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$\mathcal{H}$-inverses for RBF interpolation

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Abstract
We consider the interpolation problem for a class of radial basis functions (RBFs) that includes the classical polyharmonic splines (PHS). We show that the inverse of the system matrix for this interpolation problem can be approximated at an exponential rate in the block rank in the $\mathcal{H}$-matrix format, if the block structure of the $\mathcal{H}$-matrix arises from a standard clustering algorithm.

Keywords Radial basis function interpolation · H-matrices · Approximability · Non-uniform point clouds

Mathematics Subject Classification (2010) Primary: 65F50, Secondary: 65F30

1 Introduction

Radial basis functions (RBFs) have become an important tool in computational mathematics. Starting from the general question of interpolating scattered data, they found their way into statistics applications (see, e.g., [29, 42] or [7] for application in machine learning) and, as a specific instance of meshfree methods, into the realm of numerical methods for partial differential equations [33, 43].

The analysis of the approximation properties of a variety of RBFs is by now rather mature, [16, 33, 43]. The classical interpolation problem with RBFs leads to linear systems with fully populated system matrices, which brings their efficient solution

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to the fore. A basic question in this connection is that of an efficient representation of the system matrix. Many RBFs such as polyharmonic splines, multiquadrics, and Gaussians are "asymptotically smooth" so that the system matrix can very efficiently be approximated by blockwise low rank matrices, [11, 14] and [32, 38]. This observation opened the door for log-linear complexity matrix–vector multiplication and a subsequent iterative solution. Nevertheless, good preconditioning strategies are necessary. Domain decomposition techniques [8, 35], possibly combined with multilevel techniques [33, 37], are an option. A recent alternative is the use of the arithmetic that comes with rank-structured matrices. Here, we consider specifically $\mathcal{H}$-matrices, [31]. These are blockwise low-rank matrices endowed with a hierarchical structure that leads to an (approximate) arithmetic including addition, multiplication, inversion, and LU-factorization in logarithmic-linear complexity. This arithmetic can thus be used as a direct solver or be employed to create preconditioners. The arithmetic is only approximate but the error can be controlled by either a priori chosen parameters [25] or adaptively [15, 28]. Yet, a basic question remains whether the target, i.e., the inverse of the system matrix or the LU-factorization, can be represented in the $\mathcal{H}$-matrix format. This question has been answered for finite element discretizations of elliptic PDEs in [6], later improved in [21] (to arbitrary accuracy) and [1, 2] (locally refined meshes), as well as for boundary element discretizations, [22, 23], and the coupling of finite elements and boundary elements, [24].

It is the purpose of the present paper to provide such an approximation result in Theorem 18 for RBFs that are fundamental solutions of certain partial differential operators with constant coefficients; in particular, the popular polyharmonic splines, introduced and analyzed in [18], are covered by the present paper (see Lemma 11). Consequently, see [4, 21], one also obtains the existence of LU-decompositions in the $\mathcal{H}$-matrix format, which gives mathematical underpinning to their observed good performance in black-box preconditioning, [36, 37].

The present paper is structured as follows: In Section 2 we formulate our model problem, the so called *interpolation problem* together with the corresponding *native space* $(V, a(\cdot,\cdot))$. We then reformulate the problem as a linear system of equations (LSE) and introduce the *interpolation matrix*. Moreover, some basic definitions concerning $\mathcal{H}$-matrices are given and, finally, we state the main result, Theorem 18. Section 3 is devoted to the proof of our main result: First, in Section 3.1, we investigate the native space V and introduce the *homogeneous native space* $V_0 \subseteq V$. In Section 3.2, we reformulate the original interpolation problem with an orthogonality condition in terms of $a(\cdot,\cdot)$ and the space V_0. In Section 3.4, Corollary 37, we derive a representation formula for the inverse system matrix to reduce the original "matrix approximation problem" to a "function approximation problem". Then, the main step in the proof is the construction of low dimensional spaces from which solutions to the interpolation problem can be approximated at an exponential rate, Section 3.9. Finally, Section 4 provides some numerical examples.

Concerning notation: We write "$a \lesssim b$" iff there exists a constant $C > 0$ such that $a \leq Cb$. The constant might depend on the space dimension d, the orders k and $k_{\min}$ of the native space V, the coefficients σ_l of the bilinear form $a(\cdot,\cdot)$ et cetera. However, it does *not* depend on critical parameters such as the problem size N. We write $a \approx b$, if both $a \lesssim b$ and $a \gtrsim b$ are satisfied. Matrices and vectors in linear systems of equations are expressed in boldface letters, e.g., $\boldsymbol{A} \in \mathbb{R}^{N\times N}$ and $\boldsymbol{f} \in \mathbb{R}^N$. The only exception to this rule is the set $\boldsymbol{C} \subseteq \mathbb{R}^N$ introduced in Definition 8, from which coefficient vectors $\boldsymbol{c} \in \boldsymbol{C}$ are drawn. For all $x \in \mathbb{R}^d$ and $r > 0$, we write $\mathrm{Ball}(x, r) := \{y \in \mathbb{R}^d \mid \|y - x\|_2 < r\}$ for the Euclidean ball of radius r, centered at x. The diameter of a set $\Omega \subseteq \mathbb{R}^d$ is denoted by $\mathrm{diam}_2(\Omega) := \sup_{x,y\in\Omega} \|x - y\|_2$ and the distance between two sets $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ is given by $\mathrm{dist}_2(\Omega_1, \Omega_2) := \inf_{x_1\in\Omega_1, x_2\in\Omega_2} \|x_2 - x_1\|_2$. To treat derivatives in all space dimensions $d \geq 1$ alike, we adopt the usual multi-index notation $\mathrm{D}^\alpha := \partial_d^{(\alpha_d)} \circ \cdots \circ \partial_1^{(\alpha_1)}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_d$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. We frequently drop the supplement "for all $\alpha \in \mathbb{N}_0^d$" and use shorthand phrases like "for all $|\alpha| = k$" to describe the set $\{\alpha \in \mathbb{N}_0^d \mid |\alpha| = k\}$.

As for specific function spaces, we work with the following definitions (all function spaces are meant to be real-valued): Let $d \geq 1$ and $\Omega \subseteq \mathbb{R}^d$ be an open set. For all $p \in \mathbb{N}_0$, we use the polynomial space $\mathbb{P}^p(\Omega) := \mathrm{span}\{x^\alpha \mid |\alpha| \leq p\}$, and we also write $\mathbb{P}^{-1}(\Omega) := \{0\}$. We write $L^p(\Omega)$, $p \in [1, \infty]$, for the classical Lebesgue spaces. Moreover, a measurable function $f : \Omega \longrightarrow \mathbb{R}$ belongs to the space $L^p_{\mathrm{loc}}(\Omega)$ iff it satisfies $f|_\omega \in L^p(\omega)$ for all bounded open sets $\omega \subseteq \mathbb{R}^d$ satisfying $\overline{\omega} \subseteq \Omega$. For all $k \in \mathbb{N}_0$ and $p \in [1, \infty]$, the Sobolev space $W^{k,p}(\Omega)$ consists of all k-times weakly differentiable functions $f \in L^1_{\mathrm{loc}}(\Omega)$, whose weak derivatives $(\mathrm{D}^\alpha f)_{|\alpha|\leq k}$ lie in $L^p(\Omega)$. Similarly, a function $f \in L^1_{\mathrm{loc}}(\Omega)$ belongs to the space $W^{k,p}_{\mathrm{loc}}(\Omega)$ if it is k-times weakly differentiable and if there holds $\mathrm{D}^\alpha f \in L^p_{\mathrm{loc}}(\Omega)$ for all $|\alpha| \leq k$. In the case $p = 2$, we write $H^k(\Omega) := W^{k,2}(\Omega)$ and $H^k_{\mathrm{loc}}(\Omega) := W^{k,2}_{\mathrm{loc}}(\Omega)$.

2 Main results

2.1 The interpolation problem

We start our presentation with the definition of the *native space* V, which forms the basis of our functional-analytic framework:

Definition 1 *Let $d, k \in \mathbb{N}$ with $k > d/2$. Furthermore, let $k_{\min} \in \{0, \ldots, k\}$ and $\sigma_{k_{\min}}, \ldots, \sigma_k \geq 0$ be given coefficients with $\sigma_{k_{\min}}, \sigma_k > 0$. We define the* native space *V and the polynomial space P by:*

$$
\begin{aligned}
V &:= \{v \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \mid \forall l \in \{k_{\min}, \ldots, k\} : \forall |\alpha| = l : \mathrm{D}^\alpha v \in L^2(\mathbb{R}^d)\},\\
P &:= \mathbb{P}^{k_{\min}-1}(\mathbb{R}^d).
\end{aligned}
$$

In other words, a function $v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to V if, for all $l \in \{k_{\min}, \ldots, k\}$ and all $|\alpha| = l$, the α-th weak derivative $\mathrm{D}^\alpha v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ exists and lies in $L^2(\mathbb{R}^d)$. For all $u, v \in V$, we define

$$
a(u, v) := \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \langle \mathrm{D}^\alpha u, \mathrm{D}^\alpha v\rangle_{L^2(\mathbb{R}^d)}, \qquad |v|_a := \sqrt{a(v, v)}.
$$

Furthermore, for every open subset $\Omega \subseteq \mathbb{R}^d$*, we set*

$$|v|_{a,\Omega} := \Big(\sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \|D^\alpha v\|^2_{L^2_{(\Omega)}} \Big)^{1/2}.$$

Note that the assumption $k > d/2$ guarantees that the Sobolev embedding $H^k(\Omega) \subseteq C^0(\overline{\Omega})$ is continuous for all bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^d$ as well as $\Omega = \mathbb{R}^d$ (e.g., [30, Chapter 7]). In particular, $\|v\|_{C^0(\overline{\Omega})} \leq C(d, k, \Omega)\|v\|_{H^k(\Omega)}$, for all $v \in H^k(\Omega)$. We will make use of this fact on several occasions throughout this work.

For the basic properties of V, P, $a(\cdot, \cdot)$ and $|\cdot|_a$, we refer the reader to Lemma 20 in Section 3 below. In brief, $a(\cdot, \cdot)$ defines a symmetric positive semi-definite bilinear form on V and $|\cdot|_a$ is a seminorm with kernel P. There holds $V \subseteq C^0(\mathbb{R}^d)$, so that every $v \in V$ has well-defined point-values $v(x)$, $x \in \mathbb{R}^d$. In the extreme case $k_{\min} = k$, there holds $V = \mathrm{BL}^k(\mathbb{R}^d)$, which is the well-known *homogeneous Sobolev space* or *Beppo-Levi space* (e.g., [17], [41]). In the other extreme case $k_{\min} = 0$, however, there holds $V = H^k(\mathbb{R}^d)$, the standard Sobolev space. The norms are equivalent, and the polynomial space becomes trivial, $P = \{0\}$. In particular, $(V, a(\cdot, \cdot))$ is then a proper Hilbert space.

Definition 2 *Let* $N_{\min} := \dim P = \binom{d+k_{\min}-1}{d}$*. For each* $N \in \mathbb{N}$ *with* $N \geq N_{\min}$*, let* $\{x_1, \ldots, x_N\} \subseteq \mathbb{R}^d$ *be a set of pairwise distinct* interpolation points. *We denote their* separation distance *by*

$$h_{\min,N} := h_{\min} := \frac{1}{2} \min_{n \neq m \in \{1,\ldots,N\}} \|x_n - x_m\|_2.$$

We make the following assumptions about the family $(\{x_1, \ldots, x_N\})_{N \geq N_{\min}}$*:*

1. Boundedness: *There exists a constant* $C > 0$ *such that*

$$\forall N \geq N_{\min} : \forall n \in \{1, \ldots, N\} : \qquad \|x_n\|_2 \leq C.$$

2. Unisolvent subset: *There exists a set of points* $\{\xi_\alpha \mid |\alpha| < k_{\min}\} \subseteq \mathbb{R}^d$*, unisolvent for the space* P*, such that*

$$\forall N \geq N_{\min} : \qquad \{\xi_\alpha \mid |\alpha| < k_{\min}\} \subseteq \{x_1, \ldots, x_N\}.$$

3. Balance $N \leftrightarrow h_{\min}$: *There exist constants* $C, \sigma_{\mathrm{card}} \geq 1$ *such that*

$$\forall N \geq N_{\min} : \qquad 1 \leq C N^{\sigma_{\mathrm{card}}} h^d_{\min,N}.$$

In the proof of Theorem 50, the assumption of boundedness will allow us to apply some Poincaré-type inequality on a fixed, bounded subset of $\mathbb{R}^d$ that is independent of the problem size N. Furthermore, as a minor technical detail, it guarantees that $h_{\min,N} \lesssim 1$ for all $N \geq N_{\min}$.

The unisolvency assumption, on the other hand, allows us to infer the following implication: If a polynomial $p \in P$ satisfies $p(\xi_\alpha) = 0$ for all $|\alpha| < k_{\min}$, then already $p = 0$. This argument will be used on numerous occasions. Note that *all* sets of interpolation points $\{x_1, \dots, x_N\}$, $N \geq N_{\min}$, must contain *the same* set of unisolvent points $\{\xi_\alpha \mid |\alpha| < k_{\min}\}$. This assumption is necessary to ensure that the aforementioned Poincaré constant in Theorem 50 is independent of the problem size N. Clearly, if the family $(\{x_1, \dots, x_N\})_{N \geq N_{\min}}$ is constructed by an algorithm that successively adds more points to some initial point set, never deleting or modifying existing ones, then this assumption is satisfied.

The asserted balance between the problem size N and the separation distance $h_{\min,N}$ is fulfilled for a wide variety of families of interpolation points. As an example, consider the case where a bounded domain $\Omega \subseteq \mathbb{R}^d$ is given and where $\{x_1, \dots, x_N\} \subseteq \Omega$ for all $N \geq N_{\min}$. Denote by $\mathcal{T}_N := \{T_1, \dots, T_N\}$ the corresponding Voronoi decomposition of Ω, i.e., $T_n = \{x \in \Omega \mid \|x - x_n\|_2 = \min_{m \in \{1,\dots,N\}} \|x - x_m\|_2\}$, and by $h_{\max,N} := \max_{n \in \{1,\dots,N\}} \operatorname{diam}_2(T_n)$ the maximal cell diameter. Now suppose that

$$h_{\max,N}^{\sigma_{\text{card}}} \leq C h_{\min,N},$$

which is satisfied, e.g., for *uniform* and *algebraically graded* families of interpolation points. Then,

$$\begin{aligned} 1 \lesssim |\Omega|^{\sigma_{\text{card}}} &= \Big(\sum_{n=1}^{N} |T_n|\Big)^{\sigma_{\text{card}}} \leq \Big(\sum_{n=1}^{N} \operatorname{diam}_2(T_n)^d\Big)^{\sigma_{\text{card}}} \\ &\leq (N h_{\max,N}^d)^{\sigma_{\text{card}}} \leq C^d N^{\sigma_{\text{card}}} h_{\min,N}^d. \end{aligned}$$

Although Definition 2 is phrased in terms of a *family* of interpolation points, most of the upcoming results deal with a single set $\{x_1, \dots, x_N\} \subseteq \mathbb{R}^d$ of interpolation points. For the most part, one can therefore think of N as being “fixed” throughout this work.

Definition 3 *The* evaluation operator $E_N : C^0(\mathbb{R}^d) \longrightarrow \mathbb{R}^N$ *is defined by* $E_N v := (v(x_n))_{n=1}^N$ *for all* $v \in C^0(\mathbb{R}^d)$.

Recall that $V \subseteq C^0(\mathbb{R}^d)$, so that $E_N v$ is well-defined for all $v \in V$.

Now that the native space V and the interpolation points $x_1, \dots, x_N$ are fixed, let us pose the *interpolation problem*:

Problem 4 *Let $f \in V$. Find a function $u \in V$ that satisfies the following interpolation and minimization properties:*

$$E_N u = E_N f, \qquad |u|_a \leq \inf_{\substack{\tilde{u} \in V: \\ E_N \tilde{u} = E_N f}} |\tilde{u}|_a.$$

Clearly, the interpolation condition “$E_N u = E_N f$” can also be written as “$\forall n \in \{1, \dots, N\} : u(x_n) = f(x_n)$”. In other words, we are looking for a minimizer u of the seminorm $|\cdot|_a$ over the set of interpolants of f. While the interpolation conditions fix the values of u at the interpolation points $x_1, \ldots, x_N$, the minimization property determines the behavior of u on the rest of $\mathbb{R}^d$.

In Section 3.2 below we will see that, for every given $f \in V$, this interpolation problem has a unique solution $u \in V$. It turns out that the mapping $f \mapsto u$ is linear and that there holds the *a priori* bound $|u|_a \le |f|_a$, i.e., the problem is well-posed.

2.2 The fundamental solution

At first glance, for given $f \in V$, the solution $u \in V$ of Problem 4 looks like an infinite-dimensional object. However, since the set of interpolants $\tilde{u} \in V$ of f is so large, the minimization property contains a lot of information about u. In fact, we shall shortly see that u can be written in the form $u = \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n + \sum_{|\alpha|<k_{\min}} \boldsymbol{d}_\alpha \pi_\alpha$, where $\boldsymbol{c}_n, \boldsymbol{d}_\alpha \in \mathbb{R}$ are certain coefficients and where $\{\pi_\alpha \mid |\alpha| < k_{\min}\} \subseteq P$ is a polynomial basis. The functions $\phi_n \in C^0(\mathbb{R}^d)$, on the other hand, have the particularly simple structure of translates, i.e., $\phi_n = \phi(\cdot - x_n)$. Here, $x_n \in \mathbb{R}^d$ is the n-th interpolation point and $\phi \in C^0(\mathbb{R}^d)$ is a specific function that is intimately linked to the native space $(V, a(\cdot,\cdot))$ from Definition 1. More precisely, ϕ is a fundamental solution of the differential operator that is associated with the bilinear form $a(\cdot,\cdot)$. The above representation will then allow us to rephrase Problem 4 as a linear system of equations (LSE) for the coefficients $\boldsymbol{c}_n, \boldsymbol{d}_\alpha \in \mathbb{R}$.

Before we treat the general setting $k_{\min} \ge 0$, let us have a look at the much simpler case $k_{\min} = 0$ first. Recall that then $V = H^k(\mathbb{R}^d)$ with equivalent norms.

Lemma 5 *If $k_{\min} = 0$, then there exists a unique function $\phi \in V$, such that the following equality holds true:*

$$\forall x_0 \in \mathbb{R}^d : \forall v \in V : \qquad a(\phi(\cdot - x_0), v) = v(x_0).$$

Proof The continuous Sobolev embedding $H^k(\mathbb{R}^d) \subseteq C^0(\mathbb{R}^d)$ guarantees that the linear form $v \mapsto v(0)$ is continuous. Then, according to the Riesz-Fréchet Representation Theorem, there exists a unique function $\phi \in V$ such that $a(\phi, v) = v(0)$ for all $v \in V$. Since the coefficients $\sigma_l l!/\alpha!$ of $a(\cdot,\cdot)$ are spatially constant, a simple integral transformation yields the desired formula for general $x_0 \in \mathbb{R}^d$. □

For $k_{\min} = 0$ and given data $f \in V$, we make the ansatz $u := \sum_{n=1}^{N} \boldsymbol{c}_n \phi(\cdot - x_n) \in V$ for the solution of Problem 4. The coefficients $\boldsymbol{c}_n \in \mathbb{R}$ can be chosen such that the interpolation conditions $E_N u = E_N f$ are satisfied (cf. Lemma 30). On the other hand, for all $v \in V$ with $E_N v = \mathbf{0}$, the defining equation of ϕ tells us that $a(u, v) = \sum_{n=1}^{N} \boldsymbol{c}_n v(x_n) = 0$. In Lemma 25 further below, it will be argued in more detail that this orthogonality implies the required minimization property. We conclude that $u = \sum_{n=1}^{N} \boldsymbol{c}_n \phi(\cdot - x_n)$ is indeed the unique solution of Problem 4.

Now let us return to the general case $k_{\min} \ge 0$. Since $a(\cdot,\cdot)$ need not be *strictly* positive definite, the existence of a "basis function" ϕ is not so straightforward any more. Therefore, we have to take a different approach.

Definition 6 *Denote by* $\sigma_{k_{\min}}, \ldots, \sigma_k \geq 0$ *the coefficients of the bilinear form* $a(\cdot, \cdot)$ *from Definition 1. We define a differential operator* $\mathrm{L}^{2k} : C_0^\infty(\mathbb{R}^d) \longrightarrow C_0^\infty(\mathbb{R}^d)$ *via*

$$\mathrm{L}^{2k} v := \sum_{l=k_{\min}}^{k} \sigma_l (-\Delta)^l v.$$

The precise relationship between $a(\cdot, \cdot)$ and L^{2k} is described in the subsequent lemma. Anticipating Lemma 20, we claim that there holds $C_0^\infty(\mathbb{R}^d) \subseteq V$, so we can plug any given $v \in C_0^\infty(\mathbb{R}^d)$ into $a(\cdot, \cdot)$. Furthermore, we will prove that every $u \in V$ lies in $H^k_{\mathrm{loc}}(\mathbb{R}^d)$, so that the integrals in $a(u, v)$ are amenable to successive partial integrations over bounded subsets of $\mathbb{R}^d$.

Lemma 7 *For all* $u \in V$ *and* $v \in C_0^\infty(\mathbb{R}^d)$*, there holds the identity*

$$a(u, v) = \int_{\mathbb{R}^d} u(\mathrm{L}^{2k} v) \,\mathrm{d}x.$$

Proof The relation follows readily from successive partial integrations and the identity $\sum_{|\alpha|=l} (l!/\alpha!) \mathrm{D}^{2\alpha} v = \Delta^l v$. The compact support of $v \in C_0^\infty(\mathbb{R}^d)$ and the fact that $\mathrm{D}^\alpha u \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ for all $|\alpha| \leq k$ guarantee that all integrals involved in the computation are well-defined. □

Definition 8 *We define the set of coefficient vectors* $\boldsymbol{C} := \{\boldsymbol{c} \in \mathbb{R}^N \mid \forall p \in P : \langle \boldsymbol{c}, E_N p \rangle_2 = 0\}$.

