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The Shigesada–Kawasaki–Teramoto cross-diffusion system beyond detailed balance ☆

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Abstract

The existence of global weak solutions to the cross-diffusion model of Shigesada, Kawasaki, and Teramoto for an arbitrary number of species is proved. The model consists of strongly coupled parabolic equations for the population densities in a bounded domain with no-flux boundary conditions, and it describes the dynamics of the segregation of the population species. The diffusion matrix is neither symmetric nor positive semidefinite. A new logarithmic entropy allows for an improved condition on the coefficients of heavily nonsymmetric diffusion matrices, without imposing the detailed-balance condition that is often assumed in the literature. Furthermore, the large-time convergence of the solutions to the constant steady state is proved by using the relative entropy associated to the logarithmic entropy.

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1. Introduction

The Shigesada–Kawasaki–Teramoto (SKT) system was introduced in [19] to describe the dynamics of two competing population species. In this model, the diffusion rate of each species depends on the gradients of the densities of both species, expressed by cross-diffusion terms. They give rise to a repulsive effect leading to spatial segregation. The stationary model has been extended to three species in [17], while the time-dependent system for an arbitrary number of species was investigated in [20]. The existence of global weak solutions to the transient model has been proved only under detailed balance or imposing bounds on the self-diffusion coefficients [7]; see below for details. In this paper, we suggest a new condition on the self-diffusion coefficients, which is significantly weaker than that one in [7] in the case of heavily nonsymmetric diffusion matrices.

The SKT model consists of the following cross-diffusion equations for the population densities u_i:

$$\partial_t u_i = \operatorname{div}\left(\sum_{j=1}^n A_{ij}(u)\nabla u_j\right), \quad A_{ij}(u) = \delta_{ij}a_{i0} + \delta_{ij}\sum_{k=1}^n a_{ik}u_k + a_{ij}u_i, \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \le 3$) for $t > 0$, where $i, j = 1, \dots, n$, and δ_{ij} is the Kronecker symbol, supplemented by the initial and no-flux boundary conditions

$$u_i(0) = u_i^0 \quad \text{in } \Omega, \quad \sum_{j=1}^n A_{ij}(u)\nabla u_j \cdot \nu = 0 \quad \text{on } \partial\Omega,\ t > 0,\ i = 1, \dots, n. \tag{2}$$

The diffusion coefficients a_{ij} are nonnegative numbers. We call a_{ii} the self-diffusion coefficients and a_{ij} for $i \ne j$ the cross-diffusion coefficients. The original model for $n = 2$ species in [19] also contains a drift term involving the environmental potential and Lotka–Volterra reaction terms. We have neglected these terms to simplify the presentation. Our technique is able to treat these terms; see, e.g., [5,6].

While the global existence analysis in the two-species model is quite well understood [5,6,10], the global existence of weak solutions to the n-species system has been proven only in the so-called detailed-balance case (see below) [7] and in the case of large self-diffusion coefficients; see, e.g., [8,9,16]. Another approach was suggested by Amann [1], who proved that a priori estimates in the $W^{1,p}(\Omega)$ norm with $p > d$ are sufficient for the solutions to general quasilinear parabolic systems to exist globally in time, and he applied his result to the triangular case, where $a_{ij} = 0$ for $i > j$. However, $W^{1,p}(\Omega)$ estimates with $p > d$ for solutions to (1) under general conditions seem to be out of reach.

The main difficulty in the analysis of (1)–(2) is the fact that the diffusion matrix is generally neither symmetric nor positive semidefinite. This issue was overcome in [7] by exploiting the entropy structure of (1). This means that there exists a so-called entropy density $h : [0, \infty) \to \mathbb{R}$ such that (1) can be written in terms of the entropy variables $w_i = \partial h/\partial u_i$ as

$$\partial_t u_i(w) = \operatorname{div}\left(\sum_{j=1}^n B_{ij}(w)\nabla w_j\right), \quad i = 1, \dots, n,$$

where u_i is interpreted as a function of w and $B(w) = A(u(w))h''(u(w))^{-1}$ with $B(w) = (B_{ij}(w)) \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix, called the mobility matrix. Here, $w = (w_1, \ldots, w_n)$ and $u = (u_1, \ldots, u_n)$ are vector-valued functions. For instance, we introduce the entropy density

$$\widetilde{h}(u) = \sum_{i=1}^{n} \pi_i u_i (\log u_i - 1)dx,$$

where $\pi_i > 0$ for $i = 1, \ldots, n$ are assumed to satisfy $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i, j = 1, \ldots, n$. These equations constitute the detailed-balance condition for the Markov chain associated to (a_{ij}), and $(\pi_1, \ldots, \pi_n)$ is the corresponding invariant measure. Under this assumption, a formal computation shows that, along solutions to (1)–(2),

$$\frac{d}{dt}\int_\Omega \widetilde{h}(u(t))dx + 4\sum_{i=1}^{n}\pi_i a_{i0}\int_\Omega |\nabla\sqrt{u_i}|^2 dx + 2\sum_{i=1}^{n}\pi_i a_{ii}\int_\Omega |\nabla u_i|^2 dx \le 0,$$

which provides suitable gradient estimates. It was shown in [7] that the detailed-balance condition is not necessary for a global existence analysis. If self-diffusion dominates cross-diffusion in the sense

$$4a_{ii} > \sum_{j=1}^{n}\left(\sqrt{a_{ij}} - \sqrt{a_{ji}}\right)^2 \quad \text{for all } i = 1, \ldots, n, \tag{3}$$

then the global existence of weak solutions follows. If $a_{ii} > 0$, this condition is satisfied if the matrix (a_{ij}) is nearly symmetric.

The goal of this paper is to prove the global existence of weak solutions without imposing detailed balance under a condition that is weaker than (3) for (heavily) nonsymmetric matrices (a_{ij}). The key idea of our analysis is the observation that the entropy density

$$h(u) = \sum_{i=1}^{n} \pi_i (u_i - \log u_i) \tag{4}$$

formally satisfies the inequality

$$\frac{d}{dt}\int_\Omega h(u(t))dx + \sum_{i=1}^{n}\pi_i a_{i0}\int_\Omega |\nabla \log u_i|^2 dx + \sum_{i=1}^{n}\left(8\pi_i a_{ii} - \sum_{j\neq i}\pi_j a_{ji}\right)\int_\Omega |\nabla\sqrt{u_i}|^2 dx \le 0. \tag{5}$$

(The computation is made rigorous for approximate solutions in (13) below.) Thus, we obtain a gradient estimate for $\sqrt{u_i}$ if

$$\kappa := \min_{i=1,\ldots,n}\left(8\pi_i a_{ii} - \sum_{j=1,\, j\neq i}^{n} \pi_j a_{ji}\right) > 0 \tag{6}$$

is satisfied (we allow for $a_{i0} \geq 0$). If (a_{ij}) is almost symmetric, condition (3) outperforms (6). However, condition (6) is generally weaker than (3) if a_{ij} and a_{ji} differ significantly.

We underline this statement by the following example. Let $n = 3$, $a_{13} = a_{21} = a_{32} = 1$, and $a_{12} = a_{23} = a_{31} = 0$. Since $|a_{ij} - a_{ji}| = 1$ for $i \neq j$, the matrix (a_{ij}) is nonsymmetric. Condition (3) from [7] is equivalent to $a_{ii} > 1/2$ for $i = 1, 2, 3$, while condition (6) is equivalent to $a_{11}a_{22}a_{33} > 8^{-3}$ (see Lemma 15 in the Appendix). This is significantly weaker than $a_{11}a_{22}a_{33} > 8^{-1}$ (which follows from $a_{ii} > 1/2$) and, moreover, we only need *one* self-diffusion coefficient to be sufficiently large.

In the literature, the functional (4) has been identified as an entropy (i.e. a Lypunov functional) mainly for higher-order parabolic equations via the method of systematic integration by parts [14]. A similar functional was used to prove the convergence of solutions to the two-species SKT model to a steady state under quite particular conditions on the coefficients a_{ij} [15]. Up to our knowledge, the use of (4) in the global existence analysis of cross-diffusion systems is new.

We impose the following assumptions:

(A1) Domain: $\Omega \subset \mathbb{R}^d$ is a bounded domain with $\partial\Omega \in C^2$, $d \leq 3$, and $T > 0$.
(A2) Initial datum: $u^0 = (u_1^0, \ldots, u_n^0) \in L^1(\Omega; \mathbb{R}^n)$ satisfies $u_i > 0$ in Ω for $i = 1, \ldots, n$, $\int_\Omega h(u^0)dx < \infty$ if $d \leq 3$, and moreover $\int_\Omega (u_i^0)^2 dx < \infty$ if $d = 2, 3$.
(A3) Coefficients: $a_{ij} \geq 0$, $a_{i0} \geq 0$ for all $i, j = 1, \ldots, n$, and there exist $\pi_1, \ldots, \pi_n > 0$ such that (6) holds.

The boundary regularity in Assumption (A1) is needed to apply an $H^2(\Omega)$ elliptic regularity result for the duality method (see the proof of Lemma 6 below). The restriction to at most three space dimensions comes from the continuous embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, which is needed to conclude the weak convergence of $(u_i^\delta)^3$ in $L^2(\Omega)$, where u_i^δ are some approximate solutions; see Section 3, step 2. In view of the entropy inequality (5), we need the regularity $\int_\Omega h(u^0)dx$ for the initial datum in Assumption (A2). In two and three space dimensions, we need more integrability to deal with the quadratic nonlinearity. As already mentioned, Assumption (A3) is a relaxed "self-diffusion > cross-diffusion" condition. Note that the diffusion coefficients a_{i0} are allowed to vanish.

Notation. A vector-valued function $v : \Omega \to \mathbb{R}^n$ has the components $v_1, \ldots, v_n$. We denote the entries of a matrix $A \in \mathbb{R}^{n\times n}$ by A_{ij}. We set $Q_T = \Omega \times (0, T)$ for the space-time cylinder. Furthermore, we need the space of test functions

$$W_\nu^{2,p}(\Omega) = \{\phi \in W^{2,p}(\Omega) : \nabla\phi \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad p \geq 2,$$

and we set $H_\nu^2(\Omega) = W_\nu^{2,2}(\Omega)$.

Our first main result is as follows.

