D I P L O M A R B E I T

Log-Concavity of Mean Section Operators

zur Erlangung des akademischen Grades
Diplom-Ingenieur

im Rahmen des Studiums
Technische Mathematik

eingereicht von
Leo Brauner BSc

ausgeführt am Institut für Diskrete Mathematik und Geometrie
der Fakultät für Mathematik und Geoinformation
der Technischen Universität Wien

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Wien, am 30. August 2021

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Introduction

One of the most important results in convex geometry is the Brunn-Minkowski inequality. In its multiplicative formulation, it states that

$$\mathrm{vol}_n((1-\lambda)K+\lambda K) \geq \mathrm{vol}_n(K)^{1-\lambda}\mathrm{vol}_n(L)^{\lambda} \tag{1}$$

for all convex bodies $K, L \subseteq \mathbb{R}^n$ and $\lambda \in [0,1]$ and the equality conditions are well-understood. The Brunn-Minkowski inequality is closely related to other important geometric inequalities and has significant generalizations. For instance, the volume functional vol_n in (1) can be replaced by the intrinsic volumes V_i for $i \in \{0,\ldots,n\}$.

In the past two decades, efforts were made to relate the Brunn-Minkowski inequality to Minkowski valuations. These are operators Φ mapping convex bodies of $\mathbb{R}^n$ to convex bodies of $\mathbb{R}^n$ which satisfy the equation

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L),$$

whenever K, L, and $K \cup L$ are convex bodies. Minkowski valuations arise naturally in convex geometry, such as the projection body operator in the context of the famous Shephard problem. Hence, it is natural to ask whether the volume functional in (1) can be replaced by $\mathsf{V}_i \circ \Phi_j$, where Φ_j is a Minkowski valuation homogeneous of degree $j \in \{0,\ldots,n\}$. The most recent result on this question is from 2014 and due to A. Berg, F. Schuster, L. Parapatits, and M. Weberndorfer, and gives a positive answer for a fairly large class $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$ of Minkowski valuations of order j. More precisely, if $i, j \in \{0,\ldots,n\}$ and $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$, then

$$\mathsf{V}_i(\Phi_j((1-\lambda)K+\lambda L)) \geq \mathsf{V}_i(\Phi_j(K))^{1-\lambda}\mathsf{V}_i(\Phi_j(L))^{\lambda} \tag{2}$$

for all convex bodies $K, L \subseteq \mathbb{R}^n$ and $\lambda \in [0,1]$.

There are, however, some relevant homogeneous Minkowski valuations that are not yet covered by this result. Explicit examples of such Minkowski valuations are provided by the mean section operators, which were first introduced by P. Goodey and W. Weil in 1992. For $k \in \{0,\ldots,n\}$, the k-th mean section operator $\tilde{M}_k$ maps a convex body to its k-th mean section body, which is an (infinitesimal) Minkowski sum of its sections with

k-dimensional flats. In 2014, P. Goodey and W. Weil gave an integral representation of the support function of $\tilde{M}_k(K)$ in terms of the $(n-k+1)$-th area measure of K that involve the so-called Berg functions. These functions were originally introduced in the context of Christoffel's problem, which concerns the characterization of the first area measures of convex bodies. Said integral representation ultimately implies that mean sections operators are generally not of class $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$ if $i > j+1$. Therefore the question whether (2) holds if $i > j+1$ and we put a mean section operator for Φ_j remains open.

In Chapter 1 and Chapter 2, we summarize the standard theory on convex geometry and integral geometry, which we will later use repeatedly in our arguments. In Chapter 3 we introduce mean section operators and expound P. Goodey and W. Weil's proof for the integral representation mentioned before. In Chapter 4, we give an exposition of the research done on Brunn-Minkowski inequalities for Minkowski valuations including the proof for (2). In Chapter 5, we explain why (2) is not known for mean section operators and give a brief outlook on how this open problem could perhaps be tackled.

Acknowledgments

At this point, I want to express my gratitude to my supervisor Franz Schuster for proposing the topic, formulating a clear structure, and giving valuable advice throughout the process of writing this thesis.

I would also like to thank Oscar Moreno for the intensive discussions helping me a lot to improve this thesis.

Moreover, I want to say thank you to my family, friends, and Stefan for their constant support.

Conventions and Notation

In this thesis, we mainly follow the conventions and notations of [21].

In order to avoid ambiguity on whether zero is a natural number, we exclusively use the notations

$$\mathbb{N}_0 := \{0, 1, 2, \ldots\} \quad \text{and} \quad \mathbb{N}_+ := \{1, 2, 3, \ldots\}.$$

We shall work in n-dimensional real Euclidean vector space $\mathbb{R}^n$ with origin o, inner product $x \cdot y$ and induced norm $\|\cdot\|$. Furthermore, we fix the notation $A^\perp := \{x \in \mathbb{R}^n \mid \forall y \in A : x \cdot y = 0\}$ for subsets $A \subseteq \mathbb{R}^n$.

In the entire thesis, we assume $n \geq 2$.

We denote by $\mathbb{S} \subseteq \mathbb{R}^n$ the *unit sphere* in $\mathbb{R}^n$ and by $B \subseteq \mathbb{R}^n$ the *closed unit ball* in $\mathbb{R}^n$, that is

$$\mathbb{S} := \{x \in \mathbb{R}^n : \|x\| = 1\} \quad \text{and} \quad B := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

For a linear subspace $E \subseteq \mathbb{R}^n$, we denote by $\mathbb{S}(E)$ the unit sphere of E.

For $x \in \mathbb{R}^n$ and a linear subspace $E \subseteq \mathbb{R}^n$, we denote the orthogonal projection of x onto E by $x|E$. For a linear subspace $E \subseteq \mathbb{R}^n$, and $u \in \mathbb{S} \backslash E^\perp$ we define $\mathrm{pr}_E u := \|u|E\|^{-1} u|E$.

We denote by $\mathrm{co}(A)$, $\mathrm{aff}(A)$, and $\mathrm{lin}(A)$ the convex, affine, and linear hull of a sbuset $A \subseteq \mathbb{R}^n$, respectively.

Moreover, we fix the notations

$$\omega_n := \lambda_{\mathbb{S}}(\mathbb{S}) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)} \quad \text{and} \quad \kappa_n := \lambda_n(B) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)},$$

where λ_n denotes the n-dimensional Lebesgue measure, $\lambda_{\mathbb{S}}$ denotes the spherical Lebesgue measure (which is precisely the $(n-1)$-dimensional Hausdorff measure on $\mathbb{S}$) and Γ denotes the Gamma function. Furthermore, we define the *flag coefficient* as

$$\begin{bmatrix} n \\ j \end{bmatrix} := \binom{n}{j} \frac{\kappa_n}{\kappa_j \kappa_{n-j}}$$

for $j \in \{0, \ldots, n\}$.

Throughout this thesis, $\overline{e} \in \mathbb{S}$ will be an arbitrary but fixed point in $\mathbb{S}$.

When working with symmetric functions of multiple arguments, we use the notation

$$K^{[j]} = \underbrace{K, \ldots, K}_{j \text{ times}}$$

to indicate that j of the arguments that we plug in shall be K.

For a subset $A \subseteq \mathbb{R}^n$, we denote by $\mathbb{1}_A$ the indicator function of A, that is $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \in \mathbb{R}^n \backslash A$.

When working with topological vector spaces, we will only consider Hausdorff spaces over $\mathbb{R}$ and we will always consider the topology of a certain topological vector space as fixed. If $\mathbf{X}$ is a topological vector space, we denote its topological dual space by $\mathbf{X}^*$. We denote the canonical evaluation map by

$$\langle \cdot, \cdot \rangle : \mathbf{X} \times \mathbf{X}^* \to \mathbb{R} : \langle x, x^* \rangle := x^*(x).$$

Later on, we will expand this notation to objects that we chose to identify with elements of $\mathbf{X}^*$. If $T : \mathbf{X} \to \mathbf{Y}$ is a continuous linear operator between two topological vector spaces $\mathbf{X}$ and $\mathbf{Y}$, we denote its adjoint operator by

$$T^* : \mathbf{Y}^* \to \mathbf{X}^* : \langle x, T^* y^* \rangle := \langle Tx, y^* \rangle$$

If $\mathbf{X} = \mathbf{Y}$, the linear space $\mathbf{X}$ can be naturally identified with a subspace of $\mathbf{X}^*$, and $T : \mathbf{X} \to \mathbf{X}$ is self-adjoint in the sense that $\langle Tx_1, x_2 \rangle = \langle x_1, Tx_2 \rangle$ for all $x_1, x_2 \in \mathbf{X}$, then we write $T^* = T$.

We denote by $\mathrm{GL}(n)$ the group of bijective linear maps, by $\mathrm{O}(n)$ the group of linear isometries, by $\mathrm{SO}(n)$ the group of proper rotations, and by $\mathrm{E}(n)$ the group of rigid motions in $\mathbb{R}^n$. We denote by $\mathrm{SO}(n-1, \bar{e})$ the subgroup of rotations in $\mathrm{SO}(n)$ that leave $\bar{e}$ fixed. We denote by $\mathrm{G}(n,k)$ the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^n$ and by $\mathrm{A}(n,k)$ the affine Grassmannian of k-dimensional affine subspaces of $\mathbb{R}^n$. Moreover, for $E \in \mathrm{G}(n,k)$ and $j \leq k$, we denote by $\mathrm{G}(E,j) \cong \mathrm{G}(k,j)$ the Grassmannian of j-dimensional subspaces of E and for $E \in \mathrm{A}(n,k)$ and $j \leq k$, we denote by $\mathrm{A}(E,j)$ the affine Grassmannian of j-dimensional affine subspaces of E.

The spaces $\mathrm{G}(n,k)$ and $\mathrm{A}(n,k)$ will be endowed with the usual topologies. $\mathrm{G}(n,k)$ will be equipped with a suitably normalized $\mathrm{SO}(n)$ invariant Borel measure and $\mathrm{A}(n,k)$ with a suitably normalize $\mathrm{E}(n)$ invariant Borel measure.

Since we fix the measures on $\mathbb{S}$, $\mathrm{G}(n,k)$, and $\mathrm{A}(n,k)$, and we will mostly use them in the context of integration (and not in the context of expressing the measure of a Borel set), we will simply use the notations

$$\int_{\mathbb{S}} f(u)\, du, \qquad \int_{\mathrm{G}(n,k)} g(E)\, dE, \qquad \text{and} \qquad \int_{\mathrm{A}(n,k)} h(E)\, dE.$$

We denote the k-dimensional Hausdorff measure by $\mathcal{H}^k$.

The word “if” in a definition is to be understood as “if and only if”.

Chapter 1

Analytic Preliminaries

In this chapter, we give an overview of the analytic notions and results from the standard literature that will be employed in this thesis.

1.1 Homogeneous spaces

As a reference for this section, we cite [21].

If M is a non-empty set and G is a group, a *(left) group action* of G on M is a function

$$G \times M \to M : (g, p) \mapsto g \cdot p$$

such that

- $e \cdot p = p$ for all $p \in M$ and
- $g \cdot (h \cdot p) = (gh) \cdot p$ for all $g, h \in G$ and $p \in M$.

If an action of a group G on M is fixed, a mapping $f : M \to N$, where N is a non-empty set, is called *G-invariant*, if $f(g \cdot x) = f(x)$ for all $x \in M$ and $g \in G$. If actions of a group G are fixed on M and N, a mapping $f : M \to N$ is called *G-equivariant* or *G-intertwining*, if $f(g \cdot x) = g \cdot f(x)$ for all $x \in M$ and $g \in G$.

We say that G acts *transitively* on M if for every $p, q \in M$ there exists some $g \in G$ such that $g \cdot p = q$.

A non-empty set M equipped with a group action of G is called a *homogeneous space for G*, if G acts transitively on M. A topological space M equipped with a group action of G is called a *homogeneous space for G*, if G acts transitively on M and the maps $M \to M : p \mapsto g \cdot p$ are continuous for all $g \in G$.

The Unit Sphere The unit sphere $\mathbb{S}$, endowed with the usual group action of $\mathrm{SO}(n)$, is a homogeneous space for $\mathrm{SO}(n)$. Note that the spherical Lebesgue measure $\lambda_{\mathbb{S}}$ is invariant under the group action of $\mathrm{SO}(n)$, that is $\lambda_{\mathbb{S}}(\vartheta A) = \lambda_{\mathbb{S}}(A)$ for every $\vartheta \in \mathrm{SO}(n)$ and for every Borel set $A \subseteq \mathbb{S}$. Moreover, the measure $\lambda_{\mathbb{S}}$ is uniquely determined by its $\mathrm{SO}(n)$-invariance and normalization.

The Grassmannian The usual group action of $\mathrm{SO}(n)$ on $\mathrm{G}(n,k)$ acts transitively on $\mathrm{G}(n,k)$. Now fix some $\overline{E} \in \mathrm{G}(n,k)$ and consider the final topology on $\mathrm{G}(n,k)$ induced by the map

$$f_{n,k} : \mathrm{SO}(n) \to \mathrm{G}(n,k) : \vartheta \mapsto \vartheta \overline{E}.$$

This topology is independent of the choice of $\overline{E}$ and makes $\mathrm{G}(n,k)$ a second countable compact Hausdorff topological space. Equipped with this topology, $\mathrm{G}(n,k)$ becomes a homogeneous space for $\mathrm{SO}(n)$.

Denote by $\lambda_{\mathrm{SO}(n)}$ the Haar measure on $\mathrm{SO}(n)$ with total mass 1. Then the image measure $\mu_k^n := f_{n,k}(\lambda_{\mathrm{SO}(n)})$ is a Borel measure on $\mathrm{G}(n,k)$ with total mass 1 which is invariant under the group action of $\mathrm{SO}(n)$. Moreover, the measure μ_k^n is uniquely determined by its $\mathrm{SO}(n)$-invariance and normalization.

The Affine Grassmannian We proceed similarly with the affine Grassmannian $\mathrm{A}(n,k)$. The usual group action of $\mathrm{E}(n)$ on $\mathrm{A}(n,k)$ acts transitively on $\mathrm{A}(n,k)$. Obviously, the group $\mathrm{E}(n)$ of rigid motions is isomorphic to $\mathrm{SO}(n) \times \mathbb{R}^n$ (endowed with suitable operations) and the left group action of $\mathrm{E}(n)$ on $\mathrm{A}(n,k)$ can be described as

$$(\mathrm{SO}(n) \times \mathbb{R}^n) \times \mathrm{A}(n,k) \to \mathrm{A}(n,k) : ((\vartheta, x), E) \mapsto \vartheta E + x.$$

Now fix some $\overline{E} \in \mathrm{A}(n,k)$ and consider the final topology on $\mathrm{A}(n,k)$ induced by the map

$$g_{n,k} : \mathrm{SO}(n) \times \overline{E}^{\perp} \to \mathrm{A}(n,k) : (\vartheta, x) \mapsto \vartheta(\overline{E} + x).$$

This topology is independent of the choice of $\overline{E}$ and makes $\mathrm{A}(n,k)$ a second countable locally compact Hausdorff topological space. Equipped with this topology, $\mathrm{A}(n,k)$ becomes a homogeneous space for $\mathrm{E}(n)$.

The image measure $\nu_k^n := g_{n,k}(\lambda_{\mathrm{SO}(n)} \otimes \lambda_n)$ of the product measure is a Borel measure on $\mathrm{A}(n,k)$ that is invariant under the group action of $\mathrm{E}(n)$. Moreover, the measure ν_k^n is uniquely determined by its $\mathrm{E}(n)$-invariance and normalization.

1.2 Integral Formulas for the Unit Sphere

For $k \in \{1, \ldots, n\}$, $E \in \mathrm{G}(n,k)$, and an integrable function $f : \mathbb{S} \to \mathbb{R}$, we have

$$\int_{\mathbb{S}} f(u)\, du = \int_{\mathbb{S}(E)} \int_{\mathbb{H}(E,u)} f(v)(u \cdot v)^{k-1}\, dv\, du,$$

where

$$\mathbb{H}(E,u) := \{v \in \mathbb{S}\backslash E^{\perp} \mid \mathrm{pr}_E v = u\}.$$

This integral formula is proven in [27, Lemma 2.1] using spherical coordinates. In the special case where $u_0 \in \mathbb{S}$ and $E = u_0^{\perp} \in \mathrm{G}(n, n-1)$, we obtain a formula for spherical cylinder coordinates, namely

$$\int_{\mathbb{S}} f(u)\ du = \int_{-1}^{1} \int_{\mathbb{S}(u_0^{\perp})} f(tu_0 + \sqrt{1-t^2}v)\ dv\ (1-t^2)^{(n-3)/2}\ dt$$

for every integrable function $f : \mathbb{S} \to \mathbb{R}$. Putting $f \equiv 1$ yields

$$\int_{-1}^{1} (1-t^2)^{(n-3)/2} = \frac{\omega_n}{\omega_{n-1}}. \tag{1.1}$$

1.3 Integral Formulas for the Affine Grassmannian

As references for this section, we cite [12, 18, 22].

The measure ν_k^n is normalized so that

$$\nu_k^n(\{E \in \mathrm{A}(n,k) \mid E \cap B \neq \emptyset\}) = \kappa_{n-k}.$$

Moreover it is homogeneous of degree $n-k$ in the sense that for a Borel subset $A \subseteq \mathrm{A}(n,k)$ and $\lambda > 0$, we have

$$\nu_k^n(\lambda A) = \lambda^{n-k} \nu_k^n(A).$$

Hence, integration by substitution yields

$$\int_{\mathrm{A}(n,k)} f(\lambda^{-1}E)\ dE = \lambda^{n-k} \int_{\mathrm{A}(n,k)} f(E)\ dE$$

for all $f \in \mathbf{C}(\mathrm{A}(n,k))$ and $\lambda > 0$.

If $j \leq k$, then

$$\int_{\mathrm{A}(n,k)} \int_{A(F,j)} f(E)\ dE\ dF = \int_{\mathrm{A}(n,j)} f(E)\ dE \tag{1.2}$$

for all $f \in \mathbf{C}(\mathrm{A}(n,j))$. Due to the uniqueness of the measure ν_k^n, both sides of the equation must be equal up to some multiplicative constant. The constant can be computed from Crofton's intersection formula (2.10) by putting $f(E) := \mathsf{V}_j(B \cap E)$.

If $F \in \mathrm{A}(n,j)$, then for almost all $E \in \mathrm{A}(n,k)$, we have $E \cap F \in \mathrm{A}(n, k+j-n)$. It holds that

$$\int_{\mathrm{A}(n,k)} f(E \cap F)\ dE = \frac{\left[{k \atop k+j-n}\right]}{\left[{n \atop j}\right]} \int_{\mathrm{A}(F,k+j-n)} f(E)\ dE \tag{1.3}$$

for all $f \in \mathbf{C}(\mathrm{A}(n, k+j-n))$. Again, we immediately obtain equality up to some multiplicative constant which can be computed from Crofton's intersection formula (2.9) by putting $f(E) := \mathsf{V}_{k+j-n}(B \cap E)$.

If $k \geq 1$, then for almost all $(E_1, \ldots, E_k) \in \mathrm{A}(n, n-1)^k$, we have $E_1 \cap \cdots \cap E_k \in \mathrm{A}(n, n-k)$. It holds that

$$\int_{\mathrm{A}(n,n-1)} \cdots \int_{\mathrm{A}(n,n-1)} f(E_1 \cap \cdots \cap E_k)\, dE_1 \cdots dE_k = \frac{k!\kappa_k \begin{bmatrix} n \\ k \end{bmatrix}}{2^k \begin{bmatrix} n \\ 1 \end{bmatrix}^k} \int_{\mathrm{A}(n,n-k)} f(E)\, dE \quad (1.4)$$

for all $f \in \mathbf{C}(\mathrm{A}(n, n-k))$. This formula follows from (1.3) by induction on k, where (1.2) is needed in the induction step.

1.4 Group Representations

A *representation* of a group G on a topological vector space $\mathbf{X}$ is a group homomorphism

$$\varrho : G \to \mathrm{Aut}(\mathbf{X}),$$

where $\mathrm{Aut}(\mathbf{X})$ denotes the group of all automorphisms of $\mathbf{X}$.

A representation ϱ of a group G on a topological vector space $\mathbf{X}$ induces a group action

$$G \times \mathbf{X} \to \mathbf{X} : (g, x) \mapsto \varrho(g)x$$

of G on $\mathbf{X}$.

A *subrepresentation* of a representation ϱ of a group G on a topological vector space $\mathbf{X}$ is a representation $\varrho|_{\mathbf{Y}}$ of G on a closed subspace $\mathbf{Y}$ of $\mathbf{X}$ such that $(\varrho|_{\mathbf{Y}})(g) = \varrho(g)|_{\mathbf{Y}}$ for every $g \in G$.

A representation $\varrho : G \to \mathrm{Aut}(\mathbf{X})$ is called *irreducible*, if it only has the trivial subrepresentations 0 and ϱ. Otherwise, it is called *reducible*.

A *homomorphism* between two representations $\varrho : G \to \mathrm{Aut}(\mathbf{X})$ and $\pi : G \to \mathrm{Aut}(\mathbf{Y})$ is a linear map $T : \mathbf{X} \to \mathbf{Y}$ such that $T \circ \varrho(g) = \pi(g) \circ T$ for all $g \in G$. If two representations ϱ and π are fixed, a homomorphism is a G-equivariant continuous linear map $T : \mathbf{X} \to \mathbf{Y}$.

Example 1.1. Suppose that a compact Hausdorff space M is a homogeneous space for a group G, and denote by $\mathbf{C}(M)$ the space of all continuous functions on M, endowed with the maximum norm. Then

$$\varrho : G \to \mathrm{Aut}(\mathbf{C}(M)) : (\varrho(g)\phi)(p) := \phi(g^{-1}p)$$

is a representation of G on $\mathbf{C}(M)$.

For instance, the usual group action of $\mathrm{SO}(n)$ on the unit sphere $\mathbb{S}$ induces a representation of $\mathrm{SO}(n)$ on $\mathbf{C}(\mathbb{S})$ and $\mathbf{L}^2(\mathbb{S})$. Furthermore, the maps $f \mapsto \vartheta f$ are isometric automorphisms on $\mathbf{C}(\mathbb{S})$ and $\mathbf{L}^2(\mathbb{S})$, respectively.

Zonal Vectors We fix a point $\bar{e} \in \mathbb{S}$, and denote by $\mathrm{SO}(n-1, \bar{e})$ the subgroup of $\mathrm{SO}(n)$ fixing $\bar{e}$, that is

$$\mathrm{SO}(n-1, \bar{e}) := \{\vartheta \in \mathrm{SO}(n) \mid \vartheta \bar{e} = \bar{e}\}.$$

Clearly, $\mathrm{SO}(n-1, \bar{e}) \cong \mathrm{SO}(n-1)$. Given a representation ϱ of $\mathrm{SO}(n)$ on a topological vector space $\mathbf{X}$, we call a vector $x \in \mathbf{X}$ *zonal* if it is invariant under the action of $\mathrm{SO}(n-1, \bar{e})$. The zonal vectors form a closed subspace of $\mathbf{X}$.

1.5 Spherical Harmonics

As references for this section, we cite [15, 21].

For a function $f : \mathbb{S} \to \mathbb{R}$, we define its *radial extension* $\overline{f} : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ by

$$\overline{f}(x) := f\left(\frac{1}{\|x\|} x\right), \qquad x \in \mathbb{R}^n \backslash \{o\}.$$

The unit sphere $\mathbb{S} \subseteq \mathbb{R}^n$ is an $(n-1)$-dimensional smooth manifold. For $k \in \mathbb{N}_0 \cup \{\infty\}$ and a smooth atlas $\{(U_i, \varphi_i) : i \in I\}$ on $\mathbb{S}$, we denote by $\mathbf{C}^k(\mathbb{S})$ the space of all functions $f : \mathbb{S} \to \mathbb{R}$, for which $f \circ \varphi_i \in \mathbf{C}^k(U_i)$ for all $i \in I$. It is easy to see that this definition is independent of the choice of the smooth atlas.

Since the radial projection $\mathbb{R}^n \backslash \{o\} \to \mathbb{S} : x \mapsto \frac{1}{\|x\|} x$ is a smooth map, it follows that $f \in \mathbf{C}^k(\mathbb{S})$ implies $\overline{f} \in \mathbf{C}^k(\mathbb{R}^{n-1})$. This allows us to make the following definition:

The *Beltrami operator* or *Laplace-Beltrami operator* is defined by

$$\Delta_\circ f := (\Delta \overline{f})|_{\mathbb{S}}, \qquad f \in \mathbf{C}^2(\mathbb{S}),$$

where Δ denotes the usual Laplace operator on $\mathbb{R}^n$.

The Beltrami operator is self-adjoint in the sense that

$$\langle \Delta_\circ f, g \rangle = \langle f, \Delta_\circ g \rangle$$

for all $f, g \in \mathbf{C}^2(\mathbb{S})$.

A *spherical harmonic (of degree k)* is the restriction of a harmonic polynomial in n variables (of degree k) to $\mathbb{S}$. We denote by $\mathcal{H}^n$ the space of all spherical harmonics and by $\mathcal{H}^n_k$ the space of all spherical harmonic of degree k.

Due to the maximum principle, the map $f \mapsto f|_{\mathbb{S}}$ is injective on harmonic functions. Hence the *degree* or *order* of a spherical harmonic is well-defined and $\mathcal{H}^n_k \cap \mathcal{H}^n_\ell = \{o\}$ whenever $k \neq \ell$. In particular,

$$\mathcal{H}^n = \bigoplus_{k \in \mathbb{N}_0} \mathcal{H}^n_k$$

in the sense of a direct sum of vector spaces.

Moreover, due to the $\mathrm{SO}(n)$-equivariance of the Laplace operator, each space $\mathcal{H}^n_k$ is $\mathrm{SO}(n)$ invariant. In the language of group representations, this means that each space $\mathcal{H}^n_k$ admits a subrepresentation of $\mathrm{SO}(n)$.

Theorem 1.2. *The dimension of the spaces* $\mathcal{H}_k^n$ *is given by*

$$N(n,k) := \dim \mathcal{H}_k^n = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}.$$

We denote by $\mathbf{L}^2(\mathbb{S})$ the Hilbert space of all square integrable Lebesgue functions on the measure space $\mathbb{S}$, endowed with the usual inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|_2$.

Theorem 1.3. *The spaces* $\mathcal{H}_k^n$ *are pairwise orthogonal and* $\mathcal{H}^n$ *is dense in* $\mathbf{L}^2(\mathbb{S})$. *In particular,*

$$\mathbf{L}^2(\mathbb{S}) = \bigoplus_{k \in \mathbb{N}_0} \mathcal{H}_k^n$$

in the sense of a direct sum of Hilbert spaces.

Theorem 1.4. *For* $k \in \mathbb{N}_0$, *the only* $\mathrm{SO}(n)$ *invariant subspaces of* $\mathcal{H}_k^n$ *are* $\{0\}$ *and* $\mathcal{H}_k^n$.

In the language of group representations, that means that the representation of $\mathrm{SO}(n)$ on $\mathcal{H}_k^n$ is irreducible.

A *standard sequence of (n-dimensional) spherical harmonics* is a union of orthogonal bases of the spaces $\mathcal{H}_k^n$, $k \in \mathbb{N}_0$. Due to the previous theorem, every standard sequence of spherical harmonics is an orthogonal basis of $\mathbf{L}^2(\mathbb{S})$.

If $h_0, h_1, \ldots$ is a standard sequence of n-dimensional spherical harmonics and $f \in \mathbf{L}^2(\mathbb{S})$, then the Fourier series of f is given by $\sum_{j=0}^{\infty} c_j h_j$, where $c_j = \langle f, h_j \rangle / \|h_j\|_2$. To indicate that this is the Fourier series of f, we write

$$f \sim \sum_{j=0}^{\infty} c_j h_j,$$

and we call $\sum_{j=0}^{\infty} c_j h_j$ a *harmonic expansion* of f.

From now on, we denote by

$$\pi_k : \mathbf{L}^2(\mathbb{S}) \to \mathcal{H}_k^n$$

the orthogonal projection onto $\mathcal{H}_k^n$.

If $f \in \mathbf{L}^2(\mathbb{S})$ and $f_k = \pi_k f$, we call $\sum_{k=0}^{\infty} f_k$ the *condensed harmonic expansion* of f and we write again

$$f \sim \sum_{k=0}^{\infty} f_k.$$

Obviously, the condensed harmonic expansion of f can be constructed from any harmonic expansion of f by grouping terms of the same order.

Theorem 1.5. *If* $h_k \in \mathcal{H}^n_k$*, then*

$$\Delta_\circ h_k = -k(n+k-2)h_k.$$

Furthermore, if $f \in \mathbf{C}^2(\mathbb{S})$ *has the condensed harmonic expansion*

$$f \sim \sum_{k=0}^{\infty} f_k,$$

then $\Delta_\circ f \in \mathbf{C}(\mathbb{S})$ *has the harmonic expansion*

$$f \sim -\sum_{k=0}^{\infty} k(n+k-2)f_k.$$

Theorem 1.6. *For every* $n \geq 2$ *and* $k \geq 0$*, there exists exactly one polynomial* P^n_k *(in one variable) such that for every orthogonal basis* $(h_1, \dots, h_{N(n,k)})$ *of* $\mathcal{H}^n_k$*,*

$$\sum_{i=1}^{N(n,k)} h_i(u)h_i(v) = \frac{N(n,k)}{\omega_n} P^n_k(u \cdot v), \qquad u, v \in \mathbb{S}.$$

Furthermore, $\deg P^n_k = k$ *and* $P^n_k(\bar{e}\cdot .)$ *is, up to a multiplicative constant, the unique zonal harmonic in* $\mathcal{H}^n_k$*.*

The polyomial P^n_k is called the *Legendre polynomial of dimension* n *and degree* k.

Due to Theorem 1.6, for every $f \in \mathbf{C}(\mathbb{S})$,

$$(\pi_k f)(u) = \sum_{i=1}^{N(n,k)} \langle f, h_i \rangle h_i(u) = \frac{N(n,k)}{\omega_n} \int_{\mathbb{S}} P^n_k(u \cdot v) f(v)\, dv, \qquad u \in \mathbb{S}. \tag{1.5}$$

A linear map $T : \mathcal{H}^n \to \mathcal{H}^n$ is called a *multiplier transform* if it acts as a multiple of the identity on each space $\mathcal{H}^n_k$. Its eigenvalues on $\mathcal{H}^n_k$ are called *multipliers* and denoted by $\lambda^n_k[T]$. Later on, we will extend this notion to continuous linear operators on certain topological vector spaces.

Multiplier transforms often arise naturally in convex geometry, and they are very pleasant to work with. For instance, in order to show that a multiplier transform $T : \mathcal{H}^n \to \mathcal{H}^n$ is injective, it suffices to show that of all of its multipliers are non-zero. A central result on the existence of multiplier transforms is the Funk-Hecke theorem. It states that for every $F \in \mathbf{C}[-1,1]$, the integral transform

$$T : \mathcal{H}^n \to \mathcal{H}^n : (Th)(u) := \int_{\mathbb{S}} F(u \cdot v) h(v)\, dv$$

is a multiplier transform and it gives an explicit formula for the multipliers.

Theorem 1.7 (Funk-Hecke)**.** *If* $F \in \mathbf{C}[-1,1]$ *and* $h_k \in \mathcal{H}_k^n$*, then*

$$\int_{\mathbb{S}} F(u \cdot v) h_k(v)\ dv = a_k^n[F] h_k(u),$$

where

$$a_k^n[F] := \omega_{n-1} \int_{-1}^{1} F(t) P_k^n(t)(1-t^2)^{(n-3)/2}\ dt.$$

1.6 Function Spaces on the Unit Sphere

For the following two sections, we give [15, 19, 21] as general references.

Duality If $\mathbf{X}$ is a topological vector space, the *weak* topology* on $\mathbf{X}$ is defined as the initial topology on $\mathbf{X}^*$ induced by the maps $\langle x, \cdot \rangle : \mathbf{X}^* \to \mathbb{R}$, $x \in X$. Endowed with the weak* topology, $\mathbf{X}^*$ becomes a locally convex space and its topological dual space $(\mathbf{X}^*)^*$ can be canonically identified with $\mathbf{X}$.

Whenever $T : \mathbf{X} \to \mathbf{Y}$ is a continuous linear operator between two topological vector spaces and we endow their topological dual spaces $\mathbf{X}^*$ and $\mathbf{Y}^*$ each with the respective weak* topology, then the adjoint operator $T^* : \mathbf{Y}^* \to \mathbf{X}^*$ is again continuous.

The space $\mathbf{L}^2(\mathbb{S})$ We have already introduced the Hilbert space $\mathbf{L}^2(\mathbb{S})$ with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|_2$.

Theorem 1.8 (Riesz)**.** *The map*

$$\mathbf{L}^2(\mathbb{S}) \to \mathbf{L}^2(\mathbb{S})^* : g \mapsto \langle \cdot, g \rangle$$

is an isometric isomorphism between Banach spaces.

The Riesz representation theorem yields a natural identification of $\mathbf{L}^2(\mathbb{S})$ with its topological dual space that makes our double-use of the notation $\langle \cdot, \cdot \rangle$ both for the inner product and for the canonical evaluation map consistent.

The spaces $\mathbf{C}(\mathbb{S})$ **and** $\mathbf{M}(\mathbb{S})$ We denote by $\mathbf{C}(\mathbb{S})$ the space of all continuous real valued functions on $\mathbb{S}$. Endowed with the maximum norm $\|\cdot\|_\infty$, the space $\mathbf{C}(\mathbb{S})$ becomes a Banach space. Furthermore, we denote by $\mathbf{M}(\mathbb{S})$ the space of all finite signed Borel measures on $\mathbb{S}$. Endowed with the total variation norm, $\mathbf{M}(\mathbb{S})$ also becomes a Banach space.

Theorem 1.9 (Riesz-Markov-Kakutani)**.** *The map*

$$\mathbf{M}(\mathbb{S}) \to \mathbf{C}(\mathbb{S})^* : \mu \mapsto \int_{\mathbb{S}} \cdot\ d\mu$$

is an isometric isomorphism of Banach spaces.

The Riesz-Markov-Kakutani representation theorem yields a natural identification of $\mathbf{C}(\mathbb{S})^*$ with $\mathbf{M}(\mathbb{S})$ that motivates the notation

$$\langle \cdot, \cdot \rangle : \mathbf{C}(\mathbb{S}) \times \mathbf{M}(\mathbb{S}) \to \mathbb{R} : \langle f, \mu \rangle := \int_{\mathbb{S}} f \, d\mu.$$

From now on, we endow $\mathbf{C}(\mathbb{S})$ with the topology induced by the maximum norm and we endow $\mathbf{M}(\mathbb{S})$ with the weak* topology.

It follows from (1.5) that the projection $\pi_k : \mathbf{C}(\mathbb{S}) \to \mathbf{C}(\mathbb{S})$ is a continuous linear operator. Since π_k is self-adjoint in the sense that $\langle \pi_k f, g \rangle = \langle f, \pi_k g \rangle$ for all $f, g \in \mathbf{C}(\mathbb{S})$, projection operator π_k extends to a continuous linear operator $\pi_k : \mathbf{M}(\mathbb{S}) \to \mathbf{M}(\mathbb{S}) : \langle f, \pi_k \mu \rangle := \langle \pi_k f, \mu \rangle$.

One can easily see that for all $\mu \in \mathbf{M}(\mathbb{S})$,

$$(\pi_k \mu)(u) = \frac{N(n,k)}{\omega_n} \int_{\mathbb{S}} P_k^n(u \cdot v) \, d\mu(v), \qquad u \in \mathbb{S}.$$

A representation of $\mathrm{SO}(n)$ on $\mathbf{C}(\mathbb{S})$ is given by means of $(\vartheta f)(u) := f(\vartheta^{-1} u)$ for $f \in \mathbf{C}(\mathbb{S})$. We obtain a representation of $\mathrm{SO}(n)$ on $\mathbf{M}(\mathbb{S})$ by defining $\langle f, \vartheta \mu \rangle := \langle \vartheta^{-1} f, \mu \rangle$ for $f \in \mathbf{C}(\mathbb{S})$, $\mu \in \mathbf{M}(\mathbb{S})$, and $\vartheta \in \mathrm{SO}(n)$. Clearly, $\vartheta \mu$ is the image measure $\vartheta(\mu)$.