A careful analysis of the function ϕ from the case $k_{\min} = 0$ above leads us to the following set of assumptions:

Assumption 9 *We assume that the coefficients* $\sigma_{k_{\min}}, \ldots, \sigma_k \geq 0$ *from Definition 1 are such that there exists a function* $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ *with the following properties:*

1. Regularity: *There holds* $\phi \in C^0(\mathbb{R}^d)$.
2. Fundamental solution: ϕ *is a fundamental solution of* L^{2k}*, i.e.,*

$$\forall x_0 \in \mathbb{R}^d : \forall v \in C_0^\infty(\mathbb{R}^d) : \quad \int_{\mathbb{R}^d} \phi(x - x_0)(\mathrm{L}^{2k} v)(x) \,\mathrm{d}x = v(x_0).$$

3. Conformity: *For all* $\boldsymbol{c} \in \boldsymbol{C}$*, there holds* $\sum_{n=1}^N \boldsymbol{c}_n \phi(\cdot - x_n) \in V$.

Note that we *did not* assume that ϕ lies in the native space V. Instead, we only require certain linear combinations of its translates to do so. Combining Lemma 7 with (2) and (3), we obtain the important orthogonality $a(u, v) = \sum_{n=1}^N \boldsymbol{c}_n v(x_n) = 0$ for all u of the form $u = \sum_{n=1}^N \boldsymbol{c}_n \phi(\cdot - x_n)$, $\boldsymbol{c} \in \boldsymbol{C}$, and for all $v \in C_0^\infty(\mathbb{R}^d)$ with $E_N v = \mathbf{0}$. Using a density argument, the orthogonality can then be extended to the space of functions $v \in V$ with $E_N v = \mathbf{0}$. A detailed proof of these facts will be given later in Lemma 28.

In contrast to the case $k_{\min} = 0$, a function ϕ satisfying Assumption 9 need not be unique if $k_{\min} > 0$. In fact, if ϕ is a valid choice and if $p \in P$, then $\phi + p$ is valid as well.

According to the much celebrated Malgrange-Ehrenpreis Theorem, [39], [19], every non-trivial differential operator with constant coefficients has a fundamental solution in the distributional sense. Since $\mathrm{L}^{2k} = \sum_{l=k_{\min}}^{k} \sigma_l (-\Delta)^l$ falls into this category, this theorem provides us with a distribution $\phi \in (C_0^\infty(\mathbb{R}^d))'$ that satisfies assumption (2) in the distributional sense. However, to the best of our knowledge, there is no guarantee that this ϕ satisfies (1) and (3), if no further assumptions on the coefficients σ_l are made.

Lemma 5 is somewhat unsatisfactory, since it does not provide an explicit formula for the basis function ϕ. However, for a specific choice of the coefficients $\sigma_0, \dots, \sigma_k$, we have the following result, the proof of which we postpone to Section 3.11.

Lemma 10 *Let $b \in (0, \infty)$. Consider the case where $k_{\min} := 0$ and $\sigma_l := \binom{k}{l} b^{2(k-l)} > 0$ for all $l \in \{0, \dots, k\}$. The differential operator from Definition 6 then takes the form*

$$\mathrm{L}^{2k} v = \sum_{l=0}^{k} \binom{k}{l} b^{2(k-l)} (-\Delta)^l v = (b^2 - \Delta)^k v.$$

Define the function

$$\forall x \in \mathbb{R}^d: \qquad \phi(x) := \frac{(4\pi)^{-d/2}}{\Gamma(k)} \int_0^\infty t^{k-d/2-1} e^{-b^2 t} e^{-\|x\|_2^2/(4t)} \,\mathrm{d}t.$$

Then, ϕ satisfies Assumption 9.

We mention that ϕ can also be written in the form

$$\phi(x) = \frac{(2\pi)^{-d/2}}{2^{k-1}\Gamma(k)} \left(\frac{\|x\|_2}{b}\right)^{k-d/2} K_{k-d/2}(b\|x\|_2),$$

which goes by the name of *Matérn function*, *Sobolev spline* or *Bessel potential* in the literature (e.g., [3]). Here, $K_\nu(r) := \int_0^\infty e^{-r\cosh(s)} \cosh(\nu s)\,\mathrm{d}s$ is the well-known *modified Bessel function of the second kind*. To see the identity, one writes $K_\nu(r) = \int_{-\infty}^{\infty} e^{-r(e^s+e^{-s})/2} e^{\nu s}/2\,\mathrm{d}s$, plugs in $\nu = k - d/2$ and $r = b\|x\|_2$, and then substitutes $s = \ln(2bt/\|x\|_2)$. Furthermore, in the case where $d \in \{1, 3, 5, \dots\}$, there holds the representation

$$\phi(x) = \frac{(4\pi)^{(1-d)/2}}{\Gamma(k)(2b)^{2L+1}} \sum_{l=0}^{L} \frac{(2L-l)!}{l!(L-l)!} (2b\|x\|_2)^l e^{-b\|x\|_2}, \qquad L := k - d/2 - 1/2 \in \mathbb{N}_0.$$

This follows easily from the known identity $K_{L+1/2}(r) = \pi^{1/2}e^{-r}\sum_{l=0}^{L}(L+l)!/(l!(L-l)!(2r)^{l+1/2})$, which can be found, e.g., in [27, Page 925].

We finish this section with another example for a function ϕ satisfying Assumption 9. This time, we look at the other extreme case, $k_{\min} = k$. The proof will be delayed to Section 3.11 again.

Lemma 11 *Consider the case where $k_{\min} = k$ and $\sigma_{k_{\min}} = \sigma_k = 1$. The differential operator from Definition 6 then reads*

$$\mathrm{L}^{2k} v = (-\Delta)^k v.$$

Define the thin-plate spline

$$\begin{aligned} (d \in \{1,3,5,\ldots\})\ \phi(x) &:= C_1 \|x\|_2^{2k-d}, \\ (d \in \{2,4,6,\ldots\})\ \phi(x) &:= C_2 \|x\|_2^{2k-d} \ln \|x\|_2, \end{aligned}$$

where $C_1 := \Gamma(d/2-k)/(4^k \pi^{d/2}(k-1)!)$ and $C_2 := (-1)^{k+(d-2)/2}/(2^{2k-1}\pi^{d/2}(k-1)!(k-d/2)!)$. Then, ϕ satisfies Assumption 9.

2.3 The linear system of equations (LSE)

We already suggested that the solution $u \in V$ of Problem 4 can be written in the form $u = \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n + \sum_{|\alpha|<k_{\min}} \boldsymbol{d}_\alpha \pi_\alpha$, where the coefficients $\boldsymbol{c}_n, \boldsymbol{d}_\alpha \in \mathbb{R}$ satisfy a certain linear system of equations (LSE). The corresponding *interpolation matrix* is the main object of interest of the present work.

Definition 12 *Denote by $x_1, \ldots, x_N \in \mathbb{R}^d$ the interpolation points from Definition 2 and by $\phi \in C^0(\mathbb{R}^d)$ the fundamental solution from Assumption 9. For every $n \in \{1, \ldots, N\}$, we define the translate*

$$\phi_n := \phi(\cdot - x_n) \in C^0(\mathbb{R}^d).$$

Furthermore, let $\{\pi_\alpha \mid |\alpha| < k_{\min}\} \subseteq P$ be the Lagrange basis associated with the unisolvent point set $\{\xi_\alpha \mid |\alpha| < k_{\min}\} \subseteq \mathbb{R}^d$ from Definition 2, i.e., $\pi_\beta(\xi_\alpha) = \delta_{\alpha\beta}$ (Kronecker δ) for all $|\alpha|, |\beta| < k_{\min}$.

Definition 13 *Recalling $N_{\min} = \dim P$, we define the following matrices:*

$$\boldsymbol{A} := (\phi_n(x_m))_{m,n=1}^{N} \in \mathbb{R}^{N\times N}, \qquad \boldsymbol{B} := (\pi_\beta(x_n))_{|\beta|<k_{\min}, n\in\{1,\ldots,N\}} \in \mathbb{R}^{N_{\min}\times N}.$$

Furthermore, we define the interpolation matrix

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{pmatrix} \in \mathbb{R}^{(N+N_{\min})\times(N+N_{\min})}.$$

As will be shown later in Lemma 29, the interpolation matrix is invertible. Now, let $f \in V$ and set $\boldsymbol{f} := E_N f \in \mathbb{R}^N$. Denote by $\boldsymbol{c} \in \mathbb{R}^N$ and $\boldsymbol{d} \in \mathbb{R}^{N_{\min}}$ the unique solution of the following LSE:

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{pmatrix}.$$

Then, the unique solution $u \in V$ of Problem 4 can be written in the following form:

$$u = \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n + \sum_{|\alpha|<k_{\min}} \boldsymbol{d}_\alpha \pi_\alpha.$$

Once again, we postpone the derivation of this identity to the proof section below, Lemma 30. The first row of the LSE encodes the interpolation conditions $E_N u = E_N f$, whereas the second row guarantees that the coefficient vector $\boldsymbol{c} \in \mathbb{R}^N$ lies in the set $\boldsymbol{C}$ from Definition 8. Owing to Assumption 9, this implies the conformity $\sum_{n=1}^{N} \boldsymbol{c}_n \phi_n \in V$.

2.4 Hierarchical matrices

An extensive discussion of hierarchical matrices can be found in the books [5, 12, 31]. Here, we only provide a bare minimum of definitions that are need for the subsequent analysis.

Definition 14 *We define the* bubbles

$$\forall n \in \{1, \ldots, N\}: \qquad \Omega_n := \mathrm{Ball}(x_n, h_{\min}) \subseteq \mathbb{R}^d.$$

Similarly, for every subset $I \subseteq \{1, \ldots, N\}$, we set $\Omega_I := \bigcup_{n \in I} \Omega_n \subseteq \mathbb{R}^d$.

Note that the definition of the separation distance $h_{\min}$ in Definition 2 guarantees that the bubbles Ω_n are pairwise disjoint, i.e., $\Omega_n \cap \Omega_m = \emptyset$ whenever $n \neq m$.

Definition 15 *A subset $B \subseteq \mathbb{R}^d$ is called* (axes-parallel) box, *if it has the form $B = \times_{i=1}^{d} (a_i, b_i)$ with $a_i \leq b_i$.*

Definition 16 *Let $\sigma_{\mathrm{small}}, \sigma_{\mathrm{adm}} > 0$. A tuple (I, J) with $I, J \subseteq \{1, \ldots, N\}$ is called* small, *if there holds $\min\{\#I, \#J\} \leq \sigma_{\mathrm{small}}$. It is called* admissible, *if there exist boxes $B_I, B_J \subseteq \mathbb{R}^d$ such that $\Omega_I \subseteq B_I$, $\Omega_J \subseteq B_J$ and*

$$\mathrm{diam}_2(B_I) \leq \sigma_{\mathrm{adm}} \mathrm{dist}_2(B_I, B_J).$$

A set $\mathbb{P}$ of tuples (I, J) with $I, J \subseteq \{1, \ldots, N\}$ is called sparse hierarchical block partition, *if the following assumptions are satisfied:*

1. *The system $\{I \times J \mid (I, J) \in \mathbb{P}\}$ forms a partition of $\{1, \ldots, N\} \times \{1, \ldots, N\}$.*

2. *There holds* $\mathbb{P} = \mathbb{P}_{\text{small}} \mathbin{\dot\cup} \mathbb{P}_{\text{adm}}$, *where every* $(I, J) \in \mathbb{P}_{\text{small}}$ *is small and every* $(I, J) \in \mathbb{P}_{\text{adm}}$ *is admissible.*
3. *There exists a constant* $C > 0$, *such that*

$$\forall \boldsymbol{M} \in \mathbb{R}^{N\times N}: \qquad \|\boldsymbol{M}\|_2 \leq C \ln(N) \max_{(I,J)\in\mathbb{P}} \|\boldsymbol{M}|_{I\times J}\|_2.$$

While items (1) and (2) are more or less standard in the literature on hierarchical matrices, item (3) might seem odd to the informed reader. Usually, this inequality is proved, rather than assumed (e.g., [31, Lemma 6.5.8]). However, its proof typically requires a rigorous introduction of *(hierarchical) cluster trees* and *(hierarchical) block cluster trees*, two types of data structures that are described in great detail in [31, Chapter 5]. Here, we largely avoid this tedious task and only give a brief overview:

In essence, a *(hierarchical) cluster tree* $\mathbb{T}_N$ is a system of index *clusters* $I \subseteq \{1, \ldots, N\}$ that contains the full index set $\{1, \ldots, N\}$ as its root. Furthermore, for every $I \in \mathbb{T}_N$ with $\#I > \sigma_{\text{small}}$, the cluster tree also contains two non-empty, disjoint clusters I_1, I_2 with $I = I_1 \cup I_2$. Starting at the tree root, these *sons* are typically generated by a predefined *clustering strategy* that takes into account the geometric positions of the interpolation points $x_1, \ldots, x_N \in \mathbb{R}^d$. An example for a *geometrically balanced clustering strategy*, which uses a hierarchy of axes-parallel boxes $B_I \subseteq \mathbb{R}^d$, can be found in [26]. Finally, we denote by $\operatorname{depth}(\mathbb{T}_N) \in \mathbb{N}$ the tree's *depth*, and assume that the reader is familiar with this notion.

On the other hand, a *(hierarchical) block cluster tree* $\mathbb{T}_{N\times N}$ consists of tuples (I, J), where I and J are clusters from a given cluster tree $\mathbb{T}_N$. This time, the tuple $(\{1, \ldots, N\}, \{1, \ldots, N\})$ is the tree's root and, for every $(I, J) \in \mathbb{T}_{N\times N}$ with $\operatorname{diam}_2(B_I) > \sigma_{\text{adm}} \operatorname{dist}_2(B_I, B_J)$, all four pairs of sons (I_1, J_1), (I_1, J_2), (I_2, J_1), (I_2, J_2) lie in $\mathbb{T}_{N\times N}$. Here, the sets $B_I, B_J \subseteq \mathbb{R}^d$ are the boxes that were used to build the cluster tree $\mathbb{T}_N$. Finally, the well-known *sparsity constant* $\sigma_{\text{sparse}}(\mathbb{T}_{N\times N}) \in \mathbb{N}$ counts the maximum number of "partners" $(I, J) \in \mathbb{T}_{N\times N}$ that each cluster I or J can have (see, e.g., [31, Section 6.3]).

The geometrically balanced clustering strategy from [26] guarantees the bounds $\operatorname{depth}(\mathbb{T}_N) \lesssim \ln(h_{\min}^{-1})$ and $\sigma_{\text{sparse}}(\mathbb{T}_{N\times N}) \lesssim 1$. We plug in the relation $1 \lesssim N^{\sigma_{\text{card}}} h_{\min}^d$ from Definition 2 and get $\operatorname{depth}(\mathbb{T}_N) \lesssim \ln(N)$. Now, using [31, Lemma 6.5.8], we obtain the following inequality:

$$\begin{aligned} \forall \boldsymbol{M} \in \mathbb{R}^{N\times N}: \qquad \|\boldsymbol{M}\|_2 &\leq \sigma_{\text{sparse}}(\mathbb{T}_{N\times N}) \operatorname{depth}(\mathbb{T}_N) \max_{(I,J)\in\mathbb{P}} \|\boldsymbol{M}|_{I\times J}\|_2 \\ &\lesssim \ln(N) \max_{(I,J)\in\mathbb{P}} \|\boldsymbol{M}|_{I\times J}\|_2. \end{aligned}$$

These considerations justify the assumptions that we made in Definition 16.

Definition 17 *Let* $\mathbb{P}$ *be a sparse hierarchical block partition and* $r \in \mathbb{N}$ *a given* block rank bound. *We define the set of* $\mathcal{H}$-matrices *by*

$$\mathcal{H}(\mathbb{P}, r) := \{\boldsymbol{M} \in \mathbb{R}^{N\times N} \mid \forall (I, J) \in \mathbb{P}_{\text{adm}}: \exists \boldsymbol{X} \in \mathbb{R}^{I\times r}, \boldsymbol{Y} \in \mathbb{R}^{J\times r}: \boldsymbol{M}|_{I\times J} = \boldsymbol{X}\boldsymbol{Y}^T\}.$$

We mention that, according to [31, Lemma 6.3.6], the memory requirements to store an $\mathcal{H}$-matrix $\boldsymbol{M} \in \mathcal{H}(\mathbb{P}, r)$ can be bounded by $\sigma_{\text{sparse}}(\mathbb{T}_{N\times N})(\sigma_{\text{small}}+r)\text{depth}(\mathbb{T}_N)N$. Inserting the relations $\sigma_{\text{sparse}}(\mathbb{T}_{N\times N}) \lesssim 1$ and $\text{depth}(\mathbb{T}_N) \lesssim \ln(N)$ from before, we get an overall bound of $\mathcal{O}(r\ln(N)N)$ for the storage complexity.

2.5 The main result

We are finally in the position to formulate our main result.

Theorem 18 *Let $a(\cdot,\cdot)$ be the bilinear form from Definition 1 and consider a set of interpolation points $\{x_1,\dots,x_N\} \subseteq \mathbb{R}^d$ satisfying Definition 2. Denote by $\phi \in C^0(\mathbb{R}^d)$ the fundamental solution from Assumption 9. Furthermore, let $\left(\begin{smallmatrix}\boldsymbol{A} & \boldsymbol{B}^T\\ \boldsymbol{B} & \boldsymbol{0}\end{smallmatrix}\right)$ be the interpolation matrix from Definition 13. Finally, let $\mathbb{P}$ be a sparse hierarchical block partition as in Definition 16. Write the inverse of the interpolation matrix in the block form $\left(\begin{smallmatrix}\boldsymbol{A} & \boldsymbol{B}^T\\ \boldsymbol{B} & \boldsymbol{0}\end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix}\boldsymbol{S_{11}} & \boldsymbol{S_{12}}\\ \boldsymbol{S_{21}} & \boldsymbol{S_{22}}\end{smallmatrix}\right)$ with matrices $\boldsymbol{S_{11}} \in \mathbb{R}^{N\times N}$, $\boldsymbol{S_{21}} \in \mathbb{R}^{N_{\min}\times N}$, $\boldsymbol{S_{12}} \in \mathbb{R}^{N\times N_{\min}}$ and $\boldsymbol{S_{22}} \in \mathbb{R}^{N_{\min}\times N_{\min}}$. Then, there exists a constant $\sigma_{\exp} > 0$ such that the following holds true: For every block rank bound $r \in \mathbb{N}$, there exists an $\mathcal{H}$-matrix $\boldsymbol{M} \in \mathcal{H}(\mathbb{P}, r)$ such that*

$$\|\boldsymbol{S_{11}} - \boldsymbol{M}\|_2 \lesssim \ln(N)N^{\sigma_{\text{card}}(3k-d)/d}\exp(-\sigma_{\exp}r^{1/(d+1)}).$$

We note that the previous theorem only provides a low rank approximation to the principal subblock $\boldsymbol{S_{11}}$ of the inverse of the interpolation matrix. As $N_{\min} := \dim P = \binom{d+k_{\min}-1}{d}$ is independent of the number of interpolation points and, typically, small, the subblocks $\boldsymbol{S_{21}}, \boldsymbol{S_{12}}, \boldsymbol{S_{22}}$ can already be stored efficiently.

3 Proof of main results

3.1 The native space *V*

We start the proof of the main result with a discussion of the native space V from Definition 1. Given a function $v \in V$ and a multi-index $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \in \{k_{\min},\dots,k\}$, we know that there exists an α-th weak derivative $\mathrm{D}^\alpha v \in L^1_{\text{loc}}(\mathbb{R}^d)$ that happens to lie in $L^2(\mathbb{R}^d)$. Naively, the definition of V does not tell us anything about the existence and regularity of lower-order derivatives $\mathrm{D}^\alpha v \in L^1_{\text{loc}}(\mathbb{R}^d)$, where $|\alpha| < k_{\min}$. However, as shown below, it turns out that these lower-order derivatives do exist and lie in $L^2_{\text{loc}}(\mathbb{R}^d)$.

Our proof of this fact uses a Poincaré-type inequality of the form $\|\cdot\|_{H^k(\Omega)} \lesssim |\cdot|_{H^k(\Omega)} + \|\cdot\|_{L^1(\Omega)}$, where $\Omega \subseteq \mathbb{R}^d$ is a ball. The following, slightly more general result, will cover all the occurrences of Poincaré-type inequalities in the present work.

Lemma 19 *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Let $k \in \mathbb{N}_0$ and Z be a normed vector space. Furthermore, let $\iota_Z : H^k(\Omega) \longrightarrow Z$ be a linear continuous operator* *satisfying the implication* ($\iota_Z p = 0 \Rightarrow p = 0$) *for every* $p \in \mathbb{P}^{k-1}(\Omega)$. *Then, there holds the following Poincaré-type inequality:*

$$\forall v \in H^k(\Omega): \qquad \|v\|_{H^k(\Omega)} \leq C(d, k, \Omega, Z, \iota_Z)(|v|_{H^k(\Omega)} + \|\iota_Z v\|_Z).$$

Proof This can be proved by contradiction using arguments similar to [20, Section 5.8.1.]. □

Now let us return to the question of existence of the lower-order derivatives $\mathrm{D}^\alpha v$, $|\alpha| < k_{\min}$, if $v \in V$ is given. Apart from a Poincaré inequality, the subsequent proof sports a standard mollification argument with $C_0^\infty(\mathbb{R}^d)$-functions, as can be found in many textbooks treating Sobolev spaces (e.g., [30], [20]). In brief, given a "standard mollifier" $\mu \in C_0^\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \mu(x)\,\mathrm{d}x = 1$, one defines a sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^d)$ via $\mu_n(x) := n^d \mu(nx)$. Then, for every function $v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, one can show that $\mu_n * v \in C^\infty(\mathbb{R}^d)$ with $\mathrm{D}^\alpha(\mu_n * v) = (\mathrm{D}^\alpha \mu_n) * v$ for all $\alpha \in \mathbb{N}_0^d$. Additionally, if v has an α-th weak derivative $\mathrm{D}^\alpha v \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for some $\alpha \in \mathbb{N}_0^d$, and if there holds $(\mathrm{D}^\alpha v)|_\Omega \in L^p(\Omega)$ for some $p \in [1, \infty)$ and some open subset $\Omega \subseteq \mathbb{R}^d$, then one can prove that $(\mathrm{D}^\alpha(\mu_n * v))|_\Omega \in L^p(\Omega)$ and that $\|\mathrm{D}^\alpha v - \mathrm{D}^\alpha(\mu_n * v)\|_{L^p(\Omega)} \overset{n}{\to} 0$.

