'} D_{kk'} \int_\tau [d\tau]\, \partial_\tau \left(I_1^\circ \cdot I_2^\circ\right) \tag{3.86}$$

$$T_b = 16c\, \tilde{D}_{E_xE_y} \sum_{legs} D_{qq'} \tilde{D}_{kk'} \int_\tau [d\tau]\, \left(J_1^\circ \cdot J_2^\circ\right) \tag{3.87}$$

where the definitions (3.50) and (3.51) were used. As the generators $E_x, E_y$ have been linearly absorbed into $Q$ and $K$ (cf. (3.62),(3.71)) we can equally write these operators as

$$D_{E_xE_y} := 7\partial_Q^4 - 6\partial_Q^3\partial_K + 6\partial_Q^2\partial_K^2 - 6\partial_Q\partial_K^3 + 7\partial_K^4 \tag{3.88}$$

$$\tilde{D}_{E_xE_y} := \partial_Q^4 - \partial_K^4 \tag{3.89}$$

and set $E_x = 0 = E_y$. Note however, that this is only true if they are evaluated before the sign sums as otherwise the redefinitions (3.88)-(3.89) would be singular for terms involving $Q = 0$ or $K = 0$. We therefore pull the energy differentials into the $\tau$ integral and evaluate them next.

At this point it becomes quite important to only focus on terms that still lead to non-zero results after evaluating all these sums an differentiations. All other terms will be referred to as "harmless" and are dropped before the sums are evaluated. This helps to reduce the size of the expressions quite a lot. Still, most of the following part of the calculation was done with *Mathematica* as there are many remaining terms to be taken into account. Before tackling the integration we investigate which terms count as harmless.

### 3.4.1. Harmlessness

For the symmetric vertex we get terms of the form

$$D_{qq'} L_i^Q \sum_{qq'} f(Q, x)\ , \qquad i = 5, 6, 7 \tag{3.90}$$

and vice versa for $K$. Depending on the operator $L_i^Q$ the following relations are found to hold

$$D_{qq'} L_5^Q \sum_{qq'} Q^3 f(x) = 0 \tag{3.91}$$

$$D_{qq'} L_5^Q \sum_{qq'} \big(Q f(x) + g(x)\big) = 0 \tag{3.92}$$

$$D_{qq'} L_6^Q \sum_{qq'} Q^4 f(x) = 0 \tag{3.93}$$

$$D_{qq'} L_6^Q \sum_{qq'} \big(Q^2 h(x) + Q f(x) + g(x)\big) = 0 \tag{3.94}$$

$$D_{qq'} L_7^Q \sum_{qq'} Q^5 f(x) = 0 \tag{3.95}$$

$$D_{qq'} L_7^Q \sum_{qq'} \big(Q^3 l(x) + Q^2 h(x) + Q f(x) + g(x)\big) = 0 \tag{3.96}$$

with some arbitrary functions $l, h, f, g$ which do not depend on $Q$. The same relations hold for the case with $K$. Thus, all those special polynomial combinations are dropped when they appear in the following computations.

For the asymmetric vertex the $Q$-dependent sign sum and operators essentially fulfill the same relations as in the symmetric case with $L_i^Q$ exchanged by $M_{i-1}^Q$. Only the $K$-dependent part differs because there $\tilde{D}_{kk'}$ features instead of $D_{kk'}$. The following relations hold:

$$\tilde{D}_{kk'} M_4^K \sum_{kk'} \big(K f(x) + g(x)\big) = 0 \tag{3.97}$$

$$\tilde{D}_{kk'} M_5^K \sum_{kk'} \big(K^2 h(x) + K f(x) + g(x)\big) = 0 \tag{3.98}$$

$$\tilde{D}_{kk'} M_6^K \sum_{kk'} \big(K^3 l(x) + K^2 h(x) + K f(x) + g(x)\big) = 0\ . \tag{3.99}$$

Therefore, in the computation of the asymmetric amplitude all these terms are dropped as soon as they appear.

### 3.4.2. Symmetric vertex

Starting from (3.86) with $D_{E_x E_y}$ pulled in

$$T_a = c \sum_{legs} D_{qq'} D_{kk'} \int_\tau [d\tau]\, \partial_\tau\, D_{E_x E_y} \big(I_1^\circ\, I_2^\circ\big) \tag{3.100}$$

we take the results (3.69)-(3.70) and rewrite

$$\int_\tau [d\tau]\, \partial_\tau\, D_{E_xE_y}\big(I_1^\circ\, I_2^\circ\big) = \frac{1}{16} \lim_{\varepsilon,\epsilon\to 0} \sum_{i,j=5}^{7} L_i^Q L_j^K \sum_{qq'} \sum_{kk'} A_{ij} \tag{3.101}$$

where

$$\begin{aligned} A_{ij} := (-1)^{i+j} \int_{-\infty}^{\infty} \Big(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\Big)\, \partial_\tau\, D_{E_xE_y} \times \\ \times (Q-\tau)^i (K-\tau)^j \ln(Q-\tau+i\varepsilon) \ln(K-\tau-i\varepsilon)\ . \end{aligned} \tag{3.102}$$

These 9 integrals each are divergent as they stand which can easily be seen by looking at the integrands: After evaluating the derivatives there are still polynomial and logarithmic terms in $\tau$ left which do not die off at infinity. However it turns out that when they are summed in the end all divergences cancel leaving a finite result. To make this more plausible the value of the integrand in (3.100) which already contains the full sum was evaluated in a large $\tau$ expansion yielding

$$\lim_{\tau\to\infty} \partial_\tau\, D_{E_xE_y}\big(I_1^\circ\, I_2^\circ\big) = \frac{kk' + qq'}{63\,\tau} + \mathcal{O}\Big(\frac{1}{\tau^2}\Big) \tag{3.103}$$

which indeed dies off[3] for $\tau \to \infty$. Thus, infinities appearing because of large values of $\tau$ in the single integrals $A_{ij}$ are more an artefact of our strategy of integration than physical divergences in the amplitude. To handle them we will introduce a cutoff $R$ for each integral and see in the end that all cutoff dependent terms are either harmless or cancel for $R \to \infty$. As the integrals are quite similar in their form we only show the evaluation for one of them and will jump to the final results for the rest. Throughout the way it is constantly made use of harmlessness in order to drop terms.