Theorem 1 *(Global existence). Let Assumptions (A1)–(A3) hold. Then there exists a weak solution $u = (u_1, \ldots, u_n)$ to (1)–(2) satisfying $u_i(t) > 0$ a.e. in Ω, $\int_\Omega h(u(t))dx < \infty$ for $0 < t < T$, the regularity*

$$u_i \in L^\infty(0, T; L^1(\Omega)) \cap L^3(Q_T), \quad \sqrt{u_i} \in L^2(0, T; H^1(\Omega)),$$
$$\partial_t u_i \in L^{4/3}(0, T; W_\nu^{2,4}(\Omega)'),$$

u satisfies the initial conditions in the sense of $W_\nu^{2,4}(\Omega)'$, and it holds for all $\phi \in L^4(0,T; W_\nu^{2,4}(\Omega;\mathbb{R}^n))$ and $i = 1,\ldots,n$ that

$$\int_0^T \langle \partial_t u_i, \phi_i\rangle dt = \int_0^T \int_\Omega u_i p_i(u)\Delta\phi_i dx dt, \qquad p_i(u) = a_{i0} + \sum_{k=1}^n a_{ik}u_k, \tag{7}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $W_\nu^{2,4}(\Omega)'$ and $W_\nu^{2,4}(\Omega)$.

Observe that the weak formulation is weaker than the traditional one. We can change it, after an integration by parts, to the usual weak formulation

$$\int_0^T \langle \partial_t u_i, \phi_i\rangle dt = -\int_0^T \int_\Omega \big(p_i(u)\nabla u_i + u_i\nabla p_i(u)\big)\cdot\nabla\phi_i dx dt$$

for all $\phi_i \in L^\infty(0,T; W^{1,\infty}(\Omega))$, since $u_i\nabla u_k = u_i\sqrt{u_k}\nabla\sqrt{u_k} \in L^1(Q_T)$. The regularity is generally lower compared to the results in [7], where $u_i \in L^2(0,T; H^1(\Omega))$ has been proven. The reason is that the logarithmic entropy (4) can be interpreted to be of "zero order" with respect to u_i, while the Boltzmann entropy $\widetilde{h}(u)$, which was used in [7], is of "order one" in u_i.

Theorem 1 is shown by using the entropy method; see, e.g., [13]. Since the entropy variable $w_i = (\partial h/\partial u_i)(u) = \pi_i(1 - 1/u_i)$ is not invertible for every $w_i \in \mathbb{R}$, we regularize the entropy density by $h_\varepsilon(u) = h(u) + \varepsilon h^0(u)$ for $\varepsilon > 0$, where $h^0(u) = \sum_{i=1}^n u_i(\log u_i - 1)$. Then $h_\varepsilon^{-1} : \mathbb{R} \to (0,\infty)$ is well-defined. However, this generally destroys the entropy structure in the sense that $A(u)h_\varepsilon''(u)^{-1}$ or, equivalently, $h_\varepsilon''(u)A(u)$ may be not positive semidefinite. Therefore, we also regularize $A(u)$ by $A_\varepsilon(u) = A(u) + \varepsilon A^0(u)$, where $A^0(u)$ is a diagonal matrix with entries $(\mu_i/\pi_i)u_i^2$ and sufficiently large numbers $\mu_i > 0$. Lemma 4 below shows that $h_\varepsilon''(u)A_\varepsilon(u)$ is positive definite, which yields some $L^2(\Omega)$ gradient bounds. Note that our regularization is simpler than that one used in [7].

The estimates from the entropy inequality (5) are not sufficient to define $u_i p_i(u)$ from (7) in $L^1(Q_T)$ in the three-dimensional case, since the Gagliardo–Nirenberg inequality yields $u_i \in L^{1+2/d}(Q_T)$ only (see (19)). To obtain better regularity, we exploit the fact that the SKT model can be written as $\partial_t u_i = \Delta(u_i p_i(u))$, which allows us to use the duality method. Basically, we use $(-\Delta)^{-1}u_i$ as a test function, which leads to an estimate for $u_i^2 p_i(u)$ in $L^1(Q_T)$ and, because of $a_{ii} > 0$ due to (6), an estimate for u_i in $L^3(Q_T)$.

Theorem 2 *(Large-time behavior). Let Assumptions (A1)–(A3) hold and suppose that $d = 1$ and $a_{i0} > 0$ for all $i = 1,\ldots,n$. Let u be the weak solution to (1)–(2) constructed in Theorem 1 and let $\bar{u}_i = \mathrm{meas}(\Omega)^{-1}\int_\Omega u_i dx$ for $i = 1,\ldots,n$. Then*

$$\lim_{t\to\infty} \|u_i(t) - \bar{u}_i\|_{L^1(\Omega)} = 0.$$

Since u_i conserves the mass, $\bar{u}_i$ is independent of time. The proof of this result is surprisingly delicate in spite of our restriction to one space dimension. It is needed to guarantee the continuous embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$; see the proof of Lemma 13. The usual idea is to show that the relative entropy, associated to the entropy density (4), satisfies an inequality similar to (5) and to estimate the entropy production term (the gradient bounds) in terms of the relative entropy. Unfortunately, we have not been able to prove this entropy inequality, since the deregularization limit in the logarithmic term $\log u_i$ is difficult and the low integrability of $\partial_t u_i$ and u_i does not allow us to use $\partial h(u)/\partial u_i$ as a test function in equation (7) to derive an entropy inequality. We circumvent this issue by regularizing the relative entropy:

$$\mathcal{H}_\eta(u|\bar{u}) = \sum_{i=1}^n \pi_i \int_\Omega (\log(\bar{u}_i + \eta) - \log(u_i + \eta))dx, \quad \eta > 0.$$

The difficult part is to estimate the matrix product $h''_\varepsilon(u+\eta)A_\varepsilon(u)$. We are able to show that this matrix is positive definite up to a term of order $O(\sqrt{\varepsilon})$, which vanishes when $\varepsilon \to 0$. This shows that in the limit $\varepsilon \to 0$, for $0 \le s < t$,

$$\mathcal{H}_\eta(u(t)|\bar{u}) + C\sum_{i=1}^n \int_s^t \int_\Omega |\nabla\sqrt{u_i+\eta}|^2 dx d\sigma \le \mathcal{H}_\eta(u(s)|\bar{u}), \quad 0 < s < t.$$

The entropy production can be estimated as (see Lemma 14)

$$\int_0^\infty \|\sqrt{u_i(t)} - \sqrt{\bar{u}_i}\|^2_{L^2(\Omega)} dt \le C \int_0^\infty \int_\Omega |\nabla\sqrt{u_i}|^2 dx dt \le C(u^0).$$

Note that the Poincaré–Wirtinger inequality would only yield the difference $\sqrt{u_i(t)} - \overline{\sqrt{u_i}}$. The previous inequality implies the existence of a sequence $t_k \to \infty$ as $k \to \infty$ such that $\|\sqrt{u_i(t_k)} - \sqrt{\bar{u}_i}\|_{L^2(\Omega)} \to 0$. We will show that this implies the convergence $\mathcal{H}_\eta(u(t_k)|\bar{u}) \to 0$ as $k \to \infty$, and since the relative entropy is bounded and nonincreasing, this convergence holds for any sequence $t \to \infty$. Finally, the Csiszár–Kullback inequality (Proposition 16 in the Appendix) concludes the proof.

The paper is organized as follows. We prove the positive definiteness of $h''(u)A(u)$ and $h''_\varepsilon(u)A_\varepsilon(u)$ in Section 2. Theorem 1 is proved in Section 3, while Section 4 is devoted to the proof of Theorem 2. Some auxiliary results are collected in Appendix A.

2. Positive definiteness of mobility matrices

We introduce the Hessian matrix of $h(u)$, defined in (4), by $H(u) = h''(u)$ with entries $H_{ij}(u) = \delta_{ij}\pi_i u_i^{-2}$ for $i, j = 1, \ldots, n$.

Lemma 3. *It holds for any $z \in \mathbb{R}^n$ and $u \in (0,\infty)^n$ that*

$$z^T H(u)A(u)z \ge \sum_{i=1}^n \pi_i a_{i0}\frac{z_i^2}{u_i^2} + \frac{1}{4}\sum_{i=1}^n \bigg(8\pi_i a_{ii} - \sum_{j=1,\, j\neq i}^n \pi_j a_{ji}\bigg)\frac{z_i^2}{u_i}.$$

Proof. The elements of the matrix $H(u)A(u)$ equal

$$\begin{aligned}(H(u)A(u))_{ij} &= \delta_{ij}\pi_i\frac{a_{i0}}{u_i^2} + \delta_{ij}\sum_{k=1}^n \pi_i a_{ik}\frac{u_k}{u_i^2} + \pi_i\frac{a_{ij}}{u_i} \\ &= \delta_{ij}\pi_i\frac{a_{i0}}{u_i^2} + \delta_{ij}\bigg(2\pi_i\frac{a_{ij}}{u_i} + \sum_{k=1,\,k\neq i}^n \pi_i a_{ik}\frac{u_k}{u_i^2}\bigg) + (1-\delta_{ij})\pi_i\frac{a_{ij}}{u_i}.\end{aligned}$$

This gives for all $z\in\mathbb{R}^n$:

$$\begin{aligned}z^T H(u)A(u)z &= \sum_{i=1}^n \pi_i a_{i0}\frac{z_i^2}{u_i^2} + 2\sum_{i=1}^n \pi_i a_{ii}\frac{z_i^2}{u_i} \\ &\quad + \sum_{i=1}^n\sum_{k=1,\,k\neq i}^n \pi_i a_{ik}u_k\frac{z_i^2}{u_i^2} + \sum_{i,j=1,\,i\neq j}^n \pi_i a_{ij}\frac{z_i z_j}{u_i}.\end{aligned} \tag{8}$$

We use Young's inequality to estimate the last term:

$$\begin{aligned}\sum_{i,j=1,\,i\neq j}^n \pi_i a_{ij}\frac{z_i z_j}{u_i} &\geq -\sum_{i,j=1,\,i\neq j}^n \pi_i a_{ij}\bigg(\frac{u_j}{u_i^2}z_i^2 + \frac{1}{4}\frac{z_j^2}{u_j}\bigg) \\ &= -\sum_{i,j=1,\,i\neq j}^n \pi_i a_{ij}\frac{u_j}{u_i^2}z_i^2 - \frac{1}{4}\sum_{i,j=1,\,i\neq j}^n \pi_j a_{ji}\frac{z_i^2}{u_i}.\end{aligned}$$

The first term on the right-hand side cancels with the third term on the right-hand side of (8). Therefore,

$$z^T H(u)A(u)z \geq \sum_{i=1}^n \pi_i a_{i0}\frac{z_i^2}{u_i^2} + \sum_{i=1}^n\bigg(2\pi_i a_{ii} - \frac{1}{4}\sum_{i,j=1,\,i\neq j}^n \pi_j a_{ji}\bigg)\frac{z_i^2}{u_i}$$

which finishes the proof. □

For $\varepsilon>0$, we define the approximate entropy density

$$h_\varepsilon(u) = h(u) + \varepsilon h^0(u), \quad \text{where } h^0(u) = \sum_{i=1}^n u_i(\log u_i - 1). \tag{9}$$

We set $H^0(u) = (h^0)''(u)$ with entries $H^0_{ij}(u) = \delta_{ij}u_i^{-1}$, $i,j=1,\ldots,n$, and

$$H_\varepsilon(u) = H(u) + \varepsilon H^0(u), \quad \text{where } H_{\varepsilon,ij}(u) = \delta_{ij}\bigg(\frac{\pi_i}{u_i^2} + \frac{\varepsilon}{u_i}\bigg).$$

We also need to approximate the diffusion matrix:

$$A_\varepsilon(u) = A(u) + \varepsilon A^0(u), \quad \text{where } A^0_{ij}(u) = \delta_{ij}\frac{\mu_i}{\pi_i}u_i^2,$$

imposing that $\mu_i \geq \sum_{j\neq i}(a_{ij}+a_{ji})/2$. The latter condition is necessary to prove that the product $H_\varepsilon(u)A_\varepsilon(u)$ is also positive definite.