Lemma 20

1. *The function* $a(\cdot,\cdot)$ *defines a symmetric, positive semi-definite bilinear form on* V *and* $|\cdot|_a$ *defines a seminorm.*
2. *There holds* $P \subseteq V$.
3. *Let* $u, v \in V$ *such that* $u \in P$ *or* $v \in P$. *Then,* $a(u, v) = 0$.
4. *For all* $v \in V$, *there holds* $|v|_a = 0$ *if and only if* $v \in P$.
5. *For all* $u, v \in V$, *there holds the Cauchy-Schwarz inequality* $|a(u, v)| \leq |u|_a |v|_a$.
6. *There hold the following inclusions (as sets):*

$$C_0^\infty(\mathbb{R}^d) \subseteq H^k(\mathbb{R}^d) \subseteq V \subseteq H^k_{\mathrm{loc}}(\mathbb{R}^d) \subseteq C^0(\mathbb{R}^d).$$

   In particular, every $v \in V$ *is* k*-times weakly differentiable and there holds* $\mathrm{D}^\alpha v \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ *for all* $|\alpha| < k_{\min}$ *as well as* $\mathrm{D}^\alpha v \in L^2(\mathbb{R}^d)$ *for all* $|\alpha| \in \{k_{\min}, \ldots, k\}$.
7. *For all* $v \in V$, *there hold the bounds*

$$|v|_a \lesssim \sum_{l=k_{\min}}^{k} |v|_{H^l(\mathbb{R}^d)} \lesssim |v|_{H^{k_{\min}}(\mathbb{R}^d)} + |v|_{H^k(\mathbb{R}^d)} \lesssim |v|_a.$$

Proof Items (1) – (5) and the inclusions $C_0^\infty(\mathbb{R}^d) \subseteq H^k(\mathbb{R}^d) \subseteq V$ from (6) are elementary. The inclusion $H^k_{\mathrm{loc}}(\mathbb{R}^d) \subseteq C^0(\mathbb{R}^d)$ follows from $k > d/2$ and a well-known Sobolev embedding. Item (7) follows from the well-known characterization of Sobolev spaces by Fourier transforms (e.g., [20, Section 5.8.4.]). Finally, let us sketch the proof of $V \subseteq H^k_{\mathrm{loc}}(\mathbb{R}^d)$: Given $v \in V$, we define the approximants $v_n := \mu_n * v \in C^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, where $(\mu_n)_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^d)$ is the sequence of mollifiers from above. We then consider a sequence of bounded Lipschitz domains $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots \subseteq \mathbb{R}^d$, say, $B_i := \mathrm{Ball}(0, i)$. For fixed $i \in \mathbb{N}$, we can use Lemma 19, the convergence

$\|v - v_n\|_{L^1(B_i)} + |v - v_n|_{H^k(B_i)} \xrightarrow{n} 0$, and a Cauchy sequence argument to prove that $v|_{B_i} \in H^k(B_i)$. One can then show that the global functions $\mathrm{D}^\alpha v : \mathbb{R}^d \longrightarrow \mathbb{R}$, $|\alpha| \leq k$, defined via $(\mathrm{D}^\alpha v)|_{B_i} := \mathrm{D}^\alpha(v|_{B_i})$ lie in $L^2_{\text{loc}}(\mathbb{R}^d)$ and represent the α-th weak derivative of v. In other words, v must lie in $H^k_{\text{loc}}(\mathbb{R}^d)$. □

The next lemma establishes the fact that $C_0^\infty(\mathbb{R}^d) \subseteq V$ is dense. We remind the reader of Definition 8, where the set $\boldsymbol{C} \subseteq \mathbb{R}^d$ was introduced.

Lemma 21 *The subset $C_0^\infty(\mathbb{R}^d) \subseteq V$ is dense in the following sense: For every $v \in V$, there exists a sequence $(v_n)_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^d)$ such that $|v - v_n|_a \xrightarrow{n} 0$ as well as $|\langle \boldsymbol{c}, E_N(v - v_n)\rangle_2| \xrightarrow{n} 0$ for all $\boldsymbol{c} \in \boldsymbol{C}$.*

Proof A straightforward mollification argument shows that $V \cap C^\infty(\mathbb{R}^d) \subseteq V$ is dense in the stated sense. Therefore, it suffices to show that $C_0^\infty(\mathbb{R}^d)$ is dense in $V \cap C^\infty(\mathbb{R}^d)$. First, we prove the case where $d \geq 2$: Let $v \in V \cap C^\infty(\mathbb{R}^d)$ and $\kappa \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with $\operatorname{supp}(\kappa) \subseteq B_2$ and $\kappa \equiv 1$ on B_1, where $B_n := \operatorname{Ball}(0, n) \subseteq \mathbb{R}^d$ denotes the ball with integer radius $n \in \mathbb{N}$, centered at the origin. Clearly, the scaled version $\kappa_n := \kappa(\cdot/n) \in C_0^\infty(\mathbb{R}^d)$ satisfies $\operatorname{supp}(\kappa_n) \subseteq B_{2n}$ and $\kappa_n \equiv 1$ on B_n. Furthermore, $|1 - \kappa_n|_{W^{l,\infty}(\mathbb{R}^d)} \leq |1|_{W^{l,\infty}(\mathbb{R}^d)} + |\kappa_n|_{W^{l,\infty}(\mathbb{R}^d)} \lesssim n^{-l}$ for all $l \in \mathbb{N}_0$. Next, denote by $\hat{A} := B_2 \backslash B_1$ the "reference annulus" and let $\hat{\Pi} : H^{k_{\min}}(\hat{A}) \longrightarrow \mathbb{P}^{k_{\min}-1}(\hat{A})$ be the corresponding orthogonal projection. Let $F_n : \hat{A} \longrightarrow B_{2n} \backslash B_n$, $F_n(x) := nx$, and consider the polynomial $p_n := \hat{\Pi}(v \circ F_n) \circ F_n^{-1} \in P$ (implicitly extended to $\mathbb{R}^d$). Since we are in the case $d \geq 2$ at the moment, the annulus $\hat{A}$ is connected and we can apply the Poincaré-type inequality from Lemma 19 to the bounded Lipschitz domain $\Omega = \hat{A}$, the normed vector space $Z = H^{k_{\min}}(\hat{A})$, and the continuous mapping $\iota_Z = \hat{\Pi} : H^{k_{\min}}(\hat{A}) \longrightarrow H^{k_{\min}}(\hat{A})$. Then, using a standard scaling argument, we find that

$$\sum_{j=0}^{k_{\min}} n^j |v - p_n|_{H^j(B_{2n}\backslash B_n)} \lesssim n^{d/2} \|v \circ F_n - \hat{\Pi}(v \circ F_n)\|_{H^{k_{\min}}(\hat{A})} \lesssim n^{d/2} |v \circ F_n|_{H^{k_{\min}}(\hat{A})}$$
$$= n^{k_{\min}} |v|_{H^{k_{\min}}(B_{2n}\backslash B_n)}.$$

For every $n \in \mathbb{N}$, we define the approximation $v_n := \kappa_n(v - p_n) \in C_0^\infty(\mathbb{R}^d)$. Exploiting that $\operatorname{supp}(\mathrm{D}^\alpha \kappa_n) \subseteq B_{2n} \backslash B_n$ for all $\alpha \neq 0$ and that $\operatorname{supp}(1 - \kappa_n) \subseteq \mathbb{R}^d \backslash B_n$, we estimate

$$|v - v_n|_a \lesssim \sum_{l=k_{\min}}^{k} |v - v_n|_{H^l(\mathbb{R}^d)} \overset{p_n \in P}{=} \sum_{l=k_{\min}}^{k} |(1 - \kappa_n)(v - p_n)|_{H^l(\mathbb{R}^d)}$$
$$\overset{\text{Leibniz}}{\lesssim} \sum_{l=k_{\min}}^{k} \Bigg(\sum_{j=0}^{k_{\min}-1} |\kappa_n|_{W^{l-j,\infty}(\mathbb{R}^d)} |v - p_n|_{H^j(B_{2n}\backslash B_n)}$$

$$
\begin{aligned}
&\quad + \sum_{j=k_{\min}}^{l} |1-\kappa_n|_{W^{l-j,\infty}(\mathbb{R}^d)} |v|_{H^j(\mathbb{R}^d \setminus B_n)} \Big) \\
&\lesssim \sum_{l=k_{\min}}^{k} \Big(n^{k_{\min}-l} |v|_{H^{k_{\min}}(B_{2n}\setminus B_n)} + \sum_{j=k_{\min}}^{l} n^{j-l} |v|_{H^j(\mathbb{R}^d\setminus B_n)} \Big) \\
&\lesssim \sum_{l=k_{\min}}^{k} |v|_{H^l(\mathbb{R}^d\setminus B_n)} \xrightarrow{n} 0.
\end{aligned}
$$

The convergence in the last step follows from the fact that $\sum_{l=k_{\min}}^{k} |v|_{H^l(\mathbb{R}^d)} < \infty$ since $v \in V$. Finally, let $\boldsymbol{c} \in \boldsymbol{C}$. Clearly, there exists an index $n_0 \in \mathbb{N}$, such that $\{x_1, \ldots, x_N\} \subseteq B_n$ for all $n \geq n_0$. Since $\kappa_n \equiv 1$ on B_n, we get the following identity:

$$
\forall n \geq n_0 : \qquad \langle \boldsymbol{c}, E_N(v - v_n)\rangle_2 = \langle \boldsymbol{c}, E_N(v - \kappa_n(v - p_n))\rangle_2 = \langle \boldsymbol{c}, E_N p_n \rangle_2 = 0.
$$

This settles the question of density in the case $d \geq 2$. We turn our attention to the case $d = 1$: Since the reference annulus $\hat{A} = (-2, -1) \cup (1, 2)$ is not connected, we do not have the Poincaré inequality at our disposal. Instead, we show that the inclusions $C_0^\infty(\mathbb{R}) \subseteq \{v \in C^\infty(\mathbb{R}) \mid v^{(k_{\min})} \in C_0^\infty(\mathbb{R})\} \subseteq V \cap C^\infty(\mathbb{R})$ are both dense. To see the latter one, let $v \in V \cap C^\infty(\mathbb{R})$ and consider the Taylor-type approximants

$$
v_n(x) := \sum_{l=0}^{k_{\min}-1} \frac{v^{(l)}(0)}{l!} x^l + \frac{1}{(k_{\min}-1)!} \int_0^x (x-t)^{k_{\min}-1} \kappa_n(t) v^{(k_{\min})}(t) \, \mathrm{d}t,
$$

where $\kappa_n \in C_0^\infty(\mathbb{R})$ is the smooth cut-off function from before. Clearly, $v_n \in C^\infty(\mathbb{R})$ and $v_n^{(k_{\min})} = \kappa_n v^{(k_{\min})} \in C_0^\infty(\mathbb{R})$, meaning that v_n lies in the supposedly dense subset. To bound the error $|v - v_n|_a$, we apply Leibniz' differentiation rule to the product $\kappa_n v^{(k_{\min})}$ and exploit the support properties of κ_n. Skipping the steps that are similar to the multivariate case, we obtain

$$
\begin{aligned}
|v - v_n|_a &\lesssim \sum_{l=k_{\min}}^{k} \|v^{(l)} - v_n^{(l)}\|_{L^2(\mathbb{R})} = \sum_{l=k_{\min}}^{k} \|v^{(l)} - (\kappa_n v^{(k_{\min})})^{(l-k_{\min})}\|_{L^2(\mathbb{R})} \\
&\lesssim \cdots \lesssim \sum_{l=k_{\min}}^{k} |v|_{H^l(\mathbb{R}\setminus B_n)} \xrightarrow{n} 0.
\end{aligned}
$$

On the other hand, since $\kappa_n|_{B_n} \equiv 1$, Taylor's Theorem tells us that $v_n(x) = v(x)$ for all $x \in B_n$. Consequently, for sufficiently large $n \in \mathbb{N}$, there must hold $E_N v_n = (v_n(x_i))_{i=1}^N = (v(x_i))_{i=1}^N = E_N v$. This implies $|\langle \boldsymbol{c}, E_N(v - v_n)\rangle_2| \xrightarrow{n} 0$, even for all $\boldsymbol{c} \in \mathbb{R}^N$, and finishes the proof of the density of $\{v \in C^\infty(\mathbb{R}) \mid v^{(k_{\min})} \in C_0^\infty(\mathbb{R})\}$ in $V \cap C^\infty(\mathbb{R})$.

Finally, let $v \in C^\infty(\mathbb{R})$ with $v^{(k_{\min})} \in C_0^\infty(\mathbb{R})$. Let $R > 0$ such that $\operatorname{supp}(v^{(k_{\min})}) \subseteq (-R, R)$. Clearly, there must hold $v \in \mathbb{P}^{k_{\min}-1}((-\infty, -R))$ and $v \in \mathbb{P}^{k_{\min}-1}((R, \infty))$. In particular, for all integers $n \geq R$ and all $j \in \mathbb{N}_0$, we can bound

$$\|v^{(j)}\|_{L^2(B_{2n}\setminus B_n)} \leq C(v) n^{k_{\min}-1-j+d/2} = C(v) n^{k_{\min}-j-1/2}.$$

We then define the approximants $v_n := \kappa_n v \in C_0^\infty(\mathbb{R})$. Using Leibniz' product rule and the stability $|\kappa_n|_{W^{j,\infty}(\mathbb{R}^d)} \lesssim n^{-j}$ once again, we estimate

$$\begin{aligned} |v - v_n|_a &\lesssim \sum_{l=k_{\min}}^{k} \Big(\sum_{j=0}^{l-1} \|\kappa_n^{(l-j)} v^{(j)}\|_{L^2(B_{2n}\setminus B_n)} + \|(1-\kappa_n) v^{(l)}\|_{L^2(\mathbb{R}\setminus B_n)} \Big) \\ &\lesssim n^{-1/2} + \sum_{l=k_{\min}}^{k} |v|_{H^l(\mathbb{R}\setminus B_n)} \xrightarrow{n} 0. \end{aligned}$$

Furthermore, since $v_n|_{B_n} = v|_{B_n}$, and $\{x_1, \ldots, x_N\} \subseteq B_n$ for all sufficiently large $n \in \mathbb{N}$, it is clear that $|\langle \boldsymbol{c}, E_N(v - v_n)\rangle_2| \xrightarrow{n} 0$, where $\boldsymbol{c} \in \mathbb{R}^N$. This finishes the proof. □

Unless $k_{\min} = 0$, it is now clear that $(V, a(\cdot,\cdot))$ is not an inner product space, let alone a Hilbert space. However, using the evaluation operator $E_N : C^0(\mathbb{R}^d) \longrightarrow \mathbb{R}^N$ from Definition 3, it is not difficult to find a useful subspace of V, where $a(\cdot,\cdot)$ is strictly positive definite:

Definition 22 *We define the* homogeneous native space $V_0 := \{v \in V \mid E_N v = \mathbf{0}\} \subseteq V$.

Recall that "$E_N v = \mathbf{0}$" is equivalent to "$\forall n \in \{1, \ldots, N\} : v(x_n) = 0$." In particular, the space V_0 depends on the number N of interpolation points $x_1, \ldots, x_N$, as well as their positions.

Lemma 23 *There holds* $V_0 \cap P = \{0\}$. *Furthermore, the tuple* $(V_0, a(\cdot,\cdot))$ *constitutes a Hilbert space.*

Proof The identity $V_0 \cap P = \{0\}$ follows from the unisolvency assumption in Definition 2. Since P coincides with the kernel of $|\cdot|_a$ (cf. Lemma 20), the bilinear form $a(\cdot,\cdot)$ is positive definite on V_0. Finally, let us sketch the proof of completeness: Consider a Cauchy sequence $(v_n)_{n\in\mathbb{N}} \subseteq V_0$ with respect to $|\cdot|_a$ and pick a sequence of bounded Lipschitz domains $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots \subseteq \mathbb{R}^d$, e.g., $B_i := \operatorname{Ball}(0, i)$. For fixed $i \in \mathbb{N}$, we use Lemma 19 and the identity $E_N v_n = \mathbf{0} = E_N v_m$ to estimate

$$\begin{aligned} \|v_n - v_m\|_{H^k(B_i)} &\leq C(d, k, B_i, (x_l)_l)(|v_n - v_m|_{H^k(B_i)} + \|E_N(v_n - v_m)\|_{\ell^2(N)}) \\ &\leq C(d, k, B_i, (x_l)_l)|v_n - v_m|_a. \end{aligned}$$

We obtain limits $v_{B_i} \in H^k(B_i)$, $i \in \mathbb{N}$, and piece them together to define global functions $\mathrm{D}^\alpha v : \mathbb{R}^d \longrightarrow \mathbb{R}$, $|\alpha| \leq k$, by setting $(\mathrm{D}^\alpha v)|_{B_i} := \mathrm{D}^\alpha(v_{B_i})$. One can then prove that v lies in V_0 and that $|v - v_n|_a \xrightarrow{n} 0$. □

3.2 An equivalent interpolation problem

The homogeneous native space V_0 allows us to rewrite the minimization property in Problem 4 in terms of an orthogonality relation for $a(\cdot,\cdot)$.

Problem 24 *Let $f \in V$. Find a function $u \in V$ that satisfies the following interpolation and orthogonality conditions:*

$$E_N u = E_N f, \qquad \forall v \in V_0 : \quad a(u, v) = 0.$$

Using standard variational techniques, it is not difficult to show that Problems 4 and 24 are indeed equivalent:

Lemma 25 *Let $f \in V$. A function $u \in V$ solves Problem 4 if and only if it solves Problem 24.*

The orthogonality relation in Problem 24 is more suitable for our upcoming error analysis and we will not have to deal with the original minimization property in Problem 4 any further. The next lemma answers the question of solvability and uniqueness of solutions (effectively for both problems):

Lemma 26 *For every $f \in V$, Problem 24 has a unique solution $u \in V$. The mapping $f \mapsto u$ is linear, and there holds the* a priori *bound $|u|_a \le |f|_a$.*

Proof According to the Riesz-Fréchet Representation Theorem there exists a unique element $u_0 \in V_0$ such that $a(u_0, v) = -a(f, v)$ for all $v \in V_0$. Then, $u := u_0 + f \in V$ is the sought solution. □

3.3 Properties of the LSE

In this section, we show that the LSE from Section 2.3 is indeed regular. Furthermore, we prove the asserted representation $u = \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n + \sum_{|\alpha|<k_{\min}} \boldsymbol{d}_\alpha \pi_\alpha$ of the solution $u \in V$ of Problem 4 (or, equivalently, Problem 24).

Definition 27 *Denote by $\{\phi_1, \dots, \phi_N\} \subseteq C^0(\mathbb{R}^d)$ and $\{\pi_\alpha \mid |\alpha| < k_{\min}\} \subseteq P$ the ansatz functions from Definition 12. We define the corresponding coordinate mappings*

$$\Phi : \begin{cases} \mathbb{R}^N \longrightarrow C^0(\mathbb{R}^d) \\ \boldsymbol{c} \longmapsto \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n \end{cases}, \qquad \Pi : \begin{cases} \mathbb{R}^{N_{\min}} \longrightarrow P \\ \boldsymbol{d} \longmapsto \sum_{|\alpha|<k_{\min}} \boldsymbol{d}_\alpha \pi_\alpha \end{cases}.$$

Recalling from Lemma 20 that $P \subseteq V$, it is clear that the operator Π always maps into the native space V, regardless of the input vector $\boldsymbol{d} \in \mathbb{R}^{N_{\min}}$. As for the operator Φ, on the other hand, we remind the reader of Assumption 9: The fundamental solution $\phi \in C^0(\mathbb{R}^d)$ is not necessarily an element of V, so we cannot guarantee $\Phi\boldsymbol{c} \in V$ for *all* inputs $\boldsymbol{c} \in \mathbb{R}^N$. However, for the coefficient vectors $\boldsymbol{c}$ from the subset $\boldsymbol{C} \subseteq \mathbb{R}^N$ (cf. Definition 8), the conformity of $\Phi\boldsymbol{c}$ is provided by Assumption 9.