For the integral $A_{55}$ one finds after evaluating $\partial_\tau\, D_{E_xE_y}$

$$\begin{aligned} A_{55} = -5! \lim_{R\to\infty} \int_{-R}^{R} \Big(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\Big) \times \\ \times \Big[(Q-\tau)(K-\tau)^4 \ln(Q-\tau+i\varepsilon) + (Q-\tau)^4 (K-\tau) \ln(K-\tau-i\varepsilon) \\ + \Big\{7(K-\tau)^5 + 5(Q-\tau)^4(K-\tau) + 5(Q-\tau)(K-\tau)^4 + 7(Q-\tau)^5\Big\} \times \\ \times \ln(Q-\tau+i\varepsilon) \ln(K-\tau-i\varepsilon)\Big] \end{aligned} \tag{3.104}$$

[3] In fact, the $\mathcal{O}(\tau^{-1})$ term could still lead to logarithmic divergences but note that it only depends linearly on the momenta and is thus harmless.

where there are terms with single logarithms and bilogarithmic ones. If we continue $\tau$ analytically we see that there are poles at $\tau = \pm i\epsilon$ and branch points of the complex logarithms at $\tau = Q + i\varepsilon$ and $\tau = K - i\varepsilon$. The integral is split into three parts, two single logarithmic ones

$$A_{55}^{Q} = -5! \lim_{R\to\infty} \int_{-R}^{R} \Big(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\Big)(Q-\tau)(K-\tau)^4 \ln(Q - \tau + i\varepsilon) \tag{3.105}$$

$$A_{55}^{K} = -5! \lim_{R\to\infty} \int_{-R}^{R} \Big(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\Big)(Q-\tau)^4(K-\tau) \ln(K - \tau - i\varepsilon) \tag{3.106}$$

and a bilogarithmic one

$$\begin{aligned} A_{55}^{bil} = -5! \lim_{R\to\infty} \int_{-R}^{R} \Big(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\Big) \ln(Q - \tau + i\varepsilon) \ln(K - \tau - i\varepsilon) \times \\ \times \Big\{7(K-\tau)^5 + 5(Q-\tau)^4(K-\tau) + 5(Q-\tau)(K-\tau)^4 + 7(Q-\tau)^5\Big\} \; . \end{aligned} \tag{3.107}$$

Also, in addition to the harmless terms in (3.91)-(3.92) we drop terms

- of $\mathcal{O}(\varepsilon)$,
- of $\mathcal{O}(R^{-1})$,
- of $\mathcal{O}(\epsilon)$

as they do not yield non-zero results in the given limits. The three contributions $A_{55}^{Q}$, $A_{55}^{K}$ and $A_{55}^{bil}$ are now evaluated with the following procedure: Split each one into its two constituents corresponding to a part with the pole at $\tau = +i\epsilon$ and a part with the pole at $\tau = -i\epsilon$. Whenever there is a logarithmic branch cut in the half space complementary to the one with the pole we use one of the two contours depicted in figures 3.1 and 3.2 depending on which half space it is. This allows us to kill that logarithm and reduce the integral to a contribution along the branch cut. The argument around (3.103) justifies neglecting all auxiliary paths at infinity. More explicitly, the contour in figure 3.1 gives us[4]

$$\begin{aligned} \int_{-R}^{R} &= -\int_{R}^{K} \{\tau = e^{0\pi i}|\tau|\} - \int_{K}^{R} \{\tau = e^{2\pi i}|\tau|\} \\ &= \int_{K}^{R} \Big(\{\tau = e^{0\pi i}|\tau|\} - \{\tau = e^{2\pi i}|\tau|\}\Big) \end{aligned} \tag{3.108}$$

which kills the $K$-logarithm by

$$\ln(K - e^{0\pi i}|\tau|) - \ln(K - e^{2\pi i}|\tau|) = -2\pi i \; . \tag{3.109}$$

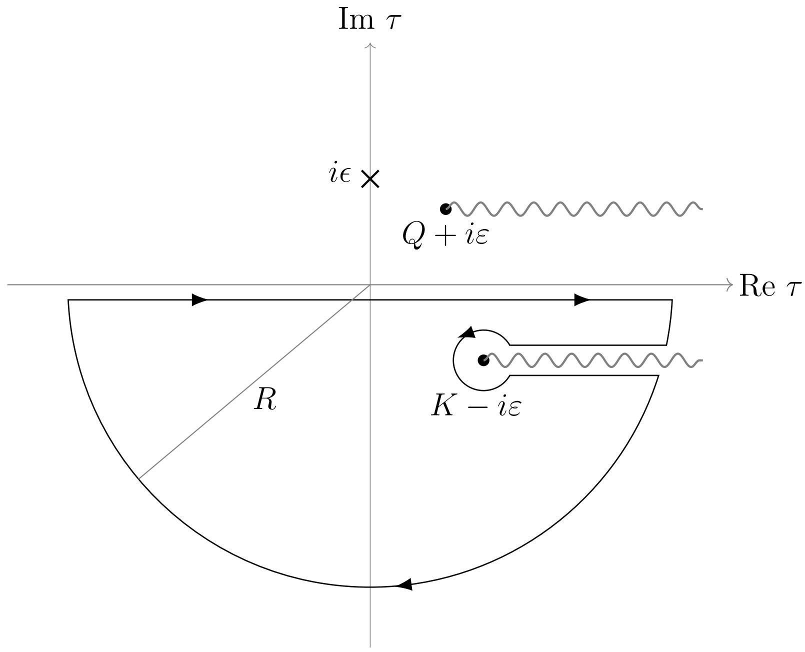


Figure 3.1.: Integration contour for circling the $K$-branch cut.

In the same way we can work through the case in figure 3.2 and find

$$\begin{aligned}\int_{-R}^{R} &= -\int_{R}^{Q}\{\tau = e^{2\pi i}|\tau|\} - \int_{Q}^{R}\{\tau = e^{0\pi i}|\tau|\} \\ &= \int_{Q}^{R}\Big(\{\tau = e^{2\pi i}|\tau|\} - \{\tau = e^{0\pi i}|\tau|\}\Big) \qquad (3.110)\end{aligned}$$

killing the $Q$-logarithm by

$$\ln(Q - e^{2\pi i}|\tau|) - \ln(Q - e^{0\pi i}|\tau|) = 2\pi i \ . \qquad (3.111)$$

The remaining integrals, which are the ones with a single logarithm in the same half space as the pole are integrated over the real axis in a standard manner.

[4]The $i\varepsilon$-contributions in the $K$-logarithms were here neglected for a moment as for the following considerations it does not make a difference whether the branch cut is on the real axis or not.