Lemma 4. *It holds for any $z \in \mathbb{R}^n$ and $u \in (0,\infty)^n$ that*

$$z^T H_\varepsilon(u)A_\varepsilon(u)z \geq z^T H(u)A(u)z + 2\varepsilon\sum_{i=1}^n a_{ii}z_i^2 + \varepsilon^2\sum_{i=1}^n \frac{\mu_i}{\pi_i}u_i z_i^2.$$

Proof. We decompose the product $H_\varepsilon(u)A_\varepsilon(u)$ as

$$H_\varepsilon(u)A_\varepsilon(u) = H(u)A(u) + \varepsilon\big(H^0(u)A(u) + H(u)A^0(u)\big) + \varepsilon^2 H^0(u)A^0(u).$$

We compute first the terms of order ε:

$$(H^0(u)A(u))_{ij} = \delta_{ij}\left(\frac{a_{i0}}{u_i} + \sum_{k\neq i}a_{ik}\frac{u_k}{u_i} + 2a_{ii}\right) + (1-\delta_{ij})a_{ij},$$

$$(H(u)A^0(u))_{ij} = \delta_{ij}\mu_i,$$

which yields

$$\begin{aligned} z^T\big(H^0(u)A(u) + H(u)A^0(u)\big)z &= \sum_{i=1}^n\left(\frac{a_{i0}}{u_i} + \sum_{k\neq i}a_{ik}\frac{u_k}{u_i} + 2a_{ii} + \mu_i\right)z_i^2 \\ &\quad + \sum_{i=1}^n\sum_{j=1,\,j\neq i}^n a_{ij}z_iz_j \\ &\geq \sum_{i=1}^n\Big(2a_{ii}+\mu_i\Big)z_i^2 + \sum_{i=1}^n\sum_{j=1,\,j\neq i}^n a_{ij}z_iz_j. \end{aligned}$$

The last term is estimated by using Young's inequality $z_iz_j \geq -(z_i^2+z_j^2)/2$:

$$\begin{aligned} \sum_{i=1}^n\sum_{j=1,\,j\neq i}^n a_{ij}z_iz_j &\geq -\frac{1}{2}\sum_{i=1}^n\sum_{j=1,\,j\neq i}^n (a_{ij}z_i^2 + a_{ij}z_j^2) \\ &= -\frac{1}{2}\sum_{i=1}^n\sum_{j=1,\,j\neq i}^n a_{ij}z_i^2 - \frac{1}{2}\sum_{j=1}^n\sum_{i=1,\,i\neq j}^n a_{ji}z_i^2, \end{aligned}$$

which, because of the choice of μ_i, shows that

$$z^T\big(H^0(u)A(u)+H(u)A^0(u)\big)z \ge 2\sum_{i=1}^n a_{ii}z_i^2+\sum_{i=1}^n\Bigg(\mu_i-\frac{1}{2}\sum_{j=1,\,j\neq i}^n(a_{ij}+a_{ji})\Bigg)z_i^2$$
$$\ge 2\sum_{i=1}^n a_{ii}z_i^2.$$

The ε^2-term becomes $z^T H^0(u)A^0(u)z=\sum_{i=1}^n(\mu_i/\pi_i)u_iz_i^2$. Collecting these terms, the proof follows. □

3. Proof of Theorem 1

Let $T>0$, $N\in\mathbb{N}$, $\tau=T/N>0$, $\delta>0$, and $\varepsilon>0$. Let $u_\varepsilon^0=(u_{\varepsilon,1}^0,\dots,u_{\varepsilon,n}^0)$ be a componentwise bounded sequence of functions with positive lower bounds satisfying $h(u_\varepsilon^0)\to h(u^0)$ strongly in $L^1(\Omega)$ and $u_\varepsilon^0\to u^0$ strongly in $L^2(\Omega)$ as $\varepsilon\to 0$.

Step 1: Solution of an approximated problem. We introduce the entropy variables $w_i=(\partial h_\varepsilon/\partial u_i)(u)=\pi_i(1-1/u_i)+\varepsilon\log u_i$, $i=1,\dots,n$. Since the range of h_ε' is $\mathbb{R}^n$, the transformation $u:\mathbb{R}^n\to(0,\infty)^n$, $u(w)=(h_\varepsilon')^{-1}(w)$, is well defined. Furthermore, we introduce the mobility matrix $B_\varepsilon(w)=A_\varepsilon(u(w))H_\varepsilon(u(w))^{-1}$. By construction of u_ε^0, we can define $w^0=h_\varepsilon'(u_\varepsilon^0)$, and this is an element of $L^\infty(\Omega;\mathbb{R}^n)$. Then $u(w^0)=u_\varepsilon^0$. Let $m=1$ if $d=1$ and $m=2$ if $d=2,3$. Given $k\in\mathbb{N}$ and $w^{k-1}\in L^\infty(\Omega;\mathbb{R}^n)$, we wish to find $w^k\in H^m(\Omega;\mathbb{R}^n)$ solving

$$\frac{1}{\tau}\int_\Omega(u(w^k)-u(w^{k-1}))\cdot\phi dx+\int_\Omega\nabla\phi:B_\varepsilon(w^k)\nabla w^k dx \qquad (10)$$
$$+\delta\int_\Omega\Bigg(\sum_{|\alpha|=m}D^\alpha w^k\cdot D^\alpha\phi+w^k\cdot\phi\Bigg)dx=0$$

for all $\phi\in H^m(\Omega;\mathbb{R}^n)$, where $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}_0^d$ is a multiindex and D^α equals the partial derivative $\partial^{|\alpha|}/\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}$.

We claim that the existence of a weak solution w^k follows from [12, Lemma 5]. The construction of h_ε ensures that Hypothesis H1 of [12] is satisfied. Lemma 4 shows that Hypothesis H2 holds as well. Also Hypothesis H3 is fulfilled since (1) does not contain any source terms. We deduce from [12, Lemma 5] that there exists a weak solution $w^k\in H^m(\Omega;\mathbb{R}^n)$ to (10), satisfying the discrete entropy inequality

$$\int_\Omega h_\varepsilon(u(w^k))dx+\tau\int_\Omega\nabla w^k:B_\varepsilon(w^k)\nabla w^k dx \qquad (11)$$
$$+\delta\tau\int_\Omega\Bigg(\sum_{|\alpha|=m}|D^\alpha w^k|^2+|w^k|^2\Bigg)dx\le\int_\Omega h_\varepsilon(u(w^{k-1}))dx.$$

We derive some estimates for w^k and $u^k:=u(w^k)$. According to Lemma 4, the second term in (11) can be estimated as follows:

$$\int_\Omega \nabla w^k : B_\varepsilon(w^k)\nabla w^k dx = \int_\Omega \nabla u^k : H_\varepsilon(u^k)A_\varepsilon(u^k)\nabla u^k dx \tag{12}$$
$$\geq \sum_{i=1}^n \int_\Omega \Big\{\kappa|\nabla(u_i^k)^{1/2}|^2 + 2\varepsilon\Big(\min_{i=1,\dots,n} a_{ii}\Big)|\nabla u_i^k|^2\Big\}dx,$$

recalling definition (6) of κ. Therefore, since $a_{ii} > 0$ by Assumption (A3), summing (11) over $k = 1,\dots,j$,

$$\int_\Omega h_\varepsilon(u^j)dx + C\tau\sum_{k=1}^j\sum_{i=1}^n\int_\Omega \big(|\nabla(u_i^k)^{1/2}|^2 + \varepsilon|\nabla u_i^k|^2\big)dx \tag{13}$$
$$+\,\delta\tau\sum_{k=1}^j\int_\Omega\Big(\sum_{|\alpha|=m}|D^\alpha w^k|^2 + |w^k|^2\Big)dx \leq \int_\Omega h_\varepsilon(u_\varepsilon^0)dx \leq C,$$

where $C > 0$ denotes here and in the following a constant which is independent of δ, ε, and τ with values changing from line to line.

To derive bounds in $H^1(\Omega)$, we apply the Poincaré–Wirtinger inequality for which we need a uniform estimate for u_i^j. We take the test function $\phi = (\delta_{i1},\dots,\delta_{in})$ in (10) and sum the resulting equation over $k = 1,\dots,j$. Then, taking into account (13),

$$0 \leq \int_\Omega u_i^j dx = \int_\Omega u_i^0 dx - \delta\tau\sum_{k=1}^j\int_\Omega w_i^k dx \tag{14}$$
$$\leq \int_\Omega u_i^0 dx + \frac{\delta}{2}\tau\sum_{k=1}^j\int_\Omega((w_i^k)^2 + 1)dx \leq C(u^0, T, \Omega).$$

We infer that

$$\|u^k\|_{H^1(\Omega)} \leq C(\varepsilon,\tau), \quad \sqrt{\delta}\|w^k\|_{H^m(\Omega)} \leq C(\tau). \tag{15}$$

Step 2: Limit $\delta \to 0$. Let $w^\delta := w^k$ and $u^\delta := u^k$. Before we pass to the limit $\delta \to 0$, we derive a very weak formulation for u^δ. It holds that

$$(B_\varepsilon(w^\delta)\nabla w^\delta)_i = (A_\varepsilon(u^\delta)\nabla u^\delta)_i = \varepsilon(A^0(u^\delta)\nabla u^\delta)_i + \nabla(u_i^\delta p_i(u^\delta))$$
$$= \frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}\nabla(u_i^\delta)^3 + \nabla(u_i^\delta p_i(u^\delta)).$$