Lemma 28 *The operator* $\Pi : \mathbb{R}^{N_{\min}} \longrightarrow P$ *is bijective. For all* $\boldsymbol{c} \in \boldsymbol{C}$*, there holds* $\Phi\boldsymbol{c} \in V$ *with* $|\Phi\boldsymbol{c}|_a \gtrsim h_{\min}^{k-d/2}\|\boldsymbol{c}\|_2$*. Finally, the operators* Φ *and* E_N *(cf. Definition 3) are transposed in the following sense:*

$$\forall \boldsymbol{c} \in \boldsymbol{C} : \forall v \in V : \qquad a(\Phi\boldsymbol{c}, v) = \langle \boldsymbol{c}, E_N v\rangle_2.$$

Proof The bijectivity of Π follows from the fact that $\{\pi_\alpha \mid |\alpha| < k_{\min}\} \subseteq P$ is a basis. Next, let us prove the transposition formula: For every $\boldsymbol{c} \in \boldsymbol{C}$, we know from Assumption 9 that $\Phi\boldsymbol{c} = \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n \in V$. Now, for all $v \in C_0^\infty(\mathbb{R}^d)$, we compute

$$\begin{aligned} a(\Phi\boldsymbol{c}, v) &\overset{\text{Lem. 7}}{=} \int_{\mathbb{R}^d} (\Phi\boldsymbol{c})(\mathrm{L}^{2k} v)\,\mathrm{d}x = \sum_{n=1}^{N} \boldsymbol{c}_n \int_{\mathbb{R}^d} \phi(x - x_n)(\mathrm{L}^{2k} v)(x)\,\mathrm{d}x \\ &\overset{\text{Ass. 9}}{=} \sum_{n=1}^{N} \boldsymbol{c}_n v(x_n) = \langle \boldsymbol{c}, E_N v\rangle_2. \end{aligned}$$

To extend this identity from $C_0^\infty(\mathbb{R}^d)$ to all of V, we use a simple density argument: Let $\boldsymbol{c} \in \boldsymbol{C}$ and $v \in V$. We know from Lemma 21 that there exists a sequence $(v_n)_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^d)$ such that $|v - v_n|_a \overset{n}{\to} 0$ and $|\langle \boldsymbol{c}, E_N v\rangle_2 - \langle \boldsymbol{c}, E_N v_n\rangle_2| \overset{n}{\to} 0$. Using the Cauchy-Schwarz inequality from Lemma 20, we find that $|a(\Phi\boldsymbol{c}, v) - a(\Phi\boldsymbol{c}, v_n)| \leq |\Phi\boldsymbol{c}|_a |v - v_n|_a \overset{n}{\to} 0$ as well. Then, since $v_n \in C_0^\infty(\mathbb{R}^d)$,

$$a(\Phi\boldsymbol{c}, v) = \lim_{n\to\infty} a(\Phi\boldsymbol{c}, v_n) = \lim_{n\to\infty} \langle \boldsymbol{c}, E_N v_n\rangle_2 = \langle \boldsymbol{c}, E_N v\rangle_2.$$

Finally, let $\boldsymbol{c} \in \boldsymbol{C}$ and denote by $\Lambda : \mathbb{R}^N \longrightarrow V$ the operator from Definition 34 further below. Anticipating the results from Lemma 35, we know that $E_N \Lambda\boldsymbol{c} = \boldsymbol{c}$ and that $|\Lambda\boldsymbol{c}|_a \lesssim h_{\min}^{d/2-k}\|\boldsymbol{c}\|_2$. Then, using the transposition formula from above and the Cauchy-Schwarz inequality from Lemma 20, we obtain the bound

$$\|\boldsymbol{c}\|_2^2 = \langle \boldsymbol{c}, \boldsymbol{c}\rangle_2 = \langle \boldsymbol{c}, E_N \Lambda\boldsymbol{c}\rangle_2 = a(\Phi\boldsymbol{c}, \Lambda\boldsymbol{c}) \leq |\Phi\boldsymbol{c}|_a |\Lambda\boldsymbol{c}|_a \lesssim h_{\min}^{d/2-k} |\Phi\boldsymbol{c}|_a \|\boldsymbol{c}\|_2.$$

We divide both sides by $h_{\min}^{d/2-k}\|\boldsymbol{c}\|_2$ and get the desired inequality. □

In the case $k_{\min} = 0$, the transposition formula in Lemma 28 reduces to $a(\phi_n, v) = v(x_n)$ for all $n \in \{1, \ldots, N\}$ and all $v \in V$. This identity is reminiscent of Lemma 5, where the identity $a(\phi, v) = v(0)$, $v \in V$, was used to *define* the basis function ϕ. Furthermore, let it be said that the transposition formula must be handled with care if $k_{\min} > 0$. The left-hand side reads $a(\Phi\boldsymbol{c}, v) = a(\sum_{n=1}^{N} \boldsymbol{c}_n \phi_n, v)$, with the sum *inside* of $a(\cdot, \cdot)$. This is a perfectly valid expression, because Assumption 9 guarantees that $\sum_{n=1}^{N} \boldsymbol{c}_n \phi_n$ lies in V, whenever $\boldsymbol{c} \in \boldsymbol{C}$. However, we *cannot* pull the sum out of $a(\cdot, \cdot)$. In fact, the expression $\sum_{n=1}^{N} \boldsymbol{c}_n a(\phi_n, v)$ is not well-defined, since ϕ_n need not lie in V and may not be plugged into $a(\cdot, \cdot)$.

Next, let us have a look at the properties of the matrices $\boldsymbol{A} \in \mathbb{R}^{N\times N}$, $\boldsymbol{B} \in \mathbb{R}^{N_{\min}\times N}$ and $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right) \in \mathbb{R}^{(N+N_{\min})\times(N+N_{\min})}$ from Definition 13. Using the evaluation operator $E_N : C^0(\mathbb{R}^d) \longrightarrow \mathbb{R}^N$ from Definition 3 once again, we see that $\boldsymbol{A}$ can be written in the form $\boldsymbol{A} = (E_N\phi_1|\ldots|E_N\phi_N)$ and that $\boldsymbol{B}^T$ can be written as $\boldsymbol{B}^T = (E_N\pi_1|\ldots|E_N\pi_{N_{\min}})$ if a linear ordering of the polynomial basis is assumed. The next lemma uses these representations.

Lemma 29

1. *For all $\boldsymbol{d} \in \mathbb{R}^{N_{\min}}$, there holds the identity $\boldsymbol{B}^T\boldsymbol{d} = E_N\Pi\boldsymbol{d}$. In particular, $\boldsymbol{B}^T$ is injective and $\boldsymbol{B}$ is surjective. The kernel of $\boldsymbol{B}$ is given by* $\ker \boldsymbol{B} = \boldsymbol{C}$.
2. *For all $\boldsymbol{c} \in \mathbb{R}^N$, there holds the identity $\boldsymbol{A}\boldsymbol{c} = E_N\Phi\boldsymbol{c}$. Furthermore,*

$$\forall \boldsymbol{c} \in \boldsymbol{C} : \qquad \langle \boldsymbol{A}\boldsymbol{c}, \boldsymbol{c}\rangle_2 \gtrsim h_{\min}^{2k-d}\|\boldsymbol{c}\|_2^2.$$

3. *The interpolation matrix $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)$ is invertible.*

Proof For all $\boldsymbol{d} \in \mathbb{R}^{N_{\min}}$, we have $\boldsymbol{B}^T\boldsymbol{d} = \sum_{|\alpha|<k} \boldsymbol{d}_\alpha E_N\pi_\alpha = E_N(\sum_{|\alpha|<k} \boldsymbol{d}_\alpha\pi_\alpha) = E_N\Pi\boldsymbol{d}$. In particular, using the bijectivity of Π and Lemma 23, we find that $\ker \boldsymbol{B}^T = \ker E_N \circ \Pi = \Pi^{-1}(V_0 \cap P) = \{0\}$. This proves that $\boldsymbol{B}^T$ is injective and hence that $\boldsymbol{B}$ is surjective. Next, we have

$$\begin{aligned} \ker \boldsymbol{B} &= \{\boldsymbol{c} \in \mathbb{R}^N \mid \boldsymbol{B}\boldsymbol{c} = \boldsymbol{0}\} = \{\boldsymbol{c} \mid \forall \boldsymbol{d} \in \mathbb{R}^{N_{\min}} : \langle \boldsymbol{B}\boldsymbol{c}, \boldsymbol{d}\rangle_2 = 0\} \\ &= \{\boldsymbol{c} \mid \forall \boldsymbol{d} \in \mathbb{R}^{N_{\min}} : \langle \boldsymbol{c}, E_N\Pi\boldsymbol{d}\rangle_2 = 0\} = \boldsymbol{C}. \end{aligned}$$

For all $\boldsymbol{c} \in \mathbb{R}^N$, there holds $\boldsymbol{A}\boldsymbol{c} = \sum_{n=1}^N \boldsymbol{c}_n E_N\phi_n = E_N(\sum_{n=1}^N \boldsymbol{c}_n\phi_n) = E_N\Phi\boldsymbol{c}$. In particular, for all $\boldsymbol{c} \in \boldsymbol{C}$,

$$\langle \boldsymbol{A}\boldsymbol{c}, \boldsymbol{c}\rangle_2 = \langle E_N\Phi\boldsymbol{c}, \boldsymbol{c}\rangle_2 \overset{\text{Lem. 28}}{=} a(\Phi\boldsymbol{c}, \Phi\boldsymbol{c}) = |\Phi\boldsymbol{c}|_a^2 \overset{\text{Lem. 28}}{\gtrsim} h_{\min}^{2k-d}\|\boldsymbol{c}\|_2^2.$$

Finally, consider coefficient vectors $\boldsymbol{c} \in \mathbb{R}^N$ and $\boldsymbol{d} \in \mathbb{R}^{N_{\min}}$ such that $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)\left(\begin{smallmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{smallmatrix}\right) = \left(\begin{smallmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{smallmatrix}\right)$. The second row tells us that $\boldsymbol{B}\boldsymbol{c} = \boldsymbol{0}$, so that $\boldsymbol{c} \in \ker \boldsymbol{B} = \boldsymbol{C}$. Then, we multiply the first row with $\boldsymbol{c}$ and get $0 = \langle \boldsymbol{A}\boldsymbol{c} + \boldsymbol{B}^T\boldsymbol{d}, \boldsymbol{c}\rangle = \langle \boldsymbol{A}\boldsymbol{c}, \boldsymbol{c}\rangle + \langle \boldsymbol{d}, \boldsymbol{B}\boldsymbol{c}\rangle = \langle \boldsymbol{A}\boldsymbol{c}, \boldsymbol{c}\rangle \gtrsim h_{\min}^{2k-d}\|\boldsymbol{c}\|_2^2$, which implies $\boldsymbol{c} = \boldsymbol{0}$. Then, the first row simplifies to $\boldsymbol{B}^T\boldsymbol{d} = \boldsymbol{0}$, from which we obtain $\boldsymbol{d} = \boldsymbol{0}$. □

At this point we have all the ingredients to show that the solution $u \in V$ of Problem 24 can be represented by means of the ansatz functions ϕ_n and π_α from Definition 12.

Lemma 30 *Let $f \in V$ and set $\boldsymbol{f} := E_N f \in \mathbb{R}^N$. Denote by $\boldsymbol{c} \in \mathbb{R}^N$ and $\boldsymbol{d} \in \mathbb{R}^{\mathbb{N}_{\min}}$ the unique solution of the following LSE:*

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{pmatrix}.$$

Then, the function

$$u := \Phi \boldsymbol{c} + \Pi \boldsymbol{d} = \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n + \sum_{|\alpha|<k_{\min}} \boldsymbol{d}_\alpha \pi_\alpha \in V$$

coincides with the unique solution of Problem 24.

Proof The second row of the LSE tells us that $\boldsymbol{Bc} = \boldsymbol{0}$, so that $\boldsymbol{c} \in \ker \boldsymbol{B} = \boldsymbol{C}$, by Lemma 29. Owing to Lemmas 20 and 28, the function $u := \Phi\boldsymbol{c} + \Pi\boldsymbol{d}$ then lies in V. Using Lemma 29 again, the first row of the LSE yields $E_N u = E_N \Phi \boldsymbol{c} + E_N \Pi \boldsymbol{d} = \boldsymbol{Ac} + \boldsymbol{B}^T \boldsymbol{d} = \boldsymbol{f} = E_N f$. Furthermore, for all $v \in V_0$, there holds the following orthogonality:

$$a(u, v) = a(\Phi\boldsymbol{c} + \Pi\boldsymbol{d}, v) \overset{\text{Lem. 20}}{=} a(\Phi\boldsymbol{c}, v) \overset{\text{Lem. 28}}{=} \langle \boldsymbol{c}, E_N v\rangle_2 \overset{\text{Def. 22}}{=} \langle \boldsymbol{c}, \boldsymbol{0}\rangle_2 = 0.$$

Therefore, the function u is a solution of Problem 24 with respect to the data f. By uniqueness, u is *the* solution. This concludes the proof. □

3.4 The representation formula for the inverse matrix

Earlier, in Lemma 29, we developed the fact that the interpolation matrix $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)$ from Definition 13 is invertible. Analogous to Theorem 18, let us write its inverse in the form $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{21} & \boldsymbol{S}_{22} \end{smallmatrix}\right)$ with matrices $\boldsymbol{S}_{11} \in \mathbb{R}^{N\times N}$, $\boldsymbol{S}_{21} \in \mathbb{R}^{N_{\min}\times N}$, $\boldsymbol{S}_{12} \in \mathbb{R}^{N\times N_{\min}}$ and $\boldsymbol{S}_{22} \in \mathbb{R}^{N_{\min}\times N_{\min}}$. In this section, we develop a representation formula for the main block $\boldsymbol{S}_{11}$.

Recall from Lemma 26 that the mapping $f \mapsto u$ of data $f \in V$ to the solution $u \in V$ of Problem 24 is a linear operator. Clearly, this abstract operator must play a prominent role in the sought representation formula.

Definition 31 *For every $f \in V$, denote by $S_N f \in V$ the unique solution of Problem 24, i.e.,*

$$E_N S_N f = E_N f \qquad \text{and} \qquad \forall v \in V_0 : \quad a(S_N f, v) = 0.$$

The linear operator $S_N : V \longrightarrow V$ is called solution operator.

Next, we need a way to map a given coefficient vector $\boldsymbol{f} \in \mathbb{R}^N$ to a function $f \in V$ with $E_N f = \boldsymbol{f}$. Since $C_0^\infty(\mathbb{R}^d) \subseteq V$ (cf. Lemma 20), such a function f can easily be constructed from a family of *dual functions* $\lambda_1, \ldots, \lambda_N \in C_0^\infty(\mathbb{R}^d)$ with $\lambda_m(x_n) = \delta_{mn}$ (Kronecker-δ). In fact, the function $f := \sum_{n=1}^N \boldsymbol{f}_n \lambda_n \in C_0^\infty(\mathbb{R}^d) \subseteq V$ does the job. It is not surprising that the definition of the dual functions depends on the separation distance $h_{\min}$ from Definition 2:

Definition 32 *Let $\lambda \in C_0^\infty(\mathbb{R})$ be a function with* $\operatorname{supp}(\lambda) \subseteq \operatorname{Ball}(0, 1)$ *and* $\lambda(0) = 1$. *For every $m \in \{1, \ldots, N\}$, we define the function $\lambda_m(x) := \lambda((x - x_m)/h_{\min})$. We refer to $\{\lambda_1, \ldots, \lambda_N\}$ as the set of* dual functions.

The basic properties of the dual functions λ_m are summarized in the next lemma. Due to its simplicity, the proof is omitted. We remind the reader of Definition 14, where the bubbles $\Omega_m \subseteq \mathbb{R}^d$ were introduced.

Lemma 33 *For all $m, n \in \{1, \dots, N\}$, there holds $\lambda_m \in C_0^\infty(\mathbb{R}^d) \subseteq V$, $\mathrm{supp}(\lambda_m) \subseteq \Omega_m$ and $\lambda_m(x_n) = \delta_{mn}$. Furthermore, we have the stability bound $|\lambda_m|_a \lesssim h_{\min}^{d/2-k}$.*

Now that the dual functions $\lambda_m \in C_0^\infty(\mathbb{R}^d)$ are properly defined, let us introduce a name for the mapping $\boldsymbol{f} \mapsto \sum_{n=1}^N \boldsymbol{f}_n \lambda_n$ from before.

Definition 34 *We define the operators*

$$\Lambda : \begin{cases} \mathbb{R}^N \longrightarrow V \\ \boldsymbol{f} \longmapsto \sum_{n=1}^N \boldsymbol{f}_n \lambda_n \end{cases}, \qquad \Lambda^T : \begin{cases} V \longrightarrow \mathbb{R}^N \\ v \longmapsto (a(v, \lambda_n))_{n=1}^N \end{cases}.$$

Recall from Lemma 20 that V is *not* necessarily a Hilbert space, so that Λ^T cannot be the proper Hilbert space transpose of Λ. However, as discussed below, the defining equation for transposed operators is still satisfied.

In the subsequent lemma, we use $\mathrm{supp}(\boldsymbol{f}) := \{n \in \{1, \dots, N\} \mid \boldsymbol{f}_n \neq 0\}$ to denote the *support* of a vector $\boldsymbol{f} \in \mathbb{R}^N$. Remember that $\Omega_I = \bigcup_{n \in I} \Omega_n \subseteq \mathbb{R}^d$ denotes a union of bubbles (cf. Definition 14). Furthermore, we make use of the local seminorms $|\cdot|_{a,\Omega}$, $\Omega \subseteq \mathbb{R}^d$, from Definition 1.

Lemma 35 *The operators Λ and Λ^T are transposed in the following sense:*

$$\forall v \in V : \forall \boldsymbol{f} \in \mathbb{R}^N : \qquad a(v, \Lambda \boldsymbol{f}) = \langle \Lambda^T v, \boldsymbol{f} \rangle_2.$$

Both Λ and Λ^T preserve locality: For all $\boldsymbol{f} \in \mathbb{R}^N$, $v \in V$ and $I \subseteq \{1, \dots, N\}$, there holds

$$\mathrm{supp}(\Lambda \boldsymbol{f}) \subseteq \Omega_{\mathrm{supp}(\boldsymbol{f})}, \qquad |\Lambda \boldsymbol{f}|_a \lesssim h_{\min}^{d/2-k} \|\boldsymbol{f}\|_2, \qquad \|\Lambda^T v\|_{\ell^2(I)} \lesssim h_{\min}^{d/2-k} |v|_{a,\Omega_I}.$$

Finally, for all $\boldsymbol{f} \in \mathbb{R}^N$, there holds $E_N \Lambda \boldsymbol{f} = \boldsymbol{f}$. In particular, we have $v - \Lambda E_N v \in V_0$ for all $v \in V$.

Proof Let $v \in V$ and $\boldsymbol{f} \in \mathbb{R}^N$. The transposition formula is straightforward: $a(v, \Lambda \boldsymbol{f}) = a(v, \sum_{n=1}^N \boldsymbol{f}_n \lambda_n) = \sum_{n=1}^N a(v, \lambda_n) \boldsymbol{f}_n = \langle \Lambda^T v, \boldsymbol{f} \rangle_2$. Next, in order to determine the support of $\Lambda \boldsymbol{f}$, we remember from Lemma 33 that $\mathrm{supp}(\lambda_n) \subseteq \Omega_n$. Abbreviating $I := \mathrm{supp}(\boldsymbol{f})$, we find that $\mathrm{supp}(\Lambda \boldsymbol{f}) = \mathrm{supp}(\sum_{n \in I} \boldsymbol{f}_n \lambda_n) \subseteq \bigcup_{n \in I} \mathrm{supp}(\lambda_n) \subseteq \Omega_I$.

In order to see the upper bound for $|\Lambda \boldsymbol{f}|_a$, we argue that the disjointness of the bubbles Ω_n implies the orthogonality $a(\lambda_n, \lambda_m) = \sum_{l,\alpha} (\sigma_l l!/\alpha!) \langle \mathrm{D}^\alpha \lambda_n, \mathrm{D}^\alpha \lambda_m \rangle_{L^2(\Omega_n \cap \Omega_m)} = 0$ for all $m \neq n$. Therefore, with Lemma 33, we obtain

$$|\Lambda \boldsymbol{f}|_a^2 = a(\Lambda \boldsymbol{f}, \Lambda \boldsymbol{f}) = \sum_{n,m=1}^N \boldsymbol{f}_n \boldsymbol{f}_m a(\lambda_n, \lambda_m) = \sum_{n=1}^N \boldsymbol{f}_n^2 |\lambda_n|_a^2 \lesssim h_{\min}^{d-2k} \|\boldsymbol{f}\|_2^2.$$

Next, consider an arbitrary index set $I \subseteq \{1, \dots, N\}$. To show the upper bound for $\|\Lambda^T v\|_{\ell^2(I)}$, we use a localized version of the Cauchy-Schwarz inequality from Lemma 20 and exploit the disjointness of the bubbles once again:

$$\|\Lambda^T v\|_{\ell^2(I)}^2 = \sum_{n \in I} a(v, \lambda_n)^2 \leq \sum_{n \in I} |v|_{a,\Omega_n}^2 |\lambda_n|_a^2 \lesssim h_{\min}^{d-2k} \sum_{n \in I} |v|_{a,\Omega_n}^2 = h_{\min}^{d-2k} |v|_{a,\Omega_I}^2.$$

Finally, for all $\boldsymbol{f} \in \mathbb{R}^N$, the Kronecker-$\delta$-property from Lemma 33 gives us the identity $E_N \Lambda \boldsymbol{f} = (\sum_m \boldsymbol{f}_m \lambda_m(x_n))_{n=1}^N = (\boldsymbol{f}_n)_{n=1}^N = \boldsymbol{f}$. In particular, for all $v \in V$, this implies $E_N(v - \Lambda E_N v) = E_N v - E_N \Lambda E_N v = E_N v - E_N v = \boldsymbol{0}$, i.e., $v - \Lambda E_N v \in V_0$. □

In Lemma 30 we demonstrated how to derive the solution $u \in V$ of Problem 24 from the solution $(\boldsymbol{c}, \boldsymbol{d}) \in \mathbb{R}^N \times \mathbb{R}^{N_{\min}}$ of the LSE in Section 2.3. In the next lemma, we go in the opposite direction, i.e., we construct the coefficient vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ from a given solution u. Once we have this result, the representation formula for the inverse interpolation matrix is merely a byproduct.

Lemma 36 *Let $f \in V$ and denote by $u \in V$ the unique solution of Problem 24. Set $\boldsymbol{f} := E_N f \in \mathbb{R}^N$. Then, the coefficient vectors*

$$\boldsymbol{c} := \Lambda^T u \in \boldsymbol{C}, \qquad \boldsymbol{d} := \Pi^{-1}(\mathrm{id} - \Phi\Lambda^T)u \in \mathbb{R}^{N_{\min}}$$

solve the LSE in Lemma 30.

Proof First, let us show that indeed $\boldsymbol{c} \in \boldsymbol{C}$: For every $q \in P$, there holds $q - \Lambda E_N q \in V_0$ (cf. Lemma 35) and thus

$$\langle \boldsymbol{c}, E_N q \rangle_2 = \langle \Lambda^T u, E_N q \rangle_2 \overset{\text{Lem. 35}}{=} a(u, \Lambda E_N q) \overset{\text{Pr. 24}}{=} a(u, q) \overset{\text{Lem. 20}}{=} 0.$$

Lemma 28 tells us that $\Phi \boldsymbol{c} \in V$, so that $p := u - \Phi \boldsymbol{c} \in V$. Using $p - \Lambda E_N p \in V_0$ (cf. Lemma 35, again) and the orthogonality of u, we get

$$\begin{aligned} |p|_a^2 = a(p, p) = a(u, p) - a(\Phi\Lambda^T u, p) &\overset{\text{Lem. 28}}{=} a(u, p) - \langle \Lambda^T u, E_N p \rangle_2 \\ &\overset{\text{Lem. 35}}{=} a(u, p - \Lambda E_N p) \overset{\text{Pr. 24}}{=} 0. \end{aligned}$$

Due to Lemma 20, this implies $p \in P$, so that $\boldsymbol{d} := \Pi^{-1} p = \Pi^{-1}(\mathrm{id} - \Phi\Lambda^T)u \in \mathbb{R}^{N_{\min}}$ is well-defined.