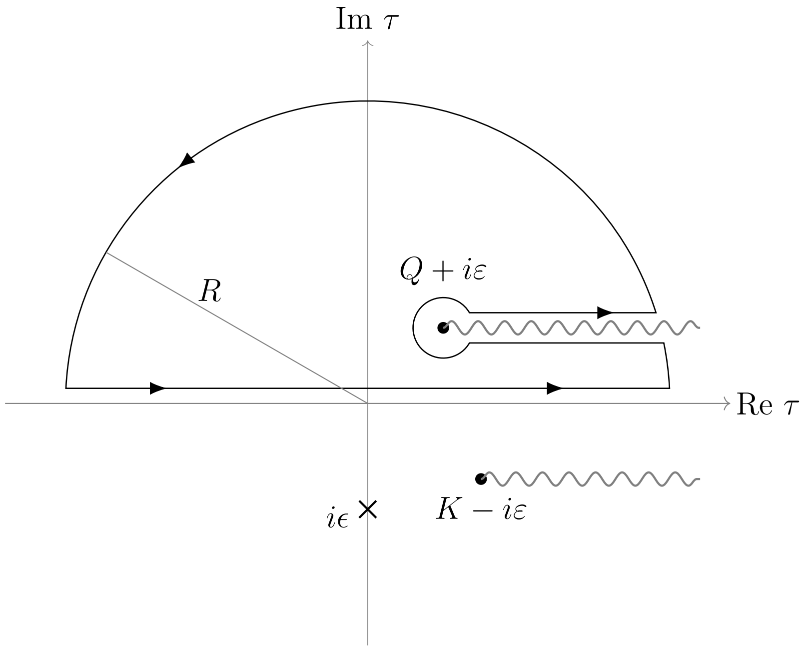


Figure 3.2.: Integration contour for circling the $Q$-branch cut.

Altogether the three contributions to $A_{55}$ then read

$$A_{55}^{Q} = -\,5!\lim_{R\to\infty}\int_{-R}^{R}\frac{d\tau}{\tau - i\epsilon}(Q-\tau)(K-\tau)^4\ln(Q-\tau+i\varepsilon)$$

$$-\,5!2\pi i\lim_{R\to\infty}\int_{Q}^{R}\frac{d\tau}{\tau + i\epsilon}(Q-\tau)(K-\tau)^4 \tag{3.112}$$

$$A_{55}^{K} = -\,5!\lim_{R\to\infty}\int_{-R}^{R}\frac{d\tau}{\tau + i\epsilon}(Q-\tau)^4(K-\tau)\ln(K-\tau-i\varepsilon)$$

$$+\,5!2\pi i\lim_{R\to\infty}\int_{K}^{R}\frac{d\tau}{\tau - i\epsilon}(Q-\tau)^4(K-\tau) \tag{3.113}$$

$$A_{55}^{bil} = 5!2\pi i\lim_{R\to\infty}\left[\int_{K}^{R}\frac{d\tau}{\tau - i\epsilon}\ln(Q-\tau+i\varepsilon) - \int_{Q}^{R}\frac{d\tau}{\tau + i\epsilon}\ln(K-\tau-i\varepsilon)\right]\times$$

$$\times\Big\{7(K-\tau)^5 + 5(Q-\tau)^4(K-\tau)$$

$$+\,5(Q-\tau)(K-\tau)^4 + 7(Q-\tau)^5\Big\} \tag{3.114}$$

and can be seen to be either purely polynomial or logarithmic but not bilogarithmic which simplifies the integration a lot. Performing the integrals with *Mathematica* yields many harmless terms, a contribution independent of $R$ and terms linear in $R$ which do not cancel and diverge in the limit $R \to \infty$, i.e.

$$A_{55} = A^{\circ}_{55} + 1440 K^2 Q^2 R \ . \tag{3.115}$$

The finite contribution $A^{\circ}_{55}$ shall not be presented here because of its size. It contains no polynomial terms but only logarithmic and Dilogarithmic contributions, the latter being always of the form

$$[polynomial] \cdot \Big[ Li_2\Big(\frac{Q}{K}\Big) - Li_2\Big(\frac{K}{Q}\Big)\Big] \tag{3.116}$$

which is reminiscent of the 4-dimensional case [7].

With the same procedure for the other eight integrals $A_{ij}$ one obtains more contributions linear in $R$ as well as a large number of finite terms. The former ones read altogether

$$\begin{aligned}
A^R_{55} &= 1440 K^2 Q^2 R & A^R_{67} &= -7800 K^4 Q^3 R \\
A^R_{56} &= -2640 K^3 Q^2 R & A^R_{76} &= -7800 K^3 Q^4 R \\
A^R_{65} &= -2640 K^2 Q^3 R & A^R_{57} &= 4320 K^4 Q^2 R \\
A^R_{66} &= 4800 K^3 Q^3 R & A^R_{75} &= 4320 K^2 Q^4 R \\
A^R_{77} &= 12600 K^4 Q^4 R
\end{aligned} \tag{3.117}$$

and cancel each other as expected when inserted into (3.101),

$$\sum_{i,j=5}^{7} L^Q_i L^K_j \sum_{qq'} \sum_{kk'} A^R_{ij} = 0 \ . \tag{3.118}$$

We can therefore drop all $R$-dependent terms and immediately perform the limit $R \to \infty$. The other terms are collected and we finally arrive at

$$T_a = \frac{c}{16} \sum_{legs} D_{qq'} D_{kk'} \lim_{\varepsilon \to 0} \sum_{i,j=5}^{7} L^Q_i L^K_j \sum_{qq'} \sum_{kk'} A^{\circ}_{ij} \tag{3.119}$$

where $A^{\circ}_{ij}$ are nine integrated expressions with all harmless terms dropped. We also took the limit $\epsilon \to 0$ which can be done safely after having performed the $\tau$-integrals.

### 3.4.3. Asymmetric vertex

Expression (3.87) with the energy differentials pulled in reads

$$T_b = 16c \sum_{legs} D_{qq'} \tilde{D}_{kk'} \int_\tau [d\tau] \left(\partial_Q^4 - \partial_K^4\right)\left(J_1^\circ J_2^\circ\right) \tag{3.120}$$

where $J_{1,2}^\circ$ are given in (3.83)-(3.84). We split $(J_1^\circ J_2^\circ)$ into its nine components like we did in the symmetric case yielding

$$\int_\tau [d\tau] \left(\partial_Q^4 - \partial_K^4\right)\left(J_1^\circ J_2^\circ\right) = \frac{1}{16} \lim_{\varepsilon,\epsilon\to 0} \sum_{i,j=4}^{6} M_i^Q M_j^K \sum_{qq'} \sum_{kk'} B_{ij} \tag{3.121}$$

where

$$\begin{aligned} B_{ij} := (-1)^{i+j} \int_{-\infty}^{\infty} \left(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\right) \left(\partial_Q^4 - \partial_K^4\right) \times \\ \times (Q-\tau)^i (K-\tau)^j \ln(Q-\tau+i\varepsilon) \ln(K-\tau-i\varepsilon) \; . \end{aligned} \tag{3.122}$$