Therefore, in view of (10), (u^δ, w^δ) solves for all $\phi \in H_\nu^2(\Omega;\mathbb{R}^n)$,

$$\frac{1}{\tau}\int_\Omega (u^\delta - u^{k-1})\cdot\phi dx - \sum_{i=1}^n\int_\Omega\Big(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^\delta)^3 + u_i^\delta p_i(u^\delta)\Big)\Delta\phi_i dx \tag{16}$$

$$+\delta\int_\Omega\bigg(\sum_{|\alpha|=m}D^\alpha w^\delta\cdot D^\alpha\phi+w^\delta\cdot\phi\bigg)dx=0.$$

In view of the uniform bounds (15) and using the compact embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$ (if $d\le 3$), there exist subsequences of (u^δ) and (w^δ), which are not relabeled, such that, as $\delta\to 0$,

$$u^\delta\to u\quad\text{strongly in }L^4(\Omega),\qquad \delta w^\delta\to 0\quad\text{strongly in }H^m(\Omega).$$

It follows from the linearity of p_i that $u_i^\delta p_i(u^\delta)\to u_ip_i(u)$ strongly in $L^2(\Omega)$. Moreover, up to a subsequence, $u^\delta\to u$ a.e. in Ω and, because of the continuous embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ for $d\le 3$, $(u_i^\delta)^3\rightharpoonup u_i^3$ weakly in $L^2(\Omega)$. Thus, passing to the limit $\delta\to 0$ in (16), we find that, for all $\phi\in H^2_\nu(\Omega;\mathbb{R}^n)$,

$$\frac{1}{\tau}\int_\Omega(u^k-u^{k-1})\cdot\phi dx=\sum_{i=1}^n\int_\Omega\bigg(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^k)^3+u_i^kp_i(u^k)\bigg)\Delta\phi_i dx, \tag{17}$$

where we have set $u^k:=u$.

Step 3: Bounds uniform in (ε,τ). We introduce piecewise in time constant functions and formulate some bounds uniform in (ε,τ). Let $u^{(\tau)}(x,t)=u(x)$ for $x\in\Omega$, $t\in((k-1)\tau,k\tau]$. At time $t=0$, we set $u^{(\tau)}(\cdot,0)=u^0_\varepsilon$. Furthermore, let $u^{(\tau)}=(u_1^{(\tau)},\ldots,u_n^{(\tau)})$. We define the backward shift operator $(\sigma_\tau u^{(\tau)})(x,t)=u^{k-1}(x)$ for $x\in\Omega$, $t\in((k-1)\tau,k\tau]$. In view of (17), $u^{(\tau)}$ solves

$$\begin{aligned}&\frac{1}{\tau}\int_0^T\int_\Omega(u^{(\tau)}-\sigma_\tau u^{(\tau)})\cdot\phi dxdt \qquad (18)\\ &=\sum_{i=1}^n\int_0^T\int_\Omega\bigg(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^3+u_i^{(\tau)}p_i(u^{(\tau)})\bigg)\Delta\phi_i dxdt\end{aligned}$$

for piecewise constant functions $\phi:(0,T)\to H^2_\nu(\Omega;\mathbb{R}^n)$. By a density argument, this equation also holds for all $\phi\in L^2(0,T;H^2_\nu(\Omega;\mathbb{R}^n))$ [18, Prop. 1.36].

We conclude from the summarized discrete entropy inequality (13), the $L^1(\Omega)$ estimate (14), and the Poincaré–Wirtinger inequality the following (ε,τ)-independent bounds.

Lemma 5. *There exists a constant $C>0$, which is independent of ε and τ, such that for all $i=1,\ldots,n$,*

$$\|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}+\|(u_i^{(\tau)})^{1/2}\|_{L^2(0,T;H^1(\Omega))}+\sqrt{\varepsilon}\|u_i^{(\tau)}\|_{L^2(0,T;H^1(\Omega))}\le C.$$

The Gagliardo–Nirenberg inequality for $p=1+2/d$ and $\theta=d/(2+d)$ gives

$$\|u_i^{(\tau)}\|^p_{L^p(Q_T)}=\int_0^T\|(u_i^{(\tau)})^{1/2}\|^{2p}_{L^{2p}(\Omega)}dt\le C\int_0^T\|(u_i^{(\tau)})^{1/2}\|^{2p\theta}_{H^1(\Omega)}\|(u_i^{(\tau)})^{1/2}\|^{2p(1-\theta)}_{L^2(\Omega)}dt \tag{19}$$

$$\le C\|(u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega)}^{p(1-\theta)} \int_0^T \|(u_i^{(\tau)})^{1/2}\|_{H^1(\Omega)}^2 dt \le C,$$

since $2p\theta = 2$. As we need at least a uniform estimate for $u_i^{(\tau)}$ in $L^{2+\eta}(Q_T)$ for $\eta > 0$ to pass to the limit in (18), the above $L^p(Q_T)$ bound is not sufficient except for $d = 1$. We need an additional estimate, which is provided by the following lemma.

Lemma 6. *There exists a constant $C > 0$, which is independent of ε and τ, such that*

$$\|u_i^{(\tau)}\|_{L^3(Q_T)} + \varepsilon^{1/4}\|u_i^{(\tau)}\|_{L^4(Q_T)} \le C, \quad i = 1, \dots, n.$$

Proof. We use the duality method. For this, let $\psi_i^k \in \{\psi \in H_\nu^2(\Omega) : \int_\Omega \psi dx = 0\}$ be the unique solution to

$$-\Delta\psi_i^k = u_i^k - \fint_\Omega u_i^k dx \quad \text{in } \Omega, \quad \nabla\psi_i^k \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{20}$$

where $\fint u_i^k dx = \text{meas}(\Omega)^{-1} \int_\Omega u_i^k dx$. This problem is well-posed since $u_i^k \in L^2(\Omega)$ and $\partial\Omega \in C^2$. We use ψ_i^k as a test function in the weak formulation (20):

$$\begin{aligned}\int_\Omega |\nabla\psi_i^k|^2 dx &= \int_\Omega u_i^k\psi_i^k dx - \fint_\Omega u_i^k dx \int_\Omega \psi_i^k dx = \int_\Omega u_i^k\psi_i^k dx \\ &\le \|u_i^k\|_{L^2(\Omega)}\|\psi_i^k\|_{L^2(\Omega)} \le C\|u_i^k\|_{L^2(\Omega)}\|\nabla\psi_i^k\|_{L^2(\Omega)},\end{aligned}$$

where we applied the Poincaré–Wirtinger inequality in the last step. Thus, $\|\nabla\psi_i^k\|_{L^2(\Omega)} \le C\|u_i^k\|_{L^2(\Omega)}$ and, by the Poincaré inequality again, $\|\psi_i^k\|_{L^2(\Omega)} \le C\|u_i^k\|_{L^2(\Omega)}$. Hence,

$$\|\psi_i^k\|_{H^1(\Omega)} \le C\|u_i^k\|_{L^2(\Omega)}.$$

Now, taking $\phi_i = \psi_i^k$ and $\phi_j = 0$ for $j \ne i$ as a test function in the weak formulation of (17) and using equation (20) for u_i^k and the property $\int_\Omega \psi_i^k dx = 0$,

$$\begin{aligned}-\frac{1}{\tau}\int_\Omega (\Delta\psi_i^k - \Delta\psi_i^{k-1})\psi_i^k dx &= \frac{1}{\tau}\int_\Omega (u_i^k - u_i^{k-1})\psi_i^k dx - \frac{1}{\tau}\fint_\Omega (u_i^k - u_i^{k-1})dx \int_\Omega \psi_i^k dx \\ &= \int_\Omega \left(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^k)^3 + u_i^k p_i(u^k)\right)\Delta\psi_i^k dx \\ &= -\int_\Omega \left(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^k)^3 + u_i^k p_i(u^k)\right)u_i^k dx + \fint_\Omega u_i^k dx \int_\Omega \left(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^k)^3 + u_i^k p_i(u^k)\right)dx.\end{aligned}$$

Summing this identity over $k = 1, \dots, N$ and observing that

$$-\frac{1}{\tau}\int_\Omega (\Delta\psi_i^k - \Delta\psi_i^{k-1})\psi_i^k dx = \frac{1}{\tau}\int_\Omega (|\nabla\psi_i^k|^2 - \nabla\psi_i^{k-1}\cdot\nabla\psi_i^k)dx$$
$$\geq \frac{1}{2\tau}\int_\Omega (|\nabla\psi_i^k|^2 - |\nabla\psi_i^{k-1}|^2)dx,$$

we obtain

$$\frac{1}{2}\int_\Omega (|\nabla\psi_i^N|^2 - |\nabla\psi_i^0|^2)dx \leq -\int_0^T\int_\Omega \left(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^4 + (u_i^{(\tau)})^2 p_i(u^{(\tau)})\right)dxdt \tag{21}$$
$$+\int_0^T\left(\fint_\Omega u_i^{(\tau)}dx\right)\int_\Omega\left(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^3 + u_i^{(\tau)}p_i(u^{(\tau)})\right)dxdt.$$

As $u_i^{(\tau)}$ is bounded in $L^\infty(0,T;L^1(\Omega))$ by Lemma 5, we can estimate the last term on the right-hand side by

$$\int_0^T\left(\fint_\Omega u_i^{(\tau)}dx\right)\int_\Omega\left(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^3 + u_i^{(\tau)}p_i(u^{(\tau)})\right)dxdt$$
$$\leq C(u^0)\int_0^T\int_\Omega\left(\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^3 + u_i^{(\tau)}p_i(u^{(\tau)})\right)dxdt.$$

We deduce from Young's inequality $ab \leq (\delta a)^p/p + (b/\delta)^q/q$ for $a, b \geq 0$ and $1/p + 1/q = 1$ for suitable $\delta > 0$ that

$$C(u^0)\frac{\varepsilon}{3}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^3 \leq \frac{\varepsilon}{6}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^4 + C_1,$$
$$C(u^0)u_i^{(\tau)}p_i(u^{(\tau)}) \leq \frac{1}{2}(u_i^{(\tau)})^2 p_i(u^{(\tau)}) + C_2 p_i(u^{(\tau)}).$$

The first terms on the right-hand sides can be absorbed by the first term on the right-hand side of (21), leading to

$$\frac{1}{2}\int_\Omega (|\nabla\psi_i^N|^2 - |\nabla\psi_i^0|^2)dx \leq -\int_0^T\int_\Omega\left(\frac{\varepsilon}{6}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^4 + \frac{1}{2}(u_i^{(\tau)})^2 p_i(u^{(\tau)})\right)dx$$
$$+\int_0^T\int_\Omega (C_1 + C_2 p_i(u^{(\tau)}))dx.$$