As for the first row of the LSE, we compute

$$\boldsymbol{A}\boldsymbol{c} + \boldsymbol{B}^T \boldsymbol{d} \overset{\text{Lem. 29}}{=} E_N(\Phi \boldsymbol{c} + \Pi \boldsymbol{d}) \overset{\text{Def.}\boldsymbol{d}}{=} E_N(\Phi \boldsymbol{c} + p) \overset{\text{Def.}p}{=} E_N u \overset{\text{Pr. 24}}{=} E_N f = \boldsymbol{f}.$$

Finally, using Lemma 29 again, we have $\boldsymbol{c} \in \boldsymbol{C} = \ker \boldsymbol{B}$. In other words, $\boldsymbol{B}\boldsymbol{c} = \boldsymbol{0}$, which is precisely the second row of the LSE. □

We close this section with the promised representation formula. The formula establishes a relationship between the interpolation matrix $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)$ from Definition 13, the coordinate mappings $\Phi : \mathbb{R}^N \longrightarrow C^0(\mathbb{R}^d)$ and $\Pi : \mathbb{R}^{N_{\min}} \longrightarrow P$ from Definiton 27, the solution operator $S_N : V \longrightarrow V$ from Definiton 31, and the operators $\Lambda : \mathbb{R}^N \longrightarrow V$ and $\Lambda^T : V \longrightarrow \mathbb{R}^N$ from Definition 34.

Corollary 37 *For all $\boldsymbol{f} \in \mathbb{R}^N$, there holds the identity*

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \Lambda^T S_N \Lambda \boldsymbol{f} \\ \Pi^{-1}(\mathrm{id} - \Phi\Lambda^T) S_N \Lambda \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ \boldsymbol{0} \end{pmatrix}.$$

In particular, the main block $\boldsymbol{S_{11}} \in \mathbb{R}^{N\times N}$ from the representation $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} \boldsymbol{S_{11}} & \boldsymbol{S_{12}} \\ \boldsymbol{S_{21}} & \boldsymbol{S_{22}} \end{smallmatrix}\right)$ satisfies the relation

$$\boldsymbol{S_{11}} \boldsymbol{f} = \Lambda^T S_N \Lambda \boldsymbol{f} \in \boldsymbol{C}.$$

Proof Let $\boldsymbol{f} \in \mathbb{R}^N$ and set $f := \Lambda \boldsymbol{f} \in V$. According to Definition 31, the function $u := S_N f \in V$ is the unique solution of Problem 24. Now Lemma 36 tells us that the coefficient vectors $\boldsymbol{c} := \Lambda^T u = \Lambda^T S_N \Lambda \boldsymbol{f} \in \boldsymbol{C}$ and $\boldsymbol{d} := \Pi^{-1}(\mathrm{id} - \Phi\Lambda^T)u = \Pi^{-1}(\mathrm{id} - \Phi\Lambda^T) S_N \Lambda \boldsymbol{f} \in \mathbb{R}^{N_{\min}}$ solve the LSE from Lemma 30, where the right-hand side is given by $\tilde{\boldsymbol{f}} := E_N f \in \mathbb{R}^N$. However, we know from Lemma 35 that $\tilde{\boldsymbol{f}} = E_N f = E_N \Lambda \boldsymbol{f} = \boldsymbol{f}$. This proves the asserted identity.

3.5 Reduction from matrix-level to function-level

In our main result, Theorem 18, we claimed that the main diagonal block $\boldsymbol{S_{11}}$ of the inverse interpolation matrix can be approximated well by $\mathcal{H}$-matrices. The goal of the subsequent lemma is to reduce this "matrix-level" question to an analogous question on the "function-level." The majority of the work has already been done by achieving Corollary 37, where the action of the matrix $\boldsymbol{S_{11}}$ on any given vector $\boldsymbol{f} \in \mathbb{R}^N$ was characterized by the abstract solution operator $S_N : V \longrightarrow V$ from Definition 31. Furthermore, thanks to the asserted inequality (cf. Definition 16)

$$\|\boldsymbol{M}\|_2 \leq C \ln(N) \max_{(I,J)\in\mathbb{P}} \|\boldsymbol{M}|_{I\times J}\|_2 \qquad \forall \boldsymbol{M} \in \mathbb{R}^{N\times N},$$

it suffices to derive an error bound for each matrix block $\boldsymbol{S_{11}}|_{I\times J}$, where (I, J) is any given cluster tuple from the sparse hierarchical block partition $\mathbb{P}$.

Lemma 38 *Let $\boldsymbol{S_{11}} \in \mathbb{R}^{N\times N}$ be the main diagonal block of the inverse interpolation matrix from Definition 13. Let $I, J \subseteq \{1, \ldots, N\}$ and $W \subseteq V$ be a finite-dimensional subspace. Then, there exist an integer $r \leq \dim W$ and matrices $\boldsymbol{X} \in \mathbb{R}^{I\times r}$ and $\boldsymbol{Y} \in \mathbb{R}^{J\times r}$ such that there holds the following error bound:*

$$\|\boldsymbol{S_{11}}|_{I\times J} - \boldsymbol{X}\boldsymbol{Y}^T\|_2 \lesssim h_{\min}^{d-2k} \sup_{\substack{f\in V: \\ \mathrm{supp}(f)\subseteq\Omega_J}} \inf_{w\in W} \frac{|S_N f - w|_{a,\Omega_I}}{|f|_a}.$$

Proof Follows from Lemma 35 and Corollary 37. We omit a detailed proof and refer the reader to [1, Lemma 3.13], where a similar bound was derived in terms of the L^2-norm. □

Lemma 38 leads us to the following question: Given boxes $B, D \subseteq \mathbb{R}^d$ (cf. Definition 15), a free parameter $L \in \mathbb{N}$ and a function $f \in V$ with $\operatorname{supp}(f) \subseteq D$, can we construct a subspace $V_{B,D,L} \subseteq V$ such that the dimension $\dim V_{B,D,L}$ and the best-approximation error $\inf_{w \in V_{B,D,L}} |S_N f - w|_{a,B}$ are both "small" at the same time? More precisely, is it possible to build $V_{B,D,L}$ in a way such that $\dim V_{B,D,L} \lesssim L^{c_1}$ and $\inf_{w \in V_{B,D,L}} |S_N f - w|_{a,B} \lesssim 2^{-c_2 L} |f|_a$ for some constants $c_1, c_2 > 0$?

Over the course of the subsequent sections, we will show that the answer to this question is affirmative (up to a factor $L^k / h_{\min}^k$ in the error bound), if the domains B, D satisfy the *admissibility condition* $\operatorname{diam}(B) \leq \sigma_{\mathrm{adm}} \operatorname{dist}(B, D)$ from Definition 16.

Let us briefly give an overview of the construction of the space $V_{B,D,L}$:

1. Due to the admissibility condition, we can safely "inflate" the box B in L increments of identical size, before it touches the box D. This generates a family of L concentric boxes between B and D, i.e., $B \subseteq B_1 \subseteq \cdots \subseteq B_L \subseteq \mathbb{R}^d \backslash D$.
2. Starting at the outermost layer B_L, we set $u_L := S_N f \in V$. Exploiting the facts $\operatorname{supp}(f) \subseteq D$ and $B_L \cap D = \emptyset$, one can then prove that u_L lies in a certain subspace $V_{\mathrm{harm}}(B_L) \subseteq V$ associated with the box B_L. The properties of this subspace allow us to construct a function $u_{L-1} \in V$ from a space with $\mathcal{O}(L^d)$ degrees of freedom that is a good approximation to u_L on the box B_{L-1}. More precisely, we can show that
$$\sum_{l=0}^{k} (\delta/\sigma_{\mathrm{sco}})^l |u_L - u_{L-1}|_{H^l(B_{L-1})} \leq \frac{1}{2} \sum_{l=0}^{k} (\delta/\sigma_{\mathrm{sco}})^l |u_L|_{H^l(B_L)},$$
where $\delta > 0$ is a parameter proportional to L^{-1} and where $\sigma_{\mathrm{sco}} > 0$ is a certain constant.
3. Our construction of the approximant u_{L-1} will guarantee that $u_{L-1} \in V_{\mathrm{harm}}(B_{L-1})$, and, since these subspaces are nested, we find that the error $u_L - u_{L-1} \in V_{\mathrm{harm}}(B_{L-1})$ as well. Again, the properties of this subspace allow us to construct a function $u_{L-2} \in V_{\mathrm{harm}}(B_{L-2})$ with $\mathcal{O}(L^d)$ degrees of freedom that approximates $u_L - u_{L-1}$ well on the box B_{L-2}. Proceeding inwards from the largest box B_L to the smallest box B_1, this procedure generates a finite sequence of functions $u_{L-1}, u_{L-2}, \ldots, u_1, u_0 \in V$ such that
$$\sum_{l=0}^{k} (\delta/\sigma_{\mathrm{sco}})^l |S_N f - (u_{L-1} + u_{L-2} + \cdots + u_1 + u_0)|_{H^l(B)} \leq 2^{-L} \sum_{l=0}^{k} (\delta/\sigma_{\mathrm{sco}})^l |S_N f|_{H^l(B_L)}.$$
From here, it is then not difficult to get an error estimate in the native space seminorm $|\cdot|_{a,B}$.

3.6 The cut-off operator

Let us now define precisely what we mean by an *inflated box*:

Definition 39 *Let* $B = \times_{i=1}^{d}(a_i, b_i)$ *with* $a_i \leq b_i$ *be a box as in Definition 15. For every* $\delta \geq 0$, *we introduce the* inflated box

$$B^{\delta} := \bigtimes_{i=1}^{d}(-\delta + a_i, b_i + \delta) \subseteq \mathbb{R}^d.$$

Note that B^{δ} is again a box. In particular, we can iterate $(B^{\delta})^{\delta} = B^{2\delta}$, $((B^{\delta})^{\delta})^{\delta} = B^{3\delta}$, et cetera.

In order for our construction of the space $V_{B,D,L}$ to work, we need a way to "restrict" the support of a given function $v \in V$ to a box $B \subseteq \mathbb{R}^d$ without destroying its global smoothness. This can be achieved by multiplying v with a smooth *cut-off function*:

Lemma 40 *Let* $B \subseteq \mathbb{R}^d$ *be a box and* $\delta > 0$. *Then, there exists a smooth* cut-off function κ_B^{δ} *with the following properties:*

$$\kappa_B^{\delta} \in C_0^{\infty}(\mathbb{R}^d), \quad \mathrm{supp}(\kappa_B^{\delta}) \subseteq B^{\delta}, \quad \kappa_B^{\delta}|_B \equiv 1, \quad 0 \leq \kappa_B^{\delta} \leq 1, \quad \forall l \in \mathbb{N}_0 : |\kappa_B^{\delta}|_{W^{l,\infty}(\mathbb{R}^d)} \lesssim \delta^{-l}.$$

Proof Write $B = \times_{i=1}^{d}(a_i, b_i)$ and pick a univariate function $g \in C^{\infty}(\mathbb{R})$ with $0 \leq g \leq 1$, $g|_{(-\infty,0]} \equiv 1$ and $g|_{[1/2,\infty)} \equiv 0$. Then, the function $\kappa_B^{\delta}(x) := \prod_{i=1}^{d} g((a_i - x_i)/\delta)g((x_i - b_i)/\delta)$, $x \in \mathbb{R}^d$, is a valid choice. □

It is convenient to introduce a name for the cut-off process described earlier:

Definition 41 *Let* $B \subseteq \mathbb{R}^d$ *be a box,* $\delta > 0$ *and* $\kappa_B^{\delta} \in C_0^{\infty}(\mathbb{R}^d)$ *be the smooth cut-off function from Lemma 40. We define the corresponding* cut-off operator

$$K_B^{\delta} : \begin{cases} V \longrightarrow H^k(\mathbb{R}^d) \\ v \longmapsto \kappa_B^{\delta} v \end{cases}.$$

In the following, we summarize the most important facts about the cut-off operator.

Lemma 42 *Let* $B \subseteq \mathbb{R}^d$ *be a box and* $\delta > 0$. *For all* $v \in V$, *the linear operator* K_B^{δ} *has the cut-off property* $\mathrm{supp}(K_B^{\delta}v) \subseteq B^{\delta}$ *and the local projection property* $(K_B^{\delta}v)|_B = v|_B$. *Furthermore, there holds the stability bound*

$$\sum_{l=0}^{k} \delta^l |K_B^{\delta} v|_{H^l(\mathbb{R}^d)} \leq C(d,k) \sum_{l=0}^{k} \delta^l |v|_{H^l(B^{\delta})}.$$

Proof Denote by $\kappa_B^{\delta} \in C_0^{\infty}(\mathbb{R}^d)$ the smooth cut-off function from Lemma 40, and let $v \in V$. The relations $\mathrm{supp}(K_B^{\delta}v) \subseteq B^{\delta}$ and $(K_B^{\delta}v)|_B = v|_B$ readily follow from Lemma 40. As for the stability bound, we have that $v \in V \subseteq H^k_{\mathrm{loc}}(\mathbb{R}^d)$ (cf. Lemma 20). Then, using Leibniz' product rule for derivatives, we obtain

$$\sum_{l=0}^{k} \delta^l |K_B^\delta v|_{H^l(\mathbb{R}^d)} = \sum_{l=0}^{k} \delta^l |\kappa_B^\delta v|_{H^l(B^\delta)} \lesssim \sum_{l=0}^{k} \delta^l \sum_{i=0}^{l} |\kappa_B^\delta|_{W^{l-i,\infty}(\mathbb{R}^d)} |v|_{H^i(B^\delta)}$$
$$\lesssim \sum_{l=0}^{k} \delta^l \sum_{i=0}^{l} \delta^{i-l} |v|_{H^i(B^\delta)} \lesssim \sum_{l=0}^{k} \delta^l |v|_{H^l(B^\delta)}.$$

□

3.7 The space $V_{\mathrm{harm}}(B)$

The key components of our proof are the subspaces $V_{\mathrm{harm}}(B) \subseteq V$ to be defined now. They exhibit a number of properties that allow us to find good low-dimensional approximants to their members.

Definition 43 *Let $B \subseteq \mathbb{R}^d$ open. We say that a function $u \in V$ is* homogeneous and harmonic on B, *if it satisfies the following conditions:*

1. *For all $n \in \{1, \dots, N\}$ with $x_n \in B$, there holds $u(x_n) = 0$.*
2. *For all $v \in V_0$ with* $\mathrm{supp}(v) \subseteq B$, *there holds $a(u, v) = 0$.*

The subspace of functions $u \in V$ that are homogeneous and harmonic functions on B is denoted by $V_{\mathrm{harm}}(B) \subseteq V$.

Loosely speaking, the space $V_{\mathrm{harm}}(B)$ consists of all functions $u \in V$ that vanish on $\{x_1, \dots, x_N\} \cap B$ and satisfy $\mathrm{L}^{2k} u = 0$ on $B \backslash \{x_1, \dots, x_N\}$ in a weak sense (cf. Definition 6, Lemma 7). Furthermore, note that $V_{\mathrm{harm}}(B)$ is an infinite-dimensional space, in general.

The next lemma establishes the fact that these spaces are nested and interact nicely with the solution operator $S_N : V \longrightarrow V$ from Definition 31 and the cut-off operator $K_B^\delta : V \longrightarrow H^k(\mathbb{R}^d)$ from Definition 41.

Lemma 44

1. *For all $B \subseteq B^+ \subseteq \mathbb{R}^d$, there holds $V_{\mathrm{harm}}(B^+) \subseteq V_{\mathrm{harm}}(B)$.*
2. *For all $B, D \subseteq \mathbb{R}^d$ with $B \cap D = \emptyset$ and all $f \in V$ with* $\mathrm{supp}(f) \subseteq D$, *there holds $S_N f \in V_{\mathrm{harm}}(B)$.*
3. *For all $B \subseteq \mathbb{R}^d$, $\delta > 0$ and $u \in V_{\mathrm{harm}}(B)$, there holds $K_B^\delta u \in V_{\mathrm{harm}}(B)$.*

Proof Item (1) is obvious. In order to see (2), consider subsets B, D and functions f as above. Then, for all $n \in \{1, \dots, N\}$ with $x_n \in B$, Definition 31 and the assumptions on B, D tell us that $(S_N f)(x_n) = f(x_n) = 0$. Furthermore, for all $v \in V_0$ (even with arbitrary support), $a(S_N f, v) = 0$. This proves $S_N f \in V_{\mathrm{harm}}(B)$. Finally, let us prove item (3): Let $B \subseteq \mathbb{R}^d$, $\delta > 0$ and $u \in V_{\mathrm{harm}}(B)$. For all $n \in \{1, \dots, N\}$ with

$x_n \in B$, we have $(K_B^\delta u)(x_n) = u(x_n) = 0$, owing to the fact that $(K_B^\delta u)|_B = u|_B$ (cf. Lemma 42). On the other hand, for all $v \in V_0$ with $\operatorname{supp}(v) \subseteq B$, we have

$$a(K_B^\delta u, v) \overset{\text{Def. 1}}{=} \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \langle \mathrm{D}^\alpha (K_B^\delta u), \mathrm{D}^\alpha v \rangle_{L^2(B)}$$
$$\overset{\text{Lem. 42}}{=} \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \langle \mathrm{D}^\alpha u, \mathrm{D}^\alpha v \rangle_{L^2(B)} = a(u, v) = 0.$$

□

Note that the membership $S_N f \in V_{\text{harm}}(B)$ only uses the interpolation and orthogonality conditions from Definition 31. The explicit representation $S_N f = \sum_{n=1}^{N} \boldsymbol{c}_n \phi_n + \sum_{|\alpha|<k_{\min}} \boldsymbol{d}_\alpha \pi_\alpha$ from Lemma 30 is completely irrelevant in this context. In fact, Lemma 38 already contained the last (implicit) occurrence of the fundamental solution ϕ for the remainder of this paper.

While Lemma 44 covers the basic, quick-to-prove aspects of the spaces $V_{\text{harm}}(B)$, we still need two more ingredients. Most importantly, we need a *Caccioppoli-type inequality* for functions $u \in V_{\text{harm}}(B^\delta)$, i.e., we want to bound the k-th derivatives of u on the box B by its lower-order derivatives on the slightly larger box B^δ.

Lemma 45 *Let $B \subseteq \mathbb{R}^d$ be a box and $\delta > 0$ with $\delta \lesssim 1$. Then, there holds the* Caccioppoli inequality*:*

$$\forall u \in V_{\text{harm}}(B^\delta): \qquad \delta^k |u|_{H^k(B)} \leq C(d, k) \sum_{l=0}^{k-1} \delta^l |u|_{H^l(B^\delta)}.$$

Proof Let us abbreviate $\kappa := \kappa_B^\delta \in C_0^\infty(\mathbb{R}^d)$ for the smooth cut-off function from Lemma 40. Recall that $|\kappa|_{W^{l,\infty}(\mathbb{R}^d)} \lesssim \delta^{-l}$ and $|\kappa^2|_{W^{l,\infty}(\mathbb{R}^d)} \lesssim \sum_{i=0}^{l} |\kappa|_{W^{i,\infty}(\mathbb{R}^d)} |\kappa|_{W^{l-i,\infty}(\mathbb{R}^d)} \lesssim \delta^{-l}$. Furthermore, $\kappa(x_n) = 0$ for all $n \in \{1, \ldots, N\}$ satisfying $x_n \notin B^\delta$.

Let $u \in V_{\text{harm}}(B^\delta)$. From Definition 43 we know that $u(x_n) = 0$ for all $n \in \{1, \ldots, N\}$ with $x_n \in B^\delta$. In particular, the product $v := \kappa^2 u \in V$ satisfies $v(x_n) = 0$ for *all* $n \in \{1, \ldots, N\}$. In other words, $v \in V_0$. Furthermore, using Lemma 42, we have $\operatorname{supp}(v) \subseteq \operatorname{supp}(\kappa) \subseteq B^\delta$. This proves that v is an admissible test function for Definition 43, i.e., $a(u, v) = 0$. Plugging in Definition 1, we get the identity

$$0 = a(u, \kappa^2 u) = \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \langle \mathrm{D}^\alpha u, \mathrm{D}^\alpha (\kappa^2 u) \rangle_{L^2(\mathbb{R}^d)}$$
$$= \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \langle \mathrm{D}^\alpha u, \mathrm{D}^{\alpha-\beta}(\kappa^2) \mathrm{D}^\beta u \rangle_{L^2(\mathbb{R}^d)}.$$

We transfer the summands with $\beta < \alpha$ to the other side of the equality and obtain the following expression:

$$\begin{aligned}\sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \|\kappa \mathrm{D}^\alpha u\|^2_{L^2(\mathbb{R}^d)} &= -\sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \sum_{\beta<\alpha} \binom{\alpha}{\beta} \langle \mathrm{D}^\alpha u, \mathrm{D}^{\alpha-\beta}(\kappa^2)\mathrm{D}^\beta u\rangle_{L^2(\mathbb{R}^d)} \\ &\lesssim \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \Big[\sum_{i=1}^{d} |\langle \mathrm{D}^\alpha u, \partial_i(\kappa^2)\mathrm{D}^{\alpha-e_i} u\rangle_{L^2(\mathbb{R}^d)}| \\ &\quad + \sum_{\substack{\beta<\alpha,\\ |\beta|\le|\alpha|-2}} |\langle \mathrm{D}^\alpha u, \mathrm{D}^{\alpha-\beta}(\kappa^2)\mathrm{D}^\beta u\rangle_{L^2(\mathbb{R}^d)}| \Big].\end{aligned}$$

For the summands in the first sum, we use Young's inequality (with arbitrary $\epsilon > 0$):

$$\begin{aligned}|\langle \mathrm{D}^\alpha u, \partial_i(\kappa^2)\mathrm{D}^{\alpha-e_i} u\rangle_{L^2(\mathbb{R}^d)}| &= 2|\langle \kappa \mathrm{D}^\alpha u, (\partial_i \kappa)\mathrm{D}^{\alpha-e_i} u\rangle_{L^2(B^\delta)}| \\ &\lesssim \|\kappa \mathrm{D}^\alpha u\|_{L^2(\mathbb{R}^d)} \|\partial_i \kappa\|_{L^\infty(\mathbb{R}^d)} \|\mathrm{D}^{\alpha-e_i} u\|_{L^2(B^\delta)} \\ &\lesssim \|\kappa \mathrm{D}^\alpha u\|_{L^2(\mathbb{R}^d)} \delta^{-1} |u|_{H^{|\alpha|-1}(B^\delta)} \\ &\lesssim \epsilon \|\kappa \mathrm{D}^\alpha u\|^2_{L^2(\mathbb{R}^d)} + \epsilon^{-1}\delta^{-2}|u|^2_{H^{|\alpha|-1}(B^\delta)}.\end{aligned}$$

Note that, by choosing ϵ sufficiently small, we can absorb the $\mathcal{O}(\epsilon)$-term in the left-hand side of the overall inequality.