Again, each of these integrals is divergent by itself. It can however be expected that summing all of them gives a finite result as we get for the integrand evaluated at large $\tau$

$$\lim_{\tau\to\infty} \left(\partial_Q^4 - \partial_K^4\right)\left(J_1^\circ J_2^\circ\right) = \frac{kk' - qq'}{90\tau^2} + \mathcal{O}\left(\frac{1}{\tau^3}\right) , \tag{3.123}$$

i.e. a fall-off at infinity. The integrals are therefore evaluated by introducing a cutoff $R$ and checking that all cutoff-dependent terms cancel modulo harmlessness in the end. By taking a look at the first integral which reads after evaluating the energy differentials

$$\begin{aligned} B_{44} = \lim_{R\to\infty} \int_{-R}^{R} \left(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\right) \times \\ \times \Big[50(K-\tau)^4 \ln(K-\tau-i\varepsilon) - 50(Q-\tau)^4 \ln(Q-\tau+i\varepsilon) \\ + \left\{24(K-\tau)^4 - 24(Q-\tau)^4\right\} \ln(Q-\tau+i\varepsilon) \ln(K-\tau-i\varepsilon)\Big] \end{aligned} \tag{3.124}$$

it can be seen that both single logarithmic terms are harmless as they depend only on one momentum combination each. We therefore concentrate on the bilogarithmic contribution and split into a part with the pole at $\tau = +i\epsilon$ and one at $\tau = -i\epsilon$. The former is evaluated with the contour in figure 3.1 and the latter with the contour in figure 3.2 leading to

$$B_{44} = 2\pi i \lim_{R\to\infty} \Big[ \int_Q^R \frac{d\tau}{\tau + i\epsilon} \ln(K - \tau - i\varepsilon) - \int_K^R \frac{d\tau}{\tau - i\epsilon} \ln(Q - \tau + i\varepsilon) \Big] \times$$
$$\times \left( 24(K-\tau)^4 - 24(Q-\tau)^4 \right) \tag{3.125}$$

where (3.108) and (3.110) have been used. There is a finite contribution $B_{44}^{\circ}$ consisting of polynomial, logarithmic and Dilogarithmic terms as well as a divergent one

$$B_{44} = B_{44}^{\circ} + 288 i\pi K^2 Q^2 \ln R \ , \tag{3.126}$$

a pattern which stays the same for all the other integrals. Also, the Dilogarithmic contribution is again of the form (3.116). The $R$-divergent parts collected together read

$$\begin{aligned}
B_{44}^R &= 288 i\pi K^2 Q^2 \ln R & B_{56}^R &= -1200 i\pi K^4 Q^3 \ln R \\
B_{45}^R &= -480 i\pi K^3 Q^2 \ln R & B_{65}^R &= -1200 i\pi K^3 Q^4 \ln R \\
B_{54}^R &= -480 i\pi K^2 Q^3 \ln R & B_{46}^R &= 720 i\pi K^4 Q^2 \ln R \\
B_{55}^R &= 800 i\pi K^3 Q^3 \ln R & B_{64}^R &= 720 i\pi K^2 Q^4 \ln R \\
B_{66}^R &= 1800 i\pi K^4 Q^4 \ln R & &
\end{aligned} \tag{3.127}$$

and like before cancel under evaluation of the sign sum and differential operators, i.e.

$$\sum_{i,j=4}^{6} M_i^Q M_j^K \sum_{qq'} \sum_{kk'} B_{ij}^R = 0 \ . \tag{3.128}$$

After finally taking the limits $R \to \infty$ and $\epsilon \to 0$ we are done with the $\tau$-integration of the asymmetric part and arrive at

$$T_b = c \sum_{legs} D_{qq'} \tilde{D}_{kk'} \lim_{\varepsilon\to 0} \sum_{i,j=4}^{6} M_i^Q M_j^K \sum_{qq'} \sum_{kk'} B_{ij}^{\circ} \ , \tag{3.129}$$

where the $B_{ij}^{\circ}$ have been evaluated with *Mathematica*. Like for the symmetric amplitude we refrain from displaying them here as the size of the expressions is quite large.

## 3.5. Summations and Differentiations

The final aim is to evaluate all the differential operators $D_{xy}$, $\tilde{D}_{xy}$, $L_i^x$ and $M_i^x$ as well as the leg and sign sums. A rough estimation gives of the order $10^4$ terms after performing all these operations which makes it far from possible to do any computations by hand at this point. However, we can still use that several of the remaining terms are harmless which helps to reduce the expressions to some extent. When doing all these simplifications an important point to take care of is the treatment of the complex logarithms. We shall restrict all of them to the principal branch

$$\ln z = \ln|z| + i\text{Arg}(z), \qquad -\pi < \text{Arg}(z) \leq \pi \tag{3.130}$$

which sometimes makes it necessary to introduce terms proportional to $2\pi i$ when a logarithm is simplified (see appendix A). Also, similarly to the case $D = 4$ new divergences appear when the sign sum is evaluated. We will primarily focus on the cancellation process of these divergent terms between the symmetric and asymmetric amplitude in this last section and show that this cancellation indeed occurs at the highest degree of transcendentality[5] $s = 2$. Regarding the size of the intermediate results this is found a non-trivial result already. The cancellation of the rest of the divergent terms with $s = 1$ still remains to be checked and is left open for future work.

### 3.5.1. Extracting the divergent terms

The terms in $A_{ij}^\circ$ and $B_{ij}^\circ$ which cause the divergences are of the form

- $Q^n \ln(K)$
- $Q^n \ln^2(K)$
- $Q^n \text{Li}_2(Q/K)$
- $K^n \ln(Q)$
- $K^n \ln^2(Q)$
- $K^n \text{Li}_2(K/Q)$

meaning that if $Q = 0$ or $K = 0$ we get logarithmic divergences. To isolate them we split the sign sum into four parts

$$\sum_{qq'} \sum_{kk'} = \sum_{Q\neq 0\neq K}^{9} + \sum_{Q=0,K\neq 0}^{3} + \sum_{Q\neq 0,K=0}^{3} + \sum_{Q=0=K}^{1} \tag{3.131}$$

[5] The degree of transcendentality $s \in \mathbb{N}$ results from the definition of the Lerch zeta function $\Phi(s,z,c) = \sum_{n=0}^{\infty} \frac{z^n}{(n+c)^s}$ which generates the Polylogarithm of degree $s$ by $Li_s(z) = z\Phi(s,z,1)$ [20]. The divergent terms are classified by their transcendentality which for $z \to \infty$ corresponds to the highest power of logarithmic terms in an expression as $\lim_{z\to\infty} Li_n(z) = \ln(-z)^n + \mathcal{O}(\ln(-z)^{n-1})$. [21]

where the two sums in the middle contain the divergent terms and the first one yields a finite result. Examining the very last sum which only consists of the term $K = 0 = Q$ it turns out that it in fact does not cause any divergences as there is always a polynomial zero compensating the logarithm.