The spaces $\mathbf{C}^\infty(\mathbb{S})$ and $\mathbf{C}^{-\infty}(\mathbb{S})$ We denote by $\mathbf{C}^\infty(\mathbb{S})$ the space of all smooth functions on $\mathbb{S}$.

A sequence $(a_k)_{k \in \mathbb{N}_0}$ of real numbers is called *rapidly decreasing* if $\sup\{k^m |a_k| : k \in \mathbb{N}_0\} < \infty$ for every $m \in \mathbb{N}_0$.

Lemma 1.10 ([19, Theorem 2.45]). *If $f \in \mathbf{C}^\infty(\mathbb{S})$, then the sequence $(\|\pi_k f\|_\infty)_{k \in \mathbb{N}_0}$ is rapidly decreasing.*

Conversely, if $f_k \in \mathcal{H}_k^n$, $k \in \mathbb{N}_0$, is a sequence of spherical harmonics such that $(\|f_k\|_\infty)_{k \in \mathbb{N}_0}$ is rapidly decreasing, then there is a unique function $f \in \mathbf{C}^\infty(\mathbb{S})$ such that $\pi_k f = f_k$ for $k \in \mathbb{N}_0$.

If $f \in \mathbf{C}^\infty(\mathbb{S})$ has the condensed harmonic expansion

$$f \sim \sum_{k=0}^{\infty} f_k,$$

we define

$$(-\Delta_\circ)^{\frac{m}{2}} f := \sum_{k=0}^{\infty} (k(n+k-2))^{\frac{m}{2}} f_k$$

for $m \in \mathbb{N}_0$. Due to the previous lemma, $(-\Delta_\circ)^{\frac{m}{2}}$ maps $\mathbf{C}^\infty(\mathbb{S})$ functions to $\mathbf{C}^\infty(\mathbb{S})$ functions.

For every $m \in \mathbb{N}_0$, define

$$p_m : \mathbf{C}^\infty(\mathbb{S}) \to \mathbb{R} : p_m(f) := \|(-\Delta_\circ)^{\frac{m}{2}} f\|_\infty.$$

Then $\mathfrak{p} := (p_m)_{m\in\mathbb{N}_0}$ is a separating family of seminorms on $\mathbf{C}^\infty(\mathbb{S})$. We endow $\mathbf{C}^\infty(\mathbb{S})$ with the locally convex topology induced by this family of seminorms. This topology is also induced by the metric $d_\mathfrak{p}$ defined by

$$d_\mathfrak{p}(f,g) := \sum_{m=0}^{\infty} 2^{-m} \frac{p_m(f-g)}{1+p_m(f-g)}, \qquad f,g \in \mathbf{C}^\infty(\mathbb{S}).$$

In particular, $\mathbf{C}^\infty(\mathbb{S})$ is a metrizable topological space and a sequence $(f_i)_{i\in\mathbb{N}}$ converges to f in $\mathbf{C}^\infty(\mathbb{S})$ if and only if $\lim_{i\to\infty} p_m(f_i - f) = 0$ for all $m \in \mathbb{N}_0$.

A topological vector space is a *Fréchet space* if it is complete and if its topology may be induced by countable separating family of seminorms. The above characterization of convergence in $\mathbf{C}^\infty(\mathbb{S})$ shows that $\mathbf{C}^\infty(\mathbb{S})$ is a complete topological space. Hence $\mathbf{C}^\infty(\mathbb{S})$ is a Fréchet space.

Furthermore, if $f \in \mathbf{C}^\infty(\mathbb{S})$, then any harmonic expansion of f as well as the condensed harmonic expansion of f converge to f in $\mathbf{C}^\infty(\mathbb{S})$. Consequently, $\mathcal{H}^n$ is dense in $\mathbf{C}^\infty(\mathbb{S})$; in particular, the space $\mathbf{C}^\infty(\mathbb{S})$ is separable.

Note also that $\Delta_\circ : \mathbf{C}^\infty(\mathbb{S}) \to \mathbf{C}^\infty(\mathbb{S})$ is a continuous linear map with respect to the Fréchet topology.

A *distribution* or *generalized function* on $\mathbb{S}$ is a continuous linear functional on $\mathbf{C}^\infty(\mathbb{S})$. We denote the space of distributions on $\mathbb{S}$ by $\mathbf{C}^{-\infty}(\mathbb{S}) := (\mathbf{C}^\infty(\mathbb{S}))^*$ and endow it with the weak* topology.

Since $\mathbf{C}^\infty(\mathbb{S})$ is separable, the space $\mathbf{C}^{-\infty}(\mathbb{S})$ is metrizable.

From the fact that $\pi_k : \mathbf{C}(\mathbb{S}) \to \mathbf{C}(\mathbb{S})$ is a continuous linear operator one can easily deduce that $\pi_k : \mathbf{C}^\infty(\mathbb{S}) \to \mathbf{C}^\infty(\mathbb{S})$ is also a continuous linear operator. Like we did before, π_k extends to a continuous linear operator $\pi_k : \mathbf{C}^{-\infty}(\mathbb{S}) \to \mathbf{C}^{-\infty}(\mathbb{S}) : \langle f, \pi_k\eta\rangle := \langle \pi_k f, \eta\rangle$.

A sequence $(a_k)_{k\in\mathbb{N}_0}$ of real numbers is *slowly increasing* if $\sup\{k^{-m}\,|a_k| : k \in \mathbb{N}_+\} < \infty$ for some $m \in \mathbb{N}_0$.

Lemma 1.11 ([19, Theorem 2.50]). *If $\eta \in \mathbf{C}^{-\infty}(\mathbb{S})$, then $\pi_k\eta \in \mathcal{H}^n_k$ for $k \in \mathbb{N}_0$ and the sequence $(\|\pi_k\eta\|_\infty)_{k\in\mathbb{N}_0}$ is slowly increasing.*

Conversely, if $f_k \in \mathcal{H}^n_k$, $k \in \mathbb{N}_0$, is a sequence of spherical harmonics such that $(\|f_k\|_\infty)_{k\in\mathbb{N}_0}$ is slowly increasing, then there is a unique distribution $\eta \in \mathbf{C}^{-\infty}(\mathbb{S})$ such that $\pi_k\eta = h_k$ for $k \in \mathbb{N}_0$. Moreover, the series $\sum_{k=0}^\infty f_k$ converges to η in $\mathbf{C}^{-\infty}(\mathbb{S})$.

As a consequence of this lemma, $\mathcal{H}^n$ is dense in $\mathbf{C}^{-\infty}(\mathbb{S})$, so $\mathbf{C}^{-\infty}(\mathbb{S})$ is separable.

Like before, a representation of $\mathrm{SO}(n)$ on $\mathbf{C}^\infty(\mathbb{S})$ is given by means of $(\vartheta f)(u) := f(\vartheta^{-1}u)$. We obtain a representation of $\mathrm{SO}(n)$ on $\mathbf{C}^{-\infty}(\mathbb{S})$ by defining $\langle f, \vartheta\eta\rangle := \langle \vartheta^{-1} f, \eta\rangle$. for $f \in \mathbf{C}^\infty(\mathbb{S})$, $\eta \in \mathbf{C}^{-\infty}(\mathbb{S})$, and $\vartheta \in \mathrm{SO}(n)$.

Natural Inclusions Of all the function spaces on $\mathbb{S}$ introduced so far, some can be identified with subspaces of others in a natural way. Making these natural identifications, one obtains the following chain of inclusions:

$$\mathbf{C}^\infty(\mathbb{S}) \subseteq \mathbf{C}(\mathbb{S}) \subseteq \mathbf{L}^2(\mathbb{S}) \subseteq \mathbf{M}(\mathbb{S}) \subseteq \mathbf{C}^{-\infty}(\mathbb{S}). \tag{1.6}$$

Moreover, all of these inclusions are to be understood as continuous maps (but not embeddings) between topological vector spaces.

It is obvious that the first three inclusions in (1.6) are valid and continuous. The validity and continuity of the fourth inclusion follows from the validity and continuity of the first one by duality.

Spaces of centered and zonal functions We say that a (generalized) function f on $\mathbb{S}$ is *centered* if $\pi_1 f = 0$. If $\mathbf{X}(\mathbb{S})$ is one of the function spaces in (1.6), we define

$$\mathbf{X}_o(\mathbb{S}) := \{f \in \mathbf{X}(\mathbb{S}) \mid \pi_1 f = 0\}$$

as the space of centered $\mathbf{X}(\mathbb{S})$ functions. Since $\pi_1 : \mathbf{X}(\mathbb{S}) \to \mathbf{X}(\mathbb{S})$ is a continuous linear operator, $\mathbf{X}_o(\mathbb{S})$ is a closed subspace of $\mathbf{X}(\mathbb{S})$ (in the respective topology of $\mathbf{X}(\mathbb{S})$).

Similarly, we define

$$\mathbf{X}(\mathbb{S}, \bar{e}) := \{f \in \mathbf{X}(\mathbb{S}) \mid \forall \vartheta \in \mathrm{SO}(n-1, \bar{e}) : \vartheta f = f\}$$

as the space of all zonal $\mathbf{X}(\mathbb{S})$ functions. Since $\mathbf{X}(\mathbb{S}) \to \mathbf{X}(\mathbb{S}) : f \mapsto \vartheta f$ is a continuous linear operator, $\mathbf{X}(\mathbb{S}, \bar{e})$ is a closed subspace of $\mathbf{X}(\mathbb{S})$.

Moreover, we define

$$\mathbf{X}_o(\mathbb{S}, \bar{e}) := \mathbf{X}_o(\mathbb{S}) \cap \mathbf{X}(\mathbb{S}, \bar{e})$$

as the space of all centered zonal $\mathbf{X}(\mathbb{S})$ functions. As the intersection of two closed subspaces of $\mathbf{X}(\mathbb{S})$, this is again a closed subspace of $\mathbf{X}(\mathbb{S})$.

For instance, $\mathbf{L}^2_o(\mathbb{S}) = (\mathcal{H}^n_1)^\perp$ is the Hilbert space of all centered $\mathbf{L}^2(\mathbb{S})$ functions on $\mathbb{S}$, the space $\mathbf{M}(\mathbb{S}, \bar{e})$ is the topological vector space of all zonal signed Borel measures on $\mathbb{S}$, and $\mathbf{C}^\infty_o(\mathbb{S}, \bar{e})$ is the Fréchet space of all centered zonal smooth functions on $\mathbb{S}$.

1.7 Zonal Functions and Functions on $[-1, 1]$

Obviously, zonal functions on $\mathbb{S}$ naturally correspond to functions on the interval $[-1, 1]$.

We define the *zonal lifting* of a function $F : [-1, 1] \to \mathbb{R}$ as the function

$$\breve{F} : \mathbb{S} \to \mathbb{R} : \breve{F}(u) := F(\bar{e} \cdot u).$$

Obviously, the *zonal lifting operator*

$$\breve{\cdot} : \mathbf{C}[-1, 1] \to \mathbf{C}(\mathbb{S}, \bar{e}) : F \mapsto \breve{F}$$

is an isometric isomorphism of Banach spaces that extends to an isomorphism

$$\breve{\cdot} : \mathbf{M}[-1,1] \to \mathbf{M}(\mathbb{S}, \bar{e}) : \mu \mapsto \breve{\mu}.$$

For $F \in \mathbf{C}[-1,1]$ and $\mu \in \mathbf{M}[-1,1]$, we have

$$\langle \breve{F}, \breve{G} \rangle = \omega_{n-1} \int_{-1}^{1} F(t)(1-t^2)^{(n-3)/2} \, d\mu(t).$$

We define $\mathbf{L}^2[-1,1]_n$ as the Hilbert space of all Lebesgue functions on $[-1,1]$ that are square-integrable with respect to the measure $(1-t^2)^{(n-3)/2}dt$ and we denote its inner product by $[\cdot,\cdot]_n$, that is

$$[F,G]_n = \int_{-1}^{1} F(t)G(t)(1-t^2)^{(n-3)/2} \, dt.$$

The zonal lifting operator extends to an isomorphism

$$\breve{\cdot} : \mathbf{L}^2[-1,1]_n \to \mathbf{L}^2(\mathbb{S}, \bar{e}) : F \mapsto \breve{F}$$

of Hilbert spaces, and

$$\langle \breve{F}, \breve{G} \rangle = \omega_{n-1}[F,G]_n$$

for $F, G \in \mathbf{L}^2[-1,1]_n$.

The Legendre polynomials P_k^n, $k \in \mathbb{N}_0$, form an orthogonal basis of $\mathbf{L}^2[-1,1]_n$ and

$$[P_k^n, P_k^n]_n = \frac{\omega_n}{\omega_{n-1} N(n,k)}.$$

For $\mu \in \mathbf{M}[-1,1]$, we fix the notation

$$a_k^n[\mu] := \omega_{n-1} \int_{-1}^{1} P_k^n(t)(1-t^2)^{(n-3)/2} \, d\mu(t).$$

Moreover, we use the notation

$$\mu \sim \sum_{k=0}^{\infty} c_k P_k^n$$

to indicate that $c_k = \omega_n^{-1} N(n,k) a_k^n[\mu]$ for all $k \in \mathbb{N}_0$.

1.8 Spherical Convolution

For this section, we give [14, 24] as references.

The *spherical convolution* of a continuous function $f \in \mathbf{C}(\mathbb{S})$ with a zonal signed measure $\mu \in \mathbf{M}(\mathbb{S}, \bar{e})$ is defined as

$$(f * \mu)(\vartheta\bar{e}) := \langle f, \vartheta\mu\rangle = \langle \vartheta^{-1} f, \mu\rangle = \int_{\mathbb{S}} f(\vartheta u)\, d\mu(u), \qquad \vartheta \in \mathrm{SO}(n).$$

The spherical convolution is a continuous function, that is $f * \mu \in \mathbf{C}(\mathbb{S})$. For every $\vartheta \in \mathrm{SO}(n)$, we have

$$(\vartheta f) * \mu = \vartheta(f * \mu).$$

Moreover,

$$\|f * \mu\|_\infty \leq \|f\|_\infty \|\mu\|_{\mathrm{TV}},$$

where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm. This shows in particular that the convolution operator

$$T_\mu : \mathbf{C}(\mathbb{S}) \to \mathbf{C}(\mathbb{S}) : f \mapsto f * \mu$$

is a $\mathrm{SO}(n)$ intertwining continuous linear operator on $\mathbf{C}(\mathbb{S})$. This operator is also self-adjoint in the sense that

$$\langle f * \mu, g\rangle = \langle f, g * \mu\rangle \tag{1.7}$$

for all $f, g \in \mathbf{C}(\mathbb{S})$ and $\mu \in \mathbf{M}(\mathbb{S}, \bar{e})$. Hence, T_μ extends to an $\mathrm{SO}(n)$ intertwining continuous linear operator

$$T_\mu : \mathbf{M}(\mathbb{S}) \to \mathbf{M}(\mathbb{S}) : \nu \mapsto \nu * \mu,$$
$$\langle f, \nu * \mu\rangle := \langle f * \mu, \nu\rangle, \qquad f \in \mathbf{C}(\mathbb{S}).$$

Consequently, the spherical convolution of a signed measure $\mu \in \mathbf{M}(\mathbb{S})$ and a zonal continuous function $f \in \mathbf{C}(\mathbb{S}, \bar{e}) \subseteq \mathbf{M}(\mathbb{S}, \bar{e})$ is given by

$$(\mu * f)(\vartheta\bar{e}) = \langle \vartheta f, \mu\rangle = \langle f, \vartheta^{-1}\mu\rangle = \int_{\mathbb{S}} f(\vartheta^{-1} u)\, d\mu(u).$$

In particular, $\mu * f \in \mathbf{C}(\mathbb{S})$ and $\|\mu * f\|_\infty \leq \|\mu\|_{\mathrm{TV}}\|f\|_\infty$ like before. If we write $f = \breve{F}$, we obtain the representation

$$(\mu * \breve{F})(u) = \int_{\mathbb{S}} F(u \cdot v)\, d\mu(v), \qquad u \in \mathbb{S}. \tag{1.8}$$

With the representation formula (1.8) and an argument of density and continuity, one can show that any two convolution operators T_μ and T_ν commute. More precisely,

$$(\rho * \mu) * \nu = (\rho * \nu) * \mu \tag{1.9}$$

for all $\rho \in \mathbf{M}(\mathbb{S})$ and $\mu, \nu \in \mathbf{M}(\mathbb{S}, \bar{e})$. Note also that the Dirac measure $\delta_{\bar{e}}$ is the right neutral element, that is

$$\mu * \delta_{\bar{e}} = \mu$$

for all $\mu \in \mathbf{M}(\mathbb{S})$.

Moreover, if $\mu, \nu \in \mathbf{M}(\mathbb{S}, \bar{e})$, then

$$\mu * \nu = \nu * \mu \in \mathbf{M}(\mathbb{S}, \bar{e}).$$

Again, this can easily be shown for continuous functions. Consequently, $\delta_{\bar{e}}$ acts as a left neutral element on zonal measures, that is

$$\delta_{\bar{e}} * \mu = \mu$$

for all $\mu \in \mathbf{M}(\mathbb{S}, \bar{e})$.

The representation formula (1.8) lets us express the orthogonal projection $\pi_k : \mathbf{M}(\mathbb{S}) \to \mathcal{H}_k^n$ as a convolution operator:

$$\pi_k \mu = \frac{N(n,k)}{\omega_n} \mu * \breve{P}_k^n.$$

Due to (1.9), it follows that

$$(\pi_k \rho) * \mu = \pi_k(\rho * \mu)$$

for all $k \in \mathbb{N}_0$, $\rho \in \mathbf{M}(\mathbb{S})$, and $\mu \in \mathbf{M}(\mathbb{S}, \bar{e})$.

This observation leads to the following addition to the Funk-Hecke theorem (for which we will give a proof later on):

Theorem 1.12 ([23, Theorem 5.1])**.** *If $T : \mathbf{C}(\mathbb{S}) \to \mathbf{C}(\mathbb{S})$ is an* $\mathrm{SO}(n)$*-equivariant bounded linear operator, then $T = . * \mu$ for some $\mu \in \mathbf{M}(\mathbb{S}, \bar{e})$, and T is a multiplier transform.*

Obviously, the concept of spherical convolutions can be extended to smooth functions and distributions. However, the definitions made above will suffice for our purposes because almost all distributions we will be working with that will appear on the right hand side of convolutions can also be looked at as finite signed measures.

Chapter 2

Geometric Preliminaries

For this whole chapter, we give [21] as a general reference.

2.1 Convex Bodies

A *convex body* is a non-empty convex compact subset of $\mathbb{R}^n$. The space of convex bodies will be denoted by $\mathcal{K}^n$. For $j \in \{0, \ldots, n\}$ and a j-dimensional affine subspace $E \cong \mathbb{R}^j$ of $\mathbb{R}^n$, we denote by $\mathcal{K}^j(E) \subseteq \mathcal{K}^n$ the space of convex bodies in E.

The *dimension* of a convex body is defined as the dimension of its affine hull $\operatorname{aff} K$. An n-dimensional convex body $K \in \mathcal{K}^n$ is also called *full dimensional*.

The *relative interior* $\operatorname{int} K \subseteq \mathbb{R}^n$ of a convex body $K \in \mathcal{K}^n$ is defined as the interior of K relative to its affine hull $\operatorname{aff} K$ and is always non-empty. The *relative boundary* $\operatorname{bd} K$ of a convex body $K \in \mathcal{K}^n$ is defined as the boundary of K relative to its affine hull $\operatorname{aff} K$ and is always non-empty, except if $\dim K = 0$.

A full dimensional convex body is *strictly convex*, if $(1-\lambda)x + \lambda y \in \operatorname{int} K$ for every $x, y \in K$ and $\lambda \in (0,1)$.

A *face* of a convex body $K \in \mathcal{K}^n$ is a subset $F \subseteq K$ with the property that $x, y \in K$ and $\frac{1}{2}(x+y) \in F$ implies $x, y \in F$. Every non-empty face of K is again a convex body. A non-empty face of dimension $\dim K - 1$ is referred to as a *facet*. An *i-face* is an i-dimensional non-empty face of K. The set of all non-empty faces of K is denoted by $\mathcal{F}(K)$ and the set of all i-faces of K is denoted by $\mathcal{F}_i(K)$. Clearly, $\emptyset$ and K are faces of K; every other face of K is called a *proper face* of K.

For $K \in \mathcal{K}^n$ and $u \in \mathbb{R}^n \backslash \{o\}$, we define the *support plane* of K with outer normal vector u as

$$H(K,u) := \{y \in \mathbb{R}^n \mid x \cdot u = \max_{z \in K} z \cdot u\}$$

and for $x \in \mathbb{R}^n$, we define the *normal cone* of K at x as

$$N(K,x) := \{u \in \mathbb{R}^n \backslash \{o\} \mid x \in H(K,u)\} \cup \{o\}.$$

For a face F of K, we define $N(K,F) := N(K,x)$, where $x \in \operatorname{int} F$.

The space $\mathcal{K}^n$ The space of convex bodies $\mathcal{K}^n$ and can be equipped with some interesting structures.

For $K, L \in \mathcal{K}^n$ and $\lambda \geq 0$, define

$$K + L := \{x + y \mid x \in K, y \in L\} \quad \text{and} \quad \lambda K := \{\lambda x \mid x \in K\}.$$

This defines operations on the space $\mathcal{K}^n$, called the *Minkowski operations* with the former being called *Minkowski addition*. Endowed with Minkowski addition, the space $(\mathcal{K}^n, +)$ becomes a commutative semigroup with neutral element $\{o\}$ and cancellation law. Moreover, the Minkowski operations are compatible in the sense that $\lambda(\mu K) = (\lambda\mu)K$, $\lambda(K + L) = \lambda K + \lambda L$ and $(\lambda + \mu)K = \lambda K + \mu K$ for $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$.

We equip $\mathcal{K}^n$ with the *Hausdorff metric*, which is defined as

$$d(K,L) := \max\left\{\min_{x\in K}\max_{y\in L}\|x-y\|, \min_{x\in L}\max_{y\in K}\|x-y\|\right\}, \qquad K, L \in \mathcal{K}^n,$$

and can also be expressed as

$$d(K,L) = \min\{\lambda \geq 0 \mid K \subseteq \lambda L, L \subseteq \lambda K\}, \qquad K, L \in \mathcal{K}^n.$$

Due to Blaschke's selection theorem, $(\mathcal{K}^n, d)$ is a locally compact complete metric space, and the Minkowski operations are continuous on $(\mathcal{K}^n, d)$.

Endowed with set inclusion, $(\mathcal{K}^n, \subseteq)$ is a partially ordered set. Clearly this partial order is closed in $\mathcal{K}^n \times \mathcal{K}^n$ and is respected by the Minkowski operations.

The *support function* $h(K,\cdot) = h_K$ of a convex body $K \in \mathcal{K}^n$ is defined as

$$h(K,u) := \max\{u \cdot x \mid x \in K\}, \qquad u \in \mathbb{S},$$

where we also make the convention $h(\emptyset,\cdot) \equiv 0$. Its homogeneous extension of degree one $\widehat{h}(K,\cdot)$ is a sublinear function on $\mathbb{R}^n$, that is

$$\widehat{h}(K,\lambda x) = \lambda\widehat{h}(K,x) \quad \text{and} \quad \widehat{h}(K,x+y) \leq \widehat{h}(K,x) + \widehat{h}(K,y)$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \geq 0$. Consequently, $\widehat{h}(K,\cdot)$ is a convex function and thus continuous. In particular, the support function $h(K,\cdot)$ is continuous. Conversely, every sublinear function on $\mathbb{R}^n$ is the support function of a unique convex body. [1]

By means of the support function, the space $\mathcal{K}^n$ can be identified with a closed convex cone of $\mathbf{C}(\mathbb{S})$. More precisely, the map

$$\iota : \mathcal{K}^n \to \mathbf{C}(\mathbb{S}) : K \mapsto h(K,\cdot)$$

[1] The support function can naturally be defined as a function on $\mathbb{R}^n$. However, defining the support function as a function on $\mathbb{S}$ is more suitable to our purposes.

is an embedding of metric spaces such that $h(\lambda K + \mu L, \cdot) = \lambda h(K, \cdot) + \mu h(L, \cdot)$ for all $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$. Moreover, $K \subseteq L$ if and only if $h(K, u) \leq h(L, u)$ for all $u \in \mathbb{S}$.

For every $f \in \mathbf{C}^2(\mathbb{S})$, there is a convex body $K \in \mathcal{K}^n$ and some $r > 0$ such that $f = h(K, \cdot) - rh(B, \cdot)$. Hence, $\iota(\mathcal{K}^n) - \iota(\mathcal{K}^n)$ is dense in $\mathbf{C}(\mathbb{S})$.

Moreover, note that a group action of $\mathrm{SO}(n)$ on $\mathcal{K}^n$ is given in a natural way.

Centered Convex Bodies The *Steiner point* of a convex body $K \in \mathcal{K}^n$ is defined as

$$s(K) := \frac{1}{\kappa_n} \int_{\mathbb{S}} h(K, u) u \, du \in \mathbb{R}^n.$$

The map

$$s : \mathcal{K}^n \to \mathbb{R}^n : K \mapsto s(K)$$

is called the *Steiner point map*. Clearly, the Steiner point map is continuous and Minkowski linear, that is $s(\lambda K + \mu L) = \lambda s(K) + \mu s(L)$ for all $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$. Moreover, $s(\{x\}) = \{x\}$ for all $x \in \mathbb{R}^n$.

A convex body $K \in \mathcal{K}^n$ is called *centered* if $s(K) = o$. It is easy to see that a convex body is centered if and only if its support function is centered. Moreover, the centered convex bodies form a closed subspace of $\mathcal{K}^n$ that is also closed under the Minkowski operations.

We also introduce the *centering operator*

$$J : \mathcal{K}^n \to \mathcal{K}^n : JK := K - s(K)$$

and the *centered support function*

$$h_o(K, \cdot) := (I - \pi_1) h(K, \cdot) \in \mathbf{C}_o(\mathbb{S}).$$

With these definitions in place,

$$h_o(K, u) = h(JK, u) = h(K, u) - s(K) \cdot u, \qquad u \in \mathbb{S}.$$

Smooth Convex Bodies A convex body is called smooth if its support function is smooth. The space of smooth convex bodies is dense in $\mathcal{K}^n$ and also closed under the Minkowski operations.

Polytopes The *convex hull* of a subset of $\mathbb{R}^n$ is the intersection of all of its convex supersets. The convex hull of a finite subset of $\mathbb{R}^n$ is called a *polytope*. Equivalently, polytopes are precisely the bounded finite intersections of closed half-spaces.

We denote the space of all non-empty polytopes by $\mathcal{P}^n$. The space $\mathcal{P}^n$ is dense in $\mathcal{K}^n$ and also closed under the Minkowski operations. For $P \in \mathcal{P}^n$, the sets $\mathcal{F}_k(P)$, $k \in \{0, \dots, n\}$, are finite.

For a polytope $P \in \mathcal{P}^n$ and $F \in \mathcal{F}_k(P)$, we define the *external angle* of P at its face F as

$$\gamma(F,P) := \frac{1}{\omega_{n-k}} \mathcal{H}^{n-k-1}(N(P,F) \cap \mathbb{S}).$$

Zonoids A Minkowski sum of finitely many closed line segments is a *zonotope.* A limit of zonotopes in the Hausdorff metric is a *zonoid.*

Theorem 2.1 ([21, Theorems 3.5.3, 3.5.4])**.** *A convex body $K \in \mathcal{K}^n$ is a centered zonoid if and only if there exists an even positive measure $\mu \in \mathbf{M}(\mathbb{S})$ such that*

$$h(K,u) = \int_{\mathbb{S}} |u \cdot v| \; d\mu(v), \qquad u \in \mathbb{S}. \tag{2.1}$$

In this case, the even positive measure μ is uniquely determined by K.

If the support function of a zonoid K can be represented in the form (2.1) with an even positive measure μ, then μ is called the *generating measure* of K.

2.2 Mixed Volumes

The volume of convex bodies behaves in some sense like a polynomial. More precisely, there is unique a symmetric function $\mathsf{V} : (\mathcal{K}^n)^n \to \mathbb{R}$, the *mixed volume*, such that, for $m \in \mathbb{N}_0$,

$$\mathsf{V}_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \mathsf{V}(K_{i_1}, \ldots, K_{i_n})$$

for all $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$.

The mixed volume $\mathsf{V} : (\mathcal{K}^n)^{n-1} \to \mathbb{R}$ is continuous, non-negative, invariant under independent translations of its arguments, monotone, Minkowski linear in each argument, and $\mathsf{V}(K^{[n]}) = \mathsf{V}_n(K)$ for all $K \in \mathcal{K}^n$. Moreover, the mixed volume is $\mathrm{SO}(n)$-invariant in the sense that

$$\mathsf{V}(\vartheta K_1, \ldots, \vartheta K_n) = \mathsf{V}(K_1, \ldots, K_n)$$

for all $K_1, \ldots, K_n \in \mathcal{K}^n$ and $\vartheta \in \mathrm{SO}(n)$.

If convex bodies $K_1, \ldots, K_n \in \mathcal{K}^n$ are fixed, then $\mathsf{V}(K_1, \ldots, K_n) > 0$ if and only if $\dim(K_{i_1} + \cdots + K_{i_k}) \geq k$ for all $k \in \{1, \ldots, n\}$ and for each choice of indices $1 \leq i_1 < \cdots < i_k \leq n$.

For fixed $i \in \{0, \ldots, n\}$ and convex bodies $C_{i+1}, \ldots C_n \in \mathcal{K}^n$, the function

$$\varphi : \mathcal{K}^n \to \mathbb{R} : \varphi(K) := \mathsf{V}(K^{[i]}, C_{i+1}, \ldots, C_n)$$

has the property that $\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$ for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$. Although we have excluded the empty set from the space $\mathcal{K}^n$, we make the convention $\varphi(\emptyset) = 0$.

We fix the notations

$$\mathsf{W}_{n-i-1}(K, L) := \mathsf{V}(K^{[i]}, B^{[n-i-1]}, L)$$

and

$$\mathsf{W}_{n-i}(K) := \mathsf{W}_{n-i}(K, K) = \mathsf{V}(K^{[i]}, B^{[n-i]}).$$

The functions $\mathsf{W}_{n-i} : \mathcal{K}^n \to \mathbb{R}$ are also called *quermassintegrals*. Note that $\mathsf{W}_{n-i}(K) > 0$ if and only if $\dim K \geq i$.

We define the *i-th intrinsic volume* as

$$\mathsf{V}_i(K) := \binom{n}{i} \frac{1}{\kappa_{n-i}} \mathsf{W}_{n-i}(K) = \binom{n}{i} \frac{1}{\kappa_{n-i}} \mathsf{V}(K^{[i]}, B^{[n-i]}).$$

Then V_n is the usual volume and V_0 is the Euler characteristic. Moreover, these functions are intrinsic in the sense that their definition does not depend on the ambient space. More precisely, for $k \in \{0, \ldots, n\}$, $E \in \mathrm{A}(n, k)$, and $i \in \{0, \ldots, k\}$, denote by V_i^E the i-th intrinsic volume on the space $\mathcal{K}^k(E) \subseteq \mathcal{K}^n$. Then $\mathsf{V}_i^E(K) = \mathsf{V}_i(K)$ for every $K \in \mathcal{K}^k(E)$. Consequently, $\mathsf{V}_i(K)$ is the i-dimensional volume of any i-dimensional convex body $K \in \mathcal{K}^n$.

Example 2.2. For the intrinsic volumes of the unit ball, one has

$$\mathsf{V}_i(B) = \binom{n}{i} \frac{\kappa_n}{\kappa_{n-i}} = \begin{bmatrix} n \\ i \end{bmatrix} \kappa_i.$$

Example 2.3. For a polytope $P \in \mathcal{P}^n$, the intrinsic volumes are given by

$$\mathsf{V}_i(P) = \sum_{F \in \mathcal{F}_i(P)} \gamma(F, P) \mathcal{H}^i(F).$$

2.3 Area Measures

By means of the Riesz-Markov-Kakutani representation theorem, it can be shown that there is a unique symmetric map $S : (\mathcal{K}^n)^{n-1} \to \mathbf{M}(\mathbb{S})$, the *mixed area measure*, such that

$$\mathsf{V}(K_1, \ldots, K_n) = \frac{1}{n} \langle h(K_1, \cdot), S(K_2, \ldots, K_n; \cdot) \rangle$$

for all $K_1, \ldots, K_n \in \mathcal{K}^n$.

The mixed area measure $S(K_1, \ldots, K_{n-1}; \cdot)$ of convex bodies $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ is always a centered positive measure. The mixed area measure $S : (\mathcal{K}^n)^{n-1} \to \mathbf{M}_o(\mathbb{S})$ is continuous, Minkowski linear in each argument, and invariant under independent translations of its arguments.

The mixed area measure is SO(n) invariant in the sense that

$$S(\vartheta K_1, \dots, \vartheta K_{n-1}; \vartheta A) = S(K_1, \dots, K_{n-1}; A)$$

for all $K_1, \dots, K_{n-1} \in \mathcal{K}^n$, Borel subsets $A \subseteq \mathbb{S}$, and $\vartheta \in \mathrm{SO}(n)$.

For fixed $i \in \{0, \dots, n-1\}$ and convex bodies $C_{i+1}, \dots C_{n-1} \in \mathcal{K}^n$, the function

$$\varphi : \mathcal{K}^n \to \mathbf{M}(\mathbb{S}) : \varphi(K) := S(K^{[i]}, C_{i+1}, \dots, C_{n-1}; \cdot)$$

has the property that $\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$ for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$.

For $i \in \{0, \cdots, n-1\}$, we define the *area measure* of K of order i as

$$S_i(K, \cdot) := S(K^{[i]}, B^{[n-i-1]}; \cdot).$$

The area measure $S_{n-1}(K, \cdot)$ of order $n-1$ is also called the *surface area measure*, since for an n-dimensional convex body $K \in \mathcal{K}^n$ and a Borel subset $A \subseteq \mathbb{S}$, we have $S_{n-1}(K, A) = \mathcal{H}^{n-1}(\tau(K, A))$, where $\tau(K, A) \subseteq \mathrm{bd}\, K$ denotes the set of all boundary points of K at which there exists a unit normal vector belonging to A.

By definition, we have

$$\mathsf{W}_{n-i}(K, L) = \frac{1}{n} \langle h(L, \cdot), S_{i-1}(K, \cdot) \rangle$$

and

$$\mathsf{W}_{n-i}(K) = \mathsf{W}_{n-i}(K, K) = \frac{1}{n} \langle h(K, \cdot), S_{i-1}(K, \cdot) \rangle.$$

Example 2.4. If $P \in \mathcal{P}^n$ is an n-dimensional polytope with facets $F_1, \dots, F_k \in \mathcal{F}_{n-1}(P)$ and corresponding outer unit normals $u_1, \dots, u_k \in \mathbb{S}$, its surface area measure is given by

$$S_{n-1}(P, \cdot) = \sum_{i=1}^{k} \mathcal{H}^{n-1}(F_i) \delta_{u_i}, \tag{2.2}$$

where δ_{u_i} denotes the Dirac measure with mass concentrated in u_i.

Example 2.5. The mixed area measure of smooth convex bodies has a smooth density function. In order to describe this density function, we need to make two definitions.