For the summands in the second sum, we can pick an index $i \in \{1, \ldots, d\}$ with $\alpha_i \geq 1$ (in the case $|\alpha| = 0$, the sum is empty anyway). Then, we perform partial integration with respect to the i-th coordinate:

$$\begin{aligned}&|\langle \mathrm{D}^\alpha u, \mathrm{D}^{\alpha-\beta}(\kappa^2)\mathrm{D}^\beta u\rangle_{L^2(\mathbb{R}^d)}| = |\langle \mathrm{D}^{\alpha-e_i} u, \mathrm{D}^{\alpha-\beta+e_i}(\kappa^2)\mathrm{D}^\beta u + \mathrm{D}^{\alpha-\beta}(\kappa^2)\mathrm{D}^{\beta+e_i} u\rangle_{L^2(B^\delta)}| \\ &\le \|\mathrm{D}^{\alpha-e_i} u\|_{L^2(B^\delta)} (\|\mathrm{D}^{\alpha-\beta+e_i}(\kappa^2)\|_{L^\infty(\mathbb{R}^d)} \|\mathrm{D}^\beta u\|_{L^2(B^\delta)} \\ &\quad + \|\mathrm{D}^{\alpha-\beta}(\kappa^2)\|_{L^\infty(\mathbb{R}^d)} \|\mathrm{D}^{\beta+e_i} u\|_{L^2(B^\delta)}) \\ &\lesssim |u|_{H^{|\alpha|-1}(B^\delta)} (\delta^{-|\alpha|+|\beta|-1} |u|_{H^{|\beta|}(B^\delta)} + \delta^{-|\alpha|+|\beta|} |u|_{H^{|\beta|+1}(B^\delta)}) \\ &= \delta^{-2|\alpha|} (\delta^{|\alpha|-1} |u|_{H^{|\alpha|-1}(B^\delta)}) (\delta^{|\beta|} |u|_{H^{|\beta|}(B^\delta)} + \delta^{|\beta|+1} |u|_{H^{|\beta|+1}(B^\delta)}) \\ &\lesssim \delta^{-2|\alpha|} \sum_{i=0}^{|\alpha|-1} \delta^{2i} |u|^2_{H^i(B^\delta)}.\end{aligned}$$

Finally, we put everything together (exploiting $\kappa \equiv 1$ on B):

$$\delta^{2k}|u|^2_{H^k(B)} \overset{\sigma_k>0}{\lesssim} \delta^{2k} \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \|\kappa D^\alpha u\|^2_{L^2(\mathbb{R}^d)}$$

$$\lesssim \sum_{l=k_{\min}}^{k} \sigma_l \sum_{|\alpha|=l} \frac{l!}{\alpha!} \underbrace{\delta^{2(k-|\alpha|)}}_{\lesssim 1} \sum_{i=0}^{|\alpha|-1} \delta^{2i} |u|^2_{H^i(B^\delta)} \lesssim \sum_{l=0}^{k-1} \delta^{2l} |u|^2_{H^l(B^\delta)}.$$

This concludes the proof. □

We end the discussion of the spaces $V_{\mathrm{harm}}(B)$ with the following observation: If $k_{\min} = 0$, we remember from Section 2.1 that $(V, a(\cdot,\cdot))$ is a Hilbert space. One can then show that $V_{\mathrm{harm}}(B) \subseteq V$ is a closed subspace with respect to the norm $|\cdot|_a$ so that the orthogonal projection from V to $V_{\mathrm{harm}}(B)$ is well-defined. However, in the general case $k_{\min} \geq 0$, this line of reasoning is not possible anymore, and we need to find a different projection $P_{B,H} : V \longrightarrow V_{\mathrm{harm}}(B)$ that is in some sense stable. The idea here is to replace $a(\cdot,\cdot)$ by a *strictly* positive definite inner product $b(\cdot,\cdot)$ and then use the corresponding orthogonal projection. The new inner product is weighted with a free parameter $H > 0$ that will be important for the definition of the low-rank approximation operator $\Pi_H : H^k(\mathbb{R}^d) \longrightarrow H^k(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ in Lemma 47 below. More precisely, the parameter H will then be the meshsize of a coarse grid used to define a low rank approximation operator later on.

Lemma 46 *Let $B \subseteq \mathbb{R}^d$ be a box with $|B| > 0$ and let $H > 0$. There exists a linear operator*

$$P_{B,H} : V \longrightarrow V_{\mathrm{harm}}(B)$$

with the following properties:

1. Projection: *For all $v \in V_{\mathrm{harm}}(B)$, there holds $P_{B,H}v = v$.*
2. Stability: *For all $v \in V$, there holds the stability bound*

$$\sum_{l=k_{\min}}^{k} H^l |P_{B,H}v|_{H^l(\mathbb{R}^d)} + \sum_{l=0}^{k} H^l |P_{B,H}v|_{H^l(B)} \leq C(d,k)\Big(\sum_{l=k_{\min}}^{k} H^l |v|_{H^l(\mathbb{R}^d)} + \sum_{l=0}^{k} H^l |v|_{H^l(B)}\Big).$$

Proof We equip the native space V from Definition 1 with the following bilinear form:

$$\forall u, v \in V: \qquad b(u,v) := \sum_{l=k_{\min}}^{k} H^{2l} \sum_{|\alpha|=l} \langle D^\alpha u, D^\alpha v\rangle_{L^2(\mathbb{R}^d)} + \sum_{l=0}^{k} H^{2l} \sum_{|\alpha|=l} \langle D^\alpha u, D^\alpha v\rangle_{L^2(B)}.$$

We remind the reader of Lemma 20, where the inclusion $V \subseteq H^k_{\mathrm{loc}}(\mathbb{R}^d)$ was derived. In particular, the local quantities $|u|_{H^l(B)}$ are finite for *all* $l \in \{0, \dots, k\}$, so that $b(\cdot,\cdot)$ is indeed well-defined. Note that the assumption $|B| > 0$ guarantees the *strict* positive definiteness of $b(\cdot,\cdot)$ on all of V. The proof of completeness of $(V, b(\cdot,\cdot))$ is very similar to the one of Lemma 23 and will therefore be omitted. Finally, using the continuous Sobolev embedding $H^k(B) \subseteq C^0(B)$ and the Cauchy-Schwarz inequality for $a(\cdot,\cdot)$ (cf. Lemma 20), one can show that $V_{\mathrm{harm}}(B)$ is a closed subspace of V with respect to $b(\cdot,\cdot)$. Consequently, the $b(\cdot,\cdot)$-orthogonal projection $P_{B,H} : V \longrightarrow V_{\mathrm{harm}}(B)$ is well-defined. The asserted stability bound follows immediately from the fact that $\|P_{B,H}v\|_b \leq \|v\|_b$ for all $v \in V$. This finishes the proof. □

3.8 The low-rank approximation operator

We remind the reader of Section 3.5, where we argued that we need to construct a certain subspace $V_{B,D,L} \subseteq V$ of low dimension. More precisely, our goal was to achieve an algebraic dimension bound of the form $\dim V_{B,D,L} \lesssim L^{c_1}$, where $c_1 > 0$ is some constant. For this purpose, we use an approximation operator of moderate rank with good local approximation properties. Our construction is a "partition of unity" method (see, e.g., [40] and [10]) on a perfect tensor-product grid with meshsize $H > 0$ that spans all of $\mathbb{R}^d$. A pleasing side effect of this method is the fact that the rank of this operator varies "smoothly" with respect to the parameter H. Once again, we make use of the axes-parallel boxes $B \subseteq \mathbb{R}^d$ and their inflated relatives $B^\delta \subseteq \mathbb{R}^d$, $\delta > 0$ for the proof (cf. Definitions 15 and 39).

Lemma 47 *Let $H > 0$ be a free parameter. Then, there exists a linear operator*

$$\Pi_H : H^k(\mathbb{R}^d) \longrightarrow H^k(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$$

with the following properties:

1. Local rank: *For every box $B \subseteq \mathbb{R}^d$, there holds the dimension bound*

$$\dim \{\Pi_H v \mid v \in H^k(\mathbb{R}^d) \text{ with } \operatorname{supp}(v) \subseteq B\} \leq C(d,k)(1 + \operatorname{diam}(B)/H)^d.$$

2. Stability/error bound: *For all $v \in H^k(\mathbb{R}^d)$, there hold the following stability and error estimates:*

$$\sum_{l=0}^k H^l |\Pi_H v|_{H^l(\mathbb{R}^d)} \leq C(d,k) \sum_{l=0}^k H^l |v|_{H^l(\mathbb{R}^d)},$$
$$\sum_{l=0}^k H^l |v - \Pi_H v|_{H^l(\mathbb{R}^d)} \leq C(d,k) H^k |v|_{H^k(\mathbb{R}^d)}.$$

Proof Let $\mu \in C_0^\infty(\mathbb{R})$ be a "mollifier" with $\operatorname{supp}(\mu) \subseteq [-1/4, 1/4]$, $\mu \geq 0$ and $\int_{\mathbb{R}} \mu(x)\,\mathrm{d}x = 1$. Consider the mollified characteristic function of the interval $[0,1) \subseteq \mathbb{R}$, i.e., $\hat{g}_1(x) := (\mu * \mathbb{I}_{[0,1)})(x) = \int_{\mathbb{R}} \mu(y)\mathbb{I}_{[0,1)}(x-y)\,\mathrm{d}y$. There holds $\hat{g}_1 \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}(\hat{g}_1) \subseteq [-1/4, 5/4]$, $\hat{g}_1 \geq 0$ and $\sum_{m\in\mathbb{Z}} \hat{g}_1(x+m) = \int_{\mathbb{R}} \mu(y) \sum_{m\in\mathbb{Z}} \mathbb{I}_{y-x+[0,1)}(m)\,\mathrm{d}y = \int_{\mathbb{R}} \mu(y)\,\mathrm{d}y = 1$ for all $x \in \mathbb{R}$. (Note that, for every given x, the sum $\sum_{m\in\mathbb{Z}} \hat{g}_1(x+m)$ contains at most two non-zero summands.) Set $\hat{\Omega} := [-1/4, 5/4]^d \subseteq \mathbb{R}^d$ ("reference patch") and $\hat{g}(x) := \prod_{i=1}^d \hat{g}_1(x_i)$ for all $x \in \mathbb{R}^d$ ("reference bump function"). There holds $\hat{g} \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}(\hat{g}) \subseteq \hat{\Omega}$, $\hat{g} \geq 0$ and $\sum_{m\in\mathbb{Z}^d} \hat{g}(x+m) = 1$ for all $x \in \mathbb{R}^d$ (at most 2^d non-zero summands). Furthermore, denote by $\hat{\Pi} : H^k(\hat{\Omega}) \longrightarrow \mathbb{P}^{k-1}(\hat{\Omega})$ the orthogonal projection with respect to $\langle \cdot,\cdot \rangle_{H^k(\hat{\Omega})}$.

Next, let $H > 0$ be a given parameter ("meshwidth"). For every $m \in \mathbb{Z}^d$, consider the affine transformations $F_m, F_m^{-1} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, given by $F_m(\hat{x}) := H(\hat{x} + m)$ and $F_m^{-1}(x) := x/H - m$. The m-th patch and m-th bump function are defined by $\Omega'_m := F_m(\hat{\Omega}) \subseteq \mathbb{R}^d$ and $g_m(x) := \hat{g}(F_m^{-1}(x))$ for all $x \in \mathbb{R}^d$. There holds $g_m \in C_0^\infty(\mathbb{R}^d)$, $\operatorname{supp}(g_m) \subseteq \Omega'_m$, $g_m \geq 0$ and $\sum_{m \in \mathbb{Z}^d} g_m(x) = 1$ for all $x \in \mathbb{R}^d$ (at most 2^d non-zero summands). Furthermore, for all $l \in \mathbb{N}_0$, we have the relation $|g_m|_{W^{l,\infty}(\mathbb{R}^d)} = H^{-l} |\hat{g}|_{W^{l,\infty}(\mathbb{R}^d)} \eqsim H^{-l}$.

Now we have everything we need to define the asserted operator:

$$\Pi_H : \begin{cases} H^k(\mathbb{R}^d) \longrightarrow & H^k(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \\ \quad v \quad \longmapsto & \sum_{m \in \mathbb{Z}^d} (\hat{\Pi}(v \circ F_m) \circ F_m^{-1}) g_m \end{cases} .$$

Note that we implicitly restricted $v \circ F_m$ from $\mathbb{R}^d$ to $\hat{\Omega}$ and extended the polynomial $\hat{\Pi}(v \circ F_m) \circ F_m^{-1}$ from Ω'_m to $\mathbb{R}^d$. Furthermore, for every compact $K \subseteq \mathbb{R}^d$, the sum in $(\Pi_H v)|_K$ contains only finitely many non-zero summands. In particular, since $\hat{\Pi}(v \circ F_m) \circ F_m^{-1} \in \mathbb{P}^{k-1}(\mathbb{R}^d) \subseteq C^\infty(\mathbb{R}^d)$ and $g_m \in C_0^\infty(\mathbb{R}^d)$, we have $\Pi_H v \in C^\infty(\mathbb{R}^d)$, indeed. The fact that $\Pi_H v \in H^k(\mathbb{R}^d)$ follows from the stability bound below.

As for the dimension bound, let $B \subseteq \mathbb{R}^d$ be a box and denote by $ms(B) := \{m \in \mathbb{Z}^d \mid B \cap \Omega'_m \neq \emptyset\}$ the set of indices of the adjacent patches. We will need an upper bound for the cardinality of $ms(B)$ in terms of $\operatorname{diam}(B)$ and H. To this end, we use appropriate sub- and supersets of $\bigcup_{m \in ms(B)} \Omega'_m$ and exploit the monotonicity of the d-dimensional Lebesgue measure, denoted by $|\cdot|$. On the one hand, since every patch Ω'_m is itself a box of side length $3H/2$, it is not surprising that $\bigcup_{m \in ms(B)} \Omega'_m \subseteq B^{3H/2}$, where $B^{3H/2}$ denotes the inflated box in the sense of Definition 39. On the other hand, for every $m \in ms(B)$, consider the m-th subpatch $\omega_m := F_m([0,1]^d) \subseteq \Omega'_m$. Clearly, these subpatches are pairwise disjoint and fulfill $\bigcup_{m \in ms(B)} \Omega'_m \supseteq \bigcup_{m \in ms(B)} \omega_m$. Combining both inclusions, we get the desired bound for $\# ms(B)$:

$$\begin{aligned} \# ms(B) H^d &= \sum_{m \in ms(B)} H^d = \sum_{m \in ms(B)} |\omega_m| = \left| \bigcup_{m \in ms(B)} \omega_m \right| \leq \left| \bigcup_{m \in ms(B)} \Omega'_m \right| \\ &\leq |B^{3H/2}| \leq C(d)(\operatorname{diam}(B) + H)^d. \end{aligned}$$

Now, for every $v \in H^k(\mathbb{R}^d)$ with $\operatorname{supp}(v) \subseteq B$, the support properties of the bump functions g_m guarantee that the sum in $\Pi_H v$ only ranges over $ms(B)$, rather than all of $\mathbb{Z}^d$. Therefore,

$$\begin{aligned} &\dim \{\Pi_H v \mid v \in H^k(\mathbb{R}^d), \operatorname{supp}(v) \subseteq B\} \\ &\qquad = \dim \{\textstyle\sum_{m \in ms(B)} (\hat{\Pi}(v \circ F_m) \circ F_m^{-1}) g_m \mid v \in H^k(\mathbb{R}^d), \operatorname{supp}(v) \subseteq B\} \\ &\qquad \leq \dim \{\textstyle\sum_{m \in ms(B)} v_m g_m \mid v_m \in \mathbb{P}^{k-1}(\mathbb{R}^d)\} \\ &\qquad \leq \# ms(B) \cdot \dim \mathbb{P}^{k-1}(\mathbb{R}^d) \\ &\qquad \leq C(d,k)(1 + \operatorname{diam}(B)/H)^d. \end{aligned}$$

Now on to the stability and error estimates. The derivation is based on the following facts:

1. *Partition of unity:* For every $v \in H^k(\mathbb{R}^d)$ there holds $v = v \cdot 1 = \sum_{m\in\mathbb{Z}^d} v g_m$.
2. *Scaling argument:* For all $v \in H^k(\Omega'_m)$, there holds $v \circ F_m \in H^k(\hat{\Omega})$ with $H^l |v|_{H^l(\Omega_m)} \eqsim H^{d/2} |v \circ F_m|_{H^l(\hat{\Omega})}, l = 0, \ldots, k$.
3. *Bramble-Hilbert:* For all $v \in H^k(\hat{\Omega})$, there holds $\|v - \hat{\Pi} v\|_{H^k(\hat{\Omega})} \leq C(d, k, \hat{\Omega})$ $|v|_{H^k(\hat{\Omega})}$.

Let $v \in H^k(\mathbb{R}^d)$ and $n \in \mathbb{Z}^d$. Again, we denote by $ms(n) := \{m \in \mathbb{Z}^d \mid \|m - n\|_\infty \leq 1\}$ the indices of the patches touching Ω'_n. Using (1), (2) and (3), we can establish a *local* error bound first:

$$
\begin{aligned}
\sum_{l=0}^k H^l |v - \Pi_H v|_{H^l(\Omega'_n)} &\overset{1.}{=} \sum_{l=0}^k H^l \left| \sum_{m\in ms(n)} (v - \hat{\Pi}(v \circ F_m) \circ F_m^{-1}) g_m \right|_{H^l(\Omega'_n)} \\
&\leq \sum_{l=0}^k H^l \sum_{m\in ms(n)} |(v - \hat{\Pi}(v \circ F_m) \circ F_m^{-1}) g_m|_{H^l(\Omega'_n \cap \Omega'_m)} \\
&\lesssim \sum_{l=0}^k H^l \sum_{m\in ms(n)} \sum_{j=0}^l |v - \hat{\Pi}(v \circ F_m) \circ F_m^{-1}|_{H^j(\Omega'_m)} |g_m|_{W^{l-j,\infty}(\mathbb{R}^d)} \\
&\eqsim \sum_{l=0}^k H^l \sum_{m\in ms(n)} \sum_{j=0}^l |v - \hat{\Pi}(v \circ F_m) \circ F_m^{-1}|_{H^j(\Omega'_m)} H^{j-l} \\
&\lesssim \sum_{m\in ms(n)} \sum_{j=0}^k H^j |v - \hat{\Pi}(v \circ F_m) \circ F_m^{-1}|_{H^j(\Omega'_m)} \\
&\overset{2.}{\eqsim} \sum_{m\in ms(n)} H^{d/2} \|v \circ F_m - \hat{\Pi}(v \circ F_m)\|_{H^k(\hat{\Omega})} \\
&\overset{3.}{\lesssim} \sum_{m\in ms(n)} H^{d/2} |v \circ F_m|_{H^k(\hat{\Omega})} \\
&\overset{2.}{\lesssim} H^k \sum_{m\in ms(n)} |v|_{H^k(\Omega_m)}.
\end{aligned}
$$

To get the desired *global* error bound, we exploit the covering property $\mathbb{R}^d \subseteq \bigcup_{n\in\mathbb{Z}^d} \Omega_n$ and sum up the local error contributions from above. Additionally, we use $\#ms(n) = 3^d$ and the fact that $\#\{n \in \mathbb{Z}^d \mid x \in \Omega'_n\} \leq 2^d$ for all $x \in \mathbb{R}^d$:

$$
\begin{aligned}
\sum_{l=0}^k H^{2l} |v - \Pi_H v|^2_{H^l(\mathbb{R}^d)} &\leq \sum_{n\in\mathbb{Z}^d} \sum_{l=0}^k H^{2l} |v - \Pi_H v|^2_{H^l(\Omega'_n)} \\
&\lesssim \sum_{n\in\mathbb{Z}^d} H^{2k} \sum_{m\in ms(n)} |v|^2_{H^k(\Omega'_m)} \\
&= 3^d H^{2k} \sum_{n\in\mathbb{Z}^d} |v|^2_{H^k(\Omega'_n)} \leq 6^d H^{2k} |v|^2_{H^k(\mathbb{R}^d)}.
\end{aligned}
$$

Finally, the stability bound can be shown via a simple triangle inequality and the previously established error bound. This concludes the proof. □

3.9 The single- and multi-step coarsening operators

This section contains the heart of our proof. Using the subspaces $V_{\mathrm{harm}}(B) \subseteq V$ from Definition 43, let us briefly recapitulate the proof outline from Section 3.5: Given a function $u \in V_{\mathrm{harm}}(B^\delta)$, our goal is to construct a function $\tilde{u} \in V_{\mathrm{harm}}(B)$ of low "dimension" that is a good approximation to u on the box $B \subseteq \mathbb{R}^d$. To this end, we concatenate the cut-off operator $K_B^{\delta/2} : V \longrightarrow H^k(\mathbb{R}^d)$ from Definition 41, the low-rank approximation operator $\Pi_H : H^k(\mathbb{R}^d) \longrightarrow H^k(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ from Lemma 47, and the projection $P_{B,H} : V \longrightarrow V_{\text{harm}}(B)$ from Lemma 46. The cut-off operator guarantees that the right-hand side of the error estimate is a *local* quantity, i.e., $|\cdot|_{H^l(B^\delta)}$. The low-rank operator Π_H is responsible for the reduction of the "dimension" of $\tilde{u}$. Finally, the projection $P_{B,H}$ maps the output of Π_H back into the space $V_{\text{harm}}(B)$.