Since $p_i(u^{(\tau)})$ depends linearly on $u_i^{(\tau)}$ and this function is uniformly bounded in $L^\infty(0,T; L^1(\Omega))$, we conclude that

$$\frac{1}{2}\int_\Omega |\nabla\psi_i^N|^2 dx + \int_0^T\int_\Omega \left(\frac{\varepsilon}{6}\frac{\mu_i}{\pi_i}(u_i^{(\tau)})^4 + \frac{1}{2}(u_i^{(\tau)})^2 p_i(u^{(\tau)})\right)dx$$
$$\le \frac{1}{2}\int_\Omega |\nabla\psi_i^0|^2 dx + C_3(u^0) \le C\|u^0_{\varepsilon,i}\|^2_{L^2(\Omega)} + C_3(u^0) \le C(u^0).$$

Taking into account the inequality $(u_i^{(\tau)})^2 p_i(u^{(\tau)}) \ge a_{ii}(u_i^{(\tau)})^3$, this finishes the proof. □

Lemma 7. *There exists a constant $C > 0$, which is independent of ε and τ, such that*

$$\|u_i^{(\tau)}\|_{L^{3/2}(0,T;W^{1,3/2}(\Omega))} \le C.$$

Proof. This estimate follows directly from Lemmas 5–6 and Hölder's inequality:

$$\|\nabla u_i^{(\tau)}\|_{L^{3/2}(Q_T)} = 2\|(u_i^{(\tau)})^{1/2}\nabla(u_i^{(\tau)})^{1/2}\|_{L^{3/2}(Q_T)},$$
$$\le 2\|u_i^{(\tau)}\|^{1/2}_{L^3(Q_T)}\|\nabla(u_i^{(\tau)})^{1/2}\|_{L^2(Q_T)} \le C,$$

as well as the bound for $u_i^{(\tau)}$ in $L^3(Q_T)$ and also in $L^{3/2}(Q_T)$. □

The improved integrability of $u^{(\tau)}$ provides a uniform bound for the discrete time derivative.

Lemma 8. *There exists a constant $C > 0$, which is independent of ε and τ, such that*

$$\tau^{-1}\|u^{(\tau)} - \sigma_\tau u^{(\tau)}\|_{L^{4/3}(0,T;W_\nu^{2,4}(\Omega)')} \le C.$$

Proof. Lemma 6 shows that

$$\|u_i^{(\tau)} p_i(u^{(\tau)})\|_{L^{3/2}(Q_T)} \le C, \quad \varepsilon\|(u_i^{(\tau)})^3\|_{L^{4/3}(Q_T)} = \varepsilon\|u_i^{(\tau)}\|^3_{L^4(Q_T)} \le \varepsilon^{1/4}C.$$

Let $\phi \in L^4(0,T; W_\nu^{2,4}(\Omega;\mathbb{R}^n))$. Then, using (18), we can estimate as follows:

$$\frac{1}{\tau}\left|\int_0^T\int_\Omega (u^{(\tau)} - \sigma_\tau u^{(\tau)})\cdot\phi dx dt\right| \le \sum_{i=1}^n \|u_i^{(\tau)} p_i(u^{(\tau)})\|_{L^{3/2}(Q_T)}\|\Delta\phi_i\|_{L^3(Q_T)}$$
$$+ \frac{\varepsilon}{3}\sum_{i=1}^n \frac{\mu_i}{\pi_i}\|(u_i^{(\tau)})^3\|_{L^{4/3}(Q_T)}\|\Delta\phi_i\|_{L^4(Q_T)} \le C(1+\varepsilon^{1/4})\|\phi\|_{L^4(0,T;W^{2,4}(\Omega))}.$$

This finishes the proof. □

Step 4: Limit $(\varepsilon, \tau) \to 0$. Lemmas 7 and 8 allow us to apply the lemma of Aubin–Lions in the version of [11], yielding the existence of a subsequence of $(u^{(\tau)})$, which is not relabeled, such that, as $(\varepsilon, \tau) \to 0$,

$$u^{(\tau)} \to u \quad \text{strongly in } L^{3/2}(Q_T) \text{ and a.e. in } Q_T.$$

It follows from Lemmas 5 and 8 that

$$\begin{aligned} (u_i^{(\tau)})^{1/2} &\rightharpoonup u_i^{1/2} \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \\ \tau^{-1}(u^{(\tau)} - \sigma_\tau u^{(\tau)}) &\rightharpoonup \partial_t u \quad \text{weakly in } L^{4/3}(0,T;W_\nu^{2,4}(\Omega)'). \end{aligned}$$

The a.e. convergence of $(u^{(\tau)})$ implies that $u_i^{(\tau)} p_i(u^{(\tau)}) \to u_i p_i(u)$ a.e. in Q_T. Since $u_i^{(\tau)} p_i(u^{(\tau)})$ is bounded in $L^{3/2}(Q_T)$, we infer that

$$u_i^{(\tau)} p_i(u^{(\tau)}) \to u_i p_i(u) \quad \text{strongly in } L^{4/3}(Q_T).$$

Furthermore, taking into account Lemma 6, as $\varepsilon \to 0$,

$$\varepsilon \|(u_i^{(\tau)})^3\|_{L^{4/3}(Q_T)} = \varepsilon^{1/4}\big(\varepsilon^{1/4}\|u_i^{(\tau)}\|_{L^4(Q_T)}\big)^3 \le C\varepsilon^{1/4} \to 0.$$

Thus, performing the limit $(\varepsilon, \tau) \to 0$ in (18) shows that u solves (7). As $u_i \in W^{1,4/3}(0,T; W_\nu^{2,4}(\Omega)') \hookrightarrow C^0([0,T]; W_\nu^{2,4}(\Omega)')$, the initial condition is satisfied in the sense of $W_\nu^{2,4}(\Omega)'$.

Remark 9 *(One-dimensional case).* The additional regularity from the duality method is not needed in the one-dimensional case. In that case, the proof simplifies. First, we may choose $\delta = \varepsilon$. Second, the Gagliardo–Nirenberg inequality (19) shows that $u_i^{(\tau)}$ is uniformly bounded in $L^3(Q_T)$. Furthermore, by estimate (12), $\sqrt{\varepsilon} u_i^{(\tau)}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. Hence, using the Gagliardo–Nirenberg inequality with $\theta = 1/2$ and the uniform bounds in Lemma 5,

$$\begin{aligned} \varepsilon \|u_i^{(\tau)}\|_{L^4(Q_T)}^4 &\le \varepsilon C \int_0^T \|u_i^{(\tau)}\|_{H^1(\Omega)}^{4\theta} \|u_i^{(\tau)}\|_{L^1(\Omega)}^{4(1-\theta)} dt \\ &\le \varepsilon C \|u_i^{(\tau)}\|_{L^\infty(0,T;L^1(\Omega))}^2 \int_0^T \|u_i^{(\tau)}\|_{H^1(\Omega)}^2 dt \le C. \end{aligned}$$

This shows that $\varepsilon^{1/4} u_i^{(\tau)}$ is uniformly bounded in $L^4(Q_T)$, and we obtain the same estimates as in Lemma 6, which allow us to conclude. □

4. Large-time behavior

In this section, we prove Theorem 2. First, we show an entropy inequality which gives time-uniform estimates.

Lemma 10 *(Entropy inequality I). It holds for all $t > 0$ that*

$$\int_\Omega h(u(t))dx + C\sum_{i=1}^n \int_0^t \int_\Omega |\nabla\sqrt{u_i}|^2 dx \le \int_\Omega h(u^0)dx.$$

Proof. We find from (13) that

$$\int_\Omega h_\varepsilon(u^{(\tau)}(t))dx + C\sum_{i=1}^n \int_0^t \int_\Omega |\nabla(u_i^{(\tau)})^{1/2}|^2 dx \le \int_\Omega h_\varepsilon(u^{(\tau)}(0))dx,$$

where $t \in ((j-1)\tau, j\tau]$. Recalling that $h_\varepsilon(u) = h(u) + \varepsilon h^0(u)$ and $h^0(u) \ge -n$ (see (9)), it follows that

$$\int_\Omega h(u^{(\tau)}(t))dx + C\sum_{i=1}^n \int_0^t \int_\Omega |\nabla(u_i^{(\tau)})^{1/2}|^2 dx \le \int_\Omega h_\varepsilon(u_\varepsilon^0)dx + C\varepsilon. \tag{22}$$

Because of the a.e. convergence of $(u^{(\tau)})$, we have $h(u^{(\tau)}(t)) \to h(u(t))$ in Ω for a.e. $t \in (0,T)$, such that Fatou's lemma implies that

$$\int_\Omega h(u(t))dx \le \liminf_{\tau\to 0} \int_\Omega h(u^{(\tau)}(t))dx$$

for a.e. $t > 0$. Then, using the weak lower semicontinuity of the $L^2(Q_T)$ norm, we infer from (22) in the limit $(\varepsilon,\tau) \to 0$ the conclusion. □

The following lemma is a consequence of Lemma 10. Both Lemma 10 and 11 are valid in several space dimensions.

Lemma 11. *There exists a constant $C > 0$, only depending on u^0, such that*

$$\|u_i\|_{L^\infty(0,\infty;L^1(\Omega))} + \|\log u_i\|_{L^\infty(0,\infty;L^1(\Omega))} + \|\nabla\sqrt{u_i}\|_{L^2(0,\infty;L^2(\Omega))} \le C.$$

Proof. The elementary inequalities $z - \log z \ge |\log z|$ and $z - \log z \ge z/2$ for $z > 0$, together with Lemma 10, imply that

$$\|\log u_i\|_{L^\infty(0,\infty;L^1(\Omega))} \le \frac{1}{\pi_i}\int_\Omega h(u^0)dx, \quad \|u_i\|_{L^\infty(0,\infty;L^1(\Omega))} \le \frac{2}{\pi_i}\int_\Omega h(u^0)dx.$$

The bound on $\nabla\sqrt{u_i}$ is a consequence of Lemma 10. □

It is essential to use the entropy at time $s = 0$ in Lemma 10 because it is unclear how to pass to limit $(\tau, \varepsilon) \to 0$ in the entropy at time $s > 0$, as $u_i(s)$ may vanish on a set of zero measure. We overcome this issue by using the test function

$$\left(\pi_i\left(1 - \frac{1}{u_i^k + \eta}\right) + \varepsilon \log(u_i^k + \eta)\right)_{i=1,\ldots,n}$$

for $\eta > 0$ in (10). This means that we need to estimate the matrix $H_\varepsilon(u+\eta)A_\varepsilon(u)$, where $u+\eta = (u_1 + \eta, \ldots, u_n + \eta)$, similarly as we estimated $H_\varepsilon(u)A_\varepsilon(u)$ in Lemma 4. This is done in the following lemma.