If f is a $\mathbf{C}^2(\mathbb{S})$ function, then its 1-homogeneous extension $\widehat{f}$ is a $\mathbf{C}^2(\mathbb{R}^n \backslash \{o\})$ function. For every $u \in \mathbb{S}$, the Hessian $\nabla^2 \widehat{f}(u)$ is a self-adjoint linear map on $\mathbb{R}^n$ that maps u to o and maps $u^\perp$ into itself. We define

$$D^2 f(u) := (\nabla^2 \widehat{f}(u))|_{u^\perp} \in \mathrm{End}(u^\perp).$$

For an $(n-1)$-dimensional real vector space X, the *mixed discriminant* of linear maps $f_1, \dots, f_{n-1} \in \mathrm{End}(X)$ is defined as

$$\mathsf{D}(f_1, \dots, f_{n-1}) := \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial\lambda_1 \cdots \partial\lambda_{n-1}} \det(\lambda_1 f_1 + \cdots + \lambda_{n-1} f_{n-1}).$$

The mixed discriminant $\mathsf{D} : \mathrm{End}(X)^{n-1} \to \mathbb{R}$ is symmetric, linear in each argument, and uniquely determined by its property that

$$\det(\lambda_1 f_1 + \cdots + \lambda_m f_m) = \sum_{i_1, \dots, i_{n-1}=1}^{m} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \mathsf{D}(f_{i_1}, \dots, f_{i_{n-1}})$$

for all $m \in \mathbb{N}_+$, $f_1, \dots, f_m \in \mathrm{End}(X)$, and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Moreover, $\mathsf{D}(f^{[n-1]}) = \det f$ for $f \in \mathrm{End}(X)$.

With these definitions in place, the mixed area measure of smooth convex bodies $K_1, \dots, K_{n-1} \in \mathcal{K}^n$ has a smooth density given by

$$\mathsf{D}(D^2 h_{K_1}, \dots, D^2 h_{K_{n-1}}).$$

Minkowski's Existence Theorem It is a natural question to ask which spherical measures appear as surface area measures of convex bodies. A characterization is provided by *Minkowski's existence theorem*, which is stated below.

Theorem 2.6 (Minkowski). *A positive measure $\mu \in \mathbf{M}(\mathbb{S})$ is the surface area measure of an n-dimensional convex body if and only if it is centered and if $\mu(\mathbb{S} \cap u^\perp) < \mu(\mathbb{S})$ for all $u \in \mathbb{S}$.*

As a consequence, the linear space spanned by surface area measures,

$$\{S_{n-1}(K, \cdot) - S_{n-1}(L, \cdot) \mid K, L \in \mathcal{K}^n\}$$

is dense in $\mathbf{M}_o(\mathbb{S})$. Consequently, if $f \in \mathbf{C}(\mathbb{S})$ such that $\langle f, S_{n-1}(K, \cdot)\rangle = 0$ for all $K \in \mathcal{K}^n$, then f is a linear harmonic.

Christoffel's Problem It is natural to ask for a characterization of the other area measures $S_j(K, \cdot)$. For the area measure of order $j = 1$, this is known as *Christoffel's problem.*

A central role in Christoffel's problem plays the Helmholtz-like operator

$$\square_n : \mathbf{C}^\infty(\mathbb{S}) \to \mathbf{C}^\infty(\mathbb{S}) : \square_n := \frac{1}{n-1}\Delta_\circ + I.$$

Namely, if $K \in \mathcal{K}^n$ is a smooth convex body then all of its area measures have a smooth density and

$$\square_n h(K, \cdot) = S_1(K, \cdot).$$

Let us therefore further investigate this operator. Since the Beltrami operator $\Delta_\circ$ is an $\mathrm{SO}(n)$-equivariant continuous linear operator on $\mathbf{C}^\infty(\mathbb{S})$, the operator $\Box_n$ is too. Since $\Box_n$ is self-adjoint in the sense that $\langle \Box_n f, g\rangle = \langle f, \Box_n g\rangle$ for all $f, g \in \mathbf{C}^\infty(\mathbb{S})$, it extends to an $\mathrm{SO}(n)$-equivariant continuous linear operator

$$\Box_n : \mathbf{C}^{-\infty}(\mathbb{S}) \to \mathbf{C}^{-\infty}(\mathbb{S}).$$

Clearly, $\Box_n$ is a multiplier map with its multipliers given by

$$\lambda_k^n[\Box_n] = 1 - \frac{k(n+k-2)}{n-1} = -\frac{(k-1)(n+k-1)}{n-1}.$$

Note that $\lambda_k^n[\Box_n] = 1$, $\lambda_k^n[\Box_n] = 0$, and $\lambda_k^n[\Box_n] < 0$ for $k \geq 2$. In particular, all multipliers except for $k = 1$ are non-zero, so $\Box_n$ is injective on centered functions and distributions. Since the multipliers $\lambda_k^n[\Box_n]$ are slowly decreasing, it follows from Lemma 1.10 that

$$\Box_n : \mathbf{C}_o^\infty(\mathbb{S}) \to \mathbf{C}_o^\infty(\mathbb{S})$$

is an $\mathrm{SO}(n)$-equivariant automorphism of $\mathbf{C}_o^\infty(\mathbb{S})$. Analogously to Theorem 1.12, there exists a zonal centered distribution $\psi_n \in \mathbf{C}_o^{-\infty}(\mathbb{S}, \bar{e})$ such that $\Box_n^{-1} f = f * \psi_n$ for all $f \in \mathbf{C}_o^\infty(\mathbb{S})$. In the context of Christoffel's problem, this means that $h_o(K, \cdot) = S_1(K, \cdot) * \psi_n$ for every smooth convex body $K \in \mathcal{K}^n$.

These zonal distributions ψ_n were described by C. Berg in [5]. He constructed inductively a sequence g_n, $n \geq 2$, of smooth functions $g_n : (-1, 1) \to \mathbb{R}$, known as *Berg's functions*, with the property that $g_n \in \mathbf{L}^1[-1, 1]_n$ and that

$$\frac{1}{\omega_{n-1}} g_n \quad \sim \quad P_0^n + \sum_{k=2}^{\infty} \frac{N(n,k)}{\omega_n} \frac{n-1}{(k-1)(n+k-1)} P_k^n.$$

Consequently, $\psi_n = \omega_{n-1}^{-1} \breve{g}_n \in \mathbf{M}_o(\mathbb{S}, \bar{e}) \subseteq \mathbf{C}_o^{-\infty}(\mathbb{S}, \bar{e})$. Christoffel's problem was solved by C. Berg in the sense that he showed that a positive measure $\mu \in \mathbf{M}(\mathbb{S})$ is the first area measure of a convex body if and only if $\omega_{n-1}^{-1} \mu * \breve{g}_n$ is the support function of a convex body.

We also want to note that the linear span of first area measures,

$$\{S_1(K, \cdot) - S_1(L, \cdot) \mid K, L \in \mathcal{K}^n\}$$

is dense in $\mathbf{M}_o(\mathbb{S})$. This follows from the fact that $\mathbf{C}_o^\infty(\mathbb{S})$ is spanned by support functions of convex bodies and that $\Box_n$ is an automorphism of $\mathbf{C}_o^\infty(\mathbb{S})$.

2.4 Valuations on Convex Bodies

A function $\varphi : \mathcal{K}^n \to G$, where $(G, +)$ is a commutative semigroup, is called a *(G-valued) valuation* if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L)$$

for all $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$.

If $(G, +)$ has a neutral element 0_G, then $\varphi : \mathcal{K}^n \to G : \varphi(K) := 0_G$ is called the *trivial (G-valued) valuation* and we fix the convention $\varphi(\emptyset) = 0_G$ for all valuations $\varphi : \mathcal{K}^n \to G$.

If X is a vector space and φ is an X-valued valuation, then φ is called *homogeneous of degree i*, if $\varphi(\lambda K) = \lambda^i \varphi(K)$ for all $K \in \mathcal{K}^n$ and $\lambda \geq 0$. Furthermore, φ is called *even* if $\varphi(-K) = \varphi(K)$ for all $K \in \mathcal{K}^n$ and *odd* if $\varphi(-K) = -\varphi(K)$ for all $K \in \mathcal{K}^n$.

A real valued valuation is just called a *valuation* and a $\mathcal{K}^n$-valued valuation (where $\mathcal{K}^n$ is endowed with Minkowski addition) is called a *Minkowski valuation*.

Example 2.7. For $i \in \{0, \dots, n\}$ and fixed convex bodies $C_{i+1}, \dots, C_n \in \mathcal{K}^n$, the map

$$\varphi : \mathcal{K}^n \to \mathbb{R} : \varphi(K) := \mathsf{V}(K^{[i]}, C_{i+1}, \dots, C_n)$$

is a continuous translation invariant valuation, homogeneous of degree i. In particular, the intrinsic volumes V_i are examples of rigid motion invariant continuous valuations.

McMullen's decomposition Much of the modern theory on valuations concerns translation invariant continuous valuations taking values in a topological vector space. A very important theorem on these valuations is the following polynomiality result:

Theorem 2.8 (McMullen)**.** *If* $\mathbf{X}$ *is a topological vector space, then for every continuous translation invariant valuation* $\varphi : \mathcal{K}^n \to \mathbf{X}$*, there exists a unique continuous symmetric mapping* $\overline{\varphi} : (\mathcal{K}^n)^n \to \mathbf{X}$ *which is translation invariant and Minkowski linear in each variable such that*

$$\varphi(\lambda_1 K_1 + \dots + \lambda_m K_m) = \sum_{i_1, \dots, i_n = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} \overline{\varphi}(K_{i_1}, \dots, K_{i_n}) \tag{2.3}$$

for all $K_1, \dots, K_m \in \mathcal{K}^n$ *and* $\lambda_1, \dots, \lambda_m \geq 0$*. Moreover, the mapping* $K \mapsto \overline{\varphi}(K^{[i]}, \mathcal{C})$ *is a continuous translation invariant valuation, homogeneous of degree i, for each* $i \in \{0, \dots, n\}$ *and fixed family of convex bodies* $\mathcal{C} = (C_{i+1}, \dots, C_n) \in (\mathcal{K}^n)^{n-i}$.

A direct consequence is *McMullen's decomposition theorem*, which can be stated as follows:

Theorem 2.9 (McMullen)**.** *If* $\mathbf{X}$ *is a topological vector space, then for every continuous translation invariant valuation* $\varphi : \mathcal{K}^n \to \mathbf{X}$*, there exist continuous translation invariant valuations* $\varphi_i : \mathcal{K}^n \to \mathbf{X}$*, homogeneous of degree* $i \in \{0, \dots, n\}$*, such that* $\varphi = \varphi_0 + \dots + \varphi_n$.

Hence, the space $\mathbf{XVal}$ of all translation invariant continuous $\mathbf{X}$-valued valuations can be written as

$$\mathbf{XVal} = \bigoplus_{i=0}^{n} \mathbf{XVal}_i,$$

where $\mathbf{XVal}_i$ denotes the space of valuations in $\mathbf{XVal}$ homogeneous of degree i. This direct sum is called the *McMullen decomposition* of the space $\mathbf{XVal}$. We also call $\varphi = \varphi_0 + \cdots + \varphi_n$ the *McMullen decomposition* of φ.

If $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ is a normed space, then $\mathbf{XVal}$ can be made a normed space my means of

$$\|\varphi\| := \max\{\|\varphi(K)\|_{\mathbf{X}} \mid K \in \mathcal{K}^n, K \subseteq B\}.$$

Moreover, if $\mathbf{X}$ is a Banach space, then $\mathbf{XVal}$, endowed with this norm, also becomes a Banach space.

The Steiner Formula A special case of (2.3) is Steiner's formula for valuations:

Theorem 2.10 (Steiner's formula)**.** *If* $\mathbf{X}$ *is a topological vector space, then for every continuous translation invariant valuation* $\varphi : \mathcal{K}^n \to \mathbf{X}$*, there exist continuous translation invariant valuations* $\varphi^{(j)} : \mathcal{K}^n \to \mathbf{X}$ *such that*

$$\varphi(K + \varepsilon B) = \sum_{j=0}^{n} \varepsilon^{n-j} \varphi^{(j)}(K)$$

for all $K \in \mathcal{K}^n$ *and* $\varepsilon \geq 0$.

Historically, Steiner's formula for the usual volume V_n was one of the most influential results of the early days of convex geometry.

The Spaces Val, MVal, and CVal We denote by $\mathbf{Val}$ the space of translation invariant continuous real valued valuations and by $\mathbf{Val}_i$ the space of valuations in $\mathbf{Val}$ homogeneous of degree i. Furthermore, we denote by $\mathbf{Val}^+$ and $\mathbf{Val}^-$ the spaces of even and odd valuations in $\mathbf{Val}$, respectively. Then due to McMullens decomposition theorem, we have

$$\mathbf{Val} = \bigoplus_{i=0}^{n} \mathbf{Val}_i = \bigoplus_{i=0}^{n} (\mathbf{Val}_i^+ \oplus \mathbf{Val}_i^-),$$

where $\mathbf{Val}_i^\pm := \mathbf{Val}_i \cap \mathbf{Val}^\pm$. We endow $\mathbf{Val}$ with the Banach space topology defined above.

We denote by $\mathbf{CVal}$ the space of translation invariant, continuous $\mathbf{C}(\mathbb{S})$ valued valuations and by $\mathbf{CVal}_i$ the space of valuations in $\mathbf{CVal}$ homogeneous of degree i. Then

$$\mathbf{CVal} = \bigoplus_{i=0}^{n} \mathbf{CVal}_i.$$

We endow **CVal** with the Banach space topology defined above.

Moreover, we denote by $\mathbf{CVal}^{\mathrm{SO}(n)}$ and $\mathbf{CVal}_i^{\mathrm{SO}(n)}$ the space of $\mathrm{SO}(n)$-equivariant valuations in **CVal** and $\mathbf{CVal}_i$, respectively.

We denote by **MVal** the space of translation invariant, continuous Minkowski valuations and by $\mathbf{MVal}_i$ the space of valuations in **MVal** homogeneous of degree i. Note that since $\mathcal{K}^n$ is not a vector space, **MVal** is also not a vector space and we can not apply McMullen's decomposition theorem on **MVal**.

Moreover, we denote by $\mathbf{MVal}^{\mathrm{SO}(n)}$ and $\mathbf{MVal}_i^{\mathrm{SO}(n)}$ the space of $\mathrm{SO}(n)$-equivariant Minkowski valuations in **MVal** and $\mathbf{MVal}_i$, respectively. The $\mathrm{SO}(n)$-equivariance implies that for every $\Phi \in \mathbf{MVal}$, there is a unique number $r(\Phi) \geq 0$ such that $\Phi B = r(\Phi)B$.

We can consider **MVal** a subspace of **CVal** by means of the support function. As mentioned before, the mapping

$$\iota : \mathcal{K}^n \to \mathbf{C}(\mathbb{S}) : K \mapsto h(K, \cdot)$$

is an embedding of metric spaces that respects the algebraic structure. Then

$$\eta : \mathbf{MVal} \to \mathbf{CVal} : \Phi \mapsto \eta \circ \Phi$$

is also injective and respects the algebraic structure. We endow **MVal** with the metric induced by the norm on **CVal** and η. Then η automatically becomes an embedding of metric spaces. In particular, every Minkowski valuation can be considered as a valuation in **CVal**. Conversely, a valuation in **CVal** can be considered as a Minkowski valuation if an only if it only takes support functions as values.

For a valuation $\Phi \in \mathbf{CVal}$, $\Phi : K \mapsto g(K, \cdot)$, we define its *associated real valued valuation* $\varphi \in \mathbf{Val}$ by

$$\varphi : \mathcal{K}^n \to \mathbb{R} : \varphi(K) := g(K, \overline{e}).$$

It is easy to see that φ is zonal if and only if Φ is $\mathrm{SO}(n)$-equivariant.

We denote the space of zonal valuations in **Val** by $\mathbf{Val}^{\overline{e}}$, and for a zonal valuation $\varphi \in \mathbf{Val}^{\overline{e}}$, we define its *associated* $\mathbf{C}(\mathbb{S})$ *valued valuation* $\Phi \in \mathbf{CVal}^{\mathrm{SO}(n)}$ by

$$\Phi : \mathcal{K}^n \to \mathbf{C}(\mathbb{S}) : K \mapsto g(K, \cdot),$$
$$g(K, \vartheta\overline{e}) := \varphi(\vartheta^{-1}K), \qquad \vartheta \in \mathrm{SO}(n).$$

These definitions give rise to an isometric isomorphism

$$\mathbf{Val}^{\overline{e}} \to \mathbf{CVal}^{\mathrm{SO}(n)}.$$

The notion of an associated real valued valuation is very helpful when dealing with valuations in **CVal** and **MVal**, because it allows us to reduce questions concerning such valuations to questions about real valued valuations, on which there exists a rich theory that is discussed in the following paragraphs.

Characterization Theorems In this paragraph we list some important characterization results for translation invariant continuous valuations. It is easy to see that the space $\mathbf{Val}_0$ is spanned by the Euler characteristic V_0. The non-trivial counterpart to this theorem for $\mathbf{Val}_n$ is known as *Hadwiger's volume characterization theorem*:

Theorem 2.11 (Hadwiger)**.** *The space $\mathbf{Val}_n$ is spanned by the volume V_n.*

The following useful description of the space $\mathbf{Val}_{n-1}$ is due to P. McMullen:

Theorem 2.12 (McMullen)**.** *For every $\varphi_{n-1} \in \mathbf{Val}_{n-1}$, there exists a unique $f \in \mathbf{C}_o(\mathbb{S})$ such that*

$$\varphi_{n-1}(K) = \langle f, S_{n-1}(K, \cdot)\rangle, \qquad K \in \mathcal{K}^n.$$

Another famous result is the following characterization of the intrinsic volumes due to H. Hadwiger:

Theorem 2.13 (Hadwiger)**.** *If $\varphi \in \mathbf{Val}$ is rigid motion invariant, then φ is a linear combination of the intrinsic volumes.*

As was shown by R. Schneider, the Steiner point map has th following important uniqueness property:

Theorem 2.14 (Schneider)**.** *The Steiner point map $s : \mathcal{K}^n \to \mathbb{R}^n$ is the unique vector valued, rigid motion equivariant, and continuous valuation.*

As a consequence of this characterization theorem, we will obtain later on that all Minkowski valuations in $\mathbf{MVal}^{\mathrm{SO}(n)}$ map convex bodies to centered convex bodies.

Smooth Valuations A group action of $\mathrm{GL}(n)$ is given on $\mathbf{Val}$ by means of

$$(A\varphi)(K) := \varphi(A^{-1}K), \qquad A \in \mathrm{GL}(n), \quad K \in \mathcal{K}^n.$$

We call a valuation $\varphi \in \mathbf{Val}$ *smooth* if the mapping

$$\mathrm{GL}(n) \to \mathbf{Val} : A \mapsto A\varphi$$

is smooth. Note that this notion makes sense because $\mathrm{GL}(n)$ can be considered as an open subset of $\mathbb{R}^{n\times n}$ and $\mathbf{Val}$ is a Banach space. We denote the space of smooth valuations by $\mathbf{Val}^\infty$. Consider the linear map

$$\iota^\infty : \begin{cases} \mathbf{Val}^\infty & \to & \mathbf{C}^\infty(\mathrm{GL}(n); \mathbf{Val}) \\ \varphi & \mapsto & (A \mapsto A\varphi) \end{cases}$$

where $\mathbf{C}^\infty(\mathrm{GL}(n); \mathbf{Val})$ denotes the space of smooth functions $\mathrm{GL}(n) \to \mathbf{Val}$, endowed with the usual Fréchet topology. The map ι^∞ is well-defined and injective, so it allows us to identify $\mathbf{Val}^\infty$ with a linear subspace of $\mathbf{C}^\infty(\mathrm{GL}(n); \mathbf{Val})$. The initial topology on $\mathbf{Val}^\infty$ with respect to ι^∞ is called the *Gårding topology* and is finer than the trace topology of the norm topology. Endowed with the Gårding topology, $\mathbf{Val}^\infty$ becomes a Fréchet space, the inclusion map $\mathbf{Val}^\infty \subseteq \mathbf{Val}$ is continuous, and $\mathbf{Val}^\infty$ is dense in $\mathbf{Val}$.

We obtain the McMullen decomposition

$$\mathbf{Val}^\infty = \bigoplus_{i=0}^{n} \mathbf{Val}_i^\infty = \bigoplus_{i=0}^{n} (\mathbf{Val}_i^{\infty,+} \oplus \mathbf{Val}_i^{\infty,-}).$$

Example 2.15. The intrinsic volumes are examples of smooth translation invariant valuations.

Example 2.16. If $K \in \mathcal{K}^n$ is strictly convex and smooth, then

$$\eta_K : \mathcal{K}^n \to \mathbb{R} : \eta_K(C) := \mathsf{V}_n(C + K)$$

is a smooth translation invariant valuation, that is $\eta_K \in \mathbf{Val}^\infty$.

We define the space of *generalized valuations* as $\mathbf{Val}^{-\infty} := (\mathbf{Val}^\infty)^*$ and endow it with the weak* topology. Furthermore, for $i \in \{0, \dots, n\}$, we define the space of *generalized valuations of degree* i as $\mathbf{Val}_i^{-\infty} := (\mathbf{Val}_{n-i}^\infty)^*$ and endow it with the weak* topology. This choice of notation will become plausible after the introduction of the Alesker product.

Alesker's Irreducibility Theorem A group action of $\mathrm{GL}(n)$ on $\mathbf{Val}$ is given by means of

$$(A\varphi)(K) := \varphi(A^{-1}K), \qquad A \in \mathrm{GL}(n), \quad K \in \mathcal{K}^n.$$

Clearly, the spaces $\mathbf{Val}_i^\pm$ are closed in $\mathbf{Val}$ and $\mathrm{GL}(n)$ invariant. A deep result of Alesker, known as *Alesker's Irreducibility Theorem*, states that these subspaces are also irreducible:

Theorem 2.17. *For $i \in \{0, \dots, n\}$, the only closed $\mathrm{GL}(n)$ invariant subspaces of $\mathbf{Val}_i^\pm$ are $\{0\}$ and $\mathbf{Val}_i^\pm$.*

In the language of group representations, a natural continuous representation of $\mathrm{GL}(n)$ on $\mathbf{Val}$ is given and the spaces $\mathbf{Val}_i^\pm$ are irreducible subrepresentations of $\mathbf{Val}$.

An analogous result holds for smooth valuations:

Theorem 2.18. *For $i \in \{0, \dots, n\}$, the only closed $\mathrm{GL}(n)$ invariant subspaces of $\mathbf{Val}_i^{\infty,\pm}$ are $\{0\}$ and $\mathbf{Val}_i^{\infty,\pm}$.*

Again, in the language of group representations, this means that $\mathbf{Val}_i^{\infty,\pm}$ are irreducible subrepresentations of $\mathbf{Val}^\infty$.

The Alesker Product The space $\mathbf{Val}^\infty$ can be equipped with an interesting multiplicative structure, which is the content of the following theorem.

Theorem 2.19. *There exists a bilinear product*

$$\mathbf{Val}^\infty \times \mathbf{Val}^\infty \to \mathbf{Val}^\infty : (\varphi, \psi) \mapsto \varphi \cdot \psi$$

uniquely determined by the following two properties:

(i) *The product is continuous in the Gårding topology.*

(ii) *If $K, L \in \mathcal{K}^n$ are strictly convex and smooth, then*

$$(\eta_K \cdot \eta_L)(C) = \mathsf{V}_{2n}(\iota(C) + K \times L),$$

where $\iota : \mathbb{R}^n \to \mathbb{R}^{2n}$ is defined as $\iota(x) := (x, x)$.

Endowed with this multiplicative structure, the space $\mathbf{Val}^\infty$ becomes an associative and commutative algebra with unit given by the Euler characteristic V_0 which is graded by the degree of homogeneity, that is $\mathbf{Val}_i^\infty \cdot \mathbf{Val}_j^\infty \subseteq \mathbf{Val}_{i+j}^\infty$.

The product defined by the theorem above is called the *Alesker product*. This product is somewhat non-trivial in the sense that it does not extend continuously to $\mathbf{Val}$.

Due to H. Hadwiger's classical result that $\mathbf{Val}_n$ is spanned by the usual volume V_n, we may define a bilinear map $\langle \cdot, \cdot \rangle$ by

$$\langle \cdot, \cdot \rangle : \mathbf{Val}_i^\infty \times \mathbf{Val}_{n-i}^\infty \to \mathbb{R} : \varphi \cdot \psi = \langle \varphi, \psi \rangle \mathsf{V}_n$$

for $i \in \{0, \ldots, n\}$. This map is clearly continuous. Moreover, the Alesker product has the property that for every non-zero $\varphi_i \in \mathbf{Val}_i^\infty$ there exists some $\psi \in \mathbf{Val}_{n-i}^\infty$ such that $\varphi \cdot \psi \neq 0$. Hence, the linear map

$$\mathbf{Val}_i^\infty \to \mathbf{Val}_i^{-\infty} = (\mathbf{Val}_{n-i}^\infty)^* : \varphi \mapsto \langle \varphi, \cdot \rangle$$

is injective, continuous, and has dense image due to Alesker's irreducibility theorem. By means of this linear map, we can consider $\mathbf{Val}_i^\infty$ as a dense linear subspace of $\mathbf{Val}_i^{-\infty}$, and the inclusion map

$$\mathbf{Val}_i^\infty \subseteq \mathbf{Val}_i^{-\infty}$$

is continuous.

2.5 Bivaluations on convex bodies

A map $\phi : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{R}$ is called a *(real valued) bivaluation* if it is a valuation in each argument. A bivaluation ϕ is called *translation biinvariant* if ϕ is invariant under independent translations of its arguments and ϕ is called $\mathrm{O}(n)$-*invariant* if $\phi(\vartheta K, \vartheta L) = \phi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $\vartheta \in \mathrm{O}(n)$. We say that ϕ has *bidegree* (i, j) if $\phi(\lambda K, \mu L) = \lambda^i \mu^j \phi(K, L)$ for all $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$.

Example 2.20. An important example of a bivaluation can be constructed from mixed volumes. For $i, j \in \{0, \dots, n\}$, $i + j \leq n$, the map

$$\phi : \mathcal{K}^n \times \mathcal{K}^n \to \mathbb{R} : \phi(K, L) := \mathsf{V}(K^{[i]}, L^{[j]}, B^{[n-i-j]})$$

defines a $\mathrm{O}(n)$-invariant, translation biinvariant, and continuous bivaluation of bidegree (i, j). [2]

We denote by **BVal** the space of all continuous, translation biinvariant bivaluations and we denote by $\mathbf{BVal}_{i,j}$ the space of all bivaluations in **BVal** of bidegree (i, j).Then the space **BVal** has the McMullen decomposition

$$\mathbf{BVal} = \bigoplus_{i,j=0}^{n} \mathbf{BVal}_{i,j}.$$

A very important tool that we will use at two crucial points is the following symmetry result by A. Alesker, A. Bernig, and F. Schuster, which we call the *symmetry theorem on bivaluations*:

Theorem 2.21 ([3, Theorem 6.4])**.** *If* $\phi \in \mathbf{BVal}_{j,j}$*,* $j \in \{0, \dots, n\}$*, is* $\mathrm{O}(n)$*-invariant, then*

$$\phi(K, L) = \phi(L, K)$$

for all $K, L \in \mathcal{K}^n$*.*

Unfortunately, in general the condition of $\mathrm{O}(n)$-invariance can not be replaced by $\mathrm{SO}(n)$-invariance, as was shown in [3]. However, many important bivaluations, like in the example above, are $\mathrm{O}(n)$-invariant, and for translation invariant, continuous Minkowski valuations, $\mathrm{SO}(n)$-equivariance already implies $\mathrm{O}(n)$-equivariance:

Theorem 2.22 ([3, Theorem 7.1])**.** *Every Minkowski valuation* $\Phi \in \mathbf{MVal}^{\mathrm{SO}(n)}$ *is* $\mathrm{O}(n)$*-equivariant.*

In conclusion, we obtain a symmetry theorem for mixed volumes of Minkowski valuations. If $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$, $j \in \{0, \dots, n-1\}$, then

$$\mathsf{W}_{n-j-1}(K, \Phi_j L) = \mathsf{W}_{n-j-1}(L, \Phi_j K) \tag{2.4}$$

for all $K, L \in \mathcal{K}^n$.

[2] We have only discussed the action of $\mathrm{SO}(n)$ and $\mathrm{SO}(n)$-invariance, but the required arguments regarding $\mathrm{O}(n)$ work analogously.

2.6 Inequalities for mixed volumes

In this section we state the classical geometric inequalities from the Brunn-Minkowski theory. Recall that two convex bodies K and L are called *homothetic* if there is some $x \in \mathbb{R}^n$ and $\lambda > 0$ such that $L = \lambda K + x$.

The *Brunn-Minkowski inequality* states that

$$\mathsf{V}_n(K+L)^{1/n} \geq \mathsf{V}_n(K)^{1/n} + \mathsf{V}_n(L)^{1/n} \tag{2.5}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ with equality if and only if K and L are homothetic. The multiplicative Brunn-Minkowski inequality states that

$$\mathsf{V}_n((1-\lambda)K + \lambda L) \geq \mathsf{V}_n(K)^{1-\lambda}\mathsf{V}_n(L)^{\lambda}$$

for all $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$ with equality if and only if K and L are translates of each other. An immediate consequence of the Brunn-Minkowski inequality is the *Minkowski inequality* which states that

$$\mathsf{V}(K^{[n-1]}, L)^n \geq \mathsf{V}_n(K)^{n-1}\mathsf{V}_n(L)$$

for all n-dimensional $K, L \in \mathcal{K}^n$ with equality if and only if K and L are homothetic. The famous isoperimetric inequality follows immediately from the Minkowski inequality.

The inequalities listed above have significant generalizations. The *general Brunn-Minkowski* inequality states that for $i \in \{2, \dots, n\}$, we have

$$\mathsf{V}((K+L)^{[i]}, \mathcal{C})^{1/i} \geq \mathsf{V}(K^{[i]}, \mathcal{C})^{1/i} + \mathsf{V}(L^{[i]}, \mathcal{C})^{1/i} \tag{2.6}$$

for all $K, L \in \mathcal{K}^n$ and $\mathcal{C} = (C_{i+1}, \dots, C_n)$, where $C_{i+1}, \dots, C_n \in \mathcal{K}^n$. The multiplicative form states that

$$\mathsf{V}(((1-\lambda)K + \lambda L)^{[i]}, \mathcal{C}) \geq \mathsf{V}(K^{[i]}, \mathcal{C})^{1-\lambda}\mathsf{V}(L^{[i]}, \mathcal{C})^{\lambda}$$

for all $K, L, \mathcal{C}$ as before and for all $\lambda \in (0,1)$.

Putting $\mathcal{C} = B^{[n-i]}$ for $i \in \{2, \dots, n\}$, we obtain the *Brunn-Minkowski inequality for quermassintegrals*, which states that

$$\mathsf{W}_{n-i}(K+L)^{1/i} \geq \mathsf{W}_{n-i}(K)^{1/i} + \mathsf{W}_{n-i}(L)^{1/i} \tag{2.7}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$ with equality if and only if K and L are homothetic. The multiplicative form states that

$$\mathsf{W}_{n-i}((1-\lambda)K + \lambda L) \geq \mathsf{W}_{n-i}(K)^{1-\lambda}\mathsf{W}_{n-i}(L)^{\lambda}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ with equality if and only if K and L are translates of each other.

The *Minkowski inequality for quermassintegrals* states that for $i \in \{2, \dots, n\}$,

$$\mathsf{W}_{n-i}(K, L)^i \geq \mathsf{W}_{n-i}(K)^{i-1}\mathsf{W}_{n-i}(L) \tag{2.8}$$

for all $K, L \in \mathcal{K}^n$ of dimension at least i with equality if and only if K and L are homothetic.

The *Alexandrov-Fenchel inequality* states that

$$\mathsf{V}(K, K, \mathcal{C})^2 \geq \mathsf{V}(K, L, \mathcal{C})\mathsf{V}(L, L, \mathcal{C})$$

for all $K, L \in \mathcal{K}^n$ and $\mathcal{C} = (C_{i+1}, \dots, C_n)$, where $C_{i+1}, \dots, C_n \in \mathcal{K}^n$.

2.7 Crofton's Intersection Formula

As references for this section, we cite [18, 21].

As a consequence of Hadwiger's characterization of rigid motion invariant, continuous valuations, we obtain Crofton's intersection formula, namely

$$\int_{\mathrm{A}(n,k)} \mathsf{V}_j(K \cap E)\, dE = \frac{\left[{k \atop j}\right]}{\left[{n \atop k-j}\right]} \mathsf{V}_{n+j-k}(K) \tag{2.9}$$

for all $j, k \in \{0, \dots, n\}$ and $K \in \mathcal{K}^n$. In the special case where $k = j$, we obtain

$$\int_{\mathrm{A}(n,k)} \mathsf{V}_k(K \cap E)\, dE = \mathsf{V}_n(K). \tag{2.10}$$

In the special case where $j = 0$, we obtain

$$\int_{A(n,k)} \mathsf{V}_0(K \cap E)\, dE = \frac{1}{\left[{n \atop k}\right]} \mathsf{V}_{n-k}(K). \tag{2.11}$$

Chapter 3

Mean Section Operators

In this chapter, we will introduce the mean section operator and centered mean section operators, and ultimately derive a practical representation formula for these operators.

First, we want to give an intuition for mean section bodies. To that end, let $K \in \mathcal{K}^n$ and consider the space $\mathrm{A}(n,k;K)$ of all k-flats $E \in \mathrm{A}(n,k)$ that intersect K. Due to its rigid motion invariance, the measure ν_k^n restricted to $\mathrm{A}(n,k;K)$ and normalized to a probability measure, can be thought of as a uniform distribution on $\mathrm{A}(n,k;K)$. Let $E_1, E_2, \ldots \in \mathrm{A}(n,k;K)$ be a sequence of independent uniform random k-flats, and define

$$K_n := \frac{1}{n}((K \cap E_1) + \cdots + (K \cap E_n)).$$

Then K_n, $n \in \mathbb{N}_+$, is a sequence of convex bodies that can be thought of as arithmetic means (in the sense of Minkowski addition) of random sections of K. The mean section body of K can be thought of as the limit of the sequence $(K_n)_{n\in\mathbb{N}_+}$. The formal definition of mean section bodies, which was first made by P. Goodey and W. Weil in 1992, is as follows:

Definition ([11, p. 420]). For $k \in \{0, \ldots, n\}$, we define the *k-th mean section body* $\tilde{M}_k(K)$ of a convex body $K \in \mathcal{K}^n$ by

$$h(\tilde{M}_k(K), u) := \int_{\mathrm{A}(n,k)} h(K \cap E, u)\, dE, \qquad u \in \mathbb{S}.$$

and we call the map

$$\tilde{M}_k : \mathcal{K}^n \to \mathcal{K}^n$$

the *k-th mean section operator*

It is not immediately obvious that the mean section body is well-defined. However, that follows from the continuity of intersections (cf. [21, Theorem 1.8.10]) and the fact that convex bodies can be defined by means of support functions.

In the cases where $k = 0$ and $k = n$, the k-th mean section operator $\tilde{M}_k$ operates in a very simple way. Since $\mathrm{A}(n,n) = \{\mathbb{R}^n\}$, we have

$$\tilde{M}_n(K) = K.$$

Since $\mathrm{A}(n,0) = \{\{x\} \mid x \in \mathbb{R}^n\}$, we have

$$\tilde{M}_0(K) = \left\{ \int_K x \; dx \right\}$$

if $\dim K = n$ and $\tilde{M}_o(K) = \{o\}$ if $\dim K < n$.

For our purposes it will be more convenient to work with the centered versions of these objects.