Theorem 48 *Let $B \subseteq \mathbb{R}^d$ be a box with $|B| > 0$ and $\delta > 0$ be a free parameter with $\delta \lesssim 1$. Then, there exist a constant $\sigma_{\text{sco}} \geq 1$ and a linear* single-step coarsening operator

$$Q_B^\delta : V_{\text{harm}}(B^\delta) \longrightarrow V_{\text{harm}}(B)$$

with the following properties:

1. Rank bound: *The rank is bounded by*

$$\operatorname{rank}(Q_B^\delta) \leq C(d,k)(1 + \operatorname{diam}(B)/\delta)^d.$$

2. Approximation error: *For all $u \in V_{\text{harm}}(B^\delta)$, there holds the error bound*

$$\sum_{l=0}^{k} (\delta/\sigma_{\text{sco}})^l |u - Q_B^\delta u|_{H^l(B)} \leq \frac{1}{2} \sum_{l=0}^{k} (\delta/\sigma_{\text{sco}})^l |u|_{H^l(B^\delta)}.$$

Proof Let $B \subseteq \mathbb{R}^d$ and $\delta > 0$ be as above. The asserted single-step coarsening operator is composed of three operators: First, we need the cut-off operator $K_B^{\delta/2} : V \longrightarrow H^k(\mathbb{R}^d)$ from Definition 41. Second, let $H > 0$ and denote by $\Pi_H : H^k(\mathbb{R}^d) \longrightarrow H^k(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ the operator from Lemma 47. The precise value of H will be chosen during the proof (it will be $H := \delta/\sigma_{\text{sco}}$ for some specific constant $\sigma_{\text{sco}} \geq 1$). Third, let $P_{B,H} : V \longrightarrow V_{\text{harm}}(B)$ be the projection from Lemma 46. The single-step coarsening operator is then defined as

$$Q_B^\delta := P_{B,H} \Pi_H K_B^{\delta/2} : V_{\text{harm}}(B^\delta) \longrightarrow V_{\text{harm}}(B).$$

Recall from Lemma 20 that $H^k(\mathbb{R}^d) \subseteq V$, so that the output of Π_H is indeed a valid input for $P_{B,H}$.

We start the discussion of Q_B^δ with the error estimate: Let $u \in V_{\text{harm}}(B^\delta)$. On the one hand, we know from Lemma 44 that $u \in V_{\text{harm}}(B)$ and that $K_B^{\delta/2} u \in V_{\text{harm}}(B)$. Since $P_{B,H} : V \longrightarrow V_{\text{harm}}(B)$ is a projection, we know that $P_{B,H} K_B^{\delta/2} u = K_B^{\delta/2} u$. It follows that $u|_B = (K_B^{\delta/2} u)|_B = (P_{B,H} K_B^{\delta/2} u)|_B$, because the cut-off operator $K_B^{\delta/2}$ leaves u untouched on the box B (cf. Lemma 42). On the other hand, we know that $u \in V_{\text{harm}}(B^{\delta/2})$ (again by Lemma 44). Since $0 < \delta/2 \lesssim 1$, we may apply the Caccioppoli inequality from Lemma 45 to the box $B^{\delta/2}$, the parameter $\delta/2$, and the function $u \in V_{\text{harm}}(B^{\delta/2})$:

$$\begin{aligned}
\textstyle\sum_{l=0}^{k} H^l |u - Q_B^{\delta} u|_{H^l(B)} &= \textstyle\sum_{l=0}^{k} H^l |P_{B,H} K_B^{\delta/2} u - P_{B,H} \Pi_H K_B^{\delta/2} u|_{H^l(B)} \\
&= \textstyle\sum_{l=0}^{k} H^l |P_{B,H} (\text{id} - \Pi_H) K_B^{\delta/2} u|_{H^l(B)} \\
&\overset{\text{Lem. 46}}{\lesssim} \textstyle\sum_{l=0}^{k} H^l |(\text{id} - \Pi_H) K_B^{\delta/2} u|_{H^l(\mathbb{R}^d)} \\
&\overset{\text{Lem. 47}}{\lesssim} H^k |K_B^{\delta/2} u|_{H^k(\mathbb{R}^d)} \\
&\overset{\text{Lem. 42}}{\lesssim} (H/\delta)^k \textstyle\sum_{l=0}^{k} \delta^l |u|_{H^l(B^{\delta/2})} \\
&\overset{\text{Lem. 45}}{\lesssim} 2^{-1} (H/\delta)^k \textstyle\sum_{l=0}^{k-1} \delta^l |u|_{H^l(B^{\delta})}.
\end{aligned}$$

Now, denote by $\sigma_{\text{sco}} \geq 1$ the implicit cumulative constant. We choose $H := \delta/\sigma_{\text{sco}} > 0$. Then,

$$\begin{aligned}
\sum_{l=0}^{k} H^l |u - Q_B^{\delta} u|_{H^l(B)} &\leq (\sigma_{\text{sco}}/2)(H/\delta)^k \sum_{l=0}^{k-1} \delta^l |u|_{H^l(B^{\delta})} = \frac{1}{2} \sum_{l=0}^{k-1} (1/\sigma_{\text{sco}})^{k-1-l} H^l |u|_{H^l(B^{\delta})} \\
&\leq \frac{1}{2} \sum_{l=0}^{k-1} H^l |u|_{H^l(B^{\delta})}.
\end{aligned}$$

Finally, we turn our attention to the rank bound. For every $u \in V_{\text{harm}}(B^{\delta})$, we know from Lemma 42 that $K_B^{\delta/2} u \in H^k(\mathbb{R}^d)$ and that $\text{supp}(K_B^{\delta/2} u) \subseteq B^{\delta/2}$. Using the local rank bound of the operator Π_H (cf. Lemma 47), we get

$$\begin{aligned}
\text{rank}(Q_B^{\delta}) &= \dim \{ P_{B,H} \Pi_H K_B^{\delta/2} u \mid u \in V_{\text{harm}}(B^{\delta}) \} \\
&\leq \dim \{ \Pi_H v \mid v \in H^k(\mathbb{R}^d) \text{ with } \text{supp}(v) \subseteq B^{\delta/2} \} \\
&\overset{\text{Lem. 47}}{\leq} C(d,k)(1 + \text{diam}(B^{\delta/2})/H)^d \overset{\delta \eqsim H}{\leq} C(d,k)(1 + \text{diam}(B)/H)^d.
\end{aligned}$$

This finishes the proof. □

A closer inspection of the previous proof tells us that the sum on the right-hand side of the approximation error actually ranges from 0 to $k-1$, rather than k. However, we will not exploit this fact any further.

Next, let us combine $L \in \mathbb{N}$ single-step coarsening operators into one *multi-step coarsening operator*:

Theorem 49 *Let $B \subseteq \mathbb{R}^d$ be a box with $|B| > 0$ and $\delta > 0$ be a free parameter with $\delta \lesssim 1$. Furthermore, let $L \in \mathbb{N}$. Then, there exists a linear* multi-step coarsening operator

$$Q_B^{\delta,L} : V_{\text{harm}}(B^{\delta L}) \longrightarrow V_{\text{harm}}(B)$$

with the following properties:

1. Rank bound: *The rank is bounded by*

$$\operatorname{rank}(Q_B^{\delta,L}) \leq C(d,k)(L + \operatorname{diam}(B)/\delta)^{d+1}.$$

2. Approximation error: *For all $u \in V_{\text{harm}}(B^{\delta L})$, there holds the error bound*

$$\|u - Q_B^{\delta,L} u\|_{H^k(B)} \leq C(d,k)\delta^{-k} 2^{-L} \|u\|_{H^k(B^{\delta L})}.$$

Proof The construction is identical to the one in [1, Theorem 3.32], and consists only of iteratively applying the single-step approximation result k-times on nested boxes. The additional factor δ^{-k} in the error bound stems from the norm equivalence $\delta^k \|\cdot\|_{H^k(B)} \lesssim \sum_{l=0}^{k} (\delta/\sigma_{\text{sco}})^l |\cdot|_{H^l(B)} \lesssim \|\cdot\|_{H^k(B)}$, which makes use of the relations $\delta^k \lesssim \delta^l \lesssim 1$. □

Looking at the right-hand side of the error bound in Lemma 38, it would be more natural to look for an error bound in the local "energy"-seminorm $|\cdot|_{a,B}$ from Definition 1, i.e., an upper bound for $|u - Q_B^{\delta,L} u|_{a,B}$ in terms of $|u|_{a,B^{\delta L}}$. In fact, if we *could* prove $|u - Q_B^{\delta} u|_{a,B} \leq 2^{-1} |u|_{a,B^{\delta}}$ in Theorem 48, then we would obtain $|u - Q_B^{\delta,L} u|_{a,B} \leq 2^{-L} |u|_{a,B^{\delta L}}$ without the extra factor δ^{-k}. However, in the proof of Theorem 48 the estimate $\delta^k |K_B^{\delta/2} u|_{H^k(\mathbb{R}^d)} \lesssim \sum_{l=0}^{k} \delta^l |u|_{H^l(B^{\delta/2})}$ from Lemma 42 was essential. Here, Leibniz' product rule introduced lower-order derivatives of u, irrespective of the degree $k_{\min} \in \{0, \ldots, k\}$ and the coefficients $\sigma_l \geq 0$ from Definition 1. Since the right-hand side of the overall inequality now contains all $l \in \{0, \ldots, k\}$ orders of derivatives, we might as well work with a *full* H^k-norm (rather than $|\cdot|_{a,B}$) right from the beginning. It seems that this minor inconvenience cannot be avoided with our method of proof.

3.10 Putting everything together

We are finally in position to prove our main result, Theorem 18. In Section 3.5, we reduced the original problem of approximating the inverse of the interpolation matrix from Definition 13 to the problem of finding a low-dimensional subspace $V_{B,D,L} \subseteq V$ with certain approximation properties. Here, it was assumed that the boxes $B, D \subseteq \mathbb{R}^d$ satisfy some admissibility condition. As we shall prove next, the range of the multi-step coarsening operator from Theorem 49 is a valid choice for $V_{B,D,L}$.

Theorem 50 *Let $B, D \subseteq \mathbb{R}^d$ be two boxes with $B \cap \{x_1, \ldots, x_N\} \neq \emptyset$, $D \cap \{x_1, \ldots, x_N\} \neq \emptyset$ and $h_{\min} \leq \operatorname{diam}(B) \leq \sigma_{\text{adm}} \operatorname{dist}(B, D)$. Furthermore, let $L \in \mathbb{N}$. Then, there exists a subspace*

$$V_{B,D,L} \subseteq V$$

with the following properties:

1. Dimension bound: *There holds the dimension bound*

$$\dim V_{B,D,L} \leq C(d,k,\sigma_{\text{adm}}) L^{d+1}.$$

2. Approximation property: *For every* $f \in V$ *with* $\operatorname{supp}(f) \subseteq D$*, there holds the error bound*

$$\inf_{v \in V_{B,D,L}} \|S_N f - v\|_{H^k(B)} \le C(d, k, \sigma_{\text{adm}}, (\sigma_l), (\xi_\alpha)_\alpha)(L/h_{\min})^k 2^{-L} |f|_a.$$

Proof Let $B, D \subseteq \mathbb{R}^d$ and $L \in \mathbb{N}$ as above. Set $\delta := \operatorname{diam}(B)/(2\sqrt{d}\sigma_{\text{adm}} L) > 0$ and denote by $Q_B^{\delta,L} : V_{\text{harm}}(B^{\delta L}) \longrightarrow V_{\text{harm}}(B)$ the multi-step coarsening operator from Theorem 49. We choose the space

$$V_{B,D,L} := \operatorname{ran}(Q_B^{\delta,L}) \subseteq V_{\text{harm}}(B) \subseteq V.$$

Using Theorem 49 and the definition of δ, we can bound the dimension as follows:

$$\dim V_{B,D,L} = \operatorname{rank}(Q_B^{\delta,L}) \le C(d,k)(L + \operatorname{diam}(B)/\delta)^{d+1} \le C(d,k,\sigma_{\text{adm}}) L^{d+1}.$$

Finally, let $f \in V$ with $\operatorname{supp}(f) \subseteq D$. In order to show that the error bound from Theorem 49 is applicable to $S_N f \in V$, we first need to establish the fact that $S_N f \in V_{\text{harm}}(B^{\delta L})$. According to Lemma 44, it suffices to prove that the sets $B^{\delta L}$ and D are disjoint. To that end, we choose a point $z \in \overline{B^{\delta L}}$ with $\operatorname{dist}(B^{\delta L}, D) = \operatorname{dist}(z, D)$. Then, $\operatorname{dist}(B, D) \le \operatorname{dist}(B, z) + \operatorname{dist}(z, D) \le \sqrt{d}\delta L + \operatorname{dist}(B^{\delta L}, D)$. Combined with the definition of δ and the admissibility condition, this yields

$$\begin{aligned} \operatorname{dist}(B^{\delta L}, D) &\ge \operatorname{dist}(B, D) - \sqrt{d}\delta L = \operatorname{dist}(B, D) - \operatorname{diam}(B)/(2\sigma_{\text{adm}}) \\ &\ge \operatorname{diam}(B)/(2\sigma_{\text{adm}}) \ge h_{\min} > 0. \end{aligned}$$

Now Lemma 44 implies $S_N f \in V_{\text{harm}}(B^{\delta L})$, so that $Q_B^{\delta,L}(S_N f) \in V_{B,D,L}$. Hence, the error bound from Theorem 49 is applicable to the function $S_N f$. Then, exploiting $\operatorname{supp}(f) \subseteq D$ and $\delta \gtrsim \operatorname{diam}(B)/L \ge h_{\min}/L$, we can estimate

$$\begin{aligned} \inf_{v \in V_{B,D,L}} \|S_N f - v\|_{H^k(B)} &\le \|S_N f - Q_B^{\delta,L}(S_N f)\|_{H^k(B)} \lesssim \delta^{-k} 2^{-L} \|S_N f\|_{H^k(B^{\delta L})} \\ &\overset{\operatorname{supp} f \subset D}{\lesssim} (L/h_{\min})^k 2^{-L} \|f - S_N f\|_{H^k(B^{\delta L})}. \end{aligned}$$

To bound the remaining local H^k-norm, we pick a suitable superset $\Omega \supseteq B^{\delta L}$ and use the Poincaré-type inequality from Lemma 19. To that end, recall from Definition 2 that there exists a constant $C > 0$, such that $\|x_n\|_2 \le C$ for *all* values of $N \ge N_{\min}$ and $n \in \{1, \dots, N\}$. Exploiting the admissibility condition and the fact that both B and D contain at least one of the interpolation points $x_1, \dots, x_N$, it is not difficult to see that $\operatorname{diam}(B) \le 2\sigma_{\text{adm}} C$. We obtain the bound

$$\sup_{x \in B^{\delta L}} \|x\|_2 \le \sup_{x \in B} \|x\|_2 + \sqrt{d}\delta L \le \max_{n \in \{1,\dots,N\}} \|x_n\|_2 + \operatorname{diam}(B) + \sqrt{d}\delta L \le 2(1 + \sigma_{\text{adm}})C,$$

which tells us that $B^{\delta L} \subseteq \operatorname{Ball}(0, 2(1 + \sigma_{\text{adm}})C) =: \Omega$. Note that Ω is independent of N and contains the unisolvent points $\{\xi_\alpha \mid |\alpha| < k_{\min}\} \subseteq \mathbb{R}^d$ from

Definition 2, which are independent of N as well. Proceeding similarly to the proof of Lemma 23, we intend to apply the Poincaré-type inequality from Lemma 19 to the bounded Lipschitz domain Ω, the normed vector space $Z = \ell^2(\{\alpha \mid |\alpha| < k_{\min}\})$, and the operator $\iota_Z : H^k(\Omega) \longrightarrow Z$, $\iota_Z g := (g(\xi_\alpha))_{|\alpha|<k_{\min}}$, which is a relative of the evaluation operator E_N from Definition 3. The continuity of ι_Z follows from the Sobolev embedding $H^k(\Omega) \subseteq C^0(\overline{\Omega})$ and the implication ($\iota_Z p = \mathbf{0} \Rightarrow p = 0$) for all $p \in \mathbb{P}^{k_{\min}-1}(\Omega)$ can be argued by unisolvency. Then, by Lemma 19 and Definition 1, we have the following bound:

$$\forall g \in H^k(\Omega) \text{ with } \iota_Z g = \mathbf{0}: \qquad \|g\|_{H^k(\Omega)} \leq C(d, k, \Omega, (\xi_\alpha)_\alpha)|g|_{H^k(\Omega)} \\ \leq C(d, k, \Omega, (\xi_\alpha)_\alpha, \sigma_k)|g|_{a,\Omega}.$$

Now, according to Definition 31, the function $f - S_N f$ vanishes on *all* interpolation points $\{x_1, \ldots, x_N\}$. Owing to Definition 2, we know that $\{\xi_\alpha \mid |\alpha| < k_{\min}\} \subseteq \{x_1, \ldots, x_N\}$, which implies $\iota_Z((f - S_N f)|_\Omega) = \mathbf{0}$. Therefore, we may apply the aforementioned Poincaré inequality to the function $(f - S_N f)|_\Omega$. Finally, using the *a priori* estimate $|S_N f|_a \leq |f|_a$ from Lemma 26, we get the following overall bound:

$$\inf_{v \in V_{B,D,L}} \|S_N f - v\|_{H^k(B)} \lesssim (L/h_{\min})^k 2^{-L} \|f - S_N f\|_{H^k(\Omega)} \\ \lesssim (L/h_{\min})^k 2^{-L} |f - S_N f|_{a,\Omega} \lesssim (L/h_{\min})^k 2^{-L} |f|_a.$$

This concludes the proof. □

Finally, we have everything we need for the proof of Theorem 18.

Proof Let $\boldsymbol{S}_{\mathbf{11}} \in \mathbb{R}^{N \times N}$ be the matrix from Theorem 18 and $r \in \mathbb{N}$ a given block rank bound. We define the asserted $\mathcal{H}$-matrix approximant $\boldsymbol{M} \in \mathbb{R}^{N \times N}$ in a block-wise fashion:

First, consider an admissible block $(I, J) \in \mathbb{P}_{\text{adm}}$. From Definition 16 we know that there exist boxes $B_I, B_J \subseteq \mathbb{R}^d$ with $\Omega_I \subseteq B_I$, $\Omega_J \subseteq B_J$ and $\operatorname{diam}(B_I) \leq \sigma_{\text{adm}} \operatorname{dist}(B_I, B_J)$. In particular, B_I and B_J both contain at least one interpolation point and there holds $\operatorname{diam}(B_I) \geq \operatorname{diam}(\Omega_I) \geq h_{\min}$ by Definition 14. This means that Theorem 50 is applicable to B_I and B_J. Now, denote by $C > 0$ the constant from the dimension bound in Theorem 50. We set $\sigma_{\exp} := 1/(2C^{1/(d+1)}) > 0$ and $L := \lfloor (r/C)^{1/(d+1)} \rfloor \in \mathbb{N}_0$. Then, Theorem 50 provides a subspace $V_{I,J,r} \subseteq V$. We apply Lemma 38 to this subspace and get an integer $\tilde{r} \leq \dim V_{I,J,r}$ and matrices $\boldsymbol{X}_{I,J,r} \in \mathbb{R}^{I \times \tilde{r}}$ and $\boldsymbol{Y}_{I,J,r} \in \mathbb{R}^{J \times \tilde{r}}$. We set

$$\boldsymbol{M}|_{I \times J} := \boldsymbol{X}_{I,J,r}(\boldsymbol{Y}_{I,J,r})^T.$$

Second, for every small block $(I, J) \in \mathbb{P}_{\text{small}}$, we make the trivial choice

$$\boldsymbol{M}|_{I \times J} := \boldsymbol{S}_{\mathbf{11}}|_{I \times J}.$$

By Definition 17, we have $\boldsymbol{M} \in \mathcal{H}(\mathbb{P}, \tilde{r})$ with a block rank bound

$$\tilde{r} \leq \dim V_{I,J,r} \overset{\text{Def. } C}{\leq} C L^{d+1} \overset{\text{Def. } L}{\leq} r.$$

For the error we get

$$\begin{aligned}
\|\boldsymbol{S}_{11} - \boldsymbol{M}\|_2 &\overset{\text{Def.16}}{\leq} \ln(N) \max_{(I,J)\in\mathbb{P}_{\text{adm}}} \|\boldsymbol{S}_{11}|_{I\times J} - \boldsymbol{X}_{I,J,r}(\boldsymbol{Y}_{I,J,r})^T\|_2 \\
&\overset{\text{Lem. 38}}{\lesssim} \ln(N) h_{\min}^{d-2k} \max_{(I,J)\in\mathbb{P}_{\text{adm}}} \sup_{\substack{f\in V:\\ \operatorname{supp}(f)\subseteq B_J}} \inf_{w\in V_{I,J,r}} \frac{|S_N f - w|_{a,B_I}}{|f|_a} \\
&\overset{\text{Thm. 50}}{\lesssim} \ln(N) h_{\min}^{d-3k} L^k 2^{-L} \\
&\overset{\text{Def. } L}{\lesssim} \ln(N) h_{\min}^{d-3k} r^{k/(d+1)} \exp(-\ln(2)(r/C)^{1/(d+1)}) \\
&\overset{\text{Def. } \sigma_{\exp}}{\lesssim} \ln(N) h_{\min}^{d-3k} \exp(-\sigma_{\exp} r^{1/(d+1)}) \\
&\overset{\text{Def. 2}}{\lesssim} \ln(N) N^{\sigma_{\text{card}}(3k-d)/d} \exp(-\sigma_{\exp} r^{1/(d+1)}).
\end{aligned}$$

□

3.11 The fundamental solution (proofs)

In this subsection we provide the proofs of Lemmas 10 and 11.

Proof of Lemma *10* Recall that $d \geq 1$, $k \in \mathbb{N}$ with $k > d/2$, $b \in (0, \infty)$ and that ϕ was given as an integral expression:

$$\forall x \in \mathbb{R}^d: \qquad \phi(x) := \frac{(4\pi)^{-d/2}}{\Gamma(k)} \int_0^\infty t^{k-d/2-1} e^{-b^2 t} e^{-\|x\|_2^2/(4t)} \,\mathrm{d}t.$$

The trivial bound $e^{-\|x\|_2^2/(4t)} \leq 1$, the substitution $t = s/b^2$, and the assumption $k > d/2$ show that the integral is indeed well-defined with a uniform upper bound $|\phi(x)| \leq C(d,k)\Gamma(k-d/2)b^{d-2k}$. In particular, Lebesgue's Dominated Convergence Theorem tells us that $\phi \in C^0(\mathbb{R}^d)$ as well.