The explicit computation of the divergent terms becomes a little more economic if we go back to the $\tau$-integrals $A_{ij}$, $B_{ij}$ and drop all terms in the integrands which cannot lead to any infinities of the mentioned type after the integration is performed. As we already know that no divergences for $\tau \to \infty$ can occur we only need to take into account the parts which diverge at $\tau \to 0$. For the explicit example of $A_{55}$ in (3.104) we are then left with

$$A_{55}^{div} = -5! \lim_{R\to\infty} \int_{-R}^{R} \Big( \frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon} \Big) \times \\ \times \Big[ 7K^5 + 7Q^5 \Big] \ln(Q - \tau + i\varepsilon) \ln(K - \tau - i\varepsilon) \tag{3.132}$$

as all other terms are of higher polynomial order in $\tau$ and thus stay finite. Note also, that the single logarithmic contributions of $A_{55}$ do not lead to divergences of this type and were therefore neglected. The same is done for all other $A_{ij}$ as well as for the $B_{ij}$, the resulting integrands of $A_{ij}^{div}$ and $B_{ij}^{div}$ are summarized in appendix C. An interesting thing to note is that only the lowest order integrals $A_{5j}$, $A_{j5}$ resp. $B_{4j}$, $B_{j4}$ contribute to the divergences, all integrands of higher order in $Q$ or $K$ automatically are of higher order in $\tau$ and therefore yield only finite results. Performing the $\tau$ integrals again in the same way as explained in section 3.4 and dropping harmless terms as well as terms like $Q^n K^m \ln(K - Q)$ which stay finite for $Q$ or $K = 0$ we arrive at the results summarized in appendix C. The limit $\lim_{\epsilon,\varepsilon\to 0}$ can be taken safely as there are no more ambiguities concerning logarithmic branches.

The next step is to evaluate the divergent parts with

$$T_a^{div} = \frac{c}{16} \sum_{legs} D_{qq'} D_{kk'} \sum_{i,j=5}^{7} L_i^Q L_j^K \Big[ \sum_{Q=0,K\neq 0}^{3} + \sum_{Q\neq 0,K=0}^{3} \Big] A_{ij}^{div} \tag{3.133}$$

$$T_b^{div} = c \sum_{legs} D_{qq'} \tilde{D}_{kk'} \sum_{i,j=4}^{6} M_i^Q M_j^K \Big[ \sum_{Q=0,K\neq 0}^{3} + \sum_{Q\neq 0,K=0}^{3} \Big] B_{ij}^{div} \, . \tag{3.134}$$

In order to be able to work with the expressions we regularize the vanishing momentum combination in each 3-sum by introducing a small parameter $\delta$. The sums then read

$$\sum_{Q=\delta,K\neq 0}^{3} f(Q,K) = f(\delta, 2k+2k') - f(\delta, 2k) - f(\delta, 2k') \tag{3.135}$$

$$\sum_{Q\neq 0,K=\delta}^{3} f(Q,K) = f(2q+2q', \delta) - f(2q, \delta) - f(2q', \delta) \tag{3.136}$$

and have to be understood in a limit $\lim_{\delta\to 0}$. This regularization can be justified by remembering that the expression $Q = q \pm q + q' \pm q'$ originates from terms $j_0(qr)e^{iqr}$ in the mode expansion (3.2) of the asymptotic states rewritten with (3.60). This means that one of the $q$'s actually corresponds to the energy whereas the other one corresponds to the momentum. As we are dealing with a massless scalar in 1+1 dimensions their difference cancels on-mass-shell. Consequently, regularizing $Q = \delta$ in the first sum above can be understood physically as a slight deviation from the mass-shell which indeed could happen by quantum mechanical effects violating energy conservation on short time scales. The same argumentation holds for $K$.

Now the sums and differential operators are evaluated with *Mathematica* and all terms which are finite in the limit $\delta \to 0$ are dropped as we are only interested in the divergent terms. Using the expansions (A.20)-(A.21) helps to further simplify the expressions and one finally arrives at divergent terms of two degrees of transcendentality: There are several terms with $s = 1$ of the form $(\,\text{polynomial}\,) \cdot \ln(\delta)$ and many terms of $s = 2$ which contain bilogarithmic expressions. The size of the terms becomes quite large at this point which is why we refrain from showing them explicitly. Keeping the focus on the part $s = 2$ we are now ready to add the two contributions and find

$$T^{div}_{s=2} = T^{div}_{a,s=2} + T^{div}_{b,s=2} = 0 \tag{3.137}$$

where it was made use of the momentum conserving delta function in the prefactor to write $q' = k + k' - q$.

# 4. Conclusion

During the course of this thesis it was at first outlined how the path integration of the geometric sector of the Einstein-massless-Klein–Gordon model could be performed along the lines of [5, 7]. Starting with the classical theory the important point here was the restriction to the s-wave sector which allowed reformulating the higher dimensional theory as a specific model of two-dimensional dilaton gravity with non-minimally coupled matter. Having the scalar field as the only local degree of freedom this theory is already non-trivial enough to permit an investigation of scattering processes. A transition to the first order formulation and the choice of temporal gauge for the geometric variables made it possible to perform the geometric sector of the path integral exactly leaving only the integration over a non-local effective action of the matter field. We were then interested in the four-point tree-level vertices of the scalar field arising from this action which at this point were pure quantum interactions mediated by gravitational backreaction. Computing the matter dependent effective fields with certain boundary conditions restricting global quantum fluctuations revealed that the intermediate scattering state corresponds to a Schwarzschild–Tangherlini black hole. This is the expected generalisation of the results for $D = 4$ found in the above references.

In chapter 3 we looked at the scattering amplitude corresponding to the mentioned vertices. It was known from previous results that for any dimension of the original non-reduced spacetime there are two contributing non-local diagrams. The chosen boundary conditions made it possible to construct an asymptotic Fock space over a flat background using a mode expansion of the scalar field in terms of Bessel functions, the featuring of the latter being typical for s-wave scattering [22].

From this point on we restricted to even dimensions of the original spacetime. Odd dimensions had to be left out because several properties of the Bessel functions made it unclear how to treat that case analytically (see the discussion in section 3.3). Also, all the higher even-dimensional cases turned out to be increasingly complicated and the appearing bookkeeping problems eventually forced a restriction to $D = 6$ which is the first interesting case after $D = 4$ already treated in [7]. The calculation was split into two parts corresponding to the two diagrams. It was found that each of them is divergent which is not a surprising result when one compares with $D = 4$. Inspired by that case we as well should expect a "miraculous" cancellation process of these divergences. The proof of this, however, turned out to be quite non-trivial and a first step in this direction was made at the end of chapter 3 where the divergences were classified according to their degree of transcendentality. It could be shown explicitly that the divergences with the highest degree $s = 2$ indeed cancel. All infinities with $s = 1$ still remain and their cancellation process together with the computation of the finite part is left to be shown in future work.

# A. Collection of formulae

Here I collect several formulae used in the main text, especially in the computation of the scattering amplitude. Partially they are just adopted from [7].