Definition. For $k \in \{0, \dots, n\}$, we define the *k-th centered mean section body* $M_k(K)$ of a convex body $K \in \mathcal{K}^n$ by

$$h(M_k(K), u) := h_o(\tilde{M}_k(K), u) = \int_{\mathrm{A}(n,k)} h_o(K \cap E, u) \; dE, \qquad u \in \mathbb{S},$$

and we call the map

$$M_k : \mathcal{K}^n \to \mathcal{K}^n$$

the *k-th centered mean section operator*.

The mean section bodies and centered mean section bodies are obviously translates of each other, namely

$$\tilde{M}_k(K) = M_k(K) + \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n-k+1}{2})}{\sqrt{\pi}\Gamma(\frac{n+1}{2})} \mathsf{V}_{n-k}(K) s(K)$$

for $k \in \{0, \dots, n\}$ and $K \in \mathcal{K}^n$, as was shown in [9, p. 165].

Lemma 3.1. *For $k \in \{1, \dots, n\}$, the k-th centered mean section operator M_k is a continuous, translation invariant,* SO(n)*-equivariant Minkowski valuation homogeneous of degree $n - k + 1$, that is*

$$M_k \in \mathbf{MVal}_{n-k+1}^{\mathrm{SO}(n)}.$$

Proof. The valuation property can be shown by an elementary computation, the translation invariance is obvious, the SO(n)-equivariance follows from the rigid motion invariance of the measure ν_k^n on $\mathrm{A}(n,k)$, the homogeneity follows from the homogeneity of ν_k^n, and the continuity follows from the continuity of intersections (cf. [21, Theorem 1.8.10]). □

Example 3.2. Due to their SO(n)-equivariance, we must have $M_k(B) = r(M_k)B$ for some radius $r(M_k) \geq 0$. It was shown in [13, Lemma 2.1] that

$$M_k(B) = \frac{\kappa_{n-k+1}}{2} \frac{\left[{k \atop 1}\right]}{\left[{n \atop 1}\right]} B. \tag{3.1}$$

As mentioned in the beginning of this chapter, we seek a practical representation formula for the centered mean section operators. As was shown by P. Goodey and W. Weil in 2014, the support function of the k-th centered mean section body can be expressed as

$$h(M_k, \cdot) = m_{n,k} S_{n-k+1}(-K, \cdot) * \breve{g}_k, \tag{3.2}$$

where g_k denotes Berg's functions related to the Christoffel problem and $m_{n,k}$ is a constant that will be computed explicitly. We will follow their proof for this formula which was given in [13].

In Section 3.1, we will first prove a representation formula for polytopes by combining two translational integral geometric formulas. Then, in Section 3.2, other than one might expect, we will not directly apply an argument of density for all $k \in \{1, \ldots, n\}$, but only for $k = 2$ to obtain a general representation formula for the operator M_2.

From there the idea is to express general mean sections in terms of mean sections with planes. To that end, we will introduce so-called spherical lifting and projection operators in Section 3.3 that will allow us to relate convex geometry in subspaces to convex geometry in the ambient space. In Section 3.4, we will investigate how these operators interact with Berg's functions. Ultimately, we will derive the desired formula (3.2) by using spherical liftings and projections as a bridge to connect mean section operators to Berg's functions.

3.1 Mean Section Bodies of Polytopes

The objective of this section is to derive a formula for mean section bodies of polytopes from two intersectional integral geometric formulas. First, we prove a general integral geometric formula.

Proposition 3.3 ([10, Theorem 1]). *For $j \in \{0, \ldots, n\}$ and two convex bodies $K, L \in \mathcal{K}^n$,*

$$\int_{\mathbb{R}^n} \mathsf{V}_j(K \cap (L+x))\, dx = \left[{n \atop j}\right] \sum_{i=j}^{n} \binom{n}{i-j} \int_{\mathrm{A}(n,n-j)} \mathsf{V}((K \cap E)^{[i-j]}, (-L)^{[n+j-i]})\, dE, \tag{3.3}$$

with the convention that $\mathsf{V}(\emptyset^{[0]}, C^{[n]}) = \mathsf{V}(C^{[n]}, \emptyset^{[0]}) = 0$.

Proof. Due to Crofton's intersection formula (2.11) and Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \mathsf{V}_j(K \cap (L+x))\, dx &= \int_{\mathbb{R}^n} \begin{bmatrix} n \\ j \end{bmatrix} \int_{\mathrm{A}(n,n-j)} \mathsf{V}_0(K \cap (L+x) \cap E)\, dE\, dx \\ &= \begin{bmatrix} n \\ j \end{bmatrix} \int_{\mathrm{A}(n,n-j)} \int_{\mathbb{R}^n} \mathsf{V}_0((K \cap E) \cap (L+x))\, dx\, dE.\end{aligned}$$

Observe that

$$\{x \in \mathbb{R}^n \mid (K \cap E) \cap (L+x) \neq \emptyset\} = (K \cap E) + (-L).$$

Since V_0 is just the Euler characteristic, we obtain

$$\int_{\mathbb{R}^n} \mathsf{V}_j(K \cap (L+x))\, dx = \begin{bmatrix} n \\ j \end{bmatrix} \int_{\mathrm{A}(n,n-j)} \mathsf{V}_n((K \cap E) + (-L))\, dE.$$

By the definition of mixed volumes,

$$\mathsf{V}_n((K \cap E) + (-L)) = \sum_{i=j}^{n+j} \binom{n}{i-j} \mathsf{V}((K \cap E)^{[i-j]}, (-L)^{[n+j-i]}).$$

Since $\dim(K \cap E) \leq n-j$ for $E \in \mathrm{A}(n, n-j)$, all terms for $i > n$ vanish. □

Corollary 3.4. *For two convex bodies $K, L \in \mathcal{K}^n$,*

$$\int_{\mathbb{R}^n} \mathsf{V}_n(K \cap (L+x))\, dx = \mathsf{V}_n(K)\mathsf{V}_n(L).$$

Proof. From (the proof of) Proposition 3.3 and the fact that $A(n, 0)$ is naturally isomorphic to $\mathbb{R}^n$, it follows that

$$\int_{\mathbb{R}^n} \mathsf{V}_n(K \cap (L+x))\, dx = \int_{\mathbb{R}^n} \mathsf{V}_n((K \cap \{x\}) + (-L))\, dx = \int_{\mathbb{R}^n} \mathbb{1}_K(x)\mathsf{V}_n(L)\, dx.$$

□

In order to prove a second translational integral geometric formula specifically for polytopes, we need to make some preparations. First, we want to give a description of the facial structure of intersections of polytopes. To that end, we will make use of the following intersectional description of faces of polytopes.

Lemma 3.5. *If a polytope $P \in \mathcal{P}^n$ is given as a finite intersection of closed half-spaces*

$$P = \bigcap_{i \in I} H_i^-,$$

then its faces are precisely the empty face and the sets

$$P \cap \bigcap_{i \in J} H_i, \qquad J \subseteq I.$$

Proof. We may assume that $I \neq \emptyset$, P is full-dimensional, and $o \in \operatorname{int} P$. Then, we may also assume that $H_i^- = H_{u_i,1}^-$, where $u_i \in \mathbb{R}^n \backslash \{o\}$ for $i \in I$.

For a subset $J \subseteq I$, we define

$$F_J := P \cap \bigcap_{i \in J} H_{u_i,1}^-.$$

First, we show that F_J is a face of P. If $F_J \neq \emptyset$, we define $u := \frac{1}{|J|} \sum_{i \in J} u_i$. Clearly, $P \subseteq H_{u,1}^-$. It is also easy to see that a vector $x \in P$ satisfies $x \cdot u = 1$ if and only if $x \cdot u_i = 1$ for all $i \in J$. Hence, $F_J = P \cap H_{u,1}$ is a face of P.

For the other direction, let F be a non-empty face of P. Then there is some $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that $P \subseteq H_{u,\alpha}^-$ and $F = P \cap H_{u,\alpha}$. Since P is full-dimensional and $o \in \operatorname{int} P$, $u \neq o$ and we may further assume that $\alpha = 1$. It follows that $u \in P^\circ = \operatorname{co}\{u_i \mid i \in I\}$, so there is a non-empty subset $J \subseteq I$ such that

$$u = \sum_{i \in J} \lambda_i u_i, \qquad \sum_{i \in J} \lambda_i = 1, \qquad \text{and} \qquad \lambda_i \in (0,1],\ i \in J.$$

It is now easy to see that a vector $x \in P$ satisfies $x \cdot u = 1$ if and only if $x \cdot u_i = 1$ for all $i \in J$. Hence $F = F_J$. ☐

Now we can describe the facial structure of the intersection of two polytopes. Intuitively, one might expect that the j-faces of the intersection of two polytopes P and P' are precisely the intersections of the i-faces of P with the $(n+j-i)$-faces of P'. The following lemma states that for almost all translates of P' that intersect P, this is the case.

Lemma 3.6. *Let $P, P' \in \mathcal{P}^n$ be two polytopes and let $j \in \{0, \dots, n\}$. Then for almost all $x \in \mathbb{R}^n$ for which $P \cap (P' + x) \neq \emptyset$, the mapping*

$$\Upsilon_x : \begin{cases} \bigcup_{i=j}^n \{(F, F') \in \mathcal{F}_i(P) \times \mathcal{F}_{n+j-i}(P') \mid F \cap (F' + x) \neq \emptyset\} & \to \quad \mathcal{F}_j(P \cap (P' + x)) \\ \qquad\qquad (F, F') & \mapsto \quad F \cap (F' + x) \end{cases}$$

is a well-defined bijection.

Proof. For convenience, we use the notation $P'_x := P' + x$, $F'_x := F' + x$. Note that, as a consequence of Lemma 3.5, the faces of $P \cap P'_x$ (if non-empty) are precisely the intersections of faces of P with faces of P'_x. Our strategy is now to define a set $Z \subseteq \mathbb{R}^n$ with $\lambda_n(Z) = 0$ and then show that Υ_x is a well-defined bijection for all $x \in \mathbb{R}^n \backslash Z$ with $P \cap P'_x \neq \emptyset$.

▷ Define A as the set of all $x \in \mathbb{R}^n$ such that

- $P \cap P'_x \neq \emptyset$, and
- for all faces $F \in \mathcal{F}(P)$, $F' \in \mathcal{F}(P')$ for which $F \cap F'_x \neq \emptyset$, it holds that $\operatorname{int} F \cap \operatorname{int} F'_x \neq \emptyset$, and that $\dim(F + F'_x) = n$.

We claim that the set $Z := \{x \in \mathbb{R}^n \mid P \cap P'_x \neq \emptyset\} \backslash A$ has Lebesgue measure zero. In order to see this, take $F \in \mathcal{F}(P)$, $F' \in \mathcal{F}(P')$ and observe that

$$\{x \in \mathbb{R}^n \mid F \cap F'_x \neq \emptyset\} = F - F'.$$

Hence, the set

$$\{x \in \mathbb{R}^n \mid F \cap F'_x \neq \emptyset, \dim(F + F'_x) < n\} \subseteq \operatorname{aff} F + \operatorname{aff} F'$$

is contained in a hyperplane and the set

$$\{x \in \mathbb{R}^n \mid F \cap F'_x \neq \emptyset, \operatorname{int} F \cap \operatorname{int} F' = \emptyset\} = (F - F') \backslash (\operatorname{int} F - \operatorname{int} F') = \operatorname{bd}(F - F')$$

is the relative boundary of a polytope. Since Z is the finite union of such sets, it has Lebesgue measure zero.

$\triangleright$ Next, we show that for $x \in A$, the mapping Υ_x is a well-defined bijection. In order to see the well-definedness, take $i \in \{j, \ldots, n\}$, $F \in \mathcal{F}_i(P)$ and $F' \in \mathcal{F}_{n+j-i}(P')$ such that $F \cap F'_x \neq \emptyset$. Then $\operatorname{int} F \cap \operatorname{int} F' \neq \emptyset$ and $\dim(F + F'_x) = n$. This implies that

$$\dim(F \cap F'_x) = \dim F + \dim F' - \dim(F + F'_x) = i + (n + j - i) - n = j,$$

and hence $F \cap F' \in \mathcal{F}_j(P \cap P'_x)$.

For showing surjectivity, take $\widehat{F} \in \mathcal{F}_j(P \cap P'_x)$. Then there are faces $F \in \mathcal{F}(P)$, $F' \in \mathcal{F}(P')$ such that $\widehat{F} = F \cap F'_x$. Since $F \cap F'_x \neq \emptyset$ it follows again that $\operatorname{int} F \cap \operatorname{int} F' \neq \emptyset$ and $\dim(F + F') = n$, implying that

$$\dim F + \dim F' = \dim(F + F'_x) + \dim(F \cap F'_x) = n + j.$$

Hence, $F \in \mathcal{F}_i(P)$ and $F' \in \mathcal{F}_{n+j-i}(P')$ for some $i \in \{j, \ldots, n\}$.

For showing injectivity, take $F, \tilde{F} \in \mathcal{F}(P)$ and $F', \tilde{F}' \in \mathcal{F}(P')$ such that

$$F \cap F' = \tilde{F} \cap \tilde{F}' = \widehat{F} \in \mathcal{F}_j(P \cap P'_x).$$

Note that whenever $K, L \in \mathcal{K}^n$ and $\operatorname{int} K \cap \operatorname{int} L \neq \emptyset$ then $\operatorname{int} K \cap \operatorname{int} L = \operatorname{int}(K \cap L)$. Hence,

$$\operatorname{int} F \cap \operatorname{int} F' = \operatorname{int} F' \cap \operatorname{int} \tilde{F}' = \operatorname{int} \widehat{F} \neq \emptyset.$$

In particular $\operatorname{int} F \cap \operatorname{int} F' \neq \emptyset$ and $\operatorname{int} \tilde{F} \cap \operatorname{int} \tilde{F}' \neq \emptyset$. However, this is only possible if $F = \tilde{F}$ and $F' = \tilde{F}'$. $\square$

We have already laid the basis for proving the desired integral representation formula for mean sections of polytopes, but we need to introduce some notion of angle between linear subspaces of $\mathbb{R}^n$.

Definition. Let E, E' be subspaces of $\mathbb{R}^n$ such that $E + E' = \mathbb{R}^n$. A *tilting map for* (E, E') is a linear map $f \in \mathrm{GL}(n)$ that

(i) acts as the identity on E, and

(ii) maps $E^\perp$ isometrically onto $E \cap (E \cap E')^\perp$.

Lemma 3.7. *Let E, E' be subspaces of $\mathbb{R}^n$ such that $E + E' = \mathbb{R}^n$. Then there exists a tilting map for (E, E').*

If f is a tilting map for (E, E') and $\tilde{f}$ is a tilting map for (E, E') or (E', E), then $f^{-1} \circ \tilde{f}$ is an isometry on $\mathbb{R}^n$.

Proof. Note that $\mathbb{R}^n$ admits the two decompositions

$$\mathbb{R}^n = E \oplus E^\perp = E \oplus [E' \cap (E \cap E')^\perp],$$

where the first one is even an orthogonal decomposition. Since the subspaces $E^\perp$ and $E' \cap (E \cap E')^\perp$ have the same dimension, there exists a linear isometric bijection g between them. Then $f := \mathrm{id}_E \oplus g$ is clearly a tilting map for (E, E').

If $f, \tilde{f}$ are both tilting maps for (E, E'), then $f^{-1} \circ \tilde{f}$ maps each of the subspaces E and $E^\perp$ isometrically onto itself. Since these subspaces make up an orthogonal decomposition of $\mathbb{R}^n$, the linear map $f^{-1} \circ \tilde{f}$ is an isometry on $\mathbb{R}^n$.

Now we consider the case where f is a tilting map for (E, E') and $\tilde{f}$ is a tilting map for (E', E). Note that we can decompose $\mathbb{R}^n$ further into

$$\mathbb{R}^n = [E \cap E'] \oplus [E' \cap (E \cap E')^\perp] \oplus E'^\perp = [E \cap E'] \oplus [E \cap (E \cap E')^\perp] \oplus E^\perp,$$

and these are both orthogonal decompositions. Obviously, $f^{-1} \circ \tilde{f}$ maps $E \cap E'$ isometrically onto itself. Moreover, $f^{-1} \circ \tilde{f}$ maps $E' \cap (E \cap E')^\perp$ isometrically onto $E^\perp$ and $E'^\perp$ isometrically onto $E \cap (E \cap E')^\perp$. Hence, the map $f^{-1} \circ \tilde{f}$ is an isometry on $\mathbb{R}^n$. □

Definition. For two subspaces E, E' of $\mathbb{R}^n$ such that $E + E' = \mathbb{R}^n$, we define $[E, E'] := |\det f|$, where f is a tilting map for (E, E').

Due to the previous lemma, this is a consistent definition and $[E, E'] = [E', E]$. We can now extend this notation to faces of polytopes. Also we introduce the notion of a common exterior angle for pairs of faces and pairs of polytopes.

Definition. For $P, P' \in \mathcal{P}^n$, $F \in \mathcal{F}_i(P)$, and $F' \in \mathcal{F}_j(P')$, such that $\dim(F + F') = n$, we define $[F, F'] := [E, E']$, where $E \in \mathrm{G}(n, i)$ is parallel to $\operatorname{aff} F$ and $E' \in \mathrm{G}(n, j)$ is parallel to $\operatorname{aff} F'$.

For $P, P' \in \mathcal{P}^n$, $F \in \mathcal{F}(P)$, and $F' \in \mathcal{F}(P')$, we define the *common exterior angle* as

$$\gamma(F, F', P, P') := \gamma(F \cap (F' + x), P \cap (P' + x)),$$

where $x \in \mathbb{R}^n$ is chosen in such a way that $\operatorname{int} F \cap \operatorname{int} F'_x \neq \emptyset$.

Moreover, for $P \in \mathcal{P}^n$, $F \in \mathcal{F}(P)$ and $u \in \mathbb{S}$, we define

$$\gamma(F, P, u) := \gamma(F, F', P, P'),$$

where an n-dimensional polytope $P' \in \mathcal{P}^n$ and a facet $F' \in \mathcal{F}_{n-1}(P')$ are chosen in such a way that u is an outer normal vector of P' at every $x \in \operatorname{int} F'$.

With these definitions and notations in place, we can formulate a translative integral geometric formula for polytopes.

Proposition 3.8 ([21, Theorem 4.4.3]). *For two polytopes $P, P' \in \mathcal{P}^n$ and $j = 0, \dots, n$, we have*

$$\int_{\mathbb{R}^n} \mathsf{V}_j(P \cap (P' + x))\, dx = \sum_{i=j}^{n} \sum_{\substack{F \in \mathcal{F}_i(P) \\ F' \in \mathcal{F}_{n+j-i}(P')}} \gamma(F, F', P, P')[F, F']\mathsf{V}_i(F)\mathsf{V}_{n+j-i}(F'). \tag{3.4}$$

Proof. For convenience, we use the notation $P'_x := P' + x$, $F'_x := F' + x$. Applying the representation of the intrinsic volumes of polytopes, the previous lemma, and the definition of the common external angle, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathsf{V}_j(P \cap P'_x)\, dx &= \sum_{\widehat{F} \in \mathcal{F}_j(P \cap P'_x)} \int_{\mathbb{R}^n} \gamma(\widehat{F}, P \cap P'_x)\mathsf{V}_j(\widehat{F})\, dx \\
&= \sum_{i=j}^{n} \sum_{\substack{F \in \mathcal{F}_i(P) \\ F' \in \mathcal{F}_{n+j-i}(P')}} \int_{\mathbb{R}^n} \gamma(F \cap F'_x, P \cap P'_x)\mathsf{V}_j(F \cap F'_x)\, dx \\
&= \sum_{i=j}^{n} \sum_{\substack{F \in \mathcal{F}_i(P) \\ F' \in \mathcal{F}_{n+j-i}(P')}} \gamma(F, F', P, P') I_j(F, F'),
\end{aligned}$$

where

$$I_j(F, F') := \int_{\mathbb{R}^n} \mathsf{V}_j(F \cap F'_x)\, dx.$$

It remains to show that $I_j(F, F') = [F, F']\mathsf{V}_i(F)\mathsf{V}_{n+j-i}(F')$. To that end, let us fix some $i \in \{j, \dots, n\}$, $F \in \mathcal{F}_i(P)$, and $F' \in \mathcal{F}_{n+j-i}(P')$. Since $I_j(F, F')$ is invariant under translations of F and F', we may assume that $o \in \operatorname{int} F \cap \operatorname{int} F'$.

In the case where $\dim(F + F') < n$, the set $\{x \in \mathbb{R}^n \mid F \cap F'_x \neq \emptyset\} = F - F' \subseteq E + E'$ has Lebesgue measure zero and so $I_j(F, F') = 0 = [F, F']\mathsf{V}_i(F)\mathsf{V}_{n+j-i}(F')$.

If $\dim(F + F') = n$, then $E + E' = \mathbb{R}^n$, where $E := \operatorname{lin} F$ and $E' := \operatorname{lin} E'$. Let us further define three spaces $E_0 := E \cap E'$, $E_1 := E \cap E_0^\perp$, and $E_2 := E' \cap E_0^\perp$. These make up an orthogonal decomposition of $\mathbb{R}^n$. Then, by applying a change of variables on $I_j(F,F')$ with respect to a tilting map for (E,E'), and Fubini's theorem, we obtain

$$I_j(F,F') = [F,F'] \int_{E_2} \int_{E_1} \int_{E_0} \mathsf{V}_j(F \cap (F' + y_0 + y_1 + y_2))\, dy_0\, dy_1\, dy_2.$$

The inner integral can be rewritten in such a form that we can apply the translative formula from Corollary 3.4 on $(F - y_1) \cap E_0$ and $(F' + y_1) \cap E_0$, considered as convex bodies in the j-dimensional Euclidean space E_0:

$$\begin{aligned}
&\int_{E_0} \mathsf{V}_j(F \cap (F' + y_0 + y_1 + y_2))\, dy_0 \\
&\quad = \int_{E_0} \mathsf{V}_j(\underbrace{(F - y_1)}_{\in E_0 + E_1} \cap \underbrace{(F' + y_0 + y_2)}_{\in E_0 + E_2})\, dy_0 \\
&\quad = \int_{E_0} \mathsf{V}_j(((F - y_1) \cap E_0) \cap ((F' + y_0 + y_2) \cap E_0))\, dy_0 \\
&\quad = \mathsf{V}_j((F - y_1) \cap E_0)\mathsf{V}_j((F' + y_2) \cap E_0).
\end{aligned}$$

If we plug this into the iterated integral above, and apply Fubini's theorem twice, we end up with

$$\begin{aligned}
I_j(F,F') &= [F,F'] \int_{E_1} \mathsf{V}_j((F - y_1) \cap E_0)\, dy_1 \int_{E_2} \mathsf{V}_j((F' + y_2) \cap E_0)\, dy_2 \\
&= [F,F']\mathsf{V}_i(F)\mathsf{V}_{n+j-i}(F')
\end{aligned}$$

which completes the proof. □

By combining the two translative integral geometric formulas (3.3) and (3.4), we can now derive a representation formula for mean section bodies of polytopes.

Theorem 3.9 ([11, Theorem 2]). *For $k \in \{1, \dots, n\}$ and a polytope $P \in \mathcal{P}^n$, there is a vector $w_k(P) \in \mathbb{R}^n$ such that*

$$h(M_k(P), u) = \frac{1}{\left[n \atop k \right]} \sum_{F \in \mathcal{F}_{n-k+1}(P)} \gamma(F, P, -u)[F, u^\perp]\mathsf{V}_{n-k+1}(F) + w_k(P) \cdot u, \qquad u \in \mathbb{S}.$$

Proof. Let $P' \in \mathcal{P}^n$ be another polytope. Then applying both translative formulas from Proposition 3.3 and Corollary 3.4 to P and $-P'$ gives

$$\begin{aligned}
&\sum_{i=j}^{n} \left[n \atop j \right] \binom{n}{i-j} \int_{\mathsf{A}(n,n-j)} \mathsf{V}((P \cap E)^{[i-j]}, (P')^{[n+j-i]})\, dE \\
&= \sum_{i=j}^{n} \sum_{\substack{F \in \mathcal{F}_i(P) \\ F' \in \mathcal{F}_{n+j-i}(P')}} \gamma(F, -F', P, -P')[F, -F']\mathsf{V}_i(F)\mathsf{V}_{n+j-i}(F').
\end{aligned}$$

Observe that the the i-th summands of these sums have homogeneity $(i, n+j-i)$ each, and thus must coincide. Choosing $j := n-k$ and $i := n-k+1$, we obtain

$$\begin{aligned} n \int_{\mathrm{A}(n,k)} &\mathsf{V}(P \cap E, -P', \dots, -P')\, dE \\ &= \frac{1}{\left[{n \atop k}\right]} \sum_{\substack{F \in \mathcal{F}_{n-k+1}(P) \\ F' \in \mathcal{F}_{n-1}(P')}} \gamma(F, -F', P, -P')[F, -F']\mathsf{V}_{n-k+1}(F)\mathsf{V}_{n-1}(F') \\ &= \int_{\mathbb{S}} \frac{1}{\left[{n \atop k}\right]} \sum_{F \in \mathcal{F}_{n-k+1}(P)} \gamma(F, P, -u)[F, u^{\perp}]\mathsf{V}_{n-k+1}(F)\, dS_{n-1}(P', u). \end{aligned}$$

Integrating the support function of $M_k(P)$ with respect to the surface area measure of P', applying Fubini's theorem, and applying the formula from above then yields

$$\begin{aligned} \int_{\mathbb{S}} &h(M_k(P), u)\, dS_{n-1}(P', u) \\ &= \int_{\mathbb{S}} \int_{\mathrm{A}(n,k)} h(P \cap E, u)\, dE\, dS_{n-1}(P', u) \\ &= \int_{\mathrm{A}(n,k)} \int_{\mathbb{S}} h(P \cap E, u)\, dS_{n-1}(P', u)\, dE \\ &= n \int_{\mathrm{A}(n,k)} \mathsf{V}(P \cap E, P', \dots, P')\, dE \\ &= \int_{\mathbb{S}} \frac{1}{\left[{n \atop k}\right]} \sum_{F \in \mathcal{F}_{n-k+1}(P)} \gamma(F, P, -u)[F, u^{\perp}]\mathsf{V}_{n-k+1}(F)\, dS_{n-1}(P', u). \end{aligned}$$

Because the choice of $P' \in \mathcal{P}^n$ was arbitrary and the space of surface area measures of polytopes is dense in $\mathbf{M}_o(\mathbb{S})$, the claim follows. □

It can be shown that in fact, $w_k(P) = o$. However, this is not necessary for our further reasoning.

3.2 Sections with Planes

From the description of mean section bodies of polytopes we can directly derive a description of 2-nd mean section bodies of general convex bodies. This is the content of the following theorem.

Theorem 3.10 ([11, Corollary 2]). *For every $K \in \mathcal{K}^n$,*

$$h(M_2(K), u) = \frac{1}{2\pi(n-1)} \int_{\mathbb{S}} (\pi - \arccos(u \cdot v))\sqrt{1 - (u \cdot v)^2}\, dS_{n-1}(-K, v) \qquad (3.5)$$

for all $u \in \mathbb{S}$.

Proof. This follows immediately from Theorem 3.9 and the description of the surface area measure of polytopes given in (2.2) by an argument of density and continuity. Note that we do not need the information that $w_k(P) = o$ since both sides of the stated formula have no linear component. □

3.3 Spherical Liftings and Projections

We aim to derive a formula for $M_k(K)$ from the formula (3.5) for $M_2(K)$, that is, to express general mean sections in terms of mean sections with planes. In order to do so, we need to relate convex geometry of subspaces to convex geometry of the ambient space. The technical tools to manage this are provided by so-called weighted spherical projections and liftings, which are the objects of investigation in this section.

Definition. For $k \in \{1, \dots, n\}$, $E \in \mathrm{G}(n,k)$, and $u \in \mathbb{S}$, we denote the $(n-k)$-dimensional half-sphere induced by E and u as

$$\mathbb{H}(E,u) := \{v \in \mathbb{S} \backslash E^\perp \mid \mathrm{pr}_E v = u\}.$$

Definition ([16, Section 3.3])**.** For $k \in \{1, \dots, n\}$, $E \in \mathrm{G}(n,k)$, and $m \geq 1-k$, the *m-weighted spherical projection operator* $\pi_{E,m}$ acts on a continuous function $f \in \mathbf{C}(\mathbb{S})$ by

$$(\pi_{E,m} f)(u) := \int_{\mathbb{H}(E,u)} f(v)(u \cdot v)^{k+m-1}\, dv, \qquad u \in \mathbb{S}(E).$$

Lemma 3.11. *For* $k \in \{1, \dots, n\}$*,* $E \in \mathrm{G}(n,k)$*, and* $m \geq 1-k$*,*

$$\pi_{E,m} : \mathbf{C}(\mathbb{S}) \to \mathbf{C}(\mathbb{S}(E))$$

is a bounded linear operator.

Proof. Let $f \in \mathbf{C}(\mathbb{S})$ and $\vartheta \in \mathrm{SO}(k, E^\perp) \cong \mathrm{SO}(E)$. Then, using that $\mathrm{pr}_E(\vartheta v) = \vartheta \mathrm{pr}_E v$ for all $v \in \mathbb{S} \backslash E^\perp$, we obtain

$$\begin{aligned}(\pi_{E,m} f)(\vartheta u) &= \int_{\mathbb{H}(E,\vartheta u)} f(v)(\vartheta u \cdot v)^{k+m-1}\, dv = \int_{\vartheta \mathbb{H}(E,u)} f(v)(\vartheta u \cdot v)^{k+m-1}\, dv \\ &= \int_{\mathbb{H}(E,u)} f(\vartheta v)(\vartheta u \cdot \vartheta v)^{k+m-1}\, dv = \int_{\mathbb{H}(E,u)} f(\vartheta v)(u \cdot v)^{k+m-1}\, dv\end{aligned}$$

for $u \in \mathbb{S}(E)$. It follows that

$$|(\pi_{E_m} f)(\vartheta u) - (\pi_{E,m} f)(u)| \leq \int_{\mathbb{H}(E,u)} |f(\vartheta v) - f(v)|\, dv \leq \frac{\omega_{n-k-1}}{2} \sup_{v \in \mathbb{S}} |f(\vartheta v) - f(v)|\,.$$

Since f is uniformly continuous on $\mathbb{S}$, the right hand side tends to zero for $\vartheta \to \mathrm{id}_{\mathrm{SO}(E)}$. This shows that $\pi_{E,m}f \in \mathbf{C}(\mathbb{S}(E))$.

The linearity of $\pi_{E,m}$ is obvious. Since

$$\|\pi_{E,m}f\|_{\mathbf{C}(\mathbb{S}(E))} \leq \sup_{u\in\mathbb{S}(E)} \int_{\mathbb{H}(E,u)} |f(v)| \; dv \leq \frac{\omega_{n-k-1}}{2}\|f\|_{\mathbf{C}(\mathbb{S})},$$

it follows that $\pi_{E,m}$ is a bounded linear operator. □

Definition ([16, Section 3.3])**.** For $k \in \{0,\dots,n\}$, $E \in \mathrm{G}(n,k)$, and $m \geq 1-k$, the adjoint operator

$$\pi^*_{E,m} : \mathbf{M}(\mathbb{S}(E)) \to \mathbf{M}(\mathbb{S})$$

of $\pi_{E,m}$ is called the *m-weighted spherical lifting operator.*

Due to the previous lemma, the m-weighted spherical lifting operator is a continuous linear operator with respect to the involved weak* topologies. In order to describe its action on continuous functions, we need the following integration formula for the unit sphere, which can easily be proven using spherical coordinates.

Lemma 3.12 ([27, Lemma 2.1])**.** *For $k \in \{1,\dots,n\}$, $E \in \mathrm{G}(n,k)$ and an integrable function $f : \mathbb{S} \to \mathbb{R}$, we have*

$$\int_{\mathbb{S}} f(u)\; du = \int_{\mathbb{S}(E)} \int_{\mathbb{H}(E,u)} f(v)(u\cdot v)^{k-1}\; dv\; du. \tag{3.6}$$

Lemma 3.13. *Let $k \in \{1,\dots,n\}$, $E \in \mathrm{G}(n,k)$, and $m \geq 1-k$.*

(i) *For $f \in \mathbf{C}(\mathbb{S}(E))$, we have*

$$(\pi^*_{E,m}f)(u) = \|u|E\|^m f(\mathrm{pr}_E u), \qquad u \in \mathbb{S}\backslash E^\perp. \tag{3.7}$$

(ii) *If $m > 0$, then*

$$\pi^*_{E,m} : \mathbf{C}(\mathbb{S}(E)) \to \mathbf{C}(\mathbb{S})$$

is a bounded linear operator.

Proof. For all $f \in \mathbf{C}(\mathbb{S}(E))$ and $g \in \mathbf{C}(\mathbb{S})$, we have

$$\begin{aligned}\langle \pi^*_{E,m}f, g\rangle = \langle f, \pi_{E,m}g\rangle &= \int_{\mathbb{S}}\int_{\mathbb{H}(E,u)} f(u)g(v)(u\cdot v)^{k+m-1}\; dv\; du\\ &= \int_{\mathbb{S}}\int_{\mathbb{H}(E,u)} \|v|E\|^m f(\mathrm{pr}_E v)g(v)(u\cdot v)^{k-1}\; dv\; du = \int_{\mathbb{S}} \|u|E\|^m f(\mathrm{pr}_E u)g(u)\; du\end{aligned}$$

due to (3.6). This shows that (3.7) holds almost everywhere. Since $\|\cdot|E\|^m f(\mathrm{pr}_E\ \cdot)$ is well-defined and continuous on $\mathbb{S}\backslash E^\perp$, we obtain (3.7).

If $m > 0$, then $\|\cdot|E\|^m f(\mathrm{pr}_E\ \cdot)$ extends to a continuous function on $\mathbb{S}$ and (3.7) yields $\|\pi^*_{E,m}f\|_{\mathbf{C}(\mathbb{S})} \leq \|f\|_{\mathbf{C}(\mathbb{S}(E))}$. □

A direct consequence of (3.7) is that for a convex body $K \in \mathcal{K}^k(E) \subseteq \mathcal{K}^n$, we have

$$h(K,\cdot) = \pi^*_{E,1} h^E(K,\cdot),$$

where $h^E(K,\cdot)$ denotes the support function of K relative to E.

We will now use the previous lemma on spherical liftings to obtain via duality information on spherical projections. To that end, define the following family of subspaces of $\mathbf{M}(\mathbb{S})$.

Definition. For $1-k \leq m \leq 0$, we define

$$\mathbf{M}_{E,m}(\mathbb{S}) := \left\{ \mu \in \mathbf{M}(\mathbb{S}) \;\middle|\; |\mu|\,(\mathbb{S}(E^\perp)) = 0,\ \int_{\mathbb{S}\setminus E^\perp} \|u|E\|^m \, d\,|\mu|\,(u) < +\infty. \right\}$$

For $m > 0$, we define $\mathbf{M}_{E,m}(\mathbb{S}) := \mathbf{M}(\mathbb{S})$.

If $1-k \leq m \leq 0$, $\mu \in \mathbf{M}_{E,m}(\mathbb{S})$, and $f \in \mathbf{C}(\mathbb{S}(E))$, then due to (3.7), we have

$$\left|\langle \pi^*_{E,m} f, \mu\rangle\right| = \left| \int_{\mathbb{S}\setminus E^\perp} \|u|E\|^m f(\mathrm{pr}_E u)\; d\mu(u) \right| \leq \|f\|_{\mathbf{C}(\mathbb{S}(E))} \int_{\mathbb{S}\setminus E^\perp} \|u|E\|^m \; d\,|\mu|\,(u).$$

Hence, for $m \geq 1-k$, due to the previous lemma, the m-weighted spherical projection operator extends to a linear operator

$$\pi_{E,m} : \mathbf{M}_{E,m}(\mathbb{S}) \to \mathbf{M}(\mathbb{S}(E)),$$
$$\langle f, \pi_{E,m}\mu\rangle_{\mathbb{S}(E)} := \langle \pi^*_{E,m} f, \mu\rangle_{\mathbb{S}}, \qquad f \in \mathbf{C}(\mathbb{S}(E)).$$

For $m > 0$, we even obtain a bounded linear operator

$$\pi_{E,m} : \mathbf{M}(\mathbb{S}) \to \mathbf{M}(\mathbb{S}(E))$$

with respect to the involved total variation norms.