Lemma 12. *There exists $\eta_0 > 0$ such that for all $0 < \eta \le \eta_0$, $u \in (0,\infty)^n$, and $z \in \mathbb{R}^n$, it holds that*

$$z^T H_\varepsilon(u+\eta)A_\varepsilon(u)z \ge \frac{\kappa}{4}\sum_{i=1}^n \frac{z_i^2}{u_i+\eta} - \eta\varepsilon C_1 \sum_{i=1}^n \frac{z_i^2}{u_i+\eta} - \eta\varepsilon^2 C_2 \sum_{i=1}^n z_i^2,$$

where $C_1 > 0$ depends on (a_{ij}), (μ_i) and $C_2 > 0$ depends on (μ_i/π_i).

Proof. We decompose the matrix $A_\varepsilon(u) = A(u) + \varepsilon A^0(u)$ as follows:

$$A_\varepsilon(u) = A_{\varepsilon,\eta}(u) - \eta A^1, \quad \text{where } A_{\varepsilon,\eta}(u) := A(u) + \eta A^1 + \varepsilon A^0(u+\eta) - \varepsilon A^2(u),$$

$$A^1_{ij} := \delta_{ij}\left(\sum_{k=1}^n a_{ik} + a_{ij}\right), \quad A^2_{ij}(u) := \delta_{ij}\eta\mu_i\pi_i^{-1}(2u_i + \eta).$$

Note that we have written the matrix $A^0(u)$ as $A^0(u) = A^0(u+\eta) - A^2(u)$ and that we have added and subtracted the matrix ηA^1. We wish to estimate

$$\begin{aligned} H_\varepsilon(u+\eta)A_\varepsilon(u) = \big(H(u+\eta) + \varepsilon H^0(u+\eta)\big)\big((A(u)+\eta A^1) + \varepsilon A^0(u+\eta) - \varepsilon A^2(u)\big) \\ - \eta\big(H(u+\eta) + \varepsilon H^0(u+\eta)\big)A^1 =: K^1 + \ldots + K^5, \end{aligned} \tag{23}$$

where

$$\begin{aligned} K^1 &= H(u+\eta)(A(u) + \eta A^1), \\ K^2 &= \varepsilon H^0(u+\eta)(A(u)+\eta A^1) + \varepsilon H(u+\eta)A^0(u+\eta), \\ K^3 &= \varepsilon^2 H^0(u+\eta)A^0(u+\eta), \\ K^4 &= -\eta\big(H(u+\eta) + \varepsilon H^0(u+\eta)\big)A^1, \\ K^5 &= -\varepsilon\big(H(u+\eta) + \varepsilon H^0(u+\eta)\big)A^2(u). \end{aligned}$$

In the following, let $z \in \mathbb{R}^n$ be fixed.

Step 1: Estimate of $z^T K^1 z$. Since $H_{ij}(u+\eta) = \delta_{ij}\pi_i(u_i+\eta)^{-2}$ and

$$\begin{aligned} A_{ij}(u) + \eta A^1_{ij} &= \delta_{ij}a_{i0} + \delta_{ij}\sum_{k=1}^n a_{ik}(u_k+\eta) + a_{ij}(u_i+\eta\delta_{ij}) \\ &= \delta_{ij}a_{i0} + \delta_{ij}\sum_{k=1}^n a_{ik}(u_k+\eta) + \delta_{ij}a_{ij}(u_i+\eta) + a_{ij}(1-\delta_{ij})u_i, \end{aligned}$$

we obtain

$$K^1_{ij} = \delta_{ij}\frac{\pi_i a_{i0}}{(u_i+\eta)^2} + \delta_{ij}\sum_{k=1}^n \pi_i a_{ik}\frac{u_k+\eta}{(u_i+\eta)^2} + \delta_{ij}\frac{\pi_i a_{ij}}{u_i+\eta} + (1-\delta_{ij})\frac{\pi_i a_{ij}u_i}{(u_i+\eta)^2}$$

and

$$\begin{aligned} z^T K^1 z = \sum_{i=1}^n \frac{\pi_i a_{i0}}{(u_i+\eta)^2}z_i^2 + 2\sum_{i=1}^n \frac{\pi_i a_{ii}}{u_i+\eta}z_i^2 + \sum_{i,k=1,\,i\neq k}^n \pi_i a_{ik}\frac{u_k+\eta}{(u_i+\eta)^2}z_i^2 \\ + \sum_{i,j=1,\,i\neq j}^n \frac{\pi_i a_{ij}u_i}{(u_i+\eta)^2}z_i z_j. \end{aligned} \tag{24}$$

We estimate the last term by Young's inequality:

$$\begin{aligned} \sum_{i,j=1,\,i\neq j}^n \frac{\pi_i a_{ij}u_i}{(u_i+\eta)^2}z_i z_j &\geq -\sum_{i,j=1,\,i\neq j}^n \frac{\pi_i a_{ij}}{u_i+\eta}|z_i z_j| \\ &\geq -\sum_{i,j=1,\,i\neq j}^n \pi_i a_{ij}\frac{u_j+\eta}{(u_i+\eta)^2}z_i^2 - \frac{1}{4}\sum_{i,j=1,\,i\neq j}^n \frac{\pi_i a_{ij}}{u_j+\eta}z_j^2 \\ &= -\sum_{i,j=1,\,i\neq j}^n \pi_i a_{ij}\frac{u_j+\eta}{(u_i+\eta)^2}z_i^2 - \frac{1}{4}\sum_{i,j=1,\,i\neq j}^n \frac{\pi_j a_{ji}}{u_i+\eta}z_i^2. \end{aligned}$$

The first term on the right-hand side cancels with the third term on the right-hand side of (24). Therefore,

$$\begin{aligned} z^T K^1 z &\geq \sum_{i=1}^n \frac{\pi_i a_{i0}}{(u_i+\eta)^2}z_i^2 + \frac{1}{4}\sum_{i=1}^n \bigg(8\pi_i a_{ii} - \sum_{j=1,\,j\neq i}^n \pi_j a_{ji}\bigg)\frac{z_i^2}{u_i+\eta} \\ &\geq \sum_{i=1}^n \frac{\pi_i a_{i0}}{(u_i+\eta)^2}z_i^2 + \frac{\kappa}{4}\sum_{i=1}^n \frac{z_i^2}{u_i+\eta}, \end{aligned}$$

where κ is defined in (6).

Step 2: Estimate of $z^T K^2 z$. It follows from

$$\varepsilon^{-1} K^2_{ij} = \delta_{ij} \frac{a_{i0}}{u_i+\eta} + \delta_{ij} \sum_{k=1}^n a_{ik} \frac{u_k+\eta}{u_i+\eta} + \delta_{ij} a_{ij} + (1-\delta_{ij}) \frac{a_{ij} u_i}{u_i+\eta} + \delta_{ij} \mu_i$$

that

$$z^T(\varepsilon^{-1} K^2) z = \sum_{i=1}^n \left(\frac{a_{i0}}{u_i+\eta} + \sum_{k=1,\, k\neq i}^n a_{ik} \frac{u_k+\eta}{u_i+\eta} + 2a_{ii} + \mu_i \right) z_i^2 + \sum_{i,j=1}^n (1-\delta_{ij}) \frac{a_{ij} u_i}{u_i+\eta} z_i z_j .$$

Using Young's inequality $z_i z_j \geq -(z_i^2 + z_j^2)/2$ and taking into account our choice of μ_i in Section 2, we find that

$$\begin{aligned} \sum_{i=1}^n \mu_i z_i^2 + \sum_{i,j=1}^n (1-\delta_{ij}) \frac{a_{ij} u_i}{u_i+\eta} z_i z_j &\geq \sum_{i=1}^n \mu_i z_i^2 - \frac{1}{2} \sum_{i=1}^n \sum_{j=1,\, j\neq i}^n a_{ij} (z_i^2 + z_j^2) \\ &\geq \sum_{i=1}^n \left(\mu_i - \frac{1}{2} \sum_{j=1,\, j\neq i}^n (a_{ij} + a_{ji}) \right) z_i^2 \geq 0. \end{aligned}$$

This shows that

$$z^T K^2 z \geq 2\varepsilon \sum_{i=1}^n a_{ii} z_i^2 \geq 0.$$

Step 3: Computation of K^3, K^4, and K^5. The definitions of the matrices yield

$$\begin{aligned} z^T K^3 z &= \varepsilon^2 \sum_{i=1}^n \frac{\mu_i}{\pi_i} (u_i+\eta) z_i^2 \geq 0, \\ z^T K^4 z &= -\eta \sum_{i=1}^n \left(\sum_{k=1}^n a_{ik} + a_{ii} \right) \left(\frac{\pi_i}{(u_i+\eta)^2} + \frac{\varepsilon}{u_i+\eta} \right) z_i^2 \\ &= -\eta \sum_{i,j=1}^n \pi_i a_{ij} \frac{(1+\delta_{ij}) z_i^2}{(u_i+\eta)^2} - \eta\varepsilon \sum_{i,j=1}^n a_{ij} \frac{(1+\delta_{ij}) z_i^2}{u_i+\eta}, \\ z^T K^5 z &= -\varepsilon\eta \sum_{i=1}^n \frac{\mu_i}{\pi_i} \left(\pi_i \frac{2u_i+\eta}{(u_i+\eta)^2} + \varepsilon \frac{2u_i+\eta}{u_i+\eta} \right) z_i^2 \\ &\geq -2\eta\varepsilon \sum_{i=1}^n \mu_i \left(\frac{1}{u_i+\eta} + \frac{\varepsilon}{\pi_i} \right) z_i^2. \end{aligned}$$

Step 4: End of the proof. We insert the estimates for $K^1, \dots, K^5$ into (23):

$$z^T H_\varepsilon(u+\eta)A_\varepsilon(u+\eta)z \ge \sum_{i=1}^n \pi_i\Big(a_{i0} - \eta\sum_{j=1}^n a_{ij}(1+\delta_{ij})\Big)\frac{z_i^2}{(u_i+\eta)^2}$$
$$+ \frac{\kappa}{4}\sum_{i=1}^n \frac{z_i^2}{u_i+\eta} - \eta\varepsilon\sum_{i=1}^n\Big(\sum_{j=1}^n a_{ij}(1+\delta_{ij}) + 2\mu_i\Big)\frac{z_i^2}{u_i+\eta} - 2\eta\varepsilon^2\sum_{i=1}^n\frac{\mu_i}{\pi_i}z_i^2.$$