## Identities involving Bessel functions

$$\int_0^\infty dk\, k\, J_\alpha(kr) J_\alpha(kr') = \frac{\delta(r-r')}{r} \tag{A.1}$$

$$J_\alpha(x) = (-1)^\alpha J_\alpha(-x) \tag{A.2}$$

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) \qquad n \in \mathbb{N}^* \tag{A.3}$$

$$j_n(x) = (-x)^n \left(\frac{1}{x}\frac{d}{dx}\right)^n j_0(x) \tag{A.4}$$

$$j_0(x) = \frac{\sin x}{x} \tag{A.5}$$

where $J_\alpha(x)$ denote the Bessel functions of the first kind of order $\alpha$ and $j_n(x)$ are the spherical Bessel functions of order $n$ [23].

## Differential operators

In the main text we use the following definitions of differential operators:

- $D_{xy} := xy\,\partial^2_{xy} - y\,\partial_y - x\,\partial_x + 1$

  It acts on polynomials like

  $$D_{xy}(abx^n y^m) = ab(m-1)(n-1)x^n y^m \tag{A.6}$$

  and therefore kills all momentum combinations in the main text that are linear in at least one of the momenta. For more properties the reader shall be referred to appendix F in [7].

- $\tilde{D}_{xy} = xy\,\partial_y - x$

  Acting on a term in a polynomial yields

$$\tilde{D}_{xy}(abx^n y^m) = ab(m-1)x^{n+1}y^m \tag{A.7}$$

  which still kills linear terms in $y$ but not in $x$.

# Path integral identities

We can express the formal path integral of a linear functional $L(\phi, \mathcal{X}) = \int \phi M \mathcal{X}$ as

$$\int (\mathcal{D}\phi)(\mathcal{D}\mathcal{X}) \exp i \int \phi M \mathcal{X} = \int (\mathcal{D}\mathcal{X})\ \delta(M\mathcal{X}) = \big(\mathrm{Det}\, M\big)^{-1} \tag{A.8}$$

where $\phi$ and $\mathcal{X}$ are Grassmann even fields. If we take both arguments of the functional to be $\phi$ we get the Gaussian integral formula

$$\int (\mathcal{D}\phi) \exp i \int \phi M \phi = \big(\mathrm{Det}\, M\big)^{-\frac{1}{2}}\ . \tag{A.9}$$

If we take $\phi$ and $\mathcal{X}$ to be Grassmann odd fields instead we get

$$\int (\mathcal{D}\phi)(\mathcal{D}\mathcal{X}) \exp i \int \phi M \mathcal{X} = \int (\mathcal{D}\mathcal{X})\ \delta(M\mathcal{X}) = \big(\mathrm{Det}\, M\big)\ . \tag{A.10}$$

# Complex logarithm

The restriction to the principal branch of the complex logarithm

$$\ln(z) = \ln|z| + i\mathrm{Arg}(z) \qquad\qquad -\pi < \mathrm{Arg}(z) \le \pi \tag{A.11}$$

for some $z \in \mathbb{C}^\times$ makes it necessary to modify some rules involving the function $\ln(z)$. The reason is the behaviour of the function $\mathrm{Arg}(z)$ when it comes to multiplication and division of two complex numbers $z_1, z_2 \in \mathbb{C}^\times$. We have

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) + 2\pi N_+ \tag{A.12}$$
$$\mathrm{Arg}(z_1/z_2) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2) + 2\pi N_- \tag{A.13}$$

where the integers $N_\pm$ are determined by

$$N_\pm = \begin{cases} -1, & \text{if} \quad \mathrm{Arg}(z_1) \pm \mathrm{Arg}(z_2) > \pi \\ 0, & \text{if} \quad -\pi < \mathrm{Arg}(z_1) \pm \mathrm{Arg}(z_2) \le \pi \\ 1, & \text{if} \quad \mathrm{Arg}(z_1) \pm \mathrm{Arg}(z_2) \le -\pi\ . \end{cases} \tag{A.14}$$

This leads to the following rules for the complex logarithm

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2) + 2\pi i N_+ \tag{A.15}$$
$$\ln(z_1/z_2) = \ln(z_1) - \ln(z_2) + 2\pi i N_- \tag{A.16}$$

which have as a special case

$$\ln(1/z) = \begin{cases} -\ln(z) + 2\pi i & \text{if} \quad z \text{ real and negative} \\ -\ln(z) & \text{otherwise} . \end{cases} \tag{A.17}$$

## Dilogarithm

The Dilogarithm is defined as the following integral

$$Li_2(z) = -\int_0^z \frac{\ln(1-u)}{u} du \tag{A.18}$$

where the path is chosen in the complex plane and must not cross the branch cut of the integrand lieing along the real axis at $[1, \infty)$ for our conventions. For evaluating the divergent terms we need the derivative

$$\frac{d}{dz} Li_2(z) = -\frac{\ln(1-z)}{z} \tag{A.19}$$

and the expansion for large $z$ with small imaginary part

$$\lim_{z\to\infty\pm i\epsilon} Li_2(z) = -\frac{1}{2}\ln^2|z| \mp i\pi \ln|z| + \mathcal{O}(1) \tag{A.20}$$
$$\lim_{z\to-\infty\pm i\epsilon} Li_2(z) = -\frac{1}{2}\ln^2|z| + \mathcal{O}(1) . \tag{A.21}$$

The Dilogarithm also fulfils the following identities

$$Li_2(z) + Li_2(1-z) = \frac{\pi^2}{6} - \ln z \ln(1-z) \tag{A.22}$$
$$Li_2(1-z) + Li_2\left(1 - \frac{1}{z}\right) = -\frac{\ln^2 z}{2} \tag{A.23}$$

which were useful for the simplification of the amplitude.

## Other identities

It is made extensive use of the regularized Fourier transformation of $\Theta(x)x^\lambda$ with some parameter $\lambda \in \mathbb{Z}$ [24]

$$\int_0^\infty x^\lambda e^{i(\sigma+i\varepsilon)x} dx = i e^{\frac{i\lambda\pi}{2}} \Gamma(\lambda+1)(\sigma+i\varepsilon)^{-\lambda-1} \tag{A.24}$$

where $\varepsilon$ is some regulator. We actually only need the singular limits

$$\lambda = \lim_{\delta\to 0}(-n+\delta), \qquad n \in \mathbb{N} \tag{A.25}$$

which can be evaluated using the expansions

$$\lim_{\delta\to 0}\Gamma(-n+\delta) = \frac{(-1)^n}{n!}\left[\frac{1}{\delta} + \psi_1(n+1) + \mathcal{O}(\delta)\right] \tag{A.26}$$

$$\psi_1(n+1) := \frac{d\ln\Gamma(z)}{dz}\Big|_{z=n+1} = -\gamma + \sum_{k=1}^{n}\frac{1}{k} \tag{A.27}$$

and

$$(\sigma+i\varepsilon)^{-\delta} = 1 - \delta\Big(\ln|\sigma| + i\pi\Theta(-\sigma) + \mathcal{O}(\varepsilon)\Big) + \mathcal{O}(\delta^2) \tag{A.28}$$

$$e^{i\pi(-n+\delta)/2} = e^{-in\pi/2}\big(1 + \delta\frac{i\pi}{2} + \mathcal{O}(\delta^2)\big) \ . \tag{A.29}$$

Here $\gamma$ is the Euler–Mascheroni constant and $\psi_1(z)$ the first logarithmic derivative of $\Gamma(z)$. The branch cut of complex logarithms is, unless stated differently, assumed to lie along the negative real axis.