From operators that are called liftings and projections, one might intuitively expect some associativity property in the sense that projecting onto a subspace and then projecting from this subspace onto a smaller subspace is the same as projecting onto this smaller subspace directly. The following lemma states that this intuitive expectation is indeed met by spherical lifting and projection operators.

Lemma 3.14 ([16, Lemma 3.12])**.** *Let $0 < k \leq k' \leq n$, $E' \in \mathrm{G}(n,k')$, $E \in \mathrm{G}(E',k)$, and $m \geq 1-k$.*

(i) *If $\mu \in \mathbf{M}_{E,m}(\mathbb{S})$, then also $\mu \in \mathbf{M}_{E',m}(\mathbb{S})$.*

(ii) *If $\mu \in \mathbf{M}_{E',m}(\mathbb{S})$ and $\pi_{E',m}\mu \in \mathbf{M}_{E,m}(\mathbb{S}(E'))$, then also $\mu \in \mathbf{M}_{E,m}(\mathbb{S})$ and*

$$\pi^{E'}_{E,m}\pi_{E',m}\mu = \pi_{E,m}\mu. \tag{3.8}$$

(iii) *If* $\mu \in \mathbf{M}(\mathbb{S}(E))$*, then*

$$\pi^*_{E',m}(\pi^{E'}_{E,m})^*\mu = \pi^*_{E,m}\mu. \tag{3.9}$$

Here and in the following, $\pi^{E'}_{E,m} : \mathbf{M}_{E,m}(\mathbb{S}(E')) \to \mathbf{M}(\mathbb{S}(E))$ *denotes the* m*-weighted spherical projection operator relative to* E'*.*

Proof. We start with proving (iii). For all $f \in \mathbf{C}(\mathbb{S}(E))$ and $u \in \mathbb{S}\backslash E'^{\perp}$, we have

$$\begin{aligned}(\pi^*_{E',m}(\pi^{E'}_{E,m})^* f)(u) &= \|u|E\|^m((\pi^{E'}_{E,m})^* f)(\mathrm{pr}_{E'}u)\\ &= \|u|E\|^m\|(\mathrm{pr}_{E'}u)|E\|^m f(\mathrm{pr}_E(\mathrm{pr}_{E'}u)) = \|u|E\|^m f(\mathrm{pr}_E u) = (\pi^*_{E,m}f)(u),\end{aligned}$$

so (3.9) holds on $\mathbf{C}(\mathbb{S}(E))$. All of the spherical liftings involved are continuous with respect to the w^*-topologies involved and $\mathbf{C}(\mathbb{S}(E))$ is dense in $\mathbf{M}(\mathbb{S}(E))$ with respect to the w^*-topology, hence (3.9) follows.

Next, we show (ii). To that end, let $\mu \in \mathbf{M}_{E'm}(\mathbb{S})$ such that $\pi_{E',m}\mu \in \mathbf{M}_{E,m}(\mathbb{S}(E'))$. For an arbitrary $f \in \mathbf{C}(\mathbb{S}(E))$, we can apply (3.9) to obtain

$$\begin{aligned}\langle f, \pi^{E'}_{E,m}\pi_{E',m}\mu\rangle_{\mathbb{S}(E)} &= \langle(\pi^{E'}_{E,m})^* f, \pi_{E',m}\mu\rangle_{\mathbb{S}(E')}\\ &= \langle \pi^*_{E',m}(\pi^{E'}_{E,m})^* f, \mu\rangle_{\mathbb{S}}\\ &= \langle \pi^*_{E,m} f, \mu\rangle_{\mathbb{S}}\\ &= \int_{\mathbb{S}} \|u|E\|^m f(\mathrm{pr}_E u)\, d\mu(u).\end{aligned}$$

In particular, the integral in the last line exists for all $f \in \mathbf{C}(\mathbb{S}(E))$. If $m \leq 0$, this shows that $\mu \in \mathbf{M}_{E,m}(\mathbb{S})$. If $m > 0$, this is clear anyway. In both cases, we obtain

$$\langle f, \pi^{E'}_{E,m}\pi_{E',m}\mu\rangle_{\mathbb{S}(E)} = \langle \pi^*_{E,m} f, \mu\rangle_{\mathbb{S}} = \langle f, \pi_{E,m}\mu\rangle_{\mathbb{S}(E)}$$

for all $f \in \mathbf{C}(\mathbb{S}(E))$, proving (3.8).

Lastly, we show (i). The claim is trivial for $m > 0$. Thus, we consider some $\mu \in \mathbf{M}_{E,m}(\mathbb{S})$ for $m \leq 0$. Then $|\mu|\,(\mathbb{S}(E'^{\perp})) \leq |\mu|\,(\mathbb{S}(E^{\perp})) = 0$ due to the monotonicity of $|\mu|$, and

$$\int_{\mathbb{S}\backslash E'^{\perp}} \|u|E'\|^m\, d\,|\mu|\,(u) = \int_{\mathbb{S}\backslash E^{\perp}} \|u|E'\|^m\, d\,|\mu|\,(u) \leq \int_{\mathbb{S}\backslash E^{\perp}} \|u|E\|^m\, d\,|\mu|\,(u) < +\infty.$$

Hence, $\mu \in \mathbf{M}_{E',m}(\mathbb{S})$. □

Area measures of convex bodies are non-intrinsic in the sense that for a convex body K lying in a proper subspace $E \subseteq \mathbb{R}^n$, its j-tharea measures $S^E_j(K,\cdot)$ relative to E in general does not coincide with its j-th area measures $S_j(K,\cdot)$ relative to the ambient space. The following lemma shows how the area measures $S^E_j(K,\cdot)$ and $S_j(K,\cdot)$ are related via weighted spherical liftings.

Lemma 3.15 ([16, Lemma 3.15]). *Let* $k \in \{1, \dots, n\}$, $E \in \mathrm{G}(n,k)$, $K \in \mathcal{K}^k(E)$, *and* $j \in \{0, \dots, k-1\}$. *Then*

$$S_j(K, \cdot) = \frac{\binom{k-1}{j}}{\binom{n-1}{j}} \pi^*_{E,-j} S^E_j(K, \cdot). \tag{3.10}$$

Proof. Since $S_j(K, \cdot)$ and $S^E_j(K, \cdot)$ depend both continuously on $K \in \mathcal{K}(E)$ with respect to the weak* topology and $\pi^*_{E,-j} : \mathbf{M}(\mathbb{S}(E)) \to \mathbf{M}(\mathbb{S})$ is continuous with respect to the weak* topologies involved, it suffices to prove the claim for polytopes.

In order to do so, let $P \in \mathcal{P}(E)$ and note that

$$S^E_j(P, \cdot) = \frac{k\kappa_{k-j}}{\binom{k}{j}} \sum_{F \in \mathcal{F}_j(P)} \frac{\mathcal{H}^{k-j-1}|_{N(E;F,P)}}{\omega_{k-j}} \mathsf{V}_j(F)$$

and

$$S_j(P, \cdot) = \frac{n\kappa_{n-j}}{\binom{n}{j}} \sum_{F \in \mathcal{F}_j(P)} \frac{\mathcal{H}^{n-j-1}|_{N(F,P)}}{\omega_{n-j}} \mathsf{V}_j(F),$$

where $N(E; F, P)$ denotes the normal cone of the face $F \in \mathcal{F}_j(P)$ relative to E. Note that $N(F, P) = N(E; F, P) + E^\perp$.

We define $E' := \operatorname{lin} N(E; F, P) \subseteq E$. Then for any $F \in \mathcal{F}_j(P)$ and $f \in \mathbf{C}(\mathbb{S})$, we have

$$\begin{aligned}
\langle f, \pi^*_{E,-j}(\mathcal{H}^{k-j-1}|_{N(E;F,P)}) \rangle_{\mathbb{S}} &= \int_{N(E;F,P)} (\pi_{E,-j} f)(v) \; d\mathcal{H}^{k-j-1}(v) \\
&= \int_{\mathbb{S}(E')} (\pi_{E,-j} f)(v) \mathbb{1}_{N(E;F,P)}(v) \; dv \\
&= \int_{\mathbb{S}(E')} \int_{\mathbb{H}(E,v)} f(y) \mathbb{1}_{N(E;F,P)}(v) (v \cdot y)^{k-j-1} \; dy \; dv \\
&= \int_{\mathbb{S}(E')} \int_{\mathbb{H}(E,v)} f(y) \mathbb{1}_{N(F,P)}(y) (v \cdot y)^{k-j-1} \; dy \; dv.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\langle f, \mathcal{H}^{n-j-1}|_{N(F,P)} \rangle_{\mathbb{S}} &= \int_{N(F,P)} f(u) \; d\mathcal{H}^{n-j-1}(u) \\
&= \int_{\mathbb{S}(E' \oplus E^\perp)} f(v) \mathbb{1}_{N(F,P)}(v) \; dv \\
&= \int_{\mathbb{S}(E')} \int_{\mathbb{H}(E' \oplus E^\perp; E', v)} f(y) \mathbb{1}_{N(F,P)}(y) (v \cdot y)^{k-j-1} \; dy \; dv,
\end{aligned}$$

where $\mathbb{H}(E' \oplus E^\perp; E', v) := \{y \in \mathbb{S}(E' \oplus E^\perp) \backslash E^\perp \mid \mathrm{pr}_E y = v\}$. It is easy to see that $\mathbb{H}(E' \oplus E^\perp; E', v) = \mathbb{H}(E, v)$ for every $v \in \mathbb{S}(E')$. Then the statement follows from the linearity of $\pi^*_{E,-j}$. $\square$

The following lemma achieves for mean section bodies what the previous one did for area measures of convex bodies. The case where $j = n - k + 2$ will be of particular interest for us, since it directly relates the k-th mean section body to the 2-nd mean section body relative to a subspace.

Lemma 3.16 ([13, Lemma 3.3])**.** *If $j \in \{0, \dots, n\}$, $E \in \mathrm{A}(n, j)$, $C \in \mathcal{K}^j(E)$, $k \in \{n - j, \dots, n\}$, then*

$$h(M_k(C), u) = \frac{\left[{k \atop k+j-n}\right]}{\left[{n \atop j}\right]} [\pi^*_{E,1} h(M^E_{k+j-n}(C), \cdot)](u), \tag{3.11}$$

where $M^E_{k+j-n}(C)$ denotes the $(k + j - n)$-th centered mean section body of C relative to E.

Proof. Due to (1.3), we have

$$\begin{aligned} h(M_k(C), u) &= \int_{\mathrm{A}(n,k)} h_o(C \cap F, u)\, dF \\ &= \int_{\mathrm{A}(n,k)} h_o(C \cap (E \cap F), u)\, dF \\ &= \frac{\left[{k \atop k+j-n}\right]}{\left[{n \atop j}\right]} \int_{\mathrm{A}(E,k+j-n)} h_o(C \cap F)\, dF \\ &= \frac{\left[{k \atop k+j-n}\right]}{\left[{n \atop j}\right]} [\pi^*_{E,1} h(M^E_{k+j-n}(C), \cdot)](u). \end{aligned}$$

□

The following lemma entails that weighted spherical projections of linear functions are again linear functions.

Lemma 3.17 ([13, Lemma 2.3])**.** *Let $k \in \{1, \dots, n\}$, $E \in \mathrm{G}(n, k)$, and $m \geq 1 - k$. Then for every $u \in \mathbb{S}$ and $v \in \mathbb{S}(E)$, we have*

$$[\pi_{E,m}(u \cdot .)](v) = \frac{\omega_{n+m+1}}{\omega_{k+m+1}} u \cdot v = \frac{\omega_{n+m+1}}{\omega_{k+m+1}} \|u|E\| \mathrm{pr}_E u \cdot v.$$

Proof. We fix $m \geq 1 - k$ and prove the statement by a downward induction on $k \in \{1 - m, \dots, n\}$.

The base case, where $k = n$, is trivial.

For the induction step, let $k > 1 - m$ and $E \in \mathrm{G}(n, n - k - 1)$. Then we can choose $E' \in \mathrm{G}(n, k)$ and $u_0 \in \mathbb{S}(E')$ such that $E = E' \cap u_0^\perp$. By our induction hypothesis,

$[\pi_{E',m}(u\cdot .)](v) = c_{n,k} u\cdot v$, where $c_{n,k} := \frac{\omega_{n+m+1}}{\omega_{k+m+1}}$. Then Lemma 3.14 (ii) yields

$$\begin{aligned}
[\pi_{E,m}(u\cdot .)](v) &= [\pi^{E'}_{E,m}\pi_{E',m}(u\cdot .)](v)\\
&= c_{n,k}[\pi^{E'}_{E,m}(u\cdot .)](v)\\
&= c_{n,k}\int_{\mathbb{H}(E';E,v)} (u\cdot y)(v\cdot y)^{k+m-2}\, dy\\
&= c_{n,k}\int_{-1}^{1} (tu\cdot u_0 + \sqrt{1-t^2}u\cdot v)(1-t^2)^{(k+m-3)/2}\, dt\\
&= c_{n,k}\int_{-1}^{1} (1-t^2)^{(k+m-4)/2}\, dt\; u\cdot v.
\end{aligned}$$

The fact that $\int_{-1}^{1}(1-t^2)^{(k+m-4)/2}\, dt = \frac{\omega_{k+m+1}}{\omega_{k+m}}$ concludes the argument. □

We now combine the concepts of weighted spherical lifting and projection operators to obtain a new kind of operator. Imagine taking a function $f\in \mathbf{C}(\mathbb{S})$, for every k-dimensional subspace E projecting it onto $\mathbf{C}(\mathbb{S}(E))$, lifting it back up to $\mathbf{C}(\mathbb{S})$, and then averaging all of these lifted projections over $E\in \mathrm{G}(n,k)$. This is essentially the idea of the following definition:

Definition ([16])**.** For $k\in\{1,\dots,n\}$, $E\in \mathrm{G}(n,k)$, $m>0$, and $j\geq 1-k$, the *mean lifted projection operator* $\pi^{[k]}_{m,j}$ acts on a continuous function $f\in\mathbf{C}(\mathbb{S})$ by

$$(\pi^{[k]}_{m,j}f)(u) := \int_{\mathrm{G}(n,k)} (\pi^{*}_{E,m}\pi_{E,j}f)(u)\, dE, \qquad u\in\mathbb{S}.$$

In order to see that this a sensible definition, note that

$$(\pi^{*}_{\vartheta E,m}\pi_{\vartheta E,j}\vartheta f)(\vartheta u) = (\pi^{*}_{E,m}\pi_{E,j}f)(u)$$

for $\vartheta\in \mathrm{SO}(n)$. From this fact, one can easily deduce that the integrand depends continuously on $E\in\mathrm{G}(n,k)$ and the resulting function $\pi^{[k]}_{m,j}f$ depends continuously on $u\in\mathbb{S}$, so $\pi^{[k]}_{m,j}$ is a well-defined linear operator on $\mathbf{C}(\mathbb{S})$. Moreover, it is easy to see that $\pi^{[k]}_{m,j}$ is $\mathrm{SO}(n)$-equivariant and also bounded, since $\pi_{E,j}$ and $\pi^{*}_{E,m}$ are bounded. In conclusion, the mean lifted projection operator

$$\pi^{[k]}_{m,j} : \mathbf{C}(\mathbb{S})\to\mathbf{C}(\mathbb{S})$$

is an $\mathrm{SO}(n)$-equivariant bounded linear operator. Later on, we will be particularly interested in some of its injectivity properties.

Theorem 3.18 ([23, Theorem 5.1])**.** *If $T:\mathbf{C}(\mathbb{S})\to\mathbf{C}(\mathbb{S})$ is an $\mathrm{SO}(n)$-equivariant bounded linear operator, then $T = .*\mu$ for some $\mu\in\mathbf{M}(\mathbb{S},\overline{e})$ and T is a multiplier transform.*

Proof. Define $\mu := T^*\delta_{\overline{e}} \in \mathbf{M}(\mathbb{S})$. Observe that the adjoint operator T^* is also rotation intertwining, so if $\vartheta \in \mathrm{SO}(n-1, \overline{e})$, then

$$\vartheta\mu = \vartheta T^*\delta_{\overline{e}} = T^*\vartheta\delta_{\overline{e}} = T^*\delta_{\overline{e}} = \mu,$$

since $\delta_{\overline{e}}$ is obviously zonal. Hence, μ is zonal as well.

For every $f \in \mathbf{C}(\mathbb{S})$ and $\vartheta \in \mathrm{SO}(n)$, we have

$$(Tf)(\vartheta\overline{e}) = \langle Tf, \vartheta\delta_{\overline{e}}\rangle = \langle f, T^*\vartheta\delta_{\overline{e}}\rangle = \langle f, \vartheta T^*\delta_{\overline{e}}\rangle = \langle f, \vartheta\mu\rangle = (f * \mu)(\vartheta\overline{e}),$$

proving $T = . * \mu$.

Since all convolution operators commute, $T\pi_k = \pi_k T$, so T maps the space $\mathcal{H}_k^n$ of spherical harmonics of degree k into itself. Because of its $\mathrm{SO}(n)$-equivariance, the operator T maps zonal functions to zonal functions. Due to Theorem 1.6, the subspace of $\mathcal{H}_k^n$ consisting of zonal spherical harmonics of degree k is one-dimensional, hence T has an eigenfunction $f_k \in \mathcal{H}_k^n$. Denote by λ_k the associated eigenvalue. Then the space

$$V_k := \mathcal{H}_k^n \cap \ker(T - \lambda_k) = \{f \in \mathcal{H}_k^n \mid Tf = \lambda_k f\}$$

is a non-trivial $\mathrm{SO}(n)$-invariant subspace of $\mathcal{H}_k^n$. Due to Theorem 1.4, it follows that $V_k = \mathcal{H}_k^n$, which means precisely that $Tf = \lambda_k f$ for all $f \in \mathcal{H}_k^n$. □

Due to this theorem and our previous considerations, the mean lifted projection operator $\pi_{m,j}^{[k]}$ can be expressed as a convolution operator $f \mapsto f * \nu_{m,j}^{[k]}$, and it is a multiplier transform. Consequently, it is self adjoint in the sense that $\langle \pi_{m,j}^{[k]} f, g\rangle = \langle f, \pi_{m,j}^{[k]} g\rangle$ for all $f, g \in \mathbf{C}(\mathbb{S})$. At the same time, a simple computation shows that $\langle \pi_{m,j}^{[k]} f, g\rangle = \langle f, \pi_{j,m}^{[k]} g\rangle$ for all $f, g \in \mathbf{C}(\mathbb{S})$. Hence,

$$\pi_{m,j}^{[k]} = \pi_{j,m}^{[k]}.$$

Moreover, the mean lifted projection operator extends to an $\mathrm{SO}(n)$-equivariant continuous linear operator

$$\pi_{m,j}^{[k]} : \mathbf{M}(\mathbb{S}) \to \mathbf{M}(\mathbb{S}) : \mu \mapsto \mu * \nu_{m,j}^{[k]}.$$

We now aim to compute this signed measure $\nu_{m,j}^{[k]}$ at play (which will turn out to be a continuous function). To that end, we need two auxiliary lemmas and fix the notations $E \vee v := \mathrm{lin}(E \cup \{v\})$ and $u \vee v := \mathrm{lin}\{u, v\}$ for a subspace $E \subseteq \mathbb{R}^n$ and vectors $u, v \in \mathbb{R}^n$.

Lemma 3.19 ([7, Equation 28])**.** *Let $k \in \{1, \dots, n\}$, $u \in \mathbb{S}$, and $h \in \mathbf{C}(\mathrm{G}(n,k))$. Then*

$$\int_{\mathrm{G}(n,k)} h(E)\, dE = \frac{\omega_k}{2\omega_n} \int_{\mathrm{G}(u^\perp, k-1)} \int_{\mathbb{S}(E^\perp)} h(E \vee v)\, |u \cdot v|^{k-1}\, dv\, dE. \tag{3.12}$$

Lemma 3.20 ([16, Lemma 3.17])**.** *Let* $k \in \{1, \dots, n\}$*,* $u \in \mathbb{S}$*, and* $g \in \mathbf{C}(\mathbb{S})$*. Then*

$$\int_{\mathbb{S}} g(v)\, dv = \frac{\omega_{n-1}}{\omega_{n-k-1}} \int_{\mathrm{G}(u^\perp,k-1)} \int_{\mathbb{S}(E^\perp)} g(v)(1-(u\cdot v)^2)^{(k-1)/2}\, dv\, dE. \tag{3.13}$$

Proof. For fixed $t \in (-1,1)$, we have

$$\int_{\mathrm{G}(u^\perp,k-1)} \int_{\mathbb{S}(E^\perp\cap u^\perp)} g(tu+\sqrt{1-t^2}y)\, dy\, dE = \frac{\omega_{n-k-1}}{\omega_{n-1}} \int_{\mathbb{S}(u^\perp)} g(tu+\sqrt{1-t^2}v)\, dv$$

due to the uniqueness of the $\mathrm{SO}(n)$-invariant Borel measure on $\mathbb{S}(u^\perp)$ up to a multiplicative constant. Hence, using Fubini's theorem and spherical cylinder coordinates, we obtain

$$\begin{aligned}
&\int_{\mathrm{G}(u^\perp,k-1)} \int_{\mathbb{S}(E^\perp)} g(y)(1-(u\cdot v)^2)^{(k-1)/2}\, dv\, dE \\
&\quad= \int_{-1}^{1} \int_{\mathrm{G}(u^\perp,k-1)} \int_{\mathbb{S}(E^\perp\cap u^\perp)} g(tu+\sqrt{1-t^2}w)\, dy\, dE\, (1-t^2)^{(n-4)/2} dt \\
&\quad= \frac{\omega_{n-k-1}}{\omega_{n-1}} \int_{-1}^{1} \int_{\mathbb{S}(u^\perp)} g(tu+\sqrt{1-t^2}v)\, dv\, (1-t^2)^{(n-4)/2} dt \\
&\quad= \frac{\omega_{n-k-1}}{\omega_{n-1}} \int_{\mathbb{S}} g(v)\, dv
\end{aligned}$$

which completes the proof. □

We are now able to describe $\pi^{[k]}_{m,j}$ as a convolution operator.

Lemma 3.21 ([16, Satz 3.18])**.** *Let* $k \in \{1, \dots, n\}$*,* $E \in \mathrm{G}(n,k)$*,* $m > 0$*,* $j \geq 1-k$*, and* $f \in \mathbf{C}(\mathbb{S})$*. Then*

$$\pi^{[k]}_{m,j}\mu = \alpha_{n,k,m,j}\ \mu * \breve{A}^{[k]}_{m,j}, \tag{3.14}$$

where

$$A^{[k]}_{m,j}(t) := (1-t^2)^{(1-k)/2} \int_0^{\pi-\arccos t} \sin(\theta)^{k+m-1} \sin(\theta+\arccos t)^{k+j-1}\, d\theta$$

and

$$\alpha_{n,k,m,j} := \frac{\omega_{n+k+m+j-1}\omega_{n-k-1}\omega_k}{\omega_{2k+m+j}\omega_n\omega_{n-1}}.$$

Proof. Due to (3.13), we obtain

$$\begin{aligned}
(\pi^{[k]}_{m,j} f)(u) &= \int_{\mathrm{G}(n,k)} (\pi^*_{E,m}\pi_{E,j} f)(u)\, dE \\
&= \frac{\omega_k}{2\omega_n} \int_{\mathrm{G}(u^\perp,k-1)} \int_{\mathbb{S}(E^\perp)} (\pi^*_{E\vee v,m}\pi_{E\vee v,j} f)(u)\, |u\cdot v|^{k-1}\, dv\, dE \\
&= \frac{\omega_k}{\omega_n} \int_{\mathrm{G}(u^\perp,k-1)} \int_{\mathbb{S}(E^\perp)} \mathbb{1}_{u^+}(v)(\pi^*_{E\vee v,m}\pi_{E\vee v,j} f)(u)(u\cdot v)^{k-1}\, dv\, dE.
\end{aligned}$$

For the integrand, we have

$$(\pi^*_{E\vee v,m}\pi_{E\vee v,j}f)(u) = \|u|(E\vee v)\|^m \int_{\mathbb{H}(E\vee v,\mathrm{pr}_{E\vee v}u)} f(y)(\mathrm{pr}_{E\vee v}u\cdot y)^{k+j-1}\,dy$$
$$= (u\cdot v)^m \int_{\mathbb{S}(E^\perp)} \mathbb{1}_{v^+}(y)f(y)(v\cdot y)^{k+j-1}\,dy,$$

where $v^+ := \{y\in\mathbb{R}^n \mid v\cdot y\geq 0\}$. Hence Fubini's theorem yields

$$(\pi^{[k]}_{m,j}f)(u) = \frac{\omega_k}{\omega_n}\int_{\mathrm{G}(u^\perp,k-1)}\int_{\mathbb{S}(E^\perp)} f(y)I(E,u,y)\,dy\,dE,$$

where

$$I(E,u,y) := \int_{\mathbb{S}(E^\perp)} \mathbb{1}_{u^+\cap y^+}(v)(u\cdot v)^{k+m-1}(y\cdot v)^{k+j-1}\,dv$$

for $E\in\mathrm{G}(u^\perp,k-1)$ and $y\in\mathbb{S}(E^\perp)$. Due to (3.6), we obtain

$$I(E,u,y) = \int_{\mathbb{S}(u\vee y)}\int_{\mathbb{H}(E^\perp;u\vee y,w)} \mathbb{1}_{u^+\cap y^+}(z)(u\cdot z)^{k+m-1}(y\cdot z)^{k+j-1}(w\cdot z)\,dz\,dw.$$

Take $w\in\mathbb{S}(u\vee y)$ and $w\in\mathbb{H}(E^\perp;u\vee y,w)$. Then clearly $z|(u\vee v) = (z\cdot w)w$, and so $z|(u\vee v)^\perp = z-(z\cdot w)w$. In particular, $z-(z\cdot w)w$ is perpendicular to u and y, which yields

$$u\cdot z = (w\cdot z)(u\cdot w) \qquad \text{and} \qquad y\cdot z = (z\cdot w)(y\cdot w).$$

Moreover, since $z\cdot w>0$, it follows that $z\in u^+\cap y^+$ if and only if $w\in u^+\cap y^+$. We obtain

$$I(E,u,y) = \int_{\mathbb{S}(u\vee y)\cap u^+\cap y^+}(u\cdot w)^{k+m-1}(y\cdot w)^{k+j-1}\int_{\mathbb{H}(E^\perp;u\vee y,w)}(w\cdot z)^{2k+m+j-1}\,dz\,dw.$$

The value of the inner integral is independent of w and can be computed to be

$$\int_{\mathbb{H}(E^\perp;u\vee y,w)}(w\cdot z)^{2k+m+j-1}\,dz = \frac{\omega_{n+k+m+j-1}}{\omega_{2k+m+j}}$$

with the help of (3.6). Thus, we end up with an integral over $\mathbb{S}(u\vee y)\cap u^+\cap y^+$, which is just a circular arc. A parametrization of $\mathbb{S}(u\vee y)\cap u^+\cap y^+$ is given by

$$(0,\pi-\angle(u,y))\ni\theta\mapsto(\cos\theta)(y-(y\cdot u)u)+(\sin\theta)u,$$

where $\angle(u,y) := \arccos(u\cdot y)$ denotes the angle between u and y. By a change of variables, we obtain

$$\int_{\mathbb{S}(u\vee y)\cap u^+\cap y^+}(u\cdot w)^{k+m-1}(y\cdot w)^{k+j-1}\,dw = \int_0^{\pi-\angle(u,y)}(\sin\theta)^{k+m-1}\sin(\theta+\angle(u,y))^{k+j-1}\,d\theta.$$

In conclusion,

$$\begin{aligned}(\pi_{m,j}^{[k]}f)(u) &= \frac{\omega_k\omega_{n+k+m+j-1}}{\omega_n\omega_{2k+m+j}}\int_{\mathrm{G}(u^\perp,k-1)}\int_{\mathbb{S}(E^\perp)} f(y)B_{m,j}^{[k]}(u\cdot y)(1-(u\cdot y)^2)^{(k-1)/2}\,dy\,dE\\ &= \alpha_{n,k,m,j}\int_{\mathbb{S}} f(v)B_{m,j}^{[k]}(u\cdot v)\,dv\end{aligned}$$

which completes the proof. □

Now that we have expressed the mean lifted projection operator explicitly as a convolution operator with a continuous function, we are able to employ the Funk-Hecke theorem for the following injectivity result:

Proposition 3.22 ([16, Satz 3.20]). *Let $k \in \{2,\dots,n-1\}$. Then $\pi_{1,1-k}^{[k]}$ is injective on* $\mathbf{C}(\mathbb{S})$.

Proof. Due to Theorem 3.18, the operator $\pi_{1,1-k}^{[k]}$ is a multiplier transform, so in order to show that it is injective, it suffices to show that all multipliers are non-zero. The Funk-Hecke theorem provides an explicit description of the multipliers by means of the representation of $\pi_{1,1-k}^{[k]}$ given in (3.14). Consequently, we will show injectivity by constructing a sequence equivalent to the multipliers, deducing a recursion formula for this sequence, and then dealing with the initial values of this sequence.

▷ Note that in the special case of the previous lemma, where $m = 1$ and $j = 1-k$,

$$A_{k,1-k}^{[k]}(t) = (1-t^2)^{(1-k)/2}\int_0^{\pi-\arccos t}(\sin\theta)^k\,d\theta = (1-t^2)^{(1-k)/2}F_k(t),$$

where

$$F_k(t) := \int_0^{\pi-\arccos t}(\sin\theta)^k\,d\theta = \int_{-1}^{t}(1-s^2)^{(k-1)/2}\,ds.$$

For our purposes, it is practical to rescale the Legendre polynomials to

$$C_j^{(n-2)/2} := \binom{n+j-2}{j}P_j^n.$$

In fact, these polynomials are also known as the Gegenbauer polynomials and can be defined more broadly. For simplicity we fix the notation $\nu := \frac{n-2}{2}$. The Gegenbauer polynomials satisfy the following recursions (cf. [8, §10.9]):

$$(j+1)C_{j+1}^\nu(t) = (n+2j-2)tC_j^\nu(t) - (n+j-3)C_{j-1}^\nu(t), \tag{3.15}$$

$$(1-t^2)\frac{d}{dt}C_j^\nu(t) = (n+j-2)tC_j^\nu(t) - (j+1)C_{j+1}^\nu(t). \tag{3.16}$$

Due to the Funk-Hecke theorem, as we have argued above, it suffices to show that the numbers

$$a_j := [A_{k,1-k}^{[k]}, C_j^{\nu}]_n = \int_{-1}^{1} F_k(t) C_j^{\nu}(t)(1-t^2)^{(n-k-2)/2} \, dt$$

are non-zero for $j \in \mathbb{N}_0$.

▷ From (3.15) we obtain

$$(j+1)a_{j+1} = (n+2j-2)\int_{-1}^{1} F_k(t) C_j^{\nu}(t) t (1-t^2)^{(n-k-2)/2} \, dt - (n+j-3)a_{j-1}. \quad (3.17)$$

The involved integral can be rewritten as

$$\begin{aligned} I_j &:= \int_{-1}^{1} F_k(t) C_j^{\nu}(t) t (1-t^2)^{(n-k-2)/2} \, dt \\ &= \frac{1}{n-k} \int_{-1}^{1} \frac{d}{dt}\left[F_k(t) C_j^{\nu}(t)\right](1-t^2)^{(n-k)/2} \, dt \\ &= \frac{1}{n-k}\left(\int_{-1}^{1} C_j^{\nu}(t)(1-t^2)^{(n-2)/2} \, dt + \int_{-1}^{1} F_k(t) \frac{d}{dt} C_j^{\nu}(t)(1-t^2)^{(n-k)/2} \, dt\right) \end{aligned}$$

by partial integration. The first term in the line above is simply

$$\int_{-1}^{1} C_j^{\nu}(t)(1-t^2)^{(n-2)/2} \, dt = [(1-t^2), C_j^{\nu}(t)]_n,$$

and hence vanishes whenever $j \notin \{0,2\}$. Therefore, if $j \in \mathbb{N}_0 \backslash \{0,2\}$, then

$$\begin{aligned} (n-k)I_j &= \int_{-1}^{1} F_k(t) \frac{d}{dt} C_j^{\nu}(t)(1-t^2)^{(n-k)/2} \, dt \\ &= \int_{-1}^{1} F_k(t)(1-t^2) \frac{d}{dt} C_j^{\nu}(t)(1-t^2)^{(n-k-2)/2} \, dt \\ &= (n+j-2)\int_{-1}^{1} F_k(t) C_j^{\nu}(t) t (1-t^2)^{(n-k-2)/2} \, dt - (j+1)a_{j+1} \\ &= (n+j-2)I_j - (j+1)a_{j+1}, \end{aligned}$$

where we applied partial integration. A quick rearrangement of this equation gives

$$I_j = \frac{j+1}{k+j-2} a_{j+1}.$$

Plugging this expression for I_j into (3.17) and rearranging the resulting equation yields the recursion formula

$$a_{j+1} = \frac{(k+j-2)(n+j-3)}{(n+j-k)(j+1)} a_{j-1}, \qquad j \in \mathbb{N}_0 \backslash \{0,2\}.$$

▷ Due to this recursion formula, it only remains to show that a_0, a_1, and a_3 are non-zero. From the definition of F_k, it is easy to see that

$$a_0 = \int_{-1}^{1} F_k(t)(1-t^2)^{(n-k-2)/2}\, dt > 0.$$

Moreover, using partial integration, we obtain

$$a_1 = (n-2)\int_{-1}^{1} t(1-t^2)^{(n-k-2)/2} F_k(t)\, dt = \frac{n-2}{n-k}\int_{-1}^{1}(1-t^2)^{(n-1)/2}\, dt > 0.$$

The Gegenbauer polynomial C_3^ν has the explicit representation

$$C_3^\nu(t) = \frac{n(n-2)}{6} t((n-1)-(n+2)(1-t^2)).$$

With some partial integration, one arrives at

$$\begin{aligned} a_3 &= \frac{n(n-2)}{6}\left(\frac{n-1}{n-k}\int_{-1}^{1}(1-t^2)^{(n-1)/2}\, dt - \frac{n+2}{n-k+2}\int_{-1}^{1}(1-t^2)^{(n+1)/2}\, dt\right) \\ &= \frac{n(n-1)\omega_{n+2}}{3(n-k)(n-k+2)\omega_{n+1}} > 0. \end{aligned}$$

This shows that $a_j \neq 0$ for all $j \in \mathbb{N}_0$, proving that $\pi_{1,1-k}^{[k]}$ is injective on $\mathbf{C}(\mathbb{S})$. □

We end this section with another injectivity result.