Choosing

$$0 < \eta \le \eta_0 := \min_{i=1,\dots,n} a_{i0}\Big(\sum_{j=1}^n a_{ij}(1+\delta_{ij})\Big)^{-1},$$

the first term on the left-hand side is nonnegative, and we obtain

$$z^T H_\varepsilon(u+\eta)A_\varepsilon(u)z \ge \frac{\kappa}{4}\sum_{i=1}^n\frac{z_i^2}{u_i+\eta} - \eta\varepsilon C_1\sum_{i=1}^n\frac{z_i^2}{u_i+\eta} - \eta\varepsilon^2 C_2\sum_{i=1}^n z_i^2,$$

where $C_1 = 2\max_{i=1,\dots,n}(\sum_{j=1}^n a_{ij} + \mu_i)$ and $C_2 = 2\max_{i=1,\dots,n}(\mu_i/\pi_i)$. This finishes the proof. □

Lemma 13 *(Entropy inequality II). Let $d = 1$ and let $0 < \eta \le \eta_0$ (see Lemma 12). Then there exists $C > 0$ independent of η such that for $0 \le s < t$,*

$$\int_\Omega h(u(t)+\eta)dx + C\sum_{i=1}^n\int_s^t\int_\Omega |\nabla\sqrt{u_i+\eta}|^2 dx d\sigma \le \int_\Omega h(u(s)+\eta)dx.$$

Proof. We use $v^k = (v_1^k, \dots, v_n^k)$ with

$$v_i^k = \frac{\partial h_\varepsilon}{\partial u_i}(u^k+\eta) = \pi_i\Big(1 - \frac{1}{u_i^k+\eta}\Big) + \varepsilon\log(u_i^k+\eta)$$

as a test function in the weak formulation of the approximate equations (10):

$$\frac{1}{\tau}\int_\Omega (u^k - u^{k-1})\cdot v^k dx + \int_\Omega \nabla v^k : B_\varepsilon(w^k)\nabla w^k dx$$
$$+ \varepsilon\int_\Omega\Big(\sum_{|\alpha|=1} D^\alpha w^k\cdot D^\alpha v^k + w^k\cdot v^k\Big)dx = 0.$$

Note that we have chosen $\delta = \varepsilon$; see Remark 9. The convexity of h_ε implies that

$$(u^k - u^{k-1})\cdot v^k = ((u^k+\eta) - (u^{k-1}+\eta))\cdot h'_\varepsilon(u^k+\eta) \ge h_\varepsilon(u^k+\eta) - h_\varepsilon(u^{k-1}+\eta).$$

Furthermore, by the definition of v^k,

$$\sum_{|\alpha|=1} D^\alpha w^k \cdot D^\alpha v^k = \nabla w^k \cdot \nabla v^k = \left(\frac{\pi_i}{(u_i^k)^2} + \frac{\varepsilon}{u_i^k}\right)\left(\frac{\pi_i}{(u_i^k+\eta)^2} + \frac{\varepsilon}{u_i^k+\eta}\right)|\nabla u_i^k|^2 \geq 0.$$

Note that $u_i^k > 0$, so quotients of the type $\pi_i/(u_i^k)^2$ are well-defined. It follows from $\nabla v^k = H_\varepsilon(u^k+\eta)\nabla u^k$ and Lemma 12 that

$$\begin{aligned}\nabla v^k : B_\varepsilon(w^k)\nabla w^k &= \nabla u^k : H_\varepsilon(u^k+\eta)A_\varepsilon(u^k)\nabla u^k \\ &\geq \frac{\kappa}{4}\sum_{i=1}^n \frac{|\nabla(u_i^k+\eta)|^2}{u_i^k+\eta} - \eta\varepsilon C_1 \sum_{i=1}^n \frac{|\nabla u_i^k|^2}{u_i^k+\eta} - \eta\varepsilon^2 C_2 \sum_{i=1}^n |\nabla u_i^k|^2.\end{aligned}$$

Summarizing, this gives

$$\begin{aligned}\int_\Omega h_\varepsilon(u^k+\eta)dx + \kappa\tau\sum_{i=1}^n |\nabla(u_i^k+\eta)^{1/2}|^2 dx &\leq \int_\Omega h_\varepsilon(u^{k-1}+\eta)dx \\ &\quad - \varepsilon\tau\int_\Omega w^k\cdot v^k dx + \eta\varepsilon\tau C_1\sum_{i=1}^n\int_\Omega \frac{|\nabla u_i^k|^2}{u_i^k+\eta} + \eta\varepsilon^2\tau C_2\sum_{i=1}^n|\nabla u_i^k|^2.\end{aligned}$$

We sum this inequality from $k = j, \ldots, \ell$ for $j < \ell$:

$$\begin{aligned}&\int_\Omega h_\varepsilon(u^\ell+\eta)dx + \kappa\sum_{i=1}^n\sum_{k=j}^\ell \tau\int_\Omega |\nabla(u_i^k+\eta)^{1/2}|^2 dx \\ &\leq \int_\Omega h_\varepsilon(u^{j-1}+\eta)dx - \varepsilon\sum_{k=j}^\ell \tau\int_\Omega w^k\cdot v^k dx \\ &\quad + \eta\varepsilon C_1\sum_{i=1}^n\sum_{k=j}^\ell \tau\int_\Omega \frac{|\nabla u_i^k|^2}{u_i^k+\eta}dx + \eta\varepsilon^2 C_2\sum_{i=1}^n\sum_{k=j}^\ell \tau\int_\Omega |\nabla u_i^k|^2 dx.\end{aligned} \tag{25}$$

We know from Lemma 5 that

$$\sum_{k=0}^N \tau\|(u_i^k)^{1/2}\|_{H^1(\Omega)}^2 + \varepsilon\sum_{k=0}^N \tau\|u_i^k\|_{H^1(\Omega)}^2 \leq C.$$

Since $|\nabla u_i^k|^2/(u_i^k+\eta) = 4u_i^k|\nabla(u_i^k)^{1/2}|^2/(u_i^k+\eta) \leq 4|\nabla(u_i^k)^{1/2}|$, the last two terms on the right-hand side of (25) are bounded from above by $\eta\varepsilon C$. Thus, it remains to estimate the first term on the right-hand side of (25). We write

$$-\varepsilon\sum_{k=j}^{\ell}\tau\int_\Omega w^k\cdot v^k dx = I_1+I_2, \quad \text{where}$$

$$I_1 := -\varepsilon\sum_{i=1}^n\sum_{k=j}^{\ell}\tau\int_\Omega \pi_i\Big(1-\frac{1}{u_i^k+\eta}\Big)w_i^k dx,$$

$$I_2 := -\varepsilon^2\sum_{i=1}^n\sum_{k=j}^{\ell}\tau\int_\Omega \log(u_i^k+\eta)w_i^k dx.$$

Since estimate (13) shows that

$$\varepsilon\sum_{i=1}^n\sum_{k=1}^N\tau\|w_i^k\|_{H^1(\Omega)}^2 \le C, \tag{26}$$

we obtain

$$I_1 \le \varepsilon\sum_{i=1}^n\sum_{k=j}^{\ell}\tau\int_\Omega \pi_i\Big(1+\frac{1}{\eta}\Big)|w_i^k|dx \le C(\eta,T)\sqrt{\varepsilon}.$$

To estimate I_2, we first compute

$$\int_\Omega |\log(u_i^k+\eta)|dx \le \int_\Omega |\log(u_i^k+\eta)-\log\eta|dx + \int_\Omega|\log\eta|dx$$

$$= \int_\Omega\bigg|u_i^k\int_0^1\frac{d\theta}{\theta u_i^k+\eta}\bigg|dx + |\log\eta|\operatorname{meas}(\Omega) \le \frac{1}{\eta}\|u_i^k\|_{L^1(\Omega)} + |\log\eta|\operatorname{meas}(\Omega).$$

By Lemma 5, $\|u_i^k\|_{L^1(\Omega)}$ is bounded uniformly in k (and (ε,τ)). We conclude from (26) and the continuous embedding $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ in one space dimension that

$$I_2 \le \varepsilon^2\sum_{i=1}^n\sum_{k=j}^{\ell}\tau\|\log(u_i^k+\eta)\|_{L^1(\Omega)}\|w_i^k\|_{L^\infty(\Omega)}$$

$$\le \varepsilon^2 C\sum_{i=1}^n\max_{k=1,\ldots,N}\|\log(u_i^k+\eta)\|_{L^1(\Omega)}\sqrt{T}\sum_{k=1}^N\tau\|w_i^k\|_{H^1(\Omega)}^2 \le C(\eta,T)\varepsilon^{3/2}.$$

Summarizing these estimates, we infer from (25), using the notation from Section 3, that

$$\int_\Omega h_\varepsilon(u^{(\tau)}(t)+\eta)dx + \kappa\sum_{i=1}^n\int_s^t\int_\Omega|\nabla(u_i^{(\tau)}+\eta)^{1/2}|^2 dx d\sigma \tag{27}$$

$$\le \int_\Omega h_\varepsilon(u^{(\tau)}(s-\tau)+\eta)dx + \eta\varepsilon C + C(\eta,T)\sqrt{\varepsilon}(1+\varepsilon),$$

where $s \in ((j-1)\tau, j\tau]$, $t \in ((\ell-1)\tau, \ell\tau]$. Since

$$|\nabla(u_i^{(\tau)}+\eta)^{1/2}| = \left|\frac{\nabla u_i^{(\tau)}}{2(u_i^{(\tau)}+\eta)^{1/2}}\right| \le \left|\frac{\nabla u_i^{(\tau)}}{2(u_i^{(\tau)})^{1/2}}\right| = |\nabla(u_i^{(\tau)})^{1/2}|,$$

it follows from estimate (13) that $\sum_{i=1}^n \|\nabla(u_i^{(\tau)}+\eta)^{1/2}\|_{L^2(Q_T)} \le C$. We have already proved that, up to a subsequence, $u_i^{(\tau)} \to u_i$ strongly in $L^{3/2}(Q_T)$ as $(\varepsilon,\tau)\to 0$. We infer that $\nabla(u_i^{(\tau)}+\eta)^{1/2} \rightharpoonup \nabla(u_i+\eta)^{1/2}$ weakly in $L^2(Q_T)$. Therefore, $u_i^{(\tau)}(t) \to u_i(t)$ strongly in $L^{3/2}(\Omega)$ for a.e. $t \in (0,T)$ and

$$\int_\Omega \log(u_i^{(\tau)}(t)+\eta)dx \to \int_\Omega \log(u_i(t)+\eta)dx.$$

By the weak lower semicontinuity of the norm,

$$\int_s^t\int_\Omega |\nabla\sqrt{u_i+\eta}|^2 dx d\sigma \le \liminf_{(\tau,\varepsilon)\to 0}\int_s^t |\nabla(u_i^{(\tau)}+\eta)^{1/2}|^2 dx.$$

The limit $(\tau,\varepsilon)\to 0$ in (27) concludes the proof. □

Next, we introduce for $0<\eta\le\eta_0$ the relative entropy

$$\begin{aligned}\mathcal{H}_\eta(u|\bar u) &= \int_\Omega \big(h(u+\eta) - h(\bar u+\eta) - h'(\bar u+\eta)\cdot((u+\eta)-(\bar u+\eta))\big)dx \\ &= \sum_{i=1}^n \pi_i \int_\Omega \left(\frac{u_i+\eta}{\bar u_i+\eta} - \log\frac{u_i+\eta}{\bar u_i+\eta} - 1\right)dx.\end{aligned}$$

Because of mass conservation, we have $\int_\Omega((u_i+\eta)/(\bar u_i+\eta)-1)dx = 0$, implying that

$$\mathcal{H}_\eta(u|\bar u) = \sum_{i=1}^n \pi_i \int_\Omega \big(\log(\bar u_i+\eta) - \log(u_i+\eta)\big)dx. \tag{28}$$

In view of Lemma 13, we can formulate the relative entropy inequality as

$$\mathcal{H}_\eta(u(t)|\bar u) + C\sum_{i=1}^n \int_s^t\int_\Omega |\nabla\sqrt{u_i+\eta}|^2 dx \le \mathcal{H}_\eta(u(s)|\bar u), \quad 0<s<t. \tag{29}$$

We claim that the relative entropy decays to zero as $t \to \infty$. To prove this, we need some preparation.