We also use the Fourier transform of $\Theta(x)$

$$\Theta(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\tau}{\tau - i\epsilon}e^{i\tau x} \ . \tag{A.30}$$

with some regulator $\epsilon$.

# B. Spherical wave equation

For solving the massless Klein–Gordon equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0 \tag{B.1}$$

in flat space and spherical coordinates we can use the fact that the wave operator still splits into a sum

$$\eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \Delta \tag{B.2}$$

where $\Delta$ is the Laplacian in spherical coordinates. We can therefore separate $\phi(t,r) = \mathcal{X}(t)\varphi(r)$ where it was assumed that we are in the pure s-wave sector, i.e. no dependence of $\phi$ on the angular coordinates is present. The equation separates with

$$\frac{\partial_t^2\mathcal{X}}{\mathcal{X}} = \frac{\Delta\varphi}{\varphi} = -k^2 \tag{B.3}$$

where $k$ is some constant which can be chosen to be strictly positive without loss of generality. The time dependent equation is immediately solved

$$\mathcal{X}(t) = Ae^{ikt} + Be^{-ikt}, \qquad A, B \in \mathbb{R} \tag{B.4}$$

and the spatial one can be shown to be of the form

$$r^2\partial_r^2\varphi + (1+2\alpha)r\partial_r\varphi + k^2r^2\varphi = 0, \qquad \alpha = \frac{D-3}{2}\ . \tag{B.5}$$

By the redefinition

$$\phi(r) = r^{-\alpha}f(r) \tag{B.6}$$

it can be recast as

$$r^2\partial_r^2 f + r\partial_r f + (k^2r^2 - \alpha^2)f = 0 \tag{B.7}$$

which is the Bessel differential equation [23]. For a fixed $k$ its general solution is given by a superposition of two linearly independent functions

$$f(r) = \begin{cases} c_1 J_\alpha(kr) + c_2 J_{-\alpha}(kr), & \alpha \in \{\frac{1}{2}, \frac{3}{2}, ...\} \quad [D \text{ even}] \\ c_1 J_\alpha(kr) + c_2 Y_\alpha(kr), & \alpha \in \{1, 2, ...\} \quad [D \text{ odd}] \end{cases} \tag{B.8}$$

where $c_1, c_2 \in \mathbb{R}$ and $J_\alpha(kr), Y_\alpha(kr)$ are the Bessel functions of the first resp. second kind. Going back to $\varphi(r) = r^{-\alpha} f(r)$ it can be shown that this is only regular at $r \to 0$ if $c_2 = 0$ in both cases. Then for $r \to \infty$ the solutions have the fall-off behaviour $\varphi \sim \mathcal{O}\big(r^{-\alpha-\frac{1}{2}}\big)$. Restricting to this case and summing up solutions for different $k$ we arrive at

$$\phi(t,r) = \int_0^\infty dk\, c(k) r^\alpha J_{-\alpha}(kr)\big(a_k^+ e^{ikt} + a_k^- e^{-ikt}\big) \tag{B.9}$$

where $c(k)$ is a still undetermined normalization function which is chosen to be $c(k) = \sqrt{k^{D-2}/2}$ in the main part. We also introduced the coefficient functions $a_k^\pm$ whose quantized versions are used to build the Fock space of the free theory.

Note, that the massless dispersion relation in this two dimensional setting reduces with the chosen units to

$$E^2 = \omega^2 = \vec{k}^2 \tag{B.10}$$

where $\vec{k} := k$ is just a scalar number in one spatial dimension. We implicitly used this definition for $k$ above and it should not be confused with the Lorentz invariant length of the two-vector $k^\mu$ which is of course zero for a massless field.

# C. Intermediate results

Here some results from the extraction of the divergent terms in section 3.5.1 are collected.

## Divergent integrands

The divergent parts[1] $A_{ij}^{div}$, $B_{ij}^{div}$ of the $\tau$ integrals (3.102), (3.122) are of the form

$$A_{ij}^{div} = 5!7 \lim_{R\to\infty} \int_{-R}^{R} \left(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\right) a_{ij} \ln(Q - \tau + i\varepsilon) \ln(K - \tau - i\varepsilon) \tag{C.1}$$

$$B_{ij}^{div} = 4! \lim_{R\to\infty} \int_{-R}^{R} \left(\frac{d\tau}{\tau - i\epsilon} + \frac{d\tau}{\tau + i\epsilon}\right) b_{ij} \ln(Q - \tau + i\varepsilon) \ln(K - \tau - i\varepsilon) \tag{C.2}$$

with

$$\begin{aligned}
a_{55} &= -K^5 - Q^5 & b_{44} &= K^4 - Q^4 \\
a_{56} &= K^6 & b_{45} &= -K^5 \\
a_{65} &= Q^6 & b_{54} &= Q^5 \\
a_{57} &= -K^7 & b_{46} &= K^6 \\
a_{75} &= -Q^7 & b_{64} &= -Q^6
\end{aligned} \tag{C.3}$$

and the rest of the combinations being zero.

[1]As explained in the main text we are only interested into the small-$\tau$ divergences. It has already been shown that divergences for $\tau \to \infty$ are harmless.