Lemma 3.23 ([13, Lemma 2.4]). *Let $k \in \{1,\dots,n\}$, $j \geq 1-k$, and $f \in \mathbf{C}(\mathbb{S})$. If $\pi_{E,j} f$ is a linear harmonic on $\mathbb{S}(E)$ for every $E \in \mathrm{G}(n,k)$, then $\pi_{1,j}^{[k]} f$ is a linear harmonic on $\mathbb{S}$.*

Proof. For every $E \in \mathrm{G}(n,k)$ and $w \in \mathbb{S}(E)$, the Funk-Hecke theorem yields

$$\int_{\mathbb{S}(E)} (\pi_{E,j} f)(v) w\cdot v\, dv = \frac{\omega_k}{k}(\pi_{E,j} f)(w).$$

Applying this and Lemma 3.17, we obtain

$$\begin{aligned} \int_{\mathbb{S}} (\pi_{1,j}^{[k]} f)(v) u\cdot v\, dv &= \int_{\mathrm{G}(n,k)} \int_{\mathbb{S}} (\pi_{E,1}^{*}\pi_{E,j} f)(v) u\cdot v\, dv\, dE \\ &= \int_{\mathrm{G}(n,k)} \int_{\mathbb{S}(E)} (\pi_{E,j} f)(v)[\pi_{E,1}(u\cdot .)](v)\, dv\, dE \\ &= \frac{\omega_{n+2}}{\omega_{k+2}} \int_{\mathrm{G}(n,k)} \|u|E\| \int_{\mathbb{S}(E)} (\pi_{E,j} f)(v) \mathrm{pr}_E u\cdot v\, dv\, dE \\ &= \frac{\omega_{n+2}}{\omega_{k+2}}\frac{\omega_k}{k} \int_{\mathrm{G}(n,k)} \|u|E\| (\pi_{E,j} f)(\mathrm{pr}_E u)\, dE \\ &= \frac{\omega_{n+2}}{\omega_{k+2}}\frac{\omega_k}{k} (\pi_{1,j}^{[k]} f)(u) \end{aligned}$$

for all $u \in \mathbb{S}$. Hence, $\pi_{1,j}^{[k]} f$ is a linear harmonic on $\mathbb{S}$. □

Note that the last two results combined show that if $f \in \mathbf{C}(\mathbb{S})$ such that $\pi_{E,1-k} f$ is a linear harmonic for all $E \in \mathrm{G}(n,k)$ then $\pi_{1,1-k}^{[k]} f$ is a linear harmonic, and hence f is a linear harmonic.

3.4 Berg's Functions

Now we come to the last ingredient for our proof of a representation formula for the mean section operators. In Section 2.3, we have already given a brief introduction into Christoffel's problem, which is the question how the first area measures of convex bodies can be characterized. An important part in the solution to Christoffel's problem are *Berg's functions*, which are defined inductively as follows:

Definition ([5, Théorème 3.3]). *Berg's functions* are the functions $g_n \in \mathbf{C}^\infty(-1,1)$, $n \geq 2$, defined by the following recursion:

$$\begin{aligned} g_2(t) &:= \frac{1}{\pi}(\pi - \arccos t)\sqrt{1-t^2} - \frac{1}{2\pi} t, \\ g_3(t) &:= 1 + t\log(1-t) + \left(\frac{4}{3} - \log 2\right) t, \\ g_{n+2}(t) &:= \frac{n+1}{(n-1)^2} t g_n'(t) + \frac{n+1}{n-1} g_n(t) + \frac{(n+1)\omega_{n+1}}{(n+2)\omega_{n+2}} t. \end{aligned}$$

We summarize the most important properties of Berg's functions in the following theorem, which was proved inductively by C. Berg:

Theorem 3.24 ([5, Théorème 3.3]).

(i) *If $n \geq 2$, then $g_n \in \mathbf{C}^\infty(-1,1)$ and $g_n \in \mathbf{L}^1[-1,1]_n$. In particular, $\breve{g}_n \in \mathbf{M}(\mathbb{S}, \bar{e})$.*

(ii) *If $n \geq 3$, then $g_n(t)\sqrt{1-t^2} \in \mathbf{L}^1[-1,1]_n$.*

(iii) *If $n \geq 2$ then*

$$\frac{1}{\omega_{n-1}} g_n \quad \sim \quad P_0^n + \sum_{k=2}^{\infty} \frac{N(n,k)}{\omega_n} \frac{n-1}{(k-1)(n+k-1)} P_k^n.$$

As a consequence of Theorem 3.24 (iii), if $K \in \mathcal{K}^n$, then

$$h_o(K,\cdot) = \frac{1}{\omega_{n-1}} S_1(K,\cdot) * \breve{g}_n \tag{3.18}$$

in the sense of measures and distributions, respectively. Since the measure $\breve{g}_n$ has a density, the right hand side has the density given by $\omega_{n-1}^{-1} \int_{\mathbb{S}} g_n(u \cdot v)\, dS_1(K,v)$ as a function in $u \in \mathbb{S}$. This means that for almost all $u \in \mathbb{S}$, this integral exists and takes the value

$h_o(K,u)$. By developing and applying a certain procedure, known as *Berg's averaging procedure*, C. Berg was able to show that in fact the integral exists for all $u \in \mathbb{S}$ and depends continuously on $u \in \mathbb{S}$, which leads to the following result:

Theorem 3.25 ([5, Théorème 5.1]). *For all $K \in \mathcal{K}^n$,*

$$h_o(K,u) = \frac{1}{\omega_{n-1}} \int_{\mathbb{S}} g_n(u \cdot v)\, dS_1(K,u), \qquad u \in \mathbb{S}.$$

As a corollary, C. Berg was able to give the following answer to Christoffel's problem:

Corollary 3.26 ([21, Theorem 8.3.8]). *A positive measure $\mu \in \mathbf{M}(\mathbb{S})$ is the first area measure of a convex body if and only if $\mu * \breve{g}_n$ is the support function of a convex body.*

Now that we have discussed the definition, some properties, and the historical relevance of Berg's functions, it is time to relate them to the main objective of this chapter. We have already seen in (3.5) that

$$h(M_2(K),\cdot) = \frac{1}{2(n-1)} S_{n-1}(K,\cdot) * \breve{g}_2. \tag{3.19}$$

In the previous section, we have seen how spherical liftings and projections relate convex geometry in subspaces to convex geometry in the ambient space and also relate the k-th mean section operators M_k to M_2. Interestingly, the characteristic property of Berg's functions, stated in (3.18), causes them to interact with spherical liftings and projections exactly in such a way that everything fits together, casually speaking.

In order to be able to work with Berg's functions, we need the following integrability result:

Lemma 3.27 ([13]). *Let $k \in \{3,\dots,n\}$ and $E \in \mathrm{G}(n,n-1)$. Then for all $u \in \mathbb{S}$, we have that $g_k(u \cdot .) \in \mathbf{M}_{E,-n+k-1}(\mathbb{S})$ and the measure $\pi_{E,-n+k-1} g_k(u \cdot .) \in \mathbf{M}(\mathbb{S}(E))$ has a density that is continuous on $\mathbb{S}(E) \backslash \{\mathrm{pr}_E u\}$.*

Proof. First, note that $d\mu(v) := \|v|E\|^{-1} dv$ is a finite measure on $\mathbb{S}$. Due to Theorem 3.24 (i), the function $\breve{g}_n$ is the density of a zonal finite signed measure on $\mathbb{S}$. Thus, $\mu * \breve{g}_n$ is again a finite signed measure on $\mathbb{S}$ that has the density

$$\int_{\mathbb{S}} g_n(. \cdot v) \|v|E\|^{-1}\, dv.$$

We choose $u_0 \in \mathbb{S}$ in such a way that $E = u_0^{\perp}$. Then for almost all $u \in \mathbb{S}$, we can make the computation

$$\begin{aligned}
&\int_{\mathbb{S}} \|v|E\|^{-1} |g_n(u\cdot v)| \ dv \\
&\qquad = \int_{\mathbb{S}(E)} \int_{\mathbb{H}(E,v)} |g_n(u\cdot y)| \,(y\cdot v)^{n-3} \ dy \ dv \\
&\qquad = \int_{\mathbb{S}(E)} \int_{-1}^{1} \left| g_n(tu\cdot u_0 + \sqrt{1-t^2}\sqrt{1-(u\cdot u_0)^2}\mathrm{pr}_E u\cdot v)\right| (1-t^2)^{(n-4)/2} \ dt \ dv \\
&\qquad = \int_{\mathbb{S}(E)\times[-1,1]} \left| g_n(tu\cdot u_0 + \sqrt{1-t^2}\sqrt{1-(u\cdot u_0)^2}\mathrm{pr}_E u\cdot v)\right| (1-t^2)^{(n-4)/2} \ d(v,t)
\end{aligned}$$

(with the convention that $\mathrm{pr}_E u_0 = \mathrm{pr}_E(-u_0) = o$). Clearly all of the integrals are finite if $u = \pm u_0$. Due to our previous considerations, this must also be the case for some $u \neq \pm u_0$. In that case, observe that the integrand in the last line has its sole singularity at $(v,t) = (\mathrm{pr}_E u, u\cdot u_0)$ and is continuous everywhere else. It can easily be deduced that for all $(w,\lambda) \in \mathbb{S}(E)\times(-1,1)$, the double integral

$$\int_{\mathbb{S}(E)} \int_{-1}^{1} \left| g_n(t\lambda + \sqrt{1-t^2}\sqrt{1-\lambda^2}w\cdot v)\right| (1-t^2)^{(n-4)/2} \ dt \ dv$$

must be finite. Since every $u \neq \pm u_0$ can be written as $\lambda u_0 + \sqrt{1-\lambda^2}w$ for some $(w,\lambda) \in \mathbb{S}(E)\times(-1,1)$, it follows that $g_n(u\cdot .) \in \mathbf{M}_{E,-1}(\mathbb{S})$ for all $u \in \mathbb{S}$. Moreover,

$$d[\pi_{E,-1} g_n(u\cdot .)](v) = \int_{\mathbb{H}(E,v)} g_n(u\cdot v)(v\cdot y)^{n-3} \ dy \ dv,$$

so $\pi_{E,-1} g_n(u\cdot .)$ has a density that is continuous on $\mathbb{S}(E)\backslash\{\mathrm{pr}_E u\}$. □

Lemma 3.28 ([13, Lemma 4.2]). *Let $E \in \mathrm{G}(n,n-1)$. Then for all $u \in \mathbb{S}$ and $v \in \mathbb{S}(E)\backslash\{\mathrm{pr}_E u\}$, we have*

$$[\pi_{E,-1} g_n(u\cdot .)](v) = \frac{(n-1)\omega_{n-1}}{(n-2)\omega_{n-2}}[\pi^*_{E,1} g_{n-1}(.\cdot v)](u) + w_{E,u}\cdot v, \tag{3.20}$$

where $w_{E,\cdot} : \mathbb{S} \to \mathbb{R}^n$ is a continuous function depending only on E.

Proof. For a smooth convex body $K \in \mathcal{K}^{n-1}(E)$, the characteristic property of Berg's function g_n yields

$$\begin{aligned}
h_o(K,u) &= \frac{1}{\omega_{n-1}} \int_{\mathbb{S}} g_n(u\cdot v) \ dS_1(K,v) \\
&= \frac{n-2}{(n-1)\omega_{n-1}} \int_{\mathbb{S}} g_n(u\cdot v) \ d[\pi^*_{E,-1} S^E_1(K,\cdot)](v) \\
&= \frac{n-2}{(n-1)\omega_{n-1}} \int_{\mathbb{S}(E)} [\pi_{E,-1} g_n(u\cdot .)](v) \ dS^E_1(K,v).
\end{aligned}$$

On the other hand, applying the characteristic property of g_{n-1} to K as a convex body in the space E yields

$$\begin{aligned} h_o(K,u) &= [\pi^*_{E,1} h^E_o(K,\cdot)](u) \\ &= \|u|E\| h^E_o(K, \mathrm{pr}_E u) \\ &= \frac{1}{\omega_{n-2}} \int_{\mathbb{S}(E)} \|u|E\| g_{n-1}(\mathrm{pr}_E u \cdot v)\, dS^E_1(K,v) \\ &= \frac{1}{\omega_{n-2}} \int_{\mathbb{S}(E)} [\pi^*_{E,1} g_{n-1}(.\cdot v)](u)\, dS^E_1(K,v). \end{aligned}$$

Consequently, for every $u \in \mathbb{S}$, there is a vector $w_{E,u} \in \mathbb{R}^n$ such that (3.20) holds for almost all $v \in \mathbb{S}$. For this fixed $u \in \mathbb{S}$, the expressions $[\pi_{E,-1} g_n(u \cdot .)](v)$ and $[\pi^*_{E,1} g_{n-1}(. \cdot v)](u)$ are both continuous at every $v \in \mathbb{S} \backslash \{\mathrm{pr}_E u\}$, which shows that (3.20) holds for all $v \in \mathbb{S}\backslash\{\mathrm{pr}_E u\}$.

For fixed $v \in \mathbb{S}(E)$, both $[\pi_{E,-1} g_n(u \cdot .)](v)$ and $[\pi^*_{E,1} g_{n-1}(. \cdot v)](u)$ are continuous at every $u \in \mathbb{S}\backslash\mathbb{H}(E,v)$, so $w_{E,u}$ must also depend continuously on u. □

With a clever trick, the result can be generalized to other Berg functions g_k.

Lemma 3.29 ([13, Lemma 4.2]). *Let $k \in \{3,\ldots,n\}$ and $E \in \mathrm{G}(n,n-1)$. Then for all $u \in \mathbb{S}$ and $v \in \mathbb{S}(E)\backslash\{\mathrm{pr}_E u\}$, we have*

$$[\pi_{E,-n+k-1} g_k(u \cdot .)](v) = \frac{(k-1)\omega_{k-2}}{(k-2)\omega_{k-2}} [\pi^*_{E,1} g_{k-1}(. \cdot v)](u) + w_{E,u} \cdot v,$$

where $w_{E,\cdot} : \mathbb{S} \to \mathbb{R}^n : u \mapsto w_{E,u}$ is a continuous function.

Proof. By expanding the expressions in the statement of the previous lemma, we obtain that, for all $u \in \mathbb{S}$ and $v \in \mathbb{S}\backslash\{\mathrm{pr}_E u\}$,

$$\begin{aligned} \int_{-1}^{1} & g_n(tu \cdot u_0 + \sqrt{1-t^2}\sqrt{1-(u\cdot u_0)^2}\mathrm{pr}_E u \cdot v)(1-t^2)^{(n-4)/2}\, dt \\ &= \int_{\mathbb{H}(E,v)} g_n(u \cdot y)(y \cdot v)^{n-3}\, dy \\ &= [\pi_{E,-1} g_n(u \cdot .)](v) \\ &= c_n [\pi^*_{E,1} g_{n-1}(. \cdot v)](u) + w_{E,u} \cdot v \\ &= c_n \|u|E\| g_{n-1}(\mathrm{pr}_E u \cdot v) + w_{E,u} \cdot v, \end{aligned}$$

where $w_{E,u} \in \mathbb{R}^n$ depends continuously on u and where $c_n := \frac{(n-1)\omega_{n-1}}{(n-2)\omega_{n-2}}$. For a fixed $u \in \mathbb{S}$, observe that all expressions involved are constant along the sets $\mathrm{pr}_E \cdot v = c$, hence $w_{E,u} = f_E(u)\mathrm{pr}_E u$ for some continuous function $f_E : \mathbb{S} \to \mathbb{R}$.

Now observe that for any $\alpha \in [-1,1]$ and $\beta \in [-1,1)$, we can find $u \in \mathbb{S}$ and $v \in \mathbb{S}\backslash\{\mathrm{pr}_E u\}$ such that $\alpha = u \cdot u_0$ and $\beta = \mathrm{pr}_E u \cdot v$. Thus we obtain that for all $\alpha \in [-1,1]$ and $\beta \in [-1,1)$,

$$\int_{-1}^{1} g_n(\alpha t + \beta\sqrt{1-\alpha^2}\sqrt{1-t^2})(1-t^2)^{(n-4)/2}\, dt = c_n\sqrt{1-\alpha^2}g_{n-1}(\beta) + f_n(\alpha)\beta, \quad (3.21)$$

where $f_n : [-1,1] \to \mathbb{R}$ is a continuous function depending only on n.

It follows that for all $u \in \mathbb{S}$ and $v \in \mathbb{S}(E)\backslash\{\mathrm{pr}_E u\}$,

$$\begin{aligned}
&[\pi_{E,-n+k-1} g_k(u \cdot .)](v) \\
&\quad = \int_{\mathbb{H}(E,v)} g_k(u \cdot y)(v \cdot y)^{k-3}\, dv \\
&\quad = \int_{-1}^{1} g_k(tu \cdot u_0 + \sqrt{1-t^2}\sqrt{1-(u\cdot u_0)^2}\mathrm{pr}_E u \cdot v)(1-t^2)^{(k-4)/2}\, dt \\
&\quad = c_k\sqrt{1-(u \cdot u_0)^2} g_{k-1}(\mathrm{pr}_E u \cdot v) + f_k(u \cdot u_0)\mathrm{pr}_E u \cdot v \\
&\quad = c_k[\pi^*_{E,1} g_{k-1}(. \cdot v)](u) + w_{E,u} \cdot v,
\end{aligned}$$

where $w_{E,u} := f_k(u \cdot u_0)\mathrm{pr}_E u$. □

Proposition 3.30. *Let $k \in \{3,\dots,n\}$, $j \in \{0,\dots,k-2\}$, and $E \in \mathrm{G}(n,n-j)$. Then for every $f \in \mathbf{C}(\mathbb{S})$, we have*

$$\pi_{E,-n+k-1}(f * \breve{g}_k) = \frac{(k-1)\omega_{k-1}}{(k-j-1)\omega_{k-j-1}}(\pi_{E,1} f) * \breve{g}_{k-j} + w_{f,E} \cdot . \quad (3.22)$$

for some $w_{f,E} \in \mathbb{R}^n$.

Proof. We fix $k \in \{3,\dots,n\}$ and prove the statement by induction on $j \in \{0,\dots,k-2\}$.

▷ $j = 0$: The base case is trivial, since $\pi_{E,m}$ is just the identity map for $E = \mathbb{R}^n$.

▷ $j \mapsto j+1$: For carrying out the induction step, suppose that the statement holds for some $j \in \{0,\dots,k-3\}$ and all $E' \in \mathrm{G}(n,n-j)$. Then, take an arbitrary $E \in \mathrm{G}(n,n-j-1)$ and $f \in \mathbf{C}(\mathbb{S})$. We may choose some $E' \in \mathrm{G}(n,n-j)$ such that $E \subseteq E'$. By applying Lemma 3.14 (ii) and the induction hypothesis, we obtain

$$\begin{aligned}
\pi_{E,-n+k-1}(f * \breve{g}_k) &= \pi^{E'}_{E,-n+k-1}\pi_{E',-n+k-1}(f * \breve{g}_k) \\
&= c_{k,j}\pi^{E'}_{E,-n+k-1}((\pi_{E',1} f) * \breve{g}_{k-j}) + \pi^{E'}_{E,-n+k-1}(w_{f,E'} \cdot .),
\end{aligned}$$

where $c_{k,j} := \frac{(k-1)\omega_{k-1}}{(k-j-1)\omega_{k-j-1}}$.

By applying Lemma 3.29 to the first summand and using the integrability properties established in Lemma 3.27, we obtain

$$
\begin{aligned}
&[\pi^{E'}_{E,-n+k-1}((\pi_{E',1}f) * \breve{g}_{k-j})](v) \\
&\quad = \int_{\mathbb{H}(E';E,v)} \int_{\mathbb{S}(E')} (\pi_{E',1}f)(u) g_{k-j}(u \cdot y)\, du\, (v \cdot y)^{k-j-3}\, dy \\
&\quad = \int_{\mathbb{S}(E')} (\pi_{E',1}f)(u) \int_{\mathbb{H}(E';E,v)} g_{k-j}(u \cdot y)(v \cdot y)^{k-j-3}\, dy\, du \\
&\quad = \int_{\mathbb{S}(E')} (\pi_{E',1}f)(u) [\pi^{E'}_{E,-n+k-1} g_{k-j}(u \cdot .)](v)\, du \\
&\quad = c_{k-j} \int_{\mathbb{S}(E')} (\pi_{E',1}f)(u) [(\pi^{E'}_{E,1})^* g_{k-j-1}(. \cdot v)](u)\, du + w'_{f,E'} \cdot v \\
&\quad = c_{k-j} [(\pi_{E,1}f) * \breve{g}_{k-j-1}](v) + w'_{f,E'} \cdot v,
\end{aligned}
$$

where $c_{k-j} := \frac{(k-j-1)\omega_{k-j-1}}{(k-j-2)\omega_{k-j-2}}$ and $w'_{f,E'} := \int_{\mathbb{S}(E')} (\pi_{E',1}f)(u) w_{E',u}\, du$.

Due to Lemma 3.17, there is a vector $w''_{f,E'} \in \mathbb{R}^n$ such that $\pi^{E'}_{E,-n+k-1}(w_{f,E'} \cdot .) = (w''_{f,E'} \cdot .)$. Then, defining $w_{f,E'} := w'_{f,E'} + w''_{f,E'}$ and observing that $c_{k,j} c_{k-j} = c_{k,j+1}$ completes the proof. □

Now we have all tools available for proving the desired formula for mean section operators. Note that a crucial ingredient will be the symmetry result on bivaluations.

Theorem 3.31 ([13, Theorem 3.4 and 4.3]). *If $k \in \{2, \dots, n\}$ and $K \in \mathcal{K}^n$, then*

$$h(M_k(K), u) = m_{n,k} \int_{\mathbb{S}} g_k(u \cdot v)\, dS_{n-k+1}(-K, v)$$

for almost all $u \in \mathbb{S}$, where

$$m_{n,k} := \frac{1}{2(n-k+1)\pi^{(k-1)/2}} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{n-k+2}{2})}.$$

Proof. We already know from Theorem 3.10 and Theorem 3.25 that this is true in the cases where $k = 2$ and $k = n$, so only the cases where $k \in \{3, \dots, n-1\}$ remain.

Note that the statement means exactly that

$$h(M_k(K), \cdot) = m_{n,k} S_{n-k+1}(-K, \cdot) * \breve{g}_k$$

in the sense of measures and distributions, respectively. Since both sides depend continuously on K, it suffices to prove this for smooth convex bodies $K \in \mathcal{K}^n$. Recall that all area measures of a smooth convex body have smooth densities.

Next, let $E \in \mathrm{G}(n, n-k+2)$ and $C \in \mathcal{K}^{n-k+2}(E)$ be arbitrary. Then

$$\begin{aligned}
&\langle h(M_k(K), \cdot), S_{n-k+1}(C, \cdot)\rangle_{\mathbb{S}} \\
&\quad = c_{n,k}\langle h(M_k(K), \cdot), \pi^*_{E,-n+k-1} S^E_{n-k+1}(C, \cdot)\rangle_{\mathbb{S}} && \text{due to (3.10)} \\
&\quad = c_{n,k}\langle \pi_{E,-n+k-1} h(M_k(K), \cdot), S^E_{n-k+1}(C, \cdot)\rangle_{\mathbb{S}(E)}
\end{aligned}$$

for some constant $c_{n,k}$.

On the other hand,

$$\begin{aligned}
&\langle h(M_k(K), \cdot), S_{n-k+1}(C, \cdot)\rangle_{\mathbb{S}} \\
&\quad = \langle h(M_k(C), \cdot), S_{n-k+1}(K, \cdot)\rangle_{\mathbb{S}} && \text{due to (2.4)} \\
&\quad = c_{n,k}\langle \pi^*_{E,1} h(M^E_2(C), \cdot), S_{n-k+1}(K, \cdot)\rangle_{\mathbb{S}} && \text{due to (3.11)} \\
&\quad = c_{n,k}\langle h(M^E_2(C), \cdot), \pi_{E,1} S_{n-k+1}(K, \cdot)\rangle_{\mathbb{S}(E)} \\
&\quad = c_{n,k}\langle S^E_{n-k+1}(-C, \cdot) * \breve{g}_2, \pi_{E,1} S_{n-k+1}(K, \cdot)\rangle_{\mathbb{S}(E)} && \text{due to (3.19)} \\
&\quad = c_{n,k}\langle S^E_{n-k+1}(C, \cdot) * \breve{g}_2, \pi_{E,1} S_{n-k+1}(-K, \cdot)\rangle_{\mathbb{S}(E)} \\
&\quad = c_{n,k}\langle (\pi_{E,1} S_{n-k+1}(-K, \cdot)) * \breve{g}_2, S^E_{n-k+1}(C, \cdot)\rangle_{\mathbb{S}(E)} && \text{due to (1.7)} \\
&\quad = c_{n,k}\langle \pi_{E,-n+k-1}(S_{n-k+1}(-K, \cdot) * \breve{g}_k), S^E_{n-k+1}(C, \cdot)\rangle_{\mathbb{S}(E)} && \text{due to (3.22)}
\end{aligned}$$

for some constant $c_{n,k}$.

Since C and E were chosen arbitrarily, it follows that for all $E \in \mathrm{G}(n, n-k+2)$, the function $\pi_{E,-n+k-1} f$ is a linear harmonic on E, where

$$f := h(M_k(K), \cdot) - c_{n,k} S_{n-k+1}(-K, \cdot) * \breve{g}_k \in \mathbf{C}(\mathbb{S}) \tag{3.23}$$

and $c_{n,k}$ is some constant. Now our injectivity results on mean lifted projections come into play. Due to Lemma 3.23, the function $\pi^{[n-k+2]}_{1,-n+k-1} f$ is a linear harmonic on $\mathbb{S}$. The mean lifted projection operator $\pi^{[n-k+2]}_{1,-n+k-1}$ is an injective multiplier transform due to Proposition 3.22, hence f is also a linear harmonic on $\mathbb{S}$. Since f is also centred, $f = 0$.

The constant $c_{n,k}$ in (3.23) can be effectively computed from (3.10), (3.11), and (3.22). Alternatively, it can also be computed from (3.1). □

Finally, we want to mention that with Berg's averaging procedure, one can prove that the formula obtained above is true for all $u \in \mathbb{S}$, whenever $\dim K \geq n-k+1$.

Theorem 3.32 ([13, Theorem 4.4]). *If $k \in \{2, \ldots, n\}$ and $K \in \mathcal{K}^n$ with $\dim K \geq n-k+1$, then*

$$h_o(M_k(K), u) = m_{n,k} \int_{\mathbb{S}} g_k(u \cdot v)\, dS_{n-k+1}(-K, v), \qquad u \in \mathbb{S},$$

where $m_{n,k}$ is as in the previous theorem.

Chapter 4

Log-Concavity of Minkowski Valuations

The Brunn-Minkowski inequality for intrinsic volumes states that for $i \in \{2, \dots, n\}$,

$$\mathsf{V}_i(K+L)^{1/i} \geq \mathsf{V}_i(K)^{1/i} + \mathsf{V}_i(L)^{1/i}$$

for all n-dimensional convex bodies $K, L \in \mathcal{K}^n$ with equality if and only if K and L are homothetic. It is natural to ask to what extent this is true for the composition of intrinsic volumes with homogeneous Minkowski valuations.

Question 1. For which $i, j \in \{1, \dots, n\}$ and $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$ does the *generalized Brunn-Minkowski inequality for Minkowski valuations*

$$\mathsf{V}_i(\Phi_j(K+L))^{1/ij} \geq \mathsf{V}_i(\Phi_j(K))^{1/ij} + \mathsf{V}_i(\Phi_j(L))^{1/ij} \tag{4.1}$$

hold and what are the respective equality conditions?

The aim of this chapter is to present the research done on this question. In Section 4.1 we start off with a result from Schuster in 2006 that covers the case where $j = n-1$, then in Section 4.2 and Section 4.3, we discuss a result from L. Parapatits and F. Schuster from 2011 that covers the cases where $i \leq j+1$, and in Section 4.4 we state the most recent result from 2014 which is due to A. Berg, L. Parapatits, F. Schuster, and M. Weberndorfer. Note that in order to shorten some arguments, the exhibition of the research will not be entirely chronological.

First of all, we want to discuss some trivial cases. If $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$ is the trivial Minkowski valuation, that is, if $\Phi_j K = \{o\}$ for all $K \in \mathcal{K}^n$, then equality holds in (4.1). For the marginal cases where $i = 1$ or $j \in \{1, n\}$, the following is known:

Lemma 4.1. *Let $i, j \in \{1, \dots, n\}$ and let $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$ be non-trivial. If $i = 1$ or $j \in \{1, n\}$, then*

$$\mathsf{V}_i(\Phi_j(K+L))^{1/ij} \geq \mathsf{V}_i(\Phi_j K)^{1/ij} + \mathsf{V}_i(\Phi_j L)^{1/ij} \tag{4.2}$$

for all n-dimensional $K, L \in \mathcal{K}^n$.

(i) *If $i = j = 1$, then equality always holds in* (4.2)*.*

(ii) *If $i = 1$ and $j > 1$, then equality holds in* (4.2) *if and only if K and L are homothetic.*

(iii) *If $i > 1$ and $j = n$, then equality holds in* (4.2) *if and only if K and L are homothetic.*

We will give a short proof of this theorem at the very end of the chapter. Until then, we will focus on the non-trivial cases where $i \in \{2, \dots, n\}$ and $j \in \{2, \dots, n-1\}$.

Before we exhibit the research done on Question 1, we want to remark that geometric inequalities of this kind are often stated in different forms. Therefore we provide a lemma that states in a very general way that these kind of formulations are always equivalent.

Lemma 4.2. *Let $\zeta : \mathcal{K}^n \to [0, \infty)$ be a continuous translation invariant function homogeneous of degree 1 such that $\zeta(K) > 0$ for every n-dimensional $K \in \mathcal{K}^n$. Then the following statements are equivalent:*

(a) *For all n-dimensional $K, L \in \mathcal{K}^n$,*

$$\zeta(K+L) \geq \zeta(K) + \zeta(L) \tag{4.3}$$

with equality if and only if K and L are homothetic.

(b) *For all n-dimensional $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$,*

$$\zeta((1-\lambda)K + \lambda L) \geq (1-\lambda)\zeta(K) + \lambda\zeta(L) \tag{4.4}$$

with equality if and only if K and L are homothetic.

(c) *For all n-dimensional $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$,*

$$\zeta((1-\lambda)K + \lambda L) \geq \zeta(K)^{1-\lambda}\zeta(L)^{\lambda} \tag{4.5}$$

with equality if and only if K and L are translates of each other.

Proof. First we show that the three inequalities (4.3), (4.4), and (4.5) are equivalent, then we show that the respective equality conditions are equivalent.

▷ Clearly, the implication (4.3)⇒ (4.4) follows from the homogeneity of ζ.

In order to see that (4.4)⇒(4.5), define $f(\lambda) := \zeta((1-\lambda)K + \lambda L)$ and then observe that (4.4) states that f is concave and (4.5) states that f is log-concave.

In order to show (4.5)⇒(4.3), assume that (4.5) holds for all n-dimensional $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$. Then

$$\begin{aligned}\frac{\zeta(K+L)}{\zeta(K)+\zeta(L)} &= \zeta\left(\frac{K+L}{\zeta(K)+\zeta(L)}\right) = \zeta\left(\frac{\zeta(K)}{\zeta(K)+\zeta(L)}\frac{K}{\zeta(K)} + \frac{\zeta(L)}{\zeta(K)+\zeta(L)}\frac{L}{\zeta(L)}\right)\\ &\geq \zeta\left(\frac{K}{\zeta(K)}\right)^{\frac{\zeta(K)}{\zeta(K)+\zeta(L)}} \zeta\left(\frac{L}{\zeta(L)}\right)^{\frac{\zeta(L)}{\zeta(K)+\zeta(L)}} = 1\end{aligned} \tag{4.6}$$

for all n-dimensional $K, L \in \mathcal{K}^n$.

▷ Now we show that the stated equality conditions are equivalent.

Clearly, if K and L are homothetic, then equality holds in (4.3) and (4.4). Moreover, equality holds in (4.3) for K and L if and only if equality holds in (4.4) for $(1-\lambda)K$ and λL. Hence, one equality implies the homothety of K and L if and only if the other does. This shows that the equality conditions in (a) and (b) are equivalent.

Next, we show that the equality condition in (c) implies the equality condition in (a). To that end, suppose that the equality condition in (c) is true and that equality holds in (4.3) for K and L. Then our computation in (4.6) entails that equality holds in (4.5) for $\zeta(K)^{-1}K$ and $\zeta(L)^{-1}L$. Then these are translates of each other, so K and L are homothetic.

Lastly, we show that the equality condition in (b) implies the equality condition in (c). Obviously, if K and L are translates of each other, then equality holds in (4.5). Now suppose that the equality condition in (a) is true and that for all $\lambda \in (0,1)$, equality holds in (4.5) for K and L. Then the concave function $f(\lambda) := \zeta((1-\lambda)K + \lambda L)$ is of the form $f(\lambda) = \zeta(K)(\zeta(L)/\zeta(K))^\lambda$. This is only possible if f is constant, which implies that K and L are homothetic, that is $L = \alpha K + x$ for some $\alpha > 0$ and $x \in \mathbb{R}^n$. Plugging this into (4.5) yields $1 - \lambda + \lambda\alpha = \alpha^\lambda$ for all $\lambda \in (0,1)$, which is only possible if $\alpha = 1$. Thus K and L are translates of each other. □

The following lemma provides general information about homogeneous Minkowski valuations in $\mathbf{MVal}^{\mathrm{SO}(n)}$ that will be used throughout this chapter.

Lemma 4.3. *Let $j \in \{0, \dots, n\}$ and $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$.*

(i) *The convex body $\Phi_j K$ is centered for all $K \in \mathcal{K}^n$.*

(ii) *There exists $r(\Phi_j) \geq 0$ such that $\Phi_j B = r(\Phi_j)B$. If Φ_j is non-trivial, then $r(\Phi_j) > 0$.*

(iii) *We have*

$$\mathsf{W}_{n-1}(\Phi_j K) = r(\Phi_j)\mathsf{W}_{n-j}(K) \tag{4.7}$$

for all $K \in \mathcal{K}^n$.

Proof. For (i), define

$$\tilde{s} : \mathcal{K}^n \to \mathbb{R}^n : \tilde{s}(K) := s(K) - s(\Phi_j K),$$

where $s : \mathcal{K}^n \to \mathbb{R}^n$ denotes the Steiner point map. Observe that $\tilde{s}$ is a rigid motion invariant and continuous vector valued valuation. Thus $\tilde{s}$ is the Steiner point map, which implies that $s(\Phi_j K) = 0$ for all $K \in \mathcal{K}^n$.

For (iii), observe that $K \mapsto \mathsf{W}_{n-1}(\Phi_j K)$ is a continuous and rigid motion invariant valuation on $\mathcal{K}^n$, homogeneous of degree j. Hence, Hadwiger's characterization of the intrinsic volumes entails that there is some constant $c \in \mathbb{R}$ such that $\mathsf{W}_{n-1}(\Phi_j K) = c\mathsf{W}_{n-j}(K)$ for all $K \in \mathcal{K}^n$. Putting $K = B$ shows that $c = r(\Phi_j)$.

For (ii), note that if $r(\Phi_j) = 0$, then (4.7) implies that $\mathsf{W}_{n-1}(\Phi_j K) = 0$ for all $K \in \mathcal{K}^n$. Thus $\dim(\Phi_j K) = 0$ for all $K \in \mathcal{K}^n$, which implies that Φ_j is trivial. □

4.1 Homogeneous Minkowski Valuations of Degree $n-1$

A first answer to Question 1 posed at the beginning of the chapter was given by F. Schuster in [23], which is the content of this section. The first step is to derive a representation theorem for $\mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$ from McMullen's representation theorem for $\mathbf{Val}_{n-1}$.