Lemma 14. *Let $g \in L^\infty(0,\infty; L^1(\Omega))$ with $g \geq 0$ and $\nabla\sqrt{g} \in L^2(0,\infty; L^2(\Omega))$ be such that $\bar{g} := \fint_\Omega g(x,t)dx$ is independent of $t > 0$ (i.e., g conserves the mass). Then there exists a constant $C > 0$ independent of g such that for $t > 0$,*

$$\|\sqrt{g(t)} - \sqrt{\bar{g}}\|_{L^2(\Omega)} \leq C\|\nabla\sqrt{g(t)}\|_{L^2(\Omega)}.$$

Proof. The proof is similar to that one in [4, Lemma 7] but some arguments are different. We argue by contradiction. Assume that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$n\|\nabla\sqrt{g(t_n)}\|_{L^2(\Omega)} < \|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)} \quad \text{for all } n \in \mathbb{N}. \tag{30}$$

This implies that $\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)} > 0$ and we can define

$$v_n := \frac{\sqrt{g(t_n)} - \sqrt{\bar{g}}}{\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)}}, \quad n \in \mathbb{N}.$$

It follows from (30) that

$$\|\nabla v_n\|_{L^2(\Omega)} = \frac{\|\nabla\sqrt{g(t_n)}\|_{L^2(\Omega)}}{\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)}} < \frac{1}{n},$$

such that $\nabla v_n \to 0$ strongly in $L^2(\Omega)$ as $n \to \infty$. By definition, $\|v_n\|_{L^2(\Omega)} = 1$ for all $n \in \mathbb{N}$, i.e., (v_n) is bounded in $H^1(\Omega)$. Taking into account the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, there exists a subsequence, which is not relabeled, such that $v_n \to v$ strongly in $L^2(\Omega)$ and $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$ as $n \to \infty$. We deduce from $\nabla v_n \to 0$ strongly in $L^2(\Omega)$ that v is a constant and, because of $\|v_n\|_{L^2(\Omega)} = 1$, we have $v \neq 0$.

Now, we show that $\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Otherwise, by contradiction, there exists a subsequence of $(g(t_n))_{n\in\mathbb{N}}$ (not relabeled) and $c > 0$ such that $\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)} \geq c$ for every $n \in \mathbb{N}$. Because of

$$\frac{\sqrt{g(t_n)} - \sqrt{\bar{g}}}{\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)}} \to v \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \tag{31}$$

Egorov's theorem [2, Theorem 4.29] shows that, for any $\varepsilon > 0$, there exists $\Omega_\varepsilon \subset \Omega$ such that $\mathrm{meas}(\Omega \setminus \Omega_\varepsilon) < \varepsilon$ and

$$\frac{\sqrt{g(t_n)} - \sqrt{\bar{g}}}{\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)}} \to v \quad \text{strongly in } L^\infty(\Omega_\varepsilon).$$

Since v is a nonzero constant, there exist $c > 0$ and $N_\varepsilon \in \mathbb{N}$ such that for all $n > N_\varepsilon$,

$$\sqrt{g(t_n)} \geq \sqrt{\bar{g}} + \frac{c}{2}v \quad \text{a.e. in } \Omega_\varepsilon \text{ if } v > 0,$$
$$\sqrt{g(t_n)} \leq \sqrt{\bar{g}_i} - \frac{c}{2}(-v) \quad \text{a.e. in } \Omega_\varepsilon \text{ if } v < 0.$$

Thus, there exist $K_1, K_2 > 0$ independent of ε such that in Ω_ε, $g(t_n) \geq \bar{g} + K_1$ if $v > 0$ and $g(t_n) \leq \bar{g} - K_2$ if $v < 0$. As the integral is absolutely continuous and $\varepsilon > 0$ is arbitrary, this contradicts the constraint $⨍ g(t_n)dx = \bar{g}$. We infer that

$$\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty$$

and consequently,

$$\frac{g(t_n) - \bar{g}}{\sqrt{g(t_n)} - \sqrt{\bar{g}}} = \sqrt{g(t_n)} + \sqrt{\bar{g}} \to 2\sqrt{\bar{g}} \quad \text{strongly in } L^2(\Omega).$$

Then the previous result and convergence (31) imply that

$$\frac{g(t_n) - \bar{g}}{\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)}} = \frac{g(t_n) - \bar{g}}{\sqrt{g(t_n)} - \sqrt{\bar{g}}} \frac{\sqrt{g(t_n)} - \sqrt{\bar{g}}}{\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)}} \to 2\sqrt{\bar{g}}v$$

strongly in $L^1(\Omega)$. However, this gives

$$\int_\Omega \frac{g(t_n) - \bar{g}}{\|\sqrt{g(t_n)} - \sqrt{\bar{g}}\|_{L^2(\Omega)}} dx \to \int_\Omega 2\sqrt{\bar{g}} v dx \neq 0,$$

which violates the conservation of mass and ends the proof. □

The previous lemma and the entropy inequality in Lemma 10 imply that

$$\int_0^\infty \|\sqrt{u_i(t)} - \sqrt{\bar{u}_i}\|_{L^2(\Omega)}^2 dt \leq C \int_0^\infty \|\nabla\sqrt{u_i(t)}\|_{L^2(\Omega)}^2 dt \leq C(u^0).$$

Consequently, there exists a sequence $(t_k) \subset [0, \infty)$ satisfying $t_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} \|\sqrt{u_i(t_k)} - \sqrt{\bar{u}_i}\|_{L^2(\Omega)} = 0.$$

This shows that

$$\begin{aligned}\lim_{k\to\infty} \|u_i(t_k) - \bar{u}_i\|_{L^1(\Omega)} &\leq \lim_{k\to\infty} \|\sqrt{u_i(t_k)} + \sqrt{\bar{u}_i}\|_{L^2(\Omega)} \|\sqrt{u_i(t_k)} - \sqrt{\bar{u}_i}\|_{L^2(\Omega)} \\ &\leq C(u^0) \lim_{k\to\infty} \|\sqrt{u_i(t_k)} - \sqrt{\bar{u}_i}\|_{L^2(\Omega)} = 0.\end{aligned}$$

In particular, we obtain, for any fixed $\eta > 0$,

$$\lim_{k\to\infty}\int_\Omega \log(u_i(t_k)+\eta)dx = \int_\Omega \log(\bar u_i+\eta)dx,$$

and in view of definition (28) of the relative entropy, this implies that

$$\lim_{k\to\infty}\mathcal{H}_\eta(u(t_k)|\bar u)=0.$$

Since $t\mapsto \mathcal{H}_\eta(u(t)|\bar u)$ is bounded and nonincreasing by (29), the convergence holds for all sequences $t\to\infty$:

$$\lim_{t\to\infty}\mathcal{H}_\eta(u(t)|\bar u)=0.$$

Finally, by the Csiszár–Kullback inequality (see Proposition 16 in the appendix),

$$\lim_{t\to\infty}\|u_i(t)-\bar u_i\|_{L^1(\Omega)} \le C\|\bar u_i+\eta\|_{L^2(\Omega)}\lim_{t\to\infty}\mathcal{H}_\eta(u(t)|\bar u)^{1/2}=0$$

for all $0<\eta\le\eta_0$, which ends the proof.

Data availability

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix A. Auxiliary results

Lemma 15. *Let $n=3$, $a_{13}=a_{21}=a_{32}=1$, and $a_{12}=a_{23}=a_{31}=0$. Then there exist $\pi_1,\pi_2,\pi_3>0$ satisfying $\kappa>0$ (see (6)) if and only if $a_{11}a_{22}a_{33}>8^{-3}$.*

Proof. The condition $\kappa>0$ is equivalent to $8\pi_1a_{11}>\pi_2$, $8\pi_2a_{22}>\pi_3$, and $8\pi_3a_{33}>\pi_1$. Multiplying these inequalities immediately gives $8^3a_{11}a_{22}a_{33}>1$. On the other hand, if this inequality is satisfied, we set

$$\pi_1=1,\quad \pi_2=\frac12\left(8a_{11}+\frac{1}{8^2a_{22}a_{33}}\right),\quad \pi_3=\frac12\left(8\pi_2a_{22}+\frac{1}{8a_{33}}\right).$$

Then $8\pi_1a_{11}>\pi_2$ is equivalent to $8^3a_{11}a_{22}a_{33}>1$, and both $8\pi_2a_{22}>\pi_3$ and $8\pi_3a_{33}>\pi_1$ are equivalent to $8^2\pi_2a_{22}a_{33}>1$, which, by definition of π_2, is equivalent to $8^3a_{11}a_{22}a_{33}>1$ again. □

The following result is proved in [3, Section 4.3, page 71, example (c)].

Proposition 16 *(Csiszár–Kullback inequality). Let $\Omega\subset\mathbb{R}^d$ be a domain and $u\in L^1(\Omega)$. We set $\bar u = ⨍_\Omega u dx$ and $\mathcal{H}(u|\bar u)=\int_\Omega(\log(\bar u+\eta)-\log(u+\eta))dx$. Then*

$$\|u-\bar u\|_{L^1(\Omega)}\le\sqrt{8}\|\bar u\|_{L^2(\Omega)}\mathcal{H}(u|\bar u)^{1/2}.$$

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