## Divergent $\tau$-integrals evaluated

Evaluating the divergent $\tau$-integrals (C.1)-(C.2) as explained in section 3.5.1 and dropping harmless terms one finds

$$A_{55}^{div} = 5!7\pi\Big[Q^5\big(2\pi - i\ln K\big)\ln K + K^5\big(2\pi + i\ln Q\big)\ln Q + 2i\big(K^5+Q^5\big)\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.4}$$

$$A_{56}^{div} = 5!7\pi\Big[K^6\big(-2\pi - i\ln Q\big)\ln Q - 2iK^6\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.5}$$

$$A_{65}^{div} = 5!7\pi\Big[Q^6\big(-2\pi + i\ln K\big)\ln K - 2iQ^6\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.6}$$

$$A_{57}^{div} = 5!7\pi\Big[K^7\big(2\pi + i\ln Q\big)\ln Q + 2iK^7\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.7}$$

$$A_{75}^{div} = 5!7\pi\Big[Q^6\big(2\pi - i\ln K\big)\ln K + 2iQ^7\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.8}$$

as well as

$$B_{44}^{div} = 4!\pi\Big[Q^4\big(2\pi - i\ln K\big)\ln K + K^4\big(-2\pi - i\ln Q\big)\ln Q - 2i\big(K^4-Q^4\big)\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.9}$$

$$B_{45}^{div} = 4!\pi\Big[K^5\big(2\pi + i\ln Q\big)\ln Q + 2iK^5\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.10}$$

$$B_{54}^{div} = 4!\pi\Big[Q^5\big(-2\pi + i\ln K\big)\ln K - 2iQ^5\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.11}$$

$$B_{46}^{div} = 4!\pi\Big[K^6\big(-2\pi - i\ln Q\big)\ln Q - 2iK^6\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] \tag{C.12}$$

$$B_{64}^{div} = 4!\pi\Big[Q^6\big(2\pi - i\ln K\big)\ln K + 2iQ^6\Big(Li_2\Big(\frac{K}{Q}\Big) - Li_2\Big(\frac{Q}{K}\Big)\Big)\Big] . \tag{C.13}$$

# Bibliography


[1] Steven B. Giddings. "The gravitational S-matrix: Erice lectures". In: *Subnucl. Ser.* 48 (2013). Ed. by Antonino Zichichi, pp. 93–147. DOI: 10.1142/9789814522489_0005. arXiv: 1105.2036 [hep-th].

[2] Daniel Grumiller, Wolfgang Kummer, and Dmitri V. Vassilevich. "Dilaton Gravity in Two Dimensions". In: *Phys.Rept.* 369 (2002). arXiv:hep-th/0204253v9, pp. 327–430.

[3] W. Kummer, H. Liebl, and D. V. Vassilevich. "Exact path integral quantization of generic 2-D dilaton gravity". In: *Nucl. Phys. B* 493 (1997), pp. 491–502. DOI: 10.1016/S0550-3213(97)00143-0. arXiv: gr-qc/9612012.

[4] W. Kummer, H. Liebl, and D. V. Vassilevich. "Nonperturbative path integral of 2-D dilaton gravity and two loop effects from scalar matter". In: *Nucl. Phys. B* 513 (1998), pp. 723–734. DOI: 10.1016/S0550-3213(97)00796-7. arXiv: hep-th/9707115.

[5] W. Kummer, H. Liebl, and D. V. Vassilevich. "Integrating geometry in general 2-D dilaton gravity with matter". In: *Nucl. Phys. B* 544 (1999), pp. 403–431. DOI: 10.1016/S0550-3213(99)00047-4. arXiv: hep-th/9809168.

[6] Karen Crowther and Niels Linnemann. "Renormalizability, Fundamentality, and a Final Theory: The Role of UV-Completion in the Search for Quantum Gravity". In: *The British Journal for the Philosophy of Science* 70 (May 2017). DOI: 10.1093/bjps/axx052.

[7] Daniel Grumiller. "Quantum Dilaton Gravity in Two Dimensions with Matter". arXiv:gr-qc/0105078v4 . PhD thesis. Technische Universitat Wien, 2001.

[8] Peter Fischer. *Vertices in spherically reduced quantum gravity.* eng. Wien, 2001.

[9] Carlo Rovelli. "Notes for a brief history of quantum gravity". In: *9th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories (MG 9).* June 2000, pp. 742–768. arXiv: gr-qc/0006061.

[10] F. R. Tangherlini. "Schwarzschild field in n dimensions and the dimensionality of space problem". In: *Nuovo Cim.* 27 (1963), pp. 636–651. DOI: 10.1007/BF02784569.

[11] Thomas Strobl. "Gravity in Two Spacetime Dimensions". In: (2000). arXiv:hep-th/0011240 .

[12] Peter Schaller and Thomas Strobl. "Poisson sigma models: A Generalization of 2-d gravity Yang-Mills systems". In: *Conference on Integrable Systems*. July 1994, pp. 181–190. arXiv: hep-th/9411163.

[13] M. Henneaux and C. Teitelboim. *Quantization of gauge systems*. 1992. ISBN: 978-0-691-03769-1.

[14] H. Kleinert. *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*. EBL-Schweitzer. World Scientific, 2009. ISBN: 9789814273572.

[15] D. Grumiller, W. Kummer, and D. V. Vassilevich. "Virtual black holes in generalized dilaton theories and their special role in string gravity". In: *Eur. Phys. J. C* 30 (2003), pp. 135–143. DOI: 10.1140/epjc/s2003-01258-5. arXiv: hep-th/0208052.

[16] D. Grumiller. "Virtual black holes and the S-matrix". In: *Int. J. Mod. Phys. D* 13 (2004), pp. 1973–2002. DOI: 10.1142/S0218271804006437. arXiv: hep-th/0409231.

[17] D. Grumiller and W. Kummer. "Absolute conservation law for black holes". In: *Phys. Rev. D* 61 (2000), p. 064006. DOI: 10.1103/PhysRevD.61.064006. arXiv: gr-qc/9902074.

[18] D. Grumiller, W. Kummer, and D. V. Vassilevich. "The Virtual black hole in 2-D quantum gravity". In: *Nucl. Phys. B* 580 (2000), pp. 438–456. DOI: 10.1016/S0550-3213(00)00231-5. arXiv: gr-qc/0001038.

[19] P. Fischer et al. "S-matrix for s-wave gravitational scattering". In: *Phys. Lett. B* 521 (2001). [Erratum: Phys.Lett.B 532, 373–373 (2002)], pp. 357–363. DOI: 10.1016/S0370-2693(01)01227-8. arXiv: gr-qc/0105034.

[20] Jeffrey C. Lagarias and Wen-Ching Winnie Li. "The Lerch Zeta function III. Polylogarithms and special values". In: *Research in the Mathematical Sciences* 3.1 (Feb. 2016). ISSN: 2197-9847. DOI: 10.1186/s40687-015-0049-2. URL: http://dx.doi.org/10.1186/s40687-015-0049-2.

[21] D. Wood. "The Computation of Polylogarithms". In: 1992.

[22] Wolfgang Bietenholz and J. J. Giambiagi. "Solutions of the spherically symmetric wave equation in (p+q)-dimensions". In: *J. Math. Phys.* 36 (1995), pp. 383–387. DOI: 10.1063/1.531313. arXiv: hep-th/9402065.

[23] B.G. Korenev. *Bessel Functions and Their Applications*. Analytical Methods and Special Functions. CRC Press, 2002. ISBN: 9780203216927.

[24] I.M. Gelfand and G.E. Shilov. *Generalized Functions: Properties and operations. Volume 1*. Academic Press, 1964. ISBN: 9781483229768.