Theorem 4.4 ([23, Theorem 1.2]). *For every $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$ there exists a unique $f \in \mathbf{C}_o(\mathbb{S}, \bar{e})$ such that*

$$h(\Phi_{n-1}K, \cdot) = S_{n-1}(K, \cdot) * f, \qquad K \in \mathcal{K}^n. \tag{4.8}$$

Proof. For showing existence, let $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$ and let $\varphi_{n-1} \in \mathbf{Val}_{n-1}^{\bar{e}}$ denote its associated zonal real valued valuation, that is

$$\varphi_{n-1}(K) := h(\Phi_{n-1}K, \bar{e}), \qquad K \in \mathcal{K}^n.$$

Due to Theorem 2.12, there exists a function $f \in \mathbf{C}_o(\mathbb{S})$ such that

$$\varphi_{n-1}(K) = \langle f, S_{n-1}(K, \cdot)\rangle, \qquad K \in \mathcal{K}^n.$$

Since φ_{n-1} is zonal, we have that for all $\vartheta \in \mathrm{SO}(n-1, \bar{e})$ and $K \in \mathcal{K}^n$,

$$\langle \vartheta f, S_{n-1}(K, \cdot)\rangle = \langle f, S_{n-1}(\vartheta^{-1}K, \cdot)\rangle = \varphi_{n-1}(\vartheta^{-1}K) = \varphi_{n-1}(K) = \langle f, S_{n-1}(K, \cdot)\rangle.$$

It follows that $f \in \mathbf{C}_0(\mathbb{S}, \bar{e})$. Moreover, using the $\mathrm{SO}(n)$-equivariance of Φ_{n-1}, we obtain that for all $\vartheta \in \mathrm{SO}(n)$,

$$\begin{aligned} h(\Phi_{n-1}K, \vartheta\bar{e}) &= h(\Phi_{n-1}\vartheta^{-1}K, \bar{e}) = \varphi_{n-1}(\vartheta^{-1}K) \\ &= \langle \vartheta f, S_{n-1}(K, \cdot)\rangle = (S_{n-1}(K, \cdot) * f)(\vartheta\bar{e}), \end{aligned}$$

showing (4.8).

For showing uniqueness, suppose that we have two functions $f, g \in \mathbf{C}_o(\mathbb{S}, \overline{e})$ that satisfy Theorem 4.4. Then

$$\langle f, S_{n-1}(K, \cdot)\rangle = \varphi_{n-1}(K) = \langle g, S_{n-1}(K, \cdot)\rangle$$

for all $K \in \mathcal{K}^n$, which implies that $f = g$. □

This representation theorem motivates the following definition:

Definition. If $j \in \{1, \ldots, n-1\}$, $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$, and $f \in \mathbf{C}_o(\mathbb{S}, \overline{e})$ such that

$$h(\Phi_j(K), \cdot) = S_j(K, \cdot) * f, \qquad K \in \mathcal{K}^n,$$

then we call f the *generating function* for Φ_j.

Generating functions are always unique if they exist. Hence this notion makes sense.

The representation theorem above states that every $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$ has a unique generating function.

Fix a Minkowski valuation $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$ with generating function $f \in \mathbf{C}_o(\mathbb{S}, \overline{e})$. Note that for convex bodies $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, the mixed area measure $S(K_1, \ldots, K_{n-1}; \cdot)$ satisfies the conditions of Minkowski's existence theorem, so there exists a *mixed convex body* $[K_1, \ldots, K_{n-1}] \in \mathcal{K}^n$ such that

$$S(K_1, \ldots, K_{n-1}; \cdot) = S_{n-1}([K_1, \ldots, K_{n-1}], \cdot).$$

Hence we we may define

$$\Phi : (\mathcal{K}^n)^{n-1} \to \mathcal{K}^n : \Phi(K_1, \ldots, K_{n-1}) := \Phi_{n-1}([K_1, \ldots, K_{n-1}])$$

and this map satisfies

$$h(\Phi(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}; \cdot) * f.$$

In particular, for $j \in \{0, \ldots, n-1\}$ we may define

$$\Phi_j : \mathcal{K}^n \to \mathcal{K}^n : h(\Phi_j K, \cdot) := S_j(K, \cdot) * f.$$

Clearly, $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$ and it admits the generating function f. Let us fix these Minkowski valuations Φ_j, $j \in \{0, \ldots, n-1\}$, for the remaining section.

An important feature of these Minkowski valuations is the symmetry and monotonicity properties discussed in the following two propositions.

Proposition 4.5 ([23, Lemma 6.2])**.** *Let $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, n-1\}$. Then*

$$\mathsf{W}_{n-i}(L, \Phi_j K) = \mathsf{W}_{n-j-1}(K, \Phi_{i-1} L) \tag{4.9}$$

for all $K, L \in \mathcal{K}^n$.

Proof. Denote by $f \in \mathbf{C}_o(\mathbb{S}, \bar{e})$ the generating function of Φ_j and Φ_{i-1}. Due to (1.7), we obtain

$$\begin{aligned} \mathsf{W}_{n-i}(L, \Phi_j K) &= \tfrac{1}{n}\langle h(\Phi_j L), S_{i-1}(K, \cdot)\rangle \\ &= \tfrac{1}{n}\langle S_j(L, \cdot) * f, S_{i-1}(K, \cdot)\rangle \\ &= \tfrac{1}{n}\langle S_{i-1}(L, \cdot) * f, S_j(K, \cdot)\rangle \\ &= \tfrac{1}{n}\langle h(\Phi_{i-1}L, \cdot), S_j(K, \cdot)\rangle \\ &= \mathsf{W}_{n-j-1}(K, \Phi_{i-1}L) \end{aligned}$$

for all $K, L \in \mathcal{K}^n$. □

Proposition 4.6. *Let $i \in \{2, \dots, n\}$, $j \in \{2, \dots, n-1\}$ and suppose that Φ_j is non-trivial. Then*

$$K \subseteq L \quad \Longrightarrow \quad \mathsf{W}_{n-i}(\Phi_j K) \leq \mathsf{W}_{n-i}(\Phi_j L) \tag{4.10}$$

for all $K, L \in \mathcal{K}^n$.

In particular, if $K \in \mathcal{K}^n$ is n-dimensional, then $\Phi_j K$ is at least i-dimensional.

Proof. Let $K, L \in \mathcal{K}^n$ with $K \subseteq L$ and define $L_\varepsilon := L + \varepsilon B$ for $\varepsilon > 0$. Then

$$h(\Phi_j L_\varepsilon, \cdot) = S_j(L + \varepsilon B, \cdot) * f = \sum_{\ell=0}^{j} \binom{j}{\ell} \varepsilon^{j-\ell} S_\ell(L, \cdot) * f = \sum_{\ell=0}^{j} \binom{j}{\ell} \varepsilon^{j-\ell} h(\Phi_\ell L, \cdot).$$

Since $\Phi_0 L = \Phi_0 B = \Phi_j B = r(\Phi_j)B$, it follows from Lemma 4.3 (ii) that $\Phi_j L_\varepsilon$ is full-dimensional. Due to (4.9) and the monotonicity of mixed volumes, we have for every $C \in \mathcal{K}^n$,

$$\mathsf{W}_{n-i}(C, \Phi_j L_\varepsilon) = \mathsf{W}_{n-j-1}(L_\varepsilon, \Phi_{i-1}C) \geq \mathsf{W}_{n-j-1}(K, \Phi_{i-1}C) = \mathsf{W}_{n-i}(C, \Phi_j K).$$

Thus, taking $C = \Phi_j L_\varepsilon$ and using the general Minkowski inequality (2.8) yield

$$\mathsf{W}_{n-i}(\Phi_j L_\varepsilon)^i \geq \mathsf{W}_{n-i}(\Phi_j L_\varepsilon, \Phi_j K)^i \geq \mathsf{W}_{n-i}(\Phi_j L_\varepsilon)^{i-1}\mathsf{W}_{n-i}(\Phi_j K).$$

Since $\mathsf{W}_{n-i}(L_\varepsilon) > 0$, we obtain $\mathsf{W}_{n-i}(\Phi_j K) \leq \mathsf{W}_{n-i}(\Phi_j L_\varepsilon)$. Due to Φ_j and W_{n-i} being continuous, (4.10) follows by taking the limit $\varepsilon \to 0$.

If $K \in \mathcal{K}^n$ is full-dimensional, then $rB + x \subseteq K$ for some $r > 0$ and $x \in \mathbb{R}^n$. Thus (4.10) implies that $\mathsf{W}_{n-i}(\Phi_j K) \geq \mathsf{W}_{n-i}(\Phi_j(rB + x)) > 0$, so $\dim K \geq i$. □

Note that this proposition entails that the Minkowski valuations Φ_j, $j \in \{2, \dots, n-1\}$, map n-dimensional convex bodies to n-dimensional convex bodies. It does, however, not directly follow that this is true for all non-trivial Minkowski valuations in $\mathbf{MVal}_j^{\mathrm{SO}(n)}$, since the Φ_j were not chosen arbitrarily. In fact, the question whether this is the case is currently an open problem.

The symmetry property (4.9) is the crucial ingredient in the proof of the following inequality. Proposition 4.6 will be important for establishing the equality conditions.

Theorem 4.7 ([23, Theorem 6.8]). *Let $i \in \{2, \dots, n\}$, $j \in \{2, \dots, n-1\}$, and suppose that Φ_j is non-trivial. Then*

$$\mathsf{V}_i(\Phi_j(K+L))^{1/ij} \geq \mathsf{V}_i(\Phi_j K)^{1/ij} + \mathsf{V}_i(\Phi_j L)^{1/ij} \tag{4.11}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ with equality if and only if K and L are homothetic.

Proof. First we prove the inequality, then we treat the equality cases.

▷ Define $C := \Phi_j(K+L)$ and denote by Φ_{i-1} the Minkowski valuation in $\mathbf{MVal}_{i-1}^{\mathrm{SO}(n)}$ with the same generating function as Φ_j. The general Brunn-Minkowski inequality (2.6) in combination with (4.9) yields

$$\begin{aligned}\mathsf{W}_{n-i}(C)^{1/j} &= \mathsf{W}_{n-i}(C, \Phi_j(K+L))^{1/j} \\ &= \mathsf{W}_{n-j-1}(K+L, \Phi_{i-1}C)^{1/j} \\ &\geq \mathsf{W}_{n-j-1}(K, \Phi_{i-1}C)^{1/j} + \mathsf{W}_{n-j-1}(L, \Phi_{i-1}C)^{1/j} \\ &= \mathsf{W}_{n-i}(C, \Phi_j K)^{1/j} + \mathsf{W}_{n-i}(C, \Phi_j)^{1/j}.\end{aligned}$$

Due to the general Minkowski inequality (2.8), we have

$$\mathsf{W}_{n-i}(C, \Phi_j K)^i \geq \mathsf{W}_{n-i}(C)^{i-1}\mathsf{W}_{n-i}(\Phi_j K), \tag{4.12}$$

$$\mathsf{W}_{n-i}(C, \Phi_j L)^i \geq \mathsf{W}_{n-i}(C)^{i-1}\mathsf{W}_{n-i}(\Phi_j L). \tag{4.13}$$

By combining these two inequalities with the inequality above, we obtain

$$\mathsf{W}_{n-i}(C)^{1/j} \geq \mathsf{W}_{n-i}(C)^{1-1/ij}\mathsf{W}_{n-i}(\Phi_j K)^{1/ij} + \mathsf{W}_{n-i}(C)^{1-1/ij}\mathsf{W}_{n-i}(\Phi_j L)^{1/ij}.$$

Due to the previous proposition, $\mathsf{W}_{n-i}(C) > 0$, hence

$$\mathsf{W}_{n-i}(C)^{1/ij} \geq \mathsf{W}_{n-i}(\Phi_j K)^{1/ij} + \mathsf{W}_{n-i}(\Phi_j L)^{1/ij}.$$

Putting $C = \Phi_j(K+L)$ and $\mathsf{W}_{n-i} = \kappa_{n-i}\binom{n}{i}^{-1}\mathsf{V}_i$ yields the desired inequality.

▷ Suppose now that equality holds in (4.11). Then equality must also hold in (4.12) and (4.13). Due to the previous proposition, $\Phi_j(K+L)$, $\Phi_j K$, and $\Phi_j L$ all have dimension at least i. Thus the equality condition of (2.8) in combination with Lemma 4.3 (ii) entails that there exist $\lambda, \mu > 0$ such that

$$\Phi_j K = \lambda \Phi_j(K+L) \quad \text{and} \quad \Phi_j L = \mu \Phi_j(K+L). \tag{4.14}$$

Plugging (4.14) into (4.11) gives

$$\lambda^{1/j} + \mu^{1/j} = 1$$

and plugging (4.14) into (4.7) gives

$$\mathsf{W}_{n-j}(K) = \lambda \mathsf{W}_{n-j}(K+L) \quad \text{and} \quad \mathsf{W}_{n-j}(L) = \mu \mathsf{W}_{n-j}(K+L).$$

Combining the two equations above yields

$$\begin{aligned}\mathsf{W}_{n-j}(K+L)^{1/j} &= (\lambda^{1/j}+\mu^{1/j})\mathsf{W}_{n-j}(K+L)^{1/j} \\ &= \mathsf{W}_{n-j}(K)^{1/j}+\mathsf{W}_{n-j}(L)^{1/j},\end{aligned}$$

so the equality condition for (2.7) implies that K and L are homothetic. □

In the special case, where $j = n-1$, we obtain:

Corollary 4.8 ([23, Theorem 6.8])**.** *Let $i \in \{2,\dots,n\}$ and let $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$ be non-trivial. Then*

$$\mathsf{V}_i(\Phi_{n-1}(K+L))^{1/i(n-1)} \geq \mathsf{V}_i(\Phi_{n-1}K)^{1/i(n-1)} + \mathsf{V}_i(\Phi_{n-1}L)^{1/i(n-1)}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ with equality if and only if K and L are homothetic.

Note that for $j \in \{2,\dots,n-2\}$ we can not derive such a corollary from Theorem 4.7 since the representation theorem at the beginning of this section only treats the case where $j = n-1$, and the Minkowski valuations $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$, $j \in \{2,\dots,n-2\}$, were not chosen arbitrarily.

4.2 The Steiner Formula for Minkowski Valuations

We have seen in the previous section that for every $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$, we can derive Minkowski valuations $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$, $j \in \{0,\dots,n-1\}$, such that

$$\Phi_{n-1}(K+\varepsilon B) = \sum_{j=0}^{n-1} \varepsilon^{n-1-j}\binom{n-1}{j}\Phi_j K$$

for all $K \in \mathcal{K}^n$ and $\varepsilon \geq 0$. Thus, Minkowski valuations in $\mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$ satisfy some kind of Steiner formula. In some sense, the fact that the Minkowski valuations Φ_j were derived from Φ_{n-1} in this way brought about the symmetry property (4.9), which in turn was the key to Theorem 4.7. This observation motivates the general Steiner formula for Minkowski valuations, which was proved by L. Parapatits and F. Schuster in 2011 and will be discussed in this section.

In fact, we have even obtained a polynomiality result similar to Theorem 2.8 for valuations $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^{\mathrm{SO}(n)}$, which is much stronger than Steiner's formula. Hence, one could ask whether such a polynomiality theorem can be proven for every $\Phi \in \mathbf{MVal}$, or equivalently, whether a McMullen decomposition theorem can be proven for every $\Phi \in \mathbf{MVal}$. In general, this is not the case, as was shown by T. Wannerer and L. Parapatits.

However, from the McMullen decomposition of **CVal** and the characterizations of V_0 and V_n, we obtain the following decomposition theorem for Minkowski valuations:

Lemma 4.9. *For every $\Phi \in \mathbf{MVal}$, there exist convex bodies $L_0, L_n \in \mathcal{K}^n$, and for every $j \in \{1, \dots, n-1\}$, a valuation in $\mathbf{CVal}_j$, denoted by $K \mapsto g_j(K, \cdot)$, such that*

$$h(\Phi K, \cdot) = h(L_0, \cdot) + \sum_{j=1}^{n-1} g_j(K, \cdot) + \mathsf{V}_n(K) h(L_n, \cdot), \qquad K \in \mathcal{K}^n. \tag{4.15}$$

Proof. The proof will be carried out in three steps: finding $g_j(K, \cdot)$, $j \in \{0, \dots, n\}$, showing that $g_0(K, \cdot)$ and $g_n(K, \cdot)$ are support functions, and finding L_0 and L_n.

▷ For the first step, consider $\overline{\Phi} : K \mapsto h(\Phi K, \cdot)$, the valuation in **CVal** associated with Φ. Via the McMullen decomposition of $\overline{\Phi}$, we obtain for every $j \in \{0, \dots, n\}$ a valuation in $\mathbf{CVal}_j$, denoted by $K \mapsto g_j(K, \cdot)$, such that

$$h(\Phi K, \cdot) = g_0(K, \cdot) + \sum_{j=1}^{n-1} g_j(K, \cdot) + g_n(K, \cdot), \qquad K \in \mathcal{K}^n.$$

▷ It only remains to take care of $g_0(K, \cdot)$ and $g_n(K, \cdot)$. To that end, extend all $g_j(K, \cdot)$ to 1-homogeneous functions on $\mathbb{R}^n$. Next, fix some $x, y \in \mathbb{R}^n$ and define a valuation $\varphi \in \mathbf{Val}$ by

$$\varphi(K) := h(\Phi K, x + y) - h(\Phi K, x) - h(\Phi K, y), \qquad K \in \mathcal{K}^n.$$

Then φ is always non-negative and the j-homogeneous component in its McMullen decomposition is given by

$$\varphi_j(K) = g_j(K, x + y) - g_j(K, x) - g_j(K, y), \qquad K \in \mathcal{K}^n. \tag{4.16}$$

For $K \in \mathcal{K}^n$ and $\lambda > 0$, we have

$$0 \le \varphi(\lambda K) = \varphi_0(K) + \sum_{j=1}^{n-1} \lambda^j \varphi_j(K) + \lambda^n \varphi_n(K).$$

Taking the limit $\lambda \to 0$ shows that $\varphi_0(K) \ge 0$; dividing by λ^n and taking the limit $\lambda \to +\infty$ shows that $\varphi_n(K) \ge 0$. Due to (4.16), it follows that $g_0(K, \cdot)$ and $g_n(K, \cdot)$ are support functions.

▷ Lastly, we need to find L_0 and L_n. To that end, fix $u \in \mathbb{S}$ and define

$$\psi_0(K) := g_0(K, u), \qquad \psi_n(K) := g_n(K, u), \qquad K \in \mathcal{K}^n.$$

Then $\psi_0 \in \mathbf{Val}_0$ and $\psi_n \in \mathbf{Val}_n$. Since $\mathbf{Val}_0$ and $\mathbf{Val}_n$ are spanned by V_0 and V_n, respectively, there are functions $h_0, h_n : \mathbb{S} \to \mathbb{R}$ such that

$$g_0(K, u) = h_0(u) \quad \text{and} \quad g_n(K, u) = \mathsf{V}_n(K) h_n(u).$$

From the fact that $g_0(K, \cdot)$, $g_n(K, \cdot)$ are support functions follows that h_0, h_n are the support functions of some convex bodies $L_0, L_n \in \mathcal{K}^n$. This completes the proof. □

Hadwiger's volume characterization theorem entails that for every valuation $\varphi_j \in \mathbf{Val}_j$, $j \in \{0, \ldots, n\}$, and every subspace $E \in \mathrm{G}(n, j)$, the restricted valuation $\varphi_j|_E : \mathcal{K}^j(E) \to \mathbb{R}$ is a multiple of the intrinsic volume V_j. This observation gives rise to the following map:

Definition ([17])**.** Let $j \in \{1, \ldots, n-1\}$. The *Klain map*

$$\mathrm{Kl}_j : \mathbf{Val}_j^+ \to \mathbf{C}(\mathrm{G}(n, j))$$

is defined by

$$\varphi|_E = (\mathrm{Kl}_j \varphi)(E)\mathsf{V}_j, \qquad \varphi \in \mathbf{Val}_j^+, \; E \in \mathrm{G}(n, j).$$

We call $\mathrm{Kl}_j\varphi : \mathrm{G}(n, j) \to \mathbb{R}$ the *Klain function* of φ.

Theorem 4.10 ([21, Theorem 6.4.11])**.** *For $j \in \{1, \ldots, n-1\}$, the Klain map Kl_j is injective.*

This means that every valuation $\varphi \in \mathbf{Val}_j^+$, $j \in \{1, \ldots, n-1\}$, is determined by its Klain function $\mathrm{Kl}_j\varphi$. Thus, it is natural to ask how a valuation $\varphi \in \mathbf{Val}_j^+$ can be reconstructed by its Klain function. The following theorem gives such a description for zonoids:

Theorem 4.11 ([21, Theorem 6.4.12])**.** *Let $\varphi_j \in \mathbf{Val}_j$, $j \in \{1, \ldots, n-1\}$. If $Z \in \mathcal{K}^n$ is a zonoid with generating measure $\mu \in \mathbf{M}(\mathbb{S})$, then*

$$\varphi_j(Z) = \frac{2^j}{j!} \int_{\mathbb{S}} \cdots \int_{\mathbb{S}} (\mathrm{Kl}_j\varphi)(\mathrm{lin}\{u_1, \ldots, u_j\})[u_1, \ldots, u_j] \; d\mu(u_1) \cdots d\mu(u_j),$$

where $[u_1, \ldots, u_j]$ denotes the j-dimensional volume of the parallelepiped spanned by the vectors $u_1, \ldots, u_j$.

As a consequence, we are able to prove a McMullen decomposition like theorem for Minkowski valuations for zonoids.

Theorem 4.12 ([20, Theorem 4.2])**.** *For every $\Phi \in \mathbf{MVal}$ and zonoid $Z \in \mathcal{K}^n$, there exist convex bodies $\Phi_0(Z), \ldots, \Phi_n(Z) \in \mathcal{K}^n$ such that*

$$\Phi(\lambda Z) = \sum_{j=0}^{n} \lambda^j \Phi_j(Z), \qquad \lambda > 0.$$

Proof. The strategy will be to define a suitable valuation, show that its homogeneous components are non-negative for zonoids, and then deduce the statement from that.

$\triangleright$ Let $\Phi \in \mathbf{MVal}$ and $K \in \mathcal{K}^n$. Then, by Lemma 4.9, there exist convex bodies $L_0, L_n \in \mathcal{K}^n$ and continuous functions $g_j(K, \cdot) \in \mathbf{C}(\mathbb{S})$, $j \in \{1, \ldots, n-1\}$, such that

$$h(\Phi K, \cdot) = h(L_0, \cdot) + \sum_{j=1}^{n-1} \lambda^j g_j(K, \cdot) + \lambda^n \mathsf{V}_n(K) h(L_n, \cdot), \qquad \lambda > 0.$$

Extend the $g_j(K,\cdot)$ to 1-homogeneous functions on $\mathbb{R}^n$, and for fixed $x, y \in \mathbb{R}^n$, define a valuation $\varphi \in \mathbf{Val}$ by

$$\varphi(K) := h(\Phi K, x+y) - h(\Phi K, x) - h(\Phi K, y), \qquad K \in \mathcal{K}^n.$$

Then φ is always non-negative and for $j \in \{0, \dots, n\}$, the j-homogeneous component in its McMullen decomposition is given by

$$\varphi_j(K) = g_j(K, x+y) - g_j(K,x) - g_j(K,y), \qquad K \in \mathcal{K}^n, \tag{4.17}$$

where $g_0(K,\cdot) := h(L_0,\cdot)$ and $g_n(K,\cdot) := \mathsf{V}_n(K)h(L_n,\cdot)$. In particular, φ_0 and φ_n are non-negative.

$\triangleright$ In order to show that the components φ_j, $j \in \{1, \dots, n-1\}$, are also non-negative, we will make use of the Klain map. To that end, take $j \in \{1, \dots, n-1\}$ and $E \in \mathrm{G}(n,j)$. Then, $\lambda B \cap E \in \mathcal{K}^j(E)$ for $\lambda > 0$. For every $i > j$, choose some $E' \in \mathrm{G}(n,i)$ such that $E \subseteq E'$ and note that due to Hadwiger's volume characterization theorem, $\varphi_i|_{E'}$ is a scalar multiple of V_i. Consequently, $\varphi_i(\lambda B \cap E) = 0$ for all $i > j$, so we obtain

$$0 \le \varphi(\lambda B \cap E) = \sum_{i=0}^{j-1} \lambda^i \varphi_i(B \cap E) + \lambda^j \varphi_j(B \cap E), \qquad \lambda > 0.$$

Dividing by λ^j and taking the limit $\lambda \to +\infty$ shows that $\varphi_j(B \cap E) \ge 0$. Denote by $\varphi_j^{\pm}$ the even and odd part of φ_j, respectively. Since $B \cap E$ is centrally symmetric, $\varphi^-(B \cap E) = 0$ and thus

$$(\mathrm{Kl}_j \varphi_j^+)(E)\kappa_j = (\mathrm{Kl}_j \varphi_j^+)(E)\mathsf{V}_j(B \cap E) = \varphi_j^+(B \cap E) = \varphi_j(B \cap E) \ge 0.$$

This argument shows that $\mathrm{Kl}_j \varphi_j^+$ is a non-negative function. Hence, Theorem 4.11 entails that $\varphi_j^+(Z) \ge 0$ for all zonoids $Z \in \mathcal{K}^n$. Since all zonoids are point symmetric, it follows that $\varphi_j^-(Z) = 0$ and thus $\varphi_j(Z) \ge 0$.

$\triangleright$ Due to (4.17), it follows that whenever $Z \in \mathcal{K}^n$ is a zonoid, $g_j(Z,\cdot)$ is a support function for $j \in \{0, \dots, n\}$. Hence, there exist convex bodies $\Phi_j(Z)$, $j \in \{0, \dots, n\}$, such that $g_j(Z,\cdot) = h(\Phi_j(Z),\cdot)$ which completes the proof. $\square$

As a consequence, we obtain the *Steiner formula for Minkowski valuations*:

Theorem 4.13 ([20, Theorem 2]). *For every $\Phi \in \mathbf{MVal}$, there are unique $\Phi^{(j)} \in \mathbf{MVal}$, $j \in \{0, \dots, n\}$, such that*

$$\Phi(K + \varepsilon B) = \sum_{j=0}^{n} \varepsilon^{n-j} \Phi^{(j)}(K), \qquad K \in \mathcal{K}^n, \ \varepsilon \ge 0. \tag{4.18}$$

Proof. Let $K \in \mathcal{K}^n$ be fixed and define

$$\Phi^K : \mathcal{K}^n \to \mathcal{K}^n : \Phi^K(L) := \Phi(K+L), \qquad L \in \mathcal{K}^n.$$

Observe that $\Phi^K \in \mathbf{MVal}$. By Theorem 4.12, there exist convex bodies $\Phi_j^K(B)$, $j \in \{0,\dots,n\}$, such that

$$\Phi(K+\varepsilon B) = \Phi^K(\varepsilon B) = \sum_{j=0}^{n} \varepsilon^j \Phi_j^K(B), \qquad \varepsilon > 0.$$

Define $\Phi^{(j)}$ by

$$\Phi^{(j)} : \mathcal{K}^n \to \mathcal{K}^n : \Phi^{(j)}(K) := \Phi_{n-j}^K(B).$$

Then clearly (4.18) is satisfied and it remains only to show that $\Phi^{(j)} \in \mathbf{MVal}$. To that end, consider the valuation in $\mathbf{CVal}$ associated with Φ, denoted by $\overline{\Phi} : K \mapsto h(\Phi K, \cdot)$. From applying the Steiner formula to $\overline{\Phi}$ and the uniqueness of its derived valuations $\overline{\Phi}^{(j)}$ follows that $\overline{\Phi^{(j)}} = \overline{\Phi}^{(j)} \in \mathbf{CVal}$, and thus $\Phi^{(j)} \in \mathbf{MVal}$. □

4.3 A Derivation Operator on Minkowski Valuations

The Steiner formula for valuations in **Val** gives rise to a derivation operator:

Definition. We define the *derivation operator*

$$\Lambda : \mathbf{Val} \to \mathbf{Val},$$
$$(\Lambda\varphi)(K) := \left.\frac{d}{dt}\right|_{t=0} \varphi(K+tB).$$

Note that Λ is linear and $\Lambda(\mathbf{Val}_j) \subseteq \mathbf{Val}_{j-1}$ for $j \in \{1,\dots,n\}$.

Now that we also have a Steiner formula for Minkowski valuations in **MVal** available, we can analogously define such a derivation operator on **MVal**:

Definition. We define the *derivation operator*

$$\Lambda : \mathbf{MVal} \to \mathbf{MVal},$$
$$h((\Lambda\Phi)(K), u) := \left.\frac{d}{dt}\right|_{t=0} h(\Phi(K+tB), u), \qquad u \in \mathbb{S}.$$

Note that $\Lambda(\mathbf{MVal}_j) \subseteq \mathbf{MVal}_{j-1}$ for $j \in \{1,\dots,n\}$.

Observe that the derivation operator on **MVal** corresponds to the derivation operator on **Val** by means of the associated real valued valuation of Minkowski valuation. That is to say, the diagram

$$\begin{array}{ccc} \mathbf{MVal}_j & \xrightarrow{\Lambda} & \mathbf{MVal}_{j-1} \\ \downarrow & & \downarrow \\ \mathbf{Val}_j & \xrightarrow{\Lambda} & \mathbf{Val}_{j-1} \end{array}$$

commutes, where the downward arrows represent the map assigning to a Minkowski valuation its associated real valued valuation.

This derivation operator on **MVal** will allow us to generalize the symmetry property (4.9). Recall that (4.9) was a consequence of the self-adjointness of convolution operators. Now that we do not have this tool available, we need to base our argument on a stronger tool, which will be provided by the symmetry theorem for bivaluations. In order to use this tool, we need to define some operators on the space **BVal**.

Definition. We define *derivation operators* $\Lambda_1, \Lambda_2 : \mathbf{BVal} \to \mathbf{BVal}$ by

$$(\Lambda_1\phi)(K,L) := \frac{d}{dt}\Big|_{t=0} \phi(K+tB, L), \qquad (\Lambda_2\phi)(K,L) := \frac{d}{dt}\Big|_{t=0} \phi(K, L+tB).$$

We define an operator $T : \mathbf{BVal} \to \mathbf{BVal}$ by

$$(T\phi)(K,L) := \phi(L,K).$$

Note that for $i,j \in \{0,\dots,n\}$, we have $\Lambda_1(\mathbf{BVal}_{i,j}) \subseteq \mathbf{BVal}_{i-1,j}$ if $i > 0$ and $\Lambda_2(\mathbf{BVal}_{i,j}) \subseteq \mathbf{BVal}_{i,j-1}$ if $j > 0$. Moreover, T is an isometric automorphism on the Banach space **BVal**, acting as an isomorphism between $\mathbf{BVal}_{i,j}$ and $\mathbf{BVal}_{j,i}$ for $i,j \in \{0,\dots,n\}$. These operators interact expectantly, which is expressed in the following lemma:

Lemma 4.14 ([20, Corollary 6.3])**.** *For all $k \in \mathbb{N}_0$, we have $T\Lambda_1^k = \Lambda_2^k T$.*

Proof. By the definition of Λ_1, T, and Λ_2,

$$(\Lambda_1\phi)(K,L) = \frac{d}{dt}\Big|_{t=0} \phi(K+tB, L) = \frac{d}{dt}\Big|_{t=0} (T\phi)(L, K+tB) = (\Lambda_2 T\phi)(L,K),$$

showing that $\Lambda_1 = T\Lambda_2 T$. From this equality and the fact that T is an involution, we deduce that

$$T\Lambda_1^k = T(T\Lambda_2 T)^k = TT\Lambda_2^k T = \Lambda_2^k T$$

for all $k \in \mathbb{N}_0$. □

Using the symmetry theorem for bivaluations, we are now able to prove a symmetry property of Minkowski valuations that generalizes (4.9).

Proposition 4.15 ([20, Corollary 6.4]). *Let* $i \in \{1, \dots, n\}$, $j \in \{2, \dots, n-1\}$, *and* $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$. *If* $i \leq j+1$, *then*

$$\mathsf{W}_{n-i}(K, \Phi_j L) = \frac{(i-1)!}{j!} \mathsf{W}_{n-j-1}(L, (\Lambda^{j-i+1}\Phi_j)(K)) \tag{4.19}$$

for all $K, L \in \mathcal{K}^n$.

Proof. Define $\phi \in \mathbf{BVal}_{j,j}$ by

$$\phi(K, L) := \mathsf{W}_{n-j-1}(K, \Phi_j L), \qquad K, L \in \mathcal{K}^n.$$

Due to (2.4) , $T\phi = \phi$, so the previous lemma yields

$$(\Lambda_1^k \phi)(K, L) = (\Lambda_2^k \phi)(L, K)$$

for all $k \in \mathbb{N}_0$. Let us now compute both sides of this equation. By induction on $k \in \{0, \dots, j\}$, we obtain

$$(\Lambda_1^k \phi)(K, L) = \frac{j!}{(j-k)!} \mathsf{W}_{n-j+k-1}(K, \Phi_j L)$$

and

$$(\Lambda_2^k \phi)(L, K) = \mathsf{W}_{n-j-1}(L, (\Lambda^k \Phi_j)(K)).$$

Hence,

$$\mathsf{W}_{n-j+k-1}(K, \Phi_j L) = \frac{(j-k)!}{j!} \mathsf{W}_{n-j-1}(L, (\Lambda^k \Phi_j)(K)).$$

Putting $k = j - i + 1$ yields (4.19). □

Proposition 4.16. *Let* $i \in \{2, \dots, n\}$, $j \in \{2, \dots, n-1\}$ *and let* $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$ *be non-trivial. If* $i \leq j+1$, *then*

$$K \subseteq L \quad \Longrightarrow \quad \mathsf{W}_{n-i}(\Phi_j K) \leq \mathsf{W}_{n-i}(\Phi_j L) \tag{4.20}$$

for all $K, L \in \mathcal{K}^n$.

In particular, if $K \in \mathcal{K}^n$ *is* n*-dimensional, then* $\Phi_j K$ *is at least* i*-dimensional.*

This proposition can be proven similarly as Proposition 4.6, by using the Steiner formula for Minkowski valuations and (4.19) in place of (4.9).

The symmetry and monotonicity properties above again provide a Brunn-Minkowski inequality for Minkowski valuations.

Theorem 4.17 ([20, Theorem 7.1])**.** *Let $i \in \{2, \ldots, n\}$, $j \in \{2, \ldots, n-1\}$, and let $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n)}$ be non-trivial. If $i \leq j+1$, then*

$$\mathsf{V}_i(\Phi_j(K+L))^{1/ij} \geq \mathsf{V}_i(\Phi_j K)^{1/ij} + \mathsf{V}_i(\Phi_j L)^{1/ij} \tag{4.21}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ with equality if and only if K and L are homothetic.

This theorem can be proven analogously as Corollary 4.8, with (4.19) in place of (4.9) and Proposition 4.16 in place of Proposition 4.6.

4.4 Smooth Minkowski Valuations

Observe that the two results regarding Question 1, that we have discussed so far, Theorem 4.7 and Theorem 4.17, were obtained in a similar way: Derive Minkowski valuations of lesser degrees from other Minkowski valuations, proof a symmetry and monotonicity theorem, and then obtain a generalized Brunn-Minkowski inequality. Hence, in order to achieve further generalizations of the past results, it seems natural to ask for a process that reverses the action of the derivation operator in some way. It turns out that we can achieve that by imposing a condition of smoothness on the Minkowski valuations involved.

Recall that for a Minkowski valuation $\Phi \in \mathbf{MVal}$, its associated real valued valuation $\varphi \in \mathbf{Val}$ is defined as

$$\varphi : \mathcal{K}^n \to \mathbb{R} : \varphi(K) := h(\Phi K, \bar{e})$$

and note that if $\Phi \in \mathbf{MVal}^{\mathrm{SO}(n)}$, then it is uniquely determined by its associated real valued valuation.

Definition. A Minkowski valuation $\Phi \in \mathbf{MVal}$ is *smooth* if its associated real valued valuation is smooth.

We denote by $\mathbf{MVal}^\infty$ the space of smooth Minkowski valuations in $\mathbf{MVal}$ is and define $\mathbf{MVal}_j^\infty := \mathbf{MVal}_j \cap \mathbf{MVal}^\infty$.

Moreover, we denote by $\mathbf{MVal}^{\mathrm{SO}(n),\infty}$ and $\mathbf{MVal}_j^{\mathrm{SO}(n),\infty}$ the spaces of $\mathrm{SO}(n)$-equivariant Minkowski valuations in $\mathbf{MVal}^\infty$ and $\mathbf{MVal}_j^\infty$, respectively.

A recent result by F. Schuster and T. Wannerer characterizes zonal, translation invariant, smooth valuations.