In order to show that ϕ is a fundamental solution of $\mathrm{L}^{2k} = (b^2 - \Delta)^k$, we use standard Fourier techniques. First, using Fubini's Theorem, a straightforward computation reveals $\phi \in L^1(\mathbb{R}^d)$. To compute the Fourier transform $\hat{\phi}$ of ϕ, recall that the Gaussian kernel $e^{-\|\cdot\|_2^2/2}$ is a fixed point of the Fourier transformation and that there holds the relation $\mathcal{F}(e^{-\|\cdot\|_2^2/(4t)})(y) = (2t)^{d/2} e^{-t\|y\|_2^2}$ for all $t > 0$ and $y \in \mathbb{R}^d$. Using the substitution $t = s/(b^2 + \|y\|_2^2)$, we obtain the following expression:

$$\widehat{\phi}(y) = \frac{(4\pi)^{-d/2}}{\Gamma(k)} \int_0^\infty t^{k-d/2-1} e^{-b^2 t} \mathcal{F}(e^{-\|\cdot\|_2^2/(4t)})(y)\,\mathrm{d}t$$
$$= \frac{(2\pi)^{-d/2}}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-(b^2+\|y\|_2^2)t}\,\mathrm{d}t = \frac{(2\pi)^{-d/2}}{(b^2 + \|y\|_2^2)^k}.$$

Now, for all $x_0 \in \mathbb{R}^d$ and $v \in C_0^\infty(\mathbb{R}^d)$, we have $\mathcal{F}^{-1}(\mathrm{L}^{2k} v)(y) = \mathcal{F}^{-1}((b^2 - \Delta)^k v)(y) = (b^2 + \|y\|_2^2)^k \check{v}(y)$, where $\mathcal{F}^{-1}$ and $\check{\cdot}$ denote the inverse Fourier transform. We know that $(b^2 + \|\cdot\|_2^2)^k \check{v} \in \mathcal{S} \subseteq L^1\mathbb{R}^d$, where $\mathcal{S} \subseteq C^\infty(\mathbb{R}^d)$ is the usual *Schwartz class* of rapidly decreasing functions. Then, using the unit imaginary number $\mathbf{i} \in \mathbb{C}$, the usual Fourier computation rules, and the duality formula $\int_{\mathbb{R}^d} u\widehat{v}\,\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{u} v\,\mathrm{d}x$ for $u, v \in L^1(\mathbb{R}^d)$, we obtain

$$\int_{\mathbb{R}^d} \phi(x - x_0)(\mathrm{L}^{2k} v)(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \phi(x - x_0)\mathcal{F}((b^2 + \|\cdot\|_2^2)^k \check{v})(x)\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \widehat{\phi}(y) e^{-\mathbf{i}\langle x_0, y\rangle} (b^2 + \|y\|_2^2)^k \check{v}(y)\,\mathrm{d}y$$
$$= (2\pi)^{-d/2} \int_{\mathbb{R}^d} \check{v}(y) e^{-\mathbf{i}\langle x_0, y\rangle_2}\,\mathrm{d}y = v(x_0).$$

Finally, let us prove the asserted conformity. Since $k_{\min} = 0$, we have $V = H^k(\mathbb{R}^d) = \{v \in L^2(\mathbb{R}^d) \mid (1 + \|\cdot\|_2^2)^{k/2}\widehat{v} \in L^2(\mathbb{R}^d)\}$, a well-known characterization of Sobolev spaces (e.g., [20, Section 5.8.4.]). In fact, the function ϕ itself lies in V, because $(1+\|\cdot\|_2^2)^{k/2}\widehat{\phi} \approx (1+\|\cdot\|_2^k)^{-1} \in L^2(\mathbb{R}^d)$. This directly implies $\sum_{n=1}^N \boldsymbol{c}_n \phi(\cdot - x_n) \in V$ for all $\boldsymbol{c} \in \mathbb{R}^N$, which concludes the proof. □

Proof of Lemma *11* The continuity of the thin-plate spline ϕ is obvious, and the fact that ϕ is a fundamental solution of $(-\Delta)^k$ was shown in [43, Theorem 10.36]. It remains to show that the function $v := \sum_{n=1}^N \boldsymbol{c}_n \phi(\cdot - x_n)$ lies in the native space V, whenever $\boldsymbol{c} \in \boldsymbol{C}$. To do so, we use Fourier techniques: According to [43, page 161], the function ϕ has a *generalized Fourier transform*, given by $\widehat{\phi}(y) := (2\pi)^{-d/2}\|y\|_2^{-2k}$. This means that $\int_{\mathbb{R}^d} \phi\widehat{w}\,\mathrm{d}x = \int_{\mathbb{R}^d} \widehat{\phi} w\,\mathrm{d}y$ for all *homogeneous Schwartz functions* $w \in \mathcal{S}_0^{2k}$, where $\mathcal{S}_0^{2k} := \{w \in \mathcal{S} \mid \forall |\beta| < 2k : (\mathrm{D}^\beta w)(0) = 0\}$.

Now, consider the auxiliary function $c := \sum_{n=1}^N \boldsymbol{c}_n e^{-\mathbf{i}\langle x_n, \cdot\rangle_2} \in C^\infty(\mathbb{R}^d)$. For all $|\beta| < k$, there holds $(\mathrm{D}^\beta c)(0) = \sum_{n=1}^N \boldsymbol{c}_n (-\mathbf{i}x_n)^\beta = (-\mathbf{i})^\beta \langle \boldsymbol{c}, E_N(\cdot)^\beta\rangle_2 = 0$, because $\boldsymbol{c} \in \boldsymbol{C}$ and $(\cdot)^\beta \in \mathbb{P}^{k-1}(\mathbb{R}^d) = P$. Using Taylor's Theorem, it follows that $|c(y)| \lesssim \min\{\|y\|_2^k, 1\}$ for all $y \in \mathbb{R}^d$. Then, for all $|\alpha| = k$, it is not difficult to see that $(\mathbf{i}\,\cdot)^\alpha c\widehat{\phi} \in L^2(\mathbb{R}^d)$, so that $\mathcal{F}^{-1}((\mathbf{i}\,\cdot)^\alpha c\widehat{\phi}) \in L^2(\mathbb{R}^d)$ is well-defined (and also real-valued). Furthermore, for all $w \in C_0^\infty(\mathbb{R}^d)$, we have $(-\mathbf{i}\,\cdot)^\alpha c\check{w} \in \mathcal{S}_0^{2k}$. We use the standard Fourier computation rules to derive the following identity with $\check{w}$ denoting the inverse Fourier transform of w

$$\int_{\mathbb{R}^d} v(\mathrm{D}^\alpha w)\,\mathrm{d}x = \int_{\mathbb{R}^d} \phi \mathcal{F}((-\mathbf{i}\,\cdot)^\alpha c\check{w})\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \widehat{\phi}(-\mathbf{i}\,\cdot)^\alpha c\check{w}\,\mathrm{d}y = (-1)^\alpha \int_{\mathbb{R}^d} \mathcal{F}^{-1}((\mathbf{i}\,\cdot)^\alpha c\widehat{\phi})w\,\mathrm{d}y.$$

This proves that the function v has an α-th derivative, given by $\mathrm{D}^\alpha v = \mathcal{F}^{-1}((\mathbf{i}\,\cdot)^\alpha c\widehat{\phi}) \in L^2(\mathbb{R}^d)$. Since this is true for all $|\alpha| = k$, we conclude that $v \in \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \mid \forall|\alpha| = k : \exists \mathrm{D}^\alpha u \in L^2(\mathbb{R}^d)\} = V$. □

4 Numerical examples

We finish this paper with a few numerical examples that were performed in `MATLAB` and `H2Lib`, [13].

In Fig. 1, we used roughly $N \approx 30.000$ interpolation points in $d = 2$ space dimensions. The domain of interest was the exterior of the *TU Wien* logo, i.e., the complement of the letters in the unit square $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$. The interpolation points were uniform in the open and algebraically graded towards reentrant corners with a grading exponent $\sigma_{\mathrm{card}} = 2$ (cf. Definition 2). As for the basis function, we used the thin-plate spline $\phi(x) = \|x\|_2^2 \ln \|x\|_2$ from Lemma 11, i.e., $k = k_{\min} = 2$. The left image shows the positions x_n of the interpolation points and the one in the middle depicts the pairs $(x_n, f(x_n))$, where the data function $f \in V$ is a smoothed indicator function of the letters. On the right-hand side, the solution $u \in V$ of the interpolation problem, Problem 4, is rendered.

Figure 2 shows the results of a problem in space dimension $d = 2$. The $N = 900$ interpolation points x_n produced a uniform 30×30 grid in the unit square $[0, 1] \times [0, 1] \subseteq \mathbb{R}^2$, i.e., $\sigma_{\mathrm{card}} = 1$. Once again the thin plate-spline $\phi(x) = \|x\|_2^2 \ln \|x\|_2$ with $k = k_{\min} = 2$ was employed. In the left image, we can see a “typical” sparse hierarchical block partition $\mathbb{P}$ in the sense of Definition 16. The somewhat “fractal” pattern of *small* and *admissible* cluster blocks arises from the fact that we ordered

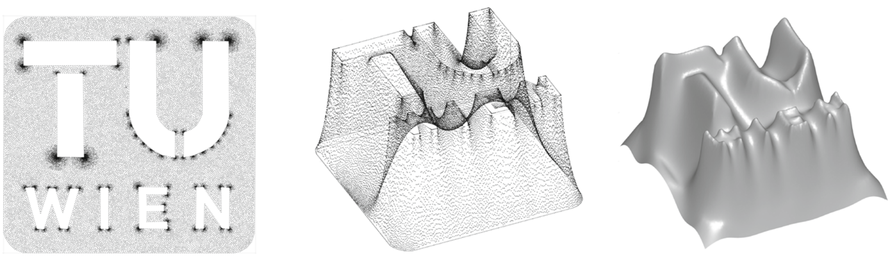

Fig. 1 Interpolation of smooth data on a non-uniform point distribution

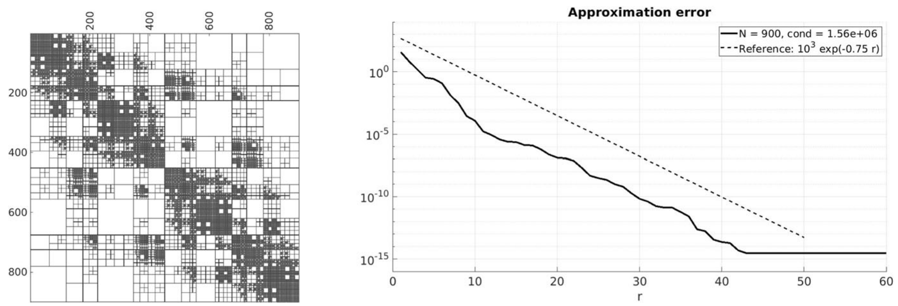

Fig. 2 A “typical” hierarchical block partition and a “typical” error plot in 2D

the interpolation points in a row-wise fashion, i.e., $x_1 = (0, 0/29)$, $x_{31} = (0, 1/29)$, $x_{61} = (0, 2/29)$, et cetera.

The right-hand image is empirical evidence that the error bound in our main theorem, Theorem 18, is correct. To generate this plot, the main block $\boldsymbol{S_{11}} \in \mathbb{R}^{N\times N}$ of the inverse $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)^{-1}$ was computed exactly using `MATLAB`'s built-in inversion routine `inv(...)`. Next, for each admissible block $\boldsymbol{S_{11}}|_{I\times J}$, $(I, J) \in \mathbb{P}_{\text{adm}}$, we used `svds(...)` to compute the corresponding singular values $\sigma_r(\boldsymbol{S_{11}}|_{I\times J})$, $r \in \{1, 2, \dots, 60\}$. Truncating the blockwise SVDs at any given rank $r \in \mathbb{N}$, we then assembled the $\mathcal{H}$-matrix $\boldsymbol{M}_r \in \mathcal{H}(\mathbb{P}, r)$ (cf. Definition 17), the best rank-r-approximation to $\boldsymbol{S_{11}}$. As discussed in our previous work [1, Section 4], this approach leads to the *computable* error bound

$$\|\boldsymbol{S_{11}} - \boldsymbol{M}_r\|_2 \lesssim \text{depth}(\mathbb{T}_N) \max_{(I,J)\in\mathbb{P}} \sigma_{r+1}(\boldsymbol{S_{11}}|_{I\times J}),$$

where $\text{depth}(\mathbb{T}_N)$ denotes the cluster tree depth previously mentioned in Section 2.4. The semi-logarithmic error plot depicts the computable error bound along with a dashed reference line. The apparent similarity suggests a relation of the form $\|\boldsymbol{S_{11}} - \boldsymbol{M}_r\|_2 \lesssim C(N)\exp(-\sigma_{\text{exp}} r)$, which is even better than our theoretical prediction $C(N)\exp(-\sigma_{\text{exp}} r^{1/3})$.

As a side note, we mention that the standard 16-digit precision arithmetic in `MATLAB` was not enough to generate a conclusive error plot. As is well-established in the literature (see, e.g., [43, Chapter 12]), the condition number of the interpolation matrix $\left(\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{smallmatrix}\right)$ scales very poorly with respect to the separation distance $h_{\min}$ introduced in Definition 2. To overcome this fundamental problem, we used `MATLAB`'s *variable-precision arithmetic* `vpa(...)` with 32 digits. This brute-force approach allowed us to carry out the explicit matrix inversion with sufficient accuracy.

The next example, Fig. 3, covers the case $d = 3$ and a uniform point distribution in the unit cube $[0, 1] \times [0, 1] \times [0, 1] \subseteq \mathbb{R}^3$, visualized in the left image. This time, we set $k = 2$ and $k_{\min} = 0$ and used the Bessel potential $\phi(x) = e^{-\|x\|_2}$ from Lemma 10 as the basis function. The error plot shows a comparison between $N \approx 10.000$, $N \approx 15.000$ and $N \approx 20.000$ interpolation points, as well as a reference curve of the form $r \mapsto C\exp(-\sigma_{\text{exp}} r^{1/4})$. In accordance with Theorem 18, the empirical decay rate seems to be independent of the problem size N.

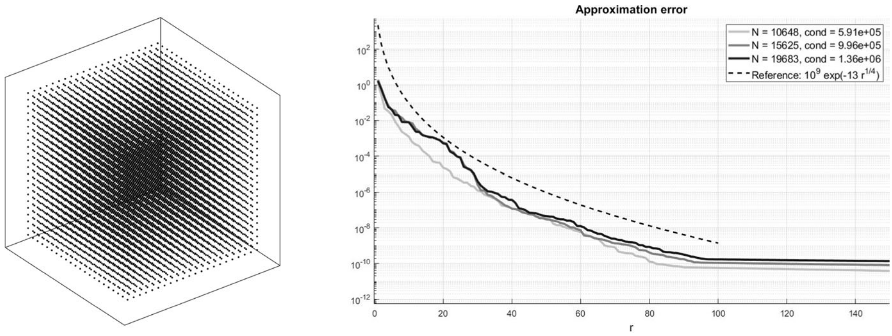

Fig. 3 Error versus block rank for different problem sizes N for a uniform 3D grid

In Fig. 4, we investigated the influence of the grading exponent σ_{card} from Definition 2 on the error decay rate in $d = 3$ space dimensions. We set $k = 2$ and $k_{\min} = 0$ and used $\phi(x) = e^{-\|x\|_2}$ as the basis function. The error plot compares the cases $\sigma_{\mathrm{card}} \in \{1, 2, 3\}$, where $\sigma_{\mathrm{card}} = 1$ is a uniform grid and $\sigma_{\mathrm{card}} = 3$ is "strongly" graded towards the origin $0 \in \mathbb{R}^3$. The problem size $N \approx 10.000$ was held constant throughout all three runs. The plot suggests that the constant $\sigma_{\exp}$ from the error bound $\exp(-\sigma_{\exp} r^{1/4})$ in Theorem 18 is independent of the grading parameter $\sigma_{\mathrm{card}} \in \{1, 2, 3\}$.

In the numerical example shown in Fig. 5, we choose a quasi-random set of points in the unit cube generated from the Halton sequence, [34], which is provided in MATLAB for arbitrary spatial dimension. Again, we set $k = 2$ and $k_{\min} = 0$ and used $\phi(x) = e^{-\|x\|_2}$ as the basis function. The error plots for $N \approx 10.000$, $N \approx 15.000$ and $N \approx 20.000$ interpolation points are very similar to the case of uniformly distributed points and are in accordance with the exponential convergence shown in Theorem 18.

Finally, we perform experiments using the $\mathcal{H}$-matrix arithmetic to approximate the inverse system matrix using the library H2Lib. As mentioned in the introduction, previous numerical results have established that the $\mathcal{H}$-matrix arithmetic is a viable tool for solving RBF interpolation problems, [36–38]. In the following, we compute a Cholesky-decomposition in the $\mathcal{H}$-matrix format by approximating the system matrix by an $\mathcal{H}$-matrix with good accuracy and then perform the Cholesky-algorithm using $\mathcal{H}$-

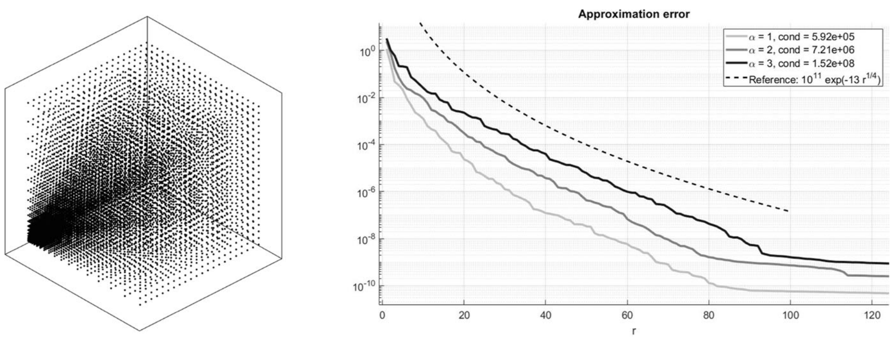

Fig. 4 Error versus block rank for an algebraically graded grid in 3D

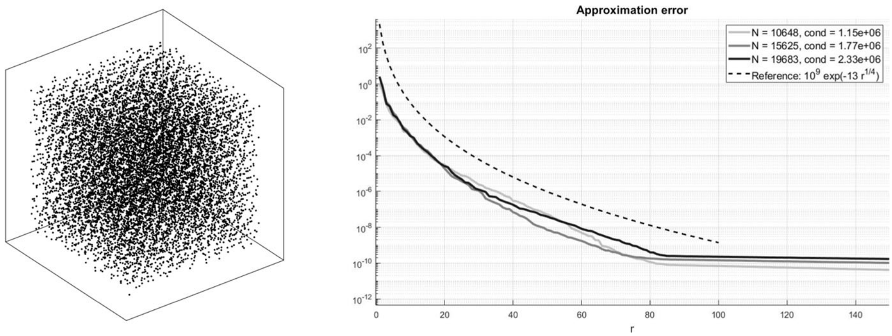

Fig. 5 Error versus block rank for pseudo-random Halton point distribution in 3D

arithmetic as described, e.g., in [4]. Afterwards, we project the matrix to a prescribed rank r. Finally, an approximate inverse can be obtained by inverting the Cholesky factors.

However, a complication arises in the *conditionally positive definite* case of polyharmonic splines as the main block of the system matrix is not invertible so that the Cholesky-decomposition cannot be computed. In fact, the system matrix has saddle point structure. Like Gaussian elimination, $\mathcal{H}$-matrix inversion of such matrices would require pivoting. To avoid pivoting it was advocated in [9, 36] to consider instead the augmented Lagrangian $\boldsymbol{A} + \gamma \boldsymbol{B}^T \boldsymbol{B}$ ($\gamma > 0$), which is SPD and therefore amenable to an $\mathcal{H}$-matrix inversion.

In the numerical example shown in Fig. 6, we compute an approximate inverse using $\mathcal{H}$-matrix arithmetic as described above for the case of thin-plate splines in space dimension $d = 2$ (uniform point distribution in the unit square, $N = 10000$, $k = k_{\min} = 2$, augmented Lagrangian approach with $\gamma = 1$) and the Bessel potential in space dimension $d = 3$ (uniform point distribution in unit cube, $N = 4096$, $k = 2$, $k_{\min} = 0$). In contrast to the previous examples performed in `MATLAB`, we

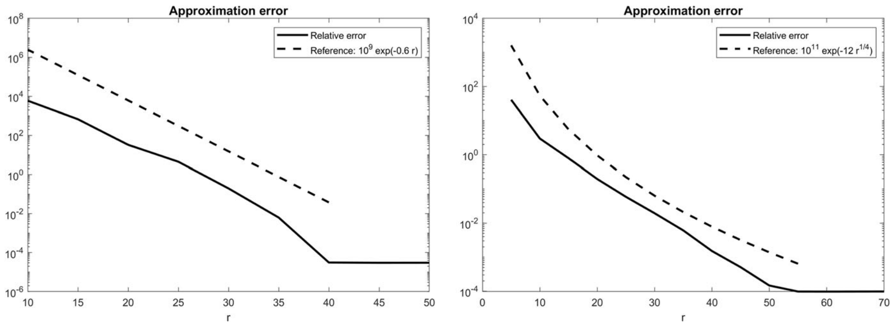

Fig. 6 Error of $\mathcal{H}$-matrix Cholesky decomposition of $\boldsymbol{A} + \gamma \boldsymbol{B}^T \boldsymbol{B}$. Left: 2D thin-plate splines, right: 3D Bessel potential

do not compute the exact inverse matrix and perform an SVD to compute an upper bound of the absolute error, but rather use the error measure $\|\boldsymbol{I} - (\boldsymbol{L}_{\mathcal{H}}\boldsymbol{U}_{\mathcal{H}})^{-1}\boldsymbol{A}\|_2$, which is an upper bound for the *relative error*. Once again, we observe exponential convergence as predicted by our main result Theorem 18. However, we mention that the error flattens out earlier than machine precision due to our method of computation, as the approximation of the initial stiffness matrix using interpolation determines the achievable accuracy.

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Declarations

Conflict of interest The authors declare no competing interests.

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