Theorem 4.18 ([22, Proposition 3.6, Theorem 5.1])**.** *For every $\varphi \in \mathbf{Val}_j^{\infty,\bar{e}}$, $j \in \{1, \ldots, n-1\}$, there is a unique $f \in \mathbf{C}_o^\infty(\mathbb{S}, \bar{e})$ such that*

$$\varphi(K) = \langle f, S_j(K, \cdot)\rangle, \qquad K \in \mathcal{K}^n.$$

An immediate consequence is a representation theorem for smooth Minkowski valuations, which can be derived from the theorem above in the same way that Theorem 4.4 was derived from Theorem 2.12.

Theorem 4.19 ([22, Theorem 4.2]). *For every $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n),\infty}$, $j \in \{1, \ldots, n-1\}$, there exists a unique $f \in \mathbf{C}_o^\infty(\mathbb{S}, \overline{e})$, such that*

$$h(\Phi_j K, \cdot) = S_j(K, \cdot) * f, \qquad K \in \mathcal{K}^n.$$

Now that we have this representation theorem at hand, it is natural to ask how the derivation operator Λ acts on generating functions. It turns out that its action on generating functions has a very nice description which also entails injectivity.

Lemma 4.20. *Let $j \in \{2, \ldots, n-1\}$. If $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n),\infty}$ has the generating function $f \in \mathbf{C}_o(\mathbb{S}, \overline{e})$, then $\Lambda\Phi_j \in \mathbf{MVal}_{j-1}^{\mathrm{SO}(n),\infty}$ and its generating function is given by jf.*

In particular, $\Lambda : \mathbf{MVal}_j^{\mathrm{SO}(n),\infty} \to \mathbf{MVal}_{j-1}^{\mathrm{SO}(n),\infty}$ is injective.

Proof. For every $u \in \mathbb{S}$, we have

$$h((\Lambda\Phi_j)(K), u) = \left.\frac{d}{dt}\right|_{t=0} [S_j(K + tB, \cdot) * f](u) = [S_{j-1}(K, \cdot) * jf](u),$$

as can easily be seen by expanding the term $S_j(K + tB, \cdot)$. □

Consequently, the diagram

$$\begin{array}{ccc} \mathbf{MVal}_j^{\mathrm{SO}(n),\infty} & \xrightarrow{\Lambda} & \mathbf{MVal}_{j-1}^{\mathrm{SO}(n),\infty} \\ \downarrow & & \downarrow \\ \mathbf{C}_o^\infty(\mathbb{S}) & \xrightarrow{jI} & \mathbf{C}_o^\infty(\mathbb{S}) \end{array}$$

commutes, where the downward arrows represent the map assigning to a smooth homogeneous Minkowski valuation its generating function.

We define some further subspaces of $\mathbf{MVal}^{\mathrm{SO}(n)}$.

Definition. For $i \in \{1, \ldots, n\}$ and $j \in \{0, \ldots, n\}$, we define

$$\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)} := \begin{cases} \mathbf{MVal}_j^{\mathrm{SO}(n)}, & i \leq j+1, \\ \Lambda^{i-j-1}(\mathbf{MVal}_{i-1}^{\mathrm{SO}(n)}), & i > j+1, \end{cases}$$

and

$$\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n),\infty} := \begin{cases} \mathbf{MVal}_j^{\mathrm{SO}(n),\infty}, & i \leq j+1, \\ \Lambda^{i-j-1}(\mathbf{MVal}_{i-1}^{\mathrm{SO}(n),\infty}), & i > j+1. \end{cases}$$

Note that $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)} \subseteq \mathbf{MVal}_j^{\mathrm{SO}(n)}$ and $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n),\infty} \subseteq \mathbf{MVal}_j^{\mathrm{SO}(n),\infty}$. Moreover, the operator

$$\Lambda^{j-i+1} : \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n),\infty} \to \mathbf{MVal}_{i-1}^{\mathrm{SO}(n),\infty}$$

is well-defined.

For the purpose of the injectivity result on Λ, we restricted ourselves to smooth Minkowski valuations. In order to generalize the results to non-smooth Minkowksi valuations, it is natural to make an argument of density and continuity. The density result required for that is provided by the following theorem:

Theorem 4.21 ([22, Corollary 5.4])**.** *Let $i \in \{1,\dots,n\}$ and $j \in \{0,\dots,n\}$. Then $\mathbf{MVal}_j^{\mathrm{SO}(n),\infty}$ is dense in $\mathbf{MVal}_j^{\mathrm{SO}(n)}$ and $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n),\infty}$ is dense in $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$.*

Using the way Λ acts on generating functions and the self-adjointness of convolution operators, we can give a short proof of a symmetry property for $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n),\infty}$.

Proposition 4.22 ([20, Corollary 6.4])**.** *Let $i \in \{1,\dots,n\}$, $j \in \{1,\dots,n-1\}$, and $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n),\infty}$. Then*

$$\mathsf{W}_{n-i}(K,\Phi_j L) = \frac{(i-1)!}{j!}\mathsf{W}_{n-j-1}(L,(\Lambda^{j-i+1}\Phi_j)(K)) \tag{4.22}$$

for all $K, L \in \mathcal{K}^n$.

Proof. Due to (1.7) and Lemma 4.20, we have

$$\begin{aligned}
\mathsf{W}_{n-i}(L,\Phi_j K) &= \tfrac{1}{n}\langle h(\Phi_j L), S_{i-1}(K,\cdot)\rangle \\
&= \tfrac{1}{n}\langle S_j(L,\cdot) * f, S_{i-1}(K,\cdot)\rangle \\
&= \tfrac{1}{n}\langle S_{i-1}(L,\cdot) * f, S_j(K,\cdot)\rangle \\
&= \tfrac{(i-1)!}{j!}\tfrac{1}{n}\langle h((\Lambda^{j-i+1}\Phi_j)(K),\cdot), S_j(K,\cdot)\rangle \\
&= \tfrac{(i-1)!}{j!}\mathsf{W}_{n-j-1}(K,(\Lambda^{j-i+1}\Phi_j)(K)).
\end{aligned}$$

□

Moreover, we obtain a monotonicity result for Minkowski valuations in $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$.

Proposition 4.23. *Let $i \in \{2,\dots,n\}$, $j \in \{2,\dots,n-1\}$ and let $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$ be non-trivial. Then*

$$K \subseteq L \quad \Longrightarrow \quad \mathsf{W}_{n-i}(\Phi_j K) \leq \mathsf{W}_{n-i}(\Phi_j L). \tag{4.23}$$

for all $K, L \in \mathcal{K}^n$.

In particular, if $K \in \mathcal{K}^n$ is n-dimensional, then $\Phi_j K$ is at least i-dimensional.

The proof is analogous to the proof of Proposition 4.6, where the previous proposition plays the role of Proposition 4.5. The smoothness condition on Φ_j can be dropped by an argument of density and continuity employing Theorem 4.21.

Combining the symmetry and monotonicity results stated above, we obtain the most recent result regarding Question 1.

Theorem 4.24 ([4, Theorem 6.5]). *Let $i \in \{2, \dots, n\}$, $j \in \{2, \dots, n-1\}$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$ be non-trivial. Then*

$$\mathsf{V}_i(\Phi_j(K+L))^{1/ij} \geq \mathsf{V}_i(\Phi_j K)^{1/ij} + \mathsf{V}_i(\Phi_j L)^{1/ij}. \tag{4.24}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ with equality if and only if K and L are homothetic.

The proof is analogous to the proof of (4.11), with (4.22) in place of (4.9) and Proposition 4.23 in place of Proposition 4.6. Like before, the smoothness condition on Φ_j can be dropped by an argument of density and continuity employing Theorem 4.21.

By virtue of Lemma 4.2, we can formulate a concavity and a multiplicative log-concavity version of this theorem. The multiplicative version stated below is (almost) how this result was originally formulated in [4].

Theorem 4.25. *Let $i \in \{2, \dots, n\}$, $j \in \{2, \dots, n-1\}$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$ be non-trivial. Then*

$$\mathsf{V}_i(\Phi_j((1-\lambda)K + \lambda L))^{1/ij} \geq (1-\lambda)\mathsf{V}_i(\Phi_j K)^{1/ij} + \lambda \mathsf{V}_i(\Phi_j L)^{1/ij}. \tag{4.25}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$ with equality if and only if K and L are homothetic.

Theorem 4.26. *Let $i \in \{2, \dots, n\}$, $j \in \{2, \dots, n-1\}$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$ be non-trivial. Then*

$$\mathsf{V}_i(\Phi_j((1-\lambda)K + \lambda L)) \geq \mathsf{V}_i(\Phi_j K)^{1-\lambda} \mathsf{V}_i(\Phi_j L)^{\lambda}. \tag{4.26}$$

for all n-dimensional $K, L \in \mathcal{K}^n$ and $\lambda \in (0,1)$ with equality if and only if K and L are translates of each other.

We conclude this chapter by giving the proof for the trivial cases regarding Question 1.

Proof of Lemma 4.1. If $j = 1$, then Theorem 4.19 in combination with Theorem 4.21 entail that every $\Phi_1 \in \mathbf{MVal}_1^{\mathrm{SO}(n)}$ is Minkowski linear, that is

$$\Phi_1(\lambda K + \mu L) = \lambda \Phi_1 K + \mu \Phi_1 L$$

for all $K, L \in \mathcal{K}^n$ and $\lambda, \mu \geq 0$. Consequently, if $i = j = 1$, then equality always holds in (4.2). If $i > 1$ and $j = 1$, then (4.2) reduces to the Brunn-Minkowski inequality for intrinsic volumes.

If $i = 1$ and $j > 1$, then (4.2) also reduces to the Brunn-Minkowski inequality for intrinsic volumes due to (4.7).

If $j = n$, then Hadwiger's volume characterization theorem entails that $\Phi_n \in \mathbf{MVal}_n^{\mathrm{SO}(n)}$ can be represented as $\Phi_n K = c_n \mathsf{V}_n(K) B$ for some constant $c_n > 0$. Consequently, (4.2) reduces to the classical Brunn-Minkowski inequality. □

Chapter 5

Log-Concavity of Mean Section Operators

As was discussed in the previous chapter, a satisfactory answer to Question 1 is provided by Theorem 4.24 in the cases where $i \in \{1, \dots, n\}$, $j \in \{2, \dots, n-1\}$, and $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$. However, in general

$$\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)} \subsetneq \mathbf{MVal}_{j}^{\mathrm{SO}(n)},$$

so Theorem 4.24 does not cover all non-trivial cases. For $i = n$, an explicit example of a Minkowski valuation beyond the covered cases is provided by the centered mean section operators. That is, if $j \in \{2, \dots, n-2\}$, then

$$M_{n-j+1} \in \mathbf{MVal}_{j}^{\mathrm{SO}(n)} \backslash \mathbf{MVal}_{j,n-1}^{\mathrm{SO}(n)}.$$

This follows from Theorem 4.4 and the fact that for $k \geq 3$, Berg's function g_k does not extend to a $\mathbf{C}[-1, 1]$ function.

In this chapter we will present some of the most recent research on this topic in Section 5.1 and then give a brief outlook on a possible strategy for tackling the open cases in Section 5.2

5.1 An Integration Operator on Minkowski Valuations

We have already seen that the derivation operator Λ is injective on $\mathbf{MVal}_{j}^{\mathrm{SO}(n),\infty}$ for $j \in \{2, \dots, n-1\}$. This fact helped us to obtain a general Brunn-Minkowsi inequality for Minkowski valuations in $\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n)}$ that can be looked at as derivatives of a sufficiently high order, since these are (up to a topological closure) precisely the Minkowski valuations on which we can apply the inverse of the derivative operator sufficiently often.

Therefore, it is natural to look for an integration operator on $\mathbf{MVal}^{\mathrm{SO}(n)}$ which, when applied to a homogeneous Minkowski valuation, increases its degree of homogeneity, and which allows for a symmetry formula dual to Proposition 4.22. Like we did with the derivation operator Λ, we first consider an operator on **Val**.

Definition ([1, 2])**.** We define the *integration operator*

$$\mathfrak{L} : \mathbf{Val} \to \mathbf{Val},$$
$$(\mathfrak{L}\varphi)(K) := \begin{bmatrix} n \\ 1 \end{bmatrix} \int_{\mathrm{A}(n,n-1)} \varphi(K \cap E)\, dE.$$

It is not hard to see that $\mathfrak{L}$ is well-defined. If $\varphi \in \mathbf{Val}$, then obviously $\mathfrak{L}\varphi$ is a valuation. The translation invariance follows from the translation invariance of the measure ν_k^n on $\mathrm{A}(n, n-1)$ and the continuity follows from the continuity of intersections (cf. [21, Theorem 1.8.10]). Moreover, due to the homogeneity of ν_k^n, we have $\mathfrak{L}(\mathbf{Val}_j) \subseteq \mathbf{Val}_{j+1}$ for $j \in \{0, \ldots, n-1\}$.

On smooth valuations, the action of $\mathfrak{L}$ can be described by means of the Alesker product:

Theorem 5.1 ([6, Theorem 3])**.** *$\mathfrak{L}$ acts on $\mathbf{Val}^\infty$ by*

$$\mathfrak{L} : \mathbf{Val}^\infty \to \mathbf{Val}^\infty : \mathfrak{L}\varphi = \mathsf{V}_1 \cdot \varphi. \tag{5.1}$$

Historically, the original definition of $\mathfrak{L}$ was given by (5.1) and then it was proved by A. Bernig that this equals the integral representation. However, we prefer defining $\mathfrak{L}$ via the integral formula for the benefit of having the operator defined on the whole space **Val** to begin with.

In the same way that we lifted Λ from a derivation operator on **Val** to a derivation operator on **MVal**, we can deal with $\mathfrak{L}$.

Definition. We define the *integration operator*

$$\mathfrak{L} : \mathbf{MVal} \to \mathbf{MVal},$$
$$h((\mathfrak{L}\Phi)(K), u) := \begin{bmatrix} n \\ 1 \end{bmatrix} \int_{\mathrm{A}(n,n-1)} h(\Phi(K \cap E), u)\, dE.$$

Again, it is not hard to see that this operator is well defined, and that $\mathfrak{L}(\mathbf{MVal}_j) \subseteq \mathbf{MVal}_{j+1}$ for $j \in \{0, \ldots, n-1\}$.

Moreover, by definition the diagram

$$\begin{array}{ccc} \mathbf{MVal}_j & \xrightarrow{\mathfrak{L}} & \mathbf{MVal}_{j+1} \\ \downarrow & & \downarrow \\ \mathbf{Val}_j & \xrightarrow{\mathfrak{L}} & \mathbf{Val}_{j+1} \end{array}$$

commutes, where the downward arrows represent the map assigning to a Minkowski valuation its associated real valued valuation.

Like we did with the derivation operator Λ in Section 4.4, we seek to describe the action of the integration operator $\mathfrak{L}$ on generating functions of smooth Minkowski valuations. It turns out that this is much more involved than it was for Λ and interestingly, Berg's functions come up.

Theorem 5.2 ([4, Theorem 4.4]). *Let $j \in \{2, \dots, n-1\}$. If $k \neq 1$, then*

$$a_k^n[g_j] = -\frac{\pi^{(n-j)/2}(j-1)}{4} \frac{\Gamma\left(\frac{n-j+1}{2}\right)\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{k+j-1}{2}\right)}{\Gamma\left(\frac{n+k-j+1}{2}\right)\Gamma\left(\frac{n+k+1}{2}\right)}$$

and $a_1^n[g_j] = 0$.

Note that in particular, $a_0^n[g_j] > 0$ and $a_k^n[g_j] < 0$ for all $k \geq 2$. Hence the multipliers $a_k^n[g_j]$, $k \neq 1$, are non-zero and they are slowly decreasing, so as an immediate consequence of the Funk-Hecke theorem and Lemma 1.11, we obtain:

Theorem 5.3 ([4, Theorem 4.3]). *For $j \in \{2, \dots, n\}$, the convolution operator*

$$. * \breve{g}_j : \mathbf{C}_o^\infty(\mathbb{S}) \to \mathbf{C}_o^\infty(\mathbb{S}) : f \mapsto f * \breve{g}_j$$

is an automorphism of the Fréchet space $\mathbf{C}_o^\infty(\mathbb{S})$.

This allows us to make the following generalization of the operator $\square_n$:

Definition. For $j \in \{2, \dots, n\}$, we define

$$\square_j : \mathbf{C}_o^\infty(\mathbb{S}) \to \mathbf{C}_o^\infty(\mathbb{S})$$

as the inverse operator of $. * \breve{g}_j$.

Next, we take a look at the action of $\mathfrak{L}$ on centered mean section operators.

Example 5.4 ([25, Chapter 6]). For $\Phi \in \mathbf{MVal}$ and $j \in \{1, \dots, n\}$, we have

$$\begin{aligned} h((\mathfrak{L}^j\Phi)(K), u) &= \begin{bmatrix} n \\ 1 \end{bmatrix}^j \int_{\mathrm{A}(n,n-1)} \cdots \int_{\mathrm{A}(n,n-1)} h(\Phi(K \cap E_1 \cap \cdots \cap E_j), u)\, dE_1 \cdots dE_j \\ &= \frac{j!\kappa_j}{2^j} \begin{bmatrix} n \\ j \end{bmatrix} \int_{\mathrm{A}(n,n-j)} h(\Phi(K \cap E), u)\, dE \end{aligned}$$

due to (1.4). Putting $\Phi = J : K \mapsto K - s(K)$, we obtain

$$\mathfrak{L}^j J = \frac{j!\kappa_j}{2^j} \begin{bmatrix} n \\ j \end{bmatrix} M_{n-j}.$$

Consequently,

$$\mathfrak{L} M_{n-j+1} = \frac{2j\kappa_j \begin{bmatrix} n \\ j \end{bmatrix}}{\kappa_{j-1} \begin{bmatrix} n \\ j-1 \end{bmatrix}} M_{n-j}. \tag{5.2}$$

A crucial tool for describing the action of $\mathfrak{L}$ on generating functions will be the following theorem:

Theorem 5.5 ([22, Theorem 5.1])**.** *The map*

$$E_j : \mathbf{C}_o^\infty(\mathbb{S}) \to \mathbf{Val}_j^\infty : (E_j f)(K) := \langle f, S_j(K, \cdot)\rangle$$

is an $\mathrm{SO}(n)$*-equivariant embedding of Fréchet spaces, and extends by continuity in the weak* topologies to an* $\mathrm{SO}(n)$*-equivariant embedding*

$$\overline{E}_j : \mathbf{C}_o^{-\infty}(\mathbb{S}) \to \mathbf{Val}_j^{-\infty}.$$

Theorem 5.6 ([25, Theorem 6.3])**.** *Let* $j \in \{1, \dots, n-2\}$*. If* $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n),\infty}$ *has the generating function* $f \in \mathbf{C}_o(\mathbb{S}, \bar{e})$*, then* $\mathfrak{L}\Phi_j \in \mathbf{MVal}_{j+1}^{\mathrm{SO}(n),\infty}$ *and its generating function is given by* $q_{n,j}\square_{n-j+1} f * \breve{g}_{n-j}$*, where*

$$q_{n,j} := \frac{2j\kappa_j \left[{n \atop j}\right]}{\kappa_{j-1}\left[{n \atop j-1}\right]} \frac{m_{n,n-j}}{m_{n,n-j+1}}$$

with $m_{n,k}$ *as in Theorem 3.31.*

In particular, $\mathfrak{L} : \mathbf{MVal}_j^{\mathrm{SO}(n),\infty} \to \mathbf{MVal}_{j+1}^{\mathrm{SO}(n),\infty}$ *is injective.*

Proof. Denote by $\mathbf{Val}_j^{\infty,\mathrm{sph}}$ the image of E_j. Then $\mathbf{Val}_j^{\infty,\bar{e}} \subseteq \mathbf{Val}_j^{\infty,\mathrm{sph}}$ due to Theorem 4.19. In [22], a description of the image of E_j in the language of representation theory was given, that implies that $\mathfrak{L}$ maps the space $\mathbf{Val}_j^{\infty,\mathrm{sph}}$ into the space $\mathbf{Val}_{j+1}^{\infty,\mathrm{sph}}$. Hence, we may define

$$T_j := E_{j+1}^{-1} \circ \mathfrak{L} \circ E_j : \mathbf{C}_o^\infty(\mathbb{S}) \to \mathbf{C}_o^\infty(\mathbb{S}).$$

Then the diagram

$$\begin{array}{ccc}
\mathbf{MVal}_j^{\mathrm{SO}(n),\infty} & \xrightarrow{\mathfrak{L}} & \mathbf{MVal}_{j+1}^{\mathrm{SO}(n),\infty} \\
\downarrow & & \downarrow \\
\mathbf{Val}_j^{\infty,\mathrm{sph}} & \xrightarrow{\mathfrak{L}} & \mathbf{Val}_{j+1}^{\infty,\mathrm{sph}} \\
{\scriptstyle E_j}\uparrow{\scriptstyle\cong} & & {\scriptstyle E_{j+1}}\uparrow{\scriptstyle\cong} \\
\mathbf{C}_o^\infty(\mathbb{S}) & \xrightarrow{T_j} & \mathbf{C}_o^\infty(\mathbb{S})
\end{array}$$

commutes, where the downward arrows represent the map assigning to a smooth homogeneous Minkowski valuation its generating function. Hence it suffices to show that

$$T_j f = q_{n,j}\square_{n-j+1} f * \breve{g}_{n-j} \tag{5.3}$$

for all $f \in \mathbf{C}_o^\infty(\mathbb{S})$.

Denote by $\tilde{T}_j$ the operator defined by the right hand side in (5.3). Due to the properties of $\mathfrak{L}$, the operator T_j is an $\mathrm{SO}(n)$-equivariant continuous linear operator on $\mathbf{C}_o^\infty(\mathbb{S})$, as is the operator $\tilde{T}_j$. Consequently, both T_j and $\tilde{T}_j$ are multiplier transforms and both T_j and $\tilde{T}_j$ extend to $\mathrm{SO}(n)$-equivariant continuous linear operators on $\mathbf{C}_o^{-\infty}(\mathbb{S})$.

Therefore, it suffices to prove $T_j\psi = \tilde{T}_j\psi$ for a single distribution $\psi \in \mathbf{C}_o^{-\infty}(\mathbb{S})$ with condensed harmonic expansion $\psi \sim \sum_{k=0}^\infty h_k$, where all non-linear harmonics do not vanish. Two such distributions are given by

$$\psi := m_{n,n-j+1}\breve{g}_{n-j+1} \in \mathbf{C}_o^{-\infty}(\mathbb{S}) \quad \text{and} \quad \eta := m_{n,n-j}\breve{g}_{n-j} \in \mathbf{C}_o^{-\infty}(\mathbb{S}).$$

Due to Theorem 3.31, these two distributions ψ and η generate (up to a reflection at the origin) the mean section operators M_{n-j+1} and M_{n-j}, respectively. Thus, (5.2) and Theorem 5.5 imply that

$$(\overline{E}_{j+1}^{-1} \circ \mathfrak{L} \circ \overline{E}_j)\psi = \frac{2j\kappa_j \left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]}{\kappa_{j-1}\left[\begin{smallmatrix} n \\ j-1 \end{smallmatrix}\right]}\eta.$$

Clearly, the linear operator $\overline{E}_{j+1}^{-1} \circ \mathfrak{L} \circ \overline{E}_j$ is continuous on its domain with respect to the topology of $\mathbf{C}_o^{-\infty}(\mathbb{S})$ and coincides with T_j on $\mathbf{C}_o^\infty(\mathbb{S})$, hence by density and continuity we have

$$T_j(m_{n,n-j+1}\breve{g}_{n-j+1}) = \frac{2j\kappa_j \left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]}{\kappa_{j-1}\left[\begin{smallmatrix} n \\ j-1 \end{smallmatrix}\right]} m_{n,n-j}\breve{g}_{n-j}.$$

On the other hand,

$$\begin{aligned}\tilde{T}_j(m_{n,n-j+1}\breve{g}_{n-j+1}) &= q_{n,j}m_{n,n-j+1}\square_{n-j+1}\breve{g}_{n-j+1} * \breve{g}_{n-j} \\ &= q_{n,j}m_{n,n-j+1}\delta_{\bar{e}} * \breve{g}_{n-j} \\ &= q_{n,j}m_{n,n-j+1}\breve{g}_{n-j}.\end{aligned}$$

□

Like we did in Section 4.4, we will use this description of the action of $\mathfrak{L}$ on generating functions in order to establish a symmetry property analogous to Proposition 4.22. To that end, let us fix some notations.

Definition. For $j \in \{1, \dots, n-2\}$, we define

$$T_j : \mathbf{C}_o^\infty(\mathbb{S}) \to \mathbf{C}_o^\infty(\mathbb{S}) : T_j f := q_{n,j}\square_{n-j+1} f * \breve{g}_{n-j}.$$

For $i \in \{2, \dots, n-1\}$ and $j \in \{1, \dots, n-2\}$ with $i > j+1$, we define

$$T_{i,j} : \mathbf{C}_o^\infty(\mathbb{S}) \to \mathbf{C}_o^\infty(\mathbb{S}) : T_{i,j} f := T_{i-2}T_{i-1}\cdots T_j f = q_{n,i,j}\square_{n-j+1} f * \breve{g}_{n-i+2},$$

where $q_{n,i,j} := q_{n,i-2}q_{n,i-1}\cdots q_{n,j}$.

Note that T_j and $T_{i,j}$ are $\mathrm{SO}(n)$-equivariant automorphisms of the space $\mathbf{C}_o^\infty(\mathbb{S})$ that extend to $\mathrm{SO}(n)$-equivariant automorphisms of the space $\mathbf{C}_o^{-\infty}(\mathbb{S})$. Moreover, they are multiplier transforms.

With these notations in place, the diagram

$$\begin{array}{ccc} \mathbf{MVal}_j^{\mathrm{SO}(n),\infty} & \xrightarrow{\mathfrak{L}} & \mathbf{MVal}_{j+1}^{\mathrm{SO}(n),\infty} \\ \downarrow & & \downarrow \\ \mathbf{C}_o^\infty(\mathbb{S}) & \xrightarrow{T_j} & \mathbf{C}_o^\infty(\mathbb{S}) \end{array}$$

commutes, where the downward arrows represent the map assigning to a smooth homogeneous Minkowski valuation its generating function.

Since the operator $\mathfrak{L}$ is much more complicated than Λ (regarding its action on generating functions), the W_{n-j-1}-term in Proposition 4.22 must now be replaced by a more complicated object that somehow cancels out the operator $\mathfrak{L}$. This consideration motivates the following definition:

Definition. For $i \in \{2, \dots, n-1\}$ and $j \in \{1, \dots, n-2\}$ with $i > j+1$, we define the $\mathbf{Val}_1^{-\infty}$-valued valuation

$$\gamma_{i,j} : \mathcal{K}^n \to \mathbf{Val}_1^{-\infty} : L \mapsto \gamma_{i,j}(L, \cdot),$$
$$\gamma_{i,j}(L, \cdot) := \overline{E}_1 \left(\tfrac{1}{\omega_{n-1}} (T_{i,j}^{-1} S_j(L, \cdot)) * \breve{g}_n \right).$$

Observe that $\gamma_{i,j}$ is a continuous, translation invariant $\mathbf{Val}_1^{-\infty}$-valued valuation of order j and that $\gamma_{i,j}(L, \cdot)$ is a generalized valuation of order 1 for every $L \in \mathcal{K}^n$. Due to Theorem 5.5, we can evaluate $\gamma_{i,j}(L, \cdot)$ on a smooth convex body $C \in \mathcal{K}^n$ in the following way:

$$\begin{aligned} \gamma_{i,j}(L, C) &= \langle S_1(C, \cdot), \tfrac{1}{\omega_{n-1}} (T_{i,j}^{-1} S_j(L, \cdot)) * \breve{g}_n \rangle \\ &= \langle \tfrac{1}{\omega_{n-1}} S_1(C, \cdot) * \breve{g}_n, T_{i,j}^{-1} S_j(L, \cdot) \rangle \\ &= \langle h(C, \cdot), T_{i,j}^{-1} S_j(L, \cdot) \rangle \\ &= \langle T_{i,j}^{-1} h(C, \cdot), S_j(L, \cdot) \rangle. \end{aligned} \tag{5.4}$$

With these definitions and notations in place, the desired symmetry result can be obtained within a few lines:

Proposition 5.7 ([4, Theorem 3]). *Let $i \in \{1, \dots, n\}$, $j \in \{2, \dots, n-2\}$, and $\Phi_j \in \mathbf{MVal}_j^{\mathrm{SO}(n),\infty}$. If $i > j+1$, then*

$$\mathsf{W}_{n-i}(K, \Phi_j L) = \gamma_{i,j}(L, (\mathfrak{L}^{i-j-1} \Phi_j)(K))$$

for all $K, L \in \mathcal{K}^n$.

Proof. Since $\mathfrak{L}^{i-j-1}\Phi_j$ is a smooth Minkowski valuation, $(\mathfrak{L}^{i-j-1}\Phi_j)(K)$ is a smooth convex body. Hence, due to (5.4), the self-adjointness of $T_{i,j}$, and (1.7), we obtain

$$\begin{aligned}\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) &= \langle T_{i,j}^{-1}h((\mathfrak{L}^{i-j-1}\Phi_j)(K),\cdot),S_j(L,\cdot)\rangle\\ &= \langle T_{i,j}^{-1}(S_{i-1}(K,\cdot)*T_{i,j}f),S_j(L,\cdot)\rangle\\ &= \langle S_{i-1}(K,\cdot)*f,S_j(L,\cdot)\rangle\\ &= \langle S_j(L,\cdot)*f,S_{i-1}(K\cdot)\rangle\\ &= \langle h(\Phi_jL,\cdot),S_{i-1}(K,\cdot)\rangle\\ &= \mathsf{W}_{n-i}(K,\Phi_jL),\end{aligned}$$

where in the third equality we used the fact that $T_{i,j}$, $T_{i,j}^{-1}$, and convolution operators are multiplier transforms in order to get $T_{i,j}$ and $T_{i,j}^{-1}$ to cancel each other out. □

As a corollary of Proposition 4.22 and Proposition 5.7, we obtain:

Corollary 5.8. *Let* $i\in\{1,\dots,n\}$, $j\in\{2,\dots,n-1\}$, *and* $\Phi_j\in\mathbf{MVal}_{j,i-1}^{\mathrm{SO}(n),\infty}$. *If* $i>j+1$, *then*

$$\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K))=\frac{(i-1)!}{j!}\mathsf{W}_{n-j-1}(L,(\Lambda^{j-i+1}\Phi_j)(K))$$

for all $K,L\in\mathcal{K}^n$.

5.2 A Brief Outlook on Future Research

Now we come back to Question 1. Note that if we were able to adapt the proof of Theorem 4.7 to our situation based on the proposition above, then the result would cover all of the missing cases. In order to do so, we would need a Brunn-Minkowski type inequality on $\gamma_{i,j}$. This prompts the following question:

Question 2. For which $i\in\{1,\dots,n\}$ and $j\in\{2,\dots,n-2\}$ with $i>j+1$ is it true that

$$\gamma_{i,j}(K+L,C)^{1/j}\geq\gamma_{i,j}(K,C)^{1/j}+\gamma_{i,j}(L,C)^{1/j} \tag{5.5}$$

for all smooth convex bodies $K,L,C\in\mathcal{K}^n$?

Note that for fixed i and j, an positive answer to Question 2 implies an positive answer to Question 1. In order to outline how (5.5) could be shown, we introduce a mixed valuation associated with $\gamma_{i,j}$.

Definition. For $i\in\{2,\dots,n-1\}$ and $j\in\{1,\dots,n-2\}$ with $i>j+1$, we define

$$\begin{gathered}\overline{\gamma}_{i,j}:(\mathcal{K}^n)^{n-1}\to\mathbf{Val}_1^{-\infty},\\ \overline{\gamma}_{i,j}(K_1,\dots,K_{n-1};\cdot):=\overline{E}_1\left(\tfrac{1}{\omega_{n-1}}(T_{i,j}^{-1}S(K_1,\dots,K_{n-1};\cdot))*\breve{g}_n\right).\end{gathered}$$

Clearly, $\overline{\gamma}_{i,j}$ is a continuous, $\mathbf{Val}_1^{-\infty}$-valued valuation of order 1 in each of its arguments, invariant under independent translations of its arguments, and

$$\overline{\gamma}_{i,j}(K^{[j]}, B^{[n-j-1]}; \cdot) = \gamma_{i,j}(K, \cdot)$$

for all $K \in \mathcal{K}^n$.

Now suppose that the Alexandrov-Fenchel type inequality

$$\begin{aligned} &\overline{\gamma}_{i,j}(K, L, C_3, \dots, C_{n-1}; C)^2 \\ &\qquad \geq \overline{\gamma}_{i,j}(K, K, C_3, \dots, C_{n-1}; C)\overline{\gamma}_{i,j}(L, L, C_3, \dots, C_{n-1}; C) \end{aligned} \tag{5.6}$$

holds for all smooth convex bodies $K, L, C_3, \dots, C_{n-1}, C \in \mathcal{K}^n$. Then (5.5) could be easily deduced from (5.6) in a similar way the general Brunn-Minkowski inequality can be derived from the Alexandrov-Fenchel inequality (cf. [21, Theorem 7.4.5]). Thus, let us take a closer look at (5.6). Observe that for smooth convex bodies $K, L, C_3, \dots, C_{n-1}, C \in \mathcal{K}^n$, we have

$$\begin{aligned} &\overline{\gamma}_{i,j}(K, L, C_3, \dots, C_{n-1}; C) \\ &\quad = \langle T_{i,j}^{-1} h_C, S(K, L, C_3, \dots, C_{n-1}; \cdot)\rangle \\ &\quad = q_{n,i,j}^{-1} \int_{\mathbb{S}} \square_{n-i+2} h_C * \breve{g}_{n-j+1} \mathsf{D}(D^2 h_K, D^2 h_L, D^2 h_{C_3} \dots, D^2 h_{C_{n-1}})\, d\lambda_{\mathbb{S}}. \end{aligned}$$

Now fix $C_3, \dots, C_{n-1}, C \in \mathcal{K}^n$ and define

$$\beta(f, g) := \int_{\mathbb{S}} \square_{n-i+2} h_C * \breve{g}_{n-j+1} \mathsf{D}(D^2 f, D^2 g, D^2 h_{C_3} \dots, D^2 h_{C_{n-1}})\, d\lambda_{\mathbb{S}}$$

for $f, g \in \mathbf{C}^2(\mathbb{S})$. Then $\beta(\cdot, \cdot)$ is a symmetric bilinear form on $\mathbf{C}^2(\mathbb{S})$, and clearly (5.6) is equivalent to the *reverse Cauchy-Schwarz inequality*

$$\beta(f, g)^2 \geq \beta(f, f)\beta(g, g)$$

for $f, g \in \mathbf{C}^2(\mathbb{S})$. The reverse Cauchy-Schwarz inequality inequality in turn can be reformulated as a spectral property. In this way, the original problem of proving a general Brunn-Minkowski inequality for Minkowski valuations can be reduced to proving a general Brunn-Minkowski inequality for $\gamma_{i,j}$ which can then be reduced to a question of spectral theory.

The procedure we have described here is known as the *Bochner method*. Most recently, the Bochner method got refined by Y. Shenfeld and R. van Handel in [26], in order to give an elegant and concise proof of the classical Alexandrov-Fenchel inequality. The bilinear form for which they proved a reverse Cauchy-Schwarz inequality is given by

$$\alpha(f, g) := \int_{\mathbb{S}} h_C \mathsf{D}(D^2 f, D^2 g, D^2 h_{C_3} \dots, D^2 h_{C_{n-1}})\, d\lambda_{\mathbb{S}}$$

Due to the operators $\square_{n-i+2}$ and $. * \breve{g}_{n-j+1}$ involved in the definition of β, it seems like the situation described here is significantly more complex than that for α and that adapting Y. Shenfeld and R. van Handel's argument to β is highly non-trivial.

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